On the convergence of the mKdV linearization transform in asymptotic spaces

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I would like to dedicate my dissertation to my mother Teuta Kacaku. Being a single parent she worked very hard throughout her life to make sure that I had nothing missing growing up. The fact that she has dementia means that she cannot comprehend any of this. But I am sure that if she could understand she would be very proud of me. If it wasn’t for my desire to make her proud I wouldn’t had the will to finish.
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Abstract of Dissertation

We prove that the modified KdV linearization transform on the line is a local diffeomorphism in an open neighborhood of zero for a class of asymptotic spaces. A function belongs to an asymptotic space of the considered class if it allows a partial asymptotic expansion at infinity. This result is related to the well-posedness of the mKdV equation in asymptotic spaces.
# Table of Contents

Dedication ii

Acknowledgements iii

Abstract of Dissertation iv

Table of Contents v

Introduction 1

Chapter 1: Formal analysis of the linearization transforms 3
   Review of the Poincaré theory of normal forms 3
   The infinite dimensional case 6

Chapter 2: The integral form of the linearization transforms $C$ and $D$ 10
   The distributions $\hat{F}_n$ and $\hat{G}_n$ 10
   Regularization of $\hat{F}_n$ and $\hat{G}_n$ 12
   The integral form of $C^-$ and $C^+$ 12
   The integral form of $D$ 16

Chapter 3: Convergence of the linearization transforms 20
   Discussion of the function spaces 20
   Convergence of the “direct” transform $D$ 22
   Convergence of the “inverse” transform $C$ 26
   Convergence on the whole line and applications to well-posedness 48

References 51

Appendix A 52

Appendix B 55
Introduction

This thesis studies the properties of the linearization transform of the (defocusing) modified Korteweg-de Vries equation (mKdV) on the line,

$$\begin{cases}
    u_t = -u_{xxx} + 6u^2u_x, \\
    u|_{t=0} = u_0.
\end{cases} \quad (1)$$

Note that the mKdV is one of the most studied non-linear partial differential equations (PDEs). It was the first example (together with the Korteweg-de Vries equation (KdV)) of a completely integrable PDE (see e.g. [9] and the references therein). The mKdV equation can be solved by a variety of methods, including the inverse method, in various function spaces including spaces of periodic functions or spaces of functions on the line decaying at infinity. We refer to [9] for the case of the line, as well as to [7] (and the references therein) for the periodic case.

In the 1980s Bondareva and Shubin [1, 2, 3] (following Menikoff [8]) studied the KdV equation in spaces of $C^\infty$-functions on the line admitting asymptotic expansion at infinity of infinite order, as well as functions on the line that not necessarily decay as $|x| \to \infty$. Bondareva and Shubin used the method of [8] that is a variant of the finite differences method. Using the results of Bondareva and Shubin as well as the property of the Miura transform, Kappeler, Perry, Shubin, and Topalov proved an analog of Bondareva and Shubin’s result for the mKdV equation [5]. The focussing mKdV equation was studied by J. Gonzalez using the same method. An essential restriction of Menikoff’s method is that it does not allow proving continuous dependence of solutions on the initial data.

A new approach for studying spacial asymptotics of non-linear PDEs was recently developed by McOwen and Topalov in [6]. The method in [6] is based on the introduction of a group of asymptotic diffeomorphisms that consists of diffeomorphisms on the line with a partial asymptotic expansion at infinity. Note that the group of asymptotic diffeomorphisms is an infinite dimensional topological group. The asymptotic spaces $A^{m,2}_N(\mathbb{R})$, $m \geq 1$, $N \geq 1$, studied in this thesis play the role of the Lie algebra of the groups of asymptotic diffeomorphisms. The elements of $A^{m,2}_N(\mathbb{R})$ consists of continuous functions $u : \mathbb{R} \to \mathbb{R}$ such that,

$$u(x) = \sum_{k=2}^{N} a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} + o \left( \frac{1}{\langle x \rangle^N} \right), \quad |x| \to \infty,$$

where $a_k$, $b_k$, $k = 2, \ldots, N$, are given constants. We refer to the beginning of Chapter 3 for the precise definition of the asymptotic spaces and the other spaces considered in this thesis. Note that the mKdV is not well suited for applying the method in [6]. The reason is that when re-written as a spray on the tangent space of the group of asymptotic diffeomorphisms the mKdV equation does not produce a Lipschitz vector field.

This is one of the reasons why in this thesis we take another approach: In order to study the well-posedness properties of the mKdV equation in spaces of functions with prescribed asymptotic behavior at infinity we first show that the mKdV linearization transform preserves these spaces and defines a (local) diffeomorphism. More specifically, assume that $E$ is one of the following Banach spaces: $H^{m,1}_N(\mathbb{R}) \cap H^{m,2}_N(\mathbb{R})$, $H^{m,2}_N(\mathbb{R})$, $H^{m,2}_N((-\infty, T))$, $H^{m,2}_N([T, \infty))$, $T \in \mathbb{R}$, $m \geq 0$, $N \geq 1$, as well as their asymptotic analogs $A^{m,2}_N((-\infty, T))$ and $A^{m,2}_N([T, \infty))$, $T \in \mathbb{R}$, $m \geq 1$, $N \geq 2$. Recall also that by definition the linearization transform of the mKdV equation is a (partially defined)
non-linear map \( v = C(u), \ u, v \in E \), that transforms the mKdV equation into
the linear equation \( v_t = -v_{xxx} \). The following statement follows from
Theorem 5 and Theorem 6 in Chapter 3 and the inverse function theorem in
Banach spaces.

**Theorem.** Let \( E \) be one of the spaces specified above. Then there exist open neighborhoods of zero \( \mathcal{U} \) and \( \mathcal{V} \) in \( E \) such that
the mKdV linearization transform \( C : \mathcal{U} \to \mathcal{V} \) is a diffeomorphism.

As the scale of Sobolev spaces \( H^{m,2}_N(\mathbb{R}), \ m \geq 0, \ N \geq 1 \) is preserved by the flow of the linear Airy equation \( u_t = -u_{xxx}, \ u|_{t=0} = u_0 \), we see that the mKdV equation is locally well-posed in an open neighborhood of zero of the scale of spaces \( H^{m,2}_N(\mathbb{R}) \). Note that this improves a result of Zhou [10],
who proved the (set theoretic) bijectivity of the linearization transform using
Riemann-Hilbert’s approach. In particular, the result in [10] does not imply continuous dependence on the initial data.

We expect that Theorem 6 holds also in the asymptotic space \( A^{m,2}_N(\mathbb{R}) \). Note also that the case of
the space \( H^{m,1}_N(\mathbb{R}) \cap H^{m,2}_N(\mathbb{R}) \) with \( m = N = 0 \) was studied in [4]. The result in [4] is global in the
sense that \( C \) is a diffeomorphism defined in the whole phase space. We expect that the same holds
also for the spaces we consider.

**Organisation of the work:** In Chapter 1 and Chapter 2 we apply formal analysis and calculate
the linearization transforms \( C^\pm \) and \( D \). The novelty here in comparison with previous works is the
connection between this formal approach and the classical Poincaré theory of normal forms. We
show that the two approaches coincide. The maps \( D \) and \( C^\pm \) are not new. They coincide with the
mKdV direct and inverse linearization maps obtained by the inverse scattering of the Zakharov-
Shabat operator. The main result of the thesis is proved in Chapter 3. The Chapter begins with
the definition of the functional spaces. The main technical part is the proof of Theorem 5. Theorem
6 is proved at the end of Chapter 2. There are two Appendices where we prove several auxiliary
algebraic identities used in Chapter 2.
Chapter 1: Formal analysis of the linearization transforms

1.1 Review of the Poincaré theory of normal forms

A way to solve differential equations is to first transform them into a simpler form. Poincaré’s theory of normal forms gives a way in which one can transform a differential equation to a simpler form in the neighborhood of an equilibrium position. Poincaré’s theory of normal forms is done in the finite dimensional case. Since in this part of the thesis we will only be concerned with the formal (algebraic) part of the method, the differential equation at hand looks as follows:

\[
\begin{pmatrix}
\frac{dx_1}{dt} \\
\vdots \\
\frac{dx_n}{dt}
\end{pmatrix}
= \begin{pmatrix}
f_1(x_1, \ldots, x_n) \\
\vdots \\
f_n(x_1, \ldots, x_n)
\end{pmatrix},
\]

where \(f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)\) are formal power series. Written in vector notation:

\[
\dot{x} = f(x).
\]

Our goal is to transform the right hand side of this equation into a simpler form. Since \(f_1, \ldots, f_n\) are power series we can separate the linear from the non-linear part and get:

\[
\begin{pmatrix}
\frac{dx_1}{dt} \\
\vdots \\
\frac{dx_n}{dt}
\end{pmatrix}
= A \begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix} + \begin{pmatrix}
q^1(x_1, \ldots, x_n) \\
\vdots \\
q^n(x_1, \ldots, x_n)
\end{pmatrix},
\]

where \(q^1, \ldots, q^n\) are power series of degree higher than 2. In vector notation:

\[
\dot{x} = Ax + v(x) = Ax + v_2(x) + v_3(x) + \ldots,
\]

where \(A\) is an \(n \times n\) matrix and \(v_2, v_3, \ldots\) are homogeneous polynomials of degree 2,3,\ldots respectively. Poincaré’s goal was to construct a formal power series \(C : \mathbb{R}^n \rightarrow \mathbb{R}^n\) which will serve as a formal change of variables that annihilates the non-linear term \(v(x) = v_2(x) + v_3(x) + \ldots\).

To state the Poincaré theorem we need the following definition:

**Definition 1.** The \(n\)-tuple \(\lambda = (\lambda_1, \ldots, \lambda_n)\) of eigenvalues is said to be resonant if among the eigenvalues there exists an integral relation of the form \(\lambda_s = (m, \lambda)\), where \(m = (m_1, \ldots, m_n) \in \mathbb{Z}^n\) with \(m_k \geq 0\). Such a relation is called a resonance. The number \(|m| = \sum m_k\) is called the order of the resonance.

The Poincaré’s theorem states:

**Theorem 1.** If the eigenvalues of the operator \(A\) are non-resonant, then the above equation can be transformed into \(\dot{y} = Ay\) by a formal change of variables \(x = C(y) = y + C_2(y) + C_3(y) + \ldots\)
By a formal change of variables we mean:

\[
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix} = \begin{pmatrix}
  y_1 \\
  \vdots \\
  y_n
\end{pmatrix} + \begin{pmatrix}
  C_2^1(y_1, \ldots, y_n) \\
  \vdots \\
  C_n^1(y_1, \ldots, y_n)
\end{pmatrix} + \ldots,
\]

where \(C_i^j(y_1, \ldots, y_n)\) are homogeneous polynomials of degree \(i\) in the variables \(y_1, \ldots, y_n\).

We will not give a complete proof of Poincaré’s theorem. Instead, we will describe the main idea as similar approach will be used in the infinite dimensional case. The formal change of variables will be chosen to transform the linear vector field \(Ay\) into the nonlinear vector field \(Ax + v(x)\).

Let’s see what are the conditions this requirement puts on the homogeneous terms of the map \(C(y) = y + C_2(y) + C_3(y) + \ldots\). One needs

\[
\frac{d}{dt}(C(y)) = A(C(y)) + v(C(y)).
\]

On one side we have,

\[
\frac{d}{dt}(C(y)) = \frac{d}{dt}\left(y + C_2(y) + C_3(y) + \ldots\right) = \dot{y} + \frac{\partial C_2}{\partial y}(y) \dot{y} + \frac{\partial C_3}{\partial y}(y) \dot{y} + \ldots = \left(I + \frac{\partial C_2}{\partial y}(y) + \frac{\partial C_3}{\partial y}(y) + \ldots\right) \dot{y} = \left(I + \frac{\partial C_2}{\partial y}(y) + \frac{\partial C_3}{\partial y}(y) + \ldots\right) Ay.
\]

On the other side,

\[
A(C(y)) + v(C(y)) = A\left(y + C_2(y) + C_3(y) + \ldots\right) + v\left(y + C_2(y) + C_3(y) + \ldots\right).
\]

Setting both sides equal to each other we get,

\[
\left(I + \frac{\partial C_2}{\partial y}(y) + \frac{\partial C_3}{\partial y}(y) + \ldots\right) Ay = A\left(y + C_2(y) + C_3(y) + \ldots\right) + v\left(y + C_2(y) + C_3(y) + \ldots\right).
\]

Comparing the coefficients in each degree we have:

Degree 1 terms:

\[
Ay = Ay.
\]

There is no requirement put on \(C(y)\) from this.

Degree 2 terms:

\[
\frac{\partial C_2}{\partial y}(y) Ay = AC_2(y) + v_2(y)
\]

\[
\frac{\partial C_2}{\partial y}(y) Ay - AC_2(y) = v_2(y)
\]

Degree 3 terms:

\[
\frac{\partial C_3}{\partial y}(y) Ay = AC_3(y) + v_3(y) + v_2(y, C_2(y)) + v_2(C_2(y), y)
\]
\[ \frac{\partial C_3}{\partial y}(y)Ay - AC_3(y) = v_3(y) + v_2(y, C_2(y)) + v_2(C_2(y), y) \]

Now for degree \( n \) terms we get:

\[ \frac{\partial C_n}{\partial y}(y)Ay - AC_n(y) = \sum_{i_1 + \ldots + i_k = n} v_k(C_{i_1}(y), \ldots, C_{i_k}(y)) \]

Notice that for each degree \( n \) the expression that appear on the left hand side is: \( \frac{\partial C_n}{\partial y}(y)Ay - AC_n(y) \)

Introduce an operator \( L_A : \text{polynomials} \to \text{polynomials} \),

\[ L_A(P)(y) := \frac{\partial P}{\partial y}(y)Ay - A(P(y)). \]

Clearly, \( L_A \) takes polynomials of degree \( n \) to polynomials of degree \( n \).

For simplicity assume that \( A \) is diagonal in the basis \( e_1, \ldots, e_n \). Then consider the basis for the vector space of polynomials of degree \( m \) given by \( y^m e_s \), where \( y^m = y_1^{m_1} \ldots y_n^{m_n} \), \( \sum m_j = m \). One has,

\[ L_A(y^m e_s) = \frac{\partial y^m e_s}{\partial y}(y)Ay - Ay^m e_s(y). \]

The only non-zero component of \( \frac{\partial y^m e_s}{\partial y}(y)Ay \) is the \( s \)-th component and it’s equal to:

\[ \frac{\partial y^m}{\partial y} Ay = \sum \frac{m_i}{y_i} y^m \lambda_s y^m = (m, \lambda) y^m. \]

On the other side:

\[ A(y^m e_s) = \lambda_s y^m e_s. \]

So we get that:

\[ L_A(y^m e_s) = [(m, \lambda) - \lambda_s] y^m e_s. \]

This means that if \( A \) is diagonal and its eigenvalues are non-resonant then the operator \( L_A \) is diagonal with non-zero eigenvalues on the vector space of polynomials of degree \( n \). In particular, \( L_A \) is invertable which means that the equations for each degree are solvable for. Hence, we can solve for each of the terms \( C_n(y) \) by inverting \( L_A \).

Let’s see what is the process of constructing a formal power series \( D(x) = x + D_2(x) + D_3(x) + \ldots \) which takes the solutions of the vector field \( Ax + v(x) \) to solutions of the vector field \( Ay \), where \( D_2, D_3, \ldots \) are vectors of homogeneous polynomials of degree \( 2, 3, \ldots \) respectively. We have,

\[ \frac{d}{dt} D(x) = A(D(x)). \]

Hence,

\[ \frac{d}{dt} D(x) = \frac{d}{dt} \left( x + D_2(x) + D_3(x) + \ldots \right) \]

\[ = \dot{x} + \frac{\partial D_2}{\partial x}(x) \dot{x} + \frac{\partial D_3}{\partial x}(x) \dot{x} + \ldots \]

\[ = \left( I + \frac{\partial D_2}{\partial x}(x) + \frac{\partial D_3}{\partial x}(x) + \ldots \right) \dot{x} \]
\[ \begin{aligned}
&= \left( I + \frac{\partial D_2}{\partial x}(x) + \frac{\partial D_3}{\partial x}(x) + \ldots \right) \left( Ax + v(x) \right) \\
&= \left( I + \frac{\partial D_2}{\partial x}(x) + \frac{\partial D_3}{\partial x}(x) + \ldots \right) \left( Ax + v_2(x) + v_3(x) + \ldots \right).
\end{aligned} \]

On the other side,

\[ A(D(x)) = A\left( x + D_2(x) + D_3(x) + \ldots \right) \]
\[ A(D(x)) = Ax + A(D_2(x)) + A(D_3(x)) + \ldots \]

Putting the two sides equal to each other and comparing the terms of the same degree we get:

Degree 1 terms:

\[ Ax = Ax \]

Degree 2 terms:

\[ \frac{\partial}{\partial x} D_2(x) Ax + v_2(x) = A(D_2(x)) \]
\[ \frac{\partial}{\partial x} D_2(x) Ax - A(D_2(x)) = v_2(x) \]

Degree 3 terms:

\[ \frac{\partial}{\partial x} D_3(x) Ax + \frac{\partial}{\partial x} D_2(x) v_2(x) + v_3(x) = A(D_3(x)) \]
\[ \frac{\partial}{\partial x} D_3(x) Ax - A(D_3(x)) = -\left( \frac{\partial}{\partial x} D_2(x) v_2(x) + v_3(x) \right) \]

Degree \( n \) terms:

\[ \frac{\partial}{\partial x} D_n(x) Ax - A(D_n(x)) = -\left( \frac{\partial}{\partial x} D_{n-1}(x) v_2(x) + \ldots + \frac{\partial}{\partial x} D_2(x) v_{n-1}(x) + v_n(x) \right) \]

i.e

\[ L_A(D_n)(x) = -\left( \frac{\partial}{\partial x} D_{n-1}(x) v_2(x) + \ldots + \frac{\partial}{\partial x} D_2(x) v_{n-1}(x) + v_n(x) \right) \]

By the same reasoning as above one sees that the solutions to such equations depend in the same way on the invertability of \( L_A \) which depends on eigenvalues of \( A \) being non-resonant.

The fact that we choose \( A \) to be in diagonal form was for simplicity, and everything can be proved if \( A \) is just invertable and with non-resonant eigenvalues.

1.2 The infinite dimensional case

We will study the modified Korteweg-de Vries equation (mKdV),

\[ u_t(x, t) = -u_{xxx}(x, t) + 6u^2(x, t)u_x(x, t). \] (2)

Since in this part of the thesis we are not concerned with convergence issues we will assume for a while that \( u \) belongs to an unspecified function space \( E \) of functions \( f : \mathbb{R} \to \mathbb{R} \) on the line and will consider the “vector field” \( X(u) := -u_{xxx} + 6u^2u_x \) on \( E \). Note that in general \( X(u) \notin E \), and hence strictly speaking \( X \) is not a vector field on \( E \). Moreover, as \( X \) is non-linear there are
no general methods or theorems that would guarantee existence or uniqueness of solutions of the corresponding non-linear PDE. In the present section we will ignore all these important issues and will work only with formal geometric objects – maps, vector fields, coordinates, power series, etc. Our main task will be to linearize the (formal) vector field $X$ by a (formal) transformation. This means that we want to construct new coordinates in $E$ that will transform $X$ into its linear part $X_0(u) := -u_{xxx}$, that is much easier to solve. So our goal then becomes to construct the “inverse” map $C : E \to E$ which takes the solutions of $X_0$ to the solutions of $X$, and the “direct” map $D : E \to E$ which takes the solutions of $X$ to the solutions of $X_0$. We will proceed as in the finite dimensional case.

Suppose that $u(t)$ is a curve in $E$ which is a solution of the vector field $X_0$. We want the “inverse” transform $C : E \to E$ to be such that $C(u)(t) = C(u(t))$ is the curve on $E$ which is a solution of $X$. Because $C(u)(t)$ is a solution of $X$ it must satisfy

$$
\frac{d}{dt} \left(C(u)(t)\right) = -C(u)_{xxx} + 6C^2(u)C(u)_x.
$$

Consider the power series $C(u) = u + C_2(u,u) + C_3(u,u,u) + \ldots$, where $C_n : E^n \to E$ are n-linear functionals. We have

$$
\frac{d}{dt} \left(C(u)(t)\right) = \frac{d}{dt} \left(u + C_2(u,u) + C_3(u,u,u) + \ldots\right).
$$

The derivative of the n-linear functionals is

$$
\frac{d}{dt} \left(C_n(u_1, \ldots, u_n)(t)\right) = \frac{\partial}{\partial u_1} C_n(u_1, \ldots, u_n)(\dot{u}_1) + \ldots + \frac{\partial}{\partial u_n} C_n(u_1, \ldots, u_n)(\dot{u}_n).
$$

Since $C_n$ is linear in each of the $u_i$’s,

$$
\frac{\partial}{\partial u_i} C_n(u_1, \ldots, u_n)(\dot{u}_i) = C_n(u_1, \ldots, \ddot{u}_i, \ldots, u_n).
$$

Finally, we get

$$
\frac{d}{dt} \left(C(u)(t)\right) = \ddot{u} + C_2(\ddot{u}, u) + C_2(u, \ddot{u}) + C_3(\ddot{u}, u, u) + C_3(u, \ddot{u}, u) + C_3(u, u, \ddot{u}) + \ldots
$$

Note that $\ddot{u} = -u_{xxx}$ since $u(t)$ solves $X_0$. Hence,

$$
\frac{d}{dt} \left(C(u)(t)\right) = -u_{xxx} - C_2(u_{xxx}, u) - C_2(u, u_{xxx}) - C_3(u_{xxx}, u, u) - C_3(u, u_{xxx}, u) - C_3(u, u, u_{xxx}) - \ldots
$$

The other side of the equation is

$$
-C(u)_{xxx} + 6C^2(u)C(u)_x = -\left(u + C_2(u, u) + C_3(u, u, u) + \ldots\right)_{xxx} +
$$

$$
+6 \left(u + C_2(u, u) + C_3(u, u, u) + \ldots\right)^2 \left(u + C_2(u, u) + C_3(u, u, u) + \ldots\right)_x.
$$

Now just like we did in the proof of the Poincaré’s theorem we compare terms of the same degree. We have:
Degree 1 terms:
\[-u_{xxx} = -u_{xxx}\]

Degree 2 terms:
\[-\left( C_2(u_{xxx}, u) + C_2(u, u_{xxx}) \right) = -C_2(u, u)_{xxx}\]
\[C_2(u, u)_{xxx} - \left( C_2(u_{xxx}, u) + C_2(u, u_{xxx}) \right) = 0\]

Degree 3 terms:
\[C_3(u, u, u)_{xxx} - C_3(u_{xxx}, u, u) - C_3(u, u_{xxx}, u) - C_3(u, u, u_{xxx}) = 6u^2u_x\]

Degree n terms:
\[C_n(u, ..., u)_{xxx} - \left( C_n(u_{xxx}, u, ..., u) + ... + C_n(u, ..., u, u_{xxx}) \right) = \sum_{i_1 + i_2 + i_3 = n} C_{i_1}(u, ..., u)C_{i_2}(u, ..., u)C_{i_3}(u, ..., u)_{xxx}\]

Now going in the other direction, suppose that \(u(t)\) is a curve in \(E\) which is a solution of the vector field \(X\). We want to calculate the “direct” transform \(D : E \rightarrow E\), such that the curve \(D(u)(t)\) is a solution the vector field \(X_0\). Because \(D(u)(t)\) is a solution of \(X_0\) it must satisfy
\[\frac{d}{dt}(D(u)(t)) = -D(u)_{xxx}.\]

Consider the power series form of \(D(u) = u + D_2(u, u) + D_3(u, u, u) + ...\), where \(D_n : E^n \rightarrow E\) are \(n\)-linear functionals.

\[\frac{d}{dt}(D(u)(t)) = \frac{d}{dt}\left(u + D_2(u, u) + D_3(u, u, u) + ...\right)\]

Similarly as for \(C(u)\) we get
\[\frac{d}{dt}\left(D(u)(t)\right) = \dot{u} + D_2(\dot{u}, u) + D_2(u, \dot{u}) + D_3(\dot{u}, u, u) + D_3(u, \dot{u}, u) + D_3(u, u, \dot{u}) + ...\]

Remember that \(\dot{u} = -u_{xxx} + 6u^2u_x\) since \(u(t)\) solves \(X\), hence we get
\[\frac{d}{dt}\left(D(u)(t)\right) = -u_{xxx} + 6u^2u_x + D_2(-u_{xxx} + 6u^2u_x, u) + D_2(u, -u_{xxx} + 6u^2u_x) + D_3(-u_{xxx} + 6u^2u_x, u, u) + D_3(u, -u_{xxx} + 6u^2u_x, u) + D_3(u, u, -u_{xxx} + 6u^2u_x) + ...

The other side of the equation is
\[-D(u)_{xxx} = -\left(u + D_2(u, u) + D_3(u, u, u) + ...\right)_{xxx}.\]

Now just like we did in the proof of the Poincaré’s theorem we have to compare terms of the same degree.

Degree 1 terms:
\[-u_{xxx} = -u_{xxx}\]
Degree 2 terms:

\[ D_2(u,u)_{xxx} - D_2(u_{xxx}, u) - D_2(u,u_{xxx}) = 0 \]

Degree 3 terms:

\[ D_3(u,u,u)_{xxx} - D_3(u_{xxx}, u,u) - D_3(u,u,u_{xxx}) - D_3(u,u,u_{xxx}) = 6u^2u_x \]

Degree n terms:

\[ D_n(u,...,u)_{xxx} - (D_n(u_{xxx},...,u)+...+D_n(u,...,u_{xxx})) = D_{n-2}(6u^2u_x,...,u)+...+D_{n-2}(u,...,6u^2u_x) \]

It's now equations (2) and (3) that we have to solve. But the n-linear forms \( C_n \) and \( D_n \) are too general to do any computations with. Following [11] we chose the linear forms being the following convolutions:

\[
C_n(u,...,u)(x) = \int F_n(-y_1,...,-y_n)u(x+y_1)...u(x+y_n)dy = [\tau F_n \ast (u \otimes ... \otimes u)] \circ i_n,
\]

\[
D_n(u,...,u)(x) = \int G_n(-y_1,...,-y_n)u(x+y_1)...u(x+y_n)dy = [\tau G_n \ast (u \otimes ... \otimes u)] \circ i_n,
\]

where \( \tau \) is the involution \( \tau H_n(y) := H_n(-y), \, y \in \mathbb{R}^n \). \( F_n \) and \( G_n \) are called the kernels of the “direct” transform \( C \) and “inverse” transform \( D \) respectively.
Chapter 2: The integral form of the linearization transforms $C$ and $D$

Since our solutions of equations (2) and (3) will be computed using the convolution form of the maps $C$ and $D$, let’s first express $6u^2 u_x$ (the non-linear part of the vector field $X$) as a convolution of $T_3(y_1, y_2, y_3)$ with $u \otimes u \otimes u$. 

\[
6u^2 u_x = 3(u^2 u_x + u^2 u_x) = 3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(y_1, y_2, y_3) \left( \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3} \right) \left( u(y_1 + x)u(y_2 + x)u(y_3 + x) \right) dy_1 dy_2 dy_3
\]

\[
-3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3} \right) \delta(y_1, y_2, y_3) u(y_1 + x)u(y_2 + x)u(y_3 + x) dy_1 dy_2 dy_3
\]

So $6u^2 u_x = \left[ T_3(y_1, y_2, y_3) * (u \otimes u \otimes u) \right] \circ i_n$ where $T_3(y_1, y_2, y_3) = -3 \left( \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3} \right) \delta(y_1, y_2, y_3)$. In order to solve the differential equations (2) and (3) we need to take their Fourier transforms. This will turn them into algebraic equations, which we will solve recurrently. But before we do that and start solving the equation we need to study how these special forms of the $n$-linear forms act under the operations that we are acting on them with. What is evident in equations (2) and (3) is that we have compositions of maps. In Appendix B we derive the following formula which describes the behavior of the Fourier transforms of the kernels when we have composition. More specifically if:

\[
M_{m+n}(u_1 \otimes \ldots \otimes u_m \otimes u_{n+1} \otimes \ldots \otimes u_{m+n}) := M_{n+1}(M_m(u_1 \otimes \ldots \otimes u_m) \otimes u_{n+1} \otimes \ldots \otimes u_{m+n})
\]

such that $M_k = [\tau H_k * (u_1 \otimes \ldots \otimes u_k)] \circ i_k$ is in the convolution form described in the previous section then the Fourier transforms of their corresponding kernels are related in the following way.

\[
\widehat{H_{m+n}(k_1, k_2)} = \widehat{H_{n+1}(k_1 + \ldots + k_m, k_2)} \widehat{H_m(k_1)}
\]

(5)

Special polynomials that come up when we compute the Fourier transforms of (2) and (3) are:

\[
P_n(k_1, k_2, \ldots, k_n) := (k_1 + k_2 + \ldots + k_n)^3 - (k_1^3 + k_2^3 + \ldots + k_n^3).
\]

The most crucial part of solving the recurrent equations (2) and (3) comes down to the following three algebraic identities for $P_n$ which we prove in Appendix A:

\[
P_n(k_1, k_2, \ldots, k_n) = 3 \sum_{i=1}^{n-1} (k_1 + \ldots + k_i)(k_i + k_{i+1})(k_{i+1} + \ldots + k_n)
\]

(6)

\[
P_n(k_1, k_2, \ldots, k_n) = 3 \sum_{i=2}^{n-1} (k_1 + \ldots + k_{i-1})(k_{i-1} + k_{i+1})(k_{i+1} + \ldots + k_n)
\]

\[
+ \sum_{i=2}^{n-1} (k_1 + \ldots + k_i)(k_{i-1} + k_{i+1})(k_i + \ldots + k_n)
\]

(7)

For $n$ an odd positive integer we have,

\[
P_n(k_1, k_2, \ldots, k_n) = 3 \sum_{i_1 + i_2 + \ldots + i_n = n, i_j \geq 1} (k_1 + \ldots + k_{i_1} + k_{i_1+i_2+\ldots+k_n})(k_{i_1} + k_{i_1+1})(k_{i_2} + k_{i_2+1})
\]

(8)
2.1 The distributions $\hat{F}_n$ and $\hat{G}_n$

Taking the Fourier transform of the both sides of (3) and using identity (4) we get,

$$-iP_n(k_1, ..., k_n)\hat{G}_n(k_1, ..., k_n) =$$

$$\hat{G}_{n-2}(k_1 + k_2 + k_3, k_4, ..., k_n)\hat{T}_3(k_1, k_2, k_3) + ... + \hat{G}_{n-2}(k_1, ..., k_{n-3}, k_{n-2}, k_{n-1}, k_n)\hat{T}_3(k_{n-2}, k_{n-1}, k_n)$$

$$-iP_n(k_1, ..., k_n)\hat{G}_n(k_1, ..., k_n) =$$

$$3i\left(\hat{G}_{n-2}(k_1 + k_2 + k_3, k_4, ..., k_n)(k_1 + k_3) + ... + \hat{G}_{n-2}(k_1, ..., k_{n-3}, k_{n-2}, k_{n-1}, k_n)(k_{n-2} + k_n)\right)$$

$$-P_n(k_1, ..., k_n)\hat{G}_n(k_1, ..., k_n) =$$

$$3\left(\hat{G}_{n-2}(k_1 + k_2 + k_3, ..., k_n)(k_1 + k_3) + ... + \hat{G}_{n-2}(k_1, ..., k_{n-2}, k_{n-1} + k_n)(k_{n-2} + k_n)\right)$$ \hspace{1cm} (9)

Solving these equations recurrently we get

**Theorem 2.**

$$\hat{G}_n(k - 1, ..., k_n) = \begin{cases} 
- \frac{1}{(k_1 + k_2)(k_2 + ..., + k_n)(k_1 + k_2 + k_3 + k_4)(k_4 + ..., + k_n) ... (k_1 + ..., + k_{n-1} + k_n)} & \text{n odd} \\
0 & \text{n even} 
\end{cases}$$

**Proof.** We prove by induction. For n=1 we can pick $\hat{G}_1(k) = -1$. For n=2, (8) forces us to pick $\hat{G}_2(k_1, k_2) = 0$. Because (8) involves only $G_{n-2}$ the claim is clear for n even. Using equation (8) and putting the right side on common denominator we get:

$$3\frac{\sum_{i=2}^{n-1} (k_1 + ... + k_{i-1})(k_{i-1} + k_{i+1})(k_{i+1} + ... + k_n) + \sum_{i=2}^{n-1} (k_1 + ... + k_i)(k_{i-1} + k_{i+1})(k_i + ... + k_n)}{(k_1 + k_2)(k_{n-1} + k_n)(k_1 + k_2 + k_3 + k_4)(k_{n-2} + k_{n-1} + k_n) ... (k_1 + ... + k_{n-1})(k_2 + ... + k_n)}$$

Using identity (6) we clearly get the desired result.

Now we take the Fourier transform of both sides of (2) and using identity (4) we get the following equation:

$$-iP_n(k_1, ..., k_n)\hat{F}_n(k_1, ..., k_n) = 3i\sum_{i_1 + i_2 + i_3 = n} \hat{T}_3(k_1 + ... + k_{i_1}, k_{i_1+1} + ... + k_{i_1+i_2}, k_{i_1+i_2+1} + ... + k_n)$$

...$\hat{F}_{i_1}(k_1, ..., k_{i_1})\hat{F}_{i_2}(k_{i_1+1}, ..., k_{i_1+i_2})\hat{F}_{i_3}(k_{i_1+i_2+1}, ..., k_n)$
\[-P_n(k_1, \ldots, k_n) \hat{F}_n(k_1, \ldots, k_n) =
3 \sum_{i_1 + i_2 + i_3 = n \atop i_j \geq 0} (k_1 + \ldots + k_{i_1} + k_{i_1+i_2+1} + \ldots + k_n) \hat{F}_{i_1}(k_1, \ldots, k_{i_1}) \hat{F}_{i_2}(k_{i_1+1}, \ldots, k_{i_1+i_2}) \hat{F}_{i_3}(k_{i_1+i_2+1}, \ldots, k_n)\]

Solving these equations recurrently we get

**Theorem 3.**
\[
\hat{F}_n(k_1, \ldots, k_n) = \begin{cases} 
-\frac{1}{(k_1+k_2)\ldots(k_{n-1}+k_n)} & n \text{ odd} \\
0 & n \text{ even} 
\end{cases}
\]

**Proof.** For \(n=1\) we choose \(\hat{F}_1(k) = 1\). For \(n=2\) we are forced to pick \(\hat{F}_2(k_1, k_2) = 0\). Since a partition of an even number of 3 parts contains at least an even part and \(\hat{F}_2(k_1, k_2) = 0\) this means that \(\hat{F}_n(k_1, \ldots, k_n) = 0\) for \(n\) even. Using this fact and putting the right side of equation (9) on common denominator we get:

\[
3 \sum_{i_1 + i_2 + i_3 = n \atop i_j \geq 1 \atop i_j \text{ odd}} (k_1 + \ldots + k_{i_1} + k_{i_1+i_2+1} + \ldots + k_n)(k_{i_1} + k_{i_1+1})(k_{i_2} + k_{i_2+1})
\]

By identity (7) we clearly get the desired result for \(n\) odd.

\[\square\]

### 2.2 Regularization of \(\hat{F}_n\) and \(\hat{G}_n\)

Notice that from Theorem 2 and 3 the distributions \(\hat{F}_n\) and \(\hat{G}_n\) have a denominator and hence they are not integrable on their poles. There is a standard way to regularize them so they become proper distribution. On each of the zeros of the denominator we add \(+i0\) or \(-i0\). We are going to focus on two important regularization for \(\hat{F}_n\):

\[
\hat{F}_n^-(k_1, \ldots, k_n) = \begin{cases} 
-\frac{1}{(k_1+k_2-i0)\ldots(k_{n-1}+k_n-i0)} & n \text{ odd} \\
0 & n \text{ even} 
\end{cases}
\]

\[
\hat{F}_n^+(k_1, \ldots, k_n) = \begin{cases} 
-\frac{1}{(k_1+k_2+i0)\ldots(k_{n-1}+k_n+i0)} & n \text{ odd} \\
0 & n \text{ even} 
\end{cases}
\]

For \(\hat{G}_n\) we consider only the following regularization:

\[
\hat{G}_n(k_1, \ldots, k_n) = \begin{cases} 
-\frac{1}{(k_1+k_2+0i)\ldots(k_{n-1}+0i)} & n \text{ odd} \\
0 & n \text{ even} 
\end{cases}
\]
2.3 The integral form of $C^-$ and $C^+$

Let’s recall that:

\[
\hat{F}_n^- (k_1, \ldots, k_n) = \begin{cases} 
-\frac{1}{(k_1+i2-\i0) \cdots (k_{n-1}+i2-\i0)} & \text{n odd} \\
0 & \text{n even}
\end{cases}
\]

\[
\hat{F}_n^+ (k_1, \ldots, k_n) = \begin{cases} 
-\frac{1}{(k_1+i2+i0) \cdots (k_{n-1}+i2+i0)} & \text{n odd} \\
0 & \text{n even}
\end{cases}
\]

Recall that

\[
C_n^- (u, \ldots, u) (x) = \int F_n^- (-y_1, \ldots, -y_n) u(x + y_1) \cdots u(x + y_n) dy = [\tau F_n^- * (u \otimes \cdots \otimes u)] \circ i_n
\]

and

\[
C_n^+ (u, \ldots, u) (x) = \int F_n^+ (-y_1, \ldots, -y_n) u(x + y_1) \cdots u(x + y_n) dy = [\tau F_n^+ * (u \otimes \cdots \otimes u)] \circ i_n
\]

We will focus on $C^- (u)$ because the calculations for $C^+ (u)$ are completely analogous.

We first start by calculating $C_3^- (u)$. We have to calculate the inverse Fourier transform of $\hat{F}_3^- (k_1, k_2, k_3)$.

Let’s recall that,

\[
\hat{F}_3^- (k_1, k_2, k_3) = \frac{-1}{(k_1 + k_2 - i0)(k_2 + k_3 - i0)} = \frac{i}{(k_1 + k_2 - i0)(k_2 + k_3 - i0)}.
\]

Now let’s take the inverse Fourier transform of $\hat{F}_3^- (k_1, k_2, k_3)$:

\[
F_3^- (x_1, x_2, x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{i}{(k_1 + k_2 - i0)(k_2 + k_3 - i0)} e^{ik_1 x_1 + ik_2 x_2 + ik_3 x_3} dk_1 dk_2 dk_3
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-1}{(k_1 + k_2 - i0)} e^{ik_1 x_1 + ik_2 x_2} \left( \int_{-\infty}^{\infty} \frac{i}{k_2 + k_3 - i0} e^{ik_3 x_3} dk_3 \right) dk_2 dk_1
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-1}{(k_1 + k_2 - i0)} e^{ik_1 x_1 + ik_2 x_2} e^{-ik_3 x_3} \theta (x_3) dk_2 dk_1
\]

\[
= \theta (x_3) \int_{-\infty}^{\infty} \frac{i}{(k_1 + k_2 - i0)} e^{ik_2 (x_2 - x_3)} dk_2
\]

\[
= \frac{\theta (x_3)}{\theta (x_3)} \int_{-\infty}^{\infty} e^{ik_1 x_1} e^{-ik_2 (x_2 - x_3)} \theta (x_2 - x_3) dk_1 = \theta (x_2 - x_3) \theta (x_3) \int_{-\infty}^{\infty} e^{ik_1 (x_1 - x_2 + x_3)} dk_1.
\]

Finally we get,

\[
F_3^- (x_1, x_2, x_3) = \delta (x_1 - x_2 + x_3) \theta (x_2 - x_3) \theta (x_3).
\]
Now we move to the case $n = 5$:

$$\hat{F}_5^-(k_1, ..., k_5) = \frac{1}{(k_1 + k_2 - i0)(k_2 + k_3 - i0)(k_3 + k_4 - i0)(k_4 + k_5 - i0)}.$$  

We first express $\hat{F}_5^-(k_1, ..., k_5)$ in terms of $\hat{F}_3^-(k_1, k_2, k_3)$:

$$\hat{F}_5^-(k_1, ..., k_5) = \hat{F}_3^-(k_1, k_2, k_3) \frac{i}{(k_3 + k_4 - i0)(k_4 + k_5 - i0)}.$$  

Using this recurrent expression of $\hat{F}_5^-(k_1, ..., k_5)$ we take the inverse Fourier transform:

$$F_5^-(x_1, ..., x_5) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{F}_3^-(k_1, k_2, k_3) e^{ik_1 x_1 + ik_2 x_2 + ik_3 x_3} ...$$  

$$... \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{i}{(k_3 + k_4 - i0)(k_4 + k_5 - i0)} e^{ik_4 x_4 + ik_5 x_5} dk_4 dk_5 \right) dk_3 dk_2 dk_1$$  

Doing the integral in parenthesis first we get:

$$= \theta(x_4 - x_5) \theta(x_5) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{F}_3^-(k_1, k_2, k_3) e^{ik_1 x_1 + ik_2 x_2 + ik_3 (x_3 - x_4 + x_5)} dk_3 dk_2 dk_1$$  

Now we use the expression of $F_3^-(x_1, x_2, x_3)$ to get by induction,

$$F_5^-(x_1, ..., x_5) = \delta(x_1 - x_2 + x_3 - x_4 + x_5) \theta(x_2 - x_3 + x_4 - x_5) \theta(x_3 - x_4 + x_5) \theta(x_4 - x_5) \theta(x_5).$$  

Similarly, using induction we get,

$$F_{2n+1}^-(x_1, ..., x_{2n+1}) = \delta(x_1 - x_2 + ... + x_{2n+1}) \theta(x_2 - x_3 + ... - x_{2n+1}) ... \theta(x_{2n} - x_{2n+1}) \theta(x_{2n+1})$$  

Now that we have the formula for the kernel we calculate the integral expression of the functional acting on a test function,

$$\langle F_{2n+1}^-, \phi \rangle = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \delta(x_1 - x_2 + ... + x_{2n+1}) \theta(x_2 - x_3 + ... - x_{2n+1}) ... \theta(x_{2n} - x_{2n+1}) \theta(x_{2n+1}) \phi(x_1, ..., x_{2n+1}) dx_1 ... dx_{2n+1}.$$  

We express $\delta(x_1 - x_2 + ... + x_{2n+1}) \theta(x_2 - x_3 + ... - x_{2n+1}) ... \theta(x_{2n} - x_{2n+1}) \theta(x_{2n+1})$ in the region of integration $S$,

$$\langle F_{2n+1}^-, \phi \rangle = \int_S \phi(x_1, ..., x_{2n+1}) dS$$  

where

$$S = \begin{cases} x_1 - x_2 + ... + x_{2n+1} = 0 \\ x_{2n+1} \geq 0 \\ x_{2n} - x_{2n+1} \geq 0 \\ x_2 - x_3 + ... - x_{2n+1} \geq 0 \end{cases}$$
So now we evaluate the functionals $F_{2n+1}^-$ on $u \otimes ... \otimes u$. We start with $n = 3$ case:

$$\left\langle F_3^-, u \otimes u \otimes u \right\rangle = \int_{x_1-x_2+x_4=0 \atop x_3 \geq 0} \int_{x_2-x_3 \geq 0} u(x_1)u(x_2)u(x_3)dS$$

Let $x_1 = x_2 - x_3$, then we get

$$\left\langle F_3^-, u \otimes u \otimes u \right\rangle = \int_{0}^{\infty} \int_{x_3}^{\infty} u(x_3)u(x_2)u(x_2-x_3)dx_2dx_3$$

Shifting the functions $u$ by $x$ to express $C_3^-(u)$ in integral form prescribed.

$$C_3^-(u) = \left\langle F_3^-, \tau_{-x}u \otimes \tau_{-x}u \otimes \tau_{-x}u \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x_1)u(x_2)u(x-x_2+x_3)dx_2dx_3$$

Finally making the change of variables $y_2 = x - x_2$, $y_1 = x - x_3$ we get

$$C_3^-(u) = \left\langle F_3^-, \tau_{-x}u \otimes \tau_{-x}u \otimes \tau_{-x}u \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(y_1)u(y_2)u(x+y_2-y_1)dy_2dy_1.$$

Similarly, in the general case by making the substitution $x_1 = x_2 - x_3 + x_4 + ... - x_{2n} + x_{2n+1}$ we get

$$\left\langle F_{2n+1}^-, u \otimes ... \otimes u \right\rangle = \int_{0}^{\infty} \int_{x_{2n+1}}^{\infty} ... \int_{x_4-x_5+...-x_{2n+1}}^{\infty} \int_{x_3-x_4+...+x_{2n+1}}^{\infty} u(x_{2n+1})u(x_{2n})...$$

...$u(x_2)u(x_2-x_3+x_4+...-x_{2n}+x_{2n+1})dx_1...dx_{2n+1}$

$$C_{2n+1}^-(u)(x) = \left\langle F_{2n+1}^-, \tau_{-x}u \otimes ... \otimes \tau_{-x}u \right\rangle =$$

$$\int_{0}^{\infty} \int_{x_{2n+1}}^{\infty} ... \int_{x_4-x_5+...-x_{2n+1}}^{\infty} \int_{x_3-x_4+...+x_{2n+1}}^{\infty} u(x-x_{2n+1})...$$

...$u(x-x_2)u(x-x_2+x_3-x_4+...+x_{2n}-x_{2n+1})dx_1...dx_{2n+1}$

Finally, after the substitutions $y_1 = x - x_{2n+1}, ..., y_{2n} = x - x_2$ we get,

$$C_{2n+1}^-(u)(x) = \left\langle C_{2n+1}^-(u, ..., u) \right\rangle = \int_{-\infty \leq y_{2n} \leq x} \int_{-\infty \leq y_2 \leq y_1} u(y_1)u(y_2)...u(y_{2n})u(x+y_{2n}-y_{2n-1}+...-y_1)dy_{2n}...dy_1.$$  

(11)
Using equation (12) we get the derivatives of $C_{2n+1}^-(u)(x)$.

$$\frac{\partial^i}{\partial x^i} C_{2n+1}^-(u)(x) = \sum_{i_1+\ldots+i_{2n+1}=1 \atop i_j \geq 0} d_{i_1,\ldots,i_{2n+1}} C_{2n+1}^-(u^{i_1},\ldots,u^{i_{2n+1}})(x)$$

Analogous calculations for $C_{2n+1}^+(u) = \left( F_{2n+1}^+, \tau_{-x}(u) \otimes \ldots \otimes \tau_{-x}(u) \right)$ give

$$C_{2n+1}^+(u)(x) = C_{2n+1}^-(u,\ldots,u)(x) = \int u(y_1)u(y_2)\ldots u(y_{2n})u(x + y_{2n} - y_{2n-1} + \ldots - y_1)dy_{2n}\ldots dy_1.$$  \hspace{1cm} (13)

Using equation (12) we get the derivatives of $C_{2n+1}^+(u)(x)$.

$$\frac{\partial^i}{\partial x^i} C_{2n+1}^+(u)(x) = \sum_{i_1+\ldots+i_{2n+1}=1 \atop i_j \geq 0} d_{i_1,\ldots,i_{2n+1}} C_{2n+1}^+(u^{i_1},\ldots,u^{i_{2n+1}})(x)$$

2.4 The integral form of $D$

First let’s recall that the Fourier transform of the kernels of $D_{2n+1}(u)$ are:

$$\hat{G}_n(k_1,\ldots,k_n) = \begin{cases} \frac{1}{(k_1+k_2+\ldots+k_n-0i)(k_1+k_2+k_3+\ldots+k_{n-1}+0i)(k_{n-1}+k_n-0i)} & \text{n odd} \\ 0 & \text{n even} \end{cases}$$

Now we calculate the inverse Fourier transform of $\hat{G}_n$. First we notice that there is a relation that $\hat{G}_n$ satisfy. The recurrence relation is the following:

$$\hat{G}_{2n+1}(k_1,\ldots,k_{2n+1}) = \frac{1}{(k_2+k_{2n+1}+0i)(k_1+\ldots+k_{2n}+0i)} \hat{G}_{2n-1}(k_1-k_{2n+1},\ldots,k_{2n-2}+k_{2n+1};k_{2n-1}+k_{2n})$$  \hspace{1cm} (14)

Let’s start with the case $n = 3$:

$$\hat{G}_3(k_1, k_2, k_3) = \frac{-1}{(k_1+k_2+i0)(k_2+k_3-i0)} = \frac{i}{(k_1+k_2+i0)(k_2+k_3-i0)}$$

Let’s take the inverse Fourier transform of $\hat{G}_3(k_1, k_2, k_3)$:

$$G_3(x_1, x_2, x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{i}{(k_1+k_2+i0)(k_2+k_3-i0)} e^{ik_1x_1+ik_2x_2+ik_3x_3} dk_3 dk_2 dk_1$$

$$G_3(x_1, x_2, x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{i}{(k_1+k_2+i0)} e^{ik_1x_1+ik_2x_2} \left( \int_{-\infty}^{\infty} \frac{i}{(k_2+k_3-i0)} e^{ik_3x_3} dk_3 \right) dk_2 dk_1$$

16
\[ G_3(x_1, x_2, x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{i}{(k_1 + k_2 + i0)} e^{ik_1x_1+ik_2x_2} \left( \theta(x_3) e^{-ik_3x_3} \right) dk_2dk_1 \]

\[ G_3(x_1, x_2, x_3) = \theta(x_3) \int_{-\infty}^{\infty} e^{ik_1x_1} \left( \int_{-\infty}^{\infty} \frac{i}{(k_1 + k_2 + i0)} e^{ik_2(x_2-x_3)} dk_2 \right) dk_1 \]

\[ G_3(x_1, x_2, x_3) = \theta(x_3 - x_2) \theta(x_3) \int_{-\infty}^{\infty} e^{ik_1(x_1-x_2+x_3)} dk_1 \]

\[ G_3(x_1, x_2, x_3) = \theta(x_3 - x_2) \theta(x_3) \delta(x_1 - x_2 + x_3) \]

Now we evaluate the functional \( G_3(x_1, x_2, x_3) \) on a generic function \( \phi \):

\[ \left\langle G_3, \phi \right\rangle = \int_{y \in \mathbb{R}^3} \theta(x_3 - x_2) \theta(x_3) \delta(x_1 - x_2 + x_3) \phi(x_1, x_2, x_3) dy \]

\[ = \int_{0}^{\infty} \int_{-\infty}^{x_3} \phi(x_2 - x_3, x_2, x_3) dx_2 dx_3. \]

Now taking \( \phi = u \otimes u \otimes u \) we get,

\[ \left\langle G_3, u \otimes u \otimes u \right\rangle = \int_{0}^{\infty} \int_{-\infty}^{x_3} u(x_2)u(x_3)u(x_2 - x_3) dx_2 dx_3. \]

By definition,

\[ D_3(u)(x) = \left\langle G_3, \tau_{-x}u \otimes \tau_{-x}u \otimes \tau_{-x}u \right\rangle = \int_{0}^{\infty} \int_{-\infty}^{x_3} u(x - x_2)u(x - x_3)u(x - x_2 + x_3) dx_2 dx_3. \]

By change of variables \( y_1 = x - x_3, y_2 = x - x_2 \), we get:

\[ D_3(u)(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} u(y_1)u(y_2)u(x + y_2 - y_1) dy_2 dy_1. \]

From this we get the basis of our induction on \( D_3(u)(0) = \left\langle G_3, u \otimes u \otimes u \right\rangle \):

\[ \left\langle G_3, u \otimes u \otimes u \right\rangle = \int_{y_1 \leq y_2 \leq y_3 \atop y_1 - y_2 + y_3 = 0} u(y_1)u(y_2)u(y_3) dS_2. \]

By the definition of the Fourier transform,

\[ \hat{G}_3(k_1, k_2, k_3) = \left\langle G_3, e^{-i(k_1y_1+k_2y_2+k_3y_3)} \right\rangle = \int_{y_1 \leq y_2 \leq y_3 \atop y_1 - y_2 + y_3 = 0} e^{-ik_1y_1}e^{-ik_2y_2}e^{-ik_3y_3} dS_2. \]
We will use this integral form of $\hat{G}_3(k_1, k_2, k_3)$ as the basis of the induction. We will prove that

$$\left\langle G_{2n+1}, u \otimes \ldots \otimes u \right\rangle = \int_{y_1 \leq y_2 \leq \ldots \leq y_{2n+1}} u(y_1) \ldots u(y_{2n+1}) dS_{2n}.$$  

Since $\frac{i}{(k_1 + \ldots + k_{2n}) + 0} = \int_{-\infty}^{0} e^{-i\zeta(k_1 + \ldots + k_{2n})} d\zeta$ when we substitute it in our recurrence relation we get:

$$\hat{G}_{2n+1}(k_1, \ldots, k_{2n+1}) =$$

$$= \frac{i}{(k_2n + k_{2n+1})} - 0i \left( \int_{-\infty}^{0} e^{-i\zeta(k_1 + \ldots + k_{2n})} \left( \int \ldots e^{-i(k_{2n-2} + k_{2n+1})y_{2n-2}e^{-i(k_{2n-1} + k_{2n})y_{2n-1}} dS_{2n-2} \right) d\zeta. \right)$$

Now we use the inductive assumption on $\hat{G}_{2n-1}(k_1, \ldots, k_{2n-1})$ to get,

$$= \frac{i}{(k_2n + k_{2n+1})} - 0i \left( \int_{-\infty}^{0} e^{-i\zeta(k_1 + \ldots + k_{2n})} \left( \int \ldots e^{-i(k_{2n-2} + k_{2n+1})y_{2n-2}e^{-i(k_{2n-1} + k_{2n})y_{2n-1}} dS_{2n-2} \right) d\zeta. \right)$$

The change of variables $y'_1 = y_1 + \zeta$, $\ldots$, $y'_{2n-1} = y_{2n-1} + \zeta$ gives,

$$= \frac{i}{(k_2n + k_{2n+1})} - 0i \left( \int_{-\infty}^{0} e^{-i\zeta(k_1 + \ldots + k_{2n})} \left( \int \ldots e^{-i(k_{2n-2} + k_{2n+1})y_{2n-2}e^{-i(k_{2n-1} + k_{2n})y_{2n-1}} dS_{2n-2} \right) d\zeta. \right)$$

$$= \frac{i}{(k_2n + k_{2n+1})} - 0i \left( \int_{-\infty}^{0} e^{-i(k_1 + \ldots + k_{2n})y_1} e^{-i(k_2 + k_{2n+1})y_2} \ldots e^{-i(k_{2n-2} + k_{2n+1})y_{2n-2}e^{-i(k_{2n-1} + k_{2n})y_{2n-1}} dS'_{2n-2} \right)$$

$$= \frac{i}{(k_2n + k_{2n+1})} - 0i \left( \int_{-\infty}^{0} e^{-i(k_1 + \ldots + k_{2n})y_1} e^{-i(k_2 + k_{2n+1})y_2} \ldots e^{-i(k_{2n-2} + k_{2n+1})y_{2n-2}e^{-i(k_{2n-1} + k_{2n})y_{2n-1}} dS'_{2n-2} \right)$$

$$= \frac{i}{(k_2n + k_{2n+1})} - 0i \left( \int_{-\infty}^{0} e^{-i(k_1 + \ldots + k_{2n})y_1} e^{-i(k_2 + k_{2n+1})y_2} \ldots e^{-i(k_{2n-2} + k_{2n+1})y_{2n-2}e^{-i(k_{2n-1} + k_{2n})y_{2n-1}} dS'_{2n-2} \right)$$
Notice that \( \frac{i}{(k_{2n} + k_{2n+1} - i0)} = \int_0^\infty e^{-i(k_{2n}+k_{2n+1})y_{2n}} dy_{2n} \) and if we multiply it by \( e^{-i(k_{2n}+k_{2n+1})y_{2n-1}} \):

\[
\frac{i}{(k_{2n} + k_{2n+1} - i0)} e^{-i(k_{2n}+k_{2n+1})y_{2n-1}} = \int_0^\infty e^{-i(k_{2n}+k_{2n+1})y_{2n}} dy_{2n} e^{-i(k_{2n}+k_{2n+1})y_{2n-1}} \]

\[
= \int_0^\infty e^{-i(k_{2n}+k_{2n+1})(y_{2n}+y_{2n-1})} dy_{2n}
\]

The change of variables \( y'_{2n} = y_{2n} + y_{2n-1} \) gives

\[
\frac{i}{(k_{2n} + k_{2n+1} - i0)} e^{-i(k_{2n}+k_{2n+1})y_{2n-1}} = \int_{y_{2n}}^\infty e^{-i(k_{2n}+k_{2n+1})y_{2n}} dy_{2n}. 
\]

By substituting this in the expression of \( \widetilde{G}_{2n-1} \) we get:

\[
\theta (-y_1 + y_2 - ... - y_{2n-1}) e^{-i(k_1-k_{2n+1})y_1} e^{-i(k_2+k_{2n+1})y_2} \]

\[
... e^{-i(k_{2n-1}-k_{2n+1})y_{2n-1}} \int_{y_{2n}}^\infty e^{-i(k_{2n}+k_{2n+1})y_{2n}} dy_{2n} dS'_{2n-2} 
\]

By introducing the new variable \( y_{2n+1} = y_{2n} - y_{2n-1} + ... - y_1 \) we notice that \( y_{2n+1} \geq y_{2n} \), and hence

\[
\widetilde{G}_{2n+1}(k_1, ..., k_{2n+1}) := \int_{y_1 \leq y_2 \leq ... \leq y_{2n} \leq y_{2n} \leq y_{2n+1} \leq 0} e^{-ik_1 y_1} ... e^{-ik_{2n+1} y_{2n+1}} dS_{2n}. 
\]

Taking Fourier inverse we get,

\[
\langle G_{2n+1}, u \otimes ... \otimes u \rangle = \int_{y_1 \leq y_2 \leq ... \leq y_{2n+1} \leq 0} u(y_1) ... u(y_{2n+1}) dS_{2n} \quad (15)
\]

and

\[
D_{2n+1}(u)(x) = \langle G_{2n+1}, \tau_{-x} u \otimes ... \otimes \tau_{-x} u \rangle = \int_{y_1 \leq y_2 \leq ... \leq y_{2n+1} \leq x} u(y_1) ... u(y_{2n+1}) dS_{2n}. \quad (16)
\]

Equations (11), (12) and (15) (resp.) give us the integral forms of the terms of the power series \( C^{-}(u) \), \( C^{+}(u) \) and \( D(u) \) (resp.).
Chapter 3: Convergence of the linearization transforms

3.1 Discussion of the function spaces

We will first introduce the function spaces that we will be concerned with. Notice that even though the maps C and D go in opposite direction to one another they do not converge in the same spaces. This happens because they are not the inverses of each other.

3.1.1 The Space $H^{m,1}_N \cap H^{m,2}_N$

The first space we consider is $H^{m,1}_N \cap H^{m,2}_N$, where $u \in H^{m,1}_N$ if

$$\alpha_{N,i}(u) = \|u^{(i)}\|_{L^1_N} = \int_{-\infty}^{\infty} \langle \zeta \rangle^N |(u^{(i)}(\zeta))| d\zeta$$

converges for all $i = 0, 1, \ldots, m$, and $u \in H^{m,2}_N$ if

$$\gamma_{N,i}(u) = \|u^{(i)}\|_{L^2_N} = \left( \int_{-\infty}^{\infty} \langle \zeta \rangle^{2N} (u^{(i)}(\zeta))^2 d\zeta \right)^{\frac{1}{2}}$$

converges for all $i = 0, 1, \ldots, m$. Denote $\langle x \rangle = \sqrt{x^2 + 1}$. The weight function $\langle x \rangle$ satisfies the following inequality,

$$\langle x + y \rangle \leq 2 \langle x \rangle \langle y \rangle.$$

The norms in $H^{m,1}_N$ and $H^{m,2}_N$ are

$$\|u\|_{H^{m,1}_N} = \sum_{i=0}^{m} \alpha_{N,i}(u)$$

$$\|u\|_{H^{m,2}_N} = \sum_{i=0}^{m} \gamma_{N,i}(u)$$

Note that $H^{m,1}_N$ and $H^{m,1}_N$ are Banach spaces. Our goal is to prove that $u \in H^{m,1}_N \cap H^{m,2}_N$ implies that $C^-(u), C^+(u), D(u), \in H^{m,1}_N \cap H^{m,2}_N$.

3.1.2 The Spaces $H^{m,2}_N$, $H^{m,2}_N((-\infty, T])$ and $H^{m,2}_N([T, -\infty))$

We already introduce the Banach space $H^{m,2}_N$. The norm for a function $f \in H^{m,2}_N$ is

$$\|f\|_{H^{m,2}_N} = \sum_{i=0}^{m} \|f^{(i)}\|_{L^2_N} = \sum_{i=0}^{m} \left( \int_{-\infty}^{\infty} \langle x \rangle^{2N} (f^{(i)}(x))^2 dx \right)^{\frac{1}{2}}.$$

We define also the analogous norms in $H^{m,2}_N((-\infty, T])$ and $H^{m,2}_N([T, +\infty))$, where $T$ is a fixed real number. By definition, for $g \in H^{m,2}_N((-\infty, T])$ and $h \in H^{m,2}_N([T, +\infty))$,

$$\|g\|_{H^{m,2}_N((-\infty, T])} = \sum_{i=0}^{m} \|g^{(i)}\|_{L^2_N((-\infty, T])} = \sum_{i=0}^{m} \left( \int_{-\infty}^{T} \langle x \rangle^{2N} (g^{(i)}(x))^2 dx \right)^{\frac{1}{2}}.$$
and

$$||h||_{H^{m,2}_N(T,\infty)} := \sum_{i=0}^{m} ||h^{(i)}||_{L^2_N(T,\infty)} = \sum_{i=0}^{m} \left( \int_T ^\infty (\langle x \rangle^{2N} h^{(i)}(x))^2 dx \right)^{\frac{1}{2}}.$$

From the definition of these spaces it is clear that we have the following inclusions of sets:

$$H^{m,2}_N \subset H^{m,2}_N((-\infty,T])$$
$$H^{m,2}_N \subset H^{m,2}_N([T,-\infty))$$

These inclusions are continuous mappings. We will use this fact when we deal with the convergence for $C^-(u)$ and $C^+(u)$.

We will need the following auxiliary lemmas. The proofs are based on the proof of Lemma 6.3 in [6].

**Lemma 1.** Let $m \geq 1$ and $N \in \mathbb{Z}^+$. Then there exists $L > 0$ such that $\forall f \in H^{m,2}_N$ and $\forall 0 \leq j \leq m - 1$,

$$\sup_{x \in \mathbb{R}} \left| f^{(j)}(x) \right| \langle x \rangle^N \leq L ||f||_{H^{m,2}_N}$$

**Proof.** Denote $g(x) = \langle x \rangle^{2N} f^2(x)$. We have

$$|g'| = 2N(|f|^2 \langle x \rangle^{2N-1}) + 2(|f| \langle x \rangle^N)(|f'| \langle x \rangle^N) \in L^1$$

Since $g' \in L^1$, $g(x) = c_- + \int_{-\infty}^{x} g'(y)dy = c_+ + \int_{x}^{\infty} g'(y)dy$ where $c_+ \geq 0$ is a non-negative constants. In particular,

$$\lim_{|x| \to \infty} g(x) = c_\pm.$$

Now assume that $c_- > 0$. Then, there exist $\alpha \geq 0$ and $\epsilon > 0$ such that $g(x) \geq \epsilon^2 > 0$ for any $x \leq \alpha$. In particular, for any $x \leq \alpha$,

$$|f(x)| \langle x \rangle^N > \frac{\epsilon}{\langle x \rangle^{\frac{1}{2}}}$$

that contradicts $\langle x \rangle^N f \in L^2$. Hence, $c_- = 0$. In the same way we prove that $c_+ = 0$. In particular,

$$g(x) = \int_{-\infty}^{x} g'(y)dy = \int_{x}^{\infty} g'(y)dy$$

and

$$\lim_{|x| \to \infty} g(x) = 0.$$

It follows that

$$|g(x)| \leq \int_{-\infty}^{\infty} |g'(y)|dy \leq (2N + 2)||f||^2_{H^{m,2}_N}.$$

Hence,

$$\langle x \rangle^N |f(x)| \leq L_0 ||f||_{H^{m,2}_N}.$$
We have the same estimates for all \( f^{(j)} \) for \( 0 \leq j \leq m - 1 \). Hence,
\[
(x)^N |f^{(j)}(x)| \leq L_j ||f||_{H_N^{m,2}}.
\]
Taking \( L = \max L_j \) proves the theorem.

In a similar way we one proves analogous lemmas for the spaces \( H_N^{m,2}((-\infty, T]) \) and \( H_N^{m,2}([T, +\infty)) \).

**Lemma 2.** Let \( m \geq 1 \) and \( N \in \mathbb{Z}^+ \). Then there exists \( L > 0 \) such that \( \forall f \in H_N^{m,2}((-\infty, T]) \) and \( \forall 0 \leq j \leq m - 1 \),
\[
\sup_{x \in (-\infty, T]} |f^{(j)}(x)| (x)^N \leq L ||f||_{H_N^{m,2}}((-\infty, T]).
\]

**Lemma 3.** Let \( m \geq 1 \) and \( N \in \mathbb{Z}^+ \). Then there exists \( L > 0 \) such that \( \forall f \in H_N^{m,2}([T, \infty)) \) and \( \forall 0 \leq j \leq m - 1 \),
\[
\sup_{x \in [T, \infty)} |f^{(j)}(x)| (x)^N \leq L ||f||_{H_N^{m,2}}([T, \infty)).
\]

### 3.1.3 Asymptotic Spaces

In this subsection we introduce the asymptotic spaces \( \mathbb{A}_N^{m,2}(\mathbb{R}) \equiv \mathbb{A}_{H_N^{m,2}}, \mathbb{A}_{H_N^{m,2}((-\infty, T])} \), and \( \mathbb{A}_{H_N^{m,2}([T, \infty))} \) considered in [6].

First we introduce the space \( \mathbb{A}_{H_N^{m,2}((-\infty, T])} \). By definition, \( u \in \mathbb{A}_{H_N^{m,2}((-\infty, T])} \) if
\[
u(x) = \sum_{k=2}^{N} \left( a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + f(x),
\]
where \( f \in H_N^{m,2}((-\infty, T]) \) is the remainder and \( H_N^{m,2}((-\infty, T]) \) is the remainder space. Here \( \langle x \rangle = \sqrt{1 + x^2} \). Note that \( \mathbb{A}_{H_N^{m,2}((-\infty, T])} \) is a Banach space supplied with the norm,
\[
||u||_{\mathbb{A}_{H_N^{m,2}((-\infty, T])}} = \sqrt{\sum_{k=2}^{N} (a_k^2 + b_k^2)} + ||f||_{H_N^{m,2}((-\infty, T])}
\]

The space \( \mathbb{A}_{H_N^{m,2}([T, \infty))} \) is defined in a similar way. The only difference is that the expansion is at \(+\infty\) and the remainder function \( f \) is in the remainder space \( H_N^{m,2}([T, \infty)) \). Functions \( u \) in the space \( \mathbb{A}_N^{m,2}(\mathbb{R}) \) have expansion both at \(-\infty\) and \(+\infty\). They can be written as
\[
u(x) = \sum_{k=2}^{N} \left( a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + f(x),
\]
where \( f \in H_N^{m,2}(\mathbb{R}) \). In the following sections we prove the convergence of the maps \( C^-(u) \) and \( C^+(u) \) (resp.) in the asymptotic spaces \( \mathbb{A}_{H_N^{m,2}((-\infty, T])} \) and \( \mathbb{A}_{H_N^{m,2}([T, \infty))} \) (resp.).
3.2 Convergence of the “direct” transform $D$

Before we start doing estimates about the convergence of $D$ we first take a look at what the hyper-plane,

$$S_{2k}^x = \{ y_1 \leq y_2 \leq \ldots \leq y_{2k} \leq y_{2k+1} \} \subset \mathbb{R}^{2k+1},$$

over which we are integrating the terms $D_{2k+1}$ of $D$, looks like. First we start to write the term $D_3$ in more explicit integral form:

$$S_2^x = \{ y_1 \leq y_2 \leq y_3 \}$$

Denoting $y_3 = x + y_2 - y_1$, we get that $S_2^x = \{(y_1, y_2) \in \mathbb{R}^2 | -\infty \leq y_1 \leq x \text{ and } y_1 \leq y_2 \leq \infty \}$. So from this $D_3$ written explicitly is:

$$D_3(u)(x) = \int_{-\infty}^{x} \int_{y_1}^{x+y_2-y_1} \int_{y_2}^{x+y_2-y_1} \int_{y_3}^{x+y_2-y_1} u(y_1)u(y_2)u(x + y_2 - y_1)dy_1dy_2dy_3$$

By similar considerations for $S_4^x$ we write $D_5$ explicitly as:

$$D_5(u)(x) = \int_{-\infty}^{x} \int_{y_1}^{x+y_2-y_1} \int_{y_2}^{x+y_2-y_1} \int_{y_3}^{x+y_2-y_1} \int_{y_4}^{x+y_2-y_1} u(y_1)\ldots u(y_4)u(x + y_4 - y_3 + y_2 - y_1)dy_1\ldots dy_4$$

By an induction argument we get the explicit expression for the general term:

$$D_{2k+1}(u)(x) = \int_{-\infty}^{x} \int_{y_1}^{x+y_2+y_2k-2-y_2k-3-\ldots-y_1} \int_{y_2}^{x+y_2+y_2k-2-y_2k-3-\ldots-y_1} \int_{y_{k-1}}^{x+y_2+y_2k-2-y_2k-3-\ldots-y_1} u(y_1)\ldots u(y_{2k})u(x + y_{2k} - y_{2k-1} + \ldots - y_1)dy_{2k}\ldots dy_1$$

We are going to consider the convergence of $D$ in the space $H_{m,1}^N \cap H_{m,2}^N$. Knowing that the norms in such a space involve derivatives of $D_{2k+1}$ we first will try to have a more compact way of computing these derivatives. To get an idea we first start computing the derivatives of $D_3$.

$$\frac{d}{dx} D_3(u)(x) = \int_{-\infty}^{x} \int_{y_1}^{x+y_2-y_1} \frac{d}{dx} u(y_1)u(y_2)u^{(1)}(x + y_2 - y_1)dy_2dy_1 + u(x) \int_{x}^{\infty} u(y_2)u(y_2)dy_2$$

$$\frac{d^2}{dx^2} D_3(u)(x) = \int_{-\infty}^{x} \int_{y_1}^{x+y_2-y_1} \frac{d^2}{dx^2} u(y_1)u(y_2)u^{(2)}(x + y_2 - y_1)dy_2dy_1 + u(x) \int_{x}^{\infty} u(y_2)u^{(1)}(y_2)dy_2$$

$$+ u^{(1)}(x) \int_{x}^{\infty} u(y_2)u(y_2)dy_2 - u(x)u(x)u(x)$$

Now by induction we get:

$$\frac{d^i}{dx^i} D_3(u)(x) = \int_{-\infty}^{x} \int_{y_1}^{x+y_2-y_1} u(y_1)u(y_2)u^{(i)}(x + y_2 - y_1)dy_2dy_1 + \sum_{i_1+i_2 = i-1} d_{i_1+i_2}^{i} u^{(i_1)}(x) \int_{x}^{\infty} u(y_2)u^{(i_2)}(y_2)dy_2$$

$$- \sum_{i_1+i_2+i_3 = i} d_{i_1+i_2+i_3}^{i} u^{(i_1)}(x)u^{(i_2)}(x)u^{(i_3)}(x).$$
Next we calculate the first derivative of $D_5$, and from it we notice an important property for the general case. We have,

$$\frac{d}{dx} D_5(u)(x) = \int \int \int \int u(y_1) \ldots u(y_d) u^{(1)}(x + y_d - y_1 \ldots + y_2 - y_1) dy_d \ldots dy_1$$

$$+ \int \int u(y_1) u(y_2) \left( u(x + y_2 - y_1) \int \int u(y_4) u(y_4) dy_4 \right) dy_2 dy_1$$

The first thing to notice about this calculation is that the derivative with respect to $x$ appearing on the outer integral of $C_5$ goes to zero as. The only non-zero derivatives is the derivative with respect to $x$ appearing on the inner integral limit and the one appearing inside the last function $u$. This fact is true in general for all $D_{2k+1}$, i.e we have only to worry about these two derivatives.

The second thing that we have to notice about the above calculation is that on the second summand the last part $\left( u(x + y_2 - y_1) \int \int u(y_4) u(y_4) dy_4 \right)$ is the same as the integral that we got when we computed the first derivative of $D_3$ just evaluated at $x + y_2 - y_1$.

Now we have noticed enough pattern and are ready to compute the derivatives of the general case $D_{2k+1}$. But first we need to introduce some other operators that make our calculations easier.

Define:

$$D_{2k+1}(A_1, \ldots, A_{2k}, A_{2k+1})(x) = \int_{S_{2k}^{2k}} A_1(y_1) \ldots A_{2k}(y_{2k}) A_{2k+1}(x + y_{2k} - y_{2k-1} + \ldots - y_1) dS_{2k}$$

$$T(A_1, A_2, A_3)(x) = A_1(x) \int_x^\infty A_2(y) A_3(y) dy$$

$$P(A_1, A_2, A_3)(x) = A_1(x) A_2(x) A_3(x)$$

Using this notation we get:

$$\frac{d}{dx} D_{2k+1}(A_1, \ldots, A_{2k}, A_{2k+1})(x) = D_{2k+1}(A_1, \ldots, A_{2k}, A_{2k+1}^{(1)})(x)$$

$$+ D_{2k-1}(A_1, \ldots, A_{2k-2}, T(A_{2k-1}, A_{2k}, A_{2k+1}))(x)$$

$$\frac{d}{dx} T(A_1, A_2, A_3)(x) = T(A_1^{(1)}, A_2, A_3)(x) + P(A_1, A_2, A_3)(x)$$

$$\frac{d}{dx} P(A_1, A_2, A_3)(x) = P(A_1^{(1)}, A_2, A_3)(x) + P(A_1, A_2^{(1)}, A_3)(x) + P(A_1, A_2, A_3^{(1)})(x)$$

$$\frac{d^2}{dx^2} D_{2k+1}(A_1, \ldots, A_{2k}, A_{2k+1})(x) = D_{2k+1}(A_1, \ldots, A_{2k}, A_{2k+1}^{(2)})(x)$$

$$+ D_{2k-1}(A_1, \ldots, A_{2k-2}, T(A_{2k-1}, A_{2k}, A_{2k+1}^{(1)}))(x) + D_{2k-1}(A_1, \ldots, A_{2k-2}, T^{(1)}(A_{2k-1}, A_{2k}, A_{2k+1}))(x)$$

$$+ D_{2k-3}(A_1, \ldots, A_{2k-4}, T(A_{2k-3}, A_{2k-2}, T(A_{2k-1}, A_{2k}, A_{2k+1}))))$$

$$= D_{2k+1}(A_1, \ldots, A_{2k}, A_{2k+1}^{(2)})(x) + D_{2k-1}(A_1, \ldots, A_{2k-2}, T(A_{2k-1}, A_{2k}, A_{2k+1}^{(1)}))(x)$$

$$+ D_{2k-1}(A_1, \ldots, A_{2k-2}, T(A_{2k-1}, A_{2k}, A_{2k+1}^{(1)}))(x) + D_{2k-1}(A_1, \ldots, A_{2k-2}, P(A_{2k-1}, A_{2k}, A_{2k+1}))(x)$$
From this two derivatives we can get the general pattern for the general derivative:

\[
\frac{d^k}{dx^k} D_{2k+1}(A_1, ..., A_{2k+1}) = D_{2k+1}(A_1, ..., A^{(i)}_{2k+1}) + \sum_{i_1 + i_2 = 1} D_{2(k-1)+1}(A_1, ..., A_{2k-2}, T(A^{(i_1)}_{2k-1}, A_{2k}, A^{(i_2)}_{2k+1}))
\]

\[+ ... + D_{2(k-i)+1}(A_1, ..., A_{2(k-i)}, T(A_{2k-2i+1}, A_{2k-i+2}, T(..., T(A_{2k+1}, A_{2k}, A_{2k+1}))))\]

All the summands in-between are \( D_{2k-j+1} \) evaluated on different permutations of \( T, P \) and derivatives of \( A \). We have,

\[
\|D_{2k+1}(A_1, ..., A_{2k+1})\|_L^N = \int_{-\infty}^{\infty} \langle x \rangle^N \left| D_{2k+1}(A_1, ..., A_{2k+1}) \right| dx
\]

\[= \int_{-\infty}^{\infty} \langle x \rangle^N \int_{S_{2k}^x} A_1(y_1) ... A_{2k}(y_{2k}) A_{2k+1}(x + y_{2k} - y_{2k-1} + ... - y_1)\, dS_{2k}^x \, dx
\]

\[\leq \int_{-\infty}^{\infty} \langle x \rangle^N \int_{S_{2k}^x} A_1(y_1) ... A_{2k}(y_{2k}) A_{2k+1}(x + y_{2k} - y_{2k-1} + ... - y_1)\, dS_{2k}^x \, dx
\]

\[\leq \frac{1}{(2k)!} \int_{-\infty}^{\infty} \langle x \rangle^N \int_{y \in \mathbb{R}^{2k}} A_1(y_1) ... A_{2k}(y_{2k}) \left( \int_{-\infty}^{\infty} \langle x + y_{2k} - ... - y_1 \rangle^N A_{2k+1}(x + y_{2k} - y_{2k-1} + ... - y_1) \, dx \right) \, dy
\]

We get

\[
\|D_{2k+1}(A_1, ..., A_{2k+1})\|_L^N \leq \frac{2^{2k}}{(2k)!} \|A_1\|_{L_x^N} ... \|A_{2k+1}\|_{L_x^N}
\]

(17)

\[
\|D_{2k+1}(A_1, ..., A_{2k+1})\|^2_{L^2} = \int_{-\infty}^{\infty} \langle x \rangle^{2N} \left( D_{2k+1}(A_1, ..., A_{2k+1}(x) \right)^2 \, dx
\]

\[= \int_{-\infty}^{\infty} \langle x \rangle^{2N} \left( \int_{S_{2k}^x} A_1(y_1) ... A_{2k}(y_{2k}) A_{2k+1}(x + y_{2k} - y_{2k-1} + ... - y_1)\, dS_{2k}^x \right)^2 \, dx
\]

\[\leq \left( \frac{2^{2k}}{(2k)!} \right)^2 \int_{-\infty}^{\infty} \langle y \rangle^N A_1(y_1) ... A_{2k}(y_{2k}) \langle x + y_{2k} - y_{2k-1} + ... - y_1 \rangle^N A_{2k+1}(x + y_{2k} - y_{2k-1} + ... - y_1) \, dy \]

\[\leq \left( \frac{2^{2k}}{(2k)!} \right)^2 \int_{-\infty}^{\infty} \langle y \rangle^N A_1(y_1) ... A_{2k}(y_{2k}) \, dy
\]

25
\[
\left( \int_{y \in \mathbb{R}^{2k}} \left( \langle y_1 \rangle^N A_1(y_1) \ldots \langle y_{2k} \rangle^N A_{2k}(y_{2k}) \langle x+y_{2k}-y_{2k-1}+\ldots-y_1 \rangle^{2N} A_{2k+1}^2(x+y_{2k}-y_{2k-1}+\ldots-y_1) dy \right) dx \right)
\]

Hence,

\[
||D_{2k+1}(A_1, \ldots, A_{2k+1})||_{L_N^3} \leq \frac{2^{2k}}{(2k)!} ||A_1||_{L_N^1} \ldots ||A_{2k}||_{L_N^1} ||A_{2k+1}||_{L_N^3} 
\]  \hspace{1cm} (18)

\[
||T(A_1, A_2, A_3)||_{L_N^1} = \int_{-\infty}^{\infty} \langle x \rangle^N A_1(x) \int_{-\infty}^{\infty} A_2(y) A_3(y) dy \, dx \leq \int_{-\infty}^{\infty} \langle x \rangle^N A_1(x) A_2(y) A_3(y) \, dy \, dx 
\]

\[
||T(A_1, A_2, A_3)||_{L_N^1} \leq ||A_1||_{L_N^3} ||A_2||_{L_N^2} ||A_3||_{L_N^2} 
\]  \hspace{1cm} (19)

Similarly,

\[
||T(A_1, A_2, A_3)||_{L_N^3} \leq ||A_1||_{L_N^3} ||A_2||_{L_N^2} ||A_3||_{L_N^2} 
\]  \hspace{1cm} (20)

Notice that all the functions \(A_j\) on which the operator \(P\) is evaluated on, are lesser than the \(m^{th}\) derivatives of functions on \(H_N^{m,2}\). Hence on all of the \(A_j\) Lemma 1 applies. From this we get

\[
||P(A_1, A_2, A_3)||_{L_N^1} = \int_{-\infty}^{\infty} \langle x \rangle^N A_1(x) A_2(x) A_3(x) \, dx \leq L^3 \int_{-\infty}^{\infty} \langle x \rangle^N \frac{||A_1||_{L_N^2} ||A_2||_{L_N^2} ||A_3||_{L_N^2}}{\langle x \rangle^{3N}} \, dx.
\]

\[
||P(A_1, A_2, A_3)||_{L_N^1} \leq L^3 \int_{-\infty}^{\infty} \frac{1}{\langle x \rangle^{4N}} \, dx ||A_1||_{L_N^2} ||A_2||_{L_N^2} ||A_3||_{L_N^2} 
\]  \hspace{1cm} (21)

Similarly we get

\[
||P(A_1, A_2, A_3)||_{L_N^2} \leq L^3 \int_{-\infty}^{\infty} \frac{1}{\langle x \rangle^{4N}} \, dx ||A_1||_{L_N^2} ||A_2||_{L_N^2} ||A_3||_{L_N^2} 
\]  \hspace{1cm} (22)

Using these estimates and the calculated derivatives of \(D_{2k+1}\) we prove that \(D(u) \in H_N^{m,1} \cap H_N^{m,2}\). One has,

\[
||D_{2k+1}(u)||_{H_N^{m,1}} \leq K_1 \frac{2^{2k}}{(2k)!} ||u||_{H_N^{m,1}}^{2k+1}
\]

and

\[
||D_{2k+1}(u)||_{H_N^{m,2}} \leq K_2 \frac{2^{2k}}{(2k)!} ||u||_{H_N^{m,2}}^{2k+1},
\]

where \(K_1, K_2\) are constants. Hence,

\[
||D(u)||_{H_N^{m,1}} \leq K_1 ||u||_{H_N^{m,1}}^{2} ||u||_{H_N^{m,1}} \epsilon^{m,1}
\]

\[
||D(u)||_{H_N^{m,2}} \leq K_2 ||u||_{H_N^{m,2}}^{2} ||u||_{H_N^{m,2}} \epsilon^{m,2}.
\]

**Theorem 4.** If \(u \in H_N^{m,1} \cap H_N^{m,2}\) then \(D(u) \in H_N^{m,1} \cap H_N^{m,2}\).
3.3 Convergence of the “inverse” transform $C$

3.3.1 Convergence in $H_N^{m,1} \cap H_N^{m,2}$

In what follows we estimate the norms $\|C^- (u)\|_{H_N^{m,1}}$ and $\|C^- (u)\|_{H_N^{m,2}}$. The estimates for $\|C^+ (u)\|_{H_N^{m,1}}$ and $\|C^+ (u)\|_{H_N^{m,2}}$ are exactly analogous that’s why we only focus on $C^-$. To illustrate our estimates we start with $C_3^-(u)(x)$,

$$
\alpha_{N,0}(C_3^-(u)(x)) = \int_{-\infty}^{\infty} \langle x \rangle^N \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(y_1)u(y_2)u(x+y_2-y_1)dy_2dy_1 \right| dx
$$

$$
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{y_1} \left| u(y_1) \right| \left| u(y_2) \right| \left| u(x+y_2-y_1) \right| dy_2dy_1 dx
$$

As $\langle x+y \rangle \leq 2\langle x \rangle \langle y \rangle$ we have,

$$
\alpha_{N,0}(C_3^-(u)(x)) \leq 2^{2N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle y_1 \rangle^N \langle y_2 \rangle^N \left| u(y_1) \right| \left| u(y_2) \right| \left\langle x+y_2-y_1 \right\rangle^N \left| u(x+y_2-y_1) \right| dy_2dy_1 dx
$$

$$
\leq 2^{2N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle y_1 \rangle^N \langle y_2 \rangle^N \left| u(y_1) \right| \left| u(y_2) \right| \left\langle x+y_2-y_1 \right\rangle^N \left| u(x+y_2-y_1) \right| dy_2dy_1 dx
$$

Now we use Fubini’s theorem to switch the order of integration,

$$
\leq 2^{2N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle y_1 \rangle^N \langle y_2 \rangle^N \left| u(y_1) \right| \left| u(y_2) \right| \left( \int_{-\infty}^{\infty} \left\langle x+y_2-y_1 \right\rangle^N \left| u(x+y_2-y_1) \right| dx \right) dy_2dy_1.
$$

Interpreting the integrals as norms we get

$$
\alpha_{N,0}(C_3^-(u)(x)) \leq 2^{2N} \alpha_{N,0}(u)^3 \leq 2^{2N} \|u\|_{H_N^{m,1}}^3.
$$

Now we go to the derivatives of $C_3^-(u)$. We have

$$
\alpha_{N,i}(C_3^-(u)(x)) \leq \sum_{i_1+i_2+i_3=i, i_j \geq 0} d_{i_1,i_2,i_3} \alpha_{N,0}(C_3^-(u^{(i_1)}, u^{(i_2)}, u^{(i_3)}))
$$

$$
\leq 2^{2N} \sum_{i_1+i_2+i_3=i, i_j \geq 0} d_{i_1,i_2,i_3} \alpha_{N,0}(u^{(i_1)}) \alpha_{N,0}(u^{(i_2)}) \alpha_{N,0}(u^{(i_3)})
$$

$$
\alpha_{N,i}(C_3^-(u)(x)) \leq 3^i 2^{2N} \|u\|_{H_N^{m,1}}^3.
$$

Finally, we have to add up all the estimates for the $L^1$-norms of the derivatives of $C_3^-$ to get:

$$
\|C_3^- (u)\|_{H_N^{m,1}} = \sum_{i=0}^{m} \alpha_{N,i}(C_3^- (u)) \leq \sum_{i=0}^{m} 3^i 2^{2N} \|u\|_{H_N^{m,1}}^3 = 2^{2N-1}(3^{m+1}-1)\|u\|_{H_N^{m,1}}^3.
$$
Similarly as we did for the $\alpha$ and then interpret the integrals as norms:

$$
\alpha_{N,0}(C^-_{2n+1}(u)) = \int \langle x \rangle^N \int u(y_1)...u(y_{2n})u(x+y_{2n} - y_{2n-1} + ... - y_1)dy_{2n}...dy_1 \, dx
$$

Summing over all the $L^1$-norms of the derivatives of $C^-_{2n+1}(u)$ we get:

$$
\|C^-_{2n+1}(u)\|_{H^{m,1}_N} \leq 2^{2nN} \frac{(2n+1)^{m+1}-1}{2n} \|u\|_{H^{m,1}_N}^{2n+1}.
$$

Finally summing over all the estimates for each $\|C^-_{2n+1}(u)\|_{H^{m,1}_N}$ we get

$$
\|C^- (u)\|_{H^{m,1}_N} \leq \sum_{n=1}^{\infty} 2^{2nN} \frac{(2n+1)^{m+1}-1}{2n} \|u\|_{H^{m,1}_N}^{2n+1}.
$$

To calculate the radius of convergence for the infinite series above we use the ratio test,

$$
R = \lim_{n \to \infty} \frac{\|u\|_{H^{m,1}_N}^{2n+3}2^{2nN+2N}((2n+3)^{m+1}-1)}{2(n+1)\|u\|_{H^{m,1}_N}^{2n+1}2^{2nN}((2n+1)^{m+1}-1)} = \|u\|_{H^{m,1}_N}^2 2^{2N} < 1
$$
Conclusion: If \( \|u\|_{H_N^{m,1}} < \frac{1}{2\gamma} \), then \( C^{-}(u) \) and \( C^{+}(u) \) converge in \( H_N^{m,1} \).

Now we give estimates for \( \|C^{-}(u)\|_{H_N^{m,2}} \). We start with \( C_3^{-}(u) \),

\[
\gamma_{N,0}(C_3^{-}(u))^2 = \int_{-\infty}^{\infty} \langle x \rangle^2 \left( \int_{-\infty}^{x} \int_{-\infty}^{y_1} u(y_1)u(y_2)u(x + y_2 - y_1)dy_2dy_1 \right)^2 dx.
\]

It follows from \( \langle x + y \rangle \leq 2\langle x \rangle \langle y \rangle \) and the Cauchy-Schwarz inequality that

\[
\gamma_{N,0}(C_3^{-}(u))^2 \leq 2^{4N} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{x} \int_{-\infty}^{y_1} \tilde{u}(y_1)\tilde{u}(y_2)dy_2dy_1 \right)^2 dx,
\]

where \( \tilde{u}(x) = \langle x \rangle^N |u(x)| \). Hence,

\[
\gamma_{N,0}(C_3^{-}(u))^2 \leq 2^{4N} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{u}(y_1)\tilde{u}(y_2)dy_2dy_1 \right)^2 dx
\]

\[
= 2^{4N} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{u}(y_1)\tilde{u}(y_2)dy_2dy_1 \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{u}(y_1)\tilde{u}(y_2) \left( \int_{-\infty}^{\infty} \tilde{u}^2(x + y_2 - y_1)dx \right)dy_2dy_1
\]

Finally we get

\[
\gamma_{N,0}(C_3^{-}(u))^2 \leq 2^{4N} \|u\|_{H_N^{m,1}}^4 \|u\|_{H_N^{m,2}}^2.
\]

Arguing in a similar way we get estimates for the derivative of \( C_3(u) \),

\[
\gamma_{N,1}(C_3^{-}(u))^2 \leq 2^{4N} \|u\|_{H_N^{m,1}}^4 \|u\|_{H_N^{m,2}}^2.
\]

Summing up the estimates of all the derivative of \( C_3(u) \) we get,

\[
||C_3^{-}(u)||_{H_N^{m,2}}^2 \leq 2^{4N} \frac{(3)^{m+1} - 1}{2} \|u\|_{H_N^{m,1}}^4 \|u\|_{H_N^{m,2}}^2.
\]

These calculations done for \( C_3(u) \) hold also for \( C_{2n+1}^{-}(u) \). In what follows we obtain,

\[
\gamma_{N,0}(C_{2n+1}^{-}(u))^2 = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} u(y_1)...u(y_2n)u(x + y_2n - y_{2n-1} + ... - y_1)dy_2...dy_1 \right)^2 dx
\]

Overestimating the integrals by taking the upper bound of the integrals to be +\( \infty \) we get:

\[
\gamma_{N,0}(C_{2n+1}^{-}(u))^2 \leq \int_{-\infty}^{\infty} \left( \int_{R^{2n}} u(y_1)...u(y_2n)u(x + y_2n - y_{2n-1} + ... - y_1)dy_2...dy_1 \right)^2 dx
\]

Using the identity \( \langle x + y \rangle \leq 2\langle x \rangle \langle y \rangle \) and the Cauchy-Schwartz inequality we get,

\[
\leq \int_{-\infty}^{\infty} \left( \int_{R^{2n}} \tilde{u}(y_1)...\tilde{u}(y_2n)dy_2...dy_1 \right)^2 \left( \int_{R^{2n}} \tilde{u}(y_1)...\tilde{u}(y_2n) \tilde{u}^2(x + y_2n - y_{2n-1} + ... - y_1)dy_2...dy_1 \right) dx.
\]
Summing up over the estimates of all the derivatives of $u$ for $C$ we get

$$\gamma_{N,0}(C_{2n+1}(u))^2 \leq 2^{4N}||u||_{H^m_N,1}^4||u||_{H^m_N,2}^2.$$ 

Doing similar manipulations we get the estimates for the derivative of $C_{2n+1}(u)$,

$$\gamma_{N,i}(C_{2n+1}(u))^2 \leq 2^{4N}(2n+1)^i||u||_{H^m_N,1}^4||u||_{H^m_N,2}^2$$

Summing up over the estimates of all the derivatives of $C_{2n+1}(u)$ we get

$$||C_{2n+1}(u)||_{H^m_N,2}^2 \leq 2^{4N}(2n+1)^{m+1} - 1||u||_{H^m_N,1}^4||u||_{H^m_N,2}^2.$$ 

Finally summing up over all $n$ we have,

$$||C(u)||_{H^m_N,2}^2 \leq \sum_{n=1}^{\infty} ||C_{2n+1}(u)||_{H^m_N,2}^2 \leq \sum_{n=1}^{\infty} 2^{2nN}\sqrt{(2n+1)^{m+1} - 1}||u||_{H^m_N,1}^4||u||_{H^m_N,2}^2.$$ 

Now we apply the ratio test to determine the radius of convergence.

$$R = \lim_{n \to \infty} \frac{||u||_{H^m_N,2}^2}{||u||_{H^m_N,2}^2} \cdot \frac{\sqrt{2n}}{\sqrt{2(n+1)}} \cdot \frac{\sqrt{2n}}{\sqrt{2n+1}^m - 1} = \frac{1}{2^{N/2}}.$$ 

**Conclusion:** If $||u||_{H^m_N} < \frac{1}{2^{N/2}}$, then $C(u)$ and $C^+(u)$ converge in $H^m_N$. Note that the same condition was sufficient for $C(u)$ and $C^+(u)$ to converge in $H^m_N$. Putting these two facts together we get that $C(u)$ and $C^+(u)$ converge in $H^m_N$ and $H^m_N$ if $||u||_{H^m_N} < \frac{1}{2^{N/2}}$.

**3.3.2 Convergence in $H^m_N((\infty, T])$ and $H^m_N([T, \infty))$**

Now we study the convergence of $C(u)$ on $H^m_N((\infty, T])$, and everything would be analogous for $C^+(u)$ on $H^m_N([T, \infty))$ where $T$ is any fixed real number. Let’s remind that the norm of a function $u \in H^m_N((\infty, T])$ is:

$$||u||_{H^m_N((\infty, T])}^2 = \sum_{i=0}^{\infty} \left( \int_{-\infty}^{T} \langle x \rangle^{2N} (u^{(i)}(x))^2 \, dx \right)^{1/2},$$

where we denote: $||u||_{L_N^2((\infty, T])} = \left( \int_{-\infty}^{T} \langle x \rangle^{2N} (u^{(i)}(x))^2 \, dx \right)^{1/2}$. 

Now we start by examining $||C_C(u^{(i_1)}, u^{(i_2)}, u^{(i_3)})||_{L_N^2((\infty, T])}$.

$$||C_C(u^{(i_1)}, u^{(i_2)}, u^{(i_3)})||_{L_N^2((\infty, T])}^2 = \int_{-\infty}^{T} \langle x \rangle^{2N} \left( \int_{-\infty}^{x} \int_{-\infty}^{x} u^{(i_1)}(y_1)u^{(i_2)}(y_2)u^{(i_3)}(x + y_2 - y_1) \, dy_2dy_1 \right)^{2} \, dx$$

30
Now notice that:

\[ \int \langle x \rangle^{2N} \left( \int \int u^{(i_1)}(y_1)u^{(i_2)}(y_2)u^{(i_3)}(x + y_2 - y_1)dy_2dy_1 \right)^2 dx \]

\[ + \int \langle x \rangle^{2N} \left( \int \int u^{(i_1)}(y_1)u^{(i_2)}(y_2)u^{(i_3)}(x + y_2 - y_1)dy_2dy_1 \right)^2 dx \]

These two summands need to be treated with different methods. We start with the first.

Since \( i_1 + i_2 + i_3 = i \leq m \) we get two cases:

**Case 1**) \( i_3 < m \)

In this case we use Lemma 2 on \( u^{(i_3)} \) from which we deduce that:

\[ u^{(i_3)}(x + y_2 - y_1) \leq \frac{L\|u\|_{H_N^{m,2}(-\infty,T)}^{i_3}}{(x + y_2 - y_1)^N} \leq \frac{L\|u\|_{H_N^{m,2}((-\infty,T})}{(x)^N} \]

The last inequality because \( |x + y_2 - y_1| > |x| \) in the domain of integration. So we get:

\[ \int \langle x \rangle^{2N} \left( \int \int u^{(i_1)}(y_1)u^{(i_2)}(y_2)u^{(i_3)}(x + y_2 - y_1)dy_2dy_1 \right)^2 dx \]

\[ \leq L^2\|u\|^2_{H_N^{m,2}((-\infty,T}) \int \int \left( \int \int u^{(i_1)}(y_1)u^{(i_2)}(y_2)dy_2dy_1 \right)^2 dx \]

\[ \leq L^2\|u\|^2_{H_N^{m,2}((-\infty,T}) \int \int \left( \int \int u^{(i_1)}(y_1)u^{(i_2)}(y_2)dy_2dy_1 \right)^2 dx \]

\[ \leq L^2\|u\|^2_{H_N^{m,2}((-\infty,T}) \int \int \left( \int u^{(i_1)}(y)dy \right)^2 \left( \int u^{(i_2)}(y)dy \right)^2 dx \]

One of the \( i_1 \) or \( i_2 \) is \( < m \). Let’s assume without loss of generality that \( i_2 < m \) and the worst case scenario that \( i_1 = m \). Then using Lemma 2 on \( u^{(i_2)} \) we get:

\[ \leq L^4\|u\|^4_{H_N^{m,2}((-\infty,T}) \int \int \left( \int u^{(m)}(y)dy \right)^2 \left( \int \frac{1}{(y)^N}dy \right)^2 dx \]

Now notice that: \( \frac{d}{dx} \int \int u^{(m)}(y)dy = u^{(m)}(x) \) from which we deduce that:

\[ \int u^{(m)}(y)dy = u^{(m-1)}(x) + K_u \]

Also notice that: \( \int \frac{1}{(y)^N}dy = \frac{-1}{N-1}(x)^{N-1} \). Now substituting back into the inequality we get:

\[ \int \langle x \rangle^{2N} \left( \int \int u^{(i_1)}(y_1)u^{(i_2)}(y_2)u^{(i_3)}(x + y_2 - y_1)dy_2dy_1 \right)^2 dx \]

31
\[ \leq L^4 ||u||^4_{H^m_N((-\infty, T])} \int_{-\infty}^{-1} \left( u^{(m-1)}(x) + K_u \right)^2 \frac{1}{\langle x \rangle^{2N-2}} dx \]

Now using Lemma 2 on \( u^{(m-1)} \) we get:

\[ \leq L^4 ||u||^4_{H^m_N((-\infty, T])} \int_{-\infty}^{-1} \left( \frac{L ||u||_{H^m_N((-\infty, T])}}{\langle x \rangle^N} + K_u \right)^2 \frac{1}{\langle x \rangle^{2N-2}} dx \]

\[ \leq L^4 ||u||^4_{H^m_N((-\infty, T])} \int_{-\infty}^{-1} \left( L ||u||_{H^m_N((-\infty, T])} + K_u \right)^2 \frac{1}{\langle x \rangle^{2N-2}} dx \]

It’s clear that for \( N \geq 2 \) it converges.

**Case 2)** \( i_3 = m \)

Notice that \( i_3 = m \) implies \( i_1, i_2 < m \). In this case the change of variables \( y'_2 = x + y_2 - y_1 \) transforms the first summand into:

\[ \int_{-\infty}^{-1} \langle x \rangle^{2N} \left( \int_{-\infty}^x \int_{-\infty}^x u^{(i_1)}(y_1)u^{(i_3)}(y_2)u^{(i_2)}(y_2 + y_1 - x)dy_2dy_1 \right)^2 dx \]

Now we use Lemma 2 on \( u^{(i_2)} \), and since \( |y_2 + y_1 - x| > |x| \) we get:

\[ \leq L^2 ||u||^2_{H^m_N((-\infty, T])} \int_{-\infty}^{-1} \left( \int_{-\infty}^x \int_{-\infty}^x u^{(i_1)}(y_1)u^{(i_3)}(y_2)dy_2dy_1 \right)^2 dx \]

Which similarly like Case 1) we finally get:

\[ \leq L^4 ||u||^4_{H^m_N((-\infty, T])} \int_{-\infty}^{-1} \left( L ||u||_{H^m_N((-\infty, T])} + K_u \right)^2 \frac{1}{\langle x \rangle^{2N-2}} dx \]

Now we focus our attention on the second summand which is just an integral on the finite interval \((-1, T)\),

\[ \int_{-1}^{T} \langle x \rangle^{2N} \left( \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} u^{(i_1)}(y_1)u^{(i_2)}(y_2)u^{(i_3)}(x + y_2 - y_1)dy_2dy_1 \right)^2 dx \]

So we try to find a supremum for the inner part of the integral.

\[ \langle x \rangle^{2N} \left( \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} u^{(i_1)}(y_1)u^{(i_2)}(y_2)u^{(i_3)}(x + y_2 - y_1)dy_2dy_1 \right) \]

Using the fact that \( \frac{1}{\langle x \rangle} \leq 1 \), we also have two cases:

**Case 1)** \( i_3 < m \)

We use Lemma 2 on \( u^{(i_3)} \).

\[ \int_{-1}^{T} \langle x \rangle^{2N} \left( \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} u^{(i_1)}(y_1)u^{(i_2)}(y_2)u^{(i_3)}(x + y_2 - y_1)dy_2dy_1 \right)^2 dx \]
\[
\leq L^2 ||u||^2_{H^m_N((\infty, T))} \int_{-1}^{T} \langle x \rangle^{2N} \left( \int_{-\infty}^{x} \int_{-\infty}^{x} u^{(i_1)}(y_1)u^{(i_2)}(y_2)dy_2dy_1 \right)^2 dx
\]

Using the same approximation as in the first summand we get:

\[
\leq L^4 ||u||^4_{H^m_N((\infty, T))} \int_{-1}^{T} \left( L||u||_{H^m_N((\infty, T))} + K_u \right)^2 \langle x \rangle^2 dx
\]

**Case 2) i_3 = m**

We do the same substitution as before \( y_2' = x + y_2 - y_1 \) to make the second summand look like:

\[
\int_{-1}^{T} \langle x \rangle^{2N} \left( \int_{-\infty}^{x} \int_{-\infty}^{x} u^{(i_1)}(y_1)u^{(i_3)}(y_2)u^{(i_2)}(y_2 + y_1 - x)dy_2dy_1 \right)^2 dx
\]

Using Lemma 2 on \( u^{(i_2)} \) we get:

\[
\leq L^2 ||u||^2_{H^m_N((\infty, T))} \int_{-1}^{T} \langle x \rangle^{2N} \left( \int_{-\infty}^{x} \int_{-\infty}^{x} u^{(i_1)}(y_1)u^{(i_3)}(y_2)dy_2dy_1 \right)^2 dx
\]

\[
\leq L^4 ||u||^4_{H^m_N((\infty, T))} \int_{-1}^{T} \left( L||u||_{H^m_N((\infty, T))} + K_u \right)^2 \langle x \rangle^2 dx
\]

So in conclusion we get:

\[
||C_3^- (u^{(i_1)}, u^{(i_2)}, u^{(i_3)})||_{L_N^2((\infty, T))}
\]

\[
\leq L^2 ||u||^2_{H^m_N((\infty, T))} \int_{-\infty}^{T} \left( L||u||_{H^m_N((\infty, T))} + K_u \right)^2 \frac{1}{\langle x \rangle^{2N-2}} dx + \int_{-1}^{T} \left( L||u||_{H^m_N((\infty, T))} + K_u \right)^2 \langle x \rangle^2 dx
\]

Using the fact that:

\[
||C_3^- (u)||_{H^m_N((\infty, T))} = \sum_{i=0}^{m} \sum_{i_1+i_2+i_3=i} d_i \int_{-\infty}^{T} C_3^- (u) \mid L_N^2((\infty, T)) = \sum_{i=0}^{m} \mid C_3^- (u^{(i_1)}, u^{(i_2)}, u^{(i_3)})\mid_{L_N^2((\infty, T))}
\]

we get:

\[
||C_3^- (u)||_{H^m_N((\infty, T))}
\]

\[
\leq \sum_{i=0}^{m} 3^i L^2 ||u||^2_{H^m_N((\infty, T))} \int_{-\infty}^{T} \left( L||u||_{H^m_N((\infty, T))} + K_u \right)^2 \frac{1}{\langle x \rangle^{2N-2}} dx + \int_{-1}^{T} \left( L||u||_{H^m_N((\infty, T))} + K_u \right)^2 \langle x \rangle^2 dx
\]

\[
\leq \frac{3^{m+1}}{2} L^2 ||u||^2_{H^m_N((\infty, T))} \int_{-\infty}^{T} \left( L||u||_{H^m_N((\infty, T))} + K_u \right)^2 \frac{1}{\langle x \rangle^{2N-2}} dx + \int_{-1}^{T} \left( L||u||_{H^m_N((\infty, T))} + K_u \right)^2 \langle x \rangle^2 dx
\]
Now we do the case of $C^*_5(u)$, which makes it completely clear how to argue in the general case.

$$
||C^*_5(u^{(i_1)}, u^{(i_2)}, u^{(i_3)}, u^{(i_4)}, u^{(i_5)})||_{L^2_N((\infty, T))}^2 = \int_{-\infty}^{T} (x)^{2N} \left( \int \int \int \int \int u^{(i_1)}(y_1) \cdots u^{(i_4)}(y_4) u^{(i_5)}(x + y_4 - y_3 + y_2 - y_1) dy_4 \cdots dy_1 \right)^2 dx
$$

$$
= \int_{-\infty}^{-1} (x)^{2N} \left( \int \int \int \int \int u^{(i_1)}(y_1) \cdots u^{(i_4)}(y_4) u^{(i_5)}(x + y_4 - y_3 + y_2 - y_1) dy_4 \cdots dy_1 \right)^2 dx
$$

$$
+ \int_{-1}^{T} (x)^{2N} \left( \int \int \int \int \int u^{(i_1)}(y_1) \cdots u^{(i_4)}(y_4) u^{(i_5)}(x + y_4 - y_3 + y_2 - y_1) dy_4 \cdots dy_1 \right)^2 dx
$$

Now we treat each of the two summands like the case for the $C^*_3$. For the first summand:

$$
\int_{-\infty}^{-1} (x)^{2N} \left( \int \int \int \int \int u^{(i_1)}(y_1) \cdots u^{(i_4)}(y_4) u^{(i_5)}(x + y_4 - y_3 + y_2 - y_1) dy_4 \cdots dy_1 \right)^2 dx
$$

we have the following two cases:

**Case 1** $i_5 < m$

In this case we use Lemma 2 on $u^{(i_5)}$ for which we deduce that:

$$
u^{(i_5)}(x + y_4 - y_3 + y_2 - y_1) \leq \frac{L|u||H_N^{m,2}((\infty, T))}{(x + y_4 - y_3 + y_2 - y_1)^N} \leq \frac{L|u||H_N^{m,2}((\infty, T))}{(x)^N}
$$

The last inequality follows as $|x + y_4 - y_3 + y_2 - y_1| > |x|$ in the domain of integration.

$$
\int_{-\infty}^{1} (x)^{2N} \left( \int \int \int \int \int u^{(i_1)}(y_1) \cdots u^{(i_4)}(y_4) u^{(i_5)}(x + y_4 - y_3 + y_2 - y_1) dy_4 \cdots dy_1 \right)^2 dx
$$

$$
\leq L^2|u|^2_{H_N^{m,2}((\infty, T))} \int_{-\infty}^{T} (x)^{2N} \left( \int \int \int \int \int u^{(i_1)}(y_1) \cdots u^{(i_4)}(y_4) dy_4 \cdots dy_1 \right)^2 dx
$$

$$
\leq L^2|u|^2_{H_N^{m,2}((\infty, T))} \int_{-\infty}^{-1} (x)^{2N} \left( \int \int \int \int \int u^{(i_1)}(y_1) \cdots u^{(i_4)}(y_4) dy_4 \cdots dy_1 \right)^2 dx
$$

Three of the $i_1, \ldots, i_4$ are $< m$. The other one is potentially $= m$. So we use Lemma 2 on those three functions and on the other the fact that $\int_{-\infty}^{x} u^{(m)}(y) dy = u^{(m-1)}(x) + K$. By substitution we get:

$$
\leq L^8|u|^8_{H_N^{m,2}((\infty, T))} \int_{-\infty}^{T} (u^{(m-1)}(x) + K_u)^2 \frac{1}{(x)^{6N-6}} dx
$$

34
Now using Lemma 2 on $u^{(m-1)}$ we get:

$$
\leq L^8 ||u||^8_{H^2((-\infty,T))} \int_{-\infty}^{-1} \left( \frac{L ||u||_{H^2((-\infty,T))}}{\langle x \rangle^N} + K_u \right)^2 \frac{1}{\langle x \rangle^{6N-6}} \, dx
$$

$$
\leq L^8 ||u||^8_{H^2((-\infty,T))} \int_{-\infty}^{-1} \left( \frac{L ||u||_{H^2((-\infty,T))}}{\langle x \rangle^N} + K_u \right)^2 \frac{1}{\langle x \rangle^{6N-6}} \, dx
$$

Its clear that for $N \geq 2$ it converges.

**Case 2)** $i_5 = m$ implies $i_1, ..., i_4 < m$. In this case the change of variables $y_2 = x + y_4 - y_3 + y_2 - y_1$ changes the first summand into:

$$
\int_{-\infty}^{-1} \left( \int_{-\infty}^{x} \int_{-\infty}^{y_1} \int_{-\infty}^{x+y_2-y_1} \int_{-\infty}^{x} u^{(i_1)}(y_1)u^{(i_2)}(y_4)y_4 + y_3 - y_2 + y_1 - x \right) dy_1 \, dx
$$

Now we use Lemma 2 on $u^{(i_4)}$, and since $|y_4 + y_3 - y_2 + y_1 - x| > |x|$ we get:

$$
\leq L^2 ||u||^2_{H^2((-\infty,T))} \int_{-\infty}^{-1} \left( \int_{-\infty}^{x} \int_{-\infty}^{y_1} \int_{-\infty}^{x+y_2-y_1} \int_{-\infty}^{x} u^{(i_1)}(y_1)u^{(i_2)}(y_4)y_4 + y_3 - y_2 + y_1 - x \right) dy_1 \, dx
$$

Which similarly like Case 1) we finally get:

$$
\leq L^8 ||u||^8_{H^2((-\infty,T))} \int_{-\infty}^{-1} \left( \frac{L ||u||_{H^2((-\infty,T))}}{\langle x \rangle^N} + K_u \right)^2 \frac{1}{\langle x \rangle^{6N-6}} \, dx
$$

Now we focus our attention to the second summand which is just an integral on the finite interval $(-1, T)$.

$$
\int_{-1}^{T} \left( \int_{-\infty}^{x} \int_{-\infty}^{y_1} \int_{-\infty}^{x+y_2-y_1} \int_{-\infty}^{x} u^{(i_1)}(y_1)u^{(i_2)}(y_4)y_4 + y_3 - y_2 + y_1 - x \right) dy_1 \, dx
$$

So we try to find a supremum for the inner part of the integral:

$$
\int_{-\infty}^{T} \left( \int_{-\infty}^{x} \int_{-\infty}^{y_1} \int_{-\infty}^{x+y_2-y_1} \int_{-\infty}^{x} u^{(i_1)}(y_1)u^{(i_2)}(y_4)y_4 + y_3 - y_2 + y_1 - x \right) dy_1 \, dx
$$

Using the fact that $\frac{1}{\langle x \rangle} \leq 1$, we consider two cases:

**Case 1)** $i_5 < m$

We use Lemma 2 on $u^{(i_5)}$.

$$
\int_{-1}^{T} \left( \int_{-\infty}^{x} \int_{-\infty}^{y_1} \int_{-\infty}^{x+y_2-y_1} \int_{-\infty}^{x} u^{(i_1)}(y_1)u^{(i_2)}(y_4)y_4 + y_3 - y_2 + y_1 - x \right) dy_1 \, dx
$$
Using the fact that:

\[\sum_{i} \text{summand look like:} \]

Using the same approximation as in the first summand we get:

\[
\leq L^2 ||u||_{H^m_N((-\infty, T])}^2 \int_{-\infty}^{-1} \left( \int \int \int \int u^{(i_1)}(y_1) \ldots u^{(i_4)}(y_4)dy_4 \right)^2 dx
\]

Using the same approximation as in the first summand we get:

\[
\leq L^8 ||u||_{H^m_N((-\infty, T])}^8 \int_{-1}^{T} \left( L ||u||_{H^m_N((-\infty, T])} + K_u \right)^2 \langle x \rangle^6 dx
\]

Case 2) We do the same substitution as above \(y'_2 = x + y_4 - y_3 + y_2 - y_1\) which makes the second summand look like:

\[
\int_{-1}^{T} \langle x \rangle^{2N} \left( \int \int \int \int u^{(i_1)}(y_3)(y_1)u^{(i_5)}(y_4)(y_4 + y_3 - y_2 + y_1 - x)dy_4 \right)^2 dx
\]

Now using Lemma 2 on \(u^{(i_5)}\) we get:

\[
\leq L^8 ||u||_{H^m_N((-\infty, T])}^8 \int_{-1}^{T} \left( L ||u||_{H^m_N((-\infty, T])} + K_u \right)^2 \langle x \rangle^6 dx
\]

So in conclusion we get:

\[||C^-_5 (u^{(i_1)}, u^{(i_2)}, u^{(i_3)}, u^{(i_4)}, u^{(i_5)})||_{L^2_N((-\infty, T])}\]

\[
\leq L^4 ||u||_{H^m_N((-\infty, T])}^4 \int_{-\infty}^{-1} \left( L ||u||_{H^m_N((-\infty, T])} + K_u \right)^2 \frac{1}{\langle x \rangle^{6N-6}} dx + \int_{-1}^{T} \left( L ||u||_{H^m_N((-\infty, T])} + K_u \right)^2 \langle x \rangle^6 dx
\]

Using the fact that:

\[||C^-_5 (u)||_{H^m_N((-\infty, T])} = \sum_{i=0}^{m} \left| \frac{d^i}{dx^i} C^-_5 (u) \right|_{L^2_N((-\infty, T])}\]

\[= \sum_{i=0}^{m} \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} ||C^-_5 (u^{(i_1)}, u^{(i_2)}, u^{(i_3)}, u^{(i_4)}, u^{(i_5)})||_{L^2_N((-\infty, T])}\]

we get:

\[||C^-_5 (u)||_{H^m_N((-\infty, T])} \leq \sum_{i=0}^{m} 5^i L^4 ||u||_{H^m_N((-\infty, T])}^4 \ldots \]

\[
\ldots \int_{-\infty}^{-1} \left( L ||u||_{H^m_N((-\infty, T])} + K_u \right)^2 \frac{1}{\langle x \rangle^{6N-6}} dx + \int_{-1}^{T} \left( L ||u||_{H^m_N((-\infty, T])} + K_u \right)^2 \langle x \rangle^6 dx
\]

36
Summing up we get:
\[
\frac{5^{m+1} - 1}{4} L^4 |u|^{4}_{H^m_N((−∞, T))} \ldots \\
\left\{ \int_{−∞}^{−1} \left( L |u|^{m,2}_{H^m_N((−∞, T])} + K_u \right)^{2} \frac{1}{(x)^{2N−6}} \, dx + \int_{−1}^{T} \left( L |u|^{m,2}_{H^m_N((−∞, T])} + K_u \right)^{2} \langle x \rangle^6 \, dx \right\}^{\frac{1}{2}}
\]

Completely analogously we get the bounds for \( C_{2n+1}^- \),
\[
||C_{2n+1}^- (u)||_{H^m_N((−∞, T])} \leq \frac{(2n+1)^{m+1} - 1}{2n} L^{2n} |u|^{2n}_{H^m_N((−∞, T])} \ldots \\
\left\{ \int_{−∞}^{−1} \left( L |u|^{m,2}_{H^m_N((−∞, T])} + K_u \right)^{2} \frac{1}{(x)^{(2n−1)(2N−2)}} \, dx + \int_{−1}^{T} \left( L |u|^{m,2}_{H^m_N((−∞, T])} + K_u \right)^{2} \langle x \rangle^{(2n−1)^2} \, dx \right\}^{\frac{1}{2}}
\]

Now summing over all \( n \) we finally get the estimate for \( C^- (u) \),
\[
||C^- (u)||_{H^m_N((−∞, T])} \leq \sum_{n=1}^{∞} ||C_{2n+1}^- (u)||_{H^m_N((−∞, T])}
\]

Now we use ratio test to get the radius of convergence,
\[
lim_{n→∞} \frac{(2n+3)^{m+1} - 1}{2n+2} L^{2n+2} |u|^{2n+2}_{H^m_N((−∞, T])} \ldots \\
\left\{ \int_{−∞}^{−1} \left( L |u|^{m,2}_{H^m_N((−∞, T])} + K_u \right)^{2} \frac{1}{(x)^{(2n+1)(2N−2)}} \, dx + \int_{−1}^{T} \left( L |u|^{m,2}_{H^m_N((−∞, T])} + K_u \right)^{2} \langle x \rangle^{(2n+1)^2} \, dx \right\}^{\frac{1}{2}}
\]

\[
\left\{ \int_{−∞}^{−1} \left( L |u|^{m,2}_{H^m_N((−∞, T])} + K_u \right)^{2} \frac{1}{(x)^{(2n−1)(2N−2)}} \, dx + \int_{−1}^{T} \left( L |u|^{m,2}_{H^m_N((−∞, T])} + K_u \right)^{2} \langle x \rangle^{(2n−1)^2} \, dx \right\}^{\frac{1}{2}}
\]

\[
= L^2 |u|^2_{H^m_N((−∞, T])} < 1
\]

Notice that even though the estimates for each of the \( C_{2n+1}^- (u) \) depend on \( K_u \) which is a constant which is finite and depends on the function \( u \) in the ratio test \( K_u \) cancels out and so the radius of convergence doesn’t depend on \( u \).

**Conclusion:** \( C^- (u) \) converges in \( H^m_N((−∞, T]) \) if \( |u|^{m,2}_{H^m_N((−∞, T])} < \frac{1}{L} \).

**Note:** Completely analogous calculations can be done for \( C^+ (u) \) in the space \( H^m_N([T, ∞)) \).

### 3.3.3 Asymptotic Spaces \( \tilde{A}_{H^m_N((−∞, T])} \) and \( \tilde{A}_{H^m_N([T, ∞))} \)

We will study functions that have asymptotic expansion at \( −∞ \). More precisely, we consider functions which have the following expansion:
\[
u(x) = \frac{c_2}{x^2} + \frac{c_3}{x^3} + \ldots + \frac{c_N}{x^N} + o(|x|^{-N}), \quad x \to −∞
\]

We will choose a more specific remainder part \( H^m_N((−∞, T]) \). Notice that,
\[
f \in H^m_N((−∞, T]) \quad \Rightarrow \quad f, f^{(1)}, \ldots, f^{(m)} \in o(|x|^{-N}) \quad \text{for} \quad x \to −∞.
\]

37
The functions $\frac{1}{x^2}$ are not defined at $x = 0$, so to fix this problem consider,

$$u(x) = \sum_{k=2}^{N} \left( a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + f(x)$$

where $f \in H^{m,2}_N((-\infty, T])$ is the remainder and $H^{m,2}_N((-\infty, T])$ is the remainder space. Here $\langle x \rangle = \sqrt{1 + x^2}$.

Our goal is to prove that $C^- (u)$ has a similar expansion at $-\infty$. Recall that $C^- (u)$ is the following power series:

$$C^- (u) = \sum_{n=0}^{\infty} C^-_{2n+1} (u) = \sum_{n=0}^{\infty} C^-_{2n+1} (u, ..., u)$$

$$= \sum_{n=0}^{\infty} C^-_{2n+1} \left( \sum_{k=2}^{N} \left( a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + f(x), ..., \sum_{k=2}^{N} \left( a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + f(x) \right)$$

Using the multilinearity of $C^-_{2n+1}$ we have:

$$C^- (u) = \sum_{n=0}^{\infty} \sum_{u_i \in \{ a_k \langle x \rangle^k, b_k x \langle x \rangle^{k+1}, f \mid k \in \{2, ..., N\} \}} C^-_{2n+1} (u_1, ..., u_{2n+1})$$

where the second sum runs over all the possible permutations, where $u_i \in \{ a_k \langle x \rangle^k, b_k x \langle x \rangle^{k+1}, f \mid k \in \{2, ..., N\} \}$. So we will have to calculate the asymptotic expansion of each $C^-_{2n+1} (u_1, ..., u_{2n+1})$ and then sum those expansions and prove that they converge.

We start by calculating the asymptotic expansion of $C^-_3 \left( \frac{1}{\langle x \rangle^1}, \frac{1}{\langle y \rangle^1}, \frac{1}{\langle z \rangle^1} \right)$. Remember that:

$$C^-_3 \left( \frac{1}{\langle x \rangle^1}, \frac{1}{\langle y \rangle^1}, \frac{1}{\langle z \rangle^1} \right) (x) = \int_{-\infty}^{y_1} \int_{-\infty}^{x} \int_{-\infty}^{1} \frac{1}{1 + y_1^{1} x} \frac{1}{1 + y_2^{1} x} \frac{1}{1 + (x + y_2 - y_1)^{1/2}} dy_2 dy_1$$

Notice that for any finite $x$, $C^-_3 \left( \frac{1}{\langle x \rangle^1}, \frac{1}{\langle y \rangle^1}, \frac{1}{\langle z \rangle^1} \right)$ takes a finite value. This means that if we calculate its asymptotic expansion for all $x < a$ for some fixed $a$, then we can extend this expansion to the whole interval $(-\infty, T]$. Let’s pick $a = -1$; i.e we will calculate the expansion of $C^-_3 \left( \frac{1}{\langle x \rangle^1}, \frac{1}{\langle y \rangle^1}, \frac{1}{\langle z \rangle^1} \right)$ for $x < -1$. Notice that this implies that $-\frac{\sqrt{x}}{(x)} < -\frac{1}{\sqrt{2}}$. By this remark it is clear that we can restrict our asymptotic space to $A_{N}^{m,2}((-\infty, -1])$.

If $x < -1$ by the change of variables $u_i = \frac{y_i}{x}$ we get:

$$C^-_3 \left( \frac{1}{\langle x \rangle^1}, \frac{1}{\langle y \rangle^1}, \frac{1}{\langle z \rangle^1} \right) (x) = \frac{1}{\langle x \rangle^{1+2+3-2}} \int_{-\infty}^{\frac{x}{\langle x \rangle}} \int_{-\infty}^{\frac{x}{\langle x \rangle} + u_1} \int_{-\infty}^{1} \frac{1}{1 + \frac{1}{\langle x \rangle}^{1} \frac{1}{\langle y \rangle^1} \frac{1}{\langle z \rangle^1}} du_2 du_1$$

Denote by: $a = a(x) = \frac{1}{\langle x \rangle^1}$ and $b = b(x) = 1 + \frac{x}{\langle x \rangle}$. Clearly $a \in A_{N}^{m,2}((-\infty, T])$. Also notice that:

$$\left( \frac{x}{\langle x \rangle} \right)^2 = \frac{x^2}{1 + x^2} = 1 - \frac{1}{\langle x \rangle^2} \implies 1 - \left( \frac{x}{\langle x \rangle} \right)^2 = \left( 1 - \frac{x}{\langle x \rangle} \right) \left( 1 + \frac{x}{\langle x \rangle} \right) = \frac{1}{\langle x \rangle^2}$$
Since, $1 - \frac{x}{\langle x \rangle} > 1 \implies b = 1 + \frac{x}{\langle x \rangle^2} < \frac{1}{\langle x \rangle^2} \implies b \in H_N^{m,2}((-\infty, t])$

denote by:

\[ I_3(a, b) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{(a + u_1^2)^{\frac{i}{2}}} \frac{1}{(a + u_2^2)^{\frac{j}{2}}} \frac{1}{(a + \frac{1}{2} + b + u_2 - u_1)^{\frac{k}{2}}} d\mu dt \]

Notice that $I_3(0, 0) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{(a + u_1^2)^{\frac{i}{2}}} \frac{1}{(a + u_2^2)^{\frac{j}{2}}} \frac{1}{(a + \frac{1}{2} + b + u_2 - u_1)^{\frac{k}{2}}} d\mu dt < \infty$ is bounded and $I_3(a, b)$ is analytic with respect to the variables $a$ and $b$. Also clearly its partial derivatives $\frac{\partial^i \partial^j}{\partial a^i \partial b^j} I_3(a, b)$ are bounded at $(0, 0)$. This implies that on any closed ball $B_r(0, 0)$ centered at $(0, 0)$, the partials $\frac{\partial^i \partial^j}{\partial a^i \partial b^j} I_3(a, b)$ are also bounded. Now we use Taylor’s theorem to calculate the expansion of $I_3(a, b)$ in $H_N^{m,2}((-\infty, T])$.

First notice that since $a = \frac{1}{(x)^2}$ and $0 < b < \frac{1}{(x)^2}$ and if $i_1, i_2 \geq \lceil \frac{N+1}{2} \rceil$ implies:

\[ \left( \frac{d^i}{dx^i} a(x) \right) \left( \frac{d^j}{dx^j} b(x) \right) \in H_N^{m,2}((-\infty, T]) \] for $j_1, j_2 \geq 0$

Now we state the Taylor’s theorem for $I_3(a, b)$.

\[ I_3(a, b) = \frac{\partial^i \partial^j}{\partial a^i \partial b^j} I_3(0, 0) + \sum_{i+j=\lceil \frac{N+1}{2} \rceil} R_{i,j}(a, b) a^i b^j \]

where the Taylor remainder is

\[ \sum_{i+j=\lceil \frac{N+1}{2} \rceil} R_{i,j}(a, b) a^i b^j = \sum_{i+j=\lceil \frac{N+1}{2} \rceil} \frac{\partial^i \partial^j}{\partial a^i \partial b^j} \left( I_3(ta, tb) \right) dt a^i b^j. \]

Since $\frac{\partial^i \partial^j}{\partial a^i \partial b^j} I_3(0, 0)$ is bounded and $i + j < \lceil \frac{N+1}{2} \rceil$ this implies that $\frac{\partial^i \partial^j}{\partial a^i \partial b^j} I_3(0, 0)$ is in the asymptotic part of $I_3(a, b)$.

Now we prove that Taylor remainder $\sum_{i+j=\lceil \frac{N+1}{2} \rceil} R_{i,j}(a, b) a^i b^j$ is in the remainder space $H_N^{m,2}((-\infty, T])$.

First notice that since

\[ R_{i,j}(a, b) = \frac{i+j}{i!j!} \int_0^1 (1-t)^{i+j-1} \partial^i \partial^j \left( I_3(ta, tb) \right) dt \]

The integral is evaluated on the interval $[0, 1]$ and $a, b \leq \frac{1}{(x)^2}$ and $\partial^i \partial^j I_3(a, b)$ is bounded on any closed ball centered at $(0, 0)$. This implies $R_{i,j}(a, b)$ is bounded and so are all its partial derivatives $\frac{\partial^i \partial^j}{\partial a^i \partial b^j} R_{i,j}(a, b)$. Since $a$ and $b$ are functions of $x$ this means that all the derivatives with respect to $x$ of $R_{i,j}(a, b)$ are also bounded. For simplicity, denote this bounds as follows:

\[ \frac{d^p}{dx^p} R_{i,j}(a, b) < R_{i,j}^p \] where $R_{i,j}^p$ is its constant bound

\[ \left\| \sum_{i+j=\lceil \frac{N+1}{2} \rceil} \frac{i+j}{i!j!} R_{i,j}(a, b) a^i b^j \right\|_{H_N^{m,2}((-\infty, -1])} = \sum_{p=0}^m \left\| \frac{d^p}{dx^p} \left( \sum_{i+j=\lceil \frac{N+1}{2} \rceil} \frac{i+j}{i!j!} R_{i,j}(a, b) a^i b^j \right) \right\|_{L_N^2((-\infty, -1])} \]
Let's start with

Now we consider the mixed terms. First we consider terms where one of the functions is in the

By Fubini's theorem we change the order of integration to get

This proves that the Taylor remainder is in the remainder space and that $I_3(a, b) \in \mathcal{A}_N^{m, 2}((-\infty, -1])$ and by just a shift $C_3^{-}\left(\frac{1}{|\langle \cdot \rangle|}, \frac{1}{|\langle \cdot \rangle|}, \frac{1}{|\langle \cdot \rangle|}\right) \in \mathcal{A}_N^{m, 2}((-\infty, -1])$.

Notice: In the previous calculation we could replace any of the $\frac{1}{|\langle \cdot \rangle|^k}$ with $\frac{x}{|\langle \cdot \rangle|^{k+1}}$ and the estimates would work out the same for $C_3^{-}$.

Now we consider the mixed terms. First we consider terms where one of the functions is in the remainder space. Let's start with $C_3^{-}\left(\frac{1}{|\langle \cdot \rangle|}, \frac{1}{|\langle \cdot \rangle|}, \frac{1}{|\langle \cdot \rangle|}, f\right)$.

Now we use the Cauchy-Schwartz inequality to get:

By Fubini's theorem we change the order of integration to get

where
Since in the region of integration $y_2 - y_1 \leq 0$ and $x \leq 0$ this implies that $\langle x \rangle \leq \langle x + y_2 - y_1 \rangle$, hence:

$$\leq \left( \int_{-\infty}^{-1} \frac{y_1}{y_1^1} \frac{1}{y_2^1} dy_2 dy_1 \right) \left( \int_{-\infty}^{-1} \frac{y_1}{y_1^2} \frac{1}{y_2^2} \int_{-\infty}^{-1} (x + y_2 - y_1) 2N f^2(x + y_2 - y_1) dx dy_2 dy_1 \right).$$

Hence

$$\left\| C_3^- \left( \frac{1}{\langle \cdot \rangle_1}, \frac{1}{\langle \cdot \rangle_2}, f \right) \right\|^2_{L_N^2((\langle \cdot \rangle_1), (-\infty, -1])} \leq \left\| f \right\|^2_{L_N^2((\langle \cdot \rangle_1), (-\infty, -1])} \left( \int_{-\infty}^{-1} \frac{y_1}{y_1^1} \frac{1}{y_2^1} \int_{-\infty}^{-1} dy_2 dy_1 \right)^2.$$

In exactly the same way we can prove that:

$$\left\| C_3^- \left( \frac{1}{\langle \cdot \rangle_1}, \frac{1}{\langle \cdot \rangle_2}, f(i) \right) \right\|^2_{L_N^2((\langle \cdot \rangle_1), (-\infty, -1])} \leq \left\| f(i) \right\|^2_{L_N^2((\langle \cdot \rangle_1), (-\infty, -1])} \left( \int_{-\infty}^{-1} \frac{y_1}{y_1^1} \frac{1}{y_2^1} \int_{-\infty}^{-1} dy_2 dy_1 \right)^2 \text{ for } i \in [0, ..., m].$$

Since,

$$\left\| C_3^- \left( \frac{1}{\langle \cdot \rangle_1}, \frac{1}{\langle \cdot \rangle_2}, f \right) \right\|^2_{H_N^{2,2}((-\infty, -1]), 1} = \sum_{l_1, l_2 \geq 0} d_{l_1, l_2} \left\| C_3^- \left( \frac{1}{\langle \cdot \rangle_1}, \frac{1}{\langle \cdot \rangle_2}, f(i) \right) \right\|^2_{L_N^2((\langle \cdot \rangle_1), (-\infty, -1])},$$

and all the summands converge, we get that $C_3^- \left( \frac{1}{\langle \cdot \rangle_1}, \frac{1}{\langle \cdot \rangle_2}, f \right) \in H_N^{m,2}((-\infty, -1]).$

Now we consider $C_3^- \left( \frac{1}{\langle \cdot \rangle_1}, f, \frac{1}{\langle \cdot \rangle_2} \right).$ We have,

$$\left\| C_3^- \left( \frac{1}{\langle \cdot \rangle_1}, f, \frac{1}{\langle \cdot \rangle_2} \right) \right\|^2_{L_N^2((\langle \cdot \rangle_1), (-\infty, -1])} \leq \int_{-\infty}^{-1} \langle x \rangle^{2N+2-l_1-l_2} \left( \int_{-\infty}^{-1} \frac{1}{(x)^2 + u_1^2} \frac{1}{(x)^2 + u_2^2} f(\langle x \rangle u_2) \right)^2 \frac{1}{(x)^2 + u_2^2} \frac{1}{(x)^2 + u_2^2 - u_1^2} \frac{1}{(x)^2 + u_2^2 - u_1^2} dx$$

$$\leq \int_{-\infty}^{-1} \langle x \rangle^{2N+2-l_1-l_2} \left( \int_{-\infty}^{-1} \frac{1}{u_1^2} f(\langle x \rangle u_2) \frac{1}{(x)^2 + u_2^2} \frac{1}{(x)^2 + u_2^2 - u_1^2} \right)^2 \frac{1}{(x)^2 + u_2^2} \frac{1}{(x)^2 + u_2^2 - u_1^2} dx$$

Since $-\infty \leq u_2 \leq \frac{x}{(x)}$ implies $-\infty \leq \langle x \rangle u_2 \leq x$, and hence $\langle \langle x \rangle u_2 \rangle \geq \langle x \rangle$, so we get

$$\left\| C_3^- \left( \frac{1}{\langle \cdot \rangle_1}, f, \frac{1}{\langle \cdot \rangle_2} \right) \right\|^2_{L_N^2((\langle \cdot \rangle_1), (-\infty, -1])} \leq \int_{-\infty}^{-1} \int_{-\infty}^{-1} \frac{1}{u_1^2} \frac{1}{u_1^2} \frac{1}{(x)^2 + u_2^2} \frac{1}{(x)^2 + u_2^2 - u_1^2} \langle x \rangle^{2N+2-l_1-l_2} f(\langle x \rangle u_2) \frac{1}{(x)^2 + u_2^2} \frac{1}{(x)^2 + u_2^2 - u_1^2} dx$$

Using the Cauchy-Schwartz we get,

$$\leq \int_{-\infty}^{-1} \int_{-\infty}^{-1} \frac{1}{u_1^2} \frac{1}{u_1^2} \frac{1}{(x)^2 + u_2^2} \frac{1}{(x)^2 + u_2^2 - u_1^2} (\langle x \rangle u_2)^{2N+2-l_1-l_2} f^2(\langle x \rangle u_2) dx$$

41
Finally using Fubini's theorem

\[ \left\| C_3^-(\frac{1}{\langle \cdot \rangle^{d_1}}, f, \frac{1}{\langle \cdot \rangle^{d_2}}) \right\|^2_{L_N^2((\infty,-1])} \leq \left\| f \right\|^2_{L_N^2((\infty,-1])} \left( \int_{-\infty}^{\infty} \frac{1}{u_1} \left( \frac{1}{\sqrt{2}} + u_2 - u_1 \right)^{d_2} du_2 du_1 \right)^2 < \infty \]

In exactly the same way one can prove

\[ \left\| C_3^- \left( \frac{1}{\langle \cdot \rangle^{d_1}}, f(i), \frac{1}{\langle \cdot \rangle^{d_2}} \right) \right\|^2_{L_N^2((\infty,-1])} \leq \left\| f(i) \right\|^2_{L_N^2((\infty,-1])} \left( \int_{-\infty}^{\infty} \frac{1}{y_1} \left( \frac{1}{\sqrt{2}} + u_2 - u_1 \right)^{d_2} dy_2 dy_1 \right)^2 \]

for \( i \in [0, ..., m] \)

Since,

\[ \left\| C_3^- \left( \frac{1}{\langle \cdot \rangle^{d_1}}, f, \frac{1}{\langle \cdot \rangle^{d_2}} \right) \right\|^2_{H_N^{m,2}((\infty,-1])} = \sum_{i_1, i_2, i_3} d_{i_1, i_2, i_3} \left\| C_3^- \left( \frac{1}{\langle \cdot \rangle^{d_1}}, f^{(i_2)}, \frac{1}{\langle \cdot \rangle^{d_2}} \right) \right\|^2_{L_N^2((\infty,-1])}, \]

and all the summands converge, we see that \( C_3^- \left( \frac{1}{\langle \cdot \rangle^{d_1}}, f, \frac{1}{\langle \cdot \rangle^{d_2}} \right) \in H_N^{m,2}((\infty,-1]). \)

In the same way it can be proved that \( C_3^- \left( f, \frac{1}{\langle \cdot \rangle^{d_1}}, \frac{1}{\langle \cdot \rangle^{d_2}} \right) \in H_N^{m,2}((\infty,-1]). \)

Now we consider the case where two of the functions are in the remainder space, i.e the cases \( C_3^- \left( \frac{1}{\langle \cdot \rangle^{d_1}}, f, f \right), C_3^- \left( f, \frac{1}{\langle \cdot \rangle^{d_1}}, f \right) \) and \( C_3^- \left( f, f, \frac{1}{\langle \cdot \rangle^{d_1}} \right). \) Let's first consider:

\[ \left\| C_3^- \left( \frac{1}{\langle \cdot \rangle^{d_1}}, f, f \right) \right\|^2_{L_N^2((\infty,-1])} \leq L \left\| f \right\|^2_{H_N^{m,2}((\infty,-1])} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{1}{y_1} \frac{1}{y_2} f(x+y_2-y_1) dy_2 dy_1 \right)^2 dx \]

We can apply Lemma 2 to either \( f(y_2) \) or to \( f(x+y_2-y_1) \) because it doesn’t make any difference in the rest of the calculations. Without loss of generality we apply it to \( f(y_2) \) to get

\[ \left\| C_3^- \left( \frac{1}{\langle \cdot \rangle^{d_1}}, f, f \right) \right\|^2_{L_N^2((\infty,-1])} \leq L \left\| f \right\|^2_{H_N^{m,2}((\infty,-1])} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{1}{y_1} \frac{1}{y_2} f(x+y_2-y_1) dy_2 dy_1 \right)^2 dx \]

By Cauchy-Schwartz inequality,

\[ \leq L \left\| f \right\|^2_{H_N^{m,2}((\infty,-1])} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{1}{y_1} \frac{1}{y_2} f(x+y_2-y_1) dy_2 dy_1 \right)^2 dx \]

Now we apply Fubini's theorem.

\[ \leq L \left\| f \right\|^2_{H_N^{m,2}((\infty,-1])} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{1}{y_1} \frac{1}{y_2} f^2(x+y_2-y_1) dx \right)^2 \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{1}{y_1} \frac{1}{y_2} dy_2 dy_1 \right)^2 \]

\[ \leq L \left\| f \right\|^2_{H_N^{m,2}((\infty,-1])} \left\| f \right\|^2_{L_N^2((\infty,T])} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{1}{y_1} \frac{1}{y_2} dy_2 dy_1 \right)^2 < \infty \]
When we consider

\[
\left\| \frac{d^i}{dx^i} C_3^{-} \left( \frac{1}{\langle \cdot \rangle_{i_1}}, f, f \right) \right\|_{L_N^2((-\infty,-1])} = \sum_{i_2, i_3} d_{i_1, i_2, i_3} \left\| \frac{d^i}{dx^i} C_3^{-} \left( \left( \frac{1}{\langle \cdot \rangle_{i_1}} \right)^{(i_2)}, f^{(i_3)} \right) \right\|_{L_N^2((-\infty,-1])}
\]

for \( i \in [0, ..., m] \), at least one of \( i_2 \) or \( i_3 \) is \(< m \). So we can apply Lemma 2 to at least one of them. So this case is reduced to the previous case where only one of the functions was in the remainder space.

The last case we have to account for is \( C_3^{-} (f, f, f) \), but this was already considered in the previous section where we proved that \( C_3^{-} \) is convergent in \( H_{N,2}^{m,2}((-\infty, T]) \).

Now we calculate the asymptotic expansion of \( C_{2n+1}^{-} \left( \frac{1}{\langle \cdot \rangle_{2}}, \frac{1}{\langle \cdot \rangle_{2}} \right) \) for a generic \( n \):

\[
C_{2n+1}^{-} \left( \frac{1}{\langle \cdot \rangle_{2}}, \frac{1}{\langle \cdot \rangle_{2}} \right) (x) = \int_{-\infty}^{\infty} \frac{1}{\langle x \rangle_{2}^n} \frac{1}{u_1^2} \frac{1}{y_1} \frac{1}{1 + y_2^2} \frac{1}{1 + y_2^2} \frac{1}{1 + (x + y_2n - y_2n - 1 + ... - y_1)^2} dy_{2n} dy_1
\]

Clearly this takes finite value for \( x \) finite. So now we focus on calculating the asymptotic expansion for \( x < -1 \). Just like above we make the change of variables \( u_i = \frac{y_i}{\langle x \rangle} \):

\[
C_{2n+1}^{-} \left( \frac{1}{\langle \cdot \rangle_{2}}, \frac{1}{\langle \cdot \rangle_{2}} \right) (x) = \int_{-\infty}^{\infty} \frac{1}{\langle x \rangle_{2+1}^n} \frac{1}{u_1^2} \frac{1}{y_1} \frac{1}{1 + u_2^2} \frac{1}{1 + u_2^2} \frac{1}{1 + (x + u_2n - u_2n - 1 + ... - u_1)^2} du_{2n} du_1
\]

If \( 2n + 2 < N + 1 \) we use the same method as before, and apply Taylor’s theorem as in the previous case to calculate the asymptotic expansion. The same method is applied when we evaluate \( C_{2n+1}^{-} \) on \( \frac{a}{\langle x \rangle^2}, \frac{b}{\langle x \rangle^2} + \). If \( 2n + 2 > N + 1 \) we claim that \( C_{2n+1}^{-} \left( \frac{1}{\langle \cdot \rangle_{2}}, \frac{1}{\langle \cdot \rangle_{2}} \right) \) is already in the remainder space \( H_{N}^{m,2}((-\infty, -1]) \).

To prove this we rewrite:

\[
C_{2n+1}^{-} \left( \frac{1}{\langle \cdot \rangle_{2}}, \frac{1}{\langle \cdot \rangle_{2}} \right) (x) = \int_{-\infty}^{\infty} \frac{1}{\langle x \rangle_{2+1}^n} \frac{1}{u_1^2} \frac{1}{y_1} \frac{1}{1 + u_2^2} \frac{1}{1 + u_2^2} \frac{1}{1 + (x + u_2n - u_2n - 1 + ... - u_1)^2} du_{2n} du_1
\]

where \( I_{2n+1}(a, b) = \int_{-\infty}^{\infty} \frac{1}{\langle x \rangle_{2+1}^n} \frac{1}{u_1^2} \frac{1}{a + u_2^2} \frac{1}{a + u_2^2} \frac{1}{1 + (a + u_2n - a + (-1 + b + u_2n - a + ... - u_1)^2} du_{2n} du_1 \)

and \( a = \frac{1}{\langle x \rangle^2} \) and \( b = 1 + \frac{x}{\langle x \rangle} \).

Clearly \( I_{2n+1}(a, b) \) is analytic with respect to \( a \) and \( b \), and all its partial derivatives \( \frac{\partial^p \partial^q}{\partial a^p \partial b^q} I_{2n+1}(0, 0) < \infty \). This implies that \( \frac{\partial^p \partial^q}{\partial a^p \partial b^q} I_{2n+1}(0, 0) \) are bounded by \( (a, b) \) in a compact set in \( \mathbb{R}^2 \). Denote those bounds by the constants \( I_{2n+1}^p < \frac{\partial^p \partial^q}{\partial a^p \partial b^q} I_{2n+1}(a, b) \). Then we have:

\[
\left\| \frac{1}{\langle x \rangle_{2+1}^n} I_{2n+1}(a, b) \right\|_{H_{N}^{m,2}((-\infty, -1])} = \sum_{p=0}^{m} \left\| \frac{d^p}{dx^p} \left( \frac{1}{\langle x \rangle_{2+1}^n} I_{2n+1}(a, b) \right) \right\|_{L_N^2((-\infty, -1])}
\]
\[
\sum_{p=0}^{m} \left\| \sum_{k+l=p} \frac{d^k}{dx^k} \left( \frac{1}{\langle x \rangle^{2n+2}} \right) \frac{d^l}{dx^l} \left( I_{2n+1}(a, b) \right) \right\|_{L^2_N((\infty, -1])}
\leq \sum_{p=0}^{m} \left( \sum_{k+l=p} (I^2_{2n+1})^2 \int_{-\infty}^{-1} \frac{1}{\langle x \rangle^2} dx \right)^{\frac{1}{2}} < \infty
\]

Hence \( C_{2n+1}^{-} \left( \frac{1}{\langle x \rangle}, ..., \frac{1}{\langle x \rangle} \right) \in H^{m,2}_N((\infty, -1]) \) for \( 2n + 2 > N + 1 \).

Notice that if \( f \in H^{m,2}_N((\infty, -1]) \) then \( f^{(0)}, ..., f^{(m-1)} \in L^2_N((\infty, -1]) \) and they are differentiable and hence bounded. On the other hand \( f^{(m)} \in L^2_N((\infty, -1]) \), but it's not necessarily bounded. By definition a function \( u \in A^1_{H^{m,2}_N((\infty, -1])} \) has an expansion:

\[
u(x) = c_2 \left\langle \frac{1}{\langle x \rangle} \right\rangle + ... + c_N \left\langle \frac{1}{\langle x \rangle} \right\rangle^N + f \text{ for } x \in (\infty, -1] \text{ where } f \in H^{m,2}_N((\infty, -1])
\]

Factoring \( \frac{1}{\langle x \rangle^2} \) in each of the derivatives of \( u \), from the boundness of the derivatives of \( f \) we can deduce that,

\[
u^{(i)}(x) = \frac{w_i(x)}{\langle x \rangle^2} \text{ where } w_i \text{ are bounded for } i < m \text{ and } w_m \in L^2_N((\infty, -1])
\]

Denote \( ||u||_\infty = \max_{x \in (\infty, T)} \sup_{x \in (-\infty, T]} w_i(x) \). We claim that \( C_{2n+1}^{-}(u, ..., u) \) has the following expression when \( 2n + 2 > N + 1 \),

\[
C_{2n+1}^{-}(u, ..., u)(x) = \frac{1}{\langle x \rangle^{2n+2}} \sum_{-\infty \leq u_i \leq \frac{1}{\langle x \rangle}} \sum_{-\infty \leq u_{n-1} \leq \frac{1}{\langle x \rangle}} \sum_{-\infty \leq u_{n-2} \leq \frac{1}{\langle x \rangle}} ... \sum_{-\infty \leq u_1 \leq \frac{1}{\langle x \rangle}} \frac{w_0(\langle x \rangle u_1) ... w_{2n}(\langle x \rangle u_{2n})}{\langle x \rangle^{2n}} + \frac{w_0(\langle x \rangle u_{2n-1})}{\langle x \rangle^{2n-1}} + \frac{w_0(\langle x \rangle u_{2n-2})}{\langle x \rangle^{2n-2}} + \frac{w_0(\langle x \rangle u_{2n-3})}{\langle x \rangle^{2n-3}} + ... + \frac{w_0(\langle x \rangle u_1)}{\langle x \rangle^{2n-1}} + \frac{w_0(\langle x \rangle u_0)}{\langle x \rangle^{2n-2}} + \frac{w_0(\langle x \rangle u_{-1})}{\langle x \rangle^{2n-3}} + ... + \frac{w_0(\langle x \rangle u_{-n})}{\langle x \rangle^{2n-1}}
\]

We also have similar expressions for its derivatives for \( i < m \).

\[
\frac{d^i}{dx^i} C_{2n+1}^{-}(u)(x) = \sum_{i_1 + ... + i_{2n+1} = i} d_{i_1, ..., i_{2n+1}} C_{2n+1}^{-}(u^{(i_1)}, ..., u^{(i_{2n+1}))}(x)
\]

Taking the \( L^2_N((\infty, -1]) \)-norm of it we get,

\[
\left|\left| \frac{d^i}{dx^i} C_{2n+1}^{-}(u) \right|\right|_{L^2_N((\infty, -1])}^2 \leq (2n + 1)^i ||u||_\infty^{2n+1} \left( \int_{-\infty}^{-1} \frac{1}{\langle x \rangle^{4n+4}} dx \right)^{\frac{1}{2}}
\]

\[
... \left( \int_{-\infty}^{-1} \frac{1}{\langle x \rangle^{2n}} \frac{1}{\langle x \rangle^{2n-1}} \frac{1}{\langle x \rangle^{2n-2}} \frac{1}{\langle x \rangle^{2n-3}} ... \frac{1}{\langle x \rangle^{2n-1}} dx \right)
\]
Hence,

$$\sum_{i=0}^{m-1} \left\| \frac{d^i}{dx^i} C_{2n+1}^-(u) \right\|_{L^2_N((-\infty,-1])} \leq \frac{(2n+1)^{m+1} - 1}{2n} ||u||_{\infty}^{2n+1} \left( \int_{-\infty}^{-1} \left( \frac{1}{4n+1} \frac{1}{x^{2n+2}} \right)^{\frac{1}{2}} \left( \frac{1}{u_2^2} - u_2^2 \right)^2 \frac{1}{u_2^2} \frac{1}{u_2^2} \frac{1}{u_2^2} (\frac{1}{\sqrt{x}} + u_2 - u_2 - u_2 - \ldots - u_1)^2 \right) du_2 \ldots du_1.$$ 

Notice that if $n > \left[ \frac{N-1}{2} \right]$ then $\int_{-\infty}^{-1} \left( \frac{1}{4n+1} \frac{1}{x^{2n+2}} \right)^{\frac{1}{2}} \left( \frac{1}{u_2^2} - u_2^2 \right)^2 \frac{1}{u_2^2} \frac{1}{u_2^2} \frac{1}{u_2^2} (\frac{1}{\sqrt{x}} + u_2 - u_2 - u_2 - \ldots - u_1)^2 \right) du_2 \ldots du_1 < \infty$

is a constant.

Now we will use the ratio test to see when

$$\sum_{n=\left[ \frac{N-1}{2} \right]}^{\infty} \sum_{i=0}^{m-1} \left\| \frac{d^i}{dx^i} C_{2n+1}^-(u) \right\|_{L^2_N((-\infty,-1])}$$

converges. We have,

$$R = \lim_{n \to \infty} \frac{(2n+1)^{m+1} - 1}{2n} ||u||_{\infty}^{2n+1} \left( \int_{-\infty}^{-1} \left( \frac{1}{4n+1} \frac{1}{x^{2n+2}} \right)^{\frac{1}{2}} \left( \frac{1}{u_2^2} - u_2^2 \right)^2 \frac{1}{u_2^2} \frac{1}{u_2^2} \frac{1}{u_2^2} (\frac{1}{\sqrt{x}} + u_2 - u_2 - u_2 - \ldots - u_1)^2 \right) du_2 \ldots du_1.$$ 

Since $R \leq ||u||_{\infty}^2$, for $||u||_{\infty} < 1$, we have

$$\sum_{n=\left[ \frac{N-1}{2} \right]}^{\infty} \sum_{i=0}^{m-1} \left\| \frac{d^i}{dx^i} C_{2n+1}^-(u) \right\|_{L^2_N((-\infty,-1])}$$

converges.

The case of the m-th derivative we have to consider the norm differently. Notice that also $u^{(m)} = \frac{w_m(x)}{(x)^2}$, but $w_m(x)$ is not bounded. But nevertheless it should be noted that $w_m \in L^2$. The m-th derivative of $C_{2n+1}^-(u, \ldots, u)$ is:

$$\frac{d^m}{dx^m} C_{2n+1}^-(u, \ldots, u) = C_{2n+1}^-(u^{(m)}, \ldots, u) + \ldots + C_{2n+1}^-(u, \ldots, u^{(m)})$$

We will consider only the term $C_{2n+1}^-(u^{(m)}, \ldots, u)$ as for all the other summands the argument is similar.

$$C_{2n+1}^-(u^{(m)}, \ldots, u) = \frac{1}{(x)^{2n+2}} \int_{-\infty}^{-1} w_m((x)u_1) \ldots w_0((x)u_{2n}) w_0((x)(\frac{x}{(x)})^{2} + u_2 - u_2 - u_2 - \ldots - u_1)^2 \right) du_2 \ldots du_1.$$
Taking its \(L^2_N((-\infty, -1])\)-norm we get,

\[
\left\| C_{2n+1}(u^{(m)}, \ldots, u) \right\|_{L^2_N((-\infty, -1])} \\
\leq \left\| u \right\|_\infty^{2n} \left( -\infty \right)^{-1} \left( \frac{\langle x \rangle^{2N}}{\langle x \rangle^{4n+4}} \left( \int \frac{w_m(\langle x \rangle u_1)}{u_1^2} \frac{1}{u_2} \ldots \frac{1}{u_{2n}} \frac{1}{(-\frac{1}{\sqrt{2}} + u_{2n} - u_{2n-1} + \ldots - u_1)^2} du_{2n} \ldots du_1 \right)^2 \right)^{\frac{1}{2}} 
\]

Using the Cauchy-Schwartz inequality we get:

\[
\left\| C_{2n+1}(u^{(m)}, \ldots, u) \right\|_{L^2_N((-\infty, -1])} \\
\leq \left\| u \right\|_\infty^{2n} \left( -\infty \right)^{-1} \left( \frac{\langle x \rangle^{2N}}{\langle x \rangle^{4n+4}} \left( \int \frac{w_m(\langle x \rangle u_1)}{u_1^2} \frac{1}{u_2} \ldots \frac{1}{u_{2n}} \frac{1}{(-\frac{1}{\sqrt{2}} + u_{2n} - u_{2n-1} + \ldots - u_1)^2} du_{2n} \ldots du_1 \right) \right) \cdot \ldots \cdot \left( \int \frac{1}{u_1^2} \frac{1}{u_2} \ldots \frac{1}{u_{2n}} \frac{1}{(-\frac{1}{\sqrt{2}} + u_{2n} - u_{2n-1} + \ldots - u_1)^2} du_{2n} \ldots du_1 \right)^{\frac{1}{2}} .
\]

So summing over all the summands we get the estimate:

\[
\left\| \frac{d^m}{dx^m} C_{2n+1}(u, \ldots, u) \right\|_{L^2_N((-\infty, -1])} \leq (2n + 1) \left\| u \right\|_\infty^{2n} \left( \int \frac{\langle x \rangle^{2N} w_m^{2}(\langle x \rangle u_1)}{\langle x \rangle^{4n+4}} \langle x \rangle^{-1} \right)^{\frac{1}{2}} .
\]
Consider the ratio:

\[ R = \lim_{n \to \infty} \frac{(2n + 3)||u||_{2n+2}^{2n+2} \left( \int_{-\infty}^{1} \left( \frac{1}{(x)^{2n}} \right)^{2} dx \right)^{\frac{1}{2}} \left( \int_{-\infty}^{1} \frac{1}{(x)^{2n+2}} \left( \frac{1}{\sqrt{2}} + u_{2n+2} - u_{2n+1} + \ldots - u_{1} \right)^{2} du_{2n+2} \ldots du_{1} \right)}{(2n + 1)||u||_{2n+1}^{2n} \left( \int_{-\infty}^{1} \left( \frac{1}{(x)^{2n}} \right)^{2} dx \right)^{\frac{1}{2}} \left( \int_{-\infty}^{1} \frac{1}{(x)^{2n}} \left( \frac{1}{\sqrt{2}} + u_{2n} - u_{2n-1} + \ldots - u_{1} \right)^{2} du_{2n} \ldots du_{1} \right)} \]

So \( R \leq ||u||_{\infty}^{2} \) implies that if \( ||u||_{\infty}^{2} < 1 \) then \( \sum_{n=\left[ \frac{2n+1}{2} \right]}^{\infty} \left| \frac{d^m}{dx^m} C_{2n+1}^{-1}(u) \right| L_{N}^{2}\left((-\infty,-1)\right) \) also converges. So putting together these two infinite sums we have:

\[ \text{Conclusion: If} \max_{1} \left( \sup_{x \in (-\infty,t]} w_{i}(x) \right) < 1 \text{ then} \sum_{n=\left[ \frac{2n+1}{2} \right]}^{\infty} \left| C_{2n+1}^{-1}(u) \right| H_{m}^{2}\left((-\infty,-1)\right) \text{ converges.} \]

Hence we prove the following theorem:

**Theorem 5.** For any fixed \( T \in \mathbb{R} \) we have:

(a) If \( u \in H_{N}^{m,1} \cap H_{N}^{m,2} \) with \( ||u||_{H_{m}^{m,1}} \) small enough then \( C^{-}(u) \) and \( C^{+}(u) \) converge in \( H_{N}^{m,1} \cap H_{N}^{m,2} \) for \( m \geq 0 \) and \( N \geq 1 \) and \( C^{\pm}(u) \in H_{N}^{m,1} \cap H_{N}^{m,2} \).

(b) If \( u \in H_{N}^{m,2}\left((-\infty,T]\right) \) with \( ||u||_{H_{N}^{m,2}\left((-\infty,T]\right)} \) small enough then \( C^{-}(u) \) converge in \( H_{N}^{m,2}\left((-\infty,T]\right) \) for \( m \geq 0 \) and \( N \geq 1 \) and \( C^{-}(u) \in H_{N}^{m,2}\left((-\infty,T]\right) \).

(c) If \( u \in H_{N}^{m,2}\left([T,\infty)\right) \) with \( ||u||_{H_{N}^{m,2}\left([T,\infty)\right)} \) small enough then \( C^{+}(u) \) converge in \( H_{N}^{m,2}\left([T,\infty)\right) \) for \( m \geq 0 \) and \( N \geq 1 \) and \( C^{+}(u) \in H_{N}^{m,2}\left([T,\infty)\right) \).

(d) For \( u \in A_{H_{N}^{m,2}\left((-\infty,T]\right)} \) with \( \max_{1} \left( \sup_{x \in (-\infty,T]} w_{i}(x) \right) < 1 \) then \( C^{-}(u) \) converges in \( A_{H_{N}^{m,2}\left((-\infty,T]\right)} \) for \( m \geq 1 \) and \( N \geq 2 \) and \( C^{-}(u) \in A_{H_{N}^{m,2}\left((-\infty,T]\right)} \).

(e) For \( u \in A_{H_{N}^{m,2}\left([T,\infty)\right)} \) with \( \max_{1} \left( \sup_{x \in ([T,\infty)} w_{i}(x) \right) < 1 \) then \( C^{-}(u) \) converges in \( A_{H_{N}^{m,2}\left([T,\infty)\right)} \) for \( m \geq 1 \) and \( N \geq 2 \) and \( C^{+}(u) \in A_{H_{N}^{m,2}\left([T,\infty)\right)} \).
3.4 Convergence on the whole line and applications to well-posedness

The aim of this section is to prove that the power series defining the linearized transform \( C := C^+ \) converges in an open neighborhood \( \mathcal{U} \) of zero in the asymptotic space \( \mathbb{A}_N^{m,2}(\mathbb{R}) \) and defines a diffeomorphism

\[
C : \mathcal{U} \to \mathcal{V}
\]

where \( \mathcal{V} \) is an open neighborhood of zero in \( \mathbb{A}_N^{m,2}(\mathbb{R}) \), \( m \geq 1, N \geq 2 \). The main idea of the proof is based on the following commutative diagram (see e.g. [4]):

\[
\begin{array}{ccc}
X_1 & \rightarrow & X \\
\downarrow & & \downarrow \\
\hat{X}_1 & \rightarrow & \hat{X}_1
\end{array}
\]

where

\[
X := L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \quad X_1 := \{ u \in X | \| \hat{u} \|_{\infty} < 1 \},
\]

\( C^\pm \) are bi-analytic diffeomorphisms and \( I \) is an analytic involution (see Theorem 1.1 and Lemma 3.4 in [4]). Recall that \( \hat{u} := \mathcal{F}(u) \) where \( \mathcal{F} \) denotes the Fourier transform

\[
\mathcal{F}(u)(s) := \int_{-\infty}^{\infty} u(x) e^{-ixs} \, dx, \quad s \in \mathbb{R}.
\]

Note that \( X_1 \) is an open subset in \( X \). We postpone for a moment the definition of the map \( I : X_1 \to X_1 \) and concentrate on the main idea of the proof of the analyticity of (23). In order to further simplify the exposition, we will first prove that there exist open neighborhoods \( \mathcal{U} \) and \( \mathcal{V} \) of zero in \( H_N^{m,2}(\mathbb{R}) \), \( N \geq 1, m \geq 0 \) so that (23) is a diffeomorphism. As \( H_N^{m,2}(\mathbb{R}) \subseteq X \) is a continuous embedding this statement follows directly from items (b) and (c) of Theorem 5, together with the commutative diagram (24) and the following Proposition.

**Proposition 1.** There is an open neighborhood \( \mathcal{W} \) of zero in \( H_N^{m,2}(\mathbb{R}) \) such that \( I : \mathcal{W} \to \mathcal{W} \) is a real-analytic diffeomorphism.

**Proof.** Instead of working with the map \( I : X_1 \to X_1 \) we consider its Fourier conjugate

\[
\hat{I} : \hat{X}_1 \to \hat{X}_1, \quad \hat{I} := \mathcal{F} \circ I \circ \mathcal{F}^{-1},
\]

where \( \hat{X}_1 \subseteq L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) is the Fourier image of \( X_1 \). In order to prove Proposition 2 we prove the following equivalent statement: There exists an open neighborhood \( \hat{\mathcal{W}} \) of zero in \( H_N^{m,2}(\mathbb{R}) \) such that \( \hat{I} : \hat{\mathcal{W}} \to \hat{\mathcal{W}} \) is an analytic diffeomorphism. By definition, for any complex-valued function, \( r = r(s), s \in \mathbb{R}, r \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \| r \|_{\infty} < 1, \)

\[
\hat{I} : r \mapsto -\bar{r}(1 - r\bar{r}) (a(r))^2,
\]

where \( a(r) \) is a complex-valued function on \( \mathbb{R} \),

\[
(a(r))(s) := e^{-K_+ \log(1 - r\bar{r})}, \quad K_+ := \mathcal{F} \circ \chi_+ \circ \mathcal{F}^{-1},
\]

\[^1\]This section was done with the essential help of my advisor
Lemma 4. The lemma follows directly from the sequence of continuous embeddings, $\chi$ denote the complex conjugate of $r$ and $a(r)$ respectively.

Now, assume that $r \in H^{N,2}_m(\mathbb{R}) \subseteq \hat{X} := \mathcal{F}(X)$, $\|r\|_\infty < 1$. As $\|r\|_\infty < 1$, the power series

$$\log(1 - r\bar{r}) = -\sum_{k \geq 1} (r\bar{r})^k / k$$

(27)

converges in $L^\infty(\mathbb{R})$. As $H^{N,2}_m(\mathbb{R})$ is a Banach algebra, for any $k \geq 1$,

$$\|(r\bar{r})^k\|_{H^N_m} \leq C^k \|r\|_{H^N_m}^{2k}$$

with a constant $C > 0$ independent of the choice of $r \in H^{N,2}_m(\mathbb{R})$ and $k \geq 1$. Hence, for $\|r\|_{H^N_m} < 1/C$ the series (27) converges is $H^{N,2}_m(\mathbb{R})$. In view of the continuity of the embedding $H^{N,2}_m(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$ we see that $\log(1 - r\bar{r}) \in H^{N,2}_m(\mathbb{R})$ and that the mapping,

$$r \mapsto \log(1 - r\bar{r})$$

defines a real-analytic map $\mathcal{O} \to H^{N,2}_m(\mathbb{R})$ where $\mathcal{O}$ is an open neighborhood of zero in $H^{N,2}_m(\mathbb{R})$. Next, consider the Hardy’s projector $K_+ = \mathcal{F} \circ \chi_+ \circ \mathcal{F}^{-1}$ (see (26)). Lemma 4 below implies that the map,

$$r \mapsto -K_+ \left( \log(1 - r\bar{r}) \right), \quad \mathcal{O} \to H^N(\mathbb{R}),$$

is real-analytic. Hence, the mapping,

$$r \mapsto a(r) := e^{-K_+ \left( \log(1 - r\bar{r}) \right)}, \quad \mathcal{O} \to 1 + H^N(\mathbb{R}),$$

is real-analytic. In particular, the complex-valued function on the line $(a(r))(s)$ belongs to $1 + H^N(\mathbb{R})$ and $(a(r))(s) \neq 0$ for any $s \in \mathbb{R}$. In view of the discussion above,

$$a(r) = 1 + A_2(r), \quad \text{where} \quad \|A_2(r)\|_{H^N} \leq C \|r\|_{H^N_m},$$

for some positive constant $C > 0$ independent of $r \in \mathcal{O}$. This together with Lemma 5 below implies that $\hat{I} : \mathcal{O} \to H^{N,2}_m(\mathbb{R})$ is a real-analytic map and one has the following expansion

$$\hat{I}(r) = -\bar{r}(1 - r\bar{r})(1 + A_2(r)) = -\bar{r} - \bar{r}(A_2(r) - r\bar{r}) + r\bar{r}^2 A_2(r).$$

(28)

Hence, the linear part of $r \mapsto \hat{I}(r)$, $\mathcal{O} \to H^{N,2}_m(\mathbb{R})$, is given by the $\mathbb{R}$-linear isomorphism $r \mapsto -\bar{r}$, $H^{N,2}_m(\mathbb{R}) \to H^{N,2}_m(\mathbb{R})$. Finally, the statement of the Proposition follows from the inverse function theorem in Banach spaces. \hfill $\square$

Lemma 4. $K_+ : H^{N,2}_m(\mathbb{R}) \to H^N(\mathbb{R})$ is a bounded map.

Proof. The lemma follows directly from the sequence of continuous embeddings,

$$H^{N,2}_m(\mathbb{R}) \overset{\mathcal{F}^{-1}}{=} H^{m,2}_N(\mathbb{R}) \overset{\chi_+}{\longrightarrow} H^{0,2}_N(\mathbb{R}) \equiv L^2_N(\mathbb{R}) \overset{\mathcal{F}}{\equiv} H^N(\mathbb{R}).$$

$\square$
Lemma 5. The pointwise multiplication

$$(f, g) \mapsto f \cdot g, \quad H^{N,2}_m(\mathbb{R}) \times H^N(\mathbb{R}) \rightarrow H^{N,2}_m(\mathbb{R})$$

is a bounded bilinear map.

The proof of this lemma follows directly from the product rule and Lemma 6.4 in [6].

Summarizing the above and using the inverse function theorem in Banach spaces one more time we prove the following theorem.

Theorem 6. For any $m \geq 0, N \geq 1$, there exist open neighborhoods of zero $U$ and $V$ in $H^{m,2}_N(\mathbb{R})$ such that the linearization transform $C : U \rightarrow V$ is a diffeomorphism.

Following the lines of the proof of Proposition 1 one can prove:

There is an open neighborhood $W$ of zero in $\mathcal{A}^{m,2}_N(\mathbb{R})$ such that $I : W \rightarrow W$ is a real-analytic diffeomorphism.

This together with the commutative diagram (24) and items (d) and (e) of Theorem 5 would imply:

For any $m \geq 1, N \geq 2$, there exist open neighborhoods of zero $U$ and $V$ in $\mathcal{A}^{m,2}_N(\mathbb{R})$ such that the linearization transform $C : U \rightarrow V$ is a diffeomorphism.

As the scale of asymptotic spaces is preserved by the flow of the linear Airy equation $v_t = -v_{xxx}$, $v|_{t=0} = v_0$, we would see that the mKdV equation is locally well-posed in an open neighborhood of zero in the scale of asymptotic spaces $\mathcal{A}^{m,2}_N(\mathbb{R}), \ m \geq 1, \ N \geq 2.$
References


Appendix A

Recall the three algebraic identities:

\[ P_n(k_1, k_2, \ldots, k_n) = 3 \sum_{i=1}^{n-1} (k_1 + \ldots + k_i)(k_i + k_{i+1})(k_{i+1} + \ldots + k_n) \]

\[ P_n(k_1, k_2, \ldots, k_n) = 3 \sum_{i=2}^{n-1} (k_1 + \ldots + k_{i-1})(k_{i-1} + k_{i+1})(k_{i+1} + \ldots + k_n) \]

\[ + \sum_{i=2}^{n-1} (k_1 + \ldots + k_i)(k_{i-1} + k_{i+1})(k_i + \ldots + k_n) \]

And for \( n \) an odd positive integer we have also:

\[ P_n(k_1, k_2, \ldots, k_n) = 3 \sum_{i_1+i_2+i_3=n \atop i_j \geq 1 \atop i_j \text{ odd}} (k_1 + \ldots + k_{i_1} + k_{i_1+i_2+1} + \ldots + k_n)(k_{i_1} + k_{i_1+1})(k_{i_2} + k_{i_2+1}) \]

The way we are going to prove these identities is by comparing the monomials on both sides and see that they do appear with the same coefficients. It is clear that the monomials that appear on \( P_n(k_1, k_2, \ldots, k_n) = (k_1 + k_2 + \ldots + k_n)^3 - (k_1^3 + k_2^3 + \ldots + k_n^3) \) with the appropriate coefficients are:

3\( k_a^2 k_b \) and 6\( k_a k_b k_c \), where \( a \neq b \neq c \in 1, \ldots, n \).

**Proof of Identity (5)**

We start out by expanding the right hand side of identity (5).

\( (k_1 + \ldots + k_i)(k_i + k_{i+1})(k_{i+1} + \ldots + k_n) = k_i^2 (k_{i+1} + \ldots + k_n) + (k_1 + \ldots + k_i)k_{i+1}^2 + (k_1 + \ldots + k_{i-1})k_i(k_{i+1} + \ldots + k_n) + (k_1 + \ldots + k_i)k_{i+1}k_{i+2} + \ldots + k_n) \)

\( k_a^2 k_b \) are found in the first and the second summands.

\( k_a k_b k_c \) with \( a \neq b \neq c \) are found appearing in the third and the fourth summands clearly with a coefficient of 2.

In conclusion the factor of 3 appearing in front of the right hand side makes the identity true.

\( \square \)

**Proof of Identity (6)**

We start out by expanding the right hand side of identity (6). We sort the product in the following groups:

\( (I) \ i \text{ odd } (k_1 + \ldots + k_{i-1})k_{i-1}(k_{i+1} + \ldots + k_n) = k_{i-1}^2 (k_{i+1} + \ldots + k_n) + (k_1 + \ldots + k_{i-2})k_{i-1}(k_{i+1} + \ldots + k_n) \)
(II) \( i \) odd \((k_1 + \ldots + k_{i-1})k_{i+1}(k_{i+1} + \ldots + k_n) = (k_1 + \ldots + k_{i-1})k_{i+1}^2 + (k_1 + \ldots + k_{i-1})k_{i+1}(k_{i+2} + \ldots + k_n)\)

(III) \( i \) even \((k_1 + \ldots + k_i)k_{i-1}(k_1 + \ldots + k_n) = k_{i-1}k_i^2 + k_{i-1}(k_1 + \ldots + k_n) + (k_1 + \ldots + k_{i-2})k_{i-1}(k_i + \ldots + k_n) + k_{i-1}k_i(k_{i+1} + \ldots + k_n)\)

(IV) \( i \) even \((k_1 + \ldots + k_i)k_{i+1}(k_1 + \ldots + k_n) = k_i k_{i+1}^2 + (k_1 + \ldots + k_i)k_{i+1}^2 + (k_1 + \ldots + k_i)k_{i+1}(k_{i+2} + \ldots + k_n) + (k_1 + \ldots + k_{i-1})k_i k_{i+1}\)

Now we identify the monomials \( k_a k_b k_c \) case by case where there are in these 4 products:

\( k_a^2 k_b \) where \( a \) is even are found in the first summands of the products (I), (II), (III) and (IV).

\( k_a^2 k_b \) where \( a \) is odd are found in the second summands of the products (III) and (IV).

\( k_a k_b k_c \) with \( a \neq b \neq c \) with \( b \) even are found in the second summands of the products (I) and (II), and in the forth summand of the products (III) and (IV). They each clearly appear with a coefficient of 2.

\( k_a k_b k_c \) with \( a \neq b \neq c \) with \( b \) odd are found in the third summands of the products (III) and (IV). They each clearly appear with a coefficient of 2.

In conclusion the factor of 3 appearing in front of the right hand side of identity (6) makes the identity true. \( \square \)

**Proof of identity (7)**

The right side of identity (7) is a sum of products

\((k_1 + \ldots + k_i + k_{i_1+i_2+1} + \ldots k_n)(k_{i_1} + k_{i_1+1})(k_{i_2} + k_{i_2+1})\)

where the sum runs over partitions \( i_1 + i_2 + i_3 = n \) of \( n \) where \( i_j \geq 1 \) and \( i_j \) odd.

We are now going to investigate the monomials that come out of each of these products depending on the partition \( i_1 + i_2 + i_3 = n \). By multiplying out we break the above product in 8 type of simpler products:

(I) \((k_1 + \ldots + k_i)k_{i_1}k_{i_1+i_2}\)

(II) \(k_{i_1}k_{i_1+i_2}(k_{i_1+i_2+1} + \ldots + k_n)\)

(III) \((k_1 + \ldots + k_i)k_{i_1}k_{i_1+i_2+1}\)

(IV) \(k_{i_1}k_{i_1+i_2+1}(k_{i_1+i_2+1} + \ldots + k_n)\)

(V) \((k_1 + \ldots + k_{i_1})k_{i_1+1}k_{i_1+i_2}\)
(VI) \(k_{i_1+1}k_{i_1+i_2}(k_{i_1+i_2+1} + \ldots + k_n)\)

(VII) \((k_1 + \ldots + k_{i_1})k_{i_1+1}k_{i_1+i_2}\)

(VIII) \(k_{i_1+1}k_{i_1+i_2}(k_{i_1+i_2+1} + \ldots + k_n)\)

Now we examine the occurrence of the monomial of the type \(k_\alpha k_\beta k_\gamma\) in each of our products types. We focus our attention on the parity of the middle variable \(k_\beta\) and the differences \(b - a\) and \(c - b\). Here is the summary of these data:

(I) \(k_\alpha k_\beta k_\gamma\) where \(b = i_1\) odd, \(0 \leq b - a \leq i_1 - 1\), \(c - b = i_2\) odd

(II) \(k_\alpha k_\beta k_\gamma\) where \(b = i_1 + i_2\) even, \(b - a = i_2\) odd, \(1 \leq c - b \leq i_3\)

(III) \(k_\alpha k_\beta k_\gamma\) where \(b = i_1\) odd, \(0 \leq b - a \leq i_1 - 1\), \(c - b = i_2 + 1\) even

(IV) \(k_\alpha k_\beta k_\gamma\) where \(b = i_1 + i_2 + 1\) odd, \(b - a = i_2 + 1\) even, \(0 \leq c - b \leq i_3 - 1\)

(V) \(k_\alpha k_\beta k_\gamma\) where \(b = i_1 + 1\) even, \(1 \leq b - a \leq i_1\), \(c - b = i_2 - 1\) even

(VI) \(k_\alpha k_\beta k_\gamma\) where \(b = i_1 + i_2\) even, \(b - a = i_2 - 1\) even, \(1 \leq c - b \leq i_3\)

(VII) \(k_\alpha k_\beta k_\gamma\) where \(b = i_1 + 1\) even, \(1 \leq b - a \leq i_1\), \(c - b = i_2\) odd

(VIII) \(k_\alpha k_\beta k_\gamma\) where \(b = i_1 + i_2 + 1\) odd, \(b - a = i_2\) odd, \(0 \leq c - b \leq i_3 - 1\)

There are two cases of \(k_\alpha^2 k_\beta\):

When \(a\) is even the monomials \(k_\alpha^2 k_\beta\) appears in part (V) and (VI) where the partition has \(i_2 = 1\) as \(i_1\) runs through all the possible odds.

When \(a\) is odd the monomials \(k_\alpha^2 k_\beta\) all appear on (I) and (III) when \(a \leq b\) and on (IV) and (VII) when \(a \geq b\).

These two cases take care of the monomial of the form \(k_\alpha^2 k_\beta\). Now we focus on monomials of the form \(k_\alpha k_\beta k_\gamma\) with \(a \neq b \neq c\).

We are going to illustrate how we find in our 8 types of products certain type of monomial. Let’s assume \(b\) is odd, \(b - a\) odd and \(c - b\) odd. It appears once on (I) when \(i_1 = b\), \(i_1 + i_2 = c\) and once more on (VIII) when \(i_1 + 1 = a\), \(i_1 + i_2 + 1 = b\). So its clear that it appears with coefficient 2. The same is true for the other cases. They all appear with coefficient 2.

In conclusion the factor of 3 in front of the right hand side makes the identity true. \(\square\)
Appendix B

Auxiliary Identity

Our aim in this section is to derive a formula that describes the behavior of the Fourier transforms of the kernels of the convolution maps under composition. The question is when we have the following composition:

\[ M_{m+n}(u_1 \otimes ... \otimes u_m \otimes u_{m+1} \otimes ... \otimes u_{m+n}) := M_{n+1}(M_m(u_1 \otimes ... \otimes u_m) \otimes u_{m+1} \otimes ... \otimes u_{m+n}) \]

Where here \( M_k \) are k-linear maps that are in the convolution form described above with the kernels corresponding to them being \( H_k \). The question here is how are \( \hat{H}_k \) (the Fourier transforms of the kernels \( H_k \)) related to each other.

By definition this means:

\[ M_{m+n}(u_1 \otimes ... \otimes u_{m+n})(x) = \left[ \tau \hat{H}_{m+n} * (u_1 \otimes ... \otimes u_{m+n}) \right]_{(x,...x)} \]

Denote: \( k = (k_1,...k_{m+n}), \overline{k_1} = (k_1,...k_m), \overline{k_2} = (k_{m+1},...,k_{m+n}) \)

So taking Fourier transform of the left hand side we get:

\[ \hat{M}_{m+n}(u_1 \otimes ... \otimes u_{m+n})(k) = \int_{-\infty}^{\infty} e^{-2\pi ikx} M_{m+n}(u_1 \otimes ... \otimes u_{m+n})(x) dx \]

Now we replace \( M_{m+n}(u_1 \otimes ... \otimes u_{m+n})(x) \) by its convolution form:

\[ = \int_{-\infty}^{\infty} e^{-2\pi ikx} \left( \int_{y \in \mathbb{R}^n} \hat{H}_{m+n}(-y_1,...,-y_{m+n})u_1(x+y_1)...u_{m+n}(x+y_{m+n}) dy \right) dx \]

Using Fubini’s theorem to switch the order of integration we get:

\[ = \int_{y \in \mathbb{R}^n} \hat{H}_{m+n}(-y_1,...,-y_{m+n}) \left( \int_{-\infty}^{\infty} e^{-2\pi ikx} u_1(x+y_1)...u_{m+n}(x+y_{m+n}) dx \right) dy \]

Using the fact that the Fourier transform of multiplication of functions is convolution of their Fourier transform we get:

\[ = \int_{y \in \mathbb{R}^n} \hat{H}_{m+n}(-y_1,...,-y_{m+n}) \left( e^{2\pi i k y_1} \hat{u}_1(k) * ... * e^{2\pi i k y_{m+n}} \hat{u}_{m+n}(k) \right) dy \]

Now we rewrite this convolution in integral form:

\[ = \int_{y \in \mathbb{R}^n} \hat{H}_{m+n}(-y_1,...,-y_{m+n}) \left( \int_{S: k_1+...+k_{m+n}=k} e^{2\pi i k_1 y_1} \hat{u}_1(k_1) ... e^{2\pi i k_{m+n} y_{m+n}} \hat{u}_{m+n}(k_{m+n}) dS \right) dy \]
We switch the order on integration once more:

\[
\int_{S: k_1 + \ldots + k_{m+n} = k} \widehat{u_1(k_1)} \ldots \widehat{u_{m+n}(k_{m+n})} \left( \int_{y \in \mathbb{R}^n} H_{m+n}(-y_1, \ldots, -y_{m+n}) e^{2\pi i k_1 y_1} \ldots e^{2\pi i k_{m+n} y_{m+n}} dy \right) dS
\]

So we finally get:

\[
\bar{M}_{m+n}(u_1 \otimes \ldots \otimes u_{m+n})(k) = \int_{S: k_1 + \ldots + k_{m+n} = k} \widehat{H_{m+n}(k_1, k_2)} \widehat{u_1(k_1)} \ldots \widehat{u_{m+n}(k_{m+n})} dS
\]

Now taking the Fourier transform of the right hand side we get:

\[
\widehat{M}_{n+1}(M_m(u_1 \otimes \ldots \otimes u_m) \otimes u_{m+1} \otimes \ldots \otimes u_{m+n})(k) = e^{-2\pi i k x} M_{n+1}(M_m(u_1 \otimes \ldots \otimes u_m) \otimes u_{m+1} \otimes \ldots \otimes u_{m+n})(x) dx
\]

Using the formula we got from transforming the left hand side we immediately get:

\[
= \int_{S_1: t + k_{m+1} + \ldots + k_{m+n} = k} \widehat{H_{n+1}(t, -k_2)} M_m(u_1 \otimes \ldots \otimes u_m)(t) \widehat{u_{m+1}(k_{m+1})} \ldots \widehat{u_{m+n}(k_{m+n})} dS_1
\]

\[
= \int_{S_1: t + k_{m+1} + \ldots + k_{m+n} = k} \widehat{H_{n+1}(t, -k_2)} \left( \int_{S_2: k_1 + \ldots + k_m = t} \widehat{H_m(-k_1)} \widehat{u_1(k_1)} \ldots \widehat{u_m(k_m)} dS_2 \right) \widehat{u_{m+1}(k_{m+1})} \ldots \widehat{u_{m+n}(k_{m+n})} dS_1
\]

Combining the region of integration we get:

\[
= \int_{S: k_1 + \ldots + k_{m+n} = k} \widehat{H_{n+1}(-k_1 + \ldots + k_m, -k_2)} \widehat{H_m(-k_1)} \widehat{u_1(k_1)} \ldots \widehat{u_{m+n}(k_{m+n})} dS
\]

Since:

\[
M_{m+n}(u_1 \otimes \ldots \otimes u_m \otimes u_{m+1} \otimes \ldots \otimes u_{m+n}) = M_{n+1}(M_m(u_1 \otimes \ldots \otimes u_m) \otimes u_{m+1} \otimes \ldots \otimes u_{m+n}) \quad \forall u_1, \ldots, u_{m+n} \in \mathbb{E}
\]

Then

\[
\int_{S: k_1 + \ldots + k_{m+n} = k} \widehat{H_{m+n}(k_1, k_2)} \widehat{u_1(k_1)} \ldots \widehat{u_{m+n}(k_{m+n})} dS
\]

\[
= \int_{S: k_1 + \ldots + k_{m+n} = k} \widehat{H_{n+1}(-k_1 + \ldots + k_m, -k_2)} \widehat{H_m(-k_1)} \widehat{u_1(k_1)} \ldots \widehat{u_{m+n}(k_{m+n})} dS
\]

Must hold for all \( u_1, \ldots, u_{m+n} \in \mathbb{E} \). So we can set equal the inner parts of the integrals of the Fourier transforms of the left hand side and the right hand side. So in conclusion we get that if:

\[
M_{m+n}(u_1 \otimes \ldots \otimes u_m \otimes u_{m+1} \otimes \ldots \otimes u_{m+n}) := M_{n+1}(M_m(u_1 \otimes \ldots \otimes u_m) \otimes u_{m+1} \otimes \ldots \otimes u_{m+n})
\]

such that \( M_k \) is in the convolution form described in the previous section then the Fourier transforms of their corresponding functionals are related in the following way:

\[
\widehat{H_{m+n}(k_1, k_2)} = \widehat{H_{n+1}(-k_1 + \ldots + k_m, -k_2)} \widehat{H_m(-k_1)}
\]