Stabilization, Estimation and Control of Linear Dynamical Systems
with Positivity and Symmetry Constraints

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To my parents for their endless love and support
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Abstract of the Dissertation

Stabilization, Estimation and Control of Linear Dynamical Systems with
Positivity and Symmetry Constraints

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Positive systems are rapidly gaining more attention and popularity due to their appearance in numerous applications. The response of these systems to positive initial conditions and inputs remains in the positive orthant of state-space. They offer nice robust stability properties which can be employed to solve several control and estimation problems. Due to their specific structural as well as stability properties, it is of particular interest to solve constrained stabilization and control problems for general dynamical systems such that the closed-loop system admits the same desirable properties. However, positive systems are not the only special class of systems with lucrative features. The class of symmetric systems with eminent stability properties is another important example of structurally constrained systems. It has been recognized that they are appearing combined with the class of positive systems. The positive symmetric systems have found application in diverse area ranging from electromechanical systems, industrial processes and robotics to financial, biological and compartmental systems. This dissertation is devoted to separately analyzing positivity and symmetry properties of two classes of positive and symmetric systems. Based on this analysis, several critical problems concerning the constrained stabilization, estimation and control have been formulated and solved. First, positive stabilization problem with maximum stability radius is tackled and the solution
is provided for general dynamical systems in terms of both LP and LMI. Second, the symmetric positive stabilization is considered for general systems with state-space parameters in regular and block controllable canonical forms. Next, the positive unknown input observer (PUIO) is introduced and a design procedure is provided to estimate the state of positive systems with unknown disturbance and/or faults. Then, the PI observer is merged with UIO to exploit their benefits in robust fault detection. Finally, the unsolved problems of positive eigenvalue assignment (which ties to inverse eigenvalue problem) and symmetric positive control are addressed.
Chapter 1

Introduction

System and control theory has played a vital role in studying and improving the performance of many dynamical systems that appear in engineering and science. Most of these systems share a principal intrinsic property which has been neglected. For example, there is a major class of systems known as positive systems whose inputs, state variables, and outputs take only nonnegative values. This implies that the response trajectory of such systems remains in nonnegative orthant of state space at all times for any given nonnegative input or initial conditions. A variety of Positive Systems can be found in electromechanical systems, industrial processes involving chemical reactors, heat exchanges, distillation columns, compartmental systems, population models, economics, biology and medicine, etc. (see [1–3] and the references therein)

The continuous-time positive systems are referred to as Metzlerian Systems. They inherit this name from Metzler matrix, a matrix with positive off-diagonal elements and in strict sense with negative diagonal entries. Metzlerian systems have a Metzler matrix and nonnegative input-output coefficient matrices. On the other hand, all the coefficients matrices of discrete-time positive systems are element-wise nonnegative. A thorough review of Metzlerian and nonnegative matrices is conducted in [4].

Positive systems and in particular Metzlerian systems not only appear in wide variety of applications but also provide impressive stability properties which can be employed to solve several control and estimation problems. For instance, it is well-known that positive stabilization is possible for any given linear systems via state and/or output feedback and various LP and LMI techniques have been proposed for this purpose [5–8].

Furthermore, the linear quadratic optimal control problem with a positivity constraint as the admissible control was addressed in [9], [10]. However, the problem of stabilization and control
CHAPTER 1. INTRODUCTION

under positivity constraints of states became apparent through the study of positive systems. Recently
efforts were devoted to solve the constrained stabilization problems based on structural characteristics
of positive systems. The main idea behind this approach is to consider special properties of positive
system and design controllers for general systems such that the closed-loop systems are stabilized and
at the same time maintains those desirable properties. The robust stability of non-negative systems
and robust stabilization with non-negativity constraints have been tackled for both conventional and
delay dynamical systems by a number of researchers [11, 15]. The solution for this category of
problem can be obtained using linear programming (LP) or linear matrix inequalities (LMI) [16, 17].

Apart from positive stabilization that can be employed for a general system, the problem of
observer design has additional restrictions when positivity constraints are imposed. Observers have
found broad application in estimation and control of dynamic systems [18–20]. A major advantage
of observers is in disturbance estimation and fault detection [20–21]. Among different observer
structures, UIO and PIO are well-qualified candidates for this purpose. Although UIO and PIO are
designed for standard linear systems [22–27], it is not obvious how to design these types of observers
for the class of positive systems. Since the response of such systems to positive initial conditions
and positive inputs should be positive, it make sense to design positive observer for positive systems. So
far, the design of positive observers was performed to estimate the states of positive systems [28–31].
However, the available positive observer designs cannot be used to estimate the states of positive
systems with unknown disturbances or faults. A recent publication provides a preliminary design of
PUIO for positive systems [32].

There is another special class of systems which appears in diverse applications which has
transfer functions or state-space representations with symmetric structures. Frequently, such systems
admit a positivity constraint as well which makes their stability and control problem even more
challenging. A great deal of effort was devoted at early stage of system development to understand
the concepts of symmetry and passivity [33–38]. Although both positive and symmetric systems have
been tackled separately and employed in system theory, the combined presence of them in control
application has not been thoroughly investigated. In fact, due to impressive robustness properties
of positive symmetric systems, we are motivated to seek procedures for stabilization of general
dynamical systems such that the closed-loop system admit positivity and symmetry structures. A
natural way of stabilizing a system with structural constraint of positivity and symmetry leads to
solving an LMI or equivalently through an LP. We consider two classes of symmetric positive systems.
The first class is a system with a symmetric positive structure, i.e. $A = AT$ is a Metzler matrix
and $B = CT \geq 0$ or a symmetric transfer function matrix that has a positive symmetric realization.
CHAPTER 1. INTRODUCTION

Using the stability properties of this class, one can perform symmetric positive stabilization of a system regardless of being positive symmetric or not. The second class is a generalized symmetric system which is defined through a block controllable canonical form in which the block sub-matrices are Metzlerian symmetric. This class of system appear in a natural way by electromechanical system, which are constructed with components that manifest a combination of inertial, compliant, and dissipative effects.

This dissertation will explore some exciting properties of positive and symmetric systems along with proposing design procedures for various types of constrained stabilization problems with application to robust control, observers and fault detection. We will start with reviewing the essential matrix analysis background for the purpose of studying positive and symmetric systems in Chapter 2. In Chapter 3, positive systems are defined and several application examples representing them are provided. Important stability properties of positive systems are introduced in this Chapter. Among these nice properties the stability radius, which is a robust stability measure, is defined and elaborated further since it plays a key role in the robust stabilization discussed in this dissertation. Two type of symmetric systems are also defined in Chapter 3 and their stability properties are explored thoroughly. Chapter 4 derives both linear programming (LP) and Linear Matrix Inequality (LMI) approaches for solving various constrained stabilization problems. The problem of constrained stabilization with maximum stability radius for positive systems is also solved in this chapter. A thorough study of diverse positive observer designs is conducted in Chapter 5. Chapter 5 starts with conventional positive observer design and then the positive unknown input observer is introduced. The PI observer that was successfully used in the past and offered several advantages [20, 39] is combined with UIO to solve the robust fault detection problem for regular and positive systems. Chapter 6 considers both symmetry and positivity with the aim of solving the symmetric positive stabilization problem. Two different methods are proposed for two symmetric structures introduced earlier. The constrained control problem is tackled in Chapter 7 and design strategies are provided to solve robust, optimal problem with positivity and symmetry constraints. Finally, after a parallel treatment for discrete-time systems, the unsolved problem of eigenvalue assignment for positive systems is investigated in Chapter 8. The dissertation will end with conclusion and future research directions in positive systems.
Chapter 2

Matrices with Special Structures

In this chapter, we are going to discuss various matrix structures which are essential to defining and analyzing special classes of systems studied in the following chapters. Certain matrices of special forms arise frequently in science and engineering with important properties. They are referred throughout the chapters wherever it is necessary. Since the main purpose of our research is linked towards two major classes of positive and symmetric matrices, we focus our attention on them. Thus, it is required to study the properties of these classes of matrices prior to propose and solve stabilization and control problems of dynamic systems with positivity and symmetry constraints.

We start with defining positive and Metzler matrices. These matrices and the mathematical background corresponding to them are needed for subsequent chapters. Symmetric matrices are discussed to prepare bedrock for defining the positive symmetric matrices. Positive symmetric matrices and their properties are reviewed in the final section of this chapter.

2.1 Nonnegative (Positive) and Metzler Matrices

Following definitions and lemmas are standard and can be found in [1-4]. Let $\mathbb{R}^{n \times m}$ be the set of $n \times m$ matrices with entries from the real field $\mathbb{R}$.

**Definition 2.1.1.** A matrix is called the monomial matrix (or generalized permutation matrix) if its every row and its every column contains only one positive entry and the remaining entries are zero.

The permutation matrix is a special case of monomial matrix. Every row and every column of the permutation matrix has only one entry equal to 1 and the remaining entries are zero. A monomial matrix is the product of a permutation matrix and a nonsingular diagonal matrix.
CHAPTER 2. MATRICES WITH SPECIAL STRUCTURES

The inverse matrix of a monomial matrix is also a monomial matrix. The inverse matrix of a permutation matrix $P$ is equal to the transpose matrix $P^T$, i.e. $P^{-1} = P^T$. The inverse matrix $A^{-1}$ of a monomial matrix $A$ is equal to the transpose matrix in which every nonzero entry is replaced by its inverse. For example the inverse matrix $A^{-1}$ of the monomial matrix

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 5 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

(2.1)

has the form

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$$

(2.2)

**Definition 2.1.2.** A matrix $A$ is called nonnegative if its entries $a_{ij}$ are nonnegative ($a_{ij} \geq 0$) and it is denoted by $A \geq 0$. Furthermore, it is called strictly positive or simply positive if all its entries are positive denoted by $A > 0$.

The set of all $n \times m$ nonnegative matrices $A \geq 0$ is defined by $\mathbb{R}^{n \times m}_{+}$, which includes the zero matrix and the set of all positive matrices $A > 0$ is a subset of $\mathbb{R}^{n \times m}_{+}$.

**Theorem 2.1.1.** The inverse matrix of a positive matrix $A \in \mathbb{R}^{n \times n}_{+}$ is a positive matrix if and only if $A$ is a monomial matrix.

**Theorem 2.1.2.** Let $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ be a monomial matrix. Then the matrix $B = P^{-1}AP$ is a positive matrix ($B > 0$) for every positive matrix $A > 0$.

The next definition and theorem are stated for general real matrices $A \in \mathbb{R}^{n \times n}$ which will be utilized for all special structure matrices. They are used to unify the standard terminology throughout the dissertation.

**Definition 2.1.3.** If $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{C}^n$, we consider the eigenvalue $\lambda$ and eigenvector $x$ of $A$ in the equation $Ax = \lambda x$ where $\Delta(\lambda) = \det(\lambda I - A)$ is the characteristic polynomial of $A$, $\lambda(A) = \{\lambda_i : \Delta(\lambda_i) = 0, \forall i = 1, \ldots, n\}$ is the set of all eigenvalues of $A$ or the spectrum of $A$, $\sigma(A) = \{\sigma_i : \sqrt{\lambda_i(A^T A)}, \forall i = 1, \ldots, n\}$ is the set of singular values of $A$, $\rho(A) = \max \{|\lambda| : \lambda \in \lambda(A)\}$ is the spectral radius of $A$, and $\alpha(A) = \max \{\Re \lambda : \lambda \in \lambda(A)\}$ is the spectral abscissa of $A$. 
CHAPTER 2. MATRICES WITH SPECIAL STRUCTURES

Theorem 2.1.3. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A \in \mathbb{R}^{n \times n}$ and define $\Delta(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$ by

$$\Delta(\lambda) = \lambda^n - S_1(\lambda_1, \ldots, \lambda_n)\lambda^{n-1} + S_2(\lambda_1, \ldots, \lambda_n)\lambda^{n-2} - \cdots \pm S_n(\lambda_1, \ldots, \lambda_n) \quad (2.3)$$

Then $S_k(\lambda_1, \ldots, \lambda_n) \triangleq E_k(A)$, the $k$-th elementary symmetric function of the eigenvalues of $A$ is the sum of the $k$-by-$k$ principal minors of $A$, i.e. $S_k(\lambda_1, \ldots, \lambda_n) = E_k(A) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \lambda_{i_j}$. In particular $S_1 = E_1 = \text{tr}A = \sum \lambda_i$ and $S_n = E_n = \det A = \prod \lambda_i$.

2.1.1 Nonnegative Matrices and Eigenvalue Characterization

Definition 2.1.4. A matrix $A \in \mathbb{R}^{n \times n}$, $n \geq 2$ is called reducible if there exists a permutation matrix $P$ such that

$$P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \quad \text{or} \quad P^T AP = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix} \quad (2.4)$$

where $B$ and $D$ are nonzero square matrices. Otherwise the matrix is called irreducible or indecomposable.

Theorem 2.1.4. The matrix $A \in \mathbb{R}^{n \times n}_+$ is irreducible if and only if

1. The matrix $(I + A)^{n-1}$ is strictly positive

$$(I + A)^{n-1} > 0 \quad (2.5)$$

2. or equivalently if

$$I + A + \ldots + A^{n-1} > 0 \quad (2.6)$$

Proof. For every vector $x > 0$

$$(I + A)^{n-1}x > 0 \quad (2.7)$$

holds if the matrix $A \in \mathbb{R}^{n \times n}_+$ is irreducible. Let $x = e_i$, where $e_i$ is the $i$th column, $i = 1, 2, \ldots, n$ of the $n \times n$ identity matrix $I$. From equation (2.7) we have $(I + A)^{n-1}e_i > 0$ for $i = 1, 2, \ldots, n$, i.e. the columns of the matrix $(I + A)^{n-1}$ are strictly positive. If matrix $A$ is reducible then equation (2.4) holds. Then the matrix $(I + A)^{n-1}$ is also reducible since

$$(I + A)^{n-1} = \begin{bmatrix} (B + I)^{n-1} & \tilde{C} \\ 0 & (D + I)^{n-1} \end{bmatrix}$$
CHAPTER 2. MATRICES WITH SPECIAL STRUCTURES

and the condition in equation (2.5) is not satisfied. The equivalence of the conditions in equations (2.5) and (2.6) follows from the relation

\[(I + A)^n = I + C_{1}^{n-1}A + C_{2}^{n-1}A^2 + \cdots + C_{n-2}^{n-1}A^{n-2} + A^{n-1}\]  

(2.8)

since \(C_{k}^{n-1} = \frac{(n-1)!}{k!(n-k-1)!}\) are positive coefficients.

Lemma 2.1.1. Let \(\lambda\) be an eigenvalue of \(A \in \mathbb{R}^{n \times n}_{+}\) and \(x\) be its corresponding eigenvector

\[Ax = \lambda x\]  

(2.9)

Then a nonnegative eigenvector \(x\) of an irreducible matrix \(A \geq 0\) is strictly positive.

Proof. From equation (2.9) it follows that if \(A > 0\) and \(x > 0\) then \(\lambda \geq 0\) and

\[(I + A)x = (1 + \lambda)x\]  

(2.10)

Let us presume that the vector \(x \geq 0\) has \(k, 1 \leq k \leq n\) zero components. Then the vector \((1 + \lambda)x\) has also \(k\) zero components. It is clear that the vector \((I + A)x\) has less than \(k\) zero components. Therefore, we obtain the contradiction and \(x\) is strictly positive.

The following theorem is the most important part of Perron-Frobenius theory [4].

Theorem 2.1.5. The strictly positive matrix \(A > 0\) has exactly one real eigenvalue \(r = \rho(A)\) such that

\[r \geq |\lambda_i|, \quad i = 1, 2, \ldots, n - 1\]  

(2.11)

to which corresponds a strictly positive eigenvector \(x\), where \(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}\), and \(\lambda_n\) are eigenvalues of \(A\). Furthermore, if \(A \geq 0\) is an irreducible nonnegative matrix. Then it satisfies the same characteristics of positive matrices with respect to its maximal eigenvalue \(r\) and its corresponding eigenvector \(x\).

The eigenvalue \(r\) is called the Perron root or Perron-Frobenius eigenvalue, which is the maximal eigenvalue of the matrix \(A\) and the vector \(x\) is its maximal eigenvector.

Theorem 2.1.6. The maximal eigenvalue of an irreducible positive matrix is larger than the maximal eigenvalue of its principal submatrices.

Theorem 2.1.7. Let \(A, B \in \mathbb{R}^{n \times n}_{+}\). If \(B \geq A \geq 0\), then \(\rho(B) \geq \rho(A)\).
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Proof. The proof follows from Wielandt’s Theorem, which states that:

If \( A, B \in \mathbb{R}^{n \times n} \) with \( B \geq |A| \), then \( \rho(B) \geq \rho(|A|) \geq \rho(A) \). This can be seen from the fact that for every \( m = 1, 2, \ldots \) we have \( |A|^m \leq |A|^m \leq B^m \) or equivalently \( \|A^m\|_2 \leq \|A|^m\|_2 \leq \|B^m\|_2 \). Thus, if we let \( m \to \infty \) and using the properties \( \rho(A) = \lim_{k \to \infty} \|A^k\|_1 \) and \( \rho(A) \leq \|A\|_2 \), we deduce \( \rho(B) \geq \rho(|A|) \geq \rho(A) \) and for \( B \geq A \geq 0 \) we have \( \rho(B) \geq \rho(A) \). ■

Theorem 2.1.8. Let \( A \in \mathbb{R}^{n \times n} \). If the row sums of \( A \) are constant, then \( \rho(A) = \|A\|_\infty \) and if the column sums of \( A \) are constant, then \( \rho(A) = \|A\|_1 \).

Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n}_+ \) be a positive or an irreducible nonnegative matrix. Denote by

\[
r_i = \sum_{j=1}^{n} a_{ij}, \quad c_j = \sum_{i=1}^{n} a_{ij}
\]

the \( i \)th row sum and the \( j \)th column sum of \( A \) respectively.

Theorem 2.1.9. If \( r \) is a maximal eigenvalue of \( A \) then

\[
\min_i r_i \leq r \leq \max_i r_i \quad \text{and} \quad \min_j c_j \leq r \leq \max_j c_j
\]

If \( A \) is irreducible then equality can hold on either side of equations \( (2.13) \) if and only if \( r_1 = r_2 = \cdots = r_n \) and \( c_1 = c_2 = \cdots = c_n \) respectively.

Theorem 2.1.10. If for positive matrix \( A = [a_{ij}] \)

\[
r_i = \sum_{j=1}^{n} a_{ij} > 0 \quad \text{for} \quad i = 1, 2, \ldots, n
\]

then its maximal eigenvalue \( r \) satisfies the inequality

\[
\min_i \left( \frac{1}{r_i} \sum_{j=1}^{n} a_{ij} r_j \right) \leq r \leq \max_i \left( \frac{1}{r_i} \sum_{j=1}^{n} a_{ij} r_j \right)
\]

2.1.2 Metzler Matrices

Definition 2.1.5. A matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) is called the Metzler matrix if its off-diagonal entries are nonnegative, \( a_{ij} \geq 0 \) for \( i \neq j; i, j = 1, 2, \ldots, n \).

Theorem 2.1.11. Let \( A \in \mathbb{R}^{n \times n} \). Then

\[
e^{At} > 0 \quad \text{for} \quad t \geq 0
\]

if and only if \( A \) is a Metzler matrix.
Proof. Necessity: From the expansion
\[ e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots \]
it follows that equation [2.16] holds for small \( t > 0 \) only if \( A \) is the Metzler matrix. Sufficiency: Let \( A \) be the Metzler matrix. The scalar \( \lambda > 0 \) is chosen so that \( A + \lambda I > 0 \). Taking into account that
\[ (A + \lambda I)(-\lambda I) = (-\lambda I)(A + \lambda I) \]
we obtain
\[ e^{At} = e^{(A+\lambda I)t-\lambda It} = e^{(A+\lambda I)t}e^{-\lambda It} > 0 \]
since \( e^{(A+\lambda I)t} > 0 \) and \( e^{-\lambda It} > 0 \).

Remark 2.1.1. Every Metzler matrix \( A \in \mathbb{R}^{n \times n} \) has a real eigenvalue \( \alpha = \max_{i} \Re(\lambda_i) \) and \( \Re(\lambda_i) < 0 \) for \( i = 1, \ldots, n \) if \( \alpha < 0 \), where \( \lambda_i = \lambda_i(A), i = 1, \ldots, n \) are the eigenvalues of \( A \).

Since the class of nonnegative matrices denoted by \( N \) is defined by \( a_{ij} \geq 0 \) for all \( i, j = 1, \ldots, n \), it is clear that this class may be regarded as a subset of the class of Metzler matrices denoted by \( \bar{M} \) with nonnegative diagonal elements, i.e. \( N \in \bar{M} \). Their spectral properties can also be related as follows. For every Metzler matrix \( A \) there exists a real number \( \alpha \) such that \( N = \alpha I + A \in \mathbb{R}^{n \times n} \). By Theorem 2.1.4 the matrix \( N \) has a real eigenvalue equal to its spectral radius \( \rho(N) = \max_{i} |\lambda_i(N)| \). Hence the matrix \( A \) has the real eigenvalue \( \rho(N) - \alpha = \mu \) and \( \lambda_i(N) < 0 \) for \( i = 1, \ldots, n \) if \( \mu < 0 \). Suppose \( \gamma = \min_{i} a_{ii} \), then there exists a real number \( \eta \geq |\gamma| \) such that the matrix \( \eta I + A = N \) is a nonnegative matrix. Let \( \lambda(A) \) be any eigenvalue of \( A \), then \( \lambda(N) - \eta = \lambda(A) \). Thus spectrum of \( A \) is copy of spectrum \( N \) shifted by \( \eta \) and vice versa.

From the matrix stability analysis point of view, a Metzler matrix \( A \) is Hurwitz stable if and only if all of its eigenvalues lie strictly in the left half of complex plane. On the other hand, a nonnegative matrix \( N \) is Schur stable if and only if all of its eigenvalues lie strictly inside of unit circle in the complex plane. It is not difficult to show that if \( A \) is a Hurwitz stable Metzler matrix, then its characteristic polynomial \( \Delta(A) \) has positive coefficients. Similarly, one can show that the characteristic polynomial of \( N - I \), where \( N \) is a Schur stable nonnegative matrix, has positive coefficients. Now, if \( \lambda_i(N), i = 1, \ldots, n \) are eigenvalues of a nonnegative matrix \( N \), then \( \lambda_i(N) - 1 \) for all \( i = 1, \ldots, n \) are eigenvalues of \( N - I \). Thus, the eigenvalues of a Schur stable matrix \( N \) are located inside the unit circle (i.e. \( |\lambda_i(N)| < 1 \)) if and only if the characteristic polynomial \( \Delta(N - I) \) has zeros with negative real parts. This establishes the fact that all equivalent stability properties of a
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Hurwitz stable Metzler matrix $A$ remain the same for a Schur stable nonnegative matrix $N$ through $N - I$ as will be further elaborated in Chapter 3.

### 2.1.3 Z-Matrices

The class of Z-matrices are defined by those matrices whose off-diagonal entries are less than or equal to zero i.e. if $A = [a_{ij}]$ is a Z-matrix, then it satisfies $a_{ij} \leq 0$ for all $i \neq j$. No restriction is put on its diagonal elements.

Note that the negated class of Z-matrices become the class of Metzler matrices. Although one can define the set of Metzler matrices by $Z^-$, we define the set by $\bar{M}$ to recognize the name Metzler and distinguish it from the class of M-matrices, which will be discussed next. A subset of the set of Z-matrices with nonnegative/positive diagonal elements play an important role in further theoretical development of this dissertation. The general Z-matrices can be both singular and nonsingular. However, the nonsingular subset of Z-matrices with nonnegative/positive diagonal elements have interesting and useful properties (see [4]).

### 2.1.4 M-Matrices

**Definition 2.1.6.** A matrix $A \in \mathbb{R}^{n \times n}$ is called an M-matrix if (1) its entries of the main diagonal are nonnegative and its off-diagonal entries are nonpositive i.e. $A \in Z$ with $a_{ij} \leq 0$ and $a_{ii} \geq 0$ and (2) there exist a positive matrix $B \in \mathbb{R}^{n \times n}_+$ with maximal eigenvalue $r$ such that

$$A = cI - B \quad (2.17)$$

where $c \geq r$.

The set of M-matrices of dimension $n$ will be denoted by $M$, and $Z$ denotes the set of Z-matrices with nonpositive off-diagonal entries. Note that from equation (2.17) it follows that if $A$ is an M-matrix then $-A$ is a Metzler matrix. From Theorem 2.1.11 it follows that for every matrix $A \in M$ we have

$$e^{-At} > 0 \quad \text{for} \quad t \geq 0 \quad (2.18)$$

**Theorem 2.1.12.** A matrix $A \in Z$ with $a_{ij} \leq 0$ and $a_{ii} \geq 0$ is an M-matrix if and only if all its eigenvalues have nonnegative real parts.

**Proof.** Let a matrix $A = [a_{ij}] \in Z$ have all eigenvalues with negative real parts and $a_{mm} = \max_i a_{ii}$. Then $B \triangleq a_{mm}I - A \in \mathbb{R}^{n \times n}_+$. Let $r$ be the maximal eigenvalue of the matrix $B$. Then $a_{mm} - r$ is a
real nonnegative eigenvalue of the matrix $A = a_{mm}I - B$ or $a_{mm} \geq r$. Therefore, $A = a_{mm}I - B$ is the M-matrix. Now let us assume that $A = cI - B$ is an M-matrix and $r$ is the maximal eigenvalue of the matrix $B$. Hence $c \geq r$. Let $\lambda_k$ be an eigenvalue of the matrix $A$ and $\text{Re}(\lambda_k)$ be its negative real part. Then

$$0 = \det [I\lambda_k - A] = \det [I\lambda_k - cI + B] = \det [I(c - \lambda_k) - B] \quad (2.19)$$

From equation (2.19) it follows that $c - \lambda_k$ is an eigenvalue of the matrix $B$. But $c \geq 0$ and $-\text{Re}(\lambda_k) > 0$. Therefore, $|c - \lambda_k| \geq c - \text{Re}(\lambda_k) > c \geq r$, which contradicts the assumption that $r$ is the maximal eigenvalue of $B$. 

Theorem 2.1.13. A matrix $A \in \mathbb{Z}$ with $a_{ij} \leq 0$ and $a_{ii} \geq 0$ is an M-matrix if and only if all principal minors of $A$ are nonnegative.

Thus, the above definition and theorems can be combined to define the class of M-matrices that are not necessarily nonsingular.

Definition 2.1.7. Suppose $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ satisfies $a_{ij} \leq 0$ for $i \neq j$ and $a_{ii} \geq 0$ for all $i = 1, \ldots, n$. Then $A$ is called an M-matrix if it satisfies any one of the following conditions:

1. $A = cI - B$ for some nonnegative matrix $B$ and some $c \geq r$, where $r = \rho(B)$.

2. The real part of each nonzero eigenvalue of $A$ is positive.

3. All principal minors of $A$ are nonnegative.

Let us also define the class of monotone matrices as follows and denote the set of monotone matrices with $M_o$.

Definition 2.1.8. A square matrix $A$ is called monotone if it satisfies any one of the following equivalent conditions:

1. $Ax \geq 0$ implies $x \geq 0$ and there exists a vector $x > 0$ such that $Ax > 0$.

2. $A^{-1}$ exists and $A^{-1} \geq 0$.

Obviously, one can define the class of nonsingular M-matrices by modifying Definition 2.1.6 and Theorems 2.1.12, 2.1.13 with $a_{ij} \leq 0$ and $a_{ii} > 0$. If so, $c \geq r$ in Definition 2.1.6 is replaced by $c > r$ and eigenvalues with nonnegative real parts become strictly positive in Theorem...
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2.1.12 Also, all principal minors in Theorem 2.1.13 should be positive instead of nonnegative. However, combining Definitions 2.1.7 and 2.1.8, one can compactly define the nonsingular M-matrices as follows. We denote the set of nonsingular M-matrices by \( M_n \).

**Definition 2.1.9.** Suppose \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) satisfies \( a_{ij} \leq 0 \) for \( i \neq j \) and \( a_{ii} > 0 \) for all \( i = 1, \ldots, n \). Then \( A \) is called a nonsingular M-matrix if it satisfies any one of the following conditions.

1. All eigenvalues of \( A \) have positive real parts.
2. \( A \) is nonsingular and \( A^{-1} \geq 0 \).
3. All leading principal minors of \( A \) are positive.
4. \( A = cI - B \) for some nonnegative matrix \( B \) and some \( c > r \), where \( r = \rho(B) \).
5. \( Ax > 0 \) if and only \( x > 0 \).

Note that there are more equivalent conditions that can be added to Definition 2.1.8 (see [4]).

According to the above definitions it can be concluded that \( M_n \subset M_o \) and \( M_n \subset M \).

Furthermore, when \( A \in M \) and \( A \) is nonsingular then \( A \in M_n \). Thus, the relationship between the set of M- and Z-matrices can be related by \( M_n \subset M \subset Z \). The following example clarifies the class of nonsingular M-matrices which should not be confused with the nonsingularity of the Z-matrices.

**Example 2.1.1.** Consider the following two matrices

\[
A_1 = \begin{bmatrix}
2 & -1 & -1 \\
-2 & 3 & -4 \\
-1 & -2 & 5 \\
\end{bmatrix} \quad A_2 = \begin{bmatrix}
2 & -1 & -1 \\
-1 & 3 & -2 \\
0 & -1 & 4 \\
\end{bmatrix}
\]

(2.20)

Although both matrices have the same structure with \( a_{ii} > 0 \) and \( a_{ij} \leq 0 \), matrix \( A_1 \) is a nonsingular Z-matrix and should not be confused with a nonsingular M-matrix because of the fact that one of its eigenvalue is negative and \( A_1^{-1} < 0 \). On the other hand, \( A_2 \) is a nonsingular M-matrix since it satisfies all necessary conditions of Definition 2.1.9.

2.1.5 Totally Nonnegative (Positive) Matrices and Strictly Metzler Matrices

In this section we shall consider nonnegative matrices with all their minors of all orders being nonnegative.
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**Definition 2.1.10.** A matrix $A \in \mathbb{R}^{m \times n}$ is called totally nonnegative (positive) if and only if all its subdeterminants of all orders are nonnegative (positive).

The Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_1^2 & a_2^2 & \cdots & a_{n-1}^2 & a_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_{n-1}^{n-1} & a_n^{n-1} \end{bmatrix}$$

is an example of a square totally positive matrix if $0 < a_1 < a_2 < \cdots < a_n$ since the matrix has positive determinant and all of its submatrices have positive determinants, too.

**Definition 2.1.11.** A square matrix $A \in \mathbb{R}^{n \times n}$ is called strictly Metzler matrix if all of its diagonal entries are negative and all of its off-diagonal elements are nonnegative, i.e. $a_{ii} < 0$, $a_{ij} \geq 0$, $\forall i \neq j$, $i, j = 1, 2, \ldots, n$.

Note that the Metzler matrix is conventionally defined as a matrix with nonnegative off-diagonal elements. Here, we define it in a strict sense to satisfy the necessary condition of stability, namely $a_{ii} < 0$. Metzler matrices are closely related to the class of M-matrices. An M-matrix has positive diagonal entries and negative off-diagonal entries. Thus, if $A$ is a Metzler matrix, then $-A$ is an M-matrix. Furthermore an M-matrix is called a nonsingular M-matrix if $M^{-1} \geq 0$. The nonsingular M-matrix has several nice properties. One can deduce that stable Metzler matrices admit similar properties, i.e., if $A$ is a stable Metzler matrix then $-A$ is a nonsingular M-matrix. The underlying theory of such matrices stems from the theory of nonnegative (positive) matrices based on Frobenius-Perron Theorem as stated in Section 2.1.1. The spectral radius of an irreducible non-negative matrix $N$, denoted by $\rho(N)$ is positive and real. An irreducible Metzler matrix can be written as $A = N - \alpha I$ for some nonsingular matrix $N$ and a scalar $\alpha$. Thus $A$ is Hurwitz stable if and only if $\alpha > \rho(N)$, and its largest eigenvalue $\mu(A) = \rho(N) - \alpha$. Note that every Metzler matrix $A$ has a real eigenvalue $\mu = \max \text{Re}(\lambda_i)$ and if $\mu < 0$, then $\text{Re}(\lambda_i) < 0$ for $i = 1, 2, \ldots, n$, where $\lambda_i$s are the eigenvalues of $A$.

Due to the connection of these matrices with the corresponding models of continuous-time and discrete-time systems, one can similarly define Metzlerian and non-negative (positive) systems which are going to be introduced and discussed in next chapter.
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2.2 Symmetric Matrices

Definition 2.2.1. A matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is called symmetric if $A = A^T$. It is skew-symmetric if $A = -A^T$.

Note that when $A$ is a general complex matrix, then it is called Hermitian if $A$ is equal to its complex conjugate transpose i.e. $A = A^*$. Since this dissertation is concentrating on real matrices associated with systems under study, the Hermitian matrices are not discussed. However, most of properties associated with symmetric matrices carry over for Hermitian case with minor adjustment. The set of all symmetric and skew-symmetric matrices are denoted by $S$ and $T$, respectively.

Theorem 2.2.1. Let $A$ be a symmetric matrix i.e. $A \in S$. Then

1. All the eigenvalues of $A$ are real.

2. The eigenvectors of $A$ corresponding to different eigenvalues are orthogonal.

3. The Jordan form representation of $A$ is a diagonal matrix.

4. $A$ can be transformed by an orthogonal matrix $Q$ consisting of its eigenvectors to a diagonal matrix $\hat{A}$, i.e. $\hat{A} = Q^{-1}AQ$, where $Q^{-1} = Q^T$.

2.2.1 Properties of Symmetric Matrices

The following list summarize some symmetric matrices properties that is needed for next chapters discussions.

1. If $A$ is a symmetric matrix, then $A + A^T$ and $AA^T$ are symmetric.

2. If $A$ is a symmetric matrix, then $A^K$ is symmetric for all $k = 1, 2, 3, \ldots$.

3. If $A$ is a symmetric nonsingular matrix, then $A^{-1}$ is symmetric.

4. If $A$ and $B$ are symmetric matrices, then $aA + bB$ is symmetric for all real scalars $a$ and $b$.

5. If $A$ is a symmetric matrix, then $PA^TP$ is symmetric for all $P \in \mathbb{R}^{n \times n}$.

If $A$ and $B$ are both square, we know that although $AB$ and $BA$ do not commute, i.e. $AB \neq BA$, their products have exactly the same eigenvalues. It is also easy to see that if $A$ (or $B$) is nonsingular, then $AB$ and $BA$ are similar by $A$ (or $B$), i.e. $AB = A(BA)A^{-1}$. On the other hand,
if $A$ and $B$ belong to community family, i.e. $AB = BA$, the family is simultaneously diagonalizable by a single nonsingular matrix $Q$ i.e. $\hat{A} = Q^{-1}AQ$ and $\hat{B} = Q^{-1}BQ$. If the community family is symmetric, then there exists an orthogonal matrix $Q$ such that $Q^T AQ$ is diagonal for all $A$ belonging to the family.

**Definition 2.2.2.** A matrix $A$ is said to be normal if $AA^T = A^T A$, i.e. if $A$ commutes with its transpose matrix.

Based on the above definition one can immediately conclude that all symmetric and orthogonal matrices are normal, since $AA^T = A^T A = A^2$ when $A = A^T$ and $AA^T = A^T A = I$ when $A^{-1} = A^T$.

### 2.2.2 Symmetrizer and Symmetrization

From the properties of symmetric matrices, it is evident that the sum of two symmetric matrices remain symmetric. However, the product of two symmetric matrices will no longer be symmetric. In spite of this fact, it is possible to prove that every matrix can be decomposed as a product of two symmetric matrices. The proof of this result requires construction of symmetrizers based on the elegant theorem of Olga Taussky.

**Definition 2.2.3.** A symmetrizer of an arbitrary square matrix $A$ is a symmetric matrix $S$ such that $A^T = S^{-1}AS$.

**Theorem 2.2.2.** Every matrix $A$ can be transformed to $A^T$ by a nonsingular symmetric matrix $S$ (symmetrizer), i.e. $A^T = S^{-1}AS$, if and only if $A$ is non-derogatory.

Recall that a matrix is non-derogatory if its characteristic polynomial is the same as its minimal polynomial (i.e. the matrix has only one Jordan block associated with each repeated eigenvalue, or every eigenvalue of $A$ has geometric multiplicity equal to one).

To prove the above theorem we take advantage of companion matrices. The companion matrix $C$ associated with its characteristic polynomial $\Delta(\lambda) = \det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ given by

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \quad (2.21)$$
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is invertible if and only if $a_n \neq 0$. This is a fact based on the construction of $c^{-1}$, which requires $a_n \neq 0$. Now, let $C$ be a companion matrix satisfying the invertibility condition $a_n \neq 0$. Then there exists an invertible symmetric matrix $X$ such that $XCX^{-1} = C^T$ given by

$$X = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

(2.22)

It is also well-known that matrix $A$ can be transformed to $C$ by a nonsingular transformation matrix defined by $P^{-1} = UU^{-1}$ where $U = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$ with $b$ a generator vector such that $\rho[U] = n$ or $\det U \neq 0$ and $U^{-1} = X$. Thus, we have $C = PAP^{-1}$ and since $XCX^{-1} = C^T$ we get

$$X \left[ PAP^{-1} \right] X^{-1} = \left[ PAP^{-1} \right]^T$$

(2.23)

or

$$XPAP^{-1}X^{-1} = P^{-T}A^TP^T$$

(2.24)

Multiplying both sides from left by $P^T$ and from right by $P^{-T}$ we obtain

$$S^{-1}AS = A^T$$

(2.25)

where $S^{-1} = P^TXP$, which is a symmetric matrix by property 5.

**Theorem 2.2.3.** Every matrix $A$ can be decomposed as a product of two symmetric matrices

$$A = S_1S_2$$

(2.26)

where $S_i = S_i^T$ for $i = 1, 2$ and either $S_1$ or $S_2$ may be chosen to be nonsingular.

**Proof.** Using Theorem 2.2.2, one can write $A = SATS^{-1}$. Since $AS = SAT$ or $(SAT)^T = SAT$, it follows that $SAT$ is symmetric. Thus, $A = S_1S_2$ where $S_1 = SAT$ and $S_2 = S^{-1}$ are both symmetric matrices.

An alternative proof without using Theorem 2.2.2 is based on diagonal transformation of $A$. Suppose $A$ is a real matrix with distinct eigenvalues. Then $A$ is diagonalizable by a matrix $Q$ i.e. $A = Q^{-1}AQ$ or $A = Q\hat{A}Q^{-1}$, which can be written as

$$A = Q\hat{A}(Q^TQ^{-T})^{-1} = S_1S_2$$

where $S_1 = Q\hat{A}Q^T$ and $S_2 = Q^{-T}Q^{-1}$ are both symmetric matrices.
Note that in this case if we define \( S = S_2^{-1} = QQ^T \), then \( SA^T = Q\hat{A}Q^T = S_1 \) and it justifies \( QT A^T Q^{-T} = \hat{A} = \hat{A}^T \) or \( Q^{-1}AQ = \hat{A} \).

**Remark 2.2.1.** Theorem 2.2.3 asserts that every matrix \( A \) can be decomposed as a product of two symmetric matrices. Thus, the same is true for a companion matrix. Let \( C = HX \) and \( C^T = XH \), then \( CX^{-1} = HXX^{-1} = H \) and \( XCX^{-1} = XH = C^T \). So, \( X = S_1 \) and \( CX^{-1} = S_2 \).

**Corollary 2.2.1.** Let \( A \) be a nonsingular symmetric matrix so that its singular value decomposition is \( A = U\Sigma U^T \) where \( U \) consists of orthogonal eigenvectors of \( AA^T \). Then \( A \) can be decomposed as \( SS^T \) where \( S = UD \) with \( D = \text{diag}\{\sqrt{\sigma_1}, \sqrt{\sigma_2}, \ldots, \sqrt{\sigma_n}\} \) and \( \sigma_i \)'s are singular values of \( A \) \( (\sigma_i = \sqrt{\lambda_i(AA^T)}) \) and \( S = U\Sigma^\frac{1}{2} \).

The following example illustrates the application of Theorems 2.2.2 and 2.2.3.

**Example 2.2.1.** Consider the Metzler matrix

\[
A = \begin{bmatrix}
-3 & 1 \\
2 & -4
\end{bmatrix}
\]  

(2.27)

with \( \Delta(\lambda) = \lambda^2 + 7\lambda + 10 \). Defining the generator vector \( b = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \) we obtain

\[
P^{-1} = U\bar{U}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}
\]  

(2.28)

where \( \bar{U}^{-1} = X \) and

\[
\bar{A} = PAP^{-1} = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} \triangleq C
\]  

(2.29)

First, it is easy to check that \( XCX^{-1} = C^T \). Next, we compute

\[
S^{-1} = P^T XP = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}
\]  

(2.30)

and obtain

\[
A^T = S^{-1}AS = \begin{bmatrix}
-3 & 2 \\
1 & -4
\end{bmatrix}
\]  

(2.31)
Finally,

\[ A = S A^T S^{-1} = S_1 S_2 \]  

(2.32)

where

\[ S_1 = S A^T = \begin{bmatrix} 1 & -4 \\ -4 & 6 \end{bmatrix} \quad \text{and} \quad S_2 = S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \]  

(2.33)

Note that the Metzler matrix is represented as a product of two symmetric matrices whereby one of them is a nonsingular Z-matrix and the other is a nonsingular nonnegative N-matrix.

**Theorem 2.2.4.** A real matrix \( A \) is symmetrizable to \( A_s \) by a similarity transformation if and only if it can be factored as the product of two symmetric matrices, one of which is positive definite.

**Proof.** Theorem 2.2.3 shows that every real matrix \( A \) can be represented as a product of two real symmetric matrices i.e. \( A = S_1 S_2 \), \( S_i = S_i^T \). This followed from the fact that \( A \) is similar to \( A^T \) via a real symmetric matrix \( S \) i.e. \( A^T = S^{-1} A S \), \( S = S^T \). Thus, \( A = \underbrace{S A^T}_{S_1} \cdot \underbrace{S^{-1}}_{S_2} = S_1 S_2 \) with both factors symmetric. \( S \) can be chosen in different ways obtaining \( (S_1, S_2) \) is not unique. However, if \( S \) is chosen such that it is positive definite, then \( S \) can be factored using Cholesky decomposition i.e. \( S = T T^T \) and we have

\[ A = T T^T A^T (T T^T)^{-1} \quad \text{or} \quad A^T = (T T^T)^{-1} A (T T^T) \]  

(2.34)

Hence

\[ T T^T A^T T^{-T} = T T \left[ (T T^T)^{-1} A (T T^T) \right] T^{-T} = T^{-1} A T = (T^T A T)^T = A_s \]  

(2.35)

This implies that \( A \) is necessarily similar to a symmetric matrix. The converse follows easily. If then \( A = S_1 S_2 \) and say \( S_1 > 0 \), \( S_1^T = S_1 \), then

\[ S_1^{-\frac{1}{2}} A S_1^{\frac{1}{2}} = S_1^\frac{1}{2} S_2 S_1^\frac{1}{2} \]  

(2.36)

Showing that \( A \) has real characteristic roots. Furthermore, using quadratic form concept these roots have the same signs as the roots of \( S_2 \). 

**Example 2.2.2.** Consider the matrix \( A \) in companion form as

\[ A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \]  

(2.37)
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Following the one alternative procedure provided in Theorem 2.2.3 is to define the matrix $S \triangleq QQ^T$ where $Q$ consists of eigenvectors associated with the eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$ given by

$$Q = \begin{bmatrix} 0.7071 & -0.4472 \\ -0.7071 & 0.8944 \end{bmatrix}$$ (2.38)

Thus,

$$S = QQ^T = \begin{bmatrix} 0.7 & -0.9 \\ -0.9 & 1.3 \end{bmatrix}$$ (2.39)

which leads to symmetric factorization of $A$ as $A = SATS^{-1} = S_1 S_2$ where

$$S_1 = SAT = \begin{bmatrix} -0.9 & 1.3 \\ 1.3 & -2.1 \end{bmatrix} \quad \text{and} \quad S_2 = S^{-1} = \begin{bmatrix} 13 & 9 \\ 9 & 7 \end{bmatrix}$$ (2.40)

Now, applying a Cholesky decomposition of $S$ we get $S = TT^T$ where

$$T = \begin{bmatrix} 0.8367 & 0 \\ -1.0757 & 0.3780 \end{bmatrix}$$ (2.41)

which leads to the symmetric transformation of $A$

$$A_s = T^{-1}AT = \begin{bmatrix} -1.2857 & 0.4518 \\ 0.4518 & -1.7143 \end{bmatrix}$$ (2.42)

2.2.3 Quadratic Form and Eigenvalues Characterization of Symmetric Matrices

Symmetric matrices appear in many diverse applications as will be elaborated in the next chapter. Their direct connections in mathematical analysis have been found in the theory of optimization because they can be used to determine if a critical point is maximum or minimum of functions with several variables by checking definiteness of the symmetric Hessian matrix. Another important venue of symmetric matrices is their direct tie to quadratic form. This plays an important role in stability and robustness analysis of dynamic system through Lyapunov equation. Given a quadratic function $Q(x)$, one can rewrite it as

$$Q(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = x^T A x$$ (2.43)
which is a quadratic form in terms of the matrix $A$. It is not difficult to show that

$$Q(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} (a_{ij} + a_{ji}) x_i x_j = x^T \left[ \frac{1}{2} (A + A^T) \right] x \quad (2.44)$$

Thus, $A$ and $\frac{1}{2} (A + A^T)$ both generate the same quadratic form, and the latter matrix is symmetric. Therefore, it suffices to study quadratic form of $A$ by only considering the quadratic form associated with its symmetric part $A_s = \frac{1}{2} (A + A^T)$. This fact allows one to check the positivity of a quadratic form (i.e. $Q(x) > 0$ for all $x$) through the positive definiteness of its associated matrix $A$ (i.e. $A \succ 0$) or by checking the positivity of principal minors of its symmetric part $A_s$. An equivalent condition for positive definiteness of $A$ or $A_s$ is that all their eigenvalues should be positive. Similar statements can be written for nonnegativity of $Q(x) \geq 0$ in terms of semi-definiteness of its associated matrices (i.e. $A \succeq 0$ or $A_s \succeq 0$).

An important fact about symmetric matrices in conjunction with quadratic form is that if $A$ is positive definite and $C$ is a nonsingular matrix defined by a congruent transformation $x = Cy$, then $x^T A x = y^T C^T A C y$ and $B = C^T A C$ is also positive definite associated with the quadratic form $y^T B y$. Consequently, the Sylvester’s law of inertia states that the matrix $B = C^T A C$ has the same number of positive, negative, and zero eigenvalues as $A$.

Since the eigenvalues of a symmetric matrix $A$ are real, we adopt the convention that they are labeled according to increasing order:

$$\lambda_{\min} = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n = \lambda_{\max} \quad (2.45)$$

The smallest and largest eigenvalues are easily characterized as the solutions to a constrained minimum and maximum problem by the following result known as Rayleigh-Ritz Theorem.

**Theorem 2.2.5.** Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and let the eigenvalues of $A$ be ordered as $(2.45)$. Then

$$\lambda_1 x^T x \leq x^T A x \leq \lambda_n x^T x \quad \forall x \in \mathbb{R}^n \quad (2.46)$$

$$\lambda_{\max} = \lambda_n = \max_{x \neq 0} \frac{x^T A x}{x^T x} = \max_{x^T x = 1} x^T A x \quad (2.47)$$

$$\lambda_{\min} = \lambda_1 = \min_{x \neq 0} \frac{x^T A x}{x^T x} = \min_{x^T x = 1} x^T A x \quad (2.48)$$

Based on the Rayleigh-Ritz Theorem above and its generalization by Courant-Fischer Theorem, Weyl proved an important result as follows.
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Theorem 2.2.6. Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric matrices and let the eigenvalues $\lambda_i(A), \lambda_i(B)$, and $\lambda_i(A + B)$ be arranged in increasing order as (2.45). Then for each $i = 1, 2, \ldots, n$ we have

$$\lambda_i(A) + \lambda_1(B) \leq \lambda_i(A + B) \leq \lambda_i(A) + \lambda_n(B) \tag{2.49}$$

and

$$\lambda_1(B) \leq \lambda_i(A + B) - \lambda_i(A) \leq \lambda_n(B) \tag{2.50}$$

or equivalently

$$|\lambda_i(A + B) - \lambda_i(A)| \leq \rho(B) \tag{2.51}$$

which is a simple example of a perturbation theorem for symmetric matrices.

Furthermore, if $B$ is positive semidefinite. Then

$$\lambda_i(A) \leq \lambda_i(A + B) \tag{2.52}$$

which is known as the monotonicity result and together with (2.50) or (2.51) can be used in robustness analysis of symmetric matrices.

The following results provide relationship between the eigenvalues of a general matrix and its associated symmetric matrix, which will be useful in connection to robust stability analysis of linear systems.

Theorem 2.2.7. Let $A \in \mathbb{R}^{n \times n}$ and denote $\lambda_i(A_s) = \lambda_i \left( \frac{A + A^T}{2} \right)$ as the eigenvalue of its symmetric part. Then

$$\lambda_i(A_s) \leq \sigma_i(A) \quad \forall i = 1, \ldots, n \tag{2.53}$$

where $\sigma_i(A) = \sqrt{\lambda_i(AA^T)}$ are the singular values of $A$.

The above theorem is due to Fan and Hoffman. It is interesting to point out that the inequality becomes equality when $A$ is a positive semidefinite matrix. Finally, we state a theorem by Bendixon and Hirsch.

Theorem 2.2.8. The real part of the eigenvalue of a matrix $A \in \mathbb{R}^{n \times n}$ are bounded by the minimum and maximum eigenvalues of its symmetric part $A_s$ i.e.

$$\lambda_1(A_s) \leq r_i \leq \lambda_n(A_s) \quad \forall i = 1, \ldots, n \tag{2.54}$$

where $r_i = \Re[\lambda_i(A)]$. 

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2.3 Nonnegative and Metzler Symmetric Matrices

Section 2.1 defined the classes of nonnegative and Metzler matrices with their properties. Section 2.2 considered the class of symmetric matrices with important properties that led to symmetrizer and symmetrization of matrices. The eigenvalue characterization of symmetric matrices with useful bounds on them were also outlined. In this section we combine both classes of sections 2.1 and 2.2 to elaborate further on the usefulness of matrices that admit joint properties of symmetry and positivity.

Definition 2.3.1. A matrix \( A \) is called symmetric nonnegative if its entries \( a_{ij} \) are nonnegative \((a_{ij} \geq 0)\) and satisfy symmetry constraint \( a_{ij} = a_{ji} \). Similarly, a matrix \( A \) is called symmetric Metzler if \( a_{ij} \geq 0 \) for all \( i \neq j \), and \( a_{ij} = a_{ji} \). Furthermore, the matrix \( A \) is strictly symmetric Metzler if in addition \( a_{ii} < 0 \).

Note that a strictly Metzler matrix \( A \) satisfies the necessary condition of Hurwitz stable with \( a_{ii} < 0 \). It is also well-known that the same necessary condition applies for a stable symmetric matrix. Thus, we have the following result.

Theorem 2.3.1. Let \( A \) be a symmetric Metzler matrix i.e. \( A \in \bar{M} \) with \( a_{ii} < 0 \) and \( a_{ij} = a_{ji} \geq 0 \). Then \( A \) is Hurwitz stable if and only if one of the following equivalent conditions is satisfied.

1. All eigenvalues of \( A \) are real and negative.
2. \( A \) is nonsingular and \( -A^{-1} \geq 0 \).
3. All principal minors of \( -A \) are positive.

Proof. The proof of the theorem is a straightforward consequence of Theorem 2.1.9 associated with nonsingular M-matrices.

An interesting by-product of symmetrization of a matrix that we discussed before is captured in the following result for stable Metzler matrices.

Theorem 2.3.2. Let \( A \) be a Hurwitz stable diagonally dominant Metzler matrix with distinct eigenvalues. Then there always exists a similarly transformation that can transform \( A \) to a symmetric Hurwitz stable Metzler matrix.

Proof. Since \( A \) is a stable Metzler matrix with distinct eigenvalues, it can be decomposed as the product of two symmetric matrices, one of which is guaranteed to be positive definite. Due to the
fact that $A$ is diagonalizable by a nonsingular matrix $Q$ consisting of its eigenvectors, we have $A = Q\hat{A}Q^{-1} = Q\hat{A}Q^T Q^{-T} Q^{-1} = S_1 S_2$ where $S_1 = Q\hat{A}Q^T$ and $S_2 = Q^{-T} Q^{-1}$ are both symmetric with $S_2$ being positive definite. By setting $Q\hat{A}Q^T = SA^T$ and $Q^{-T} Q^{-1} = S^{-1}$, one can deduce $A = SA^T S^{-1}$ where the matrix $S$ is a symmetrizer. The Cholesky decomposition of $S$ i.e. $S = TT^T$ defines the transformation matrix $T$ which yields a symmetric Hurwitz stable Metzler matrix $A_s = T^{-1} AT$.

Example 2.3.1. Consider the following stable Metzler matrix

$$A = \begin{bmatrix} -6 & 1 & 2 & 3 \\ 2 & -7 & 1 & 4 \\ 3 & 4 & -8 & 1 \\ 1 & 2 & 3 & -15 \end{bmatrix}$$

(2.55)

with eigenvalues located at $\{-2.3131, -7.7133, -10, -15.9735\}$. The matrices $Q$ and $S$ are obtained as

$$Q = \begin{bmatrix} 0.6169 & 0.3515 & 0.5164 & -0.2410 \\ 0.5782 & -0.8919 & 0.2582 & -0.3518 \\ 0.4713 & 0.2718 & -0.7746 & 0.0214 \\ 0.2512 & -0.0846 & -0.2582 & 0.9043 \end{bmatrix},$$

(2.56)

and $S = TT^T$ determines $T$

$$T = \begin{bmatrix} 0.9104 & 0 & 0 & 0 \\ 0.2870 & 1.1125 & 0 & 0 \\ -0.0208 & -0.1542 & 0.9339 & 0 \\ -0.2483 & -0.0834 & 0.3177 & 0.8861 \end{bmatrix}$$

(2.57)

which transforms $A$ to $A_s$

$$A_s = T^{-1} AT = \begin{bmatrix} -6.5487 & 0.6087 & 3.0986 & 2.9198 \\ 0.6087 & -7.5953 & 1.1823 & 2.4325 \\ 3.0986 & 1.1823 & -7.3957 & 1.4152 \\ 2.9198 & 2.4325 & 1.4152 & -14.4603 \end{bmatrix}$$

(2.58)
An important requirement in symmetric positive stabilization of dynamic systems is to construct stable symmetric nonnegative and Metzler matrices from the set of desired eigenvalues. This problem is a subclass of the so-called Inverse Eigenvalue Problem (IEP): Given a set of real or complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, determine the necessary and sufficient conditions for the set to be the eigenvalue of a matrix. It turns out that if the set $\lambda_i$’s has closed property under complex conjugation, then there always exist at least one real matrix $A$ with spectrum $\lambda(A) = \{\lambda_i : i = 1, \ldots, n\}$. This is easy to see since from the polynomial

$$
\Delta(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0
$$

one can construct a companion matrix $A$. Then by using a nonsingular transformation matrix $P$, one can obtain $\bar{A} = PAP^{-1}$ and consequently other matrices with the same set of eigenvalues.

On the other hand, the Nonnegative Inverse Eigenvalue Problem (NIEP) and Metzler Inverse Eigenvalue Problem (MIEP) are far more difficult. The NIEP for the case of complex eigenvalues has not been solved for $n \geq 4$ and it is open for further investigation. Since this chapter is devoted to the symmetric case, the real NIEP (RNIEP) and real MIEP (RMIEP) are considered.

Problem 1 (RNIEP): Determine necessary and sufficient conditions for a set of real numbers $\lambda_i$’s, $i = 1, \ldots, n$ to be the eigenvalue of a nonnegative matrix of order $n$ and find an algorithm to obtain one or more such a matrix.

Problem 2 (RMIEP): Determine necessary and sufficient conditions for a set of real numbers $\lambda_i$’s, $i = 1, \ldots, n$ to be the eigenvalue of a Metzler matrix of order $n$ and find an algorithm to obtain one or more such a matrix.

Problem 1 has been solved for $n = 2$ and 3 relatively simple and for $n = 4$ partial solution is available. The case of $n \geq 5$ is complex and has not been solved. Problem 2 is very much related to Problem 1, however, it has not been tackled separately. One may refer to [40] and the references therein for a detailed theory, algorithms, and applications of various IEPs.

**Theorem 2.3.3.** For any given set of real numbers $\lambda_i$’s, $i = 1, \ldots, n$, the sufficient condition that this set has at least one nonnegative matrix $A$ is

$$
(-1)^{k+1} (S_k(\lambda_1, \ldots, \lambda_n)) \geq 0 \quad \text{for} \quad k = 1, \ldots, n
$$

where $S_k$’s are defined as the elementary symmetric functions of the eigenvalues of $A$, i.e.

$$
S_k(\lambda_1, \ldots, \lambda_n) = E_k(A) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \lambda_{i_j}
$$
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Proof. Given \( \lambda_i \)'s, the characteristic polynomial \( \Delta(\lambda) \) can be constructed as

\[
\Delta(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0
\]

\( (2.62) \)

\[
= \lambda^n - S_1(\lambda_1, \ldots, \lambda_n)\lambda^{n-1} + S_2(\lambda_1, \ldots, \lambda_n)\lambda^{n-2} - \cdots \pm S_n(\lambda_1, \ldots, \lambda_n)
\]

where

\[
-a_{n-k} = (-1)^{k+1}S_k(\lambda_1, \ldots, \lambda_n) \geq 0
\]

(2.63)

guarantees that all coefficients \( a_i \)'s of \( \Delta(\lambda) \) to be negative and at least one nonnegative companion matrix \( A \) is generated.

Furthermore, by monomial similarity transformation matrices, one can construct nonnegative matrices with the same \( \lambda_i \)'s. Also, if the condition (2.60) is satisfied with strict inequality for a set of stable eigenvalues then the resulting \( A \) becomes a Schur stable matrix.

The Theorem 2.3.3 is a sufficient condition for the existence of RNIEP. A similar result can be written for the complex eigenvalue case provided that the largest eigenvalue among the given \( \lambda_i \)'s to be real and positive and the complex eigenvalues appear as conjugate pairs.

For \( n = 2 \) we have the following necessary and sufficient conditions for the existence of NIEP.

**Theorem 2.3.4.** Given the spectrum \( \lambda(A) = \{\lambda_1, \lambda_2\} \). Then the necessary and sufficient conditions for \( \lambda(A) \) to realize a nonnegative matrix \( A \) is

1. \( \lambda_1 \) and \( \lambda_2 \) to be real
2. \( \lambda_1 + \lambda_2 \geq 0 \) where \( \lambda_1 \geq |\lambda_2| \)

Furthermore, the following (symmetric) nonnegative matrix realizes \( \lambda(A) \).

\[
A = \frac{1}{2} \begin{bmatrix}
\lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\
\lambda_1 - \lambda_2 & \lambda_1 + \lambda_2
\end{bmatrix}
\]

(2.64)

Proof. Since the spectral radius of \( A \) must be real and positive, based on the Perron-Frobenius Theorem only real eigenvalues are allowed for \( n = 2 \). Thus (1) is a necessary condition and (2) is a sufficient condition which can be seen from the fact that if \( \lambda_1 = \rho(A) > 0 \), then \( \lambda_2 \) must be negative so that \( \lambda_1\lambda_2 < 0 \) (see Theorem 2.3.3 for general \( n \)).
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It should be noted that for a set of stable eigenvalues to be Schur stable of a realizable nonnegative matrix we have the following inequalities from the Jury test of stability.

\[ |\lambda_1 + \lambda_2| < 1 + \lambda_1 \lambda_2 \quad \text{and} \quad |\lambda_1 \lambda_2| < 1 \] (2.65)
or simply by using condition 2 of Theorem 2.3.4 \( \lambda_1 \lambda_2 < 1 \).

**Theorem 2.3.5.** Given the spectrum \( \lambda(A) = \{\lambda_1, \lambda_2\} \). Then the necessary and sufficient conditions for \( \lambda(A) \) to realize a stable Metzler matrix \( A \) are

1. \( \lambda_1 \) and \( \lambda_2 \) to be real
2. \( a_1^2 \geq 4a_0 \)

Furthermore, a set of Metzler stable matrices \( \bar{A} = PAP^{-1} \) can be realized with any one of the following Metzler matrices where \( P \) is a \( 2 \times 2 \) monomial matrix

\[
A_1 = \begin{bmatrix}
-a & a_1 a - a^2 - a_0 \\
1 & a - a_1
\end{bmatrix}, \quad
A_2 = \begin{bmatrix}
-a & 1 \\
-a_1 a - a^2 - a_0 & a - a_1
\end{bmatrix}
\] (2.66)

\[ 0 < a < a_1 \quad a_1 a - a^2 - a_0 \geq 0 \]

**Proof.** Condition (1) is obvious from the fact that a Metzler matrix has always a real eigenvalue. \( \mu = \max \Re \{\lambda_i\} \) and since \( n = 2 \) both \( \lambda_1 \) and \( \lambda_2 \) must be real. Constructing a stable characteristic polynomial \( \Delta(\lambda) = \lambda^2 + \lambda a + a_0 \) for a Metzler matrix

\[
A = \begin{bmatrix}
-a_{11} & a_{12} \\
-a_{21} & -a_{22}
\end{bmatrix} \quad a_{ij} \geq 0
\] (2.67)

It is easy to show that \( A \) must satisfy the condition \( a_1^2 - 4(a_0 + a_{12}a_{21}) \geq 0 \) or equivalently \( a_1^2 \geq 4a_0 \). Both \( A_1 \) and \( A_2 \) satisfy this condition and any monomial matrix \( P \) maintains the Metzler structure.
Chapter 3

Positive and Symmetric Systems

This chapter defines two important classes of dynamical systems that appear in various applications. Frequently, they are also appearing in a combined form. Each class has important stability and robustness properties that encourages the researchers to further investigate their usefulness in the analysis and design of general dynamical systems by imposing the positivity and symmetry constraints. The subsequent sections summarize important results on these two classes of systems which play vital role in the stabilization and observer design of the following chapters.

3.1 Positive Systems

3.1.1 Externally Positive Systems

Consider the linear continuous-time system described by the equations

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{3.1}
\]
\[
y(t) = Cx(t) + Du(t)
\]

where \(x(t) \in \mathbb{R}^n\) is the state vector at the instant \(t\), \(u(t) \in \mathbb{R}^m\) is the input vector, \(y(t) \in \mathbb{R}^p\) is the output vector, \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\), \(C \in \mathbb{R}^{p \times n}\), \(D \in \mathbb{R}^{p \times m}\). Let \(\mathbb{R}_+^{n \times m}\) be the set of \(n \times m\) matrices with nonnegative entries and \(\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}\).

**Definition 3.1.1.** The system (3.1) is called externally positive if and only if for every input \(u \in \mathbb{R}_+^m\) and \(x_0 = 0\) the output \(y \in \mathbb{R}_+^p\) for all \(t \geq 0\).

The impulse response \(g(t)\) of single-input single-output system is called the output of the system for the input equal to the Dirac impulse \(\delta(t)\) with zero initial conditions. In a similar way
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assuming successively that only one input is equal to $\delta(t)$ and the remaining inputs are zero, we may define the matrix of impulse responses $g(t) \in \mathbb{R}^{p \times m}$ of a system with $m$-inputs and $p$-outputs.

**Theorem 3.1.1.** The system (3.1) is externally positive if and only if its matrix of impulse responses is nonnegative, i.e.

$$g(t) \in \mathbb{R}^{p \times m}_+ \text{ for all } t \geq 0 \quad (3.2)$$

*Proof.* The necessity of the condition in (3.2) follows immediately from definition 3.1.1. The output of the system in (3.1) with zero initial conditions for any input $u(t)$ is given by the formula

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau \quad (3.3)$$

If the condition in (3.2) is satisfied and $u \in \mathbb{R}^m_+$, then from (3.3) we have $y \in \mathbb{R}^p_+$ for $t \geq 0$. ■

**Theorem 3.1.2.** The continuous-time system with the transfer function

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0} \quad (3.4)$$

is externally positive if $a_i \leq 0$ and $b_i \geq 0$ for $i = 1, 2, \ldots, n$.

*Proof.* We shall show that if the conditions are satisfied then $g(t) \in \mathbb{R}_+$ for $t \geq 0$. The transfer function can be expanded in the series

$$G(s) = g_1s^{-1} + g_2s^{-2} + \cdots \quad (3.5)$$

From comparison of the right hand side of transfer function and (3.5) we have

$$b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1s + b_0 = \left(s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0\right)\left(g_1s^{-1} + g_2s^{-2} + \cdots\right) \quad (3.6)$$

Comparing the coefficients at the same powers of $s$ of the equality (3.6) we obtain

$$g_1 = b_{n-1}, \quad g_2 = b_{n-2} - a_{n-1}g_1, \quad \cdots, \quad g_k = b_{n-k} - a_{n-1}g_{k-1} - a_{n-2}g_{k-2} - \cdots - a_{n-k+1}g_1 \quad (3.7)$$

From equation (3.7) it follows that if the Theorem conditions are satisfied then $g_k \in \mathbb{R}_+$ for $k = 1, 2, \ldots$. It is well-known that the impulse response $g(t)$ is the original of the transfer function $g(t) = \mathcal{L}^{-1}[G(s)]$, where $\mathcal{L}^{-1}$ is the inverse Laplace operator. From (3.5) we have $g(t) = g_1 + g_2t + g_3\frac{t^2}{2!} + \cdots$. Hence, if the conditions are satisfied then $g(t) \in \mathbb{R}_+$ for $t \geq 0$ and the system described by the transfer function is externally positive. ■

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3.1.2 Internally Positive Systems

Consider the continuous-time system described by (3.1).

Definition 3.1.2. The system (3.1) is called internally positive (shortened to positive or Metzlerian) if and only if for any \( x_0 \in \mathbb{R}_+^n \) and every \( u \in \mathbb{R}_+^n \) we have \( x \in \mathbb{R}_+^n \) and \( y \in \mathbb{R}_+^p \) for all \( t \geq 0 \).

From definition 3.1.2 it follows that the system (3.1) is internally positive only if its matrix of impulse responses is nonnegative i.e. the condition in (3.2) is satisfied. This condition in general case is not sufficient for the internal positivity of the system in (3.1). From definition 2.1.5, the matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) is a Metzler matrix if \( a_{ij} \geq 0 \) for \( i \neq j; i, j = 1, 2, \ldots, n \).

Theorem 3.1.3. The continuous-time system (3.1) is internally positive if and only if the matrix \( A \) is a Metzler matrix and \( B \in \mathbb{R}_+^{n \times m}, C \in \mathbb{R}_+^{p \times n} \) and \( D \in \mathbb{R}_+^{p \times m} \).

Proof. Sufficiency: The solution of state equation in (3.1) has form

\[
x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau
\]

By Theorem 2.1.1, the matrix \( e^{At} \in \mathbb{R}_+^{n \times n} \) if and only if \( A \) is Metzler matrix. If \( A \) is the Metzler matrix and \( B \in \mathbb{R}_+^{n \times m}, x_0 \in \mathbb{R}_+^n, u(t) \in \mathbb{R}_+^m \) for \( t \geq 0 \), then from (3.8) we obtain \( x(t) \in \mathbb{R}_+^n \) for \( t \geq 0 \) and from equation (3.1), \( y(t) \in \mathbb{R}_+^p \) since \( C \in \mathbb{R}_+^{p \times n} \) and \( D \in \mathbb{R}_+^{p \times m} \).

Necessity: Let \( u(t) = 0 \) for \( t \geq 0 \) and \( x_0 = e_i \) (the \( i \)th column of \( I_n \)). The trajectory does not leave the quarter \( \mathbb{R}_+^n \) only if \( \dot{x}(0) = Ae_i \geq 0 \), which implies \( a_{ji} \geq 0 \) for \( i \neq j \). The matrix \( A \) has to be the Metzler matrix. For the same reasons, for \( x_0 = 0 \) we have \( \dot{x}(0) = Bu(0) \geq 0 \) which implies \( B \in \mathbb{R}_+^{n \times m} \) since \( u(0) \in \mathbb{R}_+^m \) may be arbitrary. From equation (3.1) for \( u(0) = 0 \) we have \( y(0) = Cx_0 \geq 0 \) and \( C \in \mathbb{R}_+^{p \times n} \), since \( x_0 \in \mathbb{R}_+^n \) may be arbitrary. In a similar way, assuming \( x_0 = 0 \) we get \( y(0) = Du(0) \geq 0 \) and \( D \in \mathbb{R}_+^{p \times m} \), since \( u(0) \in \mathbb{R}_+^m \) may be arbitrary.

The matrix of impulse responses of the system in (3.1) is given by

\[
g(t) = Ce^{At}B + D\delta(t) \quad \text{for} \quad t \geq 0
\]

This formula may be obtained by substitution of equation (3.8) into the output equation of (3.1) and taking into account that for \( x_0 = 0 \) and \( u(t) = \delta(t), y(t) = g(t) \). If \( A \) is the Metzler matrix and \( B \in \mathbb{R}_+^{n \times m}, C \in \mathbb{R}_+^{p \times n}, D \in \mathbb{R}_+^{p \times m} \), then from (3.9) it follows that \( g(t) \in \mathbb{R}_+^{p \times m} \) for all \( t \geq 0 \). We have two important corollaries.
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Corollary 3.1.1. The matrix of impulse responses of the internally positive system in (3.1) satisfies the condition in (3.2).

Corollary 3.1.2. Every continuous-time internally positive system is also externally positive.

Note that the internally positive continuous-time system, also known as Metzlerian system, is denoted as positive system from now on in this dissertation. Here we provide some example of positive systems.

Example 3.1.1. Given the circuit shown in Figure 3.1 with known resistances $R_1$, $R_2$, $R_3$, inductances $L_1$, $L_2$ and source voltages $e_1 = e_1(t)$, $e_2 = e_2(t)$. The currents $i_1 = i_1(t)$, $i_2 = i_2(t)$ in the inductances are chosen as state variables and $y = y(t) = \begin{bmatrix} R_1 i_1 & R_2 i_2 \end{bmatrix}^T$ is chosen as the output. Using the Kirchhoff law we may write the equations in the following state space format

\[
\begin{align*}
\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} &= A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \\
y &= C \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}
\end{align*}
\](3.10)

where

\[
A = \begin{bmatrix} -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}, \quad C = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}
\](3.12)

From equation (3.12) it follows that $A$ is the Metzler matrix and $B$ and $C$ have nonnegative entries. The circuit is an example of continuous-time positive system.
**Example 3.1.2.** Another example of positive system is the two nation model of Richardsons Theory of arms force [41]. In this model two competing nations (or perhaps two competing coalitions of nations) are denoted X and Y. The variables $x(t)$ and $y(t)$ represents, respectively, the armament levels of the nations X and Y at time $t$. the general form of the model is

$$
\dot{x}(t) = ky(t) - \alpha x(t) + g \tag{3.13}
$$

$$
\dot{y}(t) = lx(t) - \beta y(t) + h \tag{3.14}
$$

In this model, the terms $g$ and $h$ are called grievances. They encompass the wide assortment of psychological and strategic motivations for changing armament levels, which are independent of existing levels of either nation. Roughly speaking, they are motives of revenge or dissatisfaction, and they may be due to dissatisfaction with treaties or other past political negotiations. The terms $k$ and $l$ are called defense coefficients. They are nonnegative constants that reflect the intensity of reaction by one nation to the current armament level of rivalry that can cause the exponential growth of armaments commonly associated with arms race. Finally, $\alpha$ and $\beta$ are called fatigue coefficients. They are nonnegative constants that represent the fatigue and expense of the effect of causing a nation to retard the growth of its own armament level; the retardation effect increasing as the level increase. The system matrix is

$$
A = \begin{bmatrix}
-\alpha & k \\
l & -\beta 
\end{bmatrix} \tag{3.15}
$$

which is a Metzler matrix.

**Example 3.1.3.** [42] (Bone Scanning) The procedure for taking a scintigram is the following: the patient receives an injection of a radionuclide, which, transported by the blood, collects in the bones. More of it tends to collect in so-called hot spots, areas where there is increased metabolic activity (which in simple terms means that the bone is breaking down, or repairing itself). The gamma rays generated by the radionuclide are captured by a specific camera that provides the image. Deciding when is the adequate time-after-injection for the scan is tricky: From a purely imaging point of view, the optimal time is the instant when the maximum contrast in the image between the hot spots and the background is obtained. Unfortunately, the evolution of the contrast of the scintigram varies intrapatient. Therefore estimating this contrast is crucial for clinicians. Based on clinical measurements, the portion of the administered dose of this radionuclide in some compartments of the
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human body was determined to be quite precisely given by the following dynamical model

\[
d\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} = \begin{bmatrix} -k_{21} - k_{41} - k_{51} & k_{12} & 0 & k_{14} & k_{15} \\ k_{21} & -k_{21} - k_{32} & k_{23} & 0 & 0 \\ 0 & k_{32} & -k_{23} & 0 & 0 \\ k_{41} & 0 & 0 & -k_{14} & 0 \\ k_{51} & 0 & 0 & 0 & -k_{15} - k_{05} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}
\]  
(3.16)

\[
y(t) = \begin{bmatrix} c & 0 & 0 & 0 & 0 \end{bmatrix} x(t)
\]  
(3.17)

where the states \(x_i(t)\)s correspond to the portion of the dose of Tc-MDP (Tc-99m(Sn)Methylene Diphosphonate, i.e. a chemical affecting the contrast) in the different compartments: \(x_1(t)\) is the portion of the dose in the blood, \(x_2(t)\) in the extracellular fluid of the bone, \(x_3(t)\) in cellular bone, \(x_5(t)\) in the tubular urine and \(x_4(t)\) in the rest of the body. Some values for the parameters of the model were obtained from physiological data. Based on clinical measurements, the following parameters for the compartmental model were obtained, with some uncertainty that represents inter-patient variations:

\[
k_{12} = 0.540 \pm 0.038, k_{21} = 0.095 \pm 0.003, k_{14} = 0.277 \pm 0.007, k_{41} = 0.431 \pm 0.011
\]  
(3.18)

\[
k_{15} = 0.233, k_{05} = 0.749, k_{23} = 0.049 \pm 0.001, k_{32} = 1.055 \pm 0.0037
\]  
(3.19)

It can be seen that this system was also a positive system with Metzlerian matrix system and positive input output matrices.

Example 3.1.4. In chemical plants, it is often necessary to maintain the levels of liquids. A simplified model of a connection of two tanks can be described as follow

\[
d\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{A_2 R_2} & \frac{1}{A_1 R_1} \\ \frac{1}{A_2 R_1} & -\frac{1}{A_2 R_1 + A_2 R_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} u
\]  
(3.20)

In this model, \(u\) is the inflow perturbation of the first tank which will cause variations in liquid level \(x_1\) that will indirectly cause variations in liquid level \(x_2\) and outflow variation \(y\) in the second tank. \(R_i\)s are the flow resistances that can be controlled by valves and \(A_i\)s are the cross section of tanks for \(i = 1, 2\). Since all \(R_i\)s and \(A_i\)s are positive it can be seen that the system matrix is a Metzler matrix and therefore this chemical plant is also a positive system.
3.1.3 Asymptotic Stability

Consider a continuous-time internally positive system described by the equation

\[
\dot{x}(t) = Ax(t), \quad x(0) = x_0
\]  

(3.21)

where \( A \in \mathbb{R}^{n \times n} \) is the Metzler matrix. The solution of equation (3.21) has the form

\[
x(t) = e^{At}x_0
\]  

(3.22)

**Definition 3.1.3.** The internally positive system in (3.21) is called asymptotically stable if and only if the solution in (3.22) satisfies the condition

\[
\lim_{t \to \infty} x(t) = 0 \quad \text{for every} \quad x_0 \in \mathbb{R}_+^n
\]  

(3.23)

The roots \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of the equation \( \det[\lambda I - A] \) are called the eigenvalues of the matrix \( A \) and their set is called the spectrum of \( A \).

**Theorem 3.1.4.** The internally positive system in (3.21) is called asymptotically stable if and only if all eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of the Metzler matrix \( A \) have negative real parts.

**Proof.** Proof can be found in [3].

**Theorem 3.1.5.** The internally positive system in (3.21) is asymptotically stable if and only if all coefficients \( a_i \) \((i = 0, 1, \ldots, n-1)\) of the characteristic polynomial

\[
\Delta(A(\lambda)) = \det[\lambda I - A] = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0
\]  

(3.24)

are positive \((a_i > 0)\).

**Proof.** Necessity: The eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of \( A \) are real or complex conjugate since the coefficients \( a_i \) of \( \Delta(A(\lambda)) \) are real. Hence if \( \text{Re} \lambda_i < 0, \ i = 0, 1, \ldots, n-1 \) then all coefficients of the polynomial \( \Delta(A(\lambda)) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \) are positive, \( a_i > 0 \) for \( i = 0, 1, \ldots, n-1 \).

Sufficiency: This will be proved by contradiction. If \( A \) is Metzler matrix then by Remark 2.1.1 \( \alpha = \max_i \text{Re} \lambda_i \) is its eigenvalue and \( \text{Re} \lambda_i < 0 \) if \( \alpha < 0 \). For \( a_i > 0 \) for \( i = 0, 1, \ldots, n-1 \) and real \( \lambda \) we have \( \Delta(A(\lambda)) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \) and \( A \) has no real nonnegative eigenvalue. Thus, we get the contradiction and \( \alpha < 0 \).
To test the asymptotic stability of the system in (3.21) we do not need to know the characteristic polynomial (3.24) and we may use the following theorem.

Theorem 3.1.6. The internally positive system in (3.21) is asymptotically stable if and only if all principal minors \( n \) of the matrix \(-A\) are positive, i.e.

\[
\begin{vmatrix} -a_{11} & a_{12} \\ a_{21} & -a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} -a_{11} & a_{12} & a_{13} \\ a_{21} & -a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \ldots, \det [-A] > 0 \quad (3.25)
\]

Proof. Note that the characteristic polynomial (3.24) maybe written as

\[
\Delta(A(\lambda)) = \det [\lambda I - A] = \det [\lambda e_1 - a_1, \lambda e_2 - a_2, \ldots, \lambda e_n - a_n] \quad (3.26)
\]

where \( a_i \) and \( e_i \) are the \( i \)-th columns of \( A \) and the \( n \times n \) identity matrix \( I \) respectively. The decomposition of the determinant on the sum of \( 2^n \) yields determinants whose columns are \( a_1, a_2, \ldots, a_n \) or \( \lambda e_1, \lambda e_2, \ldots, \lambda e_n \). Among them we have \( \frac{n!}{(n-i)!i!} \) determinants, which contains \( i \) columns of the form \( \lambda e_i \), \( i \in (1, 2, \ldots, n) \). Every such determinant is equal to the principal minor of the \( (n-i) \)-th order of the matrix \( A \). The sum of those determinants is equal to the term \( a_i \lambda^i \), \( i = 0, 1, \ldots, n - 1 \) of \( \Delta(A(s)) \). From the properties of nonnegative matrices in [4], it follows that if the conditions in (3.25) are satisfied then all principal minors are positive, since the matrix \(-A\) has all nonpositive off-diagonal entries for the Metzler matrix \( A \). Therefore all coefficients of \( \Delta(A(\lambda)) \) are positive if and only if the conditions in (3.25) are satisfied. \( \blacksquare \)

3.1.4 Bounded-Input Bounded-Output (BIBO) Stability

A signal (input, output) \( s(t) \) is called bounded if and only if its value (or the norm \( \|s\| \)) is bounded for all \( t \in [0, +\infty) \).

Definition 3.1.4. The internally positive system in (3.1) is called BIBO stable if and only if its output is bounded for any bounded input and all \( t \in [0, +\infty) \).

Let \( g(t) \) be the impulse response of the system (3.1) that is the output of the system with zero initial conditions \((x_0 = 0)\) for Dirac impulse \( \delta(t) \) input. The output \( y(t) \) of the system (3.1) with zero initial conditions for any input \( u(t) \) is given by

\[
y(t) = \int_0^t g(t - \tau)u(\tau)d\tau = \int_0^t g(\tau)u(t - \tau)d\tau \quad (3.27)
\]
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Theorem 3.1.7. The internally positive system (3.1) is BIBO stable if and only if

\[
\int_0^t g(\tau)d\tau < \infty \quad \text{for all} \quad t \in [0, +\infty)
\] (3.28)

Proof. Taking into account that the impulse response \(g(t)\) of internally positive system is nonnegative from (3.27), for a bounded \(u(t) \in \mathbb{R}_+\) we obtain

\[
y(t) = \int_0^t g(\tau)u(t - \tau)d\tau \leq \int_0^t g(\tau)d\tau \bar{u}
\] (3.29)

where \(\bar{u} \geq u(t)\) for \(t \in [0, +\infty)\). From (3.29) it follows that the output \(y(t)\) is bounded for any bounded input \(u(t)\) and all \(t \in [0, +\infty)\) if and only if the condition in (3.28) is satisfied. ■

Let \(h(t)\) be the unit response of the (3.1) system that is the output of the system with zero initial conditions for the unit step

\[
u(t) = \begin{cases} 
1 & \text{for } t > 0 \\
0 & \text{for } t < 0 
\end{cases}
\] (3.30)

The unit response \(h(t)\) and the impulse response \(g(t)\) of the (3.29) system are related by the following formula.

\[g(t) = \frac{dh(t)}{dt}, \quad h(0) = 0
\] (3.31)

or

\[h(t) = \int_0^t g(\tau)d\tau
\] (3.32)

Using equation (3.32) we may reformulate the Theorem 3.1.7 as follows.

Theorem 3.1.8. The internally positive system in (3.1) is BIBO stable if and only if its unit response is bounded for all \(t \in [0, +\infty)\).

Let \(G(s)\) be the transfer function of the system in (3.1)

\[G(s) = C(I - A)^{-1}B + D = \frac{N(s)}{D(s)}
\] (3.33)

where

\[N(s) = b_{n'}s^{n'} + b_{n'-1}s^{n'-1} + \cdots + b_1s + b_0
\] (3.34)

\[D(s) = s^{n'} + a_{n'-1}s^{n'-1} + \cdots + a_1s + a_0
\] (3.35)
where $n' \leq n$ (the equality holds if the system does not have decoupling zeros).

It is assumed that the zeros $z_1, z_2, \ldots, z_{n'}$ (the roots of $N(s) = 0$) are different from the poles $s_1, s_2, \ldots, s_n$ (the roots of $D(s) = 0$) of the transfer function in (3.33). The impulse response $g(t)$ is the original of $G(s)$, i.e.

$$g(t) = L^{-1}[G(s)] = Ce^{At}B + D\delta(t)$$  \hfill (3.36)

where $L^{-1}$ is the inverse Laplace transform operator. Without loss of generality we may assume that the poles $s_1, s_2, \ldots, s_n'$ of transfer function in equation (3.33) are distinct ($s_i \neq s_j$ for $i \neq j$). Then using (3.36) we obtain

$$g(t) = \sum_{k=1}^{n'} A_k e^{s_k t}$$  \hfill (3.37)

where $A_k = \frac{N(s_k)}{D'(s_k)}$, $D'(s_k) = (s_k - s_1) \cdots (s_k - s_{k-1})(s_k - s_{k+1}) \cdots (s_k - s_{n'})$.

From equation (3.37) it follows that the condition in (3.28) is satisfied if and only if $\text{Res}_{s_k} < 0$ for $k = 1, 2, \ldots, n'$. By Theorem 3.1.4 the condition in equation (3.28) is satisfied if and only if the denominator $D(s)$ of the transfer function in (3.33) has all positive coefficients. Therefore the following theorem has been proved.

**Theorem 3.1.9.** The internally positive system described in (3.1) is BIBO stable if and only if the denominator $D(s)$ of the transfer function (3.33) has all positive coefficients, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n - 1$.

The question then arises as to what the relationship is between the BIBO stability and the asymptotic stability of the internally positive system in (3.1). The answer is given by the following theorem.

**Theorem 3.1.10.** If the internally positive system in (3.1) is asymptotically stable then it is also BIBO stable.

**Proof.** It is well-known the set of poles $s_1, s_2, \ldots, s_{n'}$ is a subset of the eigenvalues of the matrix $A$ (the roots of the equation $\det [\lambda I - A] = 0$). The set of zeros of the minimal polynomial $\Psi(s)$ of $A$ contains all poles of the transfer function (3.33). Therefore, the system (3.1) is asymptotically stable only if the system is BIBO stable.

The considerations can be easily extended for systems with $m$ inputs and $p$ outputs by considering in turn the $mp$ suitable single-input single-output subsystems. For multi-input multi-output systems the impulse response and the unit response in Theorem 3.1.7 and 3.1.8 should be
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replaced by the matrix of impulse responses and the matrix of unit responses, respectively, and the
transfer function in the Theorem 3.1.9 should be replaced by the transfer matrix.

To conclude this section, we summarize all the above theorems in one Lemma which
can be used in the following chapters as the main stability Lemma for positive stabilization of
continuous-time systems.

Lemma 3.1.1. Let the system (3.1) be a positive continuous-time system (Metzlerian System). Then
the system (3.1) is asymptotically stable if and only if one of the following equivalent conditions is
satisfied:

1. All eigenvalues of $A$ have negative real parts.

2. All coefficients of the characteristic equation

$$\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \quad (3.38)$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, 2, \ldots, n$.

3. All principal minors of the matrix $-A$ are positive.

4. The matrix $A$ is nonsingular and $-A^{-1}$.

5. There exist a positive definite (possibly diagonal) matrix $P$ such that $A^TP + PA < 0$.

6. There exists a positive vector $v \in \mathbb{R}^n_+$ such that $Av < 0$.

Note that the above stability condition can easily be written for discrete-time systems with
respect to $A - I$.

3.1.5 Asymptotic Stability using Lyapunov Equation

In the stability analysis of Metzlerian systems, it is of interest to find conditions under
which the solution of the Lyapunov equation

$$A^TP + PA = -Q \quad (3.39)$$

is a positive matrix $P > 0$ in addition to its positive definiteness, i.e. $P > 0$. The following Lemma
is useful for one of our main results.
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Lemma 3.1.2. If a matrix $A$ is Metzler and stable, then for any positive and positive definite symmetric matrix $Q$, there is a positive and positive definite symmetric matrix $P$ as a solution of Lyapunov equation (3.39).

Proof. The Lyapunov matrix equation (3.39) can be rewritten as a linear matrix equation

$$Mp = -q$$

where $p$ and $q$ are vectors whose elements are constructed from the components $p_{ij}$ and $q_{ij}$ of $P$ and $Q$, and

$$M = A^T \otimes I + I \otimes A^T$$

is an $n^2 \times n^2$ matrix with $\otimes$ denoting the Kronecker product. The matrix $M$ is stable and by construction it is also Metzlerian. Since for any Metzler matrix $-M^{-1} > 0$, we conclude that for any $q > 0$ we have $p > 0$. The positive definiteness of $P$ follows directly from stability result of the Lyapunov matrix equation.

Corollary 3.1.3. Let the matrix $A = [a_{ij}]$ be any stable Metzler matrix. Then the following statements are equivalent:

1. There exists a positive diagonal matrix $D$ such that $A^T D + DA$ (or alternatively $AD + DA^T$) is negative definite, i.e., $A^T D + DA < 0$.

2. There exists a positive diagonal matrix $D$ such that $x^T A^T D x < 0$ (or alternatively $x^T AD x < 0$) for all $x \neq 0$.

Furthermore, if $B = [b_{ij}]$ is a matrix with negative diagonal elements $b_{ii}$ with $b_{ii} \leq a_{ii}$ and $|b_{ij}| \leq a_{ij}$. Then $B^T D + DB < 0$. A direct consequence of Lemma 3.1.2 and Corollary 3.1.3 is the fact that it is always possible to find a diagonal positive matrix $P = D$ for any Hurwitz stable Metzler matrix.

3.1.6 Robust Stability of Perturbed Systems

3.1.6.1 General Uncertain Systems

Consider the general continuous-time system with uncertainty structures defined by

$$\dot{x}(t) = (A + \Delta A(t)) x(t) + (B + \Delta B(t)) u(t)$$

(3.42)
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where

\[ \Delta A(t) = E \Delta(t) F_1 \]  \hspace{1cm} (3.43) \\
\[ \Delta B(t) = E \Delta(t) F_2 \]

with \( E \in \mathbb{R}^{n \times e} \), \( F_1 \in \mathbb{R}^{f \times n} \), \( F_2 \in \mathbb{R}^{f \times m} \) and

\[ \Delta(t) \in \Delta = \left\{ \Delta(t) \in \mathbb{R}^{e \times f} : \|\Delta(t)\| \leq 1 \right\} \]  \hspace{1cm} (3.44)

Note that the elements of \( \Delta(t) \) are Lebesgue measurable and admissible uncertainties are such that \( \Delta^T(t) \Delta(t) \leq I \).

Now, for the purpose of defining positive uncertain system let the Metzler matrix \( A \) be associated with the system \( \dot{x}(t) = Ax(t) \). Let the perturbed system be defined by affine perturbation of the form shown in equations (3.42) and (3.43) but for simplicity let assume \( F_1 = F_2 \). Then the perturbed system will look as follow.

\[ \dot{x}(t) = (A + E \Delta F) x(t) \]  \hspace{1cm} (3.45)

where \( E \in \mathbb{R}^{n \times e}_+, F \in \mathbb{R}^{f \times n}_+ \) represent the structure of uncertainties and \( \Delta \in \mathbb{R}^{e \times f}_+ \) is unknown uncertainty matrix. The following results provide robustness measures that will be used in the robustness analysis of the optimal constrained stabilization.

**Lemma 3.1.3.** Assume that the matrix \( A \) is Hurwitz stable and for any \( Q > 0 \), the solution \( A^T P + PA = -Q \) is given by \( P > 0 \). Then the perturbed system

\[ \dot{x}(t) = (A + H) x(t) \]  \hspace{1cm} (3.46)

is asymptotically stable provided that

\[ \|H\| < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \]  \hspace{1cm} (3.47)

Furthermore, if \( A \) is a Metzler stable matrix, the matrix \( P \) in (3.47) can be replaced by a diagonal positive matrix \( D \) obtained from the solution of \( A^T D + DA = -Q \).

Note that the bound (3.47) is well-known in the literature (see for example [43]). A more general measure of stability robustness is the stability radius defined in the next section.
3.1.7 Stability Radius

The stability radius can be defined for any objects such as a system, a function or a matrix. Stability radius at a given point is the radius of the largest ball, centered at the nominal point, all of whose elements satisfy pre-determined stability conditions. Stability radius is a more general measure of stability robustness [44, 45].

We suppose the perturbed system can be described in the form of equation (3.45). Therefore, $A$ is the nominal system matrix which under perturbation will be $A + E\Delta F$ where $E$ and $F$ are given perturbation structure matrices and $\Delta$ is unknown uncertainty which is allowed to be real or complex. In each case we try to find the smallest matrix $\Delta$ that makes the system unstable. In the literature, it is common to measure the size of matrix $\Delta$ by its norm. The system equations will be

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

$$u(t) = \Delta y(t)$$

In this model signals $u$ and $y$ are fictitiously introduced and are not necessarily input or output of the system. We measure size of $\Delta$ using the following norm

$$\|\Delta\| = \sup \left\{ \|\Delta y\|_F : y \in \mathbb{F}^f, \|y\|_F \leq 1 \right\}$$

(3.49)

where $\mathbb{F}$ could be either field of real or complex numbers and $e$ and $f$ are size of signals $u$ and $y$, respectively.

Let $\mathbb{S}$ denotes the stability region in the complex plane and $\lambda(M)$ denotes the eigenvalue of matrix $M$. Since we assumed that matrix $A$ is stable we can write $\lambda(A) \subset \mathbb{S}$.

**Definition 3.1.5.** The stability radius, in the field $\mathbb{F}$, of $A$ with respect to the perturbation structure $(E, F)$ is defined as

$$r = \inf \left\{ \|\Delta\| : \Delta \in \mathbb{F}^{exf}, \lambda(A + E\Delta F) \cap \mathbb{U} \neq \emptyset \right\}$$

(3.50)

The operator norm of $\Delta$ is most often measured by its maximum singular value, i.e. $\|\Delta\| = \bar{\sigma}$. In this case it can easily be established by continuity of the eigenvalues of $A + E\Delta F$.
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on $\Delta$, and the stability of $A$, that

$$ r = \inf \left\{ \bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{e \times f}, \quad \lambda(A + E\Delta F) \cap \mathcal{U} \neq \emptyset \right\} $$

$$ = \inf \left\{ \bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{e \times f}, \quad \lambda(A + E\Delta F) \cap \partial \mathcal{S} \neq \emptyset \right\} $$

$$ = \inf_{s \in \partial \mathcal{S}} \inf \left\{ \bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{e \times f}, \quad \det(sI - A - E\Delta F) = 0 \right\} $$

$$ = \inf_{s \in \partial \mathcal{S}} \inf \left\{ \bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{e \times f}, \quad \det \left( I - \Delta F(sI - A)^{-1}E \right) = 0 \right\} $$

$$ = \inf_{s \in \partial \mathcal{S}} \inf \left\{ \bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{e \times f}, \quad \det(I - \Delta G(s)) = 0 \right\} $$

For a fixed $s \in \partial \mathcal{S}$, write $G(s) = M$. Then the calculation above reduces to solving the optimization problem

$$ \inf \left\{ \bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{e \times f}, \quad \det(I - \Delta M) = 0 \right\} $$

(3.52)

When $\Delta$ is a complex matrix, the solution for this optimization problem is obtained as

$$ \bar{\sigma}(\Delta) = [\bar{\sigma}(M)]^{-1} $$

(3.53)

When $\Delta$ is constrained to be a real matrix the solution is much more complicated due to the fact that $M$ is a complex matrix.

### 3.1.7.1 Complex Stability Radius

Consider the case where all the system matrices are complex. Let us also assume that the unknown uncertainty is complex, too, i.e. $\Delta \in \mathbb{C}$. In this case we denote $r$ as $r_C$ and call it complex stability radius. The following theorem enable us to compute $r_C$ assuming that the transfer function associated with perturbed triple $(A, E, F)$ is

$$ G(s) = F(sI - A)^{-1}E $$

(3.54)

**Theorem 3.1.11.** If $A$ is stable with respect to $\mathbb{S}s$, then

$$ r_C = \frac{1}{\sup_{s \in \partial \mathcal{S}} \|G(s)\|} $$

(3.55)

where $\|G(s)\|$ denotes the operator norm of $G(s)$ and by definition $0^{-1} = \infty$.

**Proof.** The proof is an immediate consequence of equations (3.51)–(3.53).
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When \( \| \Delta \| = \bar{\sigma}(\Delta) \), the complex stability radius is obtained as

\[
r_C = \frac{1}{\sup_{s \in \partial S} \bar{\sigma}(s)}
\]  

(3.56)

By choosing \( E = F = I \) we can find the unstructured stability radius which for the case of Hurwitz stability can be found as follow.

\[
r_C = \frac{1}{\| G(s) \|_{\infty}}
\]  

(3.57)

3.1.7.2 Real Stability Radius

Consider the case where all the system matrices are real. Let us also assume that the unknown uncertainty is constrained to be real, too, i.e. \( \Delta \in \mathbb{R} \). In this case we denote \( r \) as \( r_R \) and call it real stability radius. To compute \( r_R \) we need to solve a two parameter optimization problem as discussed in the following theorem.

**Theorem 3.1.12.** The real stability radius is given by

\[
r_R = \inf_{s \in \partial S} \inf_{\gamma \in (0,1]} \sigma_2 \left( \begin{bmatrix} \text{Re}G(s) & -\gamma \text{Im}G(s) \\ \gamma^{-1} \text{Im}G(s) & \text{Re}G(s) \end{bmatrix} \right)
\]  

(3.58)

An important feature of this formula is the fact that the function

\[
\sigma_2 \left( \begin{bmatrix} \text{Re}G(s) & -\gamma \text{Im}G(s) \\ \gamma^{-1} \text{Im}G(s) & \text{Re}G(s) \end{bmatrix} \right)
\]

(3.59)

is unimodal over \( \gamma \in (0,1] \).

The computation of stability radius requires the solution of an iterative global optimization problem. Note that, computing the real stability radii is more complicated comparing to the complex stability radii and there is no closed form for none of them for the general systems. However for the class of positive systems the complex and real stability radii coincide and can be computed by closed form expressions for both continuous-time and discrete-time cases [46,47]. For this class of system one can employ the Perron-Frobenius Theorem to derive the Lemma 3.1.4.

**Theorem 3.1.13.** (Perron-Frobenius) If the matrix \( A \) is nonnegative, then

1. \( A \) has a positive eigenvalue \( r \) equal to the spectral radius of \( A \).
2. There is a positive eigenvector associated with the eigenvalue \( r \).
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3. The eigenvalue $r$ has algebraic multiplicity 1.

The eigenvalue $r$ will be called Perron-Frobenius eigenvalue.

The following lemma sums up all of our discussion about robust stability of positive systems. This Lemma is an example of nice stability properties that holds only for positive system. This is going to be a keystone for our controller design in the following chapters.

**Lemma 3.1.4.** [46] Let the Metzler matrix $A$ associated with the perturbed system (3.21) be Hurwitz stable. Then, the real and complex stability radii of the uncertain Metzlerian system $\dot{x}(t) = (A + E\Delta F)x(t)$ coincide and given by the following formulas depending on the characterization of $\Delta$.

1. Let $\|\cdot\|$ denotes the Euclidean norm in characterization of $\Delta$, then

$$r_C = r_R = \frac{1}{\|FA^{-1}E\|}$$  \hspace{1cm} (3.60)

2. Let $\Delta$ be defined by the set $\Delta = \{So\Delta : S_{ij} \geq 0\}$ with $\|\delta\| = \max \{|\delta_{ij} : \delta_{ij} \neq 0\}$ where $[So\Delta]_{ij} = S_{ij}\delta_{ij}$ represents the Schur product, then

$$r_C = r_R = \frac{1}{\rho(FA^{-1}ES)}$$  \hspace{1cm} (3.61)

where $\rho(\cdot)$ denotes the spectral radius of a matrix.

Furthermore, if the affine uncertainty structure is defined by $A(\delta) = A + \sum_{r=1}^{q} \delta_r E_r$, where $\delta = [\delta_1 \delta_2 \cdots \delta_q]^T$ is a vector of uncertain parameters confined within a prescribed set of interest $\Omega$, i.e. $\delta \in \Omega$. Then the real and complex stability radii of uncertain Metzlerian system $\dot{x}(t) = (A + \sum_{r=1}^{q} \delta_r E_r) x(t)$ coincide and it is given by

$$r_C = r_R = \frac{1}{\rho (-A^{-1} \sum_{r=1}^{q} E_r)}$$  \hspace{1cm} (3.62)

3.2 Symmetric Systems

There are many systems with symmetric structure that are modeled by state-space representation or transfer function. In this section we will focus on studying symmetric systems. However, frequently, a positivity constraint is also observed in these systems which makes their stability and control problem even more challenging. This class of positive symmetric systems is discussed in next section and their associated stabilization problem is also solved in Chapter [5]
Consider a linear time-invariant system described by
\[
\dot{x}(t) = Ax(t) + Bu(t) \quad (3.63)
\]
\[
y(t) = Cx(t) + Du(t) \quad (3.64)
\]
where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), and \( y(t) \in \mathbb{R}^p \) represents state, input, and output of the system, respectively. Note that we assume \( m = p \) for symmetric reason in subsequent discussion. The corresponding transfer function matrix is
\[
G(s) = C(sI - A)^{-1}B + D \quad (3.65)
\]
where \( G(s) \in \mathbb{R}^{m \times m} \).

The following definitions and lemmas are standard and can be found in \([48–50]\).

**Definition 3.2.1.** The system (3.63), (3.64) or equivalently (3.65) is said to be externally passive if
\[
\int_0^t u^T(\tau)y(\tau)d\tau \geq 0 \quad \text{for all inputs } u(t) \quad \text{and it is called internally passive if}
\]
\[
\begin{bmatrix}
A^T + A & B - C^T \\
B^T - C & -D - D^T
\end{bmatrix} \preceq 0 \quad (3.66)
\]

**Lemma 3.2.1.** Let (3.63), (3.64) be a minimal realization of (3.65). Then the system is externally passive if and only if \( G(s) \) is positive real or equivalently there exists a solution \( P = P^T > 0 \) to the following LMI
\[
\begin{bmatrix}
A^TP + PA & PB - C \\
B^TP - C & -D - D^T
\end{bmatrix} \preceq 0 \quad (3.67)
\]

Alternatively, there exists a solution \( Q \) to the following LMI
\[
\begin{bmatrix}
AQ + QA^T & QC^T - B \\
CQ - B^T & -D - D^T
\end{bmatrix} \preceq 0 \quad (3.68)
\]
where \( Q = P^{-1} \). Furthermore, if the system represented by \( G(s) \) is externally passive, then it admits an internally passive minimal realization satisfying (3.67).

Note that (3.68) follows from the fact that the positive realness of \( G(s) \) is equivalent to the positive realness of \( G^T(s) \).

The above result provides a method for passive realizations of positive real transfer function matrices. Given any minimal realization \( \{A, B, C, D\} \) of \( G(s) \), one can solve LMI(s) for some
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...symmetric matrix $P$ and by factoring $P^{-1} = TT^T$ the passive realization $A_p = T^{-1}AT$, $B_p = T^{-1}B$, $C_p = CT$, and $D_p = D$ can be obtained. It is also possible to obtain positive real balanced realizations by using an initial minimal realization of $G(s)$ and employing LMIs (3.67), (3.68). Once $P$ and $Q$ are obtained and factorized as $P = L^T L$ and $Q = R^T R$ one can perform singular value decomposition on $LR^T = U \Sigma V^T$ and the required transformation is determined by $T = R^T V \Sigma^{-\frac{1}{2}}$, which leads to $A_b = T^{-1}AT$, $B_b = T^{-1}B$, $C_b = CT$, and $D_b = D$.

**Definition 3.2.2.** The system (3.63), (3.64) is called symmetric with respect to state space parameter if $A^T = A$ and $C^T = B$. Similarly, the system represented by $G(s) \in \mathbb{R}^{m \times m}$ is said to be symmetric with respect to transfer function if $G(s) = G^T(s)$.

Although symmetric systems are defined in a general setting with signature matrix $\Sigma$ (i.e. a diagonal matrix with diagonal entries $+1$ or $-1$) and defined as $\Sigma G(s) = G^T(s) \Sigma$, we simplified the definition by $G(s) = G^T(s)$.

**Lemma 3.2.2.** Let $\{A, B, C, D\}$ be a minimal realization of a symmetric transfer function matrix $G(s)$. Then $\{A, B, C, D\}$ is symmetric if and only if there exists a nonsingular symmetric matrix $T_s$ such that $A = T_s^{-1}AT_s$, $B = T_s^{-1}C^T$, and $D = D^T$. Moreover, $T_s$ is unique.

To see the broad aspect and diverse applications of symmetric systems one can refer to [33–38]. The symmetric characteristic of transfer function as explored in Lemma 3.2.2 can be illustrated through the model of a space structure with collocated sensors and actuators [51] described by

$$A_0 \ddot{q}(t) + A_1 \dot{q}(t) + A_2 q(t) = Lu(t)$$

$$y(t) = L^T q(t)$$

(3.69)

where $q(t), \dot{q}(t), \ddot{q}(t) \in \mathbb{R}^n$ are the displacement, velocity, and acceleration vectors with $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^n$ representing the input and output of the system. The matrices $A_0$, $A_1$, and $A_2$ are symmetric matrices representing the mass, damping and stiffness, respectively. The transfer function of the system is $G(s) = L^T (A_0 s^2 + A_1 s + A_2)^{-1} L$, which is obviously symmetric. It has a realization given by

$$A = \begin{bmatrix} 0 & I \\ -A_0^{-1} A_2 & -A_0^{-1} A_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ A_0^{-1} L \end{bmatrix}$$

$$C = \begin{bmatrix} L^T & 0 \end{bmatrix}$$

(3.70)
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and its symmetric state-space characteristic can be established by

\[
T_s = \begin{bmatrix}
A_1 & A_0 \\
A_0 & 0
\end{bmatrix}
\]  

(3.71)

Another example is a special class of system with zeros interlacing the poles described by

\[
G(s) = K \prod_{j=1}^{n-1} (s + z_j) \prod_{i=1}^{n} (s + p_i)
\]  

(3.72)

where \(0 < p_i < z_i < p_{i+1}\) for \(i = 1, \ldots, n-1\). It is not difficult to show that \(G(s)\) can be written as

\[
G(s) = \sum_{i=1}^{n} \frac{q_i}{s + p_i}
\]  

(3.73)

with distinct poles \(-p_i < 0\) and \(q_i > 0\), which has a diagonal realization with \(A = \text{diag}\{-p_i; i = 1, \ldots, n\}\) and \(B^T = C = \begin{bmatrix} \sqrt{q_1} & \sqrt{q_2} & \cdots & \sqrt{q_n} \end{bmatrix}\). This transfer function has also a symmetric balanced realization which can be derived from the symmetric diagonal form using symmetric transformation \(T_s = T_s^T \Sigma T_s\) with \(T_s^T = T_s^{-1}\) where \(T_s\) satisfies both controllability and observability Lyapunov equations \(AT_s + T_s A^T + BB^T = 0, A^T T_s + T_s A + C^T C = 0\) leading to \(A_b = T_s A T_s^T\), \(B_b = T_s B\), and \(C_b = C T_s^T\).

The above development can be summarized in the following theorems.

**Theorem 3.2.1.** Let \(\{A, B, C\}\) be a stable minimal realization of a symmetric system \(G(s)\) satisfying state-space symmetric condition \(A = A^T, B = C^T\). Then any balanced realization obtained by \(A_b = T A T^{-1}, B_b = T B, C_b = C T^{-1}\) is also symmetric where \(T\) is an orthogonal matrix, i.e. \(T^{-1} = T^T\).

**Theorem 3.2.2.** A system with symmetric transfer function \(G(s)\) has a symmetric state-space realization if and only if it has a diagonal realization \(\{A, B, C\}\) with \(A = \text{diag}\{\lambda_i; i = 1, 2, \ldots, n\}\) and \(B = C^T\).

**Theorem 3.2.3.** Let \(G(s)\) be an \(m \times m\) symmetric transfer function matrix with distinct poles given by

\[
G(s) = \frac{N(s)}{d(s)}
\]  

(3.74)

where \(d(s)\) is the least common multiple of the denominators of the entries with partial fraction expansion

\[
G(s) = D + \sum_{i=1}^{r} \frac{W_i}{s - \lambda_i}
\]  

(3.75)
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Suppose \( \text{rank } W_i = k_i \) and let \( B_i \in \mathbb{R}^{k_i \times m} \) and \( C_i \in \mathbb{R}^{m \times k_i} \) be two constant matrices such that \( W_i = C_iB_i \), where \( C_i = B_i^T \). Then \( G(s) \) has a symmetric Gilbert realization with \( A = \text{diag} \{ \lambda_i I_{k_i} \} \).

The proofs of the above theorems can be established by construction and they are omitted for brevity.

Combining passivity and symmetry, we have the following result.

**Lemma 3.2.3.** Assume \( G(s) \) be passive and symmetric and let \( \{ A, B, C, D \} \) be a minimal realization of \( G(s) \). Then there exists a solution \( P = P^T > 0 \) of \( (3.67) \) such that \( P = T_s P^{-1} T_s \), where \( T_s \) is defined as in Lemma 3.2.2.

### 3.3 Positive Symmetric Systems

Now, we are ready to introduce the class of continuous-time symmetric positive systems or symmetric Metzlerian systems, which has a direct tie to systems with passivity and symmetry properties discussed in previous section. In this section, we extend the conventional definition of positive systems [3] to positive symmetric systems.

We consider two classes of symmetric positive systems. The first class is a system with a symmetric positive structure, i.e. \( A = A^T \) is a Metzler matrix and \( B = C^T \geq 0 \) or a symmetric transfer function matrix that has a positive symmetric realization. Using the stability properties of this class, one can perform symmetric positive stabilization of a system regardless of being positive symmetric or not.

The second class is a generalized symmetric system which is defined through a block controllable canonical form in which the block submatrices are Metzlerian symmetric. This class of system appear in a natural way by electromechanical system, which are constructed with components that manifest a combination of inertial, compliant, and dissipative effects. For example, a lumped model of a drive train containing three flywheels, two dash pots and two spring can be written as \( J\ddot{\theta} + D\dot{\theta} + S\theta = \tau \) where \( \theta \) is a vector of angular displacement, \( \tau \) is a column vector representing the torque applied to the flywheel, \( J \) is a diagonal inertia matrix, and \( D \) and \( S \) are symmetric damping and stiffness matrices, respectively. In a similar fashion one encounters electromechanical systems described by \( M\ddot{x} + D\dot{x} + Sx = u \) where \( M, D, S \) are symmetric matrices of mass, dashpot friction and spring constant, and \( u \) is the forcing vector function. Note that the coefficient matrices not only are symmetric but also admit special structure of M-matrices.
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Definition 3.3.1. The system \( (3.63), (3.64) \) is called symmetric Metzlerian system if and only if it is simultaneously symmetric as defined in Definition 3.2.2 and Metzlerian.

Theorem 3.3.1. The system \( (3.63), (3.64) \) is internally symmetric positive (symmetric Metzlerian) if and only if

\[
A \text{ is a symmetric Metzler matrix, } B = C^T \in \mathbb{R}^{n \times m}, \text{ and } D = D^T \in \mathbb{R}^{m \times m}.
\]

Remark 3.3.1. The passivity, symmetry, and positivity have an interesting connection which is subject of further investigation. One immediate connection is the relationship between relaxed system in which the impulse response is a completely monotonic function as it is in the case of positive system. For this class of system the state-space symmetric realization has the property that

\[
A = A^T \prec 0, \quad B = C^T, \quad \text{and } D = D^T
\]

whereby the system is stable with Hankel matrix satisfying \( H = H^T \succeq 0 \).

Many dynamical systems are modeled by a second or higher order vector differential equations of the form

\[
\sum_{j=0}^{r} A_{r-j} \frac{d^j z(t)}{dt^j} = u(t)
\]

where \( z(t) \in \mathbb{R}^m, A_j \in \mathbb{R}^{m \times m} \) for \( j = 0, 1, \ldots, r \) with \( A_0 = I_m \) and \( u(t) \in \mathbb{R}^m \). This type systems can be realized into Block Controllable Canonical Form (BCCF)

\[
\dot{x}(t) = Ax(t) + Bu(t) \quad y(t) = Cx(t)
\]

where

\[
A = \begin{bmatrix}
O_m & I_m & O_m & \cdots & O_m \\
O_m & O_m & I_m & \cdots & O_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O_m & O_m & O_m & \cdots & I_m \\
-A_r & -A_{r-1} & -A_{r-2} & \cdots & -A_1
\end{bmatrix},
B = \begin{bmatrix}
O_m \\
\vdots \\
O_m \\
I_m
\end{bmatrix},
C = \begin{bmatrix}
C_0 & C_1 & C_2 & \cdots & C_{r-1}
\end{bmatrix}
\]

(3.78)

with the state vector \( x = \begin{bmatrix} z \, \dot{z} \, \ldots \, z^{(r-1)} \end{bmatrix}^T \in \mathbb{R}^n, C_j \in \mathbb{R}^{m \times m}, \) and \( n = rm \). The associated polynomial matrix of \( (3.76) \) is given by

\[
P(s) = \sum_{j=0}^{r} A_{r-j} s^j
\]

(3.79)

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**Definition 3.3.2.** The BCCF (3.78) is called Metzlerian BCCF if and only if $-A_i$’s are Metzlerian matrices. Furthermore, it is called symmetric Metzlerian BCCF if and only if $-A_i$’s are symmetric Metzlerian matrices.

The poles of the system (3.77) are the latent roots of the polynomial matrix $P(s)$ defined as $\lambda(P) = \{s \in \mathbb{C} : \det P(s) = 0\}$. This is the same as the spectrum of the matrix $A$ since $\det(P(s)) = \det(\lambda I - A)$, i.e. $\lambda(P(s)) = \lambda(A)$. Furthermore, the system (3.77) is stable if all eigenvalues of the matrix $A$ or equivalently all latent roots of $P(s)$ lie in the open left half of s-plane.

The connection between the stability of the polynomial matrix $P(s)$ and the matrix $A$ plays an important role. In particular, if in the expansion (3.79) associated with (3.76) $A_0 = I_m$, then there is a one to one correspondence between the coefficient matrices of (3.79) and the block companion structure of the matrix $A$. However, if $A_0 \neq I_m$, then appropriate adjustment should be performed to find this correspondence.

The stability of single-input single-output systems can be analyzed by Ruth-Hurwitz algorithm through the coefficient of the characteristic polynomial $a_j$’s. However, the stability of dynamical systems modeled by (3.76) in terms of its coefficient matrices $A_j$ is not obvious. The best known results have been established only for second-order vector differential equation (see [52] and [53]). The following theorem is one possible stability result which we state for the special class of second-order vector differential systems with symmetric coefficient matrices.

**Theorem 3.3.2.** Let the dynamical system (3.76) be of second order, i.e.

$$A_0 \ddot{z} + A_1 \dot{z} + A_2 = u$$

(3.80)

where the coefficient matrices are symmetric with $A_0$ nonsingular. Then (3.80) is asymptotically stable if and only if $A_i$’s are nonsingular M-matrices or equivalently $-A_i$’s are stable Metzlerian matrices.

**Proof.** The second-order system (3.80) is asymptotically stability if $A_0 \succ 0$, $A_1 \succ 0$, and $A_2 \succ 0$ or $A_0^{-1}A_1 \succ 0$, and $A_0^{-1}A_2 \succ 0$. Now it is not difficult to show that these conditions are automatically satisfied if and only if $A_i$’s are nonsingular M-matrices. ■
Chapter 4

Positive Stabilization of Dynamic Systems

This chapter considers the problem of constrained stabilization of linear continuous-time systems by state feedback control law. The goal is to solve this problem under positivity constraint which means that the resulting closed-loop systems are not only stable, but also positive. We focus on the class of linear continuous-time positive systems (Metzlerian systems) and use the interesting properties of Metzler matrix discussed earlier. First, some necessary and sufficient conditions are presented for the existence of controllers satisfying the Metzlerian constraint, and the constrained stabilization is solved using linear programming (LP) or linear matrix inequality (LMI). A major objective is to formulate the constrained stabilization problem with the aim of maximizing the stability radius. We show how to solve this problem with an additional LMI formulation.

4.1 Metzlerian Stabilization

Let the state feedback control law of the form

\[ u(t) = v + Kx \quad (4.1) \]

be applied to the system (3.1). Then we get the following closed-loop system

\[ \dot{x} = (A + BK)x + Bv \quad (4.2) \]

Thus, in the first stage of our design procedure we need to find \( K \in \mathbb{R}^{m \times n} \) such that \( A + BK \) is a Metzler matrix and \( A + BK \) is a Hurwitz stable matrix. There are many ways to achieve this goal by
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using the equivalent conditions of Lemma 3.1.1. For example, using the property 3 that the leading
principal minors of \( M = -(A + BK) \) to be positive i.e. \( |M(\alpha)| > 0 \) where \( \alpha \in \mathbb{N} \) defines the order
of the minors and \( M_{ii} > 0, M_{ij} \leq 0 \), one can find the gain matrix \( K \) through a linear programming
(LP) set-up [6]. Alternatively, one can construct an LP by using the property 6 of Lemma 3.1.1
applied to \( A + BK \) as outlined in the following theorem, which is more compact [17, 54].

Theorem 4.1.1. There exist a state feedback control law (4.1) for the system (3.1) such that the
closed-loop system (4.2) becomes strictly Metzlerian stable if the following LP has a feasible solution
with respect to the variables \( w = \left[ \begin{array}{c} w_1 \\ w_2 \\ \vdots \\ w_n \end{array} \right] \in \mathbb{R}^n \) and \( z_i \in \mathbb{R}^m, \forall i = 1, \ldots, n \)

\[
Aw + B \sum_{i=1}^{n} z_i < 0, \quad w > 0 \tag{4.3}
\]

\[
a_{ij}w_j + b_i z_j \geq 0 \quad \text{for } i \neq j \tag{4.4}
\]

\[
a_{ii}w_i + b_i z_i < 0 \tag{4.5}
\]

with \( A = [a_{ij}] \) and \( B = \left[ \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right]^T \). Furthermore, the gain matrix \( K \) is obtained from

\[
K = \left[ \begin{array}{ccc} \frac{z_1}{w_1} & \frac{z_2}{w_2} & \cdots & \frac{z_n}{w_n} \end{array} \right] \tag{4.6}
\]

Proof. Imposing the structural constraint of Metzler matrix for \( A + BK \) implies that for \( i \neq j \)
we have \((A + BK)_{ij} = a_{ij} + b_j K_j = a_{ij} + b_j \frac{z_j}{w_j} \geq 0 \) and since \( w_j > 0 \), it leads to (4.4). Similarly, one can construct the strict Metzlerian diagonal condition (4.5). The equivalent stability condition \((A + BK)w < 0 \) for a positive vector \( w > 0 \) can be written as \( Aw + BKw < 0 \) or 
\( Aw + B \sum_{i=1}^{n} z_i < 0 \) with the aid of (4.6), which is (4.3). \( \blacksquare \)

The following theorem uses condition 5 of Lemma 3.1.1 along with structural constraint of
Metzler matrix.

Theorem 4.1.2. There exist a state feedback control law (4.1) for the system (3.1) such that the
closed-loop system (4.2) becomes strictly Metzlerian stable if the following LMI has a feasible solution
with respect to the variables \( Y \) and \( Z \)

\[
Z A^T + Y^T B^T + AZ + BY \prec 0 \tag{4.7}
\]

\[
(AZ + BY)_{ij} \geq 0 \quad \text{for } i \neq j \tag{4.8}
\]

\[
(AZ + BY)_{ii} < 0 \tag{4.9}
\]

where \( Y \succ 0 \) and \( Z \succ 0 \) is diagonal positive definite matrix.
Proof. Let $K$ yet to be determined such that $A + BK$ is a Metzler and stable matrix. Then using Corollary 3.1.3 the Lyapunov inequality
\[ Z(A + BK)^T + (A + BK)Z < 0 \] (4.10)
must have a positive definite diagonal solution for $Z$ and the matrix $K$ such that (4.10) is satisfied with its off-diagonal elements being non-negative. Since $Z > 0$ is diagonal, this condition holds if and only if all the off-diagonal entries of $(A + BK)Z$ are non-negative. Therefore, with the change of variable $Y = KZ$, the asymptotic stability of the closed-loop Metzlerian system is equivalent to the LMI (4.7) along with the structural constraints (4.8) and (4.9), which are similarly obtained from $(A + BK)_{ij} \geq 0$ and $(A + BK)_{ii} < 0$ by multiplying both sides with $Z$ and using the same change of variable $Y = KZ$. ■

4.2 Maximizing the stability radius by state feedback

In this section we show how to use the Lemma 3.1.1 in maximizing the stability radius by state feedback. Let the uncertain closed-loop system be written as
\[ \dot{x} = \underbrace{(A + BK) + E\Delta F}_{A_c}x \] (4.11)
Then, we seek to find a feedback controller such that the stability radius of the closed-loop system is maximized. Applying Lemma 3.1.1 for the closed-loop system (4.11) we need to solve the following problem
\[ \max_k r = \frac{1}{\|F(A + BK)^{-1}E\|} \] (4.12)
subject to LMI constraint (4.7) - (4.9). However, it is not convenient to solve the above optimization problem.

To elaborate on this, let us consider the case of unstructured stability radius i.e. $E = F = I$. This makes it possible to simplify the above optimization problem. Let $A_c(K) = A + BK$, then the objective function (4.12) reduces to
\[ \max_k r(A_c) = \min_k \|A_c^{-1}(k)\| = \min_k \frac{1}{\sigma_{\min}(A_c(k))} = \max_k [\sigma_{\min}(A_c(k))] = \sqrt{\max_k \{\lambda_{\min}[A_c^T(k)A_c(K)]\}} \] (4.13)
Thus, we need to solve the problem of maximizing the smallest eigenvalue $\lambda_{\text{min}}$ of a symmetric matrix $A_s(k) = A_c^T(k)A_c(k)$, which can be reformulated in terms of an auxiliary variable $\alpha$

$$\max_{k} \alpha \quad \text{subject to} \quad \lambda_i[A_s(k)] \geq \alpha; \quad i = 1, 2, \ldots, n$$

(4.14)

The strategy is to bound the spectrum of $A_s(k)$ from below and to maximize the lower bound $\alpha$. Since $A_sv = \lambda v$ implies $(A_s - \alpha I)v = (\lambda - \alpha)v$, the condition $\lambda_i - \alpha > 0 (\lambda_i > \alpha)$ dictates that $A_s - \alpha I > 0$ or $A_c^T(k)A_c(k) - \alpha I > 0$. Therefore (4.14) is equivalent to

$$\max_{k} \alpha \quad \text{subject to} \quad (A + BK)^T(A + BK) - \alpha I > 0$$

and combining the stability condition (4.10) along with the condition imposed by the Metzlerian structure $(A + BK)_{ij} \geq 0, (A + BK)_{ii} < 0$ lead to a nonlinear optimization problem. Note that the positive definite condition of a symmetric matrix $A_s(k) - \alpha I > 0$ can be replaced by its successive principal minors $\det \{ [A_s(k) - \alpha I] \}_{i} > 0$ for all $i = 1, \ldots, n$. Although, we still have a nonlinear programming problem, the determinants are themselves smooth and facilitate the solution process with the standard and efficient gradient based algorithm.

Realizing the computational burden encountered in solving the above optimization problem, we propose to solve it as follows. Since the real and complex stability radii of Metzlerian systems coincide, we avoid using the expression derived for real stability radius in the above optimization problem. Instead, we use the complex stability radius, which can conveniently be reformulated in terms of LMI.

**Theorem 4.2.1.** The complex stability radius of the controlled perturbed system with $G(s) = F(sI - A_c)^{-1}E$ is given by

$$r_C(A_c, E, F) = \left[ \max_{s \in \delta C_+} \| G(s) \| \right]^{-1}$$

(4.15)

where $\delta C_+$ is the boundary of $C_+$ (the closed right half of $s$ plane).
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With the aid of bounded real Lemma [55], the complex stability radius of the controlled system is the inverse of the $H_\infty$ norm of $G(s)$ and can be recast as the following optimization problem.

$$\min \gamma$$

subject to

$$\begin{bmatrix} A_TP_c + P_cA_c & P_cE & F^T \\ ETP_c & -\gamma I & 0 \\ F & 0 & -\gamma I \end{bmatrix} < 0$$

(4.16)

with the variables $P_c = P_c^T > 0$, $K$, and $\gamma$ where $A_c = A + BK$.

Note that the bounded real Lemma is employed here for (4.11), which can be regarded as a system with state space parameters $\{A_c, E, F\}$ and stable matrix $A_c$. Thus, the equivalent of $\|F(sI - A_c)^{-1}E\|_\infty < \gamma$ and the LMI (4.16) is evident. Furthermore, using the Schur complement one can write the LMI (4.16) as the following Riccati inequality

$$A_c^TP_c + P_cA + \gamma^{-1}(F^TF + P_cEE^TP_c) < 0, \quad \gamma > 0$$

(4.17)

which provides an alternative way to solve the optimization problem through its associated Hamiltonian.

In order to guarantee that the above optimization problem is formulated in terms of an LMI, we use the usual congruent transformation and pre- and post- multiply the inequality (4.16) by $\text{diag}\{Q_c, I, I\}$, $Q_c = P_c^{-1}$ and changing the variable $Y_c = KQ_c$.

Thus, the problem can alternatively be formulated in terms of the following LMI with respect to the variables $Q_c$ and $Y_c$.

$$\min \gamma$$

subject to

$$\begin{bmatrix} W_c & E & Q_cF^T \\ ET & -\gamma I & 0 \\ FQ_c & 0 & -\gamma I \end{bmatrix} < 0$$

(4.18)

where $W_c = Q_cA^T + Y_c^TB^T + AQ_c + BY_c$ and the controller gain can be obtained by $K = Y_cQ_c^{-1}$.

The above development can be summarized in the following theorem.
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Theorem 4.2.2. There exist a state feedback control law (4.1) for the system (3.1) such that the closed-loop system (4.2) becomes strictly Metzlerian stable with maximum stability radius if the LMI (4.18) along with the structural constraints

\[
(AQ_c + BY_c)_{ij} \geq 0 \quad i \neq j \quad (4.19)
\]

\[
(AQ_c + BY_c)_{ii} < 0 \quad (4.20)
\]

has a feasible solution with respect to the variable \(Y_c\) and \(Q_c\). Furthermore, the feedback gain is obtained by 

\[
K_1 = Y_c Q_c^{-1}
\]

4.3 Illustrative Examples

Example 4.3.1. Consider the following unstable MIMO system

\[
A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 1 & 3 \\ 2 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}
\]

with the eigenvalues \(\{3.7427, -1.8713 \pm i0.7112\}\). Assuming \(E_1\) and \(F_1\) are structure matrices defining perturbation given by

\[
E_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, F_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\]

Using the result of Theorem 4.2.2 the state feedback gain \(K_1\) which maximizes the stability radius of the closed-loop system is obtained as

\[
K_1 = \begin{bmatrix} -1.5606 & -0.5262 & 0.0006 \\ -0.4394 & -1.4744 & -2.9994 \end{bmatrix}
\]

and the close-loop system matrix \(A_{c1}\) becomes a Metzlerian stable with the eigenvalues located at \(\{-3.6445, -1.9988, -0.3905\}\), achieving the maximum stability radius of \(r_{max1} = 3\).

\[
A_{c1} = A + BK = \begin{bmatrix} -3.5606 & 0.4744 & 0.0006 \\ 0.5606 & -0.4744 & 0.0006 \\ 0 & 0 & -1.9988 \end{bmatrix}
\]

Next, we consider the same system with different structure matrices \(E_2\) and \(F_2\) given by

\[
E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T, F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
which leads to

\[
K_2 = \begin{bmatrix}
-1.2328 & 0.4764 & 0.0007 \\
-0.7672 & -2.4764 & -2.9993
\end{bmatrix}
\]  

(4.26)

with \( r_{\text{max}2} = 2.1213 \).

Finally, considering structure matrices \( E_3 = F_3 = I \), the unstructured stability radius is obtained as \( r_{\text{max}3} = 2 \) with the corresponding state feedback gain

\[
K_3 = \begin{bmatrix}
-1.2639 & 9.9100 & 0 \\
-0.7360 & -11.9095 & -3
\end{bmatrix}
\]  

(4.27)

It is interesting to perform a post robust stability analysis by considering a rank one perturbation as in case one on the last two Metzlerian stabilized systems. It turns out that for both cases the maximum parameter perturbation of \((1, 1)\) element that preserves stability is exactly 3 which is the stability radius of the first case.

**Example 4.3.2.** Consider the following example taken from [56],

\[
A = \begin{bmatrix}
-6.8101 & 2.1767 & 1.8666 \\
3.0130 & -4.2386 & 4.5973 \\
3.6283 & 5.6047 & 2.5210
\end{bmatrix}, B = \begin{bmatrix}
0.1240 \\
0.4318 \\
0.7602
\end{bmatrix}
\]  

(4.28)

with \( E = F = I \). Applying the two step procedure of LQR-based Metzlerian stabilization provided in [56] one can easily compute the feedback gain \( K = \begin{bmatrix}
-3.9256 & -6.3936 & -8.8368
\end{bmatrix} \) leading to the Metzlerian stable matrix \( A + BK \). The associated robust stability radius is obtained as \( r = 3.6832 \). Using the LMI based approach of Theorem 4.2.2 results in the feedback gain \( K = \begin{bmatrix}
-4.7728 & -7.3727 & -10.6468
\end{bmatrix} \) and the corresponding maximum stability radius becomes \( r_{\text{max}} = 5.5193 \). An important exercise is to investigate the robust stability radius of LQR by choosing proper \( Q \) and \( R \) matrices in order to achieve the maximum stability radius.
Chapter 5

Positive Observer Design for Positive Systems

Observers have found broad application in estimation and control of dynamic systems. A major advantage of observers is in disturbance estimation and fault detection. Among different observer structures, UIO and PIO are well-qualified candidates for this purpose. Although UIO and PIO are designed for standard linear systems, it is not obvious how to design these types of observers for the class of positive systems. Since the response of such systems to positive initial conditions and positive inputs should be positive, it makes sense to design positive observer for positive systems. It is well known that positive stabilization by state and output feedback for linear systems (regardless of being positive or not) is possible and various design techniques based on LP and LMI are proposed. So far, the design of positive observers was performed to estimate the states of positive systems. However, the available positive observer designs cannot be used to estimate the states of positive systems with unknown disturbances or faults.

The main goal of this chapter is twofold. First, we enhance the design of PUIO first introduced in [32] and show that it is possible to estimate the states of a positive system with a modified version of UIO without requiring strict positivity assumptions of certain design parameters. A subsequent step is also used to estimate the unknown input if desired. Second, we integrate PI observer structure with UIO and derive a procedure to design PIUIO for robust fault detection. Although, PI observer can not be constructed to estimate the states of positive systems, it is possible to take advantage of its capability to detect both constant and nonlinear faults. Design procedure with high PI gains are outlined through parametrized eigenvalue assignment and LMI for the case of
nonlinear faults. Finally, we extend the structure of PIUIO with a fading term and demonstrate that the same goal can be achieved with low proportional integral gain.

5.1 Problem Formulation and Previous Design Approaches

Consider the general linear time invariant system

\begin{align}
\dot{x}(t) &= Ax(t) + Bu(t) + E_d d(t) + E_a f_a(t) \\
y(t) &= Cx(t) + E_s f_s(t)
\end{align}

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( y \in \mathbb{R}^p \), \( d \in \mathbb{R}^r \), \( f_a \in \mathbb{R}^{q_a} \) and \( f_s \in \mathbb{R}^{q_s} \) are state, input, output, disturbance, actuator fault and sensor fault vectors; respectively and \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \), \( E_d \in \mathbb{R}^{n \times r} \), \( E_a \in \mathbb{R}^{n \times q_a} \) and \( E_s \in \mathbb{R}^{p \times q_s} \) are associated system matrices with \( \{A, B\} \) controllable and \( \{A, C\} \) observable pairs.

In the absence of disturbance or faults, a full order Luenberger observer for the above system is defined by

\begin{equation}
\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu
\end{equation}

Assuming the pair \( \{A, C\} \) is observable, it is always possible to find \( L \in \mathbb{R}^{n \times p} \) to guarantee the stability of \( A - LC \). Defining the error vector \( e = x - \hat{x} \), we have \( \dot{e} = (A - LC)e \) and \( \lim_{e \to \infty} e(t) = 0 \).

The situation with positive observers is different due to the structural constraint of the matrix \( A - LC \) to be Metzler. The design of positive observer makes only sense when (5.1),(5.2) is a positive system.

There are several established results based on LP and LMI to design positive observers. Here we state a result, which is the dual version of LMI-based positive stabilization [8] (see also [31]).

**Theorem 5.1.1.** Consider a positive system defined by (5.1), (5.2) without the disturbance and fault \( d(t) \), \( f_a(t) \) and \( f_s(t) \). Then there exists a positive observer of the form (5.3) if and only if the following LMI is feasible

\begin{align}
A^T P + PA - C^T Y^T - Y C &< 0 \\
A^T P - C^T Y^T + I &\geq 0 \\
Y C &\geq 0 \\
P &> 0
\end{align}
where $P \in \mathbb{R}^{n \times n}$ is a diagonal positive definite matrix and $Y \in \mathbb{R}^{n \times p}$.

The proof of this theorem can be found in the aforementioned references. Here we only emphasize that $A - LC$ requires to be stable Metzler matrix and $LC \geq 0$. The Metzler stability of $A - LC$ translates to the condition (5.4) with structural constraint (5.5). Furthermore, the positivity constraint of $LC$ is needed as shown in [13], which is equivalent to (5.6). Moreover, the gain matrix $L$ can be obtained as

$$L = P^{-1}Y$$ (5.8)

Two other distinct approaches have been reported in [29] and [30]. Unfortunately, none of these approaches are able to estimate the disturbances or faults appearing on positive systems.

### 5.2 Positive Observer Design for Systems with Known Disturbance Model

The previous section discussed a general framework for positive observer design when $d(t) = 0$, $f_a(t) = 0$ and $f_s(t) = 0$. In this section we focus on faultless system with disturbance, i.e. $d(t) \neq 0$ but $f_a(t) = f_s(t) = 0$. Note that the disturbance needs to be positive in order to be estimated using a positive observer.

Assuming the disturbance model is known and positive, the states of the system and the disturbance can be estimated using a positive Luenberger observer for an augmented system formed by combining system and disturbance models.

Let the disturbance model be represented by a positive system as follows

$$\dot{d}(t) = M_d d(t)$$ (5.9)

where the matrix $M_d$ in (5.9) is assumed to be a Metzler matrix. Then, the augmented system can be constructed by combining disturbance and states of the system as follows.

$$\dot{x}_a = A_a x_a + B_a u$$ (5.10)

where $x_a = \begin{bmatrix} \dot{x} \\ d \end{bmatrix}$, $A_a = \begin{bmatrix} A & 0 \\ 0 & M_d \end{bmatrix}$ and $B_a = \begin{bmatrix} B \\ 0 \end{bmatrix}$. Since the augmented system (5.10) is a positive system (i.e. $A_a$ is a Metzler matrix and $B_a$ is nonnegative), the same positive Luenberger observer defined in Theorem 5.1.1 can be used to estimate the states and the disturbance. Note that the constant disturbance is just a special case where $M_d = 0$. 

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5.3 Positive Unknown Input Observer (PUIO)

In previous section we assumed the disturbance model is known, however, this is not the case in general. Unfortunately, when \( d(t) \neq 0 \), none of the available approaches are applicable for the design of a positive observer to estimate the states of a positive system (5.1), (5.2) decoupled from the unknown input \( d(t) \).

In this section we solve the problem of estimating the state of a positive system in the presence of a positive unknown disturbance. A tailored positive unknown input observer will be defined and used to achieve this goal.

It is well known that an UIO can be designed under the conditions \( \text{rank}(C) = p \), \( \text{rank}(E_d) = r \), where \( p \geq r \) and \( \text{rank}(CE_d) = \text{rank}(E_d) \). Let us define the UIO structure as it is usually represented for regular systems, i.e.

\[
\begin{align*}
\dot{z} &= Fz + Gy + Hu \\
\hat{x} &= Mz + Ny
\end{align*}
\]  

(5.11)

where for simplicity we assume \( M = I \). To facilitate the derivation of the design, we decompose \( G = G_1 + G_2 \) and let \( H = TB \) with \( T \) being a design parameter. This will be clarified in the next section. Note that the UIO only uses the available input and output in order to estimate the states.

**Definition 5.3.1.** The UIO (5.11) is called PUIO for positive system (5.1), (5.2), if for all initial state \( z(0) \in \mathbb{R}_+^n \) we have \( z(t) \in \mathbb{R}_+^n \) and \( \hat{x}(t) \in \mathbb{R}_+^n \) for all \( t \geq 0 \).

**Lemma 5.3.1.** Let the UIO (5.11) be an internally positive or Metzlerian system (PUIO). Then the PUIO is internally positive if and only if \( \tilde{A} \) is a Metzler matrix, \( G \in \mathbb{R}_+^{n \times p} \), \( H \in \mathbb{R}_+^{n \times m} \), and \( N \in \mathbb{R}_+^{n \times p} \) are nonnegative matrices.

**Proof.** It is a trivial consequence of Metzlerian system definition, Definition 5.3.1 and Lemma 3.1.1.

One of the key requirement in the design of PUIO is to guarantee that the generalized inverse of a certain design matrix to be positive. If \( \tilde{A} \) is a nonnegative matrix then its generalized inverse denoted by \( \tilde{A}^g \) is not necessarily nonnegative. For a square non-singular and nonnegative matrix \( \tilde{A} \), we have \( \tilde{A}^g = \tilde{A}^{-1} \geq 0 \) if and only if \( \tilde{A} \) is monomial (or generalized permutation matrix). Since a monomial matrix can be expressed as a product of a diagonal matrix and a permutation matrix, its inverse can be expressed as \( \tilde{A}^{-1} = DA^{-1}T \) for some diagonal matrix \( D \) with positive diagonal.
elements. The main goal here is to provide necessary and sufficient conditions for a nonnegative matrix \( \tilde{A} \geq 0 \) to have \( \tilde{A}^g \geq 0 \). This is required in the procedure of PUIO design. The following results from [4] provides a positive answer to this requirement.

**Lemma 5.3.2.** Let \( \tilde{A} \) be an \( m \times n \) nonnegative matrix of rank \( r \). Then the following statements are equivalent:

1. \( \tilde{A}^g \) is nonnegative.
2. There exists a permutation matrix \( \tilde{P} \) such that \( \tilde{P} \tilde{A} \) has the form \( \tilde{P} \tilde{A} = [\tilde{B}_1^T \ldots \tilde{B}_r^T 0]^T \), where each \( \tilde{B}_i \) has rank 1 and the rows of \( \tilde{B}_i \) are orthogonal to the rows of \( \tilde{B}_j \) for \( i \neq j \).
3. \( \tilde{A}^g = D \tilde{A}^T \) for some diagonal matrix \( D \) with positive diagonal elements.

With the aid of this result one can construct \( \tilde{A}^g \). Assuming 2 holds, let \( \tilde{B} = \tilde{P} \tilde{A} \) have the form specified above. Then for \( 1 \leq i \leq r \), there exists column vectors \( x_i, y_i \) such that \( \tilde{B}_i = x_i y_i^T \). Furthermore, \( \tilde{B}_i^g \) is the nonnegative matrix:

\[
\tilde{B}_i^g = (||x_i||^2 ||y_i||^2)^{-1} \tilde{B}_i^T
\]

and moreover \( \tilde{B}^g = (\tilde{B}_1^g, \ldots, \tilde{B}_r^g, 0) \), since \( \tilde{B}_i \tilde{B}_j = 0 \) for \( i \neq j \). In particular, \( \tilde{B}^g = D \tilde{B}^T \) where \( D \) is a diagonal matrix with positive diagonal elements and thus \( \tilde{A}^g = D \tilde{A}^T \).

### 5.3.1 Design of PUIO

Using Lemma 5.3.1, Lemma 5.3.2, and [32] the design of PUIO for estimating the states of positive systems decoupled from the disturbance can be established as follows.

**Theorem 5.3.1.** Consider positive system (5.1), (5.2) with the unknown input term \( d(t) \neq 0 \) but \( f_a(t) = f_s(t) = 0 \). Assuming the generalized left inverse of \( CE_d \) is nonnegative, there exists a PUIO of the form (5.11) if and only if \( F \) is a stable Metzler matrix and the following conditions are satisfied

\[
F = A - NCA - G_1 C \quad (5.12)
\]
\[
T = I - NC \geq 0 \quad (5.13)
\]
\[
G_2 = FN \quad (5.14)
\]
\[
G = G_1 + G_2 \geq 0 \quad (5.15)
\]
\[
H = TB \quad (5.16)
\]
\[
(\text{NC} - I)E_d = 0, \quad N \geq 0 \quad (5.17)
\]
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Proof. Using (5.1), (5.2) and (5.11) the error dynamics of the PUIO is easily derived as
\[
\dot{e} = (A - NA - G_1C)e + [F - (A - NA - G_1C)]z
+ [G_2 - (A - NA - G_1C)]y + [T - (I - NC)]Bu
+ (NC - I)E_d d
\]

It is straightforward to show that if (5.12) - (5.17) are satisfied, then \(\dot{e} = Fe\) and \(\lim_{t \to \infty} e(t) = 0\). Equivalently, choosing a Lyapunov function \(V(t) = e_t P e_t\) we have \(\dot{V}(t) = e_t Q e_t\) and \(Q = F_t P + PF \prec 0\) implies that \(e(t)\) tends to zero asymptotically for any initial value \(e(0)\). Note that when \(F\) is a stable Metzler matrix, then there exists a positive definite diagonal matrix \(P \succ 0\), which is also positive, satisfying the Lyapunov inequality.

The equation (5.17) is solvable if and only if the condition \(\operatorname{rank}(CE_d) = \operatorname{rank}(E_d) = r\) is satisfied provided that a nonnegative left inverse of \(CE_d\) exists and guarantees the nonnegativity of \(N\) from

\[
N = E_d(CE_d)^g + S[I - (CE_d)(CE_d)^g]
\]

where \(S \in \mathbb{R}^{n \times p}\) is an arbitrary matrix. For simplicity we assume \(S = 0\) and require that

\[
N = E_d(CE_d)^g \geq 0
\]

Since it is assumed that \((CE_d)^g \geq 0\), the matrix \(N\) becomes nonnegative. Furthermore, the nonnegativity of \(T\) in (5.13) is satisfied with \(N \geq 0\). If (5.19) fails to achieve this, (5.18) can be used with the free parameter \(S\). Consequently, the PUIO can be designed with the design procedure outlined below.

Design Procedure

1. Check the condition \(\operatorname{rank}(CE_d) = \operatorname{rank}(E_d) = r\) to ensure the existence of UIO.

2. Based on Lemma 5.3.2, if a nonnegative left inverse of \(CE_d\) exists, then compute \(N\) from (5.18) or (5.19) such that (5.13) is satisfied.

3. Define \(A_1 = A - NA\) in (5.12) such that \(\{A_1, C\}\) is observable or detectable.
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4. Solve the following LMI for $P \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{n \times p}$ with respect to $F = A_1 - G_1C$

$$
\begin{align*}
A_1^T P + PA_1 - C^T Y^T - YC &\prec 0 \\
A_1^T P - C^T Y^T + I &\geq 0 \\
Y + PA_1 N - YCN &\geq 0 \\
P &> 0
\end{align*}
$$

(5.20)

and obtain $G_1$ as follows

$$
G_1 = P^{-1} Y 
$$

(5.21)

5. Compute $T = I - NC$ and evaluate

$$
F = A_1 - G_1 C \\
G = G_1 + A_1 N - G_1 CN \\
H = TB
$$

To justify the above procedure, one requires to find $N \geq 0$, $G_1$, and $P$ such that

$$
F^T P + PF \prec 0 \\
(NC - I)E_d = 0
$$

(5.22)

(5.23)

with the constraint that $F = A_1 - G_1 C$ is a stable Metzler matrix. Note that we have $A_1 = (I - NC) A = TA$ from (5.13) and (5.14). The solution of (5.23) given by (5.19) specifies $A_1$ and substituting $F = A_1 - G_1 C$ into (5.22), we get

$$
A_1^T P + PA_1 - C^T Y^T - YC \prec 0
$$

(5.24)

where $Y = PG_1$. The structural constraint of a Metzler matrix is justified by

$$
A_1^T P - C^T Y^T + I \geq 0.
$$

(5.25)

Also note that (5.14) and (5.15) leads to

$$
G = G_1 + A_1 N - G_1 CN \geq 0
$$

(5.26)

which can be rewritten with the aid of $PG \geq 0$ as

$$
Y + PA_1 N - YCN \geq 0
$$

(5.27)
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5.3.2 Determination of Unknown Input:

The UIO and PIO are two types of observers that can be used to estimate the unknown disturbance or fault. In this paper we provided the design of an UIO for positive system, which we called PUIO, to estimate the states decoupled from the unknown input. Then it is possible to estimate the disturbance in a subsequent step based on PUIO design which can be determined from

\[
\dot{d} = (CE_d)^T \left[ \dot{y} - CA\hat{x} - CBu \right]
\]  

(5.28)

where

\[
\dot{y} = CA\hat{x} + CBu + CE_d\dot{d}
\]  

(5.29)

Remark 5.3.1. The structure of PUIO (5.11) was defined through Definition 5.3.1 and Lemma 5.3.1. However, it is possible to avoid the restriction of positivity on \(G\) in (5.11) and yet to obtain a positive estimate of states \(\hat{x}(t)\) as the following theorem asserts.

Theorem 5.3.2. Consider a positive faultless (i.e. \(f_a(t) = f_s(t) = 0\)) system defined by (5.1), (5.2) with \(d(t) \neq 0\) and let the generalized left inverse of \(CE_d\) be nonnegative. Then there exists an UIO of the form (5.11) such that \(\hat{x}(t) \in \mathbb{R}^n_+\) for all \(t \geq 0\), if and only if \(F\) is a stable Metzler matrix and

\[
F = A - NCA - G_1C
\]  

(5.30)

\[
T = I - NC
\]  

(5.31)

\[
G_2 = FN
\]  

(5.32)

\[
G = G_1 + G_2
\]  

(5.33)

\[
H = TB \geq 0
\]  

(5.34)

\[
(NC - I)E_d = 0, \quad N \geq 0
\]  

(5.35)

\[
NCA - FNC + GC \geq 0
\]  

(5.36)

Proof. Using the proof of Theorem 5.3.1, the observer dynamics can alternatively be written directly in terms of \(\hat{x}\) as

\[
\dot{\hat{x}} = F\hat{x} + (NCA - FNC + GC)x 
\]  

(5.37)

\[+(NCB + H)u + NCE_d\dot{d}\]

It is clear that (5.37) represents a positive system with (5.30) - (5.36).
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A quick comparison between (5.12)-(5.17) with (5.30)-(5.36) reveals that the positivity constraint of $T$ and $G$ are relaxed in Theorem 5.3.2. However, it is required that (5.34) to be positive.

Thus, the design procedure of PUIO can be modified by replacing step 4 with

$$
\begin{cases}
A^T_1 P + PA_1 - C^T Y^T - Y C \prec 0 \\
A^T_1 P - C^T Y^T + I \geq 0 \\
PNCA + Y C \geq 0 \\
P \succ 0 
\end{cases}
$$

(5.38)

Note that, the third inequality in (5.38) can easily be derived from the constraint $NCA - FNC + GC \geq 0$ in (5.36).

Remark 5.3.2. It is of particular interest to avoid condition (5.36) and yet to obtain the positivity of (5.37). Defining $\Gamma = NCA - FNC + GC$ and $\hat{F} = F + \Gamma$, it is not difficult to show through a monotonicity argument that as long as $\hat{F} > F$ is Metzlerian stable, positivity of (5.37) is guaranteed. This will be illustrated in Example 5.7.2.

5.4 Positive Observer for Faulty Systems

So far we have assumed that there is no fault in the system. In this section, we focus on a faulty system when there is no disturbance, i.e. $f_a(t) \neq 0$ and/or $f_s(t) \neq 0$ but $d(t) = 0$. It should be clear that in the absence of the sensor fault ($f_s = 0$), the actuator fault ($f_a \neq 0$) can be considered as an unknown input and we may use the same PUIO design procedure of previous section. Thus, let us concentrate on the case of positive system (5.1), (5.2) with $d = 0$, $f_a = 0$ and $f_s \neq 0$. There are two approaches that can be used to transfer the sensor fault problem to an equivalent actuator fault problem. In the first approach, we define an auxiliary positive system, which can be constructed as a filtered version of the output

$$\dot{v} = M(y - v) = -Mv + MCx + ME_s f_s$$

(5.39)

where $M$ is an arbitrary non-singular M-matrix such that $-M$ becomes a stable Metzler matrix, $MC \geq 0$, and $ME_s \geq 0$. Although one can select $M$ through an inverse eigenvalue problem, the simplest choice is a diagonal positive matrix, i.e. $M = \text{diag}\{m_i; i = 1, \ldots, p\}$, which obviously satisfies the positivity conditions $MC \geq 0$, $ME_s \geq 0$. Combining (5.39) with (5.1), (5.2), we get
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the following positive augmented system. Note that by using \([5.39]\) the sensor fault \(f_s\) appears as an actuator fault.

\[
\dot{x}_{aug} = A_{aug}x_{aug} + B_{aug}u + E_{aug}f_{aug} \tag{5.40}
\]

\[
y_{aug} = C_{aug}x_{aug} \tag{5.41}
\]

where \(x_{aug} = \begin{bmatrix} x^T & v^T \end{bmatrix}^T\) and \(f_{aug} = f_s\) with

\[
A_{aug} = \begin{bmatrix} A & 0 \\ MC & -M \end{bmatrix}, \quad B_{aug} = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{5.42}
\]

\[
C_{aug} = \begin{bmatrix} 0 & I \end{bmatrix}, \quad E_{aug} = \begin{bmatrix} 0 \\ ME_s \end{bmatrix}
\]

Now a PUIO can be designed using the approach described in Section 5.3 for the positive augmented system \([5.40], (5.41)\) in which \(f_{aug}\) is treated as an actuator fault. Thus, one can replace the parameters \(\{A, B, C, E_d\}\) by \(\{A_{aug}, B_{aug}, C_{aug}, E_{aug}\}\) when applying PUIO design procedure.

In the second approach, the function \(f_s\) is unknown but it is assumed to be bounded and smooth. The objective is to design a PUIO to reconstruct the fault \(f_s\) using only \(y(t)\) and \(u(t)\). Thus, let us define

\[
\phi(t) := \dot{f}_s(t). \tag{5.43}
\]

where the sensor faults are considered incipient \([21]\) and hence \(\|\phi(t)\|\) is small but overtime the effects of the fault increment and become significant. Without loss of generality, we assume that the output of the system is ordered as follows

\[
E_s = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{5.44}
\]

Now combining \([5.43]\) with \([5.1], (5.2)\) leads to an augmented system of the form

\[
\dot{z} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} \phi \tag{5.45}
\]

\[
y = \begin{bmatrix} C & E_s \end{bmatrix} z \tag{5.46}
\]

where \(z = \begin{bmatrix} x^T & f_s^T \end{bmatrix}^T\).
Then one can apply the PUO for the above augmented system driven by the unknown signal \( \phi(t) \) provided that the augmented system is observable. It can be shown that the augmented system is observable if the pair \((A, C_1)\) does not have an unobservable mode at zero or if the open-loop system is stable.

Finally, we consider the case where both actuator and sensor faults appear in the system, i.e. \( f_a \neq 0 \) and \( f_s \neq 0 \). Applying the above filtering approach, the actuator fault and sensor fault will be combined in \( f_{aug} = \begin{bmatrix} f_a^T & f_s^T \end{bmatrix}^T \) and the augmented system described in (5.40), (5.41) can be adjusted by simply replacing \( E_{aug} \) with

\[
E_{aug} = \begin{bmatrix} E_a & 0 \\ 0 & ME_s \end{bmatrix}
\]

(5.47)

where \( A_{aug}, B_{aug}, \) and \( C_{aug} \) remains the same. Again the PUO can be designed using the approach described in Section 5.3 for the adjusted positive augmented system.

5.5 PI Observer Design

5.5.1 PI Observer for General Linear Systems

A PI observer for the standard linear system (5.1), (5.2) can be defined by [20].

\[
\dot{\hat{x}} = (A - LP)\hat{x} + Bu + LPy + Ed\hat{d}
\]

(5.48)

\[
\dot{\hat{d}} = LI(y - C\hat{x})
\]

(5.49)

where \( \hat{x} \) is the state estimate and \( \hat{d} \) denotes the estimate of the disturbance or fault.

To construct the PI observer (5.48), (5.49) the following assumptions are necessary.

**Assumption 5.5.1.** \( \text{rank}(C) = p, \text{rank}(Ed) = r, \) where \( p \geq r \) and \( \text{rank}(CE_d) = \text{rank}(Ed) \).

**Assumption 5.5.2.** For every \( \lambda \) with nonnegative real part

\[
\text{rank} \begin{bmatrix} A - \lambda I & Ed \\ C & 0 \end{bmatrix} = n + r
\]

Defining the estimation error of state and disturbance vectors as \( e = \hat{x} - x \) and \( \epsilon = \hat{d} - d \), the error dynamics of the extended system becomes

\[
\begin{bmatrix} \dot{e} \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} A - LP C & Ed \\ -LI C & 0 \end{bmatrix} \begin{bmatrix} e \\ \epsilon \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{d} \end{bmatrix}
\]

(5.50)
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Let us first assume that the disturbance \( d \) is an unknown constant. Thus, \( \dot{d} = 0 \) in (5.50). The case of general nonlinear disturbance \( d \) will be discussed in Section 5.6.

**Theorem 5.5.1.** Let the system (5.7), (5.2) without fault be positive and observable satisfying the Assumptions 5.5.1 and 5.5.2. Assuming that \( \dot{d}(t) \) is an unknown constant disturbance, then there exists a PI observer (5.48), (5.49) such that \( \lim_{t \to \infty} e(t) = 0 \) and \( \lim_{t \to \infty} \varepsilon(t) = 0 \) for any initial conditions \( x(0), \dot{x}(0) \) and \( \hat{d}(0) \). Furthermore, the gains of PI observer can be obtained by applying any eigenvalue assignment technique to \( A_x - L_x C_x \) where

\[
A_x = \begin{bmatrix} A & E_d \\ 0 & 0 \end{bmatrix}, \quad L_x = \begin{bmatrix} L_P \\ L_I \end{bmatrix}, \quad C_x = \begin{bmatrix} C & 0 \end{bmatrix}
\]

or equivalently through the feasible solution of the following LMI

\[
A_x^T P_x - C_x^T G_x^T + P_x A_x - G_x C_x \prec 0 \quad (5.51)
\]

where \( G_x = P_x L_x \).

**Proof.** It is constructive and is omitted for brevity.

\[\blacksquare\]

5.5.2 PI Observer for Positive Linear Systems

In section 5.1, we have shown how to design a proportional positive observer with Luenberger structure for positive systems using LMI as obtained in Theorem 5.1.1. It is obvious that a PI observer should also be positive when it is designed for a positive system. However, in this section we prove that it is impossible to design positive PI observers with structure (5.48), (5.49) for positive systems.

**Theorem 5.5.2.** Let the system (5.1), (5.2) without fault be positive and observable satisfying assumptions 5.5.1 and 5.5.2. Then a positive PI observer (5.48), (5.49) does not exist for positive system (5.1), (5.2).

**Proof.** Constructing the augmented system consisting of positive system and the PI observer yields

\[
\begin{bmatrix}
\dot{x} \\
\dot{\hat{x}} \\
\dot{\hat{d}}
\end{bmatrix} = \begin{bmatrix}
A & 0 & 0 \\
L_P C & A - L_P C & E_d \\
L_I C & -L_I C & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\dot{x} \\
\hat{d}
\end{bmatrix}
+ \begin{bmatrix}
B \\
0
\end{bmatrix} u + \begin{bmatrix}
E_d \\
0
\end{bmatrix} d
\]

(5.52)
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Since $L_1C$ term appears with opposite signs in (5.52), it violates the Metzlerian structure.

Although positive PI observer cannot be constructed for positive systems, one can use it to estimate the unknown input or fault, even though the state estimate may leave the nonnegative orthant. Let us formally state this in the following theorem.

**Theorem 5.5.3.** Let the system (5.1), (5.2) be positive and observable satisfying Assumptions 5.5.1 and 5.5.2. Then the disturbance or fault of positive system (5.1), (5.2) can be estimated by a non-positive PI observer (5.48), (5.49). Furthermore, the gains of PI observer can be obtained by using the LMI formulation of Theorem 5.5.1.

Theorem 5.5.1 is stated for constant disturbance estimation by using PI observer. However, it is possible to reconstruct the states of the system (5.1), (5.2) as well as the nonlinear disturbances or faults by using a PI observer with high proportional and integral gains [39, 57, 58]. A simple procedure to design high gain PI observer is to define $C_x$ with an adjustable parameter $\rho$ as $C_x = \begin{bmatrix} \rho C & 0 \end{bmatrix}$ and apply a parametrized eigenvalue assignment procedure to the pair $(A_x, C_x)$. A satisfactory transient response is achieved by increasing $\rho$ to an acceptable level. Alternatively, one can also formulate a parametrized version of LMI (5.51) and solve the problem.

5.6 Robust Fault Detection for Positive Systems using PIUIO

Consider a positive observable system given by (5.1), (5.2) with fault and disturbance, $d(t) \neq 0$, $f_a(t) \neq 0$ but $f_s(t) = 0$. To decouple the unknown disturbance from the estimated fault, we employ a robust fault detection procedure by combining two strategies. The unknown input observer strategy is used for disturbance decoupling and the PI observer capability achieves the fault detection. By defining a new output $\tilde{y} = \dot{y} - CBu$ and $\tilde{C} = CA$, an extended PI observer (PIUIO) can be constructed as follows

\[
\dot{x} = A\dot{x} + N(\tilde{y} - \tilde{C}\dot{x}) + G_1(y - C\dot{x}) + Bu + J\dot{f}_a
\]

\[
\dot{f}_a = L(y - C\dot{x}) + R\dot{f}_a
\]

where $G_1$ and $N$ are associated gains with respect to $y$ and $\tilde{y}$.

**Theorem 5.6.1.** Consider a positive system defined by (5.1), (5.2) with unknown disturbance $d(t) \neq 0$ and constant actuator fault $f_a(t)$. Assume $R = 0$ and let the Assumptions 5.5.1 and 5.5.2 be satisfied.
with respect to both disturbance and fault. Then there exists a PIUIO of the form
\[
\dot{z} = Fz + Gy + Hu + J\hat{f}_{a} \tag{5.55}
\]
\[
\dot{w} = M_{1}z + M_{2}y \tag{5.56}
\]
if and only if the following conditions are satisfied
\[
\hat{F} = \begin{bmatrix} A_{1} - G_{1}C & J \\ M_{1} & 0 \end{bmatrix} \quad \text{Hurwitz Stable} \tag{5.57}
\]
\[
F = A_{1} - G_{1}C \tag{5.58}
\]
\[
T = I - NC \tag{5.59}
\]
\[
G_{2} = FN \tag{5.60}
\]
\[
G = G_{1} + G_{2} \tag{5.61}
\]
\[
H = TB \tag{5.62}
\]
\[
J = TE_{a} \tag{5.63}
\]
\[
(NC - I)E_{d} = 0 \tag{5.64}
\]
where \(A_{1} = A - NC A\), \(M_{1} = -LC\), and \(M_{2} = L(I - CN)\).

Proof. Using (5.53), (5.54) and defining \(e = \hat{x} - x\) and \(\varepsilon = \hat{f}_{a} - f_{a}\), the error dynamics of the PIUIO is easily derived as
\[
\dot{e} = [A_{1} - G_{1}C]e - [A_{1} - G_{1}C - F]z \\
- [(A_{1} - G_{1}C)N - G_{2}]y + [(NC - I)E_{d}]d \\
-[((I - NC) - T)B]u - [(I - NC)E_{a}]f_{a} + J\hat{f} \tag{5.65}
\]
\[
\dot{\varepsilon} = M_{1}e \tag{5.66}
\]
Defining \(\tilde{\varepsilon} = [e^{T} \varepsilon^{T}]^{T}\), if the conditions (5.57)-(5.64) are satisfied then \(\dot{\tilde{\varepsilon}} = \hat{F}\tilde{\varepsilon}\) and \(\lim_{t \to \infty} \tilde{\varepsilon}(t) = 0\). Note that when \(\hat{F}\) is a stable matrix, then there exists a positive definite matrix \(\hat{P} > 0\) satisfying the Lyapunov inequality \(\hat{F}^{T}\hat{P} + \hat{P}\hat{F} < 0\).

\[\blacksquare\]

**Design Procedure**

1. Check the Assumptions 5.5.1 and 5.5.2 in terms of both \(E_{d}\) and \(E_{a}\).

2. Compute \(N\) form (5.19) without requiring to be nonnegative. Then compute \(T\) from (5.59) and determine \(H\) and \(J\) from (5.62) and (5.63) respectively.
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3. Define $A_1 = A - NC$ such that $\{\hat{A}_1, \hat{C}\}$ is observable or detectable where

$$
\hat{A}_1 = \begin{bmatrix} A_1 & J \\ 0 & 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C & 0 \end{bmatrix}
$$

(5.69)

4. Obtain the gains $G_1$ and $L$ by solving an eigenvalue assignment technique applied to $\hat{A}_1 - \hat{L}\hat{C}$ where

$$
\hat{L} = \begin{bmatrix} G_1 \\ L \end{bmatrix}
$$

or through the following LMI

$$
\hat{A}_1^T \hat{P} - \hat{C}^T \hat{G}^T + \hat{P} \hat{A}_1 - \hat{G} \hat{C} \prec 0
$$

(5.70)

where $\hat{G} = \hat{P} \hat{L}$.

5. Compute $F$ from (5.58) and obtain $G_2$ from (5.60).

6. Construct the observer from (5.55), (5.56).

The above theorem considered constant fault estimation by using PIUIO. However, to detect nonlinear fault decoupled from unknown disturbance, we refer back to the general structure (5.53), (5.54) with its fading term $R \neq 0$. In this case, the error dynamics becomes

$$
\begin{bmatrix} \dot{e} \\ \dot{\varepsilon} \end{bmatrix} = \begin{bmatrix} A_1 - G_1 C & J \\ -LC & R \end{bmatrix} \begin{bmatrix} e \\ \varepsilon \end{bmatrix} + \begin{bmatrix} 0 \\ R \end{bmatrix} f_a - \begin{bmatrix} 0 \\ I \end{bmatrix} \dot{f}_a
$$

(5.71)

The Laplace transform of the expression (5.71) can be written as

$$
e(s) = G_P(s)J\varepsilon(s)
$$

(5.72)

$$
\varepsilon(s) = - \left[ sI + LCG_P(s)J + R \right]^{-1} (sI - R) f_a(s)
$$

(5.73)

where $G_P(s) = [sI - (A_1 - G_1 C)]^{-1}$.

To minimize the effect of unknown fault on the estimation errors $e(t)$ and $\varepsilon(t)$, we require

$$
\| (sI + LCG_P(s)J + R)^{-1} \|_\infty < \gamma, \quad \gamma \to 0
$$

This leads to the condition that $\|L\| \gg \|G_1\|$ when designing the stable PIUIO.

One can use high gain $L$ by defining $\bar{J} = \eta J = \eta TD$ and let $\eta \gg 1$. It is evident that increasing $L$ is equivalent to the increase of $\bar{J}$. This can be realized by an eigenvalue assignment to $A_x - L_x C_x$ where

$$
A_x = \begin{bmatrix} A & \eta J \\ 0 & R \end{bmatrix}, \quad C_x = \begin{bmatrix} C & 0 \end{bmatrix}
$$

(5.74)
by increasing $\eta$ such that the satisfactory performance is achieved.

The additional fading term in PI observer allows to improve its stability properties. This is due to the extra degrees of freedom offered by the fading gain $R$ in the augmented matrix $A_x - L_x C_x$. Furthermore, the fading term can be tuned so that the effect of transient on the integral action decays over time. Faster suppression of transients is achieved by increasing $R$. However, this increase should not oppose the effect of integral action.

## 5.7 Illustrative Examples

**Example 5.7.1.** Consider the following third order observable positive system in which actuator fault appears as follow.

\[
\dot{x} = \begin{bmatrix}
-1 & 1 & 1 \\
1 & -2 & 0 \\
0 & 1 & -3
\end{bmatrix} x + \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} u + \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} f_a
\]

\[
y = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} x
\]

Theorem 5.3.2 can be applied to find an UIO observer of the form (5.11) which can estimate states and the fault simultaneously. Following the design procedure discussed in Section 5.3, we construct a positive matrix $N$ and obtain $T$ such that $TE_a = 0$,

\[
N = E_a (C E_a)^g = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \geq 0
\]

\[
T = I - NC = \begin{bmatrix}
0 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Defining $A_1$ using step 3, we obtain

\[
A_1 = (I - NC)A = \begin{bmatrix}
0 & -1 & 3 \\
1 & -2 & 0 \\
0 & 1 & -3
\end{bmatrix}
\]
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Solving the LMI in step 4, we obtain $P$ and $Y$ as

$$P = \begin{bmatrix} 0.44 & 0 & 0 \\ 0 & 0.27 & 0 \\ 0 & 0 & 0.26 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.96 & -0.68 \\ 0 & 0.26 \\ 0 & 0.10 \end{bmatrix}$$

Then compute the remaining observer parameters as

$$G_1 = P^{-1}Y = \begin{bmatrix} 2.20 & -1.56 \\ 0 & 0.94 \\ 0 & 0.41 \end{bmatrix}$$

$$F = A_1 - G_1C = \begin{bmatrix} -2.20 & 0.56 & 0.80 \\ 1 & -2.94 & 0 \\ 0 & 0.59 & -3 \end{bmatrix}$$

$$G_2 = FN = \begin{bmatrix} -2.20 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$G = G_1 + G_2 = \begin{bmatrix} 0 & -1.56 \\ 1 & 0.94 \\ 0 & 0.41 \end{bmatrix}, \quad H = TB = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

which leads to the desired UIO (5.11). The state responses of this system and their estimates are depicted in Fig. 5.1 along with the decoupled actuator fault estimation. Note that not only $H$ is positive but also the term $NCA - FNC + GC$ is a positive matrix which means all conditions of Theorem 5.3.2 are satisfied and the UIO designed here can be used for positive system as shown.

Example 5.7.2. Consider again the system in Example 5.7.1 but this time with both actuator and sensor fault as follows,

$$\dot{x} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} f_a$$

$$y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} f_s$$
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According to the final augmentation approach discussed in Chapter 5.4, we can form the augmented system described in (5.40), (5.41) with $A_{\text{aug}}$, $B_{\text{aug}}$ and $C_{\text{aug}}$ from (5.42) and $E_{\text{aug}}$ from (5.47) where $M = \text{diag}\{1, 2\}$.

Now, we can apply PUIO design based on Theorem 5.3.2 to the above augmented system to obtain the desired PUIO (5.11) where $G_1$ is selected such that $F = A_1 - G_1 C$ is a stable Metzler matrix as follows

$$
F = \begin{bmatrix}
-1.23 & 0.55 & 1.02 & 0.02 & 0.05 \\
1.04 & -1.85 & 0.04 & 0.04 & 0.15 \\
0.15 & 0.59 & -2.60 & 0.40 & 0.09 \\
0.01 & 0.05 & 0.51 & -1.49 & 1.05 \\
0.03 & 0.08 & 1.03 & 1.03 & -1.92 \\
\end{bmatrix}
$$

The remaining parameters can be determined from (5.32)-(5.34). Note that this example is constructed
Figure 5.2: The estimates of states, sensor and actuator faults for Example 5.7.2.

Based on Remark 5.3.2, the state responses of this system and their estimates are depicted in Fig. 5.2 along with the actuator and sensor fault estimation.

Example 5.7.3. Consider the following observable positive system with actuator fault and disturbance.

$$\dot{x} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d + \begin{bmatrix} 1 \\ 0 \end{bmatrix} f_a$$

$$y = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} x$$

Using Theorem 5.6.1, one can design a PIUIO of form (5.55), (5.56) via design procedure introduced.
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in Section 5.6 with following parameters:

\[
F = \begin{bmatrix}
-3.49 & -1.94 & 3.40 \\
2.17 & -3.67 & -1.70 \\
-1.20 & 0.98 & -2.84
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
-0.64 & -2.08 \\
-3.12 & 6.35 \\
2.13 & -3.40
\end{bmatrix}
\]

\[
J = H = \begin{bmatrix}
-0.4 \\
-1.4 \\
1
\end{bmatrix}
\]

\[
M_1 = \begin{bmatrix}
-5.73 & 0.14 & 6.01
\end{bmatrix}
\]

\[
M_2 = \begin{bmatrix}
4.05 & -12.16
\end{bmatrix}
\]

Note that it is also possible to use the PIUIO structure described in (5.53), (5.54) where

\[
N = \begin{bmatrix}
0.3 & 0.1 \\
0.3 & 0.1 \\
0 & 0
\end{bmatrix}
\]

\[
G_1 = \begin{bmatrix}
0.99 & -1.54 \\
-2.67 & 6.50 \\
2.20 & -3.38
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
5.73 & -11.60
\end{bmatrix}
\]

Both of these observer will estimate the states of the system and capture the actuator fault as it is illustrated in Fig. 5.3 However, the latter structure needs the derivative of output which could be difficult to implement.

Example 5.7.4. Consider the system

\[
\dot{x} = \begin{bmatrix}
-1 & 1 & 1 \\
1 & -2 & 0 \\
0 & 1 & -3
\end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d + \begin{bmatrix} 1 \\ 0 \end{bmatrix} f_a
\]

\[
y = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} x
\]
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Figure 5.3: The estimate of states, sensor and actuator faults for system in Example 5.7.3.

with a nonlinear sinusoid fault. We obtained two observers with $\|L\| = 873.36$ and $262.01$ by using the high gain design procedure of Theorem 5.6.1. A PIUIO with fading term is also designed to better capture the nonlinear fault. Fig. 5.4 illustrates the fault estimation of these three observers. The figures clearly shows that with very high gain, fast estimation with intense oscillation in transient response is achieved. By reducing the gain, oscillations amplitude is decreased with slower convergence rate. Finally, by adding the fading term $R$ and its proper adjustment, we have obtained a fast and smooth estimation as depicted in Fig. 5.4.
Figure 5.4: The estimates of sinusoid fault with different gains.
Chapter 6

Symmetric Positive Stabilization

The stability robustness properties of the positive systems and the class of symmetric systems are motivating factors to look into the constrained stabilization problem. Due to the fact that many dynamical systems consist of having a combined structure of positivity and symmetry we provide a procedure to achieve stabilization with both structural constraints. Regardless of having a positive system being symmetric or not, it is of particular interest to achieve constrained stabilization such that the closed-loop system becomes symmetric positive stable [59].

6.1 Symmetric Metzlerian Stabilization

In this section we consider the constrained symmetric Metzlerian stabilization for system (3.63), (3.64) by a state feedback control law. This control law must be designed in such a way that the resulting closed-loop system is Metzlerian, symmetric and asymptotically stable.

Let the state feedback control law

\[ u(t) = v + Kx \quad (6.1) \]

be applied to the system (3.63), (3.64). Then the closed-loop system is written as

\[ \dot{x} = (A + BK)x + Bv \quad (6.2) \]

Thus, in our design procedure we need to find \( K \in \mathbb{R}^{m \times n} \) such that \( A + BK \) is a stable symmetric Metzler matrix. There are many ways to achieve this goal by applying the equivalent conditions of positive stability theorem to \( A + BK \). For example, one can find the gain matrix \( K \) through a linear programming (LP) set-up [6] by inclusion of symmetry constraint. Alternatively, one can construct
an LP or an LMI with additional symmetry constraint as outlined in the following theorem, which is generalization of previous work \[7, 8\].

**Theorem 6.1.1.** There exist a state feedback control law (6.1) for the system (3.63) such that the closed-loop system (6.2) becomes symmetric Metzlerian stable if and only if

1. The following LP has a feasible solution with variables \( z = \left[ z_1 \cdots z_n \right]^T \in \mathbb{R}^n \) and \( y_i \in \mathbb{R}^m, \forall i = 1, \ldots, n \)

\[
A z + B \sum_{i=1}^{n} y_i < 0, \quad z > 0 \quad (6.3)
\]

\[
a_{ij} z_j + b_i y_j \geq 0 \quad \text{for } i \neq j \quad (6.4)
\]

\[
a_{ij} z_j + b_i y_j = a_{ji} z_i + b_j y_i \quad \text{for } i \neq j \quad (6.5)
\]

with \( A = [a_{ij}] \) and \( B = \left[ b_1 \ b_2 \ \cdots \ b_n \right]^T \). Furthermore, the gain matrix \( K \) is obtained from

\[
K = \left[ \begin{array}{cccc}
z_1 & \frac{y_1}{z_1} & \frac{y_2}{z_1} & \cdots & \frac{y_n}{z_1} \\
z_2 & \frac{y_1}{z_2} & \frac{y_2}{z_2} & \cdots & \frac{y_n}{z_2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{array} \right] \quad (6.6)
\]

or

2. The following LMI has a feasible solution with respect to the variables \( Y \) and \( Z \)

\[
Z A^T + Y^T B^T + AZ + BY \prec 0 \quad (6.7)
\]

\[
(AZ + BY)_{ij} \geq 0 \quad \text{for } i \neq j \quad (6.8)
\]

\[
(AZ + BY)_{ij} = (AZ + BY)_{ji} \quad (6.9)
\]

where \( Z \succ 0 \) is diagonal positive definite matrix. Furthermore, the gain matrix \( K \) is obtained from

\[
K = Y Z^{-1}.
\]

The above LP or LMI solve the problem of symmetric Metzlerian stabilization for a conventional state equation. Next, we provide solution to the problem of generalized symmetric Metzlerian stabilization for (3.77) and (3.78).

### 6.2 Generalized Symmetric Metzlerian Stabilization

In this section we consider the problem of constrained stabilization of systems represented by BCCF. We initially provide a solution to generalized Metzlerian stabilization for this class of system. Subsequently, we extend our result to achieve a generalized symmetric Metzlerian stabilization for BCCF.
Let the state feedback control law
\[ u = v + Kx = v + \begin{bmatrix} K_r & K_{r-1} & \cdots & K_1 \end{bmatrix} x \]  
be applied to the controllable system (3.77) and (3.78). Then the closed-loop system preserves the BCCF with
\[ -\hat{A}_i = -A_i + K_i \quad \text{for} \quad i = 1, \ldots, r \]  
Clearly, the corresponding matrix polynomial is
\[ \hat{D}(s) = \sum_{i=0}^{r} \hat{A}_i s^{r-i} \]  
with \( \hat{A}_0 = I_m, \hat{A}_i \in \mathbb{R}^{m \times m} \), and \( \hat{A} \) being the corresponding block companion matrix. Let us define
\[ F = \begin{bmatrix} F_1 & I_m & 0 & \cdots & 0 \\ 0 & F_2 & I_m & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & F_{r-1} & I_m \\ 0 & \cdots & \cdots & 0 & F_r \end{bmatrix} \]  
Then the matrix \( F \) is a linearization of \( \hat{D}(s) \) and the coefficient of \( \hat{D}(s) \) can be specified by \( F_i \)'s according to the following theorem.

**Theorem 6.2.1.** The block companion matrix \( \hat{A} \) defined by \( \hat{A}_i \)'s is similar to the matrix \( F \), that is \( F = P \hat{A} P^{-1} \) where the transformation matrix \( P \) is a lower triangular matrix with \((i,j)\)-th block
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\[ P_{i,j} = I_m \text{ for } i = j \text{ and } P_{i,j} \text{ for } i > j \text{ satisfying the following set of chain equations:} \]

\[
\begin{align*}
F_1 + P_{2,1} &= 0 \\
F_2 P_{2,1} + P_{3,1} &= 0 \\
P_{2,1} - F_2 - P_{3,2} &= 0 \\
&\vdots \\
F_{r-1} P_{r-1,1} + P_{r,1} &= 0 \\
P_{r-1,1} - F_{r-1} P_{r-1,2} - P_{r,2} &= 0 \\
&\vdots \\
P_{r-1,r-3} - F_{r-1} P_{r-1,r-2} - P_{r,r-2} &= 0 \\
P_{r-1,r-2} - F_{r-1} - P_{r,r-1} &= 0 \\
&\vdots
\end{align*}
\]

Furthermore,

\[ \hat{A}_{r-i} = P_{r,i} - F_r P_{r,i+1} \text{ for } i = 0, 1, \ldots, r - 1 \]  

(6.14)

where \( P_{r,0} = 0 \).

Proof. The proof of this theorem can be established by substituting the defined matrices \( P, F \) and \( \hat{A} \) into \( FP = P\hat{A} \). □

Using the above theorem, the following result is immediate.

Lemma 6.2.1. Let \( F_i \in \mathbb{R}^{m \times m}, i = 1, \ldots, r \) be chosen such that

\[ \sigma(F_1) \cup \sigma(F_2) \cup \ldots \cup \sigma(F_r) = \Lambda \]  

(6.15)

where \( \Lambda \) is the prescribed set of desired eigenvalues. Then,

\[ \hat{A}_k = (-1)^k TR_k[F] \quad k = 1, 2, \ldots, r \]  

(6.16)

where the corresponding matrix polynomial is

\[ I_m \lambda^r + \hat{A}_1 \lambda^{r-1} + \hat{A}_2 \lambda^{r-2} + \cdots + \hat{A}_r \]  

(6.17)

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and $TR_k[F]$ represents the block trace of the matrix $F$

$$TR_k[F] = \sum \begin{bmatrix} F_1 & I \\ F_2 & \ddots \\ \vdots & \ddots & I \\ F_k \end{bmatrix}$$  \hfill (6.18)

**Proof.** The proof of this lemma can be established by evaluating the product of the sequence of first order matrix polynomials.

$$\hat{D}(\lambda) = (\lambda I - F_1)(\lambda I - F_2) \cdots (\lambda I - F_r)$$

$$= \lambda^r I + \hat{A}_1 \lambda^{r-1} + \hat{A}_2 \lambda^{r-2} + \cdots + \hat{A}_r$$  \hfill (6.19)

Then (6.16) immediately follows by coefficient comparison (see Example 6.3.4 for $r = 4$).

Based on the above polynomial matrix $\hat{D}(\lambda)$, one can deduce the following important result.

**Lemma 6.2.2.** Let $F_i \in \mathbb{R}^{m \times m}$ for $i = 1, \ldots, r$ be a set of block diagonal stable Metzler matrices in (6.12), each with multiple Metzler blocks of order 2 or 1 such that for even or odd $m$, the blocks are properly distributed to construct (6.12). Then $-\hat{A}_k$ for $k = 1, \ldots, r$, where $\hat{A}_k = (-1)^k TR_k[F]$ with $TR_k[F]$ defined by (6.18), is also Metzler.

**Proof.** It is not difficult to show that for $m = 2$ the product of two Metzler matrices becomes an M-matrix and the product of an M-matrix with a Metzler matrix remains a Metzler matrix. Also, the sum of two or more Metzler matrices (alternatively M-matrices) maintain its structure. Since $TR_k[F]$ is constructed using sum and product of Metzlerian structures, $-\hat{A}_k$ remains Metzlerian. For $m > 2$, one requires to use Metzler matrices $F_i$’s of size 2 or 1 and construct $TR_k[F]$. Using the product and sum operations of block diagonal matrices, one can show that $-\hat{A}_k$ remains Metzlerian.

The above result can be summarized as an algorithm for generalized Metzlerian stabilization of systems represented by BCCF.

**Algorithm:**

1. Select the set of desired eigenvalues and define $\Lambda$.

2. Using lemma 6.2.2 construct symmetric stable Metzler matrices $F_i$’s.
3. Construct the matrix $F$ such that (6.15) holds.

4. Compute $\hat{A}_k$, $k = 1, 2, \ldots, r$ by (6.16).

5. Compute the state feedback gain matrix $K$ using (6.11).

The above algorithm solves the problem of generalized Metzlerian stabilization. Obviously, Metzler matrices with specified eigenvalues can be constructed by diagonal or upper (lower) triangle structures. However, constructing general Metzler matrices with specified eigenvalues is not trivial. This is required for generalized symmetric Metzlerian stabilization. Recently, an elegant procedure has been proposed to achieve this goal in [60].

Thus, the above algorithm can effectively be used to solve the generalized symmetric Metzlerian stabilization by a minor refinement. Although we can use [60] to construct distinct stable symmetric Metzler matrices $F_i$’s, one possible selection of $F_i$’s such that it satisfies the constraints of Lemma 6.2.1 is by choosing $F_1$ to be a stable symmetric Metzlerian matrix and then construct the remaining $F_i$’s according to $F_i = \alpha_i F_1$, $\alpha_i > 0$ for $i = 2, \ldots, r$. Integrating this $F_i$’s in the solution procedures of the above algorithm guarantees generalized symmetric Metzlerian stabilization of BCCF.

### 6.3 Illustrative Examples

**Example 6.3.1.** Consider the following controllable system

$$
\dot{x}(t) = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)
$$

The goal is to find the feedback gain $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ such that the closed loop system becomes stable symmetric Metzlerian. To satisfy the stability constraints we need

$$
4K_1 + 3K_2 < -22 \quad K_1 + K_2 < -3
$$

To satisfy the Metzlerian structural condition of the closed loop system we have additional constraints

$$
-4 < K_1 < -1 \quad -6 < K_2 < -2
$$

Finally the condition $K_1 - K_2 = 2$ should be satisfied to guarantee the symmetric structure of the closed-loop system. The feasible region without the symmetric constraint is shown by shaded area in
Incorporating the symmetric constraint, the line segment between the points \((-4, -6),\) \((-\frac{16}{7}, -\frac{30}{7})\) is the feasible solution for this problem. We can choose \(K = \begin{bmatrix} -3.5 & -5.5 \end{bmatrix}\) as one possible solution to obtain symmetric Metzlerian stabilization with a closed-loop system matrix as

\[
A_{cl} = A + BK = \begin{bmatrix} -2.5 & 0.5 \\ 0.5 & -3.5 \end{bmatrix}
\]

**Example 6.3.2.** Now consider the following unstable controllable system

\[
\dot{x}(t) = \begin{bmatrix} 2 & 2 & 7 \\ 1 & -3 & 0 \\ 6 & 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t)
\]

Using LP or LMI design procedures of Theorem 6.1.1 one can readily obtain the feasible solutions for \(K\). One possible solution for \(K\) is obtained as

\[
K = \begin{bmatrix} -4 & -1 & -5 \end{bmatrix}
\]

which will results in the following closed-loop system

\[
A_{cl} = A + BK = \begin{bmatrix} -2 & 1 & 2 \\ 1 & -3 & 0 \\ 2 & 0 & -4 \end{bmatrix}
\]

It is evident that the closed-loop system matrix, \(A + BK\) is a symmetric Metzler matrix with eigenvalues located at \((-5.36, -3.17, -0.47)\).
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Example 6.3.3. Let the controllable pair \((A, B)\) of the system (3.77) be represented by (3.78) with

\[
A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}
\]

which are symmetric and Metzlerian. The goal is to stabilize the system with desired eigenvalues \(\Lambda = \{-1, -1, -2, -6\}\) while maintaining the structure of block coefficient matrices. By distributing the desired eigenvalues to the matrices

\[
F_1 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, F_2 = \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix}
\]

one can construct \(F\) in (6.12). Then using lemma 6.2.1 we have

\[
\hat{A}_1 = \begin{bmatrix} 4 & -3 \\ -3 & 6 \end{bmatrix}, \hat{A}_2 = \begin{bmatrix} 6 & -6 \\ -6 & 8 \end{bmatrix}
\]

which are symmetric M-matrices. Then the feedback gain \(K\) is computed from (6.11) as

\[
K = \begin{bmatrix} -3 & 4 & -3 & 2 \\ 4 & -7 & 2 & -5 \end{bmatrix}
\]

This example shows that non-symmetric stable Metzlerian matrices \(F_1\) and \(F_2\) leads to stable symmetric Metzlerian \(-\hat{A}_i\)'s.

Example 6.3.4. Let the controllable pair \((A, B)\) of the system (3.77) be represented by (3.78) with

\[
A_1 = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}
\]

\[
A_3 = \begin{bmatrix} -1 & -1 \\ 3 & -2 \end{bmatrix}, A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}
\]

which are neither symmetric nor Metzlerian. The goal is to stabilize the system such that the closed-loop system becomes symmetric Metzlerian BC-CF. With the aid of the algorithm discussed in Section 6.2 and its refinement we construct an initial symmetric stable Metzler matrix

\[
F_1 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}
\]
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and obtain $F_i = \alpha_i F_1$ for $i = 2, 3, 4$ where $\alpha_2 = 0.2$, $\alpha_3 = 0.4$, and $\alpha_4 = 0.6$. Then (6.16) for $r = 4$ yields

\[
\hat{A}_1 = -TR_1[F] = -\sum F_i = -(F_1 + F_2 + F_3 + F_4)
\]
\[
\hat{A}_2 = TR_2[F] = \sum_{i<j} F_i F_j
\]
\[
= +(F_1 F_2 + F_1 F_3 + F_1 F_4 + F_2 F_3 + F_2 F_4 + F_3 F_4)
\]
\[
\hat{A}_3 = -TR_3[F] = -\sum_{i<j<q} F_i F_j F_q
\]
\[
= -(F_1 F_2 F_3 + F_1 F_2 F_4 + F_1 F_3 F_4 + F_2 F_3 F_4)
\]
\[
\hat{A}_4 = TR_4[F] = \sum_{i<j<q<s} F_i F_j F_q F_s = F_1 F_2 F_3 F_4
\]

leading to

\[
\hat{A}_1 = \begin{bmatrix} 4.4 & -2.2 \\ -2.2 & 4.4 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 8.2 & -6.6 \\ -6.6 & 8.2 \end{bmatrix}
\]
\[
\hat{A}_3 = \begin{bmatrix} -6.83 & 6.34 \\ 6.34 & -6.83 \end{bmatrix}, \quad \hat{A}_4 = \begin{bmatrix} 1.97 & -1.92 \\ -1.92 & 1.97 \end{bmatrix}
\]

Therefore, from equation (6.11), the feedback gain can be obtained as $K =$

\[
\begin{bmatrix}
-5.4 & 4.2 & -6.2 & 11.56 & 5.83 & -7.34 & -0.97 & 2.92 \\
5.2 & -9.4 & 9.56 & -9.2 & -3.34 & 4.83 & 1.92 & 0.03
\end{bmatrix}
\]
Chapter 7

Positive and Symmetric Control

This chapter is concerned with the control of Linear Time-Invariant (LTI) continuous-time systems with positivity and symmetry constraints. First, the problem of Linear Quadratic Regulator (LQR) under positivity constraint is formulated and solved. This problem by itself is not trivial and becomes even unsolvable when positive observer-based LQR is considered. As a consequence of this restriction, one can attempt to find possible solutions based on static and dynamic output feedback. It is well-known that strict stabilization condition of static output feedback problem can be alleviated by dynamic output feedback for general LTI system due to extra degree of freedom provided by the controller parameters. However, one can show that if there exists a dynamic controller such that the closed-loop system is positive and stable, then there exists a static controller such that the closed-loop system is positive and stable. Thus, the following sections consider the positive static state and output feedback control problems with optimality criterion. Similarly, symmetric control problem is considered and solution strategies are provided for static state and output feedback cases. This chapter is benefited from some of the recent results concerning stabilization and control of positive and symmetric systems [8, 13, 56, 61, 70]. The goal is to show the link among the results, clarify the missing links, and fill in the gaps. At the same time, new results are provided and possible extensions are suggested.
CHAPTER 7. POSITIVE AND SYMMETRIC CONTROL

7.1 Positive LQR

Consider again the unstable continuous-time linear system

\[
\dot{x}(t) = Ax(t) + Bu(t) \quad (7.1)
\]
\[
y(t) = Cx(t) \quad (7.2)
\]

where \(x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p\) are the state, input, and output vectors; respectively and \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}\) are the associate system matrices with \(\{A, B\}\) controllable and \(\{A, C\}\) observable pairs.

Our goal in this section is to stabilize the system with the condition that the closed-loop system becomes stable and admits a special structure of continuous-time positive system known as Metzlerian systems. In addition, we would like to achieve this goal with optimal LQR criterion in order to enhance the robustness properties of the feedback systems. This constrained Metzlerian stabilization with optimal criterion consists of two steps. First, we perform Metzlerian stabilization by a preliminary state feedback control law, which can be obtained using LP or LMI design equations as discussed in Chapter 4. Then we provide condition under which the Metzlerian structure of the first step is maintained while the optimality of LQR is guaranteed.

Let the state feedback control law of the form

\[
u(t) = v + K_1 x \quad (7.3)
\]

be applied to the system (7.1), (7.2). Then we get the following closed-loop system

\[
\dot{x} = (A + BK_1)x + Bv \quad (7.4)
\]

Thus, in the first stage of our design procedure we need to find \(K_1 \in \mathbb{R}^{m \times n}\) such that \(A + BK_1\) is a Metzler stable matrix. This gain can be found via either a linear programming approach as discussed in Theorem 4.1.1 or a linear matrix inequality as outlined in Theorem 4.1.2.

Now, let the Metzlerian stabilized system (7.4) in the first stage be represented by the pair \(\{A_1, B\}\) where \(A_1 = A + BK_1\). Then the minimization problem of the quadratic cost

\[
J = \int_{t_0}^\infty (v^TRv + x^TQx) \, dt \quad (7.5)
\]

subject to

\[
\dot{x} = A_1 x + Bv \quad (7.6)
\]
has an optimal control solution given by
\[ v = K_2 x = -R^{-1}B^T P x \] (7.7)
where \( P \) is the positive semidefinite solution of the Algebraic Riccati Equation (ARE).

\[ A_1^T P + PA_1 - PBR^{-1}B^T P + Q = 0 \] (7.8)

with \( R \in \mathbb{R}^{m \times m} \) and \( Q \in \mathbb{R}^{n \times n} \) assumed to be positive definite and positive semidefinite symmetric matrices, respectively. Using (7.7) in (7.6), we get the closed loop system
\[ \dot{x} = (A_1 + BK_2)x \] (7.9)
for which its solution is given by
\[ x(t) = \Phi(t, t_0)x(t_0) \] (7.10)
where
\[ \Phi(t, t_0) = \exp [(A_1 + BK_2)t] \]

Substituting for \( v \) and \( x \) from (7.7) and (7.10) into the performance index (7.5) yields
\[ J = x^T(t_0) \left[ \int_{t_0}^{\infty} \Phi^T(t, t_0)(Q + K_2^T R K_2)\Phi(t, t_0) \, dt \right] x(t_0) = x^T(t_0) P x(t_0) \] (7.11)

where it can be easily verified that \( P \) is the solution of the Lyapunov equation.

\[ (A_1 + BK_2)^T P + P(A_1 + BK_2) + (Q + K_2^T R K_2) = 0 \] (7.12)

Note that (7.12) becomes ARE if we substitute \( K_2 = -R^{-1}B^T P \), which yields the minimum value of (7.11).

The following lemma is useful for the subsequent result.

**Lemma 7.1.1.** Let \( Q_1 \) and \( Q_2 \) be two strictly positive and positive define matrices such that \( Q_1 > Q_2 \) and suppose the Riccati equation
\[ A^T P + PA - PBR^{-1}B^T P + Q_1 = 0 \]
has a positive definite solution \( P = P_1 \) with \( Q_1 \). Then the Riccati equation
\[ A^T \hat{P} + \hat{P}A - \hat{P}BR^{-1}B^T \hat{P} + Q_1 = 0 \]
has a positive definite solution \( \hat{P} = P_2 \) such that \( P_2 < P_1 \).
Theorem 7.1.1. Consider the Metzlerian stabilized system \((7.4)\) with the controllable pair \((A_1, B)\) and assume that there exist positive definite matrices \(Q\) and \(R\) such that \(Q > 0\) is strictly positive, \(BR^{-1}B^T \geq 0\) and \(A_1 - BR^{-1}B^TP_1\) is a stable Metzler matrix where \(P_1\) is the solution of the Lyapunov equation \(A_1^TP_1 + P_1A_1 = -Q\). Then there exists a sequence of decreasing positive and positive definite matrices \(P_k > 0\) for all \(k \geq 1\) satisfying the following iterative Lyapunov equation
\[
(A_1^T - P_k BR^{-1}B^T)P_{k+1} + P_{k+1}(A_1 - BR^{-1}B^TP_k) = -P_k BR^{-1}B^TP_k - Q
\] (7.13)
with \(P_0 = 0\).

Proof. Let \(P_0 = 0\) and \(Q = Q_1 > 0\), then one can construct the Lyapunov equation \(A_1^TP_1 + P_1A_1 = -Q_1\). Since \(A_1\) is a Metzler stable matrix, there exists a positive and positive definite matrix \(P_1 > 0\) for \(Q_1 > 0\). Using the assumption of the theorem, let \(P_1\) be such that \(A_2 = A_1 - BR^{-1}B^TP_1\) is a Metzler stable matrix. This can be realized by observing that \(BR^{-1}B^TP_1 \geq 0\) and using the properties of Metzler stable matrices. Next, we construct \(P_2\) based on \(P_1\) as \(A_2^TP_2 + P_2A_2 = -Q_2\), where \(Q_2 = P_1BR^{-1}B^TP_1 + Q_1 > 0\), since the sum of two positive definite matrices remains positive definite. Due to the monotonicity argument of Metzler stable matrices and Lemma 7.1.1, it is not difficult to see that \(P_1 > P_2\). Continuing this process, one can take the limit of (7.13), which corresponds to the solution of ARE \(A_1^TP + PA_1 - PBR^{-1}B^TP + Q = 0\). Indeed this ARE can be rewritten as \(A_1^TP + PA_2 = -Q\) which represents a Sylvester type matrix equation. This can compactly be written as \(\hat{M}\hat{p} = -\hat{q}\) where \(\hat{p}\) and \(\hat{q}\) are vectors whose elements are constructed from the components \(p_{ij}\) and \(q_{ij}\) of \(P\) and \(Q\), and \(\hat{M} = [A_1^T \otimes I + I \otimes A_2^T]\) is a Metzler stable matrix since \(A_1\) and \(A_2\) are both Metzlerian stable matrices. 

Lemma 7.1.2. The following statements are equivalent

1. The ARE
\[
A^TP + PA - PBR^{-1}B^TP + Q = 0
\]
has a solution \(P = P^T > 0\) with \(Q = C^TC\).

2. The Hamiltonian matrix
\[
H = \begin{bmatrix}
A & -BR^{-1}B^T \\
-Q & -A^T
\end{bmatrix}
\]
has no pure imaginary eigenvalues.
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3. The inequality

\[ A^T P + PA + Q + K^T B^T P + PBK + K^T RK \preceq 0 \]

is feasible with variables \( P \) and \( K \).

4. The LMI

\[
\begin{bmatrix}
A^T \hat{Q} + \hat{Q}A + BY + Y^T B^T & -\hat{Q}C^T & -Y^T \\
-C\hat{Q} & -I & 0 \\
-Y & 0 & -R^{-1}
\end{bmatrix} \preceq 0
\]

is feasible, where \( \hat{Q} = P^{-1} \) and \( Y = K\hat{Q} \).

Proof. The equivalence of 1 and 2 is known from LQR theory. The equivalence of 3 and 4 can be established by multiplying both sides of inequality in 3 by \( \hat{Q} \) and applying Schur complement.

An alternative and direct solution to the optimization problem above is given by the following LMI formulation.

**Theorem 7.1.2.** Consider the Metzlerian stabilized system \((7.4)\) and assume that there exist positive definite matrices \( Q \) and \( R \) such that \( Q > 0 \) is strictly positive, \( BR^{-1}B^T \succeq 0 \) and \( A_1 - BR^{-1}B^T P_1 \) is a stable Metzlerian matrix, where \( P_1 \) is the solution of the Lyapunov equation \( A^T P_1 + P_1 A_1 = -Q \). Then the optimal constrained stabilization can be obtained by solving the following LMI

\[
\min x_0^T P x_0 \quad \text{subject to}
\begin{bmatrix}
A^T P + PA + Q & PB \\
(PB)^T & -R
\end{bmatrix} \prec 0
\]

for which the optimal gain is given by

\[ K = -R^{-1}B^T P \]

Proof. The proof is trivial by observing the equivalence of ARE with the above LMI.

**Example 7.1.1.** Consider the unstable system with

\[
A = \begin{bmatrix}
-6.8101 & 2.1767 & 1.8666 \\
3.0130 & -4.2386 & 4.5973 \\
3.6283 & 5.6047 & 2.5210
\end{bmatrix}, \quad B = \begin{bmatrix}
0.1240 \\
0.4318 \\
0.7602
\end{bmatrix}
\]

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Using the LMI approach of Theorem 4.1.2 we obtain $K_1 = [-3.7102 - 6.1753 - 8.5390]$, which achieves Metzlerian stabilization of the first step, where

$$
A_1 = A + BK_1 = \begin{bmatrix}
-7.2702 & 1.4109 & 0.8078 \\
1.4109 & -6.9051 & 0.9102 \\
0.8078 & 0.9102 & -3.9703
\end{bmatrix}
$$

In the second step, we apply the procedure of Theorem 7.1.1 with $R = 1$ and

$$
Q = \begin{bmatrix}
1.9015 & 1.0011 & 1.4432 \\
1.0011 & 1.4293 & 1.3220 \\
1.4432 & 1.3220 & 1.5017
\end{bmatrix}
$$

and obtain the stabilizing feedback gain $K_2 = [-0.2154 - 0.2183 - 0.2978]$. The iterative procedure converges to

$$
P = \begin{bmatrix}
0.1719 & 0.1216 & 0.1864 \\
0.1216 & 0.1490 & 0.1828 \\
0.1864 & 0.1828 & 0.2578
\end{bmatrix}
$$

The overall feedback gain is $K = K_1 + K_2 = [-3.9256 - 6.3936 - 8.8368]$ with the closed-loop system matrix

$$
A_c = A + BK = \begin{bmatrix}
-7.2969 & 1.3839 & 0.7708 \\
1.3181 & -6.9991 & 0.7818 \\
0.6441 & 0.7443 & -4.1966
\end{bmatrix}
$$


7.2 Failure of Separation Principle in Positive Observer-based Controller

In Chapter 6 positive stabilization was discussed and methods based on LP and LMI were employed to construct the controller and in optimal fashion using LQR as described in previous section. Positive Observer was also treated in Chapter 5 by duality of state feedback using LP and LMI. However, unlike the conventional observer-based controller design that makes the combined design of observer and state feedback control possible with the aid of separation principle, it is not possible to make the same conclusion under positivity constraint.
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Let a positive Luenberger type observer be designed for the positive unstable system (7.1), (7.2) described by

\[
\dot{x}(t) = (A - LC)\dot{x}(t) + Ly(t) + Bu(t)
\]  

(7.14)

and let a feedback control law \( u(t) = K\dot{x}(t) \) be employed in conjunction with (7.14), where \( K \) is obtained such that \( A + BK \) is positive and stable. Then we have the following augmented system

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\hat{x}}(t)
\end{bmatrix} = 
\begin{bmatrix}
A & BK \\
LC & A - LC + BK
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\hat{x}(t)
\end{bmatrix}
\]

(7.15)

Now, assume \( A \) is not Hurwitz stable and that there exist matrices \( L \) and \( K \) which fulfill the positivity of \( x(t) \) and \( \hat{x}(t) \). One can easily show that such statement leads to a contradiction. Since the augmented system must be positive, it is necessary that \( BK \geq 0 \). Note that \( LC \geq 0 \) through the positive observer (7.14). Using a similarity transformation it is possible to transform the augmented system to

\[
\begin{bmatrix}
\dot{e}(t) \\
\dot{\hat{x}}(t)
\end{bmatrix} = 
\begin{bmatrix}
A - LC & 0 \\
LC & A + BK
\end{bmatrix}
\begin{bmatrix}
e(t) \\
\hat{x}(t)
\end{bmatrix}
\]

(7.16)

where \( e(t) = x(t) - \hat{x}(t) \). The fact that \( A + BK \) is Hurwitz stable and Metzler leads to the existence of \( v > 0 \) such that \( (A + BK)v < 0 \). Since \( BK \geq 0 \), it follows that \( Av < 0 \) and using stability condition 6 of Lemma 3.1.1 one can conclude that \( A \) is necessarily a Hurwitz stable matrix, which contradicts with the assumption of \( A \) being unstable matrix. Thus, separation principle does not hold. This fact caused the researchers to consider the static or dynamic output feedback for positive stabilization and control.

7.3 Positive Static Output Feedback Stabilization and Control

Consider the system (7.1), (7.2) with an output feedback control law

\[
u(t) = Hy(t)
\]

(7.17)

Then the closed-loop system can be written as

\[
\dot{x}(t) = (A + BHC)x(t)
\]

\[
y(t) = Cx(t)
\]

(7.18)
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The output feedback stabilization requires that

\[ \text{Re}[\lambda_i(A + BHC)] < 0 \quad \forall i = 1, \ldots, n \] (7.19)

to achieve asymptotic stability of the closed-loop system.

It is well-known that this problem is not trivial and certainly it is more complex when desired eigenvalues of the closed-loop system matrix \( A_c = A + BHC \) is required. The solution of this problem for positive stabilization is convenient for single-output or single-input cases as follows.

**Theorem 7.3.1.** Let the system be single-output with \( c \geq 0 \). Then there exists a static output feedback control law \( u(t) = hy(t) \) such that the closed-loop system is positive and asymptotically stable if and only if there exist two vectors \( v \in \mathbb{R}^n, z \in \mathbb{R}^m \) such that the following LP is feasible.

\[
\begin{align*}
Av + Bz &< 0 \\
(cvA + Bzc + I) &\geq 0 \\
v &> 0
\end{align*}
\]

Furthermore, all stabilizing gains \( h \) are parametrized by

\[ h = \frac{1}{cv} z \]

(7.21)

**Proof.** It is not difficult to see that constraints (7.20) are obtained from the equivalent relations

\[
\begin{align*}
(A + Bhc)v &< 0 \\
A + Bhc + \frac{1}{cv} I &\geq 0 \\
v &> 0
\end{align*}
\]

where the first line is the stability condition and the second line guarantees Metzlerian structure of \( A + Bhc \). Then by using (7.21), one can immediately obtain (7.20). \( \square \)

It should be noted that if \( c \) is not sign restricted, then one can add additional constraint \( cv > 0 \). Moreover, if nonnegative control is required i.e. \( u(t) = hy(t) \geq 0 \), then one can also add the constraint \( vc \geq 0 \) in the above LP.

On the other hand, for the single-input case, we use the fact that \( A + BHC \) is Metzler and Hurwitz stable if and only if its transpose is Metzler and Hurwitz stable. Thus, the following LP can
be written for this case

\[ A^T v + C^T z < 0 \]
\[ b^T v A^T + C^T z b^T + I \geq 0 \]
\[ v > 0 \]

Finally, for multi-input multi-output system one can take advantage of the classical dyadic or rank one factorization of \( H \) and write \( u(t) = hwy(t) \) where \( h \in \mathbb{R}^{m \times 1} \) and \( w \in \mathbb{R}^{1 \times p} \) is an arbitrary fixed parameter design. Then the following LP should be solved for feasible solution

\[ Av + Bz < 0 \quad (7.22) \]
\[ wCvA + BzC + I \geq 0 \]
\[ v > 0 \]

Moreover, \( h \) can be obtained by

\[ h = \frac{1}{wCv} z \quad (7.23) \]

**Example 7.3.1.** Consider the unstable MIMO positive system

\[
\dot{x}(t) = \begin{bmatrix} -0.1 & 2 & 1.5 \\ 0.5 & -0.3 & 0.1 \\ 0.2 & 0.5 & -2.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -0.6 \\ 1.7 & 0.4 \\ 0.6 & -1.5 \end{bmatrix} u(t) \\

y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x(t)
\]

To determine the static output feedback (7.17) such that the closed-loop system (7.18) becomes positive and stable, we solve the LP (7.20) of Theorem 7.3.1 and obtain

\[ v = \begin{bmatrix} 0.32 \\ 0.02 \\ 0.03 \end{bmatrix}, \quad z = \begin{bmatrix} -0.087 \\ -0.024 \end{bmatrix} \]

Then, from (7.21) we find the static output feedback \( H \) as follows

\[ H = \frac{1}{Cv} z = \begin{bmatrix} -0.2559 \\ -0.0706 \end{bmatrix} \]
Using this static output feedback, the closed-loop system matrix
\[
A_c = A + BHC = \begin{bmatrix}
-0.3135 & 1.7865 & 1.5 \\
0.0368 & -0.7632 & 0.2 \\
0.1524 & 0.4524 & -2.5
\end{bmatrix}
\]
will become positive and stable with eigenvalues located at \{-0.08, -0.88, -2.61\}.

Remark 7.3.1. In this section we provided an output feedback stabilization scheme for positive systems. It can be shown that the above output feedback procedure can be formulated in an optimization framework to provide the optimal performance for positive systems.

7.4 LQR of Symmetric Systems

In Chapter 3, we defined symmetric systems and provided stabilization of LTI systems with positivity and symmetric constraints. In parallel to the positive LQR of Section 7.1, we consider the design of LQR of symmetric systems in this section and draw interesting conclusions. Recall that a system is symmetric with respect to transfer function representation if \( G(s) = G^T(s) \) and it is called symmetric with respect to state space representation if \( A = A^T, B = C^T \), and in general one can define it as \( A = T^{-1}A^TT, B = T^{-1}C^T, \) and \( C = B^TT \), where \( T \) is an invertible and symmetric transformation matrix. To see the connection between transfer function and state space representations of symmetric system one can write the transpose of \( G(s) = C(sI - A)^{-1}B \) as \( G^T(s) = B^T[sI - A^T]^{-1}C^T \) or with inclusion of a symmetric invertible matrix \( T \) we have
\[
G^T(s) = B^T [(sTT^{-1} - A)^{-1}] C^T = B^T T [sI - T^{-1}A^TT]^{-1} T^{-1}C^T
\]  
(7.24)

Since \( G^T(s) = G(s) \) one can obtain \( A = T^{-1}A^TT, B = B^TT, \) and \( C = B^TT \). Note that \( A^T = TAT^{-1} \) and it confirms with the theory of symmetrization of a matrix in Section 2.2.2 of Chapter 2.

Now let the controllability and observability matrices of a symmetric system be defined by \( U \) and \( V \), respectively. Then there exists a symmetric invertible matrix \( T \) defined by \( T = V^TU^{-1} \). To see this, one can easily write
\[
V^T = \begin{bmatrix}
C^T & A^TC^T & \cdots & (A^T)^{n-1}C^T \\
C^T & A^TC^T & \cdots & (TAT^{-1})^{n-1}C^T \\
TB & TAB & \cdots & TA^{n-1}B
\end{bmatrix} = TU
\]
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and deduce that \( T = V^T U^{-1} \)

The starting point of LQR of symmetric systems is similar to the conventional LQR method. So using the performance index

\[
J = \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t)) \, dt
\]

(7.25)

for a symmetric system defined by \( A = A^T, B = C^T \) leads to the solution of optimal control law \( u(t) = Kx(t) \) with \( K = -R^{-1}B^TP \) where \( P \) is the symmetric solution of the Riccati equation

\[
A^TP + PA - PBR^{-1}B^TP + Q = 0
\]

(7.26)

The closed-loop system matrix becomes \( A_c = A + BK \) with the minimum of \( J^* = x^T(0)Px(0) \).

If we let \( Q = C^TC \) and \( R = I \), then \( J \) in (7.25) can be interpreted as the sum of input and output energies, and (7.26) reduces to

\[
A^TP + PA - PBB^TP + C^TC = 0
\]

(7.27)

with the optimal gain of \( K = -B^TP \).

Unlike positive systems one can apply observer-based controller design for symmetric systems. In fact, the optimal observer gain can be obtained by solving the dual Riccati equation

\[
AM + MA^T - MCTCM + BBT = 0
\]

(7.28)

as

\[
L = MC^T
\]

(7.29)

Using the relationship established for symmetric systems, it can be shown that the matrices \( P \) and \( M \) are related by

\[
M = T^{-1}PT^{-1}
\]

(7.30)

where \( T \) can be obtained by \( T = V^T U^{-1} \) as previously defined. Thus, one can conclude that it is sufficient to solve only one Riccati equation to obtain \( K \) and the observer gain can be determined from (7.29) with the aid of (7.30).

7.5 Stabilization and \( H_\infty \) Control of Symmetric Systems

The following result is useful for subsequent theoretical development.
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Lemma 7.5.1. Consider a stable symmetric system with \( A = A^T \) and \( B = C^T \). Then the system has \( H_\infty \) norm less than \( \gamma \) if and only if

\[
\gamma A + BB^T < 0 \quad (7.31)
\]

Furthermore, \( \gamma \) can uniquely be expressed by the explicit formula \( \gamma = \rho \left( -B^T A^{-1} B \right) \).

Proof. Using the Bounded Real Lemma, it is well-known that a stable system has an \( H_\infty \) norm less than \( \gamma \) if and only if there exists a matrix \( P \) satisfying

\[
\begin{bmatrix}
A^T P + PA & PB & C^T \\
B^T P & -\gamma I & 0 \\
C & 0 & -\gamma I
\end{bmatrix} < 0 \quad (7.32)
\]

Since symmetric Lyapunov inequality admits a common quadratic Lyapunov function with \( P = I \) and replacing \((A^T, C^T)\) by \((A, B)\) we have

\[
\begin{bmatrix}
2A & B & B \\
B^T & -\gamma I & 0 \\
B^T & 0 & -\gamma I
\end{bmatrix} < 0 \quad (7.33)
\]

Applying Schur complement formula yields the required result \((7.31)\).

The proof of \( \gamma = \rho \left( -B^T A^{-1} B \right) \) is rather lengthy and requires additional lemmas to be stated. One constructive way to show the exact formula is through regular iterative procedure for finding optimal \( \gamma \) as it is usually the case for general matrices. \[\blacksquare\]

7.5.1 The Output Feedback Stabilization of Symmetric Systems

Theorem 7.5.1. Consider the symmetric system \( A = A^T \), \( B = C^T \). Then there exists a symmetric output feedback control law \( u(t) = H y(t) \) to asymptotically stabilize the closed-loop system if and only if

\[
B_o AB_o^T < 0 \quad (7.34)
\]

where \( B_o \triangleq B^\perp \) is the orthogonal complement of \( B \), i.e. \( B_o B = 0 \) and can be computed from singular value decomposition of \( B \) as follows.

\[
B = \begin{bmatrix}
U_1 & U_2 \\
\Sigma_1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
V_1^T \\
V_2^T
\end{bmatrix}
\quad (7.35)
\]

\[
B_o = U_2^T
\]
Furthermore, if the condition (7.34) is satisfied, all stabilizing symmetric output feedback gains $H = H^T$ satisfy

$$H < B_g \left[ A B_o^T (B_o A B_o^T)^{-1} B_o^T A - A \right] B_g^T$$

(7.36)

where $B_g B = I$.

Proof. The proof of this theorem can be established with the aid of Finsler’s lemma (see [61] for more detail). ■

Suppose a dynamic controller of order $n_c \leq n$ with symmetric property is used as a candidate to stabilize the symmetric system, i.e.

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t)$$

(7.37)

$$u(t) = C_c x_c(t) + D_c y(t)$$

where $A_c = A_c^T$, $B_c = C_c^T$, and $D_c = D_c^T$. Then the augmented system and the controller can be formulated as a static output feedback problem as follows

$$\dot{x}_a(t) = (A_a + B_a H_a C_a) x_a(t)$$

(7.38)

where $x_a(t) = \begin{bmatrix} x^T(t) & x_c^T(t) \end{bmatrix}^T$ and

$$A_a = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, B_a = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, C_a = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$$

(7.39)

with the unknown matrix

$$H_a = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$$

(7.40)

Due to the symmetry of the plant and the controller, the above problem is equivalent to a symmetric static output feedback stabilization, which can be solved using Theorem 7.5.1. However, one can state the following interesting result.

**Theorem 7.5.2.** If the symmetric dynamic controller (7.37) asymptotically stabilizes the symmetric system defined by $A^T = A$ and $B = C^T$. Then the symmetric static output feedback controller $u(t) = D_c y(t)$ also asymptotically stabilizes the system.
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Proof. It is easy to see that

\[
A_{cl} = A_a + B_a H_a C_a = \begin{bmatrix} A + B D_c B^T & BB^T_c \\ B_c B^T & A_c \end{bmatrix}
\] (7.41)

is a symmetric matrix and its Hurwitz stability implies \( A_{cl} \prec 0 \), which requires that \( A + B D_c B^T \prec 0 \), which is the symmetric static output feedback control law \( u(t) = D_c y(t) \).

It should be pointed out that similar conclusion can be drawn for positive systems.

7.5.2 The \( H_\infty \) Control Design of Symmetric Systems

Let us define a more general symmetric state space representation as

\[
\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\
z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
y(t) = C_2 x(t) + D_{21} w(t)
\] (7.42)

where \( A = A^T \), \( C_1 = B_1^T \), \( C_2 = B_2^T \), \( D_{11} = D_{11}^T \), and \( D_{21} = D_{12}^T \). Then the symmetric output feedback \( H_\infty \) control design problem is to find a symmetric state output feedback control law \( u(t) = Hy(t) \) with \( H = H^T \) such that the closed-loop system is stable and the \( H_\infty \) norm between \( z(t) \) and \( w(t) \) is less than \( \gamma \).

It is not difficult to verify that the closed-loop system

\[
\dot{x}_c(t) = A_c x_c(t) + B_c w(t) \\
z(t) = C_c x_c(t) + D_c w(t)
\] (7.43)

is symmetric, i.e. \( A_c = A_c^T \), \( C_c = B_c^T \), and \( D_c = D_c^T \) where \( A_c = A + B_2 H C_2 \), \( B_c = B_1 + B_2 H D_{21} \), \( C_c = C_1 + D_{12} H C_2 \) and \( D_c = D_{11} + D_{12} H D_{21} \). The solution of the design problem is captured in the following theorem.

**Theorem 7.5.3.** Consider the symmetric system (7.42) and suppose that the stabilizability condition (7.34) is satisfied. Then a static output feedback controller \( u(t) = Hy(t) \), \( H = H^T \) which makes the closed-loop system (7.43) stable with \( H_\infty \) norm less than \( \gamma \) for any \( \gamma > \gamma^* \) can be obtained by

\[
H = (G + G^T)/2
\] (7.44)
where $G$ is given by
\begin{equation}
G = -R^{-1}L^TQM^T(M^TQM)^{-1}
\end{equation}
(7.45)
and $R$ is an arbitrary positive definite matrix such that
\begin{equation}
Q = (LR^{-1}L^T - W)^{-1} > 0
\end{equation}
(7.46)
where
\begin{equation}
L = \begin{bmatrix} B_2 \\ 0 \\ D_{12} \end{bmatrix}, M = \begin{bmatrix} B_2^T & D_{12}^T & 0 \end{bmatrix}, W = \begin{bmatrix} 2A & B_1 & B_1 \\ B_1^T & -\gamma I & D_{11} \\ B_1^T & D_{11} & -\gamma I \end{bmatrix}
\end{equation}
(7.47)
Moreover, the optimal $H_\infty$ norm $\gamma^*$ is given by
\begin{equation}
\gamma^* = \lambda_{\text{max}} \left[ N_g \left( S - SN_o^T(N_oSN_o^T)^{-1}N_oS \right) N_g^T \right]
\end{equation}
(7.48)
where
\begin{equation}
N = L_oJ \quad \text{with} \quad J = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}
\end{equation}
(7.49)
and
\begin{equation}
S = L_oWL_o^T
\end{equation}
(7.50)

### 7.6 Stabilization with Optimal Performance for Positive Systems

This section provides a connection between stability radius and $L_\sigma$-gains of positive systems. The $L_1$, $L_2$, and $L_\infty$-gains of an asymptotically stable positive system are characterized in terms of stability radii and useful bounds are derived. We show that the structured perturbations of a stable matrix can be regarded as a closed-loop system with unknown static output feedback. In particular, we use the closed-form expressions for stability radii of positive systems to compute the $L_\sigma$-gains without resorting to solve optimization problems. We also show that positive stabilization with maximum stability radius can be considered as an $L_2$-gain minimization, which can be solved by LMI. This inherently achieves performance criterion and establishes a link to the reported LP formulations. Finally, we show the unique commonality among the optimal state feedback gain
matrices in obtaining $L_\sigma$-gains of the stabilized system. Numerical examples are provided to support the theoretical results. Here we define some of the extra notation used in this section. $\mathbf{1}_n \in \mathbb{R}^n$ denotes the column vector with all entries equal to 1. $\| M \|_\sigma$ for $\sigma = 1, 2, \infty$ represents induced matrix norm, and $[A]_{r,i}$ and $[A]_{c,i}$ denote the $i$-th row and the $i$-th column of $A$.

### 7.6.1 $L_\sigma$-Gains of Positive Systems

Let us consider LTI continuous-time systems of the form

\[
\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \quad (7.51)
\]
\[
z(t) = Cx(t) + Du(t) + Fw(t) \quad (7.52)
\]

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^p$, and $z(t) \in \mathbb{R}^q$. We use the same notation as in [67] for the purpose of clarity and connection to former and subsequent results. Note that in the absence of $w(t)$, one can replace $z(t)$ by $y(t)$.

**Definition 7.6.1.** [67, 71] Given an operator $T : L_\sigma^p \rightarrow L_\sigma^q$, the $L_\sigma$-gain of $T$ is defined by

\[
\| T \|_{L_\sigma-L_\sigma} = \sup_{\| w \|_{L_\sigma}=1} \| Tw \|_{L_\sigma}
\]

for all $w \in L_\sigma$, where $\sigma$ is a positive integer.

If $T$ represents an LTI system denoted by $H$, one is usually interested in defining the gains for $\sigma = 1, 2, \infty$.

**Definition 7.6.2.** [67] The $L_1$-gain and $L_\infty$-gain of an asymptotically stable LTI system $H$ with the impulse response matrix $H(t) \in \mathbb{R}^{q \times p}$ and the transfer function matrix $\hat{H}(s) = C(sI - A)^{-1}E + F$ are given by

\[
\| H \|_{L_1-L_1} = \max_{j \in \{1, \ldots, q\}} \left\{ \sum_{i=1}^{p} \int_{0}^{\infty} \left| h_{ij}(t) \right| dt \right\} \quad (7.53)
\]

and

\[
\| H \|_{L_\infty-L_\infty} = \max_{j \in \{1, \ldots, p\}} \left\{ \sum_{i=1}^{q} \int_{0}^{\infty} \left| h_{ij}(t) \right| dt \right\} \quad (7.54)
\]

where $L_1$-gain [$L_\infty$-gain] quantifies the gain of the most influential input [output] since the max is taken over the columns [rows]. Note that $h_{ij}(t)$ for all $i, j$ are elements of $H(t)$. 

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**Proposition 7.6.1.** [67] Let us consider a system $H$ with the transfer function $\hat{H}(s) = C(sI - A)^{-1}E + F$ and its transposition $\hat{H}^*(s) = E^T(sI - A)^{-1}C^T + F^T$. Then we have

$$\|H\|_{L_\infty-L_\infty} = \|\hat{H}^*\|_{L_1-L_1}.$$  \hfill (7.55)

The author of [67] elegantly characterized $L_1$- and $L_\infty$-gains of stable positive systems and provided computational procedures to obtain them by solving LPs. The following lemma summarizes the results. We assume for subsequent analysis of positive systems that $u(t) = 0$.

**Lemma 7.6.1.** Let the system (7.51) with the transfer function $\hat{H}(s) = C(sI - A)^{-1}E + F$ be positive and asymptotically stable. Then

(i) $\|H\|_{L_1-L_1} = \max \left\{ \|\hat{H}(0)\|_1 \right\}$ is the $L_1$-gain of the mapping $w \to z$ and can be computed by the optimal solution of the following LP problem

$$\min_{\lambda, \gamma} \gamma$$

subject to

$$\begin{align*}
\lambda^T A + \mathbb{1}_q^T C &< 0 \\
\lambda^T E - \gamma \mathbb{1}_p^T + \mathbb{1}_q^T F &< 0 \\
\lambda &\in \mathbb{R}^n
\end{align*}$$

(ii) $\|H\|_{L_\infty-L_\infty} = \max \left\{ \|\hat{H}(0)\|_\infty \right\}$ is the $L_\infty$-gain of the mapping $w \to z$ and can be computed by the optimal solution of the following LP problem

$$\min_{\lambda, \gamma} \gamma$$

subject to

$$\begin{align*}
A\lambda + E\mathbb{1}_p &< 0 \\
C\lambda - \gamma \mathbb{1}_q + F\mathbb{1}_p &< 0 \\
\lambda &\in \mathbb{R}^n_p
\end{align*}$$

It can be shown that the (1) $L_1$-gain [$L_\infty$-gain] of the mapping $w \to z$ smaller than $\gamma$, (2) $\mathbb{1}_q^T \hat{H}(0) < \gamma \mathbb{1}_p^T [\hat{H}(0)\mathbb{1}_p < \gamma \mathbb{1}_q]$, and (3) the existence of $\lambda$ such that it guarantees the feasibility of LP (7.56) [(7.57)] are equivalent characterizations of $L_1$-gain [$L_\infty$-gain].

Now we briefly analyze the $L_2$-gain of stable LTI system (7.51) with $u(t) = 0$. Let $\gamma > 0$ be a fixed number and suppose there exists a positive definite symmetric matrix $P$ such that $V(x) = x^T P x$ satisfies

$$\frac{\partial V}{\partial x} \left( Ax + Ew \right) \leq -\varepsilon \|x\|^2 + \gamma^2 \|w\|^2 - \|Cx + Fw\|^2$$

\hfill (7.58)
for some \( \epsilon > 0 \). Then by assuming \( w \in L_2 \), i.e., \( \int_0^\infty \|w(t)\|^2 \, dt < \infty \) and integrating the above inequality on the interval \([0, T], T < \infty \), we get

\[
V(x(T)) \leq V(x(0)) + \gamma^2 \int_0^T \|w(t)\|^2 \, dt - \int_0^T \|z(t)\|^2 \, dt \tag{7.59}
\]

and with \( x(0) = 0 \) we have \( \|z(t)\|_2 \leq \gamma \|w(t)\|_2 \) since \( V(x(t)) \geq 0 \). Thus, \( L_2 \)-gain can be interpreted as the ratio between finite energy of the output and input bounded by \( \gamma \). With the aid of [55] we have the following lemma stated in terms of our system (7.51).

**Lemma 7.6.2.** Let the system (7.51) with \( u = 0 \) be asymptotically stable and assume \( \gamma \) is a fixed number. Then there exists \( P > 0 \) and \( \bar{\gamma} < \gamma \) such that (7.58) or equivalently \( L_2 \)-gain inequality is satisfied with \( \bar{\gamma} \) if and only if

\[
P A + A^T P + C^T C + [P E + C^T F][\gamma^2 I - F^T F]^{-1}[P E + C^T F]^T < 0
\]

\[
F^T F - \gamma^2 I < 0
\]

or there exists a positive definite symmetric matrix \( X \) such that

\[
\begin{bmatrix}
A^T X + X A & X E & C^T \\
E^T X & -\gamma I & F^T \\
C & F & -\gamma I
\end{bmatrix} < 0 \tag{7.60}
\]

which is also equivalent to \( \|C(sI - A)^{-1} E + F\|_\infty < \gamma \).

Consequently, \( L_2 \)-gain can be computed by solving an optimization problem in terms of LMI (7.60) regardless of LTI system is positive or not. However, in Section 7.6.2 we show that one can obtain \( L_1 \), \( L_\infty \) and \( L_2 \)-gains for positive systems by direct formulas without the need of solving LPs (7.56), (7.57) and LMI (7.60), respectively. These formulas are related to the stability radii, which can be explored in the next section.

### 7.6.2 Stability Radii and \( L_\sigma \)-Gains for Positive Systems

Let us partition the complex plane \( \mathbb{C} \) into two disjoint subsets \( \mathbb{C}_g \) and \( \mathbb{C}_b \), whereby one can consider \( \mathbb{C}_g \triangleq \mathbb{C}_- = \{ s \in \mathbb{C} : Re(s) < 0 \} \) for the special case of conventional open left half of the complex plane and \( \mathbb{C}_b \triangleq \mathbb{C}_+ \) as its complement.
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Definition 7.6.3. Let $C_g$ be an open subset of $\mathbb{C}$. A matrix $A \in \mathbb{C}^{n \times n}$ is said to be $C_g$ stable if $\lambda(A) \subset C_g$. The $C_g$ stability radius of a $C_g$ stable matrix $A$ with respect to perturbation structure $(E, C) \in (\mathbb{R}^{n \times p}, \mathbb{R}^{q \times n})$, written as $A(\Delta) = A + E\Delta C$ is defined by

$$r(A, E, C, \|\Delta\|) = \inf \{\|\Delta\| : \Delta \in \mathbb{F}^{p \times q}, A + E\Delta C \text{ is not } C_g \text{ stable}\} \quad (7.61)$$

where $\|\cdot\|$ is a certain matrix norm of interest and $\mathbb{F}$ denotes the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

For complex $(A, E, C)$, we denote $r_C(A, E, C, \|\Delta\|)$ as complex stability radius and for real case $r_R(A, E, C, \|\Delta\|)$ denotes the real stability radius. Furthermore, when $E$ and $C$ are identity matrices of appropriate sizes, $r_C$ or $r_R$ are unstructured stability radii. The stability radius can be represented in terms of maximum singular values of $\Delta$ when the Euclidean norm $\|\Delta\|_2$ is used, i.e.,

$$r(A, E, C, \|\Delta\|_2) = \inf \{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{p \times q}, \lambda(A + E\Delta C) \cap \mathbb{C}_+ \neq \emptyset\} \quad (7.62)$$

Denoting $\partial C_g$ as the boundary of $C_g$ we have by continuity that

$$r(A, E, C; \|\Delta\|) = \inf_{s \in \partial C} \inf \{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{p \times q}, \det(sI - A - E\Delta F) = 0\} \quad (7.63)$$

where the determinant expression can be replaced by $\det(I - \Delta \hat{H}(s)) = 0$ with

$$\hat{H}(s) = C(sI - A)^{-1}E$$

Since the structured singular value is defined by

$$\mu_C(M) = \{\inf[\bar{\sigma}(\Delta) : \det(I - \Delta M) = 0]\}^{-1}$$

we have $\mu_C(M) = \bar{\sigma}(M)$ with $\hat{H}(s) = M$ at fixed $s \in \partial C_g$ and we can write

$$r_C(A, E, C; \|\Delta\|_2) = \left\{\sup_{s \in \partial C_g} \mu_C(\hat{H}(s))\right\}^{-1} \quad (7.64)$$

$$= \left\{\sup_{s \in \partial C_g} \bar{\sigma}[C(sI - A)^{-1}E]\right\}^{-1}$$

Thus, the computation of complex stability radius $r_C$ is facilitated by tools developed in $\mathcal{H}_\infty$ analysis and those for computing the structured singular value, which is obviously the reciprocal of the stability radius. On the other hand, the computation of real stability radius $r_R = \left\{\sup_{s \in \partial C_g} \mu_R[\hat{H}(s)]\right\}^{-1}$ is
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not trivial and requires the solution of an iterative global optimization problem \(^{[45]}\). It is worth pointing out that in general we have

\[ r_R(A, E, C; \| \Delta \|_2) \geq r_C(A, E, C; \| \Delta \|_2) \geq 0 \]  \hspace{1cm} (7.65)

and for both structured and unstructured cases

\[ r_C(A, E, C; \| \Delta \|_2) = \left[ \max_{s \in \partial C} \| \hat{H}(s) \| \right]^{-1} \]  \hspace{1cm} (7.66)

which can be obtained by

\[ r_C(A, E, C; \| \Delta \|_2) = \frac{1}{\max_{\omega \in \mathbb{R}} \| \hat{H}(j\omega) \|} = \frac{1}{\| \hat{H}(s) \|_{\infty}} \]  \hspace{1cm} (7.67)

with respect to conventional complex plane of Hurwitz stability. However, for the class of positive systems the complex and real stability radii coincide and can conveniently be computed by closed form expression. A complete characterization of stability radius for positive systems (continuous and discrete cases) can be found in \(^{[46]}\). Specifically, it has been shown that with respect to unstructured perturbations for \( \Delta \) with induced norm, the stability radius can be obtained by computing the largest singular value of a constant matrix, while with respect to a fairly general class of structured perturbation for \( \Delta \) it can easily be computed by the spectral radius of a certain constant matrix. Here we provide a subset of the results pertinent to the remaining discussions.

**Theorem 7.6.1.** \(^{[46]}\) Let \( A \) be a stable Metzler matrix and let \( C \geq 0, E \geq 0 \) be nonnegative matrices of appropriate sizes as specified in Definition 4. Then, the real stability radius with respect to \( \Delta \in \mathbb{R}^{p \times q} \) and Euclidean norm \( \| \Delta \|_2 \) coincide with the complex stability radius given by

\[ r_R(A, E, C; \| \Delta \|_2) = r_C(A, E, C; \| \Delta \|_2) = \frac{1}{\| CA^{-1} E \|_2}. \]  \hspace{1cm} (7.68)

**Remark 7.6.1.** It is important to point out that Theorem 7.6.1 can be extended to any induced matrix norm of \( \Delta \). Thus, the formula (7.68) is also valid with respect to \( \| \Delta \|_1 \) and \( \| \Delta \|_{\infty} \). Also, it’s worth pointing out that if \( \Delta \) is characterized by the set \( \Delta = \{ S \circ \Delta : S_{ij} \geq 0 \} \) with \( \| \Delta \| = \max \{ |S_{ij}| : S_{ij} \neq 0 \} \) where \( [S \circ \Delta]_{ij} = S_{ij} \Delta_{ij} \) represents Schur product, then

\[ r_R = r_C = \frac{1}{\rho(CA^{-1}ES)} \]

where \( \rho(\cdot) \) represents the spectral radius of a matrix.
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In order to tie the stability radius (7.68) to $L_2$-gain, we first assume $F = 0$ in (7.51) and rewrite the LTI system as

$$\dot{x}(t) = Ax(t) + Ew(t)$$
$$z(t) = Cx(t)$$  \hspace{1cm} (7.69)

As we discussed earlier in this section, the $L_2$-gain for example is defined by

$$\|H\|_{L_2-L_2} = \max_{\omega \in \mathbb{R}} \|\hat{H}(j\omega)\|$$  \hspace{1cm} (7.70)

where we now have a strictly proper transfer function $\hat{H}(s) = C(sI - A)^{-1}E$. A quick comparison between (7.70) and (7.67) reveals that one can recast the problem of $L_2$-gain computation through the stability radius or vice versa. Suppose we consider the perturbation structure $A(\Delta) = A + E\Delta C$ as a closed-loop system with unknown static output feedback as shown in Figure 7.1. It is easy to write the closed loop system as $\dot{x}(t) = (A + E\Delta C)x(t)$, which establishes the connection between stability radius and $L_2$-gain with respect to the mapping $z \rightarrow w$. Thus minimizing $L_2$ gain for performance corresponds to maximizing the stability radius. In a similar fashion we can also connect $L_1$- and $L_\infty$-gains to the corresponding stability radii. Denoting the stability radii by $r_1, r_2$ and $r_\infty$, the corresponding $L_\sigma$-gains can be defined by $g_1, g_2$ and $g_\infty$. Then we have the following result.

**Theorem 7.6.2.** Let the system (7.69) with the transfer function $\hat{H}(s) = C(sI - A)^{-1}E$ be positive and asymptotically stable. Then

$$\|H\|_{L_\sigma-L_\sigma} = \|CA^{-1}E\|_\sigma = \frac{1}{r(A, E, C; \|\Delta\|_\sigma)}$$  \hspace{1cm} (7.71)

for $\sigma = 1, 2, \text{ and } \infty$. Furthermore, defining $L_\sigma$-gains and stability radii by $\|H\|_{L_\sigma-L_\sigma} \triangleq g_\sigma$ and...
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\[ r(A, E, C; \|\Delta\|_{\sigma}) = r_{\sigma} \text{ for } \sigma = 1, 2, \text{ and } \infty, \] we have the following bounds for \( L_{\sigma} \)-gains.

\[
\begin{align*}
\frac{1}{\sqrt{q}}g_1 & \leq g_2 \leq \sqrt{p}g_1 \\
\frac{1}{\sqrt{p}}g_{\infty} & \leq g_2 \leq \sqrt{q}g_{\infty} \\
g_2 & \leq \sqrt{g_1 g_{\infty}}
\end{align*}
\] (7.72)

and \( g_{\sigma} = \frac{1}{r_{\sigma}} \).

**Proof.** From Lemma (7.6.1) we have \( \|H\|_{L_1 - L_1} = \max\{1^T \hat{H}(0)\} = \max\{1^T [C(-A)^{-1}E]\} \).

Applying stability property 4 of Lemma 3.1.1, \( -A^{-1} \geq 0 \) and with \( C \geq 0, E \geq 0, \) the matrix \( C(-A)^{-1}E \geq 0 \). Thus, \( \|H\|_{L_1 - L_1} = \|CA^{-1}E\|_1 \). Similarly, one can express \( \|H\|_{L_\infty - L_\infty} = \|CA^{-1}E\|_\infty \).

Finally, \( \|H\|_{L_2 - L_2} = \|CA^{-1}E\|_2 \), which can directly be deduced from (7.67), (7.68) and (7.70). Using Theorem 7.6.1 along with the aforementioned development, it is evident that \( L_{\sigma} \)-gain is reciprocal of stability radius and vice versa. Thus we have (7.71). To prove (7.72), let us consider the inequality involving \( g_2 \) and \( g_{\infty} \). Since \( \hat{H}(0) \geq 0, \|\hat{H}(0)\|_{\infty} = \max_i \sum_{j=1}^{\sigma} \hat{h}_{ij}(0) = \|\hat{H}(0)\|_p \leq \|\hat{H}(0)\|_2 = \sqrt{\rho(\hat{H}(0)^T \hat{H}(0))} \). The rest of inequalities in the first two lines of (7.72) can be proved in a similar manner. To prove the third line of (7.72) we know that \( \|\hat{H}(0)\|_2 = \rho(\hat{H}(0)^T \hat{H}(0)) \leq \|\hat{H}(0)^T \hat{H}(0)\|_{\infty} \leq \|\hat{H}(0)^T \|_{\infty} \|\hat{H}(0)\|_{\infty} = \|\hat{H}(0)\|_1 \|\hat{H}(0)\|_{\infty} \). Thus, we have \( g_2 \leq \sqrt{g_1 g_{\infty}}. \)  

**7.6.3 Stabilization and Performance of Unperturbed Positive Systems**

This section considers the stabilization of unperturbed positive or non-positive systems \( (7.51) \) by state feedback control law

\[ u(t) = Kx(t) \]

such that the closed-loop system

\[
\begin{align*}
\dot{x}(t) &= (A + BK)x(t) + Ew(t) \\
z(t) &= (C + DK)x(t) + Fw(t)
\end{align*}
\] (7.73)

becomes positive, asymptotically stable and the \( L_{\sigma} \)-gain of the mapping \( w \rightarrow z \) is less than \( \gamma > 0 \). Applying Lemma 7.6.1 to (7.73) one can write LPs and obtain the required \( K \) for both cases of \( \sigma = 1 \) and \( \infty \) \[^{[67]} \). Here we only write the LP for the case of \( \sigma = \infty \) since we refer to it in our
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illustrative example at the end of this section for the purpose of comparison.

\[
\min_{\lambda, \mu, \gamma} \gamma
\]  
\[
\text{subject to}
\]
\[
A\lambda + B \sum_{i=1}^{n} \mu_i + E\mathbb{1}_p < 0
\]
\[
C\lambda + D \sum_{i=1}^{n} \mu_i - \gamma \mathbb{1}_q + F\mathbb{1}_p < 0
\]
\[
[A]_{ij}\lambda_j + [B]_{r,i}\mu_j \geq 0,
\]
\[\forall i, j = 1 \ldots n \text{ and } i \neq j\]
\[
[C]_{ij}\lambda_j + [D]_{r,i}\mu_j \geq 0,
\]
\[i = 1 \ldots q, \text{ and } j = 1 \ldots n\]

Combining (7.73) with \(z = \Delta w\) and assuming \(F = 0, D = 0\) for simplicity, we get

\[\dot{x}(t) = (A_c + E\Delta C)x\]

where \(A_c = A + BK\). Since we related the \(L_\sigma\)-gain of positive systems in terms of its stability radius, it is possible to formulate the minimization problem of \(L_\sigma\)-gain as a maximization of stability radius. Although alternative optimization problems can be constructed to solve the maximization of stability radius for \(\sigma = 1\) and \(\sigma = \infty\), the LP formulations in [67] are more convenient. On the other hand, it is of particular interest to provide a solution for \(\sigma = 2\). To this end we can take advantage of maximizing the stability radius formulation using LMI. Thus we need to solve the following optimization problem

\[
\max_K r = \frac{1}{\|C(A+BK)^{-1}E\|_2}
\]
\[
\text{subject to}
\]
\[
Z(A+BK)^T + (A+BK)Z \prec 0
\]
\[
A + BK \text{ Metzler}
\]

where \(Z \succ 0\) is a diagonal positive definite matrix. Since \(Z \succ 0\) is diagonal, the Lyapunov inequality has to be satisfied with its off-diagonal elements being non-negative. This condition holds if and only if the off-diagonal elements of \((A+BK)Z\) are non-negative. Thus, the constraints in (7.75)
can be written in LMI format by the change of variable $Y = KZ$, i.e.,

$$ZA^T + Y^T B^T + AZ + BY \preceq 0$$

$$(AZ + BY)_{ij} \geq 0$$

Regarding the objective function in (7.75) we take advantage of bounded real lemma [55]. Since the complex stability radius with respect to the closed-loop system matrix $A_c = A + BK$ is the inverse of the $\mathcal{H}_\infty$ norm of $\hat{H}(s)$, the problem can be recast as the following optimization problem

$$\min_{\gamma} \gamma$$

subject to

$$\begin{bmatrix}
A_c^T P_c + P_c A_c & P_c E & C^T \\
E^T P_c & -\gamma I & 0 \\
C & 0 & -\gamma I
\end{bmatrix} \prec 0$$

(7.77)

In order to setup (7.77) in terms of LMI, one can use the usual congruent transformation by pre and post multiplying (7.77) with $\text{diag}\{Q_c, I, I\}$, where $Q_c = P_c^{-1}$ and changing the variable $Y_c = KQ_c$. Thus, we have

$$\min_{\gamma} \gamma$$

subject to

$$\begin{bmatrix}
W_c & E & Q_c C^T \\
E^T & -\gamma I & 0 \\
C Q_c & 0 & -\gamma I
\end{bmatrix} \prec 0$$

(7.78)

Where $W_c = Q_c A^T + Y_c^T B^T + AQ_c + BY_c$ and the feedback gain can be obtained by $K = Y_c Q_c^{-1}$. Due to the fact that objective function is formulated by LMI (7.78) it is noted that the Lyapunov inequality (7.16) is integrated in (7.78) by $W_c$ with the change of variables $Z \rightarrow Q_c$ and $Y \rightarrow Y_c$. Furthermore, the Metzlerian structural constraint should be written as $(AQ_c + BY_c)_{ij} \geq 0, \forall i \neq j$. The above development leads to the following result.

**Theorem 7.6.3.** Let us consider the closed-loop system (7.73) with $D = 0, F = 0$ and its equivalent representation in terms of structural perturbations $\dot{x}(t) = (A_c + E \Delta C)x(t)$, where $A_c = (A + BK)$ and $\hat{H}(s) = C(sI - A_c)^{-1}E$. Then the following statements are equivalent:

1. There exists a state feedback gain matrix $K$ such that the closed loop system is positive, asymptotically stable and $L_2$-gain of mapping $w \rightarrow z$ is less than $\gamma > 0$. 

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2. There exists a state feedback gain matrix $K$ such that the closed-loop system with its equivalent representation is positive, asymptotically stable with maximum stability radius.

3. There exists a state feedback gain matrix $K$ such that the following LMI

$$\min_{\gamma} \gamma$$

subject to

$$\begin{bmatrix} W_c & E & Q_cC^T \\ E^T & -\gamma I & 0 \\ CQ_c & 0 & -\gamma I \end{bmatrix} \prec 0 \quad (7.79)$$

$$(AQ_c + BY_c)_{ij} \geq 0 \text{ for all } i \neq j$$

is feasible with respect to diagonal positive definite matrix $Q_c \succ 0$ and the matrix $Y_c$ where $W_c = Q_cA^T + Y_c^TB^T + AQ_c + BY_c$. In such a case the feasible solution is given by $K = Y_cQ_c^{-1}$.

A final significant result of this section is the unique characteristic of the feedback gain matrices obtained from solving LPs and LMI for computing optimal $L_\sigma$-gains for $\sigma = 1, 2$ and $\infty$. This is reflected in the following theorem.

**Theorem 7.6.4.** Let the feedback gain matrices obtained from the optimal solution of LP$_1$, LP$_\infty$ and LMI be given by $K_\sigma$, $\sigma = 1, \infty, 2$, respectively. Let also the $L_\sigma$-gains be written as

$$\hat{g}_\sigma = \|C(A + BK_\sigma)^{-1}E\|_\sigma \quad \sigma = 1, 2, \infty$$

and define the cross $L_\sigma$-gains by

$$\hat{g}_{\sigma\bar{\sigma}} = \|C(A + BK_\sigma)^{-1}E\|_\sigma \quad \bar{\sigma} \neq \sigma$$

where $\bar{\sigma} = 1, 2, \infty$. Then we have

$$\hat{g}_{\sigma\bar{\sigma}} = \hat{g}_\sigma. \quad (7.80)$$

and $\hat{g}_\sigma$ admits the same inequalities as [7.72].

**Proof.** First, let us prove the theorem for the case $\hat{g}_{\sigma\bar{\sigma}} = \hat{g}_\sigma$, where $\sigma = 2$ and $\bar{\sigma} = \infty$. So, we claim that the cross $L_2$-gain with respect to $K_\infty$ is the same as $L_2$-gain, i.e., $\hat{g}_{2\infty} = \hat{g}_2$ where

$$\hat{g}_{2\infty} = \|C(A + BK_\infty)^{-1}E\|_2$$
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and

\[ \hat{g}_2 = \|C(A + BK_2)^{-1}E\|_2. \]

Suppose \( \hat{g}_2 \geq \hat{g}_{2\infty} \), then by defining \( L_\infty \)-gain

\[ \hat{g}_\infty = \|C(A + BK_\infty)^{-1}E\|_\infty, \]

it is clear from the inequality condition (7.72) that \( \hat{g}_{2\infty} \geq \frac{1}{\sqrt{p}} \hat{g}_\infty \). Thus we have

(i) \( \hat{g}_2 \geq \hat{g}_{2\infty} \geq \frac{1}{\sqrt{p}} \hat{g}_\infty \)

Now, suppose \( \hat{g}_2 \leq \hat{g}_{2\infty} \) and assume \( \hat{g}_2 \geq \frac{1}{\sqrt{p}} \hat{g}_\infty \), then it follows that

(ii) \( \hat{g}_{2\infty} \geq \hat{g}_2 \geq \frac{1}{\sqrt{p}} \hat{g}_\infty \)

It is clear that (ii) contradicts (i) and one can conclude \( \hat{g}_{2\infty} = \hat{g}_2 \). On the other hand, if one assumes \( \hat{g}_2 \leq \frac{1}{\sqrt{p}} \hat{g}_\infty \), then it follows that

(iii) \( \hat{g}_2 \leq \frac{1}{\sqrt{p}} \hat{g}_\infty \leq \hat{g}_{2\infty} \)

which also shows that (iii) contradicts with (i). This leads to the conclusion that \( \hat{g}_{2\infty} = \hat{g}_2 \). The remaining equalities \( \hat{g}_{2\sigma \bar{\sigma}} = \hat{g}_{\sigma \bar{\sigma}} \) for all \( \sigma \) and \( \bar{\sigma} \) are valid by similar reasoning. Consequently, we have \( \hat{g}_{12} = \hat{g}_{1\infty} = \hat{g}_1 \), \( \hat{g}_{21} = \hat{g}_{2\infty} = \hat{g}_2 \) and \( \hat{g}_{\infty 1} = \hat{g}_{\infty 2} = \hat{g}_\infty \). It should be pointed out that for SISO case the gains \( K_1, K_2 \) and \( K_\infty \) coincide to a unique state feedback gain \( K \), which leads to the fact that \( \hat{g}_1 = \hat{g}_2 = \hat{g}_\infty \).

\[ \blacksquare \]

Example 7.6.1. Consider the stable Metzlerian system with

\[
A = \begin{bmatrix}
-3 & 1 & 2 \\
1 & -5 & 1 \\
0 & 1 & -6
\end{bmatrix},
E = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 0
\end{bmatrix},
C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

Using the direct formula for stability radius one can obtain \( L_\sigma \)-gains from (7.71) without resorting to LP (7.56) or LP (7.57) of Lemma 7.6.1 which were only used to compute \( L_1 \)-gain and \( L_\infty \)-gains. Thus, we obtain \( g_1 = 0.5316, g_2 = 0.5020 \) and \( g_\infty = 0.6076 \) which satisfy the inequality (7.72).
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Example 7.6.2. Consider the following unstable system with $F = 0$, $D = 0$ and

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 1 & 3 \\ 2 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, 
C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying Theorem 7.6.3 by solving the LMI (7.79) we get

$$K_2 = \begin{bmatrix} -1.2328 & 0.4764 & 0.0007 \\ -0.7672 & -2.4764 & -2.9993 \end{bmatrix}$$

and the closed-loop system matrix becomes stable Metzler matrix

$$A_c = \begin{bmatrix} -3.2328 & 1.4764 & 0.0007 \\ 0.2328 & -1.4764 & 0.0007 \\ 0 & 0 & -1.9986 \end{bmatrix}$$

with maximum stability radius $r_2 = 2.1213$ and the corresponding $L_2$-gain of $\hat{g}_2 = 0.4715$. With this feedback gain, the corresponding cross $L_\sigma$-gains are obtained as $\hat{g}_\infty = 0.6668$ and $\hat{g}_{12} = 0.3335$ which are the same as $\hat{g}_\infty$ and $\hat{g}_1$. Applying the LP (7.74) confirms that $\hat{g}_\infty = \hat{g}_\infty^2 = 0.6668$. 
Chapter 8

Positive Stabilization and Eigenvalue Assignment for Discrete-Time Systems

In this chapter we formulate and solve the problem of eigenvalue assignments for discrete-time systems with positivity constraint. The goal of this chapter is to solve the stabilization problem of discrete-time positive systems under the constraint that the eigenvalues of the closed-loop system are placed in the desired location while maintaining the positivity structure. Although the problem of positive stabilization has been solved using LP and LMI methods, the problem of eigenvalue assignment with positivity constraints is complex and remains challenging. It has only been tackled for a restricted class of single-input discrete-time positive systems. This chapter aims to provide a solution for the multi-input case. After a brief review of discrete-time positive systems and their stability properties, spectral characteristics of stable positive discrete-time systems will be analyzed and the eigenvalue assignment will be achieved by solving a set of chain equations. Numerical examples are provided to support theoretical development.

8.1 Discrete-Time Positive System

Consider a linear discrete-time system described by

\[ x(k+1) = Ax(k) + Bu(k) \]  \hspace{1cm} (8.1)
\[ y(k) = Cx(k) + Du(k) \]  \hspace{1cm} (8.2)

where \( x(k) \in \mathbb{R}^n \), \( u(k) \in \mathbb{R}^m \), and \( y(k) \in \mathbb{R}^p \) represents state, input, and output of the system, respectively.
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Definition 8.1.1. The system (8.1), (8.2) is called internally positive system if for every initial condition, \( x_0 \in \mathbb{R}_+^n \) and input \( u(k) \in \mathbb{R}_+^m \), we have \( x(k) \in \mathbb{R}_+^n \) and \( y(k) \in \mathbb{R}_+^p \) for \( k \geq 0 \).

Theorem 8.1.1. The system (8.1), (8.2) is internally positive if and only if \( A \in \mathbb{R}_+^{n \times n} \), \( B \in \mathbb{R}_+^{n \times m} \), \( C \in \mathbb{R}_+^{p \times n} \), \( D \in \mathbb{R}_+^{p \times m} \) are nonnegative (positive) matrices.

According to the well-known Frobenius-Perron Theorem the spectral radius \( \rho(A) = \{ \lambda : \max |\lambda_i|, \ \forall i = 1, \ldots, n \} \) is real and the corresponding eigenvector \( v \geq 0 \).

Theorem 8.1.2. Let the system (8.1), (8.2) be Positive. Then it is asymptotically stable if and only if any one of the following equivalent conditions is satisfied:

1. The leading principal minors of \( I - A \) are positive.
2. The matrix \( I - A \) is a nonsingular M-matrix and \( [I - A]^{-1} > 0 \).
3. There exists a diagonal positive definite matrix \( P > 0 \) such that the discrete Lyapunov inequality \( A^T P A - P < 0 \) is feasible.
4. The LMI

\[
\begin{bmatrix}
-P & A^T P \\
PA & -P
\end{bmatrix} < 0
\]

is feasible which is the Schur complement of the above Lyapunov inequality.
5. There exists a vector \( z \in \mathbb{R}_+^n \) such that \( (A - I)z < 0 \).

The stability robustness properties of the positive systems [3, 5, 46] is a motivating factor to look into the positive stabilization problem of general dynamical systems. Although positive stabilization can be realized using LP and LMI, the problem of eigenvalue assignment with positivity constraints is not trivial and requires careful considerations.

8.1.1 Positive Stabilization for Discrete-Time Systems

In this section we consider the constrained positive stabilization for system (8.1), (8.2) by a state feedback control law. This control law must be designed in such a way that the resulting closed-loop system is positive and asymptotically stable. This is the first step to make sure that stabilization is possible. Then one can pursue stabilization with additional constraints of eigenvalue assignment as will be shown below.
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Let the state feedback control law
\[ u(k) = v(k) + Kx(k) \]  
be applied to the system \((8.1), (8.2)\). Then the closed-loop system is written as
\[ x(k + 1) = A_c x(k) + Bv(k) \]  
where \( A_c = A + BK \). Thus, in our design procedure we need to find \( K \in \mathbb{R}^{m \times n} \) such that \( A + BK \) is a stable nonnegative matrix. There are many ways to achieve this goal by applying the equivalent conditions of Theorem 8.1.2 to \( A + BK \). For example, using the property 1, one can find the gain matrix \( K \) through a linear programming (LP) set-up \([6]\). Alternatively, one can construct an LP by using the property 5 or an LMI by using the property 4 as outlined in the following theorem which is a generalization of previous works \([7, 54, 56, 72]\) applied to discrete-time systems.

**Theorem 8.1.3.** There exist a state feedback control law \((8.3)\) for the system \((8.1), (8.2)\) such that the closed-loop system \((8.4)\) becomes positive stable if and only if

1. The following LP has a feasible solution with respect to the variables \( y_i \in \mathbb{R}^n \), \( \forall i = 1, \ldots, n \) and \( z = [z_1 \ z_2 \ \cdots \ z_n]^T \in \mathbb{R}^n \)
\[ (A - I)z + B \sum_{i=1}^{n} y_i < 0, \quad z > 0 \]  
\[ y_i \geq 0 \quad \text{for } i = 1, \ldots, n \]  
\[ a_{ij} z_j + b_{ij} y_j \geq 0 \quad \text{for } i, j = 1, \ldots, n \]  
with \( A = [a_{ij}] \) and \( B = \begin{bmatrix} b_{11}^T & b_{12}^T & \cdots & b_{1n}^T \ b_{21}^T & b_{22}^T & \cdots & b_{2n}^T \ \vdots & \vdots & \ddots & \vdots \ b_{n1}^T & b_{n2}^T & \cdots & b_{nn}^T \end{bmatrix}^T \). Furthermore, the gain matrix \( K \) is obtained from
\[ K = \begin{bmatrix} y_1 z_1 & y_2 z_2 & \cdots & y_n z_n \end{bmatrix} \]  

or

2. The following LMI has a feasible solution with respect to the variables \( Y \) and \( Z \)
\[ \begin{bmatrix} -Z & ZA^T + Y^T B^T \\ -AZ + BY & -Z \end{bmatrix} < 0 \]  
\[ AZ + BY \geq 0 \]  
where \( Z > 0 \) is diagonal positive definite matrix. Furthermore, the gain matrix \( K \) is obtained from \( K = YZ^{-1} \).
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The above LP or LMI solve the problem of positive stabilization for a conventional discrete-time state equation. Although positive stabilization for continuous and discrete time system have been solved, the problem of eigenvalue assignment for this class of system has not been completely solved. The best known results is only available for single-input discrete-time positive systems with controllable canonical form structure [73]. As we stated in the introduction, the aim of this chapter is to provide possible solutions for more general cases of multiple-input systems. To achieve this goal, we first restate the result in [73] and then show how to generalize the eigenvalue assignment to multiple-input discrete-time positive-systems with block controllable canonical structure.

8.1.2 Eigenvalue Assignment for Single-Input Positive Discrete-Time Systems

Consider the unstable positive single-input system described by (8.1), (8.2) represented by controllable canonical form with the parameters

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1
\end{bmatrix},
B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]  

(8.11)

If \(\lambda_1, \lambda_2, \ldots, \lambda_n\) are the desired eigenvalues of the closed-loop system matrix \(A_c\) then the desired characteristic equation becomes

\[
\Delta_d(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j) = \lambda^n + \hat{a}_1 \lambda^{n-1} + \cdots + \hat{a}_n
\]  

(8.12)

where the coefficients are represented by the elementary symmetric function \(S_j\)

\[
\hat{a}_j = (-1)^j S_j(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq i_1 < \cdots < i_j \leq n} \prod_{i_k=1}^{j} \lambda_{i_k}
\]  

(8.13)

for \(j = 1, \ldots, n\)

**Theorem 8.1.4.** There exists a state feedback gain matrix given by

\[
K = \begin{bmatrix}
a_n - \hat{a}_n & \cdots & a_1 - \hat{a}_1
\end{bmatrix}
\]  

(8.14)

such that the closed-loop system is asymptotically stable and positive and the matrix \(A_c \in \mathbb{R}^{n \times n}_+\) has the desired spectrum if the following conditions are satisfied.
1. There exists a real number \( \rho(A_c) \) representing the spectral radius of the closed-loop system matrix \( A_c \).

2. The eigenvalues occur in complex conjugate pairs.

3. \((-1)^jS_j(\lambda_1, \ldots, \lambda_n) \geq 0 \) for \( j = 1, \ldots, n \).

The proof of theorem can be established by construction. (see [73]).

### 8.1.3 Eigenvalue Assignment for Multi-Input Positive Discrete-Time Systems in Block Controllable Canonical Form

Many dynamical systems are modeled by a second or higher order vector difference equations of the form

\[
\sum_{j=0}^{r} A_{r-j} z(k+j) = u(k) \tag{8.15}
\]

where \( z(k) \in \mathbb{R}^m, A_j \in \mathbb{R}^{m \times m} \) for \( j = 0, 1, \ldots, r \) with \( A_0 = I_m \) and \( u(k) \in \mathbb{R}^m \). This type of systems can be realized into Block Controllable Canonical Form (BCCF)

\[
x(k+1) = Ax(k) + Bu(k) \tag{8.16}
\]

\[
y(k) = Cx(k)
\]

where

\[
A = \begin{bmatrix}
O_m & I_m & O_m & \cdots & O_m \\
O_m & O_m & I_m & \cdots & O_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O_m & O_m & O_m & \cdots & I_m \\
-A_r & -A_{r-1} & -A_{r-2} & \cdots & -A_1
\end{bmatrix},
B = \begin{bmatrix}
O_m \\
O_m \\
\vdots \\
O_m \\
I_m
\end{bmatrix},
C = \begin{bmatrix}
C_0 & C_1 & C_2 & \cdots & C_{r-1}
\end{bmatrix}
\tag{8.17}
\]

with \( x(k) = \begin{bmatrix} z(k) & z(k+1) & \cdots & z(k+r-1) \end{bmatrix}^T \). \( C_j \in \mathbb{R}^{m \times m} \), and \( n = rm \). The associated polynomial matrix of (8.15) is given by

\[
P(z) = \sum_{j=0}^{r} A_{r-j} z^j \tag{8.18}
\]
Definition 8.1.2. The BCCF (8.17) is called Nonnegative BCCF if and only if \(-A_i\)'s are all nonnegative matrices.

The poles of the system (8.16) are the latent roots of the polynomial matrix \(P(z)\) defined as \(\lambda(P) = \{ z \in \mathbb{C} : \det P(z) = 0 \}\). This is the same as the spectrum of the matrix \(A\) since \(\det(P(z)) = \det(\lambda I - A)\) with equal argument, i.e. \(\lambda(P(z)) = \lambda(A)\). Furthermore, the system (8.16) is stable if all eigenvalues of the matrix \(A\) or equivalently all latent roots of \(P(z)\) lie in the unit disk of the z-plane.

The connection between the stability of the polynomial matrix \(P(z)\) and the matrix \(A\) plays an important role. In particular, if in the expansion (8.18) associated with (8.15) \(A_0 = I_m\), then there is a one to one correspondence between the coefficient matrices of (8.18) and the block companion structure of the matrix \(A\). However, if \(A_0 \neq I_m\), then appropriate adjustment should be performed to find this correspondence. We are not going to elaborate on this and refer interested readers to [74].

The stability of single-input single-output systems can be analyzed by Jury test of stability through the coefficient of the characteristic polynomial \(a_j\)'s. However, the stability of dynamical systems modeled by (8.15) in terms of its coefficient matrices \(A_j\) is not obvious. The best known results have been established only for second-order vector difference equation.

In this section we consider the problem of constrained stabilization of systems represented by BCCF and provide a solution for this class of system.

Let the state feedback control law

\[
\begin{align*}
    u &= v + Kx = v + \begin{bmatrix} K_r & K_{r-1} & \cdots & K_1 \end{bmatrix} x 
\end{align*}
\]

be applied to the controllable system (8.16) and (8.17). Then the closed-loop system preserves the BCCF with

\[
    -\hat{A}_i = -A_i + K_i \quad \text{for } i = 1, \ldots, r
\]

Clearly, the corresponding polynomial matrix associated with \(\hat{A}_i\) is \(\hat{D}(z) = \sum_{i=0}^r \hat{A}_i z^{r-i}\) with \(\hat{A}_0 = I_m, \hat{A}_i \in \mathbb{R}^{m \times m}\), and \(\hat{A}\) is the desired block companion matrix yet to be determined based on
the specified eigenvalues. Let us define

\[
F = \begin{bmatrix}
F_1 & I_m & 0 & \cdots & 0 \\
0 & F_2 & I_m & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & F_{r-1} & I_m \\
0 & \cdots & \cdots & 0 & F_r
\end{bmatrix}
\]  

(8.21)

such that the desired eigenvalues are distributed among \(F_i\)'s. Then the matrix \(F\) is a linearization of \(\hat{D}(z)\) and the coefficient of \(\hat{D}(z)\) can be specified by \(F_i\)'s according to the following theorem.

**Theorem 8.1.5.** The transformation matrix \(P\) that transforms the known block diagonal matrix \(F\) to the block companion matrix \(\hat{A}\), i.e. \(\hat{A} = P^{-1}FP\) is a lower-triangular matrix with \((i,j)\)-th block \(P_{i,j} = I_m\) for \(i = j\) and \(P_{i,j}\) for \(i > j\) satisfying the similar set of chain equations as (6.13). Furthermore,

\[
\hat{A}_{r-i} = P_{r,i} - F_r P_{r,i+1} \quad \text{for } i = 0, 1, \ldots, r - 1
\]  

(8.22)

where \(P_{r,0} = 0\).

By employing the same procedure discussed in Section 6.2 of Chapter 6 and applying following minor modifications we can achieve positive eigenvalue assignment for multi-input discrete-time systems.

**Lemma 8.1.1.** Let \(F_i \in \mathbb{R}^{m \times m}\) for \(i = 1, \ldots, r\) be a set of block stable matrices in (8.21), each with multiple blocks of order 2 or 1 such that for even or odd \(m\), the blocks are properly distributed to construct (8.21) provided that the following conditions are satisfied.

1. There exists a real number \(\rho(\hat{A})\) representing the spectral radius of the closed-loop system matrix \(\hat{A}\)

2. The eigenvalues occur in complex conjugate pairs

3. \(-\hat{A}_k > 0\) for \(k = 1, \ldots, r\), where \(\hat{A}_k = (-1)^k TR_k[F]\) with \(TR_k[F]\) defined by

\[
TR_k[F] = \sum \begin{bmatrix}
F_1 & I \\
F_2 & \ddots \\
\vdots & \ddots & I \\
F_k
\end{bmatrix}
\]  

(8.23)
Then the feedback gain

\[ K = \begin{bmatrix} K_r & \cdots & K_1 \end{bmatrix} \]  

(8.24)

with \( K_i = A_i - \hat{A}_i \) will result in a positive block companion matrix.

**Proof.** This Lemma is a generalization of Theorem 8.1.4 for multi-input case and the proof is constructive and directly follows from Theorem 8.1.5. It should be pointed out that based on the procedure recently proposed in [60] one can easily construct the block matrices \( F_i \)'s such that the conditions 1-3 are satisfied. ■

### 8.1.4 Alternative Method of Eigenvalue Assignment for Multi-Input Positive Discrete-Time Systems

It is well-known that for controllable multi-input systems there exist several approaches for eigenvalue assignments by state feedback without restricting the structure of \( \{A, B\} \) pair [75]. However, when positivity constraints is imposed for the closed-loop system matrix, those methods can not be employed directly. Without loss of generality, we assume that the unstable positive system with the pair \( \{A, B\} \) is monomially transformed such that \( A \) is in companion form and

\[ B = \begin{bmatrix} 0 & \beta I_m \end{bmatrix}^T. \]

Then the following lemmas [75] are useful for transforming the above multi-input system problem to a single-input one which can be solved by applying the technique of Theorem 8.1.4.

**Lemma 8.1.2.** If \( \{A, B\} \) is a controllable pair, then for almost any \( m \times n \) real constant matrix \( K_1 \), all eigenvalues of \( A + BK_1 \) are distinct and consequently \( A + BK_1 \) is cyclic.

Recall that a matrix is cyclic if its characteristic polynomial is equal to its minimal polynomial or equivalently it has only one Jordan block associated with each distinct eigenvalue.

**Lemma 8.1.3.** If \( \{A, B\} \) is a controllable pair and if \( A \) is cyclic, then for almost any \( m \times 1 \) real vector \( q \) the pair \( \{A, Bq\} \) is controllable.

It is clear that Lemma 8.1.2 allows the system matrix \( A \) of the above defined unstable positive system to be positive and cyclic by applying a preliminary state feedback. However, for simplicity we let \( A \) to be a positive unstable companion matrix which is obviously cyclic. Let us denote the columns of the input matrix by \( b_i \) and the elements of the vector \( q \) by \( q_i \)'s. Then using Lemma 8.1.3 there exists \( q_i \)'s; \( i = 1, \ldots, m \) such that the pair \( \{A, \hat{b}\} \) remains controllable.
where $\hat{b} = q_i b_i$ represents a linear combination of $b_i$’s. It is evident that there always exist $q_i$’s such that $\hat{b} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T$ due to the fact that the pair $\{A, \hat{b}\}$ must remain in controllable canonical form. Consequently, Theorem 8.1.4 can be applied to the pair $\{A, \hat{b}\}$ or equivalently a dyadic approach can be employed in which $K$ is reduced to a unit rank matrix by expressing it as a product of two vectors $K = q \kappa$ where $q \in \mathbb{R}^{m \times 1}$ is a column vector and $\kappa \in \mathbb{R}^{1 \times n}$ is a row vector. This procedure simplifies the proposed method in [76].

Avoiding the dyadic design and assuming that the system matrix $A$ is not in companion form, we provide a systematic approach for multi-input case in special input identifiable form. Thus, the system matrix in (8.16), (8.17) is assumed to be an arbitrary positive unstable matrix and the matrix $B$ remains as $B = \begin{bmatrix} 0 & I_m \end{bmatrix}^T$. From (8.4) with $A_c = A + BK$ one can write $A_c - A = BK$ and we have the following result.

**Theorem 8.1.6.** Let the closed-loop system matrix $A_c$ be a given stable nonnegative matrix with desired eigenvalues. Then, there exists a state feedback gain matrix $K$ such that $A_c = A + BK$ if and only if

$$ (A_c - A) \in R(B) $$

where $R(.)$ denotes the range space of a matrix, or equivalently

$$ (I_n - B(B^T B)^{-1}B^T)(A_c - A) = 0 $$

Furthermore, the resulting feedback gain matrix $K$ is determined by

$$ K = (B^T B)^{-1}B^T(A_c - A) $$

Let the set of desired eigenvalues be given by $\lambda_i, i = 1, \ldots, n$. Then one can use the procedure of solving nonnegative inverse eigenvalue problem (NIEP) proposed in [60] to generate stable nonnegative matrices $A_c$ such that the condition of the above Theorem is satisfied. Alternatively, the following lemma can be used to generate desired nonnegative closed-loop matrices $A_c$.

**Lemma 8.1.4.** Let us define an auxiliary system with the pair $\{\tilde{A}, \tilde{B}\}$ where

$$ \tilde{A} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}, \quad \tilde{B} = B = \begin{bmatrix} 0 \\ I_m \end{bmatrix} $$

with $A_1$ representing the first $n - m$ rows of the system matrix $A$. Then, there exists a matrix $A_2$ such that

$$ A_c = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} $$
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is Schur stable.

Proof. The proof can easily be established by writing

\[
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
= \begin{bmatrix}
A_1 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
I_m
\end{bmatrix} A_2
\] (8.30)

Then due to the controllability of the pair \( \{\tilde{A}, \tilde{B}\} \), \( A_2 \) can be determined by using any pole placement approach. 

Since the desired eigenvalues must satisfy the nonnegativity condition of NIEP, a nonnegative matrix \( A_2 \) can always be found by repeated application of the Lemma 8.1.4

8.2 Illustrative Examples

Example 8.2.1. Let the controllable pair \((A, B)\) of the system (8.16) be represented by (8.17) with

\[
A_1 = \begin{bmatrix}
-1 & -1 \\
-1 & -1
\end{bmatrix},
A_2 = \begin{bmatrix}
-3 & -2 \\
-2 & -1
\end{bmatrix}
\]

which represents a positive but unstable system. The goal is to stabilize the system with the desired eigenvalues \( \Lambda = \{-0.1, -0.2, 0.3, 0.4\} \) while maintaining the structure of block coefficient matrices. By properly choosing \( F_i \)'s as follows

\[
F_1 = \begin{bmatrix}
-0.1 & 0 \\
0 & -0.2
\end{bmatrix},
F_2 = \begin{bmatrix}
0.3 & 0.2 \\
0 & 0.4
\end{bmatrix}
\]

and using Lemma 8.1.1 we have

\[
\hat{A}_1 = \begin{bmatrix}
-0.2 & -0.2 \\
0 & -0.2
\end{bmatrix},
\hat{A}_2 = \begin{bmatrix}
-0.03 & -0.02 \\
0 & -0.08
\end{bmatrix}
\]

which clearly shows that \( -\hat{A}_1 \) and \( -\hat{A}_2 \) are positive stable matrices. Then the feedback gain \( K \) is computed from (8.20) as

\[
K = \begin{bmatrix}
-2.97 & -1.98 & -0.8 & -0.8 \\
-2 & -0.92 & -1 & -0.8
\end{bmatrix}
\]
Example 8.2.2. Consider the following unstable positive discrete-time system

\[
    x(k + 1) = \begin{bmatrix} 0.2 & 0 & 0.4 \\ 0.5 & 0.5 & 0.9 \\ 0.8 & 0.25 & 0.3 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(k)
\]

with unstable eigenvalues \{1.11, -0.4, 0.28\}. It is desired to shift the eigenvalues to \{0.1, 0.2, 0.3\} while maintaining the positivity of the closed-loop system. Since the desired eigenvalues satisfy the condition of NIEP, using the procedure of [60] the following matrix \(A_c\) is obtained.

\[
    A_c = \begin{bmatrix} 0.2 & 0 & 0.4 \\ 0 & 0.37 & 0.15 \\ 0 & 0.01 & 0.33 \end{bmatrix}
\]

Then from (8.27) we can determine the state feedback gain

\[
    K = \begin{bmatrix} -0.5 & -0.13 & -0.75 \\ -0.8 & -0.24 & 0.03 \end{bmatrix}
\]

Next, by using the procedure described in Lemma 8.1.4 we define the auxiliary system with the following pair

\[
    \tilde{A} = \begin{bmatrix} 0.2 & 0 & 0.4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

and by applying constrained eigenvalue assignment to the pair \(\tilde{A}, \tilde{B}\) we can find

\[
    A_2 = \begin{bmatrix} 0 & 0.1 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}
\]

which leads to the positive stable matrix

\[
    A_c = \begin{bmatrix} 0.2 & 0 & 0.4 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}
\]

with state feedback gain \(K = \begin{bmatrix} -0.5 & -0.4 & -0.9 \\ -0.8 & -0.25 & 0 \end{bmatrix}\) computed from (8.27).
Chapter 9

Conclusion

In this dissertation, special classes of positive and symmetric systems have been thoroughly studied. To better grasp their properties, an introduction to matrices with special structures were provided in Chapter 2. In particular, nonnegative and symmetric matrices were discussed along with their stability properties prior to a deep dive into the positive and symmetric systems definitions in Chapter 3. Robustness properties of positive systems are also explored in Chapter 3 and two type of positive symmetric systems have been introduced. In Chapter 4, the constrained stabilization problems for general dynamical systems have been solved to achieve the closed-loop system with the same desirable properties as positive systems. The dual problem of observer design for positive systems is considered in Chapter 5 in which the PUO is designed to determine the states of positive systems decoupled form the unknown inputs. Positive observer for all type of faulty systems have been discussed in the presence of both actuator and/or sensor faults. Furthermore, the PI observer is merged with UIO to achieve robust fault detection. The design of observer also is useful in connection to stabilization and control of dynamic systems. Consequently, a major thrust of this dissertation was devoted to formulate and solve the stabilization problems for aforementioned classes of positive and symmetric systems. The design of constrained symmetric Metzlerian stabilization was discussed in Chapter 6 along with its generalization for systems in BCCF. Moreover, the positive and symmetric control problems were discussed in Chapter 7. First, the problem of LQR under positivity constraints was solved. Design procedures for static and dynamic output feedback controllers with positivity and symmetry constraints were also explored. Finally, the positive stabilization and eigenvalue assignment for discrete-time systems were addressed in Chapter 8 for a special class of systems as a parallel treatment of continuous-time case.

Although a thorough study of positive and symmetric systems has been conducted in
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this dissertation, there are still more opportunities to expand the direction of this research. Several unsolved problems of interest include:

1. Eigenvalue assignment with positivity constraint for general class of systems.

2. Constrained stabilization and control problems for time-delay systems, singular systems, and fractional-order systems.

3. Generalization of positive estimation and control for nonlinear and multi-agent systems.

4. Due to the fact that positive systems appear also in biology, finance, and medicine, it is of particular interest to investigate control techniques for these and other relevant applications.
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