Sv-map between type-I and heterotic sigma models

by Wei Fan

B. S. in Applied Physics, Shandong University of Science and Technology
M. S. in Spintronics, Shandong University of Science and Technology

A dissertation submitted to

The Faculty of
the College of Science of
Northeastern University
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

March 25, 2019

Dissertation directed by

Tomasz Robert Taylor
Professor of physics
Acknowledgements

I am deeply grateful to my advisor, Professor Tomasz Robert Taylor, for bringing me to the field of scattering amplitudes and for his support of my research. Tom chose the research topic of this dissertation for me, when I was a beginner in high energy physics and could not recognize the differences among various research fields. This research topic chosen by Tom turns out to be a good fit for my personality, and now I feel extremely glad and lucky that I am working on this interesting and challenging topic. I would also like to thank Tom for setting up the regular group meeting, and for the time he spent on giving suggestions and clarifying problems during my research progress. I appreciate very much the support of Tom on my visiting at University of Warsaw, where I had a wonderful experience and progressed a lot on my research.

I am also very grateful to Dr. Stephan Stieber and Dr. Angelos Fotopoulos, without whom the dissertation would not have been possible. I enjoy very much the collaboration with Angelos, especially the discussion we have at every group meeting when we work on the topic of this dissertation. Angelos is energetic and always provides new ideas. I appreciate very much the continuous help of Stephan on clarifying the detailed mathematical aspect of the topic of this dissertation. I would also like to thank Stephan for his strong support, when I am trying to look at and solve the topic of this dissertation from a different point of view.

I would like to thank Professor Zygmunt Lalak of University of Warsaw, for his hospitality and support during my visit there. The time studying and living at Warsaw gives me a feeling of my hometown and I will treasure those memories.

I would also like to thank my committee members, Professor James Halverson, Professor Pran Nath and Professor Brent Nelson. I enjoy the time at the regular lunch meeting of the high energy theory group, which is organized by Jim. I have learned a lot from the excellent lectures provided by Pran on quantum theory and by Brent on quantum field theory.
Finally, I will express my gratitude to my wife Qianqian Ye, for her love and support during my study and research. In addition to the delicious meals she always cooks, she provides me with the great opportunity and wonderful experience, of having normal human emotional bonding with the one from the other gender.
Abstract of Dissertation

The scattering amplitudes of gauge bosons in heterotic and open superstring theories are related by the single–valued map (sv–map): heterotic amplitudes are obtained by selecting a subset of multiple zeta value (MZV) coefficients in the $\alpha'$ (string tension parameter) expansion of open string amplitudes. In this dissertation, we argue that this relation holds also at the level of low–energy expansion (or individual Feynman diagrams) of the respective effective actions, by investigating the beta functions of two–dimensional sigma models describing world–sheets of open and heterotic strings coupled with gauge backgrounds. We will analyze the sigma model Feynman diagrams generating identical effective action terms in both theories and show that the heterotic coefficients are given by the single-valued projection of the open ones. The gauge backgrounds of the nonlinear sigma model of the heterotic string is usually studied under the fermionic representation. Here we will also propose a Wilson loop representation for it. When it is under the fermionic representation, the sv–map appears as a result of summing over all radial orderings of heterotic vertices on the complex plane. When it is under the proposed Wilson loop representation, the sv–map comes from a sum of path-ordered integrals along two opposite-directed contours, which has a simple geometric origin manifested in the nonabelian Stokes’s theorem. (This dissertation is based on our published work [1,2].)
Table of Contents

Acknowledgments 2

Abstract of Dissertation 4

Table of Contents 5

1 Introduction 8

1.1 Some basics of scattering amplitudes 14
1.2 Methodology of string amplitudes 18
1.3 Methodology of Nonlinear sigma model Approach 26
1.4 Multiple zeta values (MZV) 29
1.5 Single–valued map 31

2 Open string sigma model 37

2.1 The action 37
2.1.1 Wilson loop representation 37
2.1.2 Fermionic representation 38
2.2 The two–loop beta function 39

3 Heterotic string sigma model: fermionic representation 44

3.1 The fermionic representation 44
3.2 Reorganized perturbation method 50
3.3 Heterotic two–loop beta function 55
3.4 Single–valued multiple zeta–values and general sv–map proposal for heterotic string 57
3.4.1 Single–valued multiple zeta–values 57


3.4.2 General sv–map proposal for heterotic string ............................................. 59

4 Heterotic string sigma model: Wilson loop representation ............................ 60

4.1 Reorganized perturbation method ................................................................. 61

4.2 Construct the Wilson loop ............................................................................. 62

4.2.1 The functional variation of the Wilson loop ........................................ 64

4.2.2 The exact propagator of $\psi$ .................................................................. 69

4.3 Geometry of the Wilson loop ........................................................................ 71

4.3.1 Yang-Mills case ....................................................................................... 72

4.3.2 Open string case ...................................................................................... 75

4.3.3 Heterotic string case ................................................................................ 76

4.4 Path ordering and contour direction .............................................................. 78

5 Sv map in the fermionic representation ............................................................ 79

5.1 The Gegenbauer polynomial for angular integration ..................................... 80

5.2 The sv–map at three loops ............................................................................. 81

5.2.1 Open string three–loop integral ............................................................... 81

5.2.2 Heterotic string three–loop integral ......................................................... 83

5.3 The sv–map at four loops .............................................................................. 85

5.3.1 Figure 5.2 ................................................................................................ 85

5.3.2 Figure 5.3 ................................................................................................ 88

5.3.3 Figure 5.4 ................................................................................................ 91

5.4 Summary ....................................................................................................... 94

6 Sv map in the Wilson loop representation ....................................................... 95

6.1 The zeta(2) case ........................................................................................... 96

6.2 The zeta(3) case ........................................................................................... 98

6.2.1 Case 1 ...................................................................................................... 98
6.2.2 Case 2 ................................................................. 100
6.2.3 Case 3 ................................................................. 101
6.3 Summary ............................................................... 102

7 Conclusion .............................................................. 103

Bibliography ............................................................. 105
Chapter 1

Introduction

In the 1960s, string theory originated as a candidate theory for strong interactions of protons, neutrons, quarks and gluons. At the same time, Quantum Chromodynamics (QCD) was another candidate theory of strong interactions. QCD is described by the usual 4D quantum field theory (QFT)—the same framework used in describing weak interactions and electromagnetic interactions, where quantum fields are functions of the 4D spacetime coordinates \((t, x, y, z)\) and particles are the vibration modes of quantum fields. When a particle moves in space, its spatial trajectory is a curve and its spatial position depends on time \(X^i(t) = (x(t), y(t), z(t))\), where \(x, y, z\) are the Cartesian coordinate of our 3D sensible space.

In string theory, the physical object is the spatially extended 1D string. When the 1D string moves in space, its trajectory is a 2D surface called the string world–sheet. The string world–sheet is parameterized by \((\tau, \sigma)\), where \(\tau\) is time-like and \(\sigma\) is space-like. The spacetime coordinates \(X^\mu\) of the string are functions of its world–sheet as \(X^\mu(\tau, \sigma)\), which is called the spacetime embedding of string world–sheet. Figure 1.1 is an illustration of particle trajectory and string trajectory. So physical fields in spacetime are eventually functions of the world–sheet, and the QFT in this special case is called the conformal field theory (CFT). The vibration modes of strings have the same properties, for example mass and spin, as point-like particles of quantum fields. So what we usually think to be point-like particles, are actually tiny vibrating
strings. There are two types of string, the open string and the closed string. The open string is a 1D segment whose trajectory is a 2D sheet. The closed string is a 1D loop whose trajectory is a 2D cylinder.

In the beginning of 1970s, QCD was developed into a successful theory of strong interactions. QCD, weak theory and quantum electromagnetic theory were combined together into a unified theory of elementary particles—the standard model, with a gauge group $SU(3) \times SU(2) \times U(1)$ and three generations of quarks and leptons. The standard model can describe strong, weak and electromagnetic interactions, except gravity. The heaviest standard model particle has a mass of several hundreds GeV, and all the particle’s properties can be confirmed by scattering experiments at energies of a few dozens TeV. This is the energy scale of current elementary particles.

While QCD was a great success, string theory was still under active development and the spectrum of string theory was found to also include gravitons. This fact promotes string theory to a candidate theory of quantum gravity—a unified theory of all the four known interactions, including gravity. At that time, the literature, both research-oriented and public-oriented, on
string theory started to expand quickly. The typical energy scale of quantum gravity is the Planck scale $10^{19}$ GeV, which is far beyond the reach of current scattering experiments.

In the 1980s, researcher showed that only five string theories are consistent and anomaly-free. This is called the first revolution of string theory, which means that under the framework of string theory, only five unified theories of physics can exist. They are type-I string theory, type-IIA and type-IIB string theory, and two heterotic string theories with gauge group $SO(32)$ and $E_8 \times E_8$ respectively. In all these string theories, spacetime is 10 dimensional (9D space and 1D time), which contradicts the 4 dimensional spacetime (3D space and 1D time) that our sensible impression tells us. To explain this difference between empirical intuition and physics requirement, the framework of string compactification was adopted, which involves complicated mathematical geometry of the ‘shape’ of spacetime. The idea of compactification firstly appeared in the Kaluza–Klein theory in the 1920s, which is a pure gravity theory with 5 dimensional (4D space and 1D time) spacetime. In the Kaluza–Klein theory, when the gravitational physics of the extra space dimension is integrated out, the effective physics on the remaining 4D spacetime would give us the Einstein gravity and the Maxwell electromagnetism.

This idea was generalized to string theory in the middle of 1980s. If the six extra space dimensions are integrated out, the effective physics on the remaining 4D spacetime should give us the standard model. This requirement puts strong constraints on the possible geometry of the extra 6D space. The first result of string compactification, which is also the most famous one, tells us that the extra 6D space must be a Calabi-Yau manifold, which has a special metric that meets the stringent requirements of string compactifications. Actually, it was in the 1970s that the existence of this specific metric was proven in the mathematics literature (the Calabi conjecture was proven by Yau). This is an example where string theory is tightly connected with the research in mathematics. Over the past three decades there has been an active and very fruitful interaction between various branches of mathematics and string theory. String compactification has developed into an important field in string theory, where exotic spacetime geometries are studied and tightly connected with new developments of geometry in the mathematics literature.
In the 1990s, several duality relations were found among these five string theories, which suggest that these five string theories are different aspects of a single unified theory. This is the second revolution of string theory. The so-called T-duality connects type-IIA theory with type-IIB theory, and also connects the two heterotic string theories. The so-called S-duality connects type-I theory with the heterotic theory of gauge group $SO(32)$, and also connects the type-IIB theory with a dual formulation of itself.

During this time, the study of black holes in string theory led to an important result on the laws of black holes. String theory contains gravity, so it is natural to study black holes within string theory. Since the 1970s, it is well known that the equations describing the dynamics of black hole horizons are an analog of the law of thermodynamics. For example, the area of black hole corresponds to the entropy of thermodynamics. This is called the Bekenstein-Hawking entropy and the formula is

$$S_{BH} = \frac{\text{area of event horizon}}{4G_N},$$

where $G_N$ is the Newton constant. But this analog lacks a rigorous microscopical derivation from statistical physics. Then in the middle of 1990s, a rigorous statistical derivation of the entropy of black holds was obtained by counting microstates of black holes in a special string compactification. The result shows that indeed the entropy equals the area for these special black holes. This is one example of the application of string theory in gravitation.

In the late 1990s, researchers discovered another duality relation from string theory, the AdS/CFT correspondence or the holographic duality, which is a duality between string theory and gauge theory. Basically, the tree-level approximation of the type-IIB string theory in an $AdS_5 \times S^5$ background corresponds to the planar approximation of a specific super-Yang-Mills theory in 4D spacetime. The $AdS_5 \times S^5$ background is a 5D anti-de Sitter (AdS) spacetime times a 5D compactified sphere $S^5$, and the planar limit of the super-Yang-Mills theory is a CFT. This is the reason why it is called the AdS/CFT correspondence. Using string compactification, the $AdS_5 \times S^5$ background would generate an effective 5D theory (including gravity) on the
AdS$_5$ spacetime. The 4D spacetime of the CFT is associated with the boundary of the AdS$_5$, so it looks like that the CFT lives on the boundary of the AdS theory and this is the reason why it is also called the holographic duality.

The AdS/CFT correspondence originates from string theory, but its influence is not only limited within string theory. The AdS/CFT correspondence can be generalized to a correspondence between a general D-dimensional CFT and a (D+1)-dimensional AdS theory, not necessarily related with string theory. Nowadays, the AdS/CFT correspondence itself becomes an active field of research. And it has many applications in high energy physics, condensed matter physics, and even in fluid dynamics.

One interesting application of the AdS/CFT correspondence is on the field of quantum information theory. In quantum information theory, the entanglement entropy or von Neumann entropy is used to measure the entanglement of a bipartite system. It can be understood as the entropy computed by an observer who is constrained to only one part of the bipartite system without any access to the other part. This inaccessibility of the other part of the system reminds us of black holes, where the black hole horizon limits our access to quantum systems that describe the black hole interior. So it seems that the entanglement entropy and the Bekenstein-Hawking entropy should be somehow related. In the 2000s, computations using the AdS/CFT correspondence showed that the two kinds of entropy are indeed connected. Suppose that in a D dimensional CFT, there is a (D-1) dimensional subsystem $A$ that has a (D-2) dimensional boundary $\partial A$. For this subsystem, the continuous limit of the entanglement entropy is computed using the AdS/CFT correspondence. When the (D-1) dimensional subsystem is embedded into the (D+1) dimensional AdS space, its (D-2) dimensional boundary $\partial A$ would bound a (D-1) dimensional minimal surface $\gamma_{A}^{D-1}$. The result of the entanglement entropy is a generalization of the Bekenstein-Hawking entropy of black holes, where the area of black hole is substituted
by the area of this (D-1) dimensional minimal surface.

\[ S_A = \frac{\text{area of minimal surface}}{4G_N^{D+1}}, \]

where \( G_N^{D+1} \) is the Newton constant of the (D+1) dimensional AdS space. This result of AdS/CFT correspondence makes an unexpected connection between gravity and quantum information theory. So tools from gravity and CFT are introduced into quantum information, and this area is an active field of research nowadays.

String theory developed initially as a physics theory of unified interactions, but string theory is also connected with the development of pure mathematics in the past years. String theory has many interactions with algebraic geometry, topology and algebraic number theory. The usage of Calabi-Yau manifold in string compactification mentioned above is a famous example. Recently, there are active interactions between the development of string scattering amplitudes and the algebraic number theory. For these reasons, string theory remains as the most promising candidate of a unification theory and as a significant source of new ideas for mathematics, information theory and other branches of physics.

String theory is a big topic that contains many different fields of research. The work of this dissertation focuses on the research field of string scattering amplitudes.
1.1 Some basics of scattering amplitudes

Quantum Chromodynamics (QCD) is a quantum field theory describing strong interactions of protons, neutrons, quarks and gluons in the framework of Standard Model of particle physics. It is asymptotically free, meaning that at the energies much higher than the proton mass, like in the energy range of teraelectronvolts (TeV) accessible at the Large Hadron Collider (LHC), it is a weakly coupled theory. At such high energies, the scattering amplitudes can be computed perturbatively, by using Feynman diagrams. Due to nonlinear interactions, such computations are very complicated and cumbersome. However, for the special tree level diagrams involving N external gluons with Maximal Helicity Violating (MHV) arrangement of their helicity, the result has a very simple structure, given by the famous Parke-Taylor formula:

\[ A(1^-, 2^-, 3^+, \ldots, n^+) = \frac{(12)^4}{(12)(23) \ldots (n1)} \]  

This amplitude is the first analytic formula describing the tree level process of an arbitrary number of gauge bosons in a nonabelian Yang-Mills (YM) gauge theory. The field theory amplitude is usually expressed in terms of spinor-helicity notation, which is a convenient tool in amplitudes literature and it keeps just the physical degrees of freedom in the computation. The reader can refer to the reviews for details about the methodology of field theory amplitudes.

A rigorous mathematical proof of Eq.(1) was firstly given by Berends and Giele. An even simpler proof was constructed more recently by Britto et al., using the BCFW recursion method. They shift the momentum of gluons to complex plane, then use complex contour integrals to obtain a factorization formula for the amplitudes, which proves the Parke-Taylor formula via mathematical induction. For more details of the recursion method based on the complex shift of gluon momentum, see the review for details.

The elegant structure of the Parke-Taylor formula can be extended to $N = 4$ Super Yang-
Chapter 1

Section 1.1

Mills (SYM) theory. The tree level process of all the bosons and fermions in the SYM theory, can be written in terms of a generating functional involving the Grassmann coordinates \([12]\).

For MHV amplitudes, this generating functional is an extension of the Parke-Taylor formula. It also has a simple structure when rewritten in terms of twistors, and the introduction of twistors provides a powerful tool in the analysis of amplitudes \([13]\). The method of complex shifts, like in the BCFW recursion, can be extended to SYM theory in a straightforward manner. This generating functional also has a dual superconformal symmetry \([14]\). The reader can refer to the review \([9, 15]\) for further information about loop calculations and color-kinematic duality, gauge-gravity duality and related topics.

As YM amplitudes are contained in SYM amplitudes, they are also contained in superstring amplitudes, or equivalently, superstring amplitudes contain YM amplitudes in their low energy approximation. The type I superstring theory contains gluons and the structure of its tree level MHV amplitudes for arbitrary number of gluons has been studied extensively. The zero Regge slope limit \(\alpha' \to 0\) is just the YM amplitudes, and the \(\alpha'\) expansion represents string corrections to the YM amplitudes. From these string corrections to the YM amplitudes, we can get the effective action of gauge fields, where the string corrections to the Yang-Mills action are given by higher order interaction terms. The five gluon amplitude was firstly calculated in \([16]\) and the effective action up to \(O(\alpha'^3)\) order was given in \([17]\). For six gluon amplitude, a closed expression was given by Oprisa and Stieberger \([18]\). To explicitly show the string corrections, Stieberger and Taylor \([19]\) computed the factorization of gluon amplitudes into a new form that directly exhibits the string correction to the YM amplitudes, at order \(O(\alpha'^2)\) level and up to six gluons, then they gave a conjecture of this \(O(\alpha'^2)\) string effect to arbitrary gluon numbers. Mafra, Schlotterer and Stieberger \([20, 21]\) gave the first analytic formula for an arbitrary number of gluons and to all orders of \(\alpha'\). It is obtained by using the pure spinor formalism \([22]\) and is written as a product of SYM amplitudes times kinematic terms, which clearly exhibits string corrections to SYM amplitudes. Another closed form of string amplitude was given by Stieberger and Taylor \([23]\) using a new vertex operator provided by Berkovits and
Maldacena [24]. It involves a factorization into hypergeometric integrals and kinematic terms and can be written compactly in terms of twistors. There are further studies of string amplitudes, like string loop amplitudes and amplitudes involving fermions. See the articles [25, 26].

The gauge and gravitational physics of superstring can also be studied from the nonlinear sigma model coupled with gauge and gravitational backgrounds [27, 28]. For the gravity part, the reader can refer to [28–31] for details. For the gauge field part, the $\alpha'$ expansion of the tree level string amplitude of gluons corresponds to the ordinary QFT loop expansion of this sigma model, so we can also study the string corrections to the YM action by using standard Feynman diagrams techniques. The connection between results of string amplitudes and results of the nonlinear sigma model is shown in Fig. 1.2. From ordinary Feynman diagram calculations, we can get the counter term of the gauge field of the nonlinear sigma model. Then we get the $\beta$–function of the gauge field. Conformal symmetry requires that the theory is a fixed point of renormalization group equations and this implies that the beta function of the theory should vanish $\beta = 0$. This condition gives the equations of motion of the gauge field and thus the effective action of the space–time physics. For open bosonic string, Fradkin and Tseytlin [32] considered abelian gauge field backgrounds. In the weak field approximation, the effective action was shown to be the abelian Born-Infeld action [33]. This was confirmed up to three loops by Dorn and Otto [34]. Brecher and Perry [35] studied open superstring coupled to nonabelian gauge fields. They constructed a nonabelian generalization of Born-Infeld action (NBI) up to two loops. More discussion of NBI can be found in [33, 36, 39]. For the heterotic superstring, Sen [40, 41] calculated the coupling of nonabelian gauge field at one loop, which agrees with the YM action. It is also discussed in the superspace formalism in [31, 42].

It turns out that these amplitudes usually contain Riemann zeta values and their generalization–multiple zeta values (MZV), which is related to number theory in mathematics. So there are also active studies of scattering amplitudes in the mathematics literature. For example, the amplitudes in the ordinary $\phi^4$-theory already exhibit rich mathematical structures and are studied by methods of pure mathematics. It is proven that for a subclass of Feynman diagrams in $\phi^4$-theory,
Chapter 1

Section 1.1

Spacetime effective action
\[ \mathcal{L}_{eff} = F^2 + 0 \cdot \alpha'^3 F^3 + \alpha'^2 (\zeta(2) F^4 + \ldots) + \alpha'^3 (\zeta(3) F^5 + \zeta(3) D^2 F^4 + \ldots) + \ldots \]

Figure 1.2: The equivalence between the string amplitude approach and the nonlinear sigma model approach (for the gauge physics of open superstring theory). The top line is the tree-level gluon amplitude of open superstrings, computed from gluon vertex operators. From the \( \alpha' \) expansion of the string amplitudes, we can restore the spacetime effective action (an extension of the Yang-Mills action) that generates the same tree-level gauge interactions \[17\]. Besides the string amplitudes coming from gluon vertex operators, the gauge physics in string theory can also be studied by coupling the Polyakov action with a gauge field background \( A_\mu(X) \partial_\tau X^\mu \). The field theory for \( X^\mu \) becomes a nonlinear sigma model, because \( A_\mu(X) \) is field dependent. Using the background field expansion, this model can be studied as an interacting 2d QFT. Classically this nonlinear sigma model has conformal symmetry. Quantum corrections lead to conformal anomaly that breaks the conformal symmetry. The conformal symmetry is preserved if the beta function is set to zero. The zero beta function constraint gives e. o. m. for the gauge field, from which we can recover the spacetime effective action, which is the same one obtained from string amplitudes. (In the nonlinear sigma model computation, the UV cutoff is implemented via \( a \), which is an infinitesimal distance on the world-sheet.)
amplitudes can be expressed in terms of MZV. These mathematical studies are usually done by using methods in arithmetic algebraic geometry. See the review work [43] for references and detailed mathematical techniques.

### 1.2 Methodology of string amplitudes

For string theory, there are no interaction terms in the action of its partition function. The QFT-like interaction process manifests itself in the string splitting and rejoining process, which is determined by the topology of the string world-sheet. This world-sheet can be transformed into a Riemann surface using the diffeomorphism and the Weyl symmetry, while the external string states are represented as vertex operators at punctures of this Riemann surface. For example, Figure 1.3 shows a four-point tree-level scattering process of closed string. Two closed strings join together and then split. The world-sheet can be transformed into a Riemann sphere with four punctures using the diffeomorphism and the Weyl symmetry. The initial and final states are represented by vertex operators at these four punctures. Integration of this world-sheet contains all the possible QFT-like interactions among the four external states. The vertex operators and the Riemann surface are described by complex variable coordinates. The topology of the Riemann surface is parameterized by using moduli, which are also complex variables. So the path integral is an integral over complex functions.

Locally, the string itself has a conformal symmetry, which is a residual symmetry after fixing the diffeomorphism and the Weyl symmetry. In Euclidean coordinates of the world-sheet \((\sigma^1, \sigma^2) = (\sigma, i \tau)\), the partition function of a free string, or the Polyakov path integral is

$$
Z_{Dg_{\alpha\beta}} = \frac{1}{d! \times W_{\text{Weyl}}} e^{-S_p(g_{\alpha\beta}, X)},
$$

where \(g_{\alpha\beta}\) is the world-sheet metric and \(X\) is the space-time embedding of the string world-sheet. We can always use the diffeomorphism and the Weyl symmetry to fix \(g_{\alpha\beta}\) to a specific
Figure 1.3: Tree-level four point scattering process of closed strings. Two closed string of specific initial states join together and then split into two closed string of specific final states. This process can be expressed as a correlation function in the path integral formalism and it includes all the possible interactions that contribute to the scattering of the initial states into the final states. This quantity however is generally not covariant, because the initial and the final states depend on specific spacetime positions $X^\mu(\sigma_j), j = 1, 2, 3, 4$. To get a covariant quantity, we can push the initial and the final states to spacetime infinity, then they are invariant under diffeomorphism. Now it becomes the string S-matrix from a general correlation function. The process of pushing external states to infinity is to put them on-shell. This is like what happens in the LSZ formula of QFT, where the external particles are pushed into infinity to put them on-shell. Using the diffeomorphism and the Weyl symmetry, the world-sheet is transformed into a Riemann sphere with four punctures (the crossed dot). The initial and the final states are represented by vertex operators at the four punctures.
gauge $\hat{g}_{\alpha\beta}$, for example, a conformal gauge choice $\hat{g}_{\alpha\beta} = e^{\lambda} \delta_{\alpha\beta}$ or a flat gauge choice $\hat{g}_{\alpha\beta} = \delta_{\alpha\beta}$. Intuitively, we would have

$$\int \frac{D\hat{g}_{\alpha\beta}}{V_{\text{diff}} \times V_{\text{Weyl}}} = \int \frac{D\hat{g}_{\alpha\beta} D\lambda}{V_{\text{diff}} \times V_{\text{Weyl}}} = \int \frac{D\hat{g}_{\alpha\beta}}{V_{\text{diff}}},$$

(1.3)

where $D_{\hat{g}_{\alpha\beta}} \lambda = D\delta_{zz} D\delta_{\bar{z}\bar{z}}$ is the direction perpendicular to the gauge orbit of the conformal transformation. Even though we fix the gauge $\hat{g}_{\alpha\beta}$ to, for example, the flat gauge $\delta_{\alpha\beta}$, there is still a remaining conformal symmetry, which is a combination of the diffeomorphism and the Weyl transformation that leaves the metric flat. The conformal transformation is given infinitesimally by

$$z \rightarrow z + \xi^z(z), \quad \bar{z} \rightarrow \bar{z} + \bar{\xi}^\bar{z}(\bar{z}), \quad \text{where } e^z = \sigma^1 + i\sigma^2. \quad (1.4)$$

The 'volume' of this conformal transformation is zero, so this does not affect that fact that the redundancy of $V_{\text{Weyl}}$ has been fixed. Under the flat gauge choice, it is easy to show that

$$D\delta_{zz} D\delta_{\bar{z}\bar{z}} = D\partial_z\xi^z(z) D\partial_{\bar{z}}\bar{\xi}^{\bar{z}}(\bar{z}),$$

where $\xi$ is now the coordinate transformation

$$z \rightarrow z + \xi^z(z), \quad \bar{z} \rightarrow \bar{z} + \bar{\xi}^{\bar{z}}(\bar{z}),$$

that is perpendicular to the orbit of conformal transformation. So now the integration measure after the gauge fixing becomes

$$\int \frac{D\partial_z\xi^z(z) D\partial_{\bar{z}}\bar{\xi}^{\bar{z}}(\bar{z})}{V_{\text{diff}}} = \int \frac{D\xi^z(z) D\bar{\xi}^{\bar{z}}(\bar{z})}{V_{\text{diff}}} \det \partial_z \det \partial_{\bar{z}} = \det \partial_z \det \partial_{\bar{z}}.$$
The determinant can be written in terms of $b, c$ ghosts as following

\[
\text{Det } \partial_z = \int Dc^z Db_{zz} e^{\frac{1}{\pi} \int dz^2 b_{zz} \partial_z c^z} \\
\text{Det } \partial_{\bar{z}} = \int Dc^{\bar{z}} Db_{\bar{z}\bar{z}} e^{\frac{1}{\pi} \int d\bar{z}^2 b_{\bar{z}\bar{z}} \partial_{\bar{z}} c^{\bar{z}}}. \quad (1.5)
\]

The above analysis can be formulated in a rigorous Faddeev-Popov procedure [47, Chapter 3]. The $(b, c)$ ghosts happen to be the Faddeev-Popov determinant and its role is to covariantly fix the symmetry of the string world–sheet, which is similar to the ghosts in the nonabelian field theory.

Here we are only using the bosonic string $X(\sigma)$ as the example. For complete superstrings, there are a fermionic string $\psi(\sigma)$ and a world–sheet gravitino $\chi$, which are super-partners of the bosonic string $X(\sigma)$ and of the world–sheet metric $g_{\alpha\beta}$ respectively. When gauge fixing the world–sheet gravitino $\chi$, there are also contributions to the Faddeev-Popov determinant, which finally leads to $(\beta, \gamma)$ ghosts in addition to the $(b, c)$ ghosts.

Globally, the string amplitudes is the partition function with vertex operator insertions, or the string splitting–rejoining world–sheet with specific topology. The amplitudes read as [47, Chapter 5]

\[
\sum_{\text{topology}} \int \frac{Dg_{\alpha\beta} DX}{V_{\text{diff}} \times V_{\text{Weyl}}} e^{-S_p(g_{\alpha\beta}, X)} \prod_{i=1}^{n} \int d^2 \sigma_i \sqrt{g} V_i(p_i, \sigma_i), \quad (1.6)
\]

where there are $n$ vertex operators and $p_i$ is the momentum of the states represented by the vertex operator. Now we have to consider the constraints coming from the topology of the Riemann surface, which generates global obstructions to the local conformal symmetry of free strings. The symmetry of the path integral can not be completely fixed by the previous gauge-fixing procedure. There are remnant global degrees of freedom (d. o. f), which need to be incorporated into the integral.

One kind of the global d. o. f are the (metric) moduli, which are complex parameters determined by the topological genus of the Riemann surface. The moduli can be viewed as the
space of world–sheet metrics for a given topology, modded out by diffeomorphisms and Weyl transformations. Since they are complex numbers, this is a continuous symmetry of the path integral. The parameter space of the moduli is not completely independent. The moduli itself has one global symmetry represented by a finite group and the parameter space splits into regions of equivalent classes under this group action. Usually one of these equivalent classes is chosen as the fundamental region \( F \), which contains all the independent moduli. The moduli is only integrated over the fundamental region \( \int_F \ldots \) to avoid overcounting.

The other set of global d. o. f parametrize the conformal killing transformations and generate the Conformal Killing Group (CKG), which is the residual symmetry of the local conformal symmetry. The CKG can be captured by a finite-dimensional group acting on the world–sheet, which is a subgroup of the infinite-dimensional conformal group. Again only the equivalence class under this CKG need to be considered. The gauge fixing of this CKG turns out to be equivalent to fix some of the positions of the vertex operators with the insertion of certain ghost contributions. So for each group generator of CKG, we fix one vertex operator to a specific position with a ghost insertion there. For discrete symmetries like \( \mathbb{Z}_2 \), they lead to finite overcounting by an integer \( n_R \). We usually use a factor \( 1/n_R \) for these discrete symmetries, instead of using them to constrain the position of vertex operators.

Based on these global symmetries, the Faddeev-Popov gauge fixing process of the string amplitudes leads to

\[
\int \frac{Dg_{\alpha\beta} d^{2n}\sigma}{V_{\text{diff}} \times V_{\text{Weyl}}} = \int \frac{d\mu t}{F} \int d^{2n-\kappa}\sigma \Delta_{FP}(\hat{g}, \hat{\sigma}),
\]

where \( F \) is the fundamental region and \( \kappa \) is the number of CKG that is used to fix the vertex operators. The Faddeev-Popov determinant \( \Delta_{FP}(\hat{g}, \hat{\sigma}) \) can be written in terms of a ghost path integral as usual, but with additional ghost mode insertions originating from the global symmetries

\[
\Delta_{FP}(\hat{g}, \hat{\sigma}) = \int [DbDc] e^{-S_{\phi}[b,c]} \prod_{j=1}^{\mu} \left( b_j \partial_j \hat{g} \right) \prod_{i=1}^{\kappa} c(\hat{\sigma}_i).
\]
The insertion of $b$ ghosts, $(b, \partial_j \hat{g})$, is the zero mode of the $b$ ghosts on the Riemann surface and it comes from the change of the moduli. This zero mode insertion of $b$ ghosts makes the integral measure of moduli covariant in its parameter space. The insertion of $c$ ghosts, $c(\hat{\sigma}_i)$, is the zero mode of the $c$ ghosts on the Riemann surface and it comes from fixing the positions of the vertex operators.

Now this whole process of string splitting and rejoining can be written as a Polyakov path integral that contains an integral over the moduli space and has vertex operators inserted as following

$$
\sum_{\text{topology}} \left[ \frac{d^h t}{N_R} \right] \int Db Dc DX \, e^{-S_P(\hat{g}_\alpha, X) - S_\hat{g}} \times \left( \prod_{j=1}^{n} \frac{(b, \partial_j \hat{g})}{4\pi} \right) \left( \prod_{(i,a)\in \text{fixed}} c(\hat{\sigma}_i) \right) \int \left( \prod_{(i,a)\notin \text{fixed}} d\sigma_i^a \right) \prod_{i=1}^{n} \sqrt{\hat{g}} V_i(p_i, \sigma_i). \quad (1.8)
$$

Some of these vertex operators are fixed by CKG while others need to be integrated over all the Riemann surface. The moduli is integrated only over its fundamental region. This analytic expression is termed the string S-matrix, and its calculation is the central task of string amplitudes.

To calculate the string amplitudes, the string action and the vertex operators need to be written out explicitly. There are several formalisms to do this, the RNS formalism, the GS formalism and the pure spinor formalism. We are only considering the tree-level superstring amplitudes here, so we choose the RNS formalism, which is easier for our purpose.

For the tree-level superstring amplitudes, the world–sheet is a punctured Riemann sphere for the closed string, or disk for the open string. There is no moduli for both cases, so we only need to consider the CKG to fix the corresponding vertex operators. After the string action and the vertex operators are expressed in the RNS formalism, all the elements in the string amplitudes become complex valued functions, which means we need to compute a complicated complex...
integral as following

\[
\int D\theta D\bar{\theta} D\phi D\bar{\phi} e^{-S_{\phi}(\delta_{ab},X)-S_{\bar{\phi}}(\delta_{ab}^2,\bar{X})} \int (\prod_{(i,a)\in \text{fixed}} c(\delta_i^a)) \left( \prod_{(i,a)\not\in \text{fixed}} d\sigma_i^a \right) \prod_{i=1}^{n} \sqrt{g} V_i(p_i,\sigma_i) \\
= \int \left( \prod_{i=1}^{n} d\sigma_i^2 \sqrt{g} \right) \left( \prod_{(i,a)\in \text{fixed}} \delta(\sigma_i^a - \hat{\sigma}_i^a) \right) \left( \prod_{(i,a)\in \text{fixed}} c(\hat{\sigma}_i^a) \right) \left( \prod_{i=1}^{n} V_i(p_i,\sigma_i) \right). \tag{1.9}
\]

Firstly the path integral is evaluated into a conformal correlator \( \langle \ldots \rangle \), then the amplitude is an ordinary complex integral over this correlator. The conformal correlator is easier to compute than quantum field theory correlators, because there is not interaction terms (no loops).

To calculate the correlator, we need an explicit form of the vertex operator. Vertex operators are constructed using appropriate combinations of the string and various ghosts. There is a complexity in the representation of vertex operators, coming from the ghost system and the global symmetries of the world–sheet. The free conformal field theory (CFT) of the \((b,c)\) ghosts (and of the \((\beta,\gamma)\) ghosts) has a ghost number symmetry with the Noether current

\[
j = -bc, \quad \text{or} \quad -\beta\gamma, \tag{1.10}
\]

whose conserved charge is called the ghost number or ghost charge. After the ghosts are combined with the superstrings, there is an anomaly of the ghost number current, characterized by background ghost charge \( Q \) \([46]\). For the spectrum of the superstrings, the BRST quantization requires the physical states having a total ghost charge of zero, so the ghost number current is conserved again. For the \((b,c)\) ghosts, the background ghost charge is canceled by the zero mode insertions of the \(c\) ghost in eq. (1.8). So the Faddeev-Popov determinant of path integral quantization and the zero total ghost charge of BRST quantization, give the same result of zero mode insertions of the \(c\) ghost. Actually this is one example of the operator–state correspondence of CFT, which says that the operator insertions into the partition function correspond to Fock space states of the superstrings. For the \((\beta,\gamma)\) ghosts, there is degeneracy in the states of superstrings, so there are equivalent representations of the vertex operators for each degen-
erate state. These representations of vertex operators are called ghost–pictures. It is the ghost charge \( q \) of the \((\beta, \gamma)\) ghosts that characterizes the ghost–picture of vertex operators, where each ghost–picture carry a unique value of \( q \). We have to choose the ghost–pictures of vertex operators, such that they cancel the background ghost charge \( Q \) of the \((\beta, \gamma)\) ghosts in the scattering amplitudes.

Now we can calculate the conformal correlator explicitly. The basic element is the string propagator, or the two-string-correlator, which is straightforward to get from the Polyakov action. In CFT, we use the method of Operator Product Expansion (OPE) between vertex operators, where the string propagator is just the OPE between two strings. Computing the correlator is the same things as computing the OPE of all the vertex operators contained in the correlator, which can be decomposed into a product of OPE between just two vertex oerators, like the Wick contraction in QFT. After all the possible Wick contractions, the correlator becomes a sum of meromorphic functions multiplied by the Koba-Nielsen factor. The Koba-Nielsen factor is a product of terms \( |z_{ij}|^{k_i k_j} \). The numerator of the meromorphic function contains the polarization \( \xi \) and the momentum \( k \) of the vertex operators, and the denominator contains products of differences \( z_{ij} = z_i - z_j \) between vertex operator positions \( z_i \). The string amplitude is just the multiple variable integral over this integrand and it evaluates into multiple Gaussian hypergeometric functions, which are related to MZV. So the string amplitudes, which describe physical processes, are also connected with number theory \([48]\).

Now the task is just to perform these multiple complex integrals and obtain closed expressions for the multiple hypergeometric functions. The string amplitude contains many hypergeometric functions, with each of them multiplied by its own polarization factors. It turns out that not all of these hypergeometric functions are independent. Actually, they can be reduced to a basis of just fewer hypergeometric functions, and all the others can be expressed as linear combinations of these basis hypergeometric functions, with the numerical factors being kinematic factors, i.e., momentum-dependent factors. Such simplifications are related to the kinematic-color duality.
For the multiple hypergeometric functions, it is extremely difficult to obtain closed expressions, so what mathematicians usually do is to expand them into power series over some parameters. In our case, they are expanded into series in terms of the kinematic factors, which contains the \( \alpha' \) multiplied by some momentum invariant (Mandelstahm kinematic variable). Combined with the polarization factors, this power series expansion generates a Taylor series in terms of \( \alpha' \), with the corresponding gauge-invariant factors containing kinematics and polarization. These \( \alpha' \) expansions are the string corrections to the YM action, and from these factors at each order of \( \alpha' \), we can construct the effective action.

These string corrections, or \( \alpha' \) expansions, correspond to the loop expansions of the underlying nonlinear sigma model, as explained in Figure 1.2. Since all the information of string corrections are encoded within the multiple Gaussian hypergeometric functions, it is not straightforward to see the origin of these correction terms. Instead, we can calculate these correction terms directly in the nonlinear sigma model order by order in terms of Feynman diagrams, which exhibits intuitively the correction terms. Both the string amplitude approach and the sigma model approach should yield equivalent results of string corrections to YM theory.

### 1.3 Methodology of Nonlinear sigma model Approach

When the string is coupled to backgrounds, graviton, dilaton, gauge boson, e.t.c., the action describing the string becomes a nonlinear sigma model. This nonlinear sigma model has the Weyl symmetry in the classical level. At the quantum level, however, there is a Weyl anomaly [49], which can be expressed in terms of these background fields. Consistency of the theory requires the vanishing of this anomaly, which puts constraints on these background fields: they turn out to be the classical field equations for the background fields. In this way, one can obtain the effective action of background fields at the leading \( \alpha' \) order and beyond, so the string backgrounds can be studied from the nonlinear sigma model approach.

The method of calculations in the nonlinear sigma model approach is the background field
expansion \[50\]. Originally the background field method was introduced into the nonlinear sigma model to study the gravity part of the theory \[28\]. For our purpose of gluon studies, we are only interested in the gauge part of the theory, specifically the single-trace gauge field terms. We can just put the metric to be flat and only concentrate on the gauge fields. Although there are contributions from the gauge fields to the gravity part, these contributions give double-trace and multiple-trace gauge field terms, so there is no need to worry about their back-action effect on the single-trace gauge field terms of the action. So we can neglect the gravity part and just concentrate on the gauge part without worrying about world-sheet diffeomorphisms. When we expand the string into a classical part and a quantum part, the gauge field can be expanded in terms of the quantum string part, with factors being derivatives of the gauge fields at each order of the expansion. These derivatives of gauge fields are evaluated at the classical string part, so they do not contribute to the quantum process, which makes it possible to treat them just as ’coupling constants’ rather than fields. These greatly simplifies the calculation. Actually this provides a way to do the calculation as the standard QFT loop expansions, with the propagators given by the quantum string part and the coupling constants given by some derivatives of backgrounds fields of the classical string part.

For the study of the open superstring using this approach, we need the Wilson loop expansion. The gauge fields is attached at the boundary of the open string, which is a one dimensional contour, via the Chan-Paton factors. This approach is described by the Wilson loop \[51\]. We need to be careful when studying this gauge coupling, due to its one dimensional nature, which leads into commutator terms in the expansion for nonabelian gauge fields. We have to do more mathematical manipulations to expand the Wilson loop in terms of their quantum string parts \[52–54\]. On the other hand, when doing the loop diagram calculations, we still need to deal with the path-ordered integral. The open superstring has been calculated up to two-loop level for the nonabelian case \[35\] and it agrees with string amplitudes.

The heterotic superstring also contains gauge field couplings. This comes from the interplay between the discretized momentum and the winding number, after compactifying 16 of the
26 left-moving bosonic strings. The resulting gauge field is nonabelian. For the purpose of calculations using path integral, these nonabelian gauge fields are coupled to 32 right-handed, left-moving Majorana-Weyl world-sheet fermions, which is the fermionic representation of the gauge physics.

The open superstring amplitudes of gluons are written, after the $\alpha'$ expansion, in terms of MZV. If we replace these MZV by the single-valued multiple zeta values (SVMZV) \[55\], the corresponding result is the $\alpha'$ expansion of the gauge amplitudes of heterotic superstring \[56\]. This is the sv map between type-I superstring and heterotic superstring amplitudes. Examples of the sv map are

\[
\begin{align*}
\zeta_{sv}(2n) &= 0, \\
\zeta_{sv}(2n + 1) &= 2\zeta_{2n+1}, \\
\zeta_{sv}(3, 5) &= -10\zeta_3\zeta_5,
\end{align*}
\]

(1.11)

where the definition of $\zeta(3, 5)$ will be given in the next section. By computing three–loop and four–loop diagrams of the open string sigma model and the heterotic sigma model, we will reveal the mechanism of sv map for the $\zeta(2)$ term and the $\zeta(3)$ term.

Previously the connection between open superstring amplitudes and the closed superstring amplitudes is provided by the Kawai-Lewellen-Tye relations (KLT) \[57\], which factorizes graviton amplitudes into the product of two gauge bosons amplitudes interwinned with via matrix function of the kinematic variables. Now in sv map, the open and the closed superstring amplitudes are connected in the level of $\alpha'$ expansions, so we have a much deeper relation.
1.4 Multiple zeta values (MZV)

We have been talking about MZVs since the very beginning. Here we will briefly explain the mathematical definition of MZV. It is a generalization of the Riemann zeta values as following

\[ \zeta_{n_1, \ldots, n_r} := \zeta(n_1, \ldots, n_r) = \sum_{0 < k_1 < \ldots < k_r} \prod_{l=1}^{r} k_l^{-n_l}, \quad n_l \in \mathbb{N}^+, \quad n_r \geq 2, \]  

(1.12)

with \( r \) specifying the depth and \( w = \sum_{l=1}^{r} n_l \) denoting the weight of the MZV \( \zeta_{n_1, \ldots, n_r} \). It also has an integral definition in terms of Goncharov polylogarithms, which are iterated integrals.

The Goncharov polylogarithm is defined as

\[ L_{\sigma_0'}(z) = \int_{0}^{z} \frac{dz'}{z'-\sigma} L_{\sigma'}(z') \]

\[ L_{0m}(z) = L_{0, 0, \ldots, 0}(z) = \frac{1}{m!} (\ln z)^m \]

\[ L_{a_m}(z) = L_{a, a, \ldots, a}(z) = \frac{1}{m!} (\ln \left(1 - \frac{z}{a}\right))^m. \]  

(1.13)

Then the MZV is connected with the the Goncharov polylogarithm as following

\[ \zeta_{n_1, \ldots, n_r} = (-1)^r L_{0^{n_r-1}, 0^{n_1-1}, 1}, \]  

(1.14)

where \( 0^{n_i-1} \) means \( 0 \) repeats \( n_i - 1 \) times.

The MZV can also be viewed as special values of the multiple polylogarithms (MPL), in the same way as the Riemann zeta values are special values of the Riemann zeta function. The MPL is a generalization of the dilogarithm as following

\[ \text{Li}_{n_1, \ldots, n_r}(z_1, \ldots, z_r) := \sum_{0 < k_1 < \ldots < k_r} \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1^{n_1} \cdots k_r^{n_r}}, \quad n_l \in \mathbb{N}^+, \quad n_r \geq 2. \]  

(1.15)
The MPL and the Goncharov polylogarithm are connected as following

\[
\text{Li}_{n_1, \ldots, n_r}(z_1, \ldots, z_r) = \frac{\sigma_2}{\sigma_1}L_{n_r-1, \sigma_1}^{n_1-1,1}(z)
\]

\[
(z_1, \ldots, z_r) = \left(\frac{\sigma_2}{\sigma_1}, \frac{\sigma_3}{\sigma_2}, \ldots, \frac{z}{\sigma_r}\right)
\]

\[
\sigma_i = 2\prod_{k=i}^{r} \frac{1}{z_k}.
\]

(1.16)

The MZV is just the MPL evaluated at \((1, 1, \ldots, 1)\) as \(\zeta_{n_1, \ldots, n_r} = \text{Li}_{n_1, \ldots, n_r}(1, \ldots, 1)\).
1.5 Single–valued map

Perturbative open and closed string amplitudes seem to be rather different due to distinct underlying world–sheet topologies, with or without boundaries. Their explicit computations however, reveal some unexpected connections, in particular the Kawai–Lewellen–Tye (KLT) relations \[57\]. At the tree level, the amplitudes describing the scattering of open string states appear from a disk world–sheet, with the vertex operator insertions giving rise to real iterated integrals at the boundary. In closed string theory, the vertices are inserted and integrated on the complex sphere. The latter integrals can be expressed in terms of a sum over squares of open string amplitudes by using the KLT method \[57\]. For tree–level open string gauge amplitudes, another relation with closed string amplitudes has been found in \[58–60\]. Generally, complex world–sheet integrals can be expressed as real iterated integrals by means of the single–valued projection (sv) to be explained below. As a consequence, and in contrast to \[57\], tree–level gauge amplitudes of the heterotic string \[60\] and tree–level gravity amplitudes of closed string \[58,59\] can be expressed in terms of single–valued projections of open string gauge amplitudes. Recently, it has been argued that similar relations are expected when comparing open and closed string one–loop amplitudes \[61\].

In this dissertation, we will focus on the sv of gauge amplitudes between open string and heterotic string. In \[60\] it is demonstrated that the single trace part of the \(N\)–point tree–level heterotic superstring gauge amplitudes \(A_N^{HET}\) is given by the single–valued projection of the corresponding type–I gauge amplitude \(A_N^{I}\):

\[
A_N^{HET} = \text{sv}(A_N^{I}).
\]  

The amplitudes \(A_N^{I}\) can be expanded in the inverse string tension \(\alpha'\), yielding rational functions of the kinematic invariants multiplied by periods on \(\mathcal{M}_{0,N}(\mathbb{R})\) giving rise to multiple zeta values (MZVs). \(\mathcal{M}_{0,N}(\mathbb{R})\) describes the moduli space of disks with \(N \geq 4\) ordered marked points.
modulo the action $PSL(2, \mathbb{R})$ on those points. See Ref. [48] for more details and references therein. From eq. (1.17) it follows that the $\alpha'$–expansion of the closed string amplitude can be obtained from that of the open superstring amplitude by simply replacing MZVs by their corresponding single–valued multiple zeta values (SVMZVs) according to the rules of single-valued projection sv. From the physical point of view, SVMZVs first have appeared in the computation of graphical functions (positive functions on the punctured complex plane) for certain Feynman diagrams in the $\phi^4$ theory [62] and then rigorously defined in [55]. More explicitly, these tree-level string gauge amplitudes are expressed in terms of multiple Gaussian hypergeometric functions, which contains the parameter of $\alpha'$ [18]. The Taylor expansion of them in terms of small $\alpha'$ contains coefficients of MZV at each order. Focusing on the single-trace part of the gluon amplitudes, the sv projection on the MZV coefficients of the open superstring $\alpha'$ expansion, generates the corresponding expansion of the heterotic string amplitudes.

Since all the information is encoded in the hypergeometric functions once for all, it’s hard to see the detailed origin or mechanism behind this sv. We can go to the nonlinear sigma model approach to investigate its origin for each MZV at each order of $\alpha'$ expansion. We will compute Feynman diagrams of the nonlinear sigma model corresponding to single-trace gauge terms, where the loop number of diagrams of the sigma model corresponds to the order of $\alpha'$ expansion of string amplitudes.

The $\alpha'$–expansion of the scattering amplitudes is related to two–dimensional nonlinear sigma models, in the following way. The scattering of massless gauge bosons can be described by an effective action that contains, in addition to the Yang–Mills term, an infinite series of interactions appearing order by order in $\alpha'$. It is a low energy expansion that includes the effects of heavy string modes. This action generates effective field equations. On the other hand, dynamics of strings propagating in gauge field backgrounds are described by two–dimensional nonlinear sigma models. The world–sheet conformal invariance of strings propagating in gauge field backgrounds requires, among other things, the vanishing of the beta function associated to the coupling of background fields to the string world–sheet. This requirement leads to back-
ground field equations that should be equivalent to the equations generated by the effective action. The corresponding beta functions can be computed in sigma model perturbation theory, by using Feynman diagrams. In this context, $\alpha'$ is the sigma model coupling, with $\alpha^\ell$ appearing at the $\ell$th-loop order.

In this dissertation, we will show that the sv–map can be applied to individual sigma model Feynman diagrams with appropriately regulated ultra–violet divergences. We conjecture that the sv–map of the beta function of type–I string gives the beta–function of the heterotic string. We support this conjecture by explicit three– and four–loop computations in open and heterotic sigma models. As a corollary, the effective low–energy action describing gauge fields in heterotic superstring theory can be obtained from the respective open string action by replacing MZVs by SVMZVs.

The gauge physics of the open string sigma model can be studied in terms of the Wilson loop representation \([32, 63]\). The Wilson loop is directly gauge covariant, but is hard for the perturbation calculation because the path ordering nature of the Wilson loop is highly nontrivial \([35, 53]\). Following the path-ordered Wilson loop, the Feynman integral of gauge physics is a path-ordered integral. Since the gauge physics comes from the Chan-Paton factor on the boundary of the open string world-sheet, the integration contour is just the 1D boundary itself. Due to its 1D nature, the Wilson loop can be rewritten as a functional integral of a pair of auxiliary Grassmann fields \([64, 65]\). The Wilson line segment between two points is just the exact propagator of the Grassmann fields evaluated at those two points.

The gauge physics of the heterotic string sigma model lives on the whole 2D world-sheet, and currently can only be described by the fermionic representation. The Wilson loop representation is still missing. Although the perturbation calculation is straightforward in the fermionic representation \([40]\), the result of each Feynman diagram is not gauge covariant. The gauge covariance is hard to deal with, because by definition the model only has superconformal symmetry and does not have the gauge symmetry. A gauge covariant perturbation process was given in \([42]\), but it involves a very specific nonlocal field redefinition procedure. In this dissertation,
we will propose a method of reorganized perturbation for the fermionic representation and put
the perturbation in a gauge covariant manner. Then we will use it to perform the three–loop and four–loop computations to reveal the mechanism of sv map. For a given single-trace gauge factor, the integral in the open string sigma model is a real 1D path-ordered integral. The corresponding integral in the heterotic case is a 2D integral on the whole complex plane. Using the Gegenbauer polynomial method we can integrate out the angle direction in polar coordinates. The remaining radial part of the heterotic integral contains all possible radial orderings of the gauge vertex. We find that the sv map of the iterated open string integral equals summing over all these radial ordering of the corresponding heterotic integral [1].

The absence of a Wilson loop representation for the heterotic string sigma model is still a very bothering fact, because the open string sigma model has explicit gauge covariance due to its Wilson loop formalism but the heterotic string sigma model does not have this. So in this dissertation, we will propose a Wilson loop representation for the gauge physics of the heterotic sigma model. It turns out that the Wilson line between two points on the 2D world-sheet is also the exact propagator of the fermionic field, just like the case of the open string. So now we have a complete correspondence of descriptions of gauge physics between the open string and the heterotic string, as shown in figure 1.4.

Furthermore, after the gauge physics of both the open and the heterotic string sigma model are put under the Wilson loop representation, the sv-map between them turns out to have a very simple geometric origin: the sv-map comes from the sum of path-ordered integrals along two opposite contours [2]. For the open string case, the integration contour is just the 1D boundary itself. For the heterotic case, we will show that the integration contour is a sum of two opposite-directed contours. When we sum the Feynman integral along these two opposite-directed contours, we will obtain a result which is the sv-map of the corresponding result of the open string case. Compared with the results of the fermionic representation, where we show that the sv-map comes from a sum of all radial orderings of heterotic vertices on the complex plane, indeed this sum of two opposite-directed contours has a simpler geometric picture. Using the nonabelian
Figure 1.4: The correspondence of descriptions of gauge physics between the open string and the heterotic strings. In string theory, the gauge physics in the open string is carried by the Chan-Paton factors and the gauge physics of the heterotic string is carried by the left-movers. In nonlinear sigma model approach, the fermionic representation and the Wilson loop representation are connected by the fact that the Wilson line is the exact propagator of the fermionic field, both for the 1D boundary of the world-sheet of the open string case and for the 2D world-sheet of the heterotic string case. The blue part is the Wilson loop representation we proposed.

Stokes’s theorem, the Wilson loop along this two opposite-directed contours is just the gauge field on the whole 2D world–sheet. So the geometric origin is simpler.

This dissertation is organized as follows. In Chapter 1, we provide some basics of the development of scattering amplitudes. In Chapter 2, we set up the conventions of the nonlinear sigma model of open string. Then we briefly review the one–loop and two–loop computations of its gauge physics. In Chapter 3, we set up the conventions of the nonlinear sigma model of the heterotic string, where the gauge physics is under the fermionic representation. Then we briefly review the one–loop computation of its beta function. After that we propose the reorganized perturbation method to put everything in a gauge covariant manner. And we use it to replicate the two–loop computation of the beta function. In the end of the chapter, we briefly introduce some features of single–valued multiple zeta values and propose a general sv–map for the heterotic string, which connects the beta function of the type–I superstring with the heterotic one via a simple application of the sv–map: $\beta_h = sv(\beta_o)$. In Chapter 4, we propose the Wilson loop representation for the gauge physics of the nonlinear sigma model of the heterotic string. We show that the proposed Wilson loop representation satisfies the
correspondence shown in Figure 1.4. And we use the nonabelian Stokes’s theorem to reveal its geometric nature. In Chapter 5, we perform the three–loop and four–loop computations using the fermionic representation. We introduce the integration variables and a regularization scheme such that the sv–map applies at the level of individual Feynman diagrams. We show that the sv map comes from a sum of all the radial orderings of the heterotic vertexes on the complex plane. In Chapter 6, we perform the three–loop and four–loop computations using the Wilson loop representation. We show that the sv map comes from a sum of integrals along two opposite-directed contours. Finally in Chapter 7, we give a short summary of the results.
Chapter 2

Open string sigma model

In this chapter, we will review the nonlinear sigma model description of the open superstring. We firstly review the fermionic representation and the Wilson loop representation of its gauge physics. Then we will recall the one–loop and two–loop computation of the gauge physics.

2.1 The action

2.1.1 Wilson loop representation

For the open string, the gauge degrees of freedom is manifested via the Chan-Paton factor and only lives on the boundary \( \partial \Sigma \) of the open string world-sheet \( \Sigma \), which is just a 1-dim curve (either the unit circle or the real axis depending on the choice) parameterized by \( \tau \). The Wilson loop is directly parametrized by this 1-dim curve itself. The sigma model action describing the world-sheet \( X^\mu \) of open strings propagating in a general non-abelian gauge background \( A_\mu (X) \)
can be written in a covariant manner on the Euclidean world-sheet as

\[
S = S_\Sigma + S_{\partial \Sigma} = \frac{1}{4\pi\alpha'} \int d^2\sigma_E (\partial X^\mu \partial X_\mu + \Phi^\mu \partial \Phi_\mu + \Phi_\mu \partial \Phi^\mu) \\
S_{\partial \Sigma} = \ln \text{Tr} \mathcal{P} \exp \left\{ i \oint d\tau (A_\mu(X) \partial_\tau X^\mu - \frac{1}{2} \Phi^\mu \Phi_\nu F_{\mu\nu}) \right\} = \ln W[A],
\]

(2.1)

where \( \phi = \Phi|_\Sigma = \bar{\Phi}|_\Sigma \) is the fermionic string on the boundary and the open superstring part \( S_\Sigma \) follows from [47, Chapter 12.3] and the Chan-Paton part \( S_{\partial \Sigma} \) is defined in [66].

The Wilson loop is \( W[A] = \text{Tr} V[X, \phi, \tau, \tau] \) where \( V[X, \phi, \tau_2, \tau_1] \) is the Wilson line defined as

\[
V[X, \phi, \tau_2, \tau_1] := \mathcal{P} e^{i \int_{\tau_1}^{\tau_2} d\tau (A_\mu(X) \partial_\tau X^\mu - \frac{1}{2} \Phi^\mu \Phi_\nu F_{\mu\nu})}
\]

(2.2)

For convenience, we will omit the path ordering symbol \( \mathcal{P} \) from now on. Whenever we deal with the Wilson loop, the path ordering is always implicitly there.

### 2.1.2 Fermionic representation

The path ordering nature of the Wilson loop can be written in terms of the Heaviside step function. Since this is a 1D problem, the Heaviside step function can be viewed as the free propagator of a pair of fermionic coordinates [63, 67]. Then the Wilson loop of the open string can be viewed as coming from the following ordinary action

\[
S_{\partial \Sigma} = \int d\tau \bar{\psi} (i \left( \frac{d}{d\tau} - i(A_\mu(X) \partial_\tau X^\mu - \frac{1}{2} \Phi^\mu \Phi_\nu F_{\mu\nu}) \right) \psi (\tau))
\]

(2.3)
where the pair of fermion coordinates \( \psi(\tau), \bar{\psi}(\tau) \) live on the boundary \([63]\). The Wilson line \( V[A] \) is just the exact propagator of this fermion coordinate \([64]\)

\[
V[X, \phi, \tau_2, \tau_1] = \langle \psi(\tau_2) \bar{\psi}(\tau_1) \rangle. \tag{2.4}
\]

## 2.2 The two–loop beta function

The main focus of this study is the ultraviolet singularities due to quantum fluctuations of string coordinates around the classical background. In background field expansion, the bosonic coordinates are expanded around the classical background as \( X \rightarrow X + \xi \) and the fermionic coordinate \( \phi \) itself is treated as a quantum field (expanded around zero). Note that for simplicity, we still use the notation \( X \) rather than \( \tilde{X} \) as the classical field. It is clear from now on that \( X \) is the classical field and \( \xi, \phi \) are the quantum fluctuation.

The functional variation \([53]\) of the bosonic part of the Wilson loop under \( X \rightarrow X + \delta X \) is

\[
V[X + \delta X, \tau_0, \tau_0] = e^{i \int d\tau [A_\mu(X+\delta X)\partial_\tau(X+\delta X)^\mu - V[X]] A_\mu(X(\tau_0)) \delta X^\mu(\tau_0)}
+ i \int d\tau V[X, \tau_0, \tau] F_{\mu\nu} \partial_\tau X^\nu(\tau) \delta X^\mu(\tau) V[X, \tau, \tau_0]. \tag{2.5}
\]

Combined with the fermionic part this gives \([55]\) the action under background field expansion

\[
S_{\delta \Sigma} = i \int d\tau \text{Tr} V[X \{ A_\mu(X) \partial_\tau X^\mu + \partial X^\nu [F_{\mu_1\nu}\bar{\xi}^{\mu_1} + \sum_{n=2} D_{\mu_n} \cdots D_{\mu_2} F_{\mu_1\nu}\bar{\xi}^{\mu_1} \cdots \bar{\xi}^{\mu_n}] + \frac{1}{2} F_{\mu_1\mu_2} \bar{\xi}^{\mu_1} \partial_\tau \bar{\xi}^{\mu_2} + \sum_{n=3} \frac{n-1}{n!} D_{\mu_{n-1}} \cdots D_{\mu_2} F_{\mu_1\mu_n} \bar{\xi}^{\mu_1} \cdots \bar{\xi}^{\mu_{n-1}} \partial_\tau \bar{\xi}^{\mu_n}}
- \frac{1}{2} [F_{\nu_1\nu_2} \phi^{\nu_1} \phi^{\nu_2} + \sum_{n=1} \frac{1}{n!} D_{\mu_n} \cdots D_{\mu_1} F_{\nu_1\nu_2}(X) \phi^{\nu_1} \phi^{\nu_2} \bar{\xi}^{\mu_1} \cdots \bar{\xi}^{\mu_n}] \}, \tag{2.6}
\]

where \( \xi \) and \( \phi \) are treated as quantum fields.
The loop expansion leads to background-dependent ultraviolet divergences originating from the boundary couplings. Their treatment, in particular the renormalization procedure, are rather complicated. We focus on quantum corrections to the boundary gauge coupling $A_\mu(X)\partial_\tau X^\mu$. It is known that the requirement of the vanishing of the associated field-dependent beta function, order by order in the string tension $\alpha'$, is equivalent to background field equations [28,68]. In ambient space–time, the corresponding $\alpha'$–dependent corrections to Yang–Mills action are due to heavy string modes, integrated out at low energies.

In sigma model perturbation theory, the basic quantity is the free propagator of bosonic fluctuations $\xi$:

$$ G = -\frac{\alpha'}{2} \ln |z - z'|^2 - \frac{\alpha'}{2} \ln |z - z'|^2 z = z' = \tau = \alpha' \ln |\tau - \tau'|^2, \quad (2.7) $$

where in the last equation we choose the real axis to be the boundary of the open string. In Feynman diagrams, this propagator will be represented by a wavy line. The dashed line represents the propagator of the fermionic string coordinate $\phi$. Among Feynman rules, it is also convenient to include the propagator marked by a slash, with the mark representing the derivative of $G(z,z')$ with respect $z$ or $z'$, whichever point is closer to the mark. There is an infinite set of gauge field-dependent interactions represented by Feynman vertices.

At one–loop, the coupling under consideration receives loop corrections shown in Figure 2.1. Diagram (a) contains a short-distance singularity $\lim_{\tau \to \tau'} G(\tau - \tau')$. It can be regulated by introducing a cutoff $|\tau - \tau'| > \epsilon$, so that $G(0) \to -2\alpha' \ln \epsilon$. Diagram (b) contains $\lim_{\tau \to \tau'} \partial_\tau G(\tau - \tau')$ which vanishes if one takes a symmetric combinations of the limits $\tau - \tau' = \pm \epsilon$. As a result, in the notation of [69,70]:

$$ \text{Figure 2.1} = i\alpha' \ln \epsilon \int Tr V[X] D^{\mu} F_{\mu\rho} \partial_\tau X^\rho. \quad (2.8) $$

At the one–loop level, fermion loops do not contribute because of symmetric limit mentioned
above. The one–loop divergence can be removed by redefining the background field \( A_\rho \rightarrow A_\rho + \delta A_\rho \), with

\[
\delta A_\rho^{(1)} = \alpha' D^\mu F_{\mu\rho} \ln \epsilon + \mathcal{O}(\alpha'^2),
\]

which cancels the logarithmic divergence.

The beta function associated to the boundary coupling \( A_\mu(X)\partial_\tau X^\mu \) is defined as

\[
\beta_\rho = \frac{\partial}{\partial \ln \epsilon} \delta A_\rho = \alpha' D^\mu F_{\mu\rho} + \mathcal{O}(\alpha^2).
\]

At the leading order, zero beta function requires the background to satisfy Yang-Mills field equations.

A similar analysis can be repeated at the two–loop level, with the bosonic Feynman diagrams shown in Figure 2.2 and 2.3, see [69]. There are also diagrams involving fermion loops shown in Figure 2.4, see [70]. We refer to [69, 70] for details. Note that in [70] the background field is constrained to be on–shell, such that the linear terms \( \mathcal{O}(\xi) \) are neglected. We choose to show these diagrams because we will later make a connection to similar diagrams in the heterotic case. Note that in the open string case, the solid line is the boundary, while in the heterotic case the solid line will represent a propagating fermion field carrying the gauge field.
Figure 2.2: Two–loop diagrams with bosonic loops

Figure 2.3: One–loop diagrams with one–loop counterterm vertices which contribute at two–loops

Figure 2.4: One and two–loop diagrams with fermion loops. Dashed lines are fermionic propagators. (a) and (b) are identically zero.
From the bosonic diagrams of Figure 2.2 and Figure 2.3, one obtains

\[ \beta_{\rho}^{\text{bos}} = \frac{\partial}{\partial (\log \epsilon)} \delta A_{\rho} = \alpha' D^\mu F_{\mu\rho} + i (\alpha')^2 [D_{\rho} F^{\mu\nu}, F_{\mu\nu}] + O(\alpha'^3) . \]  

(2.11)

After including fermion loops of Figure 2.4, one finds that they cancel the bosonic contributions, leaving no ultraviolet divergences at the two–loop level:

\[ \beta_{\rho}^{\text{open}} = \frac{\partial}{\partial (\log \epsilon)} \delta A_{\rho} = \alpha' D^\mu F_{\mu\rho} + O(\alpha'^3) \]  

(2.12)

Corrections of order \( O(\alpha'^3) \) correspond to three–loop effects and are expected to be non-vanishing.
Chapter 3

Heterotic string sigma model: fermionic representation

In this chapter, we will firstly review the fermionic representation of the gauge physics of the heterotic string sigma model. Then we will recall the one–loop computation of the gauge physics of heterotic string sigma model. After that, we will use a reorganized perturbation method to put the background field expansion in a gauge covariant manner and use it to replicate the two–loop computation of the gauge physics.

3.1 The fermionic representation

In this section, we will study the gauge physics of the heterotic string sigma model. For the heterotic string \([71]\), the gauge physics are generated by the 16 left-movers. By analog of the bosonization of two fermion fields, the gauge physics can be described by 32 real, anticommuting, left-moving, right-handed coordinates \(\psi^j\) with gauge indices \(j\) which transform under the fundamental representation of \(SO(32)\). These coordinates are Majorana-Weyl spinors on the world-sheet and we will just call them fermion fields for simplicity. This is the fermionic
representation of the heterotic sigma model and the action is given as [47, Chapter 12.3]

\[
S_E = \frac{1}{2\pi\alpha'} \int d^2z \{ \partial X^\mu \partial X_\mu + \phi^\mu \partial \phi_\mu + \psi^i \partial \psi^i - i\psi \left( \partial X^\nu A_\nu - \frac{1}{2} F_{\nu \nu'} \phi^\nu \phi^{\nu'} \right) \psi \}, \tag{3.1}
\]

where the equal time contour on the complex plane is the circle and \( \phi^\mu \) is the super partner of \( X^\mu \). This corresponds to the fermionic representation of the open string sigma model eq. (2.3), except that the gauge terms here is a 2D integral while in the open string case it is a 1D integral. Note that the action in [47, Chapter 12.3] has a different form compared to the action given in [72] due to a difference of constant in the definition of the fermion field and the fermionic string: A redefinition of the action in the latter \( \phi \to \sqrt{2} i \phi, \psi \to \sqrt{-2} i \psi \) maps the latter action to the former. The classical field equation are

\[
\bar{\partial} \phi^\mu = \frac{i}{2} \psi F_{\mu \nu} \phi^\nu \psi \quad 2\bar{\partial} X^\mu = -i\psi \left( F_{\mu \nu} \partial X^\nu - \frac{1}{2} D_\rho F_{\mu \nu} \phi^\rho \phi^\nu \right) \psi \quad i\bar{\partial} \psi = - \left( \partial X^\nu A_\nu - \frac{1}{2} F_{\mu \nu} \phi^\mu \phi^\nu \right) \psi. \tag{3.2}
\]

The notation for the gauge fields are \( A_\mu = A_\mu^a T^a \), \( D_\mu = \partial_\mu - i [A_\mu, \cdot] \) and \( D_z = \partial_z - i [A_z, \cdot] \).

As in the open string case, we expand the bosonic field around the classical background: \( X^\mu \to X^\mu + \xi^\mu \). Fermions are treated as in ordinary perturbation theory. The action becomes

\[
S = S_0 + S_I \tag{3.3}
\]

where \( S_0 \) is the free part:

\[
S_0 = \frac{1}{2\pi\alpha'} \int d^2z \left( \partial X^\mu \partial X_\mu + \partial \xi^\mu \partial \xi_\mu + \phi^\mu \partial \phi_\mu + \psi^i \partial \psi^i \right) \tag{3.4}
\]
and $S_I$ is the interacting part:

$$
S_I = \frac{1}{2\pi\alpha'} \int d^2z \left[ -2\partial\partial X_\mu \xi^\mu - i\psi \left( \partial X^\nu A_\nu - \frac{1}{2} F_\mu\nu \phi^\mu \phi^\nu \right) \psi 
- i\psi \left( \partial X^\nu A_{\nu,\mu_1} - \frac{1}{2} F_{\mu_1\mu_2\nu} \phi^\mu \phi^\nu \right) \xi^{\mu_1} \partial \xi^\mu A_\mu 
- \frac{i}{2} \psi \left( \partial X^\nu A_{\nu,\mu_2} - \frac{1}{2} F_{\mu_2\mu_1\nu} \phi^\mu \phi^\nu \right) \xi^{\mu_2} \partial \xi^\mu + A_{\mu_1\mu_2} \partial \xi^{\mu_1} \xi^{\mu_2} + A_{\mu_2\mu_1} \partial \xi^{\mu_2} \xi^{\mu_1} \right) \psi + \cdots 
- \sum_{j=1}^n \frac{i}{n!} \psi \left( \partial X^\nu A_{\nu,\mu_1\mu_2\ldots\mu_n} - \frac{1}{2} F_{\mu_1\mu_2\ldots\mu_n\nu} \phi^\mu \phi^\nu \right) \xi^{\mu_1} \xi^{\mu_2} \ldots \xi^{\mu_n} \psi + \cdots \right] \tag{3.5}
$$

where a hat over index indicates that it is absent. Clearly, the expression given above is not invariant under background gauge transformations. In the case of closed string background of gravitational fields, one can expand the action in terms of bosonic normal coordinates to restore general coordinate invariance. In the case of the background gauge fields, this would require an equivalent “normal” expansion of the fermionic fields $\psi^i$ (for example, in [42], a specific nonlocal field redefinition is used to get a gauge covariant perturbation), but we will not follow this route.

The quantum field propagators are given by:

$$
\langle \xi^{\mu_1} (z_1) \xi^{\mu_2} (z_2) \rangle = \eta^{\mu_1\mu_2} G (z_1, z_2) \tag{3.6}
$$

$$
\langle \psi^{j_1} (z_1) \psi^{j_2} (z_2) \rangle = \delta^{j_1j_2} K (z_1, z_2) \tag{3.7}
$$

where the bosonic propagator

$$
G (z_1, z_2) = -\frac{\alpha'}{2} \ln \left( |z_1 - z_2|^2 \right) \tag{3.8}
$$
is subject to $4 \partial \bar{\partial} G (z_1, z_2) = -2 \pi \alpha' \delta^2 (z_1 - z_2)$. The fermionic propagators are

$$K (z_1, z_2) = -\partial_1 G (z_1, z_2), \quad \bar{K} (z_1, z_2) = -\partial_1 G (z_1, z_2). \quad (3.9)$$

As in the open string case all loop divergences can be regulated by introducing a UV cut-off $|z - z'| > \epsilon$. Some loop diagrams require also IR cut-offs, but this will not be important for the computation of the beta function. As mentioned above, unlike the open string case, where (2.6) is background gauge invariant, the heterotic string expansion (3.5) is not. We will compute the one–loop correction to the heterotic string to demonstrate how target space gauge invariance can be restored. In the background field method we compute the one–loop diagrams with external $\psi$-fermionic fields and the corresponding counterterms. The vertices required for the one–loop computation, as well as for our discussion of gauge invariance below, can be read from (3.5). They have the form $-i \bar{\psi} (B_{a,b}) \psi$ with the vertex functions $B_{a,b}$

$$B_{0,1} = A_\rho \partial X^\rho \quad B_{0,2} = -\frac{1}{2} F_{\nu_1 \nu_2} \phi^{\nu_1} \phi^{\nu_2}$$
$$B_{1,1} = A_\rho \partial \tilde{\xi}^\rho \quad B_{1,2} = A_\rho_1 \partial X^{\mu_1} \tilde{\xi}^{\mu_2} \quad B_{1,3} = -\frac{1}{2} F_{\nu_1 \nu_2, \mu} \phi^{\nu_1} \phi^{\nu_2}$$
$$B_{2,1} = \frac{1}{2} \left( (A_\rho_{\mu_1 \mu_2} - A_{\mu_1, \rho \mu_2}) + i [A_\rho, A_{\mu_1, \mu_2}] \right) \partial X^{\mu_1} \tilde{\xi}^{\mu_2}$$
$$B_{2,2} = -\frac{1}{4} \left( F_{\nu_1 \nu_2, \mu_1 \mu_2} + i [F_{\nu_1 \nu_2}, A_{\mu_1, \mu_2}] \right) \phi^{\nu_1} \phi^{\nu_2} \tilde{\xi}^{\mu_1} \tilde{\xi}^{\mu_2}$$
$$B_{2,3} = \frac{1}{2} (A_{\mu_1, \mu_2} - A_{\mu_2, \mu_1}) \partial_\xi \xi^{\mu_1} \tilde{\xi}^{\mu_2} \quad (3.10)$$

where the first index in each expression above keeps track of the number of bosonic fields $\xi^{\mu}$ of the vertex. The corresponding vertices appear in Figure 3.1. These vertices appear inside one–loop diagrams shown in Figure 3.2.

For the computation of the diagrams, we use partial integrations to move as many derivatives as possible from the bosonic propagators. When the derivative is moved to the $\psi$-fermionic
Figure 3.1: Solid lines correspond to $\psi$ fermions. Dashed lines with greek letter indices correspond to the $\phi^\mu$ fermions.

Figure 3.2: One–loop contributions to the heterotic beta function
propagators, it simplifies following the identity

\[ \partial_1 K(z_1, z_2) = -\partial_2 K(z_1, z_2) = \pi \alpha' \delta^{(2)}(z_1, z_2). \] (3.11)

Or when moved to the external \( \psi \)-fermionic fields, it simplifies using the equations of motion (3.2). Note that using the equations of motion on the external \( \psi \) fields is equivalent to attaching the vertex \( B_{0,1} \) and \( B_{0,2} \) to the Feynman diagrams, which makes it one-particle-reducible. This fact is due to the renormalizability of 2d nonlinear sigma models as discussed in [73]. Usually for background field expansion method, the only thing that matters is the renormalization of the coupling constants, and there is no need to consider the renormalization of the quantum field [50]. For the 2d nonlinear sigma model, however, the renormalization of the quantum field [73] is very important and is necessary for the renormalizability of the model. This is due to the nonlinearity of the ‘coupling constant’, \( A_{\mu}(X) \) for gauge physics and \( G_{\mu\nu}(X) \) for gravity, which are actually functional of the field \( X^\mu \). The nonlinear renormalization of the quantum field \( \psi \) can be incorporated by nonlinear field redefinition of \( \psi \), which is the reason behind this one-particle-reducible issue. This also explains the reason why the nonlinear field redefinition of \( \psi \) in [42] works.

Actually, diagram (b) vanishes due to the antisymmetry of the \( B_{2,3} \) vertex function. Only the remaining five diagrams contribute. For the purpose of making contact with the open string calculation, we use world-sheet representation for the propagators (unlike the momentum space representation of [72]). We also use similar regularization for the heterotic string propagators as we did for the open string. The result is

\[
\text{Figure 6} = -\frac{1}{2\pi \alpha'} G(0) \int d^2z (-i) \psi \partial X^\rho \left( \frac{1}{2} \partial^\mu F_{\mu \rho} - i \frac{1}{2} [A_{\rho, \mu}, A^\mu] + i [A_{\rho, \mu}, A^\mu] \right) \\
- \frac{i}{2} [A_{\mu, \rho}, A^\mu] - \frac{i}{2} (A_{\rho} A_{\mu} A^\mu + A_{\mu} A_{\mu} A_{\rho}) + A_{\mu} A_{\rho} A^\mu \right) \psi \\
= -\frac{1}{2\pi \alpha'} G(0) \int d^2z (-i) \psi \frac{1}{2} (D^\mu F_{\mu \rho}) \partial X^\rho \psi. \]

(3.12)
This expression gives the same beta function as in the open string case \((2.10)\).

### 3.2 Reorganized perturbation method

One could proceed to higher loops in a similar fashion however, since the interaction vertices are not gauge covariant, the standard procedure is rather cumbersome. In \([42]\), the background field expansion is made gauge covariant using superspace formalism combined with a very special nonlinear redefinition of the fermionic field

\[
\lambda^a \rightarrow \tilde{P}e^{-iy^a} \int_0^s ds' A_\mu(X(s')) = \tilde{\lambda}^a + s\Psi^a
\]

\[
X^\mu \rightarrow \tilde{X}^\mu + sy^\mu,
\]

(3.13)

where \(s\) parameterizes the geodesic on the world--sheet, \(X^\mu\) is the superfield version of the bosonic string and the fermionic string \(\phi\), and \(\lambda^a\) is the superfield version of the fermion field \(\psi\) that contains the gauge physics. Under this nonlinear field redefinition, the ordinary derivative on gauge fields becomes covariant derivative. We could use the nonlinear renormalization of the quantum field explained in \([73]\) to interpret the effectiveness of this special nonlinear field redefinition.

In our computation, we will not adopt the above superspace method of \([42]\). Because we want to show the sv–map between the open and the heterotic sigma model on the level of Feynman diagrams, we need to write the background field expansion of the heterotic case in terms of a style that corresponds to the open case eq. \((2.6)\). The nonlinear superfield redefinition of eq.\((3.13)\) spoils this description.

We will proceed in a slightly different method where we manage to write perturbation theory in a gauge invariant manner. In order to compute the part of the beta function involving a single chain of gauge indices (single trace terms), we need correlators with two external \(\psi\) fermions, without \(\bar{\psi}\) loops that would contribute additional trace factors. In order to discuss the loops of \(\tilde{\xi}^\mu\)
and/or $\phi^\mu$ fields, it is convenient to start with the tree-level correlators involving two external $\psi$ fields and arbitrary number of $\zeta^\mu$ and/or $\phi^\mu$ fields, see Fig. 3.3. We will show that, at this level, the correlators can be reorganised in a completely gauge invariant fashion. Consequently, we will obtain a gauge invariant effective action whose diagrammatic rules can be used to compute the loop diagrams required for the beta function.

First, consider the diagrams which involve just one $\zeta^\mu$ field. It is straightforward to demonstrate that $B_{1,1}$ and $B_{1,2}$ of (3.10) combine, after an integration by parts, in a gauge invariant form which can be reproduced by the effective action term

$$- \frac{1}{2\pi a} \int d^2z (-i) \psi F_{\mu\rho} \xi^\mu \partial X^\rho \psi. \quad (3.14)$$

Next, consider the correlators with two $\psi$’s and two $\zeta^\mu$’s. In addition to the single vertex diagrams from Figure 3.1, we have the one-particle reducible diagrams of Figure 3.4. After partial
Figure 3.4: One-particle reducible diagrams combining into effective vertices.
integrations, using Eqs. (3.11), we obtain

\[ B^{(b)}_{21} = i [A_{\rho,\mu}, A_v] \partial X^\rho \bar{\xi}^\mu \bar{\xi}^v \]
\[ B^{(c)}_{21} = -\frac{1}{2} (iA_{\mu,\rho} A_v + A_{\rho} A_{\mu} A_v + A_{\mu} A_{\rho} A_v) \partial X^\rho \bar{\xi}^\mu \bar{\xi}^v \]
\[ B^{(d)}_{21} = A_{\mu} A_{\rho} A_v \partial X^\rho \bar{\xi}^\mu \bar{\xi}^v \]
\[ B^{(b)}_{23} = i \frac{1}{2} [A_{\mu}, A_v] \partial \bar{\xi}^\mu \bar{\xi}^v. \]  

(3.15)

Now combining all of the above along with the vertices from (3.10) and similar expression which include the fermion fields \( \phi^\mu \) we obtain the following effective action

\[ S_I = -\frac{1}{2\pi \alpha'} \int d^2z (-i) \psi \left( A_{\rho} \partial X^\rho - \frac{1}{2} (F_{\mu\nu} \phi^\mu \phi^\nu) + F_{\mu_1 \rho} \bar{\xi}^{\mu_1} \partial X^\rho \right. \\
\left. - \frac{1}{2} D_{\mu_1} F_{\nu_1 \nu_2} \phi^{\nu_1} \phi^{\nu_2} \bar{\xi}^{\mu_1} + \frac{1}{2} D_{\rho} F_{\mu_1 \rho} \bar{\xi}^{\mu_1} \bar{\xi}^{\mu_2} \partial X^\rho + \frac{1}{2} F_{\mu_1 \mu_2} \bar{\xi}^{\mu_1} \partial \bar{\xi}^{\mu_2} \right) \psi. \]  

(3.16)

The Feynman diagrams constructed by using this action reproduce any correlator with two \( \psi \) fields up to order \( O(\bar{\xi}^2, \phi^2) \). Through this section we consider only terms linear in \( \partial X^\rho \). Divergent higher order terms in \( \partial X^\rho \) will be cancelled by exponentiation of diagrams which include counterterms, see analogous discussion in [70] for the open string. Modulo the external \( \psi \) fields, this action is completely equivalent in structure, albeit integrated on the sphere rather than the disk boundary, to \( O(\bar{\xi}^2, \phi^2) \) terms of the action in eq. (2.6). The open string boundary has been substituted by the \( \psi \) fields. On the heterotic side the \( \psi - \psi \) lines of the Feynman diagrams play the same role as the open string boundary line.

We can construct in a similar fashion the correlators involving two \( \psi \) fields and three \( \bar{\xi}^\mu \) fields. This requires vertices from the expansion of the heterotic action (3.5) to the third order in the \( \bar{\xi}^\mu \) fields, as well as connected diagrams of lower vertices. The most general diagram we can construct to order \( \bar{\xi}^3 \) will contain three vertices \( B_{1,1} \) from (3.10) and there are \( 3! = 6 \) possible orderings of partial integrations. In general, to order \( \bar{\xi}^n \) there will be \( n! \) orderings of partial
integrations. In principle, one can use induction to recover the heterotic analogue of eq. (2.6) to all orders. This is very complex and cumbersome. Nevertheless, a simple reflection reveals that the reorganized perturbation expansion is supported by the nonlinear renormalization property of the quantum field \[73\] and the discussion of the nonlinear redefinition of $\psi$ \[42\]. So we can say that the gauge covariant background field expansion is obtained by the partial integration tricks of the bosonic propagators and the insertion of vertex $B_{0,1}$ on the internal $\psi$ lines, and that this trick of partial integrations and vertex insertions is equivalent to the nonlinear renormalization of the quantum field in \[73\] and the nonlinear field redefinition in \[42\]. This is further justified by the Wilson loop approach to be explained in chapter 4.

The gauge covariant background field expansion reads as

$$S_I = -\frac{1}{2\pi\alpha'} \int d^2z (-i) \psi \left[ A_\mu \partial X^\mu + \partial X^\mu \left( F_{\mu_1 \mu_2} \bar{\psi}^{\mu_1} + \sum_{n=2} \frac{1}{n!} D_{\mu_n} \cdots D_{\mu_2} F_{\mu_1 \mu_2} \bar{\psi}^{\mu_1} \cdots \bar{\psi}^{\mu_n} \right) \right. \right.$$  

$$+ \left. \left( \frac{1}{2} F_{\mu_1 \mu_2} \bar{\psi}^{\mu_1} \partial \bar{\psi}^{\mu_2} + \sum_{n=3} \frac{n-1}{n!} D_{\mu_n-1} \cdots D_{\mu_2} F_{\mu_1 \mu_2} \bar{\psi}^{\mu_1} \cdots \bar{\psi}^{\mu_n-1} \partial \bar{\psi}^{\mu_n} \right) \right.$$  

$$- \frac{1}{2} \left( F_{\nu_1 \nu_2} \phi^{\nu_1} \phi^{\nu_2} + \sum_{n=1} \frac{1}{n!} D_{\nu_n} \cdots D_{\nu_2} F_{\nu_1 \nu_2} (X) \phi^{\nu_1} \phi^{\nu_2} \bar{\phi}^{\nu_1} \cdots \bar{\phi}^{\nu_n} \right) \right] \psi. \quad (3.17)$$

With the exception of the external fermion field $\psi$, this background field expansion has the same structure as the background field expansion of the Wilson loop of the open string case eq. (2.6). This result suggests that there should be a Wilson loop representation for the gauge physics of the heterotic sigma model, which will be discussed in the next section. Now we can construct, to any loop order, all gauge invariant heterotic diagrams by using this action.

To summarize, we reorganized perturbation theory in a completely background gauge invariant method for the sigma model beta function computation. So we can so we can proceed to the computation of the two–loop and three–loop beta functions. After this reorganization, the heterotic string perturbation expansion is diagrammatically equivalent to the open string one.
\( (b) = (-1)^2 \times (c) \)

Figure 3.5: Bosonic diagrams in the heterotic case that contribute to the single pole/logarithmic divergence. Diagram (b) has a gauge vertex structure \( F_{\mu_1 \mu_2} F^{\mu_1 \mu_2} \) and diagram (c) has a gauge vertex structure \( F_{\mu_2 \mu_1} F^{\mu_1 \mu_2} \). For diagram (b), we do the partial integral to move the derivative from one line to the other line and we get a minus sign. This minus sign can be used to change the vertex of diagram (b), which finally gives us diagram (c).

### 3.3 Heterotic two–loop beta function

At this point, we can use eq. (3.17) to compute the two–loop beta function of the heterotic string. The relevant diagrams are the same as in Figures 2.2, 2.3, and 2.4. We start with the bosonic contributions. All the diagrams except (b) and (c) of Figure 2.2 are pretty straightforward: it is easy to see that there is no single pole/logarithmic divergence in those diagrams except (b) and (c). Let’s focus on diagrams (b) and (c). In [1], we have computed the single pole/logarithmic divergence of (b) and (c) explicitly, which corresponds to the divergence of the open case eq. (2.11). Here we will only show that the bosonic contribution and the fermionic contribution cancel at the two–loop level, and we can show this diagrammatically.

As shown in Figure 3.5, diagram (b) equals diagram (c). There is one minus sign coming from partial integration and another minus sign from the indices of the gauge vertices, which cancels each other. So we have

\[(b) + (c) = 2(c), \quad (3.18)\]

which means the single pole/logarithmic divergence of the bosonic contribution is twice the result of diagram (c) in Figure 3.5.
Figure 3.6: Fermionic diagrams in the heterotic case that contribute to the single pole/logarithmic divergence. This happens to equal the bosonic diagram (c) of Figure 3.5 by a factor of $(-2)$. The factor 2 comes from the two possible ways of connecting the fermionic string $\phi$. The minus sign comes from shifting the derivative from the first vertex to the second vertex, as in eq. (3.19).

Now let’s look at the contribution from the fermionic string loop $\phi$. The fermionic two-loop diagram differs from the bosonic diagram (c) by a factor of $(-2)$, as shown in Figure 3.6. There are two ways of connecting the fermionic string $\phi$, which contributes to the factor 2. The fermionic propagator is proportional to the derivative of the bosonic propagator, so we have

$$\bar{K}(z_1, z_2)K(z_1, z_2) = (-\partial_1 G(z_1, z_2))(-\partial_1 G(z_1, z_2)) = (-\partial_1 G(z_1, z_2))(-2\partial_2 G(z_1, z_2)),$$

(3.19)

which is exactly the minus of the bosonic propagators in diagram (c). So we have the $(-2)$ factor, and the fermionic contribution cancels the bosonic contribution, like what happens in the open string case. As a result, the two-loop contribution to the beta function vanishes and we recover the expected heterotic beta function:

$$\beta^{\text{het}}_{\rho} = \frac{\partial}{\partial (\log \epsilon)} \delta A_\rho = \alpha' D^\mu F_{\mu\rho} + \mathcal{O}(\alpha'^3).$$

(3.20)

This is consistent with the absence of $F^3$ terms in the heterotic string effective action.

The vanishing of the two-loop contribution to the beta function has been previously demonstrated in Ref. [74] by using superspace techniques and with a specific nonlinear and nonlocal redefinition of fields. For our purposes, it is preferable though to consider bosonic and fermionic
contributions separately and to evaluate world-sheet position integrals instead of momentum integrals. By using this method, it will become easier to make contact with the sv–map.

### 3.4 Single–valued multiple zeta–values and general sv–map proposal for heterotic string

#### 3.4.1 Single–valued multiple zeta–values

The analytic dependence on the inverse string tension $\alpha'$ of string tree–level amplitudes furnishes an extensive and rich mathematical structure, which is related to modern developments in number theory and arithmetic algebraic geometry, cf. Ref. [48] and references therein.

The topology of the string world–sheet describing tree–level scattering of open strings is a disk, while tree–level scattering of closed strings is computed on the complex sphere. Open string amplitudes are expressed by integrals along the boundary of the world–sheet disk (real projective line) as iterated (real) integrals on $\mathbb{RP}^1\setminus\{0, 1, \infty\}$, whose values (more precisely the coefficients in their power series expansion in $\alpha'$) are given by multiple zeta values (MZVs). On the other hand, closed string amplitudes are given by integrals over the complex world–sheet sphere as iterated integrals on $\mathbb{P}^1\setminus\{0, 1, \infty\}$ integrated independently on all choices of paths. While in the $\alpha'$–expansion of open superstring tree–level amplitudes generically the whole space of MZVs eq. (1.12) enters [58,75], closed superstring tree–level amplitudes exhibit only a subset of MZVs appearing in their $\alpha'$–expansion [58,75]. This subclass can be identified [76] as single–valued multiple zeta values (SVMZVs)

$$\zeta_{sv}(n_1, \ldots, n_r) \in \mathbb{R} \quad (3.21)$$

originating from single–valued multiple polylogarithms (SVMPs) at unity [77]. SVMZVs have been studied by Brown in [55] from a mathematical point of view. They have been identi-
fied as the coefficients in an infinite series expansion of the Deligne associator \[^78]\) in two non–commutative variables. On the other hand, in physics SVMZVs firstly appear in the computation of graphical functions for certain Feynman amplitudes of \(\phi^4\) theory \[^62]\).

The SVMZVs eq. \((3.21)\) can be obtained from the MZVs eq. \((1.12)\) by introducing the following homomorphism:

\[
sv : \zeta_{n_1, \ldots, n_r} \mapsto \zeta_{sv}(n_1, \ldots, n_r).
\]

(3.22)

The SVMZVs eq. \((3.21)\) satisfy the same double shuffle and associator relations as the usual MZVs eq. \((1.12)\) and many more relations \[^55]\). For instance we have (cf. Ref. \[^76]\) for more examples):

\[
sv(\zeta_2) = \zeta_{sv}(2) = 0,
\]

(3.23)

\[
sv(\zeta_{2n+1}) = \zeta_{sv}(2n + 1) = 2 \zeta_{2n+1}, \quad n \geq 1,
\]

(3.24)

\[
sv(\zeta_{3,5}) = -10 \zeta_3 \zeta_5, \quad sv(\zeta_{3,7}) = -28 \zeta_3 \zeta_7 - 12 \zeta_5^2,
\]

(3.25)

\[
sv(\zeta_{3,3,5}) = 2 \zeta_{3,3,5} - 5 \zeta_3^2 \zeta_5 + 90 \zeta_2 \zeta_9 + \frac{12}{5} \zeta_2^2 \zeta_7 - \frac{8}{7} \zeta_2 \zeta_5^2, \ldots.
\]

(3.26)

Strictly speaking, the map \(sv\) is defined in the Hopf algebra \(\mathcal{H}\) of motivic MZVs \(\zeta^m\), cf. \[^55]\) for more details.

In supersymmetric Yang–Mills (SYM) theory a large class of Feynman integrals in four space–time dimensions lives in the subspace of SVMZVs or SVMPs. As pointed out by Brown in \[^55]\), this fact opens the interesting possibility to replace general amplitudes with their single–valued versions (defined by the map \(sv\)), which should lead to considerable simplifications. In string theory this simplification occurs by replacing open superstring amplitudes by their single–valued versions describing closed superstring amplitudes. In fact, in this work we have detected a large class of Feynman diagrams in two dimensions, which integrate to SVMZVs by considering heterotic world–sheet beta–functions.
3.4.2 General sv–map proposal for heterotic string

The purpose of the current section is to use the two–loop computation and general results of the preceding sections, in order to establish a concrete connection between the open string beta function and the heterotic one. The proposal is the following: assume we can write the beta function of the open string to any loop order as

$$\beta_o = \sum_n F_n I_n |_{\ln \epsilon}$$

(3.27)

where the form factors $F_n = F_n(F, D)$ contain the background fields and their covariant derivatives while $I_n$ are ultra-violet divergent integrals of which we keep only the coefficients of single logarithmic divergences needed for the computation of the beta function. Then we can claim that

$$\beta_h = \sum_n F_n H_n |_{\ln \epsilon} = \sum_n F_n \text{sv}(I_n) |_{\ln \epsilon} = \text{sv}(\beta_o)$$

(3.28)

where we used the equivalence of eq. (3.17) with eq. (2.6) to show that the factors $F_n$ are the same between the open string and the heterotic string. $H_n$ are the corresponding divergent integrals of the heterotic string. This proposal clearly relies on the existence of a sv–compatible regularization scheme. Although in the following sections we will concentrate to the lowest derivative terms, $(DF)^n F^{n-1}$, based on the discussion of the previous section it should be obvious that the proposal is valid for all derivative terms.

Although our discussion is at the level of the sigma model beta functions i.e. the equations of motion, the most interesting application is at the level of effective actions. Modulo possible field redefinitions, we expect that the effective action for the single trace gauge sector of the heterotic string is generated via the sv–map acting on the open superstring effective action.
Chapter 4

Heterotic string sigma model: Wilson loop representation

In the previous chapter, the fermionic representation of the heterotic sigma model is explored. Its perturbation calculation eq. (3.5) is straightforward (without the path-ordering of the Wilson loop), however, this perturbation process is not gauge invariant for each diagram, due to the presence of the term $\partial A_\mu$ in the action. Only after combining all the diagrams at each loop can we get a gauge invariant result. If one is doing the complete renormalization, then this does not affect the final result. But if one just wants to compute a subset of all the diagrams at each loop, then the lack of gauge invariant is a trouble. In this case, a perturbation method that is gauge invariant at each diagram level is needed. In [42], a specially chosen nonlocal field transformation is used to put the perturbation into a gauge invariant form at each term. In the previous chapter, we use a reorganized perturbation method to achieve a gauge invariant perturbation without going through the nonlocal field transformation. We find that the reorganized perturbation eq. (3.17) has the same structure as the Wilson loop expansion of the open string eq. (2.6), except for the existence of the fermion field $\psi$. This suggests the possibility of constructing a Wilson loop representation for the heterotic sigma model.

Indeed, this Wilson loop representation of the gauge physics of the heterotic sigma model
can be constructed. In this chapter, we firstly revisit the reorganized perturbation method and get some hints from it. Then we will construct the Wilson loop, compute its functional derivatives and prove that it is the exact propagator of the fermion field $\psi$ of the fermionic representation. After that, we will investigate the geometry of the Wilson loop and generalize the nonabelian Stokes’s theorem to the fermionic case, from which we construct the heterotic sigma model action using the Wilson loop.

### 4.1 Reorganized perturbation method

Usually in the perturbation calculation

$$e^{-S(F_{\mu \nu})} = \int D\Phi D\psi e^{-S_{E}[X,A,\phi,\psi]},$$

we compute diagrams involving all the propagators of $X, \phi, \psi$ and combine all the diagrams at each loop to get a gauge invariant result. On the other hand, if we firstly integrate out the fermion fields $\psi$

$$e^{-S_{eff}(X,F,\phi)} = \int D\psi e^{-S_{E}[X,A,\phi,\psi]},$$

we would get a gauge invariant effective action of the gauge field strength $F_{\mu \nu}$. Then we can do the remaining functional integral perturbatively in a gauge invariant manner. Actually this effective action $S_{eff}(X,F,\phi)$ is exactly the Wilson loop representation we want to construct, which corresponds to the open string case eq. (2.1). But in practice, it is impossible to obtain a closed form for $S_{eff}$ perturbatively, since we can not we cannot integrate out $\psi$ by using brute force. The case of the open string sigma model in chapter 2.1 is an exception. There the integral eq. (2.3) is a 1D integral and we can get a closed form of $S_{eff}(X,F,\phi)$ as shown in eq. (2.4), which states that the Wilson line segment is the exact propagator of the boundary fermion field $\psi_{open}$. This is a hint for the method we plan to use in order to construct the Wilson
loop representation of the heterotic sigma model. In the next section 4.2, we will use an indirect way to show that the Wilson line in the heterotic case is also the exact propagator of the fermion field $\psi$, like the open string case eq. (2.4).

In the reorganized perturbation method, we firstly do the background field expansion $X \rightarrow X + \xi$ and treat $\xi, \phi, \psi$ as the quantum field. Then we pick all the tree-level diagrams at each order of $\alpha'$ and only integrate out the internal $\psi$ propagators. The final result is gauge invariant of the form $\psi f(\xi, \phi, F_{\mu\nu}) \psi$, where $f(\xi, \phi, F_{\mu\nu})$ contains the gauge covariant vertices having the same structure as the Wilson loop expansion eq. (2.6). This result is really the background field expansion of the exact propagator of $\psi$, as illustrated in Figure 3.3. This is another hint towards the relation between the Wilson line segment and the exact propagator of $\psi$.

**4.2 Construct the Wilson loop**

In this section, we will build up the Wilson loop of the gauge terms of the heterotic string sigma model, and show that it satisfies all the requirements. However, this is just a single quantity of Wilson loop, and we will need to construct the action using this Wilson loop. (In the next section 4.3, we will propose a way to rewrite the action of the heterotic sigma model using this Wilson loop, based on analyzing the geometry of the Wilson loop.)

So in this section we will only consider gauge terms of the heterotic sigma model

\[
L^f_E[A, \psi] = \psi D_z \psi + \frac{i}{2} \psi F_{\nu_1 \nu_2} \phi^{\nu_1} \phi^{\nu_2} \psi \\
= \psi \partial_z \psi - i \psi (A_{\mu} \partial_{\nu} X^{\mu} - \frac{1}{2} F_{\nu_1 \nu_2} \phi^{\nu_1} \phi^{\nu_2}) \psi.
\] (4.3)

The notation for the gauge fields are $A_{\mu} = A^a_{\mu} T^a$, $D_{\mu} = \partial_{\mu} - i [A_{\mu}, \cdot]$ and $D_z = \partial_z - i [A_z, \cdot]$. Gauge symmetry corresponds to the following transformation of world–sheet fields (For simplicity, we will directly say gauge transformation in the following, instead of explicitly

62
saying field transformations on the world–sheet.)

\[ A_\mu \rightarrow U A_\mu U^\dagger + i U \partial_\mu U^\dagger \]
\[ \psi \rightarrow U \psi. \]  \hspace{1cm} (4.4)

It is easy to verify that the various terms in the Lagrangian transform as

\[ F_{\mu\nu} \rightarrow UF_{\mu\nu} U^\dagger, \]
\[ D_z \psi \rightarrow UD_z \psi \]
\[ L_E^f[A, \psi] \rightarrow L_E^f[A, \psi]. \]

Notice that the heterotic string sigma model is only required to have superconformal symmetry. The gauge symmetry defined above is just a field redefinition from the point of view of the world-sheet. That’s why we did not have gauge invariant results for each diagram in the perturbation calculation. Only after integrating out the full correlator on the world–sheet, we get the space-time effective action of the gauge field with the spacetime gauge symmetry. Here the purpose of constructing the Wilson loop is to get a gauge invariant result for each diagram, so the perturbation result is easier to see and to compare with the open string case.

To describe this gauge symmetry geometrically, firstly we need to build the Wilson line for an infinitesimal distance, then extend it to a finite length via path ordering and finally obtain the Wilson loop which is gauge invariant. See Peskin and Schroeder [79, Chapter 15] for the case of ordinary quantum field theory.

Here we define the Wilson line for an infinitesimal separation \( \epsilon \) in the same way as the open string case eq. (2.2)

\[ V[z_1 + \epsilon, z_1] := \exp\{i \int_{z_1}^{z_1+\epsilon} dz [\partial_\nu X^\mu A_\mu - \frac{1}{2} F_{\nu_1 \nu_2} \phi^{\nu_1} \phi^{\nu_2}] \}, \]  \hspace{1cm} (4.5)
except that here $\epsilon$ is on the complex plane while in the open string case it is on the boundary. The parametrization of the Wilson line is actually $V[z_2, z_1] = V[X, \phi, z_2, z_1]$, but for convenience we will not explicitly distinguish them and will use whatever is convenient. Under the gauge transformation, this Wilson line transforms as

$$V[z_1 + \epsilon, z_1] \rightarrow U[z_1 + \epsilon] V[z_1 + \epsilon, z_1] U[z_1]^+, \quad (4.6)$$

which is exactly the property we expect for the Wilson line. Now we define the Wilson loop by taking the trace for a path ordered loop of this Wilson line

$$W[X, \phi, C] := \text{Tr} V[X, \phi, C, z, z], \quad (4.7)$$

where $C$ is a loop starting from $z$ and ending at $z$ and the dependence on the bosonic string $X$ and the fermionic string $\phi$ is written explicitly. Compared with the Wilson loop of the open string in section 2.1.1, the only difference is the contour on the world-sheet of the integral. For the open string, the contour is the 1-dim boundary. For the heterotic string, it can be any loop on the complex plane.

This definition of the Wilson line is simply an analog of the open string case and is not enough to justify itself. In the case of open string, the key property is that the Wilson line is the exact propagator of the fermion field eq. (2.4). To justify the definition for the heterotic case, we need to prove an analog of this property. To do this, we firstly need to investigate the background field expansion of the Wilson loop.

### 4.2.1 The functional variation of the Wilson loop

The perturbation calculation using the Wilson loop is usually done in the background field expansion. The background field expansion of the bosonic string is $X \rightarrow X + \delta X$, where $\delta X$ is a quantum field. For the fermionic string, it is taken as a quantum field itself $\phi = 0 + \phi$. In
Figure 4.1: The discretized contour of the Wilson loop in the string world-sheets. $z$ is the holomorphic coordinate of the complex plane parameterizing the string $X$. We start from an arbitrary closed contour $X(z)$ (the inner ring), then deform it to the closed contour $X(z) + \delta X(z)$ (the outer ring). The two closed contours are discretized into lattice positions from $z_0$ to $z_N$, with the periodic condition $z_0 = z_{N+1}$. Now the continuous contour is decomposed into discretized segments $b_j$ (for the original contour) or $b'_j$ (for the deformed contour). The ‘displacement’ or the deformation of the contour is characterized by $a_j$ for each lattice position $z_j$. The explicit forms of these segments are given in eq. (4.8).
this way, the fermionic string contribution is an ordinary derivative

\[ V[X + \delta X, \phi, z_0, z_0] = V[X + \delta X, z_0, z_0] \]
\[ + \int dz_1 V[X, z_0, z_1] \{ -\frac{i}{2} F_{\nu \lambda} (X(z_1)) \phi^{\nu} \phi^{\lambda} \} V[X, z_1, z_0], \]

where the bosonic part is \( V[X + \delta X, z_0, z_0] = V[X + \delta X, \phi = 0, z_0, z_0] \). The functional derivative of the bosonic string contribution is highly nontrivial. We will follow the reference [53] to explain how to compute \( V[X + \delta X, z_0, z_0] \).

For the Wilson loop, the background field expansion \( X + \delta X \) is just a variation of the contour of the loop integral and the world-sheet coordinate serves as a parametrization of this contour. To do the functional derivative, we discretized the contour by \( z_{j+1} - z_j = \epsilon \) and compute the variation, then take the limit \( \epsilon \to 0 \). The discretized loop is shown in figure 4.1.

Then we define the following quantities

\[ a_j = 1 + iA_\mu(X(z_j)) \delta X^\mu(z_j) \]
\[ a_{j+1} = 1 + iA_\mu(X(z_{j+1})) \delta X^\mu(z_{j+1}) \]
\[ b_j = 1 + iA_\mu(X(z_j)) \{ X^\mu(z_{j+1}) - X^\mu(z_j) \} \]
\[ b'_j = 1 + iA_\mu(X(z_j)) + \delta X(z_j) \{ X^\mu(z_{j+1}) + \delta X^\mu(z_{j+1}) - X^\mu(z_j) - \delta X^\mu(z_j) \}, \quad (4.8) \]

where \( b_j \) represents the infinitesimal segment of \( V[X + \delta X, z_0, z_0] \) and \( a_j \) characterizes the change of this infinitesimal segment under the variation \( X \to X + \delta X \). After a bit of algebra and just keeping the leading order of variation, we have the following result

\[ a_j^{-1} b_j^{-1} a_{j+1} b_j = 1 - iF_{\mu \nu}(X(z_j)) \{ X^\nu(z_{j+1}) - X^\nu(z_j) \} \delta X^\mu(z_j). \quad (4.9) \]
Taking its inverse, we get the infinitesimal segment of the Wilson loop after the variation

\[ b'_j = a_{j+1}b_j(1 + iF_{\mu\nu}(X(z_j))(X^\nu(z_{j+1}) - X^\nu(z_j))\delta X^\mu(z_j))a_j^{-1}. \]

Now the whole Wilson line after the variation is

\[
\prod_{j=0}^{N} b'_j = b'_N b'_{N-1} \ldots b'_0 \\
= a_{N+1}b_N(1 + iF_{\mu\nu}(X(z_N))(X^\nu(z_{N+1}) - X^\nu(z_N))\delta X^\mu(z_N))a_N^{-1} \\
\times a_Nb_{N-1}(1 + iF_{\mu\nu}(X(z_{N-1}))(X^\nu(z_N) - X^\nu(z_{N-1}))\delta X^\mu(z_{N-1}))a_{N-1}^{-1} \\
\ldots a_1b_0(1 + iF_{\mu\nu}(X(z_0))(X^\nu(z_1) - X^\nu(z_0))\delta X^\mu(z_0))a_0^{-1} \\
= a_0b_N b_{N-1} \ldots b_0a_0^{-1} \\
+ \sum_{j=0}^{N} \epsilon a_0b_N \ldots b_{j+1}[iF_{\mu\nu}(X(z_j))(X^\nu(z_{j+1}) - X^\nu(z_j))\delta X^\mu(z_j)]b_j \ldots b_0a_0^{-1}. \tag{4.10}
\]

Taking the continuous limit \( \epsilon \to 0 \), we obtain the functional variation of the bosonic part of the Wilson line

\[
V[X + \delta X, z_0, z_0] \\
= a_0V[X, z_0, z_0]a_0^{-1} + a_0 \oint dzV[X, z_0, z](iF_{\mu\nu}(X(z))\delta X^\nu(z))V[X, z, z_0]a_0^{-1} \\
= V[X, z_0, z_0] + i(A_{\mu}(X(z_0)))V[X, z_0, z_0] - V[X, z_0, z_0]A_{\mu}(X(z_0))\delta X^\mu(z_0) \\
+ \oint dzV[X, z_0, z](i\delta X^\nu(z)F_{\mu\nu}(X(z))\delta X^\mu(z))V[X, z, z_0]. \tag{4.11}
\]
Combine with the fermionic part, we get the complete variation

\[
V[X + \delta X, \phi, z_0, z_0] = V[X, z_0, z_0] + i \delta X^\mu(z_0) A_\mu V[X, z_0, z_0] - i V[X, z_0, z_0] A_\mu \delta X^\mu(z_0) \\
+ i \oint dz_1 V[X, z_0, z_1] \{ \partial_\tau X^\nu(z_1) F_{\mu\nu}(X(z_1)) \delta X^\mu(z_1) \\
- \frac{1}{2} F_{\nu_1\nu_2}(X(z_1)) \phi^{\nu_1} \phi^{\nu_2} \} V[X, z_1, z_0].
\]

(4.12)

Taking its trace we get the functional variation of the Wilson loop

\[
W[X + \delta X, \phi] = W[X] + i \oint dz_1 \text{Tr} \{ V[X] (\partial_\tau X^\nu(z_1) F_{\mu\nu}(X(z_1)) \delta X^\mu(z_1) \\
- \frac{1}{2} F_{\nu_1\nu_2}(X(z_1)) \phi^{\nu_1} \phi^{\nu_2} \},
\]

(4.13)

The first part in the integrand is the contribution from the bosonic string X as in reference [53], so the higher order functional derivatives coming from the bosonic string are the same as in reference [53]. The second part is the contribution from the fermionic string \phi.

Now let’s look at the higher order functional derivatives of the fermionic string contribution.

\[
\mathcal{F}[X, \phi] := -\frac{1}{2} \text{Tr} \{ V[X] F_{\nu_1\nu_2}(X(z_1)) \phi^{\nu_1} \phi^{\nu_2} \}.
\]

(4.14)

Now do the functional expansion again

\[
\mathcal{F}[X + \delta X, \phi] = -\frac{1}{2} \text{Tr} \{ V[X + \delta X] F_{\nu_1\nu_2}(X(z_1) + \delta X(z_1)) \phi^{\nu_1} \phi^{\nu_2} \} \\
= \mathcal{F}[X] - \frac{1}{2} \text{Tr} \{ (i A_\mu \delta X^\mu(z_1) V[X] - i V[X] A_\mu \delta X^\mu(z_1)) F_{\nu_1\nu_2}(X(z_1)) \phi^{\nu_1} \phi^{\nu_2} \\
+ \oint dz_2 V[X] \partial_\mu F_{\nu_1\nu_2}(X(z_2)) \delta(z_2 - z_1) \delta X^\mu(z_2) \phi^{\nu_2} \phi^{\nu_1} \}.
\]

(4.15)
So we obtain the functional derivative of the fermionic contribution

\[
\frac{\delta F[X(z_1)]}{\delta X^\mu(z_2)} = -\frac{1}{2} \text{Tr}\{V[X]D_\mu F_{\nu_1\nu_2}(X)\phi^{\nu_1}\phi^{\nu_2}\delta(z_2 - z_1)\}. \tag{4.16}
\]

All the higher order functional derivatives follow straightforwardly.

From this functional variation, we can obtain the background field expansion of the Wilson loop

\[
W[X + \xi, \phi] = i \int dz \text{Tr} V[X] \left\{ \partial_z X^\mu A_\mu + \partial X^\nu [F_{\mu_1\nu_1}\xi^{\mu_1} + \sum_{n=2}^{\infty} \frac{1}{n!} D_{\mu_1} \ldots D_{\mu_n} F_{\nu_1\nu_2}(X)\phi^{\nu_1}\phi^{\nu_2}\xi^{\mu_1} \ldots \xi^{\mu_n}] + \frac{1}{2} F_{\mu_1\nu_2} \xi^{\mu_1} \partial \xi^{\nu_2} + \sum_{n=3}^{\infty} \frac{n-1}{n!} D_{\mu_{n-1}} \ldots D_{\mu_2} F_{\mu_1\mu_n} \xi^{\mu_1} \ldots \xi^{\mu_n} \partial \xi^{\mu_n} \right\}, \tag{4.17}
\]

This result is the same as eq. (2.6), except the difference between parameters \(z\) and \(\tau\). Now the Wilson loop between the open string case and the heterotic case corresponds to each other exactly. In section 4.3, we will construct the action of the gauge physics using the Wilson loop for the heterotic sigma model. Then the open string sigma model and the heterotic sigma model can correspond to each other in the level of action.

### 4.2.2 The exact propagator of \(\psi\)

Now let’s prove that the Wilson loop is the exact propagator of the fermion field \(\psi\). Firstly, from the path ordering of the Wilson loop, we would have the following differential equation

\[
\frac{d}{dz_2} V[z_2, z_1] = i(\partial_{z_2} X^\mu A_\mu(X(z_2)) - \frac{1}{2} F_{\nu_1\nu_2}(X(z_2))\phi^{\nu_1}\phi^{\nu_2}) V[z_2, z_1], \tag{4.18}
\]

which is just an analog of the differential equation of the time evolution operator in quantum field theory. Integrating out this differential equation gives us the path ordered Wilson line for
a curve of finite length. This equation is essentially equivalent to the following variation

\[ V[z_2 + \epsilon_2, z_1] = V[z_2, z_1] + \epsilon_2 i(\partial z_2 X^\mu A_\mu(z_2) - \frac{1}{2} F_{\mu \nu z_2}(z_2) \phi^{\nu_1} \phi^{\nu_2}) V[z_2, z_1]. \] (4.19)

Secondly, we can obtain this variation in a different way using eq. (4.12)

\[ V[z_2 + \epsilon_2, z_1] = V[X + \delta X, \phi + \delta \phi, z_2 + \epsilon_2, z_1] \]
\[ = V[X, \phi, z_2, z_1] + \frac{\partial V[z_2, z_1]}{\partial z_2} \epsilon_2 \]
\[ + iA_\mu(z_2) \delta X(z_2) V[z_2, z_1] - iV[z_2, z_1] A_\mu(z_1) \delta X(z_1) \]
\[ + i \int dz V[z_2, z] (\partial X^\nu(z) F_{\mu \nu} \delta X^\mu(z) - F_{\nu \mu z_2} \phi^{\nu_1} \delta \phi^{\nu_2}) V[z, z_1]. \] (4.20)

Use \( \delta X(z) = \partial X(z) \delta(z - z_2) \epsilon_2 \) and \( \delta \phi = \partial z \delta(z - z_2) \epsilon_2 = 0 \), the above equation becomes

\[ V[z_2 + \epsilon_2, z_1] = V[X, \phi, z_2, z_1] + \frac{\partial V[z_2, z_1]}{\partial z_2} \epsilon_2 + iA_\mu(z_2) \partial X^\mu(z_2) \delta(0) V[z_2, z_1] \epsilon_2 \]
\[ - iV[z_2, z_1] A_\mu(z_1) \partial X^\mu(z_1) \delta(z_2 - z_1) \epsilon_2 + i \partial X^\mu(z_2) \partial X^\nu(z_2) F_{\mu \nu} V[z_2, z_1] \epsilon_2 \]
\[ = V[X, \phi, z_2, z_1] + \frac{\partial V[z_2, z_1]}{\partial z_2} \epsilon_2 + iA_\mu(z_2) \partial X^\mu(z_2) \delta(0) V[z_2, z_1] \epsilon_2 \]
\[ - iV[z_2, z_1] A_\mu(z_1) \partial X^\mu(z_1) \delta(z_2 - z_1) \epsilon_2. \] (4.21)

Combining the two different ways of doing the variation, eq. (4.19) and eq. (4.21), we get

\[ \frac{\partial}{\partial z_2} - i(\partial z_2 X^\mu A_\mu(z_2) - \frac{1}{2} F_{\mu \nu z_2}(z_2) \phi^{\nu_1} \phi^{\nu_2}) V[z_2, z_1] \]
\[ = -iA_\mu(z_2) \partial X^\mu(z_2) V[z_2, z_1] \delta(0) + iV[z_2, z_1] A_\mu(z_1) \partial X^\mu(z_1) \delta(z_2 - z_1). \] (4.22)

With the exception of the \( \delta(0) \) term, the Wilson line \( V[z_2, z_1] \) is the inverse of the differential operator

\[ \frac{\partial}{\partial z_2} - i(\partial z_2 X^\mu A_\mu(z_2) - \frac{1}{2} F_{\mu \nu z_2}(z_2) \phi^{\nu_1} \phi^{\nu_2}). \] (4.23)
Compared with eq. (4.3), we see that $V[z_2, z_1]$ is the exact propagator of the fermion field $\psi$,

$$V[z_2, z_1] = \langle \psi(z_2) \psi(z_1) \rangle_{L^E}. \quad (4.24)$$

The $\delta(0)$ term is probably a tadpole and shift only the background, or it can be related with the overall normalization of the partition function of $\psi$. So it does not affect the exact propagator and we can just throw away this infinity term from the partition function.

Now this definition of the Wilson loop for the heterotic sigma model has been justified. The relation between the Wilson loop and the fermionic representation for the heterotic sigma model, is exactly the same as that relation for the open string sigma model. The contour integral of the Wilson loop is equivalent to the ordinary perturbation in terms of the fermion field $\psi$. For the heterotic string, this relation is highly nontrivial, because the fermion field $\psi$ lives on the whole complex plane. For the open string, this relation is a trivial one, because its fermionic field $\psi_{open}$ just lives on the boundary and its propagator is just the Heaviside step function.

### 4.3 Geometry of the Wilson loop

The Wilson loop is a geometrical object that encodes gauge invariance of physical quantities. To build up the action of the heterotic string sigma model using the Wilson loop, we need to explore its geometry. Firstly we will look at how we arrive at the classical Yang-Mills action using the Wilson loop and this will serve as a protocol. Then we will discuss the open string case and the heterotic string case. In all of these theories, the gauge invariant classical action is obtained from a sum over the loop contours, which is equivalent to a sum of the gauge field strength over the spacetime.
4.3.1 Yang-Mills case

For the Yang-Mills case, we will use the lattice theory as a convenient illustration. In the lattice theory, the sum of Wilson loop over all the loops will generate the Yang-Mills action, as shown in the following equation (See Srednicki [80, Chapter 82] for details)

\[ S \approx \sum_{\text{loops}} W[\text{plaquette}], \quad (4.25) \]

where \( W[\text{plaquette}] \) is the Wilson loop associated with a specific plaquette. This equivalence of the gauge invariant action and the sum over all the Wilson loops comes from the geometric nature of the Wilson loop.

The line integral of the gauge field \( A_\mu \) in the Wilson loop is connected with the area integral of the field strength \( F_{\mu\nu} \) via

\[ \text{Tr} \mathcal{P} \exp\{i \oint_C dx^\mu A_\mu(x)\} = \text{Tr} \mathcal{P} \exp\{i \int_{\Sigma} d\sigma^{\mu\nu}(x) V[x_0, x] F_{\mu\nu}(x) V[x, x_0]\}, \quad (4.26) \]

where \( \sigma^{\mu\nu}(x) \) is the area element on the surface \( \Sigma \) bounded by the closed loop \( C \) and \( V[x_0, x] = \mathcal{P} \exp\{i \int_{x_0}^x dy^\mu A_\mu(y)\} \). In the abelian case, this is simply the Stokes’s theorem. In the non-abelian case, this is called the nonabelian Stokes’s theorem and is highly nontrivial [81–83].

Now let’s go to a lattice theory to see how this Stokes’s theorem leads to the sum over loops. For simplicity (to be able to draw the figure), let’s assume a 3D spacetime lattice. And we choose the right-hand rule to associate the direction of the area with the direction of the loop. The gauge contribution to the Yang-Mills action should be a volume integral over the 3D spacetime. However, on the lattice, the gauge field is only defined on the 1D loops (the boundary of the plaquette). By the Stokes’s theorem, we extend the gauge content from the 1D loop to the 2D area (plaquette) bounded by the loop. In this way, the volume integral of the gauge content over the unit cube becomes the sum of the area integral of the gauge content over all the plaquettes of the cube. Let’s look at figure 4.2 for illustration. The gauge content \( \exp\{i \oint_{C_{1,2}} dX^\mu A_\mu\]
Figure 4.2: The sum of the Wilson loop over both the directions. In the lattice, each plaquette $\Sigma$ has two loops of opposite directions $C_1$ and $C_2$. By the Stokes’s theorem, the line integrals over $C_1$ and $C_2$ are connected with area integrals over the two area (faces) that have opposite normal directions $\vec{n}_1$ and $\vec{n}_2$. In this way, the Wilson loop over $C_1$ and $C_2$ are connected with the gauge contribution to the left cube and the right cube respectively.
Chapter 4  
Section 4.3

is defined by the Wilson loop integral over the boundary of the plaquette $\Sigma$, where there are two opposite directions $C_1$ and $C_2$ for the 1D loop. By Stokes’s theorem, the gauge content is extended to two area integrals over the plaquette (face of the cube) $\exp[i \int_{\vec{n}_{1,2}} d\sigma_{\mu\nu} F_{\mu\nu}]$, with opposite normal directions $\vec{n}_1$ and $\vec{n}_2$ of the area. The area integral with normal direction $\vec{n}_1$ is associated with the gauge content of the left unit cube and the area integral with normal direction $\vec{n}_2$ is associated with the gauge content of the right unit cube.

In this way, the classical action $S(\Sigma)$ of this plaquette $\Sigma$, which is just the sum over both the directions of the Wilson loop $W[C_{1,2}]$, turns out to be a sum of the gauge physics from the left cube and the right cube, which are all the unit cubes that are adjacent to the plaquette

$$S(\Sigma) = \sum_{j=1}^{2} \exp[i \int_{C_{1,2}} dX^\mu A_\mu] = \sum_{j=1}^{2} \exp[i \int_{\vec{n}_{1,2}} d\sigma_{\mu\nu} F_{\mu\nu}]$$

= gauge content from the left cube + gauge content from the right cube

= sum of all the gauge content around $\Sigma$. 

(4.27)

So the lattice Yang-Mills action is just the sum of $\exp[i \int_{\vec{n}} d\sigma_{\mu\nu} F_{\mu\nu}]$ over all the elementary oriented areas of the spacetime lattice. Going to the continuum limit by taking the infinitesimal lattice spacing, the classical Yang-Mills action is equivalent to the sum of the gauge physics $\exp[i \int_{\vec{n}} d\sigma_{\mu\nu} F_{\mu\nu}]$ over all the infinitesimal oriented areas of the whole spacetime.

We should keep in mind that in 3D and higher dimensional spacetime, this geometric nature of the Wilson loop only serves as an intuitive picture of the gauge physics, because this sum over plaquettes can only be done in the lattice approximation rather than the continuum limit. However, for strings, this geometric picture is a practical method to do the calculation, because the 2D nature of world-sheet of the string, as will be explored in the following.
### 4.3.2 Open string case

For the open string sigma model, the functional derivative of the Wilson loop eq. (2.6) is

\[
W[X + \delta X, \phi] = W[X] + i \int d\tau \text{Tr} V[X] \{ \partial X^\mu F_{\mu \nu} \delta X^{\nu 1} - \frac{1}{2} F_{\nu 1 \nu 2} \phi^{\nu 1} \phi^{\nu 2} \} + O((\delta X)^2).
\]

(4.28)

If we just look at the bosonic string part (set \(\phi = 0\)), this equation is just the nonabelian Stokes’s theorem investigated in [63, 82, 83]. Choose the area element as \(\delta \sigma^\text{bosonic}_{\mu \nu} = \delta \tau \partial X^{\nu} \delta X^{\mu}\). The functional variation is an area integral of \(F_{\mu \nu}\). We obtain using the nonabelian Stokes’s theorem the following expression for the bosonic open string [63, 82]

\[
\text{Tr} \mathcal{P} \exp\{i \int d\tau \partial X^{\mu}(\tau) A_{\mu}(X)\} = \text{Tr} \mathcal{P} \exp\{i \int d\sigma^\text{bosonic}_{\mu \nu}(X) V[X_0, X] F_{\mu \nu}(X(\tau)) V[X, X_0]\},
\]

(4.29)

where \(V[X_0(\tau_0), X(\tau)] = \mathcal{P} \exp\{i \int_{\tau_0}^{\tau} d\tau' \partial X^{\mu}(\tau') A_{\mu}(X)\}\).

Now let’s turn on the fermionic string \(\phi \neq 0\) and treat \(\phi\) itself as the variation (like \(\delta X\)). By analog of the bosonic area element, we define the area element of the fermionic string to be \(\delta \sigma^\text{fermionic}_{\mu \nu} = \delta \tau \phi^{\mu} \phi^{\nu}\) in the Grassmann space. Now the fermionic part of the functional derivative also becomes an area integral of \(F_{\mu \nu}\). Like the bosonic case, we can integrate out the functional derivative and obtain a fermionic contribution to the nonabelian Stokes’s theorem. So we obtain a generalization of the nonabelian Stokes’s theorem to the superstring

\[
W[C] = \text{Tr} \mathcal{P} \exp\{i \int d\tau [\partial X^{\mu}(\tau) A_{\mu}(X) - \frac{1}{2} F_{\nu 1 \nu 2}(X) \phi^{\nu 1} \phi^{\nu 2}]\}
\]

\[
= \text{Tr} \mathcal{P} \exp\{i \int \left[ d\sigma^\text{bosonic}_{\mu \nu} - \frac{1}{2} d\sigma^\text{fermionic}_{\mu \nu}\right] V[X_0, X] F_{\mu \nu}(X) V[X, X_0]\},
\]

(4.30)

where \(V[X_0(\tau_0), X(\tau)] = \mathcal{P} \exp\{i \int_{\tau_0}^{\tau} d\tau' [\partial X^{\mu}(\tau') A_{\mu}(X) - \frac{1}{2} F_{\nu 1 \nu 2}(X) \phi^{\nu 1} \phi^{\nu 2}]\}\).

From this generalized nonabelian Stokes’s theorem, we can see the geometry of the Wilson loop in the open string case and then obtain the classical action from a sum over loops like the Yang-Mills case. Let’s look at figure 4.3 for illustration. For open string, the gauge field only
Figure 4.3: The geometry of the Wilson loop of the open string. On the left is the open string in spacetime. On the right is its conformal transformation into unit disk. The loop is $C$ and the area bounded is $\Sigma$.

lives on the boundary $C$ of the string via the Chan-Paton factors. So the only loop we have is the boundary itself. After conformal transformation into the unit disk, the area $\Sigma$ bounded by $C$ is the disk itself. So by the Stokes’s theorem, the single loop $C$ would contain all the contributions of the gauge field of the open string. This explains the fact that in the open string sigma model we only use a single Wilson loop eq. (2.1) without any sum over loops, because this already includes all the gauge contributions of the string.

### 4.3.3 Heterotic string case

For the heterotic string, the generalized nonabelian Stokes’s theorem can be obtained straightforwardly following the discussion of the open string case

$$W[C] = \text{Tr} \mathcal{P} \exp \{i \int dz [\partial X^\mu(z) A_\mu(X) - \frac{1}{2} F_{\nu_1 \nu_2}(X) \phi^{\nu_1} \phi^{\nu_2}] \}$$

$$= \text{Tr} \mathcal{P} \exp \{i \int [d\sigma_{\text{bosonic}}^{\mu \nu} - \frac{1}{2} d\sigma_{\text{fermionic}}^{\mu \nu}] V[X_0, X] F_{\mu \nu}(X) V[X, X_0] \}$$

(4.31)

where $V[X_0(z_0), X(z)] = \mathcal{P} \exp \{i \int_z^z dz' [\partial X^\mu(z') A_\mu - \frac{1}{2} F_{\nu_1 \nu_2} \phi^{\nu_1} \phi^{\nu_2}] \}$. This is nearly the same as the open string one eq. (4.30), except that here the parametrization is $z$.

Let’s look at figure 4.4 for the geometry of the Wilson loop of the heterotic string. There are two types of loops on the closed string, the longitudinal one $C_1$ and the transversal one $C_2$. 

76
Figure 4.4: The geometry of the Wilson loop of the heterotic string. On the left is the heterotic string in spacetime. There are two types of loops, the longitudinal one $C_1$ and the transversal one $C_2$. On the right are the conformal transformation of the closed string, where $C_1$ is transformed into the real axis and $C_2$ is transformed into a circle. The area bounded by the two loops are $\Sigma_1$ and $\Sigma_2$ respectively, as can be distinguished by their color.

The conformal transformation maps the closed string into the whole complex plane and $C_{1,2}$ are mapped into the real axis and the circle respectively. Let’s focus on $C_1$ first. The area bounded by $C_1$ is the upper half-plane $\Sigma_1$. By the nonabelian Stokes’s theorem, the Wilson loop of $C_1$ would give the gauge physics of the upper half-plane. If we revert the direction $-C_1$, the area bounded will be the lower half-plane and the nonabelian Stokes’s theorem would give the gauge physics of the lower half-plane. So if we sum the Wilson loop over the loop $C_1$ and $-C_1$, we will have the gauge physics of the whole complex plane, thus of the whole closed string. Now look at $C_2$. It is straightforward to see that the sum of the Wilson loop over $C_2$ and $-C_2$ will also give the gauge physics of the whole complex plane.

This result can be generalized to an arbitrary loop $C$. Because of the 2D nature of the closed string, the two areas bounded by $C$ and $-C$ are complementary and their sum is the whole complex plane. So by nonabelian Stokes’s theorem, we arrive at the following proposal for the
Wilson loop approach of the heterotic string

\[ e^{-S_{eff}[F_{\mu\nu}]} = \sum_{\pm C} \int D\phi DX e^{-S_E[X,A,\phi,C]} \]
\[ = \sum_{\pm C} \int D\phi DX \text{Tr} \mathcal{P} \exp \left\{ -\frac{1}{2\alpha'} \int d^2 z \left[ \partial X^\mu \partial X_\mu + \phi^\mu \partial \phi_\mu \right] \right\} + i \oint_C dz \left[ \partial X^\mu(z) A_\mu(X) - \frac{1}{2} F_{\nu_1 \nu_2}(X) \phi^{\nu_1} \phi^{\nu_2} \right] \].

(4.32)

This is similar to the open string case eq. (2.1), except that now we have to sum over two directions of the contour. So the background field expansion of this action parallels that of the open string case, just replace \( \tau \) in eq. (2.6) with \( z \). Unlike the Yang-Mills case where the sum over Wilson loops is only calculable in lattice theory, here for the heterotic string, the sum over Wilson loops is practical: pick an arbitrary loop \( C \), sum the Wilson loop over both directions \( \pm C \), then the result is the classical action of the gauge field in spacetime.

### 4.4 Path ordering and contour direction

Since there are two directions of the contour, we need to give a comment about its relation with the path ordering of the gauge factors. We will take the convention of distinguishing the path ordering of the gauge field and the direction of the contour, i.e., we treat them as two different kinds of ordering. Firstly, we define the direction \( C_+ \) and \( C_- \) for a contour loop. Since we are using the upper half plane for the open string world-sheet, we define \( C_+ \) to be from \( -\infty \) to \( \infty \) on the real axis and \( C_- \) is just its inverse. Then, for a given vertex structure of a Feynman diagram, the gauge factors of the vertices are defined to be along the direction of \( C_+ \). Finally, when calculate this Feynman diagram, we just compute the integrand along \( C_+ \) in the open string case, and compute the integrand along both \( C_+ \) and \( C_- \) in the heterotic string case. By this convention, we can just focus on the computation of the integrand, and leave the vertex structure of gauge factors aside.
Chapter 5

Sv map in the fermionic representation

For the bosonic open string sigma model, the three–loop diagrams of the abelian case have been studied in [69]. The world–sheet integrals for the non–abelian diagrams of open string sigma model, unlike the abelian case, are path–ordered. It is rather difficult to deal with these path-ordered integrals by using traditional momentum space integrals and dimensional regularization. Instead, we will use world-sheet position integrals and a short-distance cutoff. So when two vertices, at the boundary position $t_1$ and $t_2$, approach each other, there will be a singularity regularized by $|t_1 - t_2| \geq \epsilon$. (For simplicity, in the following we will use $t$ for coordinates on the boundary of the open string world–sheet, instead of $\tau$.) In our regularization scheme, the path-ordered integrals can be converted into integrals of hyperlogarithms [84], which are directly connected to MZVs.

For the heterotic string, the three–loop computation appears in [74], in the framework of superspace and momentum space integrals. Since our goal is to examine the sv–map between open and heterotic diagrams, we need to use the same regularization scheme for both theories. So for the heterotic diagrams, we will integrate on the complex plane directly, without going to the momentum space. We will use a brute force cutoff: when two vertices at $z_1$ and $z_2$ approach each other, the singularity is regularized by $|z_1 - z_2| \geq \epsilon$.

It is very hard to deal with the full set of diagrams in the nonabelian case, even at three–
loops. So instead of pursuing the complete renormalization program, we have a more modest goal: for a given Feynman diagram, we want to show that the coefficients of the UV divergent single logarithmic $\ln \epsilon$ terms of the open and heterotic string integrals satisfy the sv–map. This makes sense given the one-to-one correspondence between Feynman diagrams. Nevertheless, such a comparison is quite subtle because a generic diagram contains also higher powers of logarithms, so the coefficients of single logarithmic terms are regularization dependent, as is any beta function beyond two–loops. The problem essentially boils down to finding a sv–map compatible regularization prescription.

Since we are interested in the single trace terms only, we consider diagrams with one boundary, a single fermion $\psi$ line in the heterotic case, and no $\psi$ loops. We will focus on the diagrams involving bosonic loops. They are non-vanishing at any loop order, although their fermionic counterparts may eventually lead to cancellations.

5.1 The Gegenbauer polynomial for angular integration

For the heterotic case, the Feynman integrals are integrated over the complex plane and we are using the direct cutoff $|z_1 - z_2| \geq \epsilon$. To compute the integral, we use polar coordinate $z = xe^{i\alpha}$. Our algorithm for its computation is to firstly integrate over the angle variable $\alpha$ and then integrate over the radial variable $x$. We need to use the method of Gegenbauer polynomials for the angle integral.

Gegenbauer polynomials $C_m$ appear in the series expansion \([85]\):

$$\frac{1}{(1 - 2ax + x^2)^p} = \sum_{n=0}^{\infty} C_n^{(p)}(a) \ x^n , \ |x| < 1 , \ |a| \leq 1 . \quad (5.1)$$
For our purposes, we need the special case

\[
\frac{1}{|1 - xe^{i\alpha}|^2} = \frac{1}{1 - 2 \cos \alpha + x^2} = \begin{cases} 
\sum_{n=0}^{2} C_n^{(1)}(\cos \alpha) \frac{1}{x^n}, & 0 < x < 1, \\
\sum_{n=0}^{2} C_n^{(1)}(\cos \alpha) \frac{1}{x^{n+2}}, & x > 1.
\end{cases}
\]  

where (see e.g. [86])

\[C_n^{(1)}(\cos \alpha) = \sum_{k,l \geq 0: k+l=n} \cos[(k-l)\alpha].\]  

In order to compute angular integrals, we need the following formula

\[
\int_{0}^{2\pi} d\alpha \: e^{i\rho\alpha} C_n^{(1)}(\cos \alpha) = \begin{cases} 
2\pi, & n = 2j + p, \quad j \geq 0, \\
0, & \text{otherwise},
\end{cases}
\]  

which is easy to prove by using elementary methods.

### 5.2 The sv–map at three loops

In the following, we will focus on the diagram shown in Figure 5.1. It contributes to the sigma model an ultra-violet divergent Lagrangian term of the form \( \partial X^\nu D_{\mu_1} F_{\nu}^{\mu_3} F_{\mu_4}^{\mu_1} \).

#### 5.2.1 Open string three–loop integral

The open string integral associated to the Feynman diagram of Figure 5.1 is given by

\[
I_{3o} = \int_{-\infty < t_1 < t_2 < t_3 < \infty} dt_1 dt_2 dt_3 \frac{\ln t_{21}}{t_{31} t_{32}}.
\]  


Short-distance singularities appear when the segments $t_{j+1,j} = t_{j+1} - t_j$ shrink to zero, i.e. when the vertices at $t_j$ and $t_{j+1}$ become coalescent. With the following change of variables:

$$w = t_{31}, \quad u = \frac{t_{21}}{t_{31}} = \frac{t_{21}}{w} \quad (0 < u < 1), \quad (5.6)$$

we obtain

$$I_{30} = \int_{-\infty}^{\infty} dt_1 \int_{\epsilon}^{\mu} \frac{dw}{w} \int_{\epsilon}^{1-\epsilon} du \frac{\ln w + \ln u}{1 - u} \quad (5.7)$$

where we also imposed an infrared cutoff $\mu$ on $w$. We focus, however, on short-distance singularities. The leading $\ln^3 \epsilon$ singularity comes from the $\ln w$ term in the numerator. The single logarithmic term is easy to isolate:

$$I_{30} = \int_{-\infty}^{\infty} dt_1 \zeta_2 \ln \epsilon + \ldots \quad (5.8)$$

where we neglected higher logarithmic singularities as well as finite terms. The origin of $\zeta_2$ is simple to see from elementary integrals

$$\int_0^1 du \frac{\ln u}{1 - u} = \int_0^1 dx \frac{\ln (1 - x)}{x} = -\text{Li}_2(1) = -\zeta_2.$$

![Figure 5.1: The Feynman diagram corresponding to structure $\partial X^\nu D_{\mu_1} F_{\nu}^{\mu_3} F_{\mu_3}^{\mu_4} F_{\mu_4}^{\mu_1}$. For the open string case, the solid line represents the boundary and $F(j) = F(t_j)$. For the heterotic string case, the solid line represents the propagator of $\psi$ and $F(j) = F(z_j)$.](image-url)
5.2.2 Heterotic string three–loop integral

The heterotic sigma model integral corresponding to the Feynman diagram of Figure 5.1 is

\[
I_{3h} = \int d^2 z_i \prod_{i=1}^{3} \ln \left| z_{12} \right|^2 \frac{\ln \left| z_{13} \right|^2}{z_{12} z_{23} z_{23} z_{13}}.
\]  

(5.9)

We use the same type of variables as for open strings (5.6):

\[
z_{31} = r e^{i \theta}, \quad \frac{z_{21}}{z_{31}} = x e^{i \alpha}.
\]

(5.10)

The integral becomes

\[
I_{3h} = \int d^2 z_1 \int \frac{d^2 z_{31}}{|z_{31}|^2} \int dx da \frac{\ln \left| z_{31} \right|^2 + \ln x^2 \ln \left| 1 - x e^{i \alpha} \right|^2}{e^{-i \alpha} e^{-i \alpha} |1 - x e^{i \alpha}|^2}
\]

(5.11)

Note that x integration covers whole complex plane, \(0 < x < \infty\), unlike the analogous open string variable \(u\) that covers the range \(0 < u < 1\) only. With the complex coordinate system centered at \(z_1\), \(0 < x < 1\) corresponds to radial ordering \([z_2, z_3]\) while \(1 < x < \infty\) to \([z_3, z_2]\). This integration yields different results in the two radial ordering regions. It is convenient to define \(u = x^2, w = r^2\).

Using the Gegenbauer expansion eq. (5.2) the integral becomes

\[
I_{3h} = (2\pi) \int d^2 z_1 \int_{e}^{\infty} \frac{dr}{r} \int dx da \ln (r^2 x^2)^{i \alpha} \left\{ \sum_{m=0}^{\infty} C_m^{(1)} (\cos \alpha) x^m, 0 < x < 1, \right. \\
\left. \sum_{m=0}^{\infty} C_m^{(1)} (\cos \alpha) \frac{1}{x^{m+2}}, x > 1. \right. 
\]

(5.12)
Using the formula eq. (5.4) to perform the angle integration, we get the following

\[ I_{3h} = (2\pi)^2 \int d^2 z_1 \int \frac{dr}{r} \int dx \ln (r^2 x^2) \sum_{k=0}^{\infty} \frac{x^{2k+1}}{x^{2k+3}}, 0 < x < 1 \]

\[ = (2\pi)^2 \int d^2 z_1 \int \frac{dr}{r} \int dx \ln (r^2 x^2) \left\{ \begin{array}{ll} \frac{x}{1-x^2}, & 0 < x < 1, \\ \frac{1}{x(x^2-1)}, & x > 1 \end{array} \right. \]

\[ = \pi^2 \int d^2 z_1 \int \frac{dw}{w} \int du \ln uw \left\{ \begin{array}{ll} \frac{1}{1-u}, & 0 < u < 1, \\ \frac{1}{u(u-1)}, & u > 1 \end{array} \right. \]

(5.13)

The radial integral above has two integration regions. For the first integral region \( 0 < u < 1 \), we get

\[ I_{3h} = \pi^2 \int d^2 z_1 \int_{\epsilon}^{\mu} \frac{dw}{w} \int_{\epsilon}^{1-\epsilon} du \ln u \frac{1}{1-u} = \pi^2 \int d^2 z_1 (\zeta_2 \ln \epsilon), \]  

(5.14)

where the IR divergence is thrown away as usual. This is the same radial integral as the path-ordered open integral (5.7). Within our regularization, among all the radial ordered regions of the heterotic integral, there is always one radial integral that matches the open integral. For the second integral region \( u > 1 \), we get

\[ I_{3h} = \pi^2 \int d^2 z_1 \int_{\epsilon}^{\mu} \frac{dw}{w} \int_{1+\epsilon}^{\infty} du \frac{\ln u}{u(u-1)} = \pi^2 \int d^2 z_1 (-\zeta_2 \ln \epsilon), \]  

(5.15)

where the IR divergence is thrown away. The origin of \( \zeta_2 \) in the second integral is simple to get

\[ \int_{1}^{\infty} du \frac{\ln u}{u(u-1)} \bigg| _{x=1/u} = \int_{0}^{1} dx \frac{\ln x}{1-x} = \zeta_2. \]

The sum of these two integration regions cancel each other, so the final result is zero

\[ I_{3h} = 0 + \ldots, \]  

(5.16)
which is consistent with the sv–map, $sv(\zeta_2) = 0$.

5.3 The sv–map at four loops

We will be considering three representative four–loop diagrams shown in Figures 5.2, 5.3 and 5.4. The respective contributions to the beta function probe single-trace effective action terms in which the Lorentz indices of five $F_{\mu\nu}$ tensors are contracted in various ways. For each diagram, there are four vertices hence three intervals on the open string boundary. Ultra-violet singularities appear in the limit when the vertices coalesce, i.e. when one or more intervals shrink to zero size. With the short distance cutoff $\epsilon$, the logarithmic singularities can be as strong $\ln^4 \epsilon$, therefore the single logarithmic terms of interest are very sensitive to the way how this cutoff is imposed, that is to the choice of integration variables and the order in which the intervals shrink to minimum size. Open string positions are real while the heterotic ones are complex, containing radial and angular parts, therefore it is no possible to choose identical variables. There is however, a natural choice of “matching” variables, similar to what we used for three–loops, such that after integrating out the angles, one radial ordering of the heterotic string vertex positions, is the same as “time” ordering on the boundary and yields exactly the same integral as in the open string case. We will show that single value projection appears as a result of adding all radial orderings.

5.3.1 Figure 5.2

The vertex structure of this Feynman diagram is $\partial X^\nu D_{\mu_1} F_{\nu \mu_2} F_{\mu_3} F_{\mu_4} F_{\mu_5}$. 

85
Figure 5.2: The Feynman diagram contributing to $\partial X^\nu D_{\mu_1} F_\nu \mu_3 F_{\mu_3} \mu_4 F_{\mu_4} \mu_5 F_{\mu_5} \mu_1$.

(a) Open string

The open string integral is

$$I_{4o} = \int_{-\infty}^{-\infty} dt_1 dt_2 dt_3 dt_4 \frac{\ln t_{21}}{t_{41} t_{32} t_{43}}. \quad (5.17)$$

With the following change of variables

$$w = t_{41}, \quad u = \frac{t_{21}}{t_{31}}, \quad v = \frac{t_{31}}{t_{41}}, \quad (5.18)$$

we obtain

$$I_{4o} = \int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} dt_{41} \int_{0}^{t_{41}} dt_{31} \int_{0}^{t_{31}} dt_{21} \frac{\ln t_{21}}{t_{41}(t_{31} - t_{21})(t_{41} - t_{31})}$$

$$= \int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} \frac{dw}{w} \int_{0}^{1} dv \int_{0}^{1} du \frac{\ln(wuv)}{(1-u)(1-v)}. \quad (5.19)$$

It is easy to see that no single logarithmic $\ln \epsilon$ terms appear after integrations. This is consistent with the results of [17], where no effective action terms were found corresponding to the respective part of the beta function.
Chapter 5 Section 5.3

(b) Heterotic string

The heterotic string integral corresponding to this Feynman diagram is

\[
I_{4h} = \int \prod_{j=1}^{j=4} d^2 z_j \frac{1}{z_{12}z_{23}z_{34}z_{23}z_{34}z_{14}} \ln |z_{12}|^2.
\] (5.20)

We use the same type of variables as for the open string (5.18):

\[
z_{41} = re^{i\theta}, \quad \frac{z_{21}}{z_{31}} = xe^{i\alpha}, \quad \frac{z_{31}}{z_{41}} = ye^{i\beta},
\] (5.21)

The integral becomes

\[
I_{4h} = 2\pi \int d^2 z_1 \int \frac{dr}{r} \int dxdy \int d\alpha d\beta \frac{\ln r^2 + \ln x^2 + \ln y^2}{e^{-i\alpha}e^{-i\beta}|1 - xe^{i\alpha}|^2|1 - ye^{i\beta}|^2}.
\] (5.22)

The angular integrals can be performed by using the Gegenbauer method

\[
I_{4h} = 2\pi \int d^2 z_1 \int \frac{dr}{r} \int dxdy \ln (r^2 x^2 y^2) \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta e^{i\alpha} e^{i\beta}
\left\{ \begin{array}{l}
\sum_{m=0}^{\infty} C_m^{(1)} (\cos \alpha) x^m, \quad x < 1 \\
\sum_{m=0}^{\infty} C_m^{(1)} (\cos \alpha) \frac{1}{x^{m+2}}, \quad x > 1
\end{array} \right\} \times \left\{ \begin{array}{l}
\sum_{n=0}^{\infty} C_n^{(1)} (\cos \beta) y^n, \quad y < 1 \\
\sum_{n=0}^{\infty} C_n^{(1)} (\cos \beta) \frac{1}{y^{n+2}}, \quad y > 1
\end{array} \right\}.
\] (5.23)

Expressed in terms of \(u = x^2\), \(v = y^2\) and \(w = r^2\), the result after the angle variable
Figure 5.3: The Feynman diagram contributing to $\partial X^\nu D^{\mu_1} F_\nu^{\mu_3} F_{\mu_3}^{\mu_4} F_{\mu_4}^{\mu_5} F_{\mu_5}$. Integration has four radial orderings:

$$I_{4h} = \pi^3 \int d^2 z_1 \int_{\epsilon}^u \frac{dv}{w} \int du \int dv \ln(wuv) \times \left\{ \begin{array}{ll}
\frac{1}{(1-u)(1-v)}, & u, v \in (0,1), \\
\frac{1}{v(v-1)(1-u)}, & u < 1 < v, \\
\frac{1}{u(u-1)(1-v)}, & v < 1 < u, \\
\frac{1}{uv(u-1)(v-1)}, & u, v \in (1,\infty).
\end{array} \right.$$ (5.24)

It is easy to see that although all four regions contribute ultraviolet divergent terms, none of them yields a single logarithmic $\ln \epsilon$ term. Hence the diagram under consideration does not contribute to the respective effective action term neither in open nor in heterotic string theories.

5.3.2 Figure 5.3

The vertex structure of this Feynman diagram is $\partial X^\nu D^{\mu_1} F_\nu^{\mu_3} F_{\mu_3}^{\mu_4} F_{\mu_4}^{\mu_5} F_{\mu_5}$. (a) Open string

The open string integral is

$$J_{4o} = \int_{-\infty < t_1 < t_2 < t_3 < t_4 < \infty} dt_1 dt_2 dt_3 dt_4 \frac{\ln t_{21}}{t_{31} t_{42} t_{43}}. \quad (5.25)$$

88
With the same change of variables as in Eq. (5.18), we obtain

\[
J_{4o} = \int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} dt_{41} \int_{0}^{t_{41}} dt_{31} \int_{0}^{t_{31}} dt_{21} \frac{\ln t_{21}}{t_{31}(t_{41} - t_{21})(t_{41} - t_{31})}
\]

\[
= \int_{-\infty}^{\infty} dt_1 \int_{\epsilon}^{\infty} dw \int_{\epsilon}^{1-\epsilon} dv \int_{\epsilon}^{1-\epsilon} du \frac{\ln(wuv)}{(1 - uv)(1 - v)}. \tag{5.26}
\]

The single logarithmic term is easy to isolate:

\[
J_{4o} = \int_{-\infty}^{\infty} dt_1 \zeta_3 \ln \epsilon + \ldots \tag{5.27}
\]

where we neglected higher logarithmic singularities as well as finite terms.

(b) Heterotic string

The heterotic string integral corresponding to this Feynman diagram is

\[
J_{4h} = \int \prod_{j=1}^{i=4} d^2 z_j \frac{1}{z_{12}z_{23}z_{34}z_{13}z_{24}z_{34}} \ln |z_{12}|^2. \tag{5.28}
\]

We use the same integration variable as in Eq. (5.21), in terms of which

\[
J_{4h} = 2\pi \int d^2 z_1 \int \frac{dr}{r} \int dx dy d\alpha d\beta \frac{\ln r^2 + \ln x^2 + \ln y^2}{e^{-i\alpha}e^{-i\beta}(1 - xe^{-i\alpha})(1 - ye^{i\alpha}e^{i\beta})(1 - ye^{i\beta})^2}. \tag{5.29}
\]
In this case, the angular integrals are slightly harder but can be handled by using the Gegenbauer method:

\[
J_{4h} = 2\pi \int d^2 z_1 \int \frac{dr}{r} \int dxdy \ln\left(r^2 x^2 y^2\right) \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta e^{i\alpha} e^{i\beta} \times \\
\left\{ \begin{array}{l}
\sum_{l=0}^{\infty} x^l e^{-l i\alpha}, \quad x < 1 \\
\sum_{l=0}^{\infty} \frac{1}{x^{l+1}} e^{(l+1)i\alpha}, \quad x > 1
\end{array} \right\}
\times \\
\left\{ \begin{array}{l}
\sum_{m=0}^{\infty} x^m y^m e^{m i\alpha} e^{m i\beta}, \quad xy < 1 \\
\sum_{m=0}^{\infty} -\frac{1}{x^{m+1} y^{m+1}} e^{-(m+1)i\alpha} e^{-(m+1)i\beta}, \quad xy > 1
\end{array} \right\}
\times \\
\left\{ \begin{array}{l}
\sum_{n=0}^{\infty} C_n^{(1)} (\cos \beta) y^n, \quad y < 1 \\
\sum_{n=0}^{\infty} C_n^{(1)} (\cos \beta) \frac{1}{y^{n+2}}, \quad y > 1
\end{array} \right\}.
\] (5.30)

After integrating out the angle variable, they yield six radial orderings (here again, we use \( u = x^2, v = y^2 \) and \( w = r^2 \)):

\[
J_{4h} = \pi^3 \int d^2 z_1 \int_{\mu}^{\infty} \frac{dw}{w} \int du \int dv \ln(wuv) \times \\
\left\{ \begin{array}{l}
\frac{1}{(1 - v)(1 - uv)}, \quad u, v \in (0, 1) \rightarrow \bar{\zeta}_3 \ln \epsilon, \\
\frac{1}{v(v - 1)(1 - u)}, \quad u < 1/v < 1 \rightarrow -2\bar{\zeta}_3 \ln \epsilon, \\
0, \quad v < 1/u < 1 \rightarrow 0, \\
\frac{1}{uv(v - 1)(u - 1)}, \quad 1/u < v < 1 \rightarrow \bar{\zeta}_3 \ln \epsilon, \\
-\frac{1}{uv(v - 1)}, \quad 1/v < u < 1 \rightarrow -\bar{\zeta}_3 \ln \epsilon, \\
\frac{1}{uv(v - 1)(uv - 1)}, \quad u, v \in (1, \infty) \rightarrow -\bar{\zeta}_3 \ln \epsilon.
\end{array} \right\}
\] (5.31)
\[ J_{4h} = \pi^3 \int d^2z_1 \frac{2}{\zeta_3} \ln \epsilon + \ldots. \tag{5.32} \]

Comparing with Eq. (5.27), we find the result in agreement with \( \text{sv}(\zeta_3) = 2\zeta_3 \).

### 5.3.3 Figure 5.4

The vertex structure of this Feynman diagram is \( \partial X^\nu D_{\mu_1} F_{\nu} \mu_3 F_{\mu_4} \mu_5 F_{\mu_4} \mu_3 F_{\mu_5} \mu_1 \).

(a) Open string

The open string integral is

\[ K_{4o} = \int_{-\infty<t_1<t_2<t_3<t_4<\infty} dt_1 dt_2 dt_3 dt_4 \frac{\ln t_{31}}{t_{41} t_{42} t_{32}}. \tag{5.33} \]
With the same change of variables as in Eq. (5.18), we obtain

\[
K_{4o} = \int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} dt_{41} \int_{0}^{t_{41}} dt_{31} \int_{0}^{t_{31}} dt_{21} \frac{\ln t_{31}}{t_{41}(t_{41} - t_{21})(t_{31} - t_{21})}
\]
\[
= \int_{-\infty}^{\infty} dt_1 \int_{e}^{\mu} dw \int_{e}^{1-e} dv \int_{e}^{1-e} du \frac{\ln(wv)}{(1-wv)(1-u)} .
\]

(5.34)

It is easy to see that

\[
K_{4o} = -\int_{-\infty}^{\infty} dt_1 \zeta_3 \ln \epsilon + \ldots
\]

(5.35)

where we neglected higher logarithmic singularities as well as finite terms.

(b) Heterotic string

The heterotic string integral corresponding to this Feynman diagram is

\[
K_{4h} = \int \prod_{j=1}^{i=4} d^2 z_j \frac{1}{z_{12}z_{23}z_{34}} \ln |z_{13}|^2 .
\]

(5.36)

Using the variables defined in Eq. (5.21),

\[
K_{4h} = 2\pi \int d^2 z_1 \int \frac{dr}{r} \int dx dy \, d\alpha d\beta \frac{\ln r^2 + \ln y^2}{e^{-i\alpha}e^{-i\beta}(1-ye^{-i\beta})(1-xye^{i\alpha}e^{i\beta})}[1-xe^{i\alpha}]^2 .
\]

(5.37)
The Gegenbauer expansion is

\[
K_{4h} = 2\pi \int d^2z_1 \int \frac{dr}{r} \int dx dy \ln (r^2 y^2) \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta e^{i\alpha} e^{i\beta} \times
\]

\[
\left\{ \begin{array}{ll}
\sum_{l=0}^{\infty} y^l e^{-l i \beta}, & y < 1 \\
\sum_{l=0}^{\infty} -\frac{1}{y^{l+1}} e^{(l+1) i \beta}, & y > 1 \\
\end{array} \right.
\]

\[
\times \left\{ \begin{array}{ll}
\sum_{m=0}^{\infty} x^m y^m e^{m i \alpha} e^{m i \beta}, & xy < 1 \\
\sum_{m=0}^{\infty} -\frac{1}{x^{m+1} y^{m+1}} e^{-(m+1) i \alpha} e^{-(m+1) i \beta}, & xy > 1 \\
\end{array} \right.
\]

\[
\times \left\{ \begin{array}{ll}
\sum_{n=0}^{\infty} C_n^{(1)} (\cos \alpha) x^n, & x < 1 \\
\sum_{n=0}^{\infty} C_n^{(1)} (\cos \alpha) \frac{1}{x^{n+2}}, & x > 1 \\
\end{array} \right\}.
\]

(5.38)

After performing angular integrations, we obtain six radial orderings (here again, we use \(u = x^2\), \(v = y^2\) and \(w = r^2\):

\[
K_{4h} = \pi^3 \int d^2z_1 \int_0^\mu \frac{dw}{w} \int \frac{du}{u} \int \frac{dv}{v} \ln (wv) \times
\]

\[
\left\{ \begin{array}{ll}
\frac{1}{(1-u)(1-uv)}, & u, v \in (0, 1) \rightarrow -\zeta_3 \ln e, \\
0, & u < 1/v < 1 \rightarrow 0, \\
\frac{1}{u(u-1)(1-v)}, & v < 1/u < 1 \rightarrow -\zeta_3 \ln e, \\
-\frac{1}{uv(u-1)}, & 1/u < v < 1 \rightarrow -\zeta_3 \ln e, \\
\frac{1}{uv(1-u)(v-1)}, & 1/v < u < 1 \rightarrow -\zeta_3 \ln e, \\
\frac{1}{uv(u-1)(uv-1)}, & u, v \in (1, \infty) \rightarrow 2\zeta_3 \ln e,
\end{array} \right.
\]

(5.39)
After adding all orderings, we obtain

\[ K_{4h} = -\pi^3 \int d^2 z_1 \ 2\zeta_3 \ln \epsilon + \ldots. \]  

(5.40)

Here again, now comparing with Eq. (5.35), we find the result in agreement with \( sv(\zeta_3) = 2\zeta_3 \).

### 5.4 Summary

In this chapter, we have performed explicit three- and four-loop Feynman diagram computations to reveal the mechanism of the \( sv \) map. The integral in the open string sigma model is a real 1D path-ordered integral. The corresponding integral in the heterotic case is a 2D integral on the whole complex plane, which contains several radial ordering after integrating out the angle direction in polar coordinates. With our choice of variables and cutoffs, one of the radial orderings, the canonical one, yields the same integral as open strings. We showed that the single value projection from open to heterotic string sigma models appears as a result of summing over all six radial orderings of the heterotic integral.
Chapter 6

Sv map in the Wilson loop representation

In this chapter we will explore the sv-map using the Wilson loop representation for both the open string and the heterotic string. We will show how sv-map of $\zeta_2$ and $\zeta_3$ arises at the level of three and four loop of the sigma model Feynman diagrams. Our purpose is to find the mechanism of the sv-map, rather than compute the complete beta function. So instead of pursuing the complete renormalization, we only compute the single pole (single logarithmic divergence) and will show that for a given Feynman diagram of single-trace terms, the single poles of the open and the heterotic string integrals satisfy the sv-map. We will just focus on bosonic loops and compute a few representative diagrams of three loop and four loop, and show that sv-map connects the open string calculation and the heterotic case in each diagram. (Representative here means they corresponds to permutations of the vertex structure at a given loop level).

For the computation we use the following setup (this is a recall of what we get)

\[
\text{open: } = \int D\phi DX \mathcal{P} \exp\left\{ -\frac{1}{4\pi\alpha'} \int d^2z(\partial X^{\mu}\partial X_{\mu} + \Phi^{\mu}\partial \Phi_{\mu} + \Phi^{\mu'}\partial \Phi_{\mu'}) \right. \\
+ i \oint_{C_+} dt [\partial X^{\mu}(t) A_{\mu}(X) - \frac{1}{2} F_{\nu_1\nu_2}(X) \phi^{\nu_1} \phi^{\nu_2}] \\
\text{hete: } = \sum_{C_+,C_-} \int D\phi DX \mathcal{P} \exp\left\{ -\frac{2}{4\pi\alpha'} \int d^2z[\partial X^{\mu}\partial X_{\mu} + \Phi^{\mu}\partial \Phi_{\mu}] \\
+ i \oint_{C} dz [\partial X^{\mu}(z) A_{\mu}(X) - \frac{1}{2} F_{\nu_1\nu_2}(X) \phi^{\nu_1} \phi^{\nu_2}] \right\},
\]

(6.1)
where \( C_+ \) is the real axis, \( C_- \) is its inverse and we will use variable \( t \) for the real axis from now on. The background field expansion is given in eq. (2.6). Both the propagators are 
\[
\langle X^\mu(t_1) X^\nu(t_2) \rangle = -\eta^\mu\nu \alpha' \ln (t_1 - t_2)^2
\]
on the real axis. The bosonic propagators are represented by wavy lines and the contour of loop (the real axis) is represented by a solid line. A slash on the wavy line represents a derivative of the propagator, with respect to the most close vertex coordinate.

### 6.1 The \( \zeta(2) \) case

Now, let’s look at how the sv-map of \( \zeta(2) \) arises at three loop level. From mathematical analysis we expect that the sv-map of \( \zeta_2 \) satisfies the relation \( \text{sv}(\zeta_2) = 0 \). We will focus on the diagram shown in figure 5.1. It contributes to the sigma model an ultra-violet divergent Lagrangian term of the form \( \partial^\nu D_{\mu_1} F_{\mu_3} F_{\mu_4} F_{\mu_1} \). The open string integral associated to this diagram is

\[
I_{3,C_+} = (-2\alpha')^3 \int_{-\infty<t_1<t_2<t_3<\infty} dt_1 dt_2 dt_3 f(V(t_1), V(t_2), V(t_3)) \frac{\ln t_{21}}{t_{31} t_{32}}
\]
\[
= (-2\alpha')^3 \int_{-\infty}^{\infty} dt_3 f(V(t_3), V(t_3), V(t_3)) \int_{-\infty<t_1<t_2<t_3} dt_2 dt_1 \frac{\ln t_{21}}{t_{31} t_{32}}
\]
\[
\approx \int_{-\infty}^{\infty} dt_3 \int_{-\infty<t_1<t_2<t_3} dt_2 dt_1 \frac{\ln t_{21}}{t_{31} t_{32}}, \tag{6.2}
\]

where \( t_{jk} = t_j - t_k \) and \( f(V(t_1), V(t_2), V(t_3)) = \partial X^\nu(t_1) D_{\mu_1} F_{\mu_3} (t_1) F_{\mu_4} (t_2) F_{\mu_1} (t_3) \) is the vertex structure. In the last step we hide the constant factors \((-2\alpha')^3\) and the vertex structure, and just focus on the computation of the integrals. We will hide such constant factors and vertex structures in all the following computations.
To compute it, do the following change of variables

\[ w = t_{31} \]
\[ u = \frac{t_{32}}{t_{31}} = \frac{t_{32}}{w}, \quad 0 < u < 1, \]

which changes the original integral as following

\[
I_{3,C^+} = \int_{-\infty}^{\infty} dt_3 \int_{-\infty}^{0} dt_{13} \int_{t_{13}}^{0} dt_{23} \frac{\ln (t_{31} - t_{32})}{t_{31} t_{32}}
\]
\[
= \int_{-\infty}^{\infty} dt_3 \int_{0}^{\infty} dw \int_{0}^{1} du \frac{\ln w + \ln (1 - u)}{u}
\]
\[
= \int_{-\infty}^{\infty} dt_3 \left( -\zeta_2 (\ln \lambda - \ln \epsilon) \right),
\]

where we use the brute force cutoff \( \epsilon \) for the UV and \( \lambda \) for the IR. Only the single poles (single logarithmic divergence) are kept and higher order divergences are thrown away, the same for all the following computations.

For the heterotic integral we also need the other contour

\[
I_{3,C^-} = \int_{-\infty}^{-\infty} dt_1 \int_{t_{31} > t_{3} > t_{2} > t_{1}} dt_2 dt_3 \frac{\ln t_{21}}{t_{31} t_{32}},
\]

which is obtained in a similar manner as eq. (6.2). Notice that the contour \( C^- \) means that we start from \( \infty \) and go to \( -\infty \) (we will not explicitly mention this from now on). Here we do the following change of variables

\[ w = t_{31} \]
\[ u = \frac{t_{21}}{t_{31}} = \frac{t_{21}}{w}, \quad 0 < u < 1, \]
which changes the original integral as following

\[
I_{3, C_-} = - \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{0} dt_{31} \int_{t_{31}}^{0} dt_{21} \ln \frac{t_{21}}{t_{31} t_{32}}
\]

\[
= - \int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} dw \int_{0}^{1} du \ln \frac{w + \ln u}{1 - u}
\]

\[
= - \int_{-\infty}^{\infty} dt_1 \left(-\zeta_2 (\ln \lambda - \ln \epsilon)\right). \quad (6.7)
\]

The heterotic integral is zero, which is just a sum of $C_+ \text{ and } C_-$ given in eq. (6.4) and eq. (6.7).
So we have the $s\nu$-map at three loop $s\nu(\zeta_2) = 0$.

6.2 The zeta(3) case

Now let’s see how the $s\nu$-map of $\zeta(3)$ arises at four loop. The mathematical $s\nu$ of $\zeta_3$ is $s\nu(\zeta_3) = 2\zeta_3$ [59]. We choose three representative diagrams figure. 5.2, figure. 5.3 and figure. 5.4.

6.2.1 Case 1

Firstly we compute diagram of figure. 5.2, which has the vertex structure $\partial X^\nu D^\nu_\mu F^\mu_3 F^\mu_4 F^\mu_5 F^\mu_1$. The open string integral is

\[
I_{41, C_+} = \int_{-\infty}^{\infty} dt_4 \int_{-\infty}^{t_1} dt_2 dt_3 \ln \frac{t_21}{t_{41} t_{32} t_{43}}. \quad (6.8)
\]

Using the following change of variables

\[
w = t_{41}
\]
\[
v = \frac{t_{42}}{t_{41}} = \frac{t_{42}}{w}
\]
\[
u = \frac{t_{43}}{t_{42}}, 0 < u, v < 1. \quad (6.9)
\]
we get

\[
I_{41,C_+} = \int_{-\infty}^{\infty} dt_4 \int_{-\infty}^{0} dt_{14} \int_{t_{14}}^{0} dt_{24} \int_{t_{24}}^{0} dt_{34} \int_{t_{41}}^{\infty} dt_{42} \int_{t_{42}}^{\infty} dt_{43} \frac{\ln (t_{41} - t_{42})}{t_{41} (t_{42} - t_{43}) t_{43}} \\
= \int_{-\infty}^{\infty} dt_4 \int_{0}^{\infty} \frac{dw}{w} \int_{0}^{1} dv \int_{0}^{1} du \frac{\ln w + \ln (1 - v)}{uv(1 - u)} \\
= 0. \tag{6.10}
\]

There are no single poles. This is consistent with the results of \[17\], where no effective action terms were found corresponding to this respective part of the beta function.

The heterotic string integral needs in addition the other contour $C_-$

\[
I_{41,C_-} = \int_{-\infty}^{\infty} dt_1 \int_{\infty}^{0} dt_{41} \int_{t_{41}}^{0} dt_{31} \int_{t_{31}}^{0} dt_{21} \frac{\ln t_{21}}{t_{41} t_{32} t_{43}}. \tag{6.11}
\]

Using the change of variables

\[
w = t_{41} \]
\[
v = \frac{t_{31}}{t_{41}} = \frac{t_{31}}{w} \]
\[
u = \frac{t_{21}}{t_{31}}, 0 < u, v < 1. \tag{6.12}
\]

we get

\[
I_{41,C_-} = -\int_{-\infty}^{\infty} dt_1 \int_{\infty}^{0} dt_{41} \int_{t_{41}}^{0} dt_{31} \int_{t_{31}}^{0} dt_{21} \frac{\ln (t_{21})}{t_{41} (t_{31} - t_{21}) (t_{41} - t_{31})} \\
= \int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} \frac{dw}{w} \int_{0}^{1} dv \int_{0}^{1} du \frac{\ln w + \ln v + \ln u}{(1 - u)(1 - v)} \\
= 0. \tag{6.13}
\]

The heterotic integral is the sum of eq. (6.10) and eq. (6.13), while the open string integral is just eq. (6.10). So we see that $sv(0) = 0$. 

99
6.2.2 Case 2

Firstly we compute diagram of figure. \[5.3\], which has the vertex structure \(\partial X D_{\mu_1} F_{\nu}^{\mu_2} F_{\mu_3}^{\mu_4} F_{\mu_5}^{\mu_4} F_{\mu_6}^{\mu_5}\).

The open string integral is

\[ I_{42,C_+} = \int_{-\infty}^{\infty} dt_4 \int_{-\infty}^{0} dt_1 \int_{t_1}^{0} dt_2 \int_{t_2}^{0} dt_3 \frac{\ln t_21}{t_31} \cdot t_42t_43. \] \[(6.14)\]

Using the change of variables eq. (6.9), we get

\[ I_{42,C_+} = \int_{-\infty}^{\infty} dt_4 \int_{-\infty}^{0} dt_1 \int_{t_1}^{0} dt_2 \int_{t_2}^{0} dt_3 \frac{\ln t_41 - t_42}{(t_41 - t_43)t_42t_43} \]
\[ = \int_{-\infty}^{\infty} dt_4 \int_{0}^{\infty} \frac{dw}{w} \int_{0}^{1} dv \int_{0}^{1} du \frac{\ln w + \ln (1 - v)}{uv(1 - uv)} \]
\[ = \int_{-\infty}^{\infty} dt_4 ( -2\zeta_3(\ln \lambda - \ln e) ). \] \[(6.15)\]

The heterotic string integral requires that we compute the integral on \(C_-\)

\[ I_{42,C_-} = \int_{\infty}^{\infty} dt_1 \int_{\infty}^{\infty} dt_3 \int_{t_3}^{\infty} dt_2 \frac{\ln t_21}{t_3t_42t_43}. \] \[(6.16)\]

Using the change of variables eq. (6.12) we get

\[ I_{42,C_-} = -\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_3 \int_{t_3}^{\infty} dt_2 \frac{\ln t_21}{t_3(t_41 - t_21)(t_41 - t_31)} \]
\[ = \int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} \frac{dw}{w} \int_{0}^{1} dv \int_{0}^{1} du \frac{\ln w + \ln v + \ln u}{(1 - uv)(1 - v)} \]
\[ = \int_{-\infty}^{\infty} dt_1 ( -\zeta_3(\ln \lambda - \ln e) ). \] \[(6.17)\]

The heterotic integral is the sum of eq. (6.15) and eq. (6.17), while the open string integral is just eq. (6.15). We see that the sv-map for \(\zeta_3\) is recovered modulo a 3/4 factor which will be discussed later in this section \(sv(\zeta_3) = \frac{3}{4} \times 2\zeta_3\).
6.2.3 Case 3

Firstly we compute diagram of figure. 5.4, which has the vertex structure $\partial X^v D_{\mu_1} F_{\mu_3} F_{\mu_4} F_{\mu_5}$. The open string integral is

$$I_{43, C^+} = \int_{-\infty}^{\infty} dt_4 \int_{-\infty}^{0} dt_1 \int_{t_1}^{0} dt_2 \int_{t_2}^{0} dt_3 \frac{\ln t_3}{t_4 t_1 t_2 t_3 t_2 t_3}. \quad (6.18)$$

Using the change of variable eq. (6.9) we get

$$I_{43, C^+} = \int_{-\infty}^{\infty} dt_4 \int_{0}^{\infty} dt_1 \int_{t_1}^{0} dt_2 \int_{t_2}^{0} dt_3 \frac{\ln t_4 - t_3}{t_4 t_1 t_2 (t_4 - t_3)}$$
$$= \int_{-\infty}^{\infty} dt_4 \int_{0}^{\infty} \frac{dw}{w} \int_{0}^{1} dv \int_{0}^{1} du \frac{\ln w + \ln (1 - uv)}{v(1 - u)}$$
$$= \int_{-\infty}^{\infty} dt_4 (2\zeta_3 (\ln \lambda - \ln \epsilon)). \quad (6.19)$$

The heterotic string integral needs the other contour

$$I_{43, C^-} = \int_{\infty}^{-\infty} dt_1 \int_{\infty}^{t_4 > t_3 > t_2 > t_1} dt_4 dt_3 dt_2 \frac{\ln t_3}{t_4 t_1 t_2 t_3 t_2 t_3}. \quad (6.20)$$

Using the change of variables eq. (6.12) we get

$$I_{43, C^-} = -\int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} dt_4 \int_{t_4}^{0} dt_3 \int_{t_3}^{0} dt_2 \int_{t_2}^{0} dt_1 \frac{\ln t_3}{t_4 (t_4 - t_2)(t_3 - t_2)}$$
$$= \int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} \frac{dw}{w} \int_{0}^{1} dv \int_{0}^{1} du \frac{\ln w + \ln v}{(1 - u)(1 - uv)}$$
$$= \int_{-\infty}^{\infty} dt_1 (\zeta_3 (\ln \lambda - \ln \epsilon)). \quad (6.21)$$

The heterotic integral is the sum of eq. (6.19) and eq. (6.21), while the open string integral is just eq. (6.19). We see again that the sv-map for $\zeta_3$ is recovered modulo a 3/4 factor $sv(\zeta_3) = (\frac{3}{4}) \times 2\zeta_3$. So from our computation, we have $sv(\zeta_3) = (\frac{3}{4}) \times 2\zeta_3$. There is a total factor of 3/4 here,
compared with the mathematical result. We argue that this factor is just a normalization factor for all the diagrams at four loop level, so it can be incorporated into the action, because we found this same factor in both section 6.2.2 and section 6.2.3.

6.3 Summary

In this chapter, we demonstrated that the sv–map comes from a sum of two opposite-directed integral contours, using the Wilson loop representation for both the open and heterotic string sigma models. In the previous chapter 5 using the fermionic representation, the sv-map is shown to come from a sum of all radial orderings of heterotic vertices on the complex plane. So the Wilson loop representation gives to the sv-map a simpler geometric origin, which comes from the nonabelian Stokes's theorem. Based on these, we conjecture that the sv-map of a general MZV comes from this sum of contours of the Wilson loop representation.
Chapter 7

Conclusion

We addressed the question how the relations between open and heterotic superstring scattering amplitudes discovered in [60] are reflected by the properties of the effective gauge field theory. According to [60], the amplitudes describing the scattering of gauge bosons in both theories, more precisely the amplitudes involving a given single trace gauge group factor, are related by the sv map which maps open amplitudes to heterotic amplitudes order by order in the $\alpha'$ expansion. To that end, we studied the sigma models describing world–sheet dynamics of strings propagating in ambient spacetime endowed with gauge field backgrounds. This framework allows reconstructing the effective action order by order in $\alpha'$ by studying the beta functions associated to the couplings of background gauge field to the string world–sheet. The requirement of the vanishing beta functions leads to background field equations generated by the effective action.

For open strings, the string coupling to background gauge fields is described by a one dimensional Wilson loop with the vertex at the boundary. In the heterotic sigma model, there are two representations for the gauge fields, the fermionic representation and the Wilson loop representation. In the fermionic representation, the vertices spread over two–dimensional world–sheet, while in the Wilson loop representation the vertices are along two opposite-directed Wilson loop contours. The form of the vertices however, is very similar in all these cases. This al-
allows reformulating perturbation theory in terms of identical Feynman diagrams. For each loop correction to the background-string coupling in the heterotic sigma model, there is a matching open string sigma model coupling at the boundary. In open string theory, its coefficient is a path-ordered real integral over vertex positions. For the heterotic theory, the vertices are integrated over the whole complex plane in the fermionic representation and integrated along two opposite-directed contours in the Wilson loop representation. For open strings, the path order determines the gauge group factor. For the heterotic case in the fermionic representation, the gauge group factor is determined by an anti-holomorphic factor hence the vertices come in all radial orderings. For the Wilson loop representation of the heterotic case, the gauge group factor is determined by the Wilson loop itself as in the open string case.

We performed explicit three– and four–loop Feynman diagram computations supporting the conjecture that the beta functions, hence the effective background field actions of open and heterotic superstring theories, are related by the sv map. The starting point is one-to-one matching between complex and real integrands emerging from a given Feynman diagram. The position of Feynman vertices on the world-sheet can be parameterized by complex variables that match their real, open string counterparts at the boundary. We introduced a short-distance cutoff $\epsilon$ and extracted the beta functions from the ultraviolet divergent coefficients of single-logarithmic terms $\sim \ln \epsilon$. In general, these coefficients (hence the beta functions) depend on the details of the cutoff, in particular how the cutoff is imposed on the integration variables. With our choice of variables and cutoffs however, one of the radial orderings, the canonical one, yields the same integral as open strings. In the fermionic representation of the heterotic sigma model, we showed that the sv map from open to heterotic string sigma models appears as a result of summing over all radial orderings of the gauge vertices on the complex plane. In the Wilson loop representation of the heterotic sigma model, we showed that the sv map comes from the sum of Wilson loop integral along two opposite-directed contours.

It would be very interesting to find a rigorous proof of a general sv relation between all ordered and complex integrals encountered in Feynman diagrams of sigma models.
Bibliography


