Convex Polytopes and Tilings with Few Flag Orbits

by Nicholas Matteo

B.A. in Mathematics, Miami University
M.A. in Mathematics, Miami University

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Egon Schulte
Professor of Mathematics
Abstract of Dissertation

The amount of symmetry possessed by a convex polytope, or a tiling by convex polytopes, is reflected by the number of orbits of its flags under the action of the Euclidean isometries preserving the polytope. The convex polytopes with only one flag orbit have been classified since the work of Schl"afli in the 19th century. In this dissertation, convex polytopes with up to three flag orbits are classified.

Two-orbit convex polytopes exist only in two or three dimensions, and the only ones whose combinatorial automorphism group is also two-orbit are the cuboctahedron, the icosidodecahedron, the rhombic dodecahedron, and the rhombic triacontahedron. Two-orbit face-to-face tilings by convex polytopes exist on $\mathbb{E}^1$, $\mathbb{E}^2$, and $\mathbb{E}^3$; the only ones which are also combinatorially two-orbit are the trihexagonal plane tiling, the rhombille plane tiling, the tetrahedral-octahedral honeycomb, and the rhombic dodecahedral honeycomb. Moreover, any combinatorially two-orbit convex polytope or tiling is isomorphic to one on the above list.

Three-orbit convex polytopes exist in two through eight dimensions. There are infinitely many in three dimensions, including prisms over regular polygons, truncated Platonic solids, and their dual bipyramids and Kleetopes. There are infinitely many in four dimensions, comprising the rectified regular 4-polytopes, the $p, p$-duoprisms, the bitruncated 4-simplex, the bitruncated 24-cell, and their duals. There are only finitely many other examples, comprising the Gosset polytopes $k_{21}$ and their duals in five through eight dimensions, and the polytopes $k_{22}$ and their duals in five or six dimensions.

The polygons in the plane with any number $k$ of flag orbits are described. These are certain $pk$-gons, and also certain $\frac{pk}{2}$-gons if $k \geq 4$ is even, for any $p$ so that the number of sides is at least three.

The resulting flag orbits from various standard operations used to construct new polytopes from others are also discussed. Finally, it is proven that for any $k$, the polytopes with $k$ flag orbits occur in only finitely many dimensions, and that a certain class of $d$-dimensional polytopes (including most uniform polytopes) is either regular or has at least $d − 1$ flag orbits.
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CHAPTER I

Introduction

The amount of symmetry possessed by a convex polytope, or a tiling by convex polytopes, is reflected by the number of orbits of its flags under the action of the Euclidean isometries preserving the polytope. The polytopes with the fewest such orbits are the “most symmetric”. The convex polytopes with only one flag orbit have been classified since the work of Schläfli in the 19th century. In this dissertation, a classification is given for convex polytopes with up to three flag orbits. A description of the polygons in the plane with any number $k$ of flag orbits is also given. The resulting flag orbits from various standard operations used to construct new polytopes from others are also discussed. Finally, it is proven that for any $k$, the polytopes with $k$ flag orbits occur in only finitely many dimensions, and that a certain class of $d$-dimensional polytopes (including most uniform polytopes) is either regular or has at least $d - 1$ flag orbits.

A reference to a section in the current chapter appears as §1; a reference to a section in another chapter always includes the chapter number in Roman numerals, such as §II.1. A reference to an item in the current chapter, such as the first lemma in §4, has the form “Lemma IV”. A reference to another chapter, for instance the first proposition in §II.4, has the form “Proposition II.1”.

1. Convex polytopes

Convex polytopes and their symmetries are some of the oldest objects of mathematical study. The oldest extant work of axiomatic mathematics, Euclid’s *Elements*, is concerned with the construction of the Platonic solids (or as we will call them, the one-orbit 3-polytopes). Through the ensuing millennia, highly symmetric figures in the plane or 3-space, and highly symmetric tilings using these as building blocks, have been of enduring interest to artists
and mathematicians alike. In the past two centuries, mathematicians have extended the investigation into higher dimensions.

Over this time, polytopes have been given many definitions. Here, a *convex polytope* is the convex hull of finitely many points in \(d\)-dimensional Euclidean space \(\mathbb{E}^d\) (Grünbaum 2003). These are the objects which (in 2 or 3 dimensions) would have been familiar to the ancient Greeks. Throughout this dissertation, the word “polytope” without qualification means “convex polytope.” A *\(d\)-polytope* is one whose affine hull is \(d\)-dimensional.

Some authors define a *(convex) polyhedron* as the intersection of finitely many closed half-spaces in \(\mathbb{E}^d\), or equivalently, as the solution set to a finite system of linear inequalities:

\[
\{ x \in \mathbb{E}^d \mid Ax \leq b \},
\]

where \(A\) is an \(m \times d\) matrix, \(b \in \mathbb{E}^m\), and for two vectors \(a = (a_1, \ldots, a_m)^T\) and \(b = (b_1, \ldots, b_m)^T\), \(a \leq b\) if and only if \(a_i \leq b_i\) for each \(i\).

This is a convex object, but is not necessarily bounded. These authors define “(convex) polytope” to mean a bounded convex polyhedron, or equivalently, a compact convex polyhedron. This is equivalent to our earlier definition.

However, we will not use this definition of polyhedron: throughout the sequel, *polyhedron* means 3-polytope, and *polygon* means 2-polytope.

A *(proper) face* of a polytope \(P\) is the intersection of \(P\) with a *supporting hyperplane* \(H\), i.e. a hyperplane such that \(H \cap P\) is not empty, and \(P\) is contained in one of the closed half-spaces determined by \(H\). In addition to these proper faces, we admit two *improper faces*: the *empty face*, \(\emptyset\), and \(P\) itself. With the inclusion of these improper faces, the faces of \(P\), ordered by containment, form a lattice: the *face lattice* of \(P\), denoted \(\mathcal{L}(P)\). A \(j\)-face is a face whose affine hull is \(j\)-dimensional; we call 0-faces *vertices*, 1-faces *edges*, \((d-3)\)-faces *subridges*, \((d-2)\)-faces *ridges*, and \((d-1)\)-faces *facets*. The empty face is considered a \((-1)\)-face. Two faces are said to be *incident* if one contains the other.

Other definitions of “polytope” abandon convexity, and have some other data to specify what the faces are. For instance, a star-polygon, such as a pentagram, consists of vertices in
the plane and edges which are straight line segments connecting the vertices, but the edges have intersections with other edges in their interiors. Alternatively, edges can be given as straight line segments between vertices with no intersections, but with the vertices in $\mathbb{E}^3$ (or higher dimensions) not lying in any 2-dimensional subspace; this is called a skew polygon. Allowing such polygons as faces or vertex figures permits the construction of star-polyhedra, such as the Kepler-Poinsot polyhedra, and the Petrial polyhedra.

2. Tilings

A tessellation of a space $X$ is a collection $\mathcal{T}$ of subsets, called tiles, which cover the space without gaps or overlaps: i.e. such that the union of the tiles in $\mathcal{T}$ is all of $X$, and the intersection of the interiors of any two distinct tiles is empty. We shall be concerned solely with tessellations whose tiles are convex polytopes and which are face-to-face, meaning that the intersection of any two tiles is a face of each (possibly the empty face). We will also only consider locally finite tessellations, meaning that every point in $X$ has a neighborhood meeting only finitely many tiles (equivalently, every compact subset of $X$ meets only finitely many tiles). For these tessellations we will use the word tiling, which is usually a synonym for tessellation. Let us emphasize this.

**Definition.** In this dissertation, tiling means a locally finite, face-to-face tessellation by convex polytopes.

Another common synonym is honeycomb, which is used particularly for tilings of three dimensions. We will use this word primarily when it has become part of the traditional name for a particular tiling.

A convex $d$-polytope can be considered as a tiling of the $(d-1)$-sphere $S^{d-1}$ by convex $(d-1)$-polytopes. This is probably closer to how the ancient Greeks conceived of polyhedra: as an assembly of polygons, attached edge-to-edge, until they form a closed surface. There are some subtly different ways of interpreting this definition. We can consider the boundary of a $d$-polytope in $\mathbb{E}^d$; this is a $(d-1)$-dimensional surface, homeomorphic to a $(d-1)$-sphere,
tiled by \((d-1)\)-polytopes. Alternatively, we can project the boundary of the \(d\)-polytope, from a point in the interior of the polytope, onto a sphere containing it. This results in a tiling of the \((d-1)\)-sphere whose tiles are “spherical convex polytopes”: these appear curvilinear. They are a convex hull of finitely many points in spherical space; the edges are segments of geodesics in the sphere.

Generalizing this, we can regard tilings of other spaces as polytopes. Most historical interest has been in tilings of Euclidean space \(\mathbb{E}^d\) by \(d\)-polytopes (particularly for \(d \in \{2, 3\}\)). We will also consider tilings of hyperbolic space \(\mathbb{H}^d\). Many of the properties of convex polytopes essentially depend on the boundary being a simply-connected space; thus, many results generalize to tilings of any simply-connected space of constant curvature. By scaling the metric, we can assume the curvature is either 1, 0, or \(-1\), and the space is \(S^d\), \(\mathbb{E}^d\), or \(\mathbb{H}^d\), respectively.

Considered this way, some restrictions on convex polytopes start to seem rather artificial. One may tile the circle \(S^1\) by two line segments (that is, split it into two arcs), giving a “digon”; this cannot be realized as a traditional convex polygon in \(\mathbb{E}^2\), but is nothing exceptional as a tiling of \(S^1\). If we allow digons as tiles, then instead of five regular polyhedra, we have infinitely many more: a tiling can be formed on \(S^2\) with two vertices, at the north and south poles, and \(n\) digonal “lunes”, for any \(n \geq 2\).

3. Abstract polytopes

The combinatorial aspects that all of these definitions of “polytope” have in common are captured by the definition of an abstract polytope. An abstract polytope is a ranked, partially ordered set. The elements are called faces, and the maximal chains of faces are called flags. Two flags are said to be adjacent if they differ in exactly one face; if they differ in the face of rank \(j\), they are said to be \(j\)-adjacent. An abstract polytope of rank \(d\) must satisfy these four axioms (McMullen and Schulte [2002]):

(P1) There is a least face \(F_{-1}\), the empty face (of rank \(-1\)), and a greatest face \(F_d\) (of rank \(d\)).
Every flag contains $d + 2$ faces.

(Strong flag-connectivity:) For any two flags $\Phi$ and $\Psi$, there exists a sequence of flags $\Phi =: \Phi_0, \Phi_1, \ldots, \Phi_k := \Psi$, such that each flag is adjacent to its neighbors and $\Phi \cap \Psi \subseteq \Phi_i$ for each $i$.

(The diamond condition:) For any $j$, $1 \leq j \leq d$, any $j$-face $G$, and any $(j - 2)$-face $F$ contained in $G$, there are exactly two faces $H$ (of rank $j - 1$) such that $F < H < G$.

The diamond condition ensures that for each flag $\Phi$ there is a unique $i$-adjacent flag, and we denote it $\Phi^i$.

The face lattice of any convex polytope $P$ satisfies these properties (McMullen and Schulte 2002 p. 22), and we will identify $P$ with the abstract polytope consisting of $\mathcal{L}(P)$. More generally, the face lattice of a tiling of $d$-dimensional space meets the criteria defining an abstract polytope of rank $d + 1$, and we call it a rank $(d + 1)$ tiling.

4. Symmetry and flags

The symmetries of a convex polytope $P$ are the Euclidean isometries which carry $P$ to itself. These form a group, the symmetry group of $P$, denoted $G(P)$. Similarly, the automorphisms of an abstract polytope $K$ are the order-preserving bijections from $K$ to itself; these form the automorphism group of $K$, denoted $\Gamma(K)$. For a convex polytope $P$, we use $\Gamma(P)$ to denote the automorphism group of $P$ considered as an abstract polytope. This group, consisting of the lattice isomorphisms from $\mathcal{L}(P)$ to itself, might more pedantically be called $\Gamma(\mathcal{L}(P))$. Since each transformation in $G(P)$ acts as an automorphism of $\mathcal{L}(P)$, we can consider $G(P)$ as a subgroup of $\Gamma(P)$.

Definition. A flag of a $d$-polytope $P$ is a set of $d + 2$ mutually incident faces. Thus, a flag consists of the empty face, a vertex, an edge containing the vertex, a 2-face containing the edge, and so on, all the way up to $P$ itself.

The name flag is suggested by the way that a 2-face hung on a line segment erected over a vertex resembles a flag hanging on a flagpole. The flags of $P$ are exactly the maximal chains.
in $\mathcal{L}(P)$ (i.e. the maximal linearly ordered sets of faces). The set of all flags of $P$ is denoted $\mathcal{F}(P)$. Transformations in $G(P)$ (or automorphisms in $\Gamma(P)$) induce an action on $\mathcal{F}(P)$ in an obvious way.

It is frequently useful to associate with each flag $\Phi$ of a $d$-polytope $P$ a chamber, which is a $(d - 1)$-simplex in the boundary of $P$, whose vertices are interior points of each of the proper faces in $\Phi$. If we preassign some interior point to each face of $P$, then the chambers for the flags of $P$ form a tiling of the boundary of $P$, called a barycentric subdivision of $P$. (Often, some particular choice of interior points is preferable, such as the centroids mentioned below.)

Thus, we have three different ways to think of flags:

- As a collection of faces of the polytope $P$.
- As a chain snaking up the Hasse diagram of $\mathcal{L}(P)$.
- As a chamber in the boundary of $P$.

We will switch freely between these viewpoints.

The symmetry group of a convex polytope (and more generally, any group of automorphisms of an abstract polytope) acts freely (or “semiregularly”) on the set of its flags. That is, there is at most one automorphism carrying a given flag to another one; equivalently, the stabilizer of any flag is trivial.

**Lemma 1.** For any automorphism $\gamma \in \Gamma(P)$ and any flag $\Phi$, $\gamma(\Phi^i) = \gamma(\Phi)^i$ for each $i$.

**Proof.** For every $j$ not equal to $i$, $\Phi$ and $\Phi^i$ have the same $j$-face. So $\gamma(\Phi^i)$ has the same $j$-face as $\gamma(\Phi)$ and $\gamma(\Phi)^i$. Since $\Phi^i$ has a different $i$-face than $\Phi$ does and $\gamma$ is a bijection, $\gamma(\Phi^i)$ has a different $i$-face than $\gamma(\Phi)$. But this $i$-face contains the $(i - 1)$-face of $\gamma(\Phi)$, and is contained in the $(i + 1)$-face of $\gamma(\Phi)$; by the diamond property, there is a unique such $i$-face other than the $i$-face of $\gamma(\Phi)$. Hence, $\gamma(\Phi^i) = \gamma(\Phi)^i$. $\square$

**Proposition 2.** Any group of automorphisms of a polytope $P$ acts freely on $\mathcal{F}(P)$. In particular, $G(P)$ acts freely on $\mathcal{F}(P)$.
Proof. Let $\Phi$ be any flag of $P$. If $\gamma \in \Gamma(P)$ is such that $\gamma(\Phi) = \Phi$, then $\gamma(\Phi^i) = \Phi^i$ for every flag adjacent to $\Phi$. Thus, by induction, $\gamma$ fixes all flags of $P$ by flag-connectedness. So $\gamma$ is the identity. \qed

The orbits of flags under the action of $G(P)$ are called flag orbits. A polytope with $k$ distinct flag orbits is called a $k$-orbit polytope. If we wish to emphasize that the action is by Euclidean isometries, we will say that $P$ is geometrically $k$-orbit. Orbits of flags under the action of the automorphism group $\Gamma(P)$ are called combinatorial flag orbits, and a polytope with $k$ such orbits is called combinatorially $k$-orbit. In the context of abstract polytopes, this is the only definition possible and the adjectives may be dropped.

In general, a polytope has more combinatorial automorphisms of its face lattice than it has Euclidean symmetries. Hence, if the symmetry group has $k$ flag orbits and the automorphism group has $m$ flag orbits, then $m \leq k$; in fact $m | k$.

**Corollary 3.** The order of the symmetry group $G(P)$ divides the number of flags of $P$. Each flag orbit has the same size, namely $|G(P)|$, and so $P$ is a $k$-orbit polytope if and only if the number of flags is $k|G(P)|$.

Conway, Burgiel, and Goodman-Strauss (2008, p. 273) introduce the term flag rank for the number of flag orbits, and also suggest that a $k$-orbit polytope be called $\frac{1}{k}$-regular. In this dissertation we shall determine all the half-regular and $\frac{1}{3}$-regular convex polytopes.

### 5. One-orbit polytopes

One-orbit polytopes are the regular polytopes. These are among the oldest objects of mathematical study. It is well known (Coxeter 1973) that there are infinitely many regular polygons, namely the regular $p$-gon for each $p \geq 3$. These one-orbit polygons have been intensively studied from the time of the ancient Babylonians until Gauss and Wantzel characterized the constructible regular polygons.

We will use some of the geometric properties of the one-orbit polygons. Every vertex is congruent to every other, and the internal angle at every vertex is $\frac{p-2}{p} \pi$. Every edge is
the same length. If the edge length is 1, then the circumference (the distance of the vertices from the centroid of the polygon) is \( \frac{1}{2 \sin \frac{\pi}{p}} \). On the other hand, if we choose \( p \) equally spaced points on the unit circle to be our vertices, for example the points \( e^{2\pi li/p} \) in the complex plane with \( l = 1, \ldots, p \), then the circumference is 1 and the edge length is \( 2 \sin \frac{\pi}{p} \).

There are three one-orbit \( d \)-polytopes which appear in every dimension \( d \geq 3 \). These are:

- The regular \( d \)-simplex, which is the convex hull of \( d + 1 \) mutually equidistant points (which is the maximum number of mutually equidistant points in \( \mathbb{E}^d \)). Coxeter (1973, §7.2) gives the regular \( d \)-simplex the abbreviated name \( \alpha_d \).
- The \( d \)-crosspolytope, also called the \( d \)-orthoplex. This is the convex hull of \( 2d \) points, with two placed on each of \( d \) mutually perpendicular lines through a point \( o \), each point the same distance from \( o \). One possibility for the vertices are \( \pm e_i \) where \( e_1, \ldots, e_d \) is the standard basis of \( \mathbb{E}^d \). Coxeter gives the \( d \)-crosspolytope the abbreviated name \( \beta_d \).
- The \( d \)-cube, also called a \( d \)-orthotope, a hyper-cube, or a measure polytope. This can be taken as the convex hull of the \( 2^d \) points \( \pm 1, \ldots, \pm 1 \) in \( \mathbb{E}^d \). Coxeter gives the \( d \)-cube the abbreviated name \( \gamma_d \).

In three dimensions, there are two additional one-orbit polytopes, for a total of five regular polyhedra known as the Platonic solids. The 3-simplex is the tetrahedron. The 3-cube is simply called the cube. The 3-crosspolytope is the octahedron. The additional Platonic solids are the dodecahedron and the icosahedron. Although each of these names, with the exception of the cube, could be taken to refer to a wide variety of polyhedra with the appropriate number of facets (e.g., “octahedron” can refer to any polyhedron with eight facets), we will only use these unqualified names to refer to the regular polytopes. The construction of these one-orbit polyhedra and proof that there are no others is the subject of Euclid’s Elements.

In four dimensions, there are three additional one-orbit polytopes, for a total of six. These are the 24-cell, which consists of 24 octahedral facets; the 120-cell, which has 120 dodecahedral facets; and the 600-cell, which has 600 tetrahedral facets.
In five or more dimensions, there are no additional one-orbit polytopes! The classification of the one-orbit $d$-polytopes for $d \geq 4$ was accomplished by Ludwig Schlafli, and independently by several others, in the 19th century (Coxeter [1973]).

6. Symmetry equivalence

There are a variety of natural equivalence relations on polytopes. Our classifications will generally be up to *symmetry equivalence*, but will also indicate the possible variations up to *similarity*. We define both of these equivalence relations, and several others.

Two polytopes are *isomorphic* if there is an isomorphism between their face lattices.

We say that two isomorphic polytopes $P$ and $Q$ are *symmetry equivalent* if there is an isomorphism $f: \mathcal{L}(P) \to \mathcal{L}(Q)$ and a group isomorphism $h: G(P) \to G(Q)$ such that $f(\gamma(F)) = h(\gamma)(f(F))$ for every symmetry $\gamma \in G(P)$ and every face $F \in \mathcal{L}(P)$.

Symmetry equivalence is considerably stricter than isomorphism: it requires the two polytopes to have the same amount of symmetry, and to have the symmetry groups act in the same way. Thus, a cube and a rectangular box are isomorphic, but not symmetry equivalent: the cube has 48 symmetries, and the box only 16. A rectangle and a rhombus are isomorphic, and even have isomorphic symmetry groups (both are $C_2 \times C_2$), but are not symmetry equivalent since the symmetry groups act in different ways (the size of the vertex orbits differ, for example).

Two polytopes $P$ and $Q$ are *similar* if $Q$ can be obtained from $P$ by an isometry followed by a scale transformation.

Even more strictly, $P$ and $Q$ are *congruent* if there is an isometry mapping $P$ onto $Q$, and *directly congruent* if this is a direct isometry (i.e. an orientation-preserving isometry). Frequently, there is no difference between these cases. There can only be a distinction when $P$ is not directly congruent to any of its own reflections, in which case $P$ is said to posses an *enantiomorphic form*. In older terminology, such polytopes were sometimes said to be *chiral*, but more recently this term is reserved for two-orbit polytopes such that adjacent flags are always in different flag orbits.
Since, in general, we attach no significance to the scale of our spaces, similarity is sufficient for us to say that two polytopes are the same. Our classification will be up to symmetry equivalence, but when there is a family of non-similar polytopes realizing the symmetry equivalence class, we will mention this.

Very frequently for few-orbit polytopes, any two polytopes which are symmetry equivalent are also similar. Such polytopes are called perfect (Gévay 2002; Robertson 1984).

The various equivalences we have defined form a hierarchy of partitions of the space of convex $d$-polytopes. The whole space can be partitioned into isomorphism classes. Each isomorphism class is then partitioned into symmetry equivalence classes. Each symmetry equivalence class is partitioned into similarity classes. There is a unique similarity class in a symmetry equivalence class if and only if the class is perfect. A similarity class is partitioned into congruence classes. (This partition can be parameterized by a real number in $(0, \infty)$, the scale parameter.) Finally, each congruence class can be partitioned into at most two classes of directly congruent polytopes.

7. Organization

The remainder of this chapter is concerned with definitions and basic properties of polytopes and tilings which will be used in the remainder of the dissertation.

Chapter II defines orbit graphs, which show the relationship between the flag orbits of a polytope. Restrictions on the potential orbit graphs of convex polytopes will be found, enabling a complete enumeration of all the potential orbit graphs for polytopes with few flag orbits.

Chapter III discusses standard operations used to construct polytopes. The number of flags, and the resulting number of flag orbits, of the resulting polytopes are investigated. In some cases, the resulting orbit graph can be constructed. These operations are also useful to describe the polytopes we encounter in the remainder of the dissertation.

When classifying $k$-orbit polytopes for various $k$, it becomes clear that the 2-dimensional case is similar each time. Chapter IV deals with the 2-dimensional $k$-orbit polytopes for any
$k$: there are always infinitely many, which all have the same orbit graph if $k$ is odd, or one of three possible orbit graphs if $k$ is even. Of course, every polygon is combinatorially regular.

Chapter $V$ contains a classification of the (geometrically) two-orbit polytopes. Besides the combinatorially regular two-orbit polygons, these exist only in three dimensions and are also combinatorially two-orbit: they are the cuboctahedron, the icosidodecahedron, and their duals. We also classify the two-orbit tilings of Euclidean space. There are finitely many up to symmetry equivalence, tiling one, two, or three dimensions. The only two-orbit tilings which are also combinatorially two-orbit are the trihexagonal plane tiling, the rhombille plane tiling, the tetrahedral-octahedral honeycomb, and the rhombic dodecahedral honeycomb.

In Chapter $VI$ we show that any combinatorially two-orbit polytope is isomorphic to a (geometrically) two-orbit polytope (equivalently, to one whose automorphism group and symmetry group coincide). Hence, a combinatorially two-orbit convex polytope is 3-dimensional and is isomorphic to one of the cuboctahedron, icosidodecahedron, rhombic dodecahedron, or rhombic triacontahedron. The same is true of combinatorially two-orbit normal tilings, which must be one of the trihexagonal tiling, rhombille tiling, tetrahedral-octahedral honeycomb, or rhombic dodecahedral honeycomb.

Chapter $VII$ contains a classification of the three-orbit polytopes. There are infinitely many in three or four dimensions, just four each (up to similarity) in five or six dimensions, and just two each (up to similarity) in seven or eight dimensions. Any three-orbit polytope in dimension three or higher is also combinatorially three-orbit, with the exception of tetragonal disphenoids, a kind of irregular tetrahedra.

Finally, Chapter $VIII$ contains some statements about what dimensions can contain $k$-orbit polytopes, for general $k$.

8. Definitions

Definition. For a $d$-polytope $P$ and $I \subseteq \{-1, 0, \ldots, d\}$,

- A *chain of type $I$* is a chain of faces in $\mathcal{L}(P)$ with an $i$-face for each $i \in I$, and no others.
• A chain of cotype $I$ is a chain in $\mathcal{L}(P)$ with an $i$-face for each $i \notin I$, and no others.

Thus, flags are the chains of type $\{-1, 0, \ldots, d\}$ (or of cotype $\emptyset$).

**Definition.** A $d$-polytope is said to be

• **fully transitive** if its symmetry group acts transitively on its $i$-faces for every $i = 0, \ldots, d - 1$.

• **combinatorially fully transitive** if its automorphism group acts transitively on the faces of each dimension. In this case we may instead say that $\Gamma(P)$ is fully transitive.

• **$i$-transitive** if its symmetry group acts transitively on its $i$-faces.

• **$j$-intransitive** if it is $i$-transitive for every $i \neq j$, but not for $i = j$.

It is a theorem of McMullen’s thesis [1968] that fully transitive convex polytopes are regular.

**Theorem 4** (McMullen 1968, 4C6). A $d$-polytope $P$ is regular if and only if its symmetry group $G(P)$ is transitive on the $j$-faces for each $j = 0, \ldots, d - 1$.

**Definition.** A polytope is **uniform** if every face is a vertex-transitive polytope, and the 2-faces are regular.

Note that the faces are required to be vertex-transitive as polytopes in their own right; the stabilizer of the face in the symmetry group of the whole polytope need not act transitively on its vertices. Uniform polytopes are commonly defined recursively, declaring the regular polygons to be uniform, and then saying uniform polytopes in dimension 3 or higher are the vertex-transitive polytopes with uniform facets.

**Definition.** The **centroid** of a polytope is the mean of its vertices. That is, if a polytope $P$ has $n$ vertices $x_1, \ldots, x_n$, then the centroid is

$$\frac{1}{n} \sum_{i=1}^{n} x_i.$$ 

The centroid of a polytope is a fixed point of all its symmetries.
DEFINITION. Given a $d$-polytope $P$, a point $x$ in $\mathbb{E}^d$ is said to be beneath a facet $F$ of $P$ if $x$ lies in the same open half-space determined by the hyperplane $\text{aff}(F)$ as $P$ does.

If $x$ is on the far side of the hyperplane from $P$, then $x$ is said to be beyond $F$.

DEFINITION. A dihedral angle in a polytope $P$ is the angle formed by two facet-defining hyperplanes. In general, we will be interested in the dihedral angles between two facets that intersect in a ridge $R$; this is called the dihedral angle of $P$ at $R$.

If we take a 2-dimensional plane $P$ orthogonal to the codimension-2 space $\text{aff}(R)$, the hyperplanes defining the two facets incident to $R$ are orthogonal to $P$, and each meets $P$ in a line. The dihedral angle at $R$ is the angle formed in the plane by these two lines.

In a 4-polytope, it seems this angle would more properly be called a dichoral angle, or for the general case a ditopal angle. But “dihedral” seems to be the standard terminology for any number of dimensions.

8.1. Well-behaved tilings. A tessellation is normal if it satisfies three conditions:

N.1 Every tile is homeomorphic to a closed ball.
N.2 The intersection of any two tiles is connected (or empty).
N.3 The tiles are uniformly bounded. That is, there are positive numbers $u$ and $U$ such that every tile contains a ball of radius $u$ and is contained in a ball of radius $U$.

Any tessellation by closed convex sets automatically satisfies properties N.1 and N.2. So when we require a tiling to be normal, it is equivalent to require the tile sizes to be bounded. Every normal tessellation is locally finite.

Any tiling with finitely many flag orbits, under its symmetry group, is automatically normal. This is because there can be only finitely many congruence classes of tiles, establishing uniform boundedness.

It follows from Bieberbach’s theorems on crystallographic groups that a tiling of $\mathbb{E}^d$ with finitely many flag orbits is also periodic, meaning that its symmetry group contains translations in $d$ linearly independent directions (Schulte 1993). Otherwise, the fundamental
regions of the symmetry group would be unbounded, hence contain infinitely many flag chambers. In the case of a plane tiling $\mathcal{T}$, the possible symmetry groups $G(\mathcal{T})$ fall into three categories (Grünbaum and Shephard [1987b], §1.4):

- $G(\mathcal{T})$ contains no translations. Then $G(\mathcal{T})$ is one of the cyclic groups $C_n$ or one of the dihedral groups $D_n$. Each has an unbounded fundamental region.
- $G(\mathcal{T})$ contains translations in just one direction. Then $G(\mathcal{T})$ is one of the seven frieze groups, also called the strip groups. Each has an unbounded fundamental region.
- $G(\mathcal{T})$ is periodic. Then $G(\mathcal{T})$ is one of the 17 wallpaper groups, also called the planar crystallographic groups.

Hence, any $k$-orbit plane tiling $\mathcal{T}$, with $k \in \mathbb{N}$, is periodic, with one of the 17 wallpaper groups as its symmetry group.

It follows that $\mathcal{T}$ is balanced (Statement 3.3.13 of Grünbaum and Shephard [1987b]), strongly balanced and prototile balanced (Statement 3.4.8), and metrically balanced (Statement 3.7.3).

None of these conclusions hold for a tiling with finitely many flag orbits under its automorphism group. A combinatorially $k$-orbit tiling need not be normal, nor periodic.

Among tilings of Euclidean (or hyperbolic) space, a rank 3 tiling is called a plane tiling, and a rank 2 tiling is called an apeirogon. The latter necessarily consists of infinitely many edges (line segments) covering the line, and has been described as the limit of a sequence of $n$-gons as $n \to \infty$ (Coxeter [1973], p. 58).

Just as with polytopes, a rank $(d+1)$ tiling is said to be uniform if it is vertex-transitive and has uniform $d$-polytopes as tiles (Coxeter [1940a]).

8.2. Faces of polytopes. We reproduce some well-known results. See e.g. Grünbaum [2003]

**Proposition 5.** Every polyhedron has a 2-face which is a triangle, quadrilateral, or pentagon. By duality, every polyhedron has a vertex which is at most 5-valent.
Proof. Let \( v_i \) be the number of \( i \)-valent vertices, \( f_j \) be the number of \( j \)-gonal faces, and \( e \) be the number of edges of a polyhedron \( P \). By Euler’s formula,

\[
\sum_{i \in \mathbb{N}} v_i - e + \sum_{j \in \mathbb{N}} f_j = 2. 
\]

By counting the edges at each vertex, we conclude

\[
\sum_{i \in \mathbb{N}} iv_i = 2e, 
\]

and similarly, counting the edges on each face, we conclude

\[
\sum_{j \in \mathbb{N}} jf_j = 2e. 
\]

Add twice equation (2) to (3) and substitute this for \( 6e \) in six times equation (1) to obtain

\[
\sum_{i \in \mathbb{N}} 6v_i - \sum_{i \in \mathbb{N}} 2iv_i - \sum_{j \in \mathbb{N}} jf_j + \sum_{j \in \mathbb{N}} 6f_j = 12, 
\]

or

\[
2 \sum_{i \in \mathbb{N}} (3 - i)v_i + \sum_{j \in \mathbb{N}} (6 - j)f_j = 12. 
\]

Since the sum must be positive, but the only positive terms on the left hand side are contributed by \( f_3, f_4, \) and \( f_5 \), at least one of these must be nonzero. \( \square \)

It follows that every \( d \)-polytope with \( d \geq 3 \) has some 2-faces with at most 5 sides, since every 3-face is a polyhedron. In fact, it is true that every polytope in five or more dimensions has a 2-face which is a triangle or quadrilateral (Kalai \[1990\]), and by duality, every \( d \)-polytope has a subridge contained in at most 4 facets for \( d > 4 \).

A related fact for normal plane tilings is (Grüenbaum and Shephard \[1987b\] Statement 3.2.3):

**Proposition 6.** Every normal plane tiling has infinitely many tiles with at most 6 sides.

An extremely similar proof gives the following:

**Proposition 7.** Every polyhedron has either some triangular 2-faces or some 3-valent vertices.
Proof. Taking four times Euler’s formula \((1)\), substitute the sum of equations \((2)\) and \((3)\) for \(4e\) to obtain
\[
\sum_{i \in N} 4v_i - \sum_{i \in N} iv_i - \sum_{j \in N} jf_j + \sum_{j \in N} 4f_j = 8,
\]
or
\[
\sum_{i \in N} (4-i)v_i + \sum_{j \in N} (4-j)f_j = 8.
\]
The sum must be positive, but the only positive terms on the left hand side are contributed by \(v_3\) and \(f_3\). Hence at least one of these is nonzero (in fact, \(v_3 + f_3 \geq 8\)). \(\square\)

8.3. Vertex type.

Definition. A polyhedron or rank-3 tiling of the plane \(E^2\) is said to have a vertex type, denoted \((p.q.r\ldots)\), if each vertex is incident to a \(p\)-gonal face, a \(q\)-gonal face, etc., in the given cyclic order.

An exponent may be used to indicate repetition; for instance, the regular tiling by equilateral triangles, \((3.3.3.3.3)\), is denoted \((3^6)\). Any vertex-transitive polyhedron or rank-3 tiling has a vertex type. The notation may be used more generally for polyhedra or tilings which are not necessarily vertex-transitive, but where every vertex is in faces of the same size, in the same order. An example of this occurring is the pseudorhombicuboctahedron, all the vertices of which are in three squares and a triangle, so it has vertex type \((3.4.4.4)\). However, it is not vertex-transitive. (This has been the cause of various errors in classification of polyhedra, discussed in Grünbaum [2009].) It can be obtained from the rhombicuboctahedron, which is vertex-transitive and has the same type, by selecting an equatorial ring of eight squares, and rotating the square cupola above this ring by \(\frac{2\pi}{8}\).

Proposition 8. If a polyhedron \(P\) has vertex type \((k_1\ldots.k_m)\), or more generally if every vertex is incident to \(2\)-faces of sizes \(k_1,\ldots,k_m\) (not necessarily in that order), then
\[
(4) \quad \sum_{i=1}^{m} k_i - 2 < 2.
\]
\(\)
Proof. Let $g_j$ be the number of $j$-gonal faces at each vertex. Then $f_j = \frac{g_j}{j} v$, where $v$ is the number of vertices of $P$. Hence, for the total number of faces, we have

$$f = \sum f_j = v \sum \frac{g_j}{j} = v(1/k_1 + 1/k_2 + \cdots + 1/k_m),$$

where the last equality follows by expanding $g_j/j$ into $g_j$ copies of $1/j$. We also have $mv = 2e$. Hence, in Euler’s formula, we have

$$f - e + v = v(1/k_1 + 1/k_2 + \cdots + 1/k_m - m/2 + 1) = 2$$

Multiplying by $2/v$, we have

$$\sum_{i=1}^{m} 2/k_i - m + 2 = 4/v$$

$$2 - 4/v = \sum_{i=1}^{m} \left(1 - \frac{2}{k_i}\right) = \sum_{i=1}^{m} \frac{k_i - 2}{k_i}. \quad \Box$$

If $P$ is a normal plane tiling, then formula (4) holds as an equality (Grübaum and Shephard 1987b, Statement 3.5.4).

Dually, a polyhedron or plane tiling has a facet type $[p.q.r...]$ if every 2-face contains a $p$-valent vertex, a $q$-valent vertex, etc., in that cyclic order. Any facet-transitive polyhedron or plane tiling has a facet type. By identical arguments, we have

**Proposition 9.** If a polyhedron $P$ has facet type $[k_1, \ldots, k_m]$, or more generally if every facet has vertices of valence $k_1, \ldots, k_m$ (not necessarily in that order), then

$$\sum_{i=1}^{m} \frac{k_i - 2}{k_i} < 2.$$

This applies to normal plane tilings when formula (5) is made an equality (Grübaum and Shephard 1987b, Statement 3.5.1).

**8.4. Coxeter diagram.** See for instance Coxeter 1934, 1940a, 1973 for a description of the Coxeter diagram (there modestly called the “representation by graphs” for fundamental regions or for reflection groups), sometimes also known as a Dynkin diagram. This is a labeled graph associated with each reflection group, meaning a discrete group generated by
reflections. We follow Coxeter in using the term *dots*, rather than vertices or nodes, for the elements of this graph, to reduce confusion with the vertices of polytopes or the nodes of orbit graphs. Similarly, the term *link* is generally used in place of “edge”.

The diagram can be considered a representation of the fundamental region of a reflection group $G$. This fundamental region is a simplex, or a product of simplices, cut out by hyperplanes of reflection, and the dihedral angles are submultiples of $\pi$. For a finite group (such as the symmetry group of a convex polytope), it is a spherical simplex, i.e. a convex region on the surface of a sphere centered at a fixed point of $G$. The symmetry group of a tiling with no fixed points has a Euclidean simplex, or product, as its fundamental region. The diagram includes a dot for each facet of this region, with a link between any two facets whose dihedral angle is not $\pi/2$; the link is labeled $p$ if the dihedral angle is $\pi/p$, except that the label is usually omitted if $p = 3$.

The diagram can also be considered as giving a group presentation. Each dot represents a generator, which is an involution; the links give the order of each product of two generators (which is two if there is no link; in this case the generators commute). The group with this presentation is an (abstract) Coxeter group, and is isomorphic to any group generated by reflections in hyperplanes with the indicated angles between them. A Coxeter group along with distinguished generators corresponding to the dots in its Coxeter diagram is a *Coxeter system*. Two Coxeter systems are said to be *isomorphic* (as Coxeter systems) if they have the same diagram and there is an isomorphism of the groups carrying the distinguished generators of the first to the distinguished generators of the second (corresponding the same dot).

It is possible for the symmetry groups of two polytopes to be isomorphic as groups, but not isomorphic as Coxeter systems.

The diagram can also be modified to indicate the construction of a polytope. One or more dots are ringed. An *initial point* is then chosen, which is to lie in all the facets of the fundamental region except those which are ringed. Alternatively, each dot can be considered, rather than representing a facet $F$ of the fundamental region, to represent the vertex opposite
F. Then the initial point is chosen within the simplex spanned by the ringed vertices. We then take the image of the initial point under all the elements of the reflection group, and form the convex hull. This is known as Wythoff’s construction.

In the case of a regular polytope, the diagram is always a string diagram (meaning that every dot has valence at most 2) with a single endpoint ringed. Distinguished generators can be found by fixing an arbitrary base flag $\Phi$. Then the distinguished generators are $\rho_0, \ldots, \rho_{d-1}$ such that each $\rho_i$ carries $\Phi$ to its $i$-adjacent flag $\Phi^i$. Each $\rho_i$ is associated with the dot at distance $i$ from the ringed endpoint. The relations are $\rho_i^2 = 1$ for each $i$, and $(\rho_i \rho_j)^{p_{ij}} = 1$ for each distinct $i, j$, where $p_{ij}$ is the label of the link between the corresponding dots of the Coxeter diagram, or 2 if there is no link.

8.5. Schlafli symbol. A $d$-polytope possesses a Schlafli symbol $\{p_1, \ldots, p_{d-1}\}$ if for each $i = 1, \ldots, d-1$, each $(i+1)$-face $G$, and every $(i-2)$-face $F \leq G$, there are $p_i$ $i$-faces $H$ (equivalently, $(i-1)$-faces $H$) with $F < H < G$. (Equivalently, the section $G/F$, defined in §III.2, is a $p_i$-gon.) The symbol for the 1-polytope (a line segment) is taken to be $\{\}$. A polytope which has a Schlafli symbol is said to be equivelar. See e.g. McMullen and Schulte (2002, p. 11), McMullen (1967), or Coxeter (1973).

Note that, for a polytope $P$ to be equivelar, it suffices that $\Gamma(P)$ acts transitively on chains of type $\{i-2, i+1\}$ for each $i \in \{1, \ldots, d-1\}$.

For a regular $d$-polytope $P$, the entry $p_i$ can also be defined as the order of the automorphism $(\rho_{i-1}\rho_i)$, where $\rho_k$ is an involution which carries a base flag $\Phi$ to its $k$-adjacent flag $\Phi^k$.

We use these symbols in two different ways. When we say that a polytope $P$ has Schlafli type $\{p_1, \ldots, p_{d-1}\}$, it simply means that $P$ is equivelar, and the relevant sections have the given sizes. On the other hand, when a polytope or tiling is named by the symbol $\{p_1, \ldots, p_{d-1}\}$, this means the regular polytope or tiling of the given type. More precisely, when used to name a polytope, the Schlafli symbol (and various modifications with numbers on two or more lines) is regarded as shorthand for a Coxeter diagram with links labeled
by each $p_i$, and with the leftmost dot as the only ringed dot (Coxeter 1973). For example, \{p_1, p_2, \ldots \} means the polytope generated by Wythoff’s construction from the fundamental region

\[ p_1 \quad p_2 \] 

The polytope \{p_1, p_2, \ldots \} is generated by Wythoff’s construction on

\[ p_1 \quad p_2 \] 

and the symbol \{p_1, \ldots, p_k, q_1, \ldots, q_i\} corresponds to the diagram

\[ p_1 \quad p_k \] 

In the case that all the entries in a Schlӓfli symbol of this form are equal to 3 (which is always the case for convex polytopes), the polytope is given the abbreviated name $k_{ij}$ (Coxeter 1973, p. 201).

For convex polytopes, any polytope with a (single row) Schlӓfli type is isomorphic to a regular polytope with that name. For instance, there are (geometrically) three-orbit polyhedra known as tetragonal disphenoids; these have Schlӓfli type \{3, 3\}, but cannot be called “the polytope \{3, 3\}”, which is the regular tetrahedron. Among abstract polytopes, there can be a wide variety of non-isomorphic polytopes with the same Schlӓfli type, but this does not happen for convex polytopes.
CHAPTER II

Orbit Graphs

An orbit graph is a multigraph, which shows the relationship between all the flag orbits of a polytope. The graph expresses the “symmetry type” of a polytope, for which reason it is also frequently called a “symmetry type graph” (Cunningham et al. 2015; Ković 2011; Río-Francos et al. 2013). The orbit graph is also essentially the same as the Delaney-Dress symbol for tilings (Schattschneider and Senechal 1997, p. 52). We will use these graphs as a tool in classifying $k$-orbit convex polytopes. First we will determine all the possible orbit graphs on $k$ nodes, then determine all the polytopes which have that orbit graph.

1. Defining Orbit Graphs

**Definition.** The orbit graph for a $d$-polytope $P$ and a subgroup $G$ of the automorphism group $\Gamma(P)$ is a multigraph (an undirected graph allowing loops and multiple edges). Its nodes are the flag orbits of $P$ under the action of $G$. Given a flag orbit $A$ and a flag $\Phi \in A$, there is an edge labeled $i$ from $A$ to the flag orbit containing $\Phi^i$, for each $i \in \{0, \ldots, d-1\}$. This graph is denoted $G_G(P)$.

It is easy to see that the edges are well-defined, using Lemma 1.1. For, given two flags $\Phi$ and $\Psi$ in the same orbit, there is an automorphism $\gamma \in G$ such that $\gamma(\Phi) = \Psi$. Then $\gamma(\Phi^i) = \Psi^i$, so the $i$-adjacent flags to $\Phi$ and $\Psi$ are in the same orbit.

With this definition, there is a loop at a node whenever $i$-adjacent flags remain in the same orbit. The typical definition for a symmetry type graph has a semi-edge instead (an edge with only one endpoint).

The edges labeled $i$ are called $i$-edges. Two nodes connected by an $i$-edge are said to be $i$-adjacent.
When $G$ is the trivial group, the resulting graph is the *flag graph* of $P$, with one node for each flag, and an $i$-edge between each pair of $i$-adjacent flags. For a convex polytope $P$, the flag graph can be drawn on the boundary of $P$ (which is homeomorphic to the $(d-1)$-sphere), with the node for each flag placed within the simplex formed by the centroids of each of its constituent proper faces. The $i$-edge then passes through the $i$th *wall* of this simplex, which is the $(d-2)$-simplex opposite the centroid of the $i$-face. Thus, the flag graph of $P$ is “dual” to the barycentric subdivision of $P$ (see §III.7), in that it exchanges nodes for $(d-1)$-faces, and edges for $(d-2)$-faces. The flag graph of a polytope (including any abstract polytope) is an instance of a *maniplex*, introduced by Wilson (2012) to generalize both abstract polytopes and maps.

When $G$ is a larger group of automorphisms, the resulting orbit graph $G(P)$ is the quotient of the flag graph by the action of $G$. We shall generally assume that $G = G(P)$, and simply call the orbit graph $G(P)$ for this group. The *combinatorial orbit graph* is $G_{\Gamma(P)}(P)$.

### 2. Symmetry Type

**Definition.** We say that a set of polytopes have the same *symmetry type* if they all have the same orbit graph.

Note that this is different from the definition of “symmetry type” given by Grünbaum and Shephard (1987b §1.4), where two tilings are said to have the same symmetry type if their symmetry groups are isomorphic, where the isomorphism is in the strong sense of preserving a certain group diagram. In effect, this requires rotations to be carried to rotations, reflections to reflections, etc.

At the coarsest level of our hierarchy of equivalence classes (§I.6), symmetry equivalence classes may also be grouped into symmetry types. Each symmetry type includes at most one symmetry equivalence class from each isomorphism class. Symmetry equivalence classes can also be grouped into a set of all polytopes with the same symmetry group diagram, that is, symmetry types in the sense of Grünbaum and Shephard.
3. Graph terminology

To discuss orbit graphs, we introduce some standard graph terminology. We generally use the word “nodes” rather than “vertices” for graphs, to reduce confusion with 0-faces of polytopes.

**Definition.** For a graph $H$,

- The *degree*, or the *valence*, of a node of $H$ is the number of incident edges.
- $H$ is *$n$-regular* if every node has valence $n$.
- $H$ is a *path* if it has $n$ edges and $n + 1$ nodes which can be indexed $v_0, \ldots, v_n$ such that two nodes are adjacent if and only if their indices are consecutive. The *length* of the path is $n$.
- $H$ is a *cycle* if it has an equal number $n$ of nodes and edges, and the nodes can be indexed $v_1, \ldots, v_n$ such that two nodes are adjacent if and only if their indices are consecutive modulo $n$.
- $H$ is *connected* if each pair of nodes belongs to a path.
- $H$ is *acyclic* if it contains no cycle.
- $H$ is a *tree* if it is connected and acyclic.
- A *subgraph* of $H$ is a graph with a subset of the nodes and edges of $H$. The subgraph is *spanning* if it includes every node of $H$.
- A *$k$-factor* is a $k$-regular spanning subgraph of $H$.
- A *perfect matching* is a set of edges of $H$ such that every node is incident to exactly one edge in the set. A perfect matching is a 1-factor of $H$ (strictly speaking, the former is the set of edges of the latter).

Equivalently, a cycle is a connected 2-regular graph, and a path is a connected acyclic graph whose maximal node degree is 2 if it has more than two vertices.

In an orbit graph, every node is incident to a single edge labeled with each rank. So every node has the same degree, if we count loops as contributing only once to the degree. We can
thus say that the orbit graph of a $d$-polytope is $d$-regular, with a partition of its edges into $d$ perfect matchings, each determined by a rank. (Another way to say this is that the orbit graph has a 1-factorization into $d$ factors.) In a graph theoretic context, loops may or may not be allowed as elements of a matching. The full flag graph of $P$ is a simple graph (a graph without loops or multiple edges), which can be said to have a 1-factorization into $d$ factors, in the strictest sense.

REMARK. If we omit all the loops in an orbit graph, it is easy to recover the full orbit graph, by putting a loop labeled $i$ at every node which does not have any incident $i$-edge. Suppressing the loops in this manner can result in a much cleaner graph, particularly in higher dimensions.

When we say that an orbit graph is a path, or some other graph $H$ where not every node has the same degree, it should be understood that $H$ is the loop-suppressed version of the graph, and loops must be added appropriately for the full orbit graph.

In an orbit graph, the $i_1, i_2, \ldots, i_k$-walk starting at a node $v$ is a sequence of edges of the graph, starting with the $i_1$-edge incident to $v$, continuing along the $i_2$-edge, and so on, terminating after crossing the $k$th edge. A walk is closed if it ends at the starting node $v$.

On the other hand, we will say $(i, j)$-walk, at a node $v$, to refer to the walk starting at $v$ and alternating $i$-edges and $j$-edges until it returns to $v$ (across a $j$-edge). More generally, a $(i_1, \ldots, i_l)$-walk takes the edges with the indicated labels in the given order. Each sequence of $l$ edges in the walk, with labels $i_1, \ldots, i_l$, is called a step of the walk. The walk continues until it returns to $v$ at the end of a step.

4. Properties

As a quotient of the flag graph, the properties of abstract polytopes are reflected in the orbit graph. To wit, flag-connectivity tells us that $\mathcal{G}(P)$ is connected. We also know that, if $|i - j| \geq 2$, then walking from a flag $\Phi$ to its $i$-adjacent flag, then the $j$-adjacent flag, then the $i$-adjacent flag, and then $j$-adjacent again returns to $\Phi$. This is essentially a
consequence of the diamond condition, and is sometimes expressed by saying that $i$-adjacency and $j$-adjacency commute if $i$ and $j$ are not consecutive.

In the full flag graph, this tells us that in the subgraph consisting of all the $i$-edges and $j$-edges (which is a 2-factor of the graph), each connected component is a cycle of length four. In the orbit graph, the $i, j, i, j$-walks starting at every node must be closed whenever $|i - j| \geq 2$.

In $G(P)$ this forces the $j$-edges to “commute across” the $i$-edges (when $|i - j| \geq 2$) in the following sense. Suppose $u$ and $v$ are $i$-adjacent nodes. If the $j$-edge at $u$ is a loop, then the $j$-edge at $v$ is a loop, also. If the $j$-edge at $u$ leads to a node $w$ (distinct from $u$ and $v$), then the $j$-edge at $v$ leads to a node $w'$, and $w$ and $w'$ are $i$-adjacent. (Note that $w$ and $w'$ cannot coincide, unless $u$ and $v$ also coincide, since each node is incident to exactly one $j$-edge.)

Hence, if $|i - j| \geq 2$, there are just five possibilities for the closed walk on edges with label $i$ and $j$ starting at any node, shown in Figure II A. An equivalent figure is found in Cunningham et al. (2015).

![Figure II A. Possible walks on $i$-edges and $j$-edges with $|i - j| \geq 2$](image)

In order to get information about the $i$-faces of $P$, one approach is to delete all the $i$-edges from the orbit graph.

**Definition.** The graph $G_i(P)$ is the subgraph of $G(P)$ derived by deleting all $i$-edges. Similarly, the graph $G_I(P)$, for some $I \subset \{0, \ldots, d - 1\}$ is derived from $G(P)$ by deleting all edges with labels in $I$.

**Proposition 1.** For any $I \subset \{0, \ldots, d - 1\}$, the orbits of chains of type $I$ under the action of $G(P)$ are in one-to-one correspondence with the connected components of $G_I(P)$.

**Proof.** Let $\Omega$ be a chain of type $I$, and $\Phi$ be a flag containing $\Omega$. The flag $\Phi$ is in some node of $G(P)$. All the other nodes in the same connected component $C$ of $G_I(P)$ can be reached
from the node of $\Phi$ without changing the faces in $\Omega$, so each of these nodes contains flags which contain $\Omega$. Conversely, by strong flag-connectivity, every flag which contains $\Omega$ lies in a node in this component $C$. So, to each chain of type $I$ we can associate a single connected component of $\mathcal{G}_I(P)$.

Any other chain $\Lambda$ of type $I$ which is contained in any flag in any node of this component $C$ is similarly contained in some flag in every node of $C$. So, there is a flag $\Psi$ containing $\Lambda$ and in the same orbit as $\Phi$, and so there is a symmetry carrying $\Omega$ to $\Lambda$. Hence the set of chains of type $I$ lying in flags in nodes from the component $C$ are contained in a single orbit.

On the other hand, for any symmetry $\gamma \in G(P)$, $\gamma(\Omega)$ is contained in the flag $\gamma(\Phi)$, hence is one of those chains of type $I$ only contained in flags in nodes from the component $C$. Thus, the entire orbit $G(P) \cdot \Omega$ is associated with $C$. \[\square\]

In particular, the orbits of $i$-faces are in bijection with the connected components of $\mathcal{G}_i(P)$. Hence, $G(P)$ acts transitively on the $i$-faces of $P$ if and only if $\mathcal{G}_i(P)$ is connected.

**Corollary 2.** If $P$ is a $k$-orbit (convex) $d$-polytope with $k > 1$, then for some $i \in \{0, \ldots, d-1\}$, the graph $\mathcal{G}_i(P)$ is disconnected.

**Proof.** Otherwise, $P$ would be fully transitive. But then $P$ would be regular (by Theorem 1.4), contradicting that it has $k$ flag orbits. \[\square\]

A further restriction on orbit graphs for convex polytopes is very helpful.

**Definition.** An orbit graph is $(i, i+1)$-even if the $(i, i+1)$-walk starting at any node has evenly many steps (so the number of edges is a multiple of $4$).

Equivalently, the connected components of the 2-factor consisting of the $i$-edges and the $(i+1)$-edges are either cycles whose length is a multiple of four, or paths with an odd number of edges, terminated at both ends by loops.

**Proposition 3.** The orbit graph of a $d$-polytope $P$, with $d \geq 3$, cannot be both $(i-1,i)$-even and $(i,i+1)$-even for any $i \in \{1, \ldots, d-2\}$. 

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Proof. For each \( i \in \{1, \ldots, d-2\} \), let \( F_{i-2} \) and \( F_{i+2} \) be an incident pair of faces with the indicated dimensions. Then the section \( F_{i+2}/F_{i-2} \) is a polyhedron. (Sections will be defined in §III.2.) By Proposition I.7, this section has either some triangular 2-faces or some 3-valent vertices. Starting from any flag in a triangular 2-face, a 0, 1, 0, 1, 0, 1-walk in the flag graph is closed; but this is a \((i-1, i)\)-walk of 3 steps in \( P \). Similarly, starting from any flag including a 3-valent vertex, a 1, 2, 1, 2, 1, 2-walk in the flag graph is closed, and this is a \((i, i+1)\)-walk of 3 steps in \( P \).

Therefore, also in the orbit graph, there are some nodes such that either a walk alternating \((i-1)\)-edges and \( i \)-edges, or a walk alternating \( i \)-edges and \((i+1)\)-edges, return to the node after three steps. Thus, the shortest closed walks have either one or three steps: in particular, they cannot be even.

We have actually proved: for any \( i \in \{1, \ldots, d-2\} \), either some \((i-1, i)\)-walks, or some \((i, i+1)\)-walks, have one of the forms in Figure IIb (shown for the case of an \((i, i+1)\)-walk). □

Another way to say this is that if \( P \) happens to be \((i, i+1)\)-even, then there must be some \((i-1, i)\)-walks as in Figure IIb (if \( i \geq 1 \)) and also some \((i+1, i+2)\)-walks as in Figure IIb (if \( i \leq d-3 \)). This greatly cuts down on the number of potential orbit graphs for convex polytopes.

5. Covering

An orbit graph \( H \) is said to cover another orbit graph \( K \) if there is a \( k \)-to-1 mapping of the nodes of \( H \) onto the nodes of \( K \), such that nodes in \( H \) are \( i \)-adjacent if and only if their images in \( K \) are \( i \)-adjacent. This also gives a \( k \)-to-1 mapping of the edges, except

\[ \]
when mapping onto a loop, in which case $\frac{k}{2}$ edges map onto a single loop, where loops in the preimage count as half an edge. (In other words, everything is $k$-to-1 so long as we count loops on both sides as only half an element.)

These counting details can be eliminated if the orbit graphs are modified so that each $i$-edge is replaced by two directed $i$-edges, one in each direction. Loops are left as they are, but can be considered directed loops. In this case, the map from $H$ to $K$ is $k$-to-1 everywhere, and is actually a covering map in the sense of algebraic topology.

The orbit graphs which cover a given orbit graph $K$ correspond to “lower symmetry forms” of $K$. That is, for any polytope $P$ with orbit graph $K$, any “lower symmetry form” of $P$ has an orbit graph which covers $K$. A “lower symmetry form” is a polytope isomorphic to $P$, but which is preserved only by a proper subgroup of $G(P)$. These arise in several contexts.

For instance, after coloring some elements of $P$ (the vertices, say, or the facets), the color-preserving symmetries of $P$ are a smaller group, say $G_c(P)$, and the orbit graph $G_{G_c(P)}(P)$ covers $G(P)$.

When $P$ appears as a face of another polytope $Q$, not all of its symmetries necessarily extend to symmetries of $Q$. Often, the relationships of $P$ with the other faces of $Q$ restrict how it can be moved. Thus the orbit graph of $P$ as a face of $Q$ covers $G(P)$. When all the symmetries of each facet do extend to $Q$, then $Q$ is called hereditary (Mixer, Schulte, and Weiss 2014). In this case, the orbit graphs of each facet appear as the connected components of $G_{d-1}(Q)$. (That is, the connected components are not $k$-fold covering maps for any $k > 1$, but are isomorphic to the orbit graph of every facet corresponding to the component.)

Another way to find “lower symmetry” forms of $P$ is by deforming it, finding a different realization, still isomorphic to $P$, with strictly fewer symmetries.

In fact, the various orbit graphs associated with different “symmetry forms” of $P$ form a partially ordered set. Every such orbit graph covers the combinatorial orbit graph of $P$, and every such graph is covered by the flag graph of $P$. This partially-ordered set is closely related to the subgroup lattice of $\Gamma(P)$. Conjugate subgroups of $\Gamma(P)$ result in the same
Figure IIc. The lattice of orbit graphs for a quadrilateral orbit graph. An example orbit graph lattice for a quadrilateral is shown in Figure IIc. Some quadrilaterals realizing each symmetry type are shown in Figure IID. One type cannot be realized without coloring the flags as indicated.
FIGURE IID. Example quadrilaterals corresponding to the symmetry types in Figure IIC.
CHAPTER III

Standard Operations

We define various standard operations which derive new polytopes from other polytopes. These operations suffice to construct all the polytopes with few orbits, starting from the line segment and the regular polygons.

1. Duality

A \( d \)-polytope \( Q \) is said to be dual to a \( d \)-polytope \( P \) if the face lattice \( \mathcal{L}(Q) \) is anti-isomorphic to the lattice \( \mathcal{L}(P) \), that is, identical to \( \mathcal{L}(P) \) with the order reversed. A bijective, order-reversing function \( h: \mathcal{L}(P) \to \mathcal{L}(Q) \) is called a duality. A dual polytope to \( P \) is often denoted \( P^* \). Clearly, any two duals of \( P \) are combinatorially isomorphic. A dual \( P^* \) to any convex polytope \( P \) may be constructed by the process of polar reciprocation (Grünbaum 2003 §3.4): After translating \( P \), if necessary, so that its centroid is placed at the origin, let \( P^* = \bigcap_{y \in P} \{ x \mid \langle x, y \rangle \leq 1 \} \), where \( \langle x, y \rangle \) is the scalar product. Then \( (P^*)^* = P \) and \( G(P^*) = G(P) \). Thus, when necessary, we may assume that a polytope and its dual have the same symmetry group.

This operation is said to be reciprocation with respect to a reciprocating sphere, which in this case is the unit sphere in \( \mathbb{E}^d \). By changing the parameter 1 in the definition of \( P^* \), we change the reciprocating sphere, thus changing the scale of the reciprocal polytope.

The dual polytope \( P^* \) has the same number of flags, and the same number of flag orbits, as \( P \). The orbit graph of \( P^* \) is the same as that for \( P \), but with every edge-label \( i \) replaced by \( d - i - 1 \). Thus, \( P \) is \( j \)-transitive if and only if \( P^* \) is \( (d - j - 1) \)-transitive; the group \( G(P) \) acts transitively on chains of type \( I \) if and only if \( G(P^*) \) acts transitively on chains of type \( \{ d - i - 1 \mid i \in I \} \); and so on.
The dual $P^*$ formed by polar reciprocation is sometimes called the *reciprocal* polytope, rather than the dual, especially if we want to emphasize the geometric relation between elements of $P$ and $P^*$, rather than the combinatorial anti-isomorphism.

For tilings, the symmetry group of a dual cannot be assumed to be the same. For every normal tiling $\mathcal{T}$, there is a normal tessellation which is dual to $\mathcal{T}$ in the combinatorial sense (i.e. as face lattices). For example, a dual, normal tessellation can be constructed by taking the barycentric subdivision of the given tiling, and treating the union of all the chambers incident to a given vertex as a tile. But this dual tessellation need not have convex tiles, even if $\mathcal{T}$ does. If the tiles are convex, they need not meet face-to-face, even if the tiles of $\mathcal{T}$ do. If this dual tessellation is face-to-face, it need not have the same symmetry group. As an example of this failure, we will meet a three-orbit tiling $t\{3,6\}$ consisting of hexagons of two different congruence classes. Dual tilings exist to $t\{3,6\}$, which preserve all of its symmetries, but any such tiling also has additional symmetries and is actually regular! As far as the author is aware it is not known what conditions on $\mathcal{T}$ guarantee that a dual tiling exists with convex tiles meeting face-to-face and with the same symmetry group.

The symbol for duality is $d$ in the Conway polyhedron notation. The operation is called $\delta$ in McMullen and Schulte (2002, p. 192).

2. Sections

A *section* of a polytope $P$, for incident faces $F \subset G$, is the portion of the face lattice $\mathcal{L}(P)$ consisting of all the faces containing $F$ and contained in $G$, and is denoted $G/F$. So $G/F = \{ H \in \mathcal{L}(P) \mid F \leq H \leq G \}$, inheriting the order. Every such section can be realized as the face lattice of a convex polytope, and we often identify convex polytopes with their face lattices.

This section may be realized as a convex polytope by taking the polar dual $G^*$ to $G$, say with a duality $h: G \to G^*$. This contains only the faces $\{ H \in \mathcal{L}(P) \mid H \leq G \}$, with the ordering reversed. Now the face $h(F)$ corresponding to $F$ has faces $h(H)$ for each $H \in \mathcal{L}(P)$.
such that $\emptyset \leq h(H) \leq h(F)$, or equivalently, $F \leq H \leq G$. Thus the dual $h(F)^*$ to this face of $G^*$ is the desired polytope.

Now suppose $F$ is a $j$-face and $G$ is an $l$-face. A walk along adjacent flags of $G/F$ corresponds to a walk along adjacent flags of $P$, where the $i$-adjacencies in the latter are restricted to $j < i < l$.

Thus, a graph for the section $G/F$ can be derived from the orbit graph $G(P)$ by deleting all the edges with labels at most $j$, or with labels at least $l$. Pick one connected component $C$ of the resulting graph which includes flags containing $F$ and $G$. There may be several such components, but all will be identical as edge-labeled graphs.

**Lemma 1.** After deleting all $i$-edges in $G(P)$ with $i \leq j$ or $i \geq l$, any two connected components whose nodes contain flags with the $l$-face $G$ and the $j$-face $F$ are isomorphic as edge-labeled graphs.

**Proof.** Suppose that $C$ and $D$ are two such connected components. Both lie within a single component $X$ of the graph $G_{(j,l)}(P)$ obtained from $G(P)$ by deleting all $j$-edges and $l$-edges (by Proposition II.1 with $I = \{j, l\}$). Thus, there is a path in $X$ from some node of $C$ to some node of $D$.

Now, any $i$-edge on this path, with $j < i < l$, commutes with any adjacent $h$-edge on the path with $h < j$ or $h > l$, in that the two labels can be interchanged without affecting the path’s final endpoints. Indeed, suppose the $i$-edge leads from node $u$ to node $w$, and the $h$-edge from node $w$ to node $v$. If we take instead the $h$-edge from $u$ to some node $w'$, then the $i$-edge from $w'$ leads again to $v$ (cf. Figure II). Thus, we may rearrange the path so that all edges whose labels are between $j$ and $l$ are at the end. These edges lie within the connected component $D$. So, discarding these edges results in a path $p$ from a node $v$ in $C$ to a node $v'$ in $D$, all of whose labels are less than $j$ or greater than $l$; let $h$ be its sequence of edge labels.

Any other node $w \in C$ can be reached from $v$ by a path with labels between $j$ and $l$. Since the edges in $p$ commute with every such edge, the $h$-walk from each node $u$ on this
path leads to a corresponding node $u' \in D$ with an identically labeled walk from $v'$. Hence, the nodes of $C$ are in bijection with the nodes of $D$ by $h$-walks, and if two nodes in $C$ are $i$-adjacent, so are the corresponding nodes in $D$.

Thus, all the connected components remaining in $X$ after deleting the edges with labels below $j$ or above $l$ are identical. \qed

Now relabel the edges remaining in $C$ by subtracting $j + 1$ from each label. We name the resulting graph the \textit{restricted graph} for $G/F$.

This orbit graph reflects the symmetries of $G/F$ under symmetries of $P$: specifically, those symmetries in $G(P)$ which fix all the faces of $P$ which contain $G$, and all the faces of $P$ which are faces of $F$. These form a subgroup which acts faithfully on $G/F$ in a well-defined way. We call it the \textit{restricted group} and denote it by $G_P(G/F)$. If we perform the construction of the polytope realizing $G/F$ carefully, the isometries in $G_P(G/F)$ are actually isometries fixing the polytope $G/F$. Namely, within the subspace $\text{aff}(G)$, consider the centroid of $G$ to be the origin when forming $G^*$, then, within the subspace $\text{aff}(h(F))$, consider the centroid of $h(F)$ to be the origin when forming $h(F)^*$.

The restricted group $G_P(G/F)$ is a subgroup of the full symmetry group of $G/F$. In general, $G/F$ may have more symmetries. Correspondingly, the orbit graph of $G/F$ may have fewer nodes than the component $C$. The restricted graph covers the orbit graph $\mathcal{G}(G/F)$.

In fact, even the symmetries of $P$ can induce more symmetries of $G/F$ than those captured by $G_P(G/F)$. We will now form a \textit{contracted graph} corresponding to the group of symmetries of $P$ which fix both faces $G$ and $F$ (but not necessarily the faces containing $G$, or the faces contained in $F$). This group also acts on $G/F$, but not necessarily faithfully.

Take the connected component $X$ of $\mathcal{G}_{(j,l)}(P)$ corresponding to our section. Now instead of removing the edges whose labels are below $j$ or above $l$, contract these edges: that is, consider their endpoints to be the same node, and remove the edge. After performing this operation, wherever there are multiple edges with label $i$ between two nodes, we replace them with a single edge. There will only be one $i$-edge left adjacent to each remaining node $v$, for
each $i$ with $j < i < l$, since the $i$-edges at all the nodes which were contracted to $v$ “commute” with all the edges which were removed. From the proof of Lemma 1, we see that the entire connected component $X$ is contracted to the nodes of one of the connected components $C$ of $G_{(0,\ldots,j,l,\ldots,d−1)}$ by contracting the $h$-walks. Hence, the contracted graph can be derived from the restricted graph by identifying any nodes which are linked, in the original graph, by $h$-edges with $h < j$ or $h > l$. (Any edges between these nodes become loops.)

We again relabel the edges in this modified graph by subtracting $j + 1$ from each label. The result is the contracted graph for $G/F$. The contracted graph is, in general, smaller than the restricted graph. For instance, in the graph $\bullet \xrightarrow{2} \bullet \xrightarrow{3} \bullet$, the restricted graph for one orbit of 2-faces is $\bullet \xrightarrow{1} \bullet$, while the contracted graph for the same 2-faces is $\bullet$.

These two graphs give different information about the section. In the example above, the restricted graph tells us that the 2-face in question alternates edges: hence it is an even-sided polygon. The contracted graph tells us that the polygon is actually regular, and not e.g. a rectangle. Neither graph could give all this information by itself.

For each face $F$ of a polytope $P$, the section $P/F$ is called the face figure of $F$. In particular, for each vertex $v$, the vertex figure is $P/v$, and for an edge $e$, the edge figure is $P/e$. These face figures determine how the face is surrounded by other faces of $P$ and play a major role in our considerations. In the other direction, a section $F/\emptyset$ involving the empty face is identified with the face $F$; it represents $F$ considered as a polytope in its own right.

Vertex figures are ubiquitous, and are often defined slightly differently. The vertex figure at $v$ may be defined as the intersection of $P$ with a hyperplane which cuts $P$ strictly between $v$ and all the other vertices. Another definition selects the midpoint of each edge incident to $v$ as the vertices of the vertex figure; the edges are the line segments between each pair of edge midpoints on a 2-face incident to $v$; and so on. This forms, in general, a skew polytope, thus abandoning convexity. This is how the vertex figure is defined in Coxeter (1973). If all the vertices of $P$ lie on a sphere, and all the edges of $P$ are the same length, then the midpoints of the edges incident to $v$ lie on a hyperplane $H$. In this case, the skew polytope
formed from the edge midpoints is in fact convex and coincides with the intersection of $P$ with $H$, so these definitions coincide. The vertex figure of a tiling of $\mathbb{R}^d$ may be defined as a spherical polytope, i.e. a tiling of $S^{d-1}$, formed by projecting all the faces incident to $v$ on a sufficiently small sphere around $v$ (small enough so that it does not meet or contain any faces which are not incident to $v$). Vertex figures under any of these definitions are isomorphic, and often we only care about the face lattice of the vertex figure.

Note that for vertex figures and facets, there is no distinction between the restricted graph and the contracted graph, nor are there multiple connected components to choose from in Lemma 1.

3. Pyramids

The pyramid over a $d$-polytope $P$ is a $(d + 1)$-polytope formed as the convex hull of $P$ with a new vertex, the apex, not in $\text{aff}(P)$. In general, we shall assume that the apex is placed on a line passing through the centroid of $P$ and orthogonal to $\text{aff}(P)$. However, the convex hull of $P$ with any point not in $\text{aff}(P)$ is still called “a” pyramid over $P$.

Even with the more strict requirement for the apex, there are an infinite number of possible placements, all resulting in mutually dissimilar polytopes. In the event that $P$ has all edges the same length, it may or may not be possible to place the apex so that the new edges have the same length again. For instance, it is possible to do this over regular $n$-gons for $n \leq 5$, but not otherwise.

With the apex placed over the centroid of $P$, clearly all the symmetries in $G(P)$ still act on the pyramid, stabilizing the apex. Moreover, the apex $z$ being “special”, these are the only symmetries of the pyramid, unless another vertex is also an apex over a $d$-polytope congruent to $P$ and including $z$. Since any facet of the pyramid including $z$ is a pyramid over a facet of $P$, this can only happen if $P$ is itself a pyramid (with the apex at the same height).

We conclude that the number of symmetries of a $k$-fold pyramid, with the apex at the same height each time, over a base $P$ which is not a pyramid, is $k!|G(P)|$. 

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**Proposition 2.** The number of flags of a pyramid over a $d$-polytope $P$ is $(d+2)|F(P)|$.

**Proof.** A pyramid over a 1-polytope is a triangle; this has $6 = 3 \cdot 2$ flags, so the claim holds. We proceed by induction on $d$.

Let $d > 1$ and suppose that the claim holds for pyramids over $(d-1)$-polytopes. Let $P$ be a $d$-polytope with facets $F_1, \ldots, F_n$. The facets of the pyramid over $P$ are $P$ itself, and the pyramid over $F_i$ for each $i$. Hence, using the induction hypothesis, the number of flags of the pyramid is

$$|F(P)| + \sum_{i=1}^{n} (d+1)|F(F_i)| = |F(P)| + (d+1)\sum_{i=1}^{n} |F(F_i)|.$$ 

But the sum of the number of flags in each facet of $P$ is the number of flags of $P$. So this sum is $(d+2)|F(P)|$, as claimed. □

**Corollary 3.** A $k$-fold pyramid (with all apices at the same height) over an l-orbit $d$-polytope (which is not itself a pyramid) is a $(k+d)$-polytope with $\binom{d+k+1}{k}$ flag orbits.

**Proof.** By the Proposition, the number of flags of the $k$-fold pyramid is

$$(d+k+1)(d+k) \cdots (d+2)|F(P)|,$$

and by the preceding remarks, the symmetry group has order $k!|G(P)|$. It is the product of $G(P)$ with the symmetric group $S_k$ acting on the $k$ apices. Thus, there are

$$\frac{(d+k+1)!|F(P)|}{(d+1)!k!|G(P)|}$$

flag orbits. □

The orbit graph for a pyramid over a $d$-polytope $P$ can be constructed as follows. (This is the orbit graph assuming that $P$ is not itself a pyramid, or that the new apex is at a unique height if $P$ is a pyramid. In the case of a repeated pyramid with apices at the same height, this graph is a cover of the orbit graph under the full symmetry group.)

Begin with $d+2$ copies of the orbit graph $G(P)$, indexed $0, \ldots, d+1$. In the $i$th copy, delete all edges labeled $d-i$, and increase the label of all edges with labels greater than $d-i$.  

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by one. Note that the 0th copy is unchanged, and the last copy has all its edges remaining, but all the labels have been incremented.

At every node in the $i$th copy, add a new edge labeled $d - i$ leading to the corresponding node in the $(i + 1)$st copy of $\mathcal{G}(P)$. See Figures IIIA and IIIB for some examples.

**Figure IIIA.** The orbit graph of a pyramid over a regular polygon $\{p\}$

**Figure IIIB.** The orbit graph of a pyramid over a cuboctahedron $\{^3_{14}\}$

4. Cartesian product

The *Cartesian product* of an $m$-polytope $P$ and an $n$-polytope $Q$ is an $(m + n)$-polytope $P \times Q$; the set of points of $P \times Q$ is the Cartesian product of the polytopes as point sets. That is, $(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a point in $P \times Q$ if and only if $(x_1, \ldots, x_m)$ is a point of
$P$, and $(y_1, \ldots, y_n)$ is a point of $Q$. Coxeter calls this operation a \textit{rectangular product} \cite{1973} p. 124).

For each $i \in \{0, \ldots, m+n\}$, the $i$-faces of $P \times Q$ are the products $F \times G$ for every $j$-face $F$ of $P$ and $k$-face $G$ of $Q$ such that $j + k = i$. We exclude the empty face of $P$ and $Q$ from this construction. In particular, if $V(P)$ is the vertex set of the polytope $P$, then $V(P \times Q) = V(P) \times V(Q)$. So, if $f_i(P)$ is the number of $i$-faces of the polytope $P$, we have $f_0(P \times Q) = f_0(P)f_0(Q)$. More generally, the number $f_i(P \times Q)$ is determined by polynomial multiplication. If we form an “$f$-polynomial" $f(P) = \sum_{i=0}^{m} f_i(P)x^i$, then $f(P \times Q) = f(P)f(Q)$.

A basic and important instance of a Cartesian product is the product of a $d$-polytope $P$ with a line segment $\{\}$. This is a \textit{prism} over $P$, a $(d+1)$-polytope which is the convex hull of two copies of $P$, displaced in a direction orthogonal to the affine hull of $P$ in $\mathbb{E}^{d+1}$.

Another example we will meet are the $p,q$-duoprisms, the product $\{p\} \times \{q\}$ of two regular polygons. These are 4-polytopes, with one set of facets consisting of $q$ $p$-gonal prisms, stacked end-to-end and forming a ring. (This does not seem to make sense at first, but it is possible in four dimensions, just as you can stack 3 squares end-to-end in the plane, then fold them together to a closed ring.) The sides of the prisms form $p$ rings of $q$ squares each, which form $p$ $q$-gonal prisms, giving the other set of facets.

\textbf{Proposition 4.} The product $P \times Q$ of an $m$-polytope $P$ and an $n$-polytope $Q$ has $\binom{m+n}{n} |\mathcal{F}(P)||\mathcal{F}(Q)|$ flags.

\textbf{Proof.} Any product with a single point is just the identical polytope, and satisfies the claim. So the smallest non-trivial case is the product of two line segments, which is a rectangle: we verify that it has $\frac{(1+1)!}{1!1!} 2 \cdot 2$ flags. This is the only product in two dimensions. We prove the claim in higher dimensions by induction.

Let $k \geq 2$ and suppose that the claim holds for any $k$-dimensional product of two polytopes. Let $d = k+1$, $P$ be an $m$-polytope, and $Q$ be an $n$-polytope such that $m + n = d$. The facets of $P \times Q$ are $F \times Q$, for each facet $F$ of $P$, and $P \times G$, for each facet $G$ of $Q$. Index the facets of $P$ as $F_1, \ldots, F_r$ and suppose facet $F_i$ has $k_i$ flags. Similarly, index the facets of
Q as \(G_1, \ldots, G_s\) and suppose facet \(G_i\) has \(k'_i\) flags. By the induction hypothesis, \(F_i \times Q\) has \((m-1+n)k_i|\mathcal{F}(Q)|\) flags, and \(P \times G_i\) has \((m+n-1)|\mathcal{F}(P)|k'_i\) flags. Thus

\[
|\mathcal{F}(P \times Q)| = \sum_{i=1}^{r} \binom{m-1+n}{n} k_i |\mathcal{F}(Q)| + \sum_{i=1}^{s} \binom{m+n-1}{n-1} |\mathcal{F}(P)| k'_i
\]

\[
= \binom{m+n-1}{n} |\mathcal{F}(Q)| \sum_{i=1}^{r} k_i + \binom{m+n-1}{n-1} |\mathcal{F}(P)| \sum_{i=1}^{s} k'_i
\]

\[
= \binom{m+n-1}{n} |\mathcal{F}(Q)||\mathcal{F}(P)| + \binom{m+n-1}{n-1} |\mathcal{F}(P)||\mathcal{F}(Q)|
\]

\[
= \binom{m+n}{n} |\mathcal{F}(P)||\mathcal{F}(Q)|. \quad \square
\]

Also, the direct product of symmetry groups \(G(P) \times G(Q)\) acts on \(P \times Q\), with \(G(P)\) acting in the first \(m\) dimensions, and \(G(Q)\) acting in the last \(n\) dimensions. Hence

**Corollary 5.** The product \(P \times Q\) of a \(k\)-orbit \(m\)-polytope \(P\) and an \(l\)-orbit \(n\)-polytope \(Q\) has a divisor of \((m+n)kl\) flag orbits.

**Proof.** We have \(|\mathcal{F}(P)| = k|G(P)|\), and \(|\mathcal{F}(Q)| = l|G(Q)|\), so

\[
|\mathcal{F}(P \times Q)| = \binom{m+n}{n} k|G(P)| l|G(Q)|.
\]

The group \(G(P) \times G(Q)\) acts freely on the flags of \(P \times Q\), so there are \((m+n)kl\) flag orbits under this group. This group is a subgroup of the symmetry group \(G(P \times Q)\); if the symmetry group is larger, its order is a multiple of \(|G(P)||G(Q)|\). \(\square\)

By simple iteration, we see

**Corollary 6.** The product \(P_1 \times \cdots \times P_t\), where \(P_i\) is a \(k_i\)-orbit \(m_i\)-polytope, has

\[
\binom{m_1 + \cdots + m_t}{m_1, \ldots, m_t} |\mathcal{F}(P_1)| \cdots |\mathcal{F}(P_t)|
\]

flags, and the number of flag orbits is a divisor of

\[
\binom{m_1 + \cdots + m_t}{m_1, \ldots, m_t} k_1 \cdots k_t.
\]

The number \(\binom{n}{m_1, \ldots, m_t}\) is the multinomial coefficient, \(n!/(m_1! \cdots m_t!)\).
The symmetry group of a Cartesian product is larger than the direct product of the symmetry groups when there are repeated factors (i.e. two or more distinct polytopes among the \( P_i \) are congruent). In this case, if there are \( k \) copies of the repeated factor, the symmetry group grows by a factor of \( k! \). These are the \( k! \) permutations of the \( k \) sets of coordinates corresponding to each of the factors.

For instance, a triangular prism \( \{3\} \times \{\} \) has three flag orbits. A prism over this prism, \( \{3\} \times \{\} \times \{\} \), has 12 flag orbits if the two line segments are of different lengths; if both are the same length, there are only six flag orbits. A third prism, \( \{3\} \times \{\} \times \{\} \times \{\} \), has 60 orbits if all the line segments are of different lengths; it has 30 orbits if two line segments are the same; it has ten orbits if all three line segments are the same.

A \( d \)-dimensional Cartesian product can only be regular if it consists of \( d \) congruent factors, i.e. if it is \( \{\}^d \), which is the \( d \)-cube.

**4.1. The orbit graph of a Cartesian product.** Suppose \( \Phi \) is a flag in the Cartesian product of an \( m \)-polytope \( P \) with an \( n \)-polytope \( Q \). The \( l \)-face of \( \Phi \) has the form \( F_i \times G_j \) for some \( i \)-face \( F_i \) of \( P \) and some \( j \)-face \( G_j \) of \( Q \), where \( i + j = l \). Then the \((l + 1)\)-face of \( \Phi \) must either have the form \( F_{i+1} \times G_j \) or the form \( F_i \times G_{j+1} \). If we exclude the empty face, \( \Phi \) has \( m + n + 1 \) entries, starting with a vertex \( F_0 \times G_0 \) and ending with \( P \times Q = F_m \times G_n \). It follows that the faces of \( P \) which are factors of faces in \( \Phi \) are a flag \( \phi = \{F_0, \ldots, F_m\} \) of \( P \), and similarly the faces of \( Q \) which are factors are a flag \( \psi = \{G_0, \ldots, G_n\} \) of \( Q \). (For convenience, we continue excluding the empty face.)

Given these two flags \( \phi \) and \( \psi \), we construct a flag of \( P \times Q \) by starting with \( F_0 \times G_0 \) and incrementing either the index of the face from \( P \), or of the face from \( Q \), until we have reached \( m \) and \( n \) respectively. Among the \( m + n \) incrementing steps, we must choose \( m \) steps to increment the \( P \) face, and increment the \( Q \) face in the others. Thus, there are \( \binom{m+n}{m} \) different flags of \( P \times Q \) composed from the flags \( \phi \) and \( \psi \).

Let us encode these incrementation patterns by a string of \( m \) letters \( P \) and \( n \) letters \( Q \), placed where we increment the index from the corresponding face. For instance, the string
$PPQQQ$ corresponds to a flag of the form

$$\{F_0 \times G_0, F_1 \times G_0, F_2 \times G_0, F_2 \times G_1, F_2 \times G_2, F_2 \times G_3\}.$$ 

The string $QPQPQ$ corresponds to a flag of the form

$$\{F_0 \times G_0, F_0 \times G_1, F_1 \times G_1, F_1 \times G_2, F_2 \times G_2, F_2 \times G_3\}.$$ 

In growing from rank $l$ to rank $l + 2$, $\Phi$ can follow four possible patterns (namely $PP$, $PQ$, $QP$, or $QQ$). If the pattern is $PP$, the “diamond” formed in the Hasse diagram of $P \times Q$ between the $l$-face and $(l + 2)$-face of $\Phi$ is

$$\begin{array}{c}
F_{i+2} \times G_j \\
\nearrow & \nwarrow \\
F_{i+1} \times G_j & F_{i+1} \times G_j \\
\searrow & \swarrow \\
F_i \times G_j \\
\end{array}$$

The face $F_{i+1}'$ is the $(i + 1)$-face of the flag $\phi^{i+1}$ of $P$. We see that the $(l + 1)$-adjacent flag to $\Phi$ has the same incrementation pattern as $\Phi$, but applied to the flags $\phi^{i+1}$ and $\psi$.

If the pattern is $QQ$, then the diamond is

$$\begin{array}{c}
F_i \times G_{j+2} \\
\nearrow & \nwarrow \\
F_i \times G_{j+1} & F_i \times G'_{j+1} \\
\searrow & \swarrow \\
F_i \times G_j \\
\end{array}$$

The face $G'_{j+1}$ is the $(j + 1)$-face of the flag $\psi^{j+1}$ of $Q$. Here, the flag $\Phi^{i+1}$ has the same incrementation pattern, applied to the flags $\phi$ and $\psi^{j+1}$.

If the pattern is either $PQ$ or $QP$, then the diamond is

$$\begin{array}{c}
F_{i+1} \times G_{j+1} \\
\nearrow & \nwarrow \\
F_{i+1} \times G_j & F_i \times G_{j+1} \\
\searrow & \swarrow \\
F_i \times G_j \\
\end{array}$$
In this case, the \((l + 1)\)-adjacent flag to \(\Phi\) has a different incrementation pattern, transposing the letters \(P\) and \(Q\), applied to the same flags \(\phi\) and \(\psi\).

Suppose the flag orbits of \(P\) are \(\mathcal{O}_1, \ldots, \mathcal{O}_k\) and the flag orbits of \(Q\) are \(\mathcal{O}'_1, \ldots, \mathcal{O}'_{k'}\). We will organize the nodes of the orbit graph for \(P \times Q\) under the action of \(G(P) \times G(Q)\) in \(\binom{m+n}{m}\) layers, one for each incrementation pattern. (The order among the layers does not matter, except perhaps for ease of drawing.) Each layer will consist of \(kk'\) nodes, in each layer labeled as \(\mathcal{O}_i \times \mathcal{O}'_j\) for each \(i \in \{1, \ldots, k\}\), \(j \in \{1, \ldots, k'\}\).

Let us consider the edges of the orbit graph at a given node \(\nu\), labeled \(\mathcal{O}_i \times \mathcal{O}'_j\) for some fixed \(i\) and \(j\) and with the pattern \(T_1 \cdots T_{m+n}\) where each \(T_\eta\) is either \(P\) or \(Q\). The 0-edge agrees with the 0-edge in the orbit graph of \(T_1\): i.e. if \(T_1\) is \(P\) and \(\mathcal{O}_0^0\) is the 0-adjacent node to \(\mathcal{O}_i\) in \(G(P)\), then \(\nu\) is 0-adjacent to the node labeled \(\mathcal{O}_0^0 \times \mathcal{O}'_j\) (in the same layer). For each \(\eta \geq 1\), if \(T_\eta = T_{\eta+1}\) and \(\eta' := |\{\zeta \leq \eta \mid T_\zeta = T_\eta\}|\) (so \(T_\eta\) is the \(\eta'\)th occurrence of that letter), then the \(\eta\)-edge at \(\nu\) similarly agrees with the \(\eta'\)-edge in \(G(T_\eta)\). If \(T_\eta \neq T_{\eta+1}\), then the \(\eta\)-edge at \(\nu\) goes to the node labeled \(\mathcal{O}_i \times \mathcal{O}'_j\) in the layer with these two positions interchanged.

Figure IIIc shows an example of the construction for the product of a regular polygon \(P\) with a two-orbit polygon \(Q\), where the latter has two different edge lengths; for example, the product in \(\mathbb{E}^4\) of an equilateral triangle with a rectangle.

4.2. Orbit graph of a prism. In the particular case of the product of a \(d\)-polytope \(P\) with a line segment \(Q\), i.e. a prism over \(P\), the incrementation patterns are the \(d + 1\) strings consisting of \(d\) letters \(P\) and a single letter \(Q\). Now the relevant group is \(G(P) \times C_2\). Since \(Q\) has but one flag orbit, the nodes of each layer are a copy of the nodes of \(G(P)\). For a pattern with \(Q\) in the \(i\)th position, all the \(\eta\)-edges with \(\eta < i - 1\) agree with the \(\eta\)-edges of \(G(P)\). The \((i - 1)\)-edge and the \(i\)-edge go to the corresponding node of the layer with \(Q\) now in the \((i - 1)\)st position or the \((i + 1)\)st position, respectively (so long as \(Q\) is not already in the first or last position). If \(Q\) is in the first position, then the 0-edges are loops. (If \(Q\) is in the last position, then there are no \((i + 1)\)-edges.) The \(\eta\)-edges with \(\eta > i\) remain in the same layer and agree with the \((\eta - 1)\)-edges of \(G(P)\).
This construction is extremely similar to the construction of the orbit graph of a pyramid over a \(d\)-polytope in §3. Instead of \(d + 2\) copies of \(G(P)\), we have only \(d + 1\) copies, and the new 0-edges at the final layer are loops. Figure IIIc shows the orbit graph of a prism over an isosceles triangle (or any other three-orbit polygon).

**Figure IIIc.** The orbit graph of the product of a regular \(p\)-gon \(P\) with an isogonal two-orbit \(q\)-gon \(Q\), under \(G(P) \times G(Q)\)

4.3. **Duals of Cartesian products.** The dual to a Cartesian product of polytopes is the *free sum*, or *direct sum*, of the duals to the polytopes. The free sum \(P + Q\) of an \(m\)-polytope \(P\) and an \(n\)-polytope \(Q\) can be formed by placing \(P\) and \(Q\) in totally orthogonal
subspaces of $\mathbb{E}^{m+n}$ (e.g. the coordinate subspace corresponding to the first $m$ coordinates, and that corresponding to the last $n$ coordinates), with the centroids of both $P$ and $Q$ placed at the origin, and taking the convex hull. So, $(P \times Q)^* = P^* + Q^*$. By contrast with the product, which has a copy of each factor for each vertex of the other, $P$ and $Q$ do not appear as faces in the direct sum, but lie entirely in the interior. The facets of the free sum are the convex hull of a facet of $P$ with a facet of $Q$.

In particular, the dual of a prism $P \times \{\}$ is the bipyramid $P^* + \{\}$: we put a line segment through $P^*$ in a direction orthogonal to $\text{aff}(P)$, with its midpoint passing through the centroid of $P^*$, and take the convex hull. The facets of the bipyramid are pyramids over the facets of $P^*$. The vertex figure at the apices is $P^*$, and at the base vertices (i.e. the vertices of $P^*$) is a bipyramid over the original vertex figure.

The dual of a $p,q$-duoprism is called a $p,q$-duopyramid. This is the convex hull of a regular $p$-gon and a regular $q$-gon in totally orthogonal 2-planes in $\mathbb{E}^4$, both centered on the origin. So, using complex coordinates, one set of vertices for such a duopyramid would be $(e^{2ki\pi/p}, 0)$ for $k \in \{1, \ldots, p\}$ and $(0, e^{2li\pi/q})$ for $j \in \{1, \ldots, q\}$. Its facets are irregular tetrahedra, the convex hull of two line segments from totally orthogonal 2-dimensional planes.

5. Kleetopes

The Kleeotope of a polytope $P$ is a polytope formed by adding a new vertex above every facet $F$ of $P$, forming a pyramid over $F$. (As usual for pyramids, we place the apex on the normal line to $\text{aff}(F)$ through the centroid of $F$.) The apices should be placed sufficiently close that the union of $P$ with these pyramids over its facets is convex: that is, so that the line segment between two of the new apices passes through the interior of $P$. Thus, the original facets of $P$ are obliterated, and the pyramids which were formed over the ridges of $P$ become the new facets.

For instance, the Kleeotope of a cube has 24 triangular facets; the old square faces are contained in the interior of the polyhedron.
The Kleetope of a polyhedron \( P \) with \( p \)-gonal facets is called a \( p \)-akis \( P \), referring to the \( p \)-fold increase in the facets. For instance, the Kleetope of a tetrahedron is the triakis tetrahedron, and that of the cube is the tetrakis cube.

This operation is known by a wide variety of names. In the Conway polyhedron notation, it is called \( \text{kis} \), with operator \( k \) (in reference to the \( p \)-akis names). It is also known as cumulation, accretion, augmentation, akisation (another reference to “becoming \( p \)-akis”), apiculation (“to raise to a peak”), or “stellar subdivision in a facet”.

The Kleetope operation is dual to truncation (discussed in §6), in that the Kleetope of \( P \) is dual to the truncation of \( P^* \).

**Proposition 7.** The Kleetope of a \( d \)-polytope \( P \) has \( d \) times as many flags as \( P \).

**Proof.** The facets of \( P \) being \( (d - 1) \)-polytopes, by Proposition 2 the pyramid over each facet \( F \) has \( (d + 1) \) times as many flags as \( F \). Removing the base \( F \), the other facets of this pyramid have a total of \( d |F(F)| \) flags.

Then the total number of flags in the Kleetope is the sum of \( d |F(F)| \) for each facet \( F \) of \( P \), which is \( d \) times the number of flags of \( P \). \( \square \)

One way to manipulate the number of flag orbits of a polytope is to create “partial Kleetopes”, by raising pyramids on just some facets. The resulting increase in flags interacts with the resulting changes in symmetry.

For example, the 3-cube has 48 flags in one flag orbit. Raising a pyramid on one facet increases the number of flags by 16.

- Raising a pyramid on just one facet reduces the symmetries to 8, so there are 8 flag orbits.
- Raising pyramids on two opposite facets leaves 16 symmetries, with 48 + 32 flags, so gives 5 flag orbits.
- Raising pyramids on two adjacent facets leaves just 4 symmetries, so gives 20 flag orbits.
• Raising pyramids on the three facets incident to a given vertex leaves just 6 symmetries, with 96 flags, so this has 16 flag orbits.

• Raising pyramids on three facets in a row, so that two of them are opposite facets, leave just 4 symmetries, so there are 24 flag orbits.

• Raising pyramids on four facets forming a ring leaves 16 symmetries, with 112 flags, so there are 7 flag orbits.

• Raising pyramids on the four facets incident to the endpoints of a given edge leaves just 4 symmetries, so there are 28 flag orbits.

• Raising pyramids on five facets leaves 8 symmetries, with 128 flags, so there are 16 flag orbits.

• Raising pyramids on all six facets yields the Kleetope of the cube, with 48 symmetries, 144 flags, and 3 flag orbits.

A particular case of a partial Kleetope which we will encounter is the alternate Kleetope. This can be constructed if it is possible to select “every other facet” of a polytope $P$. That is, there must be a subset of the facets of $P$ such that no two of the chosen facets are adjacent to each other, but every facet outside the set is adjacent only to facets within it. Equivalently, the graph whose nodes are the facets of $P$, with edges whenever the facets are adjacent, must be a bipartite graph. This is the case exactly when each subridge of $P$ is in evenly many facets (Coxeter 1973, p. 154). When this is possible, the alternate Kleetope raises a pyramid over each of the selected facets, but not the others. In some cases, such as the cuboctahedron, each of the two partite sets of facets results in a different polytope, so the selected facets must be indicated. (This is succinctly done in this case by using the name “tetrakis cuboctahedron” or “triakis cuboctahedron”, each of which refers to a different alternated Kleetope.) In the case of the octahedron, the choice of partite set is irrelevant.

A sequence of “further” Kleetopes can be formed in the same manner—adding a new vertex over every facet—by increasing the height of the new apices. As the apices are pushed further away, the first change in the combinatorial isomorphism class of the resulting polytope...
comes when, for the first time, two pyramids formed over a ridge become coplanar. In the event that the apices may be chosen such that the two pyramids formed over every ridge are coplanar, we will call the result an extended Kleetope. In this extended Kleetope the line segment between two new vertices above adjacent old facets does not go through the interior of $P$, but passes through the ridge between the facets. This operation is called join in the Conway notation, with operator $j$. It is dual to rectification $t_1$, defined in the next section, in that the extended Kleetope of $P$ is dual to $t_1 P^*$. Unlike the basic Kleetope, the extended Kleetope may not be possible to form for an arbitrary polytope $P$. For instance, the polyhedron in Figure IIIe does not admit such an operation. The coordinates of its vertices are $(-1, 0, \pm 1)$ and $(\frac{1}{2}, \pm \sqrt{3}, \pm 2)$. The apices over the lateral, quadrilateral faces have a unique possible placement if placed orthogonally above the face centroids, namely at $(2, 0, 0)$ and $(-1, \pm \sqrt{3}, 0)$. The ridge-traversing line segments from the latter two of these points have an intersection which is not on the ridge-traversing line from the first.

![Figure IIIe. A polyhedron not admitting an extended Kleetope](image)

It suffices that $G(P)$ acts transitively on the facets and ridges of $P$ for the extended Kleetope to exist. Alternatively, having projected $P$ onto an enclosing sphere to form a sphere tiling, we can always perform an equivalent operation, dividing each tile into pyramids over its facets, and joining each pair of pyramids over the same base.
Another way to think of the Kleetope operation is as the convex hull of the polytope \( P \) together with its reciprocal \( P^* \), with respect to a reciprocating sphere chosen so that the vertices of \( P^* \) are just outside the facets of \( P \). As we increase the size of the reciprocating sphere, the resulting Kleetope changes also. We will call these operations \( K_{i,j} \) (provided they can be performed) with the subscripts to be explained. We start with \( K_0 P \), which is just \( P \) itself; the facets are the convex hull of each facet \( F \) of \( P \) with its dual face, \( F^* \), a vertex lying within \( F \). As the reciprocating sphere grows, the dual vertices rise up, and the facets are now the convex hulls of each ridge \( R \) of \( P \) with the dual vertices \( F^* \) of the facets incident to \( R \). This is the usual meaning of Kleetope, \( K \; P \; = \; K_{0,1} \; P \). When we reach the extended Kleetope \( K_1 \), the new facets are the convex hull of each ridge and the corresponding face of the dual polytope, which is an edge, orthogonal to the affine hull of the ridge. As we continue to grow the reciprocating sphere, the facets become the convex hull of each subridge and its corresponding dual face (a polygon), and so on. In general, the facets of \( K_{i,i+1} \; P \) are the convex hulls of each \((d - i - 2)\)-face \( F \) of \( P \) with the \( i \)-faces of \( P^* \) corresponding to \((d - i - 1)\)-faces incident to \( F \). The facets of \( K_i \; P \) are the convex hulls of each \((d - i - 1)\)-face of \( P \) with its own dual \( i \)-face; this occurs if the reciprocating sphere is tangent to each \((d - i - 1)\)-face. Thus we have a sequence of polytopes \( P = K_0 \; P, K_{0,1} \; P, K_1 \; P, K_{1,2} \; P, K_2 \; P, \ldots \). The indices are chosen so that \( K_{i,i+1} \) is dual to the truncation operation \( t_{i,i+1} \) described below.

6. Truncations

A very wide variety of operations on a polytope fall into the category of truncations. The most basic is vertex truncation: Chop off every vertex of the polytope with a hyperplane orthogonal to the vector to the vertex from the centroid. These hyperplanes should not meet in \( P \). For a polyhedron, each \( q \)-valent vertex is replaced by \( q \) new vertices, each of which is 3-valent. We choose the hyperplanes to be at the same distance from each vertex.

This operation is denoted \( t \), or more verbosely \( t_{0,1} \); this operator is used by Coxeter, Johnson, and Conway alike.
The facets of \( tP \) are the truncated facets, \( tF \), for each facet \( F \) of \( P \), along with a new facet around each missing vertex \( v \), which is the vertex figure at \( v \). (To be precise, these facets are symmetry-equivalent to the vertex figure \( P/v \) as we have defined it, and may even coincide. Sometimes, the vertex figure at \( v \) is defined to be this object.) Hence, the number of facets \( f_{d-1}(tP) \) is \( f_{d-1}(P) + f_0(P) \). The vertex figure of each vertex \( v \) of the truncation, lying on an edge \( e \) of \( P \), is a pyramid over the edge figure of \( e \).

The collection of truncating hyperplanes for all the vertices of \( P \) are exactly the bounding hyperplanes for a reciprocal polytope \( P^* \). So, we can consider the truncation to be the intersection of \( P \) with its reciprocal, with the reciprocating sphere chosen appropriately. (However, if \( P \) is not vertex-transitive, then the facets of \( P^* \) may not be the same distance from each corresponding vertex, as we specified above. In fact, it may not even be possible to simultaneously have each facet of \( P^* \) meet \( P \) between the corresponding vertex and the other vertices of \( P \).)

As we move the truncating planes inward from each vertex, (or, for a vertex-transitive polytope, as we shrink the reciprocating sphere), we achieve further degrees of truncation. At first, the facets which are vertex figures of the original polytope increase in size, without changing the isomorphism class of the truncation. The first change in isomorphism class comes when the points of intersection of two truncating planes with an edge coincide. If this happens on every edge simultaneously, so that the truncating plane for each vertex \( v \) passes through the midpoint of every edge incident to \( v \), the resulting truncation is called the rectification of \( P \), with operator \( t_1 \) (which is understood as “truncation to the 1-face-midpoints”). This notation is described in Coxeter [1985].

Johnson uses the operator \( r \) for this operation [1966, p. 57]. On the other hand, Coxeter uses an operator \( r \) to mean “rhombi-” in (Coxeter 1940a). But as this is only applied to the rhombicuboctahedron \( r \{3\}_{4} \) and the rhombicosidodecahedron \( r \{3\}_{5} \), and these are actually the rectification of the cuboctahedron and the icosidodecahedron respectively, there is no disagreement. The same operation is called “ambo”, or more verbosely “1-ambo”, with operator
a in the Conway polyhedron notation, apparently from the Greek αμβων, meaning “edge” or “rim” (Conway and Sloane 1991).

Although it is always possible to take the convex hull of the edge midpoints of a given polytope $P$, in general the midpoints of the edges incident to a given vertex $v$ of $P$ will not lie in a hyperplane, and so will form several facets. In order for $P$ to have a rectification, these points must form a single facet, which will be the vertex figure of $v$. This is always true if the vertices of $P$ lie on a sphere, and all the edges of $P$ have the same length.

For a polyhedron $P$, the rectification is always realizable. That is, there is a polyhedron whose vertices correspond to the edges of $P$, with each set of vertices which correspond to the edges of a facet of $P$ forming a facet, and each set of vertices which correspond to the edges at a vertex of $v$ also forming a facet. This is essentially a consequence of Steinitz’s theorem that any 3-connected planar graph is the edge graph of a polyhedron (Grünbaum 2003). For higher dimensions, we must assume that $P$ is sufficiently symmetric for rectification or other higher degrees of truncation to be defined.

A polyhedron is one instance of a map—an embedding of a graph on a 2-dimensional surface. In the context of maps, the result of rectification is called the medial map. These are examined in Río-Francos et al. (2013).

The rectification of $P$ has half as many vertices as the truncation of $P$, with one vertex per edge of $P$, rather than two. It still has the same number of facets as $tP$, consisting of the vertex figures of each vertex of $P$ and the rectification of each facet of $P$. The vertex figure at the midpoint of the edge $e$, as a vertex of the rectification, is a prism over the edge figure of $e$.

If we move the truncating planes yet further inward, they begin to cut the facets cut out by other truncating planes. Thus, the facets which arise as vertex figures become truncated themselves. The former edges of $P$ lie entirely outside the new polytope, and the vertices are on the line segment between the edge-midpoints of $P$ and the centroids of the incident 2-faces. This suggests why the resulting polytope is called $t_{1,2}P$. Johnson calls this operation
bitruncation. For a map \( M \), the resulting map is known as the leapfrog map of \( M \) (Río-Francos 2013, §4.2).

As we continue to move the truncating planes inward, they will successively encounter the centroids of the \( i \)-faces of \( P \). The result is called \( t_i P \). Truncation to the 2-face centroids, \( t_2 P \), is called birectification; \( t_3 \) can be called trirectification, and so on. When the truncating planes are between the scale for \( t_i P \) and \( t_{i+1} P \), the resulting polytope is denoted \( t_{i,i+1} P \), and called \((i + 1)\)-truncation (bitruncation, etc.). The vertices of \( t_{i,i+1} P \) lie on the line segment between \( i \)-face centroids and \((i + 1)\)-face centroids of \( P \).

This process can continue until the truncating planes pass through the facet centroids of \( P \), yielding \( t_{d-1} P \): this is the dual polytope to \( P \). (Once again, we must consider that these truncations can only be defined if \( P \) is sufficiently symmetric. While it is always possible to take the convex hull of the centroids of each facet of a polytope \( P \), the result will not necessarily be dual to \( P \).)

Inductively, we can show that \( t_i P \) has \( \binom{d-1}{i} \) times as many flags as \( P \). The facet arising at each old vertex \( v \) is \( t_{i-1}(P/v) \), and in place of each facet \( F \) we have \( t_i F \). Thus, inductively, the total number of flags is

\[
\sum_v \binom{d-2}{i-1} |\mathcal{F}(P/v)| + \sum_F \binom{d-2}{i} |\mathcal{F}(F)| = \left( \binom{d-2}{i-1} |\mathcal{F}(P)| + \binom{d-2}{i} |\mathcal{F}(P)| \right).
\]

Similarly, \( t_{i,i+1} P \) has \( \binom{d}{i+1} \) times as many flags as \( P \). The facets arising from vertex figures are \( t_{i-1,i}(P/v) \), with \( \binom{d-1}{i} |\mathcal{F}(P/v)| \) flags, and in place of each facet \( F \) we have \( t_{i,i+1} F \) with \( \binom{d-1}{i+1} |\mathcal{F}(F)| \) flags. (Here, we understand \( t_{-1,0} = t_0 \) to be identity operations.)

We can generalize these operations still further. Analogously to \( t_{i,i+1} \), we let \( t_{i,j} P \) be a polytope whose vertices are on the line segment joining the centroids of an incident \( i \)-face and \( j \)-face of \( P \). Yet more generally, for a subset \( I \subseteq \{0, \ldots, d-1\} \), \( t_I P \) has its vertices somewhere in the simplex spanned by the centroids of mutually incident \( i \)-faces for each \( i \in I \). (In other words, the vertices are chosen in the faces of the barycentric subdivision of
corresponding to each chain of type $I$ in $\mathcal{L}(P)$. For definiteness, we can specify that each vertex of $t_P$ is the centroid of the corresponding simplex of face centroids.

These generalized truncations are best understood through Wythoff’s construction. Suppose that $G$ is a finite reflection group whose fundamental region is a simplex $R$. Wythoff’s construction generates a polytope from any point $v$ chosen within the fundamental region $R$, as the convex hull of the orbit $G \cdot v$. If $P$ is a regular polytope, then the fundamental regions of its symmetry group are the chambers of a barycentric subdivision; the vertices of the fundamental region can be labeled $C_i$ if they are in the relative interior of an $i$-face of $P$. The polytope $P$ is generated by Wythoff’s construction with the initial vertex at $C_0$. The polytope $t_i P$ is generated by Wythoff’s construction with the initial vertex at $C_i$, and the polytope $t_I P$ is generated by Wythoff’s construction with the initial vertex somewhere in the face of $R$ spanned by the points $\{C_i \mid i \in I\}$.

Thus, with Coxeter diagrams, $t_I \{p, q, \ldots, r\}$ is represented by the string diagram

```
p q .......... r
```

with the $i$th dot ringed for each $i \in I$. In Conway’s notation, $t_{\{i, \ldots, j\}}$ is called “$i \ldots j$-ambo”.

When $I = \{0, \ldots, d - 1\}$, the operation is called omnitruncation. This corresponds to selecting an initial vertex for Wythoff’s construction in the interior of the fundamental region. Hence, the generating group $G$ acts simply transitively on the resulting vertices; the generated polytope $Q$ has $f_0(Q) = |G|$. For a regular polytope $P$, the omnitruncation of $P$ has one vertex for each flag of $P$: the edge graph of the omnitruncation is the flag graph of $P$. Thus, the omnitruncation of $P$ is dual to the barycentric subdivision of $P$ (discussed below).

In the case of 3-polytopes, the omnitruncation operation is called “bevel” by Conway, with symbol $b$. For polyhedra, it is achieved by “beveling” the edges (cutting off each edge by a plane parallel to it) and vertices. Johnson calls the operation for 3-polytopes “cantitruncation,” which means $t_{0,1,2}$ in general.

When $I = \{0, d - 1\}$, the vertices are chosen between the original vertex locations and the centers of each incident facet. This results in the facets of $t_{0,d-1} P$ being “shrunk” in place,
while new facets form in the spaces left between. This operation is known as *contraction.* It is equivalent to think of the facets remaining the same size, but moving each away from the center of the polytope, and again filling in the spaces left between. In this context, the operation is called *expansion* (although it has the same effect). These operations were pioneered by Stott, and allow the generation of many interesting polytopes. Conway calls this operation “expand”, e. For 3-polytopes, Coxeter preferred to refer to this operation with the “rhombi” prefix, e.g. denoting $t_{0,2}\{p, q\}$ by $r\{p\}_q$ (Coxeter 1940a, p. 394; Coxeter, Longuet-Higgins, and Miller 1954, pp. 403–404).

7. Barycentric subdivision

The barycentric subdivision $B$ of a $d$-polytope $P$ has as its vertices an internal point, say the centroid, of each proper face of $P$. Edges of $B$ connect the vertices corresponding to incident faces of $P$. The 2-faces of $B$ are triangles whose vertices correspond to each mutually incident set of three faces of $P$, and so on, so that $i$-faces of $B$ correspond to chains in $L(P)$ of cardinality $i$. The facets or *chambers* of the barycentric subdivision are $(d - 1)$-simplices corresponding to flags of $P$ (with the improper faces removed).

Normally, the barycentric subdivision of a polytope is simply the above construction, which lives in the boundary of $P$. Frequently, we identify a flag $\Phi$ of $P$ with its corresponding chamber. The $(d - 2)$-faces of the chamber $\Phi$ are called *walls*. The wall opposite the centroid of an $i$-face of $P$ is called the $i$th wall, and the chamber adjacent to the chamber $\Phi$ across the $i$th wall corresponds to the $i$-adjacent flag $\Phi^i$.

The barycentric subdivision can also be understood as an operation on the sphere tiling obtained by projecting $P$ onto an enclosing $(d - 1)$-sphere centered at a fixed point of $G(P)$. In the Conway notation, this operation on the spherical complex is called meta, with symbol $m$. For a 3-polytope $P$, this is the dual operation to beveling: $(bP)^* = mP^*$.

If we wish to regard the barycentric subdivision as a polytope operation constructing a new convex polytope from $P$, we must modify the result so that all the chambers are facets, and are not coplanar with most of their adjacent chambers. A simple way to proceed is to
project the barycentric subdivision onto a sphere enclosing $P$, then take the convex hull of the images of the vertices of $B$; however, this may not always give a convex polytope isomorphic to the barycentric subdivision. Schulte and Williams (n.d., Lemma 3) show that, if instead of simply projecting all the vertices of the subdivision onto the same sphere, we place them more carefully with respect to the supporting hyperplanes of the faces of the polytope, then the barycentric subdivision of any convex polytope can be realized as a convex polytope.

Since there is a facet for each flag of $P$, and each facet is a $(d - 1)$-simplex, $mP$ has $d!$ times as many flags as $P$ does. Thus, in general, it also has $d!$ times the number of flag orbits under $G(P)$, although it is possible to gain symmetries through this operation. For instance, $m\{3,3\}$ is a Kleetope of the cube and has cubic symmetry, so it has three flag orbits rather than six.

8. Alternation

The alternation operation requires a polytope $P$ where one can select “every other vertex”, i.e., such that the edge graph consisting of the vertices and edges of $P$ is bipartite. This is the case for any polytope of type $\{p,q,\ldots\}$ with $p$ even, or indeed whenever every 2-face has an even number of sides (Coxeter 1973, p. 154). Having selected such a subset $V$ of vertices of $P$, take the convex hull of $V$ and forget about the rest: this is the alternation of $P$.

More precisely, although the above operation can be done for any polytope with a bipartite edge graph, for the result to be called an alternation we desire the set of vertices adjacent to each discarded vertex to lie in a hyperplane, so that the vertex figures of the discarded vertices appear as facets in the result.

This operation is also referred to as halving, with symbol $\eta$ in McMullen and Schulte (2002). The Conway operation is called half, with symbol $h$. Coxeter (1973, §8.6) refers to it as partial truncation, or alternation, and also gives it the symbol $h$.

The resulting polytope has half the number of vertices of $P$. The facets are the alternated facets of $P$, along with the vertex figures of the missing vertices (these vertex figures involve the actual adjacent vertices, rather than internal points of the incident edges). Thus, the
number of facets of an alternated $d$-polytope $P$ is $\frac{1}{2}f_0(P) + f_{d-1}(P)$, unless $d = 2$, in which case the "alternated facets" are just points, or $d = 3$, in which case the "alternated facets" may just be edges if the original facets were squares.

The dual of the alternation of a polytope $P$ is a kind of extended alternate Kleetope of $P^*$. If an alternation of $P$ exists, then in $P^*$ it is possible to select alternate facets, and form an alternate Kleetope. Place apices over those facets of $P^*$ which are dual to the deleted vertices of $P$, so that the pyramids formed over the ridges are coplanar with the adjacent, unmodified facets. For greater applicability, rather than placing the apices on the normal line through the centroid of a facet, choose the intersection of the defining hyperplanes of the adjacent facets, if it exists. We denote this operation by $aK_\star$. When considering a polytope as the intersection of the closed half-spaces of its facet-defining hyperplanes, this operation amounts to discarding the defining hyperplanes of the selected set of facets. This can be done for any polytope with a bipartite facet graph, but only when the defining hyperplanes of the adjacent facets meet in a single point above each discarded facet do we call the result an extended alternate Kleetope.

We have $(hP)^* = aK_\star P^*$, where the reciprocal $P^*$ is taken with respect to the centroid of the remaining vertices in $hP$ (which may differ from the centroid of $P$ itself).

These operators disguise the fact that a choice of a partite set of vertices or facets is involved, and so should generally only be used when the outcome is independent of this choice. (For instance, if $P$ is the rhombic dodecahedron, then $hP$ could mean either a cube or an octahedron; the same is true of $aK_\star \{\frac{3}{4}\}$.)

For a polytope whose 2-faces are all 4-gons, the vertex figure of a vertex $v$ remaining in the alternation is the rectification of its original vertex figure. Each edge $e$ in the original vertex figure of $v$ corresponds to a 4-gon incident to $v$, which in the alternation becomes an edge from $v$ to its opposite corner. Thus the edge $e$ is replaced by its midpoint.

Any $d$-polytope whose 2-faces all have evenly many sides has at least some quadrilateral faces if $d \geq 3$. Hence, in any 2-transitive polytope with even-sided 2-faces, the 2-faces
are 4-gons, so the previous remark about vertex figures actually covers most cases we will encounter.

9. Remark

To justify the statement at the beginning of the chapter that every polytope with few flag orbits can be constructed by these operations, starting from polygons, it should be mentioned that the icosahedron can be formed as an alternation of a truncation of the octahedron (to a particular distance); the 24-cell is the rectification of the 4-crosspolytope; and the 600-cell can be formed as a Kleetope of an alternation of a truncation of the 24-cell.
CHAPTER IV

\textit{k}-orbit Polygons

1. In zero or one dimension

In the smallest dimensions, we do not need to separately classify the set of \textit{k}-orbit polytopes for each \textit{k}. There is only one 0-dimensional polytope, the point, and only one 1-dimensional polytope, a line segment.

The 0-polytope has also been dubbed a “monad” by Johnson (n.d.). The line segment could also be called an edge, but this name is usually reserved for a line segment which is a face of a larger polytope, just as “vertex” is reserved for 0-dimensional faces. By analogy with the names “polyhedron” and “polytope”, the 1-polytope should have a name signifying its two constituent facets, the endpoints. But “gon” is the Greek root referring to points, and by historical accident, “polygon” refers to 2-polytopes, and “digon” means a 2-polytope with 2 edges (either a degenerate case in Euclidean space, or a tiling of $S^1$). It seems that 2-polytopes could more properly be called “polytoxa”, “toxon” being the root referring to edges. This association seems to have first been coined by Grünbaum and Shephard (1978). So we could call the 1-polytope a “toxon”, but somehow, like “gon” and “hedron”, this root is only used in relation to the facets of polytopes of one dimension higher. It is called a “dyad” in Johnson (1966), inter alia, but this does not seem to have caught on. More recently, Johnson has suggested “ditel”, based on the Greek τέλος, meaning an end (Johnson 2012).

Each of these is regular; so there are no \textit{k}-orbit \textit{d}-polytopes for \textit{d} \leq 1 and \textit{k} \geq 2. A 1-polytope has two flags, so it could be colored so as to be two-orbit (by coloring its two endpoints different colors). But any realization as a line segment in $\mathbb{E}^1$ (or indeed, as a line segment in $\mathbb{E}^d$ for any \textit{d}), or as a tiling of $S^0$, is geometrically regular.
2. Two dimensions

The case of 2-polytopes is no longer trivial, but remains simple enough to classify all the
$k$-orbit polygons, for any $k \geq 1$, rather than treating each value of $k$ separately. We shall
prove that there are infinitely many $k$-orbit polygons (up to symmetry equivalence) for any
$k \geq 1$.

- If $k$ is odd, there is a single symmetry type of $k$-orbit polygon, whose orbit graph
  is a path of length $k - 1$ alternating 0-edges and 1-edges. This symmetry type is
  realized by $pk$-gons, for each $p \geq 3$ (if $k = 1$) or each $p \geq 1$ (otherwise).
- If $k$ is even and $k \geq 4$, there are three symmetry types of $k$-orbit polygon.
  (1) A path of length $k - 1$, alternating 0-edges and 1-edges:

```
1 • — — — 0 — — 1
```

This is realized by $pk$-gons, for each $p \geq 1$.

(2) Dual to (1), a path of length $k - 1$, alternating 1-edges and 0-edges:

```
0 • — — — 1 — — 0
```

This is also realized by $pk$-gons for each $p \geq 1$.

(3) A cycle with $k$ nodes, alternating 0-edges and 1-edges. This is realized by

$(p^k/2)$-gons for each $p \geq 2$ (if $k = 4$) or each $p \geq 1$ (otherwise).

- If $k = 2$, only the symmetry types (1) and (2) of the above case are found, i.e.
  $1 • — — • 1$ and $0 • — — • 0$. Each symmetry type is realized by $2p$-gons for each
  $p \geq 2$.

If we allow digons, then the restriction on $p$ for the case $k = 1$ can be loosened to $p \geq 2$.

The restriction for type $\bullet — — \bullet$ with $k = 2$ can be removed, allowing $p \geq 1$, if we have a digon
as a tiling of $S^1$ with two different edge lengths. The dual type, $\bullet — — \bullet$, needs to have two
dges of the same length, and vertices with different angle measures. This could conceivably
be realized as a tiling of a simple closed curve embedded in $\mathbb{E}^2$, homeomorphic to $S^1$, but
with only one reflectional symmetry, such as a teardrop shape. Finally, the restriction for
type (3) with \( k = 4 \), could similarly be removed, allowing \( p \geq 1 \), with a digon
tiling a homeomorphic embedding of \( S^1 \) with no symmetries.

The proof that these are the only possible orbit graphs for 2-polytopes is simple.

\[ \text{PROOF.} \] Every node of the orbit graph is incident to a 0-edge and a 1-edge, either of which
may actually be a loop.

\textit{Case 1.} If the orbit graph contains no loops, then it must be a cycle. Since the edges
alternate label, the cycle must have evenly many edges.

We cannot have a cycle with just two edges, i.e. the graph
\[
\begin{array}{c}
\text{1} \\
\bullet \\
\text{0}
\end{array}
\]

because then the polygon is both vertex-transitive and edge-transitive (by Proposition II.1),
hence regular. (So the graph would not have two nodes.)

Thus, these cycles appear with an even number \( k \) of nodes, with \( k \geq 4 \).

\textit{Case 2.} If any node is incident to two loops, then this must be the only node, so we have
\( k = 1 \) and the graph \( 0 \circ \circ 1 \) (which is vacuously “a path of length \( k - 1 \)” as described.)

\textit{Case 3.} Suppose some node \( u \) is incident to just one loop. We can start a walk from \( u \) by
traversing the single edge which is not a loop, then continue at each node by traversing the
edge other than the one we came in on, if it exists. This can only terminate when we come to
another node \( v \) incident to a loop. By connectivity, we must have traversed the entire graph,
so the graph (disregarding the loops) is a path alternating 0-edges and 1-edges.

If \( k \) is odd, then there are evenly many edges in the path. Hence, if the edge incident to
\( u \) is a 0-edge, then the edge incident to \( v \) must be a 1-edge. So, if we determine to draw this
graph with the endpoint incident to the 1-edge first, there is only one possibility.

If \( k \) is even, so that there are oddly many edges in the path, then there are two possibilities:
either the path starts and ends with 0-edges, so that the two loops in the graph are 1-loops,
or the path starts and ends with 1-edges, and the two loops in the graph are 0-loops. \( \square \)
3. Construction

We have proved that any $k$-orbit polygon must have one of the claimed orbit graphs. We now show that $pk$-gons realizing the path graphs actually exist for each of the claimed values of $p$. Subsequently, we will deal with the cyclic orbit graphs.

3.1. For $p$ at least 3. For $p \geq 3$, take a regular $p$-gon $Q$. Let $\ell = \lceil \frac{k}{2} \rceil$. Let $e$ be an edge of $Q$. Say its endpoints are $v_0$ and $v_k$. Now, above $e$, construct $k$ consecutive line segments of $\ell$ distinct lengths, beginning at $v_0$ and ending at $v_k$; label the new endpoints of these line segments $v_i$ so that the $i$th line segment joins $v_{i-1}$ to $v_i$. Choose the line segments to be preserved by the reflection through the perpendicular bisector of $e$ (i.e. so that this reflection carries each $v_i$ to $v_{k-i}$.) Also ensure that the angle between two line segments at each vertex is less than $\pi$, i.e. that the region bounded by $e$ and these new line segments is convex, and that the angle between $e$ and the line segment at $v_0$ is less than $\frac{\pi}{p}$. See Figure IVA.

![Figure IVA](image-url)

**Figure IVA.** Constructing new line segments above an edge of the regular $p$-gon $Q$

The images of these edges under the symmetries of $Q$ form an $pk$-gon $P$, with $k$ edges similarly positioned over each edge of $Q$. The polygon $P$ has all the $2p$ symmetries of $Q$, but with $2pk$ flags, resulting in $k$ orbits, as guaranteed by the construction. This is a consequence of the fact that $\ell$ distinct edge lengths were taken, since then each new edge may only be mapped to the corresponding locations over each edge of $Q$. It is also possible to achieve the same effect by having the first edge $v_0v_1$ be one length, and the other edges (besides $v_{k-1}v_k$) all of a second length, so long as $k > 2$.

The two flags of $P$ involving the vertex $v_0$ are in the same orbit. Changing to the 0-adjacent flag of each of these (i.e. considering the other endpoint of the edge) is a different flag; thus the orbit graph of $P$ begins $1c \ldots 0 \ldots$ . When $k$ is even, then when we reach
the vertex \( v_{k/2} \) (for instance, the vertex \( v_2 \) on the left of Figure IV-A), the 1-adjacent flags are in the same orbit (via the reflection bisecting \( e \)). Thus the orbit graph is on \( k \) nodes (one each for \( v_0 \) and \( v_{k/2} \), and two each for \( v_1, \ldots, v_{k-1} \)):

\[
1 \rightarrow 0 \rightarrow 1 \rightarrow 0 \cdots 0 \rightarrow 1 \rightarrow 0 \rightarrow 1
\]

When \( k \) is odd, then when we reach the edge \( v_{k-1}v_{k+1} \), the 0-adjacent flags are in the same orbit. The orbit graph is again on \( k \) nodes:

\[
1 \rightarrow 0 \rightarrow 1 \rightarrow 0 \cdots 0 \rightarrow 1 \rightarrow 0
\]

When \( k = 1 \), the construction returns the regular \( p \)-gon \( Q \). When \( k = 2 \), we have \( \ell = 1 \) and so all the edges of \( P \) are the same length; a further measure is required to ensure that \( P \) is not a regular \( 2p \)-gon. Choose \( v_1 \) such that the internal angle between the edges is not \( \frac{p-1}{p}\pi \) (the internal angle of a regular \( 2p \)-gon). Equivalently, ensure that the angle between the edge \( e \) and the new line segment added at \( v_0 \) is not \( \frac{1}{2p}\pi \).

In the case that \( k \) is even, the dual polygon to \( P \) will have a different orbit graph, namely

\[
0 \rightarrow 1 \rightarrow 0 \rightarrow 1 \cdots 1 \rightarrow 0 \rightarrow 1 \rightarrow 0
\]

3.2. For smaller values of \( p \). For \( p = 2 \), carry out the same construction as above, on either side of a single line segment. That is, we construct \( k \) edges above a line segment \( e \) in the same manner as above. Then, reflect the \( k \) edges across \( e \) to form a polygon with \( 2k \) edges.

For \( p = 1 \), a similar construction may be performed “above a point” rather than an edge, and symmetric about a ray from the point. Examples are shown in Figure IV-B.

3.3. Cyclic type. For even \( k \) greater than 2, another symmetry type exists, with a cyclic orbit graph. We show that \( (p^k) \)-gons of this type exist for each \( p \geq 1 \) (or \( p \geq 2 \) if \( k = 4 \)).

For \( p \geq 3 \), take a regular \( p \)-gon \( Q \) and fix an edge \( e \). Construct \( \frac{k}{2} \) line segments, all of distinct lengths, above \( e \), in a manner similar to the method above (ensuring that all internal
angles are smaller than \( \pi \), as before). Translate these new line segments to every edge of \( Q \) by all the rotations in \( G(Q) \). These line segments form a \( \left( \frac{nk}{2} \right) \)-gon \( P \), which is preserved only by the rotational symmetries of \( Q \), a group of order \( p \). Thus \( P \) is a \( k \)-orbit polygon (but see caveats below when \( k = 4 \)).

For \( p = 2 \), we put \( \frac{k}{2} \) line segments, all of distinct lengths, on one side of a given line segment \( e \), and add their images under a half-turn about the midpoint of \( e \), forming a \( k \)-gon.

For \( p = 1 \) and \( k \geq 6 \), we simply form a closed path of \( \frac{k}{2} \) line segments, all of different lengths.

In a polygon with this orbit graph, 0-adjacent flags are always in different orbits. For this, it suffices that the endpoints of every edge not be in the same vertex orbit. In the case that \( k \geq 6 \), the differing edge lengths are enough to guarantee this. However, when \( k = 4 \), we have just two edge lengths, and every vertex is in one edge of each length. So in this case, we must make sure that the vertices have different angles. As before, it suffices to make sure that the internal angles are not those of a regular \( 2p \)-gon, namely \( \frac{p-1}{p} \pi \).

For the sake of a definite algorithm, when \( k \geq 6 \) we can proceed as follows. Let \( \ell = \frac{k}{2} \).

We will place all the new points \( v_1, \ldots, v_{\ell-1} \) on the circumcircle of \( Q \). Divide the arc length above the edge \( e \) of \( Q \) into \( \left( \frac{\ell+1}{2} \right) \) units of the same length. The arc length of each piece will be \( \frac{2\pi}{p\left( \frac{\ell+1}{2} \right)} \). Place \( v_1 \) one unit away from \( v_0 \), \( v_2 \) two units from \( v_1 \), and so on, with each \( v_i \) placed \( i \) units distant from \( v_{i-1} \).
4. Coverings

When the restricted orbit graph for a 2-face of a higher-dimensional polytope $P$ (a connected component of the graph $G(P)$ after deleting all edges with labels above 1) is a $k$-orbit graph, the 2-face is either a $k$-orbit polygon with the given graph, or a $j$-orbit polygon with a graph covered by the given graph.

A $k$-orbit graph which is a path covers a $j$-orbit path if and only if $j \mid k$. In the case that $j$ is even, the loops of the $j$-orbit graph match those of the $k$-orbit graph. That is, if the loops incident to the endpoints of the $k$-orbit graph are labeled 0, then so are the loops at the endpoints of the $j$-orbit graph, and similarly if the loops are labeled 1.

A $k$-cycle covers a $\left(\frac{k}{2}\right)$-path. If $\frac{k}{2}$ is even, then both choices of path (with 0-loops or 1-loops) are covered by the $k$-cycle. Hence, the $k$-cycle covers any path of length $j$ where $j$ is a divisor of $\frac{k}{2}$. The $k$-cycle also covers the $j$-cycle graph for any even divisor $j$ of $k$ with $j \geq 4$.

In addition to the possible orbit graphs of polygons, the chiral graph $\bullet \xrightarrow{0} \xrightarrow{1}$ could appear as a restricted graph. This could cover any regular polygon.
CHAPTER V

Two-orbit Polytopes and Tilings

1. Introduction

Here we will classify all geometrically two-orbit convex polytopes and tilings of Euclidean space. As far as flag orbits are concerned, two-orbit polytopes are as close to regular as possible while not being regular. Two-orbit convex polytopes can either be combinatorially two-orbit, if \( G(P) = \Gamma(P) \), or combinatorially regular, in which case \( G(P) \) is a subgroup of index 2 in \( \Gamma(P) \). In the more general case of abstract polytopes, combinatorially two-orbit polyhedra were examined by Hubard (2010). The chiral polytopes are notable examples of two-orbit abstract polytopes (Schulte and Weiss 1991). However, convex polytopes cannot be chiral (Schulte and Weiss 1991, p. 496).

We shall classify two-orbit polytopes up to symmetry equivalence. We will mention whenever there is a family of non-similar examples of a type. In the other cases, which include every case which is also combinatorially two-orbit, any two examples are actually similar; thus these polytopes are perfect in the sense of Robertson (1984).

Two-orbit convex polytopes turn out to be even scarcer than one-orbit convex polytopes, and exist only in two or three dimensions. As we showed in Chapter IV, there are infinitely many in two dimensions. For each \( p \geq 2 \), a two-orbit \( 2p \)-gon may be constructed by alternating edges of two distinct lengths, with the same angle at each vertex (namely the interior angle of a regular \( 2p \)-gon, \( \frac{2p - 2}{p} \pi \)). Dual to each of these is another type of two-orbit \( 2p \)-gon, with uniform edge lengths, but alternating angle measures. Such \( 2p \)-gons are shown, with duals atop one another, in Figure VA.

In three dimensions, there are just four two-orbit polyhedra: the cuboctahedron, its dual the rhombic dodecahedron, the icosidodecahedron, and its dual the rhombic triacontahedron,
Figure VA. The first few two-orbit convex polygons, in pairs of duals

all shown in Figure VB. We summarize the results in Theorems 1 and 2. These results will appear in Matteo 2014.

Theorem 1. There are no two-orbit convex $d$-polytopes with $d \geq 4$ (or $d \leq 1$). There are exactly four, if $d = 3$: the cuboctahedron, icosidodecahedron, rhombic dodecahedron, and rhombic triacontahedron. If $d = 2$, there are two infinite series of $2p$-gons, for each $p \geq 2$. Polygons of one series alternate between two distinct edge lengths. Polygons of the other alternate between two distinct angle measures.

In §6 we classify all two-orbit tilings. See that section for a description of each of tilings named here.

Theorem 2. There are no two-orbit tilings of $\mathbb{E}^d$ if $d \geq 4$ (or if $d = 0$). If $d = 1$, there is one family: an apeirogon alternating between two distinct edge lengths. If $d = 2$, there are four: the trihexagonal tiling (3.6.3.6); its dual, the rhombille tiling; a family of tilings by translates of a rhombus; and a family of tilings by rectangles. If $d = 3$, there are two: the tetrahedral-octahedral honeycomb and its dual, the rhombic dodecahedral honeycomb.

In the above two theorems, all those examples which vary by a real parameter greater than one (both types of $2p$-gons, the apeirogon, and the tilings by rhombi and rectangles) are combinatorially regular; in each case, allowing the parameter to become one yields a
Cuboctahedron  Icosidodecahedron  Rhombic Dodecahedron  Rhombic Triacontahedron

Figure Vb. The two-orbit convex polyhedra

regular polygon or tiling, to which all other members of the family are isomorphic. The other examples, namely the four polyhedra, the trihexagonal tiling, the rhombille tiling, the tetrahedral-octahedral honeycomb, and the rhombic dodecahedral honeycomb, are all unique up to similarity, and are all combinatorially two-orbit.

2. Preliminary Facts

Note that the symmetry group of a two-orbit $d$-polytope $P$ can have at most two orbits on its $j$-faces, for any $j < d$.

**Proposition 3.** Suppose $P$ is a two-orbit $d$-polytope. If the symmetry group $G(P)$ does not act transitively on $j$-faces for some $j$, then $G(P)$ acts transitively on $i$-faces for all $i \neq j$, where $0 \leq i, j \leq d - 1$.

**Proof.** Otherwise, we have two orbit classes of $i$-faces, say class I and II, and two classes of $j$-faces, say A and B. Without loss of generality, suppose $j < i$. Let us say that a flag of
P whose $j$-face is in class A and whose $i$-face is in class I is an A-I flag, and similarly for other cases. Then we have more than two flag types, A-I, A-II, B-I, and B-II, unless the $j$-faces in class A occur only in one class of $i$-faces, say I, and $j$-faces in class B occur only in $i$-faces in class II. But this is inconsistent with the connectivity property (P3) a sequence of adjacent flags from an A-I flag to a B-II flag must pass through either a A-II flag or a B-I flag. Therefore $P$ has at least three flag orbits. \hfill \Box

Since Theorem 4 tells us a two-orbit $d$-polytope $P$ cannot be fully transitive, $P$ must be $j$-intransitive for some $j$ in \{0, \ldots, d - 1\}. The $j$-faces of $P$ fall in two orbits.

In the language of Hubard (2010), a 0-intransitive two-orbit polyhedron is of class $2_{1,2}$, a 1-intransitive two-orbit polyhedron is of class $2_{0,2}$, and a 2-intransitive two-orbit polyhedron is of class $2_{0,1}$. Proposition 3 and the above comments were proved by Hubard (2010). They are consequences of Theorem 5 therein, which we may paraphrase to say that an (abstract) two-orbit $d$-polytope $P$ is either fully transitive, or there exists a $j$ ($1 \leq j \leq d$) such that $P$ is $i$-transitive for every $i \neq j$, but not for $i = j$. In using any results about abstract two-orbit polytopes, however, we must be careful to remember that convex two-orbit polytopes may be combinatorially regular and not combinatorially two-orbit.

**Proposition 4.** If $P$ is a two-orbit $j$-intransitive $d$-polytope, and $\Phi$ is any flag, then for any $i \neq j$ the $i$-adjacent flag $\Phi^i$ is in the same orbit as $\Phi$. That is, there exists a symmetry $\rho_i \in G(P)$ such that $\rho_i(\Phi) = \Phi^i$.

**Proof.** Since there are only two flag orbits, and two classes of $j$-faces, the orbit of a given flag is determined entirely by its $j$-face. For $i \neq j$, $\Phi$ and $\Phi^i$ share their $j$-face, hence are in the same flag orbit. \hfill \Box

**Corollary 5.** If $P$ is a two-orbit $j$-intransitive $d$-polytope, and $\Phi$ is any flag, then the $j$-adjacent flag $\Phi^j$ is not in the same flag orbit as $\Phi$.

**Proof.** If $\Phi^j$ were in the same orbit as $\Phi$, then by Proposition 4, for each $i = 0, \ldots, d - 1$ there exists an isometry $\rho_i$ of $P$ such that $\rho_i(\Phi) = \Phi^i$. But if a flag is in the same orbit as all
of its adjacent flags, it follows from flag-connectedness that \( P \) is regular (see Proposition 2B4 of McMullen and Schulte (2002) or Theorem 4B1 of McMullen (1968)).

The next corollary is immediate from Corollary 5.

**Corollary 6.** If \( P \) is a two-orbit \( j \)-intransitive \( d \)-polytope, then for any \((j + 1)\)-face \( F_{j+1} \) of \( P \) and any \((j - 1)\)-face \( F_{j-1} \) contained in \( F_{j+1} \), the two \( j \)-faces \( H \) with \( F_{j-1} < H < F_{j+1} \) are in different \( j \)-face orbits.

**Proposition 7.** If \( P \) is a two-orbit \( j \)-intransitive \( d \)-polytope, then \( G(P) \) acts transitively on chains of cotype \( \{j\} \).

**Proof.** Let \( \Psi \) and \( \Omega \) be two chains of cotype \( \{j\} \). By Corollary 6, the two \( j \)-faces which are incident to the \((j - 1)\)-face and \((j + 1)\)-face of \( \Psi \) are in different \( j \)-face orbits. Recall that the orbit of a given flag is determined entirely by its \( j \)-face. So we may extend \( \Psi \) to a flag \( \Psi' \) in some flag orbit. Similarly, we may extend \( \Omega \) to a flag \( \Omega' \) in the same orbit as \( \Psi' \). Then there is a symmetry \( \gamma \in G(P) \) satisfying \( \gamma(\Psi') = \Omega' \), and thus \( \gamma(\Psi) = \Omega \).

**Proposition 8.** If \( P \) is a two-orbit \( j \)-intransitive \( d \)-polytope, then \( j = 0 \) or \( j = d - 1 \).

**Proof.** Suppose \( 1 \leq j \leq d - 2 \). Then there is a \((j - 2)\)-face \( F_{j-2} \) contained in some \((j + 2)\)-face \( F_{j+2} \) in \( P \). The section \( Q = F_{j+2}/F_{j-2} \) is a polyhedron. Recall that the restricted group \( G_P(Q) \) is the subgroup of \( G(P) \) fixing all the faces of \( F_{j-2} \) and all the faces containing \( F_{j+2} \).

By Proposition 7, \( G_P(Q) \) acts transitively on the vertices and facets of \( Q \) (corresponding to \((j - 1)\)-faces and \((j + 1)\)-faces of \( P \), respectively.) By vertex transitivity, every vertex is in the same number \( q \) of edges. By Corollary 6, the edge orbits alternate across each facet, so \( q \) is even. By facet transitivity, each facet is a \( p \)-gon for some \( p \), and again by Corollary 6 the edge orbits alternate at each vertex, so \( p \) is even.

However, this contradicts Proposition 1.7: recall that any polyhedron without triangular facets has at least eight \( 3 \)-valent vertices.

It follows that the orbit graph for a two-orbit \( d \)-polytope is either \( \bullet 0 \bullet \) or \( \bullet d - 1 \bullet \).
Proposition 9. If $P$ is a two-orbit $j$-intransitive $d$-polytope, then all $i$-faces are regular for $i \leq j$. More generally, any section $G/F$, where $G$ is a $k$-face and $F$ is an $l$-face, is regular if $j \leq l$ or $k \leq j$. If $l < j < k$, then $G/F$ has two flag orbits under the restricted group $G_P(G/F)$.

Proof. Since there are only two flag orbits, and two classes of $j$-faces, the orbit of a given flag is determined entirely by its $j$-face. Suppose that for the section $G/F$ we do not have $l < j < k$. Choose a base flag $\Phi$ of $G/F$ and extend it to a flag $\Phi'$ of $P$. Now any flag $\Psi$ of $G/F$ may be extended to a flag $\Psi'$ of $P$ which agrees with $\Phi'$ for all $i$-faces with $i \leq l$ or $i \geq k$. In particular, $\Phi'$ and $\Psi'$ share the same $j$-face, so there is an isometry $\gamma \in G(P)$ such that $\gamma(\Phi') = \Psi'$. Then $\gamma$ restricts to a symmetry of $G/F$ carrying $\Phi$ to $\Psi$. Hence $G/F$ is regular.

On the other hand, if $l < j < k$, then $G/F$ contains a $(j - 1)$-face $F_{j-1}$ of $P$ and a $(j + 1)$-face $F_{j+1}$ of $P$ which contains $F_{j-1}$. By Corollary 6 the two $j$-faces $H$ of $P$ with $F_{j-1} < H < F_{j+1}$ are in different orbits. Thus $G/F$ has at least two flag orbits under those isometries in $G(P)$ which restrict to $G/F$. On the other hand, for any two flags $\Phi$ and $\Psi$ of $G/F$ which contain the same kind of $j$-face of $P$, we may extend these to flags $\Phi'$ and $\Psi'$ of $P$ which agree on all $i$-faces with $i \leq l$ and $i \geq k$. Then an isometry $\gamma \in G(P)$ exists with $\gamma(\Phi') = \Psi'$, and this $\gamma$ restricts to $G/F$ where it takes $\Phi$ to $\Psi$. Hence $G/F$ has two flag orbits under those transformations in $G(P)$ which restrict to $G/F$. □

Note that those sections in Proposition 9 with two flag orbits under the restricted group are either two-orbit polytopes or regular. Their full group of symmetries includes the restricted group, but may be bigger. If the section is in fact two-orbit, then its symmetry group agrees with the restricted group. In particular, if a face $F$ of a two-orbit $j$-intransitive polytope is two-orbit, then $F$ is also $j$-intransitive; note than then $j = 0$, by Proposition 8.

3. Two Dimensions

The two types of two-orbit polygon found in Chapter IV are the (irregular) isogonal polygons and the (irregular) isotoxal polygons. Isogonal means vertex-transitive; these are the
polygons of type $\bullet \frac{1}{2} \bullet$. These polygons alternate between edges of two distinct lengths, and a $2p$-gon of this type (for $p \geq 3$) can be constructed by truncating a regular $p$-gon (avoiding the one choice of truncation which yields edges of the same length). (The existence of non-regular rectangles, the remaining case, is well known.) In the top row of Figure VA, you may see how the hexagon is a truncated equilateral triangle, and the octagon is a truncated square.

Isotoxal means edge-transitive; these are the polygons of type $\bullet 0 \bullet$. These polygons alternate between two distinct internal angles. These may be constructed by taking the convex hull of vertices placed at the midpoint of each edge of a polygon of the first type. (This procedure, taking the centroids of facets for vertices, can be used to form (combinatorially) dual polytopes to regular polytopes and other highly-symmetric polytopes, but does not produce a dual in general.) This produces a different polygon than polar reciprocation, but still of the correct type. Figure VC shows an example of these dual-forming procedures.

Figure VC. A rectangle, overlaid with its midpoint dual and polar dual

4. Three Dimensions

A quasiregular polyhedron may be defined as being vertex-transitive and edge-transitive with regular facets (Fejes Tóth 1964, p. 113) or as having two kinds of facets, each regular and entirely surrounded by the other kind (Coxeter, Longuet-Higgins, and Miller 1954, p. 412). By Propositions 3 and 9, any 2-intransitive two-orbit polyhedron is vertex-transitive, edge-transitive, and has regular facets in two orbits. The two types of facet must alternate around each vertex, i.e. each edge must be incident to one facet of each type, by edge-transitivity. Thus any 2-intransitive two-orbit polyhedron is quasiregular. But there are
only two quasiregular polyhedra: the cuboctahedron and the icosidodecahedron, two of the Archimedean solids (Coxeter 1973, p. 18).

We may verify that these are two-orbit polyhedra. The cuboctahedron has 12 vertices, each incident to 4 edges, and each edge is in 2 facets, so it has $12 \cdot 4 \cdot 2 = 96$ flags. The cuboctahedron may be formed by truncating each vertex of the 3-cube at the midpoints of the edges, so it retains all the symmetries of the cube, a group of order 48. Hence the cuboctahedron has at most two flag orbits, and since it is not regular, it is a two-orbit polyhedron (and also combinatorially two-orbit).

The icosidodecahedron has 30 vertices, each in 4 edges, so it has $30 \cdot 4 \cdot 2 = 240$ flags. The icosidodecahedron may be formed by truncating each vertex of the dodecahedron at the midpoints of the edges, so it retains all the symmetries of the dodecahedron, a group of order 120. Hence the icosidodecahedron has at most two flag orbits, and since it is not regular, it is a two-orbit polyhedron (and also combinatorially two-orbit).

Any two-orbit polyhedron which is 0-intransitive must be dual to one of these two, so we have the rhombic dodecahedron, dual to the cuboctahedron, and the rhombic triacontahedron, dual to the icosidodecahedron. As duals to Archimedean solids, these are Catalan solids.

Rather than using the list of quasiregular polyhedra, it is possible to arrive at candidates for 0-intransitive or 2-intransitive two-orbit polyhedra by considering all the edge-transitive polyhedra. It turns out there are only nine: the five Platonic solids, the cuboctahedron, the icosidodecahedron, the rhombic dodecahedron, and the rhombic triacontahedron (Graver and Watkins 1997; Grünbaum and Shephard 1987a).

By Proposition 8 there are no 1-intransitive two-orbit polyhedra. In fact, polyhedra which are vertex-transitive and facet-transitive have a name, the noble polyhedra, and the only non-regular ones (i.e. the 1-intransitive polyhedra) are disphenoids, which are tetrahedra with non-equilateral triangular faces (Brückner 1906, p. 26). It is not hard to see that, if not regular, a tetrahedron has at least three flag orbits (since it includes a non-equilateral triangle).
Hence the cuboctahedron, icosidodecahedron, rhombic dodecahedron, and rhombic triacontahedron are the only two-orbit polyhedra. The combinatorial equivalent to this result is found in Orbanić, Pellicer, and Weiss (2010, p. 427) as a consequence of Theorem 6.1 therein, stating that every two-orbit map on the sphere is either the medial of a regular map on the sphere, or dual to one.

5. Higher dimensions

Suppose $P$ is a $j$-intransitive two-orbit $d$-polytope with $d \geq 4$. By Proposition 8, $j$ is either 0 or $d - 1$. Any two-orbit 0-intransitive polytope is dual to a two-orbit $(d - 1)$-intransitive polytope, so we shall restrict our attention to the latter case. Such a polytope is vertex-transitive, and by Proposition 9 has regular facets. This is the definition used by Gosset (1900) for semiregular polytopes. In his 1900 paper he gives a complete list of all the semiregular polytopes. The list was proved to be complete by G. Blind and R. Blind (1991).

There are only seven semiregular convex polytopes in more than three dimensions. There are three 4-polytopes: the rectified 4-simplex, the snub 24-cell, and the rectified 600-cell. The rectified 4-simplex, which Gosset called “tetroctahedric”, is the convex hull of the midpoints of the edges of the 4-simplex. The facets are tetrahedra and octahedra. It has 360 flags, with 10 vertices, each in 6 edges, each edge in 3 ridges, and each ridge in 2 facets. It has the same symmetry group as the 4-simplex, of order 120; hence it has three flag orbits.

The rectified 600-cell, which Gosset called “octicosahedric”, is the convex hull of the midpoints of the edges of the 600-cell. The facets are octahedra and icosahedra. It has 43,200 flags, with 720 vertices, each in 10 edges, each edge in 3 ridges and each ridge in 2 facets. It has the same symmetry group as the 600-cell, of order 14,400; hence it has three flag orbits.

The snub 24-cell, which Gosset called “tetricosahedric”, has icosahedra and tetrahedra for facets. This may be constructed from the regular 24-cell in a manner analogous to this construction of the icosahedron from the octahedron (Coxeter 1973, pp. 51–52): truncating the vertices of the octahedron, yielding a polyhedron with six squares and eight hexagons, then selecting alternate vertices, so that the hexagons become triangles, the squares become
edges, and 12 new triangles appear around the deleted vertices. If the truncation is done to the appropriate distance, a regular icosahedron results. Similarly, if one selects alternate vertices from a truncated 24-cell, icosahedra are formed from the truncated octahedron cells, tetrahedra are formed from the cubic cells, and more tetrahedra appear around the deleted vertices.

It has 96 vertices, each in 9 edges; 6 of these edges are in 3 ridges, and the other 3 edges are in 4 ridges. (This already makes it clear that there are at least two orbit classes of edges, as well as at least two orbit classes of facets, so it cannot be two-orbit.) Each ridge is in 2 facets. Hence there are 5,760 flags. It has half the symmetries of the 24-cell, leaving 576. So it has ten flag orbits.

The remaining examples form Coxeter’s $k_{21}$ family (Coxeter [1973, §11.8, 1988], with one each in dimensions 5 through 8. They are the 5-demicube, or $1_{21}$, Gosset’s “5-ic Semi-regular”; 2$_{21}$ or “6-ic Semi-regular”; 3$_{21}$ or “7-ic Semi-regular”; and 4$_{21}$ or “8-ic Semi-regular”. Each of these has the preceding one for its vertex figure, starting with the rectified 4-simplex (which may also be called $0_{21}$) as the vertex figure of the 5-demicube. Of course, by Proposition 9, if any member of this family were two-orbit, then the previous member (being a section) would either be two-orbit or regular. So by induction, none of these polytopes are two-orbit. In fact, each has three flag orbits.

Thus, no two-orbit convex polytopes exist in more than three dimensions.

Conway, Burgiel, and Goodman-Strauss (2008, pp. 409–411) say that the $n$-dimensional demicube, i.e. the alternation of the $n$-cube (which they call a hemicube), has $n - 2$ flag orbits. So the 4-demicube should be two-orbit. The 4-demicube is described specifically as a 4-crosspolytope “but with only half its symmetry.” This apparently contradicts our result!

However, if the 4-cube has for its vertices the 16 points in $E^4$ with all coordinates 0 or 1, then the vertices of the 4-demicube are $(0, 0, 0, 0)$, $(1, 1, 1, 1)$, and all vectors with two 0’s and two 1’s. Hence if $x$ is a vertex, so is $1 - x$, where $1 = (1, 1, 1, 1)$. Grouping the 8 vertices in pairs $(x, 1 - x)$, we find four axes which are mutually perpendicular. Thus we have four
antipodal pairs of vertices of a regular 4-crosspolytope. Hence the “two-orbit” 4-demicube
is actually a regular 4-crosspolytope with artificially restricted symmetries, essentially by
coloring the facets depending whether they were formed inside a facet, or at a missing vertex,
of the 4-cube.

6. Tilings

Most of the results proved in this chapter for polytopes are also valid for tilings of \( \mathbb{E}^d \).

Proposition 3 still applies: if a two-orbit tiling is not fully transitive, then it is not transitive
on the faces of exactly one dimension, say \( j \), and we call it \( j \)-intransitive. However, Theorem 1.4
does not apply; the proof depends on the fact that the vertices of a vertex-transitive polytope
lie on a sphere, which is not the case for a tiling. So fully transitive two-orbit tilings are a
possibility (and some exist). Proposition 4, its corollaries, and Proposition 7 still hold for
any \( j \)-intransitive two-orbit tilings. Proposition 8 only holds for two-orbit tilings with rank
at least 4; it fails for planar tilings, because then the polyhedral section mentioned in the
proof is not proper, and not spherical. Finally, Proposition 9 applies: the faces and sections
of a two-orbit tiling have at most two orbits.

6.1. Apeirogons. Up to symmetry equivalence, there is one two-orbit tiling of the line:
an apeirogon alternating between two distinct edge lengths. Its similarity classes vary by a
single real parameter greater than one. Note that no well-behaved construction for duals of
tilings is known. For regular tilings, one may take the centroids of the facets as the vertices
of the dual, but this does not work in general. For example, if one constructs a dual to this
two-orbit apeirogon by taking edge midpoints for vertices, one obtains a regular apeirogon,
which is then self-dual!

This tiling is combinatorially regular.

6.2. Plane tilings. We consider four cases of plane tilings, based on their transitivity
properties.
6.2.1. *Fully transitive.* Grünbaum and Shephard (1987b) provide the full list of isohedral (i.e. tile-transitive) plane tilings (Table 6.2.1), isotoxal (i.e. edge-transitive) plane tilings (Table 6.4.1), and isogonal (vertex-transitive) plane tilings (Table 6.3.1). There are only four plane tilings realizable by convex tiles which have all three properties: the three regular plane tilings and a tiling by translations of a rhombus, labeled IH74 as an isohedral tiling, IG74 as an isogonal tiling, and IT20 as an isotoxal tiling. On page 311 it is confirmed that this rhombus tiling is the only non-regular fully transitive tiling realizable by convex tiles. Figure Vd shows a portion of this tiling, with flags of one orbit shaded. For a given flag $\Phi$, both the 0-adjacent flag $\Phi^0$ and the 2-adjacent flag $\Phi^2$ are in the other orbit, whereas the 1-adjacent flag $\Phi^1$ remains in the same orbit; thus with the notation of Hubard (2010) this is a two-orbit tiling in class $2_1$.

A family of non-similar versions of this tiling may be obtained by varying a single real parameter greater than one (the ratio of the diagonals of the rhombus.) The tiling is self-dual when taking tile centroids for vertices. It is combinatorially regular.

6.2.2. *2-intransitive.* The facets of a 2-intransitive two-orbit tiling must be regular, by Proposition 9. By edge-transitivity, the two facets bordering each edge are from different orbits; hence they alternate around each vertex. By vertex-transitivity, the tiling has a vertex type $(p.q.p\ldots)$.

If six facets appear at each vertex, then they must all be triangles, since after replacing any triangle by a regular $n$-gon with $n \geq 4$, the six polygons cannot fit together in the plane.
The only tiling with six equilateral triangles at every vertex is the regular tiling \((3^6)\). Hence there must be exactly four facets at each vertex.

If none of the facets are triangles, then each has at least four sides. Four squares fit exactly around a vertex, but after replacing any squares by regular \(n\)-gons with \(n \geq 5\), the four polygons will not fit in the plane. The only tiling with four squares at every vertex is the regular tiling \((4^4)\). Hence there must be at least some triangles.

If all four faces at each vertex are equilateral triangles, there is too much angular deficiency to tile the plane; indeed, the only such figure on a simply-connected surface is the regular octahedron, \((3^4)\).

If triangles alternate with squares, the resulting figure is the cuboctahedron, \((3.4.3.4)\). If triangles alternate with pentagons, the resulting figure is the icosidodecahedron, \((3.5.3.5)\).

(This is, in brief, the proof that these are the only quasiregular polyhedra.)

If triangles alternate with hexagons, we do obtain a plane tiling, denoted \((3.6.3.6)\). This is one of the 11 uniform plane tilings, also called Archimedean tilings. This tiling, seen in Figure VE, is sometimes called “trihexagonal” or “hexadeltille.”

![Figure VE. The trihexagonal tiling](image)

If we replace the hexagons by regular \(n\)-gons with \(n \geq 7\), the total angles are excessive to fit in the plane.

Hence \((3.6.3.6)\) is the unique two-orbit 2-intransitive plane tiling. Coxeter (1973 p. 60) calls it by the extended Schláfli symbol \(\{^3_6\}\), which is suggestive of the construction by taking the midpoints of the edges of the regular tiling \(\{3, 6\}\), or equivalently of its dual, the regular...
tiling \{6,3\}. He describes it as a quasiregular tessellation. It is combinatorially two-orbit. Taking the dual by using tile centroids for vertices works well and results in the rhombille tiling detailed below.

6.2.3. **1-intransitive.** By facet-transitivity, each facet has the same number of sides, say \(p\), and by vertex-transitivity, each vertex is incident to the same number of edges, say \(q\). Thus a 1-intransitive plane tiling has a Schl"afli symbol \(\{p,q\}\). Since edges of the two orbits alternate at each vertex of a tile, \(p\) and \(q\) are both even; the only possible symbol is \(\{4,4\}\). The tiles must be regular or two-orbit. The only tiling by squares is regular; so the tiles must be two-orbit 4-gons, i.e.

 rectangles or rhombi. Recall that any two-orbit face of a two-orbit \(j\)-intransitive polytope (or tiling) is also \(j\)-intransitive; so the tiles must be rectangles.

Indeed, there are just two vertex-transitive tilings by rhombi, up to symmetry equivalence. It follows from vertex-transitivity, or from adding angle defects, that rhombi must be arranged with two acute angles and two obtuse angles at each vertex. In the case that the two angle types alternate, we obtain the tiling in Figure V\(\text{D}\) which we know to be fully transitive. In the case that the obtuse angles are adjacent to each other, and the acute angles are adjacent to each other, we do obtain a 1-intransitive plane tiling. The rhombi are arranged in strips which alternate direction. However, this tiling actually has four flag orbits.

This leaves only the tiling by translates of a rectangle. This is the unique two-orbit 1-intransitive plane tiling up to symmetry equivalence. Up to similarity, there is a family varying by a single real parameter greater than one. It is self-dual by taking tile centroids and combinatorially regular, being isomorphic to the square tiling \((4^4)\). Figure V\(\text{F}\) shows a patch of this tiling, with flags of one orbit shaded.

6.2.4. **0-intransitive.** It is tempting to say that any 0-intransitive tiling must be dual to a 2-intransitive one. However, Gr"unbaum and Shephard (1987b) admonish us that for tilings, no duality theorem exists which would allow us to make such statements! Nonetheless, it turns out that the only 0-intransitive two-orbit tiling is indeed dual to the uniform tiling \((3.6.3.6)\). We can confirm this by again turning to the tables of isohedral and isotoxal tilings.
in (Grünbaum and Shephard 1987b); the only tiling realizable by convex tiles with both properties, besides those mentioned in §6.2.1, is denoted IH37 as an isohedral tiling and IT11 as an isotoxal tiling.

This is a tiling by copies of a rhombus, which can be viewed as dividing the hexagons of the regular tiling \( (6^3) \) into three rhombi each. It is called “rhombille” or “tumbling blocks,” and is familiar as the visual illusion of a stair-case of blocks which can be seen in two ways. Figure Vg shows a patch, with flags of one orbit shaded. It is combinatorially two-orbit.

**Figure Vf.** The 1-intransitive rectangle tiling

**Figure Vg.** The rhombille tiling

### 6.3. Tilings of three-space.

A rank 3 section of a tiling of \( \mathbb{R}^3 \) is either a facet or a vertex figure, and is a polyhedron in either case, so Proposition § holds: thus a two-orbit tiling of \( \mathbb{R}^3 \) is either fully transitive, 0-intransitive, or 3-intransitive.
6.3.1. **3-intransitive.** A 3-intransitive two-orbit tiling of 3-space has regular polyhedral tiles and is vertex-transitive, which means that it is a uniform tiling. Grünbaum (1994) listed all 28 uniform tilings of 3-space. We can eliminate those which contain tiles which are not regular: these include all but numbers 1, 2, and 22. Of these, only one is two-orbit: the tetrahedral-octahedral honeycomb, #1 on Grünbaum’s list, also called “alternated cubic”, “tetroctahedrille”, or “octatetrahedral”. Tiling #2 is a honeycomb similarly made of tetrahedra and octahedra, but with some triangle faces contained in two tetrahedra, and some in two octahedra; thus it is not 2-face-transitive. Tiling #22 is the regular cubic honeycomb.

The tetrahedral-octahedral honeycomb is 3-intransitive. Coxeter describes it as the unique quasiregular honeycomb (Coxeter 1973, p. 69) and assigns it the modified Schläfli symbol \{3, 3/4\} and an abbreviated symbol \(h\delta_4\) (Coxeter 1940a, p. 402). Being semiregular (with regular tiles and a vertex-transitive group), it also appears in Gosset’s list (Gosset 1900) as the “simple tetroctahedric check.” Monson and Schulte (2012) describe this tiling at length. It has 6 octahedra and 8 tetrahedra meeting at each vertex; the vertex figure is a cuboctahedron. The corresponding “net,” the 1-skeleton of the tiling, is named \(f_{cu}\) by crystallographers in (Delgado Friedrichs, O’Keeffe, and Yaghi 2002), where this tiling is conjectured to be the unique one with transitivity 1112, i.e. whose symmetry group has one orbit on vertices, edges, and 2-faces, and two orbits on tiles.

6.3.2. **0-intransitive.** A 0-intransitive tiling of 3-space has two kinds of vertices, and every edge must be incident to one of each (by edge-transitivity), so each 2-face has evenly many sides. The facets are regular or 0-intransitive two-orbit; the only possibilities are the cube, the rhombic dodecahedron, or the rhombic triacontahedron. The only face-to-face tiling by cubes is the regular one. The rhombic triacontahedron has a dihedral angle of \(4\pi/5\) (Coxeter 1973, p. 26), so it is impossible to fit an integral number of them around an edge in 3-space. However, the rhombic dodecahedron, with a dihedral angle of \(2\pi/3\), does form a two-orbit tiling of 3-space in a unique way. This tiling (called the rhombic dodecahedral honeycomb) is dual to the tetrahedral-octahedral honeycomb above. The corresponding net is named \(f_{lu}\) in
(Delgado Friedrichs, O’Keeffe, and Yaghi 2002), and described as the structure of fluorite (CaF$_2$.) It is conjectured there to be the unique tiling with transitivity 2111.

6.3.3. Fully transitive. Suppose $\mathcal{T}$ is a fully transitive two-orbit tiling; then the facets are regular or two-orbit, and all of one type. Since $\mathcal{T}$ is vertex-transitive, if the facets were regular, $\mathcal{T}$ would be uniform, and we have already checked all the uniform tilings. Thus the facets must be two-orbit. Since $\mathcal{T}$ is 2-face-transitive, every 2-face is the same, which rules out the cuboctahedron or icosidodecahedron as facets. The remaining possibilities are the rhombic dodecahedron and the rhombic triacontahedron. As mentioned above, no integral number of rhombic triacontahedra can fit around an edge, so it cannot tile 3-space. Exactly three rhombic dodecahedra can fit around an edge. So, the vertex figure of such a tiling is regular or two-orbit, with 3-valent vertices, some square faces, and some triangular faces. No such polyhedron exists.

6.4. Higher dimensions. For a rank $(d + 1)$ tiling $\mathcal{T}$ with $d \geq 4$, the facets and vertex figures are $d$-dimensional polytopes with at most two orbits. Since no two-orbit convex polytopes exist in $d \geq 4$ dimensions, the facets and vertex figures must, in fact, be regular; but then $\mathcal{T}$ itself is regular (Coxeter 1973, p. 129).

7. Conclusion

The number of half-regular convex polytopes and tilings (to use Conway’s pleasant term for “two-orbit”) is perhaps surprisingly small. Those which are also combinatorially two-orbit are simply the cuboctahedron and the icosidodecahedron (the only two quasiregular polyhedra) and their duals; the trihexagonal tiling (the only quasiregular plane tiling) and its dual; and the tetrahedral-octahedral honeycomb (the only quasiregular honeycomb) and its dual. It is notable, perhaps, that although duality is not generally well-defined for tilings, it always works well for two-orbit tilings which are combinatorially two-orbit, just as it always works well for regular tilings and uniform plane tilings. However, it does not generally work out for two-orbit tilings which are combinatorially regular!
The above seems suggestive that “quasiregular,” which has previously had rather ad-hoc definitions, could be taken to mean “facet-intransitive two-orbit.” Coxeter (1973, p. 18) defines a “quasi-regular polyhedron” as “having regular faces, while its vertex figures, though not regular, are cyclic and equi-angular (i.e., inscribable in circles and alternate-sided).” The definition of a quasiregular plane tiling does not seem to be clearly stated, but the implication (in 1973 §4.2) is that a quasiregular plane tiling is one formed, as the quasiregular polyhedra can be, by truncating the vertices of a regular tiling to the midpoints of the edges. In (Coxeter 1973 §4.7), a tiling of 3-space (or honeycomb) “is said to be quasi-regular if its cells are regular while its vertex figures are quasi-regular.” This suggests the beginning of an inductive definition for “quasiregular” in higher dimensions, which would perhaps agree with ours. Indeed, facet-intransitive two-orbit polytopes have regular facets and the vertex figures are again facet-intransitive and two-orbit. It would be good to determine if having regular facets and facet-intransitive two-orbit vertex figures implies that the polytope is two-orbit. This is vacuously true for convex polytopes, since R. Blind (1979) classified all regular-faced $d$-polytopes with $d \geq 4$, and none have two-orbit vertex figures. However, the corresponding result for abstract polytopes would clarify the agreement of the definitions.

The word “quasiregular” is also applied to some star polytopes, such as the dodecadodecahedron $\{\frac{5}{5/2}\}$ and the great icosidodecahedron $\{\frac{3}{5/2}\}$ in (Coxeter 1973 pp. 100–101); three ditrigonal forms: the ditrigonal dodecadodecahedron, small ditrigonal icosidodecahedron, and great ditrigonal icosidodecahedron (also called “triambic” instead of “ditrigonal”); and nine hemihedra: the tetrahemihexahedron, octahemioctahedron, cubohemioctahedron, small icosihemidodecahedron, small dodecahemidodecahedron, great dodecahemicosahedron, small dodecahemicosahedron, great dodecahemicosahedron, small dodecahemicosahedron, great dodecahemicosahedron, and great icosihemidodecahedron (using names from Wenninger (1974)). All of these are two-orbit facet-intransitive. It would be good to establish that these are the only two-orbit facet-intransitive star polytopes.

Remaining questions include the classification of two-orbit tilings of hyperbolic space, two-orbit star polytopes, and other non-convex two-orbit polytopes in Euclidean space. The
general abstract two-orbit polyhedra have been addressed in Hubard (2010), with extension to higher dimensions in preparation (Hubard and Schulte n.d.). An overview is provided by Helfand (2013, §1.3). The important special case of chiral polytopes have been studied extensively but many open questions remain; a recent survey is given by Pellicer (2012).

8. Summarizing table

For comparison with three-orbit polytopes, we summarize all the two-orbit polytopes in Table VA and the two-orbit tilings in Table VB. Parenthesized numbers count combinatorially regular polytopes; the others are combinatorially two-orbit.

<table>
<thead>
<tr>
<th>$d$</th>
<th>Facet-intransitive</th>
<th>Vertex-intransitive</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$(\infty)$: $2m$-gons $t{m}$ ($m \geq 2$)</td>
<td>$(\infty)$: $2m$-gons $K{m}$ ($m \geq 2$)</td>
</tr>
<tr>
<td>3</td>
<td>2: $t_1{3,4}$</td>
<td>2: $K_1{3,4}$</td>
</tr>
<tr>
<td></td>
<td>$t_1{3,5}$</td>
<td>$K_1{3,5}$</td>
</tr>
</tbody>
</table>

Table VA. Two-orbit convex $d$-polytopes

In the truncations $t\{m\}$ listed in these two tables, it should be understood that the truncating distance is chosen so that the resulting edges are not all the same length. Similarly, in the dual Kleetopes, the new vertices should be chosen inside of the circumcircle of $\{m\}$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$d$-intransitive</th>
<th>0-intransitive</th>
<th>Type 2$_{(1)}$</th>
<th>Type 2$_{(0,2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1): $\infty$-gon $t{\infty}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1: $t_1{3,6}$</td>
<td>1: $K_1{3,6}$</td>
<td>(1): IH74</td>
<td>(1): Rectangle tiling ${\infty} \times {\infty}$</td>
</tr>
<tr>
<td>3</td>
<td>1: $h{4,3,4}$</td>
<td>1: $aK_*{4,3,4}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table VB. Two-orbit tilings of $\mathbb{E}^d$
In Table V, the operator $h$ refers to alternation, and $aK_*$ to an extended alternate Kleotope operation. For tilings, this amounts to adding a new vertex at the centroid of alternately chosen tiles, and adjoining the pyramids from this apex over the facets of the tile to the adjacent, unmodified tiles. When performed on the cubic tiling, this creates rhombic dodecahedra. The tiling IH74 is the fully-transitive tiling by rhombi detailed in §6.2.1. In the product $\{\infty\} \times \{\infty\}$ it is understood that the edge lengths of the two factors are different.
CHAPTER VI

Combinatorially Two-orbit Polytopes and Tilings

1. Introduction

In the previous chapter, we found all the two-orbit convex polytopes. These exist only in the plane and in 3-space. In the plane, there are two infinite families, one consisting of the irregular isogonal polygons, and the other consisting of the irregular isotoxal polygons. In 3-space, there are only four: the two quasiregular polyhedra, namely the cuboctahedron and the icosidodecahedron, and their duals, the rhombic dodecahedron and the rhombic triacontahedron.

Not every polytope can be realized such that every automorphism is also a Euclidean isometry. Bokowski, Ewald, and Kleinschmidt (1984) construct a combinatorially 84-orbit 4-polytope $P$ which is not isomorphic to any polytope $P'$ whose symmetry group $G(P')$ is the same as the automorphism group $\Gamma(P')$. However, it is proved by McMullen (1968, Theorem 3A1) that a polytope is combinatorially one-orbit if and only if it is isomorphic to a (geometrically) one-orbit polytope. In this chapter, we show that every combinatorially two-orbit polytope is isomorphic to a (geometrically) two-orbit polytope. The converse is not quite true, since any $2n$-gon is isomorphic to a two-orbit polytope, yet is not combinatorially two-orbit; but it is true in all other cases.

In §6 we show, similarly, that every combinatorially two-orbit normal tiling is isomorphic to a two-orbit tiling. Recall that, for our tilings, normal means that the tile sizes are bounded above and below. It seems that the corresponding question for one-orbit tilings remains open, with a finite list of possible exceptions. We summarize the results in these theorems, to appear in Matteo 2015a.
Theorem 1. Any combinatorially two-orbit convex polytope is isomorphic to a (geometrically) two-orbit polytope. Hence, if $P$ is a combinatorially two-orbit $d$-polytope, then $d = 3$ and $P$ is isomorphic to one of the cuboctahedron, the icosidodecahedron, the rhombic dodecahedron, or the rhombic triacontahedron.

In fact, the theorem holds not only for convex $d$-polytopes, but also for spherical complexes, i.e. tilings of $S^{d-1}$ by convex polytopes. The key ingredients are the symmetry group being finite, the boundary being simply connected, and the Euler characteristic of rank 3 sections.

In light of the fact that, for $d > 2$, all two-orbit $d$-polytopes are also combinatorially two-orbit, and that both conditions are vacuous for $d \leq 1$, we can say that a convex $d$-polytope with $d \neq 2$ is combinatorially two-orbit if and only if it is isomorphic to a two-orbit convex polytope.

Theorem 2. A combinatorially two-orbit tiling need not be isomorphic to a two-orbit tiling. However, combinatorially two-orbit tilings of $\mathbb{E}^d$ occur only for $d = 2$ or $d = 3$.

Theorem 3. Any combinatorially two-orbit normal tiling is isomorphic to a two-orbit tiling. Hence, if $\mathcal{T}$ is a combinatorially two-orbit normal tiling of $\mathbb{E}^d$, then either

(i) $d = 2$ and $\mathcal{T}$ is isomorphic to one of the trihexagonal tiling or the rhombille tiling, or

(ii) $d = 3$ and $\mathcal{T}$ is isomorphic to one of the tetrahedral-octahedral honeycomb or the rhombic dodecahedral honeycomb.

2. Preliminaries

In this chapter, by contrast with the rest of the dissertation, orbit graphs $G(P)$ will be assumed to be under the action of the automorphism group $\Gamma(P)$.

2.1. Class. Let $I \subset \{0, 1, \ldots, d - 1\}$ and $\Phi$ be a flag of a combinatorially two-orbit $d$-polytope $P$. If the $i$-adjacent flag $\Phi^i$ is in the same orbit as $\Phi$ precisely when $i \in I$, then we say $P$ is in class $2_I$ (Hubard 2010; Hubard, Orbanić, and Weiss 2009). It is not hard to see that this class is well-defined; see Hubard (2010) for proofs of this and the following remarks.
In terms of orbit graphs, \( I \) is the set of labels of loops in the graph, which necessarily agree at both nodes. We cannot have \(|I| = d\), because then all edges are loops and the orbit graph cannot have two connected nodes. If \(|I| \leq d - 2\), then there are at least two edges joining the two nodes. Hence, the subgraph \( \mathcal{G}_i(P) \) (deleting the \( i \)-edges) remains connected for every \( i \), and so \( P \) is combinatorially fully transitive. On the other hand, if \(|I| = d - 1\), there is a single edge joining the two nodes of \( \mathcal{G}(P) \). Say it is a \( j \)-edge. Then \( \mathcal{G}_j(P) \) has two components, so \( \Gamma(P) \) has two orbits on the \( j \)-faces.

Hence, the automorphism group \( \Gamma(P) \) is fully transitive if and only if \(|I| \leq d - 2\). The only other case is that \(|I| = d - 1\), and then \( \Gamma(P) \) acts transitively on all \( i \)-faces with \( i \in I \), but has two orbits on the \( j \)-faces for the unique \( j \) not in \( I \). We shall see that all combinatorially two-orbit polytopes are \( j \)-intransitive for some \( j \).

2.2. Modified Schlafli symbol. For the purposes of this chapter, we will use a modified Schlafli symbol. It is like the standard symbol \( \{p_1, \ldots, p_{d-1}\} \) for a \( d \)-polytope \( P \), but possibly with some positions \( p_i \) replaced by a stack of two distinct numbers, \( p_i q_i \). Wherever a single number \( p_j \) appears, it means (as usual) that every section \( F_{j+1}/F_{j-2} \) is a \( p_j \)-gon. If two numbers \( p_j q_j \) appear, it means that all such sections \( F_{j+1}/F_{j-2} \) are either \( p_j \)-gons or \( q_j \)-gons. If \( P \) is a two-orbit polytope with such a symbol, then the orbit of a flag \( \Phi = \{F_{-1}, \ldots, F_d\} \) is determined by whether \( F_{j+1}/F_{j-2} \) is a \( p_j \)-gon or a \( q_j \)-gon. If it is a \( p_j \)-gon, and \( P \) is of class 2\( _I \), then the corresponding section of \( \Phi^i \) is a \( q_j \)-gon precisely when \( i \notin I \). In order for the section to have a different size, \( \Phi^i \) must differ from \( \Phi \) in either the \((j + 1)\)-face or the \((j - 2)\)-face—but by definition it differs in exactly the \( i \)-face. We conclude that only \((j + 1)\) or \((j - 2)\) (or both) are not in \( I \).

Beware that you cannot read off the symbols for sections from the symbol for \( P \), as you can with a standard Schlafli symbol, without additional information. For instance, in the type \( \{4, 3_4, 4\} \) discussed below, the facets are of type \( \{4, 3_4\} \) (the rhombic dodecahedron) and the vertex figures are of type \( \{3_4, 4\} \) (the cuboctahedron). However, in the tetrahedral-octahedral
tiling of type \(\{3, \frac{3}{4}, 4\}\), the vertex figures are cuboctahedra \(\{\frac{3}{4}, 4\}\), but the facets alternate between two types, tetrahedra \(\{3, 3\}\) and octahedra \(\{3, 4\}\).

Those polytopes with standard Schl"afl"i symbols (with just one number in each position), so that the size of every section \(F_{j+1}/F_{j-2}\) depends only on \(j\), are called *equivelar*. Equivelar convex polytopes are combinatorially regular (McMullen and Schulte 2002, Theorem 1B9).

On the other hand, in a combinatorially two-orbit polytope, obviously the sections \(F_{j+1}/F_{j-2}\) for a given \(j\) can have at most two sizes. So every combinatorially two-orbit convex polytope has a modified Schl"afl"i symbol, with at least one stack of two numbers appearing.

3. Results on combinatorially two-orbit polytopes

**Lemma 4.** If \(P\) is in class \(2_I\) and \(j \notin I\), then \(\Gamma(P)\) acts transitively on chains of cotype \(\{j\}\).

**Proof.** Let \(\Psi\) and \(\Omega\) be two chains of cotype \(\{j\}\). Each of these may be extended to two flags of \(P\) which, being \(j\)-adjacent, are in different flag orbits. Thus, we extend \(\Psi\) to a flag \(\Psi'\) and \(\Omega\) to a flag \(\Omega'\) such that both are in the same orbit; then the automorphism \(\gamma \in \Gamma(P)\) carrying \(\Psi'\) to \(\Omega'\) also takes \(\Psi\) to \(\Omega\). \(\square\)

**Corollary 5.** If \(P\) is in class \(2_I\) and \(j \notin I\), then \(P\) has a modified Schl"afl"i symbol whose entry \(p_i\) is single-valued except possibly at \(i = j - 1\) and \(i = j + 2\).

**Proof.** By Lemma 4, \(\Gamma(P)\) acts transitively on the set of sections \(F_{i+1}/F_{i-2}\) for each rank \(i\) unless \(i + 1 = j\) or \(i - 2 = j\). \(\square\)

Recall that if all entries of the Schl"afl"i symbol are single-valued, then \(P\) is combinatorially regular. But by the Corollary, if two distinct ranks \(i < j\) are missing from \(I\), then all the entries would be single-valued unless \(j = i + 3\), so that \(j - 1\) coincides with \(i + 2\). This also shows that no three ranks \(i < j < k\) can be missing from \(I\).

**Lemma 6.** If a \(d\)-polytope \(P\) is in class \(2_I\) and \(j \notin I\), then in its modified Schl"afl"i symbol the entries \(p_j\) (if \(j \geq 1\)) and \(p_{j+1}\) (if \(j \leq d - 2\)) are even.
Proof. If $1 \leq j \leq d - 1$, then consider any section $F_{j+1}/F_{j-2}$ with incident faces of the indicated ranks. This is a polygon whose edges correspond to $j$-faces of $P$. A walk along the edges of this polygon can be extended to a sequence of adjacent flags of $P$, alternately $j$-adjacent and $(j - 1)$-adjacent. The flags change orbits whenever the $j$-face is changed. But changing $(j - 1)$-faces (corresponding to vertices of the polygon) will not change the orbit, since $(j - 1)$ and $j$ do not differ by 3. Thus the polygon has evenly many sides. Hence $p_j$, the $j$th entry in the Schlafli symbol for $P$ (which is single-valued by Corollary 5) is even.

Similarly, if $0 \leq j \leq d - 2$, then any section $F_{j+2}/F_{j-1}$ is a polygon whose vertices correspond to $j$-faces of $P$. A walk along the edges of this polygon corresponds to a sequence of adjacent flags of $P$, alternately $j$-adjacent or $(j + 1)$-adjacent, with the $j$-adjacent flags in different orbits. Hence the polygon again has evenly many sides, so $p_{j+1}$ is even. \hfill \Box

Corollary 7. If a $d$-polytope $P$ is in class $2_I$ and $j \notin I$, then $j = 0$ or $j = d - 1$.

Proof. If $j \notin I$ and $0 < j < d - 1$, then both the entries $p_j$ and $p_{j+1}$ appear in the Schlafli symbol. But this contradicts Proposition L.7; a polyhedral section $F_{j+2}/F_{j-2}$ would have the symbol $\{p_j, p_{j+1}\}$ with two even entries, which is impossible. \hfill \Box

Continuing the preceding remarks, we conclude that the only way two distinct ranks can be missing from $I$, where $P$ is in class $2_I$, is if $I$ omits both 0 and $d - 1$ and $d - 1 = 0 + 3$, i.e. $P$ must be a 4-polytope in class $2_{\{1,2\}}$. We will postpone considering this special case until §5. Otherwise, $|I| = d - 1$ and any two-orbit polytope of type $2_I$ must be either vertex-intransitive or facet-intransitive. Since vertex-intransitive polytopes are the duals of the facet-intransitive polytopes, we will deal with the latter in §4.

4. Combinatorially facet-intransitive two-orbit polytopes

Suppose $P$ is a combinatorially two-orbit $d$-polytope which is facet-intransitive, i.e. it is in class $2_I$ where $I = \{0, 1, \ldots, d - 2\}$. It follows that $P$ is what is called an \emph{alternating semiregular polytope} by Monson and Schulte \cite{2012}. Fix a flag $\Phi$, the base flag. Then for each $i \in I$, there is an automorphism $\rho_i \in \Gamma(P)$ such that $\rho_i(\Phi) = \Phi^i$. There is no automorphism
carrying $\Phi$ to $\Phi^{d-1}$, which is in the other orbit. However, the flag $\Phi^{d-1,d-2,d-1}$, reached by changing the facet of $\Phi$, then changing the ridge, then flipping facets again, is in the same orbit as $\Phi$, so there is an automorphism $\rho'_{d-2}$ carrying $\Phi$ to $\Phi^{d-1,d-2,d-1}$. This automorphism is referred to as $\alpha_{d-1,d-2,d-1}$ by Hubard (2010).

**Lemma 8.** The automorphisms $\rho_i$ and $\rho'_{d-2}$ generate the whole automorphism group of $P$, so $\Gamma(P) = \langle \rho_0, \rho_1, \ldots, \rho_{d-2}, \rho'_{d-2} \rangle$.

**Proof.** Write $\Phi = \{F_-, F_0, \ldots, F_{d-1}, P\}$, and say the facet-adjacent flag $\Phi^{d-1}$ has the facet $F'_{d-1}$. First we show that the given generators suffice to carry the flag $\Phi^{d-1}$ to each of its adjacent flags $\Phi^{d-1,i}$ for $i \leq d - 2$.

Let $i \leq d - 3$. Since $\rho_i$ fixes $F_{d-2}$ and $F_{d-1}$, it must also fix $F'_{d-1}$. Hence, it fixes all faces of $\Phi^{d-1}$ except for its $i$-face $F_i$, so $\rho_i(\Phi^{d-1}) = \Phi^{d-1,i}$.

On the other hand, $\rho_{d-2}$ cannot fix $F'_{d-1}$. Since $\rho_{d-2}(\Phi) = \Phi^{d-2}$, the image of the $(d - 1)$-adjacent flag $\Phi^{d-1}$ must be $(d - 1)$-adjacent to $\Phi^{d-2}$, i.e. $\rho_{d-2}(\Phi^{d-1}) = \Phi^{d-2,d-1}$. But the automorphism $\rho'_{d-2}$ which carries $\Phi$ to $\Phi^{d-1,d-2,d-1}$ must carry $\Phi^{d-1}$ to $\Phi^{d-1,d-2}$.

Thus, the given generators carry $\Phi^{d-1}$ to each of its adjacent flags except for $\Phi$.

Now let $\gamma$ be any automorphism of $P$. The automorphism $\gamma$ is the unique one carrying $\Phi$ to $\gamma(\Phi)$. By exhibiting an automorphism in $\langle \rho_0, \rho_1, \ldots, \rho_{d-2}, \rho'_{d-2} \rangle$ carrying $\Phi$ to $\gamma(\Phi)$, we show that the arbitrary element $\gamma$ lies in this subgroup.

By the flag-connectedness property of polytopes, there is a sequence of adjacent flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_n = \gamma(\Phi)$. For each $k \in \{1, \ldots, n\}$ there is some $i_k \in \{0, \ldots, d - 1\}$ such that the flag $\Phi_k$ is $i_k$-adjacent to the preceding flag $\Phi_{k-1}$. Suppose $1 \leq k \leq n$ and we have written either $\Phi_{k-1} = \sigma(\Phi)$ or $\Phi_{k-1} = \sigma(\Phi^{d-1})$ for some $\sigma$ contained in $\langle \rho_0, \rho_1, \ldots, \rho_{d-2}, \rho'_{d-2} \rangle$.

If $i_k \leq d - 3$, then $\Phi_k$ is $\sigma(\rho_{i_k}(\Phi))$ or $\sigma(\rho_{i_k}(\Phi^{d-1}))$, respectively.

If $i_k = d - 2$, then $\Phi_k$ is $\sigma(\rho_{d-2}(\Phi))$ or $\sigma(\rho'_{d-2}(\Phi^{d-1}))$, respectively.

If $i_k = d - 1$, then $\Phi_k$ is $\sigma(\Phi^{d-1})$ or $\sigma(\Phi)$, respectively.

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Thus we continue until we have written \( \Phi_n = \sigma(\Phi) \) or \( \sigma(\Phi^{d-1}) \) for some \( \sigma \) contained in \( \langle \rho_0, \rho_1, \ldots, \rho_{d-2}, \rho'_{d-2} \rangle \). Since \( \Phi_n = \gamma(\Phi) \) is in the same orbit as \( \Phi \) and \( \Phi^{d-1} \) is not, we must in fact have \( \Phi_n = \sigma(\Phi) \) and \( \gamma = \sigma \).

By Corollary 5 with \( j = d - 1 \), \( P \) will have a modified Schlӓfli symbol of the form \( \{p_1, \ldots, p_{d-3}, p_{d-2}, p_{d-1}\} \), where \( p_{d-2} \neq q_{d-2} \), since \( P \) cannot be equivelar. Figure VIa shows the Coxeter diagram for these generators, modified by labeling the dots with the corresponding generator. Such a diagram is dubbed a “tail-triangle diagram” by Monson and Schulte (2012), making \( \Gamma(P) \) a “tail-triangle group.” Note that \( p_{d-1} \) must be even, by Lemma 6.

![Figure VIa. The Coxeter diagram of a facet-intransitive two-orbit polytope](image)

Since the generators of \( \Gamma(P) \) satisfy all the Coxeter relations implied by the diagram, \( \Gamma(P) \) is a quotient of the Coxeter group associated with the diagram. However, in principle the generators of \( \Gamma(P) \) might satisfy additional relations. We shall show that, in fact, there are no additional relations in \( \Gamma(P) \), so that \( \Gamma(P) \) is exactly the Coxeter group associated with the diagram in Figure VIa. Since \( \Gamma(P) \) is finite, we can then have recourse to the classification of finite Coxeter groups.

**Lemma 9.** The automorphism group \( \Gamma(P) \) is a Coxeter group, with Coxeter diagram as in Figure VIa.

The proof is a modification of that of McMullen (1968, Theorem 3A1), that a combinatorially regular convex polytope is isomorphic to a regular one. The method is also in Coxeter’s proof (1973, §5.3) that the Coxeter relations fully define the group generated by reflections in the walls of the fundamental region described by the diagram. The essence is that any relation in the group (i.e. a word in the generators representing the identity) can be represented as a loop in the boundary of the polytope \( P \); contracting this loop to a point
gives a guide to reducing the word, using the given relations, until it is empty. This shows that every relation in the group is a consequence of the Coxeter relations. The following proof is modeled on, and sometimes verbatim from, that of Coxeter (1973, §5.3).

Proof. We associate flags of \( P \) with chambers of a barycentric subdivision \( \mathcal{B} \) of the boundary of \( P \). Each flag \( \Omega = \{ G_{-1}, G_0, \ldots, G_{d-1}, G_d \} \) is associated to the \((d-1)\)-simplex whose vertices are “barycenters” of each proper face of \( \Omega \). These barycenters can be any preassigned points in the relative interior of each face of \( P \). So the vertices of the simplex for \( \Omega \) are the vertex \( G_0 \), the midpoint (say) of the edge \( G_1 \), and so on up an interior point of the facet \( G_{d-1} \). Each face of this simplex corresponds to a subchain of \( \Omega \). A facet of the simplex is a \((d-2)\)-simplex involving the centers of all but one of the proper faces in \( \Omega \). Say the missing face is \( G_i \). Then the facet, called the \( i \)th wall, forms the boundary between the simplex for \( \Omega \) and the simplex for the \( i \)-adjacent flag \( \Omega^i \). We identify each flag with its corresponding chamber in the boundary of \( P \).

The union of the chambers \( \Phi \) and \( \Phi^{d-1} \) constitute a “fundamental region” \( R \) for \( \Gamma(P) \), since every flag is the image of one of these. For \( 0 \leq i \leq d-3 \), the \( i \)th wall of \( \Phi \) and the \( i \)th wall of \( \Phi^{d-1} \) are contiguous, and we will call their union the \( i \)th wall of \( R \). The \((d-2)\)th wall of \( \Phi \) is called the \((d-2)\)th wall of \( R \), and the \((d-2)\)th wall of \( \Phi^{d-1} \) is called the \( z \)th wall of \( R \) (\( z \) is just a symbol distinct from \( 0, \ldots, d-1 \)). The \((d-1)\)th walls of \( \Phi \) and \( \Phi^{d-1} \) are in the interior of \( R \) and are not walls of \( R \). Thus, \( R \) has walls labeled \( 0, \ldots, d-2, z \).

Say the vertex of \( \mathcal{B} \) lying in the relative interior of the face \( F_i \) of \( \Phi \) is \( C_i \). Recall that \( F'_{d-1} \) is the facet in \( \Phi^{d-1} \); say the vertex in relint(\( F'_{d-1} \)) is \( C'_{d-1} \). Then \( R \) contains the \( d \) vertices \( C_i \), plus \( C'_{d-1} \), but \( C_{d-2} \) is on the “edge” from \( C'_{d-1} \) to \( C_{d-1} \) and is not a vertex of \( R \), so that \( R \) has \( d \) vertices and is again a simplex, with vertices \( C_0, \ldots, C_{d-3}, C_{d-1}, C'_{d-1} \). See Figure VI (Some facets may be skew, rather than linear.) In the left figure, where \( d = 3 \), the \( i \)th wall is labeled \( i \). In the right figure, where \( d = 4 \), the face \( C_0 C_1 C'_3 \) is the \( z \)th wall; the face \( C_0 C_1 C_3 \) is the 2nd wall; the face \( C_0 C'_3 C_3 \) is the 1st wall; and the face \( C_1 C'_3 C_3 \) is the 0th wall.
Figure VIb. The region $R$ composed of $(d - 1)$-adjacent chambers, in the case $d = 3$ (left) or $d = 4$ (right)

Now for $\gamma \in \Gamma(P)$, the chambers for $\gamma(\Phi)$ and $\gamma(\Phi^{d-1})$ are adjacent, and their union is called “region $\gamma$.” We pass through the $i$th wall of region $\gamma$ into region $\gamma \rho_i$ (for $i \in \{0, \ldots, d - 2, z\}$), where $\rho_z$ denotes $\rho'_{d-2}$. Each automorphism $\gamma$ carries $i$-faces to other $i$-faces, and the two orbits of $(d - 1)$-faces are carried only to themselves. Although $\gamma$ does not actually map points to other points, if we consider a vertex of $B$ as representing the face in whose relative interior it lies, it makes sense to say that each vertex of $R$ is carried only to the unique vertex of the same type in each region $\gamma$.

To a word $w = \rho_{i_1} \ldots \rho_{i_k}$, where $i_j \in \{0, \ldots, d - 2, z\}$, we associate a path from $R$ to region $\rho_{i_1} \ldots \rho_{i_k}$ passing through the $i_1$th wall of $R$, then the $i_2$th wall of region $\rho_{i_1}$, and so on. (By a path we mean a continuous curve which avoids any $(d - 3)$-face of $B$.)

If the word $w$ represents the identity, we must show that the relation $w = 1$ is a consequence of the Coxeter relations inherent in Figure VIa. These relations are $(\rho_i \rho_j)^{m_{ij}} = 1$, where $m_{ii} = 1$ for all $i$, and otherwise $m_{ij}$ is the label on the edge from $\rho_i$ to $\rho_j$, or 2 if there is no edge. If $w = 1$, the path associated to $w$ is a closed path back to $R$. Consider what happens to the expression $\rho_{i_1} \ldots \rho_{i_k}$ as the closed path is gradually shrunk until it lies wholly within region $R$. Whenever the path goes from one region to another and then immediately returns, this detour may be eliminated by canceling a repeated $\rho_i$ in the expression, in accordance with the relation $(\rho_i)^2 = 1$. The only other kind of change that can occur during the shrinking process is when the path momentarily crosses a $(d - 3)$-face $F$. 

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If $F$ is the intersection of the $i$th and $j$th walls of one region, so that it does not contains vertices of the types opposite the $i$th and $j$th walls, then $F$ does not contain vertices of those types in any region that contains it. So the walls containing $F$ alternate between $i$th walls and $j$th walls, and $F$ is contained in $2m_{ij}$ regions.

This change will replace $\rho_i\rho_j\rho_i\cdots$ by $\rho_j\rho_i\rho_j\cdots$ (or vice versa) in accordance with the relation $(\rho_i\rho_j)^{m_{ij}} = 1$. The shrinkage of the path thus corresponds to an algebraic reduction of the expression $w$ by means of the Coxeter relations. Since the boundary of $P$ is topologically a $(d - 1)$-sphere, and simply connected if $d > 2$, we can shrink the path to a point. It follows that every relation in $\Gamma(P)$ is a Coxeter relation. □

We can now complete the proof of

**Theorem 10.** Any combinatorially two-orbit facet-intransitive convex polytope is isomorphic to a two-orbit convex polytope.

Since $P$ has finitely many flags, we know that $\Gamma(P)$ is a finite Coxeter group. Consulting the list of finite Coxeter groups, we see that $p_{d-1}/2$ must be 2, since no cycles appear in diagrams of finite Coxeter groups. Furthermore, the only diagram with four or more dots that branches as in Figure VI is $D_n$, where every edge has the label 3. But we must have $p_{d-2} \neq q_{d-2}$, since $P$ is not equivelar. Hence the diagram must not have a “tail”: we must have $d = 3$, and the only admissible diagrams of finite Coxeter groups are those in Figure VIC.

![Figure VIc](image)

**Figure VIc.** Potential Coxeter diagrams for the automorphism group of a two-orbit polytope

We know that both of these Coxeter groups occur as the automorphism group of a two-orbit facet-intransitive polyhedron: $B_3$ for the cuboctahedron, and $H_3$ for the icosidodecahedron.
The next lemma will show that the isomorphism class of a two-orbit facet-intransitive polytope is determined by its automorphism group (as a Coxeter system), so these are the only possibilities.

For the purposes of the Lemma, we will fix a canonical form of the Coxeter group presentation, as encoded in the diagram of Figure VI or the Schläfli symbol \{p_1, \ldots, p_{d-3}, q_{d-2}, p_{d-1}\}, such that \( p_{d-2} < q_{d-2} \). A flag \( \Phi \) will be said to be an appropriate base flag if the generators \( \rho_i \) corresponding to \( \Phi \), defined as in Lemma 8, satisfy \( (\rho_{d-3}\rho_{d-2})^{p_{d-2}} = 1 \). We prove the Lemma generally, rather than restricting to the two presentations in Figure VII, so that it also applies to tilings, or indeed, any abstract polytopes.

**Lemma 11.** Two combinatorially two-orbit facet-intransitive polytopes \( P_1 \) and \( P_2 \) are isomorphic if and only if their automorphism groups have the same presentation (with generators as in Lemma 8, and Coxeter relations as depicted in Figure VI), if we require \( p_{d-2} < q_{d-2} \).

**Proof.** If \( h: \mathcal{L}(P_1) \rightarrow \mathcal{L}(P_2) \) is an isomorphism, let \( \Phi \) be an appropriate base flag for \( P_1 \). Then the generators of \( \Gamma(P_2) \) corresponding to the base flag \( h(\Phi) \) must satisfy the same relations that the generators of \( \Gamma(P_1) \) corresponding to \( \Phi \) do, so that the groups have the same presentation.

Conversely, suppose \( P_1 \) and \( P_2 \) are combinatorially two-orbit facet-intransitive polytopes with the same presentation. For \( i = 1, 2 \), let \( \Phi_i \) be an appropriate base flag of \( P_i \) and define the generators \( \rho^i_1, \ldots, \rho^i_{d-2}, \rho^i_{d-2} \) of \( \Gamma(P_i) \) with respect to \( \Phi_i \) as in Lemma 8. Since \( \Gamma(P_1) \) and \( \Gamma(P_2) \) have the same presentation, the map carrying \( \rho^1_2 \mapsto \rho^2_2 \) and \( \rho^1_{d-2} \mapsto \rho^2_{d-2} \) extends to a group isomorphism \( f \). Then the bijection of the sets of flags taking \( \gamma(\Phi_1) \mapsto f(\gamma)(\Phi_2) \) and \( \gamma(\Phi_1^{d-1}) \mapsto f(\gamma)(\Phi_2^{d-1}) \), for all \( \gamma \in \Gamma(P_1) \), determines the required isomorphism between the lattices \( \mathcal{L}(P_1) \) and \( \mathcal{L}(P_2) \). \( \square \)

5. Exceptional possibilities in \( \mathbb{E}^4 \)

We now return to the exceptional possibilities left open for combinatorially two-orbit 4-polytopes (see the end of §2). Recall that such a polytope \( P \) is in class \( 2_{(1,2)} \), so it is
combinatorially fully transitive. For any flag $\Phi$, the 1-adjacent flag $\Phi^1$ and 2-adjacent flag $\Phi^2$ are in the same orbit as $\Phi$, while the 0-adjacent flag $\Phi^0$ and 3-adjacent flag $\Phi^3$ are not. By 2-face-transitivity, all the 2-faces have the same number of sides, $p_1$. All the edges are in the same number of facets, $p_3$. By Lemma 6 with $j = 0$ and $j = 3$, $p_1$ and $p_3$ are even. Since $P$ is not equivelar, the Schläfli symbol has the form $\{p_1, q_2, p_3\}$ where $p_1$ and $p_3$ are even.

Each facet and vertex figure of $P$ has at most two combinatorial flag orbits, by the action of the automorphism group of $P$ restricted to these sections. Since $P$ is facet-transitive, each facet must have both $p_2$-gons and $q_2$-gons as vertex figures. Since $P$ is vertex-transitive, each vertex figure must have both $p_2$-gons and $q_2$-gons as faces. Thus the facets and vertex figures are not combinatorially regular: they are combinatorially two-orbit 3-polytopes. By the preceding proof, the facets and vertex figures are isomorphic to one of the four two-orbit polyhedra. Since all 2-faces are the same, and by the necessary compatibility of the vertex figures with the facets, the two possibilities are:

- A polytope whose facets are isomorphic to the rhombic dodecahedron, and whose vertex figures are isomorphic to cuboctahedra; the modified Schläfli symbol is $\{4, 3, 4\}$, and

- A polytope whose facets are isomorphic to the rhombic triacontahedron, and whose vertex figures are isomorphic to icosidodecahedra; the modified Schläfli symbol is $\{4, 3, 5, 4\}$.

In each case, the polytope would be combinatorially self-dual. However, we demonstrate that such polytopes cannot exist.

Suppose that $P$ has the first combinatorial type above, $\{4, 3, 4\}$. Consider the angle at a vertex $v$ in a 2-face $F$ containing $v$. That is, in the affine hull $\text{aff}(F)$, which is a plane, we take the interior angle of the quadrilateral $F$ at $v$. The sum of all these angles at the 4 vertices of $F$ is $2\pi$. So, if we take the sum of all such angles in the whole polytope $P$—i.e. the sum of the angle for every incident pair of vertex and 2-face in $P$—the sum is $2\pi f_2$, where $f_2$ is the number of 2-faces of $P$. Since every vertex is in 24 2-faces (the number of edges of the
cuboctahedron), and each 2-face has 4 vertices, \( f_2 = 6f_0 \) (where \( f_0 \) is the number of vertices of \( P \)).

On the other hand, let \( v \) be any vertex of \( P \) and consider the sum of the angles in each 2-face incident to \( v \). Each 2-face lies in exactly two facets: one where \( v \) is in 4 edges, and one where \( v \) is in 3 edges. (Correspondingly, each edge of the vertex figure, the cuboctahedron, is in one square and one triangle.) We may partition the 24 2-faces at \( v \) into 6 sets of 4, each set consisting of the 2-faces of a facet \( G \) containing \( v \) wherein \( v \) has valence 4. The sum of the angles of \( v \) within these four 2-faces must be less than \( 2\pi \) (the difference from \( 2\pi \) is the angular deficiency or defect.) Hence the sum of the angles at \( v \) in all the 2-faces containing \( v \) is less than \( 6 \cdot 2\pi \), and the sum of the angles of all incident pairs of vertices and 2-faces is therefore less than \( 6f_02\pi \).

But this contradicts the earlier conclusion that the sum is exactly \( 6f_02\pi \). Therefore, no such polytope can exist.

The same argument rules out the possibility of a polytope of the second type, \( \{4, \frac{3}{5}, 4\} \). Each vertex is in 60 2-faces (the number of edges of the icosidodecahedron), and each 2-face has 4 vertices, so we have \( f_2 = 15f_0 \), and the sum of the angles over all incident pairs of vertex and 2-face is \( 15f_02\pi \).

On the other hand, the 2-faces at each vertex \( v \) can be partitioned into 12 sets of 5, each set consisting of the 2-faces of a particular facet \( G \) containing \( v \) wherein \( v \) has valence 5. The sum of the angles at \( v \) in all these 2-faces is less than \( 2\pi \), so the sum of all the angles of \( v \) in the 60 2-faces containing \( v \) is less than \( 12 \cdot 2\pi \).

Thus we have \( 15f_02\pi < 12f_02\pi \), a contradiction, so no such polytope can exist.

With these possibilities disposed of, we have proved Theorem 1.

6. Tilings

In this section, we deal with combinatorially two-orbit tilings of Euclidean space \( \mathbb{E}^d \). Recall that tilings are assumed to be by convex polytopes, locally finite, and face-to-face.
In §3, Lemmas 4 and 6 and Corollary 5 apply also to combinatorially two-orbit tilings. Corollary 7 holds, but requires a modified proof:

**Corollary 7’.** If a rank-\(d\) tiling \(T\) is in class \(2\), and \(j \notin I\), then \(j = 0\) or \(j = d - 1\).

**Proof.** Suppose \(0 < j < d - 1\). Then there is an incident pair of faces \(F_{j+2}\) and \(F_{j-2}\). If \(F_{j+2}/F_{j-2}\) is a proper section, then it is isomorphic to a convex polytope, with symbol \(\{p_j, p_{j+1}\}\) with two even entries (by Lemma 6). It is impossible for convex polytopes to have such a symbol.

On the other hand, if \(F_{j+2}/F_{j-2}\) is all of \(T\), then \(j - 2 = -1\) and \(j + 2 = d\), i.e. \(j = 1\) and \(d = 3\), so we have an edge-intransitive planar tiling. By Corollary 5, \(T\) has type \(\{p_1, p_2\}\) with single-valued entries. Hence \(T\) is a combinatorially regular tiling, a contradiction. \(\square\)

The proof showed that \(T\) is equivelar. The previously mentioned Theorem 1B9 of McMullen and Schulte (2002), stating that equivelar convex polytopes are combinatorially regular, actually depends only on the surface being simply-connected; hence, equivelar tilings are combinatorially regular. This is also discussed by Dress and Schulte (1987), who demonstrate that any equivelar tiling of a simply-connected space is a quotient of a combinatorially regular tiling of a simply-connected space; but since a simply-connected space is its own universal cover, the two tilings must be identical.

Lemmas 8, 9, and 11 in §4 also apply to tilings, but since the automorphism group of a tiling is not finite, we get no corresponding short list of potential diagrams. If we could conclude that the automorphism group was of so-called “affine type”, then the analog to Theorem 10 would follow.

Theorem 4A4 of McMullen’s thesis (1968) says, for \(d \neq 3\), a rank-\(d\) convex polytope with combinatorially regular vertex figures and combinatorially regular facets is combinatorially regular. The proof works equally well for tilings; we sketch it here.

**Lemma 12.** A rank-\(d\) tiling \(T\), \(d \neq 3\), whose vertex figures and facets are all combinatorially regular is itself combinatorially regular.
Proof. Each vertex figure $T/v$ is combinatorially regular, hence equivelar, so for any $i \in \{2, \ldots, d - 1\}$, every incident pair of $(i + 1)$-face $G_{i+1}$ and $(i - 2)$-face $G_{i-2}$ containing $v$ gives a polygonal section $G_{i+1}/G_{i-2}$ of the same size, say $p_i(v)$. Each edge figure, being contained in a vertex figure, is also equivelar, so for any $i \in \{3, \ldots, d - 1\}$, the size of each section $G_{i+1}/G_{i-2}$ of incident faces containing a given edge is also constant. Since any two vertices of $T$ may be linked by a chain of vertices and edges, the Schläfli entries $p_i$ with $i \geq 3$ are well-defined on all of $T$.

Similarly, face-chains of facets and ridges show that $p_i$ is well-defined for $i \leq d - 3$, and face-chains of vertices and facets cover the remaining case when $d = 4$ and $i = 2$. Therefore, $T$ is equivelar, hence combinatorially regular. \hfill \Box

\textbf{Theorem 13.} All combinatorially two-orbit tilings are of $\mathbb{E}^2$ or $\mathbb{E}^3$.

Proof. A combinatorially two-orbit tiling has facets and vertex figures which are either combinatorially regular or combinatorially two-orbit. If we are tiling $\mathbb{E}^d$, and $d \geq 4$, then by Theorem \[1\] the facets and vertex figures are actually combinatorially regular, so the tiling is combinatorially regular. Thus $d < 4$.

Of course, tilings of $\mathbb{E}^0$ and $\mathbb{E}^1$ are trivial, and no combinatorially two-orbit ones exist. The remaining cases are tilings of $\mathbb{E}^2$ or $\mathbb{E}^3$. \hfill \Box

\textbf{6.1. Planar tilings.} Planar tilings are the only case, in light of Lemma \[12\] where a combinatorially two-orbit tiling can have combinatorially regular tiles and vertex figures. Indeed, any planar tiling has combinatorially regular tiles and vertex figures, since all polygons are combinatorially regular.

\textbf{Theorem 14.} There are infinitely many (isomorphism classes of) combinatorially two-orbit tilings of the plane.

To see this, first we show that there are combinatorially regular tilings by convex $p$-gons, with three tiles at each vertex, for every $p \geq 6$. This is a consequence of Statement 4.7.1
of *Tilings and Patterns* (Grünbaum and Shephard 1987b, p. 194). We paraphrase the result, taking advantage also of Statement 4.1.1 that homeomorphisms preserving a tiling are equivalent to combinatorial automorphisms, and of convexification (Grünbaum and Shephard 1987b, p. 202).

**Lemma 15** (Grünbaum and Shephard 1987b, 4.7.1). There exists a combinatorially regular tiling of type \(\{j, k\}\), for positive integers \(j, k\), if and only if \(1/j + 1/k \leq 1/2\). Such a tiling can be normal only if equality holds.

Since \(1/p + 1/3 \leq 1/2\) for every \(p \geq 6\), there is a combinatorially regular tiling of type \(\{p, 3\}\) for every such \(p\). From this tiling \(\mathcal{R}\), we can form a combinatorially two-orbit tiling by “truncating” at each vertex to the midpoints of its incident edges, analogously to the formation of the cuboctahedron from the cube, of the icosidodecahedron from the dodecahedron, or of the trihexagonal tiling from the regular hexagonal tiling. Each edge of \(\mathcal{R}\) is reduced to its midpoint. The midpoints of the three edges incident to a vertex become the vertices of a triangular tile. The midpoints of the \(p\) edges of a \(p\)-gonal tile in \(\mathcal{R}\) become the vertices of a smaller \(p\)-gonal tile. For instance, with \(p = 7\), this is a “triheptagonal” tiling. Each vertex of this new tiling (formerly an edge midpoint) is in four tiles: two triangles (the vertex figures of the endpoints of the former edge), and two \(p\)-gons. Thus the vertex type of the new tiling is \((3.p.3.p)\).

However, none of these examples are normal for \(p \geq 7\). We proceed to show

**Theorem 16.** Every normal combinatorially two-orbit planar tiling by convex polygons is isomorphic to one of the (geometrically) two-orbit planar tilings: either the trihexagonal tiling or its dual, the rhombille tiling.

A combinatorially fully-transitive planar tiling is equivelar, since every tile has the same number \(p\) of sides, and every vertex is in the same number \(q\) of faces. So any combinatorially two-orbit tiling \(\mathcal{T}\) of \(\mathbb{E}^2\) must not be fully transitive. By Corollary 7, \(\mathcal{T}\) is
either facet-intransitive, in which case $\Gamma(T)$ acts transitively on its vertices and edges, or vertex-intransitive, in which case $\Gamma(T)$ acts transitively on its facets (tiles) and edges.

In the former case, we apply the analog of Proposition 1.8 for normal tilings. If the vertex type is $(k_1 \ldots k_j)$, then
\[
\sum_{i=1}^{j} \frac{k_i - 2}{k_i} = 2.
\]
Each vertex is incident to evenly many tiles which alternate orbits; the latter follows from the edge-transitivitiy. Since each term is at least $\frac{1}{3}$, there are no solutions with $j \geq 8$. With 6 tiles at each vertex, the only solution is when all tiles are triangles, $(3^6)$; but this is a combinatorially regular tiling by triangles. So we consider 4 tiles at each vertex. If none of the tiles are triangles, the only solution is four 4-gons, $(4^4)$; but this is a combinatorially regular tiling by 4-gons. So we must have $(3.k.3.k)$. The only solution is $k = 6$, which is isomorphic to the trihexagonal tiling (see Figure V). This tiling has two triangles and two hexagons alternating at each vertex.

On the other hand, if $\Gamma(T)$ acts transitively on facets, we apply the analog of Proposition 1.9 for normal tilings. If the facet type is $[j_1 \ldots j_k]$, then
\[
\sum_{i=1}^{k} \frac{j_i - 2}{j_i} = 2.
\]
Every facet has evenly many sides and the valence of each vertex alternates, again by the edge-transitivity. Clearly, this has the same solutions as before. We have facet type $[3^6]$, a combinatorially regular tiling by hexagons; $[4^4]$, a combinatorially regular tiling by 4-gons; and $[3.6.3.6]$, isomorphic to the rhombille tiling (see Figure V). The latter has rhombus tiles, with three tiles meeting at the obtuse corners, and 6 tiles meeting at the acute corners.

6.2. Tilings of $\mathbb{E}^3$. A tiling of $\mathbb{E}^3$ has rank 4. Hence, by Lemma 12, if it has combinatorially regular facets and vertex figures, it must be combinatorially regular. So a combinatorially two-orbit tiling $T$ of $\mathbb{E}^3$ must have some tiles or some vertex figures from the list of combinatorially two-orbit polyhedra. By Corollary 7, the class of $T$ must be $2_{\{0,1,2\}}$.
(tile-intransitive), $2_{\{1,2,3\}}$ (vertex-intransitive), or $2_{\{1,2\}}$ (fully transitive). We consider these cases.

6.2.1. **Tile-intransitive.** In this case, $\mathcal{T}$ is in class $2_{\{0,1,2\}}$ and the automorphism group is transitive on the vertices, edges, and 2-faces of the tiling. There are two different tile orbits, and each tile must be combinatorially regular (since the orbit of a flag is determined entirely by which type of tile it includes). Thus, all the vertices have isomorphic vertex figures, which must be a two-orbit polyhedron; since there are two types of tile, the vertex figure must be facet-intransitive, i.e. the cuboctahedron or the icosidodecahedron.

With the cuboctahedron as vertex figure, each vertex is 3-valent in some tiles, and 4-valent in others. The only regular polyhedron with 4-valent vertices is the octahedron; the only regular polyhedron with 3-valent vertices and triangular faces (to match the octahedron) is the tetrahedron. But the tiling built from tetrahedra and octahedra in this manner is the tetrahedral-octahedral honeycomb, $\{3, \frac{3}{4}, 4\}$, one of the (geometrically) two-orbit tilings.

With the icosidodecahedron as vertex figure, each vertex is 3-valent in some tiles, and 5-valent in others. The only regular polyhedron with 5-valent vertices is the icosahedron, and the other tiles must be tetrahedra. Such a tetrahedral-icosahedral tiling has type $\{3, \frac{3}{5}, 4\}$. Indeed, a tiling can be built up in such a way, in hyperbolic space; it is known as the alternated order-5 cubic honeycomb. It can be carved out of a tiling by cubes with 5 around each edge, $\{4, 3, 5\}$, which is a regular tiling of hyperbolic space. Inscribe a tetrahedron in each cube, so that tetrahedra in adjacent cubes alternate direction. The shape left around a vertex which is not part of the tetrahedra is an icosahedron. (There are 20 cubes around each vertex in $\{4, 3, 5\}$.)

We show by contradiction that there is no normal tiling of $\mathbb{E}^3$ of this type. Suppose $\mathcal{T}$ is a normal tetrahedral-icosahedral tiling. Divide each icosahedron of $\mathcal{T}$ into twenty pyramids (over each of its 2-faces), and adjoin each of these pyramids to the tetrahedron with which it shares the 2-face. Thus we have partitioned $\mathbb{E}^3$ into tiles, one for each tetrahedron in $\mathcal{T}$, consisting of the tetrahedron with a pyramidal cap added to each of its 2-faces. These tiles
need not be convex, but are isomorphic to cubes (or rather, the simplicial complex on the
boundary of a cube determined by a tetrahedron inscribed into this cube). Each tile has six
neighboring tiles, with each of which it shares two triangular faces of its pyramidal caps; we
treat each such pair of triangular faces as a single “skew” 4-gonal face. Thus we get a tiling $C$
topologically isomorphic to the order-5 cubic tiling $\{4,3,5\}$. If we start with a normal tiling,
this one will be, also. Say each tile of $C$ contains a ball of radius $u$ and is contained in a ball
of radius $U$. Then the number of tiles in a ball of radius $r$ is at most $r^3/u^3$.

Next consider a growing sequence of patches of the tiling $C$. (For our purposes, a patch
can be defined as any finite set of tiles of $C$ whose union is homeomorphic to a ball.) Begin
with a vertex of the tiling, designated $A_0$. Let $A_1$ consist of all the tiles containing $A_0$, let $A_2$
consist of all the tiles with nonempty intersection with the union of $A_1$, and so on, so $A_{n+1}$
consists of all the tiles with nonempty intersection with the union of $A_n$.

Let us call a tile of $A_n$ which has any 2-face on the boundary of $A_n$ a $k$-tile if it has $k$
2-faces on the boundary of $A_n$. By induction, we see that all tiles on the boundary are 1-tiles,
2-tiles, or 3-tiles; that every edge on the boundary of $A_n$ is contained in either 1 or 2 tiles of
$A_n$; and that every vertex on the boundary of $A_n$ is contained in either 1, 2, or 5 tiles of $A_n$.

- A tile not in $A_n$ which contains a 2-face in $A_n$ becomes a 1-tile of $A_{n+1}$. Each of its
  four exposed edges is in two tiles of $A_{n+1}$. See the leftmost example in Figure VI
  d.
- A tile not in $A_n$ which contains only an edge in $A_n$ becomes a 2-tile of $A_{n+1}$. Six of
  its exposed edges are in two tiles of $A_{n+1}$, while one exposed edge is only in this tile.
  See the middle example in Figure VI d.
- A tile not in $A_n$ which contains only a vertex in $A_n$ becomes a 3-tile of $A_{n+1}$. Six of
  its exposed edges are in two tiles of $A_{n+1}$, while three exposed edges are only in this
tile. See the rightmost example in Figure VI d.

Let $a_n$ be the number of 1-tiles in $A_n$, $b_n$ be the number of 2-tiles, and $c_n$ be the number
of 3-tiles. The 1-tiles in $A_{n+1}$ are those tiles which share a 2-face with some $k$-tile of $A_n$, so
Over a 2-face in $A_n$

Over an edge in $A_n$

Over a vertex in $A_n$

**Figure VI D.** Tiles of $A_{n+1} \setminus A_n$. Vertices and edges on the boundary of $A_{n+1}$ are labeled by the number of tiles of $A_{n+1}$ containing them. Elements belonging to $A_n$ are darkened.

We have

$$a_{n+1} = a_n + 2b_n + 3c_n.$$  

The 2-tiles in $A_{n+1}$ are added above edges on the boundary of $A_n$. One such 2-tile is added above each edge contained in two tiles of $A_n$, and two such 2-tiles are added above each edge contained in a unique tile of $A_n$. Counting the number of edges of each type in the boundary of $A_n$ yields

$$b_{n+1} = \frac{4a_n + 6b_n + 6c_n}{2} + 2(b_n + 3c_n) = 2a_n + 5b_n + 9c_n.$$  

The 3-tiles in $A_{n+1}$ are added above vertices on the boundary of $A_n$. A vertex contained in five tiles of $A_n$ is also contained in five 1-tiles of $A_{n+1}$, added above the five incident 2-faces in the boundary of $A_n$, and in five 2-tiles of $A_{n+1}$ added above the five incident edges in the boundary of $A_n$, so we need to add five more 3-tiles to include all 20 of the incident tiles.

A vertex $v$ contained in two tiles of $A_n$, labeled by a circled 2 in **Figure VI D** is contained in two 2-faces on the boundary of $A_n$, a single edge labeled 1, and two edges labeled 2, in each of the two tiles. Thus $v$ is also contained in four 1-tiles of $A_{n+1}$, and in six 2-tiles of $A_{n+1}$: two added above each edge labeled 1, and one above each edge labeled 2. So we need to add eight 3-tiles of $A_{n+1}$ at $v$ to have all 20.
A vertex in a unique tile of $A_n$ is labeled by a circled 1 in the rightmost example of Figure VI. There are three 1-tiles of $A_{n+1}$ added above the incident 2-faces, and six 2-tiles of $A_{n+1}$, with two above each of the three incident edges. We must add ten 3-tiles of $A_{n+1}$ to have all 20 incident tiles. Hence

$$c_{n+1} = \frac{4a_n + 4b_n + 3c_n}{5} \times + \frac{2b_n + 3c_n}{2} + 10c_n = 4a_n + 12b_n + 25c_n.$$ 

Thus we have an equation for the number of $k$-tiles on the boundary of each patch $A_n$, beginning with the patch $A_1$ consisting of the 20 tiles incident to $A_0$:

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ 4 & 12 & 25 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix}.$$ 

This matrix is diagonalizable, making it straightforward to solve for the total number of tiles in the patch $A_n$:

$$|A_n| = \frac{5}{7} \left( \frac{9}{2\sqrt{14}}((15 + 4\sqrt{14})^n - (15 - 4\sqrt{14})^n) - 8n \right).$$

This is exponential in $n$; the number of tiles increases by a factor of roughly 30 in each successive patch. Now consider a ball centered at $A_0$ with radius $2nU$. This ball contains the patch $A_n$, but the number of tiles in the ball is at most $(2nU)^3/u^3$. An exponential function of $n$ cannot remain bounded by a cubic function of $n$, so there must be some $n$ such that $|A_n| > (2nU)^3/u^3$, a contradiction.

Therefore, no normal tiling of $E^3$ has type $\{4,3,5\}$, even allowing non-convex tiles. So no tetrahedral-icosahedral tiling of type $\{3, \frac{3}{5}, 4\}$ can be normal either. On the other hand, a priori there seems to be no obstruction to constructing (non-normal) tilings of these types.

6.2.2. Vertex-intransitive. In this case, the orbit of a flag is determined by the vertex it contains. So the vertex figures are combinatorially regular. The tiles are two-orbit vertex-intransitive polyhedra, i.e. the rhombic dodecahedron or rhombic triacontahedron.
With the rhombic dodecahedron, a vertex which is incident to 4 edges in a given tile has a vertex figure with a square face; hence the vertex figure is a cube. Each edge incident to the vertex is in 3 tiles. A vertex which is incident to 3 edges in a given tile has a vertex figure with triangular faces. Every edge of the tiling is incident to one vertex of each type, hence is in 3 tiles, so the second type of vertex figure must be a tetrahedron. Rhombic dodecahedra put together in this way form the rhombic dodecahedral honeycomb \( \{4, \frac{3}{4}, 3\} \), one of the (geometrically) two-orbit tilings.

With the rhombic triacontahedron as tile, any vertex which is incident to five edges in a given tile has a pentagon in its vertex figure; hence its vertex figure is a combinatorially regular dodecahedron. Every edge is incident to one vertex of this type, so every edge is in three tiles. Thus the other vertices, which are incident to three edges in each tile, have tetrahedra for vertex figures. This potential tiling has type \( \{4, \frac{3}{5}, 3\} \) and is dual to the tetrahedral-icosahedral tiling discussed above. Like that one, this tiling can be realized in hyperbolic space, with a two-orbit symmetry group. For any normal tiling there is a dual tiling which is also normal (but the tiles of which are not necessarily convex). Hence, if a normal tiling of type \( \{4, \frac{3}{5}, 3\} \) existed, we could find a normal tiling of type \( \{3, \frac{3}{5}, 4\} \), which we know to be impossible.

6.2.3. \textit{Class 2}_{1,2}. This is the same class discussed in \S\S\S\S\S and the same considerations establish that we have either a cuboctahedron vertex figure with rhombic dodecahedra as tiles, type \( \{4, \frac{3}{4}, 4\} \), or an icosidodecahedron vertex figure with rhombic triacontahedra as tiles, type \( \{4, \frac{3}{5}, 4\} \). (We note that the cuboctahedron and icosidodecahedron are non-tiles, meaning there cannot be any tiling of \( \mathbb{E}^3 \) using only tiles isomorphic to these; see Schulte [1985].)

Perhaps these types can be realized as locally finite tilings. However, there is no such normal tiling. Essentially the same proof as in \S\S\S\S applies, along with the Normality Lemma (Schattschneider and Senechal 1997, p. 45), which says that in a normal tiling, the ratio of (the number of tiles that meet the boundary of a spherical patch of the tiling) to (the number
of tiles in the patch) goes to zero as the radius of the patch grows. The two methods of
counting internal angles of 2-faces in §5 hold for all the faces in the interior of a given patch.
Discrepancies occur only at tiles on the boundary, where a vertex is not surrounded by all
the 2-faces incident to it in the tiling. Taking the limit as the patch grows, the discrepancies
go to zero and the inequality remains. The details are too tedious to include here.

With these ruled out, we have established

**Theorem 17.** Every normal combinatorially two-orbit tiling of $\mathbb{E}^3$ is isomorphic to one of
the (geometrically) two-orbit tilings: either the tetrahedral-octahedral honeycomb or its dual,
the rhombic dodecahedral honeycomb.

7. Open Questions

**Question.** Is a combinatorially regular tiling of $\mathbb{E}^d$ by convex polytopes, $d \geq 3$, necessarily
isomorphic to a regular tiling of $\mathbb{E}^d$? (Except for $d = 4$, this says that any combinatorially
regular tiling is isomorphic to the tiling by $d$-cubes.)

The answer is probably *no*, but the author does not know a counterexample.

**Question.** Is a combinatorially regular normal tiling of $\mathbb{E}^d$ necessarily isomorphic to a regular
tiling of $\mathbb{E}^d$?

The answer is probably *yes*, but the author knows a proof only for the cases $d \leq 2$.

**Question.** Are there combinatorially two-orbit tilings of $\mathbb{E}^3$ not isomorphic to any two-orbit
tiling?

Any such tiling would have one of the previously discussed types $\{3, \frac{3}{5}, 4\}$, $\{4, \frac{3}{5}, 3\}$,
$\{4, \frac{3}{4}, 4\}$, or $\{4, \frac{3}{5}, 4\}$. The author believes that non-normal tilings of these types probably do
exist.

For results in these directions, as well as other open questions of this type, see Schulte
([2011]).
CHAPTER VII

Three-orbit Polytopes and Tilings

1. Admissible orbit graphs

Three flag orbits are the first case where orbit graphs are truly useful. We will list the possible orbit graphs in each dimension, then determine which polytopes exist with a given graph. In §7 we conclude by summarizing the classification in Tables VII A and VII B.

In an orbit graph with three nodes, there can not be any node which is adjacent to two other (distinct) nodes via edges with non-consecutive labels. For, suppose there were such a situation, say with edges labeled $i$ and $j$ with $j - i \geq 2$ (all other labels are suppressed):

\[ \bullet \quad i \quad \bullet \quad j \quad \bullet \]

Recall that in any orbit graph, the $i, j, i, j$-walks starting at every node must be closed whenever $|j - i| \geq 2$. In order to complete a closed $i, j, i, j$-walk on the present orbit graph, starting from the leftmost node, the $i$-edge at the rightmost node cannot be a loop. This $i$-edge cannot go to either of the left two nodes, which already have an incident $i$-edge, so it would need to go to a fourth node.

Hence, an orbit graph on three nodes cannot contain a cycle (because some node would be adjacent to the other two nodes by non-consecutively labeled edges).

So the only possible orbit graphs are: $\bullet \quad i \quad \bullet \quad i + 1 \quad \bullet \quad i \quad \bullet \quad i - 1$ (this is equivalent to the first, with a different choice of $i$); or $\bullet \quad i \quad \bullet \quad i + 1 \quad \bullet$. Cunningham et al. (2015) assign names to the symmetry types with these orbit graphs. The first is denoted $3_{i,i+1}^{i}$. The last, with the double edge, is denoted $3_{i}^{i}$. 
2. Planar Case

As we know from §IV.2, there is only one symmetry type of three-orbit 2-polytope, with orbit graph 1–0–1–0. Such polygons may have edges of two distinct lengths, say $a$ and $b$, recurring in the pattern $a, a, b, a, a, b, \ldots$. The simplest example is an isosceles triangle. Every such polygon has $3m$ sides for some $m \geq 1$. In Figure VIIA flags are colored to match the orbit graph.

![Figure VIIA. Three-orbit polygons with flags colored by orbit](image)

In the isosceles triangle, both the edge lengths and the angles at the vertices take on two distinct values. When $m \geq 2$, it is possible to have examples with all edges the same length, but two distinct angles at the vertices, with the angle measures again appearing in the pattern $a, a, b$. It is also possible to have two distinct edge lengths, but all angles the same. Such examples are not related to each other by duality, as opposed to the two-orbit case. Since all three-orbit polygons have the same symmetry type, these are just undistinguished points in the space of all three-orbit $(3m)$-gons. Figure VIIb shows some of these singular examples of hexagons, demonstrating that their duals (whether via polar reciprocation or the convex hull of midpoints) are generic.

There is also a single symmetry equivalence class of tilings of $\mathbb{E}^1$ with this orbit graph: an apeirogon with two distinct edge lengths occurring in the pattern $a, a, b$. There is a family of non-similar versions of this tiling, varying by a single real parameter.
3. Rank 3

There are three possible symmetry types for three-orbit polyhedra. There is an infinite collection of polyhedra of type $3^{1,2}$: the $p$-gonal prisms. There are also five additional polyhedra of this type, up to symmetry equivalence (not similarity). Dually, there are infinitely many polyhedra of type $3^{0,1}$, the $p$-gonal bipyramids, and five more examples up to symmetry equivalence. There is a single polyhedron of type $3^1$, up to symmetry equivalence.

3.1. Type $3^{1,2}$. This is the type with orbit graph $\begin{array}{ccc} 2 & 1 & 2 \\ 0 & 0 & 0 \end{array}$ or, suppressing loops, $\begin{array}{cccc} 1 & 2 \\ 0 & 0 \end{array}$. The name $3^{1,2}$ for this symmetry type comes from Cunningham et al. (2015).

Observing the connected components of the graph after deleting all 2-edges, namely $\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}$ and $\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}$, and bearing in mind Proposition II.1, we see that some 2-faces are isogonal two-orbit polygons and others are regular. The polytopes are vertex-transitive, with vertex figure of type $\begin{array}{ccc} 0 & 1 \\ 0 & 0 \end{array}$. Recall that the orbit graphs of the vertex figures are obtained by deleting all 0-edges and decrementing all remaining labels by one. Thus the vertex figure is a three-orbit polygon, hence a $3m$-gon for some $m \geq 1$. Recall that every polyhedron has some vertex which is at most 5-valent; so the vertex figure must be a triangle.
Suppose the regular 2-faces are \( p \)-gons, and the isogonal 2-faces are \( 2l \)-gons. The orbit graph shows that each vertex is in two of the \( 2l \)-gons, and a single \( p \)-gon, so the vertex type is \((2l.2l.p)\). By Proposition I.8, we have \( 2 \frac{l-1}{l} + \frac{p-2}{p} < 2 \).

- If \( l = 2 \), then we have \( 1 + \frac{p-2}{p} < 2 \), and any \( p \geq 3 \) gives a solution.

These are the \( p \)-gonal prisms (see Figure VIIc). Note that the 4-gonal sides have edge-intransitive symmetry, so they must be rectangles. It is possible to place the \( p \)-gonal base faces at an appropriate distance so that the side faces are squares, but this is irrelevant to the size of the symmetry group; the sides still “act like” rectangles. There is one exception to this rule, if the base face is already a prism. In this case, taking the height of the two prisms to be the same yields additional symmetry. Of course, the only regular \( p \)-gon which is a prism is a square, so the only case where varying the prism height can yield extra symmetries is a 4-gonal prism. A square prism is three-orbit with rectangular sides, and one-orbit with square sides (in which case it is the 3-cube).

- If \( l = 3 \), then we have \( \frac{4}{3} + \frac{p-2}{p} < 2 \), so we must have \( p \in \{3, 4, 5\} \).
- If \( l = 4 \), then we have \( \frac{3}{2} + \frac{p-2}{p} < 2 \), so we must have \( p = 3 \).
- If \( l = 5 \), we have \( \frac{2}{5} + \frac{p-2}{p} < 2 \), so again \( p = 3 \) is the only solution.
- If \( l \geq 6 \), there are no solutions with \( p \geq 3 \).

These five solutions are the truncated Platonic solids (see Figure VIIc). We see from the orbit graph that the sides of the \( 2l \)-gons alternate between being adjacent to a second \( 2l \)-gon or a regular \( p \)-gon. All sides of the \( p \)-gons are adjacent to \( 2l \)-gons.

3.1.1. Tilings. A plane tiling \( T \) of this symmetry type, just as above, has some regular \( p \)-gon tiles, and some isogonal \( 2l \)-gon tiles. The tiling is vertex-transitive and the vertex figures are \( 3m \)-gons, hence must be either triangles or hexagons.

If the vertex figures are hexagons, then the vertex type is \((2l.2l.p.2l.2l.p)\), and from the analog of Proposition I.8 for normal tilings we have the equation

\[
4 \frac{l-1}{l} + 2 \frac{p-2}{p} = 2.
\]
Figure VIIc. Some polyhedra of type $3^{1,2}$, with flags colored to match the colored orbit graph.

But $l$ is at least 2, and $p$ is at least 3, and these values already give us $8/3$ on the left hand side. Thus the vertex figures cannot be hexagons.

If the vertex figures are triangles, we have a similar situation to that for polytopes, but now $2\frac{l-1}{l} + \frac{p-2}{p} = 2$. The only solutions in positive integers are $l = 3, p = 6$; $l = p = 4$; and $l = 6, p = 3$. These solutions correspond to the truncated triangular tiling $t\{3,6\}$, the truncated square tiling $t\{4,4\}$, and the truncated hexagonal tiling $t\{6,3\}$, respectively (see Figure VIIId). The truncation of a regular tiling $\mathcal{R}$ is formed as follows: choose a distance $\delta$, less than half the edge length of the tiling. Replace each vertex $v$ of the tiling by a new tile, the convex hull of the points at distance $\delta$ from $v$ along each of the edges incident to $v$. This tile will be a $p$-gon if $v$ was a $p$-valent vertex of $\mathcal{R}$. In place of each original $q$-gonal tile $F$ of $\mathcal{R}$ we have a $2q$-gon in the truncation: a portion of each of the original $q$ edges remains, along with $q$ new edges bordering the new tiles around each vertex of $F$.

Note that $t\{3,6\}$ is isomorphic to $\{6,3\}$, hence is combinatorially regular. The other two examples are combinatorially three-orbit.

3.2. Type $3^{0,1}$. This is dual to type $3^{1,2}$ above. The orbit graph is $1 \xrightarrow{2} 0 \xrightarrow{2} 1 \xrightarrow{2} 0$.

We see that any polytope $P$ with this graph is facet-transitive, with facets which are $3m$-gons for some $m \geq 1$; since every polyhedron has some 2-faces with less than six sides, the faces of $P$ must be isosceles triangles. The vertices fall in two orbits, one with a regular vertex.
Figure VIId. The tilings of type $3^{1,2}$

figure, say a $p$-gon, and the other with a 0-intransitive vertex figure, i.e. an isotoxal two-orbit polygon, say a $2l$-gon.

Similarly to the rectangular sides of the prisms above, the triangular faces of $P$ might sometimes be equilateral. Regardless, they have only the symmetries of an isosceles triangle. The apex vertex of the triangle is distinguished as the one with a regular vertex figure in $P$, and the other two vertices of the triangle have two-orbit vertex figures in $P$. An exception is possible only if the two types of vertex figure coincide, i.e. if $p = 2l$. But if every vertex of a polyhedron has the same even valence, then it must be 4 (by Proposition I.5). So the only case when the symmetry of $P$ is affected by whether its triangles are equilateral or not is when $P$ is isomorphic to $\{3,4\}$, the regular octahedron.

By dual arguments to those in the previous case, we have the same set of solutions for $l$ and $p$. When $l = 2$, so that four triangles come together at each 4-valent vertex, we have a $p$-gonal bipyramid (see Figure VIIe). When $l \in \{3,4,5\}$, we have the Kleetopes of the Platonic solids (see Figure VIIe). Say $Q = \{p,l\}$ is a Platonic solid. Adding a new vertex over the centroid of each facet of $Q$ replaces the facet by $p$ triangles around a $p$-valent vertex. The original vertices of $Q$ are now in $2l$ triangles.

With less reliance on duality, we can conclude that the polytopes with this orbit graph are Kleetopes by considering the union of the triangles incident to each $p$-valent vertex $v$. Since all the triangles are congruent and isosceles, with $v$ as the apex of each, the edges
opposite $v$ in each triangle lie in a plane, forming a $p$-gon. Truncating this $p$-gon at every $p$-valent vertex, we achieve a polyhedron whose 2-faces are all $p$-gons, and with half as many 2-faces incident to each remaining vertex, i.e. a polyhedron \( \{p,l\} \). In the case $l = 2$, we have a degenerate “dihedron”, consisting of two $p$-gons pasted to each other, and otherwise \( \{p,l\} \) is a Platonic solid.

The Kleetopes of the Platonic solids are also known as the triakis tetrahedron, triakis octahedron, triakis icosahedron, tetrakis cube, and pentakis dodecahedron. The adjective in each case refers to the number of facets each original facet was covered by, or to the valence of the added vertices. Each of these solids can also be constructed by taking one of the 0-intransitive two-orbit polyhedra (even if only realizable as regular, so one of the cube, rhombic dodecahedron, or rhombic triacontahedron), selecting a set of alternate vertices (i.e. one partite set of the bipartite edge graph) and moving these vertices in toward the center of the polyhedron, while leaving the other vertices alone.

![A triangular bipyramid](image1.png)

**Figure VIIe.** Some polyhedra of type $3^{0,1}$, with flags colored to match the colored orbit graph

3.2.1. **Tilings.** As for polytopes, we see that any tiling $\mathcal{T}$ with this orbit graph is tile-transitive, and the tiles are $3m$-gons. For normal tilings, we can have either triangular or hexagonal tiles. Say the two vertex orbits are $p$-valent and $2l$-valent, as before. For the case of hexagonal tiles, we apply the same argument as in §3.1.1 to the facet type $[2l.2l.p.2l.2l.p]$ and see that no such normal tiling exists.
For the case of triangular tiles, the same arithmetic as in §3.1.1 gives us the solutions $l = 3, p = 6$; $l = p = 4$; and $l = 6, p = 3$. These solutions correspond to the “Kleetope” of the regular hexagonal tiling, of the regular square tiling, and the regular triangular tiling, respectively. By Kleetope of a tiling $\mathcal{R}$ we mean the tiling formed by adding a new vertex in the center of each tile $F$ of $\mathcal{R}$, adjacent to all the vertices of $F$, thus dividing each $q$-gonal face into $q$ triangles. The original vertices of $\mathcal{R}$ now have twice the valence in the new tiling, being incident to each of the original edges, and also a new edge for each incident face.

In the Kleetope of the hexagonal tiling, every vertex is 6-valent, and so this tiling is isomorphic to the regular triangular tiling $\{3,6\}$. In fact, since all the triangles must be congruent, any tiling of this symmetry equivalence class is actually regular; hence this tiling is perfect. It is interesting that the dual tiling $t\{3,6\}$ is realizable with three-orbit symmetry, but this one is not, as a plane tiling.

One possibility to realize this tiling with three-orbit symmetry is as a “skew plane tiling”, sitting in $\mathbb{E}^3$. Put the hexagonal tiling on a plane inside $\mathbb{E}^3$, and then form the Kleetope with new vertices above the centroids of the tiles, outside the plane of the tiling.

The other two examples, the Kleetope of $\{4,4\}$ and the Kleetope of $\{3,6\}$, are combinatorially three-orbit.

3.3. Type $3^1$. This is the type with orbit graph $\begin{tikzpicture}[scale=0.5]
\node (0) at (0,0) [circle,fill,inner sep=2pt]{0};
\node (1) at (1,0) [circle,fill,inner sep=2pt]{1};
\node (2) at (2,0) [circle,fill,inner sep=2pt]{2};
\node (3) at (3,0) [circle,fill,inner sep=2pt]{$1$};
\draw (0) -- (1);
\draw (1) -- (2);
\draw (2) -- (3);
\end{tikzpicture}$. We see that it is both vertex-transitive and facet-transitive, yet not edge-transitive. This is the definition of an (irregular) noble polyhedron; the only noble polyhedra which are not regular are the tetragonal and rhombic disphenoids (stated without proof by Brückner [1906, p. 26]; also apparently observed by Hessel [1871]). The tetragonal disphenoids are tetrahedra all of whose 2-faces are congruent isosceles triangles. Rhombic disphenoids are tetrahedra with mutually congruent scalene faces; these have six flag orbits.

Examining the orbit graph after removing all 2-edges, we see that the 2-faces of the polyhedron are three-orbit polygons, hence (by facet-transitivity and Proposition 1.5) are isosceles triangles. The vertex figure’s orbit graph is identical $\begin{tikzpicture}[scale=0.5]
\node (0) at (0,0) [circle,fill,inner sep=2pt]{0};
\node (1) at (1,0) [circle,fill,inner sep=2pt]{1};
\node (1) again at (1,0) [circle,fill,inner sep=2pt]{$1$};
\draw (0) -- (1);
\end{tikzpicture}$, and so
for the same reason, each vertex is 3-valent. Thus any polyhedron of this type has Schläfli symbol \{3, 3\}, hence is isomorphic to a tetrahedron.

Thus, there is a single symmetry equivalence class of polyhedron with this orbit graph. The different similarity classes can be parameterized by a single real parameter, e.g. the ratio of the lengths of the legs (meaning the edges which are not the base of a triangle) and the base of the isosceles triangle faces. With this parameterization, the family has its unique one-orbit member where the parameter is 1.

3.3.1. Tilings. Any tiling of this type is tile-transitive, and the tiles are either triangles or hexagons. It is also vertex-transitive, and the vertices are either 3-valent or 6-valent. Thus, such a tiling is equivelar, and the Schläfli type is either \{3, 3\}, \{3, 6\}, \{6, 3\}, or \{6, 6\}. We can dismiss the first and last type, because they egregiously fail Proposition L,8 for tilings.

Considering a tiling by isosceles triangles, we can see from the orbit graph that these must be glued base-to-base, while across each leg we have two triangles with their apices at opposite ends of the leg. Thus, we can form a distorted, three-orbit triangular tiling of the form on the left of Figure VIIf.

![Three-orbit tilings of type $3^1$](image)

**Figure VIIf.** Three-orbit tilings of type $3^1$

In this case, taking the dual by tile centroids yields a three-orbit tiling of type \{6, 3\}, seen on the right of Figure VIIf.

These are combinatorially regular. In each case there is a family of non-similar tilings varying by a single real parameter.
3.4. Remarks. The combinatorial equivalent of this classification of three-orbit polyhedra is in Orbanić, Pellicer, and Weiss [2010, p. 424], as Theorem 5.2: Any (combinatorially) three-orbit map on the sphere is either the truncation of a regular map on the sphere, or its dual. The $q$-gonal prisms appear as the truncation of the regular polyhedron of type $\{2, q\}$, which denotes the tiling of $S^2$ by $q$ digonal lunes. These are valid maps, though not convex polytopes. The duals of these are the $q$-gonal bipyramids. The tetragonal disphenoids are not included, since they are combinatorially regular.

4. Rank 4

There are five possible orbit graphs. There are five polytopes of types $3^2^3$ and another five of type $3^0^1$, up to similarity. There are no polytopes of type $3^1^2$. Type $3^2$ has the infinite series of $p,p$-duoprisms, and two more examples, up to similarity. Type $3^1$ has the infinite series of $p,p$-duopyramids and again two more examples, up to similarity.

4.1. Type $3^2^3$. This has orbit graph $\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}$. We see that any 4-polytope of this type is vertex-transitive and edge-transitive. The vertex figure is of type $\begin{array}{cc} 1 & 2 \\ 0 & 0 \end{array}$, hence is a $p$-gonal prism or a truncated Platonic solid. The facets come in two types: one regular, and one facet-intransitive two-orbit. The latter can only be an octahedron, cuboctahedron, or icosidodecahedron.

We can dismiss the truncated Platonic solids as vertex figures, since each has a 2-face with six or more sides, but none of the candidate facets have vertices with valence more than five. Hence the vertex figure must be a triangular prism, square prism, or pentagonal prism.

Each of the candidate facets is edge-transitive, and up to similarity has a unique realization with the desired symmetry (i.e., is perfect). Hence, each has a well-defined dihedral angle. The sum of the dihedral angles must be less than $2\pi$ in order for the polyhedra to fit around an edge in 4-space. Each edge is in three facets (since each potential vertex figure has 3-valent vertices): one of the regular facets (corresponding to the base faces of the vertex figures) and two of the two-orbit facets (corresponding to the rectangular sides of the vertex figures).
We examine all the possible combinations of facets. The dihedral angles can be found in Coxeter [1973 pp. 292–293]. The dihedral angles of the regular polyhedra are given by the column labeled $\pi - 2\psi$. The dihedral angle of each quasiregular polyhedron $\{p,q\}$, where the octahedron is $\{3\}$, the cuboctahedron is $\{3\}_{4\,4}$, and the icosidodecahedron is $\{3\}_{5\,5}$, is given by $\pi - \chi$ where $\chi$ is from the entry for $\{p,q\}$ (or $\{q,p\}$). (The angle $\psi$ is defined as the angle subtended at the center of the polyhedron between an edge midpoint and a facet centroid; the angle $\chi$ is the angle subtended at the center between a vertex and a facet centroid (Coxeter 1973 p. 21).) The given angles are approximations.

\[
\begin{align*}
\text{octahedron } (109.5^\circ) + & \begin{cases} 
\text{tetrahedron } (70.5^\circ) : & 289.5^\circ \\
\text{octahedron } (109.5^\circ) : & 328.4^\circ \\
\text{icosahedron } (138.2^\circ) : & 357.1^\circ 
\end{cases} \\
\text{cuboctahedron } (125.3^\circ) + & \begin{cases} 
\text{tetrahedron } (70.5^\circ) : & 321.1^\circ \\
\text{octahedron } (109.5^\circ) : & 360^\circ \\
\text{icosahedron } (138.2^\circ) : & 388.7^\circ \\
\text{cube } (90^\circ) : & 340.5^\circ 
\end{cases} \\
\text{icosidodecahedron } (142.6^\circ) + & \begin{cases} 
\text{tetrahedron } (70.5^\circ) : & 355.8^\circ \\
\text{octahedron } (109.5^\circ) : & 394.7^\circ \\
\text{icosahedron } (138.2^\circ) : & 423.4^\circ \\
\text{dodecahedron } (116.6^\circ) : & 401.8^\circ 
\end{cases}
\end{align*}
\]

The six combinations where the sum of the dihedral angles is less than $360^\circ$ are all realizable: they are the rectifications of the regular 4-polytopes, meaning that every vertex of the regular polytope is truncated by a hyperplane passing through the midpoint of every
edge incident to the vertex. New facets are introduced, isomorphic to each vertex figure, and the old facets are also rectified.

The octahedron + tetrahedron combination is the rectified 4-simplex \( t_1\{3,3,3\} \), also known as 0\(_{21}\). It has five tetrahedra and five octahedra; the vertex figure is a triangular prism (which can be called \(-1_{21}\)). Since all the facets are regular, this is a semiregular polytope; it occurs on Gosset’s list \[(1900)\] as “tetroctahedric”. It has 10 vertices (one for each of the \( \binom{5}{2} \) edges of the 4-simplex). The triangular prism has 36 flags, so we conclude that \( t_1\{3,3,3\} \) has 360 flags. It also retains all the 5! symmetries of the 4-simplex, so it is three-orbit.

The octahedron + octahedron combination is the rectified 4-crosspolytope \( t_1\{3,3,4\} \) (each of the eight vertices of the 4-crosspolytope is replaced by its vertex figure, an octahedron, and each of the 16 tetrahedral facets is rectified in place, forming an octahedron). With every facet being an octahedron, and three octahedra around every edge, this polytope has Schläfli type \( \{3,4,3\} \), and is actually the regular 24-cell. The vertex figure is a square prism—actually a cube. Although it can be colored as a three-orbit polytope, it cannot be realized with three-orbit symmetry. By edge-transitivity, all the triangles are equilateral, and it follows that the octahedra are regular, whence the whole polytope is regular.

The octahedron + icosahedron combination is the rectified 600-cell \( t_1\{3,3,5\} \). It has 120 icosahedra, and 600 octahedra. The vertex figure is a pentagonal prism. Since all the facets are regular, it is semiregular; it occurs on Gosset’s list as “octicosahedric”. This has 720 vertices—one per edge of the 600-cell—and so \( 720 \cdot 60 = 43200 \) flags. It retains all 14400 symmetries of the 600-cell, hence is three-orbit.

The cuboctahedron + tetrahedron combination is the rectified 4-cube \( t_1\{4,3,3\} \). The facets of the 4-cube become eight cuboctahedra, and the vertices of the 4-cube yield 16 tetrahedra. The vertex figure is a triangular prism. It has 32 vertices, hence \( 32 \cdot 36 = 1152 \) flags, and all 384 symmetries of the 4-cube, hence 3 flag orbits.

The cuboctahedron + cube combination is the rectified 24-cell \( t_1\{3,4,3\} \). Rectifying the octahedral facets of the 24-cell yields 24 cuboctahedra, and around the vertices we have 24
cubes. The vertex figure is a triangular prism. There are 96 vertices, hence \(96 \cdot 36 = 3456\) flags. It retains the 1152 symmetries of the 24-cell, hence is three-orbit.

The icosidodecahedron + tetrahedron combination is the rectified 120-cell \(t_1\{5, 3, 3\}\). This has 120 icosidodecahedra and 600 tetrahedra. The vertex figure is a triangular prism. It has 1200 vertices, hence \(1200 \cdot 36 = 43200\) flags, and all 14400 symmetries of the 120-cell, hence 3 flag orbits.

4.1.1. Tilings. The cuboctahedron + octahedron combination, which is the only combination with dihedral angle sum exactly 360°, forms a tiling of 3-space. It is the rectification of the cubic tiling \(\{4, 3, 4\}\) in the sense that its vertices are the edge midpoints of \(\{4, 3, 4\}\), the octahedra are the vertex figures around the vertices of \(\{4, 3, 4\}\), and the cuboctahedra are the rectified tiles of \(\{4, 3, 4\}\). This is mentioned as a “semi-regular” tiling by Coxeter [1973] pp. 69–70) (with a sense different from that of Gosset), and assigned a modified Schlāfli symbol \(\{3, 4, 4\}\). There it notes that each vertex is incident to two octahedra and four cuboctahedra, with the vertex figure a “cuboid” or square prism, and that the ratio of the lateral edges to the base edges of this prism must be \(\sqrt{2}\).

Grünaun (1994) enumerated all the uniform tilings of 3-space. Since this tiling is vertex-transitive and has uniform tiles, it appears on this list as \#7, denoted \((3.3.3.3)^2(3.4.3.4)^4\) (a symbol recording the vertex types of the tiles meeting at each vertex).

4.2. Type 3\(^{0,1}\). This type has orbit graph \(\begin{array}{c}
1 & 0 & 3 & 3 & 3
\end{array}\). This is dual to type 3\(^{2,3}\), so there are five examples, all among the duals of the rectified regular 4-polytopes. Since the rectified 4-crosspolytope is the regular 24-cell \(\{3, 4, 3\}\), which is self-dual, the 24-cell once again shows up as a sixth example which can be colored to have this symmetry, but which is geometrically regular.

The orbit graph shows us that any 4-polytope of this symmetry type is facet-transitive, and each facet is a polyhedron of the type discussed in §3.2, hence a bipyramid or a Kleetope. The vertices come in two orbits. Vertices in one orbit are apices of all their incident facets, and have a regular vertex figure. Vertices in the other orbit are not apices of any facet, and
have a vertex-intransitive vertex figure, which must be the cube, rhombic dodecahedron, or rhombic triacontahedron. The edges also come in two orbits: the edges which are incident to apices, and the others. Edges in one of these orbits are found in three facets, and edges in the other orbit are in either three, four, or five facets.

It is possible to enumerate the possible combinations of vertex figures and find which facets can fit by an argument based on dihedral angles, as we did in the previous section. However, since the bipyramids and Kleetopes each have two dihedral angles, which are variable (one angle increases as the other decreases), this is needlessly messy. We shall content ourselves with duality to determine that only the following examples exist.

The author does not know any generally accepted nomenclature for these polytopes. They were called “extended Kleetopes” in Chapter III and are also called the \textit{join} of the regular polytopes, in Conway’s notation.

The dual to \(t_1\{3,3,3\} = 0_{21}\) consists of ten triangular bipyramids. It has ten vertices, five of which have tetrahedral vertex figures, and five of which have cubic vertex figures.

The dual to \(t_1\{3,3,5\}\) has 720 pentagonal bipyramids for facets. It has 720 vertices, 120 of which are apices of the bipyramids and have dodecahedral vertex figures, and the other 600 of which have cubic vertex figures.

The dual to \(t_1\{4,3,3\}\) has 32 triangular bipyramids for facets. It has 24 vertices, 16 of which are apices of the bipyramids and have tetrahedral vertex figures, and the other 8 of which have rhombic dodecahedra for vertex figures.

The dual to \(t_1\{3,4,3\}\) has 96 triangular bipyramids for facets. It has 48 vertices, 24 of which are apices of the bipyramids and have octahedral vertex figures, and the other 24 of which have rhombic dodecahedra for vertex figures.

The dual to \(t_1\{5,3,3\}\) has 1200 triangular bipyramids for facets. It has 720 vertices, of which 600 are apices of the bipyramids and have tetrahedra for vertex figures, and the other 120 of which have rhombic triacontahedra for vertex figures.
4.2.1. *Tilings.* The unique tiling with this symmetry type is dual to the tiling \( \{3,4\} \) of \( \mathbb{E}^3 \) seen in §4.1.1. We cannot rely on duality to determine this, however, since there is no duality theorem for tilings. Consider any tiling with this symmetry type; as before, the tiles are bipyramids. Consider a vertex \( v \) which is at the apex of all the incident bipyramids; it has a regular vertex figure \( P \). The tiling is 2-transitive, and all the 2-faces at \( v \) are congruent isosceles triangles with their apex at \( v \). If we consider the vertices of \( P \) to be the actual adjacent vertices in \( T \), then the edges in \( P \) appear as edges in \( T \), and the 2-faces of \( P \) are the bases of the incident bipyramids. Hence, we can redivide the tiling into a tiling by copies of \( P \), removing all the vertices in one orbit and retaining the vertices in the other. But the only regular polyhedron which fills space is the cube. Hence the original tiles were square bipyramids. At the remaining vertices, the original set of incident edges fell into two orbits (those going to apices, and the others). The removal of the edges in one orbit corresponds to modifying the vertex figure by removing one partite set of a bipartition of its vertices, i.e. an alternation. Hence the original vertex figures at these vertices can be alternated to give an octahedron: they must be rhombic dodecahedra.

The preceding analysis can be summarized by saying that this symmetry type describes two orbits of vertices, with one regular vertex figure \( \{p,q\} \) and one two-orbit vertex figure \( \mathbb{K}_1\{r,s\} \). (The cube is \( \mathbb{K}_1\{3,3\} \), the rhombic dodecahedron is \( \mathbb{K}_1\{4,3\} \), and the rhombic triacontahedron is \( \mathbb{K}_1\{5,3\} \)). Removing vertices in the former vertex orbit of the tiling or 4-polytope results in tiles (or facets) \( \{p,q\} \), and the vertex figure at the remaining vertices is an alternation of \( \mathbb{K}_1\{r,s\} \), which is in fact \( \{r,s\} \) (or \( \{s,r\} \)). The result has regular facets and vertex figures, hence is regular. This situation, with the result of the vertex removal being a regular polytope or tiling, characterizes the extended Kleetopes. In the rank 3 case, the vertex removal is an alternation operation.

The facets of this tiling are square bipyramids (non-regular octahedra). The vertex figure at each apex is a cube, and the vertex figure at the other vertices is a rhombic dodecahedron. Hence, each edge incident to an apex is in three tiles, and each edge along the base of the bipyramid is in four tiles.
This tiling can be formed from the cubic tiling \(\{4, 3, 4\}\) as follows. Add a new vertex in the center of each cube. Over each 2-face, form a bipyramid whose apices are the new vertices in the two incident cubes. These squat octahedra fill space and give the desired tiling.

### 4.3. Type \(3^{1,2}\)

This type has orbit graph

\[
\begin{array}{ccc}
3 & 1 & 3 \\
0 & 0 & 0
\end{array}
\]

There are no 4-polytopes with this symmetry type!

We see that such a polytope is facet-transitive and the facet is a polyhedron of type \(3^{1,2}\), hence a \(q\)-gonal prism or a truncated Platonic solid. There are two edge orbits, and two 2-face orbits. The edges in one of the edge orbits lie, in each facet, between 2-faces of different orbits; these edges are in evenly many facets.

If the facet is a prism, then the dihedral angle of the edge between its \(q\)-gonal base face and the rectangular side face is \(90^\circ\). If there are evenly many prisms around such an edge, there must be at least four; but then the total dihedral angle is \(360^\circ\). So these cannot form a 4-polytope (but potentially form a tiling of \(\mathbb{E}^3\)).

If the facet is a truncated Platonic solid \(t\{p, q\}\), the dihedral angles between a \(2p\)-gonal truncated 2-face and an adjacent \(q\)-gonal 2-face is always more than \(90^\circ\), so these cannot even potentially fit in a tiling of \(\mathbb{E}^3\), let alone a 4-polytope. These dihedral angles are the same as the dihedral angle of the quasiregular polyhedron \(\{p\}_{q}\), which can be found in \([4.1]\), they are approximately \(109.5^\circ\) for the truncated tetrahedron, \(125.3^\circ\) for the truncated cube and truncated octahedron, and \(142.6^\circ\) for the truncated dodecahedron and truncated icosahedron.

The absence of any polytopes with this orbit graph is a stroke of good fortune. It means that, for any dimension \(d \geq 4\), any polytopes of type \(3^{i,i+1}\) must have \(i = 0\) or \(i = d - 2\). For, suppose \(P\) is a \(d\)-polytope with orbit graph \(\bullet i \xrightarrow{i+1} \bullet\) where \(1 \leq i \leq d - 3\). Consider a section \(F_{i+3}/F_{i-2}\) between incident faces of the indicated ranks. This section is a 4-polytope whose orbit graph is \(\bullet 1 \xrightarrow{2} \bullet\). But these do not exist. Hence we have

**Lemma 1.** If a convex \(d\)-polytope has orbit graph \(\bullet i \xrightarrow{i+1} \bullet\), then \(i \in \{0, d - 2\}\).
4.3.1. **Tilings.** Suppose $\mathcal{T}$ is a tiling of this symmetry type. As we saw above, the facets must be $q$-gonal prisms, with four attached around each edge between the base face and a side. This implies that the prisms are just stacked.

The tiling is also vertex-transitive, and the vertex figure is a polyhedron of type $3^{0,1}$, hence a $r$-gonal bipyramid or a Kleetope of a Platonic solid. The apices of the vertex figure correspond to the edges of $\mathcal{T}$ which are only in the rectangular side faces of the prisms. The other vertices of the vertex figure correspond to the base/side edges. In a Kleetope these vertices always have valence at least 6, whereas we can only fit exactly four prisms around a base/side edge, so we can dismiss the Kleetopes as potential vertex figures.

Let us assume the prisms are oriented so that the $q$-gonal base faces are horizontal and the rectangular sides are vertical. We need to fit an integral number of prisms around each vertical edge. The dihedral angle of these edges in each prism is the same as the internal angle of the regular $q$-gon; the possibilities are exactly the same as for fitting a whole number of regular $q$-gons together in the plane.

So, there are three tilings of this symmetry type:

1. Stacked layers of prisms over the regular triangular plane tiling $\{3,6\}$.
2. Stacked layers of prisms over the regular plane tiling by squares, $\{4,4\}$.
3. Stacked layers of prisms over the regular hexagonal plane tiling $\{6,3\}$.

In each case the height of the prisms can be scaled arbitrarily, giving a family of non-similar tilings varying by a single real parameter. The height has no effect on the symmetry equivalence class of the tiling, except in the case of the tiling by square prisms, which becomes the regular cubic tiling when the height is equal to the side-length of the base squares.

Since these tilings are vertex-transitive and each tile is vertex-transitive, these tilings are uniform when the height of the prisms is chosen such that the sides are squares. Hence they appear in Grünbaum’s list (1994). The tiling by triangular prisms is #11, the tiling by hexagonal prisms is #26, and the tiling by square prisms is #22, but this is the regular tiling.

The tilings by square prisms is combinatorially regular; the other two tilings are combinatorially three-orbit.
4.4. **Type $3^2$.** This type has orbit graph $\begin{tikzpicture}[scale=0.5]
\node (a) at (0,0) {$3$};
\node (b) at (1,0) {$2$};
\node (c) at (2,0) {$3$};
\node (d) at (3,0) {$2$};
\node (e) at (0,1) {$0$};
\node (f) at (1,1) {$0$};
\node (g) at (2,1) {$0$};
\node (h) at (3,1) {$1$};
\draw (a) -- (b);
\draw (b) -- (c);
\draw (c) -- (d);
\draw (d) -- (a);
\draw (e) -- (f);
\draw (f) -- (g);
\draw (g) -- (h);
\draw (h) -- (e);
\end{tikzpicture}$. This is facet-transitive. The facets are of type $\begin{tikzpicture}[scale=0.5]
\node (a) at (0,0) {$1$};
\node (b) at (1,0) {$2$};
\node (c) at (1.5,0) {$0$};
\end{tikzpicture}$ or $3^{1,2}$, hence are $q$-gonal prisms or truncated Platonic solids $t\{p,q\}$. It is also vertex-transitive. The vertex figure is of type $\begin{tikzpicture}[scale=0.5]
\node (a) at (0,0) {$1$};
\node (b) at (1,0) {$2$};
\node (c) at (1.5,0) {$0$};
\end{tikzpicture}$ or $3^1$, hence is a tetragonal disphenoid. It is also edge-transitive, with every edge in 3 facets. The edge-transitivity implies that the facets must be in their uniform versions, with all faces regular.

One way to construct a tetragonal disphenoid is as follows. Choose an arbitrary line segment. Take a second line segment of the same length of the first, such that each is a perpendicular bisector of the other. Then translate the second line segment an arbitrary distance along the normal to the plane in which the segments lie. The convex hull of these two segments is the disphenoid; the other four edges join each pair of endpoints of the two segments. This construction explains the alternative name *digonal antiprism* for a tetragonal disphenoid: it is formed as an antiprism for a line segment.

In the vertex figure, the two special edges correspond to the $q$-gon 2-faces of our 4-polytope. The other four edges correspond to the $2p$-gon 2-faces (where we take $p = 2$ in the case of prism facets). Hence we see that each edge of the 4-polytope is in two facets between a $q$-gon and a $2p$-gon, and in one facet between two $2p$-gons. Two facets are attached across a $2p$-gon in a rotated manner, so that an edge which is between the $2p$-gon and a $q$-gon in one facet is between two $2p$-gons in the other facet, and vice versa.

Let us consider the possible dihedral angles and the total sum around an edge.

<table>
<thead>
<tr>
<th>Facet</th>
<th>$q$-gon/$2p$-gon</th>
<th>$2p$-gon/$2p$-gon</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t{3,3}$</td>
<td>109.5°</td>
<td>70.5°</td>
<td>289.5°</td>
</tr>
<tr>
<td>$t{3,4}$</td>
<td>125.3°</td>
<td>109.5°</td>
<td>360°</td>
</tr>
<tr>
<td>$t{3,5}$</td>
<td>142.6°</td>
<td>138.2°</td>
<td>423.4°</td>
</tr>
<tr>
<td>$t{4,3}$</td>
<td>125.3°</td>
<td>90°</td>
<td>340.5°</td>
</tr>
<tr>
<td>$t{5,3}$</td>
<td>142.6°</td>
<td>116.6°</td>
<td>401.8°</td>
</tr>
<tr>
<td>$t{2,q}$</td>
<td>90°</td>
<td>$180° - \frac{360°}{q}$</td>
<td>$360° - \frac{360°}{q}$</td>
</tr>
</tbody>
</table>
The only possibilities with truncated Platonic solids are built from the truncated tetrahedron or the truncated cube. A priori there are no angle restrictions for $t\{2,q\}$, the $q$-gonal prisms.

Gluing together uniform truncated tetrahedra in the prescribed way yields the bitruncated 4-simplex, $t_{1,2}\{3,3,3\}$. This is formed from the 4-simplex by taking three vertices in each triangle, one on each line from the triangle center to the midpoint of each edge. These form a truncated tetrahedron within each original facet, and the new vertices around every edge incident to an original vertex also form a truncated tetrahedron. Choose the new vertices such that all edges are the same length. This 4-polytope has 10 truncated tetrahedra for facets, and 30 vertices.

Gluing together uniform truncated cubes in the prescribed way yields the bitruncated 24-cell, $t_{1,2}\{3,4,3\}$. This is formed from the 24-cell in the same manner as the bitruncated 4-simplex. Within each original facet, the new vertices in each 2-face form a truncated cube (a bitruncated octahedron). The new vertices around every edge incident to an original vertex also form a truncated cube. This 4-polytope has 48 truncated cubes for facets, and 288 vertices.

For regular 4-polytopes, bitruncation is “truncation halfway to the dual”. If one truncates every vertex of a regular 4-polytope by a hyperplane cutting through the polytope normally to the vertex vector, starting with the hyperplane passing through the vertex and gradually moving it inwards, the resulting polytope will pass through the truncation, the rectification, the bitruncation, the rectification of the dual, the truncation of the dual, and finally result in the dual itself. Bifruncation is the midpoint of this process. In other words, the bitruncation of a regular 4-polytope is equal to the bitruncation of its dual (but is not itself self-dual, even if the original 4-polytope is self-dual). In general, the bitruncation of a regular 4-polytope has six flag orbits. In the case of a self-dual polytope, however, the duality gives an extra symmetry, which halves the number of flag orbits. In other words, in the bitruncation of a self-dual polytope, the two kinds of facets, namely the truncation of the vertex figures around each original vertex and the bitruncation of each original facet, are congruent. The symmetry
which carries the facets centered on the original vertices to those within the original facets is not a reflection; it generates a “doubled symmetry group” which cannot be generated by reflections in hyperplanes. The symmetry group of $t_{1,2}\{3,3,3\}$ is denoted $[[3,3,3]]$ (Coxeter 1985, p. 286) and is of order 240, double the size of $[3,3,3]$; the symmetry group of $t_{1,2}\{3,4,3\}$ is denoted $[[3,4,3]]$ and is of order 2304, double the size of $[3,4,3]$.

We also see that the dihedral angles of any $q$-gonal prism will fit together appropriately. These form a 4-polytope known as a (uniform) $q,q$-duoprism, also called a proprism (Conway, Burgiel, and Goodman-Strauss 2008, p. 391) or a $q$-gonal double prism (Johnson n.d.). It is the Cartesian product $\{q\} \times \{q\}$ of two regular $q$-gons. More generally, the $p,q$-duoprism is the product of a $p$-gon and a $q$-gon. Such a product is described by Coxeter (1973, p. 124).

Each $q,q$-duoprism has $2q$ facets, which are $q$-gonal prisms. It has $2q$ $q$-gon 2-faces (the Cartesian products of each $q$-gon with each of the $q$ vertices of the other) and $q^2$ square 2-faces (the Cartesian products of each of the edges of one $q$-gon with each edge of the other). There are $2q^2$ edges (the Cartesian products of each vertex of each $q$-gon with each edge of the other). There are $q^2$ vertices. Since the vertex figure has 24 flags, the $q,q$-duoprism has $24q^2$ flags. The symmetry group has order $8q^2$ (it consists of the direct product of two dihedral groups of order $2q$, along with an extra involutory symmetry which interchanges the two perpendicular subspaces containing each $q$-gon). This symmetry group, which can be denoted $[[q,2,q]]$, cannot be generated by reflections.

These are combinatorially three-orbit, except when $q = 4$. The $4,4$-duoprism is the 4-cube. The 4-cube can be colored to have this three-orbit symmetry type, but any realization having these symmetries is actually regular. By edge-transitivity, the square prisms are actually cubes, whence the whole polytope is regular.

4.4.1. Tilings. By the dihedral angles considered earlier, there is one possible tiling with this symmetry type: the tiles are truncated octahedra. Two of these octahedra are attached on a hexagonal face in a rotated fashion, so that an edge which is between two hexagons on one octahedron is between a hexagon and a square on the other, and vice versa.
This tiling can be formed as a bitruncation of the regular cubic tiling \( \{4,3,4\} \). Within each 2-face of the cubic tiling, choose four new vertices, one on each line from the center of the 2-face to the midpoint of one of its edges. The square faces are formed by the four vertices within each original 2-face and the four vertices around each original edge. The hexagonal faces are formed by the six vertices in the 2-faces incident to an original vertex in a given cube. One truncated octahedron is within each cube, and one is around each original vertex.

Being uniform, this tiling appears in Grünbaum’s list (Grünbaum 1994) as \#28. It has twice the symmetries of the cubic lattice; its symmetry group is denoted \([ [4,3,4]]\) (see Coxeter [1940a p. 404], where this tiling is denoted \(t_{1,2}\delta_4\)).

4.5. **Type 3\(^1\).** This type has orbit graph \[
\begin{array}{ccc}
3 & 1 & 2 \\
0 & 3 & 0 \\
0 & 1 & 3 \\
\end{array}
\] and is dual to type 3\(^2\). It is facet-transitive; the facets are tetragonal disphenoids. It is also vertex transitive; the vertex figures are bipyramids or Kleetopes.

The dual to a \(p,p\)-duoprism is a 4-polytope known as a \(p,p\text{-duopyramid}\). This can be formed by taking two regular \(p\)-gons, both centered at the origin in \(\mathbb{E}^4\) but in totally perpendicular subspaces of dimension 2. Their convex hull is the duopyramid. It can be denoted as \(\{p\} + \{p\}\). Its elements are \(p^2\) disphenoids, \(2p^2\) triangles, \(p^2 + 2p\) edges, and \(2p\) vertices; the vertex figure is a \(p\)-gonal bipyramid. In the case \(p = 4\), the duopyramid is a 4-crosspolytope.

The dual to the bitruncated 4-simplex \(t_{1,2}\{3,3,3\}\) is a 4-polytope with 30 disphenoids, 60 triangles, 40 edges, and 10 vertices. The vertex figure is a triakis tetrahedron, a Kleetope of a tetrahedron. This 4-polytope can be formed from a 4-simplex by adding a new vertex over the centroid of each facet, on the circumsphere (the sphere containing all the vertices of the 4-simplex). This polytope appears in Gévay’s partial classification of the perfect 4-polytopes (Gévay 2002 Table 2) with the symbol \(f_{1,2}\text{A}_4\).

The dual to the bitruncated 24-cell \(t_{1,2}\{3,4,3\}\) is a 4-polytope with 288 disphenoids, 576 triangles, 336 edges, and 48 vertices. The vertex figure is a triakis octahedron. This polytope can be formed from a 24-cell by adding a new vertex over the centroid of each facet (i.e. along
the normal vector to the facet hyperplane) at the correct distance to fall on the circumsphere
of the 24-cell. If this polytope is inscribed in the unit sphere, then the vertices, considered as
quaternions, form a discrete subgroup of the unit quaternions: the binary octahedral group.
(This is a double cover of the rotational symmetry group of the octahedron or the 3-cube;
the doubling comes from the fact that a unit quaternion \( q \) and \( -q \) both perform the same
rotation in 3-space.) This polytope appears in Gévay (2002, Table 2) with the symbol \( f_{1,2}F_4 \).

4.5.1. Tilings. The unique tiling of this symmetry type is dual to the tiling by truncated
octahedra of type \( 3^2 \). It can be formed from the cubic honeycomb by adding new vertices in
the center of each cube. An edge is added from each new vertex to each of the new vertices in
adjacent cubes, passing through the centroid of the intervening 2-face. Consider one of these
edges, along with one of the edges of the 2-face. They are in the attitude described in §4.4
for constructing a tetragonal disphenoid. The tetragonal disphenoids thus formed around
each of the new edges we defined form a tiling of \( \mathbb{E}^3 \). The vertex figure is a tetrakis cube.

5. Rank 5

In light of Lemma 1 there are just five possible orbit graphs for three-orbit 5-polytopes:
\[
\begin{array}{c}
\bullet & 3 & 4 & \bullet, & \bullet & 0 & 1 & \bullet, & \bullet & 3 & 4 & \bullet, & \bullet & 1 & 2 & \bullet, & \bullet & 2 & 3 & \bullet.
\end{array}
\]
We shall see that only the first four of these are realized, and only by a single polytope each, up to similarity.

5.1. Type \( 3^34 \). This type has orbit graph \( \begin{array}{c}
\bullet & 2 & 4 & 3 & \bullet, & \bullet & 1 & 0 & 2 & 3, & \bullet & 0 & 1, & \bullet & 2 & 3 & \bullet, & \bullet & 0 & 1, & \bullet & 2 & 3 & \bullet.
\end{array} \) The facets of any polytope
\( P \) with this orbit graph fall in two orbits. The facets in one orbit have orbit graph \( \bullet & 3 & \bullet. \) There are no 4-polytopes with two flag orbits, so these facets must be a regular 4-polytope
covered by this graph; the graph is \( (2,3) \)-even, so every edge of the 4-polytope is in evenly
many facets. The only possibility is the 4-crosspolytope. The facets in the other orbit
are regular, and must be simplicial, since they are attached to 4-crosspolytopes. So the
possibilities are the 4-simplex \( \{3,3,3\} \), the 4-crosspolytope \( \{3,3,4\} \), or the 600-cell \( \{3,3,5\} \).
The vertices fall in one orbit, and the vertex figure has orbit graph \( \bullet & 2 & 3 & \bullet \) or \( 3^{2,3} \), hence
is a rectified regular 4-polytope. The facets of this vertex figure which correspond to the
4-crosspolytopes of \( P \) must be octahedra. Hence, the possible vertex figures are \( 0_{21} \), composed of octahedra and tetrahedra; the 24-cell, composed of octahedra; or \( t_1\{3,3,5\} \), composed of octahedra and icosahedra. The second kind of facet of the vertex figure must be compatible with the regular facets of \( P \), so we have three possibilities:

<table>
<thead>
<tr>
<th>Facets</th>
<th>Vertex figure</th>
<th>Edge figure</th>
<th>2-face figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {3,3,4} &amp; {3,3,3} )</td>
<td>( 0_{21} )</td>
<td>Triangular prism</td>
<td>Isosceles triangle</td>
</tr>
<tr>
<td>( {3,3,4} )</td>
<td>24-cell</td>
<td>Cube</td>
<td>Equilateral triangle</td>
</tr>
<tr>
<td>( {3,3,4} &amp; {3,3,5} )</td>
<td>( t_1{3,3,5} )</td>
<td>Pentagonal prism</td>
<td>Isosceles triangle</td>
</tr>
</tbody>
</table>

In any case, every 2-face is in three facets: two 4-crosspolytopes, and one of the second kind of facet. Consider the dihedral angles of the potential facets. The dihedral angle of the 4-simplex is \( \arccos\left(\frac{1}{4}\right) \approx 75.5^\circ \); that of the 4-crosspolytope is \( 120^\circ \); and that of the 600-cell is \( 164.5^\circ \) (Coxeter 1973, Table I).

Hence, the first possibility in our table can fit in 5-space; the second possibility can exist only as a tiling of 4-space; and the third is an impossibility.

The 5-polytope built of 4-crosspolytopes and 4-simplices in this way can be formed as an alternation of the 5-cube \( \{4,3,3,3\} \). Around each of the 16 omitted vertices there appears a vertex figure, the 4-simplex. Each of the 10 facets is replaced by an alternated 4-cube, which is a 4-crosspolytope. The vertex figure of an alternated \( d \)-cube is the rectification of the original vertex figure. To see this, consider a vertex \( v \) remaining in the alternated \( d \)-cube. Each vertex of its original vertex figure corresponds to an incident edge in the original cube, all of which have been removed. Every original square incident to \( v \) is reduced to an edge between \( v \) and its diametric opposite in the square; the square corresponds to an edge \( e \) in the original vertex figure, and the new edge across the square corresponds to the midpoint of \( e \). Hence, the vertex figure is \( t_1\{3,3,3\} \) or \( 0_{21} \).

This polytope has the symmetry prescribed by the orbit graph, in that every flag within a given 4-simplex is in the same orbit, and the 3-faces of the 4-crosspolytopes alternate between
being adjacent to 4-simplices and being adjacent to other 4-crosspolytopes. Thus the unique polytope of this symmetry type is the 5-demicube, also known as $1_{21}$. As an alternation, it has only half the symmetries of the 5-cube. The symmetry group of the 5-cube, $[4, 3, 3, 3]$, has order $2^55!$; the order of $G(1_{21})$ is thus 1920. With 16 vertices, and recalling that $0_{21}$ has 360 flags, $1_{21}$ has 5760 flags and is indeed three-orbit.

5.1.1. Tilings. We saw that the only possible tiling of this symmetry type is made of 4-crosspolytopes, with 24-cell vertex figures. This is actually a regular tiling of 4-space, $\{3, 3, 4, 3\}$. Since the symmetry demands that all the 4-crosspolytopes actually be regular, and the 24-cells of the requisite symmetry also must be regular, the tiling can also only be realized in a regular manner (in order to have at least the symmetry prescribed by the orbit graph). It can be colored to yield this three-orbit symmetry, as follows. This tiling can be constructed as an alternation of the cubic tiling $\{4, 3, 3, 4\}$: each alternated 4-cube is a 4-crosspolytope, and the tiles which arise as vertex figures around the omitted vertices are also 4-crosspolytopes. The vertex figure at the remaining vertices is the rectification of the original vertex figure, as explained above, hence is $t_1\{3, 3, 4\} = \{3, 4, 3\}$. Simply color the tiles according to whether they arose as alternations of the original tiles or as vertex figures of the omitted vertices; the alternations represent two flag orbits and the vertex figures just one.

5.2. Type $3^{0,1}$. This type has orbit graph $\begin{array}{c} 3 \\ 2 \end{array} 4 \begin{array}{c} 0 \\ 1 \end{array} 3 \begin{array}{c} 2 \\ 0 \end{array} 1 \begin{array}{c} 4 \\ 2 \end{array} 4$. It is dual to type $3^{3,4}$. Hence the unique example is the dual of the 5-demicube.

From the orbit graph, we see that any polytope $P$ of this symmetry type is facet-transitive and the facets are 4-polytopes with orbit graph $\begin{array}{c} 0 \\ 2 \end{array} 1 \begin{array}{c} 1 \\ 0 \end{array}$ or $3^{0,1}$, hence are duals to a rectified regular 4-polytope. The vertices of $P$ fall in two orbits, one with a regular vertex figure, and the other with the 4-cube as its vertex figure. Consider an edge $e$ incident to one vertex of each orbit ($e$ must exist, by connectedness). The edge $e$ corresponds to a vertex in the 4-cube vertex figure of one of its endpoints, hence the edge figure of $e$ is a tetrahedron; it remains
a tetrahedron in the vertex figure of the other endpoint of \( e \). So, the second kind of vertex figure must be the 4-simplex \( \{3, 3, 3\} \), the 4-cube \( \{4, 3, 3\} \), or the 120-cell \( \{5, 3, 3\} \).

The facets, in order to be compatible with the 4-cube vertex figures, must have some vertex figures which are 3-cubes. The possibilities are the dual to \( 0_{21} \), the dual to \( t_1 \{3, 3, 5\} \), or the 24-cell. Since the 24-cell has a dihedral angle of 120\(^\circ\), it cannot be a facet in 5-space. If we do not wish to rely on duality for our classification, we can calculate the dihedral angle of the dual to \( t_1 \{3, 3, 5\} \) (Gawrilow and Joswig [2000]) and find it to be approximately 161.3\(^\circ\).

Thus the unique polytope of this symmetry type has facets which are dual to \( 0_{21} \) and vertex figures which are 4-simplices and 4-cubes. The author does not know of a standard name for the polytope beyond “the dual to \( 1_{21} \”).

This is a kind of alternately Klee 5-crosspolytope, i.e. a 5-crosspolytope with a pyramid erected over alternate facets, but with the apex sufficiently high so that the pyramid over a ridge is coplanar with the adjacent, unmodified facet. The facets of the end result are the “extended Kleetopes” of the 16 unmodified 4-simplex facets.

5.2.1. Tilings. By the above considerations of dihedral angles, the only potential tile is the 24-cell. Then both kinds of vertex figure are 4-cubes. So, the unique tiling of this symmetry type is the 24-cell honeycomb \( \{3, 4, 3, 3, 3\} \), the dual of the alternated cubic honeycomb \( \{3, 3, 4, 3\} \) discussed in \S5.1.1. As in that case, any realization with the prescribed symmetries is actually regular. The honeycomb may be colored to be three-orbit, as follows.

This honeycomb can be formed from the cubic honeycomb \( \{4, 3, 3, 4\} \) by adding a new vertex in the center of alternate 4-cube tiles. For each 4-cube \( C \) without a new internal vertex, we form the “extended Kleetope” as a tile, with the new vertices in each adjacent cube as the apices over the facets of \( C \). The facets of the new tile are bipyramids over each ridge of \( C \), which are squares, so we get square bipyramids: in fact, these are regular octahedra, and the 4-polytope derived from \( C \) is a 24-cell. (This is Gosset’s construction for \( \{3, 4, 3\} \), according to Coxeter [1973, p. 150].)
Having thus formed the 24-cell honeycomb, we color the vertices according to whether they are new “apex” vertices, or original vertices of the cubic honeycomb.

5.3. Type $3^3$. This type has orbit graph \[
\begin{align*}
1 &\rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow 0 \\
0 &\rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow 1
\end{align*}
\] This type is analogous to the type $3^2$ for 4-polytopes, discussed in §4.4. In fact, we shall find that just as in that case, this symmetry type consists of the polytopes which are constructed from self-dual regular polytopes by “truncation halfway to the dual”. In five dimensions, the only self-dual regular polytope is the 5-simplex, and truncation halfway to the dual means truncating to 2-face centroids, the operation $t_2$, also known as \textit{birectification}.

Suppose $P$ is a polytope with this orbit graph. We see that $P$ is 3-intransitive, that is, $G(P)$ acts transitively on faces of every rank save 3. The facets have orbit graph \[
\bullet \rightarrow 2 \rightarrow 3 \rightarrow \bullet
\] or $3^23$, hence are rectified regular 4-polytopes. By 2-face-transitivity, we can reject $t_1\{4, 3, 3\}$, $t_1\{3, 4, 3\}$, and $t_1\{5, 3, 3\}$ as potential facets, since each contains two different kinds of 2-face. The remaining possibilities are $0_{21}$, the 24-cell, or $t_1\{3, 3, 5\}$.

The vertex figures have orbit graph \[
\bullet \rightarrow 2 \rightarrow 3 \rightarrow \bullet
\] or $3^2$, hence are $p, p$-duoprisms for some $p$. We can reject the bitruncated 4-simplex $t_{1,2}\{3, 3, 3\}$ and the bitruncated 24-cell $t_{1,2}\{3, 4, 3\}$ as potential vertex figures since their facets are not prisms, but the vertex figure of each rectified regular 4-polytope is a prism.

The ridges (3-faces) of $P$ fall into two orbits. Let us call a ridge a V-ridge if it arose as a vertex figure in the rectification of a regular 4-polytope, i.e. if it is a tetrahedron in $0_{21}$, an icosahedron in $t_1\{3, 3, 5\}$, or an appropriate octahedron in the 24-cell. The ridges which arose as rectified facets of a regular 4-polytope will be called F-ridges; these are octahedra in every case.

The edge figure of $P$ is a tetragonal disphenoid, and the 2-face figure is an isosceles triangle. So every 2-face is in three ridges: in fact, it is in one V-ridge and two F-ridges. (This is encoded in the orbit graph, where the flags containing a V-ridge correspond to the leftmost node, and the other two nodes correspond to flags containing an F-ridge.)
The dihedral angle of a rectified regular 4-polytope, at a triangle between two F-ridges, is the same as the dihedral angle of the original regular 4-polytope: hence 75.5° for 0₂₁, 120° for the 24-cell, and 164.5° for t₁{3, 3, 5}. The dihedral angle at a triangle T between a V-ridge and an F-ridge can be determined as follows. The (2-dimensional) plane which is totally perpendicular to T (in the affine hull of the 4-polytope) passes through the center C₄ of the 4-polytope and the centers of the two 3-faces incident to T (this occurs because all the 3-faces are regular, and the 4-faces are uniform). Say the centroid of the F-ridge (an octahedron) is F₃, the centroid of the other 3-face is V₃, and the centroid of the triangle T is C₂. Consider the quadrilateral formed by C₄, F₃, V₃, and C₂. The angle at C₂ is the desired dihedral angle. The angles at F₃ and V₃ are right angles (again this follows from the 4-polytope being uniform). The angle at C₄ is the angle given by Coxeter (1973, Table I) as χ. It is \(\arccos \frac{1}{4} \approx 75.5°\) for the 4-simplex, 60° for the 4-crosspolytope, and \(\frac{\pi}{3} - \frac{1}{2} \arccos \frac{1}{4} \approx 22.24°\) for the 600-cell. This gives us the second dihedral angles of each potential facet of P to be 104.5° for 0₂₁, 120° for the 24-cell, and 157.8° for t₁{3, 3, 5}.

Hence the only possible 5-polytope is composed of 0₂₁ facets; the sum of dihedral angles is too large in the other cases.

There is such a 5-polytope, the birectified 5-simplex t₂{3, 3, 3, 3}, also known as 0₂₂, since its Coxeter diagram is

```
0-----1-----2
   |
   |
```

The vertex figure is a 3,3-duoprism, which can also be called \(-1₂₂\), since its Coxeter diagram is

```
0---1
  |
  |
```

The 5-polytope 0₂₂ has 12 facets; 30 tetrahedral ridges and 30 octahedral ridges; 120 triangles; 90 edges; and 20 vertices.

5.3.1. Tilings. There is one tiling of \(\mathbb{E}^4\) with this symmetry type, composed of 24-cell tiles. The vertex figure is a 4,4-duoprism, i.e. a 4-cube. This tiling is also the regular tiling
The same tiling we saw in §5.2.1! This tiling cannot be realized as a three-orbit tiling by its full symmetry group, but it can be colored in two different ways to have two different types of three-orbit symmetry. One coloring was described in §5.2.1.

The second, different coloring we use in this case is based on a different construction of \(\{3,4,3,3\}\) from the cubic honeycomb \(\{4,3,3,4\}\): place a vertex at the centroid of every 2-face of the cubic honeycomb. One set of tiles are formed by all the new vertices within a given original tile, giving the birectification \(t_2\{4,3,3\}\), which is a 24-cell. (Truncating to the 2-face centroids of the 4-cube is equivalent to truncating to the edge midpoints of its dual, the 4-crosspolytope.) The other set of tiles are formed by all the new vertices in 2-faces containing a given original vertex, yielding the rectification of the original vertex figure, again a 24-cell.

We color the octahedra one color (say red) if they arose as a rectified vertex figure of an original vertex within a given 4-cube of the original tiling, and another color (say blue) if they were formed either within a given 3-face of the original tiling, or by the new vertices in all the 2-faces incident to a given original edge (the edge figure, \(\{3,4\}\), is an octahedron). This gives the honeycomb the desired three-orbit symmetry: the flag orbits are

- Flags including a red octahedron,
- Flag with a blue octahedron and a triangle bordering a red octahedron within the flag’s 4-face (i.e. so that the 3-adjacent flag has a red octahedron), and
- Flags with a blue octahedron and a triangle bordering another blue octahedron within the flag’s 4-face (i.e. so that the 3-adjacent flag has a blue octahedron).

5.4. Type \(3^1\). The orbit graph of this type is \(\begin{array}{ccc} 3 & 4 & 1 \\ 2 & 0 & 3 \\ 0 & 3 & 4 \\ 1 & 4 & 2 \end{array}\). It is dual to the type \(3^3\) in the previous section. Hence, there is a unique 5-polytope of this symmetry type, the dual to \(0_{22}\).

Proceeding without appeal to duality, suppose that \(P\) is a 5-polytope with this orbit graph. \(P\) is facet-transitive, and the facets are either \(p,p\)-duopyramids for some \(p\), the dual to the bitruncated 4-simplex, or the dual to the bitruncated 24-cell. \(P\) is vertex-transitive,
and the vertex figure has orbit graph $\bullet^0 \overset{1}{\longrightarrow} \bullet$ or $3^{0.1}$, hence is an “extended Kleetope”, i.e. the dual of a rectified regular 4-polytope. The facets of this vertex figure are either triangular, square, or pentagonal bipyramids.

Therefore, the vertex figure of the facet must be a bipyramid: this rules out the duals to the bitruncated 4-simplex or 24-cell, whose vertex figures are a triakis tetrahedron or a triakis octahedron, respectively. The potential facets are therefore a 3,3-duopyramid; a 4,4-duopyramid, which is a 4-crosspolytope; or a 5,5-duopyramid. Recall that the dihedral angle of a 4-crosspolytope is $120^\circ$, so it cannot be the facet. We can compute the dihedral angle of a 5,5-duopyramid to be $130.9^\circ$. On the other hand, the dihedral angles of the 3,3-duopyramid are $104.5^\circ$. (To determine these dihedral angles, one may use coordinates for the vertices of a triangle in a $p, p$-duopyramid, given in $\mathbb{C}^2$: $(e^{2ki\pi/p}, 0)$, $(e^{2(k+1)i\pi/p}, 0)$, and $(0, e^{2il\pi/p})$ for some $k$ and $l$. The two disphenoids incident to this triangle have the additional vertices $(0, e^{2(i-1)\pi/p})$ or $(0, e^{2(i+1)\pi/p})$, respectively. For simplicity, choose $k = l = 0$, then determine the normal vectors to the affine hyperplanes spanned by the four points in each disphenoid.)

Thus, the only possibility for a 5-polytope of this type must have 3,3-duopyramids for facets, and the vertex figure is one of the dual to $0_{21}$, the dual to $t_1\{4, 3, 3\}$, the dual to $t_1\{3, 4, 3\}$, or the dual to $t_1\{5, 3, 3\}$. Consider the 2-face figures of $P$ in each case, i.e., the edge figures of the vertex figures.

- The vertex figures of the dual to $0_{21}$ are cubes and tetrahedra, and so the 2-face figures of $P$ in this case are triangles.
- The vertex figures of the dual to $t_1\{4, 3, 3\}$ are tetrahedra and rhombic dodecahedra, and so some 2-face figures of $P$ are triangles and some are quadrilaterals.
- The vertex figures of the dual to $t_1\{3, 4, 3\}$ are octahedra and rhombic dodecahedra, and so some 2-face figures of $P$ are triangles and some are quadrilaterals.
- The vertex figures of the dual to $t_1\{5, 3, 3\}$ are tetrahedra and rhombic triacontahedra, and so some 2-face figures of $P$ are triangles and some are pentagons.
But $P$ is 2-face-transitive. Therefore, only $0_{21}$ can be the vertex figure, and so $P$ must be the dual to $0_{22}$, as expected. This is composed of 20 3,3-duopyramids; 90 disphenoids; 120 triangles; 60 edges; and 12 vertices. This polytope can be formed as the convex hull of the vertices of a 5-simplex and the vertices of its dual, ensuring that all the vertices lie on the same circumsphere.

5.4.1. Tilings. The considerations above establish that a tiling of this type must have facets which are 4-crosspolytopes, and the vertex figure must be a dual of a rectified regular 4-polytope with square bipyramid facets: the only possibility is the 24-cell. Thus we once again meet the regular tiling $\{3,3,4,3\}$. It can be given a coloring to achieve this type of three-orbit symmetry based on the following construction.

Start with the regular cubic honeycomb of $E^4$, $\{4,3,3,4\}$. Consider it to be overlaid with the dual honeycomb whose vertices are the centers of the tiles of the first. The dual face to each 4-cube is a vertex, the dual face to each 3-cube is an edge, and the dual face to each square is another square totally orthogonal to the first. The free sum of these two squares is a 4,4-duopyramid, i.e. a 4-crosspolytope. The vertices in the centroids of the original tiles are incident to 24 of these 4-crosspolytopes, one for each 2-face of the tile. The original vertices of the cubic tiling are also incident to 24 of these 4-crosspolytopes, one for each incident 2-face of the tiling (corresponding to the 24 edges in the original tiling’s vertex figure $\{3,3,4\}$).

Color the edges of the edge-intransitive tiling according as they were original edges of one of the two dual cubic honeycombs, or new edges (which join a vertex in one honeycomb to a vertex in the dual honeycomb.) This coloring gives the desired three-orbit symmetry.

5.5. Type $3^2$. This type has orbit graph $\begin{array}{c|c|c|c|c|c|c} 0 & 1 & 2 & 3 & 4 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$. There are no 5-polytopes with this symmetry type!

Suppose $P$ is a polytope with this orbit graph. Then $P$ is facet-transitive, and the facets (of type $3^2$) are either $p,p$-duoprism for some $p$, bitruncated 4-simplices, or bitruncated 24-cells. $P$ is vertex-transitive, and the vertex figures (of type $3^1$) are either $q,q$-duopyramids for some $q$, the dual to the bitruncated 4-simplex, or the dual to the bitruncated 24-cell.
The vertex figures of $P$ are particularly well-behaved. Since $P$ is edge-transitive, all edges of $P$ are the same length, say $2l$. The midpoint of every edge of $P$ lies on a sphere $S_e$ centered at the centroid of $P$. The midpoints of every edge incident to a given vertex $v$ lie on a sphere centered at $v$. So, taking these midpoints as the vertices of the vertex figure $P/v$, we see that they lie on the intersection of two 4-spheres, which is a 3-sphere, and the intersection of a hyperplane $H$ with $S_e$. Hence the vertex figure is realized as the intersection $H \cap P$.

The potential facets of $P$ each have two kinds of 2-faces, let us say $p$-gons and $2m$-gons. (We have $m = 2$ for the $p,p$-duoprisms; $p = 3$ and $m = 3$ for the bitruncated 4-simplex; and $p = 3, m = 4$ for the bitruncated 24-cell.) An edge of the vertex figure within a regular $p$-gon has length $2l \cos \frac{\pi}{p}$.

Suppose the vertex figure of $P$ is a $q,q$-duopyramid. The edges of this vertex figure fall in two orbits: the $2q$ edges of the totally perpendicular $q$-gons, which we will call base edges, and $q^2$ edges joining each pair of vertices of the $q$-gons. The base edges correspond to the $p$-gon 2-faces of $P$. Say the circumradius of the $q$-gons is $\lambda$, so the length of these edges is $2\lambda \sin \frac{\pi}{q}$. The length of the non-base edges of a duopyramid is always $\sqrt{2}\lambda$ (one such edge joins $(\lambda, 0, 0, 0)$ to $(0, 0, \lambda, 0)$). These edges are the vertex figure of the $2m$-gons of $P$: hence $\sqrt{2} \lambda = 2l \cos \frac{\pi}{2m}$, giving us the requirement

$$\lambda = \sqrt{2} l \cos \frac{\pi}{2m} = \begin{cases} l & \text{if } m = 2 \\ \sqrt{\frac{3}{2}} l & \text{if } m = 3 \\ \sqrt{1 + \frac{1}{2} l} & \text{if } m = 4 \end{cases}$$

Here we have used $\cos \frac{\pi}{8} = \frac{\sqrt{2 + \sqrt{2}}}{2}$. We can have $m = 2$, that is $\lambda = l$, only if the vertex $v$ is the centroid of its own vertex figure, which can happen only for a tiling. Let us consider the other two possibilities for $m$. 
Suppose \( m = 3 \); then \( p = 3 \) and \( 2\lambda = \sqrt{6}l \), so we have

\[
\sqrt{6}l \sin \frac{\pi}{q} = 2l \cos \frac{\pi}{3}, \quad \text{or} \quad \\
\sin \frac{\pi}{q} = \frac{1}{\sqrt{6}}.
\]

This has no solution for integer \( q \).

Suppose \( m = 4 \); then \( p = 3 \), and we have

\[
2l \sqrt{1 + \sqrt\frac{1}{2} \sin \frac{\pi}{q}} = 2l \cos \frac{\pi}{3}, \quad \text{or} \quad \\
\sin \frac{\pi}{q} = \frac{1}{\sqrt{4 + 2\sqrt{2}}}.
\]

This has a solution with \( q = 8 \).

Suppose \( P \) has bitruncated 24-cell facets, 8,8-duopyramid vertex figures, and has \( f_0 \) vertices. Each vertex is in 64 facets, each of which has 288 vertices; each vertex is in 128 3-faces, each of which has 24 vertices; each vertex is in 16 triangles and 64 octagons; and each vertex is in 16 edges. Thus Euler’s formula is

\[
1 - \frac{64}{288} f_0 + \frac{128}{24} f_0 - \frac{16}{3} f_0 - \frac{64}{8} f_0 - \frac{16}{2} f_0 - f_0 + 1 = 0
\]

which simplifies to

\[
2 = \frac{11}{9} f_0
\]

or \( f_0 = \frac{18}{11} \), which is not possible. Therefore, the vertex figure of \( P \) cannot be a \( q, q \)-duopyramid.

There remains to consider the case when the vertex figure of \( P \) is dual to the bitruncated 4-simplex, or dual to the bitruncated 24-cell. To perform a similar analysis, we need to determine the edge lengths of these polytopes when the circumradius is \( \lambda \).

One construction for the dual to the bitruncated 4-simplex is the convex hull of the vertices of a 4-simplex and the vertices of its dual, ensuring that all the vertices lie on the same circumsphere. The base edges of the disphenoid facets are the edges of the original 4-simplex, or of its dual, and the other edges are between one vertex of each type. An easy set of coordinates for a 4-simplex of circumradius \( \lambda \) lie in a hyperplane of \( \mathbb{E}^5 \): the five vertices
are the permutations of \((\frac{2\lambda}{\sqrt{5}}, \frac{-\lambda}{2\sqrt{5}}, \frac{-\lambda}{2\sqrt{5}}, \frac{-\lambda}{2\sqrt{5}})\), and the dual vertices are the negations of these. In the convex hull, each vertex is adjacent to every other except for its antipode; the vertex figure is a Kleetope over a tetrahedron. The length of the base edges is 
\[
\sqrt{\left(\frac{2\lambda}{\sqrt{5}} + \frac{\lambda}{2\sqrt{5}}\right)^2 + \left(-\frac{\lambda}{2\sqrt{5}} - \frac{2\lambda}{\sqrt{5}}\right)^2} = \frac{\sqrt{5}}{2} \lambda
\]
and the length of the other edges is 
\[
\sqrt{\left(\frac{2\lambda}{\sqrt{5}} - \frac{\lambda}{2\sqrt{5}}\right)^2 + \left(-\frac{\lambda}{2\sqrt{5}} + \frac{2\lambda}{\sqrt{5}}\right)^2 + 3\left(-\frac{\lambda}{2\sqrt{5}} - \frac{\lambda}{2\sqrt{5}}\right)^2} = \frac{\sqrt{3}}{2} \lambda.
\]

Thus, we must have \(2l \cos \frac{\pi}{2m} = \frac{\sqrt{3}}{2} \lambda\), hence
\[
\lambda = \sqrt{\frac{8}{3}} l \cos \frac{\pi}{2m} = \begin{cases} 
\sqrt{\frac{4}{3}} l & \text{if } m = 2 \\
\sqrt{2} l & \text{if } m = 3 \\
\sqrt{\frac{4 + \sqrt{8}}{3}} l & \text{if } m = 4
\end{cases}
\]

Also, we must have \(2l \cos \frac{\pi}{p} = \sqrt{\frac{5}{2}} \lambda\). If \(m = 2\), this is
\[
2l \cos \frac{\pi}{p} = \sqrt{\frac{5}{2}} \sqrt{\frac{4}{3}} l
\]
\[
\cos \frac{\pi}{p} = \sqrt{\frac{5}{6}}
\]

But no integer value of \(p\) satisfies this equation.

If \(m = 3\), then \(p = 3\), and we have
\[
2l \cos \frac{\pi}{3} = \sqrt{\frac{5}{2}} \sqrt{2} l
\]
\[
1 = \sqrt{5}
\]

which is clearly not a solution.
If \( m = 4 \), again \( p = 3 \) and we have

\[
2l \cos \frac{\pi}{3} = \sqrt{\frac{5}{2}} \sqrt{\frac{4 + \sqrt{8}}{3}} \]

\[
1 = \sqrt{\frac{10 + 5\sqrt{2}}{3}}
\]

which is also not a solution. Therefore, the vertex figure of \( P \) cannot be the dual to the bitruncated 4-simplex.

Similarly, we can construct the dual to the bitruncated 24-cell as the convex hull of the vertices of the 24-cell and the vertices of its dual, ensuring that all vertices lie on the same circumsphere. Coordinates for a 24-cell of circumradius \( \lambda \) are the permutations of \((\pm \frac{\lambda}{\sqrt{2}}, \pm \frac{\lambda}{\sqrt{2}}, 0, 0)\), and coordinates for the reciprocal 24-cell are the permutations of \((\pm \lambda, 0, 0, 0)\) and of \((\pm \frac{\lambda}{2}, \pm \frac{\lambda}{2}, \pm \frac{\lambda}{2}, \pm \frac{\lambda}{2})\) (Coxeter [1973], p. 156). The base edges of the disphenoids are the edges of the 24-cell or its dual. One such edge is from \((\frac{\lambda}{\sqrt{2}}, \frac{\lambda}{\sqrt{2}}, 0, 0)\) to \((\frac{\lambda}{\sqrt{2}}, 0, \frac{\lambda}{\sqrt{2}}, 0)\) and this has length \( \lambda \). The other edges go between the two kinds of vertex; one such edge is from \((\lambda, 0, 0, 0)\) to \((\frac{\lambda}{\sqrt{2}}, \frac{\lambda}{\sqrt{2}}, 0, 0)\) and this has length \( \sqrt{2 - \sqrt{2}} \lambda \).

Hence

\[
\lambda = \frac{2}{\sqrt{2} - \sqrt{2}} l \cos \frac{\pi}{2m} = \begin{cases} 
\sqrt{2 + \sqrt{2}} l & \text{if } m = 2 \\
\sqrt{\frac{3}{2 - \sqrt{2}}} l & \text{if } m = 3 \\
\sqrt{3 + 2\sqrt{2}} l & \text{if } m = 4 
\end{cases}
\]

Also, we must have \( \lambda = 2l \cos \frac{\pi}{p} \). If \( m = 4 \), this is

\[
\sqrt{3 + 2\sqrt{2}} l = 2l \cos \frac{\pi}{3}
\]

\[
\sqrt{3 + 2\sqrt{2}} = 1
\]

which is not a solution.
If \( m = 3 \), this is

\[
\sqrt{\frac{3}{2 - \sqrt{2}}} l = 2l \cos \frac{\pi}{3} \\
\sqrt{\frac{3}{2 - \sqrt{2}}} = 1
\]

which is not a solution. If \( m = 2 \), this is

\[
\sqrt{2 + \sqrt{2}} l = 2l \cos \frac{\pi}{p} \\
\frac{1}{2} \sqrt{2 + \sqrt{2}} = \cos \frac{\pi}{p}
\]

which has the solution \( p = 8 \).

Suppose \( P \) has 8,8-duoprism facets, vertex figures dual to the bitruncated 24-cell, and has \( f_0 \) vertices. Each vertex is in 288 facets, each of which has 64 vertices; each vertex is in 576 3-faces, each of which has 16 vertices; each vertex is in 192 octagons and 144 squares; and each vertex is in 48 edges. Thus Euler’s formula is

\[
1 - \frac{288}{64} f_0 + \frac{576}{16} f_0 - \frac{192}{8} f_0 - \frac{144}{4} f_0 + \frac{48}{2} f_0 - f_0 + 1 = 0
\]

which simplifies to \( f_0 = \frac{4}{11} \), which is impossible. Therefore, the vertex figure of \( P \) cannot be the dual to the bitruncated 24-cell. But we have dismissed all the potential vertex figures; so no such polytope \( P \) can exist.

The absence of any such polytopes is another stroke of good fortune. It allows us to assert

\textbf{Lemma 2.} If a convex \( d \)-polytope has orbit graph \( \bullet \overset{i}{\longrightarrow} \overset{i+1}{\bullet} \overset{i-1}{\bullet} \), then \( i \in \{1, d-2\} \).

The proof is entirely similar to that of Lemma 1.

5.5.1. Tilings. The above analysis showed that a tiling of \( E^4 \) of type \( 3^2 \) with a \( q,q \)-duopyramid vertex figure must have \( m = 2 \), i.e. must have \( p,p \)-duoprisms for facets. The other potential vertex figures did not allow for \( \lambda = l \) at all, so they cannot be the vertex
figure of a tiling. In the former case, we must have

\[ 2l \sin \frac{\pi}{q} = 2l \cos \frac{\pi}{p}. \]

With integers \( p, q \geq 3 \), there are only three solutions: \( \sin \frac{\pi}{3} = \cos \frac{\pi}{6} \), \( \sin \frac{\pi}{4} = \cos \frac{\pi}{4} \), and \( \sin \frac{\pi}{6} = \cos \frac{\pi}{3} \).

The resulting possibilities are:

- A 3,3-duopyramid vertex figure with 6,6-duoprism facets.
- A 4,4-duopyramid vertex figure with 4,4-duoprism facets.
- A 6,6-duopyramid vertex figure with 3,3-duoprism facets.

The dihedral angles of the \( p, p \)-duoprisms are correct in each case. The dihedral angle at each square face of a duoprism is always 90°, and the dihedral angle at each \( p \)-gonal face is the same as the internal angle of the \( p \)-gon. The edge figure of the tiling (the vertex figure of the \( q, q \)-duopyramid) is a \( q \)-gonal bipyramid, so the 2-face figure is either a 4-gon or a \( q \)-gon. Thus the total dihedral angles are always 360° around each 2-face.

Since 4,4-duoprisms are 4-cubes and 4,4-duopyramids are 4-crosspolytopes, the tiling for the case \( p = q = 4 \) is actually the regular tiling \( \{4, 3, 3, 4\} \). The facets and vertex figures must be actually regular to have the desired symmetries, hence the tiling itself is regular. However, if we color each 4-cube as a 4,4-duoprism—that is, in the product \( \{4\} \times \{4\} \), color each of the eight squares arising as the product of a square with a vertex with one color, say red, and each of the 16 squares arising as the product of two edges with a second color, say blue—then the tiling has the desired three-orbit symmetry. The flags containing a red square correspond to the leftmost node of the orbit graph. The flags containing a blue square and an edge between blue and red squares of the flag’s 3-face are in the middle node, and the flags containing a blue square and an edge between blue squares of the flag’s 3-face are in the right node.

The other two cases are combinatorially three-orbit. Each is dual to the other.
These three tilings are the Cartesian products of each of the three regular plane tilings with themselves: \(\{3, 6\} \times \{3, 6\}\) is the tiling by 3,3-duoprisms; \(\{4, 4\} \times \{4, 4\}\) is the tiling by 4,4-duoprisms; and \(\{6, 3\} \times \{6, 3\}\) is the tiling by 6,6-duoprisms. The Cartesian product of tilings is subtly different from that of polytopes. The product \(M \times P\) of an \(m\)-polytope \(M\) and an \(n\)-polytope \(P\) has \(l\)-faces, for every \(l \leq m + n\), consisting of the products of every \(i\)-face of \(M\) and \(j\)-face of \(P\) with \(i + j = l\), including the \(m\)-face \(M\) itself, and the \(n\)-face \(P\) itself. In the product of tilings, we only allow proper faces as factors. This results in the product of two rank-3 objects having rank 5, rather than rank 6.

6. Higher dimensions

From rank 6 on, it becomes infeasible to draw full orbit graphs, so we shall only give orbit graphs with the loops suppressed. In light of Lemmas 1 and 2, there are only four possible orbit graphs to consider, in any rank, namely \(3^{d-2,d-1}\), \(3^{0,1}\), \(3^{d-2}\), and \(3^1\).

6.1. Type \(3^{d-2,d-1}\). We can prove by induction:

**Proposition 3.** There is a unique \(d\)-polytope of symmetry type \(d-2\) \(d-1\) for \(d = 5, 6, 7, 8\), denoted \((d-4)_{21}\), and a unique tiling of \(E^8\) when \(d = 9\). The facets of this polytope are \((d-1)\)-simplices and \((d-1)\)-crosspolytopes. The \((d-2)\)-faces of a given \((d-1)\)-crosspolytope facet alternate between connecting two crosspolytopes, or connecting a crosspolytope and a simplex. The vertex figure is \((d-5)_{21}\).

We have established this for \(d = 5\). Suppose that the claim is true for rank \(h \geq 5\), and let \(d = h + 1\). Suppose that \(P\) is a \(d\)-polytope of this symmetry type. The orbit graph breaks into two facet orbits: one is regular, and the other is an \(h\)-polytope whose facets are alternately in two different orbits. This latter must be an \(h\)-crosspolytope. So the other type of facet of \(P\) must be simplicial.

The vertex figure of \(P\) has orbit graph \(h-2\) \(h-1\) and so, by the induction hypothesis, must be \((d-5)_{21}\); so the facets of the vertex figure are \((d-2)\)-crosspolytopes and \((d-2)\)-simplices. The \((d-2)\)-crosspolytopes are the vertex figures of the \((d-1)\)-crosspolytope facets.
of $P$; the $(d - 2)$-simplices are the vertex figures of the other facets, which must therefore be simplices.

By induction, the $(d - 4)$-face figure (of type $3^{1,2}$) is always a triangular prism, whose squares correspond to the $(d - 1)$-crosspolytope facets of $P$, and whose triangles correspond to $(d - 1)$-simplex facets. The subridge figure (of type $3^{0,1}$) is always an isosceles triangle: each subridge is in two crosspolytope facets, and a single simplex facet. Consider the dihedral angles of these regular polytopes. The dihedral angle of an $n$-simplex is $\arccos \frac{1}{n}$, and that of an $n$-crosspolytope is $\pi - 2 \arcsin \frac{1}{\sqrt{n}}$ (Coxeter 1973, Table I (iii)).

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Simplex dihedral angle $\alpha$</th>
<th>Crosspolytope dihedral angle $\beta$</th>
<th>$2\beta + \alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$60^\circ$</td>
<td>$90^\circ$</td>
<td>$240^\circ$</td>
</tr>
<tr>
<td>3</td>
<td>$70.5^\circ$</td>
<td>$109.5^\circ$</td>
<td>$289.5^\circ$</td>
</tr>
<tr>
<td>4</td>
<td>$75.5^\circ$</td>
<td>$120^\circ$</td>
<td>$315.5^\circ$</td>
</tr>
<tr>
<td>5</td>
<td>$78.5^\circ$</td>
<td>$126.9^\circ$</td>
<td>$332.2^\circ$</td>
</tr>
<tr>
<td>6</td>
<td>$80.4^\circ$</td>
<td>$131.8^\circ$</td>
<td>$344^\circ$</td>
</tr>
<tr>
<td>7</td>
<td>$81.8^\circ$</td>
<td>$135.6^\circ$</td>
<td>$353^\circ$</td>
</tr>
<tr>
<td>8</td>
<td>$82.8^\circ$</td>
<td>$138.6^\circ$</td>
<td>$360^\circ$</td>
</tr>
</tbody>
</table>

We see that $n$-simplices and $n$-crosspolytopes can fit around a $(n - 2)$-face of an $(n + 1)$-polytope in the specified manner for $n \leq 7$, thus making our polytope $P$ possible up to $d = 8$.

Thus far, we have established that there is at most one polytope of this symmetry type in each rank $d = 5, 6, 7, 8$; that such a polytope must be as described in the Proposition; and that there can be no such $d$-polytope for $d \geq 9$. The actual existence of these polytopes $k_{21}$ is well established (Gosset 1900; G. Blind and R. Blind 1991; Coxeter 1973, §11.8, 1940b).

If we wish to double-check that these polytopes are indeed three-orbit, we can compute the number of flags of $4_{21}$. This polytope has 17280 7-simplex facets, and 2160 7-crosspolytope facets (Coxeter 1973, p. 204). Hence, it has $17280 \cdot 8! + 2160 \cdot 2^7 \cdot 7! = (192 + 384) 10!$ flags. Its symmetry group is $E_8$, with order $192 \cdot 10!$. Hence, it has three flag orbits. Since each
of the other polytopes is a face figure of $4_{21}$, and each contains a triangular prism as a face figure, each of the polytopes must be three-orbit.

6.1.1. Tilings. The unique possibility for facets and vertex figure in each dimension preclude the possibility of any tilings until rank 9. In this case, we can fit 8-simplices and 8-crosspolytopes together to fill space, with $4_{21}$ as the vertex figure. This tiling of $\mathbb{E}^8$ is known as $5_{21}$, or as “9-ic Semi-regular” in Gosset (1900). Since the Coxeter diagram is

![Diagram of Coxeter diagram]

we see that the symmetry group is the infinite group with the diagram called $T_9$ by Coxeter (1973, Table IV) or more recently $\tilde{E}_8$ (Coxeter 1988).

6.2. Type $3^{0,1}$. By duality with type $3^{d-2,d-1}$, there is a unique polytope of this symmetry type in each dimension $d = 5, 6, 7, 8$. This polytope is facet-transitive, and the facets are the dual to $(d - 5)_{21}$. Hence, the 2-faces are isosceles triangles, 3-faces are triangular bipyramids, the 4-faces are the dual to $0_{21}$, and so on. The vertex figures are $(d - 1)$-simplices and $(d - 1)$-cubes.

There is a tiling of $\mathbb{E}^8$ with this symmetry type, whose tiles are dual to $4_{21}$ and whose vertex figures are 8-simplices and 8-cubes.

6.3. The 6-polytope of type $3^4$. Consider the orbit graph for a 6-polytope $P$. This polytope is facet-transitive, and the facets (of type $3^{3,4}$) must be the 5-demicube, $1_{21}$. The polytope is also vertex-transitive, and the vertex figure (of type $3^3$) can only be the birectified 5-simplex, $0_{22}$. So the edge figure is a 3,3-duoprism: each edge is in 9 triangles, 18 tetrahedra, 6 4-simplices, 9 4-crosspolytopes, and 6 $1_{21}$s.

$P$ is 2-face-transitive, and the 2-face figure (of type $3^1$) is a tetragonal disphenoid; each triangle is in two 4-simplices and four 4-crosspolytopes, and four $1_{21}$s.

$P$ is 3-face-transitive; each 3-face is a regular tetrahedron, and each is incident to one 4-simplex and two 4-crosspolytopes, and three $1_{21}$s.
Such a polytope exists: it is known as $1_{22}$. Recall that this notation refers to its Coxeter diagram

$$\begin{array}{c}
\bullet \\
\circ \\
\bullet \\
\end{array}$$

which in turn describes how to generate the polytope by Wythoff’s construction, starting with the indicated vertex of the fundamental region for the group $E_6$. The vertex figure of a polytope with such a Coxeter diagram is determined by removing the ringed dot, and ringing the adjacent dot (Coxeter 1935 §9), which yields the Coxeter diagram for $0_{22}$; the facets can be found by removing the unringed terminal dots, which yields the Coxeter diagram for $1_{21}$, as desired. The 6-polytope $1_{22}$ is discussed by Coxeter (1940b, §6) and its discovery attributed to E. L. Elte. It has 72 vertices, 720 edges, 2160 triangles, 2160 tetrahedra, 432 4-simplices, 270 4-crosspolytopes, and 54 5-demicube facets.

6.4. The tiling of type $3^5$. Consider the orbit graph $\begin{array}{c}
5 \\
4 \\
6 \\
\end{array}$ for a 7-polytope $P$. Similarly to the case in §6.3, the facets (of type $3^4,5$) must be $2_{21}$, and the vertex figure (of type $3^4$) must be $1_{22}$. These form a tiling of 6-space, known as $2_{22}$.

There are several ways to reach this conclusion:

- To obtain a polytope whose vertex figure is $1_{22}$, we “prepend” a dot to the ringed dot on the Coxeter diagram for $1_{22}$, thus obtaining the Coxeter diagram

$$\begin{array}{c}
\circ \\
\bullet \\
\bullet \\
\end{array}$$

We then verify that the facets, whose Coxeter diagrams are obtained by removing an unringed terminal dot, are $2_{21}$. But this Coxeter diagram, called $T_7$ by Coxeter (1973 p. 297) or commonly $\widetilde{E}_6$, represents an infinite group, so this is a tiling.

- Armed with the knowledge that each vertex is in 54 facets, each with 27 vertices; in 270 5-crosspolytopes, in 432 5-simplices, in 2160 4-simplices, in 2160 tetrahedra, in 720 triangles, and in 72 edges, we apply Euler’s formula:

$$\frac{54}{27} f_0 - \frac{270}{10} f_0 - \frac{432}{6} f_0 + \frac{2160}{5} f_0 - \frac{2160}{4} f_0 + \frac{720}{3} f_0 - \frac{72}{2} f_0 + f_0 = 0.$$
For a 7-polytope, this sum would be 2; so this can only be a tiling of $\mathbb{E}^6$.

- We can calculate the dihedral angles of the facet $2_{21}$. At a ridge of $2_{21}$ between two 5-crosspolytopes, the dihedral angle is about $104.5^\circ$, and between a 5-crosspolytope and a 5-simplex, it is about $127.8^\circ$. The 4-face figure of $P$ is an isosceles triangle, with the apex corresponding to a 5-simplex face, and the other two vertices corresponding to 5-crosspolytope faces. The sum of two 5-simplex/5-crosspolytope dihedral angles and one 5-crosspolytope/5-crosspolytope dihedral angle is exactly $360^\circ$, so again we must have a tiling of $\mathbb{E}^6$, not a polytope.

This also shows that there are no $d$-polytopes of type $3^{d-2}$ for $d > 7$, since such a polytope would have a 7-dimensional section (the $(d - 8)$-face figure) of type $3^5$, which does not exist.

6.5. Type $3^1$. By duality, there is a unique 6-polytope with orbit graph $\bullet 1 \rightarrow 2 \leftarrow 0$, dual to $1_{22}$. The facets are dual to $0_{22}$ (i.e., the facets are the convex hull of a 5-simplex together with its reciprocal) and the vertex figure is the dual to $1_{21}$ (a kind of alternately Klee 5-crosspolytope).

This polytope also fills 6-space, forming the unique rank 7 tiling with the orbit graph $\bullet 1 \rightarrow 2 \leftarrow 0$. This tiling of $\mathbb{E}^6$ is dual to $2_{22}$. The vertex figures are dual to $2_{21}$.

7. Summary

We give a summary of all the (geometrically) three-orbit convex polytopes in Table VIIA and of all the three-orbit tilings in Table VIIB. Parenthesized numbers count combinatorially regular polytopes or tilings; the others are combinatorially three-orbit. Here we have used the ad-hoc operators $K$ to mean the Kleetope; $K_1$ to mean the “extended Kleetope”; aK, to mean the “extended alternate Kleetope”, with pyramids erected over alternate facets, coplanar with the remaining facets; and “$\bigcup$” to mean the convex hull of the union of a polytope with its reciprocal, with all the vertices on the same sphere. The operator $h$ means alternation, and is used thus by Coxeter. A star after a symbol means the dual. We have used the symbol $t\{2, q\}$ to mean a $q$-gonal prism, and $K\{p, 2\}$ to mean a $p$-gonal bipyramid. The Schläfli
symbol \(\{1 + 2\}\) for an isosceles triangle, and hence \(\{1 + 2, 1 + 2\}\) for a tetragonal disphenoid, is suggested by Coxeter [1973, p. 116]. By extension, we have used \(\{2 + 4\}\) for a three-orbit hexagon.

In all the entries of Table VIIA with a varying parameter, the polytope is actually combinatorially regular when the parameter is 4, and the 4-polytopes are also geometrically regular in this case. Both tables omit any polytopes or tilings with three-orbit symmetry under some subgroup of their symmetry group (i.e. with some coloring) but which are regular under the full symmetry group, with the exception of the 4-cube and 4-crosspolytope just mentioned.

<table>
<thead>
<tr>
<th>(d)</th>
<th>Type (3^{d-2, d-1})</th>
<th>Type (3^{0,1})</th>
<th>Type (3^{d-2})</th>
<th>Type (3^1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 ((\infty)): (3m)-gons ((m \geq 1))</td>
<td>(t{2, q}) ((q \geq 3))</td>
<td>(K{p, 2}) ((p \geq 3))</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(t{3, 3})</td>
<td>(K{3, 3})</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(t{3, 4})</td>
<td></td>
<td>(K{4, 3})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(t{3, 5})</td>
<td></td>
<td>(K{5, 3})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(t{4, 3})</td>
<td>(K{3, 4})</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(t{5, 3})</td>
<td>(K{3, 5})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 (\infty):</td>
<td></td>
<td>(\infty:) ({1 + 2, 1 + 2})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(t_1{3, 3, 3}) ((0_{21}))</td>
<td>(K_1{3, 3, 3})</td>
<td>({p} \times {p}) ((p \geq 3))</td>
<td>({p} + {p}) ((p \geq 3))</td>
</tr>
<tr>
<td></td>
<td>(t_1{3, 3, 5})</td>
<td>(K_1{5, 3, 3})</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(t_1{3, 4, 3})</td>
<td>(K_1{3, 4, 3})</td>
<td>(t_{1,2}{3, 3, 3})</td>
<td>(\infty:) (\bigcup{3, 3, 3})</td>
</tr>
<tr>
<td></td>
<td>(t_1{4, 3, 3})</td>
<td>(K_1{3, 3, 4})</td>
<td>(t_{1,2}{3, 4, 3})</td>
<td>(\infty:) (\bigcup{3, 4, 3})</td>
</tr>
<tr>
<td></td>
<td>(t_1{5, 3, 3})</td>
<td>(K_1{3, 3, 5})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1: (h{4, 3, 3, 3}) ((1_{21}))</td>
<td>1: (aK_+{3, 3, 3, 4})</td>
<td>1: (t_2{3, 3, 3}) ((0_{22}))</td>
<td>1: (\bigcup{3, 3, 3})</td>
</tr>
<tr>
<td>6</td>
<td>1: (2_{21})</td>
<td>1: (2_{21}^*)</td>
<td>1: (l_{22})</td>
<td>1: (l_{22}^*)</td>
</tr>
<tr>
<td>7</td>
<td>1: (3_{21})</td>
<td>1: (3_{21}^*)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1: (4_{21})</td>
<td>1: (4_{21}^*)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table VIIA.** Three-orbit convex \(d\)-polytopes
Table VII requires Kleetope operations for tilings. The Kleetope operators for a tiling \( \mathcal{T} \) always add new vertices in the center of every tile of \( \mathcal{T} \). In \( K \mathcal{T} \), the new tiles are the pyramids over the facets of each original tile (which are ridges of \( \mathcal{T} \)), whose apex is the new vertex in that tile. In \( K_1 \mathcal{T} \), the new tiles are bipyramids over the ridges of \( \mathcal{T} \). In \( K_{1,2} \mathcal{T} \), the new tiles partition the bipyramids of the previous type: Suppose \( F \) is a ridge of \( \mathcal{T} \), and \( x, y \) are the new vertices in the tiles of \( \mathcal{T} \) containing \( F \). The new tiles are the convex hull of the line segment \([x, y]\) with each facet of \( F \), i.e. with the subridges of \( \mathcal{T} \) incident to \( F \). If \( \mathcal{T} \) is a tiling of \( \mathbb{E}^3 \), the facets of \( K_{1,2} \mathcal{T} \) partition each \( q \)-gonal bipyramid in \( K_1 \mathcal{T} \) into \( q \) tetragonal disphenoids around the line segment between the two apices.
<table>
<thead>
<tr>
<th>$d$</th>
<th>Type $3^{d-1,d}$</th>
<th>Type $3^{0,1}$</th>
<th>Type $3^{1,2}$</th>
<th>Type $3^{d-1}$</th>
<th>Type $3^2$</th>
<th>Type $3^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1): “$3\infty$-gon”</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$2+(1)$: \begin{align*} &amp; \text{t}{3,6} \ &amp; \text{t}{4,4} \ &amp; \text{t}{6,3} \end{align*}</td>
<td>$2$: \begin{align*} &amp; \text{K}{4,4} \ &amp; \text{K}{3,6} \end{align*}</td>
<td>(2): \begin{align*} &amp; {1+2,2+4} \ &amp; {2+4,1+2} \end{align*}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$1$: \text{t}{4,3,4}</td>
<td>$1$: \text{K}{4,3,4}</td>
<td>$2+(1)$: \begin{align*} &amp; {3,6} \times {\infty} \ &amp; {4,4} \times {\infty} \ &amp; {6,3} \times {\infty} \end{align*}</td>
<td>$1$: \text{t}_{1,2}{4,3,4}</td>
<td></td>
<td>$1$: \text{K}_{1,2}{4,3,4}</td>
</tr>
<tr>
<td>4</td>
<td></td>
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<td>$2+(1)$: \begin{align*} &amp; {3,6} \times {3,6} \ &amp; {4,4} \times {4,4} \ &amp; {6,3} \times {6,3} \end{align*}</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$1$: $2_{22}$</td>
<td>$1$: $2_{22}^*$</td>
</tr>
<tr>
<td>8</td>
<td>$1$: $5_{21}$</td>
<td>$1$: $5_{21}^*$</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

**Table VIIb.** Three-orbit tilings of $\mathbb{E}^d$
CHAPTER VIII

Restrictions on $k$-orbit Polytopes

1. $k$-Orbit polytopes occur in only finitely many dimensions

Knowing that two-orbit and three-orbit polytopes occur in only finitely many dimensions lets us give an inductive proof that the same is true for $k$-orbit polytopes, for any $k > 1$.

Theorem 1. For any integer $k > 1$, there is $N_k \in \mathbb{N}$ such that $k$-orbit polytopes exist only in fewer than $N_k$ dimensions.

In other words, for any $d \geq N_k$, $d$-polytopes are either regular or have more than $k$ orbits.

Proof. We know that $N_2 = 4$ and $N_3 = 9$. Let $k > 3$ and suppose that such an $N_j$ exists for all $j$ with $1 < j < k$. Let $N = 2 \max(\{ N_j \mid 1 < j < k \} \cup \{k\})$.

Suppose $P$ is a $k$-orbit $d$-polytope with orbit graph $G(P)$ for any $d \geq N$. For some rank $l$, the graph $G_l(P)$ obtained from $G(P)$ by omitting the $l$-edges is disconnected, by Corollary II.2 (otherwise, $P$ is fully transitive, hence regular). If $l < \frac{d-1}{2}$, then we consider the dual polytope $P^*$, which is not transitive on its $(d - l - 1)$-faces. Thus we may assume that $l \geq \frac{d-1}{2}$. In fact, if $d$ is even, then $l \geq \frac{d}{2}$; in particular, if $d = N$, which is even, then $l \geq \frac{N}{2}$, and if $d > N$ then $\frac{d-1}{2} \geq \frac{N}{2}$. So we may assume that $l \geq \frac{N}{2}$.

Now the orbit graph of each $l$-face considered as an $l$-polytope, with its own symmetry group, is covered by the corresponding connected component of $G_l(P)$ (see Proposition II.1), which is a proper subgraph, so has fewer than $k$ flag orbits. But because $l \geq \frac{N}{2} \geq N_j$ for every $j < k$, the $l$-face cannot have fewer than $k$ flag orbits, unless it has just one. Thus, every $l$-face is a regular $l$-polytope.

Therefore, the $(l + 1)$-faces are polytopes with only regular facets. Such polytopes were classified by R. Blind (1979). The only instances in dimension $n$ greater than 4 are
• The regular polytopes.
• The Gosset polytopes \((n - 4)_{21}\), for dimensions up to 8.
• The pyramid over the \((n - 1)\)-crosspolytope.
• The bipyramid over the \((n - 1)\)-simplex.

(In the last two items, there is a unique choice of altitude yielding regular facets.)

Recall from §III.3 that a pyramid over a regular \(l\)-polytope (which is not itself a pyramid) has \(l + 2\) flag orbits, and from §III.4 that a bipyramid over a regular \(l\)-polytope (which is not itself the dual of some prism) has \(l + 1\) flag orbits. But either of these is larger than \(N/2 \geq k\), whereas the \((l + 1)\)-face, as part of a \(k\)-orbit polytope, can have at most \(k\) flag orbits. This excludes the last two items of the above list.

Also, we know that \((l + 1) > N/2 \geq N_3 = 9\), so the only remaining possibility is that the \((l + 1)\)-faces are also regular. Thus, we similarly conclude that the \((l + 2)\)-faces of \(P\) are regular, etc., and finally that \(P\) itself must be regular.

Therefore, \(k\)-orbit polytopes exist only in fewer than \(N\) dimensions, and so there is a minimum \(N_k \leq N\) such that no \(k\)-orbit polytopes exist in \(\mathbb{E}^n\) with \(n \geq N_k\).

The proof shows that \(N_k \leq 2^{k-3} \cdot 9\), for \(k \geq 3\). However, this is much larger than necessary. Already for \(k = 4\), the least dimension not admitting four-orbit polytopes is \(N_4 = 8\). On the other hand, the rectification of a regular \(d\)-polytope has \((d - 1)\) flag orbits. So \(k\)-orbit polytopes certainly appear in dimension \(k + 1\).

**Corollary 2.** For each \(k > 1\), the least dimension \(N_k\) not admitting any \(k\)-orbit polytopes satisfies \(k + 2 \leq N_k \leq 2^{k-3} \cdot 9\).

### 2. Orthoschemes

An orthoscheme is a simplex whose facets can be ordered \(F_0, \ldots, F_d\) such that each facet \(F_i\) is perpendicular to every other facet except possibly \(F_{i-1}\) and \(F_{i+1}\). In particular, \(F_0\) is perpendicular to every other facet except for \(F_1\), and \(F_d\) is perpendicular to every other facet except for \(F_{d-1}\).
An orthoscheme is *acute* if every dihedral angle between facets $F_{i-1}$ and $F_i$, for $i = 1, \ldots, d$, is acute. (Every other dihedral angle is always $\pi/2$.) An orthoscheme is *non-obtuse* if every dihedral angle is less than or equal to $\pi/2$. Every orthoscheme is acute in Euclidean space or hyperbolic space (Debrunner 1990, p. 124). In spherical space (with curvature 1), an orthoscheme $S$ is acute if and only if its diameter is less than $\pi/2$.

There are many other equivalent definitions of an orthoscheme. An *ordered simplex* is a simplex with a given ordering of its vertices, $P_0, \ldots, P_d$. Any of the following conditions on an ordered simplex define an orthoscheme:

- The Coxeter diagram representing the simplex (or, more generally, the *scheme* in the sense of Kellerhalls (1991)) is a string diagram, i.e. has no branches (no nodes of valence greater than two).
- For $j = 1, \ldots, d-1$, the affine span $\text{aff}\{P_0, \ldots, P_j\}$ is totally orthogonal to $\text{aff}\{P_j, \ldots, P_d\}$ (Debrunner 1990).
- Let $z_i$ be the “edge vector” $P_{i+1} - P_i$, for $i = 0, \ldots, d-1$. The simplex is an orthoscheme if and only if the inner product $z_i \cdot z_j$ is 0 for $i \neq j$ (Tschirpke 1993).
- The simplex is bounded by hyperplanes $H_0, \ldots, H_d$ such that $H_i \perp H_j$ for $|i - j| > 1$ (Kellerhalls 1991).
- Each triangle $P_iP_jP_k$, where $i < j < k$, is right-angled at $P_j$ (Coxeter 1989).
- Assuming that $P_0$ is the origin, the simplex is an orthoscheme if the linear span $\text{lin}\{P_j - P_i \mid i < j \leq d\}$ is orthogonal to the vector $P_i$ in $\mathbb{E}^d$ for all $1 \leq i \leq d - 1$ (Bezdek 2010).

The vertices $P_0$ and $P_d$ are called the *main vertices*. These are never in a right angle in a 2-face, whereas every other vertex is.

An orthoscheme is also called a *path simplex*. A $d$-orthoscheme can be considered as the convex hull of a graph, the path of length $d$, embedded in a $d$-dimensional space so that every edge is orthogonal to all the others. The simplex can be fully described by labeling each edge $e$ in this path with the dihedral angle opposite $e$ (that is, if the endpoints of $e$ are $P_i$ and...
It is labeled by the dihedral angle between the facets opposite $P_i$ and opposite $P_{i+1})$. All other dihedral angles are $\pi/2$.

Our interest in orthoschemes arises because they are the natural form of a flag chamber for many polytopes. Recall that the flag chambers of a $(d+1)$-polytope $P$ are $d$-simplices forming a tiling of the boundary of $P$, the barycentric subdivision, which can be formed by preassigning an interior point of each face of $P$. We generally desire the barycentric subdivision to be respected by the symmetry group $G(P)$, meaning that chambers are carried onto chambers. This means that the chosen interior point of an $i$-face $F$ must be carried by any symmetry $\gamma$ to the chosen interior point of the $i$-face $\gamma(F)$. This implies that the chosen point must be a fixed point of the stabilizer of $F$, and that for each $i$ it suffices to choose an interior point in one representative of each $i$-face orbit, determining the rest by the images of this point.

We will be interested in barycentric subdivisions of $(d+1)$-polytopes projected onto a $d$-sphere. Choose a fixed point $c$ of $G(P)$ in $\text{relint}(P)$, and project $P$ onto a sphere centered at $c$. Then we take a barycentric subdivision of the resulting sphere tiling of $S^d$.

**Definition.** An orthodivision of $P$ is a barycentric subdivision of $P$ into orthoschemes, respected by $G(P)$. A polytope possessing an orthodivision is said to be orthodivisible. Similarly, a (spherical) orthodivision of a sphere tiling $\mathcal{P}$ is a barycentric subdivision of $\mathcal{P}$ into orthoschemes, respected by $G(\mathcal{P})$. A polytope $P$ whose projection on a sphere centered at a fixed point of $G(P)$ possesses a spherical orthodivision is said to be spherically orthodivisible.

One natural choice for the interior point in each face $F$ is its centroid (the mean position of its vertices). This is always a fixed point of $G(F)$, and hence also of the stabilizer of $F$ in $G(P)$. There are other possible choices, however, which may yield an orthodivision even when the centroids do not.

If all the vertices of $P$ lie on a $d$-sphere, then the vertices of each $i$-face $F$ lie on an $(i-1)$-sphere, the circumsphere of $F$. If, furthermore, for each face $F$ the center of its circumsphere (the circumcenter of $F$) is in the relative interior of $F$, then this center is a
natural choice for the point representing \( F \) in the barycentric subdivision. We borrow some terminology for this from Hirani (2003).

**Definition.** A polytope \( P \) is *well-centered* if all its vertices lie on a sphere and the center of this sphere is in the (relative) interior of \( P \). A polytope is *completely well-centered* if every face (including \( P \) itself) is well-centered.

In a completely well-centered polytope, the flag chambers determined by circumcenters are orthoschemes. (This is because the line from the center of a sphere \( S \) to the center of any lower-dimensional sphere \( T \subset S \) is orthogonal to the subspace cutting out \( T \).) Furthermore, these points are respected by \( G(P) \).

For \( P \) to be completely well-centered, it suffices that \( P \) is vertex-transitive and every face \( F \) of \( P \) is also a vertex-transitive polytope (meaning that \( G(F) \) acts transitively on the vertices of \( F \); it is not necessary that the stabilizer of \( F \) in \( G(P) \) does so). However, this condition is not necessary. For example, a pentagonal bipyramid can be formed with all its vertices on a sphere. Then the facets are acute isosceles triangles, and their circumcenters are in their interior. However, the bipyramid is not vertex-transitive.

Yet another choice of points in the interior of faces is designed for the somewhat stricter requirement of splitting \( P \) itself into orthoschemes (rather than just the boundary). This method was developed by Rogers (1964) and the resulting chamber is dubbed a *Rogers simplex* by Bezdek (2010, Definition 7.3.4). Suppose \( o \) is an interior point of \( P \) fixed by \( G(P) \) such that the orthogonal projection of \( o \) onto the facet-defining hyperplane \( \text{aff}(F) \) for each facet \( F \) belongs to \( \text{relint}(F) \). If such a point \( o \) exists, we shall call it a *Rogers center* of \( P \). Then the chosen point in each face \( H \) is the point of \( H \) closest to \( o \).

A polytope \( P \) possesses a Rogers center if and only if the vertex truncation of some reciprocal \( P^* \) can be realized as the intersection of \( P \) with \( P^* \).

Fix a flag \( \Phi = \{F_{-1}, \ldots, F_{d+1}\} \) of a \( d \)-polytope \( P \) with a Rogers center \( o \) and let \( r_j \) be the closest point to \( o \) in \( F_{-j} \). If \( r_0, \ldots, r_i \) are linearly independent for some \( i \) (considering \( o \) to be the origin), then the simplex \( \text{conv}\{o, r_0, \ldots, r_i\} \) is called a Rogers simplex, and is actually
an orthoscheme. However, it is possible for the points to fail to be linearly independent: in fact, some of the points may coincide, so that the resulting subdivision has some degenerate simplices with zero volume.

3. Splitting polytopes

Definition. A dissection of a \(d\)-polytope \(P\) is a division of \(P\) into smaller \(d\)-polytopes. In other words, it is a finite collection of \(d\)-polytopes whose union is \(P\) and whose interiors are pairwise disjoint.

- A dissection is proper if it has more than one part.
- A dissection is respected by \(G(P)\) if the image of each of the parts under any symmetry in \(G(P)\) is again a part of the dissection.
- A subdivision of \(P\) is a dissection into polytopes which meet face-to-face. That is, the intersection of any two parts is a face of each (possibly empty).
- A triangulation of \(P\) is a subdivision into \(d\)-simplices.
- An orthoscheme triangulation is a triangulation whose simplices are orthoschemes.
- A flag triangulation is a triangulation consisting of pyramids over flag chambers, respected by \(G(P)\).

Note that by our definitions, a barycentric subdivision of \(P\) is not a subdivision of \(P\), but rather a tiling of the boundary of \(P\). However, by choosing an interior point \(v\) of \(P\) and taking the pyramids with apex \(v\) over each simplex in a barycentric subdivision of \(P\), we obtain a subdivision of \(P\). If this subdivision is respected by \(G(P)\), it is what we call a flag triangulation. In other contexts, this is also commonly called a barycentric subdivision.

Proposition 3. A \((d + 1)\)-polytope with a flag triangulation by orthoschemes is orthodivisible and spherically orthodivisible. Moreover, the \(d\)-orthoschemes of the induced orthodivisions have identical dihedral angles except between their \((d - 1)st\) and \(dth\) walls.

Proof. The intersection of each \((d + 1)\)-orthoscheme with the boundary of \(P\) is a \(d\)-orthoscheme, yielding a barycentric subdivision \(B\) into orthoschemes.
The point \( c \in \text{int}(P) \) which is a vertex of the flag triangulation is necessarily a fixed point of \( G(P) \). Let \( \mathcal{P} \) be the sphere tiling obtained by projecting \( P \) onto a sphere \( S \) centered at \( c \), and consider the barycentric subdivision \( \mathcal{B}' \) obtained by projecting \( \mathcal{B} \) onto \( S \). For each \((d + 1)\)-orthoscheme \( D \) in the flag triangulation, there is a spherical \( d \)-simplex \( D' \) in \( \mathcal{B}' \), which is the image of the \((d + 1)\)st wall of \( D \). The \( i \)th wall of \( D \), for \( i = 0, \ldots, d \), is determined by a hyperplane \( H_i \) passing through \( c \), and the facet \( F_i \) of \( D' \) is the intersection of \( H_i \) with \( S \). So the dihedral angle between any two facets \( F_i \) and \( F_j \) of \( D' \) is equal to the dihedral angle between \( H_i \) and \( H_j \), which is \( \frac{\pi}{2} \) if \(|i - j| \geq 2\); hence \( D' \) is an orthoscheme.

Moreover, say \( \hat{D} \) is the \((d + 1)\)st wall of \( D \); this is the preimage of \( D' \) in the orthodivision \( \mathcal{B} \). The wall \( \hat{D} \) is orthogonal to every other wall of \( D \) except the \( d \)th. Hence, for \( i, j < d \), the dihedral angle between the \( i \)th and \( j \)th walls of \( \hat{D} \) is identical to the angle between \( H_i \) and \( H_j \), hence is the same as the dihedral angle between \( F_i \) and \( F_j \) in \( D' \). So, the Euclidean orthoscheme \( \hat{D} \) and the spherical orthoscheme \( D' \) have all the same dihedral angles except between their \((d - 1)\)st and \( d \)th walls.

This is the dihedral angle around the only ridge of \( \hat{D} \) or \( D' \) which is contained in a subridge of \( P \) or \( \mathcal{P} \). So this says “the curvature of \( P \) is concentrated in the subridges”. □

The converse is not true. There are orthodivisible polytopes which have no flag triangulation by orthoschemes. For instance, let \( P \) be a partial Kleetope of the cube formed by adjoining square pyramids on two opposite sides, such that the new facets are equilateral triangles. Then \( P \) is orthodivisible, using the centroids of the vertices, edges, and 2-faces. But consider the cross-section of \( P \) in Figure VIII. In order to have an internal point of \( P \) whose orthogonal projection onto the top triangular facets is in the interior of the triangles, it must be within the top gray-shaded region; in order for the orthogonal projection on the bottom facets to be within the triangles, it must be in the bottom gray-shaded region.

Note that a completely well-centered polytope actually admits a flag triangulation by orthoschemes, not merely a orthodivision.
The Rogers construction will succeed in creating a flag triangulation by orthoschemes if the projection of the Rogers center \( o \) onto each facet \( F \) is a Rogers center of \( F \), and the projection of this point onto each ridge is a Rogers center of the ridge, and so on. These centers need to be fixed by the stabilizer of the face in \( G(P) \), not necessarily by the face’s own full symmetry group.

A polytope \( P \) has a flag triangulation by orthoschemes if and only if it has a Rogers center whose projection onto \( \text{aff}(F) \) belongs to \( \text{relint}(F) \) for every face \( F \) of \( P \).

4. Fundamental regions

If the symmetry group \( G \) of a \( d \)-polytope is generated by reflections, then its hyperplanes of reflection divide \( \mathbb{E}^d \) into finitely many cones, which are fundamental regions of \( G \). Taking a fixed point of \( G \) to be the origin, then \( G \) carries any sphere \( S \) centered at the origin to itself. Consider \( G \) to be acting on \( S \) (thus identifying \( G \) with a finite subgroup of the orthogonal transformations \( O(\mathbb{E}^d) \)). The fundamental regions of \( G \) on \( S \) are the intersections of \( S \) with each cone, and are spherical simplices (Coxeter 1934, Theorem 4). The vertices of this simplex, viewed as vectors, fall into mutually orthogonal sets corresponding to irreducible components of \( G \).

These components are irreducible reflection groups, whose Coxeter diagrams are from a short list of possibilities. All of these are string diagrams with the exception of \( D_n \) (for each
Let us use the term *branch* in a tree $T$ to mean a maximal path found in $T$ all of whose internal nodes have valence 2 in $T$ and one of whose endpoints is a leaf (valence 1) in $T$. Each of the exceptional Coxeter diagrams has a single dot of valence three. In $D_n$ this dot is in two branches of length one and one branch of length $n - 3$; in $E_n$ this dot is in one branch of length one, one branch of length two, and one branch of length $n - 4$.

The Coxeter diagram for $G$ has one connected component for each irreducible component of $G$. Thus, a finite reflection group $G$ has a string Coxeter diagram if and only if none of its components are $D_n$, $E_6$, $E_7$, or $E_8$.

If $\mathcal{P}$ is any sphere tiling of $S$ whose symmetry group is $G$, then any fundamental region of $G$ is a union of the flag chambers of any barycentric subdivision of $\mathcal{P}$ respected by $G$. For, if any flag chamber $\phi$ is not contained within a fundamental region, then the chamber $\phi$ crosses one of the hyperplanes of reflection of $G$; suppose the associated reflection is $\gamma$. Then $\gamma(\phi)$ overlaps with $\phi$; since $\gamma(\phi)$ is itself a chamber and the chambers form a tiling, we must have $\gamma(\phi) = \phi$, but this contradicts that $G$ acts freely on the flags of $\mathcal{P}$.

5. Splitting orthoschemes

Some attention has been paid to how one may dissect an orthoscheme into other, smaller orthoschemes. Papers such as (Brandts, Korotov, and Křížek 2007; Coxeter 1989; Debrunner 1990) are focused on finding a division into a small number of orthoschemes: namely, a $d$-dimensional orthoscheme can always be split into $d$ smaller orthoschemes. (These are not congruent, in general.) On the other hand, we want to show that a $d$-dimensional orthoscheme cannot be properly dissected into fewer than $d$ orthoschemes. We start with several preparatory lemmas.

**Lemma 4.** If a non-obtuse $d$-orthoscheme $S$ with $d \geq 3$ is properly dissected into orthoschemes, then one of its facets is properly dissected (into facets of the parts). That is, there exists a facet $F$ such that no orthoscheme of the dissection contains all of $F$. 163
Proof. Suppose $S$ is properly dissected into orthoschemes $S_1, \ldots, S_m$. Now suppose that every facet of $S$ is intact; that is, that each facet of $S$ is contained in a single orthoscheme $S_i$. Any simplex containing two of the facets of $S$ contains all the vertices of $S_i$, hence contains all of $S$. So each orthoscheme $S_i$ contains at most one facet of $S$. Thus, the dihedral angle between any two facets of $S$ is split into at least two parts. Suppose the orthoscheme $S_1$ contains the facet $F$ of $S$. Then the dihedral angle of $F$ with each other facet of $S_1$ is strictly less than the dihedral angle of $F$ along the same ridge with a facet of $S$. Hence, all of the dihedral angles of $F$ with other facets of $S_1$ are acute. But this contradicts that $S_1$ is an orthoscheme, since every facet of a $d$-orthoscheme is orthogonal to at least one other facet if $d > 2$. □

We can strengthen Lemma 4 slightly, if we begin with an acute orthoscheme.

**Lemma 5.** If an acute $d$-orthoscheme $S$ with $d \geq 3$ is properly dissected into orthoschemes, then at least one of the facets incident to each vertex must be properly dissected.

**Proof.** Index the vertices of $S$ as $P_0, \ldots, P_d$, opposite respective facets $F_0, \ldots, F_d$, as in the definitions of an orthoscheme. Let $P_j$ be any vertex, and suppose all the facets incident to $P_j$ are left intact. Some facet $F_k$ has an acute dihedral angle with the facet $F_j$. Say $T$ is the $d$-orthoscheme of the dissection containing $F_k$. As in the previous proof, $T$ has an acute dihedral angle along every ridge of $F_k$ which includes $P_j$. The only remaining ridge is the ridge between $F_k$ and $F_j$, which is already acute; so the face $F_k$ has all acute dihedral angles within $T$, a contradiction. □

Applying the Lemma inductively gives the

**Corollary 6.** If an acute orthoscheme is properly dissected into orthoschemes, then some of its 2-faces at each vertex must be properly dissected.

The following two lemmas apply to acute 2-orthoschemes, that is, right triangles whose remaining two angles are acute. This excludes spherical triangles with two or three right angles.
Lemma 7. There is a unique proper dissection of a right triangle into right triangles keeping its two legs intact.

![Figure VIIIb. The situation of Lemma VIII.7](image)

Proof. Let the triangle have vertices $A$, $B$, and $C$ with the right angle at $B$. The altitude from $B$ to the hypotenuse $AC$ divides the triangle into two right triangles with the legs $AB$ and $BC$ intact. Let $P$ be the foot of this altitude. See Figure VIIIb.

Any point in the triangle $BPC$, besides $P$ itself, forms an obtuse angle with the leg $BC$. So the point $C'$ we pick to form a right triangle over $BC$ is either $P$ or lies beyond the altitude $BP$; so the triangle $BC'C$ contains an interior point of the line $BP$. Similarly, the point $A'$ chosen to form a right triangle over the leg $AB$ is either $P$ or lies on the far side of the altitude $BP$ from the leg $AB$. So the triangle $BA'A$ contains an interior point of $BP$.

But then some portion of the line segment from $B$ to $P$ is contained in both triangles $BA'A$ and $BC'C$, contradicting that these are distinct parts of a dissection, unless this segment is in the boundary of both triangles; this means that both $A'$ and $C'$ are on the altitude $BP$, and thus both are in fact equal to $P$. □

In the Euclidean case, the locus of possible points $A'$ forming a right angle with the leg $AB$, and the locus of possible points $C'$ forming a right angle with the leg $BC$, are circles with the respective leg as a diameter, intersecting at $P$ and $B$. However, we are primarily interested in the spherical case.

Lemma 8. There is no proper dissection of a right triangle into right triangles keeping the hypotenuse intact.

Proof. Let the triangle have vertices $A$, $B$, $C$ with its right angle at $B$. Any point $B'$ of the triangle besides $B$ forms an obtuse angle with the hypotenuse $AC$, while the other two angles $B'AC$ and $B'CA$ are both acute. □
Lemma 9. An acute 3-orthoscheme $S$ cannot be properly triangulated by orthoschemes with two 2-faces left intact.

Proof. Suppose $S$ admits a proper triangulation by orthoschemes with two 2-faces left intact. The 2-faces left intact include five edges of $S$; so there is a single edge of $S$ which is divided. This edge must be the hypotenuse of both the remaining faces, otherwise they cannot be subdivided, by Lemma 8. Index the vertices of $S$ as $P_0, P_1, P_2, P_3$ as in the definition of an orthoscheme. Then the only edge which is the hypotenuse of two triangles is $P_0P_3$. See Figure VIIIc.

![Figure VIIIc. The situation of Lemma VIII.9. The shaded triangles remain intact.](image)

By Lemma 7, there is a unique way to split the triangles $P_0P_1P_3$ and $P_0P_2P_3$, by dropping altitudes onto the edge $P_0P_3$. The feet of these altitudes must coincide, since otherwise $S$ cannot be subdivided into orthoschemes. Let $z$ be the foot of these altitudes.

By construction, the triangles $P_0zP_1$, $P_1zP_3$, $P_0zP_2$, and $P_2zP_3$ are all right-angled at $z$. Also, the triangle $P_1zP_2$ must be a right triangle, being part of an orthoscheme triangulation, and the right angle must also be at $z$. But then $z$ is at a right angle in every incident 2-face, which is never the case in an acute orthoscheme.

Lemma 10. If a triangulation $\mathcal{T}$ of a simplex $S$ has $r$ new vertices (meaning vertices of $\mathcal{T}$ which are not vertices of $S$), then the triangulation has at least $r + 1$ simplices.

Proof. List the simplices of $\mathcal{T}$, so that each simplex after the first is adjacent to at least one of the previously listed simplices. If $S$ is a $d$-simplex, then the first simplex listed has $(d + 1)$
vertices of \( T \). Each subsequent simplex includes at most one vertex of \( T \) not previously seen. So, to include all \( d + r + 1 \) vertices of \( T \), we must list at least \( r \) simplices after the first (hence at least \( r + 1 \) total). \[ \square \]

We can now prove the theorem:

**Theorem 11.** A \( d \)-dimensional acute orthoscheme has no proper subdivision into fewer than \( d \) orthoschemes.

**Proof.** We proceed by induction on \( d \). For \( d = 1 \) or \( d = 2 \), the claim is trivial. Suppose \( d \geq 3 \) and that the claim is true for all orthoschemes of dimension less than \( d \).

Let \( S \) be a \( d \)-dimensional acute orthoscheme with vertices \( P_0, \ldots, P_d \) opposite facets \( F_0, \ldots, F_d \), respectively, indexed as in the definitions of an orthoscheme. Consider a proper subdivision of \( S \) into orthoschemes. By Lemma 4, some facet \( F_j \) is split, and by induction there are at least \( (d - 1) \) parts; suppose these lie in \( d \)-orthoschemes \( S_1, \ldots, S_{d-1} \) in the subdivision of \( S \).

If \( S \) has been split into only these \( d - 1 \) orthoschemes and no others, then each must be a pyramid with apex \( P_j \). Then in each triangle \( P_jP_kP_l \), with \( j < k < l \), the hypotenuse \( P_jP_l \) is intact; so by Lemma 8, the 2-face \( P_jP_kP_l \) is left intact. Similarly, every triangle \( P_hP_iP_j \), with \( h < i < j \), is left intact.

Therefore, if \( j = 0 \) or \( j = d \), all the triangles incident to \( P_j \) are intact, contradicting Corollary 6.

Otherwise, \( 0 < j < d \). Suppose some triangle \( P_iP_jP_k \) with \( i < j < l \) is left intact. Then, for any \( k \) with \( j < k < l \), the 3-orthoscheme \( P_iP_jP_kP_l \) has its 2-faces \( P_iP_jP_l \) and \( P_jP_kP_l \) intact. Hence by Lemma 9, the whole 3-orthoscheme is intact. Therefore every triangle \( P_iP_jP_k \) is intact, where \( j < k < l \). In particular, \( P_iP_jP_{j+1} \) is intact.

Now let \( k \) be any number with \( j + 1 < k \leq d \) and consider the 3-orthoscheme \( P_iP_jP_{j+1}P_k \). The 2-faces \( P_iP_jP_{j+1} \) and \( P_jP_{j+1}P_k \) are intact, so again the whole 3-orthoscheme is intact. Hence, every triangle \( P_iP_jP_k \) is intact, for any \( k > j \).
Similarly, in the 3-orthoscheme $P_iP_{j-1}P_jP_k$ (for any $k > j$), the 2-faces $P_iP_jP_k$ and $P_iP_{j-1}P_j$ are intact, so the whole 3-orthoscheme is intact; in particular, the triangle $P_{j-1}P_jP_k$ is intact.

So, for any $h < j - 1$, in the 3-orthoscheme $P_hP_{j-1}P_jP_k$, the triangles $P_hP_{j-1}P_j$ and $P_{j-1}P_jP_k$ are both intact, so the whole 3-orthoscheme is intact.

Thus, $P_hP_jP_k$ is intact, for any $h < j$ and any $k > j$. So every 2-face at $P_j$ is intact, again contradicting Corollary 6.

The only remaining possibility is that every triangle $P_iP_jP_l$ with $i < j < l$ is split. The legs $P_iP_j$ and $P_jP_l$ are left intact, so there is a unique subdivision by the altitude from $P_j$ to the hypotenuse $P_iP_l$. Hence there is at least one new vertex for every pair $(i, l)$ with $i < j < l$. There are $j(d - j) = dj - j^2$ such pairs.

Since $j$ is neither 0 nor $d$, this has its minimum when $j = 1$ or $j = d - 1$, giving us at least $d - 1$ new vertices. But then by Lemma 10 there are at least $d$ simplices in the subdivision.

6. Flag orbits of orthodivisible polytopes

**Definition.** An *SRG polytope* is one whose symmetry group has a connected string Coxeter diagram.

Recall that a symmetry group has a string diagram if it is generated by reflections and none of its irreducible components are $D_n$, $E_6$, $E_7$, or $E_8$. Equivalently, the fundamental region of the group, on a sphere preserved by the group, is an orthoscheme. This orthoscheme is acute if and only if the Coxeter diagram is connected, so that the group is irreducible. ("SRG" stands for "string reflection group"). In this case the group has exactly one fixed point.

The immediate consequence of Theorem 11 is

**Theorem 12.** A spherically orthodivisible SRG $d$-polytope is either regular or has at least $d - 1$ flag orbits.
Proof. Let $P$ be a spherically orthodivisible SRG $d$-polytope. Let $\mathcal{P}$ be the projection of $P$ onto a sphere $S$ centered at the fixed point of $G(P)$ in the interior of $P$. Let $R$ be a fundamental region of $G(P)$ on $S$; recall that $R$ is an acute orthoscheme. A barycentric subdivision of $\mathcal{P}$ respected by $G(P)$ also subdivides $R$ into orthoschemes. Either this subdivision is not proper, in which case $G(P)$ acts transitively on the flag chambers of $\mathcal{P}$ and hence on the flags of $P$, or else it must divide the $(d-1)$-orthoscheme $R$ into at least $d-1$ flag chambers. □

Since a completely well-centered polytope has a flag triangulation by orthoschemes, by Proposition 3 it is spherically orthodivisible.

**Corollary 13.** Any completely well-centered SRG $d$-polytope is either regular or has at least $d-1$ flag orbits.

**Corollary 14.** An SRG $d$-polytope, all of whose faces are vertex-transitive, is either regular or has at least $d-1$ flag orbits.

In particular, this applies to uniform polytopes (which have the additional condition that all 2-faces are regular). We introduce another important class of polytopes which are completely well-centered.

**Definition.** A polytope in $\mathbb{E}^d$ is Wythoffian if it can be made via Wythoff’s construction.

Recall that this means there is a finite group $G$, generated by reflections in $\mathbb{E}^d$, and a point $v \in \mathbb{E}^d$, such that $P$ is the convex hull of the orbit of $v$ under $G$. In this case, the images of $v$ are the vertices of $P$, and $G$ is a subgroup of the symmetry group $G(P)$.

Note that several groups $G$ can generate the same polytope in this manner. For example, the 3-cube with vertices $(\pm 1, \pm 1, \pm 1)$ can be produced by the images of any one of its vertices under the group generated by the three reflections in the $xy$ plane, the $xz$ plane, and the $yz$ plane; this group has order 8. It can also be produced by the images of one of its vertices under the full symmetry group of the cube, a group of order 48.
By construction, a Wythoffian polytope is well-centered. Furthermore, each face of a Wythoffian polytope is Wythoffian. Therefore, a Wythoffian polytope is completely well-centered.

**Corollary 15.** A Wythoffian SRG $d$-polytope is either regular or has at least $d - 1$ flag orbits.

7. **Further results**

A classification of the four-orbit convex polytopes has also been completed (see Matteo 2015b). Intriguingly, the last of these appear in dimension 7, one dimension earlier than the last three-orbit polytope!

It is possible to generalize Lemmas VI.8 and VI.9 to give a distinguished presentation for the symmetry group of any convex polytope, based on its orbit graph. In fact, it is the fundamental group of a certain 2-dimensional CW complex built on the orbit graph. The $(i - 1, i)$-cycles in the graph, for $i = 1, \ldots, d$, (or more precisely such cycles after contracting all the edges with labels at least $i + 2$ or at most $i - 3$), play the role of the links in a Coxeter diagram: associating a number to each gives the relations between distinguished generators. Just as with the Schläfli symbol composed of the link labels of a Coxeter diagram, the numbers attached to these cycles also determine any convex polytope or tiling of a simply-connected space up to isomorphism, and any abstract polytope with the same numbers is a quotient of this polytope or tiling.

It remains to investigate the behavior of $k$-orbit convex polytopes for $k > 4$, especially the size of the least dimension $N_k$ not admitting any $k$-orbit polytopes. Also of interest is the least number $k_d$ of flag orbits of a non-regular $d$-polytope. (We have $k_2 = k_3 = 2$, while $k_4$ through $k_8$ are all 3, and $k_9 \geq 5$.) Results leading toward the classification of $k$-orbit polytopes for general $k$, mostly for abstract polytopes, are found in Cunningham et al. (2015), Helfand (2013), and Orbanić, Pellicer, and Weiss (2010).
Bibliography


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