STRUCTURAL CONNECTIONS IN MECHANICAL DESIGN – FROM NANO- TO MACRO-SCALE

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ABSTRACT

Connections between structural members are ubiquitous in the field of mechanical design. In this dissertation we do several analyzes for different kinds of structural connections — a natural connection due to adhesion, a two-sided constraint connection, and a bolted jointed connection. This work ranges from the nano-scale, as in the adhesive contact, to the macro-scale, as in the bolted joint.

First consider a natural connection in which adhesive contact and the resulting morphology of micro-/nano-scale beams/plates on a sinusoidal/grooved surface are the subject. There are three different ways to model this behavior: a) without induced tension as has been studied in a previous investigation; b) with the stiffening effect due to the contact regions being prevented from slipping axially, thereby inducing tension due to transverse displacement; c) the case with frictional slip regions which reduces the induced tension compared to case (b). This analysis uses continuum beam theory with bending and induced tension as needed. The three different models show significant quantitative differences.

Now consider an elastic strip clamped between two rigid plates over a portion of its length. For this two-sided constraint connection, the correction for rotational stiffness within the support is taken into account. This rotational stiffness is determined by applying a distributed load (statically equivalent to a couple moment) to such a constrained infinite elastic strip in plane strain. The displacement profile is determined analytically and approximated by a straight line. The ratio of the applied moment to the slope of this line provides the rotational stiffness. Examples are given for a cantilever and a fixed-fixed plate/beam as well as for a clamped circular plate.

For a bolted joint connection, we treat this configuration as a compliant receding contact problem where the flexural deformation of the components induce variable-location prying forces. The design parameters are the joint dimensions, material properties, the bolt pre-load, and the applied force. We have obtained results
using an analytical approach with beam theories (Euler-Bernoulli and Timoshenko) and for comparison a sampling of results using a more comprehensive three-dimensional finite element elasticity analysis for cantilever and T-stub connections. The results of our analysis for the bolt force are also interpreted by the Goodman approach to predict needed bolt endurance strength under cyclic loading. By comparing the results of these three approaches, we are able to provide guidance on how to design the joint and apply the preload to prevent failure caused by repeated prying action.
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Chapter 1. Introduction

Connections between structural members are ubiquitous in the field of mechanical design. In this dissertation we do several analyzes for different kinds of structural connections ranging from the nano- to the macro-scale. One is a nano-scale connection due to adhesion, one is a two-sided constraint connection, and one is macro-scale bolted joint connection.

1.1 Background and Literature Review

1.1.1 A nano-scale connection due to adhesion

Adhesion is an attractive force existing between two contacting surfaces which tends to hold them together. At a macro-scale, a surface may seem to be really flat. However at a nano-scale the surface of the substrate may not be as flat as we think. For example, the surface of a table may seem smooth at a macro-scale. However, imagine if you zoom in, you may find an irregular or possibly grooved surface. When we place a carbon nanotube on a grooved or wavy surface, the nanotubes will be forced to deform because of the adhesion. This deformation in turn increases the adhesive interaction.

Ever since carbon nanotubes (CNTs) were discovered in 1991 [1], their excellent mechanical and electrical properties, i.e. small size, low density, high stiffness, high strength, high thermal conductivity [2–4], semi-conductivity and high metallic conductivity [5], and ohmic contact with Pd [6] and Au/Cr [7, 8], have made CNTs potentially desirable components in a variety of applications, such as material reinforcement, drug delivery systems and nano-electronics. Because of their biocompatibility and high strength, CNTs may also be used in biomedical applications, such as CNT artificial muscle [9]. CNTs can also be used for field emission displays [10,11], and such devices have demonstrated characteristics such as low turn-on electric field, high emission current density, and high stability.

Gao and Huang [12] analyzed the effect of surface roughness on the morphology and adhesion energy of substrate-supported multi-layer graphene membranes. A theoretical study of the deposition of a single layer graphene sheet onto a grooved substrate using a continuum model (Wagner and Vella, [13]) gave some further insight into this interaction. Rather than assuming a sinusoidal deformation, this analysis accounts for regions of
contact and separation between the graphene sheet and the substrate. Because a graphene sheet is thin and its resistance to axial motion was assumed to be negligible (i.e. no friction), the primary deformation mode was assumed to be bending. This formulation highlights the crucial role played by the adhesion energy and the groove geometry in determining the nature of the transition from non-conformal to conformal morphologies.

1.1.2 The finite rotational stiffness within a fixed support

In plate/beam theory, the nature of the boundary condition at an end depends on the relative stiffness of the plate/beam compared to that of the support. For example if the rotational stiffness of the support is much greater than the rotational stiffness of the freestanding portion of the beam/plate, the constraint can be assumed to be fixed. At the other limit, if the rotational stiffness of the support is much less than that of the freestanding portion, the constraint can be approximated as pinned. If these two stiffnesses are comparable then the support needs to be modeled as a rotational spring with finite stiffness. The analysis presented here is equally valid for either plate bending theory or beam theory with minor modifications.

It was shown by Fang and Wickert [14] using plane strain finite elements, Greek and Chitica [15] using three-dimensional finite element analysis, and by Ryan et al. [16] using modified beam theory that the fixed portion of a one-sided constraint (i.e. a portion is fixed on its lower side) has a finite rotational stiffness. Thus a bending moment $M_0$ applied to the end of the fixed portion of the plate/beam will produce a finite angle of rotation even for a perfect one-sided bond to an ideally rigid substrate. In [16] the use of a modified beam theory that accounts for bending, shear and extensional deformations in the bonded region, was used to determine this finite rotational stiffness. Examples were given for cantilever, bridge, and guided structures subjected to either transverse loads or residual stresses. The correction factors that account for displacements due to this rotational compliance were determined and found to depend on the Poisson’s ratio as well as on the ratio of the length of the suspended section of the structure to its thickness. These corrections are typically comparable to or greater than the additional corrections due to shear deformation in the suspended portion of the structure. Other papers, e.g. Haggblad and Bathe [17], recognized the crucial role of the boundary conditions in plate theory pertaining to the use of Kirchoff vs. Reissner-Mindlin theories.
One application area for this work is in the field of microelectromechanical systems (MEMS). For example, microstructures such as resonators (used in toxic gas sensors, mass and biological sensors, temperature sensors, force and acceleration sensors, and earthquake actuated switches [18], microvalves [19], and microphones [20]). The fabrication of MEMS evolved from the process technology used in semiconductor device fabrication [21], in which the basic techniques are deposition of material layers, patterning by photolithography and etching to produce the required shapes. When a layer is deposited on a substrate and another material is deposited on the layer, the result is a two-sided constraint.

1.1.3 Prying action in a macro-scale connection

Bolts are often used to hold flat components together while transmitting a tensile (separating) load. When the components are relatively rigid and the transmitted load exceeds the combined bolt preloads, complete separation of the components can occur, resulting in the load simply being shared among the bolts. But when the components are flexible, the tensile load may cause elastic partial separation, in which case there can be a large contact force at the separation boundary. This “prying” force adds significantly to the bolt tension, which then exceeds its expected share of the tensile load. This creates a risk of bolt yielding accompanied by a deleterious tension loss upon unloading.

The simplest example of prying action is a cantilevered beam bolted to a substrate with a single bolt passing into a tapped hole in the substrate. A more practical example of significant prying occurs in a T-stub, which is a T-shaped symmetric metal component made of a tensile web and bolted flange.

In most studies of a T-stub, an Euler-Bernoulli beam analysis is conducted by assuming a fixed location of the prying force. For example, in the American Institute of Steel Construction Manual [22], the design procedure is based on assuming that the location of the prying force is at each tip of the connector flange. But this assumption underestimates the prying force by artificially reducing the moment-arm ratio. Kulak et al. [23] recommended that the lift-off distance be taken as 1.25L, where L is the distance from the centerline of the bolt to the face of the web. The lift-off distance in this study is the distance from the centerline of the bolt to the tip of the flange. Swanson et al. [24] carried
out ultimate-strength tests on 48 T-stub specimens to develop a large database to calibrate simplified models usable for limit-state design. Coelho et al. [25] presented the results of 32 tests on bolted T-stub connections made up of welded plates. Abidelah et al. [26] investigates the influence of the bolt bending on the behavior of the T-stubs. A 3D finite element model was first developed and validated by comparison with experimental results found in the literature. Zhao et al. [27] conducts experiments to investigate the tensile behavior of ten T-stub joints made of different types of high performance structural steel including the Reheated, Quenched and Tempered (RQT) grade S690 high strength steel, the Thermal-Mechanical Controlled Processed (TMCP) grade S385 and S440 structural steels. Ribeiro et al. [28] describes the non-linear behavior of a T-stub joint exposed to impact loads, by exploring a finite element model developed in ABAQUS software.

1.2 Problem Statement and Research Objectives

The mechanical interactions between a micro-/nano-scale element, such as a carbon nanotube or a graphene sheet, and a sinusoidal/grooved surface are often critical for micro-/nano-electromechanical (MEM/NEM) device operation. In this thesis, the objective is to determine the conditions under which such an element will follow the topology of the surface, when it will separate and the extent of separation (shown in Fig. 1). The model uses continuum Euler-Bernoulli beam theory (but is equally applicable to its two-dimensional counterpart of cylindrical plate bending) which includes bending stiffness and induced tension stiffness. The general methodology is an extension of that of Wagner and Vella [13] in that there are separate regions of contact and separation. Three different situations are compared: a) the case without the stiffening effect (i.e. the case studied in [13]); b) the case with the contact regions axially fixed with tension induced due to transverse deflection; and c) the case with frictional slip regions which reduces the induced tension compared to case (b). The results from these cases are compared to each other. In case (b) and case (c), the transverse deflection of the CNT/GE is sufficient to produce axial stretching, and this induced tension is included in the model. It is found that this induced tension significantly affects the GE/CNT morphology.
Figure 1 Schematic representation of the three possible CNT morphologies on a sinusoidal/grooved surface. (a) point/line-contact, (b) partially conformal contact in which the CNT separates from the surface in the region $\lambda L < x < (1 - \lambda)L$, and (c) fully conformal contact.

In this thesis, we also further develop the study of an elastic plate/beam which in this case has an end perfectly bonded at its top and bottom surfaces to rigid constraints; such a configuration will be referred to here as a clamped support. It will be analyzed by utilizing simple plate/beam theory with a modified boundary condition. The use of plane strain elasticity implies that a moment $M_0$ applied to the fixed portion of the structure will produce a finite angle of rotation even for this type of clamped support. This rotational stiffness scales with the product of the shear modulus and the square of the plate thickness and further depends upon Poisson’s ratio. Examples demonstrating the importance of this effect are given for a cantilever and a clamped-clamped plate/beam and also for a clamped circular plate. For these cases the correction due to this rotational compliance is often greater than the correction due to shear deformation in the freestanding portion of the structure. Results are also obtained using two-dimensional finite element analyzes and agree well with the analytical results obtained using these modified boundary conditions.

The finite rotational stiffness within the support is determined by applying a distributed load (statically equivalent to a couple moment) to such a constrained infinite elastic strip, as illustrated in Fig. 2. The displacement profile is determined analytically and approximated by a straight line. The ratio of the applied moment to the slope of this
line provides the rotational stiffness. Examples are given for a cantilever and a fixed-fixed plate/beam as well as for a clamped circular plate.

![Diagram](image)

**Figure 2** A constrained semi-infinite elastic strip subjected to a couple moment $M_0$.

For the study of prying action, we treat the configuration as a compliant receding contact problem where the design parameters are the joint dimensions, material properties, the bolt preload, and the applied force. We study the cases of the cantilever beam (illustrated in Fig. 3) and T-stub (shown in Fig. 4). With the help of the Goodman diagram, we can also find the safe way to design these connections for fatigue.

![Diagram](image)

**Figure 3** Bolted cantilever beam.  **Figure 4** Bolted T-stub.
Chapter 2. Adhesion of a Micro-/Nano- Beam/Plate to a Sinusoidal/Grooved Surface

Three different scenarios can describe the morphology of a micro-/nano-element adhering to a sinusoidal or grooved surface as illustrated schematically in Figure 1 a, b, c. For sufficiently weak adhesion, we expect the point-contact scenario to occur in which contact between the beams/plates and the surface occurs only at the peaks of the sinusoidal surface. This configuration is denoted “non-conformal”. In the case of a groove, rather than a continuously sinusoidal surface, the contact would occur only on the flat region of the surface. We use the term “partially conforming” to indicate that the element conforms to the surface over a finite portion of the sinusoidal surface or groove length. The “conformal” scenario corresponds to sufficiently strong adhesion so that the beam/plate is deflected enough to contact the sinusoidal/grooved surface over its entire length. Here we focus on the transitions from non-conformal to partially-conformal, and on the partially-conformal to the conformal configurations. The objective is to better understand the stiffening effect and its importance in the relationship between the adhesion energy and the various morphologies.

2.1 Stiffening Effect

The beam/plate deflects in the vertical direction due to adhesion. Friction between the beam/plate and the surface will cause it to stretch due to the required increase in length, which leads to induced tension. In this section, the effect of this induced tension is analyzed. The elastic beam has flexural rigidity $EI$ and thickness $h$ and its deflected position is represented by $z = w(x)$. Consider the sinusoidal/grooved surface (Fig. 1) of characteristic
length, \( L \), and depth, \( \delta \) with the shape of the surface given by \( z = w_s(x) \). The free body diagram of a differential element of the element subjected to a shear force \( (V) \), bending moment \( (M) \) and tension \( (T) \) is shown in Fig. 5. The displacement varies only in the \( x \)-direction and the tension acts only in the \( x \)-direction. Therefore the tension is constant. The moment-curvature relation is unaffected by tension and is given by \( EI \frac{d^2w}{dx^2} = M \) [29], where \( E \) is the elastic (Young’s) modulus and \( I \) is the second moment of the beam cross sectional area.

\[
EI \frac{d^2w}{dx^2} = M
\]

where \( EI \) is the flexural rigidity.

Equilibrium of the differential element requires \( \frac{dM}{dx} = V + T \frac{dw}{dx} \) and \( \frac{dV}{dx} = 0 \) which, when combined with the moment-curvature equation, results in the fourth-order differential equation given by

\[
EIw''' - Tw'' = 0 \quad \Rightarrow \quad w''' - \beta^2 w'' = 0
\]

where

\[
\beta^2 = \frac{T}{EI}
\]

and a prime (’) indicates differentiation with respect to \( x \). It is noted that although this problem is formulated using beam theory (e.g. a CNT), it is also valid for cylindrical plate bending (e.g. GE). In the latter case the flexural rigidity \( EI \) is replaced with
\[ D = \frac{Eh^3}{12(1 - \nu^2)} \] and \( V \) and \( M \) become the shear force per unit width and bending moment per unit width respectively, and \( \nu \) is the Poisson’s ratio.

Because of symmetry, we only need to consider a half period of the beam/plate. The assumed contact region is \( 0 < x < \lambda L \) where \( \lambda \) is a priori unknown. The solution of (1) is given by

\[
 w(x) = \begin{cases} 
 w_s(x), & 0 < x < \lambda L \\
 A_1 \sinh(\beta x) + A_2 \cosh(\beta x) + A_3 x + A_4, & \lambda L < x < L/2 
\end{cases}
\]

and in the region \( (\lambda L < x < L/2) \) is subject to the four boundary conditions

\[
 w(\lambda L) = w_s(\lambda L), \quad w'(\lambda L) = w'_s(\lambda L), \quad w'(L/2) = 0, \quad V(L/2) = 0 \Rightarrow w''(L/2) = 0
\]

Application of these conditions to Eq. (3) results in

\[
 w(x) = w_s(\lambda L) + \frac{w'_s(\lambda L)}{\beta} \coth\left(\frac{\beta L}{2} - \lambda \beta L\right) - \frac{w'_s(\lambda L)}{\beta} \frac{\cosh\left(\frac{\beta L}{2} - \lambda \beta L\right)}{\sinh\left(\frac{\beta L}{2} - \lambda \beta L\right)}, \quad \lambda L < x < L/2
\]

where \( \lambda \) is at this point unknown.

The change of curvature of the CNT from the left to the right of the contact point at \( x = \lambda L \) is related to the work of adhesion (\( \gamma \)) by [30],[31] i.e.

\[
 \sqrt{2\gamma / EI} = w''(\lambda L) - w'_s(\lambda L) = \sqrt{2 / l_{ec}}
\]

where \( l_{ce} = (EI / \gamma)^{1/2} \) is the referred to as the elasto-capillary length [32]. As described by Pamp and Adams [33], there is an attractive force in the small separation region in which the separation is less than a threshold value, along with a reaction force at the contact boundary. In the limit of a surface energy model of adhesion, this action represents an ever increasing attractive force which approaches the contact boundary, i.e. a discontinuous moment \( (M_o) \). It is further noted that for a beam (e.g. CNT), as opposed to a plate (e.g. GE), the work of adhesion is per unit length, rather than per unit area, i.e. \( \gamma \) is replaced by \( \gamma c \) where \( c \) is the assumed contact width taken here to be the CNT radius and if for a rectangular cross section, it is \( \gamma b \) where \( b \) is the width. Equation (6) can also be written as
\[
\left( \frac{2L^4\gamma}{\delta^2EI} \right)^{\frac{1}{2}} = L^4 \left( \frac{L}{\delta} \right) \left[ -\beta w''(\lambda L) \cosh\left( \frac{\beta L}{2} - \lambda \beta L \right) \right] \left( \frac{L}{\delta} \right) \left( \sinh\left( \frac{\beta L}{2} - \lambda \beta L \right) \right) - w''(\lambda L)
\]

(7)

It is now necessary to relate the transverse deflection to the induced tension. For

\[
\left| \frac{dw}{dx} \right| \ll 1
\]

the change of axial position (\(\Delta\)) due to the transverse displacement is given by

\[
\Delta = \int_0^L (ds - dx) = \int_0^L \sqrt{(dx)^2 + (dw)^2} - dx \approx \frac{1}{2} \int_0^L [w'(x)]^2 dx
\]

(8)

The relation between stress and strain in uniaxial tension can be written as

\[
\frac{T}{A} = E \frac{\Delta}{L}
\]

(9)

which, when combined with Eqs. (2) and (8), leads to

\[
(\beta L)^2 = \frac{AL}{I} \Delta = \frac{AL^2}{I} \frac{1}{2L} \int_0^L [w'(x)]^2 dx
\]

(10)

As an example consider a cosine shape of amplitude \(\delta\) and length \(L\) for the surface profile, i.e.

\[
w_s(x) = \frac{\delta}{2} + \frac{\delta}{2} \cos(2\pi \frac{x}{L})
\]

(11)

\[
w'(x) = \begin{cases} 
- \frac{\delta\pi}{L} \sin(2\pi x/L), & 0 < x < \lambda L \\
\frac{\sinh \left( \frac{\beta L}{2} - \beta x \right)}{\sinh \left( \frac{\beta L}{2} - \lambda \beta L \right)} - \frac{\delta\pi}{L} \sin(2\pi \lambda) \frac{\sinh \left( \frac{\beta L}{2} - \beta x \right)}{\sinh \left( \frac{\beta L}{2} - \lambda \beta L \right)}, & \lambda L \leq x \leq L/2
\end{cases}
\]

(12)

Then the use of Eq. (10) and Eq. (12), leads to

\[
(\beta L)^2 = \frac{\delta^2 \pi^2}{r^2} \left[ \frac{\lambda}{2} - \frac{1}{8\pi} \sin(4\pi \lambda) + \frac{\sin^2(2\pi \lambda)}{2\beta L} \coth\left( \frac{\beta L}{2} - \lambda \beta L \right) + \frac{\sin^2(2\pi \lambda)}{\sinh^2 \left( \frac{\beta L}{2} - \lambda \beta L \right)} \left( \frac{\lambda}{2} - \frac{1}{4} \right) \right]
\]

(13)

where the radius of gyration \((r)\) of the cross-sectional area is defined by
\[ r = \frac{I}{\sqrt{A}} \]

(14)

and equal to \( d/4 \) for a CNT of diameter \( d \) and equal to \( h/\sqrt{12} \) for a GE sheet of thickness \( h \). Finally by combining Eqs. (7) and (12)

\[
\left( \frac{2L^4\gamma}{\delta^2 EI} \right)^{1/2} = \pi \beta L \sin(2\pi\lambda) \coth\left( \frac{\beta L}{2} \right) - \lambda \beta L + 2\pi^2 \cos(2\pi\lambda)
\]

(15)

is obtained. Equations (13) and (15) are two nonlinear algebraic equations for two dimensionless unknowns, \( \lambda \) and \( \beta L \). It is noted that in the limit as the non-dimensional tension (\( \beta L \)) goes to zero, Eq. (15) becomes

\[
\left( \frac{2L^4\gamma}{\delta^2 EI} \right)^{1/2} = \frac{2\pi \sin(2\pi\lambda)}{1 - 2\lambda} + 2\pi^2 \cos(2\pi\lambda)
\]

(16)

which agrees with the result without tension [13].

2.1.1 Results and Discussion

We will now calculate results using Eqs. (13) and (15). The procedure is as follows:

(a) Vary \( \delta/r = 1, 2, 3, 4, 5, 10, 12, 15, 20 \).

(b) For each value of \( \delta/r \), vary the value of \( \beta L \) from 0.001 to 5 or greater, and for each \( \beta L \) solve the nonlinear algebraic Eq. (13) for the corresponding value of \( \lambda \).

(c) Use Eq. (15) to obtain the value of the dimensionless adhesion energy,

\[ \left( \frac{2L^4\gamma}{\delta^2 EI} \right)^{1/2} \]

corresponding to each pair of values of \( \beta L \) and \( \lambda \) and the chosen value of \( \delta/r \).

The non-dimensional adhesion energy, \( \left( \frac{2L^4\gamma}{\delta^2 EI} \right)^{1/2} \), vs. the contact length parameter, \( \lambda \), is shown for small and large values of \( \delta/r \) in Figs. 6a and 3b respectively. These results show that when the adhesion energy is less than the threshold value of \( 2\pi^2 \), the CNT/GE remains flat on the substrate. Once the adhesion energy exceeds its peak value on a given curve, the beam/plate snaps into a fully conformal configuration. Between these two limits, the beam/plate partially conforms to the substrate. Thus the portions of
these curves for $\lambda > 0$ but less than the value corresponding to the peak adhesion energy are stable. Larger values of $\lambda$ correspond to unstable configurations. Here it has been assumed that sufficient friction exists between the beam/plate and the surface to prevent axial slip. Sufficient adhesion energy causes the beam/plate to deform toward the troughs of the sinusoidal/grooved surface. This action causes the beam/plate to stretch which, in turn, induces tension along its entire length. From Fig. 6a when the amplitude of the surface waviness is sufficiently small i.e. $\delta/r < 2$, the induced tension does not have a significant difference as the results approach those without tension. However for $\delta/r \geq 2$ the effect becomes progressively more important (Figs. 6a,b). For larger values of $\delta/r$ more energy is needed for fully conformal contact. The values of the adhesion energy vs. $\beta L$ are shown in Fig. 7. Note from Figs. 6 and 7 that as $\delta/r$ increases, the induced tension ($\beta L$) increases and the required adhesion energy to snap to the completely conformal morphology increases. If the beam/plate is micro-scale, $\delta/r$ is likely relatively small, and the relation of adhesion energy vs. contact length parameter is shown in Fig. 6a $\delta/r = 1, 2, 3, 4, 5$. For a nano-scale beam/plate, such as a CNT/GE, $\delta/r$ is likely large as shown in Fig. 6b.
Figure 6: Adhesion energy vs. $\lambda$ for different small and large $\delta/r$. 

(a) 

(b)
Figure 7 Adhesion energy vs. $\beta L$ for different small and large $\delta/r$. 

The value of $\lambda$ which maximizes the non-dimensional adhesion energy corresponding to the snap to complete conformity and the associated maximum value of this adhesion energy vs. the amplitude of the sinusoid ($\delta/r$) are shown in Figs. 8 and 9 respectively. Now consider the asymptotic behavior as $\beta L \to \infty$, in which case Eqs. (13) and (15) become

$$(\beta L)^2 = \frac{\delta^2 \pi^2}{r^2} \left[ \frac{1}{2} \frac{\lambda}{8\pi} \sin(4\pi \lambda) \right]$$ (17)

$$\left( \frac{2L^4}{\delta^2 EI} \right)^{1/2} = \pi \beta L \sin(2\pi \lambda)$$ (18)
Figure 8 The relation between $\lambda_{\text{MAX}}$ and $\delta/r$.

Figure 9 The three morphologies in terms of the adhesion energy and $\delta/r$.

Then by substituting Eq. (17) into (18)
\[
\left( \frac{2L^4 \gamma}{\delta^2 EI} \right)^{\frac{1}{2}} = \frac{\delta \pi^2}{r} \sqrt{\frac{\lambda}{2} - \frac{\sin(4\pi \lambda)}{8\pi}} \sin(2\pi \lambda) \tag{19}
\]

is obtained. Taking the derivative of Eq. (19) with respect to \( \lambda \), setting it equal to zero, and solving the resulting nonlinear algebraic equation yields \( \lambda = 0.3083 \) which is the value of \( \lambda \) which extremizes the non-dimensional adhesion energy. Finally Eq. (19) gives the corresponding adhesion energy for snap to conformity as

\[
\left( \frac{2L^4 \gamma}{\delta^2 EI} \right)^{\frac{1}{2}} = 0.3970 \frac{\delta \pi^2}{r} \tag{20}
\]

These asymptotic results are shown by the straight line in Fig. 8 and by the result that the value of the adhesion energy approaches a straight line as shown in Fig. 9.

Fig. 9 shows explicitly the three regions (point/line contact, partially conformal contact, and fully conformal contact) in terms of the adhesion energy and \( \delta/r \). Recall that the effect of \( \delta/r \) is to increase the induced tension. From Fig. 9 it is clear that this induced tension will have an extremely important effect on the resulting configuration. For \( \delta/r = 0 \), partially conformal contact occurs only if the adhesion parameter is in the narrow window between \( 2\pi^2 \) and 20.98. However for \( \delta/r = 25 \) the region of partially conformal contact is between \( 2\pi^2 \) and 97.96.

In Gao and Huang [12], the adhesion energy per unit area is \( 0.45 \text{J}/\text{m}^2 \), the flexural rigidity is 1.61 eV, the equilibrium separation is \( h_0 = 0.6 \) nm, the depth of the groove is 0.4 nm, and the thickness of the graphene \( t = 0.34 \text{nm} \), which gives \( \delta/r = 4.075 \). In our thesis, the threshold value of the dimensionless adhesion energy is \( 2\pi^2 \) and the peak value of the dimensionless adhesion energy is 23.0. Thus the threshold value corresponds to \( L = 2.06 \text{nm} \) or \( L/h_0 = 3.42 \) whereas the peak value corresponds to \( L = 2.22 \text{nm} \) or \( L/h_0 = 3.70 \). These results are consistent with those given by Gao and Huang in which the sharpest portion of the transition occurs over the somewhat wider range of \( 3 < L/h_0 < 7 \).
2.2 Frictional Slip Regions

In this section it is assumed that the induced tension $T$ acting along the $x$-direction is sufficient to cause part of the CNT/GE (beam/plate) in the contact region to slip on a portion of the surface. The two slip zones are symmetrically located. The length of the slip region is “$a$” which is a priori unknown. The slip region is characterized by friction with a constant shear stress, which is similar to a shear lag model.

The free body diagram of the slip region on the left portion of the beam/plate is shown in Fig. 10.

![Free body diagram of slipping condition.](image)

Figure 10 Free body diagram of slipping condition.

The constant tension in the separation region is denoted by $T$ whereas the tension in the slip region varies with $x$ and is denoted $N(x)$. Equilibrium in the $x$-direction of the slip region results in

$$\frac{dN}{dx} = \tau_0 c$$

(21)

where $c$ is the width of the contact region in which shear stress is assumed to act. The tension is related to the axial strain and hence to the tangential displacement ($u$) according to

$$N = \alpha A = EA \varepsilon = EA \frac{du}{dx}$$

(22)

Combining Eq. (21) and (22), yields
\[ EA \frac{d^2 u}{dx^2} = \tau_o c \]  

(23)

The solution of Eq. (23) is subjected to the boundary conditions

\[ u(0) = 0, \quad \frac{du(0)}{dx} = 0 \]  

(24)

from which

\[ u(x) = \frac{1}{2} \frac{\tau_o c}{EA} x^2 \]  

(25)

is obtained. Equation (24) pertains to the cross-section which separates the stick and slip regions. In the stick region the strain is zero and hence the tension will be zero.

Then the slip displacement at the boundary with the separation region is

\[ u(a) = \frac{1}{2} \frac{\tau_o c}{EA} a^2 \]  

(26)

and the tension is

\[ T = EA \frac{du(a)}{dx} = EA \frac{\tau_o c}{EA} a = \tau_o ca \]  

(27)

from which the slip displacement is expressed in terms of the induced tension \( T \) by

\[ u(a) = \frac{1}{2} \frac{T^2}{EA \tau_o c} \]  

(28)

2.2.1 Slip Regions

In part of each contact region between the beam/plate and the surface there will be slip and in the rest there will be stick. There is also a stretch existing in the suspended portion \( (1 - 2\lambda)L \) given by

\[ \Delta_1 = \frac{TL(1 - 2\lambda)}{EA} \]  

(29)

and the stretch in the slip regions is

\[ \Delta_2 = 2u(a) = \frac{T^2}{EA \tau_o c} \]  

(30)

Thus this total stretch is needed in order to accommodate the increase in the curved length of the beam/plate, i.e.
\[ \Delta_1 + \Delta_2 = \frac{1}{2} \int_0^t (w'(x))^2 \, dx \]  

(31)

\[ \Rightarrow \frac{TL(1-2\lambda)}{EA} + \frac{T^2}{EA\tau_c} = \int_0^{\lambda\beta} [w'(x)]^2 \, dx + \int_{L\lambda}^{L} [w'(x)]^2 \, dx \]  

(32)

As an example consider the cosine shape given by Eq. (11), in which case Eq. (32) leads to

\[ (\beta L)^2 \left( \frac{r}{L} \right)^2 \left( 1 - 2\lambda \right) + (\beta L)^2 \left( \frac{r}{L} \right)^2 \frac{EA}{\tau_c L} \]

\[ = \frac{\delta^2 \pi^2}{L^2} \left( \frac{\lambda}{2} - \frac{1}{8\pi} \sin(4\pi\lambda) + \frac{\sin^2(2\pi\lambda)}{2\beta L} \coth \left( \frac{\beta L}{2} - \frac{\lambda}{\beta L} \right) + \frac{\sin^2(2\pi\lambda)}{\sinh^2 \left( \frac{\beta L}{2} - \frac{\lambda}{\beta L} \right)} \left( \frac{\lambda}{2} - \frac{1}{4} \right) \right) \]

(33)

and Eqn. (15) is unchanged.

2.2.2 Results and Discussion

The solution procedure is as follows:

(a) Vary the shear stress \( \tau_c L/EA = 10 \ 1, 0.1, 0.01, 0.001, 0.0001 \).

(b) Start by choosing \( r/L = 0.01 \) and \( \delta/r = 10 \).

(c) For each set of values of \( \tau_c L/EA, r/L, \) and \( \delta/r \), vary \( \beta L \) and solve the algebraic Eq. (33) for the corresponding value of \( \lambda \).

(d) Eq. (15) is then used to obtain the resulting adhesion energy corresponding to each set of values of \( \tau_c L/EA, r/L, \delta/r \) and \( \lambda \).

Using this procedure we determine the non-dimensional adhesion energy, \( \left( 2L^4 \gamma/\delta^2 EI \right)^{1/2} \), vs. the contact length parameter, \( \lambda \), for different sets of values of \( \tau_c L/EA, r/L, \) and \( \delta/r \) as shown in Fig. 11. As in Figs. 6a,b the stable portions of these curves are for \( \lambda \) between zero and the value which makes the adhesion energy a maximum. Note that for the largest value of the shear stress, the length of the slip region is expected to approach zero and this slip region will shelter the stick region from stretching. Consequently if it is assumed that the stretch only exists between in the suspended portion \( (1-2\lambda)L \) then
\[
\frac{T}{A} = E \frac{\Delta}{(1 - 2\lambda)L}
\]  
(34)

Figure 11 Adhesion energy vs \( \lambda \) for different values of \( \tau_0 cL/EA \) when \( r/L = 0.01 \) and \( \delta/r = 10 \).

and

\[
(\beta L)^2 (1 - 2\lambda) = \frac{\delta^2 \pi^2}{r^2} \left[ \frac{\lambda}{2} - \frac{1}{8\pi} \sin(4\pi\lambda) + \frac{\sin^2(2\pi\lambda)}{2\beta L} \coth\left(\frac{\beta L}{2} - \lambda\beta L\right) + \frac{\sin^2(2\pi\lambda)}{\sinh^2\left(\frac{\beta L}{2} - \lambda\beta L\right)} \left(\frac{\lambda}{2} - \frac{1}{4}\right) \right]
\] 
(35)

The results of using Eq. (15) and Eq. (35) are shown in Fig. 11 and represent the limit of very high shear stress (\( \tau_0 cL/EA > 10 \)). On the other hand for \( \tau_0 cL/EA \) sufficiently small, i.e. \( \tau_0 cL/EA < 0.0001 \), the induced tension does not make a significant difference and the results are close to those without tension. With increasing \( \tau_0 cL/EA \), a larger peak value of \( (2L^4\gamma/\delta^2 EI)^{1/2} \) is required for the beam/plates to snap into full conformity with the substrate. Once \( (2L^4\gamma/\delta^2 EI)^{1/2} \) reaches the peak value, the beam/plate conforms fully.
2.3 Conclusion

The transitions of the morphologies among three states of a CNT/GE (or more generally a beam/plate) on a sinusoidal/grooved are found, i.e. the beam/plate remaining flat on the substrate, partially conformal on the substrate, and a fully conformation configuration. In addition there are three different ways to model this behavior: a) the case without the stiffening effect (studied in a previous investigation); b) the case in which the contact regions are prevented from slipping axially and thereby induce tension due to its transverse displacement; c) the case with frictional slip regions which reduces the induced tension compared to case (b). It is found that as the adhesion energy increases, the induced tension increases and the contact region decreases. The transition from non-conformal to partially conforming configurations is independent of the model. However the three different models show considerable quantitative differences for the transition from a partially to a fully conformal morphology. The minimum value of the adhesion energy required for this transition is case (a) while the maximum value is for case (b). Case (c) gives a transition between the no-slip and no-tension cases which depends upon the value of the non-dimensional shear stress.
Chapter 3. A Boundary Condition Correction for the Clamped Constraint of Elastic Plate/Beam Theory

3.1 Problem Formulation

An infinite elastic strip as shown in Fig. 12, is constrained at its top and bottom surface. In order to simulate the deformation the plate will have at its clamped end, a body force distribution is applied at \( x_1 = 0 \) which is statically equivalent to an applied bending moment. As shown in Fig. 2 half of the infinite strip is considered subjected to a bending moment \( M_0 \) equal to one-half of that applied to the full strip. Here we use plane strain linear elasticity. Hence the configuration considered is that in which the dimension of the structure in the \( x_3 \)-direction is large compared to the dimensions of the structure in the other two directions. In this sense it is compatible with the use of cylindrical plate bending theory.

![Figure 12 An infinite elastic strip of height \( h \) clamped at its top and bottom surfaces.](image)

The plane strain Navier equations of equilibrium are given by

\[
(\lambda + G)u_{i,j} + Gu_{j,i} + B_j = 0, \quad j = 1, 2 \tag{36}
\]

where \( \lambda \) is Lame’s constant, \( G \) is the shear modulus, \( u_j \) and \( B_j \) are respectively the displacement and body force in the \( x_j \)-direction, a comma \( i \) denotes differentiation with respect to \( x_i \), and the index \( i \) is summed from 1 to 2. The expanded form of Eqn. (36) is obtained as

\[
(\lambda + G)(u_{1,11} + u_{2,21}) + G(u_{1,11} + u_{1,22}) + B_{1} = 0 \tag{37}
\]

\[
(\lambda + G)(u_{1,12} + u_{2,22}) + G(u_{2,11} + u_{2,22}) + B_{2} = 0 \tag{38}
\]

Now the Fourier transform in the \( x_1 \)-direction given by
where \( \eta \) is the transform variable, is applied to Eqns. (37) and (38). This procedure leads to

\[
(\lambda + G) \left( i \eta \frac{du_1}{dx_2} \right) + G \frac{d^2 u_1}{dx_2^2} + (\lambda + 2G)(-\eta^2 \tilde{u}_1) + \tilde{B}_1 = 0
\]

(40)

\[
(\lambda + G) \left( i \eta \frac{du_1}{dx_2} \right) + (\lambda + 2G) \frac{d^2 u_2}{dx_2^2} + G(-\eta^2 \tilde{u}_2) + \tilde{B}_2 = 0
\]

(41)

where \( \tilde{B}_1 = Cx_2 \tilde{b}(x_1)/h \) and \( \tilde{B}_2 = 0 \) resulting in \( \tilde{B}_1 = Cx_2 / (\sqrt{2\pi} h) \) and \( \tilde{B}_2 = 0 \). The forms chosen for \( \tilde{B}_1 \) and \( \tilde{B}_2 \) are to create a couple moment at \( x_1 = 0 \).

The displacements \( u_1(x_1, x_2) \) and \( u_2(x_1, x_2) \) are subject to the four boundary conditions that these quantities vanish along \( x_2 = \pm h/2 \). These conditions can be applied to \( \tilde{u}_1 \) and \( \tilde{u}_2 \) as

\[
\tilde{u}_1(\eta, h/2) = 0, \quad \tilde{u}_1(\eta, -h/2) = 0, \quad \tilde{u}_2(\eta, h/2) = 0, \quad \tilde{u}_2(\eta, -h/2) = 0
\]

(42)

After applying the boundary conditions, the use of the inverse Fourier transformation along with the relation \( \lambda = 2G \nu/(1-2\nu) \), where \( \nu \) is Poisson’s ratio, the displacement fields become

\[
\left( \frac{u_1}{h} \right) / \left( \frac{C}{G} \right) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \left( \frac{\tilde{u}_1}{h^2} \right) / \left( \frac{C}{G} \right) \cos(\eta x_1) d(\eta h)
\]

\[
= \sqrt{\frac{2}{\pi}} \int_0^\infty \left( \frac{4x_2 h}{h} \cosh(\eta h) \frac{x_2}{h} \sinh(\eta h/2) + 2 \cosh(\eta h/2)(1-2\nu) \frac{x_2}{h} \eta h \cosh(\eta h/2) + 2(-1+\nu)(-1+4\nu) \sinh(\eta h/2) \right) \cos(\eta x_1) d(\eta h)
\]

(43)

\[
+ (-1+2\nu)(2(3-4\nu) \frac{x_2}{h} \sinh(\eta h) + \eta h(2 \frac{x_2}{h} + \sinh(\eta h/2) \sinh(\frac{x_2}{h} \eta h))) \right) \cos(\eta x_1) \frac{d(\eta h)}{m}
\]

where \( m = 4\sqrt{2\pi} (\eta h)^2 (-1+\nu)(\eta h + (3-4\nu) \sinh(\eta h)) \) and
\[
\left( \frac{u_x}{h} \right) / \left( \frac{C}{G} \right) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \left( \frac{\tilde{u}_x}{h^2} \right) i \sin(\eta x_1) d(\eta h)
\]

\[
= \int_0^\infty \left( 2[\eta h + (3 - 4\nu) \sinh(\eta h)] + \sinh(\frac{\eta h}{2}) \left( (4(4\nu - 3) + (\eta h)^2(2\nu - 1)) \cosh(\frac{x_2}{h}) + 4\eta \frac{x_2}{h} \sinh(\frac{x_2}{h}) \right) \right) \sin(\eta x_1) d(\eta h)
\]

\[
-2(\eta h) \cosh(\frac{\eta h}{2}) \left[ \cosh(\frac{x_2}{h}) + (2\nu - 1)\eta h \frac{x_2}{h} \sinh(\frac{x_2}{h}) \right] \frac{\sin(\eta x_1)}{m \eta h} d(\eta h)
\]

\[\text{(44)}\]

By using the symbolic interpreter language Mathematica® [34].

By using numerical integration with respect to \( \eta h \), the curves for \((u_1/h)/(C/G)\) vs. \(x_2/h\) are determined. A simple linear fit to \(u_1(0,x_2)\) can then be used to obtain the constant \(K\) defined by

\[
(u_1(0,x_2)/h)/(C/G) = K(x_2/h)
\]

\[\text{(45)}\]

Because the configuration shown in Fig. 12 is for an infinite elastic strip, half of that moment is applied to each semi-infinite strip, i.e.

\[
M = \frac{1}{2} \int B_1 dV x_2 dA = \frac{Cbh^2}{24}
\]

\[\text{(46)}\]

where \(h\) is the height and \(b\) is the width of the strip. Thus the moment per unit width is

\[
M_0 = \frac{Ch^2}{24}
\]

\[\text{(47)}\]

and the angle of rotation \((\phi)\) is

\[
\phi = \frac{CK}{G}
\]

\[\text{(48)}\]

The rotation stiffness of the support is then given by

\[
K_1 = M_0 / \phi = \frac{Gh^2}{24K}
\]

\[\text{(49)}\]

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\[
K_1 = M_0 / \phi = \frac{Gh^2}{24K}
\]

\[\text{(49)}\]

By using the symbolic interpreter language Mathematica® [34].

By using numerical integration with respect to \( \eta h \), the curves for \((u_1/h)/(C/G)\) vs. \(x_2/h\) are determined. A simple linear fit to \(u_1(0,x_2)\) can then be used to obtain the constant \(K\) defined by

\[
(u_1(0,x_2)/h)/(C/G) = K(x_2/h)
\]

\[\text{(45)}\]

Because the configuration shown in Fig. 12 is for an infinite elastic strip, half of that moment is applied to each semi-infinite strip, i.e.

\[
M = \frac{1}{2} \int B_1 dV x_2 dA = \frac{Cbh^2}{24}
\]

\[\text{(46)}\]

where \(h\) is the height and \(b\) is the width of the strip. Thus the moment per unit width is

\[
M_0 = \frac{Ch^2}{24}
\]

\[\text{(47)}\]

and the angle of rotation \((\phi)\) is

\[
\phi = \frac{CK}{G}
\]

\[\text{(48)}\]

The rotation stiffness of the support is then given by

\[
K_1 = M_0 / \phi = \frac{Gh^2}{24K}
\]

\[\text{(49)}\]
independent of the elastic constants. However here with displacements prescribed on most of the boundary both the stresses and the displacements depend on the elastic constants.

The normal stress in the $x_1$-direction is then calculated by

$$
\tau_{11} = \frac{2G}{(1-2\nu)}[(1-\nu)\frac{\partial u_1}{\partial x_1} + \nu \frac{\partial u_2}{\partial x_2}]
$$

and the internal moment at any cross-section $x_1$ is

$$
M = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{11} x_2 dx_2
$$

The process for determining the rotational stiffness can now be summarized.

(a) Vary $\nu = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$.

(b) For each value of $\nu$ set $x_1 = 0$ and vary the value of $x_2/h$ in small increments from -0.5 to 0.5. Use Eqn. (8) to determine the value of $(u_1/h)/(C/G)$ at $x_1 = 0$ by numerical integration.

(c) Use a linear fit for each of the curves of $(u_1/h)/(C/G)$ vs. $x_2/h$ to determine the corresponding value of $K$. These results are shown in Fig. 13.

(d) Use Eqn. (14) in order to obtain the values of $K_t$ as shown in Table 1.
Figure 13 The horizontal displacement \( \frac{u_1}{h} / \left( \frac{C}{G} \right) \) at \( x_1=0 \) vs. \( x_2 / h \) for various values of Poisson’s ratio and the corresponding curve-fits.

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( Kt ) (One-sided bond)</th>
<th>( k ) (Two-sided bond)</th>
<th>( Kt ) (Two-sided bond)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.745Gh2</td>
<td>0.0359</td>
<td>1.161Gh2</td>
</tr>
<tr>
<td>0.1</td>
<td>0.793Gh2</td>
<td>0.0343</td>
<td>1.215Gh2</td>
</tr>
<tr>
<td>0.2</td>
<td>0.856Gh2</td>
<td>0.0322</td>
<td>1.294Gh2</td>
</tr>
<tr>
<td>0.3</td>
<td>0.943Gh2</td>
<td>0.0295</td>
<td>1.412Gh2</td>
</tr>
<tr>
<td>0.4</td>
<td>1.063Gh2</td>
<td>0.0265</td>
<td>1.628Gh2</td>
</tr>
<tr>
<td>0.5</td>
<td>1.234Gh2</td>
<td>0.0195</td>
<td>2.137Gh2</td>
</tr>
</tbody>
</table>

Table 1 Rotational stiffness \( (Kt) \) vs. Poisson’s ratio \( (\nu) \)
The values of $K_t$ obtained by Ryan et al. [16] are also shown in Table 1 for the situation in which the plate/beam is constrained on just one side. From these results, we can conclude that the value of $K_t$ for a two-sided constraint is larger than for a one-sided bond as would be expected. It is noted that this stiffness is less than twice the stiffness with a one-sided constraint. It is also noted that the variation of $K_t$ with Poisson’s ratio is greater than it is for a one-sided constraint.

The relationship between the internal moment and $x_1/h$ has also been calculated. The moment decays very quickly with $x_1/h$ reaching a value of 0.5% of $M_0$ when $x_1/h = 1$. Thus although the analysis presented here pertains to an infinitely long clamped region, these results can also be used even if the length of the clamped region is as small as the plate/beam thickness.

3.2 Examples of Structures With Transverse Loads

3.2.1 Cantilever Plate

Consider now a cantilever plate with a concentrated line-load at its end as shown in Fig. 14. The clamped portion can be replaced by a pin support with a rotational spring as described in the previous section and as shown in Fig. 15. The rotation of the beam at this end is equal to $\psi_0 = PL/K_t$, and thus the cantilever end deflection will increase by

$$\delta_\psi = \psi_0 L = PL^2/K_t.$$  

Recall that the bending deflection for cylindrical bending is

$$\delta_B = PL^3/3D,$$

where $D = Eh^3/12(1 - \nu^2)$ and the deflection component resulting from shear is

$$\delta_s = P(L/h)/(G\kappa^2),$$

where $\kappa^2 = 10(1 + \nu)/(12 + 11\nu)$ for a rectangular cross-section [36].

![Cantilever Plate Diagram](image-url)
Figure 14 Cantilever plate subjected to a concentrated line load at its free end.

\[ K_t \]

Figure 15 A roller with a rotational spring.

Consequently the total deflection is given by

\[ \delta = \delta_b + \delta_s + \delta_r \] \hspace{1cm} (52)

Note that the deformations due to bending, rotation, and shear vary as the cube, square, and first power of \( L/h \) respectively.

In Fig. 16, the importance of shear deformation and rotational compliance are shown, each compared to the end deflection due to bending alone, as functions of the length-to-thickness ratio. Also shown is the total correction due to the combined effects of shear deformation and rotational compliance. These results are for a Poisson’s ratio of \( \nu = 0.3 \), which leads to \( \kappa^2 = 0.850 \). For a completely fixed end, the effect of shear deformation is about 3.5% of the bending deflection for \( L/h = 5 \). However, the rotational compliance of the constraint has an additional 10.0% effect compared with bending for a total correction of 13.5%.
Figure 16 The relationships among shear, rotation and bending deflections for a cantilever with a Poisson’s ratio of $\nu = 0.3$.

Also shown in Fig. 17 are the numerical results of a finite element simulation using ANSYS with 80 plane strain 4-node elements through the height. These elements were distributed uniformly along the length and thickness directions. The analytical results are compared with the finite element results in Fig. 17. The agreement is excellent indicating that the added correction due to rotation is appropriate.
3.2.2 **Clamped-Clamped Configuration**

A similar procedure is also used for the clamped-clamped configuration shown in Fig. 18. The bending deflection for a fixed-fixed plate is given by

\[ \delta_b = \frac{PL^3}{192D} \]  

(53)

The rotational deflection can be easily determined by solving

\[ Dw''' = 0, \quad w(0) = 0, \quad K_iw'(0) = Dw''(0), \quad w'(L/2) = 0, \quad Dw''(L/2) = P/2 \]  

(54)

and subtracting the bending solution. The resulting deflection at the middle of the beam due to rotation effects alone is

\[ \delta_r = \frac{PL^3/D}{32(2 + K_i)} \]  

(55)

Figure 18 Clamped-clamped configuration.

\[ K_i = K, L/D = (1 - \nu)(L/h)/4K \]  

(56)

The deflection at the mid-point caused by shear deformation is

\[ \delta_s = \frac{(P/2) L}{GK^2h^2} \]  

(57)
Figure 19 The relationships among shear, rotation and bending deflections for a clamped-clamped configuration with a Poisson’s ratio of $\nu = 0.3$.

In Fig. 19 is shown the variation of the shear and rotational deflections compared with the bending deflection for Poisson’s ratio $\nu = 0.3$. The result shows that when $L/h$ is between 5 and about 13, the correction for shear deformation is greater than the correction for rotation deformation. But when $L/h$ is greater than about 13, the effect of rotation compliance is more important than is classical shear deformation.

The numerical results using finite elements are shown in Fig. 20, for a Poisson’s ratio of 0.3. The finite element simulation parameters were similar to those of the cantilever. These numerical results agree very well with the analytical results.
Figure 20 The analytical results vs finite element results for a clamped-clamped configuration with a Poisson’s ratio of \( \nu = 0.3 \).

3.2.3 Clamped Circular Plate With Uniform Load

The clamped constraint will now be applied to a circular plate. Recall that the moment within the clamped constraint decays very quickly with distance (down to about \( 0.005M_0 \) at \( x_l = h \)). Thus the conditions for this circular geometry correspond approximately to that of plane strain. If the circular plate of radius \( a \) carries a uniform pressure of intensity \( q \) over the entire surface of the plate, the third-order differential equation for the transverse deflection \( w \) becomes [37]

\[
\frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) \right] = \frac{qr}{2D}
\]

The boundary conditions are

\[
w(a) = 0 \\
M_r = K_i \frac{dw}{dr} \bigg|_{r=a} \tag{59}
\]

where

\[
w'(0) = 0
\]
\[ M_r = -D \left( \frac{d^2w}{dr^2} + \frac{\nu dw}{r \, dr} \right) \]

resulting in

\[ w(r) = \left( \frac{1}{64D} \right) \left( \frac{q a^4}{r^4} - 1 + \frac{2K a}{D} + 2(3+\nu) \left( 1 - \left( \frac{r}{a} \right)^2 \right) \right) \]

(60)

where the term \( qa^4 / 64D \) represents the bending displacement at the center of a fixed plate. At \( r = 0 \), the ratio of the rotation displacement to the bending displacement is

\[ \delta_r = \left( \frac{q a^4}{64D} \right) = \frac{4}{K a / D + 1 + \nu} \]

(61)

By using the definitions of \( D \) and \( K \), we can rewrite Eqn. (61) as

\[ \delta_r = \left( \frac{q a^4}{64D} \right) = \frac{4}{(1-\nu)(a/h) / 4K + 1 + \nu} \]

(62)

The additional deflection of the middle surface of the plate due to the shearing deformation is [11]

\[ \delta_s = \frac{3}{2} \frac{qa^2}{4Gh} \]

(63)

and the negative deflection of the middle surface of the plate due to the lateral pressure is [11]

\[ \delta_l = -\frac{vqa^2}{2hE} \]

(64)

In Fig. 21 is shown the variation of the shear, lateral and rotational deflections compared with the bending deflection when \( \nu = 0.3 \). It is noted that as \( 2a/h \) becomes sufficiently large, the deflection approaches the value of the fixed edge. When \( 2a/h \) is greater than 5 but less than about 15, the correction due to shear and lateral is greater than the correction due to rotation, while for \( 2a/h \) greater than about 15 that trend is reversed.
The relationships among shear, lateral and rotation and bending deflections for a circular plate with a Poisson’s ratio of $\nu = 0.3$.

3.3 Conclusion

In many applications, particularly but not exclusively in MEMS and in nanomechanics, a plate/beam is clamped between rigid constraints. An elasticity analysis has shown that there will be rotation of the elastic strip within this rigid clamp. Our contribution is to allow simple beam theory to be used with a minor modification in the boundary condition in order to model this effect. The rotation at the clamp produces an added deflection in the suspended part of the structure. In the case of a cantilever with a transverse load at its end, the correction due to rotational compliance, is generally greater than the correction due to shear deformation. In a clamped-clamped configuration, the effect of rotational compliance is greater than the effect of shear deformation when $L/h$ is greater than about 13. The results for transverse loading agree well with plane strain elastic finite element modeling. For a circular plate, the rotational stiffness is also important becoming comparable to the combined deformation due to shear and lateral pressure when the diameter is about 15 times the thickness.
Chapter 4. Prying Action in a Transversely Bolted Cantilever

4.1 The Bolted Cantilever using Euler-Bernoulli Theory

Perhaps the simplest useful example of variable-contact prying is a beam fastened to a substrate with a single bolt passing into a tapped hole. This configuration can represent a beam or one portion of a plate (with periodic spacing between bolt centers) as depicted in Fig. 22a. If the bolt is sufficiently stiff, then the beam can be approximated as a cantilever. The bolt is modeled as a linear elastic spring of constant $k$ with tensile preload $F_0$, both of which affect the overall stiffness of the connection. The application of a load to the end of the cantilever induces prying action that results in a greater bolt tension than might be naively expected, with the possibility of yield (and hence a loss of preload upon unloading). Time-varying cantilever loads can obviously cause fatigue failure of the bolt.
Figure 22 (a) Schematic representation of prying action on a beam/plate; (b) Free body diagram of prying action on a beam/plate

The associated bolt stretch will allow the beam/plate to lift away from the rigid base over an initially unknown distance “a”. If there are two bolts along the beam axis, and the distance between the two bolts is greater than “a”, then the second bolt is shielded from an increase in load and becomes irrelevant. However if the axial bolt spacing is less than “a”, the situation will be different, i.e. the effect of the second bolt must be included in the analysis.

The rectangular cross-section beam has flexural rigidity $E^*I$, thickness $h$, and its transverse deflection is $w(x)$. In this case, there is no reaction force from the rigid base within the straight beam section, only a point load $F_p$ at point $x = 0$ (Fig. 22b) exists. As there is no distributed load acting in the suspended section, the Euler-Bernoulli differential equation becomes

$$E^*Iw''' = 0, \quad 0 < x < a$$

(65)
where $E^*$ represents the Young’s modulus of a beam reasonably modeled by plane stress, and it represents the plane strain modulus $E/(1-\nu^2)$ when the beam is wide (or is one of a sequence of adjoining strips), and the prime (’) denotes differentiation with respect to $x$. In any interval without loads the general solution of Eq. (1) is

$$w(x) = c_3 x^3 + c_2 x^2 + c_1 x + c_0.$$ (66)

We will apply this expression in the region $0 < x < a$. Normally only four boundary conditions are needed to solve Eq. (66), as there are only four unknown integration constants. However in this case one extra boundary condition will be needed because the lift-off distance “$a$” is also unknown.

For $x < 0$, the beam lies on the substrate, and hence the deflection, the slope and the internal bending moment are all zero. Therefore at $x = 0^+$, these quantities remain zero due to continuity. At $x = a^-$, the bending moment equals the product of the applied force and the distance between the centerline of the bolt and the position where the force is applied. Also at that point the shear force is equal to the bolt force $F_B$ minus the applied force $P$. Thus Eq. (66) is subject to the five boundary conditions

$$w'(0) = 0, \quad w''(0) = 0, \quad E^* Iw'''(a) = PL, \quad E^* Iw''(a) = F_B - P$$ (67)

Application of the first three boundary conditions leads to $c_0 = 0$, $c_1 = 0$, $c_2 = 0$, and from the fourth boundary condition $c_3 = PL/6E^* Ia$. By using the fifth boundary condition

$$F_B = P(L + a)/a$$ (68)

is obtained. This relation is simply a balance of moments about $x=0$. If the beam were considered rigid, then this moment balance would lead to the same equation as (68) but with “$a$” replaced by the distance from the bolt to the left end of the beam (Fig. 22a). Thus Eq. (68) demonstrates that when “$a$” is small, the bolt force is increased.

The bolt force is related to preload, bolt stiffness, and the bolt extension by

$$F_B = F_0 + kw(a)$$ (69)

where $k$ is the stiffness of the bolt, and the bolt extension is equal to $w(a)$, i.e. it is equal to the beam deflection at $x = a$. The effect of friction is neglected because there is
assumed to be no horizontal constraint. Replacing $F_B$ in Eq. (68) by using Eq. (69), collecting terms in $P$ and multiplying both sides by $L \beta / F_0$ leads to

$$P L \beta / F_0 = \frac{\beta a}{1 + \beta a / \beta L - (\beta a)^3}$$

(70)

where $\beta = \left(\frac{k}{6E^*I}\right)^{1/3}$ has the dimensions of reciprocal length (physically realistic magnitudes will be discussed in the next section), and $P L \beta / F_0$ represents the non-dimensional moment of the applied force. When $w(a)$ and $\beta$ are substituted, Eq. (69) for the bolt force becomes

$$F_B / F_0 = 1 + (a \beta)^2 \left(\frac{P L \beta}{F_0}\right)$$

(71)

Equations (70) and (71) can be solved for the two unknowns, $a$ and $F_B$. The solution procedure is as follows:

(a) For selected values of $L \beta$, use Eq. (70) to determine $PL \beta / F_0$ as a function of $\beta a$. The values chosen for $L \beta$ are 2, 4, 10, and 10000 (as a limiting value representing a pure moment load). This relation between the non-dimensional lift-off distance and the non-dimensional moment at $x=a$, $PL \beta / F_0$, is illustrated in Fig. 23.

(b) Eq. (71) gives the non-dimensional bolt force $F_B / F_0$ in terms of $PL \beta / F_0$ and $\beta a$. The results from a) allow the relation between the non-dimensional moment $PL \beta / F_0$ and the non-dimensional bolt force to be plotted as in Fig. 24.
Figure 23 Plot of normalized lift off length vs the normalized moment using Euler-Bernoulli beam theory.
Figure 24 Plot of normalized bolt force vs. the ratio of normalized moment for different values of $L\beta$ using Euler-Bernoulli beam theory.

The lift-off distance “$a$” is affected both by applied load and bolt preload as illustrated in Fig. 23. We observe that as the non-dimensional moment increases so does the lift-off distance but this distance asymptotes at large moment. A longer beam with the same moment as a shorter beam (i.e. with a smaller applied force) results in a smaller value of the lift-off distance. As $L\beta$ becomes very large then the asymptotic value of the lift-off distance becomes $\beta a = 1$ whereas for $L\beta = 2$, the asymptotic value becomes 1.165 as shown in Fig. 23. Note that if the preload is zero then the asymptotic value of $\beta a$ is reached immediately for the smallest value of $P$, and it remains constant at a value which is only determined by $L\beta$.

From Fig. 24 we can conclude that with each value of $L\beta$, receding contact theory predicts that the bolt force increases with the non-dimensional moment. However the increase in bolt force is proportionally less than the increase in applied load. This is
because as the applied load increases, the lift-off distance increases so the prying force is not proportional. When the normalized moment $PL\beta/F_0$ becomes large, so $a$ approaches a constant, the bolt force becomes proportional to the moment (approaching a straight line). The limiting lines for large $L\beta$ and for $L\beta = 2$ are shown in Fig. 24.

### 4.2 The Bolted Cantilever using Timoshenko Theory

Now we consider the bolted cantilever with Timoshenko beam theory which includes the effect of shear deformation. Not only does this increase flexural deformation in the suspended region, but it also results in a rapidly decaying shear at the beginning of the supported region. The positive sign convention for shear force, bending moment, slope and rotation are shown in Fig. 25. The bending moment and the shear force in the beam are related to the displacement and the rotation by [29]

![Figure 25 Sign convention for positive shear force, moment and rotation.](image)

\[ M = -E^*I \frac{dw}{dx} + \kappa^2 GA \left( \frac{dw}{dx} + \psi \right) \]

Equilibrium of the differential element gives

\[
\frac{d}{dx} \left[ \kappa^2 GA \left( \frac{dw}{dx} + \psi \right) \right] + p = 0
\]

\[
\frac{d}{dx} \left( E^*I \frac{d\psi}{dx} \right) - \kappa^2 GA \left( \frac{dw}{dx} + \psi \right) = 0
\]

where $\kappa^2$ is the Timoshenko shear coefficient usually taken as 0.833 [36], $E^*I$ is the flexural rigidity, $G$ is the shear modulus, $A$ is the cross-sectional area of the beam, $w$ is the transverse displacement of the mid-surface, and $\psi$ is the angle of the rotation of the
cross-section which was originally normal to the mid-surface of the beam. It is conventional to choose $\psi$ as positive clockwise, so that $\psi = -w'(x)$ in Euler-Bernoulli beam theory. The effect of friction is neglected as there is assumed to be no restraint to horizontal displacements. Again a concentrated vertical force $P$ is applied to the beam at a distance $L$ as shown in Fig. 22.

Define the parameter $\alpha^2 = \kappa^2 GA / E^* I = 12 \kappa^2 G / E^* h^2$, where $\alpha$ reflecting shear rigidity has the dimensions of reciprocal length. In the contact region $x < 0$, $w = 0$, leading to

$$\frac{d}{dx} \left[ \kappa^2 GA \psi \right] + p = 0$$

(74)

$$E^* I \frac{d^2 \psi}{dx^2} - \kappa^2 GA \psi = 0$$

Because of shear deformation the reaction force is not concentrated at a point, but rather it exhibits exponential decay into the contact region. This is because even with $w = 0$, the layer still has rotational deformation. From the second equation of Eq. (74), we obtain

$$\psi = c_4 e^{\alpha x} + c_5 e^{-\alpha x}, \quad x < 0$$

(75)

Solutions for $\psi$ which remain bounded as $x \rightarrow -\infty$ require that $c_0 = 0$. From the first equation of Eq. (74), the pressure decreases as $e^{\alpha x}$ as $x \rightarrow -\infty$, with the centroid of the pressure distribution at $x = -1/\alpha$. Because the contact region is of finite length (call it $c$) it is necessary that $\alpha c$ be sufficiently large such that $e^{-\alpha c} << 1$.

In the suspended region $0 < x < a$, Eq. (74) is applicable with $p = 0$. At $x = 0$, we require that $w$, $\psi$, $Q$ and $M$ be continuous, as there is no external force or moment applied at that point. Therefore the continuity conditions are

$$\psi(0-) = \psi(0+), \quad \psi'(0-) = \psi'(0+), \quad w(0) = 0, \quad w'(0) = 0$$

(76)

where $\psi'(0)$ is continuous because the internal bending moment is continuous, and $w'(0)$ is continuous from Eq. (73) because the shear force and rotation are continuous at that point. At $x = a$, the boundary conditions are
\[-E^*l\psi'(a) = PL, \quad \kappa^2GA(\psi(a) + w'(a)) = P - F_B\] (77)

Application of boundary and continuity conditions leads to the deflection and the rotation in the lift-off region

\[w(x) = \frac{PLx^2(\alpha x + 3)}{6E_I(\alpha a + 1)}, \quad 0 < x < a\] (78)

\[\psi_+ = \frac{-PL[(\alpha x)^2 + \alpha x + 1]}{2E_I(\alpha^2 a + \alpha)}, \quad 0 < x < a\] (79)

Using the boundary conditions of Eq. (76) and making the result dimensionless yields

\[PL\beta/F_0 = (1 - P/F_0)(a\beta + \gamma)/(1 - (a\beta)^3 - 3(a\beta)^2\gamma)\] (80)

where \(\gamma = \beta/\alpha\) is a dimensionless parameter representing the contribution of shear compliance. When \(\gamma = 0\), in other words for \(\alpha\) approaching infinity, Eq. (80) is equivalent to Eq. (70) of Euler-Bernoulli beam theory. Using \(\beta/\gamma\) to replace \(\alpha\), the bolt force becomes

\[F_B/P = F_0/P + (L\beta)(a\beta)^3/(a\beta + \gamma) + 3(L\beta)(a\beta)^2\gamma/(a\beta + \gamma)\] (81)

which approaches Eq. (71) of Euler-Bernoulli beam theory when \(\gamma = 0\). When \(a\beta = 0\), from Eq. (81) we can see that \(F_B/P = F_0/P\). Thus when the lift-off distance is zero, the bolt is not stretched, so the bolt force is equal to preload. In Euler-Bernoulli theory \(a\beta = 0\) only occurs when \(P = 0\), but from Eq. (80) \(a\beta = 0\) in Timoshenko theory can correspond to non-zero values of \(P\).

To understand realistic parameter ranges, we consider \(\beta, \alpha\) and \(\gamma\) in turn. In the case of \(\beta\), which is essentially the reciprocal of the liftoff distance for a Bernoulli-Euler beam restrained by a spring without a preload, we estimate bolt stiffness \(k\) as \((\pi d^2/4)E_B/h\), where \(d\) is the bolt diameter and the bolt length is taken as the beam height. This allows us to write \(\beta h = \left(\pi/2 \times E_B/E^* \times d/b \times d/h\right)^{1/3}\), where the parentheses essentially enclose the ratio of bolt axial stiffness to beam axial stiffness. In realistic situations, the modulus ratio
typically ranges from about 1 (steel bolts, steel beam) to 60 (steel bolts, polymer beam). The $d/b$ ratio tends to lie between 0.5 and 0.05 (preserving the beam section area), and $d/h$ between 4 and 0.1. For identical materials $\beta h$ will generally lie toward the lower end of 0.2–1.16, or multiply times $(64)^{1/3} = 4$ for disparate materials.

We next turn to $\alpha h = \left[12(5/6)G/E^\ast\right]^{1/2}$. For an isotropic material, in plane stress this reduces to $[5/(1+\nu)]^{1/2}$, and in plane strain to $[5(1-\nu)]^{1/2}$, which is generally in the middle of the range 1.6 to 2.2. The only substantial deviation arises for severely anisotropic materials such as wood, whose reported $G/E^\ast$ ratios vary from 0.038 (Western Hemlock) to 0.21 (Northern White Cedar), allowing $\alpha h$ to range from 0.62 to 1.45.

The upshot is that variations in $\gamma$ can be attributed mostly to the role of $\beta$, which might exhibit an eightfold range due to geometry, and another fourfold variation if the beam modulus is low. In contrast $\alpha$ tends to lie near 2, although it may drop by a factor of 2 or 3 with substantial anisotropy. For conventional materials and geometry, we expect a value of $\gamma$ near 0.2, although values exceeding 3 might be achievable.

For any selected $\gamma$, the procedure for determining the relationship between the bolt force and preload, and the relationship between lift-off distance and preload is as follows:

Given a value of $\gamma$ such as $\gamma = 0.2$, choose different values of $L\beta$, i.e. 2, 4, 10, 200. For each $L\beta$ vary the value of $\alpha \beta$ and use Eq. (80) to obtain the corresponding value of $PL \beta |F_0$, as shown in Fig. 26.

From Eq. (81), the value of $F_0/F_0$ is acquired, as illustrated in Fig. 27.
Figure 26 For $\gamma = 0.2$ and various $L_\beta$, the non-dimensional lift-off distance vs. the normalized moment using Timoshenko beam theory. Note that for this value of $\gamma$, no liftoff occurs until the non-dimensional moment equals 0.18-0.2.
For $\gamma = 0.2$ and various $L\beta$, the normalized bolt force vs. the normalized moment using Timoshenko beam theory.

As illustrated in Fig. 26, the upper limit of the dimensionless separation length (which occurs at zero preload or large applied force) decreases slightly with increasing length. An interesting feature is that the lift-off distance is zero until a certain value of the applied force is reached. The bolt force increases less than linearly with the cantilever length as shown in Fig. 27. For a certain value of applied force, as the length increases the moment increases which leads to an increase in the bolt force. This conclusion is similar to Euler-Bernoulli beam theory.

By choosing a special normalization of the forces, it is possible to capture all the influence of $L\beta$ in a single curve, so curves representing different $\gamma$ can be presented in a single universal plot (where the value $\gamma = 0$ is the Euler-Bernoulli model). Rearranging Eqs. (80) and (81), the equations become

\[
1 = (a\beta)^3 + 3(a\beta)^2 \gamma + (F_0/P-1)a\beta/L\beta + (F_0/P-1)\gamma/L\beta \quad (82)
\]

\[
(F_B/P-1)/L\beta = (F_0/P-1)/L\beta + (a\beta)^3/(a\beta+\gamma) + 3(a\beta)^2 \gamma/(a\beta+\gamma) \quad (83)
\]
which relate $a\beta$ to $(F_0/P - 1)/L\beta$ and $(F_B/P - 1)/L\beta$ to $(F_0/P - 1)/L\beta$. The solution procedure is as follows:

(a) Choose different values of $\gamma$. For each $\gamma$ vary the value of $a\beta$ and use Eq. (82) to obtain the corresponding value of $(F_0/P - 1)/L\beta$, as plotted in Fig. 28.

(b) From Eq. (83), the value of $(F_B/P - 1)/L\beta$ is acquired, as depicted in Fig. 29.

Figure 28 Universal plot of non-dimensional lift-off length vs. normalized preload encompassing all values of $L\beta$ and various values of $\gamma$ using Timoshenko beam theory. The crosses represent the result of three-dimensional FEA with $\gamma = 0.1966$. 

![Universal plot of non-dimensional lift-off length vs. normalized preload](image-url)
Figure 29 Universal plot of the normalized bolt force vs. the normalized preload for all \( L \beta \) and various \( \gamma \), using Timoshenko beam theory. The crosses represent the result of three-dimensional FEA with \( \gamma = 0.1966 \).

As shown in Fig. 28, the upper limit of the lift-off distance decreases with increasing \( \gamma \). This behavior is expected as increasing \( \gamma \) corresponds to greater shear deformation and a more compliant beam. The ratio of the bolt force due to the applied force decreases as \( \gamma \) increases, as illustrated in Fig. 29, again due to effect of shear deformation leading to a more compliant beam. Note that the value of \( a \beta \) becomes equal to zero when a certain value of \( (F_0/P - 1)/L \beta \) is exceeded due to low load \( P \) (Fig. 28), where the end point of each curve is on the linear line of \( F_B/P = F_0/P \) of Fig. 29. This result is important because it indicates that for \( P/F_0 \) sufficiently small there will be no further increase in the bolt force above the preload.

The angle of rotation in the suspended region \( 0 < x < a \) can be determined from Eq. (79). When \( x = a \)

\[
\psi(a) = \left| \frac{PL^3}{3EI} \right| \frac{3}{2(\beta L)(\beta a/\gamma + 1)} \left( \frac{\beta a}{\gamma} \right)^2/\gamma + \beta a + \gamma 
\]

(84)
is obtained. This gives the fractional increase in tip deflection due to this rotation (i.e., added deflection at the end of the cantilever due to rotation at the bolt location, divided by the end deflection without rotation (\(-\frac{PL^3}{3EI}\)).

The procedure for determining the results is as follows: Given a value of \( \gamma \) such as \( \gamma = 0.2 \), choose different values of \( L\beta \), i.e. 2, 4, 6, 8, 10. For each \( L\beta \) vary the value of \( a\beta \) and use Eq. (20) to obtain the corresponding value of \( \frac{\psi(a)L}{\left(-\frac{PL^3}{3EI}\right)} \). The relationship between \( \psi(a)L \) and \( PL\beta/F_0 \) is shown in Fig. 30.

![Figure 30](image)

Figure 30 For \( \gamma = 0.2 \), beam tip deflection due solely to rotation \( \psi \) at \( x = a \), divided by Euler-Bernoulli tip deflection (that is, \( \frac{\psi(a)L}{\left(-\frac{PL^3}{3EI}\right)} \)), plotted against the normalized moment.

Fig. 30
From Fig. 30, as $L\beta$ increases, the upper limit of the ratio of the added deflection due to rotation to the end deflection of a fixed end cantilever decreases, but the actual value of the added deflection increases. As the applied force increases, the angle of the rotation ratio increases so that the angle of rotation increases more than linearly. The smaller $L\beta$ is, the more important is the angle of rotation in affecting the added effect to the end deflection compared to the fixed end deflection. The smaller the preload is, the greater is the added deflection effect.

4.3 The Bolted Cantilever using Three-Dimensional Finite Element Analysis

Finite element analysis (FEA) is a numerical technique for solving engineering problems applied to a variety of problems in structures, heat transfer, and fluid flow etc. which finds approximate solutions to boundary values problems for partial differential equations. It is usually used in problems in which analytical solutions are not easily obtained and mathematical expressions required for solution are not simple because of complicated geometries, loadings and material properties. Euler-Bernoulli theory and Timoshenko theory are simple one-dimensional models that make simple analytical approximations which allow closed-form solutions to be obtained. Because beam theories are unable to include three-dimensional effects, such as the beam width, a finite element analysis of three-dimensional elastic behavior including frictionless contact between bodies is used in order to determine the importance of these effects. Our analysis employed the commercial software Solidworks® Simulation Professional, which meshed the model with tetrahedrons. This program has a ‘contact’ feature which we used with zero friction.

Higher mesh density is expected to improve the accuracy of the solution at the cost of computational time, In our case, we use a fine mesh in some specific regions, while a coarse mesh is used in the remainder as depicted in Figs. 31 and 32. As the model is symmetric about the mid-plane which passes through the center of the bolt, we use half the width of the layer, the base, and the bolt with appropriate symmetric boundary conditions of zero shear stress and zero normal displacement. The lift-off distance varies continuously from the mid-plane to the back-plane and is determined at those two locations. We consider the average value as the appropriate metric for the lift-off distance.
Figure 31 Three dimensional finite element model. The beam is the 0.5” thick layer at the top, in variable contact with a block of substrate. Instead of passing the elastic bolt through a hole in the beam, it extends upward to a fixed end, making an axial spring. The view given here is the midplane of the beam and bolt, with load to be applied at the beam end on the right. The back face of this model is the back face of the beam, or the midplane dividing analogous strips of plate.
Figure 32 Three dimensional model after meshing. This is a half-model of the beam, bolt, and substrate. The very fine mesh is provided to enhance accuracy in finding the edge of contact. Vertical loads are seen at the right upper edge of the beam; while the far beam end, and the end and bottom of the substrate, are fixed. Symmetry boundary conditions are used on the front and back planes.
The half-width geometry analyzed is shown in Fig. 32. The half-width is 1.5 in., the thickness is 0.5 in., the length $L$ is 10 in., and the diameter of the bolt (which is modeled extending upward to a fixed support, to avoid a hole through the beam) is $\frac{1}{4}$ inch. The materials of the bolt and substrate are both alloy steel with $E = 30 \text{ Mpsi}$ and $\nu = 0.28$. In this case $\gamma = 0.1966$, $\beta = 0.7844 \text{ m}^{-1}$, and $L\beta = 7.84$. A force of 100 lb is applied to the half-width (so the force of the entire width is 200 lb) with no bolt preload. From Timoshenko beam theory the lift-off distance should be $a\beta = 0.8797$ which corresponds to $a = 1.12 \text{ in.}$ and a bolt force of $F_B = 1658 \text{ lb}$. Different fine mesh sizes were explored, i.e. 0.02 in., 0.04 in., 0.06 in., and 0.08 in. From the results shown in Table 2, we can conclude that when the fine mesh size decreases from 0.08 in to 0.02 in, the lift-off distance and bolt force don’t differ significantly. Nonetheless we will use 0.02 in. for the fine mesh realizing that other geometries may be more sensitive to mesh size.

The largest deflections of the substrate for different mesh sizes are $-2.84 \times 10^{-5} \text{ in}$, $-2.83 \times 10^{-5} \text{ in}$, $-2.81 \times 10^{-5} \text{ in}$, $-2.82 \times 10^{-5} \text{ in}$. These magnitudes are less than 0.04% of the deflection difference between the bottom of the layer and the substrate. Thus any differences between the FEA results and those of the simpler beam theories are not attributable to substrate deflection which is neglected in the beam theories.

A preload is simulated by applying a displacement of 0.001 in. downward to the top of the bolt which is then held fixed. The corresponding preload of the bolt is 2140 lb. We then apply five different forces to the end of the layer, i.e. 100 lb, 200 lb, 400 lb, 600 lb, 800 lb to the entire width. The comparison between the FEA results and the Timoshenko beam (TB) results are shown in Fig. 28 and Fig 29. We can see that the effect of beam load on bolt force in FEA is smaller than the TB result. This is because in Timoshenko beam theory certain assumptions are made on how the beam can deform resulting in a stiffer model. In particular there are no variations across the width of the beam and thickness deformation is not allowed. For these reasons the results for the lift-off distance of three-dimensional FEA is slightly smaller than for the TB theory.

Now a 100 lb force is applied to the entire beam and the values of the full width are chosen as 0.5 in, 1 in, 2 in, 3 in, 4 in, and 5 in, without altering the material properties or other dimensions. The values of $\gamma$ and $\beta$ and the results of FEA and TB for these different widths are shown in Table 3. From Table 3, we see that the bolt force from the three-dimensional FEA is
always smaller than that using Timoshenko beam theory. As already discussed the stiffness in FEA is less than for Timoshenko beam theory. Also the lift-off distance increases with the beam width for both FEA and TB. This is because when the width increases the beam/layer stiffness increases compared to the bolt stiffness. However the effect of increasing width is more pronounced for FEA than for TB due to the added deformation of the layer through the width.
<table>
<thead>
<tr>
<th>Fine mesh size (in)</th>
<th>Bolt force from FEA model (lb)</th>
<th>TB result of bolt force (lb)</th>
<th>Difference (%)</th>
<th>Average a (in)</th>
<th>TB result of a (in)</th>
<th>Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>1472</td>
<td>1658</td>
<td>12.64%</td>
<td>1.100</td>
<td>1.121</td>
<td>1.9%</td>
</tr>
<tr>
<td>0.04</td>
<td>1474</td>
<td>1658</td>
<td>12.48%</td>
<td>1.120</td>
<td>1.121</td>
<td>0.09%</td>
</tr>
<tr>
<td>0.06</td>
<td>1480</td>
<td>1658</td>
<td>12.03%</td>
<td>1.120</td>
<td>1.121</td>
<td>0.09%</td>
</tr>
<tr>
<td>0.08</td>
<td>1484</td>
<td>1658</td>
<td>11.73%</td>
<td>1.120</td>
<td>1.121</td>
<td>0.09%</td>
</tr>
</tbody>
</table>

Table 2 The comparison between FEA results with different mesh sizes and TB (Timoshenko Beam) results for the force of 200 lb

<table>
<thead>
<tr>
<th>Width (in)</th>
<th>$\gamma$</th>
<th>$\beta$</th>
<th>Bolt force of FEA (lb)</th>
<th>Bolt force of TB (lb)</th>
<th>Difference of bolt force (%)</th>
<th>Lift off distance of FEA (in)</th>
<th>Lift-off distance of TB (in)</th>
<th>Difference of lift off distance (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.3573</td>
<td>1.4253</td>
<td>1090</td>
<td>1369</td>
<td>25.60%</td>
<td>0.480</td>
<td>0.538</td>
<td>12.1%</td>
</tr>
<tr>
<td>1</td>
<td>0.2836</td>
<td>1.1312</td>
<td>1020</td>
<td>1132</td>
<td>10.98%</td>
<td>0.620</td>
<td>0.718</td>
<td>15.8%</td>
</tr>
<tr>
<td>2</td>
<td>0.2251</td>
<td>0.8979</td>
<td>844</td>
<td>931</td>
<td>10.31%</td>
<td>0.920</td>
<td>0.953</td>
<td>3.6%</td>
</tr>
<tr>
<td>3</td>
<td>0.1966</td>
<td>0.7844</td>
<td>742</td>
<td>829</td>
<td>11.73%</td>
<td>1.120</td>
<td>1.121</td>
<td>0.09%</td>
</tr>
<tr>
<td>4</td>
<td>0.1786</td>
<td>0.7126</td>
<td>670</td>
<td>763</td>
<td>13.88%</td>
<td>1.320</td>
<td>1.258</td>
<td>-4.7%</td>
</tr>
<tr>
<td>5</td>
<td>0.1658</td>
<td>0.6616</td>
<td>618</td>
<td>715</td>
<td>15.70%</td>
<td>1.500</td>
<td>1.374</td>
<td>-8.4%</td>
</tr>
</tbody>
</table>

Table 3 The comparison between FEA results with different values of $\gamma$ and $\beta$ for different width and TB results
4.4 Conclusion

In this study we investigate the simplest example of elastic prying action, i.e. a beam/plate bolted to a substrate with a single bolt passing into a tapped hole in the substrate. This configuration can represent a beam or a plate with periodic spacing between bolt centers. We treat this as a compliant receding contact problem where the beam elastic flexural deformation induces a variable-location prying reaction. With bolt preload specified, a load applied to the end of the bolted cantilever raises the bolt force, leading to variable (i.e. repeated, alternating, or fluctuating) stress. Unlike classical bolted connections between rigid bodies, the preload does not entirely shield the bolt from force variations. In the Euler-Bernoulli model, even the smallest load overcomes the preload to stretch the bolt. The resulting beam rotation at the bolt location also alters the cantilever tip deflection.

The lift-off distance increases continuously with the applied force using Euler-Bernoulli theory. But in Timoshenko beam theory, the lift-off distance is zero until the applied force exceeds a certain value. The upper limit of the lift-off distance decreases with an increase in shear deformation. The value of the bolt force starts with the preload and then increases with the applied force. It is also affected by the lift-off distance with smaller values corresponding to a larger bolt force due to an increase in prying.

An interesting approximation to the needed bolt strength arises by noticing (Fig. 8) that the simpler Euler-Bernoulli theory over-estimates the bolt load, so it may be used for design estimates. In view of Fig. 3, a low value of preload seems to reduce bolt force, so we can make use of the upper line written as $F_B = 1.35 \ PL\beta$ (where preload is ignored). Consider the yield strength of a bolt needed to just cause bending yield in the beam. We can write $\sigma_{YB} A_B$ for the tensile yield force of the bolt, and $\sigma_{Y} Ah/6$ to represent beam yield moment (equal to $PL$). The yield equality expression then appears as $\sigma_{YB} A_B = 1.35 (\sigma_{Y} Ah/6h) (\beta h)$. The final parentheses may be replaced, for steel bolt and beam, by $2^{1/3}(A_B/A)^{1/3}$. In terms of cross section areas, $A_B/A > 0.15(\sigma_{Y} / \sigma_{YB})^{3/2}$ if the bolt is not to yield before the beam, regardless of orientation ($b<h$ or $b>h$).

In the cases investigated by three-dimensional FEA, the bolt force was smaller than that using Euler-Bernoulli and Timoshenko Beam (TB) theories, perhaps resulting from the smaller stiffness of the three-dimensional FEA. Also the lift-off distance
increased with the beam width for both FEA and for beam theories. Future work will apply these principles to the more widely used structural connection of a T-stub.
Chapter 5.  Tension T-stub

5.1 Tension T-Stub Connection Modeled with Euler-Bernoulli Beam Theory

Bolted T-stub connections are most commonly used in steel frame structures. A T-stub is a T-shaped metal component made of a tensile web and bolted flange. The distance from the centerline of the bolt to the surface of the web is \( L \), as illustrated in Fig. 33. (Treated as a horizontal beam, the web is so short and tall that it is reasonably idealized as rigid.) Each of the bolts is tightened to a preload \( F_0 \). Actually there is a resulting interface pressure, but for an Euler-Bernoulli beam this appears as a concentrated force. If the T-stub is rigid, as depicted in Fig. 33, the force per bolt when load (beyond twice the preload) is applied is simply \( F_B = P \). However if there is flexural deformation of the flange, as shown in Fig. 34, a prying force is induced at each of the separation points which results in \( F_B > P \). The smaller the distance between forces \( F_P \) and \( F_B \) the greater is the prying action. The simplest way to evaluate this prying action is by considering the effect of the applied load on the axial force in the bolt. (While distortion of the connected parts also results in bending of the bolt, we leave this out of our investigation.)

Figure 33 (a) Schematic of rigid (i.e. undeformed) T-stub (b) Equivalent periodic array of bolts
Before outlining the following beam analysis, we describe how bolt compliance is treated. The conventional ‘machine design’ model of bolted joint compliance involves the bolt shaft, plus a certain part of the clamped flange or plate sometimes called a ‘Roetscher pressure cone’ (hollow frustum) [39]. This estimate of compliance of clamped material appears sensible and well-defined only for the specific problem of bolt tightening, where bolt load is applied to the faying surface under the head (presumed rigid). The tightening problem is a matter of stiffnesses in series. However, in the case of a loaded bolted plate, it very much matters where the separating P load is applied. If at the upper surface just around the bolt head, we have a clear combination of stiffnesses in parallel. But if at the lower surface around the bolt, we have a series combination. And if near the midplane of the plate, we see elements of both series and parallel load paths. It seems reasonable to conclude that the stiffness of a bolt plus clamped material is not well defined until the geometry of load application is known – and then the problem likely requires a complex analysis.

In our beam analyses we have ignored the compliance of clamped materials, which we consider not uniquely defined. Partly this is justified by literature which estimates this compliance as typically just 5% - 10% of bolt compliance[23]. Further, with a beam whose ‘pressure cones’ may be considered rigid, it doesn’t matter whether the load path is through the top, middle or bottom of the layer. In that case external load generates no displacement (hence no bolt stretch) until it overcomes preload. The more sophisticated treatment of the bolting guide implies that the bolt could increase its tension about 5% before separation occurs, for some typical load path.
Due to the symmetry of the T-stub and the loading, only half of the T-stub needs to be analyzed (Fig. 35), and the applied force is just $P$. (In fact, with the web modeled as rigid the relevant beam length excludes web thickness.) $M_0$ is the statically indeterminate moment needed so that the beam rotation at the web vanishes. The bolt force ($F_B$), the prying force ($F_P$), and the lift-off distance “$a$” are also unknown. In the suspended region $0 \leq x \leq a$ there is no distributed load so the Euler-Bernoulli equation becomes $E^* I w^{(4)} = 0$, where the prime (‘) denotes differentiation with respect to $x$, and $E^*$ represents the Young’s modulus $E$ of a beam reasonably modeled by plane stress, or represents the plane strain modulus $E/(1-\nu^2)$ when the beam is wide (or is one of a sequence of adjoining strips). The boundary conditions on $w$ at $x=0$ are $w(0)=0$, $w'(0)=0$, $w''(0)=0$. The corresponding transverse beam deflection is then $w(x) = c_3 x^3$ ($0 \leq x \leq a$). We have illustrated the beam extending some distance in the $x$ direction. In our analyses we have assumed that stress resultants within the beam ‘die away’ in that direction, and so implicitly that the available length includes a few characteristic lengths of exponential decay. For a Timoshenko beam of isotropic material the characteristic length is about half the beam height, so a ‘flat section’ of about $2h$ should suffice. If lift-off distance $a$ approaches the edge closer than this, the analysis would not be applicable. For an Euler-Bernoulli beam the characteristic distance is zero, so any nonzero negative extent of the beam will suffice.

![Diagram](image_url)

Figure 35 (a) Schematic of T-stub joint deformation. The bolt is modeled by a linear spring, and $w(a)$ is the length bolt has been stretched. $F_B$ is the bolt force, $F_P$ is the reaction force, $P$ is applied force, and $M_0$ is the moment caused by $P$. (b) The relationship between bolt force and preload.
Two boundary conditions are applied to the above deflection expression in order to determine the two unknowns $c_3$ and $a$: the bending moment and shear force at $x = a$ as found from static equilibrium:

$$E' I w''(a) = -M_0 + PL, \quad E' I w''(a^-) = F_B - P$$  \hspace{1cm} (85)$$

where $F_B = F_0 + k\omega(a)$ . We will apply these conditions in the region $0 < x < a$. Application of the first boundary condition of Eq. (1) allows us to determine $c_3$, so $w(x)$ becomes

$$w(x) = \left( -M_0 + PL \right) x^3 / 6EIa$$  \hspace{1cm} (86)$$

Now consider the region $a < x < a+L$. The zero slope at the flange-web interface can be expressed in terms of slope at $a$ plus the slope change due to end loads $P$ and $M_0$:

$$w'(a + L) = w'(a) + PL^2 / 2E^* I - M_0 L / E^* I = 0$$  \hspace{1cm} (87)$$

Using the displacement expression of Eq. (86) to evaluate $w'(a)$ in Eq. (3) allows us to determine $M_0 = PL(a + L)/(a + 2L)$, so the displacement expression can be updated to replace $M_0$:

$$w(x) = \left( -M_0 + PL \right) x^3 / 6EIa$$  \hspace{1cm} (86)$$

The result is a relation between $(P/F_0)$ and the unknown $(a\beta)$:

$$[(L\beta)^2 - (L\beta)^2 (a\beta)^3 + (a\beta)^2 + 2(a\beta)(L\beta)](P/F_0) = (a\beta)^2 + 2(a\beta)(L\beta)$$  \hspace{1cm} (88)$$

In principle we would like to determine $(a\beta)$ for any selected $(P/F_0)$. Then the nondimensional bolt force $F_B/F_0$ is given by nondimensionalizing $F_B = F_0 + k\omega(a)$:

$$F_B / F_0 = 1 + \left[ (P/F_0)(L\beta)^2 \right] (a\beta) / (a\beta + 2L\beta)$$  \hspace{1cm} (89)$$

The procedure for plotting consequences of equations (88) and (89) is as follows:

(a) For selected values of $L\beta$, vary $a\beta$ and use Eq. (88) to determine the corresponding value of $(P/F_0)$. Then $a\beta$ can be plotted as a function of $(P/F_0)$.

(b) For each value of $(P/F_0)$, $L\beta$ and $a\beta$, use Eq. (89) to determine $F_B/F_0$, which can be plotted as a function of $(P/F_0)$.
In Fig. 36, the abscissa is the applied force (per bolt) divided by the bolt preload, and the ordinate represents the normalized lift-off distance. For low loads \((P/F_0 < 1)\), the smaller web-to-bolt distances \((L\beta)\) correspond to smaller lift-off distances \((a\beta)\). When \(P/F_0 > 1\), the smaller web-to-bolt distance gives a larger lift-off distance. When \(P/F_0=1\), Eq. (4) becomes \(a\beta = 1\). When the applied force is equal to the preload, this case can be considered as a cantilever beam loaded by a couple of forces, i.e. preload \(F_0\) and \(P\), then the stretch force without preload \(kw(a)\) with equal reaction force form another couple. Therefore, once preload is matched by \(P\), the region \(x<a\) is becomes identical to a moment-loaded bolted cantilever without preload, a case where the lift-off distance is constant. In other words, the non-dimensional lift-off distance is constant when \(P/F_0=1\), no matter the value of \(L\beta\). As the applied force increases beyond this point so does the lift-off distance but it asymptotes to an upper limit, \(a\beta\). That asymptotic value is determined by setting the coefficient in the bracket term of Eq. (4) to zero for \(P/F_0 \to \infty\). This limiting behavior is shown in Fig. 37. In Fig. 37, we can see the upper limit of the lift-off distance decreases with an increase of the distance \(L\beta\) between the bolt and the web. For large loads, we see that \(a\beta\) approaches a limit near 1 or 2. In effect, \(1/\beta\) or \(2/\beta\) gives the lift-off distance when there is high force or no preload (in the latter case, a fixed value independent of load). Small \(\beta\) (large \(EI/k\)) corresponds to large lift-off distance \(a\). This is protective of the bolt, because it implies a small prying force: taking moments about \(x=a\), it can be shown that \(F_P\) equals \(P(L/a)(L/(a+2L))\) which is unusually large only when \(a/L\) is small.
Figure 36 Normalized lift-off distance $a_{\beta}$ vs. normalized applied force $F_b / F_0$ for various values of $L_{\beta}$.

Figure 37 Normalized the uplimit value of lift-off distance $a_{\beta}$ vs normalized bolt-to-flange separation distance $L_{\beta}$. 
In Fig. 38 for bolt force, for any value of $L\beta$, receding contact theory predicts that the bolt force increases with the non-dimensional applied force. At first the bolt force increases slowly (very little stretch beyond the preloaded state) as indicated by the horizontal slope as $P/F_0 \to 0$. For greater values of $P/F_0$ the ratio of bolt force to preload becomes larger with an increase of $L\beta$. When $P/F_0 \to \infty$ in Eq. (89), as mentioned previously, $a\beta$ reaches a certain upper limit value, and with $L\beta$ constant the first term on the right side of Eq. (89) is neglected compared with the second term, which gives the asymptotic lines which converge at the origin. Consequently the bolt force increases proportionally with the applied force for large values of the applied force.

![Figure 38](image.png)

Figure 38 The normalized bolt force $F_B/F_0$ vs. the normalized applied force $P/F_0$.

5.2 Tension T-stub Connection Modeled with Timoshenko Beam Theory

We now include the effect of beam shear deformation by modeling the bolted T-stub with Timoshenko beam theory. Not only does this model increase the flexural and shear deformations of the flange, it also produces an exponentially decreasing
distributed contact pressure within the contact region, shifting the contact reaction force to a negative-x location.

The positive sign convention for shear force, bending moment, slope and rotation are shown in Fig. 39. The bending moment and the shear force in the beam are related to the displacement and the rotation by [29]

\[
M = -E^*I \frac{d\psi}{dx} \\
Q = \kappa^2 GA \left( \frac{dw}{dx} + \psi \right)
\]

(90)

where \((dw/dx + \psi)\) gives the rotation of horizontal and vertical elements toward each other.

![Figure 39 Sign convention for positive shear force, bending moment and cross-section rotation.](image)

Static equilibrium of the differential element shown gives

\[
\frac{d}{dx} \left[ \kappa^2 GA \left( \frac{dw}{dx} + \psi \right) \right] + p = 0 \\
\frac{d}{dx} \left( E^*I \frac{d\psi}{dx} \right) - \kappa^2 GA \left( \frac{dw}{dx} + \psi \right) = 0
\]

(91)

where \(p\) is the distributed load, \(\kappa^2\) is the Timoshenko shear coefficient usually taken as 0.833 [36], \(E^*I\) is the bending rigidity, \(G\) is the shear modulus, \(A\) is the cross-sectional area of the beam, \(w\) is the transverse displacement of the mid-surface, and \(\psi\) is the angle of rotation of the cross-section which was originally normal to the centroidal axis of the beam. It is conventional to choose \(\psi\) as positive clockwise, so that \(\psi = -w'(x)\) in Euler-Bernoulli beam theory. The effect of horizontal friction from the
substrate is neglected (note that if two T-stubs are bolted together, symmetry rigorously precludes it).

In the contact region \((x < 0)\), the displacement vanishes \((w = 0)\), which leads to

\[
\frac{d}{dx} \left[ \kappa^2 GA \psi \right] + p = 0
\]

\[
E^* I \frac{d^2 \psi}{dx^2} - \kappa^2 GA \psi = 0
\]

(92)

From the second of these the solution for \(\psi\) which remains bounded as \(x \to -\infty\) can be shown to be \(\psi = c_1 e^{\alpha x}\), when \(x < 0\), where \(\alpha = \sqrt{\frac{\kappa^2 GA}{E^* I}}\). Then the first of these can be solved for \(p = -\kappa^2 GA \alpha c_1 e^{\alpha x}\) when \(x < 0\).

In the suspended region \(0 < x < a\), \(p = 0\) so Eq. (91) can be written \(dQ/dx = 0\), which leads to a constant shear force. From Eq (91) we see that \(\psi'' = \alpha^2 c\), resulting in \(\psi = \alpha^2 c x^2/2 + c_2 x + c_3\).

At \(x = 0\), we require that \(w, \psi, Q\) and \(M\) be continuous, as there is no concentrated force or moment applied at that point. Therefore the continuity conditions are

\[
w(0^+) = w(0^-) = 0, \quad \psi(0^-) = \psi(0^+) = c_1, \quad w'(0^+) = w'(0^-) = 0, \quad \psi'(0^-) = \psi'(0^+) = \alpha c_1
\]

(93)

where \(\psi'(0)\) is continuous because the internal bending moment is continuous, and \(w'(0)\) is continuous because the shear force and rotation are continuous at that point. At the bolt location \((x = a)\), the boundary conditions are

\[
-E^* I \psi'(a) = -M_0 + PL, \quad \kappa^2 GA (\psi(a^-) + w'(a^-)) = P - F_B
\]

(94)

The deflection \(w(x)\) is obtained from from eqs. (7) with \(p=0\), that is

\[
w = -\alpha^2 c x^3/6 - c_2 x^2/2 - c_1 x + c_4.
\]

Then application of the boundary and continuity conditions of Eq. (93) and Eq. (94) leads to the rotation angle and the deflection in \(0 < x < a\)

\[
\psi = \frac{-(-M_0 + PL)\alpha x}{2E^* I(\alpha a + 1)} - \frac{(-M_0 + PL)x}{E^* I(\alpha a + 1)} - \frac{-M_0 + PL}{E^* I(\alpha^2 a + \alpha)}
\]

(95)
Furthermore the vanishing of flange rotation at the flange-web interface leads to
\[ \psi(a + L) = 0 \]
\[ \Rightarrow \left( -M_0 + PL \beta \right) \left( L \beta + (a \beta)^2 / 2(a \beta + \gamma) + (a \beta) \gamma / (a \beta + \gamma) + \gamma^2 / (a \beta + \gamma) \right) + \frac{P(L \beta)^2}{2E' I \beta^2} = 0 \]

(97)

where \( \gamma = \beta / \alpha = L \beta (h / L) (1 / \alpha h) \). The rotation at \( x = a + L \) includes the effects of the rotation at \( x = a \), the applied force, and the statically indeterminate moment.

Solving Eq. (97) for \( M_0 \) leads to
\[ M_0 (L \beta - (L \beta)^2 / 2(L \beta + (a \beta)^2 / 2(a \beta + \gamma) + (a \beta) \gamma / (a \beta + \gamma) + \gamma^2 / (a \beta + \gamma)) = 0 \]

(98)

From the boundary condition of Eq. (94)_2, we find
\[ (P / F_0) (L \beta)^2 \left( 1 - (a \beta)^2 - 3 \gamma (a \beta)^2 \right) / \left( 2L \beta (a \beta + \gamma) + (a \beta)^2 + 2 \gamma (a \beta) + 2 \gamma^2 \right) + 1 = 1 \]

(99)

From this expression, the liftoff distance \( a \) can be found for any value of \( P / F_0 \).

The bolt force is then obtained from \( F_b = F_0 + kw(a) \) as
\[ F_b / F_0 = 1 + \frac{(P / F_0) (L \beta)^2 ((a \beta)^2 + 3 \gamma (a \beta)^2)}{2L \beta (a \beta + \gamma) + (a \beta)^2 + 2 \gamma (a \beta) + 2 \gamma^2} \]

(100)

To generate plots of Eqs. (99) and (100) we proceed as follows:

(a) For a given value of \( \gamma \), and distinct values of \( L \beta \), vary \( a \beta \) and find \( P / F_0 \) using Eq. (99). Plot \( a \beta \) as a function of \( P / F_0 \) for each \( L \beta \).

(b) For the same values of \( \gamma \) and \( L \beta \), use corresponding pairs of \( a \beta \) and \( P / F_0 \) in Eq. (100) to find \( F_b / F_0 \). (If desired, Eq. (98) can also be used to obtain \( M_0 (L \beta) / F_0 \).)

Note that when \( \gamma = 0 \), Eq. (99) and Eq. (100) reduce to Euler-Bernoulli beam theory. In Euler-Bernoulli theory \( a \beta = 0 \) occurs only when \( P = 0 \), but from Eq. (99)
\( a\beta = 0 \) in Timoshenko theory can also occur with non-zero values of \( P \). Note that when \( a\beta = 0 \), from Eq. (100) we can see that \( F_0/F_0 = 1 \), so when the lift-off distance is zero, the bolt is not stretched, making the bolt force equal to the preload.

In order to understand realistic parameter ranges, we consider \( \beta, \alpha \) (dimensional) and \( \gamma \) (dimensionless) in turn. In the case of \( \beta \), which is essentially the reciprocal of the liftoff distance for an Euler-Bernoulli beam restrained by a spring without a preload, we approximate the bolt stiffness \( k \) as \( (\pi d^2/4)E_B/h \), where \( E_B \) is the Young’s modulus of the bolt, \( d \) is the bolt diameter, and the bolt length is taken as the beam height. This allows us to write

\[
\beta h = \left( \frac{\pi}{2} \times E_B / E^* \times d / b \times d / h \right)^{1/3},
\]

where the parentheses essentially enclose the ratio of bolt axial rigidity to beam axial rigidity. In many realistic situations, the modulus ratio is equal to one (steel bolt and steel slender beam). The \( d/b \) ratio tends to lie between 0.5 (preserving some beam cross-section area) and 0.05, and \( d/h \) is between 4 and 0.1. Thus for identical materials \( \beta h \) is bounded between 0.2 and 1.46.

We next turn to \( \alpha h = [12 \kappa^2 G/E^*]^{1/2} \) which is a function of material properties only. For steel with a Poisson’s ratio \( \nu = 0.28 \), \( \alpha h \) is 1.98 and it will be similar for other engineering metals. Due to the near-constancy of \( \alpha h \), the dimensionless ratio \( \gamma = \beta/\alpha \) partakes of the same range of magnitudes as \( \beta h \).

When contrasted with the cantilever beam of our previous chapter, the case of a Timoshenko-beam T-stub seems not to lead to a ‘universal’ plot showing all solutions. Practically speaking, one could pick several different values of \( \gamma \) and present the corresponding plots. The trouble is, \( \gamma \) covers a wide range. An alternative approach is to write \( \gamma \) as

\[
\gamma = \frac{\beta}{\alpha} = \frac{L\beta}{h/L}(1/\alpha h).
\]

For typical metal, \( \nu = 0.3 \), then \( \alpha h = 1.96 \). A plot for this single value of \( \alpha h \) can represent 5:1 ranges of \( L\beta \) and \( h/L \), without being too crowded.

For a certain value of applied force, the moment increases with the length which leads to an increase in the bolt force, as shown in Fig. 41. When \( P/F_0 \to \infty \) in Eq. (100), as mentioned before, \( a\beta \) reaches a certain upper limit value \( a_\infty \beta \), and for a given value of \( L\beta \), and \( h/L \) is constant when given the certain geometry, \( F_0/F_0 \) only depends on \( P/F_0 \). This conclusion is similar to Euler-Bernoulli beam theory.
Figure 40 When $h/L=0.2$, non-dimensional lift-off length vs. normalized applied force for various values of $L\beta$ (various $\gamma$) and using Timoshenko beam theory. The crosses represent the result of three-dimensional FEA.
Figure 41 When \( h/L = 0.2 \), plot of the normalized bolt force vs. the normalized applied force for various \( L\beta \) (various \( \gamma \)), using Timoshenko beam theory. The crosses represent the result of three-dimensional FEA.

5.3 Tension T-Stub using Three-Dimensional Finite Element Analysis

Euler-Bernoulli theory and Timoshenko theory are simple one-dimensional models that make analytical approximations which allow closed-form solutions to be obtained. Because beam theories are unable to include three-dimensional effects, such as the flange width, a finite element analysis of three-dimensional elastic behavior including frictionless contact between the two bodies is used in order to determine the importance of these effects. Finite element analysis (FEA) is a numerical technique for solving engineering problems for which analytical solutions are not easily obtained and mathematical expressions required for solution are not simple because of complicated geometries, loadings and material properties. It can be applied to a variety of problems in structures, heat transfer, and fluid flow etc. in order to find approximate solutions to boundary values problems for partial differential equations. Our analysis employed the
commercial software Solidworks® Simulation Professional, which meshed the model with tetrahedrons. This program has a ‘contact’ feature which we used with zero friction.

The geometry analyzed is shown in Fig. 42. The beam half-width b/2 is 1.5 in., the thickness is 0.75 in., the distance between the centerline of the bolt and the surface of the web is 1.5 in, and the diameter of the bolt (which is modeled extending upward to a fixed support, to avoid a hole through the beam) is 3/4 in. The materials of the bolt, substrate and T-stub are both steel with $E = 30$ Mpsi and $\nu = 0.28$. In this case $\gamma = 0.36$, $\beta = 0.950$ in$^{-1}$, and $L\beta = 1.425$.

Higher mesh density is expected to improve the accuracy of the solution at the cost of computational time. In our case, we use a fine mesh in the contact regions between the beam and substrate and between the bolt and beam, while a coarse mesh is used in the remainder as depicted in Fig. 43. As the T-stub is symmetric about the web and also symmetric about the mid-plane (extending through the middle of the bolts), we can use a quarter of the configuration in the simulation. After applying a force $2P$ to the entire T-stub, only $P/2$ is applied to this quarter configuration.

Note that in this three-dimensional elasticity model the lift-off distance $a$ is not a single number. It varies continuously from the mid-plane to the back-plane and is therefore determined at these two locations. We consider both values of $a$ to compare with the results from Timoshenko beam theory.

A preload is simulated by applying a 250 lb force downward to the bottom of the bolt. Based on relative stiffnesses of bolt-spring and beam+substrate, this results in a specific force applied to the beam surface (representing preload) which also experiences the correct vertical stiffness (bolt k) under subsequent T-stub load. For the dimensions shown, the resulting values of preload for half bolt are 120-140 lb depending upon the flange width. For the cases simulated, the applied force is taken to be five times as great as the bolt preload, i.e. $P=5F_0$.

The suite of simulations covered the following parameter ranges:

1. L from 1.5 to 5 inches
2. b (beam width = distance from bolt to bolt in an array) 1.5 to 5 inches, this affects $\beta$
3. h = 0.75 inches, only
4. Both materials: steel
5. Preloading force: 250 lb (resulting in a somewhat variable preload $F_0$ depending on $w$)

6. Loading force $P = 5F_0$ in each case

In each case we determined the nondimensional liftoff distance $a\beta$ (both front and back), and non-dimensional bolt force $F_B/F_0$. The results are shown in Table 4.

The bolt force of Timoshenko beam theory is 5% to 17% higher than that of finite element analysis, slightly increasing for the wider beams. The liftoff length $a$ of Timoshenko beam theory exceeds both front and back FEA values for a narrow beam, while it is between the two values for medium and wide beams. Roughly speaking, compared to the average of front and back values, the TB value is of order 20% high for narrow beams, close to accurate for medium, and 30% low for wide.

Figure 42 Three-dimensional finite element model. The beam is the 0.75” thick layer at the top, in variable contact with a block of substrate. Instead of passing the elastic bolt through a hole in the beam, it extends upward to a fixed end, making an axial spring. Bolt preload is modeled by the purple forces applied at the top of the beam (bottom of the bolt-spring). The view given here is the midplane of the beam and bolt, with the load to be applied at the right end of the beam (the middle of the T-stub). The back face of this model is the back face of the beam, or the midplane dividing multiple strips.
Figure 43 Three-dimensional model after meshing. This is a quarter-model of the beam, and substrate and half model of bolt. The very fine mesh is provided to enhance accuracy in finding the edge of contact. Vertical loads (based on T-stub load per bolt, halved for this split bolt model) are seen at the right surface of the beam; while the far beam end, and the end and bottom of the substrate, are fixed.
Table 4 Different width and different distance between two bolts when \( P/F_0 = 5 \) and \( h = 0.75 \) in

<table>
<thead>
<tr>
<th>Distance between bolt and web</th>
<th>( \Delta \theta ) of FEA from mid plane</th>
<th>( \Delta \theta ) of FEA from back surface</th>
<th>( \Delta \theta ) of TB</th>
<th>( \Delta \theta ) of EB</th>
<th>FB/F0 of FEA</th>
<th>FB/F0 of TB</th>
<th>FB/F0 of EB</th>
<th>Difference between bolt force (FEA and TB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5 in</td>
<td>0.92</td>
<td>1.02</td>
<td>1.11</td>
<td>1.40</td>
<td>6.59</td>
<td>6.95</td>
<td>7.31</td>
<td>-5.2%</td>
</tr>
<tr>
<td>2.5 in</td>
<td>0.72</td>
<td>0.86</td>
<td>0.92</td>
<td>1.21</td>
<td>8.44</td>
<td>9.36</td>
<td>10.14</td>
<td>-9.8%</td>
</tr>
<tr>
<td>3.75 in</td>
<td>0.71</td>
<td>0.74</td>
<td>0.84</td>
<td>1.13</td>
<td>10.94</td>
<td>12.51</td>
<td>13.80</td>
<td>-12.5%</td>
</tr>
<tr>
<td>5 in</td>
<td>0.59</td>
<td>0.68</td>
<td>0.80</td>
<td>1.10</td>
<td>13.39</td>
<td>15.71</td>
<td>17.50</td>
<td>-14.8%</td>
</tr>
</tbody>
</table>

| Full width 3 in               |                                       |                                          |               |               |             |             |             |                                          |
| 1.5 in                        | 1.18                                   | 1.75                                     | 1.30          | 1.54          | 5.94        | 6.34        | 6.50        | -6.3%                                    |
| 2.5 in                        | 0.80                                   | 1.33                                     | 1.03          | 1.28          | 7.45        | 8.27        | 8.66        | -9.9%                                    |
| 3.75 in                       | 0.70                                   | 1.16                                     | 0.92          | 1.17          | 9.49        | 10.85       | 11.52       | -12.5%                                   |
| 5 in                          | 0.66                                   | 1.04                                     | 0.87          | 1.13          | 11.53       | 13.49       | 14.44       | -14.5%                                   |

| Full width 5 in               |                                       |                                          |               |               |             |             |             |                                          |
| 1.5 in                        | 1.63                                   | 3.20                                     | 1.47          | 1.69          | 5.46        | 5.96        | 6.04        | -8.4%                                    |
| 2.5 in                        | 0.87                                   | 2.30                                     | 1.12          | 1.35          | 6.61        | 7.56        | 7.79        | -12.6%                                   |
| 3.75 in                       | 0.66                                   | 1.88                                     | 0.99          | 1.21          | 8.26        | 9.76        | 10.17       | -15.4%                                   |
| 5 in                          | 0.65                                   | 1.63                                     | 0.93          | 1.15          | 10          | 12.02       | 12.61       | -16.8%                                   |

5.4 Design Implications

We present two kinds of design discussion in this section, in each case limiting consideration to the simpler Bernoulli-Euler model for simplicity. One is a guideline for limiting the required static bolt strength by preventing significant prying force; the other is a guideline for the minimum bolt strength to resist fatigue.

If the static force per bolt required to be carried is \( P \), and we wish to design the bolts according to that magnitude of loading, what T-stub dimensions and properties will prevent the bolt force from significantly exceeding (say) 1.5\( P \). Although the information needed to discern this is available in Fig. 38, it is helpful to replot it with \( F_B/P \) as a function of \( F_0/P \).
We see in Fig. 44 below that requiring $F_B/P < 1.5$ (for example) leads to simple restrictions on $L\beta$ and $F_0/P$. First of all we would need $L\beta < 2.0$, in fact the lower the better, with target $\sim 1.0$. (For a wide range of reasonable dimensions with a steel flange and steel bolt, using the $h\beta$ expression of Timoshenko section, the target 1.0 requires that $L$ be less than a small multiple of $h$, i.e. $L/h < N$ where $1.6 < N < 3.4$.) This illustrates the benefit of large $h$ and minimum $L$.

Furthermore, the smaller $F_0$ the better. Clearly the bolt is loaded the least when $F_0 = 0$. But there are two reasons not to embrace that limit: fatigue in the case of cyclic loading (discussed below); and reduction of joint stiffness. Examining the figure, we see that the smallest values of $L\beta$ confer partial protection against bolt-force consequences of $F_0$ – the curve is close to horizontal, up to $F_0 = P$. For static loading we therefore suggest design targets of $L\beta = 1$ and $F_0 = P$, with an expectation that actual $F_B$ can thereby be restricted to $F_B/P < 1.5$ (for example). This makes bolt selection the simplest.

![Figure 44 Euler-Bernoulli T-Stub results plotted with altered non-dimensionalization.](image)

When $F_0 >> P$, $F_B$ is typically approximated by $F_0$ times a factor a little greater than 1. When $F_0 \leq P$, $F_B$ is approximated by $0.6 + 0.48L\beta$.

We execute this design strategy in detail by using allowable bolt stress $\sigma_A$ to define bolt diameter. In the case of steel bolt and flange, restricting bolt force to less
than $1.5P$ and requiring $L\beta = 1$ leads to $h > L^{0.75} \left( \frac{2(1.5)P}{b \sigma_A} \right)^{0.25}$, while of course we also need $F_0 \leq P$.

We now turn to fatigue, specifically the case where $P$ is repeatedly removed and reapplied. Then bolt force ranges from $F_0$ when $P = 0$, to its value $F_B$ predicted by prying analysis under load $P$. To determine a bolt strength that can survive cyclic tensile loading between two nominal stress levels, the well-known Goodman approach defines a straight-line failure envelope on a plot of mean tensile stress (abscissa) and cyclic tensile stress (ordinate, half the range between min and max).

Here we address the following question: According to the Goodman theory of tensile fatigue, what bolt ultimate strength should lead to infinite life, when tensile load cycles between a given maximum tension force $F_{\text{MAX}}$ and minimum tension force $F_{\text{MIN}}$. We have coined this quantity the ‘required Goodman strength, $F_G$’, and find it by plotting axial forces rather than axial stresses (which differ only by the constant factor $A_{\text{bolt}}$).

The Goodman line for a bolt has one point ($x$-intercept) defined by cyclic load $= 0$, and mean load equal to ultimate strength, $F_G$, a value yet to be determined. Another point on this line ($y$-intercept) is the endurance limit with zero mean load. For carbon steel, we may take the endurance limit as half the ultimate load, divided by the operative stress concentration factor. According to section 5.6 of Pilkey’s *Peterson’s Stress Concentration Factors* (3e Wiley, 2008) [40], values between 2.7 and 6.7 have been suggested, so we will use $K = 3$ for purposes of illustration. Thus, the ‘endurance strength’ under reversed loading of a bolt is $F_G/(2K) = F_G/6$. The bolt loading condition is plotted as a mean load $(F_{\text{MAX}} + F_{\text{MIN}})/2$ and a cyclic load $(F_{\text{MAX}} - F_{\text{MIN}})/2$, and required bolt strength $F_G$ must be chosen so this point lies on the line joining $(F_G,0)$ to $(0,F_G/6)$, most easily expressed in intercept-intercept form. This results in an expression for required $F_G$:

$$3(F_{\text{MAX}} - F_{\text{MIN}}) + (F_{\text{MAX}} + F_{\text{MIN}})/2 = 3.5F_{\text{MAX}} - 2.5F_{\text{MIN}} \quad \text{(101)}$$

Particularized for prying, we propose $F_G = 3.5F_B - 2.5F_0$. (More generally, $F_G = (K + 0.5)F_B - (K - 0.5)F_0$)

This computed quantity is plotted instead of $F_B$, to show required bolt ultimate strength in place of peak bolt load.
Figure 45 we plot the “Goodman Strength” (required bolt ultimate strength to provide infinite life) versus preload, when the bolt stress concentration factor is taken as $K = 3$.

This plot indicates firstly that $L\beta = 1$ is highly desirable. Furthermore a preload $F_0$ of order $PL\beta$ is also significantly protective. (With a smaller value of $K$ this preload effect is less dramatic.)

5.5 Conclusions

The T-stub prying action under repeated loading has been studied in this thesis. A T-stub is a T-shaped metal component made of a tensile web and bolted flange. With the bolt preload specified, a tensile load is applied to the T-stub which raises the bolt force. The bolt is modeled by a linear spring, leading to variable (i.e. repeated, alternating, or fluctuating) stress. We treat this configuration as a compliant receding contact problem where the beam elastic flexural deformation induces a variable-location prying reaction.

The lift-off distance increases continuously with the applied force using Euler-Bernoulli theory. But in Timoshenko beam theory, the lift-off distance is zero until the applied force exceeds a certain value. The upper limit of the lift-off distance decreases
with an increase in the distance between the centerline of the bolt to the web. The value of the bolt force starts at the preload and then increases with the applied force. In the cases investigated by three-dimensional FEA, the bolt force is similar as Timoshenko Beam (TB) theories when \( P/F_0 = 5 \).

According to the T-stub Goodman plot of Euler-Bernoulli beam theory, this plot indicates firstly that \( L\beta = 1 \) is highly desirable. Furthermore a preload \( F_0 \) of order \( PL\beta \) is also significantly protective. (With a smaller value of \( K \) this preload effect is less dramatic.)
Chapter 6. Bending T-stub Connection

6.1 Bending T-stub with Euler-Bernoulli Beam Theory

Another practical connection is a T-stub with an applied moment $M_0$ with respect to the center of the beam, as shown in the Fig. 46. The distance from the centerline of the bolt to the surface of the web is $L$, and the thickness of the web is $t$. $L_1$ is the distance from of the other side where it starts to be lift off to the web surface. The preload $F_0$ is applied to each bolt. The bolt force of the bolt which is lifted up is $F_B$. The bolt force ($F_B$), the prying forces ($F_{p1}$ and $F_{p2}$), and the lift-off distance “a” are also unknown. In the suspended region $0 \leq x \leq a$ there is no distributed load and the Euler-Bernoulli equation becomes $E^* I w''' = 0$, where the prime (’) denotes differentiation with respect to $x$, $E^*$ represents the Young’s modulus $E$ of a beam reasonably modeled by plane stress, or represents the plane strain modulus $E/(1-\nu^2)$ when the beam is wide (or is one of a sequence of adjoining strips).
Figure 47 Free body diagram

The boundary conditions at \( x=0 \) are \( w_1(0) = 0, \quad w'_1(0) = 0, \quad w''_1(0) = 0 \), as shown in Fig. 47. The corresponding transverse beam deflection is then

\[
w_1(x) = c_4 x^3 \quad (0 \leq x \leq a).
\]

While the boundary conditions and continuity conditions at \( x=a \), \( ELw''_2(a) = ELw'''(a) - (F_0 + kw_2(a)), \quad w_1(a) = w_2(a), \quad w'_1(a) = w'_2(a), \quad w''_1(a) = w''_2(a) \), the deflection at \( a<x<a+L \) is

\[
w_2(x) = (c_4 - (F_0 + kc_4 a^3)/6EI)(x-a)^3 + 3c_4 a(x-a)^2 + 3c_4 a^2 (x-a) + c_4 a^3.
\]

The boundary conditions at \( x=a+L+t+L_1 \) are similar to those at \( x=0 \),

\[
w_3(a+L+t+L_1) = 0, \quad w'_3(a+L+t+L_1) = 0, \quad w''_3(a+L+t+L_1) = 0, \quad w''''_3(a+L+t+L_1) = 0,
\]

so

\[
w_3(x) = c_5 (x-(a+L+t+L_1))^3, \quad a+L+t<x<a+L+t+L_1.
\]

The web region between \( a+L \) and \( a+L+t \) is rigid, and the slope is constant. Four boundary conditions are applied to all the above deflection expressions in order to determine the four unknowns, i.e. \( c_4, c_5, a \) and \( L_1 \), namely the displacement, slope, the bending moment and shear force in this region \( (a+L<x<a+L+t)\):

\[
w_3(a+L+t) = w_2(a+L) + w'_2(a+L)t, \quad w'_3(a+L) = w'_2(a+L+t),
\]

\[
w''_3(a+L) = w''_2(a+L+t),
\]

\[
ELw''_2(a+L) + ELw''_3(a+L+t) + M_0 = ELw'_3(a+L+t)
\]

(102)

From the Eq (1)3 and Eq (1)4, we obtain two nonlinear equations to find the relationship between \( a/L \) and \( L_1/L \):

\[
3(a/L+2)(L_1/L)^2 t/L - (1 + L_1/L)(1 + L_1/L + a/L)(a/L + 1 - L_1/L)(a/L + 2) = 0
\]

(103)
The nondimensional bolt force is

\[
\frac{a}{L} \left[ (1 + L_\|/L)^2 + 2 (L_\|/L) (t/L) (3 + L_\|/L) + 6 (L_\|/L) (t/L)^2 - (M_0/F_0L) (L_\|/L)^3 (3 + 2 L_\|/L + 6 t/L) \right] \\
+ (a/L)^3 [ (1 + L_\|/L) (2 + L_\|/L) + (L_\|/L) (t/L) (6 + L_\|/L) + 3 L_\|/L (t/L)^2] - (M_0/F_0L) (L_\|/L)^3 (3 + L_\|/L + 3 t/L) \\
+ (a\beta)^3 (M_0/F_0L) (L_\|/L) (1 + L_\|/L + 3 t/L) + (a/L)^3 (1 + L_\|/L) - (M_0/F_0L) (L_\|/L) (1 + L_\|/L) (t/L)] \\
- (L_\|/L) (M_0/F_0L) (1 + L_\|/L + 3 t/L) = 0
\]

(104)

The nondimensional bolt force is

\[
\frac{F_B}{F_0} = 1 + kw(a)/F_0 \\
= 1 + \frac{(1 + L_\|/L + 3 t/L)(a\beta)^3}{(1 + L_\|/L + 3 t/L) + (a/L)^3 (3 + L_\|/L + 3 t/L) + a/L (3 + 2 L_\|/L + 6 t/L) + (a/L)^3 - (a\beta)^3 (1 + L_\|/L + 3 t/L)}
\]

(105)

The procedure for plotting consequences of equations (103), (104) and (105) is as follows:

1. For selected values of \( L\beta \), choose \( t/L = 0.2 \), vary \( M_0/F_0L \) and use Eq. (103) and Eq.(104) to determine the corresponding values of \( a/L \) and \( L_\|/L \). Because \( a\beta = (a/L)(L\beta) \), the quantity \( a\beta \) can be plotted as a function of \( M_0/F_0L \).

2. For each value of \( M_0/F_0L, t/L, L\beta, a/L \) and \( L_\|/L \), use Eq. (105) to determine \( F_B/F_0 \), which can be plotted as a function of \( M_0/F_0L \).

In Fig. 48, the abscissa is the applied moment divided by the result of the bolt preload times the separation distance, and the ordinate represents the normalized lift-off distance. The lines of different \( L\beta \) crosses at different values of \( M_0/F_0L \), around 1.3 to 1.5. When \( M_0/F_0L < 1.3 \sim 1.5 \), the smaller the separation distances (\( L\beta \)) correspond to smaller lift-off distances (\( a\beta \)). When \( M_0/F_0L > 1.3 \sim 1.5 \), the smaller separation distance gives a larger lift-off distance.
Figure 48 When \( t/L = 0.2 \), normalized lift-off distance \( a\beta \) vs. normalized applied moment \( M_0/F_0L \) for various values of \( L\beta \).

In Fig. 49, with each value of \( L\beta \), receding contact theory predicts that the value of the bolt force starts with the preload and then increases with the applied moment. At first the bolt force increases slowly (very little stretch beyond the preloaded state) as indicated by the horizontal slope as \( M_0/F_0L \to 0 \). For greater values of \( M_0/F_0L \), the ratio of bolt force to preload becomes larger with an increase of \( L\beta \). The ratio of bolt force to preload becomes larger with an increase of \( L\beta \).
When $t/L = 0.2$, normalized lift-off distance $F_B/F_0$ vs. normalized applied moment $M_0/F_0L$ for various values of $L\beta$.

### 6.2 Bending T-stub with Timoshenko Beam Theory

We now include the effect of shear deformation by modeling this configuration with Timoshenko beam theory. Not only does this model increase the flexural and shear deformations of the flange, it also produces a distributed contact pressure in the contact region resulting in a rapidly decreasing contact pressure at the beginning of the supported region.

The positive sign convention for shear force, bending moment, slope and rotation are shown in Fig. 50. The bending moment and the shear force in the beam are related to the displacement and the rotation by [29]

\[
M = -E^* I \frac{d\psi}{dx} \\
Q = \kappa^2 GA \left( \frac{dw}{dx} + \psi \right)
\]  

(106)
Figure 50 Sign convention for positive shear force, bending moment and cross-section rotation.

Static equilibrium of the differential element shown gives

\[
\frac{d}{dx} \left[ \kappa^2 GA \left( \frac{dw}{dx} + \psi \right) \right] + p = 0
\]

\[
\frac{d}{dx} \left( E^* I \frac{d\psi}{dx} \right) - \kappa^2 GA \left( \frac{dw}{dx} + \psi \right) = 0
\]  \hspace{1cm} (107)

where \( p \) is the distributed load, \( \kappa^2 \) is the Timoshenko shear coefficient usually taken as 0.833 [36], \( E^* I \) is the flexural rigidity, \( G \) is the shear modulus, \( A \) is the cross-sectional area of the beam, \( w \) is the transverse displacement of the mid-surface, and \( \psi \) is the angle of the rotation of the cross-section which was originally normal to the centroidal axis of the beam. It is conventional to choose \( \psi \) as positive clockwise, so that \( \psi = -w' \) in Euler-Bernoulli beam theory. The effect of friction is neglected as there is assumed to be no restraint to horizontal displacements.

In the contact regions \( (x < 0 \text{ and } x > a+L+t+L) \), the displacement vanishes \( (w = 0) \) which leads to

\[
\frac{d}{dx} \left[ \kappa^2 GA \psi \right] + p = 0
\]

\[
E^* I \frac{d^2 \psi}{dx^2} - \kappa^2 GA \psi = 0
\]  \hspace{1cm} (108)

The solution for \( \psi \) which remains bounded as \( x \to -\infty \) is given by \( \psi_0 = c_0 e^{\alpha x} \), when \( x < 0 \). When \( x > a+L+t+L \), solution for \( \psi \) which remains bounded as \( x \to +\infty \) is given by \( \psi_4 = c_3 e^{-\alpha (x-(a+L+t+L))} \). The rotation angle \( \psi \) is exponentially decreasing, so is the shear force. In the suspended regions \( 0 < x < a+L, a+L+t < x < a+L+t+L \), Eq. (6) is applicable with \( p = 0 \), which leads to a constant shear force in the suspended regions.
and $\psi$ is deduced from Eq. (6)$_2$. The deflection can then be obtained from Eq. (6)$_1$ with $p = 0$.

So the angle of rotation and deflection in these three regions ($0 < x < a$, $a < x < a + L$ and $a + L + t < x < a + L + t + L_1$) are

$$
\psi_1 = \alpha^2 c_1 x^2 / 2 + c_2 x + c_3, \quad 0 < x < a
$$

$$
w_1 = -\alpha^2 c_1 x^3 / 6 - c_2 x^2 / 2 + (c_1 - c_3)x + c_4, \quad 0 < x < a
$$

$$
\psi_2 = \alpha^2 c_5 (x-a)^2 / 2 + c_6 (x-a) + c_7, \quad a < x < a + L
$$

$$
w_2 = -\alpha^2 c_5 (x-a)^3 / 6 - c_6 (x-a)^2 / 2 - (c_5 - c_7)(x-a) + c_8, \quad a < x < a + L
$$

$$
\psi_3 = \alpha^2 c_9 (x-(a + L + t + L_1))^2 / 2 + c_{10} (x-(a + L + t + L_1)) + c_{11}, \quad a + L + t < x < a + L + t + L_1
$$

$$
w_3 = -\alpha^2 c_9 (x-(a + L + t + L_1))^3 / 6 - c_{10} (x-(a + L + t + L_1))^2 / 2 + (c_9 - c_{11})(x-(a + L + t + L_1)) + c_{12}, \quad a + L + t < x < a + L + t + L_1
$$

At $x = 0$ and $x = a + L + t + L_1$, we require that $w$, $\psi$, $Q$ and $M$ be continuous, as there is no concentrated force or moment applied at that point. Therefore the continuity conditions are

$$
w(0^+) = w(0^-) = 0, \quad \psi(0^-) = \psi(0^+) = c_0, \quad w'(0^+) = w'(0^-) = 0, \quad \psi'(0^-) = \psi'(0^+) = \alpha c_0
$$

$$
w(a + L + t + L_1^-) = w(a + L + t + L_1^+) = 0, \quad \psi(a + L + t + L_1^-) = \psi(a + L + t + L_1^+) = c_{13},
$$

$$
w'(a + L + t + L_1^-) = w'(a + L + t + L_1^+) = 0, \quad \psi'(a + L + t + L_1^-) = \psi'(a + L + t + L_1^+) = -\alpha c_{13}
$$

where $\psi'(0)$ and $\psi'(a + L + t + L_1)$ is continuous because the internal bending moment is continuous, and $w'(0)$ and $w'(a + L + t + L_1)$ is continuous because the shear force and rotation are continuous at that point. The deflection is obtained from the equation of constant shear force

The boundary conditions and continuity conditions at $x = a$

$$\kappa^2 GA(\psi_2(a) + w_2'(a)) = \kappa^2 GA(\psi_1(a) + w_1'(a)) + (F_0 + kw_1(a)), \quad w_1(a) = w_2(a),
$$

$$\psi_1(a) = \psi_2(a), \quad \psi_1'(a) = \psi_2'(a)$$

The web region just as Euler-Bernoulli beam theory between $a + L$ and $a + L + t$ is rigid, and the angle is constant. So the boundary conditions and continuity conditions are
\[ w_3(a + L + t) = w_2(a + L) - \psi_2(a + L)t, \quad w_2'(a + L) = w_3'(a + L + t), \]

\[ \psi_2(a + L) = \psi_3(a + L + t), \]

\[-EI\psi_2'(a + L) - \kappa^2 GA\psi_2(a + L) + w_2'(a + L)t + M_o = -EI\psi_3'(a + L + t) \]

(115)

Application of Eq (112) results in

\[ w_1 = -\alpha^2 c_0 x^3 / 6 - \alpha c_0 x^2 / 2, \quad 0 < x < a \]

(116)

Using Eq (113), we can obtain

\[ w_3 = -\alpha^2 c_{13} (x - (a + L + t + L_1))^{3/2} / 6 + \alpha c_{13} (x - (a + L + t + L_1))^2 / 2, \]

\[ a + L + t < x < a + L + t + L_1 \]

(117)

The deflection between \( a \) and \( a + L_1 \) is derived from Eq (115)

\[
\begin{align*}
& w_2(x) = -\alpha^2 c_{13} (x - a)^3 / 6 + [EI\alpha c_{13} - M_o + EI\alpha^2 c_{13} (L + L_1 + t)](x - a)^2 / 2EI \\
& + [2M_o L - EL_1(2\alpha(L + L_1) + \alpha^2 (L^2 + L_1^2 + 2L(L_1 + t)))](x - a)/2EI \\
& + [3EI\alpha c_{13} (L^2 + 2LL_1 + L_1(L_1 + 2t)) + EI\alpha^2 c_{13} (L^3 + 3L^2_1 + 3L^2(L_1 + t) + L_1^2(L_1 + 3t)) + 6EIc_{15} t - 3M_o L^2] / 6EI, \\
& a < x < a + L
\end{align*}
\]

(118)

Application of Eq (114) to determine the four unknown, i.e., \( c_0, c_{13}, a \) and \( L_1 \). The Eq(114) and Eq (114) to show the relationship between \( a/L \) and \( L_1/L \):

\[
\begin{align*}
& - (4\gamma^3 + 2\gamma^2 (L\beta)(2(1 + a/L + L_1/L) + t/L) + a/L(L\beta)^2 ((1 + L_1/L)(1 + a/L + L_1/L) + t/L(2 + a/L)) \\
& + \gamma(L\beta)^3 ((a/L)^3 + (1 + L_1/L)^2 + 2t/L + 2a/L(2 + 2L_1/L + t/L))) \\
& + M_o/L - F_1(L\beta)(2\gamma(1 + a/L - L_1/L) + L\beta + a/L(L\beta)(2 + a/L + 6\gamma^3 a/L - L\beta(L_1/L)^2 \\
& + 2\gamma^2(a/L)^3(L\beta)^2)(a/L + 3L_1/L) + \gamma(a/L)^2(L\beta)^2(-3 + 2(a/L)L_1/L) + 3(L_1/L)^2) \\
& + (a/L)^3(L\beta)^4 ((L_1/L)^2 - 1) = 0
\end{align*}
\]

(119)

\[
\begin{align*}
& 12\gamma^3(t/L) + (a/L)(L\beta)^4 ((1 + L_1/L)(1 + a/L + L_1/L)(-1 + a/L(L_1/L - 1) + 2L_1/L) + 3t/L(L_1/L)^2(2 + a/L)) \\
& + 6\gamma^3(L\beta)(-1 + (a/L)^2 + 2L_1/L + (L_1/L)^2 + 2t/L(1 + a/L + L_1/L)) \\
& + 2\gamma^2 (L\beta)^2 (-2 + (a/L)^3 + 3(a/L)^2(L_1/L + t/L) + 3(a/L)(-1 + 2t/L + L_1/L(2 + L_1/L + t/L)) \\
& + L_1/L(3 + 6t/L + L_1/L(6 + L_1/L + 3t/L))) \\
& + \gamma(L\beta)^3 (-1 + 2L_1/L)(a/L)^3 + (L_1/L)^3(3 + 2L_1/L + 6t/L) \\
& + (a/L)^2(-3 + 6L_1/L(1 + L_1/L + t/L)) + 2a/L(-2 + L_1/L(3 + 6t/L + L_1/L(6 + L_1/L + 3t/L)))) = 0
\end{align*}
\]

(120)

The bolt force at \( x = a \)
\[ \frac{F_B}{F_0} = \]

\[
\frac{M_0 \beta/F_0 (a/L)^3 (L \beta)^3 (3r + (a/L) (L \beta) r^2 + 2r (L_a/L) (L \beta)) ((L_a/L) - (L \beta))}{4r^3 + 2r^2 (L_a/L) (L \beta) (L_a/L) + d/L (L \beta) (L_a/L) + a/L (L \beta) (L_a/L) + (L_a/L) (L \beta) + \gamma (L \beta)^3 ((a/L)^2 + (L_a/L)^2 + (L_a/L)^2 + 2r (L_a/L) (L \beta))}
\]

To generate plots of \( \alpha \beta \) and \( \frac{F_B}{F_0} \) we proceed as follows:

(a) For a given value of \( \gamma \), and distinct values of \( L \beta \), choose \( t/L = 0.2 \), vary \( M_0 \beta/F_0 \) using two nonlinear equations Eq. (119) and Eq. (120) to determine the corresponding values of \( a/L \) and \( L_1/L \). Then plot \( \alpha \beta \) as a function of \( M_0 \beta/F_0 \).

(b) For each value of \( M_0 \beta/F_0 \), \( L \beta \), \( t/L \), \( a/L \), and \( L_1/L \), use Eq. (121) to get \( \frac{F_B}{F_0} \).

In Fig. 51, in Euler-Bernoulli theory \( \alpha \beta = 0 \) occurs only when \( M_0 = 0 \), but in Timoshenko theory \( \alpha \beta = 0 \) can also occur with non-zero values of \( M_0 \). When \( M_0 \beta/F_0 \to \infty \), \( \alpha \beta \) reaches a certain upper limit value \( a_\infty \beta \). The upper limit value decreases with the increase of \( L \beta \). In Fig. 52, at first the bolt force increases slowly (very little stretch beyond the preloaded state) as indicated by the horizontal slope as \( M_0 \beta/F_0 \to 0 \). For greater values of \( M_0 \beta/F_0 \) the ratio of bolt force to preload becomes larger with an increase of \( L \beta \). When \( M_0 \beta/F_0 \to \infty \) in Eq. (121), as mentioned previously, \( \alpha \beta \) reaches a certain upper limit value, and with constant \( L \beta \) and \( t/L \) the first term on the right side of Eq. (121) is neglected compared with the second term, which gives the asymptotic lines. Given the same value of \( M_0 \beta/F_0 \), the bolt force decreases with the increase of \( L \beta \).
Figure 51 When $\gamma = 1.2032$, non-dimensional lift-off length vs. normalized applied moment for various values of $L\beta$ and using Timoshenko beam theory.
When \( \gamma = 1.2032 \), plot of the normalized bolt force vs. the normalized applied moment for various \( L\beta \), using Timoshenko beam theory.

6.3 T-Stub using Three-Dimensional Finite Element Analysis

Because beam theories are unable to include three-dimensional effects, such as the flange width, a finite element analysis of three-dimensional elastic behavior including frictionless contact between the two bodies is used in order to determine the importance of these effects. Finite element analysis (FEA) is a numerical technique for solving engineering problems for which analytical solutions are not easily obtained and mathematical expressions required for solution are not simple because of complicated geometries, loadings and material properties.

The geometry analyzed is shown in Fig. 53. The beam half-width \( b/2 \) is 1.5 in., the thickness is 0.75 in., the distance between the centerline of the bolt and the surface of the web is 4 in, and the diameter of the bolt (which is modeled extending upward to a fixed support, to avoid a hole through the beam) is 3/4 in. The materials of the bolt, substrate and T-stub are both steel with \( E = 30 \text{ Mpsi} \) and \( \nu = 0.28 \). In this case \( \gamma = 0.36 \), \( \beta = 0.976 \text{ in}^{-1} \), and \( L\beta = 3.9056 \).
Higher mesh density is expected to improve the accuracy of the solution at the cost of computational time. In our case, we use a fine mesh in the contact regions between the beam and substrate and between the bolt and beam, while a coarse mesh is used in the remainder as depicted in Fig. 54.

Note that in this three-dimensional elasticity model the lift-off distance $a$ is not a single number. It varies continuously from the mid-plane to the back-plane and is therefore determined at these two locations. We consider both values of $a$ to compare with the results from Timoshenko beam theory.

A preload is simulated by applying a 200 lb force to the bottom of the bolt. Based on relative stiffnesses of bolt-spring and beam+substrate, this results in a specific force applied to the beam surface (representing preload) which also experiences the correct vertical stiffness (bolt stiffness $k$) under subsequent T-stub load. For the dimensions shown, the resulting values of preload for half bolt are 100-120 lb depending upon the flange width. For the case simulated, the applied force is 100 lb, 200 lb, 300 lb, 400 lb, and 500 lb respectively at the tip of the top surface of the web to represent the moment with respect to the center of the beam.

The suite of simulations covered the following parameter ranges:

1. $L = 4$ inches only.
2. $b$ (beam width = distance from bolt to bolt in an array) 1.5 to 5 inches, this affects $\beta$
3. $h = 0.75$ inches, only
4. Both materials: steel
5. Preloading force: 200 lb (resulting in a somewhat variable preload $F_0$ depending on $w$)
6. Loading force is 100 lb, 200 lb, 300 lb, 400 lb, and 500 lb in each case of the same width to represent the moment.
Three-dimensional finite element model. The beam is the 0.75” thick layer at the top, in variable contact with a block of substrate. Instead of passing the elastic bolt through a hole in the beam, it extends upward to a fixed end, making an axial spring. Bolt preload is modeled by the purple forces applied at the top of the beam (bottom of the bolt-spring). The view given here is the midplane of the beam and bolt, with the load to be applied at the web of the middle of the T-stub. The back face of this model is the back face of the T-stub, or the midplane dividing multiple strips.
Figure 54 Three-dimensional model after meshing. This is a half-model of the T-stub, and substrate and half model of bolt. The very fine mesh is provided to enhance accuracy in finding the edge of contact.

The bolt force of Timoshenko beam theory agrees well with that of finite element analysis as shown in Fig. 56, 58, 60. So does the lift-off distance for a medium beams as illustrated in Fig. 55. Then the liftoff length $a$ of Timoshenko beam theory exceeds the values for a narrow beam (shown Fig. 57), while it is less than the values of a wide beam (shown in Fig. 59). Because when the beam becomes wider, it is easier to lift off at the back surface.
When $b=3$ in, non-dimensional lift-off length vs. normalized applied moment using Timoshenko beam theory. The crosses represent the result of three-dimensional FEA.
Figure 56 When $b=3$ in, plot of the normalized bolt force vs. the normalized applied moment using Timoshenko beam theory. The crosses represent the result of three-dimensional FEA.

Figure 57 When $b=1.5$ in, non-dimensional lift-off length vs. normalized applied moment using Timoshenko beam theory. The crosses represent the result of three-dimensional FEA.
Figure 58 When $b=1.5$ in, plot of the normalized bolt force vs. the normalized applied moment using Timoshenko beam theory. The crosses represent the result of three-dimensional FEA.

Figure 59 When $b=5$ in, non-dimensional lift-off length vs. normalized applied moment using Timoshenko beam theory. The crosses represent the result of three-dimensional FEA.
Figure 60 When b=5 in, plot of the normalized bolt force vs. the normalized applied moment using Timoshenko beam theory. The crosses represent the result of three-dimensional FEA.

6.4 Conclusion

The bending T-stub prying action has been studied in this thesis. A T-stub is a T-shaped metal component made of a web and bolted flange. With the bolt preload specified, a bending load is applied to the T-stub which raises the bolt force. The bolt is modeled by a linear spring, leading to variable (i.e. repeated, alternating, or fluctuating) stress. We treat this configuration as a compliant receding contact problem where the beam elastic flexural deformation induces a variable-location prying reaction.

The lift-off distance increases continuously with the applied moment using Euler-Bernoulli theory. But in Timoshenko beam theory, the lift-off distance is zero until the applied moment exceeds a certain value. The upper limit of the lift-off distance decreases with an increase in the distance between the centerline of the bolt to the web. The value of the bolt force starts at the preload and then increases with the applied moment. The bolt force increases with the increase of the separation distance in Euler-Bernoulli beam theory, it decreases when the separation distance increases.
Chapter 7. Conclusion

Connection are very common in the field of mechanical design. In this dissertation we have done several analyzes for different kinds of connections: a natural connection due to adhesion, a two-sided constraint connection, and a bolted joint connection. These range from the nano-scale to the macro-scale.

For adhesion effect, the transitions of the morphologies among three states of a CNT/GE (or more generally a beam/plate) on a sinusoidal/grooved are found, i.e. the beam/plate remaining flat on the substrate, partially conformal on the substrate, and a fully conformation configuration. In addition there are three different ways to model this behavior: a) the case without the stiffening effect (studied in a previous investigation); b) the case in which the contact regions are prevented from slipping axially and thereby induce tension due to its transverse displacement; c) the case with frictional slip regions which reduces the induced tension compared to case (b). It is found that as the adhesion energy increases, the induced tension increases and the contact region decreases. The transition from non-conformal to partially conforming configurations is independent of the model. However the three different models show considerable quantitative differences for the transition from a partially to a fully conformal morphology. The minimum value of the adhesion energy required for this transition is case (a) while the maximum value is for case (b). Case (c) gives a transition between the no-slip and no-tension cases which depends upon the value of the non-dimensional shear stress.

For a plate/beam which is clamped between rigid constraints, an elasticity analysis has shown that there will be rotation of the elastic strip within this rigid clamp. Our contribution is to allow simple beam theory to be used with a minor modification in the boundary condition in order to model this effect. The rotation at the clamp produces an added deflection in the suspended part of the structure. In the case of a cantilever with a transverse load at its end, the correction due to rotational compliance, is generally greater than the correction due to shear deformation. In a clamped-clamped configuration, the effect of rotational compliance is greater than the effect of shear deformation when $L/h$ is greater than about 13. The results for transverse loading agree well with plane strain elastic finite element modeling. For a circular plate, the rotational stiffness is also important becoming comparable to the combined deformation due to shear and lateral pressure when the diameter is about 15 times the thickness.
For prying action, we start with a simple problem – a bolted cantilever. We treat this as a compliant receding contact problem where the beam elastic flexural deformation induces a variable-location prying reaction. With the bolt preload specified, a load applied to the end of the bolted cantilever raises the bolt force, leading to variable (i.e. repeated, alternating, or fluctuating) stress. Unlike classical bolted connections between rigid bodies, the preload does not entirely shield the bolt from force variations. In the Euler-Bernoulli model, even the smallest load overcomes the preload to stretch the bolt. The resulting beam rotation at the bolt location also alters the cantilever tip deflection.

The lift-off distance increases continuously with the applied force using Euler-Bernoulli theory. But in Timoshenko beam theory, the lift-off distance is zero until the applied force exceeds a certain value. The upper limit of the lift-off distance decreases with an increase in shear deformation. The value of the bolt force starts with the preload and then increases with the applied force. It is also affected by the lift-off distance with smaller values corresponding to a larger bolt force due to an increase in prying.

In the cases investigated by three-dimensional FEA, the bolt force was smaller than that using Euler-Bernoulli and Timoshenko Beam (TB) theories, perhaps resulting from the smaller stiffness of the three-dimensional FEA. Also the lift-off distance increased with the beam width for both FEA and for beam theories.

The T-stub prying action under repeated loading also has been studied in this thesis. A T-stub is a T-shaped metal component made of a tensile web and bolted flange. With the bolt preload specified, a tensile load is applied to the T-stub which raises the bolt force. The bolt is modeled by a linear spring, leading to variable (i.e. repeated, alternating, or fluctuating) stress. We treat this configuration as a compliant receding contact problem where the beam elastic flexural deformation induces a variable-location prying reaction.

The lift-off distance increases continuously with the applied force using Euler-Bernoulli theory. But in Timoshenko beam theory, the lift-off distance is zero until the applied force exceeds a certain value. The upper limit of the lift-off distance decreases with an increase in the distance between the centerline of the bolt to the web. The value of the bolt force starts at the preload and then increases with the applied force. In the cases investigated by three-dimensional FEA, the bolt force is similar as Timoshenko Beam (TB) theories when $P/F_0=5$. 
According to the T-stub Goodman plot of Euler-Bernoulli beam theory, this plot indicates firstly that \( L\beta = 1 \) is highly desirable. Furthermore a preload \( F_0 \) of order \( PL\beta \) is also significantly protective. (With a smaller value of \( K \) this preload effect is less dramatic.)

For the bending T-stub connection, the lift-off distance increases continuously with the applied moment using Euler-Bernoulli theory. But in Timoshenko beam theory, the lift-off distance is zero until the applied moment exceeds a certain value. The upper limit of the lift-off distance decreases with an increase in the distance between the centerline of the bolt to the web. The value of the bolt force starts at the preload and then increases with the applied moment. The bolt force increases with the increase of the separation distance in Euler-Bernoulli beam theory, it decreases when the separation distance increases.
REFERENCES


