Packetized Wireless Communication Under Jamming:
A Game Theoretic Approach

A Dissertation Presented
by
Koorosh Firouzbakht
to
The Department of Electrical and Computer Engineering
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in
Electrical and Computer Engineering

Northeastern University
Boston, Massachusetts
August 2015
To my parents, Zohreh and Hamid.
# Contents

| List of Figures | p. v |
| List of Tables | p. vii |
| Acknowledgments | p. viii |
| Abstract of the Dissertation | p. ix |

## 1 Introduction  
1.1 Fundamentals of Game Theory  
1.2 Game Theory in Wireless Communications  
1.2.1 Spectrum Sharing in Cognitive Radio Networks  
1.2.2 Power Control in CDMA and OFDMA Networks  
1.2.3 Medium Access Control  
1.2.4 Physical Layer Jamming  
1.3 Related Work  
1.4 Motivation Behind Research  
1.5 Organization of the Dissertation  

## 2 Constrained Zero-sum Games  
2.1 System Model and Problem Statement  
2.1.1 The Channel Model  
2.1.2 The Transmitter Model  
2.1.3 The Jammer Model  
2.2 Constrained zero-sum Game Model  
2.2.1 Action Sets and Mixed-strategies  
2.2.2 The Utility Function and The Payoff Matrix  
2.3 Game Characteristics  
2.3.1 Existence of the NE and Connection to Linear Programming  
2.3.2 Existence of Jamming Power Threshold  
2.4 Game Analysis  
2.4.1 Optimal Mixed-Strategies for $J_{\text{ave}} < J_{\text{TH}}$  

---

ii
2.4.2 Optimal Mixed-Strategies for $J_{\text{ave}} \geq J_{\text{TH}}$ ........................................... 47
2.4.3 Optimal Mixed-Strategies for ......................................................... 47
2.5 Packetized Wireless AWGN Link Under Power Limited Jamming .......... 48
  2.5.1 Rate Adaptation in Packetized Links ............................................ 48
  2.5.2 Uniformly Distributed Jamming Powers ....................................... 51
2.6 Concluding Remarks ................................................................. 56

3 Constrained Bimatrix Games .................................................. 57
  3.1 General-sum Games ................................................................. 58
    3.1.1 Standard Bimatrix Games ...................................................... 59
    3.1.2 Constrained Bimatrix Games ................................................ 60
  3.2 Connection to Quadratic Programing ......................................... 64
  3.3 Packetized AWGN under Power Limited Jamming ............................ 70
  3.4 Concluding Remarks ................................................................. 77

4 Adaptive Multilayer Superposition Coding Technique .................. 78
  4.1 Superposition Coding in Broadcast Channels .................................. 79
  4.2 Adaptive multilayer Superposition Coding ..................................... 81
    4.2.1 Channel Model ................................................................. 82
    4.2.2 Jammer Model and Strategy Set ............................................ 82
    4.2.3 Transmitter Model and Strategy Set ...................................... 83
    4.2.4 Utility Function and Expected Payoff ..................................... 85
  4.3 Performance Analysis of the Adaptive Superposition Coding Technique . 87
    4.3.1 Rate Adaptation Technique .................................................. 87
    4.3.2 Superposition Coding Technique .......................................... 88
  4.4 Numerical Results ................................................................. 92
  4.5 Concluding Remarks ................................................................. 93

5 Adaptive OFDM Systems Under Jamming ................................ 95
  5.1 Introduction ................................................................. 95
  5.2 System Model ................................................................. 96
  5.3 Transmitter Model .............................................................. 97
  5.4 Jammer Model ................................................................. 101
  5.5 Average Throughput of the Adaptive OFDM ................................ 104
  5.6 Performance Analysis of IEEE 802.11 OFDM ................................ 107
  5.7 Concluding Remarks .............................................................. 109

6 Conclusion ................................................................. 110

A An Upper Bound for the Jamming Threshold .................................. 112

B Existence of the Nash Equilibrium ........................................... 116
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>A simple jamming game.</td>
<td>5</td>
</tr>
<tr>
<td>1.2</td>
<td>A layered overview of game theory applications in wireless communications.</td>
<td>7</td>
</tr>
<tr>
<td>1.3</td>
<td>Attacks related to physical layer jamming.</td>
<td>14</td>
</tr>
<tr>
<td>1.4</td>
<td>Relay channel under jamming.</td>
<td>16</td>
</tr>
<tr>
<td>1.5</td>
<td>Classification of the constrained and standard two-player games.</td>
<td>18</td>
</tr>
<tr>
<td>1.6</td>
<td>Characteristics of constrained games.</td>
<td>20</td>
</tr>
<tr>
<td>2.1</td>
<td>Packetized wireless communication link under jamming.</td>
<td>25</td>
</tr>
<tr>
<td>2.2</td>
<td>Transmitter model.</td>
<td>26</td>
</tr>
<tr>
<td>2.3</td>
<td>Jammer model.</td>
<td>28</td>
</tr>
<tr>
<td>2.4</td>
<td>One to one relation between transmission rates and jamming powers.</td>
<td>30</td>
</tr>
<tr>
<td>2.5</td>
<td>Dominated strategies vs. non-dominated strategies.</td>
<td>33</td>
</tr>
<tr>
<td>2.6</td>
<td>Connection between the constrained zero-sum games and linear programming.</td>
<td>36</td>
</tr>
<tr>
<td>2.7</td>
<td>Jammer’s randomization gain and average power.</td>
<td>52</td>
</tr>
<tr>
<td>2.8</td>
<td>Average throughput of the AWGN link under jamming.</td>
<td>55</td>
</tr>
<tr>
<td>3.1</td>
<td>System model.</td>
<td>70</td>
</tr>
<tr>
<td>3.2</td>
<td>Average throughout at the NE as a function of $J_{\text{ave}}$.</td>
<td>74</td>
</tr>
<tr>
<td>3.3</td>
<td>Average destroyed packets at the NE as a function of $J_{\text{ave}}$.</td>
<td>76</td>
</tr>
<tr>
<td>4.1</td>
<td>BSC broadcast channels.</td>
<td>80</td>
</tr>
<tr>
<td>4.2</td>
<td>Codeword space (left). Achievable rates for BSC broadcast channel (right).</td>
<td>81</td>
</tr>
<tr>
<td>4.3</td>
<td>System model.</td>
<td>82</td>
</tr>
<tr>
<td>4.4</td>
<td>Adaptive multilayer superposition coding scheme.</td>
<td>84</td>
</tr>
<tr>
<td>4.5</td>
<td>Typical Jamming Power Threshold as a Function of $\Delta \alpha$, $N_J \gg 1$.</td>
<td>88</td>
</tr>
<tr>
<td>4.6</td>
<td>Achievable Transmission Rates for multilayer Superposition Coding.</td>
<td>92</td>
</tr>
<tr>
<td>4.7</td>
<td>Performance Gain Ratio multilayer Superposition Coding Technique.</td>
<td>93</td>
</tr>
<tr>
<td>5.1</td>
<td>The transmitter model.</td>
<td>98</td>
</tr>
<tr>
<td>5.2</td>
<td>The transmitter’s adaptation block.</td>
<td>101</td>
</tr>
<tr>
<td>5.3</td>
<td>The jammer model.</td>
<td>102</td>
</tr>
<tr>
<td>5.4</td>
<td>The jammer’s adaptation block.</td>
<td>103</td>
</tr>
</tbody>
</table>
5.5 Adaptive OFDM under jamming. ........................................ 104
B.1 Single-valued function (left) vs. set-valued function (right). ......... 117
B.2 Geometric representation of upper semi-continuity. .................. 119
List of Tables

2.1 Summary of the Results .................................................. 47

3.1 Necessary and Sufficient Conditions for a Pair of strategies to be the NE. . . . 64

5.1 IEEE 801.11 modulation schemes and code rates. . . . . . . . . . . . . . . . . 108
Acknowledgments

I would like to express my sincerest gratitude to my advisors, Professor Masoud Salehi and Professor Guevara Noubir for their wonderful guidance and mentorship throughout this work and my education at Northeastern University. This entire process would have been impossible without their support and guidance throughout my years in Electrical and Computer Engineering Department. I am grateful for their ideas, encouragements, and flexibility.

I also wish to thank my dissertation committee member, Professor Kuashik Chowdhury. His feedback and support during my research have greatly improved the quality of this dissertation.

I also thank all my friends, colleagues and professors of the Communication and Digital Signal Processing Center for making these years such memorable ones.

Going through PhD years would have not been possible if not for unconditional love that I have received from my family and great friends, especially my parents, my brother Dariush and my sister Nazanin. I am thankful to all my friends in Boston for making this beautiful city my second home, and all the memories that I will cherish for the rest of my life.
Abstract of the Dissertation

Packetized Wireless Communication Under Jamming:
A Game Theoretic Approach

by

Koorosh Firouzbakht

Doctor of Philosophy in Electrical and Computer Engineering
Northeastern University, August 2015

Dr. Masoud Salehi, Adviser and Dr. Guevara Noubir, Advisor

Game theory has been widely used to study many specific jamming and interactive communication problems in wireless networks. However, these theoretical studies are only applicable to very specific problems and the study is often based on a particular features or weaknesses of the communication system under study. These features and weaknesses are in most cases unrelated to the physical layer of the protocol stack. Consequently, these studies fail to accurately model more general problems or even the same problem with minor changes in the original assumptions. This limitation demands development of a more general theoretical frameworks that can be applicable to a wide range of jamming scenarios.

In this dissertation, we develop two general game-theoretic frameworks, constrained zero-sum and constrained bimatrix, that can be used to model many interactive communication scenarios in wireless networks when physical layer jamming is present. In constrained games, players’ strategies are limited to a subset of all possible strategies and as a result, a broader class of problems can be modeled by using these frameworks.

Furthermore, we formulate the interactions between adaptive communicating nodes and smart power limited adversaries by constrained zero-sum and constrained bimatrix games and provide necessary and sufficient conditions under which existence of the Nash equilibrium solutions for these non-typical games are guaranteed.

We show that constrained zero-sum games and linear programming have deep connections. For every constrained zero-sum game there exists a linear program whose solution
yields a Nash equilibrium and every equilibrium solution of the zero-sum game is a solution of the linear program. Similarly, we show that there exists a similar relationship between the Nash equilibrium solutions of constrained bimatrix games and global solutions of a quadratic program. Every NE solution of the bimatrix game is a global maximizer of a quadratic program and every global maximum of the quadratic program yields a NE solution.

We use our constrained frameworks to study some typical jamming problems in packetized networks where we derive analytical as well as numerical results for the optimal strategies and closed form expressions for the expected value of the game at the the Nash equilibrium. Our analytical results suggest that a strategic jammer that uses optimal jamming strategies can significantly degrade the performance of packetized networks compared to non-strategic jammers. Furthermore, we prove that there exists a certain threshold on jammer’s average power, $J_{TH}$, such that if the jammer’s average power exceeds $J_{TH}$, the game value at the Nash equilibrium is the same as the case when the jammer uses his maximum power all the time. Additionally, we study the performance of an adaptive OFDM wireless communication system under power limited jamming and we show that with modest assumptions, this problem can be formulated into the constrained zero-sum or constrained bimatrix frameworks also.

Finally, inspired by the superposition coding technique used in broadcast channels, we propose an adaptive multilayer superposition coding technique to improve the performance of packetized networks. Our analytical results shows that superposition coding not only achieves better average performance under jamming, but also increases the jamming threshold.
Chapter 1

Introduction

Over the last decades, wireless communication has been established as an enabling technology to an increasingly large number of applications. The convenience of wireless and its support of mobility has revolutionized the way we access information services and interact with the physical world. Beyond enabling mobile devices to access information and data services ubiquitously, wireless technology is widely used in cyber-physical systems such as air-traffic control, power plants synchronization, transportation systems, navigation systems and human body implantable devices. This pervasiveness has elevated wireless communication systems to the level of critical infrastructure. Nevertheless, security issues of wireless communications remain a serious concern.

Radio-frequency wireless communications occur over a broadcast medium that is shared by the wireless spectrum users. Shared nature of the wireless medium necessitates that the communicating nodes interact with each other in a competitive and/or collaborative manner to access the network’s shared spectrum and resources. These interactions are often regulated by the wireless system designers or other regulatory agencies to optimize individual performances of the communicating nodes and/or overall performance of the entire network.

Moreover, because of the broadcast nature of the wireless medium, the physical layer in wireless networks is exposed to adversaries such as eavesdroppers and jammers. Unlike
 legitimate users, these malicious users pursue deliberate actions to degrade or compromise the performance, reliability, and/or security of the wireless network. Among the many security threats that the wireless networks are subject to, physical layer jamming is one of the most prominent and challenging threats. Jamming at the physical layer not only can lead to service interruption or denial of service attacks, but is often a prelude to other attacks such as spoofing, man-in-the-middle attacks, etc. Thus, it is critical to study the effect of jamming on wireless networks and determine the long-term achievable performance and the optimal strategy to achieve it.

Furthermore, interactions between legitimate and malicious users and its impact on the wireless network performance has many characteristics that would lead to a natural game-theoretic formulation as these problems cannot be well modeled by the traditional optimization tools. Not surprisingly, there are numerous works in wireless communications that use game theory to study many jamming problems and jamming scenarios. However, majority of these theoretical studies are only applicable to very specific problems and fail to accurately model more general jamming problems. In this dissertation, our aim is to develop a general game-theoretic framework for a packetized wireless communication system under power limited jamming that can be used to model a wide range of jamming scenarios.

The rest of this chapter is organized as follows: we will first briefly review fundamentals of game theory and introduce some of the key concepts and definitions that we use throughout this thesis. We will then review some of the applications of game theory in wireless networks. We limit our focus to the physical layer applications or problems that are directly related to the physical layer. In Section 1.3, we provide a brief overview of the related work and the motivation behind our work and our main contributions. Finally, we conclude the chapter by presenting the structure of the thesis.

1.1 Fundamentals of Game Theory

In this section, we give a brief review of some important definitions and concepts of the game theory, focusing on its applications to wireless communications.

Game theory is a set of mathematical and modeling tools which provide a basis for
CHAPTER 1. INTRODUCTION

“Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact.”

Martin Osborne and Ariel Rubinstein [3].

analysis of the interactions between rational decision makers. It gives a systematic way of predicting what might (or what should) happen when different players with possibly conflicting interests (goals) interact. Reference [1] provides a comprehensive review of the game theory. This book and the references therein present fundamental results of game theory and its applications in wireless communications and networking. In [2] the authors present the most fundamental concepts of non-cooperative game theory. This short tutorial is specifically written for wireless network engineers and uses examples that are focused on wireless networks.

Definitions of some basic notions in game theory and related notation are given below.

- An \( N \)-player game is defined by the triplet \( \mathcal{G} = (\mathcal{P}, A_i, U_i), i = 1, \ldots, N \); where \( \mathcal{P} \) is the set of players of the game. \( A_i \) and \( U_i, i = 1, \ldots, N \) are the set of available actions/pure strategies and the set of preference relations/utility functions for the players, respectively; as discussed below.

- Strategic game (also normal form) is a game model which all players act simultaneously without prior knowledge of other players’ actions and the outcome of the game is revealed after the game is played. Matching pennies and rock-paper-scissors are examples of strategic games.

- The players in the game theoretic context are the decision makers. In wireless communications the players could be nodes of the network such as transmitters, receivers and the intermediate relays in between. In general, it is assumed that players are rational that is, they always act in a way that maximizes their utility.

\[1\] That is, the decision makers always act in a way that maximizes their objectives.

\[2\] A game must involve at least two decision makers, this is unlike traditional optimization problems which involve only one decision maker.
• **The actions or pure-strategies** are the possible options (alternatives) available to the players; such as different modulation schemes, coding rates, transmission powers, etc. The set of all available pure-strategies of a player is called his *pure-strategy set* or action set ($A_i$). The *strategy profile* is the set of pure-strategies that the players choose and it determines the *outcome* of the game and the *payoffs* to the players.

• **The preference relation** represents the evaluation of the game outcome by the players. More specifically, it determines the more preferable outcomes of the game for each player. The preference relation is commonly represented by a *utility function* which assigns a real number to each outcome of the game. Examples of more preferable outcomes in wireless communications are: higher signal to noise/interference, higher transmission rate, higher throughput, lower bit error rate, etc.

• **The mixed-strategy** is an alternative way of playing the game. A player is said to be playing a *mixed-strategy* if he randomizes his actions over his strategy set. In other words, a mixed-strategy is a *probability distribution* over the pure-strategies and results in an *expected payoff*.

• **Non-cooperative games** is a class of games where collaboration between the players is not allowed. In non-cooperative games, players selfishly pursues their own interest which completely or partially conflicts with other players’ interests. Non-cooperative games are often used to model situations where players compete with each other to access shared resources. Such frameworks are the natural way of modeling a communication systems in the presence of an adversary.

• **Best response correspondence** of a player is a set-valued function which maps every strategy profile to a set of strategies that are optimal against other players’ strategies. In other words, the best response correspondence returns strategies that are best reply against other players’ strategies.

• **Nash equilibrium (NE)** is a well-known concept in game theory. A strategy profile is said to be *Nash equilibrium (NE)* when none of the players can unilaterally increase
CHAPTER 1. INTRODUCTION

their payoffs by changing their strategies. Alternatively, NE is a strategy profile that is the mutual best response strategy by the players.

- **Zero-sum game** is the extreme case of non-cooperative games. A game is said to be a zero-sum game if and only if for every possible strategy profile, sum of the payoffs to the players is zero. A zero-sum game is used to represent a closed system, that is, if a player gains a payoff, that payoff must have been lost by someone else. The zero-sum game is used to model situations where the players are strictly competing for resources or when they have extremely conflicting goals.

- **General-sum game** is an extension to zero-sum games. In general-sum games, it is no longer required that the sum of players’ payoffs to be zero. In general-sum games a gain by one player does not necessarily correspond to a loss by another player. A two-player general-sum game, in its normal form, is commonly referred to as a bimatrix game.

The following example shows how a simple jamming scenario can be modeled as a game and how one can make an appropriate connection between an outcome of the game and a practical problem.

**Example:** (A simple jamming game) Figure 1.1 is an example of a simple game. A transmitter and a receiver communicate over a wireless link that is subject to jamming. The wireless
CHAPTER 1. INTRODUCTION

Medium is divided into two non-overlapping channels. In each time slot, the transmitter chooses one of the available channels and sends a packet to the receiver. In the same manner, in each time slot, the jammer chooses a channel to jam. In this simple game, both players (the transmitter and the jammer) have the same action set, $A_x = A_y = \{1, 2\}_{\text{channel}}$. If the players (i.e., the transmitter and the jammer) choose the same channel, the packet is jammed and the jammer receives a payoff of (+1) and the transmitter receive (-1). If they choose different channels, the packet is successfully transmitted and the payoffs are (+1) and (-1) for the transmitter and the jammer, respectively. The optimal transmission/jamming strategies in this scenario is randomly choose a channel with probability 0.5 which results in successful transmission of half of the packets (on average). This well known frequency hopping technique corresponds to the NE solution of a zero-sum game shown in Figure 1.1. This example shows how game theory can be used to model a communication problem with interacting nodes.

1.2 Game Theory in Wireless Communications

Many modern wireless networks such as sensor, ad hoc and mesh networks often operate in a decentralized, self-configurable fashion. Network nodes are governed by a distributed protocol which allows the nodes to choose an action, i.e., make a decision, from a set of available actions based on their evaluation of the network conditions (possibly relying on the information provided by the other nodes). These decisions not only have impact on the performance of individual nodes but may also have impact on the overall performance of the entire network. Nodes may seek the greater good of the network that is, they seek actions that optimize the overall performance of the entire network, or they can act selfishly and compete with other nodes to optimize their individual performances. Additionally, nodes may act maliciously, i.e, seek actions that result in performance degradation of the individual nodes or the entire network.

All these examples have many of the characteristics that would lead to a natural game theoretic formulation as these problems cannot be completely modeled by the traditional optimization tools. Moreover, as Software Defined Radios (SDRs) and Cognitive Radios
CHAPTER 1. INTRODUCTION

Figure 1.2: A layered overview of game theory applications in wireless communications.

(CR) have become capable of more sophisticated and complicated adaptation algorithms, employing game theoretic models has become an even better match for future wireless networks.

Game theory has been used to solve problems in numerous aspects of wired and wireless communication systems — from security at the physical and MAC layers (e.g., jamming and eavesdropping) to routing and intrusion detection systems (e.g., collaborative IDS’s) at upper layers of the protocol stack. Figure 1.2 shows an overview of some of the problems and challenges in wireless networks that have been studied using game-theoretic approaches. Topics in Figure 1.2 are organized according to the protocol layer to which they are related. Reference [4], and the references therein, provide a structured and comprehensive overview of the game-theoretic approach to security and privacy in computer and communications networks. In [5], the authors use a layered approach to survey applications of game theory in wireless networking and game models that are most suitable for each problem.

Reference [6] presents a classification of applications of game theory in network security based on the game model that is used to approach the problem. The survey covers both cooperative and non-cooperative games (see Table I in [6]). In what follows, we briefly review some of the applications of game theory in wireless communications. We limit our focus to the physical layer or applications that are directly related to the physical layer. For applications of game theory in other network layers we refer the reader to [4, 5].
CHAPTER 1. INTRODUCTION

1.2.1 Spectrum Sharing in Cognitive Radio Networks

The growing demand for wireless spectrum has made the traditional fixed spectrum allocation techniques (often done by the FCC or other regulatory agencies) quite inefficient. To address this shortcoming, new spectrum allocation techniques have been proposed such as Dynamic Spectrum Access (DSA) which has become a promising approach due to the development of fast Software Defined Radios (SDR) and more powerful/capable Cognitive Radios (CR).

In cognitive radio networks, the wireless spectrum is shared between primary users (users that are licensed to operate in that spectrum band) and secondary users (unlicensed users), where the secondary users access the spectrum in an opportunistic manner. Dynamic spectrum access and spectrum sharing address the issue of how to allocate the limited available spectrum, offered by the primary users, among multiple wireless devices. This problem has two important aspects, spectrum usage efficiency and fairness to wireless users and has been modeled both as a cooperative and a non-cooperative game.

The cooperative model is often used where cooperation between the spectrum users exists and/or is allowed and the resulting equilibrium of the game is more efficient and fair [7]. Bargaining games and coalition games are two types of cooperative games that have been studied in the literature to study DSA in cognitive radio networks. In Bargaining games individual nodes have the opportunity to reach a mutually beneficial agreement with other nodes but no agreement may be imposed on any nodes without its approval. For example, reference [8] considers a case where primary users and a base station coexists with a network of secondary users in the same frequency band. The authors develop a resource allocation method in which the primary users require a minimum level of Quality of Service (QoS). Primary and secondary users inform the base station of their Channel State Information (CSI), QoS requirements, and power limitations for resource allocation for each time slot. The base station is responsible for computing transmission powers based on user priority and channel conditions for all players. This paper considers a cooperative bargaining game model where the base station acts as a marketplace in which the primary and secondary users bargain for the common resources. The solution of this bargaining game is called the Nash Bargaining Solution (NSB) and it has been shown that NSB can achieve a spectrum
CHAPTER 1. INTRODUCTION

allocation that is fair and efficient.

In coalition games, a set of players can cooperate with others by forming cooperating
groups and thus improve their payoff in a game. In [9] the authors consider the problem of
spectrum sensing in a decentralized network with \( N \) secondary users and a single primary
user. The metric that is used in this paper is the *missing probability* for secondary users which
is defined as the probability of mis-detecting the presence of the primary user. To minimize
the missing probability, the secondary users interact with each other to form collaborative
coalitions where a selected node within each coalition (coalition head) is responsible to make
the final decision on the presence or absence of the primary user based on the reports from
the coalition members. A coalition formation algorithm is also proposed in this paper where
the secondary users can merge or split coalitions to improve their detection probabilities.
The authors’ simulation results suggest that the proposed algorithm significantly reduces the
average missing probability per secondary user compared to non-cooperative models for the
same false alarm levels.

Even though cooperation among rational users in CR networks can generally improve
the overall network performance, when collaboration between network nodes is not possible
or permitted, such as, scenarios where secondary users compete with each other for channel
or spectrum access to improve their individual performances, non-cooperative frameworks
such as *auction based* games can better model the DSA problem [10]. In *auction based*
games, the primary users lease their unused spectrum to secondary users through auction
mechanisms possibly for monetary or other gains [11] [12] [13].

1.2.2 Power Control in CDMA and OFDMA Networks

Power control in CDMA networks is another example that has been studied by game
theoretic approaches [5]. In the CDMA power control problem, a player’s utility function is
usually defined such that it increases with the signal to interference plus noise ratio (SINR)
but decreases with transmission power [14]. This assumption is well justified since an
increase in the SINR results in lower error probability and hence would increase the quality
of service. Assuming all network nodes operate at fixed transmission power, an increase in
a node’s transmission power, results in higher SINR for that node. However, this increase
CHAPTER 1. INTRODUCTION

in the transmission power results in lower SINR for other users. Because most wireless nodes are battery-operated, energy management is an important consideration and if a node’s power is too high, not only it reduces other nodes’ SINR but it also wastes valuable battery life.

In [15], the CDMA power control problem is modeled as a non-cooperative game in which users selfishly decide how much power to transmit over each carrier such that their respective utilities are maximized. The utility function that is considered in this paper is defined as the number of reliable bits transmitted over all the carriers per the amount of energy consumed, measured in bits/joules. This type of utility function is particularly appropriate for networks where energy efficiency is important. This work also proposes an iterative and distributed algorithm to achieve the Nash equilibrium solution of the game (when it exists) and numerical simulations suggest that the proposed approach results in a significant improvement in the total network utility achieved at equilibrium compared to a multi-carrier system where users maximize their utilities over each carrier independently. This work and similar works (e.g., [16]) suggest that game theoretic power control strategies can achieve significant individual and/or overall performance improvements over traditional power control algorithms.

Reference [17] studies the problem of interference avoidance in a general orthogonal signaling (e.g., in time, frequency or spreading code) multiple access scheme. This paper considers a network with multiple co-located transmitters and receivers, where the interference caused to the transmission of a particular node due to other user transmissions is influenced by the correlation between the waveforms of the transmitters, transmit power levels and the channel characteristics. This work formulates the problem of interference avoidance as a potential game where at each stage of the game, players choose actions that improve their utility functions in a round-robin fashion. A better response and best response algorithms are proposed and convergence of these algorithms to the Nash equilibrium solution of the game is investigated.

Game theory has also been used to study power control in Orthogonal Frequency Division Multiple Access (OFDMA) networks. In OFDMA networks, the objective is to minimize the overall transmission power under rate and power constraints by allocating users’ rates and powers to the available sub-channels. This problem have been studied in the
CHAPTER 1. INTRODUCTION

literature in non-cooperative [18, 19] and cooperative frameworks [20]. In non-cooperative framework network nodes use local and selfish power control strategies to maximize their individual performances while in the cooperative framework, network nodes use distributed and (possibly) selfless power control strategies to optimize the overall performance and fairness of the system.

Reference [18] uses a distributive non-cooperative game to perform sub-channel assignment, adaptive modulation and power control for multi-cell and multi-user OFDMA networks. This paper considers an OFDMA network with multiple co-channel links that exist in distinct cells which share the same spectrum. The co-channel links cause interference among each other. Each link consists of a mobile user and its assigned base station and the objective of the game is to minimize the overall transmitted power under the rate and power constraints, by adjusting the rate allocation over different sub-channels for different users. To achieve this goal a virtual referee is introduced to regulate the competition for the resource usage. Numerical results suggest that compared to the iterative water-filling methods, the proposed scheme significantly reduces the overall transmission power of the network.

1.2.3 Medium Access Control

The medium access control problem arises when multiple network users attempt to access a shared communication medium. When a wireless network operates in the infrastructure mode, a base station (or access point) is often responsible for coordinating the transmission of multiple wireless users with which it is associated. However, lack of a base station in the decentralized and ad hoc wireless networks makes the medium access control problem more challenging.

Game theoretic approaches have been used extensively to study random access as this problem (many users competing with each other to access a shared medium), inherently has all the necessary elements of a game. References [21] and [22] study the medium access problem in a wireless LAN with multiple wireless nodes as random access game. At each stage of the game, the wireless nodes estimate their conditional collision probability and update their channel access probabilities (strategies) in a better response fashion (i.e.,
gradient play) to increase their respective payoffs (utility gain from channel access) or decrease their respective losses (utility cost from collisions). Simulation results suggest that game model based random access protocols can achieve significant performance gain over the standard IEEE 802.11 DCF as well as achieving better system-wide fairness and quality of service.

One of the earliest game theoretic studies in this category was characterizing the performance of slotted Aloha in the presence of selfish users [23]. In slotted Aloha, the actions that are available during each time slot are to transmit or to wait. Network users typically wish to transmit their information as soon as possible however, if multiple users transmit simultaneously all the transmissions fail due to collision. It has been shown that if a failed transmission is costly to users then, the optimal strategy for a rational selfish user is not to transmit during every time slot [24, 25].

1.2.4 Physical Layer Jamming

Among the many security threats that the wireless networks are subject to, jamming at the physical layer is one of the most prominent and challenging threats and anti-jamming has been an active area of research for more than a century. The first experiments with intentional radio jamming date to the British naval exercises in the Mediterranean in 1902, less than five years after the first shipboard wireless communication system was used [26]. But the first use of radio jamming in a combat situation took place during the Russo-Japanese War of 1904–1905 where Russian wireless telegraphy stations successfully interrupted wireless communication between a group of Japanese battleships [27]. Radio jamming was also common during World War I and II. One of the effective methods used in World War II to jam the radar signals was to drop thousands of aluminum tinsel shreds from fighter or bomber aircrafts. The first selective jamming took place in December 1944 during the battle of Bulge where American B-24 bombers used high-powered radio jamming transmitters to jam the German AM communications while keeping the American FM communication uninterrupted. By the end of the war most countries involved in the war had developed extensive jamming equipment to jam wireless communications and radar systems [26].

Various techniques for combating jamming have been developed at the physical layer
CHAPTER 1. INTRODUCTION

which include directional antennas, spread spectrum communication and power, modulation
or coding control. However, at the time, most of wireless systems were neither packetized
nor networked and therefore, reliable communication in the presence of adversaries has
regained significant interest in the last few years, as new jamming attacks as well as
need for more complex applications and deployment environments have emerged. Several
specifically crafted attacks and counter-attacks have been proposed for packetized wireless
data networks [28, 29, 30, 31], multiple access resolution [32, 33, 34, 35], multi-hop
networks [36, 30], broadcast and control communication [37, 38, 39, 40, 41, 42, 43], cross-
layer resiliency [44], wireless sensor networks [45, 46], spread-spectrum without shared
secrets [47, 48, 49], and navigation information broadcast systems [50].

Nevertheless, little work has been done on protecting adaptation algorithms, specifically,
Rate Adaptation Algorithms (RAA) against adversarial attacks. Rate adaptation plays
an important role in widely used wireless communication systems such as IEEE 802.11
standard as the link quality in a WLAN is often highly dynamic. In recent years, a number
of algorithms for rate adaptation have been proposed in the literature [51, 52, 53, 54, 55,
56, 57, 58], and some are widely deployed [59, 60]. Rate adaptation for the widely used
IEEE 802.11 protocol was investigated in [61, 62]. Experimental and theoretical analysis of
optimal jamming strategies against currently deployed rate adaptation algorithms indicate
that the performance of IEEE 802.11 can be significantly degraded with very few interfering
pulses.

For example, in [62] it is shown that for the “SampleRate” rate adaptation algorithm,
evite reactive jamming pulses every second are sufficient to achieve the same network
throughput degradation achieved by a periodic jammer with the jamming energy cost 100
times higher. Some other RAA algorithms react even worse to smart jamming attacks;
the “ONOE” algorithm in particular suffers from a phenomenon known as the congestion
collapse where the nodes fail to recover from the lowest data rate even after the jammer
stops jamming. Furthermore, the commoditization of software defined radios makes these
attacks even more practical and calls for investigation of the performance of packetized
communication under adaptive jamming.

Jamming represents a real and serious threat to wireless networks as jamming at the
physical layer not only can lead to service interruption or denial of service, but it is often
a prelude to other attacks such as spoofing, man in the middle attacks, downgrade attacks, etc. \cite{63,64,65,66,67} (see Figure 1.3). In a recent case, investigations by FCC showed that Wi-Fi jammers were used to increase the revenue of the Marriott hotels. A Marriott hotel in Tennessee used Wi-Fi jammers to “intentionally interfered with and disabled Wi-Fi” hotspot networks to “prevent consumers from connecting to the Internet via their own personal Wi-Fi networks”, forcing them to use hotel’s expensive internet service \cite{68}. In addition, a sequence of highly mediatized incidents since 2012, impacting cellular, GPS, and Wi-Fi networks, resulted in the FCC releasing an urgent customer advisory cautioning against use of jamming devices. Since then, the FCC has stepped up its education and enforcement effort, rolled out a new jammer tip-line and issued several fines. These incidents resulted in denial of service on air traffic control systems, Wi-Fi and Cellular coverage in public places and hotels, and interception of users communications through rogue base stations (see \cite{69,70}).

Game theory has been used extensively to study/model all aspects of jamming in wireless networks — from designing game-theoretic anti-jamming adaptation algorithms to combat jamming and designing smart jamming techniques to game-theoretic frameworks to model specific jamming problems. In the following sections we briefly review some related works that use game theory to study jamming in wireless communications. We will also discuss the motivation behind our work in more detail.
1.3 Related Work

Jamming at the physical layer is often modeled as a zero-sum game either with complete information \[71\] \[72\] \[73\] \[74\] or incomplete information \[75\] \[76\]. Zero-sum games are a special class of non-cooperative games. In a zero-sum game, for all strategy profiles the sum of players’ payoffs is zero and as a result, if a player gains a payoff, that payoff must have been lost by other player(s). Thus, the zero-sum framework is often used to model situations where players have extremely conflicting goals — jamming in wireless communication is one such case, where the zero-sum game framework can fully capture the conflicting goals of the players in this setting. Moreover, the equilibrium solution of the zero-sum game is in fact the worst case scenario solution, which guarantees a minimum payoff regardless of the other player’s strategy \[77\].

In \[71\] a game theoretic approach is used to study the efficiency of Frequency Hopping (FH) in wireless networks. In this works the authors study a case where both players, the wireless link and the jammer, employ frequency hopping in order to achieve their objectives. While the wireless link switches between bands to avoid jamming, the jammer hops across bands to degrade the performance of the wireless link. This paper models the interactions between the wireless link and the jammer as a two-player zero-sum game where both players have the same set of strategies (the set of available bands) and the objective of the wireless link is to maximize its average throughput. The experimental results of this work shows that frequency hopping is quite inadequate for protecting 802.11 networks against jamming.

Reference \[78\] considers a one-way time-slotted packet radio communication link (packet-switched) in the presence of a jammer. This work assumes transmission at a fixed data rate but random transmission power such that, in each time slot, the transmitter and jammer choose their respective transmission and jamming power levels in a random fashion from zero to some positive value. Additionally, both players are subject to temporal energy constraints. This paper uses a dynamic zero-sum game to formulate this problem where the transmitter’s objective is defined as a function of the players’ activities during a time slot (i.e., whether or not either of the players are transmitting). This work suggest that under certain operating conditions, this dynamic jamming game admits steady-state pure or mixed optimal strategies. However, the main result of this paper is rather trivial; a player that uses
CHAPTER 1. INTRODUCTION

Figure 1.4: Relay channel under jamming.

more power wins the game.

Zero-sum games have also been used to develop anti-jamming techniques, for example [75] propose an anti-jamming game for dynamic spectrum access in a cognitive radio network subject to jamming. The jammer frequently injects packets to the shared spectrum to prevent the secondary users utilize the spectrum opportunities efficiently. The jammer’s objective is to cause the maximum damage to the secondary network with his limited capabilities. This work uses a stochastic zero-sum framework where the game is played in a sequence of stages. At the beginning of each stage the game is in some state. After the players select and execute their actions, the game then moves to a new random state (based on the channel quality). The secondary users know the spectrum availability, the channel quality, and the attacker’s strategy by observing the status of the jammed channels and based on their observation, they decide how many channels they should reserve for transmitting control and data messages and how to switch between the different channels.

Game theory has also been used to model wireless communications under jamming. Reference [79] uses the zero-sum framework to model a wireless relay channel under jamming. This work considers a communication system with a jammer and a relay, participating in the link between a source and a destination (see Figure 1.4). This work uses the mutual information between the input and the output of the channel, \( I(X;Y) \), as the measure of the performance of the communication link. In contrast to the majority of the jamming problems studied in the literature, where the game is played between the transmitter and the jammer, this work considers a case where the game is played between the jammer and the relay. The transmitter and the receiver are assumed to be unaware of the jamming. The jammer and
CHAPTER 1. INTRODUCTION

the relay can eavesdrop the channel and use the information obtained to perform correlated jamming/relaying. The authors study the optimal strategies under various assumptions and conditions and show that in the nonfading scenario, when both players have full knowledge of the source signal, linear jamming and relaying are the optimal strategies.

1.4 Motivation Behind Research

Many other works in the literature have used game-theoretic approaches to study other specific jamming problem and/or scenario in wireless networks (e.g., [80][81]). However, many of these theoretical studies (if not all) are only applicable to a very specific problem and the study is often based on a particular feature or weakness of some communication system (often features or weaknesses that are related to the non-physical layers of the protocol stack). As a result, these frameworks fail to accurately model more general problems or even the same problem with a small change in the original assumptions or the jamming scenario. This limitation demands development of a more general theoretical framework that can be applicable to a wide range of jamming scenarios.

Furthermore, in the zero-sum game framework, it is usually assumed that the players, the communicating nodes, and the jammer, have perfect knowledge of the game and the actions that are available to the other player and players use this knowledge to compute their respective optimal strategies. Moreover, the main assumption in this model is that the players’ goals are exactly the opposite of each other, that is, it is assumed that if one player’s goal is to maximize a certain utility function and the other player’s goal is to minimize the same utility function.

However, in some jamming scenarios, perfect knowledge of the system parameters (or available actions) may not be a feasible option or too costly for a player. In addition, players may have objectives that are not exactly the opposite of each other, for example, the transmitter may wish to minimize the average error probability while the jammer wishes to minimize the average throughput of the system (as opposed to maximizing the average error probability which would naturally lead to a zero-sum framework).

In such scenarios, a more appropriate framework to model the communication system
 CHAPTER 1. INTRODUCTION

![Constrained Bimatrix Games Diagram](image)

Figure 1.5: Classification of the constrained and standard two-player games.

under jamming would be the general sum (bimatrix) game instead of a zero-sum game. In bimatrix games it is no longer required that the sum of the players’ payoffs to be zero (or a constant value). As a result, players can have different objectives and the respective payoffs can be defined based on the players’ goals and their knowledge of the game (which in general may be imperfect). It is clear that two-player, zero-sum games are a special case of the more general bimatrix game where the two payoff matrices are negative of each other.

In addition, in standard zero-sum and bimatrix games there are no additional restrictions on players’ mixed-strategies, i.e., players may choose any probability distribution over their respective action sets (pure-strategies). However, in many scenarios, due to practical restrictions, not all mixed-strategies are permitted and/or feasible. For instance, assume maximizing the average throughput of a wireless link. Maximizing the average throughput requires using higher transmission rates; but to maintain an acceptable error rate at the receiver, higher rates must be transmitted at higher transmission power. Because of battery limitation (internal limitation) or the FCC regulations (external limitations), the transmitter must keep its average transmission power below a certain value. Consequently, the wireless user cannot use actions that are more preferable to him (such as transmitting at the highest rate all the time). He may only choose actions that result in an average transmission power less than or equal to the predetermined value.

Such scenarios demand for a more general framework to overcome these limitations of the standard zero-sum and bimatrix games. To address this problem, we introduce the constrained zero-sum and constrained bimatrix games. In constrained games, the players’
CHAPTER 1. INTRODUCTION

mixed-strategies not only have to be a probability distributions but they must also satisfy some additional constraints too. It is clear that the standard zero-sum games and standard bimatrix games are in fact special cases of the more general constraint bimatrix games. Figure 1.5 shows the relation between the constrained and standard games.

In his seminal paper, “Non Cooperative Games” [82], Nash proved that for any game (including the standard zero-sum and bimatrix games) with finite set of pure strategies, there exists at least one (pure or mixed) equilibrium such that no player can increase his payoff by unilaterally deviating from his strategy. In the proof of the existence of the Nash equilibrium, no additional constraints were assumed on the mixed strategy spaces. However, in constrained games, the players are restricted to use only a subset of their respective strategy sets. That is, by imposing additional constraints on the players’ mixed-strategies, we are eliminating certain mixed-strategies that could have possibly resulted in a NE solution. Therefore, the existence of the NE for the constrained games is not trivial anymore and must be established.

To prove that a model has a Nash equilibrium, often fixed-point theorems such as Brouwers or Kakutanis fixed-point theorems are used. However, fixed-point theorems only prove the existence of a solution and do not provide a constructive way to find the solution or the optimal strategies. Therefore, it is essential to establish the necessary and sufficient conditions for the equilibrium solution(s) as well as a systematic approach to obtain a numerical solution for the constrained games when closed form solutions do not exist. To summarize, the three important aspects associated with the constrained games are (also see Figure 1.6):

1. **Existence of a solution**: Does the Nash equilibrium exists for the constrained game?

2. **Identifying the Nash equilibrium**: What are the necessary and sufficient conditions for optimality (i.e., how one can identify the equilibrium solution)?

3. **Solving the constrained game**: How one can solve the game numerically?

Moreover, many modern wireless communication systems such as IEEE 802.11 and 4G LTE networks provide connectionless service (packet-switched) at the physical layer. In packetized wireless systems, physical layer parameters such as transmission power/rate,
modulation scheme, etc. can change in real time, allowing the use of effective adaptation algorithms, previously possible only at the upper layers of the protocol stack. Therefore, it is important to study the effects of packetization on the wireless networks in adversarial environment. Another important issue is to understand the interaction between the communicating nodes and the adversary, determine the long-term achievable performance and the optimal strategy to achieve it, as well as the optimal strategy for the adversary. Our goal is to study what should (or might) happen when a packetized wireless link is subject to physical layer jamming.

To address these questions, we use a game theoretic approach to formulate the interactions between the communicating nodes and the jammer. Furthermore, we study the problem of determining the optimal transmission strategies and adaptation mechanisms for a transmitter/receiver with multiple transmission “choices/parameters” when the wireless channel is subject to jamming by a power constrained jammer. We consider a general system model where a pair of nodes (transmitter and receiver) communicate using data packets. An adversary can interfere with the communication but is constrained by an instantaneous maximum power per packet ($J_{\text{max}}$) as well as a long-run average power ($J_{\text{ave}}$).

Packets each selected with appropriate transmission parameters either overcome the interference or are lost otherwise. Inappropriate selection of the transmission parameters can
CHAPTER 1. INTRODUCTION

either increase the chance of underperforming (if the parameters are selected conservatively) or loosing a packet (if the parameters are selected aggressively). In this communication scenario, it is crucial to understand the interaction between the communicating nodes and the adversary, determine the long-term achievable maximum performance and the optimal transmitter strategy to achieve it, as well as the optimal strategy for the adversary. While, for a channel with fixed-power jammer, the optimal strategies for communication and jamming and the system performance are derived from the fundamental information theoretic results, these questions are still open for a packetized communication system.

Our contributions can be summarized as follows:

• Based on the standard zero-sum and bimatrix games, we develop a constrained zero-sum and bimatrix frameworks to model power limited jamming in packetized networks. In constrained games, players strategies are limited to only a subset of strategies that are available in standard games and as a result, more general problems can be modeled by constrained games.

• We show that interactions between adaptive communicating nodes and smart power limited adversaries can be formulated by constrained zero-sum and/or constrained bimatrix games and prove the necessary and sufficient conditions under which existence of the Nash Equilibrium for these non-typical games is guaranteed.

• We show that constrained zero-sum games and linear programming have a deep connections. For every constrained zero-sum game there exists a linear program whose solution yields a Nash equilibrium and every equilibrium solution of the zero-sum game is a solution of the linear program.

• We show that there exists a similar correspondence between the Nash equilibrium solutions of constrained bimatrix games and global solutions of a quadratic program such that every NE solution of the bimatrix game is a global maximizer of a quadratic program and every global maximum of the quadratic program yields a NE solution.

as channel capacity theorem, could be applied to each packet being transmitted. Nevertheless, our analytical study is not limited to such cases and can be easily extended.
CHAPTER 1. INTRODUCTION

- We use our constrained frameworks to study some typical jamming problems in packetized networks where we provide analytical as well as numerical results for the optimal strategies and closed form expressions for the expected value of the game at the the Nash equilibrium.

- Our analytical results suggest that a strategic jammer that uses optimal jamming strategies can significantly degrade the performance of packetized networks compared to non-strategic jammers. Furthermore, we prove that there exists a certain threshold on jammer’s average power, $J_{TH}$, such that if the jammer’s average power exceeds $J_{TH}$, the game value at the Nash equilibrium is the same as the case when the jammer uses his maximum power all the time.

- Inspired by the superposition coding technique in broadcast channels, we propose an adaptive multilayer superposition coding technique to improve the performance of packetized networks. Our analytical results show that superposition coding not only achieves better average performance under jamming, it also increases the jamming threshold.

- We show that, under moderate assumptions, the performance of an adaptive OFDM wireless communication system under power limited jamming can also be formulated into the constrained zero-sum or constrained bimatrix frameworks and as a result, all the analytical results derived for these frameworks can be applied to this problem directly.

1.5 Organization of the Dissertation

The remainder of the dissertation is organized as follows:

In Chapter 2 we introduce a general packetized wireless communication link under power limited jamming and develop a constrained two-player zero-sum framework to model this problem. We provide the necessary and sufficient conditions under which existence of a Nash equilibrium is guaranteed. Additionally, we show that there exist a one to one relation between the constrained zero-sum games and linear programming problem such that, for
CHAPTER 1. INTRODUCTION

every constrained zero-sum game there exist a unique linear program whose solution yields
an equilibrium. We then use our framework to study two jamming scenarios in a wireless
AWGN link under power limited jamming and show that game theoretic analysis of this
problem reveals rather surprising results.

In Chapter 3 we extend the framework developed in Chapter 2 to a constrained bimatrix
framework to be able to model even broader jamming scenarios. We prove the necessary
and sufficient conditions under which the existence of a Nash equilibrium is guaranteed for
this constrained game. Furthermore, we show that there exist a one-to-one correspondence
between constrained bimatrix games and a quadratic programming problem such that, every
solution of the game (if exists) is a global maximum of a quadratic program and every global
maximum of the quadratic program is a solution of the game.

In Chapter 4 inspired by the superposition coding technique in broadcast channels, we
propose an adaptive multilayer superposition coding technique to improve the performance
of packetized networks. Our analytical results shows that superposition coding not only
achieves better average performance under jamming, it also increases the jamming threshold
(which itself increase the cost of jamming for the adversary).

In Chapter 5 we study the performance of an adaptive OFDM wireless communication
system under power limited jamming and show that under modest assumptions, this problem
can also be formulated into the constrained zero-sum or constrained bimatrix frameworks.
As a result, all the analytical results that are presented in chapters 2 and 3 can be applied to
this problem directly. Finally, we conclude the dissertation in Chapter 6.
Chapter 2

Constrained Zero-sum Games

Many works in the literature have used game-theoretic approach to study some specific jamming problem and/or scenario in wireless networks (e.g., see [80, 81]). However, many of these theoretical studies (if not all) are only applicable to a very specific problem and the study is often based on a particular feature or weakness of a communication system (often features or weaknesses that are unrelated to the physical layer). As a result, these frameworks fail to accurately model more general problems or even the same problem with a small change in the original assumptions or the jamming scenario. This limitation demands development of more general theoretical frameworks that can be applicable to a wide range of jamming scenarios.

We begin this chapter by introducing our system model and problem statement; a general communication system under physical layer jamming. Then, we introduce our game model, the players and their respective strategies. Additionally, we will define a general utility function and a payoff matrix that could be used as a measure of performance in a wide range of jamming problems. At the physical layer, the interaction between the legitimate users an the adversary is often modeled as a zero-sum game in order to capture their conflicting goals [4]. We use the two-player zero-sum game framework to model the problem with the additional constraint that the jammer must maintain an overall average jamming power. We show how this additional constraint affects jammer’s mixed-strategy set and makes the game
model different from a typical zero-sum game.

Finally, as a special case, we investigate the maximum achievable throughput of a rate-adaptive wireless AWGN link under power limited jamming and we show that the framework we developed can be used to derive analytical and numerical results for this typical jamming problem.

### 2.1 System Model and Problem Statement

In this section we formally introduce our system model and define the problem under study. The corresponding system model is shown in Figure [2.1](#). The transmitter and the receiver are communicating through a packetized, wireless noisy channel. Beside the channel noise, the transmitted packets are also disrupted by an adversary, the *jammer*. The jammer’s maximum and average jamming powers are assumed to be limited to $J_{\text{max}}$ and $J_{\text{ave}}$, respectively.

Our goal is to keep our model less system-dependent, that is, we make only general assumptions that could be valid (or can be used as a basis) for a wide range of scenarios rather than depending on specific characteristics or weaknesses of a communication system.

#### 2.1.1 The Channel Model

The wireless communication link between the transmitter and the receiver is assumed to be a single-hop, noisy channel with fixed and known channel parameters, i.e., the transmitter
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

Figure 2.2: Transmitter model.

and the jammer have perfect knowledge of the channel. Furthermore, the communication link is being disrupted by an adversary, the jammer. The jammer transmits radio signals to increase the effective noise at the receiver and hence degrades the performance of the communication link (e.g., decreases the channel capacity or throughput, degrades the quality of service, etc.) between the transmitter and the receiver. We assume packet-switched communication at the physical layer, i.e., packetized communications where transmissions occur in disjoint time intervals (time slots) during which transmitter’s and jammer’s state (parameters) remain unchanged (see the discussion in Section 1.3 for more details).

Even though in practice, most packetized systems allow the packet size to be variable, in this work, whenever it simplifies the analytical results, we assume that packets are long enough that fundamental theorems on reliable communications, such as channel capacity theorem, could be applied to each packet being transmitted. This assumption is well justified since many modern wireless standards such as IEEE 802.11, IEEE 802.15 and 4G LTE networks provide only connectionless service at physical layer and many internet protocols use packet sizes of up to (and above) 1,500 bytes. Nevertheless, our study is not limited to long or fixed sized packets and it can be easily extended to communication systems with variable packet size.

2.1.2 The Transmitter Model

The transmitter has an adaptation block which enables him to change and adapt his transmission parameters (e.g., transmission power, rate, modulation, etc.). In order to combat jamming, the transmitter changes his transmission parameters according to a probability distribution (his strategy). The transmitter chooses an optimal distribution to achieve the
best average performance (or his expected payoff) which is presented by a preference/utility function. Common measures of performance in wireless networks are achievable capacity, network throughput, quality of service (QoS), power consumption, etc. [4]. As stated before, we assume transmissions are packet-based. The transmitter’s model is shown in Figure 2.2.

The interleaver block in transmitter’s model is a countermeasure to burst errors and burst jamming (transmitting a burst of white noise to disrupt a few bits in a packet). Interleaving is frequently used in digital communications and storage devices to improve the burst error correcting capabilities of a code. Burst errors are specially troublesome in short length codes as they have limited error correcting capabilities. In such codes, a few number of errors could result in a decoding failure or incorrect decoding. A few incorrectly decoded codewords within a larger frame could make the entire frame corrupted.

Combining effective interleaving schemes such as cryptographic interleaving and capacity achieving codes, such as turbo and LDPC codes, results in effective transmission schemes (see [29]) which make burst jamming ineffective. Therefore, in our study we do not consider burst jamming.

2.1.3 The Jammer Model

Radio jamming or simply jamming is deliberate transmission of radio signals with the intention of degrading performance of a communication link. A fairly large number of jamming models have been proposed in the literature [83]. The most benign jammer is the barrage noise jammer. The barrage noise jammer transmits bandlimited white Gaussian noise with flat power spectral density (psd) of $J$ Watts/Hz. It is usually assumed that the barrage noise jammer power spectrum covers exactly the same frequency range as the communicating system. This kind of jammer simply increases the Gaussian noise level from $N$ to $(N + J)$ at the receiver’s front end. Another frequently used jamming model is the pulse-noise jammer. The pulse noise jammer transmits pulses of bandlimited white Gaussian noise having total average power of $J_{ave}$ referred to the receiver’s front end. It is usually assumed that the jammer chooses the center frequency and bandwidth of the noise to be the same as the transmitter’s center frequency and bandwidth. The jammer chooses its pulse duty factor to cause maximum degradation to the communication link while maintaining the
average jamming power $J_{ave}$. For a more realistic model, the pulse-noise jammer could be subject to a maximum peak power constraint. Other jamming models, to name a few, are the partial-band jammer and single/multiple-tune jammer.

We study a more sophisticated jamming model. The jammer in study is a reactive jammer, i.e., he is only active when a packet is being transmitted and silent otherwise. We assume that the jammer uses a set of discrete jamming power levels arbitrary placed between $J = 0$ and $J = J_{max}$. The jammer can choose any jamming power level to increase the effective noise at the receiver, but he has to maintain an overall average jamming power, denoted by $J_{ave}$. The jammer uses his available power levels according to a probability distribution (his strategy), he chooses an optimal strategy to minimize the performance of the communication link while maintaining his maximum and average power constraints. As discussed in Section 2.1.2 burst jamming is not an optimal jamming scheme and hence, we assume that the jammer remains active during the entire packet transmission, i.e., the jammer transmits a continuous jamming signal with a fixed power (variance) $J \in [0, J_{max}]$ for each transmitting packet. The jammer model is shown in Figure 2.3.

2.2 Constrained zero-sum Game Model

In this section we formulate the jamming problem introduced in Section 2.1 in a constrained zero-sum game. We provide analytical and numerical results for transmitter and jammer optimal strategies and a closed form expression for the expected value of the game at the NE.
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

2.2.1 Action Sets and Mixed-strategies

The jammer has the option to select its operating power in any given packet from the set of discrete values of available jamming powers, arbitrarily placed between 0 and \( J_{\text{max}} \). We assume \((N_J + 1)\) distinct power levels, or pure strategies, are available to the jammer (the size of the jammer’s action sets). In the most general case, the jammer’s action set is a set of different jamming power levels in the interval \([0, J_{\text{max}}]\). We denote this set by \( \mathcal{J} \)

\[
\mathcal{J} = \left\{ 0 \leq J_j \leq J_{\text{max}}, \ 0 \leq j \leq N_J \text{ and } J_j \neq J_k \text{ for } j \neq k \right\}
\]

Without loss of generality, we assume the power levels are sorted in an increasing order and \([0, J_{\text{max}}] \subset \mathcal{J} \), i.e.

\[
J_0 = 0 < \cdots < J_j < J_{j+1} < \cdots < J_{N_J} = J_{\text{max}} \quad 0 < j < N_J
\]

For simplicity, we place the possible jamming power levels in a vector and form the jammer’s pure strategy column vector, \( \mathbf{J} \), where

\[
\mathbf{J}^T = \begin{bmatrix} J_0 & \cdots & J_j & \cdots & J_{N_J} \end{bmatrix}_{1 \times (N_J+1)}
\]

and \( ^T \) indicates vector transposition. Unlike typical zero-sum games in which there are no other constraints on the mixed-strategies, in our model, the jammer’s mixed-strategy must satisfy the additional average power constraint \( J_{\text{ave}} \), where \( J_{\text{ave}} < J_{\text{max}} \). Hence, in our constrained game model, not all mixed-strategies (and not even those pure strategies that are greater than \( J_{\text{ave}} \)) are feasible strategies \([77]\). If we let \( \mathbf{y} \) denote the jammer’s mixed-strategy vector\(^1\) and let \( \mathbf{Y} \) be the standard \((N_J+1)\)-simplex, for a typical zero-sum game, jammer’s mixed-strategy must satisfy the following

\[
\mathbf{y}^T = \begin{bmatrix} y_0 & \cdots & y_j & \cdots & y_{N_J} \end{bmatrix} \in \mathbf{Y} : \sum_{j=0}^{N_J} y_j = 1 \quad y_j \geq 0 \quad 0 \leq j \leq N_J
\]

By using the jammer’s pure strategy vector we define the new mixed-strategy set, \( \mathbf{Y}_{\text{LE}|J_{\text{ave}}} \), for our constrained game as

\[
\mathbf{Y}_{\text{LE}|J_{\text{ave}}} = \{ \mathbf{y} \in \mathbf{Y} \mid \mathbf{y}^T \mathbf{J} \leq J_{\text{ave}} \}
\]

\(^1\)The probability distribution vector on the set of jammer’s pure strategies.
where $Y_{LE|J_{ave}}$ is a subset of the $(N_J + 1)$-simplex which includes all mixed-strategies with an average power less than or equal to $J_{ave}$.

Since by introducing the new constrained mixed-strategy set, defined in equation (2.4), we are eliminating some mixed-strategies that could have been otherwise selected, we must first establish the existence of the Nash equilibrium for this game model. This game is not a typical zero-sum game with a finite number of pure strategies for which the existence of the NE is guaranteed. In Section 2.3.1, we prove the existence of the NE for the constrained game, where the jammer’s mixed-strategy set is limited to $Y_{LE|J_{ave}}$.

The transmitter’s pure strategy set, or equivalently his action set, is a set of $(N_T + 1)$ discrete transmission parameters (e.g., power, rate, etc.). We assume each strategy from transmitter’s action set can tolerate up to a certain level of jamming power, we indicate this jamming power level by $J_T$ to distinguish it from the jammer’s actual jamming power, $J$.

We assume the packet that is being transmitted with this strategy can be fully recovered at the receiver for any jamming power less than or equal to $J_T$ but will be completely lost for jamming powers greater than $J_T$. This assumption is inspired by Shannon’s channel capacity theorem which states that reliable communication at a given rate is possible when the noise power is below a certain level and becomes impossible if the noise power exceeds that value. For instance, consider a rate adaptive transmitter that is communicating over a stationary wireless channel that is subject to additive jamming. The effect of jamming on the wireless channel is reduction of the effective signal to noise ratio from $P_t/N$ to $P_T/(N + J)$ at the receiver’s front. Assuming that $P_T$ and $N$ are fix, the capacity of the wireless channel becomes a function of the jamming power, $J$. Therefore, for any given rate, $R_i$, there exist a unique jamming power, $J_{T,i}$, below which reliable communication is
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

possible (see Figure 2.4).

Since corresponding to each transmitter pure strategy there exists a certain jamming power below which reliable transmission is possible, we can define a one-to-one relation between transmitter pure strategies and the corresponding jamming powers denoted by $J_T$. We use these jammer power values as representatives of transmitter’s pure strategies. As a result, without loss of generality, transmitter’s pure strategy set can be represented as

$$\mathcal{J}_T = \left\{ 0 \leq J_{T,i} \leq J_{\text{max}}, \quad 0 \leq i \leq N_T \right\}$$ (2.5)

WLOG, we assume $J_{T,0} = 0$ and $J_{T,N_T} = J_{\text{max}}$, i.e., the transmitter’s highest and lowest payoffs correspond to the jammer’s lowest ($J = 0$) and highest ($J = J_{\text{max}}$) jamming powers. The transmitter uses his available transmission parameters (or the equivalent jamming powers from the set $\mathcal{J}_T$) according to a probability distribution (his mixed-strategy) and his goal is to find an optimal strategy to maximize the expected performance of the communication link. We use column vector $\mathbf{x}_T$, to indicate the transmitter’s mixed-strategy vector,

$$\mathbf{x}_T^T = \begin{bmatrix} x_0 & \cdots & x_i & \cdots & x_{N_T} \end{bmatrix}_{1 \times (N_T+1)} \in \mathbf{X}$$ (2.6)

where $\mathbf{X}$ is the standard $(N_T+1)$-simplex.

2.2.2 The Utility Function and The Payoff Matrix

Because transmissions occur in the presence of an adversary, recovery of the transmitted information/packets at the receiver is not always guaranteed. Each strategy from the transmitter’s action set can tolerate up to a certain level of jamming power (denoted by $J_T$) therefore, reliable communication is possible when the actual jamming power ($J$) is less than or equal to $J_T$. For long enough packets, we can further assume that packets can be recovered only when the actual jamming power, $J$, is less than or equal to $J_T$ (i.e., if and only if $J \leq J_T$) and packets are completely lost otherwise\footnote{For packets with short lengths, the transition from full recovery to complete loss is usually gradual.}. Therefore, the utility function of the game (the payoff to the transmitter), $Z(J_T, J)$, can be modeled as

$$Z(J_T, J) = \begin{cases} Z(J_T) & J_T \geq J \\ 0 & J_T < J \end{cases} \quad J_T \in \mathcal{J}_T, \; J \in \mathcal{J}$$ (2.7)
where \( Z(J_T) \) is an arbitrary function of \( J_T \) and represents the performance (payoff) of the communication system under jamming. The function \( Z(J_T) \) assigns a positive value to each strategy from the transmitter’s action set and is intuitively a strictly decreasing function of \( J_T \), i.e., the payoff to the transmitter decreases when the jamming power increases, i.e.,

\[
Z_0 > \cdots > Z_i > Z_{i+1} > \cdots > Z_{N_T}; \quad 0 < i < N_T \tag{2.8}
\]

where \( Z_i \triangleq Z(J_T = J_i) \) and \( J_i \in \mathcal{J}_T \). Even though \( Z(J_T) \) could be any arbitrary decreasing function of \( J_T \), defining \( Z(J_T) \) based on the channel capacity is a common practice in the games involving a transmitter-receiver pair and an adversary \([79, 81, 4]\).

Given that our game model is a constrained zero-sum two-player game, the payoff to the jammer is the negative of the transmitter’s payoff. Furthermore, we can formulate the payoffs (for each pure strategy pair) in a payoff matrix where the transmitter and the jammer would be the row and the column players respectively. The resulting payoff matrix, \( Z \), is in general a non-square matrix and from equation (2.7), we see that the non-zero elements of each row of \( Z \) are equal. We will show in Lemma 1 that WLOG, we can assume that \( Z \) is a square, lower triangular matrix with equal non-zero entries in each rows, i.e.,

\[
Z_{(N_T+1) \times (N_T+1)} = \begin{bmatrix}
Z_0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
Z_i & \cdots & Z_i & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
Z_{N_T} & \cdots & Z_{N_T} & \cdots & Z_{N_T}
\end{bmatrix} \tag{2.9}
\]

as a result, the expected payoff of the game for the mixed-strategy pair \((x, y)\) can be written as

\[
Z(x, y) = x^T Z y, \quad y \in \mathcal{Y}_{LE|J_{ave}} \quad x \in \mathcal{X} \tag{2.10}
\]

**Lemma 1.** Let \( Z \) be the payoff matrix in the constrained two-player zero-sum game defined by the utility function (2.7). The payoff matrix obtained by removing the dominated strategies is a square lower triangular matrix with size less than or equal to \( \min \{ N_T, N_J \} + 1 \).

**Proof.** Assume the jammer’s power levels are arbitrary distributed over the range \([0, J_{Max}]\) where \( N_T < N_J \). A typical case where \( N_T < N_J \) is shown in Figure 2.5 (left). In
Figure 2.5: Dominated strategies vs. non-dominated strategies.

Figure 2.5, the transmitter’s pure strategies are mapped to the jammer’s power levels for better visualization. Between some of the transmitter’s pure strategies there might be a pure strategy of the jammer but since $N_T < N_J$, according to the Pigeonhole principle, between at least two of the transmitter’s pure strategies (not necessarily any two pure strategy as sketched) there must be more than one jamming power level (shown as dashed or solid lines ending in squares). Any of these jamming power levels (or pure strategies) could be used to terminate the information transmitted by the rate corresponding to the power level immediately to the left of them (shown as solid line ending in circles). From these pure strategies, a rational jammer would choose the one with the lowest power level (the solid line) and hence, it would dominate the rest (dashed lines). Therefore, the number of non-dominated pure strategies for the jammer is at most equal to the number of the transmitter’s pure strategies (first part of the lemma).

If the pure strategies were uniformly distributed over $[0, J_{Max}]$, as sketched, for every transmitter’s pure strategy there would be exactly one non-dominated strategy for the jammer and hence, there would be no intention for the jammer to use more pure strategies than the transmitter. The same discussion can be given for the number of pure strategies a rational transmitter should use for the case $N_T > N_J$ (see Figure 2.5 (right)). Henceforward, without loss of generality, we assume $N_T = N_J$.

As a consequence of Lemma 1, we need to consider only square matrices, which simplifies further development. In the next section we will study the outcome of the game under jammer’s different values of average power. WLOG, in the following sections, we assume the size of $Z$ matrix is $N_T + 1$. 

33
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

2.3 Game Characteristics

In this Section, we study two basic properties of the game. We will show that although we have put an additional constraint on the jammer’s mixed-strategy set, the existence of the Nash equilibrium is still guaranteed. In game theory, fixed-point theorems are often used to prove that a model has an equilibrium solution, however, we use a more constructive approach to prove the existence of the Nash equilibrium. We show that there is a deep connection between constrained two-player zero-sum games and linear programming problem such that, every constrained two-player, zero-sum game there exists an equivalent linear program whose solution yields a NE for the game and every NE of the game is a solution of the linear program. Therefore, well-known and effective algorithms to solve linear programs such as the simplex and the interior point algorithms can be used to find the equilibrium points of any constrained zero-sum game numerically.

Additionally, we show that by randomizing his strategy, the jammer can force the transmitter to operate at the lowest payoff, given that the average jamming power exceeds a certain threshold, \( J_{TH} < J_{\text{max}} \). We will also derive an upper bound for this jamming power threshold.

2.3.1 Existence of the Nash Equilibrium and Connection to Linear Programming

For every zero-sum game with finite set of pure strategies, there exists at least one (pure or mixed) Nash equilibrium (NE) such that no player can do better by unilaterally deviating from his strategy [77]. In our game model, we are assuming an additional constraint on the jammer’s mixed-strategy set; the jammer must maintain a maximum average jamming power. This additional assumption changes the jammer’s mixed-strategy set from a standard \( n \)-simplex to a subset of it. Therefore, the existence of the NE for this non-typical zero-sum game must be established. In the following lemma, we show that the existence of the NE under the additional constraint is guaranteed.

**Lemma 2.** For the constrained two-player zero-sum game defined by the utility function \( Z(J_T, J) \), given in (2.7), and the payoff matrix \( Z \), given by (2.9), there exists at least one
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

NE in the form of transmitter’s mixed-strategy, \( x \in X \), and the jammer’s mixed-strategy, \( y \in Y \).

Proof. Consider a two-player zero-sum game in which, due to some practical reason, not all mixed-strategies are feasible strategies [77]. Assume the mixed-strategies \( x \in X \) and \( y \in Y \) (player I’s and player II’s mixed-strategies respectively) must be chosen from some hyper-polyhedron, i.e., from constraint sets defined by linear inequalities and equalities. If we let \( Z \) be the payoff matrix, player I’s problem is to find

\[
\max_{x \in X} \left( \min_{y \in Y} xZy^T \right)
\]

(2.11)

where \( x \) and \( y \) are row vectors and the sets \( X \) and \( Y \) in the most general case, are defined by

\[
X : \begin{cases} 
xB \leq c \\
x \geq 0
\end{cases} \quad Y : \begin{cases} 
yE^T \geq f \\
y \geq 0
\end{cases}
\]

(2.12)

Similarly, player II’s problem is to find

\[
\min_{y \in Y} \left( \max_{x \in X} xZy^T \right)
\]

(2.13)

Consider the optimization problem in (2.11), the inner minimization problem can be represented by a linear program whose objective function depends on \( x \). From the duality theorem [77] and if the program is feasible and bounded, then the two programs

\[
\minimize (xZ)y^T \quad \text{subject to:} \begin{cases} 
ymE^T \geq f \\
y \geq 0
\end{cases}
\]

(2.14)

and

\[
\maximize zf^T \quad \text{subject to:} \begin{cases} 
zE \leq xZ \\
z \geq 0
\end{cases}
\]

(2.15)

will have the same value (where column vector \( z \) is the vector of dual variables). If we plug the dual program (2.15) in (2.11), player I’s problem becomes a pure maximization problem, i.e

\[
\maximize zf^T \quad \text{subject to:} \begin{cases} 
zE - xZ \leq 0 \\
xB \leq c \\
x, z \geq 0
\end{cases}
\]

(2.16)
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

This problem can be solved by the usual linear program algorithms i.e., simplex algorithm. In a similar way, it can be shown that player II’s problem could be reduced to a pure minimization problem,

\[
\text{minimize } cs^T \quad \text{subject to: } \begin{cases} 
  sB^T - yZ^T \geq 0 \\
  yE^T \geq f \\
  y, s \geq 0
\end{cases}
\] (2.17)

where vector \( s \) is the vector of dual variables. It can be verified that the program (2.17) is the dual of the program (2.16) and therefore, if both are feasible and bounded, they will have the same value and the constrained game will have a NE in mixed-strategies. Figure 2.6 shows the connection between the constrained zero-sum games and the linear programming problem. This figure also summarizes the equivalent game theoretic and linear programming formulations of the constrained zero-sum games.

By using appropriate set of matrices and vectors, we can reformulate transmitter’s and jammer’s strategy constraint sets defined in Section 2.2 into the general format introduced in (2.12). Specifically, consider the transmitter’s constraint set, the following set of matrix, \( B \), and vector, \( c \), can be used to represent transmitter’s constraint set:

\[
B_{(N_T+1)\times2} = \begin{bmatrix}
1^T_{(N_T+1)\times1} & -1^T_{(N_T+1)\times1}
\end{bmatrix}
\] (2.18)

\[
c_{1\times2} = \begin{bmatrix}
1 & -1
\end{bmatrix}
\]

Figure 2.6: Connection between the constrained zero-sum games and linear programming.
similarly, jammer’s constraint set can be represented by the following matrix, $E$, and vector, $c$, 

$$
E^T_{(N_T+1) \times 3} = \begin{bmatrix}
1^T_{(N_T+1) \times 1} & -1^T_{(N_T+1) \times 1} & -J^T_{(N_T+1) \times 1}
\end{bmatrix} \quad (2.19)
$$

$$
f_{1 \times 3} = \begin{bmatrix}
1 & -1 & -J_{\text{ave}}
\end{bmatrix}
$$

From (2.16) and (2.19) the maximization program objective function becomes

$$
z^T f = z_1 - z_2 - J_{\text{ave}} z_3 \quad (2.20)
$$

To show that the maximization program is bounded and feasible and hence has a solution in mixed-strategies, we need to show that the objective function, given in (2.20), is bounded and the constraint set defined by the set of matrices and vectors in (2.18) and (2.19) is non-empty. Assume, for now, that the objective function is unbounded, we must have

$$
z_1 - z_2 - J_{\text{ave}} z_3 > Z_{\text{max}} = \max_{i,j} Z_{ij} \quad (2.21)
$$

for some $z = [z_1 \ z_2 \ z_3] \geq 0$ that satisfy the constraints in (2.16). Consider the first inequality in (2.16), multiplying vectors $z$ and $x$ by the first column of matrices $E$ and $Z$ results in

$$
z_1 - z_2 - J_0 z_3 - x Z_{i,:1} \leq 0 \\
\Rightarrow z_1 - z_2 \leq x Z_{i,:1} \leq Z_{\text{max}} = \max_i Z_{i,:1} \\
\Rightarrow Z_{\text{max}} \geq z_1 - z_2 \quad (2.22)
$$

where $Z_{i,:1}$ denotes the first column of $Z$ and by assumption we have $J_0 = 0$. Plugging (2.22) in (2.21) results in $z_3 < 0$ which is in contradiction with $z > 0$. As a result, the objective function cannot be greater than $Z_{\text{max}}$ and hence the program is bounded. To show that the constraint set is non-empty, we need to show that there exists at least one pair of $(x, z)$ that satisfies the constraints in (2.16). From the first inequality in (2.16) we must have

$$
z_1 - z_2 - J_i z_3 \leq x Z_{i,:} \quad \forall i : 1 \leq i \leq N_T + 1 \quad (2.23)
$$

but $0 \leq x Z_{i,:} \leq Z_{\text{max}}$ for $1 \leq i \leq N_T + 1$ and for all probability vectors $x$, therefore if we let

$$
z_1 - z_2 - J_i z_3 \leq 0 \quad (2.24)
$$
then \((2.23)\) would be satisfied for all \(1 \leq i \leq N_T + 1\). Choosing \(z_3 = 0\) and letting \(0 < z_1 < z_2\) satisfies the inequality \((2.24)\) and as a result the transmitter’s constraint set is nonempty.

Therefore, the transmitter’s maximization program is feasible and bounded and has a solution. As a result of the duality theorem, the dual of this program, jammer’s minimization problem, is also feasible and bounded has the same solution (NE in mixed-strategies).  

2.3.2 Existence of Jamming Power Threshold

A Barrage noise jammer (or a jammer that only uses pure-strategies) requires an average power equal to \(J_{\text{max}}\) to force the average performance of the wireless link to the minimum, \(Z_{N_T}\). However, as the Following theorem proves, a jammer that uses mixed-strategies can achieve the same goal with (possibly) significantly less average jamming power. Theorem 1 also gives an upper bound for this jamming threshold.

**Theorem 1.** Let us assume we have a constrained two-player zero-sum game defined by the utility function \(Z(J_T, J)\), given in \((2.7)\), the payoff matrix, \(Z\), given in \((2.9)\), the transmitter’s mixed-strategy \(x \in X\), and the jammer’s mixed-strategy \(y \in Y_{LE|J_{\text{ave}}}\), given in \((2.4)\). Then, there exists a jamming power threshold, \(J_{\text{TH}} < J_{\text{max}}\), and \(y^* \in Y_{LE|J_{\text{ave}}}\) such that if \(J_{\text{ave}} \geq J_{\text{TH}}\) then,

\[
x^{*T} = \begin{bmatrix} 0_{1 \times N_T} & 1 \end{bmatrix}_{1 \times (N_T+1)}
\]

\(Z(x^*, y^*) = Z_{N_T}\) \(\quad (2.25)\)

Where \(x^*, y^*\) are the transmitter’s and jammer’s optimal mixed-strategies, respectively, and \(Z(x^*, y^*)\) is the game value at the NE (we use these notations throughout the paper).

Theorem 1 states that there exits a jamming power threshold, \(J_{\text{TH}}\), such that if the jammer’s average power exceeds \(J_{\text{TH}}\) then the transmitter’s optimal mixed-strategy is to use the strategy corresponding to jammer’s maximum power, resulting in the lowest payoff as if the jammer was using its maximum jamming power all the time.

**Proof.** Assume the jammer is using a mixed-strategy, \(\hat{y}\), which is not necessarily optimal,
and is defined by
\[
\dot{y}^T = \begin{bmatrix} \dot{y}_0 & 0 & \cdots & 0 & \dot{y}_{N_T} \end{bmatrix}_{1 \times (N_T+1)}
\] 
(2.26)

It can easily be verified that (2.26) is indeed a valid probability distribution (since by assumption we have \(Z_{N_T} < Z_0\)). Let \(\dot{J}_{ave}\) be the average jamming power of the strategy \(\dot{y}\);
\[
\dot{J}_{ave} = \sum_{j=0}^{N_T} \dot{y}_j J_j = \left(1 - \frac{Z_{N_T}}{Z_0}\right) J_{max} < J_{max}
\] 
(2.27)

Furthermore, assume the transmitter is using an arbitrary mixed-strategy \(x \in X\) against jammer’s strategy \(\dot{y}\) defined in (2.26). Define \(Z(x, \dot{y})\) to be the expected payoff of the game for the mixed-strategy pair \((x, \dot{y})\);
\[
Z(x, \dot{y}) = x^T \dot{y} = Z_{N_T} \left[(x_0 + x_{N_T}) + \sum_{i=1}^{N_T-1} \left(\frac{Z_i}{Z_0}\right) x_i\right]
\]
\[
\leq Z_{N_T} (x_0 + x_{N_T}) \leq Z_{N_T}
\] 
(2.28)

Since by assumption we have \(\frac{Z_i}{Z_0} < 1\) for all \(0 < i < N_T\) and \(0 \leq x_i \leq 1\) for all \(0 \leq i \leq N_T\). Additionally, the equality in (2.28) holds if and only if \(x_0 + x_{N_T} = 1\) Thus, by using the mixed-strategy, \(\dot{y}\), and an average power \(\dot{J}_{ave} \leq J_{max}\) given in (2.27), the jammer can force a payoff at most equal to the transmitter’s lowest payoff, \(Z_{N_T}\).

The jamming power given in (2.27) is not necessarily the lowest possible threshold. In Section 2.4.3 we provide a closed form expression for the lowest average jamming power, \(J_{TH}\) (jamming threshold) that can force the payoff \(Z_{N_T}\).

2.4 Game Analysis

In this section we study the optimal strategies for the transmitter and the jammer. We divide this section into two subsections; the case where the average jamming power is less than the jamming threshold, defined in Section 2.3.2 (see Theorem 1), and the case where it is greater than or equal to the jamming threshold. As we showed in Section 2.3.1 the standard linear programing techniques that are used to solve standard two-player zero-sum games can be appropriately modified to solve two-player zero-sum games with linear
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

constraints. Therefore, even though all the results in this section are derived analytically, all constrained two-payer zero-sum games can also be solved numerically.

2.4.1 Optimal Mixed-Strategies for $J_{\text{ave}} < J_{\text{TH}}$

We start by developing the optimal strategies for specific values of average jamming power, denoted by $J_{\text{ave},m}$, $m = 0, \cdots, N_T - 1$. As we show, for these specific average jamming powers, the expected payoff of the game at the NE is equal to $Z_m$, $m = 0, \cdots, N_T - 1$. Later we extend our analytic results to the case where average jamming power is not necessarily equal to $J_{\text{ave},m}$.

2.4.1.1 Case 1; $J_{\text{ave}} = J_{\text{ave},m} < J_{\text{TH}}$, $m = 0, \cdots, N_T - 1$

Since the average jamming power is less than the jamming threshold, the expected value of the game at the NE is in the interval $(Z_{N_T}, Z_0]$. Assume, for now, the average jamming power, $J_{\text{ave}} < J_{\text{TH}}$, is such that the optimal strategy for the jammer is to use $(m + 1)$ of his pure strategies (the support set of the jammer’s mixed-strategy is $y_0 \leq y \leq y_m$), i.e.,

$$y^T = \left[ y_0 \cdots y_j \cdots y_m \ 0_{1 \times (N_T - m)} \right]_{1 \times (N_T + 1)} \tag{2.29}$$

where $0 \leq m < N_T$ and $0_{1 \times (N_T - m)}$ indicates a row vector of $(N_T - m)$ zeros. Intuitively, as the jammer’s average power increases, he is able to use pure strategies with higher jamming power. Consider the mixed-strategy $\hat{y}$ given by

$$\hat{y}^T = Z_m \left[ z_0^{-1} \cdots z_j^{-1} - z_{j-1}^{-1} \cdots z_{m-1}^{-1} - Z_{m-1}^{-1} 0_{1 \times (N_T - m)} \right] \tag{2.30}$$

It can be easily verified that (2.30) is indeed a probability distribution. The average power of the mixed-strategy $\hat{y}$ is

$$J_{\text{ave},m} \triangleq J^T \hat{y} = Z_m \sum_{j=1}^{m} \left( Z_j^{-1} - Z_{j-1}^{-1} \right) J_j \tag{2.31}$$
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

Assume the jammer is using the mixed-strategy \( \hat{y} \) against transmitter’s arbitrary strategy, \( x \in X \). The resulting expected payoff for the game is

\[
Z(x, \hat{y}) = x^T Z y = Z_m x^T \begin{bmatrix} 1_{(m+1) \times 1} \\ Z_{m+1} Z_m^{-1} \\ \vdots \\ Z_j Z_m^{-1} \\ \vdots \\ Z_{N_T} Z_m^{-1} \end{bmatrix} \leq Z_m \tag{2.32}
\]

since by assumption we have \( Z_i < Z_j \) for \( i > j \). It is clear from (2.32) that the optimal strategy for the transmitter, against \( \hat{y} \), is to use at most the same number of pure strategies, \((m + 1)\), i.e.

\[
x^T = \begin{bmatrix} x_0 & \cdots & x_i & \cdots & x_m & 0_{1 \times (N_T - m)} \end{bmatrix}_{1 \times (N_T + 1)} \tag{2.33}
\]

any strategy other than (2.33) results in a lower expected payoff for the transmitter.

Since \( \hat{y} \) given in (2.30) is not necessarily an optimal mixed-strategy for the jammer, the optimal strategy would result in an expected payoff less than \( Z_m \). Therefore, \( Z_m \) can be used as an upper bound for the game value at the NE and all mixed-strategies with average jamming power \( J_{ave} = J_{ave,m} \). We present this fact in the following lemma.

**Lemma 3.** Assume the jammer’s average power is given by (2.31). Then for the constrained two-player zero-sum game defined in Theorem[1] the following inequality holds

\[
Z(x^*, y^*) \leq Z(x, \hat{y}) \leq Z_m \quad J_{ave} = J_{ave,m} \quad \forall x \in X \tag{2.34}
\]

As discussed above, the optimal strategy for the transmitter against \( \hat{y} \) is to use, at most, \((m + 1)\) of his strategies. Consider the mixed-strategy \( \hat{x} \) given by

\[
\hat{x}^T = \begin{bmatrix} x_0 & \cdots & x_i & \cdots & x_m & 0_{1 \times (N_T - m)} \end{bmatrix} = \begin{bmatrix} b_0 Z_0^{-1} & \cdots & b_i Z_i^{-1} & \cdots & b_m Z_m^{-1} & 0_{1 \times (N_T - m)} \end{bmatrix} \tag{2.35}
\]

where selection of \( b_i \)'s will be discussed later. Since \( \hat{x} \) is a probability distribution, we must have

\[
\sum_{i=0}^{m} x_i = \sum_{i=0}^{m} b_i Z_i^{-1} = 1 \quad b_i > 0 \quad i = 0, \cdots, m \tag{2.36}
\]
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

Assume the transmitter is using the mixed-strategy \( \hat{\mathbf{x}} \), which is not necessarily optimal, against jammer’s arbitrary strategy \( \mathbf{y} \in \mathbf{Y}_{\text{LE}|J_{\text{ave}}} \). The expected payoff of the game is

\[
Z(\hat{\mathbf{x}}, \mathbf{y}) = \hat{\mathbf{x}}^T Z \mathbf{y} = y_0 \sum_{i=0}^{m} b_i + \cdots + y_j \sum_{i=j}^{m} b_i + \cdots + y_m b_m
\]

\[
= B \sum_{j=0}^{m} y_j - \left( y_0 b_0 + \cdots + y_j \sum_{i=0}^{j-1} b_i + \cdots + y_m \sum_{i=0}^{m-1} b_i \right)
\]

(2.37)

where \( B = \sum_{i=0}^{m} b_i \). Since we assumed \( J_{\text{ave}} < J_{\text{TH}} \), a rational jammer would use all his available power, i.e.

\[
\mathbf{J}^T \mathbf{y} = \sum_{j=1}^{m} J_j y_j = J_{\text{ave}} \quad \text{where} \quad \mathbf{y} \in \mathbf{Y}_{\text{LE}|J_{\text{ave}}}
\]

\[
J_1 < \cdots < J_j < \cdots < J_m
\]

(2.38)

Let the sum of the terms in the parentheses in relation (2.37) be proportional to the jammer’s average power, i.e.,

\[
\sum_{i=0}^{j-1} b_i = d J_j \quad \text{where} \quad d > 0
\]

\[
\Rightarrow b_i = d (J_{i+1} - J_i) \quad \text{where} \quad d > 0
\]

(2.39)

Then (2.37) becomes independent of jammer’s strategy and the expected payoff of the game is

\[
Z(\hat{\mathbf{x}}, \mathbf{y}) = \hat{\mathbf{x}}^T Z \mathbf{y} = \sum_{i=0}^{m} b_i - d J_{\text{ave}} \forall \mathbf{y} \in \mathbf{Y}_{\text{LE}|J_{\text{ave}}}
\]

\[
= b_m + d (J_m - J_{\text{ave}})
\]

(2.40)

It is clear from (2.40) that a rational jammer should use all his available power, \( J_{\text{ave}} \), to achieve the lowest possible expected payoff against \( \hat{\mathbf{x}} \). If we substitute \( b_i \)'s from (2.39)
in (2.36) we have
\[
 b_m Z_m^{-1} + d \sum_{j=0}^{m-1} (J_{j+1} - J_j) Z_j^{-1} = 1
\]

\[
 b_m + dZ_m \left( \sum_{j=0}^{m-1} J_{j+1} Z_j^{-1} - \sum_{j=0}^{m-1} J_j Z_j^{-1} \right) = Z_m
\]

\[
 b_m + dZ_m \left( \sum_{j=1}^{m} J_j Z_{j-1}^{-1} - \sum_{j=1}^{m} J_j Z_j^{-1} + J_m Z_m^{-1} \right) = Z_m
\]

\[
 b_m + d \left( J_m - \sum_{j=1}^{m} (Z_j^{-1} - Z_{j-1}^{-1}) J_j \right) = Z_m
\]

and finally from (2.31) we have
\[
 b_m + d(J_m - J_{\text{ave},m}) = Z_m
\]

If we substitute $J_{\text{ave}}$ with $J_{\text{ave},m}$ in (2.40) and use (2.42) we have
\[
 Z (\hat{x}, y) = b_m + d(J_m - J_{\text{ave},m}) = Z_m \quad \forall y \in Y_{\text{LE}|J_{\text{ave},m}}
\]

Since $\hat{x}$ given in (2.35) is not necessarily an optimal mixed-strategy, the optimal strategy for the transmitter results in an expected payoff greater than $Z_m$. Therefore, $Z_m$ could be used as a lower bound for the game value at the NE and all mixed-strategies with average jamming power $J_{\text{ave}} = J_{\text{ave},m}$. We present this result in the following lemma.

**Lemma 4.** Assume the jammer’s average power is given by (2.31). Then for the constrained two-player zero-sum game defined in Theorem 1 the following inequality holds
\[
 Z(x^*, y^*) \geq Z(\hat{x}, y) \geq Z_m \quad J_{\text{ave}} = J_{\text{ave},m}
\]
\[
 \forall y \in Y_{\text{LE}|J_{\text{ave},m}}
\]

However, from Lemma 3 and for $J_{\text{ave}} = J_{\text{ave},m}$ we know the game value cannot be more than $Z_m$ and hence the game value at the NE is indeed $Z_m$ and $\hat{x}$ and $\hat{y}$ given in (2.35) and (2.30) are optimal mixed-strategies for the transmitter and the jammer, respectively. Since by assumption, $m < N_T$, we have $J_{m+1} \in J$, therefore we can let
\[
 b_m = d (J_{m+1} - J_m)
\]
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

and from (2.43) we have

\[ d = \frac{Z - m}{J_{m+1} - J_{\text{ave},m}} \quad \text{for} \quad 0 \leq m < N_T \]

\[ b_i = \frac{J_{i+1} - J_i}{J_{m+1} - J_{\text{ave},m}} Z_m \quad i = 0, \ldots, m \]  

substituting (2.46) in (2.35), the transmitter’s optimal mixed-strategy becomes

\[ \hat{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_i \\ \vdots \\ x_m \\ 0_{N_T-m} \end{bmatrix} = \begin{bmatrix} \frac{(J_{i+1} - J_i)Z_i^{-1}}{J_{m+1} - J_{\text{ave},m}} \\ \vdots \\ \frac{(J_{m+1} - J_m)Z_m^{-1}}{J_{m+1} - J_{\text{ave},m}} \\ 0_{N_T-m} \end{bmatrix} \]  

We summarize the results we derived so far in the following theorem.

**Theorem 2.** Consider the constrained two-player zero-sum game defined by utility function (2.7), payoff matrix (2.9), and transmitter and jammer mixed-strategy sets \( X \) and \( Y \), defined in (2.6) and (2.4), respectively. Then, the expected value of the game at the Nash-Equilibrium is

\[ Z(\hat{x}^*, \hat{y}^*) = Z_m \]

and \( J_{\text{ave},m} \) is given by

\[ J_{\text{ave},m} = Z_m \sum_{j=1}^{m} (Z_j^{-1} - Z_{j-1}^{-1}) J_j \quad 0 \leq m < N_T \]

Furthermore, \( \hat{x}^* \) and \( \hat{y}^* \) are given by (2.47) and (2.30), respectively.

If we define the effective jamming power, \( J_{\text{eff}} \), to be the jamming power a reactive non-strategic jammer (i.e., a jammer that uses only pure strategies) would need to force the same operating point at the NE (\( Z_m \) in this case) then, the effective jamming power becomes

\[ J_{\text{eff}} = J_m > J_{\text{ave},m} \quad \text{for} \quad 0 < m < N_T \]  

which means that randomizing helps the jammer to achieve the same performance as the reactive non-strategic jammer with less average jamming power.
2.4.1.2 Case 2; Optimal Strategies for the General Case

In the previous section we established the optimal mixed-strategies for $J_{\text{ave}} = J_{\text{ave},m}$. Consider the more general case where the jammer’s average power is not necessarily equal to $J_{\text{ave},m}$ for some $0 \leq m < N_T$. Obviously we have $J_{\text{ave},m} \leq J_{\text{ave}} < J_{\text{ave},m+1}$ for some $m \in \{0, \ldots, N_T - 1\}$. Let

$$J_{\text{ave}} = J_{\text{ave},m} + \varepsilon = Z_m \sum_{j=1}^{m} (Z^{-1}_j - Z^{-1}_{j-1}) J_j + \varepsilon$$

for some $m \in \{0, \ldots, N_T - 1\}$

Consider the mixed-strategy $\hat{y}$ for the jammer according to

$$\hat{y} = aZ_m \begin{bmatrix} Z_0^{-1} \\ \vdots \\ Z_j^{-1} - Z_{j-1}^{-1} \\ \vdots \\ Z_m^{-1} - Z_{m-1}^{-1} \\ \hat{y}_{m+1} \\ 0_{(N_T - m - 1) \times 1} \end{bmatrix}_{(N_T+1) \times 1}$$

Since $\hat{y}$ is a probability distribution we must have

$$\sum_{j=0}^{m+1} y_j = aZ_m (Z_m^{-1} + \hat{y}_{m+1}) = 1 \Rightarrow a = \frac{1}{1 + Z_m \hat{y}_{m+1}}$$

and from (2.50) we can rewrite the expression for the jammer’s average power as

$$J_{\text{ave}} = \sum_{j=1}^{m+1} y_j J_j = aJ_{\text{ave},m} + aZ_m \hat{y}_{m+1} J_{m+1}$$

from (2.49) \( aJ_{\text{ave},m} + aZ_m \hat{y}_{m+1} J_{m+1} = J_{\text{ave},m} + \varepsilon \quad \text{and hence } a \text{ and } \hat{y}_{m+1} \text{ become} \)

$$\hat{y}_{m+1} = \frac{J_{\text{ave}} - J_{\text{ave},m}}{J_{m+1} - J_{\text{ave}}} Z_m^{-1} \text{ and } a = \frac{J_{m+1} - J_{\text{ave},m}}{J_{m+1} - J_{\text{ave}}}$$
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

Assume the jammer is using the mixed-strategy given in (2.50) against the transmitter’s arbitrary strategy. Then the expected payoff of the game is

\[
Z(x, \hat{y}) = aZm x^T \begin{bmatrix}
1_{(m+1)\times 1} \\
Zm+1 Zm^{-1} \hat{y}_{m+1} \\
\vdots
\end{bmatrix} = \frac{Jm+1 - Jave}{Jm+1 - Jave,m} Zm
\]

(2.54)

Where we have used the fact that a rational transmitter would only use up to \((m + 1)\) of his strategies since otherwise the expected payoff of the game would be even less. As before, since \(\hat{y}\) is not necessarily an optimal mixed-strategy, the corresponding expected payoff can be used as an upper bound for the game and hence, we have the following lemma.

**Lemma 5.** Let us assume that the jammer’s average power, \(J_{\text{ave}}\), satisfies \(J_{\text{ave},m} < J_{\text{ave}} < J_{\text{ave},m+1}\) for some \(m \in \{0, \ldots, N_T - 1\}\). Then for the constrained two-player zero-sum game defined in Theorem 7 and for all \(x \in X\) the following inequality holds

\[
Z(x^*, y^*) \leq Z(x, \hat{y}) \leq \frac{Jm+1 - Jave}{Jm+1 - Jave,m} Zm
\]

(2.55)

where \(\hat{y}\) is given in (2.50).

Now assume the transmitter is using the same mixed-strategy given in (2.35). From (2.40) and (2.46) we have

\[
Z(\hat{x}, y) = b_m + d (J_m - J_{\text{ave}}) = d (Jm+1 - J_{\text{ave}})
\]

\[
= \frac{Jm+1 - Jave}{Jm+1 - Jave,m} Zm
\]

(2.56)

which is the same expression for the upper bound of the game derived in Lemma 5. As a result we have the following theorem.

**Theorem 3.** Consider a constrained two-player zero-sum game defined by utility function (2.7), payoff matrix (2.9), and transmitter and jammer mixed-strategy sets \(X\) and \(Y_{\text{LE}}|J_{\text{ave}}\) given in (2.6) and (2.4), respectively. Assume the jammer’s average power \(J_{\text{ave}}\) satisfies

\[
J_{\text{ave},m} < J_{\text{ave}} < J_{\text{ave},m+1}\text{ for some } m \in \{0, \ldots, N_T - 1\}
\]

(2.57)

Then, the expected value of the game at the NE is

\[
Z(x^*, y^*) = \frac{Jm+1 - Jave}{Jm+1 - Jave,m} Zm
\]

(2.58)

46
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

Table 2.1: Summary of the Results

\[ Z(\mathbf{x}^*, \mathbf{y}^*) = \mathbf{x}^* \mathbf{Z} \mathbf{y}^* = \frac{J_{m+1} - J_{\text{ave}, m}}{J_{m+1} - J_{\text{ave}, m}} \mathbf{Z}_m \quad J_{\text{ave}, m} \leq J_{\text{ave}} < J_{\text{ave}, m+1} \]

\[ J_{\text{ave}, m} = Z_m \sum_{j=1}^m (Z_j^{-1} - Z_{j-1}^{-1}) J_j \quad m = 0, \ldots, N_T - 1 \]

<table>
<thead>
<tr>
<th>Transmitter’s Optimal Mixed-Strategy</th>
<th>Jammer’s Optimal Mixed-Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^* = \frac{Z_m}{J_{m+1} - J_{\text{ave}, m}} \begin{bmatrix} (J_1 - J_0)Z_0^{-1} \ \vdots \ (J_{i+1} - J_i)Z_i^{-1} \ \vdots \ (J_{m+1} - J_m)Z_m^{-1} \end{bmatrix} )</td>
<td>( y^* = \frac{J_{m+1} - J_{\text{ave}, m}}{J_{m+1} - J_{\text{ave}, N_T}} \mathbf{Z}<em>m \begin{bmatrix} Z_0^{-1} \ \vdots \ Z_j^{-1} - Z</em>{j-1}^{-1} \ \vdots \ Z_m^{-1} - Z_{m-1}^{-1} \end{bmatrix} )</td>
</tr>
</tbody>
</table>

where \( x^* \) and \( y^* \) are the transmitter’s and the jammer’s optimal mixed-strategies, respectively, and \( J_{\text{ave}, m} \) is given by (2.31). Furthermore, \( x^* \) and \( y^* \) are given by (2.47) and (2.50), respectively.

Table 2.4.1.2 shows the summary of the results derived so far.

2.4.2 Optimal Mixed-Strategies for \( J_{\text{ave}} \geq J_{\text{TH}} \)

2.4.3 Optimal Mixed-Strategies for

Let us assume in Lemma 3 we let \( m = N_T \), then we have

\[ Z(\mathbf{x}^*, \mathbf{y}^*) \leq Z_{N_T} \quad J_{\text{ave}} = J_{\text{ave}, N_T} \quad \forall \mathbf{x} \in X \quad (2.59) \]

Since the game value cannot be less than \( Z_{N_T} \), we conclude that \( Z(\mathbf{x}^*, \mathbf{y}^*) = Z_{N_T} \). But from Theorem 1 we know that for \( J_{\text{ave}} \geq J_{\text{TH}} \) the game value at the NE is also \( Z_{N_T} \) and since \( J_{\text{ave}, N_T} \) is the smallest average jamming power for which the game value at NE is equal to \( Z_{N_T} \), we conclude that \( J_{\text{TH}} = J_{\text{ave}, N_T} \). We summarize this result in the following theorem.
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

Theorem 4. Consider the constrained two-player zero-sum game defined by utility function (2.7), payoff matrix (2.9), and transmitter and jammer mixed-strategy sets \( X \) and \( Y_{LE} \), defined in (2.6) and (2.4), respectively. Then, there exists a jamming power threshold, \( J_{TH} < J_{max} \), such that

\[
Z(x^*, y^*) = Z_{NT} \quad \forall J_{ave} \geq J_{TH}
\]

where the value of \( J_{TH} \) is given by

\[
J_{TH} = Z_{NT} \sum_{j=1}^{N_T} (Z_j^{-1} - Z_{j-1}^{-1}) J_j \quad J_j \in J \tag{2.60}
\]

Furthermore, the jammer’s optimal mixed-strategy with the lowest average power, that can achieve the NE is given by

\[
y^T = Z_{NT} \begin{bmatrix} z_0^{-1} & \cdots & z_{j-1}^{-1} & \cdots & z_{N_T-1}^{-1} \end{bmatrix} 1 \times (N_T+1)
\]

In other words, if the average jamming power exceeds \( J_{TH} \) given by (2.60), then the optimal strategic jammer (i.e., the jammer which uses optimal mixed-strategy) can force the expected payoff equal to the transmitter’s lowest payoff at the NE. This expected payoff is equal to the reactive non-strategic jammer with average power \( J_{max} \), i.e., \( J_{eff} = J_{max} > J_{TH} \).

2.5 Packetized Wireless AWGN Link Under Power Limited Jamming

In this section we study two typical jamming scenarios and we show that the framework defined in previous sections can be used to determine the optimal transmission and jamming strategies. Even though our analytical results are not limited to the AWGN channel, for simplicity, we use the packetized AWGN channel model and we assume packets are long enough that channel capacity theorem could be applied to each packet being transmitted.

2.5.1 Rate Adaptation in Packetized Links

In this section we study a special case of the game defined in the previous sections. We assume the communication link between the transmitter and the receiver is a single-hop,
packetized (discrete-time), AWGN channel with fixed and known noise variance, $N$, and the communication link is being disrupted by an additive adversary. We assume the jammer is an additive Gaussian jammer with flat power spectral density. It can be shown [84] that in the AWGN channel with a fixed and known noise variance, an iid Gaussian jammer is the most effective jammer in minimizing the capacity between the transmitter and the receiver. The effect of the Gaussian jammer on the communication link is reduction of the effective signal to noise ratio (SNR) at the receiver from $P_T/N$ to $P_T/(N + J)$, where $J$ represents the jammer power (variance) and $P_T$ is the transmitter power, both measured at the receiver side.

The transmitter has a rate adaptation block which allows him to transmit at $(N_T + 1)$ different but fixed rates according to the system’s design specifications. Any rate other than these given rates is not a feasible option for the transmitter. Additionally, the transmission rates are bounded between a minimum and a maximum transmission rate denoted by $R_{\min}$ and $R_{\max}$, respectively, i.e., $R_{\min} \leq R_i \leq R_{\max}$ for $i = 0, \ldots, N_T$. Without loss of generality, we assume the rates are sorted in a decreasing order. Hence, the transmitter’s action set becomes

$$\mathcal{R} = \{ R_0 = R_{\max} > \cdots > R_i > \cdots > R_{N_T} = R_{\min} \}$$ (2.61)

The transmitter’s goal is to maximize the achievable expected transmission rate over the channel. Assuming that transmission at channel capacity is possible, and from the capacity of discrete-time AWGN channel, the transmission power must at least be equal to

$$R_{\max} = R_0 = \frac{1}{2} \log \left( 1 + \frac{P_T}{N} \right) \text{ (nats/transmission)} \quad (2.62)$$

$$\Rightarrow \quad P_T = N \left( e^{2R_0} - 1 \right)$$

Throughout the rest of this section, we assume transmission at channel capacity and the transmission power is fixed and given by (2.62). Given that the channel noise variance is assumed to be fixed and known, corresponding to each transmission rate $R_j \in \mathcal{R}$ there exists a certain jammer power, $\hat{J}_j \geq 0$, below which reliable transmission is possible, i.e.

$$R_j = \frac{1}{2} \log \left( 1 + \frac{P_T}{N + \hat{J}_j} \right) ; \quad j = 0, \ldots, N_T$$

$$\Rightarrow \quad \hat{J}_j = N \frac{e^{2R_0} - e^{2R_j}}{e^{2R_j} - 1} ; \quad j = 0, \ldots, N_T$$ (2.63)
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

With this notation, we can define a one to one correspondence between transmitter rates and jammer power levels. Therefore, we can use \( \hat{J}_j \) given in (2.63) and/or \( R_j \) for \( j = 0, \cdots, N_T \) to refer to transmitter strategies interchangeably.

Assume the jammer’s goal is to force the transmitter to operate at his lowest rate, \( R_{N_T} \), while keeping the lowest possible average and maximum jamming power. As a result of Lemma 1, the jammer does not need to use more strategies than the transmitter i.e., he only needs \( (N_T + 1) \) jamming power levels. Consider the following action set for the jammer

\[
\mathcal{J} = \{J_0, J_1, \cdots, J_j, \cdots, J_{N_T}\}
\]

The term \( \delta N \) with \( \delta > 0 \) is an extra added jamming power to the non-zero jamming powers to make sure that \( R_{j-1} \) is greater than channel capacity for the jamming power \( J_j \). Since the transmitter’s goal is to achieve the maximum possible expected transmission rate and the jammer’s goal is to minimize the same expected value, we can define the utility function based on the capacity of the discrete-time AWGN channel, i.e.

\[
C(R_j, J_j) = \begin{cases} 
R_j & \hat{J}_j \geq J_j \\
0 & \hat{J}_j < J_j
\end{cases} \quad R_j \in \mathcal{R} \text{ and } J_j \in \mathcal{J}
\]

The utility function defined in (2.65) has the same format of the general utility function defined in (2.7) and it can be easily verified that the \( R_j \) as defined in (2.63) is a strictly decreasing function of \( j \), or equivalently \( \hat{J}_j \). Hence, we can directly apply Theorem 4 to find the minimum average jamming power or the jamming power threshold, \( J_{TH} \). Substituting (2.63) and (2.64) in (2.60) and simplifying the result, the jamming threshold becomes

\[
J_{TH} = R_{N_T} \sum_{j=1}^{N_T} \left( \frac{1}{R_j} - \frac{1}{R_{j-1}} \right) J_j = N \delta \left( 1 - \frac{R_{N_T}}{R_0} \right) \\
+ NR_{N_T} \sum_{j=1}^{N_T} \left[ \left( \frac{1}{R_j} - \frac{1}{R_{j-1}} \right) e^{2R_0} - \frac{e^{2R_{j-1}}}{e^{2R_{j-1}} - 1} \right]
\]

(2.66)
and the jammer’s optimal mixed-strategy with the minimum average jamming power that achieves the NE is given by

\[ \mathbf{y}^T = R_{N_T} \left[ R_0^{-1} \cdots R_{j-1}^{-1} \cdots R_{N_T}^{-1} - R_{j}^{-1} \cdots R_{N_T}^{-1} \right]_{1 \times (N_T+1)} \]  

(2.67)

We define the jammer’s randomization gain to be the power advantage that he gains for switching from pure-strategies (i.e., reactive non-strategic jammer) to optimal mixed-strategies. With this definition, the randomization gain becomes the ratio of the jammer’s pure strategy power \( J_{\text{max}} \) in this case over the average power of the optimal mixed-strategy that forces the same expected payoff at the NE, i.e.

\[ \text{Randomization Gain} = \frac{J_{\text{max}}}{J_{\text{TH}}} = \frac{J_{N_T}}{J_{\text{TH}}} > 1 \]  

(2.68)

To provide a numerical example, assume, a single-hop jamming resilient communication system that uses the typical rates of the IEEE 802.11x standard i.e., the available coded data rates of the communication system are

\[ R_a = \left\{ 54, 51, 48, \cdots, 12, 9, 6 \right\} \quad \text{(Mb/s)} \]  

(2.69)

Figure 2.7(top) shows the randomization gain of the optimal jammer as a function of the expected transmission rate at the NE for this example. The figure is sketched for continuous AWGN channel and typical values of \( N \) and \( \delta \). For this typical example, the randomization gain of the jammer is

\[ 3 \text{ dB} \approx 1.8 \leq \text{Randomization Gain} \leq 16 \approx 12.5 \text{ dB} \]  

(2.70)

Figure 2.7(top) also provides a comparison between the reactive non-strategic jammer and the optimal strategic jammer. As expected, the optimal strategic jammer requires less average power than the reactive non-strategic jammer to force the same expected rate at the NE. Figure 2.7(bottom) shows a typical optimal transmission strategy for the transmitter in this case.

### 2.5.2 Uniformly Distributed Jamming Powers

Consider a communication link with the same setup defined in Section 2.5.1. Assume the jammer is using \((N_T + 1)\) discrete jamming power levels equally spaced in the interval
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

Figure 2.7: Jammer's randomization gain and average power as a function of expected rate at the NE (left), transmitter's typical optimal mixed-strategy for \( J_{\text{ave}} < J_{\text{TH}} \) (right).

\[ J = \left\{ J_j = \frac{j}{N_T} J_{\text{max}}; \quad 0 \leq j \leq N_T \right\} \quad (2.71) \]

The transmitter has a rate adaptation block which allows him to transmit at any arbitrary rate. Assume the transmitter’s goal is to maximize the achievable expected transmission rate over the discrete-time AWGN channel. As a result of Lemma 1, the optimal strategy for the transmitter is to use, at most, \((N_T + 1)\) rates where each rate corresponds to one of the jammer pure strategies. Assuming that transmission at the AWGN channel capacity is possible, we can define the achievable transmission rate based on the discrete-time AWGN channel capacity when the signal to noise ratio \( P_T/N \) is replaced by the signal to noise plus jamming power ratio \( P_T/N+J_{\text{ave}} \). In this case, the transmitter’s action set becomes

\[ \mathcal{R} = \left\{ R_i = \frac{1}{2} \log \left( 1 + \frac{P_T}{N + J_{\text{ave}}} \right); \quad 0 \leq i \leq N_T \right\} \quad (2.72) \]

In this special case, the jamming power set representing the transmitter’s action set, \( \mathcal{J}_{\text{T}} \), is identical to the jammer’s action set, \( \mathcal{J}_{\text{T}} = \mathcal{J} \). Since the transmitter’s goal is to achieve the maximum expected transmission rate over the channel, we can define the utility function of the game based on the AWGN channel capacity, i.e.,

\[ Z(J_{\text{T}}) = R(J_{\text{T}}) = \frac{1}{2} \log \left( 1 + \frac{P_T}{N + J_{\text{T}}} \right) \quad J_{\text{T}} \in \mathcal{J} \quad (2.73) \]
CHAPTER 2. CONSTRAINED ZERO-SUM GAMES

Given that at rates higher than capacity reliable communication is impossible and since \(R(J_T)\) defined in (2.73) is a strictly decreasing function, the framework defined in Section 2.2 can be applied to this special case. Thus, the results derived in Section 2.4 can be used to determine the optimal strategies and the expected value of the game at NE.

Assuming the jammer’s average power, \(J_{\text{ave}}\), satisfies \(J_{\text{ave},m} \leq J_{\text{ave}} < J_{\text{ave},m+1}\) for some \(m = 0, \ldots, N_T - 1\), the optimal mixed-strategies for the transmitter and the jammer simplify to

\[
x^* = \left( \sum_{i=0}^{m} R_i^{-1} \right)^{-1} \begin{bmatrix} R_0^{-1} & \cdots & R_i^{-1} & \cdots & R_{m+1}^{-1} 0_{1 \times (N_T-m)} \end{bmatrix}
\]

\[
y^* = \left( (m+1) - N_T \frac{J_{\text{ave}}}{J_{\text{max}}} \right) \left( \sum_{i=0}^{m} R_i^{-1} \right)^{-1} \begin{bmatrix} R_0^{-1} & \cdots & R_j^{-1} & \cdots & R_{m}^{-1} \sum_{i=0}^{m} R_{m+1}^{-1} - J_{\text{ave},m} \sum_{i=0}^{m} R_{m+1}^{-1} - J_{\text{ave},m} \end{bmatrix} \begin{bmatrix} 0_{1 \times (N_T-m-1) \times 1} \end{bmatrix}
\]

where \(J_{\text{ave},m}\) is given by

\[
J_{\text{ave},m} = \frac{1}{N_T} J_{\text{max}} \left( m + 1 \right) - R_m \sum_{i=0}^{m} R_i^{-1}
\]  

(2.74)

The expected value of the game at the NE, as function of the jammer’s average power, is given by

\[
R(J_{\text{ave}}) = \left( (m+1) - N_T \frac{J_{\text{ave}}}{J_{\text{max}}} \right) \left( \sum_{i=0}^{m} R_i^{-1} \right)^{-1}
\]  

(2.75)

In this special case it can be shown that an upper bound for jamming power threshold is given by

\[
J_{\text{TH,U}} = \frac{1}{2} N_T J_{\text{max}} \left( 1 - \frac{R_{\text{ave}}}{R_0} \right) J_{\text{max}} < \frac{1}{2} J_{\text{max}} \quad N_T \gg 1
\]  

(2.76)

and a simple strategy and an approximation to the jammer’s optimal strategy that achieves this bound is given by

\[
\dot{y}^T = [\dot{y}_0 \ \dot{y} \ \cdots \ \dot{y}] ; \ \dot{y}_0 = \frac{R_{\text{ave}}}{R_0} \text{ and } \dot{y} = \frac{1}{N_T} (1 - \dot{y}_0)
\]  

(2.77)
Proof. Assume the jammer is using a mixed-strategy, \( \hat{y} \), according to
\[
\hat{y} = \begin{bmatrix}
y_0 \\
y \\
\vdots \\
y
\end{bmatrix}_{(N_T+1) \times 1}
\]
where
\[
y_0 = 1 - \frac{2N_T}{N_T + 1} \cdot \frac{J_{\text{ave}}}{J_{\text{max}}} \\
y = \frac{2}{N_T + 1} \cdot \frac{J_{\text{ave}}}{J_{\text{max}}}
\]
(2.78)
It can easily be verified that \( \hat{y} \in Y_{\text{LE}|\text{ave}} \). Furthermore, assume the transmitter is using an arbitrary mixed-strategy in which the probability associated with the payoff \( R_i \), \( 0 \leq i \leq N_T \) is denoted by \( x_i \). Define \( R(x, \hat{y}) \) to be the expected payoff of the game for the transmitter’s arbitrary mixed-strategy, \( x \), against jammer’s mixed-strategy defined in (2.78)
\[
R(x, \hat{y}) = R_{-i,N_T} + R_{N_T} x_{N_T} + R_i x_i \Pr \left[ J \leq J_T = J_i \right]
\]
(2.79)
where \( R_{-i,N_T} \) denotes the partial expected payoff resulting from all pure strategies except for the \( i \)’th and \( N_T \)’th strategies. In order to improve his payoff, the transmitter deviates from his current strategy, \( x \), to a new strategy, \( x' \), where \( x'_{N_T} = x_{N_T} + \delta \) and \( x'_i = x_i - \delta \) and \( 0 < \delta \). Define \( R(x', \hat{y}) \) to be the expected payoff for the new strategy.
\[
R(x', \hat{y}) = R_{-i,N_T} + R_{N_T} (x_{N_T} + \delta) + R_i (x_i - \delta) \Pr \left[ J \leq J_T = J_i \right]
\]
(2.80)
\[
= R(x, \hat{y}) + \delta \left[ R_{N_T} - R_i \left( y_0 + iy \right) \right]
\]
Let \( \Delta R \) be the difference in the expected payoff of the game caused by deviating to the new strategy, i.e.,
\[
\Delta R = R(x', \hat{y}) - R(x, \hat{y})
\]
(2.81)
\[
= \delta \left[ R_{N_T} - R_i \left( 1 - 2 \cdot \frac{N_T - i}{N_T + 1} \cdot \frac{J_{\text{ave}}}{J_{\text{max}}} \right) \right]
\]
where \( \delta > 0 \) and \( 0 \leq i < N_T \). Assume (for now) that \( \Delta R \geq 0 \) then we can rewrite (2.81) as
\[
J_{\text{ave}} \geq J_{\text{max}} \left( 1 - \frac{R_{N_T}}{R_i} \right) \left( \frac{1}{2} \cdot \frac{N_T + 1}{N_T - i} \right) \triangleq U_i
\]
(2.82)
Define \( J_{\text{TH,U}} \) as
\[
J_{\text{TH,U}} = \max_{0 \leq i < N_T} J_{\text{max}} \left( 1 - \frac{R_{N_T}}{R_i} \right) \left( \frac{1}{2} \cdot \frac{N_T + 1}{N_T - i} \right)
\]
(2.83)
\(^3\)We will use the term semi-uniform to refer to this class of pmf.
then for $J_{\text{ave}} > J_{TH}$ and for all $\delta > 0$ and $0 \leq i < N_T$ the inequality in (2.82) is satisfied and hence $\Delta R > 0$. As a result, the transmitter can improve his payoff by dropping the probability of his $i$'th strategy and adding it to his $N_T$'s strategy (the strategy with the lowest payoff). Since the $i$'th strategy was chosen arbitrary, the transmitter can improve his expected payoff by dropping probability from all strategies, except the $N_T$'th strategy, and adding them to the $N_T$'th strategy. This process can be continued until all probabilities are accumulated in $x_{N_T}$ and no further improvement to the expected payoff is possible.

In general, it can be shown that (2.83) is maximized for $i = 0$ (See Appendix A) which results in the desired upper bound and the mixed-strategy given in (2.76) and (2.78), respectively. The penalty in using the semi-uniform strategy instead of the optimal mixed-strategy is that the jammer requires greater jamming power to force the same expected NE.

Figure 2.8 (top) shows the expected transmission rate at the NE as a function of the jammer’s average power for typical values of $P_T$, $N$ and $J_{\text{max}}$. The figure is sketched for four different jammers; optimal strategic jammer, semi-uniform jammer, reactive non-strategic jammer and random strategic jammer (a jammer that uses a random mixed-strategy). As it can be verified by Figure 2.8 (and numerical simulations), all non-optimal jamming strategies under-perform the optimal strategic jammer. Figure 2.8 (bottom), shows a typical
optimal mixed-strategy with the lowest average power that can force $R_{\text{min}}$ at the NE. The randomization gain of this strategy is

$$\text{Randomization Gain} \approx 2.9 = 4.6 \text{ dB} \quad (2.84)$$

Here we showed by analytical and numerical results that randomization can significantly assist a power-limited jammer. In Chapter 4, we propose the adaptive superposition coding scheme for the transmitter to recover some of the performance loss due to the jammer’s randomization. Adaptive superposition coding allows the transmitter to simultaneously transmit multiple code layers with different code rates. We further show that by randomizing the code rates across the superimposed layers better average performance is achievable.

### 2.6 Concluding Remarks

In this chapter, we formulated the interaction between an adaptive transmitter (a transmitter with multiple transmission choices) and a smart power limited jammer in a game theoretic context. We showed that packetization and adaptivity benefits a smart jammer. While the standard information-theoretic performance results for a jammed channel corresponds to pure Nash equilibrium, packetized adaptive communication leads to a lower expected value and a mixed-strategy Nash Equilibrium. Inspired by the Shannon’s capacity theorem, we defined a general utility function and a payoff matrix which may be applied to a variety of jamming problems. Furthermore, we showed the existence of optimal mixed-strategy NE for the transmitter and the jammer. We showed the existence of a threshold on jammer’s average power such that if the jammer’s average power exceeds this threshold then the expected value of the game at the NE corresponds to the transmitter’s lowest payoff, this is as if the jammer was using the maximum jamming power all the time. Finally, we studied an special case of optimal strategies in a discrete-time AWGN wireless channel under jamming and showed that randomization can significantly assist a smart jammer with limited average power.
In zero-sum game framework, it is usually assumed that the players have perfect knowledge of the game and the actions that are available to the other players, and they use this knowledge to compute their respective optimal strategies. In such a case, the zero-sum framework fully captures the conflicting goals of the players. Moreover, the equilibrium solution of the zero-sum game guarantees a minimum payoff regardless of the other player’s strategy [77].

However, in some jamming scenarios, having perfect knowledge of the system parameters (or available actions) may not be a feasible option or too costly for a player. In addition, players may have objectives that are not exactly the opposite of each other, for example, the transmitter may wish to minimize the average error probability while the jammer wishes to minimize the average throughput of the system (as opposed to maximizing the average error probability).

In such scenarios, a more appropriate framework to model the communication system under jamming would be a bimatrix game instead of a zero-sum game[1]. In bimatrix games it is no longer required that the sum of the players’ payoffs to be zero (or a constant value) [77]. As a result, players can have different objectives and the respective payoffs can be defined based on the players’ goals and their knowledge of the game (which in general may be

---

[1] It can be shown that zero-sum games are special cases of the more general bimatrix games.
imperfect). Such a formulation, encompasses a variety of situations from full competition to full cooperation.

Additionally, in standard zero-sum and bimatrix games there are no additional restrictions on players’ mixed-strategies, i.e., players may choose any probability distribution over their respective action sets (pure-strategies). However, there exist scenarios where, due to practical reasons, not all mixed-strategies are permitted and/or feasible.

Such scenarios demand for a more general framework to study them. In this paper, we study a constrained bimatrix game to overcome these limitations. In constrained games, the players’ mixed-strategies not only have to be a probability distributions but they must satisfy some additional constraints too (Figure 1.5 shows the classification of standard and constrained games). We study the necessary and sufficient conditions under which the existence of the Nash equilibrium is guaranteed as well as a systematic approach to find the NE.

The rest of this chapter is organized as follows, in Section 3.1 we will introduce the constrained bimatrix framework and provide the necessary and sufficient conditions under which the existence of a constrained NE solution is guaranteed. In Section 3.2 we show that the solution of the this constrained game corresponds to the global maximizers of a quadratic program. In Section 3.3 we will use the framework that we developed to study a typical jamming problem. Finally, we conclude this chapter in Section 3.4.

3.1 General-sum Games

We start by introducing the concept of the Nash equilibrium (NE) for standard bimatrix games. Then, we generalize the standard bimatrix framework by adding linear constraints on the players strategies and formulate the constraint bimatrix framework. We refer the reader to [2] and the references therein for an introduction to some of the most fundamental concepts of non-cooperative game theory. This tutorial is specifically written for wireless network engineers and uses intuitive examples that are focused on wireless networks. [1] provides a more comprehensive review of non-cooperative game theory and its applications in wireless communications and networking. Throughout the rest of the paper, we refer to
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

player one (row player) as the transmitter and refer to player two (column player) as the jammer. Nevertheless, applications of our framework is not limited to jamming in wireless communications.

3.1.1 Standard Bimatrix Games

Consider a game where transmitter’s action set (for instance, transmission rates) is given by
\[
\mathcal{R} = \{r_1, r_2, \ldots, r_m\} \quad r_i \in \mathbb{R}_+, \ 1 \leq i \leq m \tag{3.1}
\]
Without loss of generality assume \(\mathcal{R}\) is a sorted set, i.e., \(0 \leq r_1 < \cdots < r_{i-1} < r_i < \cdots < r_m\). Transmitter’s vector of possible actions (simply, action vector) is the column vector \(r^T\) defined as
\[
r^T = [r_1 \ r_2 \ \cdots \ r_i \ \cdots \ r_m]_{1 \times m} \tag{3.2}
\]
where \(^T\) indicates matrix transposition. Similarly, we define the jammer’s action set and action vector as
\[
\mathcal{J} = \{j_1, j_2, \ldots, j_n\} \quad j_k \in \mathbb{R}_+, \ 1 \leq k \leq n \tag{3.3}
\]
and
\[
j^T = [j_1 \ j_2 \ \cdots \ j_k \ \cdots \ j_n]_{1 \times n} \tag{3.4}
\]
where without loss of generality we assume, \(0 \leq j_1 < \cdots < j_{k-1} < j_k < \cdots < j_n\). A standard bimatrix game (also known as two-player general sum game) is defined by a pair of \(m \times n\) matrices \(A\) and \(B\) such that, if player one plays row \(i\) and player two plays column \(k\), the elements at row \(i\) and column \(k\) of the matrices \(A\) and \(B\) (i.e., \(a_{ik}\) and \(b_{ik}\)) would be the payoffs received by players one and two, respectively.

If we let the players randomize their actions (i.e., allow them to use mixed-strategies\(^2\)), the expected payoffs of the game for the mixed strategy profile \((x, y)\) are
\[
A(x, y) \triangleq x^T A \ y \quad \text{for player I} \tag{3.5}
\]
\[
B(x, y) \triangleq x^T B \ y \quad \text{for player II}
\]

\(^2\)A player is playing a mixed-strategy if he randomizes his actions over his action set according to a probability distribution.
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

where \( x \in X^m \triangleq \{ x \in \mathbb{R}^m_+ \mid \sum_{i=1}^{m} x_i = 1 \} \) and \( y \in Y^n \triangleq \{ y \in \mathbb{R}^n_+ \mid \sum_{k=1}^{n} y_k = 1 \} \) are mixed-strategy vectors of players one and two, respectively. Player one’s goal is to find an optimal strategy, \( x \), that maximize his expected payoff (given by payoff matrix \( A \)) against player two’s strategy, \( y \), i.e., player one wants to solve the following problem

\[
\max_{x \in X^m} A(x, y) \quad \text{for all} \quad y \in Y^n
\]  

(3.6)

while player two’s goal is to maximize his own payoff (given by payoff matrix \( B \)) by solving the following problem

\[
\max_{y \in Y^n} B(x, y) \quad \text{for all} \quad x \in X^m
\]  

(3.7)

The strategy profile \((x^*, y^*)\) is said to be an equilibrium pair (or equivalently the Nash equilibrium, NE) if \((x^*, y^*)\) satisfies (3.6) and (3.7) simultaneously. That is, \( x^* \) maximizes (3.6) for \( y^* \) and \( y^* \) maximizes (3.7) for \( x^* \), and therefore, no player benefits by unilaterally changing his strategy. Furthermore, following inequalities hold for an equilibrium pair \((x^*, y^*)\) of a bimatrix game.

\[
A(x^*, y^*) \geq A(x, y^*) \quad \text{for all} \quad x \in X^m
\]  

(3.8)

and

\[
B(x^*, y^*) \geq B(x^*, y) \quad \text{for all} \quad y \in Y^n
\]  

(3.9)

and finally, the following theorem states that every bimatrix game in its standard form has at least one NE.

**Theorem 5.** Every finite bimatrix game in its standard form has at least one equilibrium pair (Nash equilibrium) in mixed-strategies.

**Proof.** See Theorem 1 in [82].

\[ \square \]

### 3.1.2 Constrained Bimatrix Games

Consider a bimatrix game for which, due to practical reasons, not all mixed-strategies are permitted and/or are feasible. For instance, assume maximizing the average throughput of a wireless link. Maximizing the average throughput requires using higher transmission
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

rates; but to maintain an acceptable error rate at the receiver, higher rates must be transmitted at higher transmission power. Because of battery limitation (internal limitation) or the FCC regulations (external limitations), the transmitter must keep its average transmission power below a certain value. Consequently, the wireless user cannot use certain actions that are more preferable to him (such as transmitting at the highest rate all the time). He may only choose actions that result in an average transmission power less than or equal to a predetermined value.

Assume the mixed-strategy pair \( x \) and \( y \) must be chosen from some hyperpolyhedron defined by linear inequalities,

\[
x \in \tilde{X} \triangleq \{ x \in X^m \mid r^T x \leq r_{ave} \}
\]

and

\[
y \in \tilde{Y} \triangleq \{ y \in Y^n \mid j^T y \leq j_{ave} \}
\]

we denote this constraint game by \( G = (A, B, r, j, r_{ave}, j_{ave}) \) where \( A, B \in \mathbb{R}^{m \times n} \) are the payoff matrices for player one and two, respectively, and weight vectors \( r \) and \( j \) are given by (3.2) and (3.4), respectively. It is easily verified that for \( r_{ave} \geq \max r_i \) and \( j_{ave} \geq \max j_k \) the constrained game simplifies to the bimatrix game in its standard form; hence, the unconstrained game can be viewed as a special case of the constrained games (see Figure 1.5). Therefore, in the following we assume that at least one of the following inequalities holds

\[
r_{ave} < \max r_i \quad \text{or} \quad j_{ave} < \max j_k
\]

By introducing (3.10) and (3.11), and assuming that (3.12) holds, we are eliminating some mixed-strategies that could have been otherwise selected. Therefore, the existence of the NE solution for this constrained bimatrix game is not trivial and must be established (see Appendix B).

Assuming that in the constrained bimatrix game \( G = (A, B, r, j, r_{ave}, j_{ave}) \), the jammer is playing his optimal strategy \( y^* \). Transmitter’s optimal strategy, \( x^* \), against \( y^* \) is the

---

\( ^3 \)This is different from the bimatrix game in its standard form where the payoff matrices \( A \) and \( B \) completely define the game.
maximizer of the following problem

\[
\max_x x^T A^* y^* \quad \text{s.t.} \quad \begin{cases}
1^T x - 1 = 0 \\
r^T x - r_{\text{ave}} \leq 0 \\
-x \leq 0
\end{cases}
\] (3.13)

Similarly, jammer’s optimal strategy, \(y^*\), against \(x^*\) is the maximizer of the following problem

\[
\max_y x^* y^T B \quad \text{s.t.} \quad \begin{cases}
1^T y - 1 = 0 \\
j^T y - j_{\text{ave}} \leq 0 \\
-y \leq 0
\end{cases}
\] (3.14)

Individually, (3.13) and (3.14) are linear programs, but \(x^*\) and \(y^*\) are not known in advance or in general they may not even exist. Theorem 6 gives the necessary and sufficient conditions that any Nash equilibrium solution of \(G\) must satisfy. That is, every NE solution of \(G\) satisfies the conditions in Table 3.1 and every strategy pair \((x, y)\) that satisfies the conditions in Table 3.1 must be a NE \([86]\). Additionally, in the Appendix B, we prove the conditions under which existence of the NE for this constrained bimatrix game is guaranteed.

**Theorem 6.** Let \(G = (A, B, r, j, r_{\text{ave}}, j_{\text{ave}})\) be a constrained bimatrix game defined by matrices \(A, B \in \mathbb{R}^{m \times n}\). A strategy pair \((x^*, y^*)\) is an equilibrium pair (NE), if and only if there exists scalers \(u, v \geq 0\) and \(\alpha, \beta \in \mathbb{R}\) such that the conditions in Table 3.1 are satisfied.

**Proof.** Consider the KKT conditions for the linear program (3.13). The optimal solution \(x^*\) must satisfy the **primal feasibility** conditions given by

\[
\begin{cases}
1^T x^* - 1 = 0 \\
r^T x^* - r_{\text{ave}} \leq 0 \\
-x^* \leq 0
\end{cases}
\] (3.15)

which are identical to conditions (I.1) – (I.3) in Table 3.1. From the **dual feasibility** conditions we must have

\[
\nabla (x^* A y^*) - \sum_{i=1}^{m} \lambda_i \nabla (-x^*_i) - u \nabla (r^T x^* - r_{\text{ave}}) - \mu \nabla (1^T x^* - 1) = 0
\] (3.16)
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

such that

\[ \lambda_i \geq 0, \quad u \geq 0, \quad \mu \in \mathbb{R} \quad (3.17) \]

where \( \lambda_i, u \) and \( \mu \) are the KKT multipliers corresponding to constraints in (3.13). If we simplify (3.16) and use vector representations for KKT multipliers we get

\[ Ay^* + \lambda_{m \times 1} - ur - \mu 1_{m \times 1} = 0 \quad (3.18) \]

or, equivalently,

\[ Ay^* - ur - \mu 1_{m \times 1} \leq 0 \quad u \geq 0 \text{ and } \mu \in \mathbb{R} \quad (3.19) \]

which gives us condition (I.4) in Table 3.1 (where we have made a change of variable, \( \mu \rightarrow \alpha \), and have used the fact that \( \lambda_{m \times 1} \geq 0 \)). Finally, from the complementary slackness conditions we must have

\[ \left\{ \begin{array}{l}
\lambda^T x^* = 0 \\
u \left(r^T x^* - r_{ave} \right) = 0
\end{array} \right. \quad (3.20) \]

the second condition in (3.20) is identical to (I-6). By multiplying (3.18) by \( x^T \) and using \( \lambda^T x^* = 0 \) we have

\[ x^T Ay^* + x^T \lambda - u x^T r - \mu x^T 1 = 0 \quad (3.21) \]

which can be further simplified to

\[ x^T Ay^* - ur_{ave} - \mu = 0 \quad u \geq 0 \text{ and } \mu \in \mathbb{R} \quad (3.22) \]

which results in condition (I.5) in Table 3.1. In the exact same way, we can derive KKT’s necessary conditions of optimality for the jammer to get the conditions (II.1) – (II.7) in Table 3.1. To prove that these conditions are also sufficient, we can use the fact that the objective functions in (3.13) and (3.14) are linear (affine), and as a result, the KKT conditions are necessary and sufficient for optimality; this concludes the proof.

Table 3.1 summarizes the necessary and sufficient conditions of optimality for the constrained bimatrix game. Furthermore, the following lemma, gives the expected payoff of the players at the Nash equilibrium.
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

Table 3.1: Necessary and Sufficient Conditions for a Pair \((x^*, y^*)\) to be the NE.

<table>
<thead>
<tr>
<th>Player I</th>
<th>Player II</th>
</tr>
</thead>
<tbody>
<tr>
<td>1^T x^* - 1 = 0</td>
<td>1^T y^* - 1 = 0</td>
</tr>
<tr>
<td>r^T x^* - r_{ave} \leq 0</td>
<td>j^T y^* - j_{ave} \leq 0</td>
</tr>
<tr>
<td>-x^* \leq 0</td>
<td>-y^* \leq 0</td>
</tr>
<tr>
<td>A y^* - ur_{m\times1} - \alpha 1_{m\times1} \leq 0</td>
<td>x^T B - vJ_{n\times1} - \beta 1_{n\times1} \leq 0</td>
</tr>
<tr>
<td>x^T A y^* - ur_{ave} - \alpha = 0</td>
<td>x^T B y^* - vj_{ave} - \beta = 0</td>
</tr>
<tr>
<td>u (r^T x^* - r_{ave}) = 0</td>
<td>v (j^T y^* - j_{ave}) = 0</td>
</tr>
<tr>
<td>u \geq 0, \alpha \in \mathbb{R}</td>
<td>v \geq 0, \beta \in \mathbb{R}</td>
</tr>
</tbody>
</table>

**Lemma 6.** Consider the constrained bimatrix game \(G = (A, B, r, j, r_{ave}, j_{ave})\). The expected payoffs of the game for the equilibrium pair \((x^*, y^*)\) are

\[
A(x^*, y^*) = ur_{ave} + \alpha
\]

\[
B(x^*, y^*) = vj_{ave} + \beta
\]

for the transmitter and jammer, respectively.

**Proof.** Follows from conditions (I.5) and (II.5) in Table 3.1.

3.2 Connection to Quadratic Programming

While Theorem 6 gives the necessary and sufficient conditions for the strategy profile \((x^*, y^*)\) to be a NE of \(G\), it does not provide a constructive way to find the NE solution(s) and the equilibrium pairs of the constrained bimatrix game. We have previously shown [87] that for every constrained two-player zero-sum game there exists an equivalent linear program whose solution yields a NE for the game and every NE of the game is a solution of the corresponding linear program.

In this section, we show that there exist a similar connection between the NE solutions and equilibrium pairs of the constrained bimatrix games and global maximum(s) of a
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

quadratic program\[89\]. In the following theorem, we show that the global maximum of the quadratic program in (3.25) subject to the constraints in (3.26) satisfies all conditions of Theorem 6 and therefore, the corresponding maximizer is a NE solution of \( G \).

**Theorem 7.** Let \( G = (A, B, x, y, r_{\text{ave}}, j_{\text{ave}}) \) be a constrained bimatrix game with \( A, B \in \mathbb{R}^{m \times n} \). The strategy pair \((x^*, y^*)\) is a Nash equilibrium of \( G \) if and only if there exist scalers \( u^*, v^* \geq 0 \) and \( \alpha^*, \beta^* \in \mathbb{R} \) such that \((x^*, y^*, u^*, v^*, \alpha^*, \beta^*)\) is a global maximizer of the following quadratic program

\[
\begin{align*}
\text{maximize} & \quad x^T (A + B) y - ur_{\text{ave}} - vj_{\text{ave}} - \alpha - \beta \\
\text{subject to:} & \\
& \quad Ay - ur_{m \times 1} - \alpha 1_{m \times 1} \leq 0 \tag{3.26.1} \\
& \quad x^T B - vj_{n \times 1} - \beta 1_{n \times 1} \leq 0 \tag{3.26.2} \\
& \quad r^T x - r_{\text{ave}} \leq 0 \tag{3.26.3} \\
& \quad j^T y - j_{\text{ave}} \leq 0 \tag{3.26.4} \\
& \quad 1^T x - 1 = 0 \tag{3.26.5} \\
& \quad 1^T y - 1 = 0 \tag{3.26.6} \\
& \quad -x, -y \leq 0, -u, -v \leq 0 \text{ and } \alpha, \beta \in \mathbb{R} \tag{3.26.7}
\end{align*}
\]

**Proof.** First, notice that the constraints in (3.26) satisfy all the conditions of Table 3.1 except for (I.5), (I.6) and (II.5), (II.6). As a result, if we show that the global maximum of the quadratic program in (3.25) satisfies these additional conditions, then, by Theorem 6, it must be a NE solution of \( G \). If we premultiply (3.26.1) by \( x^T \) and use (3.26.3) to simplify the result we have

\[
x^T A y - ur_{\text{ave}} - \alpha \leq 0 \tag{3.27}
\]

since \( x^T \) is a probability vector and \( u \geq 0 \). Similarly, we can obtain the following inequality from (3.26.2) and (3.26.4):

\[
x^T B y - vj_{\text{ave}} - \beta \leq 0 \tag{3.28}
\]

by combining inequalities (3.27) and (3.28) we observe that

\[
f(x, y, u, v, \alpha, \beta) \equiv x^T (A + B) y - ur_{\text{ave}} - vj_{\text{ave}} - \alpha - \beta \leq 0
\]

\[\text{The connection between standard bimatrix games and quadratic programs was first shown in [88].}\]
Thus, any set of variables \( (x^*, y^*, u^*, v^*, \alpha^*, \beta^*) \) that satisfies
\[
 f(x^*, y^*, u^*, v^*, \alpha^*, \beta^*) = 0 \tag{3.29}
\]
is a global maximum of (3.25). Next, we will consider the KKT necessary conditions for optimality for the optimization problem in (3.25). To find the necessary KKT conditions, we stack the variables in the following vector and we take the gradients in the same order.
\[
 z_T \triangleq \begin{bmatrix} x_1^T \times m & y_1^T \times n & u & v & \alpha & \beta \end{bmatrix}^T_{(m+n+4) \times 1} \tag{3.30}
\]
From this point forward, we assume all variables are optimal and for convenience, we drop the \( * \) from the variables. Primal feasibility conditions are identical to the constraints in (3.26). The dual feasibility condition necessitates\footnote{The KKT conditions are the necessary conditions (not sufficient) since the objective function in (3.25) is non-convex.} that, for the global maximizer of (3.29), \( z \), the gradient of \( f(z) \) must be a linear combination of the gradients of the binding constraints.
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

in \((3.26)\), i.e., we must have

\[
\begin{bmatrix}
(A + B)y \\
(A + B)^T x \\
-r_{ave} \\
-j_{ave} \\
-1 \\
-1
\end{bmatrix} - \begin{bmatrix}
m \sum_{i=1}^{m} \lambda_i \\
-n \sum_{k=1}^{n} \mu_k \\
-1 \\
0
\end{bmatrix} \begin{bmatrix}
B_{i,:} \\
0 \\
-1 \\
0
\end{bmatrix} - \begin{bmatrix}
r \\
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}
\]

\[
- \begin{bmatrix}
b_1 \\
b_2 \\
a_1 \\
a_2
\end{bmatrix} = \begin{bmatrix}
\sigma_1 \\
\sigma_2
\end{bmatrix}
\]

where we have taken the gradients of the constraints in the same order as in \((3.26)\), and \(A_{i,:}\) and \(e_i\) denote the \(i\)'th row of \(A\) and the \(i\)'th basis vector, respectively. Additionally, the KKT multipliers must satisfy

\[
\begin{align*}
\lambda_i, \phi_i &\geq 0 \quad \text{for } i = 1, \ldots, m \\
\mu_k, \theta_k &\geq 0 \quad \text{for } k = 1, \ldots, n \\
b_1, b_2, \sigma_1, \sigma_2 &\geq 0 \\
a_1, a_2 &\in \mathbb{R}
\end{align*}
\]

By inspecting parts (v) and (vi) of the systems of vector equations in \((3.31)\), we observe
that for the KKT multipliers \( \lambda_i \) and \( \mu_k \) we have

\[
\sum_{i=1}^{m} \lambda_i = 1, \quad \lambda_i \geq 0 \quad \text{and} \quad \sum_{k=1}^{n} \mu_k = 1, \quad \mu_k \geq 0
\]  

(3.33)

Now, let

\[
\lambda_i = x_i \quad \text{for } i = 1, \ldots, m
\]

\[
\mu_k = y_k \quad \text{for } k = 1, \ldots, n
\]

(3.34)

and note that, because of the constraints on the KKT multipliers \( \lambda_i, \mu_k, a_1 \) and \( a_2 \), we are allowed to make these assumptions. From parts (iii) and (iv) of (3.26) we obtain

\[
j^T y - j_{ave} + \sigma_2 = 0 \quad \sigma_2 \geq 0
\]

(3.35)

and finally, from parts (i) and (ii) we have

\[
x^T B - b_2 j^T - \beta 1^T + \theta^T = 0 \quad b_2, \theta \geq 0
\]

\[
A y - b_1 r - \alpha 1 + \phi = 0 \quad b_1, \phi \geq 0
\]

(3.36)

By substituting the KKT multipliers with the variables given in (3.34) we may write the complementary slackness conditions for (3.26) as follows

\[
x^T A y - u x^T r - \alpha = 0
\]

(3.37)

\[
x^T B y - v j^T y - \beta = 0
\]

(3.38)

\[
b_1 r^T x = b_1 r_{ave}
\]

(3.39)

\[
b_2 j^T y = b_2 j_{ave}
\]

(3.40)

and

\[
\phi^T x = \theta^T y = \sigma_1 u = \sigma_2 v = 0
\]

(3.41)

Now, if we multiply the first relation in (3.35) by \( v \) and use (3.41) to simplify the result we have

\[
v(j^T y - j_{ave}) + v \sigma_2 = 0
\]

\[
\Rightarrow v(j^T y - j_{ave}) = 0
\]

(3.42)
which is identical to condition (II.6) in Table 3.1. Similarly, from the second relation in (3.35), we can obtain
\[ u(x^T r - r_{ave}) = 0 \] (3.43)
which gives us condition (I.6) in Table 3.1. Finally, if we post-multiply (3.36) by \( y \) and use (3.41) to simplify the result we have
\[ x^T B y - b_2y^T y - \beta 1^T y + \theta^T y = 0 \]
\[ \Rightarrow x^T B y - b_2y^T y - \beta = 0 \] (3.44)
comparing (3.44) with (3.38) we notice that \( b_2 = v \) and by using (3.39) we obtain the desired result:
\[ x^T B y - v_{jave} - \beta = 0 \] (3.45)
Similarly, we can show that \( b_1 = u \) and
\[ x^T A y - u_{rave} - \alpha = 0 \] (3.46)
Conditions (3.45) and (3.46) are exactly conditions (II.5) and (I.5) in Table 3.1 and as a result, the maximizer of the quadratic program in (3.25), subject to constraints in (3.26), satisfies all the conditions of Theorem 6 and, hence, is a Nash equilibrium of \( G \). The last step is to show that the set of variables \((x, y, u, v, \alpha, \beta)\) is indeed a global maximizer of (3.25). Adding (3.45) to (3.46) gives us the desired result.
\[ x^T(A + B) y - u_{rave} - v_{jave} - \alpha - \beta = 0 \] (3.47)

The converse of theorem states that if \((x^*, y^*)\) is a NE pair, then, \((x^*, y^*, u^*, v^*, \alpha^*, \beta^*)\) is a global maximizer of (3.25). By using Lemma 6 and the necessary and sufficient conditions in Table 3.1 it can be easily verified that
\[ x^{*T}(A + B) y^* - u^*r_{ave} - v^*j_{ave} - \alpha^* - \beta^* = 0 \] (3.48)
and hence the NE solution of the constrained bimatrix game \( G = (A, B, r, j, r_{ave}, j_{ave}) \) is indeed a global maximum of the quadratic program defined in (3.25). This concludes the proof. \( \square \)
3.3 Special Case: Packetized AWGN under Power Limited Jamming

In this section we use the framework we developed in the previous sections to study a typical jamming problem and show that the constrained bimatrix game can be used to formulate this typical problem.

Consider the wireless communication system shown in Figure 3.1. The communication link between a base-station (transmitter) and a mobile user (receiver) is a single-hop, packet-switched, AWGN channel with fixed and known noise variance, $N$, measured at the receiver’s side. Furthermore, assume the communication link is being disrupted by an average power limited additive Gaussian jammer with flat power spectral density. The impact of the Gaussian jammer on the communication link is the reduction of the effective signal to noise ratio (SNR) at the receiver from $P_T/N$ to $P_T/(N + J)$, where $J$ represents the jammer power (variance) and $P_T$ is the transmitter power, both measured at the receiver side.

We assume that the jammer uses a set of discrete jamming power levels denoted by $J$. The jammer may use any jamming power but must maintain an overall average power constraint, denoted by $J_{ave}$. The jammer uses his available power levels according to a probability distribution (his strategy) and his goal is to cause the maximum damage to the communication link by destroying as many packets as possible while maintaining the average power constraint.

The base-station has a rate adaptation block with $n$ different but fixed rates. Transmission
rates are bounded between minimum and maximum rates denoted by $R_{\text{min}}$ and $R_{\text{max}}$, respectively. Without loss of generality, we assume the rates are sorted in a decreasing order. Hence, the base-station’s action set, denoted by $\mathcal{R}$, becomes

$$\mathcal{R} = \{ R_0 = R_{\text{max}} > \cdots > R_i > \cdots > R_{n-1} = R_{\text{min}} \} \quad (\text{nats/\text{trans}}) \quad (3.49)$$

Assuming $R_{\text{max}}$ is feasible and packets are long enough that channel capacity theorem could be applied to each packet, it follows from the capacity of the discrete-time AWGN channel that we must have

$$P_T \geq P_{\text{min}} = N \left( e^{2R_{\text{max}}} - 1 \right) \quad (3.50)$$

to make all transmission rates viable. Throughout the rest of this section, we assume that the base-station transmits data packets at a fixed and known power, $P_T$, where $P_T \geq P_{\text{min}}$. The base-station uses the available rates according to a probability distribution (his strategy) and his goal is to find an optimal strategy to maximize the average throughput of the channel subject to jamming.

Given that the channel noise variance is fixed and known, corresponding to each transmission rate $R_j \in \mathcal{R}$ there exists a certain jammer power, $\hat{J}_j \geq 0$, such that if the actual jamming power used by the jammer is less than $\hat{J}_j$, then reliable communication is possible, i.e.,

$$R_j = \frac{1}{2} \log \left( 1 + \frac{P_T}{N + \hat{J}_j} \right)$$

$$\Rightarrow \quad \hat{J}_j = \frac{P_T}{e^{2R_j} - 1} - N \quad (j = 0, \ldots, n - 1) \quad (3.51)$$

Assuming that $\mathcal{R}$ is publicly available (such as the typical rates of IEEE 802.11 standard) and $P_T$ and $N$ could be estimated, the jammer can use (3.51) to construct his action set, specifically, consider the following action set

$$\mathcal{J} = \{ J_0, J_1, \cdots, J_j, \cdots, J_n \} \quad (3.52)$$

where $J_j$ for $0 \leq j \leq n$ is given by

$$J_j = \begin{cases} 0 & j = 0 \\ \hat{J}_{j-1} + \delta N = \frac{P_T}{e^{2R_{j-1}} - 1} + (\delta - 1)N & j = 1, \ldots, n \end{cases}$$
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

The jammer adds \( \delta N \) with \( \delta > 0 \) to his non-zero jamming powers to make sure that \( R_j \) is greater than the channel capacity corresponding to \( J_j \). Jammer’s mixed-strategy set, \( Y_{J_{ave}}^{n+1} \), is then the set of all probability vectors that result in an average power less than or equal to \( J_{ave} \), i.e.,

\[
Y_{J_{ave}}^{n+1} = \{ y_{(n+1)\times1} \in Y^{n+1} \mid y^T J \leq J_{ave} \}
\]  

(3.53)

where \( y_{(n+1)\times1} \) and \( J_{(n+1)\times1} \) are jammer’s mixed-strategy and jamming power vectors, respectively, and \( Y^{n+1} \) is a standard \((n+1)\)-simplex.

Since destroyed packets do not contribute to the average throughput of the communication system, the payoff per transmitted packet, \( C(R_i, J_j) \), for the pure-strategy pair \((R_j, J_j) \in \mathcal{R} \times \mathcal{J} \) is equal to the transmission rate of that packet if the packet is recovered, and zero if it is destroyed, i.e.,

\[
C(R_i, J_j)_{\text{nats/trans.}} = \begin{cases} 
R_i & j < i \\
0 & j \geq i 
\end{cases} \quad (R_i, J_j) \in \mathcal{R} \times \mathcal{J}
\]  

(3.54)

Therefore the payoff matrix corresponding to (3.54), where the base-station is the row player, will be an \( n \times (n+1) \) matrix with zero elements above the main diagonal, i.e.,

\[
C = \begin{bmatrix}
R_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
R_i & \cdots & R_i & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
R_{n-1} & \cdots & R_{n-1} & 0 & \cdots & 0 
\end{bmatrix}_{n \times (n+1)}
\]  

(3.55)

Let \( x_{n\times1} \) and \( y_{(n+1)\times1} \) be the base-station’s and jammer’s mixed-strategies, respectively. Then the base-station’s problem becomes the following maximization problem

\[
\text{maximize} \quad x^T C y \quad \text{for all} \ y \in Y_{J_{ave}}^{n+1}
\]  

(3.56)

It can be proved \([87]\) that this problem has closed form solution and the average throughput at the Nash equilibrium as a function of \( J_{ave} \) is given by

\[
C(x^*, y^*) = \frac{J_{m+1} - J_{ave}}{J_{m+1} - J_{ave,m}} R_m \quad J_{ave,m} \leq J_{ave} < J_{ave,m+1}
\]  

(3.57)

for \( 1 \leq m < n - 1 \) and \( J_{ave,m} \) is defined as

\[
J_{ave,m} = R_m \sum_{j=1}^{m} (R_j^{-1} - R_{j-1}^{-1}) J_j \quad 1 \leq m \leq n
\]  

(3.58)
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

Figure 3.2 shows the average throughput of the communication link at the NE as a function of jammer’s average power for a typical case. For this example, we use the range of rates from the IEEE 802.11 standard, i.e., we assume coded data rates of the base station are distributed between $R_{\text{min}} = 1 \text{ Mbps}$ and $R_{\text{max}} = 54 \text{ Mbps}$ and the channel bandwidth is 22 MHz.

Since jammer’s goal is to maximize the number of destroyed packets, we define the jammer’s payoff per packet to be 1 if the packet is destroyed and 0 if the packet is recovered. Thus, the jammer’s utility function for the pure-strategy pair $(R_i, J_j)$ becomes

$$J(R_i, J_j) = \begin{cases} 0 & j < i \\ 1 & j \geq i \end{cases} \quad (R_i, J_j) \in \mathcal{R} \times \mathcal{J}$$

(3.59)

and the payoff matrix corresponding to (3.59) becomes

$$J^T = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 \cdots 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 \cdots 0 & 1 \end{bmatrix}_{n \times (n+1)}$$

(3.60)

Comparison of the payoff matrices in (3.55) and (3.60) clearly shows base-station’s and jammer’s conflicting goals; while the base-station’s non-zero payoffs appear on or below the main diagonal of his payoff matrix ($C$), jammer’s non-zero payoffs are above the main diagonal of his respective payoff matrix ($J^T$). But in contrast to the zero-sum games, the sum of the two matrices in (3.55) and (3.60) is not zero.

Since the jammer’s utility function is not the negative of the base-station’s utility function the jammer can play two different games to cause damage to the performance of the communication link. The jammer can simply ignore base-station’s utility function and maximize his average utility based on his own payoff matrix. This game is equivalent to a constrained zero-sum game with matrix $J$ given in (3.60) and average power constraint $J_{\text{ave}}$ where the jammer is the row player (maximizer).

It can be easily verified that any row in jammer’s payoff matrix ($J$) is dominated by the last row which corresponds to his maximum jamming power ($J_n$). But because of the average jamming power constraint, $J_{\text{ave}}$, the jammer cannot use $J_n$ all the time. As a result
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

Figure 3.2: Average throughput at the NE as a function of $J_{\text{ave}}$.

the optimal strategy for the jammer is to use his maximum jamming power with probability $p = J_{\text{ave}} / J_n$ and not jam a packet with probability $(1 - p)$. Therefore, jammer’s expected payoff (average destroyed packets) as a function of his average power for the constrained zero-sum game becomes

$$J^*_{\text{zero-sum}}(J_{\text{ave}}) = \frac{1}{J_n} J_{\text{ave}} \quad 0 \leq J_{\text{ave}} \leq J_n$$

(3.61)

The optimal strategy for this zero-sum game (this strategy is called the jammer’s $\text{maxmin}$ strategy) guarantees the payoff given in (3.61) regardless of the base-station’s strategy.

An alternative approach for the jammer is to play the constrained bimatrix game $G = (C, J^T, R, R_{\text{ave}}, J, J_{\text{ave}})$, where $R$ is the base station rate vector. Since in this special case the base station does not have an average constraint on its strategies, $R_{\text{ave}}$ is an arbitrary number that satisfies $R_{\text{ave}} > \max R_i$. With this assumption, the condition (I.2) in Table 3.1 becomes redundant and from condition (I.5) in Table 3.1 it follows that $u = 0$, hence, the
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

quadratic program in (3.25) simplifies to

\[
\begin{align*}
\text{maximize} & \quad x^T (C + J^T) y - v J_{\text{ave}} - \alpha - \beta \\
\text{subject to} & \quad C y - \alpha 1 \leq 0 \\
& \quad x J^T - v J - \beta 1 \leq 0 \\
& \quad J^T y - J_{\text{ave}} \leq 0 \\
& \quad 1^T y - 1 = 0 \\
& \quad 1^T x - 1 = 0 \\
& \quad x, y, v \geq 0
\end{align*}
\] (3.62)

and the expected payoff of the jammer at the NE becomes

\[
J_{\text{bimatrix}}^* (J_{\text{ave}}) = v^* J_{\text{ave}} + \beta^*
\] (3.64)

where \( v^* \) and \( \beta^* \) are the global maximizers of (3.62).

It can be shown (see Theorem 8) that for certain values of \( J_{\text{ave}} \), the maximization problem in (3.62) has a closed form solution. For these specific values, the expected payoff of the base station is equal to \( R_m, m = 0, \ldots, n - 1 \).

**Theorem 8.** In the constrained bimatrix game \( G = (C, J^T, R, R_{\text{ave}}, J, J_{\text{ave}}) \) let

\[
J_{\text{ave}} = R_m \sum_{i=0}^{m} (R_{i-1}^{-1} - R_{i-2}^{-1}) J_i \text{ for } m = 0, \ldots, n - 1
\] (3.65)

then, equilibrium pair solution, \((x^*, y^*)\), and the optimal mixed-strategies are given by

\[
\begin{align*}
x^* &= [x_0, \ldots, x_i, \ldots, x_m, 0] \\
x_i &= J_m^{-1} (J_i - J_{i-1}) \\
y^* &= [y_0, \ldots, y_i, \ldots, y_m, 0] \\
y_i &= R_m (R_i^{-1} - R_{i-1}^{-1})
\end{align*}
\] (3.66)

for the base station and the jammer respectively (where we used \( R_{-1}^{-1} = J_{-1} = 0 \)). Furthermore, the expected payoffs of \( G \) at the NE are

\[
\begin{align*}
x^* C y^* &= R_m \\
y^* J x^* &= R_m J_m^{-1} \sum_{i=0}^{m} (R_i^{-1} - R_{i-1}^{-1}) J_i
\end{align*}
\] (3.67)
Proof. It is sufficient to show that there exist \( v > 0 \) and \( \alpha, \beta \in \mathbb{R} \) for which (3.62) is zero. Let \( v = J_m^{-1}, \alpha = R_m \) and \( \beta = 0 \), then

\[
x^* (C + J^T) y^* - J_m^{-1} J_{\text{ave}} - R_m = 0
\]

(3.68)

Analytical study and numerical simulations verify that the expected payoff at the NE for the constrained bimatrix game strictly outperforms the zero-sum game if the average jamming power is less than a jamming threshold, \( J_{\text{TH}} \). That is, the expected payoff of the bimatrix game satisfies

\[
J^*_{\text{bimatrix}} (J_{\text{ave}}) > J^*_{\text{zero-sum}} (J_{\text{ave}}) \text{ for all } 0 < J_{\text{ave}} < J_{\text{TH}}
\]

(3.69)

and

\[
J^*_{\text{bimatrix}} (J_{\text{ave}}) = J^*_{\text{zero-sum}} (J_{\text{ave}}) \text{ for all } J_{\text{ave}} \geq J_{\text{TH}}
\]

(3.70)
CHAPTER 3. CONSTRAINED BIMATRIX GAMES

The jamming threshold, $J_{\text{TH}}$, is the minimum average jamming power required to force a transmitter to operate at his lowest rate in a single-hop packetized wireless link. It can be proved (see [87] Theorem 4) that the minimum average jamming power that can force the transmitter to use his lowest rate is given by

$$J_{\text{TH}} = R_{n-1} \sum_{j=1}^{n-1} (R_j^{-1} - R_{j-1}^{-1}) J_j$$

(3.71)

Figure 3.3 shows a comparison between the expected payoff of the zero-sum game and the constrained bimatrix game for a typical case. As expected, the average payoff of the bimatrix game at the NE strictly dominates the zero-sum game for $J_{\text{ave}} < J_{\text{TH}}$ and the expected payoffs converge for $J_{\text{ave}} \geq J_{\text{TH}}$, i.e., the bimatrix game simplifies to a zero-sum game.

3.4 Concluding Remarks

In this chapter, we extended the game-theoretic framework we developed in Chapter 2 to the constrained bimatrix framework that can be used to model even more practical jamming problems in packetized wireless networks. In contrast to the standard bimatrix games, in constrained bimatrix games the players’ strategies must satisfy some additional average conditions, consequently, not all strategies are feasible and the existence of the NE is not guaranteed anymore. We provided the necessary and sufficient conditions under which the existence of the Nash equilibrium (NE) is guaranteed and showed that the equilibrium pairs and the Nash equilibrium solution of this constrained game corresponds to the global maximum of a quadratic program. Finally, we studied a typical packetized wireless link under power limited jamming and showed that the game theoretic analysis of this typical problem yields rather surprising results.

---

6Theoretical analysis suggests that for a single-hop packetized communication link under power limited jamming, such a threshold always exists [87], [90]. Experimental studies confirm the existence of such a threshold on jammer’s average power [91].
Chapter 4

Adaptive Multilayer Superposition Coding Technique

In Chapters 2 and 3 we studied a wireless communication system under power limited jamming. We used a general utility function that can be applied to a variety of jamming scenarios and used two game theoretic frameworks to model some typical jamming problems. As the numerical results show (see Sections 2.5 and 3.3), a strategic jammer that uses the optimal mixed-strategy can be much more effective than a non-strategic jammer with the same average power.

Additionally, in Section 2.3 we proved the existence of a jamming power threshold ($J_{TH}$) such that, if the jammer’s average power exceeds $J_{TH}$, the expected payoff of the game at the Nash equilibrium (NE) is equal to the transmitter’s lowest payoff. This expected NE corresponds to the case where the jammer uses his maximum jamming power ($J_{max}$) all the time (see Section 2.4.3). We also proved that this jamming threshold is strictly less than jammer’s maximum power, $J_{max}$.

In some jamming scenarios (see Section 2.5.2 for example), the jamming threshold can be up to one order of magnitude less than the maximum jamming power. That is, the strategic jammer can cause the maximum damage with an average power much less than his maximum power. Also, there exist simple and easy to implement distributions
(see Equation 2.78) which can well approximate the jammer’s complicated optimal mixed-strategy with little penalty. These suggest that in packetized networks, randomization can significantly assists a strategic jammer and motivate us to investigate more jamming resilient techniques. In this chapter we propose an *adaptive multilayer superposition coding* (SP coding for short) technique to improve the performance of the wireless link under jamming.

The remainder of this chapter is organized as follows: In Section 4.1 we introduce the *superposition coding* technique for broadcast channels. In Section 4.2, we introduce our system model and our proposed *multilayer superposition coding technique*. In Section 4.3, we study the performance of the proposed technique under power limited jamming. Additionally, we compare the performance of the superposition coding technique with rate adaptation and show that SP coding can achieve better performance under power limited jamming. In Section 4.4, we provide numerical result for a typical wireless communication link under jamming. Finally, we conclude this chapter in Section 4.5.

### 4.1 Superposition Coding in Broadcast Channels

*Superposition coding* technique was first introduced as an effective coding scheme for broadcast channels where one receiver experiences a less noisy channel than the other receivers. We introduce the superposition coding technique by providing an example.

Consider a broadcast channel with input alphabet \( X = \{0, 1\} \) and two binary symmetric channel (BSC) as shown in Figure 4.1 (left). Assume the link between \( X \) and \( Y_1 \) is a noise-free BSC channel (i.e., zero probability of error) while the link between \( X \) and \( Y_2 \) is a noisy BSC with error probability \( p \). It is clear that the corresponding channel capacities are \( C_1 = 1 \) and \( C_2 = C_\text{BSC}(p) \triangleq 1 - H(p) < 1 \) for channel 1 and 2, respectively.

By time sharing the channel, any pair of rates, \((R_1, R_2)\), that is a convex combination of \((R_1, R_2) = (C_2, C_2)\) and \((R_1, R_2) = (1, 0)\) is achievable which gives us an inner bound for the capacity region of the broadcast channel (the straight line in Figure 4.2 (right)).

Assume that the noisy BSC channel in Figure 4.1 (left) is replaced by a noisier channel shown in Figure 4.1 (right). More specifically, the new channel may be constructed by

---

1 Examples in this section are borrowed from the Cover’s original paper [92].

2 \( H(p) = -p \log(p) - (1 - p) \log(1 - p) \) is the binary entropy function.
cascading two BSC channels with error probabilities $p$ and $\alpha$, respectively. This channel is equivalent to a BSC channel with error probability $\hat{p} = \alpha \bar{p} + \bar{\alpha}p$ where $\bar{\alpha} = 1 - \alpha$ and $0 < \alpha < 1$ and $\bar{p}$ is defined similarly. The capacity of the cascade BSC channel is $C_{\text{BSC}}(\hat{p}) < C_2$.

Now, in the original noisy BSC channel, consider the set of rates that satisfy $R_2 \leq C_{\text{BSC}}(\hat{p}) < C_2$ with corresponding codewords $x \in \{0, 1\}^n$ (shown by the black dots in Figure 4.2 (left)). Obviously, such rates are achievable and even though they are less than the capacity of channel 2, they can tolerate larger noises. More specifically, codewords can tolerate noises of Hamming weight $n(\alpha \bar{p} + \bar{\alpha}p)$ compared to the noises of weight $np$ that can be tolerated by codes that are designed for the original BSC channel ($n$ is the block length of the code).

We now can take advantage of this extra tolerance in noise, which is equal to $n(\alpha \bar{p} + \bar{\alpha}p) - np = n\alpha$, by packing extra information that is intended for the receiver 1 only. We assign this extra information to codewords that are at the Hamming distance less than $n\alpha$ from the original codewords $x \in \{0, 1\}^n$ (the shaded circles in Figure 4.2 (left)). This information appears as an extra noise for receiver 2 which can be tolerated.

Assume that $x$ is the codeword that is intended for both receivers and $r$ is a codeword at the Hamming distance $n\alpha$ of $x$, the transmitter constructs the superimposed codeword, $y$, by linear superposition of $x$ and $r$,

$$y = x + r$$ \hspace{1cm} (4.1)

the receiver then transmits $y$ to both receivers. Receiver 2, receives $y$ through a noisy BSC (with error probability $p$) and treats $r$ as an extra noise, this is as if $y$ were sent through a
Chapter 4. Adaptive Multilayer Superposition Coding Technique

Figure 4.2: Codeword space (left). Achievable rates for BSC broadcast channel (right).

BSC with error probability $\hat{p} = \alpha \bar{p} + \bar{\alpha} p$. Therefore, rate $R_2 = C_{BSC}(\alpha \bar{p} + \bar{\alpha} p)$ is achievable for channel 2.

However, receiver 1, receives $y$ through a noise-free BSC channel and therefore can decode both $x$ and $r$. The number of codewords at the Hamming distance $\alpha n$ of $x$ is

$$\binom{n}{\alpha n} \approx 2^{n H(\alpha)}$$  \hspace{1cm} (4.2)

hence the following rate is also achievable for channel 1

$$R_1 = C_{BSC}(\alpha \bar{p} + \bar{\alpha} p) + \frac{1}{n} \log (2^{n H(\alpha)}) = C_{BSC}(\hat{p}) + H(\alpha)$$  \hspace{1cm} (4.3)

As a result, the pair of rates $(R_1, R_2)$ for all $0 < \alpha < 1$ is jointly achievable with this superimposition coding scheme. It can be proven that these rates (the outer curve in Figure 4.2 (right)) strictly dominate the achievable rates by time sharing. This simple example shows the benefits of using superposition coding in broadcast channels.

4.2 Adaptive multilayer Superposition Coding

The system model that we use in this chapter to analyze the performance of our proposed technique is similar to the model we used in chapters 2 and 3 (see Figure 4.3). For
convenience, we will use the framework that was developed in Chapter 2 (constrained zero-sum) to study the performance of the adaptive SP coding technique under power limited jamming. Nevertheless, extending the results presented in this chapter to the more general constrained bimatrix framework is straightforward.

4.2.1 Channel Model

The wireless communication link between the transmitter and the receiver is a single-hop, packetized, additive white Gaussian noise (AWGN) with a fixed and known noise variance, \(N\), referred to the receiver’s front end. Furthermore, the communication link is being disrupted by an additive adversary, the jammer. The jammer transmits radio signals to degrade the performance of the wireless communication link. We assume packets are long enough that channel capacity theorem could be applied to each packet being transmitted.

4.2.2 Jammer Model and Strategy Set

We assume the jammer uses a set of \((N_J + 1)\) discrete jamming power levels, \(J\), equally spaced between \(J = 0\) and \(J = J_{\text{max}}\). We denote this set (the pure strategy set) by \(J\) and, without loss of generality, assume the power levels are sorted in an increasing order and \(\{0, J_{\text{max}}\} \subset J\), i.e.,

\[
J = \left\{ 0 \leq J_j \leq J_{\text{max}}; J_j = \frac{j}{N_J}J_{\text{max}}, 0 \leq j \leq N_J \right\}
\] (4.4)
Furthermore, we define jammer’s pure strategy vector, $J$,

$$J^T = \begin{bmatrix} J_0 & \ldots & J_j & \ldots & J_{N_J} \end{bmatrix}_{1 \times (N_J+1)} \tag{4.5}$$

where $^T$ indicates transposition. The jammer uses these jamming powers according to a probability distribution (his strategy) and his goal is to find an optimal probability distribution to minimize the expected achievable transmission rate over the channel while maintaining maximum and average power constraints denoted by $J_{\text{max}}$ and $J_{\text{ave}} < J_{\text{max}}$, respectively.

As stated in Section 2.2, unlike typical zero-sum games, the jammer’s strategy must satisfy the additional average power constraint $J_{\text{ave}}$, where $J_{\text{ave}} < J_{\text{max}}$. Hence, not all mixed-strategies are feasible. Let $y$ denote the jammer’s strategy vector and $Y_{N_J+1}$ be the standard $(N_J + 1)$-simplex, then, the set of feasible jamming strategies that satisfies the average power constraint becomes

$$Y_{\text{LE}|J_{\text{ave}}} = \left\{ y \in Y_{N_J+1} \middle| y^T J \leq J_{\text{ave}} \right\} \tag{4.6}$$

Clearly $Y_{\text{LE}|J_{\text{ave}}}$ is a subset of the $(N_J + 1)$-simplex which includes all strategies with an average power less than or equal to $J_{\text{ave}}$. It can be shown that for this constrained two-player zero-sum game, which the jammer’s strategy set is reduced to (4.6), at least one Nash Equilibrium (NE) exists (see Section 2.3.1).

### 4.2.3 Transmitter Model and Strategy Set

Assume the transmitter has an adaptation block which enables him to transmit at different rates. Furthermore, assume that the adaptation block allows the transmitter to distribute his available transmission power, $P_T$, among $m$ different code rates. The transmitter then superimposes the codes and transmits the resulting signal to the receiver. Figure 4.4 shows the proposed adaptive multilayer superposition coding model.

The receiver uses successive cancellation decoding [93] to decode the $m$-layer superimposed signal; the receiver first decodes the first layer code while treating the upper layer codes as a part of noise. The receiver then subtracts off the first layer code and decodes the second layer while treating upper layers as noise. This decoding scheme is repeated until all layers are decoded. With successive cancellation decoding, an upper layer code can be recovered only if and only if all the lower layer codes have been successfully decoded.
CHAPTER 4. ADAPTIVE MULTILAYER SUPERPOSITION CODING TECHNIQUE

For the \( n \)’th layer code (\( 1 \leq n \leq m \)), the transmitter allocates fraction of his available power denoted by \( \alpha_n P_T \) where \( 0 \leq \alpha_n \leq 1 \) and \( \sum_{i=1}^{m} \alpha_i = 1 \). In presence of the jammer, and with successive cancellation decoding, maximum and minimum achievable transmission rates at the \( n \)'th layer are given by

\[
R_{n,\text{max}} = \frac{1}{2} \log \left( 1 + \frac{\alpha_n P_T}{N + P_T \sum_{i=n+1}^{m} \alpha_i} \right)
\]

(4.7)

and

\[
R_{n,\text{min}} = \frac{1}{2} \log \left( 1 + \frac{\alpha_n P_T}{N + J_{\text{max}} + P_T \sum_{i=n+1}^{m} \alpha_i} \right)
\]

(4.8)

respectively. In (4.7) and (4.8) we assume all lower layer codes have been successfully decoded and \( (P_T \sum_{i=n+1}^{m} \alpha_i) \) is the fraction of transmission power that is added to the noise for the \( n \)’th layer code.

For simplicity, we assume power allocation fractions, \( \alpha_n \) (\( 1 \leq n \leq m \)), are chosen from a discrete set, \( \mathcal{A} \), containing \( (N_{\alpha} + 1) \) distinct fractions, uniformly spaced by \( \Delta \alpha = \frac{1}{N_{\alpha}} \), i.e.,

\[
\mathcal{A} = \{0, \Delta \alpha, 2\Delta \alpha, \cdots, N_{\alpha} \Delta \alpha = 1\}
\]

(4.9)

Furthermore, we define the power allocation \( m \)-tuple, \( \alpha \), to be the ordered sequence of \( m \) fractions representing the fraction of transmission power that is allocated to the correspond-

Figure 4.4: Adaptive multilayer superposition coding scheme.
CHAPTER 4. ADAPTIVE MULTILAYER SUPERPOSITION CODING TECHNIQUE

Assuming transmission at channel capacity, possible achievable rates for the \( n \)'th layer code can be written as

\[
R_n = \frac{1}{2} \log \left( 1 + \frac{\alpha_n P_T}{N + J_{T,n} + P_T \sum_{i=n}^{m} \alpha_i} \right)
\]  

(4.11)

where

\[
0 \leq J_{T,n} \leq J_{\text{max}}
\]  

(4.12)

With these notations, Equations (4.10), (4.11) and (4.12), and for a given \( \alpha \) \( m \)-tuple, we have a one-to-one correspondence between the possible rates at \( n \)'th layer, \( R_{n,\text{min}} \leq R_n \leq R_{n,\text{max}} \), and \( J_{T,n} \). We use the term assumed jamming power for \( J_{T,n} \) to distinguish it from jammer’s actual jamming power (\( J \)). Even though the transmitter can arbitrarily choose \( J_{T,n} \) from \([0, J_{\text{max}}]\), it can be shown that it is sufficient to assume that \( J_{T,n} \in \mathcal{J} (1 \leq n \leq m) \) where \( \mathcal{J} \) is defined in (4.4) (also see [87]). Similarly, we define the assumed power \( m \)-tuple, \( J_T \), as

\[
J_T = (J_{T,1}, \ldots, J_{T,n}, \ldots, J_{T,m}) \in \mathcal{J}^m
\]  

(4.13)

Power allocation and assumed power \( m \)-tuples together, \((\alpha, J_T)\) uniquely specify the rates for all the code layers, consequently, transmitter’s strategy is a joint probability distribution \( x_{\alpha,J_T} \) over the two \( m \)-tuples given in (4.10) and (4.13) such that expected achievable transmission rate is maximized.

4.2.4 Utility Function and Expected Payoff

Consider the problem of maximizing the average throughput of the wireless link under jamming. Because the transmissions occur in the presence of the jammer, recovery of the transmitted packets at the receiver is not always guaranteed. Assume, for the \( n \)'th layer code, the transmitter is using a rate that corresponds to \( J_{T,n} \). This code can only be recovered at the receiver if the following conditions are met;

- The jammer’s actual power, \( J \), is less than or equal to \( J_{T,n} \), i.e., the code rate is less than or equal to the channel capacity.
CHAPTER 4. ADAPTIVE MULTILAYER SUPERPOSITION CODING TECHNIQUE

- All the lower level codes have been successfully decoded (successive cancellation decoding, see Section 4.2.3).

As a result, we define the payoff to the transmitter for the strategy triplet \((\alpha, J_T, J)\) to be the sum of the code rates that are recoverable at the receiver, i.e.,

\[
C(\alpha, J_T, J) = \sum_{i=1}^{m} \frac{1}{2} U_i(J_T, J) \log \left[ 1 + \frac{\alpha_i P_T}{N + j_{T,i} + (1 - \sum_{j=1}^{i} \alpha_j) P_T} \right]
\]  

(4.14)

where

\[
U_i(J_T, J) \begin{cases} 
1 & \text{if } J_{T,j} \geq J \text{ for all } j \leq i \\
0 & \text{o.w.}
\end{cases} \quad \text{for } 0 \leq i \leq m
\]  

(4.15)

Since we are using the constrained zero-sum framework to model this problem, the payoff to the jammer is the negative of the transmitter’s payoff. We can formulate the payoffs (for each pure strategy from respective strategy sets) in a payoff matrix, denoted by \(C\), where the transmitter and jammer would be the row and column players, respectively. In this case, the average throughput of the wireless link for the strategy pair \((x_\alpha, J_T, y)\) becomes

\[
C(x_\alpha, J_T, y) = x_{\alpha, J_T}^T C y
\]  

(4.16)

where \(x_{\alpha, J_T}\) is a column vector of appropriate size. The optimal strategies and the average throughput of the link at the Nash equilibrium is the solution of following \(\min \max\) problem,

\[
C(x_{\alpha, J_T}^*, y^*) = \min_{x_{\alpha, J_T} \in X} \max_{y \in \mathbb{R}^{N_{J_T} + 1} | J_{\text{ave}}} x_{\alpha, J_T}^T C y
\]  

(4.17)

where \(x_{\alpha, J_T}^*\) and \(y^*\) are the transmitter’s and jammer’s optimal strategies, respectively, and \(C(x_{\alpha, J_T}^*, y^*)\) is the expected value of the game the Nash equilibrium (i.e., expected average throughput), and \(X\) is a standard simplex of the appropriate dimension. From (4.14) it is clear that

\[
R_{\min} \leq C(x_{\alpha, J_T}^*, y^*) \leq R_{\max} \text{ for all } J_{\text{ave}} \in [0, J_{\max}]
\]  

(4.18)

where \(R_{\min}\) and \(R_{\max}\) are given by

\[
R_{\min} = \frac{1}{2} \log \left( 1 + \frac{P_T}{N + J_{\max}} \right) \quad \text{and} \quad R_{\max} = \frac{1}{2} \log \left( 1 + \frac{P_T}{N} \right).
\]
CHAPTER 4. ADAPTIVE MULTILAYER SUPERPOSITION CODING TECHNIQUE

4.3 Performance Analysis of the Adaptive Superposition Coding Technique

In this section we study the performance of the adaptive $m$-layer superposition coding technique introduced in Section 4.2 under power limited jamming. Furthermore, we show that by randomizing his strategy, the jammer can force the transmitter to operate at his lowest rate, given that the average jamming power exceeds a certain threshold, $J_{TH} < J_{max}$. We provide analytical results and closed form expressions for the jamming threshold and jammer’s optimal strategy with the lowest average power that forces the transmitter to operate at his lowest rate. To compare the performance of adaptive multilayer superposition coding scheme under smart jamming, we first provide the jamming threshold and jammer’s optimal strategy for the case where the transmitter uses only one layer of code (rate adaptation or no SP coding).

4.3.1 Rate Adaptation Technique

Consider the jamming problem introduced in Section 4.2 where the transmitter is not allowed to use the superposition coding scheme. It can be proved [87] that there exists a jamming power threshold, denoted by $J_{TH,NOSP} < J_{max}$, such that if the jammer’s average power exceeds $J_{TH,NOSP}$ then the jammer, by randomizing his strategy, can force the transmitter to operate at his lowest rate at the NE. This expected achievable rate corresponds to the case where the jammer uses his maximum power $J_{max}$ all the time (Barrage noise jammer with average power $J_{max}$). Furthermore, the value of $J_{TH,NOSP}$ is given by

$$ J_{TH,NOSP} = \frac{1}{N_J} \left( N_J + 1 - R_{N_J} \sum_{j=0}^{N_J} R_j^{-1} \right) J_{max} \quad (4.19) $$

where $R_j$ is defined by

$$ R_j = \frac{1}{2} \log \left( 1 + \frac{P_T}{N + \frac{j}{N_J} J_{max}} \right) \quad (4.20) $$
CHAPTER 4. ADAPTIVE MULTILAYER SUPERPOSITION CODING TECHNIQUE

Figure 4.5: Typical Jamming Power Threshold as a Function of $\Delta \alpha$, $N_J \gg 1$.

and the jammer’s optimal strategy with the lowest average power that can achieve this NE is given by

$$y^* = R_{N_J} \cdot \begin{bmatrix} R_0^{-1} \\ \vdots \\ (R_j^{-1} - R_{j-1}^{-1}) \\ \vdots \\ (R_{N_J}^{-1} - R_{N_J-1}^{-1}) \end{bmatrix}_{(N_J+1) \times 1}$$

(4.21)

It can also be shown that for a large enough $N_J$, the jamming threshold can be approximated by the following integral relation

$$J_{TH,NOSP} \approx J_{\text{max}} - \int_0^{J_{\text{max}}} \log \left( 1 + \frac{P_T}{N + J_{\text{max}}} \right) \log \left( 1 + \frac{P_T}{N + J} \right) dJ \quad \text{for} \quad N_J \gg 1$$

(4.22)

which does not have a solution in terms of standard mathematical functions.

4.3.2 Superposition Coding Technique

Now assume the transmitter is using an $m$-layer superposition coding technique where $m \geq 2$. In this case, it can be proved that the jamming power threshold, denoted by
CHAPTER 4. ADAPTIVE MULTILAYER SUPERPOSITION CODING TECHNIQUE

\( J_{TH,SP} < J_{max} \), is independent of \( m \) and is given by

\[
J_{TH,SP} = \frac{1}{N_J} \left( (N_J + 1) - r_{N_J} \sum_{j=0}^{N_J} r_j^{-1} \right) J_{max}
\]  

(4.23)

where

\[
r_i = \frac{1}{2} \log \left( 1 + \frac{\Delta \alpha P_T}{N + \frac{i}{N_J} J_{max}} \right) \quad \text{for} \ i = 0, \ldots, N_J
\]

(4.24)

and jammer’s optimal strategy with the lowest average power that achieves this NE is given by

\[
y^* = r_{N_J} \cdot \begin{bmatrix} r_0^{-1} \\ \vdots \\ (r_j^{-1} - r_{j-1}^{-1}) \\ \vdots \\ (r_{N_J}^{-1} - r_{N_J-1}^{-1}) \end{bmatrix}_{(N_J+1) \times 1}
\]

(4.25)

Proof (Outline). Assume, for now, that the jammer’s average power is equal to his maximum jamming power, that is, \( J_{ave} = J_{max} \). For this case and from (4.18) we know that the expected achievable transmission rate is \( R_{min} \). A possible optimal strategy for the transmitter that achieves this NE is to allocate all his power to the first layer and use his lowest rate all the time. Therefore, the jamming power threshold indeed exists and is at most equal to the maximum jamming power.

Since the receiver uses successive cancellation decoding scheme, it can be argued that \( J_T \) \( m \)-tuples that are increasing in \( n \) (see Equation (4.10)) are not optimal strategies\(^3\) and as jammer’s average power approaches \( J_{TH,SP} \) a rational transmitter would allocate smaller fraction of his available power to upper layer codes (since they could be decoded only if all lower level codes have been decoded). Given this discussion, as jammer’s average power approaches the threshold (i.e., \( J_{ave} \rightarrow J_{TH,SP} \)) a candidate for the optimal power allocation strategy would be

\[
\hat{\alpha} = (\alpha_1, \alpha_2, 0, \ldots, 0) \quad \text{where} \quad \alpha_2 = 1 - \alpha_1 = \Delta \alpha
\]

(4.26)

\(^3\)For an increasing \( J_T \), the lower level codes that must be decoded prior to \( n \)’th layer decoding would be lost for smaller jamming powers.
that is, allocating the smallest possible fraction of power ($\Delta\alpha$) to the second layer code and allocate the remaining transmission power to the first layer code. In this case the utility function defined in (4.14) simplifies to

\[
C(\hat{\alpha}, \hat{J}_T, J) = \hat{r} + \begin{cases} \frac{1}{2} \log \left(1 + \frac{\Delta\alpha P_T}{N + J_T} \right) & J_T \geq J \\ 0 & J_T < J \end{cases}
\] (4.27)

where $\hat{r}$ is given by

\[
\hat{r} = \frac{1}{2} \log \left(1 + \frac{(1 - \Delta\alpha) P_T}{N + J_{\text{max}} + \Delta\alpha P_T} \right)
\] (4.28)

and in (4.27) we have assumed $\hat{J}_T = (J_{\text{max}}, J_T, 0, \cdots)$ and $0 \leq J_T < J_{\text{max}}$. Given that the $\hat{r}$ contribution to the expected payoff of the transmitter is fixed and independent of the transmitter’s and jammer’s strategies, we can split the game matrix $C$, defined in (4.16), into a constant payoff matrix and a lower triangular payoff matrix with equal non-zero elements in each row, i.e.,

\[
C = \hat{r} \cdot 1_{(N J + 1) \times (N J + 1)} + \hat{C}
\] (4.29)

where $1_{(N J + 1) \times (N J + 1)}$ denotes a $(N J + 1)$ by $(N J + 1)$ matrix of ones and $\hat{C}$ is given by

\[
\hat{C} = \begin{bmatrix} r_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ r_i & \cdots & r_i & 0 \\ \vdots & \ddots & \ddots & \vdots \\ T_{N J} & \cdots & T_{N J} & \cdots & T_{N J} \end{bmatrix}_{(N J + 1) \times (N J + 1)}
\] (4.30)

If we plug (4.29) in (4.16) and simplify the result, it can be seen that the game decomposes to a new sub-game define by the new payoff matrix $\hat{C}$, i.e.,

\[
C(x_{\hat{\alpha}}, J_T, y) = x_{\hat{\alpha}, J_T}^T (\hat{r} \cdot 1 + \hat{C}) y
\]

\[
= \hat{r} \cdot 1 \cdot y + x_{\hat{\alpha}, J_T}^T \hat{C} \cdot y
\]

\[
= \hat{r} + x_{\hat{\alpha}, J_T}^T \hat{C} \cdot y
\] (4.31)

where 1 denotes a column vector of ones. But for $J_{\text{ave}} \rightarrow J_{\text{TH,SP}}^{-}$ we have

\[
C(x_{\hat{\alpha}, J_T}^*, y^*) \rightarrow R_{N J}^{-} \quad \text{as} \quad J_{\text{ave}} \rightarrow J_{\text{TH,SP}}^{-}
\] (4.32)
It can be easily verified that $R_{NJ} = \hat{r} + r_{NJ}$, as the result and from (4.31) we can conclude that

$$x^{T}_{\alpha,J_T} \hat{C}y^* \rightarrow r_{NJ}^{-} \quad | \quad J_{ave} \rightarrow J_{TH,SP}^{-}$$ \hspace{1cm} (4.33)

Therefore, for the jammer’s optimal strategy at the NE and for $J_{ave} = J_{TH,SP}$ we only need to consider the sub-game defined by the matrix $\hat{C}$. Assume the jammer is using the strategy defined in (4.25) against transmitter’s arbitrary strategy. If we plug (4.25) in (4.31) and simplify the result, it can be verified that

$$x^{T}_{\alpha,J_T} \hat{C}y^* = r_{NJ}^{-} \quad | \quad y^* \text{ given in (4.25)}$$ \hspace{1cm} (4.34)

In other words, if the jammer uses $y^*$, regardless of the transmitter’s strategy, the expected achievable rate at the NE would be the lowest transmission rate. The average jamming power of strategy $y^*$ is equal to the jamming power threshold and is given by

$$J_{TH,SP} = J^T y^*$$

$$= r_{NJ} \sum_{j=1}^{N_J} \left( r_j^{-1} - r_{j-1}^{-1} \right) \left( \frac{j}{N_J} J_{max} \right)$$

$$= \frac{r_{NJ}}{N_J} \left( N_J r_{N_J}^{-1} - \sum_{j=1}^{N_J-1} r_{j-1}^{-1} \right) J_{max}$$

$$= \frac{1}{N_J} \left( (N_J + 1) - r_{NJ} \sum_{i=0}^{N_J} r_{j}^{-1} \right) J_{max}$$

which is in agreement with with (4.23). \hfill \Box

Now if we let $\Delta \alpha \rightarrow 0^+$ (i.e., if we let the transmitter choose his power allocation fractions from a continuous set) then, the jamming threshold simplifies to

$$\lim_{\Delta \alpha \rightarrow 0^+} J_{TH,SP} \approx \frac{1}{2} \cdot \frac{J^2_{max}}{N + J_{max}} < \frac{1}{2} J_{max} \quad \text{for} \quad N_J \gg 1$$ \hspace{1cm} (4.36)

Proof. If we take the limit of (4.23) with respect to $\Delta \alpha \rightarrow 0^+$ we have

$$\lim_{\Delta \alpha \rightarrow 0^+} J_{TH,SP} = J_{max} \frac{N_J}{N_J} \left[ (N_J + 1) - \sum_{j=0}^{N_J} \lim_{\Delta \alpha \rightarrow 0^+} f_j(\Delta \alpha) \right]$$ \hspace{1cm} (4.37)
where $f_j(\Delta \alpha)$ is defined as

$$f_j(\Delta \alpha) = \frac{\log\left(1 + \frac{\Delta \alpha P_T}{N J + J_{\text{max}}}ight)}{\log\left(1 + \frac{\Delta \alpha P_T}{N + \frac{J_{\text{max}}}{J_j}} N J + J_{\text{max}}\right)}$$

It can be verified that

$$\lim_{\Delta \alpha \to 0^+} f_j(\Delta \alpha) = \frac{N + \frac{J_{\text{max}}}{J_j}}{N + J_{\text{max}}}$$

If we plug (4.38) in (4.37) and let $N J \gg 1$ we have

$$\lim_{\Delta \alpha \to 0^+} J_{\text{TH,SP}} = \frac{J_{\text{max}}}{N J} \left[ (N J + 1) - \frac{N J + 1}{N + J_{\text{max}}} \left( N + \frac{1}{2} J_{\text{max}} \right) \right]$$

$$= \left( \frac{N J + 1}{N J} \right) J_{\text{max}} \frac{1}{2} J_{\text{max}} \frac{1}{N + J_{\text{max}}}$$

$$\approx \frac{1}{2} J_{\text{max}}^2 \quad \text{for} \quad N J \gg 1$$

(4.39)

### 4.4 Numerical Results

Analytical and numerical results show that the adaptive multilayer superposition coding scheme introduced in Section 4.3 (assuming $m \geq 2$) can reach better expected achievable
transmission rates than no SP coding scheme. Figure 4.6 shows the expected achievable transmission rate for $m = 1, 2, 3$ and $4$ as a function of jammer’s average power for typical values of transmission power and maximum jamming power ($m = 1$ correspond to the case where the transmitter is not using SP coding scheme).

Figure 4.7 shows the performance gain of the adaptive $m$-layer SP coding scheme for $m = 2, 3$ and $4$ over the no SP coding case for the same typical values. Additionally, numerical results verify that 

$$J_{TH,SP} > J_{TH,NOSP} \quad \text{for all } m \geq 2$$

(4.40)

that is, the average jamming power required to force a transmitter with adaptive $m$-layer SP coding scheme to operate at his lowest rate is greater than the average power required for the transmitter that is not using the SP coding (the jammer needs more jamming power to force the transmitter to operate at the same NE).

4.5 Concluding Remarks

In this chapter, inspired by the superposition coding technique in broadcast channels, we propose an adaptive multilayer superposition coding technique to improve the performance of packetized networks. Our analytical results shows that, compared to rate adaptation,
CHAPTER 4. ADAPTIVE MULTILAYER SUPERPOSITION CODING TECHNIQUE

superposition coding not only achieves better average performance under jamming, it also increases the jamming threshold.
Chapter 5

Adaptive OFDM Systems Under Jamming

5.1 Introduction

The many desirable characteristics of the Orthogonal Frequency Division Multiplexing systems (OFDM) such as high spectral efficiency, high data rates, robustness in multipath fading channels and ease of implementation have made OFDM the primary physical layer solution for most modern wireless communication systems. Wireless Local Area Networks (WLAN) based on different flavors of the IEEE 802.11 or Wireless Metropolitan Area Networks (WMAN) based on IEEE 802.16 both use OFDM as the main physical layer modulation scheme. Additionally, most leading cellular technologies such as 4G LTE standard rely on OFDM at the physical layer.

Nevertheless, it has been shown in the literature that current implementations of OFDM are vulnerable to a variety of jamming attacks specially due to OFDM sensitivity to channel estimation and synchronization between the transmitter and the receiver [94]. For instance, reference [95] studies the possibility of jamming the channel estimation procedure as an efficient type of attack. This work suggests that targeting the accuracy of channel state
information estimation not only requires less power, but also more efficient than other jamming techniques.

Reference [96] studies jamming attacks that prevents the receiver from ever acquiring proper synchronization by targeting the symbol timing estimation algorithm, the first step in the synchronization process. This work suggests that a jammer who exploits the weakness in the timing estimation algorithm can cause massive errors to all synchronization estimates.

Game theory has also been used to study jamming games in OFDM systems, for instance, reference [81] considers jamming in a wireless OFDM network with transmission costs for both jammer and transmitter. This work uses the general-sum framework to model the jamming problem. The numerical example in this work suggests that when the jammer is close to the base station, the jammer should pay less attention to the subchannels with poor quality and spend more energy on the subchannels with good quality which in turn, forces the transmitter to use the resources of the bad quality subchannels.

In this chapter, we study the performance of an adaptive OFDM wireless communication system under power limited jamming. We show that with modest assumptions, this problem can be formulated into either the constrained zero-sum, or the constrained bimatrix frameworks introduced previously. As a result, all the analytical results that were presented in chapters 2 and 3 can be directly applied to this problem.

## 5.2 System Model

In this section, we briefly introduce our system model and discuss the motivation behind our work. The details of our model will be discussed in the sections that follow. The transmitter and the receiver are communicating over a wireless noisy channel that is subjected to an adaptive adversary. The communicating nodes are using an adaptive Orthogonal Frequency Division Multiplexing (adaptive OFDM) to communicate. The transmitter adaptively changes the subcarriers’ data rates such that the overall throughout of the wireless link is maximized.

On the other hand, the jammer, also adaptively, jams the OFDM subchannels with different jamming powers, in order to degrade the performance of the wireless link. We
assume that the jammer can use arbitrary jamming powers and can jam any subchannel that he wishes but for practical reasons, he must maintain a maximum jamming power and energy. Our goal is to model this jamming problem and study the long term achievable performance of this adaptive OFDM system.

5.3 Transmitter Model

Consider an OFDM wireless communication system with \( K \) subchannels where the bandwidth of each subchannel is \( \Delta f \). The transmitter has an adaptation block which enables him to jointly change/adapt his channel coding rate and modulation scheme for each subcarrier (see Figure 5.1). For convenience, we assume the transmitter uses time domain channel coding and subchannel bandwidth is sufficiently narrow such that the frequency response characteristics of the subchannels are ideal.

Without loss of generality, assume the data rates for each subcarrier are chosen from a set of \( N \) distinct data rates, denoted by \( \mathcal{R} \), i.e.,

\[
\mathcal{R} = \{ R_0 = R_{\text{max}}, \ldots, R_i, \ldots, R_{N-1} = R_{\text{min}} \} \text{ (bps) } \quad ||\mathcal{R}|| = N \quad (5.1)
\]

where \( R_{\text{min}} \) and \( R_{\text{max}} \) denotes the minimum and maximum available data rates for each subcarrier, respectively. Furthermore, assume the channel frequency response is such that (in the absence of jamming) \( R_{\text{max}} \) is feasible for all subchannels\(^1\). Obviously, to maximize the throughput of the wireless link (or equivalently, to maximize the average data rate of OFDM symbols), the maximum achievable rate, \( R_{\text{max}} \), must be used for all subcarriers. However, because of jamming, the subchannels’ capacities are not known in advanced and therefore it is not known which rates are feasible prior to transmission.

To overcome this problem, the transmitter randomly assigns the data rates to the subcarriers such that the overall throughput of the wireless link is maximized. Let the column vector \( r_{K \times 1}^{(n)} \) denote the transmitter’s strategy for the \( n \)th OFDM symbol,

\[
r_{K \times 1}^{(n)} = [r_1 \ldots r_k \ldots r_K] \quad \text{where } r_k \in \mathcal{R} \quad (5.2)
\]

\(^1\)This assumption is not particularly restrictive since any infeasible data rate can be removed from \( \mathcal{R} \).
that is, for the \(n\)th OFDM symbol, the \(k\)th element of \(r^{(n)}\) is the data rate at which the \(k\)th subcarrier is to be coded/ modulated. For each OFDM symbol, the adaptation block selects a vector of \(K\) data rates where each rate is selected from the set of available data rates given in (5.1) and passes this vector to the coding/modulation block (see Figure 5.1). The coding/modulation block transmits the \(n\)th OFDM symbol according to \(r^{(n)}\). From this point forward, all strategy vectors are assumed to be per OFDM symbol (i.e., for the \(n\)th symbol), and for convenience, we drop the \((n)\) from the vectors.

Because of jamming, reliable recovery of the transmitted data is not guaranteed for all subchannels therefore, the resulting bit rate per OFDM symbol can be written as

\[
\text{bit rate/symbol} = \sum_{k=1}^{K} \hat{r}_k \quad \text{where} \quad \hat{r}_k = \begin{cases} r_k, & \text{if } r_k \leq c_k \\ 0, & \text{if } r_k \geq c_k \end{cases} \quad (5.3)
\]

and \(c_k, k = 1, \ldots, K\) denotes the actual channel capacity for the \(k\)th subchannel, which in general, is a function of channel frequency response and the jammer power spectral density. For sufficiently narrow \(\Delta f\), we can assume that the jammer’s power spectral density is flat for all subchannels and therefore, we may express \(c_k\) as

\[
c_k = C(p_k, j_k, N_k) \quad \text{for} \quad k = 1, \ldots, K \quad (5.4)
\]

where \(C(p_k, j_k, N_k)\) denotes the channel capacity function of the wireless link and \(p_k, j_k\) and \(N_k\) (all in \(\text{W/Hz}\)) denote the power spectral densities of the transmitter, jammer and channel noise for the \(k\)th subchannel (all measured at the receiver front end), respectively.

98
CHAPTER 5. ADAPTIVE OFDM SYSTEMS UNDER JAMMING

Let \( \hat{x}_i \) denote the number of subcarriers coded/modulated with \( R_i \in \mathcal{R}, i = 0, \ldots, N - 1 \), obviously we have

\[
\sum_{i=0}^{N-1} \hat{x}_i = K \quad \text{and} \quad 0 \leq \hat{x}_i \leq K \tag{5.5}
\]

now let \( x_i \triangleq \frac{1}{K} \hat{x}_i \), i.e., \( x_i \) denotes the fraction of subcarriers transmitted at rate \( R_i \).

From (5.5) it follows,

\[
\sum_{i=0}^{N-1} x_i = 1 \quad \text{and} \quad 0 \leq x_i \leq 1 \tag{5.6}
\]

As a result, for sufficiently large \( K \), the following vector can be well approximated by the following probability vector

\[
x^T_{1 \times N} = [x_0 \ldots x_i \ldots x_{N-1}] \quad \text{for} \quad K \gg N \tag{5.7}
\]

If we assume that the subchannels have nearly same channel characteristics or when the effects of nonideal wireless channel (including different channel gains for subcarriers etc.) have been compensated by appropriate transmission power allocation at the receiver then, the probability vector in (5.7) can be used as an alternative way of representing the transmitter’s strategy. More specifically, the following two vectors may be used interchangeably, to study the optimal transmission strategy and average throughput of the adaptive OFDM system under jamming:

\[
r^T_{1 \times K} = [r_1 \ldots r_k \ldots r_K] \quad r_k \in \mathcal{R} \quad \leftrightarrow \quad x^T_{1 \times N} = [x_0 \ldots x_i \ldots x_{N-1}] \quad \text{s.t.} \quad \begin{cases}
0 \leq x_i \leq 1 \\
\sum_{i=0}^{N-1} x_i = 1
\end{cases} \tag{5.8}
\]

That is, by assuming that the wireless channel impairments have been compensated by the proper transmission power allocation, all subcarriers experience nearly the same channel characteristics across the entire frequency band and as a result, knowing \( x_{N \times 1} \) which gives the fractions of subcarriers coded/modulated at available data rates is sufficient to study the performance of the wireless OFDM system under jamming.

Even though the optimal transmission strategy can be computed in terms of \( x \), vector \( r \), which contains the actual transmission rates, must be reconstructed from \( x \) in order to

---

2Also see the discussion in Section 5.5 for more details.
code/modulate the input data. Below, we will discuss one possible approach to construct the transmission rate vector from the probability vector. First, the transmitter constructs vector \( \hat{x}_{N \times 1} \) which contains the number of subcarriers coded/modulated with the available data rates. This can be done by multiplying the probability vector \( x \) by \( K \) and rounding off the results to the closest integer such that the sum remains \( K \), i.e.,

\[
\hat{x}_{1 \times N}^T = \text{round} \, Kx_{1 \times N}^T \tag{5.9}
\]

where round() denotes the rounding to the nearest integer operation (such that the sum remains \( K \)). From vector \( \hat{x} \) the data rate assignment matrix, denoted by \( P_{N \times K} \) is constructed.

To fill out the elements of \( P \), we simply need to assign ones and zeros to the elements of \( P \) such that,

\[
\begin{align*}
\sum_{k=1}^{K} P_{ik} &= \hat{x}_i \quad \text{for} \quad i = 0, \ldots, N - 1 \\
\sum_{i=1}^{N} P_{ik} &= 1 \quad \text{for} \quad k = 1, \ldots, K
\end{align*} \tag{5.10}
\]

That is, the number of ones in each row of \( P \) is equal to the corresponding component of \( \hat{x} \) and every column contains exactly one 1. For example, one possible data rate assignment matrix is given in (5.11).

\[
P_{N \times K} = \begin{bmatrix}
111 & 00 & 0000 & \ldots \\
000 & 11 & 0000 & \ldots \\
000 & 00 & 1111 & \ldots \\
\vdots & & & \ddots
\end{bmatrix} \quad \text{← # of 1’s = } \hat{x}_0 \\
\text{← # of 1’s = } \hat{x}_1 \\
\text{← # of 1’s = } \hat{x}_2 \tag{5.11}
\]

A 1 at row \( i \) and column \( k \) of \( P \), indicates that the \( k \)th subcarrier is to be coded/modulated with the \( i \)th data rate, Finally, the rate vector for the \( n \)th OFDM symbol, \( r_{K \times 1}^{(n)} \), can be written as

\[
r_{1 \times K}^{(n)} = R_{1 \times N}^T \text{randperm}^{(n)} P_{N \times K} \tag{5.12}
\]

where \( R \) is the vector of available data rates (from (5.1)) and \( \text{randperm}^{(n)} P_{N \times K} \) denotes randomly permuting columns of \( P_{N \times K} \) for \( n \) times. As we discuss in Section 5.5, for each OFDM symbol, it is necessary to randomly permute columns of \( P \). Figure 5.2 shows the transmitter’s adaptation block.
CHAPTER 5. ADAPTIVE OFDM SYSTEMS UNDER JAMMING

5.4 Jammer Model

The jammer has an adaptation block which allows him to jam individual subchannels with (possibly) different jamming powers (see Figure 5.3). However, for practical reasons, the jammer’s maximum jamming power per subchannel is limited to $J_{\text{max}}$ (W/Hz). Furthermore, we assume the jammer’s energy budget per OFDM symbols is limited to $E_{\text{max}}$ (Joules). Finally, we assume $E_{\text{max}}$ is such that the jammer cannot use $J_{\text{max}}$ for all subchannels otherwise, the energy constraint would be redundant.

We denote the jammer’s jamming power set by $\mathcal{J}$. Without loss of generality assume, $\mathcal{J}$ is given as follows,

$$\mathcal{J} = \{J_0 = 0, \ldots, J_i, \ldots, J_{M-1} = J_{\text{max}}\} \text{ (W/Hz)} \quad \text{where} \quad ||\mathcal{J}|| = M \quad (5.13)$$

Because of the jammer’s energy constrained, $E_{\text{max}}$, the jammer’s average power, denoted by $E[J]$, is constrained to

$$E[J] \leq \frac{E_{\text{max}}}{T_s} \quad (5.14)$$

where $T_s$ is the OFDM symbol duration. Let $\mathbf{j}_{K\times1}$ denote the jammer’s strategy per OFDM symbol (equivalently, jamming vector),

$$\mathbf{j}_{K\times1}^T = [j_1 \ldots j_k \ldots j_K] \text{ (W/Hz)} \quad \text{where} \quad j_k \in \mathcal{J} \quad (5.15)$$

that is, the $k$th subchannel is jammed with $j_k \in \mathcal{J}$ (W/Hz). From (5.14) we have,

$$T_s \sum_{k=1}^{K} j_k \Delta f = T_s \Delta f \sum_{k=1}^{K} j_k \leq E_{\text{max}} \quad (5.16)$$

$$\Rightarrow \sum_{k=1}^{K} j_k \leq \frac{E_{\text{max}}}{T_s \Delta f} \quad (5.17)$$
CHAPTER 5. ADAPTIVE OFDM SYSTEMS UNDER JAMMING

Now let \( \hat{y}_i \) denote the number of subchannels jammed with \( J_i \in \mathcal{J} \), obviously, we have

\[
\sum_{k=1}^{K} j_k = \sum_{i=0}^{M-1} \hat{y}_i J_i \tag{5.18}
\]

It is also clear that,

\[
\sum_{i=0}^{M-1} \hat{y}_i = K \quad \text{and} \quad 0 \leq \hat{y}_i \leq K \tag{5.19}
\]

Define \( y_i \triangleq \frac{1}{K} \hat{y}_i \), from (5.19), we have

\[
\sum_{i=0}^{M-1} y_i = 1 \quad \text{and} \quad 0 \leq y_i \leq 1, \quad \text{for } i = 0, \ldots, M - 1 \tag{5.20}
\]

Therefore, for sufficiently large \( K \), the following vector can be well approximated by a probability vector

\[
y_{1 \times M}^T = [y_0 \ldots y_i \ldots y_{M-1}] \quad \text{for } K \gg M \tag{5.21}
\]

If we write (5.17) in terms of \( y_i, i = 1, \ldots, M \), we obtain the following constraint on \( y \).

\[
\sum_{k=1}^{K} j_k = K \sum_{i=1}^{M} y_i J_i \leq \frac{E_{\max}}{T s \Delta f} \tag{5.22}
\]

Therefore, (5.21) can be used as an alternative way of representing the jammer’s strategy. More specifically, the following two vectors can be used interchangeably to study the optimal
CHAPTER 5. ADAPTIVE OFDM SYSTEMS UNDER JAMMING

Figure 5.4: The jammer’s adaptation block.

Even though the optimal jamming strategy can be computed in terms of \( y \), as it is shown in Figure 5.3, vector \( j^{(n)} \), which contains the actual jamming powers for the \( n \)th OFDM symbol, must be passed to the jamming block in order to allocate the available jamming power accordingly. The jamming vector can be constructed from the jamming probability vector in the exact same manner that the rate vector was constructed. Here we only provide the results and refer the reader to Section 5.3 for the details.

\[
J_i^{\top} = [j_1 \ldots j_k \ldots j_K] \quad, \quad j_k \in \mathcal{J} \quad \leftrightarrow \quad y_i^{\top} = [y_0 \ldots y_i \ldots y_{M-1}] \\
\text{s.t.} \quad \begin{cases} 
0 \leq y_i \leq 1 \\
\sum_{i=0}^{M-1} y_i = 1 \\
\sum_{i=0}^{M-1} y_i J_i = y^{\top}J \leq \frac{E_{\text{max}}}{KT_s \Delta f}
\end{cases} \tag{5.23}
\]

Even though the optimal jamming strategy can be computed in terms of \( y \), as it is shown in Figure 5.3, vector \( j^{(n)} \), which contains the actual jamming powers for the \( n \)th OFDM symbol, must be passed to the jamming block in order to allocate the available jamming power accordingly. The jamming vector can be constructed from the jamming probability vector in the exact same manner that the rate vector was constructed. Here we only provide the results and refer the reader to Section 5.3 for the details.

\[
J_i^{\top} = [j_1 \ldots j_k \ldots j_K] \\
\text{randperm}(n) \\
P_M \times K \\
j_{K \times 1} = \begin{bmatrix}
j_1 \\
j_k \\
j_K 
\end{bmatrix}
\]

The random permutation block is necessary in the jammer’s adaptation block to randomize the jamming vector for every OFDM symbol before passing it to the jamming block. Otherwise, the subchannels will be jammed with a fixed power and the jamming problem simplifies to a simple water filling problem [97].

\[3\text{See the discussion in Section } 5.5 \text{ for more details.}\]
CHAPTER 5. ADAPTIVE OFDM SYSTEMS UNDER JAMMING

Figure 5.5: Adaptive OFDM under jamming.

5.5 Average Throughput of the Adaptive OFDM

We consider the problem of maximizing the average throughput of the wireless link under jamming by randomly adapting the data rates of the subcarriers. Obviously, in absence of the jammer, the optimal strategy to maximize the average throughput is to use the maximum data rate for all subcarriers. However, in the presence of the jammer, the capacities of the subcarriers are not known in advance.

By assigning different data rates to the subcarriers, the transmitter can increase the possibility that some of the subcarriers overcome the jamming (see Figure 5.5). Furthermore, data rates assignments must be done randomly, that is, for every OFDM symbol the data rate assignment pattern must be randomized. This randomization is necessary since static data rate assignment pattern would make higher data rates more vulnerable to jamming as these data rates are easier to jam.

Consider a typical wireless OFDM system such as IEEE 802.11, without loss of generality, assume the set of available data rates in the OFDM system, $\mathcal{R}$, is sorted in a decreasing

---

\[This \text{ is equivalent to maximizing the average number of data bits per OFDM symbol or the average data rate per OFDM symbol.}\]
order, i.e.,
\[ R = \{ R_0 = R_{\text{max}}, \ldots, R_i, \ldots, R_{N-1} = R_{\text{min}} \} \text{ bps} \] (5.25)

where \( R_i, i = 0, \ldots, N \) are the available data rates of the OFDM system for the subcarriers.

It can be shown that to jam a rate adaptive wireless system with \( N \) rates, it sufficient to use no more than \( N + 1 \) jamming powers \[87\]. Therefore, WLOG, we assume the jammer is using the following jamming set
\[ J = \{ J_0 = 0, \ldots, J_j, \ldots, J_N = J_{\text{max}} \} \text{ W/Hz} \] (5.26)

When the wireless channel is nearly flat or, when the jamming is the dominant cause of the noise at the receiver front end (which is the typical case for most jamming scenarios), it can be assumed that each rate in \( R \) can tolerate up to a certain level of jamming power which is the same for all subchannels and it is completely lost otherwise (since every subchannel experiences nearly the same channel or the jamming is the dominant factor of the noise). As we will see shortly, this assumption greatly simplifies the analytical results and allows us to express the results in closed form expressions.

Assume \( R_i \in R, 0 \leq i \leq N-1 \), can be recovered for any jamming power less than \( J_{i+1} \), \( 0 \leq i \leq N-1 \). That is, \( R_0 \) can only tolerate \( J_0 \) and no rate can tolerate \( J_N \). Furthermore, let the transmitter’s and the jammer’s strategies be\[5\]
\[ \hat{x}^T = [\hat{x}_0 \ldots \hat{x}_i \ldots \hat{x}_{N-1}]_{1 \times N} \text{ \hat{x}_i : # of subchannels to be sent with } R_i \] (5.27)

and
\[ \hat{y}^T = [\hat{y}_0 \ldots \hat{y}_i \ldots \hat{y}_N]_{1 \times (N+1)} \text{ \hat{y}_i : # of subchannels jammed with } J_i \] (5.28)

where \( \hat{x} \) and \( \hat{y} \) are defined and constructed as discussed in sections \[5.3\] and \[5.4\] respectively. Since the transmitter and the jammer randomize their respective strategies independently, the partial average throughput from rate \( R_0 \), and denoted by \( T_0 \), becomes
\[ T_0 = \hat{x}_0 \left( \frac{\hat{y}_0}{K} \right) R_0 \] (5.29)

\[5\] Throughout the rest of this chapter, whenever clear from the context, we refer to the vectors \( r, x \) and \( \hat{x} \) as the transmitter’s strategies interchangeably. In other cases, we explicitly mention which of the vector are being referred to.
CHAPTER 5. ADAPTIVE OFDM SYSTEMS UNDER JAMMING

that is because \( \hat{x}_0 \) is the number of subcarriers coded/modulated with \( R_0 \) and \( \hat{y}_0/K \) is the probability that a subchannel is jammed with \( J_0 \). Similarly, the partial average throughput from data rate \( R_i \) is

\[
T_i = \hat{x}_i \left( \frac{1}{K} \sum_{j=0}^{i} \hat{y}_j \right) R_i
\]

Therefore, the average throughput per OFDM symbols becomes,

\[
T = \sum_{i=0}^{N-1} T_i = \sum_{i=0}^{N-1} \hat{x}_i \left( \frac{1}{K} \sum_{j=0}^{i} \hat{y}_j \right) R_i
\]

\[
= K \sum_{i=0}^{N-1} \sum_{j=0}^{i} \hat{x}_i \hat{y}_j R_i
\]

\[
= K \sum_{i=0}^{N-1} \sum_{j=0}^{i} \hat{x}_i \hat{y}_j R_i
\]

\[
= x^T K R_{N \times (N+1)} y
\]

where, \( x \) and \( y \) are defined in (5.7) and (5.21), respectively and the matrix \( R_{N \times (N+1)} \) is a lower triangular matrix where the nonzero elements of the rows of \( R \) are equal to the data rates, i.e.,

\[
R_{N \times (N+1)} = \begin{bmatrix}
R_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
R_i & R_i & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
R_{N-1} & \cdots & R_{N-1} & 0
\end{bmatrix}
\]

therefore, the optimal transmission/jamming and the average throughput of the wireless link at the equilibrium is the solution of the following maxmin problem,

\[
T(x^*, y^*) = \max_x \min_y x^T K R_{N \times (N+1)} y \quad \text{s.t.} \quad \begin{cases} 
x^T 1 = 1 \\
y^T 1 = 1 \\
y^T J \leq \frac{E_{\text{max}}}{KT_s \Delta f}
\end{cases}
\]

where, \( x^* \) and \( y^* \) denote the optimal transmission and jamming strategies, respectively. The maxmin problem in (5.33) can be solve (both analytically and numerically) similar to Section 2.4.
5.6 Performance Analysis of IEEE 802.11 OFDM

In IEEE 802.11, the 20 MHz channel bandwidth is subdivided into 52 subchannels with a separation of $\Delta f = \frac{20}{64} = .3125$ MHz. Of the total 52 subcarriers, 48 carry data and 4 are pilot subcarriers, however, for convenience, in our analysis we assume the number of subcarriers is $K = 64$ and all the subcarriers carry data. The OFDM symbol interval is $T_s = 4 \mu s$ (which includes a 0.8 $\mu s$ cyclic prefix), which results in the symbol rate $R_s = 0.25$ MSymbols/sec. Table 5.1 shows some of the IEEE 802.11 physical layer parameters including the modulation schemes and code rates used for the subcarriers. The last column in Table 5.1 shows the resulting data rates for the subcarriers. Hence, the set of available data rates for the subcarriers is

$$R = \{ R_0 = 1.125, R_1 = 1.0, .75, .5, .375, .25, .1875, R_7 = .125 \} \text{Mbps} \quad (5.34)$$

Suppose that we have a base station that is communicating with a mobile user and the total transmission power for the 64 subcarriers is 200 mW. Then transmission power per subcarrier becomes,

$$P_T = \frac{0.2}{64} = .0031 \text{ W} = -25 \text{ dBW} \quad (5.35)$$

Furthermore, assume that the combined transmitter/receiver antenna gains and the losses from other sources result in a $L_T = 90$ dB attenuation in the received signal power. Then, the power of the received signal per subcarrier is,

$$(P_R)_{dB} = (P_T)_{dB} - (L_T)_{dB} = -25 - 90 = -115 \text{ dBW} \quad (5.36)$$

The power spectral density of additive noise at the receiver front end is $N_0 = 4.1 \times 10^{-21}$ W/Hz. Therefore the signal to noise ratio at the receiver front end becomes,

$$\text{SNR}_{dB} = (P_R)_{dB} - (\Delta f N_0)_{dB} \cong 34 \quad (5.37)$$

and from (5.37) the capacity of the subcarriers is

$$C_{AWGN} = \Delta f \log (1 + \text{SNR}) = 3.52 \text{ Mbps} \quad (5.38)$$

Since the capacity of the link is greater than $\max_i R$, all the available data rates are feasible
CHAPTER 5. ADAPTIVE OFDM SYSTEMS UNDER JAMMING

<table>
<thead>
<tr>
<th>modulation</th>
<th>code rate</th>
<th>bits/subcarrier</th>
<th>data rate/subcarrier</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPSK</td>
<td>1/2</td>
<td>1/2</td>
<td>125 Kbps</td>
</tr>
<tr>
<td>BPSK</td>
<td>3/4</td>
<td>3/4</td>
<td>187.5 Kbps</td>
</tr>
<tr>
<td>QPSK</td>
<td>1/2</td>
<td>1</td>
<td>250 Kbps</td>
</tr>
<tr>
<td>QPSK</td>
<td>3/4</td>
<td>3/2</td>
<td>375 Kbps</td>
</tr>
<tr>
<td>16-QAM</td>
<td>1/2</td>
<td>2</td>
<td>500 Kbps</td>
</tr>
<tr>
<td>16-QAM</td>
<td>3/4</td>
<td>3</td>
<td>750 Kbps</td>
</tr>
<tr>
<td>64-QAM</td>
<td>2/3</td>
<td>4</td>
<td>1 Mbps</td>
</tr>
<tr>
<td>64-QAM</td>
<td>3/4</td>
<td>9/2</td>
<td>1.125 Mbps</td>
</tr>
</tbody>
</table>

Table 5.1: IEEE 801.11 modulation schemes and code rates.

(when the jammer is not active). Now, suppose that the jammer’s combined antenna gain and losses from other sources results in a $L_J = 60$ dB attenuation in the jamming signal measured at the receivers front end. To make the channel capacity drop below the data rate (i.e., make the data rates infeasible), the jammer needs to increase the channel noise at the receiver front end by

$$P_{JR,i} = \Delta f J_i = \frac{P_R}{2R_i/\Delta f - 1} - \Delta f N_0 \quad (W)$$

(5.39)

where $R_i$’s are given in (5.34). Hence, the jammer’s transmission power becomes

$$(P_{JT,i})_{dB} = (P_{JR,i})_{dB} + (L_J)_{dB}$$

(5.40)

If we substitute the numerical values, the jammer’s strategy set becomes

$$J = \{0, 0.0003, 0.0004, 0.0007, 0.0015, 0.0024, 0.0042, 0.0061, 0.0098\} \text{ W/subchannel}$$


(5.41)

The optimal transmission and jamming strategies and the expected value of the game at the Nash equilibrium can be derived analytically (and numerically) in the same manner as was presented in Section 2.4. For instance, it can be shown that the minimum average jamming power needed to force the lowest rate for the jammer that is using optimal strategy is

$$J_{TH} = K \Delta f R_T \sum_{i=1}^{7} (R_i^{-1} - R_{i-1}^{-1}) J_j = .2133 \text{ W} = -6.7 \text{ dBW}$$

(5.42)
CHAPTER 5. ADAPTIVE OFDM SYSTEMS UNDER JAMMING

whereas the average jamming power that the Barrage noise jammer require to achieve the same average throughput is

\[ J_{\text{barrage}} = \frac{K P_R}{2 R_{\text{min}} / K \Delta f - 1} - K \Delta f N_0 = 0.39 \text{ W} = -4.1 \text{ dBW} \]  \hspace{1cm} (5.43)

This shows a gain of 2.6 dB for the strategic jammer.

5.7 Concluding Remarks

In this chapter, we studied the performance of an adaptive OFDM wireless communication system under power limited jamming. We showed that under modest assumptions, this problem can be formulated into the constrained zero-sum or constrained bimatrix frameworks introduced previously. As a result, all the analytical results that were presented in chapters 2 and 3 can be applied to this problem directly.
Chapter 6

Conclusion

In this dissertation, we developed two general game-theoretic frameworks, constrained zero-sum and constrained bimatrix, that can be used to model many interactive situations in wireless networks including physical layer jamming. In constrained games, players strategies are limited to a subset of strategies that are available in standard games and as a result, a broader class of problems can be modeled by constrained games.

In Chapter 2, we formulated the interaction between an adaptive transmitter (a transmitter with multiple transmission choices) and a smart power limited jammer in a game theoretic context. We showed that packetization and adaptivity benefits a smart jammer. While the standard information-theoretic performance results for a jammed channel corresponds to pure Nash equilibrium, packetized adaptive communication leads to a lower expected value and a mixed-strategy Nash Equilibrium. Inspired by the Shannon’s capacity theorem, we defined a general utility function and a payoff matrix which may be applied to a variety of jamming problems. Furthermore, we showed the existence of optimal mixed-strategy NE for the transmitter and the jammer. We showed the existence of a threshold on jammer’s average power such that if the jammer’s average power exceeds this threshold then the expected value of the game at the NE corresponds to the transmitter’s lowest payoff, this is as if the jammer was using the maximum jamming power all the time. Finally, we studied an special case of optimal strategies in a discrete-time AWGN wireless channel under jamming and
CHAPTER 6. CONCLUSION

showed that randomization can significantly assist a smart jammer with limited average power.

In Chapter 3, we extended the game-theoretic framework we developed in Chapter 2 to the constrained bimatrix framework that can be used to model even more practical jamming problems in packetized wireless networks. In contrast to the standard bimatrix games, in constrained bimatrix games the players’ strategies must satisfy some additional average conditions, consequently, not all strategies are feasible and the existence of the NE is not guaranteed anymore. We provided the necessary and sufficient conditions under which the existence of the Nash equilibrium (NE) is guaranteed and showed that the equilibrium pairs and the Nash equilibrium solution of this constrained game corresponds to the global maximum of a quadratic program. Finally, we studied a typical packetized wireless link under power limited jamming and showed that the game theoretic analysis of this typical problem yields rather surprising results.

In Chapter 4, inspired by the superposition coding technique in broadcast channels, we propose an adaptive multilayer superposition coding technique to improve the performance of packetized networks. Our analytical results shows that, compared to rate adaptation, superposition coding not only achieves better average performance under jamming, it also increases the jamming threshold.

In Chapter 5, we studied the performance of an adaptive OFDM wireless communication system under power limited jamming. We showed that under modest assumptions, this problem can be formulated into the constrained zero-sum or constrained bimatrix frameworks introduced previously. As a result, all the analytical results that were presented in chapters 2 and 3 can be applied to this problem directly.
Appendix A

An Upper Bound for the Jamming Threshold

In section (2.5.2) we showed, without giving a proof, that

\[
\max_{0 \leq i < N_T} \left\{ U_i \right\} \triangleq \left( 1 - \frac{R_{NT}}{R_i} \right) \left( \frac{1}{2} \frac{N_T+1}{N_T-i} \right) J_{\text{max}} = J_{TH,U} \tag{A.1}
\]

where \( R_i \) and \( J_{TH,U} \) are given by (2.72) and (2.76), respectively.

Proof. To show that (A.1) is true for all \( P_T, N, J_{\text{max}}, J_{\text{ave}} > 0 \) we need to show that \( U_i \) is indeed maximized for \( i = 0 \). First, we rewrite \( U_i \) as

\[
U_i = \frac{1}{2} J_{\text{max}} \frac{N_T+1}{N_T} \frac{1}{1 - \frac{i}{N_T}} \left( 1 - \frac{R_{NT}}{R_i} \right); \quad 0 \leq i < N_T
\]

\[
= \left( \frac{1}{2} J_{\text{max}} \frac{N_T+1}{N_T} \right) \frac{J_{\text{max}}}{J_{\text{max}} - \left( \frac{i}{N_T} J_{\text{max}} \right)} \left( 1 - \frac{R_{NT}}{R_i} \right) \tag{A.2}
\]

define \( J \) and \( R(J) \) as

\[
J = \left( \frac{i}{N_T} J_{\text{max}} \right) \quad 0 \leq i < N_T
\]

\[
R(J) = \frac{1}{2} \log \left( 1 + \frac{P_T}{N + J} \right) \quad 0 \leq J < J_{\text{max}} \tag{A.3}
\]
substituting (A.3) in (A.2) and we have

\[ U(J) = \left( \frac{1}{2} J_{\text{max}} \frac{NT + 1}{NT} \right) \frac{J_{\text{max}}}{J_{\text{max}} - J} \left[ 1 - \frac{R(J_{\text{max}})}{R(J)} \right] \]

\[ = a \times \frac{J_{\text{max}}}{J_{\text{max}} - J} \left[ 1 - \frac{R(J_{\text{max}})}{R(J)} \right] \]

\[ = a \times F(J); \quad \text{where} \quad a > 0 \quad \text{and} \quad 0 \leq J < J_{\text{max}} \]

If \( F(J) \) in (A.4) were a decreasing function of \( J \) then \( U_i \) and \( U(J) \) would also be decreasing functions of \( i \) and \( J \) respectively. Let

\[ F(J) = f(J)g(J) \quad \text{where} \quad f(J) = \frac{J_{\text{max}}}{J_{\text{max}} - J} \]

\[ g(J) = 1 - \frac{R(J_{\text{max}})}{R(J)} \]

(A.5)

For decreasing \( F(J) \) we have

\[ \frac{\partial}{\partial J} F = g \frac{\partial}{\partial J} f + f \frac{\partial}{\partial J} g < 0 \]

\[ f, g > 0 \quad \text{for} \quad 0 \leq J < J_{\text{max}} \Rightarrow \frac{\partial}{\partial J} f \frac{f}{f} < -\frac{\partial}{\partial J} g \frac{g}{g} \]

(A.6)

From (A.5) we have

\[ -\frac{\partial}{\partial J} g \frac{g}{g} = \frac{1}{N + J} \times \left( \frac{x}{1 + x} \right) \left( \frac{\log(1 + x_m)}{\log(1 + x)} \right) \times \left( \frac{1}{\log(1 + x) - \log(1 + x_m)} \right) \]

\[ \frac{\partial}{\partial J} f \frac{f}{f} = \frac{P^{-1}_T x_m x}{x - x_m} \]

(A.7)

where

\[ x = \frac{P_T}{N + J} \quad \text{and} \quad x_m = \frac{P_T}{N + J_{\text{max}}} \quad \text{and} \quad 0 < x_m < x \]

(A.8)

If we plug (A.7) and (A.8) in (A.6) and simplify the resulted inequality we have

\[ Z = \frac{x^{-1}_m x}{1 + x \log(1 + x) - \log(1 + x_m)} \log(1 + x) \log(1 + x) \]

\[ > 1 \quad \text{for} \quad 0 < x_m < x \]

(A.9)
APPENDIX A. AN UPPER BOUND FOR THE JAMMING THRESHOLD

We need to show that \((A.9)\) holds for all \(0 < x_m < x\). Notice that

\[
\lim_{x \to x^+_m} Z \sim \frac{x^{-1} \log(1 + x_m)}{x^{-1} \log(1 + x)} \to 1^+ \quad \forall \ 0 < x_m < x \quad (A.10)
\]

where we used

\[
\frac{d}{dz} z^{-1} \log(1 + z) < 0 \quad \forall \ 0 < z \quad (A.11)
\]

and the following natural logarithm property

\[
\frac{z}{1 + z} < \log(1 + z) \leq z \quad \text{for all} \quad z > 0 \quad (A.12)
\]

For simplicity we rewrite inequality in \((A.9)\) as

\[
Z_2 = [x(x - x_m) \log(1 + x_m)] - [x_m(1 + x) \log(1 + x) \log(1 + x) - \log(1 + x_m))] > 0 \quad (A.13)
\]

As a result of \((A.10)\) we have \(\lim_{x \to x^+_m} Z_2 \to 0^+\) for all \(0 < x_m < x\). Since \((A.13)\) holds for \(x \to x^+_m\), if \(Z_2\) were a strictly increasing function of \(x\) for all \(x > x_m\), \((A.13)\) and \((A.9)\) would also hold as a corollary.

To show that \(Z_2\) is strictly increasing, we first verify that

\[
\frac{\partial Z_2}{\partial x} \bigg|_{x = x_m} = 0 \quad (A.14)
\]

given that \((A.14)\) is true, an alternative way to proceed is to show that \(\frac{\partial Z_2}{\partial x}\) is itself strictly increasing function of \(x\) (strictly convex function of \(x\)). Define \(Z_3\)

\[
Z_3 = \frac{\partial^2 Z_2}{\partial x^2} \times (1 + x) = 2 \log(1 + x_m) - 2x_m + 2x \log(1 + x_m) - 2x_m \log(1 + x) + x_m \log(1 + x_m) \quad (A.15)
\]

It can be verified that for all \(x > x_m\) and \(x_m > 0\) we have \(\lim_{x \to x^+_m} Z_3 > 0\). Taking the partial derivative of \(Z_3\) with respect to \(x\) we have

\[
\frac{\partial Z_3}{\partial x} = 2 \left[ \log(1 + x_m) - \frac{x_m}{1 + x} \right] \quad (A.16)
\]

but from \((A.12)\) we have

\[
\log(1 + x_m) > \frac{x_m}{1 + x_m} > \frac{x_m}{1 + x} \quad \text{for all} \quad x > x_m \quad (A.17)
\]

\[
\Rightarrow \ 2 \left[ \log(1 + x_m) - \frac{x_m}{1 + x} \right] > 0 \quad \forall x > x_m > 0
\]
and hence we have
\[
\frac{\partial Z_3}{\partial x} > 0 \quad \text{for all } x > x_m > 0 \tag{A.18}
\]
Consequently, \(Z_2\) is indeed an increasing function of \(x\) for all \(0 < x_m < x\). Taking the reverse steps that resulted in (A.13) and (A.9) we can conclude that \(U_i\) in (A.1) is indeed a strictly decreasing function of \(i\) and hence it is maximized for \(i = 0\). \(\square\)
Appendix B

Existence of the Nash Equilibrium for the Constrained Bimatrix Games

In game theory, fixed-point theorems are commonly used to prove that a model has an equilibrium point. In particular, Brouwer’s fixed point theorem is often used to prove the existence of a solution for finite games (e.g., see [82]), however, the approach used in [82] (and similar approaches) cannot be extended to constrained game. As a consequence of the average constrains on mixed-strategies, arbitrary probabilities cannot be assigned to some pure-strategies and as a result, a more general approach is required. In our approach, we use the Kakutani’s fixed point theorem to prove that our constrained bimatrix model has at least one equilibrium point. We start this section by providing some definitions.

**Set-valued function** (set-function or correspondence): Denoted by $F : X \rightrightarrows Y$ is mapping from $X$ to non-empty subsets of $Y$, i.e., for all $x \in X$ we have $F(x) \in 2^Y - \emptyset$. As opposed to a single-valued function (or simply, a function), a set-valued function can map its input to more than one output. Figure B.1 shows a comparison of a single-valued function and a set-valued function.

**Convexed-valued function**: Let $F : X \rightrightarrows Y$ be a set-valued function then $F$ is convex-valued if $F(x)$ is a convex set for all $x \in X$.

**Upper semi-continuous**: $F$ is upper semi-continuous if the following holds: for every
sequence $x_k$ in $X$ that converges to some point $x \in X$ and for every sequence $y_k$ in $Y$ that converges to $y \in Y$, if $y_k \in F(x_k)$ for all $k \in \mathbb{N}$, then $y \in F(x)$.

**Fixed point of a set-valued function:** Let $F : Z \supseteq Z$ be a set-valued function then $x^* \in Z$ is a fixed point of $F$ if $x^* \in F(x^*)$.

The following theorem, known as Kakutani’s fixed point theorem, provides the sufficient conditions for a set-valued function defined on a subset of Euclidean space to have a fixed point.

**Theorem 9 (Kakutani Fixed Point Theorem).** Let $Z \subseteq \mathbb{R}^n$ be a nonempty compact and convex set and let $F : Z \supseteq Z$ be an upper semi-continuous and convex-valued correspondence. Then $F$ has a fixed point.

**Proof.** See [98].

**Theorem 10.** Let $\mathcal{G} = (A, B, r, j, r_{ave}, j_{ave})$ be a constrained bimatrix game, then $\mathcal{G}$ has a Nash equilibrium solution if $r_{ave} \geq \min r_i$ and $j_{ave} \geq \min j_k$.

**Proof.** Let $F$ be a set-valued function defined on $(\hat{X} \times \hat{Y})$,

$$F : (\hat{X} \times \hat{Y}) \supseteq (\hat{X} \times \hat{Y}) \quad (B.1)$$

where $\hat{X}$ and $\hat{Y}$ are defined in (3.10) and (3.11), respectively. Obviously, $\hat{X} \subseteq \mathbb{R}^m$ and $\hat{Y} \subseteq \mathbb{R}^n$ are non-empty, closed and convex subsets. (Note that $\hat{X}$ and $\hat{Y}$ are intersections of standard $k$-simplices and closed half spaces, furthermore, the intersections are non-empty since by assumption $r_{ave} \geq \min r_i$ and $j_{ave} \geq \min j_k$). Therefore, the subspace resulted by
APPENDIX B. EXISTENCE OF THE NASH EQUILIBRIUM

the Cartesian product of \( \hat{X} \) and \( \hat{Y} \), \((\hat{X} \times \hat{Y})\), is also a non-empty, closed and convex subset of \( \mathbb{R}^{m+n} \). Now define \( F \), such that,

\[
F(x, y) = F_x(y) \times F_y(x) = \{(\bar{x}, \bar{y})\} \quad (x, y) \in \hat{X} \times \hat{Y}
\]

(B.2)

where

\[
F_x(y) = \{\bar{x}_i\} \triangleq \operatorname{argmax}_{x \in \hat{X}} x^T A y
\]

(B.3)

and

\[
F_y(x) = \{\bar{y}_j\} \triangleq \operatorname{argmax}_{y \in \hat{Y}} x^T B y
\]

(B.4)

That is, \( F \) maps every strategy pair \((x, y)\) to the Cartesian product of the sets \( \{\bar{x}_i\} \) and \( \{\bar{y}_j\} \) (given by \( F_x(y) \) and \( F_y(x) \), respectively) where all \( \bar{x}_i \)'s are optimal against \( y \) and all \( \bar{y}_j \)'s are optimal against \( x \).

From (B.3) and (B.4) it is clear that for all \((x, y)\) in \( \hat{X} \times \hat{Y} \) the set valued function \( F_x(y) \) depends only on \( y \) and \( F_y(x) \) depends only on \( x \). Therefore, if we show that \( \{\bar{x}_i\} = F_x(y) \) is convexed and upper semi-continuous for every \( y \in \hat{Y} \) by extending the exact same argument to \( \{\bar{y}_j\} = F_y(x) \) we can show that \( F(x, y) \) is convexed-valued and upper semi-continuous.

Consider \( F_x(y) \) in (B.3), for any given \( y \) in \( \hat{Y} \) the problem in (B.3) is a linear program in \( x \). Therefore, the solution is always at the intersection of some binding constraints, i.e., it is a polytope at some corner of the feasible region in the direction of the gradient of \( F_x \) (see Figure B.2). As a result, the set \( \{\bar{x}_i\} \) is either a singleton in \( \hat{X} \) (when \( \nabla F_x(y) \) is not normal to some face of \( \hat{X} \) – Figure B.2 top) or a face of \( \hat{X} \) (when \( \nabla F_x(y) \) is normal to some face of \( \hat{X} \) – Figure B.2 bottom), in either case, the solution set is convex and compact for all \( y \in \hat{Y} \). By using the same argument, it is clear that \( \{\bar{y}_j\} \) is also convex and compact for all \( x \in \hat{X} \). Therefore, the set-valued function \( F(x, y) \) is also convexed-valued.

It can be shown (by contradiction) that for every sequence \( y_k \) in \( \hat{Y} \) that converges to \( y \) and for every sequence \( \bar{x}_k \) that converges to \( \bar{x} \) such that \( \bar{x}_k \in F_x(y_k) \) for all \( k \in \mathbb{N} \) then we must have \( \bar{x} \in F_x(y) \). If \( F_x \) was not upper semi-continuous then \( \bar{x} \notin F_x(y) \) for some sequence. Assume \( F_x(y) \) is a singleton in \( \hat{X} \) (Figure B.2 top), we can find \( K \) sufficiently large to make \( y_K \) arbitrarily close to \( y \) and therefore, for all \( k > K \) we have \( F_x(y_{k>K}) = F_x(y) = \bar{x} \) which is a contradiction. This argument can be easily extended to the case where \( F_x(y) \) is some face of \( \hat{X} \) (Figure B.2 bottom).
APPENDIX B. EXISTENCE OF THE NASH EQUILIBRIUM

Therefore, $F_x(y)$ is upper semi-continuous in $\hat{X}$ (so is $F_y(x)$ in $\hat{Y}$). Hence, $F$ is an upper semi-continuous function. Therefore, the set-valued function $F(x, y)$ defined in (B.2) satisfies the requirements of Kakutani’s theorem and has a fixed point $(x^*, y^*)$ such that

$$(x^*, y^*) \in F(x^*, y^*) \quad (B.5)$$

that is, there exist a strategy pair $(x^*, y^*)$ where its elements are optimal against each other and by definition, this is an equilibrium point of $G$. This concludes the proof. \hfill \square
Bibliography


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY
