Systematic Phenomenology on the Landscape of Calabi–Yau Hypersurfaces in Toric Varieties

by Ross Altman

B.S. in Applied and Engineering Physics, Cornell University
M.Eng. in Applied Physics, Cornell University
M.S. in Physics, Northeastern University

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Brent Nelson
Professor of Physics
Dedication

This thesis is enthusiastically dedicated to my mother, Lori Altman, who has always kept my feet planted firmly on the ground, while never discouraging me from lifting my head into the clouds. I am forever as grateful as I am metaphorically tall.
Acknowledgements

I could not have accomplished a fraction of this work without the help of my early collaborators James Gray, Yang-Hui He, Vishnu Jejjala, Benjamin Jurke, Joan Simón. They have been instrumental in molding me into the physicist I am today. I would also like to thank my more recent collaborator Xin Gao for all his hard work and insight. To my academic siblings Jonathan Carifio and Gustavo Salinas de Souza, we make a good team, and I look forward to many fruitful collaborations in the future. Much of this work is due in large part to Jon’s efforts. I would like to thank Lara Anderson, Per Berglund, Volker Braun, Stefan Groot Nibbelink, Seung-Joo Lee, André Lukas, Michele Cicoli, and Washington Taylor for many helpful discussions during the development of this database. I would also like to thank Balázs Szendrői for his valuable feedback on the topic of gluing Kähler cones. I must also acknowledge the Discovery cluster at the Massachusetts Green High Performance Computing Center (MGHPCC), and in particular its administrator Nilay Roy, for all his help and patience in assisting with my computational efforts. Finally, I am in debt to the pioneering work of Max Kreuzer. I have not been in the field long enough to have known him, but his work, both mathematical and computational, have been a constant inspiration. I hope that my work on this database serves his legacy well. Much of this work was funded by the U.S. National Science Foundation under the grant CCF-1048082, EAGER: CiC: A String Cartography.
Abstract of Dissertation

The largest known database of Calabi-Yau threefold string vacua was famously produced by Kreuzer and Skarke in the form of a complete construction of all 473,800,776 reflexive polyhedra that exist in four dimensions [1]. These reflexive polyhedra describe the singular limits of ambient Gorenstein toric Fano varieties in which Calabi-Yau threefolds are known to exist as the associated anticanonical hypersurfaces. In this thesis, we review how to unpack the topological and geometric information describing these Calabi-Yau threefolds using the toric construction, and provide, in a companion online database (see www.rossealtman.com), a detailed inventory of these quantities which are of interest to string phenomenologists. Many of the singular ambient varieties associated to the Kreuzer-Skarke list can be partially smoothed out into a multiplicity of distinct, terminal toric ambient spaces, each of which may embed a unique Calabi-Yau threefold. Some, however are not unique, and can be identified through topological and smoothness constraints. A distribution of the unique Calabi-Yau threefolds which can be obtained from each 4D reflexive polyhedron, will be provided up to current computational limits. In addition, we will detail the computation of a variety of quantities associated to each of these vacua, such as the Chern classes, Hodge data, intersection numbers, and the Kähler and Mori cones.

Then, moving on to actual string phenomenology on the Calabi-Yau compactification vacua, we outline the prescription for moduli stabilization with a supersymmetry breaking vacuum known as the LARGE Volume Scenario (LVS), paying particular attention to the so-called “Swiss cheese” models. It is an important open problem in string model building to identify the set of Swiss cheese solutions within the space of Calabi-Yau threefolds. In this thesis, we present an algorithm to isolate a special subset of Swiss cheese solutions that are characterized by “holes,” or small 4-cycles in homology, descending from the toric
divisors inherent to the original four dimensional reflexive polyhedra. Implementing these methods, we find 2, 313 “toric” Swiss cheese manifolds, over half of which have \( h^{1,1} = 6 \). Of these, 70 have two or more large 4-cycles and a flat direction in the effective potential. In an explicit example, we find a stable minimum for the small Kähler moduli and a flat direction in the large moduli.

Finally, we approach the subject of orientifolding the Calabi-Yau threefold vacuum of a type IIB theory in order to break \( \mathcal{N} = 2 \) supergravity down to \( \mathcal{N} = 1 \) in the low energy effective theory. To this end, we describe the process of choosing a non-trivial \( \mathbb{Z}_2 \) involution, and locating its fixed points on the compactification manifold. It will be shown that consistency of this involution across the full Kähler cone is very restrictive and results in at most O3/O7 planes in nearly every case. We also discuss the splitting of the Kähler moduli space of the orientifold into even and odd parity components, and present concrete examples demonstrating this process.
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Chapter 1

Introduction

It is well known that Calabi-Yau manifolds provide one of the simplest methods for compactifying the extra dimensions of string theory while preserving some supersymmetry in the four dimensional, low energy effective theory. Following the original idea of Kaluza and Klein [2,3], the topology and geometry of the compact extra dimensions determine the effective dynamics seen in the lower dimensional theory. Such geometrical vacuum constructions offer a rich framework for string phenomenology, as well as more formal applications, in a variety of different string theoretic contexts.

A compelling\(^1\) early example of the use of Calabi-Yau threefolds in string theory is apparent in the subject of heterotic string phenomenology, as originally described in [4]. During the last decade, semi-realistic constructions exhibiting precisely the charged matter spectrum of the Standard Model of particle physics have been derived numerous times.

\(^1\)It should be emphasized that there is little consensus within the field as to which string theoretic construction is most likely to lead to realistic low energy particle physics, with each approach having its own strengths and weaknesses.
within this context [5–11]. A unifying theme in this thesis is that, given the link between the compactification geometry and the low energy dynamics, knowledge of the geometry of the underlying Calabi-Yau threefold vacuum is crucial in order to make any progress. Over the last three decades, various large datasets of explicit Calabi-Yau constructions have been developed, ranging from the so-called complete-intersection Calabi-Yau manifolds (CICYs) [12–18], to elliptically-fibered manifolds over toric bases [19,20], and hypersurfaces in toric ambient spaces [1,21,22]. By far the largest set found to date is the latter, due to the impressive analysis of 4-dimensional, reflexive lattice polytopes by Kreuzer and Skarke in the 1990s, which allows us to realize a Calabi-Yau threefold as a hypersurface in a 4-dimensional Gorenstein toric Fano variety [1,21]. A multitude of different data sets have been extracted from this initial work. While all of the applications are too numerous to list here, some interesting examples of such work done with this data can be found in [23–30].

In a seminal work by Batyrev et al. [31], it was shown that the anticanonical hypersurface in a toric variety is generically Calabi-Yau if the object underlying the construction of the variety, a convex lattice polytope, obeys the rather strong condition of reflexivity. The classification of inequivalent reflexive polytopes is an exercise in combinatorics and is thus amenable to computer analysis. The software package PALP (Package for Analyzing Lattice Polytopes) [32] was written with precisely this goal in mind. In all, a finite, yet very impressive 473,800,776 reflexive polytopes in four dimensions were found, each of which maps to an ambient toric variety. Moreover, each of these singular varieties
admits at least one, but potentially many maximal projective crepant partial (MPCP) desingularizations, some of which represent adjacent regions in the moduli space of the same manifold, and some of which are entirely independent. This leaves us with an indeterminate, but undeniably large class of smooth Calabi-Yau threefold hypersurfaces, well in excess of the half billion reflexive polytopes\(^2\). Some important topological information including the Hodge numbers for the associated Calabi-Yau hypersurface can be found using purely combinatoric methods, and were recorded by Kreuzer and Skarke in their dataset [33]. In this thesis, and in the database [34] hosted at the accompanying website www.rossealtman.com, we wish to expand upon the Kreuzer-Skarke result, in order to fill a number of gaps in the information that is currently publicly available.

1. Although the number of reflexive polytopes in the Kreuzer-Skarke database is well understood, it is unclear precisely how many distinct Calabi-Yau threefolds emerge from this list. Some of the geometrical data of these manifolds, such as Hodge numbers, can be determined purely in terms of the ambient polytope data. Of the 473,800,776 reflexive polytopes, it has been shown that there are 30,108 distinct pairs \((h^{1,1}, h^{2,1})\) counting the Kähler and complex structure moduli of the geometries, respectively. However, different triangulations into simplexes of the same polytope can potentially give rise to different Calabi-Yau hypersurfaces, which, while agreeing on this basic geometric data, differ in more subtle ways. In short, even taking into account the redundancy present in describing the same Calabi-Yau

\(^2\)For context, one might compare this to the case of 2- and 3-dimensional toric ambient spaces giving rise to Calabi-Yau one- and two-fold hypersurfaces (i.e., the torus and K3 surfaces). In these dimensions there are only 16 and 4,319 reflexive polytopes, respectively.
Chapter 1. Introduction

manifolds as hypersurfaces in different ambient spaces, there are likely to be many
more than 473,800,776 Calabi-Yau threefolds in the Kreuzer-Skarke list. Unfortu-
nately, performing all of the necessary triangulations to access this data is extremely
computationally intensive, and PALP is only able to complete the necessary compu-
tations for relatively low values of $h^{1,1}$ [35]. Clearly, therefore, it is of use to have
as much of this data as possible pre-computed and archived in an easy-to-access
format.

2. For calculations in subjects such as model building (for example in the heterotic
case [36–38]), more than just the Hodge numbers of a Calabi-Yau threefold are
required. The Chern classes, Kähler cones, and triple intersection numbers, which
help to characterize the Calabi-Yau threefold in a more refined manner, for example,
are also necessary. These quantities can, again, be extracted using various versions,
some not yet publicly available, of PALP, but again, only up to relatively low values
of $h^{1,1}$. Therefore, as in the previous bullet point, it is clear that it would be
preferable to have as much of this data as possible pre-computed and archived in
an easy-to-access format.

3. A variety of code packages are available for calculating geometrical properties of
Calabi-Yau threefold hypersurfaces in toric varieties. Building upon the original
work of Max Kreuzer and his collaborators, there have been some updated versions
of PALP, consolidating and enhancing his contributions while providing additional
documentation for the program [35, 39]. New tools for analyzing such geometries
are now available within the context of Sage [40] and TOPCOM [41]. In addition to being computationally expensive to run over large data sets, the language of programs such as PALP assume a level of detailed mathematical knowledge which, for physicists who are unfamiliar with the subject and simply wish to extract particular properties of the Calabi-Yau manifolds, may seem somewhat cumbersome to learn. Once more we see that it would be useful to have the data that physicists are interested in pre-computed and archived in the language of a physicist.

In Chapter 2, we review the computational procedures for calculating the relevant topological and geometrical quantities of toric Calabi-Yau threefolds in a manner that is completely self-contained within the Sage computational package. Because of limitations in the computational power that has been applied to the problem to date, the database associated to this thesis will provide all systematic triangulations up to $h^{1,1} = 6$, but more will be added as it is processed. This already exceeds what can can be accessed with PALP, and importantly, the physicist accessing this data need not run or become familiar with any additional software. We expect to update the website as time goes by to accommodate ever increasing $h^{1,1}$.

With the enormous number of candidate Calabi-Yau compactifications in hand, model builders are confronted with the challenge of isolating the set of constructions which might potentially replicate physics in the real world. In type IIB string theory, the particularly difficult problem of moduli stabilization can be avoided via flux considerations in one of two prevailing Calabi-Yau threefold compactification paradigms: KKLT [42]
or the LARGE Volume Scenario (LVS) [28, 43, 44], in which both the axion-dilaton and
complex structure moduli can be assumed already stable at tree level due to flux consid-
erations. A particularly interesting subset of the latter class of solutions are the so-called
“Swiss cheese” compactifications. This name derives from the fact that a subset of the
Kähler moduli are large and control the overall volume of the manifold, while the the
rest of the Kähler moduli remain small and control the volumes of the “holes” at which
non-perturbative corrections to the superpotential, such as Euclidean D3-instantons, are
localized. In Chapter 3, we consider a special subclass of Swiss cheese compactifications
characterized by large and small cycles that descend directly from the toric divisors of
the Calabi-Yau threefold and are therefore directly encoded in the 4-dimensional reflexive
polyhedra of Kreuzer and Skarke. We detail an algorithm for identifying such geometries.
Implementing this algorithm, we have conducted a first scan of the current database of
Calabi-Yau threefolds \((h^{1,1} \leq 6)\) for the existence of the special class of Swiss cheese
geometries, which will be referred to as toric Swiss cheese solutions. When we find a
solution of this type, we compute the rotation matrices from the given basis of 2-cycle
and 4-cycle volumes (represented by \(t^i\) and \(\tau_i\), respectively) into the bases where the large
and small cycles are manifest. The result of this scan is a set of 2,313 toric Swiss cheese
Calabi-Yau vacua, over half of which have \(h^{1,1} = 6\). In addition, we perform a direct,
but partial, scan of Calabi-Yau geometries, for which the volume form can be written
explicitly and diagonally in terms of 4-cycle volumes, as this form greatly simplifies the
computation required to minimize the effective potential and find stable minima for the
Kähler moduli. We then choose a particular example of a toric Swiss cheese manifold
from our database, and perform an explicit minimization of the potential, thereby sta-
bilizing the Kähler moduli. The full set of solutions from our scan is available in the

Complementary to the problem of moduli stabilization in obtaining a realistic low energy
limit is the task of minimizing the amount of supersymmetry required by our theory.
Compactifying a type IIA or type IIB string theory on a Calabi-Yau threefold results in
an $\mathcal{N} = 2$ supersymmetric theory in four dimensions. In order to break this to an $\mathcal{N} = 1$
supersymmetric theory in four dimensions, one must perform an orientifold projection on
one of the two gravitinos of the theory. As a result, among the vast number of Calabi-Yau
threefolds, those that permit such an orientifold under some proper $\mathbb{Z}_2$ involution, $\sigma$, are
of great phenomenological interest (for a review, see [45–48]).

An orientifold projection is generally composed of two parts. One is worldsheet parity,
and another is an involution on the internal manifold. The involution $\sigma$ is a non-trivial $\mathbb{Z}_2$
action on the Calabi-Yau space such that $\sigma^2 = 1$. In order for the orientifold action to still
preserve some supersymmetry, the involution must be isometric and holomorphic [49,50].
These conditions require that the pullback $\sigma^*$ of the involution must always map $(p, q)$-
forms to $(p, q)$-forms on $X$. In particular, the Kähler $(1,1)$-form $J$ must be preserved
and the unique holomorphic $(3,0)$-form $\Omega$ must be an eigenform of $\sigma^*$ with eigenvalues
$\pm 1$ (i.e. $\sigma^*J = J$ and $\sigma^*\Omega = \pm \Omega$). The geometry may contain some fixed loci under
the involution, which will correspond to orientifold planes. If there is no fixed locus and
the orientifold is smooth, such an involution describes a freely-acting $\mathbb{Z}_2$. This, however,
is non-trivial since under the orientifold action, the Calabi-Yau threefold may acquire new singularities. In general, the involution $\sigma$ splits the cohomology groups $H^{p,q}(X/\sigma^*)$ into eigenspaces of even and odd parity. When, in addition, the orientifold manifold is smooth, the dimensions of these split cohomology classes form the new Hodge numbers of the quotient manifold $H^{p,q}(X/\sigma^*) = H^{p,q}_+(X/\sigma^*) \oplus H^{p,q}_-(X/\sigma^*)$.

In particular, Calabi-Yau orientifolds with non-trivial odd equivariant cohomology class $H^{1,1}_-(X/\sigma^*)$ play an important role in string phenomenology. In Type IIB orientifold compactifications with $O3/O7$-planes, when $h^{1,1}_-(X/\sigma^*) > 0$, there are non-trivial involutively odd moduli $(b^a, c^a)$, $a = 1, \cdots, h^{1,1}_-(X/\sigma^*)$, in the bosonic closed spectrum coming from the R-R and NS-NS two-form fields $C_2$ and $B_2$, respectively. The consequences of these odd moduli for F-term moduli stabilization has been studied extensively [25] in the context of the LARGE Volume Scenario (LVS) [51] in Type IIB. These odd moduli also play an important role in building axion inflationary models in string theory, such as axion monodromy inflation [52–61] and aligned inflation [62–65]. A great deal of progress has been made in understanding the structure of these moduli on some specific Calabi-Yau vacua without considering the orientifold involution explicitly [27,67–71], for example in terms of the axion landscape or Swiss cheese structure, which, as you will recall will also be studied at some length in this thesis. It would be a great step forward to provide, as an extension of our database of Calabi-Yau threefolds, a concrete classification of Calabi-Yau orientifold data, explicitly counting the even and odd moduli. This is a primary motivating factor for the final chapter of this thesis.
In addition to these considerations, many efforts have been made to combine the global issues of string compactification, such as moduli stabilization and tadpole/anomaly cancellation, with local issues, such as constructing realistic string models that can describe cosmological inflation and D-brane models of particle physics, simultaneously. However, there is not yet a fully realistic global string model which consistently combines these two. One of the most serious obstacles to this is the tension between chirality and moduli-fixing by non-perturbative effects, as studied in Chapter 3 of this thesis, due to extra charged zero modes from the intersecting of D7 branes and the Euclidean D3-instanton branes, which generate the non-perturbative corrections [72]. However, several potential solutions have already been proposed [26,73–77]. One approach is to study D-brane models at toric singularities where the particle physics can be described by the dual quiver gauge theory [26,76,77]. In such constructions, the toric singularity must be embedded in a compact $X$ orientifold, with $h^{1,1}(X/\sigma^*) \neq 0$ in order for the cycle to shrink to zero size. Another approach is to turn on non-trivial world-volume fluxes $\mathcal{F}_E$, supported by a divisor $D_E$ with odd cohomology $\omega_E \in H^{1,1}_-(D_E)$, which is the pullback of $\omega_a \in H^{1,1}_-(X/\sigma^*)$ to the Euclidean D3-instanton branes [74,78]. Equipped with the fluxed-instanton contribution, one can obtain a further contribution to the Euclidean D3-instanton superpotential that then cancels the extra charged zero modes. In both approaches, finding Calabi-Yau threefolds with a non-trivial odd cohomology class is a crucial ingredient.

The organization of this thesis is as follows:

1. In Chapter 2, for readers who are familiar with algebraic geometry but who have
not studied toric varieties, we aim to provide a pedagogical explanation of how practically to obtain geometrical data describing a Calabi-Yau threefold starting from an element of the dataset of [1, 21]. Results for the (favorable) triangulation and unique Calabi-Yau geometry extraction are presented in Section 2.9.

2. In Chapter 3, we begin by outlining the conditions for the possible existence of a large volume solution in the language of toric geometry. This allows us to set our notation and conventions. The conditions in the general case are summarized in Subsection 3.1.4, while the particular case of toric Swiss cheese are presented in Subsection 3.1.5. Section 3.2 contains a schematic of the algorithm used in Sage to detect and compute toric Swiss cheese solutions for the various Calabi-Yau threefold vacua. In preparation for presenting our results, we will establish some terminology on classifying large volume solutions on the basis of the form of the Calabi-Yau volume when written in terms of a basis of 4-cycles in Section 3.3. As a wide range of terminology is used throughout the literature, this classification will hopefully help to elucidate some of the key concepts. A general discussion and derivation of Kähler moduli stabilization in the context of the LARGE Volume Scenario (LVS) is presented in Section 3.4, and a concrete example is demonstrated in Section 3.5. Results of the toric Swiss cheese scan are then presented in Section 3.6 along with a short discussion of some implications.

3. In Chapter 4, we first reexamine the favorable Calabi-Yau threefolds that have been generated in terms of their polytope, triangulation, and consistent geometry
structure. Then, in Section 4.1.2, we use the Hodge numbers of all toric divisors on a given Calabi-Yau threefold in order to identify all pairs of “Non-trivial Identical Divisors (NID)” and present the (“proper”) divisor involutions in Section 4.1.3. All orientifold plane fixed-point loci are then identified in Section 4.1.4, and this information is used to classify the involutions as either non-trivially or freely acting. The cohomology class splitting under these involutions is then determined in Section 4.1.5. Several concrete examples are presented in Section 4.2, and a presentation of the results of our scans is given in Section 4.3.

4. For the reader who is familiar with differential geometry but is not so comfortable with algebraic concepts, Appendix A provides a basic introduction to some of the notions which are used in the text. And finally, Appendix B contains the derivation of some important identities for computing the effective potential in LVS models of Kähler moduli stabilization, as presented earlier in Section 3.4.
Chapter 2

A Database of Toric Calabi-Yau Threefold Hypersurfaces

In this section, we will review how to extract relevant topological and geometrical information about Calabi-Yau threefolds, starting from reflexive polytopes. Due to the work of Kreuzer and Skarke [1,33], not only do we know that the set of four-dimensional reflexive polytopes is finite, but we actually have a complete database of them to draw on. In the subsequent section we will present the results of using the methods we discuss here to convert the database of four-dimensional reflexive polytopes [33] to a database of Calabi-Yau threefold properties [34]. For a brief introduction to some of the key algebro-geometric concepts used in this section please refer to Appendix A.

In what follows, we will assume the ambient space $\mathcal{A}$ to be a Gorenstein toric Fano variety with dimension $n = 4$ (although we will sometimes present results in general $n$) whose
anticanonical divisor $X = -K_A$ is a Calabi-Yau threefold hypersurface. As such, the
Newton polytope $\Delta$, corresponding one-to-one via the symplectic moment map with $A$
is a reflexive polytope, which implies that $\Delta$ as well as its dual $\Delta^*$ are lattice polytopes
containing only the origin in their respective interiors. Also, for simplicity of notation,
we will represent all linear equivalence classes of divisors $[D]_{lin}$ by a representative divisor
$D$.

2.1 Kreuzer-Skarke Database

Kreuzer and Skarke created an algorithm to generate all reflexive polytopes in dimension
$n \geq 4$. Fortunately for us, we are only interested in dimension $n = 4$. It turns out that
there are 473,800,776 reflexive polytopes with $n = 4$.

The output of the Kreuzer-Skarke (KS) database [33] is a text file with every 5 (i.e.,
$n + 1$) lines corresponding to a reflexive polytope. The first line is a summary of certain
key information about the Newton polytope $\Delta$, its dual $\Delta^*$, and the toric variety $A$ they
encode. It reads

$$
\begin{array}{c|c|c|c|c|c}
n & |V(\Delta)| & M : |\Delta| & |V(\Delta)| & N : |\Delta^*| & |V(\Delta^*)| & H : h^{1,1}(X), h^{2,1}(X) & \chi(X) \\
\end{array}
$$

The notation $V(\Delta)$ represents the set of vertices of the reflexive polytope $\Delta$, and $|P|$
indicates the cardinality, or number of lattice points, in any subspace $P \subset M, N$, where
$M \cong N \cong \mathbb{Z}^n$ are dual 4-dimensional dual lattices.

The remaining four ($n$) lines contain a matrix whose columns are the vertices $m \in V(\Delta)$.
of the Newton polytope $\Delta$. In this section, we will occasionally refer to the 95th polytope with $h^{1,1}(X) = 3$, $\Delta_{3,95}$ as an example. It is given in the KS database as

\begin{verbatim}
 4 6 M:100 6 N:8 6 H:3,81 [-156]
 1 1 1 1 -5 -5
 0 3 0 0 -6 0
 0 0 3 0 0 -6
 0 0 0 3 -3 -3
\end{verbatim}

Table 2.1: Example of polytope input from the Kreuzer-Skarke database [33].

### 2.2 Parsing the Database using Sage

We first extract the necessary information from the database entry. The database directly gives us the vertices $V(\Delta)$ of the Newton polytope. It also gives us the Hodge numbers $(h^{1,1}(X), h^{2,1}(X))$ and the Euler number $\chi(X)$ of a Calabi-Yau hypersurface $X \subset \mathcal{A}$, but we will recalculate these later for the sake of completeness.

Sage allows us to define the polytope $\Delta$ from vertices $V(\Delta)$ using the `Polyhedron` class. The dual (or polar) polytope $\Delta^*$ can then be obtained using the `Polyhedron→polar()` method. From these, we can easily obtain the lattice points of $\Delta$ and $\Delta^*$ with `Polyhedron→integral_points()` as well as the vertices $V(\Delta^*)$ with `Polyhedron→vertices()`.

We then obtain the faces $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Delta^*)$ using the `H(ypersurface) representation()` method. From this, it is a simple matter to construct the cones $\sigma_F = \text{cone}(F)$ using the `Cone` class. The cones of $\Delta$ and $\Delta^*$ can each be joined into fans $\Sigma(\Delta)$ and $\Sigma(\Delta^*)$ using the `Fan` class.
2.3 Stringy Hodge Numbers and Euler Number

The stringy Hodge numbers for a generic Calabi-Yau hypersurface $X \subset \mathcal{A}$ can be computed using Batyrev’s well-known formulae [31]

\begin{align*}
h^{1,1}(X) &= |\Delta^*| - n - 1 - \sum_{\text{codim}(F^*) = 1} |\text{relint}(F^*)| + \sum_{\text{codim}(F^*) = 2} |\text{relint}(F^*)| \cdot |\text{relint}(F)| \\
\end{align*}

\begin{align*}
h^{2,1}(X) &= |\Delta| - n - 1 - \sum_{\text{codim}(F) = 1} |\text{relint}(F)| + \sum_{\text{codim}(F) = 2} |\text{relint}(F)| \cdot |\text{relint}(F^*)| \tag{2.1}
\end{align*}

The Euler number is then easily computed via

\begin{align*}
\chi(X) &= 2(h^{1,1}(X) - h^{2,1}(X)) \tag{2.2}
\end{align*}

2.4 MPCP Desingularization and Triangulation of $\Delta^*$

The variety $\mathcal{A}$ generated by $\Delta$ may be a singular space. If it is too singular, then our Calabi-Yau hypersurface $X \subset \mathcal{A}$ may not be smooth even though it is base point free. Therefore, we must find an appropriate resolution of singularities given by a birational morphism $\pi : \tilde{\mathcal{A}} \to \mathcal{A}$ such that the desingularized space $\tilde{\mathcal{A}}$ is smooth enough that the hypersurface $X \subset \tilde{\mathcal{A}}$ can be chosen smooth. Because $X$ has dimension 3, it can generically be transversally (i.e., smoothly) deformed around singular loci with codimension 3. Such singular loci with codimension $\geq 3$ are called terminal singularities [79]. Thus, we need only consider partial desingularizations which resolve everything up to terminal
Another important condition for $X$ to be smooth is for it to be well-defined everywhere when viewed as a Cartier divisor. For this to be true, we must be able to write $X$ uniquely in terms of a basis on every coordinate patch $U$ on the open cover of $\tilde{A}$. Because an ample (effective) Cartier divisor is defined locally on $U$ by a single regular function, this means that the regular functions on $U$ must form a unique factorization domain. When $\tilde{A}$ is smooth, all ample divisors are Cartier, and we say that $\tilde{A}$ is factorial. However, if $\tilde{A}$ contains terminal singularities (i.e. is quasi-smooth), an ample divisor may only be $\mathbb{Q}$-Cartier. In this case, we say that $\tilde{A}$ is $\mathbb{Q}$-factorial. A variety with only terminal singularities is already a normal variety, so that the regular functions on $U$ are integrally closed, however $\mathbb{Q}$-factoriality is a stronger condition. A hypersurface in a variety of this kind will be smooth [80]

Because reflexive polytopes correspond one-to-one with birational equivalence classes of Gorenstein toric Fano varieties, we are guaranteed that $X = -K_A$ is already ample, and so we need not introduce any exceptional divisors (discrepancies) to the canonical divisor in the desingularization $\pi$, i.e., $K_{\tilde{A}} = \pi^*(K_A)$, and we say that the desingularization is crepant [31,81]. As a result, the desingularized space $\tilde{A}$ will still be a Gorenstein toric Fano variety, and therefore projective.

Following Batyrev [31], we define a maximal projective crepant partial (MPCP) desingularization $\pi : \tilde{A} \to A$ to be one such that the pullback $\pi^*$ is crepant, and the desingularized space $\tilde{A}$ is $\mathbb{Q}$-factorial and has no worse than terminal singularities. Furthermore,
given any Gorenstein toric Fano variety $\mathcal{A}$, there exists at least one such MPCP desingularization $\pi$ (see [31] for the proof).

More importantly for our purposes is how this desingularization is reflected in the polytope formulation. The removal of non-terminal singularities can be approached by refining the open cover $\mathcal{U}(\tilde{\mathcal{A}})$ on $\tilde{\mathcal{A}}$ as much as possible. Because each maximal 4-cone of $\Delta^*$ corresponds to a coordinate patch $U \in \mathcal{U}(\tilde{\mathcal{A}})$, this amounts to subdividing the maximal cones as much as possible such that each subdivision is still a convex rational polyhedral cone with a vertex at the origin. We call this a fine, star subdivision with star center at the origin. The condition of $\mathbb{Q}$-factoriality requires that each new subdivided maximal cone is simplicial, i.e., each has four generating rays (since we require $n = 4$).

This kind of subdivision is in fact a triangulation into simplexes. Also, because $\tilde{\mathcal{A}}$ must be projective, these simplicial cones must be projections of cones from an embedding space. In the literature, these are referred to as regular triangulations [82–84]. Thus, in order to find an MPCP desingularization for $\mathcal{A}$, we must find at least one fine, star, regular triangulation (FSRT) of $\Delta^*$.

Before we move on, there is an important point to be made here. All lattice points other than the origin are either vertices $\mathcal{V}(\Delta^*)$, or they are not $\hat{\mathcal{V}}(\Delta^*) = \Delta^* \setminus \mathcal{V}(\Delta^*)$. Because $\Delta^*$ is reflexive and therefore contains no interior points save the origin, it must be true that both $\mathcal{V}(\Delta^*), \hat{\mathcal{V}}(\Delta^*) \subset \partial \Delta^*$. Before desingularization, the generating rays of the maximal cones $\sigma \in \Sigma_4(\Delta^*)$ are the minimal cones $\sigma \in \Sigma_4(\Delta^*)$ whose lattice points include only the origin and points in $\mathcal{V}(\Delta^*)$. But, the notion of subdivision of maximal cones implies that
before desingularization, \( \hat{V}(\Delta^*) \neq \emptyset \). In general, this is true; for each facet \( F \in \mathcal{F}_3(\Delta^*) \), there may be lattice points on the boundary such that \( \text{ske}l_2(\Delta^*) \supset \partial F \neq \emptyset \) and there may be points in the interior such that \( \text{ske}l_3(\Delta^*) \supset \text{relint}(F) \neq \emptyset \). However, in the case that \( \hat{A} \) has only terminal singularities, we may ignore points in \( \text{relint}(F) \) in the process of subdivision.

This can be explained by considering the orbifold group on \( \hat{A} \), whose construction is given by [30]

\[
\tilde{G} \cong N/\Lambda_{n-1}
\]  

(2.3)

where \( \Lambda_d \) is the lattice generated by \( \text{ske}l_d(\Delta^*) \). We now apply an important result of Hasse and Nill [85]:

\[
\text{if } n \geq 3, \quad \Lambda_{n-2} = \Lambda_{n-1}.
\]

(2.4)

Since we are working with \( n = 4 \), this result implies that \( \Lambda_2 = \Lambda_3 \), and the orbifold group depends only on \( \Lambda_2 \). Therefore, we may effective ignore points which appear in \( \text{ske}l_3(\Delta^*) \) and not in \( \text{ske}l_2(\Delta^*) \).

The underlying reason for this is that with \( \Delta^* \) reflexive, these points interior to facets correspond precisely to the Demazure roots [30, 85–87] for the orbifold automorphism group \( \tilde{G} \). Thus, to maximize computational efficiency, we need only triangulate the point configuration \( \text{ske}l_2(\Delta^*) \).
In practice, we want this point configuration to be searchable, so we choose the specific ordering of points given by

\[ \mathcal{P}(\Delta^*) = \text{sort}(\mathcal{V}(\Delta^*)) \cup \text{sort} (\text{skel}_2(\Delta^*) \setminus \mathcal{V}(\Delta^*)) . \]  

(2.5)

Effectively, then, subdivision of the fan will correspond to expanding the set of vertices of \( \Delta^* \) from \( \mathcal{V}(\Delta^*) \) to \( \mathcal{P}(\Delta^*) \). We will sometimes refer to the points in \( \mathcal{P}(\Delta^*) \) (or just \( \mathcal{P} \)) as the resolved vertices of \( \Delta^* \). Furthermore, vertices \( n_\rho \in \mathcal{P} \) correspond one-to-one with toric divisors \( D_\rho \subset \tilde{\mathcal{A}} \) with bijection \( n_\rho \rightarrow D_\rho \).

It is possible to enumerate the FSRTs of the configuration \( \mathcal{P} \) (with star center at the origin) using TOPCOM \cite{TOPCOM, TOPCOM2}, however, this becomes highly inefficient as \( |\mathcal{P}| \) becomes large. A better way to proceed, which is also inherently parallelizable, is to instead consider the configurations \( \mathcal{P} \cap \sigma \) for each \( \sigma \in \Sigma_4(\Delta^*) \). The FSRTs of each maximal cone \( \sigma \) can then be obtained separately, each using Volker Braun’s tremendously useful implementation of TOPCOM in Sage, and then recombined. For each maximal cone, TOPCOM returns a set

\[ \mathcal{T}(\mathcal{P} \cap \sigma) = \{ T_\sigma \mid T_\sigma \text{ an FSRT of } \mathcal{P} \cap \sigma \} . \]  

(2.6)

We can also define the new subdivided fan \( \Sigma(\tilde{\mathcal{A}}) \) composed of all intersections of the simplexes \( S \in T(\tilde{\mathcal{A}}) \). Then, the FSRT \( T(\tilde{\mathcal{A}}) = \Sigma_0(\tilde{\mathcal{A}}) \), and the simplexes \( S \in T(\tilde{\mathcal{A}}) \) form its maximal cones.
The trade-off for computation efficiency here is that the recombination of triangulated maximal cones\(^1\) is somewhat intricate and tricky. We use the following algorithm\(^2\):

1. Choose a triangulation \(T_\sigma \in \mathcal{T}(\mathcal{P} \cap \sigma)\) for each maximal cone \(\sigma \in \Sigma_4(\Delta^*)\).

   If all combinations have previously been checked, terminate.

2. Split up all maximal cones into pairs \((\sigma, \sigma')\) (if there is an odd number, there will be one unpaired cone).

3. For one pair \((\sigma, \sigma')\), check that:
   
   - For each simplex \(S \in T_\sigma\), there exists a simplex \(S' \in T_{\sigma'}\) such that \(S \cap S' \cap \sigma \cap \sigma' \neq \emptyset\).
   
   - For each simplex \(S' \in T_{\sigma'}\), there exists a simplex \(S \in T_\sigma\) such that \(S \cap S' \cap \sigma \cap \sigma' \neq \emptyset\).
   
   - \(T_\sigma \cup T_{\sigma'}\) is a regular triangulation (see below).

   True if \((\sigma, \sigma')\) satisfies all conditions, false otherwise.

   - If true, repeat step 3 for the next pair.
   
   - If false, repeat step 1 with a different combination of triangulations \(T_\sigma \in \mathcal{T}(\mathcal{P} \cap \sigma)\) for each maximal cone \(\sigma \in \Sigma_4(\Delta^*)\).

---

\(^1\)We would like to thank Volker Braun for suggesting to us this method of parallelization via the triangulation of maximal cones.

\(^2\)A similar algorithm was in use concurrently by Long, McAllister, and McGuirk (see [68]).
4. If there is only one pair \((\sigma, \sigma')\), then the triangulation \(T = T_\sigma \cup T_{\sigma'}\) is an FSRT of \(\mathcal{P}\), and therefore of \(\Delta^*\) (i.e., \(T \in \mathcal{T}(\mathcal{P})\)).

Repeat step 1 with a different combination of triangulations \(T_{\sigma} \in \mathcal{T}(\mathcal{P} \cap \sigma)\) for each maximal cone \(\sigma \in \Sigma_4(\Delta^*)\).

Otherwise, define new cones and triangulations by combining each pair via \(\hat{\sigma} = \sigma \cup \sigma'\) and \(T_{\hat{\sigma}} = T_\sigma \cup T_{\sigma'}\).

Split up all new cones into pairs \((\hat{\sigma}, \hat{\sigma}')\) and repeat step 3.

To check whether a triangulation \(T\) of the point set \(\mathcal{P}\) is regular, we use the following well-known algorithm [83, 84, 89]

1. Compute the Gale transform\(^3\) \(\mathcal{P}^\vee\) of \(\mathcal{P}\).

2. For each simplex \(S \in T\), define the set \(\mathcal{Q}(S) = \{n_i^\vee \in \mathcal{P}^\vee \mid n_i \in \mathcal{P} \setminus S\}\).

3. If \(\bigcap_{S \in T} \text{relint}(\text{cone}(\mathcal{Q}(S))) \neq \emptyset\), then \(T\) is a regular triangulation of \(\mathcal{P}\).

Another algorithm involves removing the origin of \(\Delta^*\) and performing a FRT (non-star) triangulation of the full set of vertices \(\mathcal{P}(\Delta^*)\). We then reintroduce the origin (i.e., star triangulation) by manually removing any triangulations with simplexes joining two or more facets. While this algorithm is generally much faster, it is not parallelizable, and therefore will run into memory constraints at higher \(h^{1,1}\). We have used both algorithms to find the FSRTs, and have obtained identical results.

\(^3\)The Gale transform \(\mathcal{P}^\vee\) of a set of points \(\mathcal{P}\) is given by constructing the set of augmented vectors \(\hat{\mathcal{P}} = \{(1, n) \mid n \in \mathcal{P}\}\), and solving the matrix equation \(\hat{\mathcal{P}} \cdot [\mathcal{P}^\vee]^T = 0\).
2.5 Weight Matrix

Recall from equation (A.37) the definition of a toric variety

$$\mathcal{A} \cong \frac{V}{(C^*)^{k-n} \times G}.$$  \hspace{1cm} (2.7)

After desingularization, we obtain a similar toric variety given by

$$\tilde{\mathcal{A}} \cong \frac{\tilde{V}}{(C^*)^{k-n} \times \tilde{G}}.$$ \hspace{1cm} (2.8)

The group $\tilde{G}$ is nothing more than the orbifold group $\tilde{G} = N/\Lambda_{n-1}$. However, we must still describe the action on $\tilde{\mathcal{A}}$ of the split $(k-n)$-torus $(C^*)^{k-n}$ given by the product of 1-tori $C^*$.

The toric variety $\tilde{\mathcal{A}}$ may be treated as a weighted projective space with respect to each of the $k-n$ split 1-tori $C^*$ with weights $w_r = (w_r^1, ..., w_r^k) \in (\mathbb{Z}_{\geq 0})^k$ such that

$$(z_1, ..., z_k) \sim (\lambda^{w_r^1} z_1, ..., \lambda^{w_r^k} z_k), \quad \lambda \in \mathbb{C}^*,$$ \hspace{1cm} (2.9)

for all $r = 1, ..., k-n$ running over the 1-tori.

It can be shown that the weights $w_r^\rho$ satisfy an equation of the form

$$\sum_{\rho=1}^{k} w_r^\rho \cdot \langle m, n_\rho \rangle = 0, \quad \forall \ m \in \Delta,$$ \hspace{1cm} (2.10)
or because the fan $\Sigma(\Delta)$ is complete, we can equivalently write

$$\sum_{\rho=1}^{k} w_{r}^{\rho} n_{\rho} = 0. \quad (2.11)$$

Recall that for $\tilde{\Delta}$, the vertices of $\Delta^*$ are given by $n_{\rho} \in P$ such that $k = |P|$. Then, equation (2.11) can be written in matrix form such that

$$[P] \cdot W^T = 0 \quad \text{with} \quad \sum_{\rho=1}^{k} w_{r}^{\rho} > 0 \quad \text{and} \quad W \geq 0, \quad (2.12)$$

where $W$ is the $(k - n) \times k$ weight matrix

$$W = \begin{pmatrix}
    w_{1} \\
    \vdots \\
    w_{k-n}
\end{pmatrix} = \begin{pmatrix}
    w_{1}^{1} & \cdots & w_{1}^{k} \\
    \vdots & \ddots & \vdots \\
    w_{k-n}^{1} & \cdots & w_{k-n}^{k}
\end{pmatrix}. \quad (2.13)$$

Thus, we see that

$$w_{1}, \ldots, w_{k-n} \in \ker([P]) \text{ linearly independent} \quad (2.14)$$

where we assume the kernel to be taken over the integers, so that the weights are well-defined as exponents in a polynomial.

The kernel $\ker([P])$ can be easily computed in Sage given $P$. However, in general, non-negativity of the entries of $W$ is not guaranteed this way. We overcome this limitation
by restricting to the positive orthant \((\mathbb{Z}_{\geq 0})^k\), such that

\[ w_1, \ldots, w_{k-n} \in \ker([P]) \cap (\mathbb{Z}_{\geq 0})^k \text{ linearly independent} \quad (2.15) \]

To perform this computation in Sage, we create two Polyhedron objects, one generated by \emph{lines} specified by the elements of \(\ker([P])\), and the other generated by \emph{rays} specified by the unit basis vectors

\begin{verbatim}
sage: kerPolyhedron = Polyhedron(lines=ker.basis());
sage: posOrthant = Polyhedron(rays=identity_matrix(ker.ngens()).columns());
\end{verbatim}

The rays of the intersection of these objects are guaranteed to be non-negative, however there may be redundant elements. Because there are \(k - n\) tori, we should find that \(\text{rank}(W) = k - n\). Again, in the interest of organization, we sort the elements in ascending order and choose the first \(k - n\) linearly independent ones. These will be the rows of the weight matrix \(W\).

### 2.6 The Chow Group and Intersection Numbers

Next, we review how to compute the Chow group \(A^1(\tilde{\mathcal{A}})\), which describes the intersection of divisors on \(\tilde{\mathcal{A}}\). Recall from Appendix A that the Chow group of Cartier divisors is given by the quotient \(A^1(\tilde{\mathcal{A}}) \cong C(\tilde{\mathcal{A}}) / \sim_{\text{lin}}\). We will therefore need to work out the ideal which generates linear equivalence classes among the divisors.
2.6.1 Linear Ideal and Stanley-Reisner Ideal

There is an analogous equation to (2.11), the defining equation of the weight matrix, relating toric divisors. It can be written

\[ \sum_{\rho=1}^{k} n_\rho \cdot D_\rho = 0. \]  \hspace{1cm} (2.16)

This gives a linear relation between the divisors and we define the linear ideal

\[ I_{lin} = \sum_{\rho=1}^{k} n_\rho \cdot D_\rho. \]  \hspace{1cm} (2.17)

In addition, there are non-linear relationships among the toric divisors. Consider a case where we have \( d \) toric divisors such that \( D_{i_1} \cdot \ldots \cdot D_{i_d} = \int_{\tilde{\mathcal{A}}} \gamma(D_{i_1}) \wedge \ldots \wedge \gamma(D_{i_d}) = 0 \). In the polytope construction, these divisors with null intersection correspond to the points \( n_{i_1}, \ldots, n_{i_d} \in \mathcal{P} \) which do not appear together as vertices of any simplex \( S \in T(\tilde{\mathcal{A}}) \) in the FSRT \( T(\tilde{\mathcal{A}}) \) corresponding to the MPCP desingularization \( \tilde{\mathcal{A}} \). The set of these null intersections forms another ideal

\[ I_{SR}(\tilde{\mathcal{A}}) = \{ D_{i_1} \cdot \ldots \cdot D_{i_d} \mid n_{i_1}, \ldots, n_{i_d} \not\in S, \forall S \in T(\tilde{\mathcal{A}}) \} \]  \hspace{1cm} (2.18)

known as the Stanley-Reisner ideal. These sets of divisors with null intersections clearly provide another constraint on the Chow group of intersections.

\footnote{Where \( \gamma(D) \) denotes the Poincaré dual in cohomology to the divisor \( D \) in homology. In some sections of this thesis, we identify the two by an abuse of notation.}
2.6.2 Chow Group

Given $I_{\text{lin}}$ and $I_{\text{SR}}(\tilde{A})$, we have all the information we need to define linear equivalence classes between the Cartier divisors in $C(\tilde{A})$. Then, we are in a position to define the Chow group as the quotient

$$A^1(\tilde{A}) \cong \frac{C(\tilde{A})}{I_{\text{lin}} + I_{\text{SR}}(\tilde{A})}. \quad (2.19)$$

In practice, we do not know what $C(\tilde{A})$ is. However, the Picard group of a toric variety is given by $\text{Pic}(\tilde{A}) \cong \mathbb{Z}^{k-n}$, and therefore we know that $C(\tilde{A})/\sim_{\text{lin}} \cong \mathbb{Z}^{k-n}$ as well. If we express $A^1(\tilde{A})$ as a polynomial ring $A^1_{\text{poly}}(\tilde{A})$, we can write

$$A^1_{\text{poly}}(\tilde{A}) \cong \frac{\mathbb{Z}[J_1, \ldots, J_{k-n}]}{I_{\text{SR}}(\tilde{A})}. \quad (2.20)$$

We also do not know what the basis elements $J_1, \ldots, J_{k-n}$ are in terms of the toric divisor classes $D_1, \ldots, D_k$. However, we can still determine the Chow group using the toric divisor classes and linear equivalence such that

$$A^1_{\text{poly}}(\tilde{A}) \cong \frac{\mathbb{Z}[D_1, \ldots, D_k]}{I_{\text{lin}} + I_{\text{SR}}(\tilde{A})}. \quad (2.21)$$

By comparing equations (2.11) and (2.16), we see that there is a correspondence between the columns $W^i$ of the weight matrix and the toric divisor classes $D_i$. We may therefore

\footnote{In fact, the row space of $W$ is identical to that of the Mori cone matrix (compare equation (2.11) to the algorithm used to compute the Mori cone matrix in Section 2.7).}
choose a “basis” of divisor classes $\tilde{J}_1, \ldots, \tilde{J}_{k-n}$ by picking a set of $k-n$ orthogonal columns of $W$. However, this “basis” is not guaranteed to be orthonormal, and will therefore only span $H^{1,1}(\tilde{A})$ if given rational coefficients. We then say that $\tilde{J}_1, \ldots, \tilde{J}_{k-n}$ is a $\mathbb{Q}$-basis.

Like the weight matrix itself, this $\mathbb{Q}$-basis is resolution-independent and therefore is valid for all desingularizations $\tilde{A}$ of $A$. This is essential, since it will enable us to accurately compare the Chern classes and intersection numbers of different desingularizations without introducing an arbitrary change of basis. The importance of this property will become clear in Section 2.8.1 when we use Wall’s theorem to glue together the various phases of the complete Kähler cone corresponding to a distinct Calabi-Yau threefold geometry.

Given the toric divisor classes, a $\mathbb{Q}$-basis of divisor classes, and the linear and Stanley-Reisner ideals, the computation of $A^1_{\text{poly}}(\tilde{A})$ in equation (2.21) is easily accomplished in Sage by defining a PolynomialRing object and an ideal. Because we will be working with a $\mathbb{Q}$-basis of $H^{1,1}(\tilde{A})$ rather than a $\mathbb{Z}$-basis (see Section 2.7), we must define our polynomial ring $A^1_{\text{poly}}(\tilde{A})$ over $\mathbb{Q}$ as follows:

```python
sage: C = PolynomialRing(QQ, names=['t']+
                      ['D'+str(i+1) for i in range(k)]
                      +['J'+str(i+1) for i in range(k-n)]);
sage: DD = list(C.gens()[1:-(k-n)]);
sage: JJ = list(C.gens()[-(k-n):]);
sage: ChowIdeal = C.ideal(Ilin+ISR);
```

Again, however, in practice we still do not know the $\mathbb{Z}$-basis $J_1, \ldots, J_{k-n}$. Later, we will be able to choose one explicitly by considering the Kähler cone constraint (see Section 2.7).
2.6.3 Intersection Numbers

Previously, we computed the Chow group $A^1_{\text{poly}}(\tilde{A})$ as the quotient group of a polynomial ring. However, in this construction, the product of elements can only take the form of polynomials. But as we know, the product of elements of the Chow group is actually an intersection product of 1-cocyles, and not a polynomial product. Moreover, the intersection product of $n$ 1-cocyles on the $n$-dimensional space $\tilde{A}$, is just an integer in $A^n(\tilde{A}) \subset \mathbb{Z}$ (or a rational number in $\mathbb{Q}$ if we have chosen a $\mathbb{Q}$-basis). Thus, we must choose a normalization for the polynomial ring such that

$$\text{norm} : A^n_{\text{poly}}(\tilde{A}) \xrightarrow{\sim} A^n(\tilde{A})$$

(2.22)

is a bijection. One such normalization choice involves the lattice volume $\text{vol}(S)$ of a simplex $S \in T(\tilde{A})$.

If the coordinate patch $U$ has no terminal singularities, i.e., corresponding to points interior to facets on $\Delta^*$ (see Section 2.4), then the corresponding simplex $S_U$ has no such interior points, and we say that it is elementary. Therefore, because all the cones of a reflexive polytope have lattice distance 1, $S_U$ must have unit volume, $\text{vol}(S_U) = 1$. If, however, the coordinate patch $U$ has terminal (i.e., orbifold) singularities, then there will be points interior to the facets of $S_U$ (see Section 2.4), and the volume of the corresponding simplex will have $\text{vol}(S_U) > 1$. 
If $\tilde{A}$ is smooth everywhere, i.e., has no terminal singularities, then the normalization is simple, and every intersection of $n$ 1-cocycles is equal to 1.

Specifically, we define the normalization as follows. Choose a set of $n$ toric divisor classes $\hat{D}_1, \ldots, \hat{D}_n$ such that they have corresponding vertices $\hat{n}_{i_1}, \ldots, \hat{n}_{i_n} \in \mathcal{P} \cap \hat{S}$ for $\hat{S} \in T(\tilde{A})$ a simplex. Then, for any set of $n$ toric divisor classes $D_{i_1}, \ldots, D_{i_n}$ corresponding to vertices $n_{i_1}, \ldots, n_{i_n} \in \mathcal{P} \cap S$ for $S \in T(\tilde{A})$, the normalization takes the form

$$\text{norm} : D_{i_1} \cdot \ldots \cdot D_{i_n} \mapsto \frac{1}{\text{vol}(S)} \frac{D_{i_1} \cdot \ldots \cdot D_{i_n}}{\hat{D}_1 \cdot \ldots \cdot \hat{D}_n}.$$  (2.23)

A Calabi-Yau hypersurface in this construction is defined to be $X = -K_{\tilde{A}} = \sum_{\rho=1}^k D_\rho$ (see Appendix A). Since it is a hypersurface, it has codimension 1, and we can therefore find the intersection numbers in the Chow group $A^n(X)$ by taking $n-1$ toric divisors $D_{i_1}, \ldots, D_{i_{n-1}}$, and intersecting them with $X$ directly, i.e., $D_{i_1} \cdot \ldots \cdot D_{i_{n-1}} \cdot X$. Because $X$ is a formal sum of toric divisor classes, we can still use the same normalization condition in equation (2.23).

### 2.6.4 Favorability

It is important to note that the toric divisor classes on $\tilde{A}$ do not always descend to a Calabi-Yau hypersurface $X$. In order to visualize this, consider the short exact sequence

$$0 \to TX \to T\tilde{A}|_X \to \mathcal{N}_{X/\tilde{A}} \to 0$$  (2.24)
with dual sequence
\[ 0 \to \mathcal{N}_{X/\mathcal{A}}^* \to T^*\mathcal{A}|_X \to T^*X \to 0. \quad (2.25) \]

This induces the long exact sequence in sheaf cohomology, part of which is given by
\[ \cdots \to H^1(X, \mathcal{N}_{X/\mathcal{A}}^*) \xrightarrow{\alpha} H^1(X, T^*\mathcal{A}|_X) \to H^1(X, T^*X) \to H^2(X, \mathcal{N}_{X/\mathcal{A}}^*) \xrightarrow{\beta} H^2(X, T^*\mathcal{A}|_X) \to \cdots \quad (2.26) \]

By Dolbeault’s theorem, \( H^1(X, T^*X) \cong H^{1,1}(X) \cong A^1(X) \). Then, by the exactness of equation (4.7), we find
\[ A^1(X) \cong \text{coker}(\alpha) \oplus \ker(\beta). \quad (2.27) \]

The cokernel of the map \( \alpha \) describes the descent of the Kähler moduli on \( \mathcal{A} \) to Kähler moduli on \( X \), while the kernel of the map \( \beta \) describes “new” Kähler moduli on \( X \) which do not descend from \( \mathcal{A} \). As long as \( \ker(\beta) = 0 \) and \( \text{coker}(\alpha) = H^1(X, T^*\mathcal{A}|_X) \), then all of the Kähler forms descend from the ambient space, and we know \( A^1(X) \) completely. Otherwise we are missing important information about \( A^1(X) \). We then say that \( X \), and by a slight abuse of terminology, also the ambient variety \( \mathcal{A} \) are unfavorable. Studying these unfavorable cases is a problem we leave for future work. In the present study we simply flag these ambient varieties as unfavorable in the database.

If \( X \) is favorable then \( \dim(A^1(X)) \cong \dim(A^1(\mathcal{A})) \). This is equivalent to \( h^{1,1}(X) = \dim(H^{1,1}(X)) \cong \dim(\text{Pic}(\mathcal{A})) \). However, for a toric variety \( \text{Pic}(\mathcal{A}) = \mathbb{Z}^{k-n} \). Thus, if \( h^{1,1}(X) \neq k-n \), then \( \mathcal{A} \) is unfavorable.
2.7 Mori and Kähler Cones

In order to be sure that the hypersurface $X$ is Calabi-Yau, we must ensure that its linear equivalence class $[X]_{\text{lin}}$ is a Kähler class, or equivalently that the cohomology class $\gamma(X)$ of its Poincaré dual is a Kähler form. This amounts to determining whether $\gamma(X)$ lies within the Kähler cone

$$ K(\tilde{A}) = \left\{ J \in H^{1,1}(\tilde{A}) \mid \int_{[C]_{\text{num}}} J > 0, [C]_{\text{num}} \in \text{NE}(\tilde{A}) \right\}, $$

where $\text{NE}(\tilde{A}) \subset N_1(\tilde{A})$ is the Mori cone (or the cone of (numerically effective) curves)

$$ \text{NE}(\tilde{A}) = \left\{ \sum_i a_i [C^i]_{\text{num}} \in N_1(\tilde{A}) \mid a_i \in \mathbb{R}_{>0} \right\} = \text{cone} \left( \{ [C^i]_{\text{num}} \} \right), $$

and where $[C^i]_{\text{num}}$ are the numerical equivalence classes of the irreducible, proper curves on $\tilde{A}$. In practice, we specify these curves via their intersections with the toric divisor classes $D_1, ..., D_k \subset \tilde{A}$. These intersections form a matrix, which we call the Mori cone matrix

$$ M^i_j = [C^i]_{\text{num}} \cdot [D_j]_{\text{num}} = \int_{[C^i]_{\text{num}}} \gamma(D_j). $$

Note that the Mori cone itself can be reconstructed from the rows of $M$, i.e.,

$$ \text{cone} \left( \{ M^1, ..., M^{k-n} \} \right) \cong \text{NE}(\tilde{A}). $$

Then, the rows of $M$ represent the generating curves $[C^i]_{\text{num}}$ of the Mori cone. In order to calculate these, we use an algorithm originally put forward by Oda and Park [87], though the following version is due to Berglund, Katz,
and Klemm [90] (see also [91] and [92]):

1. Augment each \( \mathbf{n}_\rho \in \mathcal{P} \) to a vector one dimension higher via \( \mathbf{n}_\rho \mapsto \mathbf{\bar{n}}_\rho = (1, \mathbf{n}_\rho) \).

2. Find all pairs of \( n \)-dimensional simplexes \( S_i, S_j \in T(\tilde{\mathcal{A}}) \) such that \( S_i \cap S_j \) is an \((n - 1)\)-dimensional simplex, and define the set \( S = \{(S_i, S_j)\} \).

3. For each such pair \( s \in S \), find the unique linear relation \( \sum_{\rho=1}^{k} b^s_\rho \cdot \mathbf{\bar{n}}_\rho = 0 \), such that

   3.1. All the coefficients \( b^s_\rho \) are minimal integers.

   3.2. \( b^s_\rho = 0 \) for \( \mathbf{n}_\rho \in \mathcal{P} \setminus (S_i \cup S_j) \), where \( s = (S_i, S_j) \).

   3.3. \( b^s_\rho \geq 0 \) for \( \mathbf{n}_\rho \in (S_i \cup S_j) \setminus (S_i \cap S_j) \), where \( s = (S_i, S_j) \).

4. Find a basis of minimal integer vectors \( b^s_1, ..., b^{s_{k-n}} \) such that \( b^s \) can be expressed as a positive linear combination, for all \( s \in S \).

5. The Mori cone matrix is given by \( \mathbf{M} = \begin{pmatrix} (b^s_1)^T \\ \vdots \\ (b^{s_{k-n}})^T \end{pmatrix} \).

We see from equation (2.30) that the rows of \( \mathbf{M} \) represent the curves which generate the Mori cone. Next, we compute the matrix dual to \( \mathbf{M} \) whose columns generate the Kähler cone. We first choose a basis of divisor classes \( J_1, ..., J_{k-n} \). We want these to be the generators of the Kähler cone, so from the definition of the Kähler cone in equation (2.28), they must satisfy

\[
\int_{[C]_{\text{num}}} \gamma(J_j) > 0, \quad [C]_{\text{num}} \in \overline{\mathcal{NE}(\tilde{\mathcal{A}})}.
\]  
(2.31)
But by the definition of the Mori cone in equation (2.29), we can write

$$
\int_{[C]_{num}} \gamma(J_j) = \sum_i a_i \left( \int_{[C^i]_{num}} \gamma(J_j) \right), \quad a_i \in \mathbb{R}_{>0}
$$

(2.32)

for $[C^i]_{num}$ the curves generating the Mori cone.

We see then that if equation (2.31) is satisfied for the generating curves $[C^i]_{num}$, then it must be satisfied for all $[C]_{num} \in \overline{NE}(\tilde{A})$. Thus, we can define the Kähler cone matrix

$$
K^i_j = \int_{[C^i]_{num}} \gamma(J_j).
$$

(2.33)

Comparing equations (2.30) and (2.33), we see that

$$
J_j = \sum_{i=1}^{k} b_j^i D_i \quad \text{when} \quad K^i_j = \sum_{\rho=1}^{k} b_j^\rho M^i_\rho.
$$

(2.34)

There are, in general, many choices of such a basis.

The Lefschetz theorem on (1,1)-classes tells us that $A^1(\tilde{A}) \cong \text{Pic}(\tilde{A}) \cong H^{1,1}(\tilde{A}) \subset H^2(\tilde{A}, \mathbb{Z})$. Therefore, using the above construction ensures that $J_1, ..., J_{k-n}$ generate the Kähler cone with integer coefficients (i.e. a $\mathbb{Z}$-basis). This is an important point if we wish to construct holomorphic line bundles in $\text{Pic}(\tilde{A})$. Otherwise, however, it is sufficient to construct a basis $\tilde{J}_1, ..., \tilde{J}_{k-n}$ which generates the Kähler cone with rational coefficients (a $\mathbb{Q}$-basis). This relaxes the constraint on the columns of $K$ from orthonormality to orthogonality (see Section 2.6.2). In this case, we can always choose the basis elements
\( \tilde{J}_j \) to be a subset of the toric divisor classes and define a modified Kähler cone matrix \( \tilde{K} \) such that

\[
\tilde{J}_j = \sum_{i=1}^{k} \delta^i_j \cdot D_i \quad \text{when} \quad \tilde{K}^i_j = \sum_{\rho=1}^{k} \delta^\rho_j M^i_\rho. \tag{2.35}
\]

The Kähler cone matrix \( \tilde{K} \) in this case is no longer orthonormal (only orthogonal), and the Kähler cone itself no longer trivial. \( \mathbb{Z} \) - and \( \mathbb{Q} \) - bases coincide when \( \tilde{A} \) is smooth (i.e. factorial).

### 2.8 Gluing of Kähler Cones

We have seen in Section 2.4 that each Gorenstein toric Fano variety \( \mathcal{A} \) corresponding to a reflexive polytope in the Kreuzer-Skarke database has at least one, but potentially many MPCP desingularizations \( \tilde{\mathcal{A}} \), and that these correspond exactly to FSRT subdivisions of the fan. It is not always the case, however, that these desingularizations contain distinct Calabi-Yau hypersurfaces \( X \). Rather, each desingularization \( \tilde{\mathcal{A}}_i \) yields a distinct Kähler cone in the Kähler moduli space within which the Poincaré dual \( \gamma(X_i) \) is constrained.

If the Calabi-Yau hypersurfaces of two or more desingularizations share certain key topological invariants, then it can be shown that they are topologically equivalent and can be considered representations of the same Calabi-Yau threefold. In this case, the Kähler form of this Calabi-Yau threefold is allowed to reside within the Kähler cone of either representation, and we refer to these disjoint Kähler cone chambers as its phases.

In order to allow the Kähler form to smoothly vary over its full range, the phases of
the Kähler cone must be glued together in an appropriate manner\textsuperscript{6} (see [95], Conjecture 6.2.8). Because the Kähler cone is dual to the Mori cone, this is equivalent to the less intricate task of taking the intersection of the Mori cones corresponding to each Kähler cone phase. This procedure yields a new Mori cone

\[ \text{NE}(\tilde{A}) = \bigcap_i \text{NE}(\tilde{A}_i). \]  

(2.36)

The new Mori cone matrix is then given by \( M = \left[ \text{rays} \left( \text{NE}(\tilde{A}) \right) \right]^T \). If the Chow group \( A^1(\tilde{A}_i) \) of each phase is written in the same basis, then by duality the Kähler cone can be determined using either equation (2.34) or (2.35) depending on whether it is a \( \mathbb{Z} \)-basis or a \( \mathbb{Q} \)-basis (see Sections 2.6.2 and 2.7).

It remains to determine whether some subset of the desingularizations \( \tilde{A}_i \) of \( A \) contain hypersurfaces \( X_i \) which are representations of the same Calabi-Yau threefold \( X \). In the next two sections, we present two, presumably equivalent, methods of determining the full, composite Kähler cone corresponding to a distinct Calabi-Yau threefold \( X \).

### 2.8.1 Chern Classes and Wall’s Theorem

Viewing the Calabi-Yau hypersurfaces \( X_i \) as real, \( 2(n-1) \)-dimensional (in our case \( n = 4 \)) oriented manifolds, we may use an influential theorem due to Wall [96]:

\textsuperscript{6}It has been pointed out to us by Balázs Szendrői that this construction may result in only a subspace of the full Kähler cone of the full Calabi-Yau threefold, or equivalently, only an upper bound on its full Mori cone [93,94].
Chapter 2. A Database of Toric Calabi-Yau Threefold Hypersurfaces

**Theorem 1.** The homotopy types of complex compact 3-folds are classified by the Hodge numbers, the intersection numbers, and the first Pontryagin class.

However, because we are working with Calabi-Yau threefolds we can replace the first Pontryagin class with the second Chern class.

The total Chern class of a vector bundle $V$ is given by $c(V) = \sum_i c_i(V)$, where $c_0 = 1$.

In the special case that $V$ is actually a line bundle, then $c_1(V)$ is the only non-trivial Chern class, and $c(V) = 1 + c_1(V)$. The splitting principle tells us that we can break up the Chern class of the vector bundles of interest into the product of Chern classes of line bundles. Recall that the Gorenstein toric Fano variety $\tilde{A}$ has $k$ toric divisors $D_1, ..., D_k$, each of which corresponds to a line bundle $O_{\tilde{A}}(D_1), ... O_{\tilde{A}}(D_k)$. As $\tilde{A}$ is 4-dimensional, its Chern class can be written

$$c(T\tilde{A}) = 1 + c_1(T\tilde{A}) + c_2(T\tilde{A}) + c_3(T\tilde{A}) + c_4(T\tilde{A})$$

(2.37)

The Chern classes of the ambient variety $\tilde{A}$ can be calculated easily in Sage again using the `PolynomialRing` object implemented earlier in Section 2.6.2:

```python
sage: cA = prod([(1+C.gen(0)*D) for D in DD]).reduce(ChowIdeal);
sage: cAList = [cA.coefficient({C.gen(0): i}) for i in range(cA.degree(C.gen(0)))];
```
In order to calculate the Chern classes of the Calabi-Yau hypersurface $X$, we consider the short exact sequence

$$0 \to TX \to \tilde{T}_A|_X \to \mathcal{N}_{X/\tilde{A}}|_X \cong \mathcal{O}_{\tilde{A}}(X)|_X = \mathcal{O}_X(X) \to 0. \quad (2.38)$$

Recall that $c_1(\mathcal{O}_X(X)) = X$. By the definition of the Chern class and because $X$ is 3-dimensional, we write

$$c(T \tilde{A}|_X) = c(TX) c(\mathcal{O}_X(X)) = \left(1 + c_1(TX) + c_2(TX) + c_3(TX)\right) \left(1 + X\right) \quad (2.39)$$

$$= 1 + \left(c_1(TX) + X\right) + \left(c_2(TX) + c_1(TX) X\right) + \left(c_3(TX) + c_2(TX) X\right) + \ldots .$$

$$c(TX) = \frac{c(T \tilde{A}|_X)}{c(\mathcal{O}_A(X)|_X)} = \frac{c(T \tilde{A}|_X)}{1 + c_1(\mathcal{O}_X(X))} \quad (2.40)$$

$$= 1 + \left(c_1(T \tilde{A}) - c_1(\mathcal{O}_X(X))\right) + \left(c_2(T \tilde{A}) - c_1(\mathcal{O}_X(X)) c_1(\mathcal{O}_X(X)) + c_1(\mathcal{O}_X(X))^2\right) + \ldots .$$

However, we know that $c_1(TX) = 0$ and thus, from the above, $c_1(T \tilde{A}) = c_1(\mathcal{O}_X(X))$. Therefore, we can write the second Chern class as

$$c_2(TX) = c_2(T \tilde{A}) - c_1(T \tilde{A}) c_1(\mathcal{O}_X(X)) + c_1(\mathcal{O}_X(X))^2 = c_2(T \tilde{A}) . \quad (2.41)$$

However, the Calabi-Yau condition tells us that $c_1(TX) = 0$ and thus, comparing equations (2.37) and (2.39), we find that $X = c_1(T \tilde{A}) = \sum_{i=1}^k D_i$ and therefore, after some
algebra, that

\[ c_2(TX) = c_2(T\hat{A}) \quad \text{and} \quad c_3(TX) = c_3(T\hat{A}) - c_2(T\hat{A}) c_1(T\hat{A}). \]  

(2.42)

Furthermore, the Euler number calculated in Section 2.3 can be checked by integrating the top Chern class

\[ \chi(X) = \int_X c_3(TX) \]  

(2.43)

We now have enough information to compute all of the information required by Wall’s theorem. If these quantities are identical for multiple desingularizations \( \hat{A}_i \) of \( A \), then their hypersurfaces \( X_i \) should be considered identical\(^7\) and their Kähler cone phases glued via equations (2.36), (2.34), and (2.35).

### 2.8.2 Identifying Flop Transitions

A second technique\(^8\) for determining when the Kähler cone phases of two desingularizations \( \hat{A}_i \) should be glued amounts to checking whether or not all singularities in the walls between these phases are avoided by the Calabi-Yau hypersurface. This can be done by tracing a curve with negative self-intersection through a flop in the wall of the Kähler cone and making sure that the Calabi-Yau hypersurface misses this curve on both sides.

\(^7\)We would like to thank Balázs Szendrői for pointing out that this particular technique can, in some cases, result in the gluing of inequivalent manifolds. The primary counterexample was put forward by Gross and Ruan (see [97], Theorems A.2 and A4.3), although other such explicit counterexamples are seemingly difficult to find.

\(^8\)Wall’s theorem is enough to ensure that two threefolds are homotopic as real 6-manifolds. However, it may be, for example, that the natural complex structure inherited from the ambient space for two different descriptions of a Calabi-Yau threefold are not in the same connected component of complex structure moduli space.
A curve $C$ with self-intersection $C^2 < 0$ necessarily has negative intersection with every toric divisor which contains it, i.e., $C \cdot D_{j_1} < 0$, ..., $C \cdot D_{j_d} < 0$ for $C \subset D_{j_1} \cap ... \cap D_{j_d}$.

If the transition between two phases is a flop, then such a curve $C$ will blow down to a point on the Kähler cone wall, and then blow up again to a new curve $C'$ on the adjacent phase. Then, we can use the following algorithm due to Berglund, Katz, Klemm, and Mayr [98] (see also [91]):

1. Compare the Mori cone matrices $M_1$ and $M_2$ of two phases corresponding to two desingularizations $\tilde{A}_1$ and $\tilde{A}_2$ of $A$.

   If a row of $M_1$ appears in $M_2$ with its signs flipped, then these rows represent generators $C_1$ and $C_2$ of the Mori cone $NE(\tilde{A}_1)$ and $NE(\tilde{A}_2)$, such that $C_1$ which blows down to a point in the wall of the Kähler cone and then blows up to $C_2$ on the other side.

   Equivalently, there exists a flop from $\tilde{A}_1$ to $\tilde{A}_2$.

2. Determine a subvariety $V_1 = D_{i_1} \cap \cdots \cap D_{i_d}$ ($d < n$) of $\tilde{A}_1$ from the intersection of toric divisors which have negative intersection with $C_1$ (i.e., columns $i_1, ..., i_d$ of $M_1$ with negative entries on the row corresponding to $C_1$).

   Similarly, determine a subvariety $V_2 = D_{i_1} \cap \cdots \cap D_{i_d}$ ($d < n$) of $\tilde{A}_2$ from the intersection of toric divisors which have negative intersection with $C_2$ (i.e., columns $i_1, ..., i_d$ of $M_2$ with negative entries on the row corresponding to $C_2$).
3. If $V_1 \cdot X_1 = 0$ and $V_2 \cdot X_2 = 0$, then $X = X_1 = X_2$ is a single Calabi-Yau hypersurface, and the flop does not exist in $X$.

4. Repeat steps 1 through 3 for all pairs of adjacent Kähler cone phases corresponding to desingularizations of $\mathcal{A}$.

5. Each group of desingularizations $\{\tilde{\mathcal{A}_i}\}$ which are related by flops that do not exist in the hypersurface defines a single Calabi-Yau geometry $X$.

   The Mori cone of $X$ is obtained by taking the intersection of the Mori cones of the associated desingularizations via equations (2.36).

In practice, however, we have used the gluing procedure based on Wall’s theorem discussed in Section 2.8.1.

### 2.9 Results

The results of the procedures outlined in this chapter can be queried directly using the search engine at [www.rossealtman.com](http://www.rossealtman.com). The total number of (favorable) polytopes, triangulations, and glued geometries for $1 \leq h^{1,1}(X) \leq 6$ are presented in Table 2.2, while Tables 2.3-2.6 outline the frequency with which certain toric divisor topologies occur in these manifolds (see Section 4.1.2 for more details on these divisor topologies).
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<td>13635</td>
<td>85682</td>
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<tr>
<td># of Favorable Polytopes</td>
<td>5</td>
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<td>1185</td>
<td>4897</td>
<td>16608</td>
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<tr>
<td># of Favorable Triangulations</td>
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<td>5330</td>
<td>56714</td>
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<tr>
<td># of Favorable Geometries</td>
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<td>305</td>
<td>2000</td>
<td>13494</td>
<td>84525</td>
</tr>
</tbody>
</table>

Table 2.2: Number of (Favorable) Polytopes, Geometries and Triangulations in the database.
<table>
<thead>
<tr>
<th>h$^{1,1}(X)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>del Pezzo surface dP$_n$, n $\leq$ 8</td>
<td>0</td>
<td>30</td>
<td>644</td>
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<td>2153348</td>
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<td>9883</td>
<td>139958</td>
<td>1687119</td>
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<tr>
<td>(Exact-)Wilson surface</td>
<td>0 (0)</td>
<td>5 (0)</td>
<td>137 (32)</td>
<td>1800 (621)</td>
<td>24365 (10222)</td>
<td>287121 (131681)</td>
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<tr>
<td>K3 surface</td>
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<td>573</td>
<td>4336</td>
<td>36008</td>
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<tr>
<td>Special Deformation SD1 surface</td>
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<td>227</td>
<td>2825</td>
<td>30052</td>
<td>289805</td>
</tr>
<tr>
<td>Special Deformation SD2 surface</td>
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<td>394</td>
<td>2769</td>
<td>20754</td>
<td>161544</td>
</tr>
<tr>
<td>del Pezzo and K3</td>
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<td>8</td>
<td>390</td>
<td>6423</td>
<td>87060</td>
<td>1017746</td>
</tr>
<tr>
<td>del Pezzo and (Exact-)Wilson</td>
<td>0 (0)</td>
<td>3 (0)</td>
<td>140 (19)</td>
<td>3020 (912)</td>
<td>57942 (22497)</td>
<td>924284 (402321)</td>
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<tr>
<td>K3 and (Exact-)Wilson</td>
<td>0 (0)</td>
<td>4 (0)</td>
<td>170 (51)</td>
<td>1715 (616)</td>
<td>18504 (6988)</td>
<td>174201 (72873)</td>
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<td>del Pezzo, K3 and (Exact-)Wilson</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>94 (13)</td>
<td>2127 (632)</td>
<td>37455 (12675)</td>
<td>507522 (197259)</td>
</tr>
</tbody>
</table>

Table 2.3: Numbers of triangulation-wide combinations of particular toric divisors.
<table>
<thead>
<tr>
<th>( h^{1,1}(X) )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>del Pezzo surface ( dP_n ), ( n \leq 8 )</td>
<td>0</td>
<td>30</td>
<td>442</td>
<td>5098</td>
<td>56179</td>
<td>583697</td>
</tr>
<tr>
<td>Non-Shrinkable Rigid surface ( dP_n ), ( n &gt; 8 )</td>
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<td>12</td>
<td>328</td>
<td>4600</td>
<td>52906</td>
<td>561059</td>
</tr>
<tr>
<td>(Exact-)Wilson surface</td>
<td>0 (0)</td>
<td>5 (0)</td>
<td>129 (32)</td>
<td>1629 (601)</td>
<td>20778 (9217)</td>
<td>235437 (115403)</td>
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<tr>
<td>K3 surface</td>
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<td>16</td>
<td>275</td>
<td>2504</td>
<td>24081</td>
<td>221863</td>
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<tr>
<td>Special Deformation SD1 surface</td>
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<td>123</td>
<td>1856</td>
<td>21764</td>
<td>225136</td>
</tr>
<tr>
<td>Special Deformation SD2 surface</td>
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<td>15</td>
<td>210</td>
<td>1998</td>
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<td>146462</td>
</tr>
<tr>
<td>del Pezzo and K3</td>
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<td>4</td>
<td>201</td>
<td>2320</td>
<td>23764</td>
<td>221566</td>
</tr>
<tr>
<td>del Pezzo and (Exact-)Wilson</td>
<td>0 (0)</td>
<td>3 (0)</td>
<td>104 (19)</td>
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<td>20466 (8926)</td>
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</tr>
<tr>
<td>K3 and (Exact-)Wilson</td>
<td>0 (0)</td>
<td>2 (0)</td>
<td>74 (21)</td>
<td>869 (345)</td>
<td>10030 (4339)</td>
<td>100845 (47321)</td>
</tr>
<tr>
<td>del Pezzo, K3 and (Exact-)Wilson</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>51 (8)</td>
<td>803 (289)</td>
<td>9827 (4150)</td>
<td>100587 (47068)</td>
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</table>

Table 2.4: Numbers of triangulations that admit at least one particular combination of toric divisors.
Chapter 2. *A Database of Toric Calabi-Yau Threefold Hypersurfaces*

<table>
<thead>
<tr>
<th>$h^{1,1}(X)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>del Pezzo surface $\text{dP}_n$, $n \leq 8$</td>
<td>0</td>
<td>23</td>
<td>348</td>
<td>3621</td>
<td>35045</td>
<td>297816</td>
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<tr>
<td>Non-Shrinkable Rigid surface $\text{dP}_n$, $n &gt; 8$</td>
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<td>10</td>
<td>269</td>
<td>3718</td>
<td>33781</td>
<td>251593</td>
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<tr>
<td>(Exact-)Wilson surface</td>
<td>0 (0)</td>
<td>5 (0)</td>
<td>92 (19)</td>
<td>718 (236)</td>
<td>5892 (2434)</td>
<td>40301 (19245)</td>
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<tr>
<td>K3 surface</td>
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<td>24</td>
<td>342</td>
<td>1913</td>
<td>10745</td>
<td>56702</td>
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<td>6599</td>
<td>38938</td>
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<td>Special Deformation SD2 surface</td>
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<td>235</td>
<td>1145</td>
<td>5851</td>
<td>29568</td>
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<td>del Pezzo and K3</td>
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<td>184693</td>
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<tr>
<td>del Pezzo and (Exact-)Wilson</td>
<td>0 (0)</td>
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<td>125 (31)</td>
<td>751 (286)</td>
<td>5500 (2085)</td>
<td>30345 (14038)</td>
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<td>0 (0)</td>
<td>63 (9)</td>
<td>817 (264)</td>
<td>10574 (3494)</td>
<td>82525 (35303)</td>
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</table>

**Table 2.5:** Numbers of geometry-wide combinations of particular toric divisors.
### Table 2.6: Numbers of consistent geometries that admit at least one particular combination of toric divisors.

<table>
<thead>
<tr>
<th>$h^{1,1}(X)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>del Pezzo surface $dP_n$, $n \leq 8$</td>
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<td>84328</td>
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<tr>
<td>Non-Shrinkable Rigid surface $dP_n$, $n &gt; 8$</td>
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<td>10</td>
<td>182</td>
<td>1704</td>
<td>12356</td>
<td>81224</td>
</tr>
<tr>
<td>(Exact-)Wilson surface</td>
<td>0 (0)</td>
<td>5 (0)</td>
<td>84 (19)</td>
<td>621 (228)</td>
<td>4773 (2131)</td>
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<td>29217</td>
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<td>Special Deformation SD2 surface</td>
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<td>115</td>
<td>892</td>
<td>6345</td>
<td>37481</td>
</tr>
<tr>
<td>del Pezzo and (Exact-)Wilson</td>
<td>0 (0)</td>
<td>3 (0)</td>
<td>65 (11)</td>
<td>571 (196)</td>
<td>4675 (2049)</td>
<td>31441 (16126)</td>
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<tr>
<td>K3 and (Exact-)Wilson</td>
<td>0 (0)</td>
<td>2 (0)</td>
<td>51 (13)</td>
<td>331 (143)</td>
<td>2555 (1134)</td>
<td>15216 (7852)</td>
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<td>del Pezzo, K3 and (Exact-)Wilson</td>
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<td>0 (0)</td>
<td>34 (5)</td>
<td>293 (114)</td>
<td>2484 (1073)</td>
<td>15139 (7780)</td>
</tr>
</tbody>
</table>
Chapter 3

Scanning for Toric and Explicit Swiss Cheese Solutions

3.1 Detecting Toric Swiss Cheese Solutions

Given a Calabi-Yau threefold hypersurface $X$ in an ambient 4-dimensional toric variety $\mathcal{A}$ with $k$ toric coordinates $x_1, ..., x_k$, each corresponding to a divisor $D_i = \{ x_i = 0 \}$ on $X$. The Kähler moduli space is given by $H^{1,1}(X) \cap H^2(X; \mathbb{Z})$ with dimension $h^{1,1}(X) = \dim H^{1,1}(X)$; we shall be largely concerned with the so-called favorable manifolds where all divisor classes on $X$ descend from that of the ambient $\mathcal{A}$, so that $h := h^{1,1}(X) = h^{1,1}(\mathcal{A})$. A $\mathbb{Z}$-basis of 2-form classes, corresponding to 4-cycles in homology via Poincaré duality, can be chosen as $\{ J_1, ..., J_{h^{1,1}} \} \in H^{1,1}(X) \cap H^2(X; \mathbb{Z})$ spanning the space. Because a Calabi-Yau manifold is Kähler, it is naturally equipped with a characteristic Kähler 2-form class $J \in H^{1,1}(X) \cap H^2(X; \mathbb{Z})$, which is Poincaré dual to the homology class of $X$. 
in $\mathcal{A}$. Expanding the Kähler form in a basis, we find

$$J = t^i J_i$$

(3.1)

with Kähler parameters $t^i \in \mathbb{Z}$. However, the Kähler form itself is basis independent, and we can therefore choose any basis\(^1\) \{\(J_1^A, \ldots, J_h^A\)\} $\in H^{1,1}(X) \cap H^2(X; \mathbb{Q})$, where, for the sake of computational efficiency, we have relaxed the requirement of $\mathbb{Z}$-valued coefficients to the more general case of $\mathbb{Q}$-valued ones. We can then expand the Kähler form in this new $A$-basis as follows

$$J = t^A J_i^A$$

(3.2)

Note that because $J$ is basis-independent, we can easily do this as many times as we want with new bases $B, C, \ldots$. The cohomology or Chow ring structure on $X$, however, is basis-dependent. In this chapter, we wish to identify a “Swiss cheese” basis in which the large, volume-modulating 4-cycles are manifestly separated from the small, blowup 4-cycles, which are phenomenologically useful in achieving moduli stabilization. But since we have no natural choice of basis to work with, finding one which satisfies the Swiss cheese condition [67] must involve an arbitrary basis change with many unconstrained degrees of freedom. It is therefore an extremely computationally expensive undertaking, especially when faced with higher dimensional moduli spaces. We will present a scan based on a fully general formulation, which is currently being conducted despite the

\(^1\)Note: the superscript Latin characters $A, B, C, \ldots$ are labels rather than indices, and will not obey the Einstein summation convention.
computational toll. However, in order to work around this bottleneck, the only options remaining are to narrow the scope of the search to a special case or to find a particularly natural basis to work with. Later, we will outline a technique that is a combination of these two approaches.

We now consider only the class of smooth toric Calabi-Yau threefolds [1], i.e. those obtained as the anticanonical hypersurface in a 4-dimensional toric variety with no worse than terminal singularities. A database [99] of these Calabi-Yau threefolds is available through a robust search engine at www.rossealtman.com. The topological and geometric information for these manifolds is presented in an arbitrary $\mathbb{Z}$-basis $\{J_1, \ldots, J_h\}$.

We can define the $A$-basis of the Kähler class as a linear transformation of the original basis $J_i$. This transformation should be invertible, so we define the transformation matrix $T^A \in GL_h(\mathbb{Q})$ by

$$J_i^A = (T^A)_i^j J_j.$$  \hspace{1cm} (3.3)

In the same manner, we may introduce matrices $T^B, T^C$, etc. for the $B$-, $C$-, etc. basis representations of the Kähler class.

### 3.1.1 Volume, Large Cycle, and Small Cycle Conditions

The complex subvarieties of $X$ can be written in terms of 2-cycle curves $C^i$, 4-cycle divisors $J_i$, and the compact Calabi-Yau 6-cycle $X$. Curves are dual to divisors, and can be expressed in a basis $C^1, \ldots, C^h \in \mathcal{M}(\mathcal{A})$ such that $C^i \cdot J_j = \delta^i_j$, where $\mathcal{M}(\mathcal{A})$ is called the Mori cone, or cone of curves. The Kähler class $J$ acts as a calibration 2-form on these
Chapter 3. *Scanning for Toric and Explicit Swiss Cheese Solutions* 49

2n-cycles on $X$, fixing their volumes according to\(^2\)

\[ \text{vol}(C^i) = \frac{1}{i!} \int_{C^i} J = \frac{1}{i!} \int t^i J_j = t^i \delta^i_j = t^i \tag{3.4} \]

\[ \text{vol}(J_i) = \frac{1}{2i} \int_{J_i} J = \frac{1}{2i} \int X J_i \land t^i J_j \land t^k J_k = \frac{1}{2} t^j t^k \kappa_{ijk} := \tau_i \tag{3.5} \]

\[ \text{vol}(X) = \frac{1}{3i} \int_X J \land J \land J = \frac{1}{3i} \int X t^i J_i \land t^j J_j \land t^k J_k = \frac{1}{6} t^i t^j t^k \kappa_{ijk} := \mathcal{V} \tag{3.6} \]

where $\kappa_{ijk} = \int_X J_i \land J_j \land J_k$ is the triple intersection tensor corresponding to the Chow ring structure of the Calabi-Yau threefold $X$. We can expand the volume $\mathcal{V}$ in complete generality by assuming that each of the three copies of $J$ in the integral is written in a different basis.

\[ \mathcal{V} = \frac{1}{3!} \int_X J \land J \land J = \frac{1}{3!} \int X t^{Ai} t^{Bj} t^{Ck} \int X J^A_i \land J^B_j \land J^C_k \]

\[ = \frac{1}{3!} \int X t^{Ai} t^{Bj} t^{Ck} \int X ((T^A)_i \land (T^B)_j \land (T^C)_k) \]

\[ = \frac{1}{3!} \int X t^{Ai} t^{Bj} t^{Ck} (T^A)_i \land (T^B)_j \land (T^C)_k \]

\[ = \frac{1}{3!} \int X t^{Ai} t^{Bj} t^{Ck} (T^A)_i \land (T^B)_j \land (T^C)_k \kappa_{rst} \tag{3.7} \]

\(^2\)We have slightly abused notation by writing $J_i$ for both the divisor class and its Poincaré dual in homology.
The volume of each of the 4-cycles \( \tau_i \) can then be written as the derivative of the total volume with respect to each of the 2-cycle volumes \( t^i \).

\[
\tau_i^A = \frac{dV}{dt^A_i} = \frac{d}{dt^A_i} \left[ \frac{1}{3!} t^{A'i'} t^{B'j'} t^{C'k'} \int_X J_{i'}^A \land J_{j'}^B \land J_{k'}^C \right] = \frac{1}{2} t^{Bj} t^{Ck} \int_X J^A_i \land J^B_j \land J^C_k \\
= \frac{1}{2} t^{Bj} t^{Ck} (T^A)_i^r (T^B)_j^s (T^C)_k^t \int_X J_r \land J_s \land J_t \\
= \frac{1}{2} t^{Bj} t^{Ck} (T^A)_i^r (T^B)_j^s (T^C)_k^t \kappa_{rst} . \tag{3.8}
\]

In a generic basis \( J_i^A \), the Kähler moduli may be arbitrarily large or small. When looking at phenomenological models in the large Volume Scenario (LVS), however, we wish to choose a basis consisting of a minimal set of cycles with large volume, with the remaining cycles small. Thus, in the following formulation, the number of large and small cycles will be labeled \( N_L \) and \( N_S \), respectively, such that \( h = N_L + N_S \). For compactness of notation and in analogy to computational pseudocode, we define the following index intervals

\[
I^{\text{Toric}} = [1, k] \quad \text{(Toric divisors)} \tag{3.9}
\]

\[
I = [1, h] \quad \text{(Original basis)} \tag{3.10}
\]

\[
I^A = [1, h] , \quad I_L^A = [1, N_L] , \quad \text{and} \quad I_S^A = [N_L + 1, h] \quad \text{(A-basis)} \tag{3.11}
\]

\[
I^B = [1, h] , \quad I_L^B = [1, N_L] , \quad \text{and} \quad I_S^B = [N_L + 1, h] \quad \text{(B-basis)} \tag{3.12}
\]
where $k$ is the total number of toric divisors on the resolved Calabi-Yau threefold\footnote{An $n$-dimensional toric variety $\mathcal{A}$ constructed from an $n$-dimensional reflexive lattice polytope $M$ obeys the short exact sequence

$$0 \rightarrow M \rightarrow \bigoplus_{i=1}^{k} \mathbb{Z}D_i \rightarrow \text{Pic}(\mathcal{A}) \cong H^{1,1}(\mathcal{A}) \cap H^2(\mathcal{A}; \mathbb{Z}) \rightarrow 0$$

where the $D_i$ are toric divisor classes. Therefore, $k = h^{1,1}(\mathcal{A}) + \dim(M) = h^{1,1}(\mathcal{A}) + \dim_{\mathbb{C}}(\mathcal{A})$. So, when the codimension 1 hypersurface $X \subset \mathcal{A}$ is favorable, we have $k = h^{1,1}(X) + \dim_{\mathbb{C}}(X) + 1$. In the case of a Calabi-Yau threefold, $k = h^{1,1}(X) + 4$ specifically.} $X$.

We will assume that there is a specific basis $\{J^A_i\}$ such that

$$t^A_i = \begin{cases} 
\text{Large}, & i \in I^A_L \\
\text{Small}, & i \in I^A_S 
\end{cases} \quad (3.13)$$

and a specific basis $\{J^B_i\}$ such that

$$\tau^B_i = \begin{cases} 
\text{Large}, & i \in I^B_L \\
\text{Small}, & i \in I^B_S 
\end{cases} \quad (3.14)$$

and we will work only in these two bases for the remainder of this work.

We then see that from Equations (3.7) and (3.11)-(3.13) that if we want the total volume $V$ to be large, then

$$\exists (i, j, k) \in I^A_L \times I^A \times I^A : (T^A_i)^r (T^A_j)^s (T^A_k)^t \kappa_{rst} \neq 0. \quad (3.15)$$
We also see from Equations (3.8) and (3.11)-(3.14) that

\[ \exists (j, k) \in I_B^A \times I_A^A : (T^B)_i^r (T^A)_j^s (T^A)_k^t \kappa_{rst} \neq 0, \quad \forall i \in I_B^L \]  
(3.16)

\[ (T^B)_i^r (T^A)_j^s (T^A)_k^t \kappa_{rst} = 0, \quad \forall (i, j, k) \in I_B^B \times I_A^L \times I_A^A. \]  
(3.17)

But, if \( \{J^A_i\} \) is a basis, then \( T^A \) must be full rank. This implies that we can write Equation (3.17) as

\[ (T^B)_i^r (T^A)_j^s \kappa_{rst} = 0, \quad \forall (i, j, t) \in I_B^I \times I_A^L \times I. \]  
(3.18)

### 3.1.2 Kähler Cone Condition

#### 3.1.2.1 The Mori and Kähler Cones

A Kähler manifold is defined as a symplectic manifold with a closed symplectic 2-form \( J \), which is simultaneously consistent with an almost complex and Riemannian structure. The former imposes the constraint that \( J \) is in fact a (1,1)-form, while the latter requires \( J \) to be locally positive definite. This is directly related to the fact that the volume of a curve \( \text{vol}(C) = \int_C J > 0 \). By expanding \( J = t^i J_i \) in a basis \( \{J_i\} \in H^{1,1}(X) \), we ensure that it is indeed a closed (1,1)-form, however we must also constrain it to be positive definite. To make this explicit, we check that \( J \) has positive intersection with every subvariety of
complementary codimension, i.e. curves $C$

\[
\mathcal{K}(\mathcal{A}) = \left\{ J \in H^{1,1}(X) \mid \text{vol}(C) = \int_C J > 0 \right\} \tag{3.19}
\]

Thus, the allowed values of $J$ form a convex cone in the Kähler moduli space. The curves $C$ then form a dual cone, known as the Mori cone (as discussed earlier in Section 2.7) $\mathcal{M}(\mathcal{A}) \subset \text{Hom}(H^{1,1}(\mathcal{A}), \mathbb{Q}) \cong \mathbb{Q}^h$ which is generated\footnote{Again, for the sake of computational efficiency, we have relaxed the requirement of $J \in H^{1,1}(\mathcal{A}) \cap H^2(\mathcal{A}; \mathbb{Z})$ to $J \in H^{1,1}(\mathcal{A}) \cap H^2(\mathcal{A}; \mathbb{Q})$.} by a set of extremal rays $C^1, \ldots, C^r$ such that

\[
\mathcal{M}(\mathcal{A}) = \left\{ \sum_{i=1}^r a_i C^i \mid a_i \in \mathbb{R}_{\geq 0} \right\} \tag{3.20}
\]

These extremal rays can be regarded as linear functionals on the divisors, and can therefore easily be computed in terms of the toric divisors from symplectic moment polytope information provided in the Kreuzer-Skarke database. Then, given our original basis of divisor classes $\{J_i\}_{i \in I}$ of $H^{1,1}(X) \cong H^{1,1}(\mathcal{A})$ for favorable geometries, we can define the $r \times h$ Kähler cone matrix of intersection numbers between the generating curves $C^i$ and the basis divisor classes

\[
K^i_{\ j} = \int_{C^i} J_j \tag{3.21}
\]

whose rows represent the generating curves, or equivalently, rays of the Mori cone $\mathcal{M}(\mathcal{A})$. Using this Kähler cone matrix, and referring to Equations (3.2) and (3.3), we see that

\[
\int_{C^i} J = t^{A_j} \int_{C^i} J^A_j = t^{A_j} (T^A)_j^k \int_{C^i} J_k = t^{A_j} (T^A)_j^k K^i_{\ k} \tag{3.22}
\]
where \( J \in H^{1,1}(\mathcal{A}) \) is the Kähler form on \( \mathcal{A} \). If we want \( J \in \mathcal{K}(\mathcal{A}) \), then we must satisfy

\[
\int_C J > 0, \quad \forall C \in \mathcal{M}(\mathcal{A}).
\]

This is equivalent to

\[
0 < \int_C J = \int_C \sum_i a_i \int_{C_i} J, \quad \text{with } a_i \in \mathbb{R}_{>0}, \quad \forall i \in I. \tag{3.23}
\]

Since this must be true for arbitrary \( a_i \), then each term of the sum must satisfy the inequality independently

\[
0 < \int_{C_i} J = t^{A_j} (T^A)_j^k K^i_k = (K^A)^i_j t^{A_j}, \quad \forall i \in I. \tag{3.24}
\]

This, then, is the set of conditions which must be satisfied in order for the Kähler form \( J \) to lie within the Kähler cone. Unfortunately, this procedure only tells us the Kähler cone of the ambient toric variety \( \mathcal{A} \), while that of the Calabi-Yau hypersurface may be larger. It is still, however, a sufficient condition.
3.1.2.2 Large and Small Cycle Kähler Cone Conditions

Without loss of generality, we can always rearrange the rows of $K^A = K (T^A)^T$ to put as many zero entries as possible in the lower left quadrant.

$$K^A = \begin{pmatrix} p_1^A & \cdots & p_N^A & \ddots & \vdots & (q_1^A)^T & \cdots & (q_m^A)^T \\ 0 & \ddots & \vdots & \vdots & \vdots & m & 0 \leq m \leq r \end{pmatrix}$$

$$\begin{pmatrix} p_1^A & \cdots & p_N^A & \ddots & \vdots & (q_1^A)^T & \cdots & (q_m^A)^T \end{pmatrix}$$

$N_L$ and $h-N_L$
Then, in the large volume limit where \((t^A)^1, \ldots, (t^A)^{N_L} \to \pm \infty\), the Kähler cone condition of Equation (3.24) becomes

\[
\sum_{i=1}^{N_L} \lim_{t^A \to \pm \infty} p_i^A t^A_i > 0
\]  

(3.26)

\[
q_i^A \cdot \begin{bmatrix} (t^A)^{N_L+1} \\ \vdots \\ (t^A)^h \end{bmatrix} > 0, \ \forall j \in [1, m].
\]  

(3.27)

For the first expression to be well-defined, each term must either be satisfied independently or be identically zero, so that

\[
\lim_{t^A \to \pm \infty} p_i^A t^A_i \geq 0, \ \forall i \in I_L^A \\
\Rightarrow p_i^A \geq 0 \ \text{or} \ p_i^A \leq 0, \ \forall i \in I_L^A \\
\Rightarrow \pm p_i^A \geq 0, \ \forall i \in I_L^A
\]  

(3.28)

In order to satisfy Equation (3.27), we first recognize that the rows of \(K^A\) are just the generating rays of the Mori cone \(\mathcal{M}(A) \subset \mathbb{Q}^h\), as expressed in the \(A\)-basis of \(H^{1,1}(X)\). Then, we see that cone \((q_1^A, \ldots, q_m^A)\) must be a convex subcone of at most dimension \(h - N_L\). Then, defining the dual \(\sigma^\vee\) to a \(d\)-dimensional convex cone \(\sigma\) by

\[
\sigma^\vee = \{ n \in \mathbb{Q}^d \mid (m, n) \geq 0, \ \forall m \in \sigma \subset \mathbb{Q}^d \},
\]  

(3.29)
we see that the solution space of Equation (3.27) is just the relative interior of the dual cone, where the inequality is strict

$$\left[ \begin{array}{c} (t^A)^{N_L+1} \\ \vdots \\ (t^A)^{h^{k,1}} \end{array} \right] \in \text{relint} \left( \text{cone} \left( q^A_1, \ldots, q^A_m \right) \right). \quad (3.30)$$

Thus, a solution exists if and only if

$$\dim \left[ \text{relint} \left( \text{cone} \left( q^A_1, \ldots, q^A_m \right) \right) \right] > 0. \quad (3.31)$$

### 3.1.3 Homogeneity Condition

The effective potential in the low-energy supergravity limit of a type IIB theory in the LARGE Volume Scenario (LVS) has exponential factors involving small cycle moduli that are proportional to $V$, and can often be volatile unless the terms are carefully balanced. More specifically, to have a finite minimum, each term must be of the same order in $V^{-1}$.

We refer to this property as homogeneity of the terms in the effective potential. This leads to a restrictive requirement on the Kähler potential, and in turn on the Kähler metric. Because the 4-cycle volumes obey the ordering $\tau_i^B \gg \tau_j^B$, $\forall (i, j) \in I^B_L \times I^B_S$, all terms in the effective potential involving $\tau_i^B$ are exponentially suppressed for each $i \in I^B_L$.

The requirement on the Kähler metric can then be expressed as
where the $\{h_i^{1/2}\}_{i \in I^B_S}$ are $h - N_L$ functions of degree-1/2 in the small 4-cycles $\{\tau^B_k\}_{k \in I^B_S}$.

Now, we consider the expansion of the Kähler metric $(K^{-1})_{ij}$ in $\mathcal{V}^{-1}$ (see Appendix 3.4 for details) \cite{28,43}

\[
(K^{-1})_{ij} = -2\mathcal{V} \left( \int_X J_i^B \wedge J_j^B \wedge J \right) + 2\tau^B_i \tau^B_j + \mathcal{O} \left( \mathcal{V}^{-1} \right)
\]

\[
= -2\mathcal{V} t^A_k \left( \int_X J_i^B \wedge J_j^B \wedge J^A_k \right) + 2\tau^B_i \tau^B_j + \mathcal{O} \left( \mathcal{V}^{-1} \right)
\]

\[
= -2\mathcal{V} t^A_k (T^B_i)^r (T^B_j)^s (T^A_k)^t \left( \int_X J_r \wedge J_s \wedge J_t \right) + 2\tau^B_i \tau^B_j + \mathcal{O} \left( \mathcal{V}^{-1} \right)
\]

\[
= -2\mathcal{V} t^A_k (T^B_i)^r (T^B_j)^s (T^A_k)^t \kappa_{rst} + 2\tau^B_i \tau^B_j + \mathcal{O} \left( \mathcal{V}^{-1} \right). \tag{3.33}
\]

The diagonal elements of $(K^{-1})_{ij}$ have the form

\[
\frac{(K^{-1})_{ii}}{\mathcal{V}} = -2t^A_j (T^B_i)^r (T^B_i)^s (T^A_j)^t \kappa_{rst} + 2\left( \frac{\tau^B_i}{\mathcal{V}} \right)^2 + \mathcal{O} \left( \mathcal{V}^{-1} \right). \tag{3.34}
\]

But, by definition, $\tau^B_i \ll \mathcal{V}$, $\forall i \in I^B_S$, so

\[
\frac{(K^{-1})_{ii}}{\mathcal{V}} \approx -2t^A_j (T^B_i)^r (T^B_i)^s (T^A_j)^t \kappa_{rst}, \quad \forall i \in I^B_S. \tag{3.35}
\]
Then, because the 4-cycle volumes are quadratic in the 2-cycle volumes, we have found our degree-1/2 functions \( \{h_{1/2}^i\}_{i \in I^B_S} \) from Equation (3.35)

\[
h_{1/2}^i \left( \{\tau^B_j\}_{j \in I^B_S} \right) = -2t^{Aj} (T^B)_i^r (T^B)_i^s (T^A)_j^t \kappa_{rst}, \quad \forall i \in I^B_S.
\]  

(3.36)

By inspecting Equations (3.13) and (3.36), we find the following

\[
(T^B)_i^r (T^B)_i^s (T^A)_j^t \kappa_{rst} = 0, \quad \forall (i,j) \in I^B_S \times I^A_L
\]  

(3.37)

\[
\exists j \in I^A_L : (T^B)_i^r (T^B)_i^s (T^A)_j^t \kappa_{rst} \neq 0, \quad \forall i \in I^B_S.
\]  

(3.38)

Note that because \( \kappa_{rst} \) is a symmetric tensor, Equation (3.37) is implied by Equation (3.18) and therefore redundant.

This homogeneity condition is critically important for finding a Swiss cheese solution with \( N_S = 1 \). However, when \( N_S > 1 \), the exponential factors in the effective potential have more degrees of freedom, and the necessity of this condition is loosened. However, it remains a sufficient condition in most circumstances, and we simply flag these cases in our scan when we encounter them, rather than constraining the search parameters.

### 3.1.4 General List of Conditions

In this section, we have compiled all the conditions necessary for \( X \) to have a Swiss cheese solution in the large volume scenario. For ease of notation, we make the following
Then, in order for a Swiss cheese solution to exist with $N_L$ large 4-cycles, there must exist invertible\(^5\) $A$- and $B$-bases such that

1. (Equation (3.15): Volume) \[ \exists (i, j, k) \in I_L^A \times I_A^A \times I_A^A : \kappa_{ij}^{AAA} \neq 0 \]
2. (Equation (3.16): Large Cycle) \[ \exists (j, k) \in I_L^A \times I_A^A : \kappa_{ij}^{BAA} \neq 0, \forall i \in I_L^B \]
3. (Equation (3.18): Small Cycle) \[ \kappa_{ij}^{BA0} = 0, \forall (i, j, k) \in I_S^B \times I_L^A \times I \]
4. (Equation (3.38): Homogeneity) \[ \exists j \in I_S^A : \kappa_{ij}^{BBA} \neq 0, \forall i \in I_S^B \]
5. (Equation (3.28): Kähler Cone (L)) \[ \pm p_i^A \geq 0, \forall i \in I_L^A \]
6. (Equation (3.31): Kähler Cone (S)) \[ \dim \left[ \text{relint} \left( \text{cone} \left( q_1^A, \ldots, q_m^A \right) \right) \right] > 0 \]

---

\(^5\) $T^A, T^B \in GL_h(\mathbb{Q})$ implies that both are invertible: $\det (T^A) \neq 0$ and $\det (T^B) \neq 0$. 
where

\[
\begin{pmatrix}
\begin{array}{cccc}
p_i^A & \ldots & p_{N_L}^A & \cdots \\
0 & \quad (q_i^A)^T & \quad \vdots & (q_m^A)^T \\
\end{array}
\end{pmatrix}
\]

is the Kähler cone matrix after rotation into the $A$-basis.

### 3.1.5 Special Case: Toric Swiss Cheese

Each favorable toric Calabi-Yau threefold is endowed with a set of special 2-form toric classes $\{D_i\}_{i \in I^{\text{Toric}}}$ dual to the 4-cycle toric divisors, which descend directly from the ambient space $\mathcal{A}$. The Kähler moduli space $H^{1,1}(X)$ is always spanned by these toric 2-forms, up to some redundancy. We know, therefore, than any basis expansion of a point in the moduli space may be equivalently described as a linear combination of the toric 2-forms, though it will not be unique. Practically speaking, however, this form is advantageous for a scan since the ring structure of $H^{1,1}(X)$ has already been computed for these directly via toric methods and will not cost us anything. In addition, the redundancy
of the toric divisors allows us to scan over multiple choices of basis (in particular, many
will naturally be \( \mathbb{Z} \)-bases) simply by sampling subsets of the toric divisors. So, while
there is no natural basis for our calculations, the toric divisor classes \( \{D_i\}_{i \in I_{\text{Toric}}} \) form a
natural “pseudo-basis” in spite of their redundancy.

A basis formed by a pure subset of the toric 2-forms will not always have a Swiss cheese
solution. Even if such a solution exists, an arbitrary rotation may still be required.
However, if we limit ourselves to the case in which some subset of the toric 2-forms is
already a Swiss cheese basis (i.e. \( \{J^A_i\}_{i \in I^A}, \{J^B_i\}_{i \in I^B} \subset \{D_i\}_{i \in I_{\text{Toric}}} \)), then our problem
is reduced to a relatively simple combinatorical one. In order to see this, we define the
injective maps

\[
\alpha : I^A \hookrightarrow I_{\text{Toric}} \quad \beta : I^B \hookrightarrow I_{\text{Toric}}
\]

\[
J^A_i \mapsto D_{\alpha(i)} \quad J^B_i \mapsto D_{\beta(i)} \quad (3.40)
\]

We also define the toric triple intersection tensor and the Mori cone matrix\(^6\)

\[
d_{ijk} = \int_X D_i \wedge D_j \wedge D_k \quad (3.41)
\]

\[
M^i_j = \int_{C^i} D_j \quad (3.42)
\]

\(^6\)The Mori cone matrix essentially the same as the Kähler cone matrix, but expanded in the toric
divisors rather than a basis. We give it this name because it is the object that is directly computed from
torus invariant curves viewed as linear functionals relating the toric divisors.
Then, we can rewrite

\[
\begin{align*}
\kappa_{ijk}^{AAA} &= d_{\alpha(i)\alpha(j)\alpha(k)} \\
\kappa_{ijk}^{BAA} &= d_{\beta(i)\alpha(j)\alpha(k)} \\
\kappa_{ijk}^{BA0} &= d_{\beta(i)\alpha(j)k} \\
\kappa_{ijk}^{BBA} &= d_{\beta(i)\beta(j)\alpha(k)} \\
(K^A)^i_j &= M^i_{\alpha(j)} \\
\end{align*}
\]

\[(3.43)\]

It is clear, then, that the conditions in Section 3.1.4 become purely combinatoric in nature and take the form

1. (Equation (3.15): Volume) \[\exists(i, j, k) \in \alpha(I^A_L) \times \alpha(I^A) \times \alpha(I^A) : d_{ijk} \neq 0\]
2. (Equation (3.16): Large Cycle) \[\exists(j, k) \in \alpha(I^A_L) \times \alpha(I^A) : d_{ijk} \neq 0, \ \forall i \in \beta(I^B_L)\]
3. (Equation (3.18): Small Cycle) \[d_{ijk} = 0, \ \forall(i, j, k) \in \beta(I^B_S) \times \alpha(I^A_L) \times I^\text{Toric}\]
4. (Equation (3.38): Homogeneity) \[\exists j \in \alpha(I^A_S) : d_{iij} \neq 0, \ \forall i \in \beta(I^B_S)\]
5. (Equation (3.28): Kähler Cone (L)) \[\pm p^A_i \geq 0, \ \forall i \in \alpha(I^A_L)\]
6. (Equation (3.31): Kähler Cone (S)) \[\dim \left[ \text{relint} \left( \text{cone} (q^A_1, \ldots, q^A_m)^\vee \right) \right] > 0\]
where

\[
\begin{pmatrix}
\begin{array}{ccc}
p^A_{\alpha(1)} & \cdots & p^A_{\alpha(N_L)} \\
0 & \ddots & \\
\vdots & & \ddots \\
(q^A_m)^T & & \end{array}
\end{pmatrix}
\]

\[
M^A = \begin{pmatrix}
N_L & h-N_L
\end{pmatrix}
\]

Now, instead of solving a complex linear system for two arbitrary rotation matrices \(T^A, T^B \in GL_h(\mathbb{Q})\), we simply need to choose two subsets \(\alpha(I^A), \beta(I^B) \subset I^{\text{Toric}}\). Since the toric triple intersection tensor \(d_{i,j,k}\) and the Mori cone matrix \(M'_{\cdot j}\) are basis-independent, it is a simple combinatoric matter to search \(d_{i,j,k}\) for subtensors that meet these constraints. If one is found, then we are done and the sets \(\alpha(I^A)\) and \(\beta(I^B)\) determine the bases7

\[
\{ J^A_i \}_{i \in I^A} = \{ D_j \}_{j \in \alpha(I^A)} \quad \text{and} \quad \{ J^B_i \}_{i \in I^B} = \{ D_j \}_{j \in \beta(I^B)}
\]

for which there exists such a Swiss cheese solution.

---

7Again, even if the original basis \(\{ J_i \}_{i \in I}\) is a \(\mathbb{Z}\)-basis, it is not guaranteed in our analysis that \(\{ J^A_i \}_{i \in I^A}\) and \(\{ J^B_i \}_{i \in I^B}\) are as well.
3.2 Implementing Toric Swiss Cheese Detection

Given the combinatorial conditions set forward in the previous section, it is fairly straightforward to scan the database of toric Calabi-Yau threefolds [34] for Swiss cheese solutions. The procedure we use is as follows, but there are many variations.

1. From the database, we can readily obtain the toric triple intersection tensor \( d_{ijk} \), the Mori cone matrix \( M \), and the weight matrix \( W \). The latter is defined by the conditions \( \sum_{\rho=1}^k W_r^\rho n_\rho = 0 \) and \( W \geq 0 \), where the \( k \) 4-dimensional vectors \( \{n_1, \ldots, n_k\} \) are the vertices of the dual polytope.

2. The Small Cycle condition reads

\[
d_{ijk} = 0, \quad \forall (i, j, k) \in \beta(I^B_S) \times \alpha(I^A_L) \times I^{\text{Toric}}
\]

This tells us that we can search for any row of any submatrix of \( d_{ijk} \) that contains all zeroes, and the indices of those rows and submatrices give us all possible combinations of \( \alpha(I^A_L) \) and \( \beta(I^B_S) \).

3. We then assemble all possible complementary sets of indices \( \alpha(I^A_L) \) and \( \beta(I^B_S) \) from among the \( k \) toric divisor indices to get the full sets \( \alpha(I^A) \) and \( \beta(I^B) \), each of which contain \( h \) total indices.

4. We construct the submatrices \( W_{\alpha(i)}^{\alpha(j)} \) and \( W_{\beta(i)}^{\beta(j)} \) of the weight matrix and check that both are full rank, otherwise we have chosen redundant toric divisors.
5. We then check the Volume condition

$$\exists (i, j, k) \in \alpha(I_L^A) \times \alpha(I^A) \times \alpha(I^A) : d_{ijk} \neq 0$$

6. Given the Mori cone matrix $M$ and the set of indices $\alpha(I^A)$, we construct the submatrix $M_{\alpha(j)}^i$ and reorder the rows until it takes the form

$$M_{\alpha(j)}^i = \begin{bmatrix}
    p_{\alpha(1)}^A & \cdots & p_{\alpha(N_L)}^A & \cdots \\
    \vdots & & & \vdots \\
    0 & & & (q_1^A)^T \\
    \vdots & & & \vdots \\
    0 & & & (q_m^A)^T
\end{bmatrix}, \quad 0 \leq m \leq h.$$

7. Next, we check the Large Cycle Kähler cone condition

$$\pm p_i^A \geq 0, \quad \forall i \in \alpha(I_L^A)$$

8. Then the Small Cycle Kähler cone condition

$$\dim \left[ \text{relint} \left( \text{cone} \left( q_1^A, \ldots, q_m^A \right) \right) \right] > 0$$
9. The Large Cycle condition

\[ \exists (j, k) \in \alpha(I^A_L) \times \alpha(I^A) : d_{ijk} \neq 0, \ \forall i \in \beta(I^B_L) \]

10. And finally, the Homogeneity condition

\[ \exists j \in \alpha(I^A_S) : d_{ij} \neq 0, \ \forall i \in \beta(I^B_S) \]

11. If all the conditions in Section 3.1.5 are satisfied, then the sets of indices \( \alpha(I^A) \) and \( \beta(I^B) \) are converted into rotation matrices

\[
(T^A)^j_i = \delta^{\alpha(j)}_i \quad \text{and} \quad (T^B)^j_i = \delta^{\beta(j)}_i
\]  

(3.44)

12. We also check whether the \( A \)- and \( B \)-bases are \( Z \)-bases. This is the case if and only if the remaining redundant toric divisors all intersect each other smoothly at a point on the desingularized ambient toric variety \( A \), up to an action of the fundamental group.

13. We repeat this procedure for \( N_L = 1, \ldots, h - 1 \), so that at least one 4-cycle is always large and at least one 4-cycle is always small. The results are recorded in the database [34] as well.

14. Finally, we can take multiple passes at the dataset, beginning with a randomly chosen \( GL_k(Z) \) transformation on the toric divisors \( \{D_1, \ldots, D_k\} \) each time. The full
Swiss cheese solution set should begin to converge after many loops, but it is unclear how slow that convergence should be. This is still a significant improvement over the method of solving the linear system for $T^A$ and $T^B$, as each loop will uncover a handful of solutions with purely combinatorial efficiency. We save this larger scan for a later work.

3.3 Swiss Cheese Classification

In previous studies, the majority of Swiss cheese geometries have been constructed explicitly using a top down approach. Here, working from a vast database of known candidate geometries [34,99], we attack the problem from the bottom up with the hope of identifying as many viable Swiss cheese vacua as possible. Toward this end, in this section we lay out a scheme for categorizing Swiss cheese geometries with varying degrees of generality.

The Kähler moduli $t^i$ are the natural geometrical parameters on the Calabi-Yau threefold $X$, and it is a simple matter to write the volume form in terms of these as

$$V = \frac{1}{3!} t^i t^j t^k \kappa_{ijk}, \quad (3.45)$$

where the intersection tensor $\kappa_{ijk}$ encodes the Chow ring structure on $X$. In the low energy ten dimensional supergravity limit, the relevant field parameters descending from $X$ are the complexified\footnote{The $b_i$ are axionic partners of the $\tau_i$ 4-cycle volumes.} 4-cycle volumes $T_i = \tau_i + i b_i$. There is a natural injective map
from the 2-cycles to the 4-cycles via

\[ t^i \mapsto \tau_i \equiv \frac{\partial \mathcal{V}}{\partial t^i} \quad (3.46) \]

Depending on the Chow ring structure hidden in \( \mathcal{V} \), it may be possible to choose a basis in which the map is invertible, at least on some subset of \( t^i \). If so, then it is possible to write \( \mathcal{V} \) explicitly in terms of the 4-cycle volumes (at least partially). In this case, we say that \( X \) is explicitly Swiss cheese.

In addition, the Swiss cheese condition requires that some set of large 4-cycles determine the scale of the overall volume \( \mathcal{V} \), while the remaining small 4-cycles determine the scale of the missing “holes”. This can be observed directly from the form of \( \mathcal{V} \) when each 4-cycle volume \( \tau_i \) contributes independently as its own term, such that

\[ \mathcal{V} = \sum_{i=1}^{b^{1,1}} \lambda_i \tau_i^{3/2} \quad (3.47) \]

When this is the case, we say that the volume is diagonalized, as there are no mixed terms. The conditions set forward in Section 3.1.4 guarantee that \( X \) obeys the Swiss cheese condition, but even when \( X \) is explicit, it is not always possible to find a basis that makes Equation (3.47) manifest. When it is possible, though, we say that \( X \) is diagonal.

Finally, a Swiss cheese geometry \( X \) that is both maximally explicit and maximally diagonal has special properties, and we refer to it as a strong Swiss cheese geometry. In any
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<th>Partially Explicit</th>
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<td></td>
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<td></td>
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<td></td>
<td>$\tau^{3/2}$ (Strong)</td>
</tr>
</tbody>
</table>

Table 3.1: Classification of the allowed forms of monomial terms $M_i$ in the volume polynomial $V = \sum_i \lambda_i M_i$.

In other case, $X$ is said to be *weak*.

In order to give a more thorough classification, we first define the monomial functions

- $f^d \equiv f^d(t^1, \ldots, t^{h_{1,1}})$: a degree $d$ monomial in the 2-cycle volumes.
- $g^d \equiv g^d(\tau_1, \ldots, \tau_{h_{1,1}})$: a degree $d$ monomial in the 4-cycle volumes.

With this notation, Table 3.1 enumerates the monomial forms $M_i$ that can appear in the expression for the overall volume

$$V = \sum_i \lambda_i M_i.$$  \hspace{1cm} (3.48)

Note that a special case of the $g^d$ occurs when the case in question is a function of only one 4-cycle volume. In these cases we have replaced $g^d$ with $\tau^d$ in Table 3.1.

The volume is naturally expressed in terms of 2-cycles as a sum of monomials of the form $f^3$, as in Equation (3.45). If none of the maps in Equation (3.46) are invertible, then this form of the volume must remain *implicit* only. If one or more of the maps can be inverted, then mixed terms involving monomials from the various sets $g^d$ are possible. Thus, a partially explicit case may contain terms that remain in the form $f^3$, but must
contain terms involving $g^d$. A completely explicit case involves a sum of monomials from the set $g^{3/2}$ only. We note that a fully explicit form for the volume may not always display the geometrical properties of the Calabi-Yau manifold most clearly. For example, a K3-fibration is often evidenced by terms of the form $f\tau$ in the volume [27].

We now focus our attention on the lower-right quadrant of Table 3.1. These are the cases that are designated as partially, or fully, diagonal. When restricted to these cases, we can write the most general volume form in a new basis $C$ as

$$\mathcal{V} = \sum_{i=1}^{n} \lambda_i C_i \left( \tau_i \right)^{3/2} + g^{3/2} \left( \tau_{n+1}, \ldots, \tau_h \right) + f g \left( \tau_{n+1}, \ldots, \tau_h \right) + f^2 g^{1/2} \left( \tau_{n+1}, \ldots, \tau_h \right) + f^3$$

with $n \leq h$. Thus, a partially diagonal volume form may contain terms from the general sets $f^d$ and $g^d$, but must contain at least one term of the form $\tau^{3/2}$. A circumstance such as Equation (3.47) is both fully explicit (no terms involving 2-cycle volumes) and fully diagonal.

In contrast, consider the general form of the volume in terms of the Kähler moduli given by Equation (3.7):

$$\mathcal{V} = \frac{1}{3!} \epsilon^{C_i C_j C_k} \left( T_i \right)^{r} \left( T_j \right)^{s} \left( T_k \right)^{t} \kappa_{rst}$$

$$= \frac{1}{3!} \epsilon^{C_i C_j C_k} \kappa^{C}_{ijk}.$$  (3.50)
where $\kappa_{ijk}^C = (T^C)_i \tau (T^C)_j \kappa_{rst}^C$. Then, we can recover the 4-cycle volumes

$$\tau^C_i = \frac{\partial V}{\partial t^C_i} = \frac{1}{2} t^C_j t^C_k \kappa_{ijk}^{CCC}$$

and rewrite the volume as

$$V = \frac{1}{3} t^C_i \tau^C_i.$$

We can then scan the database of Calabi-Yau vacua for cases in which the Chow ring structure allows for the identification

$$\tau^D_i = (T^D)_i \tau_j = (T^D (T^C)^{-1})_i \tau^C_j = \frac{1}{9(\lambda_i^C)^2} t^C_i t^C_i, \quad \forall i \leq n.$$}

When this is the case, the volume takes the explicit form

$$V = \sum_{i=1}^{h} \pm \lambda_i^C \tau^C_i \sqrt{\tau^D_i}$$

where the sign of each coefficient $\lambda_i^C$ can be fixed by Kähler cone and non-negative volume considerations. Furthermore, when the $C$ and $D$ bases coincide, the volume can be written in the diagonal form

$$V = \sum_{i=1}^{h} \pm \lambda_i^C (\tau^C_i)^{3/2}.$$
In this case, the LVS vacuum takes the form of a strong “Swiss Cheese” compactification, in which terms with negative sign punch out “holes” in an overall volume. Comparing Equations (3.51) and (3.53), we see that

\[ \kappa_{ijk}^{CCC} = \begin{cases} 
\frac{2}{9 \Lambda^3} , & i = j = k \leq n \\
0 , & i,j,k \leq n \\
\text{Undetermined} , & \text{otherwise}
\end{cases} \tag{3.56} \]

Therefore, we see that in the $C$-basis, $\kappa_{ijk}^{CCC} = (T^C)_i^r (T^C)_j^s (T^C)_k^t \kappa_{rst}$ is a partially-diagonal, rank three tensor. In fact, if $n = h$, then $\kappa_{ijk}^{CCC}$ is fully diagonal, and $X$ is a strong Swiss cheese geometry. This result is derived via similar methods in [27].

### 3.4 Kähler Moduli Stabilization

The bosonic field content in the string frame is as follows

- **R-R sector**: A 0-form potential $C_0$, a 2-form potential $C_2$, and a 4-form potential $C_4$.

- **NS-NS sector**: The 0-form dilaton $\phi$, the 2-form 10D graviton $g_{\mu\nu}$, and the antisymmetric Kalb-Ramond 2-form $B_2$.

- **Scalar moduli**: Kähler moduli ($\tau_i$) and complex structure moduli ($U_i$).
A constant variation in the dilaton can be shown to produce a corresponding variation in the string coupling according to $\frac{\delta g_s}{g_s} = \delta \phi$, so that we may express the coupling as $g_s \sim e^{\phi}$. Furthermore, in the Einstein frame, the 10D string-frame graviton is rescaled by $g^{(s)}_{\mu\nu} \rightarrow g^{(E)}_{\mu\nu} = g_s^{-1/2} g^{(s)}_{\mu\nu}$. This results in the rescaling of each 2-cycle volume as $t^i \rightarrow \hat{t}^i = g_s^{-1/2} t^i$, and therefore $\tau_i \rightarrow \hat{\tau}_i = g_s^{-1} \tau_i$ and $V \rightarrow \hat{V} = g_s^{-3/2} V$. It is convenient to make the following field redefinitions

- The axion-dilaton $S = g_s^{-1} + iC_0$, which corresponds to the complex structure of the elliptic fiber in the F-theory generalization.
- The complexified Kähler moduli $T_i = \int J_i (J \land J + iC_4) = \hat{\tau}_i + ib_i$.

In this section, we will show that at tree level, the scalar potential of the 4D effective supergravity theory exhibits a “no-scale” structure, at which only the axion-dilaton and complex structure moduli are stabilized. We further show that in order to break the “no-scale” structure and stabilize the volume modulus, we must include the leading $\alpha'$ correction to the volume in the Kähler potential. And finally, we show that to stabilize the remaining Kähler blowup modes, we must consider non-perturbative corrections to the superpotential resulting from the structure on the blowup cycles. In the end, we will write down the corrected scalar potential

$$V = V_{\text{tree}} + V_{\alpha'} + V_{\text{non-perturbative}}$$

(3.57)
which can be minimized to stabilize the Kähler blowup moduli, as well as the volume modulus, which gets fixed exponentially large with respect to the blowup moduli. The presence of a flat direction in the moduli space at this minimum leaves the door open for Kähler blowup moduli-mediated inflation. We will discuss this in the next section.

In the absence of flat directions, the blowup moduli are all stabilized at small values. However, this still leaves any non-blowup Kähler moduli (which correspond instead to fibration modes and typically have large values) unstabilized. This “extended no-scale” structure can only be broken by adding subleading string loop corrections. The fibration moduli can then be stabilized by minimizing the further corrected potential. The presence of a flat direction at this minimum would allow for Kähler fibration moduli-mediated inflation. This discussion we leave for a future work.

3.4.1 $V_{\text{tree}}$

At tree level in $\alpha'$, the Kähler potential for $X$ can be expressed in the separated form\footnote{Technically, the Einstein frame volume $\hat{V} = g_s^{-3/2}V$ depends on the axion-dilaton $S$ through $g_s \sim S^2$. However, we can disregard this, since we will be differentiating with respect to the Einstein frame Kähler moduli $\hat{\tau}$, which have a complementary dependence on $S$.}

$$
\mathcal{K} = \mathcal{K}_S + \mathcal{K}_T + \mathcal{K}_U
= -\ln \left( s + \bar{s} \right) + -2 \ln \left( \hat{\nabla} \right) + -\ln \left( -i \int_X \Omega \wedge \bar{\Omega} \right) \tag{3.58}
$$

where $\Omega$ is the unique, holomorphic $(3,0)$-form on $X$ which contains the dependence on the complex structure moduli $U_i$, $i = 1, ..., h^{2,1}$. If we turn on the non-trivial RR and NS-NS gauge fluxes $F_3 = dC_2$ and $H_3 = dB_2$, and construct the complexified flux
\( G_3 = F_3 + iSH_3 \), then the superpotential can be written as

\[
W = W_{\text{Gukov-Vafa-Witten}} = \int_X \Omega \wedge G_3
\]  

(3.59)

In order to obtain a warped 10D background, the Bianchi identity constrains \( G_3 \) to be imaginary self-dual (i.e. \( *_6 G_3 = iG_3 \)). It can be shown that this is equivalent to the constraints on the GVW superpotential

\[
D^A W = 0, \quad \Phi_A \in \{S, U_i\}
\]  

(3.60)

The full scalar potential in the 4D effective supergravity theory then has the form

\[
V = e^K \left[ (\mathcal{K}^{-1})_{ab} D^a W D^b W - 3|W|^2 \right], \quad \Phi_a, \Phi_b \in \{S, T_1, ..., T_{h^{1,1}}, U_1, ..., U_{h^{2,1}}\}
\]  

(3.61)

where the inverse Kähler metric is defined as

\[
(\mathcal{K}^{-1})_{ab} \equiv \left( \frac{\partial K}{\partial \Phi_a \partial \Phi_b} \right)^{-1}
\]

and the gauge connection for the covariant derivative is given by \( \partial_a K \equiv \frac{\partial K}{\partial \Phi_a} \) so that

\[
D^a W = \partial^a W + W \partial^a K
\]
Because $\mathcal{K}$ is separated into $\mathcal{K}_S$, $\mathcal{K}_T$, and $\mathcal{K}_U$, the inverse Kähler metric has a block diagonal form

$$ (\mathcal{K}^{-1})_{\bar{a}b} = \begin{pmatrix} (\mathcal{K}_S^{-1}) & 0 & 0 \\ 0 & (\mathcal{K}_T^{-1})_{ij} & 0 \\ 0 & 0 & (\mathcal{K}_U^{-1})_{AB} \end{pmatrix} $$

where $\Phi_i, \Phi_j \in \{T_1, \ldots, T_{h_1,1}\}$ and $\Phi_A, \Phi_B \in \{U_1, \ldots, U_{h_2,1}\}$. Then, the scalar potential separates

$$ V = V_S + V_T + V_U - 3e^K|W|^2 $$

where the second equality follows from the constraints in Equation (3.60). Then, focusing on $V_T$, we have

$$ V_T = e^K \left[ (\mathcal{K}^{-1})_{ij} D^i W \bar{D}^j W \right] $$

The first derivative of $\mathcal{K}_T$ is given by

$$ \mathcal{K}_T^i = \partial^i \mathcal{K}_T = \frac{\partial}{\partial T_i} \mathcal{K}_T = -2\hat{\mathcal{V}}^{-1} \frac{\partial \hat{\mathcal{V}}}{\partial T_i} = -2\hat{\mathcal{V}}^{-1} \frac{1}{2} \left( \frac{\partial \hat{\mathcal{V}}}{\partial t_i} - i \frac{\partial \hat{\mathcal{V}}}{\partial b_i} \right) $$

$$ = -\hat{\mathcal{V}}^{-1} \hat{\mathcal{V}}^i $$
Then, the second derivative is given by

$$
\mathcal{K}_{ij} = \partial^i \partial^j \mathcal{K}_T = - \left( \frac{\partial}{\partial T_i} \hat{\mathcal{V}}^{-1} \right) \hat{\mathcal{V}}^j - \hat{\mathcal{V}}^{-1} \left( \frac{\partial}{\partial T_j} \hat{\mathcal{V}}^i \right) \\
= \hat{\mathcal{V}}^{-1} \left( \frac{\partial \hat{\mathcal{V}}}{\partial \hat{\tau}_i} - \frac{i}{\partial \phi_i} \right) \hat{\mathcal{V}}^j - \hat{\mathcal{V}}^{-1} \frac{1}{2} \left( \frac{\partial \hat{\mathcal{V}}^j}{\partial \hat{\tau}_i} - \frac{i}{\partial \phi_i} \right) \\
= \frac{1}{2 \hat{\mathcal{V}}} \left( \frac{1}{\hat{\mathcal{V}}} \hat{\mathcal{V}}^i \hat{\mathcal{V}}^j - \hat{\mathcal{V}}^{ij} \right) 
$$

(3.66)

Now, we assume that \((\mathcal{K}^{-1})_{ij}\) is of the form

$$
(\mathcal{K}^{-1})_{ij} = u \hat{\mathcal{V}} \left( \hat{\mathcal{V}}^{-1} \right)_{ij} + v \hat{\mathcal{V}}^k \left( \hat{\mathcal{V}}^{-1} \right)_{ki} \hat{\mathcal{V}}^l \left( \hat{\mathcal{V}}^{-1} \right)_{lj} + O\left( \text{higher order in } \hat{\mathcal{V}}^{-1} \right) 
$$

(3.67)

Then, we have

$$
\mathcal{K}_{ij} (\mathcal{K}^{-1})_{jk} = \frac{1}{2 \hat{\mathcal{V}}} \left[ \frac{1}{\hat{\mathcal{V}}} \hat{\mathcal{V}}^i \hat{\mathcal{V}}^j - \hat{\mathcal{V}}^{ij} \right] \left[ u \hat{\mathcal{V}} \left( \hat{\mathcal{V}}^{-1} \right)_{jk} + v \hat{\mathcal{V}}^l \left( \hat{\mathcal{V}}^{-1} \right)_{lj} \hat{\mathcal{V}}^m \left( \hat{\mathcal{V}}^{-1} \right)_{mk} \right] \\
= \frac{1}{2 \hat{\mathcal{V}}} \left[ u \hat{\mathcal{V}}^i \hat{\mathcal{V}}^j \left( \hat{\mathcal{V}}^{-1} \right)_{jk} + \frac{v}{\hat{\mathcal{V}}} \hat{\mathcal{V}}^i \hat{\mathcal{V}}^l \left( \hat{\mathcal{V}}^{-1} \right)_{lj} \hat{\mathcal{V}}^m \left( \hat{\mathcal{V}}^{-1} \right)_{mk} - u \hat{\mathcal{V}} \hat{\mathcal{V}}^i \left( \hat{\mathcal{V}}^{-1} \right)_{jk} \right] \\
= \frac{1}{2 \hat{\mathcal{V}}} \left[ u \hat{\mathcal{V}}^i \hat{\mathcal{V}}^j \left( \hat{\mathcal{V}}^{-1} \right)_{jk} + \frac{v}{\hat{\mathcal{V}}} \hat{\mathcal{V}}^i \left( 3 \hat{\mathcal{V}} \right) \hat{\mathcal{V}}^m \left( \hat{\mathcal{V}}^{-1} \right)_{mk} - u \hat{\mathcal{V}} \hat{\mathcal{V}}^i \left( \hat{\mathcal{V}}^{-1} \right)_{jk} \right] \\
= \frac{1}{2 \hat{\mathcal{V}}} \left[ u \hat{\mathcal{V}}^i \hat{\mathcal{V}}^j \left( \hat{\mathcal{V}}^{-1} \right)_{jk} + 3 v \hat{\mathcal{V}}^i \hat{\mathcal{V}}^j \left( \hat{\mathcal{V}}^{-1} \right)_{jk} - u \hat{\mathcal{V}} \delta^i_k - v \delta^i_k \hat{\mathcal{V}}^m \left( \hat{\mathcal{V}}^{-1} \right)_{mk} \right] \\
= \frac{1}{2 \hat{\mathcal{V}}} \left[ (u + 2v) \hat{\mathcal{V}}^i \hat{\mathcal{V}}^j \left( \hat{\mathcal{V}}^{-1} \right)_{jk} - u \hat{\mathcal{V}} \delta^i_k \right] \\
= \left( \frac{u}{2} + v \right) \frac{1}{\hat{\mathcal{V}}} \left( \frac{1}{2} \hat{\mathcal{V}}^i \right) (2 \hat{\tau}_k) - \frac{u}{2} \delta^i_k \\
= \left( \frac{u}{2} + v \right) \frac{i \hat{\tau}_k}{\hat{\mathcal{V}}} - \frac{u}{2} \delta^i_k 
$$

(3.68)
In order for this result to be consistent and general, we must have

\[ u = -2 \quad \text{and} \quad v = 1 \] (3.69)

so that

\[ (K_T^{-1})_{ij} = -2 \hat{V} \left( \hat{V}^{-1} \right)_{ij} + \hat{V}^k \left( \hat{V}^{-1} \right)_{ki} \hat{V}^l \left( \hat{V}^{-1} \right)_{lj} + \mathcal{O} \left( \text{higher order in } \hat{V}^{-1} \right) \] (3.70)

We also know that since \( W = W_{GVW} \) is independent of the Kähler moduli, we can write

\[ D^i W = W \partial^i K_T = -W \hat{V}^{-1} \hat{V}^i \] (3.71)

Then, expanding Equation (3.64), we have

\[
V_T = e^K \left[ (K_T^{-1})_{ij} D^i W D^j W \right] \\
= e^K |W|^2 \left[ -2 \hat{V} \left( \hat{V}^{-1} \right)_{ij} + \hat{V}^k \left( \hat{V}^{-1} \right)_{ki} \hat{V}^l \left( \hat{V}^{-1} \right)_{lj} \right] \hat{V}^{-2} \hat{V}^i \hat{V}^j \\
= e^K |W|^2 \hat{V}^{-2} \left[ -2 \hat{V} \hat{V}^i \left( \hat{V}^{-1} \right)_{ij} \hat{V}^j + \hat{V}^k \left( \hat{V}^{-1} \right)_{ki} \hat{V}^i \hat{V}^l \left( \hat{V}^{-1} \right)_{lj} \hat{V}^j \right] \\
= e^K |W|^2 \hat{V}^{-2} \left[ -2 \hat{V} \left( 3 \hat{V} \right) + \left( 3 \hat{V} \right) \left( 3 \hat{V} \right) \right] \\
= 3e^K |W|^2
\] (3.72)
and the scalar potential in Equation (3.63) reduces to the “no-scale” form with

\[ V_{\text{tree}} = 0 \]  

(3.73)

Thus, at tree level, this potential cannot be minimized, and so both the volume and the individual Kähler moduli are left unstabilized.

### 3.4.2 \( V_{\alpha'} \)

In order to stabilize the volume \( \hat{V} \), we must break the “no-scale” structure. We do this by considering the leading \((\alpha')^3\) correction to \( V \), given by \( \xi \), where \( \xi = -\chi(X)\zeta(3) \). With this correction, the Einstein frame volume becomes

\[
\hat{V} = g^{-3/2}_s V \\
\rightarrow g^{-3/2}_s \left( V + \frac{\xi}{2} \right) = \hat{V} + \frac{\xi}{2} \left( \frac{S + \bar{S}}{2} \right)^{3/2}
\]  

(3.74)

Then, the Kähler potential for \( X \) can be expressed in a partially separated form as

\[
K = K_{S/T} + K_U \\
= -\ln \left( S + \bar{S} \right) - 2 \ln \left( \hat{V} + \frac{\xi}{2} \left( \frac{S + \bar{S}}{2} \right)^{3/2} \right) + \ln \left( -i \int X \Omega \wedge \bar{\Omega} \right)
\]  

(3.75)

\( \chi(X) \) is the Euler characteristic of \( X \), and \( \zeta(3) \approx 1.20206 \) is the Riemann zeta function, evaluated at 3.
For ease of notation, we also define

\[ \mathcal{K}_T = -2 \ln \left( \hat{\mathcal{V}} + \xi \left( \frac{S + \bar{S}}{2} \right)^{3/2} \right) = -2 \ln \left( \hat{\mathcal{V}} + \frac{\xi}{2} \right) \]  

(3.76)

This allows us to write \( K^{ij} = \partial^i \partial^j K \) in block diagonal form

\[
\mathcal{K}^{ab} = \begin{pmatrix}
(K_{S/T})^{00} & (K_{S/T})^{0j} & 0 \\
(K_{S/T})^{j0} & (K_{T})^{ij} & 0 \\
0 & 0 & (K_U)^{AB}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
a & b^T \\
b & C
\end{pmatrix}
\]

(3.77)

where \( \Phi_i, \Phi_j \in \{T_1, \ldots, T_{h_1,1}\} \) and \( \Phi_A, \Phi_B \in \{U_1, \ldots, U_{h_2,1}\} \). The inverse is given by

\[
(K^{-1})_{a\bar{b}} = \frac{1}{d} \begin{pmatrix}
1 & -b^T C^{-1} \\
-C^{-1}b & dC^{-1} + C^{-1}bb^TC^{-1}
\end{pmatrix}
\]

(3.78)
where \( d = a - b^T C^{-1} b \). In addition, our matrix \( C \) is diagonal, so

\[
C^{-1} = \begin{pmatrix}
(K^{-1})_{ij} & 0 \\
0 & (K^{-1})_{AB}
\end{pmatrix}
\]

so Equation (3.78) reduces to

\[
(K^{-1})_{ab} = \frac{1}{d} \times 
\begin{pmatrix}
1 & - (K^T)_{0k} (K^{-1})_{kj} & 0 \\
-(K^{-1})_{ik} (K_{S/T})^{k0} & d (K^{-1})_{ij} + (K^{-1})_{ik} (K^T)_{k0} (K_{S/T})^{0l} (K^{-1})_{lj} & 0 \\
0 & 0 & d (K^{-1})_{AB}
\end{pmatrix}
\]

(3.80)

where \( d = (K_{S/T})^{00} - (K^T)_{0i} (K^{-1})_{ij} (K_{S/T})^{j0} \).

The full scalar potential in the 4D effective supergravity theory again has the form

\[
V = e^\Phi \left[ (K^{-1})_{ab} D^a W D^b W - 3|W|^2 \right], \quad \Phi_a, \Phi_b \in \{S, T_1, ..., T_{h_1+1}, U_1, ..., U_{h_2+1} \}
\]

(3.81)

Due to the superpotential constraints (see Equation (3.60)) that stabilize the axion-dilaton and complex structure moduli at tree level, any terms proportional to \( D^S W \) or \( D^U W \) vanish, and we need only consider the center block of the Equation (3.80). This
gives us the corrected inverse Kähler metric \[43, 100\]

\[
\left( \tilde{K}_T^{-1} \right)_{ij} = (K_T^{-1})_{ij} + \frac{1}{d} (K_T^{-1})_{ik} (K^T_{S/T})^{k0} (K_{S/T})^{0l} (K_T^{-1})_{lj} \tag{3.82}
\]

where, from Equation (3.75) and Equation (3.67) with \( \hat{\nu} \rightarrow \hat{\nu} + \frac{\xi}{2} \) and \( \hat{\xi} = g_s^{-3/2} \xi \), we find that

\[
(K_T^{-1})_{ij} = - \left( 2\hat{\nu} + \hat{\xi} \right) (\hat{\nu}^{-1})_{ij} + 2 \left( \frac{2\hat{\nu} + \hat{\xi}}{4\hat{\nu} - \hat{\xi}} \right) \hat{\nu}^k (\hat{\nu}^{-1})_{ki} \hat{\nu}^l (\hat{\nu}^{-1})_{lj} \tag{3.83}
\]

\[
(K_{S/T})^{0i} = \frac{3\hat{\xi} g_s}{4 \left( 2\hat{\nu} + \hat{\xi} \right)^2} \hat{\nu}^i \tag{3.84}
\]

\[
(K_{S/T})^{i} = - \frac{2}{2\hat{\nu} + \hat{\xi}} \hat{\nu}^i \tag{3.85}
\]

and

\[
d = \frac{g_s^2 \left( \hat{\nu} - \hat{\xi} \right)}{4 \left( 4\hat{\nu} - \hat{\xi} \right)} \tag{3.86}
\]

The scalar potential then takes the form

\[
V = e^\kappa \left[ \left( \tilde{K}_T^{-1} \right)_{ij} D^i W D^j W - 3|W|^2 \right] \tag{3.87}
\]
Recall that the superpotential is independent of the Kähler moduli, so that $D^i W = W \partial^i \mathcal{K} = W \left( \mathcal{K}_{S/T} \right)^i$. Then, the scalar potential reduces to

$$V = V_{\text{tree}} + V_{\alpha'}$$

$$= 0 + 3e^K |W|^2 \xi \frac{\left( \xi^2 + 7\xi \bar{V} + \bar{V}^2 \right)}{(\bar{V} - \xi)(2\bar{V} + \xi)^2} \quad (3.88)$$

We can now find a stable minimum for the volume modulus.

### 3.4.3 $V_{\text{non-perturbative}}$

In the previous subsection, we were able to use the leading $\alpha'$ correction to the volume in order to break the “no-scale” structure of the potential and stabilize the volume modulus, but this still did not give us a mechanism for stabilizing the remaining Kähler moduli. In order to find such a mechanism, we must further consider the effect of non-perturbative features on the superpotential. The superpotential then takes the form

$$W = W_{GVW} + W_{\text{non-perturbative}}$$

$$= \int_X \Omega \wedge G_3 + A_i e^{-\hat{a}_i T_i} \quad (3.89)$$

where the scalar constants $\hat{a}_i$ depend on non-perturbative effects such as D brane instantons ($\hat{a}_i = \frac{2\pi}{g_s}$) or gaugino condensation ($\hat{a}_i = \frac{2\pi}{g_s N}$) on the corresponding 4-cycle $J_i \in H_4(X; \mathbb{Z})$, and the complex constants $A_i$ encode threshold effects depending implicitly on the complex structure and D3 brane positions.
The scalar potential still takes the form in Equation (3.87), except that
\[
D^i W = \partial^i W + W \partial^i \mathcal{K} = -\hat{a}_i A_i e^{-\hat{a}_i T_i} + W (\mathcal{K}_{ad/\xi})^i
\] (3.90)

Plugging this in, we find that
\[
V = e^K \left[ (\tilde{\mathcal{K}}^{-1})_{ij} \hat{a}_i \hat{a}_j A_i \tilde{A}_j e^{-\hat{a}_j T_j} \right. \\
- (\tilde{\mathcal{K}}^{-1})_{ij} e^{-\hat{a}_i T_i} (\mathcal{K}_{S/T})^j + (\mathcal{K}_{S/T})^i \hat{a}_j \tilde{A}_j W e^{-\hat{a}_j T_j} \right] \\
+ (\tilde{\mathcal{K}}^{-1})_{ij} |W|^2 (\mathcal{K}_{S/T})^i (\mathcal{K}_{S/T})^j - 3|W|^2
\] (3.91)

where the last line is just $V_{\alpha'}$. This then reduces to
\[
V = V_{\text{tree}} + V_{np1} + V_{np2} + V_{\alpha'}
\] (3.92)

with
\[
V_{np1} = e^K (\tilde{\mathcal{K}}^{-1})_{ij} \hat{a}_i \hat{a}_j A_i \tilde{A}_j e^{-\hat{a}_j T_j} \\
= e^K \hat{a}_i \hat{a}_j A_i \tilde{A}_j e^{-\hat{a}_i T_i + \hat{a}_j T_j} \left( 2\hat{\mathcal{V}} + \hat{\xi} \right) \left[ - (\hat{\mathcal{V}}^{-1})_{ij} + \left( \frac{2}{4\hat{\mathcal{V}} - \hat{\xi}} \right) \hat{\mathcal{V}}^k \left( \hat{\mathcal{V}}^{-1} \right)^k_{ij} \right] \\
= 2e^K \hat{a}_i \hat{a}_j |A_i A_j| e^{-(\hat{a}_i \hat{\tau}_i + \hat{a}_j \hat{\tau}_j)} e^{i(\hat{\theta}_i - \hat{\theta}_j - \hat{\tau}_i b_i + \hat{\tau}_j b_j)} \left( 2\hat{\mathcal{V}} + \hat{\xi} \right) \left[ -\kappa_{ijk} \hat{\tau}_k + \left( \frac{4}{4\hat{\mathcal{V}} - \hat{\xi}} \right) \hat{\tau}_i \hat{\tau}_j \right]
\] (3.93)
and

\[
V_{np^2} = -e^{\kappa} \left( \hat{\kappa}_T^{-1} \right) \left( \hat{a}_i \hat{A}_i \hat{\bar{W}} e^{-\hat{a}_i \hat{T}_i} (\kappa_{S/T})^{\hat{j}} + (\kappa_{S/T})^{\hat{i}} \hat{a}_j \hat{\bar{A}}_j \hat{W} e^{-\hat{a}_j \hat{T}_j} \right) = e^{\kappa} \hat{a}_i \left( \frac{4 \hat{\xi}^2 + \hat{\xi} \hat{\bar{V}} + 4 \hat{V}^2}{\hat{\bar{V}} - \hat{\xi}} \left( \frac{\hat{\xi}}{2 \hat{\bar{V}}} \right) \right) \left( A_i \hat{\tau}_i \hat{W} e^{-\hat{a}_i \hat{T}_i} + \hat{\bar{A}}_i \hat{\bar{\tau}}_i \hat{W} e^{-\hat{a}_i \hat{T}_i} \right) = 2 e^{\kappa} \hat{a}_i |A_i| W |e^{i\theta_i} - e^{i\phi}|
\]

where \( A_i = |A_i| e^{i\theta_i} \) and \( W = |W| e^{i\phi} \).

The axionic part \( b_i \) of the complexified Kähler moduli can be decoupled and stabilized independently. The result, however, depends heavily on the topology of \( X \) as encoded in the triple intersection tensor \( \kappa_{ijk} \), and therefore its complexity also scales rapidly with increasing numbers of blowup moduli. In their appendix, the authors of [28] did an excellent job of classifying the resulting axion-stabilized scalar potential for various forms of \( \kappa_{ijk} \) in the large volume limit for up to two blowup moduli, and the reader is encouraged to refer there for more detail.

In this chapter, we consider only the “Swiss cheese” case in which the small 4-cycle blowup moduli can be explicitly separated from the large moduli which control the volume and fibration structure. In addition, for the sake of simplicity, we turn our attention only to cases with one small blowup modulus \( \hat{\tau}_s \), while the rest are sent large. In this case, it is a relatively simple matter to stabilize the single axion \( b_s \), as its contribution cancels in
Chapter 3. Scanning for Toric and Explicit Swiss Cheese Solutions

We find that

\[ V_{np1} = 2e^{K\hat{a}_s^2 |A_s|^2} e^{-2\hat{a}_s \hat{r}_s} \left( 2\hat{\cal V} + \hat{\xi} \right) \left( -\kappa_{sst}\hat{t}^i + \left( \frac{4}{4\hat{\cal V} - \hat{\xi}} \right) \hat{r}_s \hat{\tau}_s \right) \]  

(3.95)

\[ V_{np2} = -2e^{K\hat{a}_s |A_s|} W |e^{-\hat{a}_s \hat{r}_s} \hat{r}_s \frac{4\hat{\xi}^2 + \hat{\xi} \hat{\cal V} + 4\hat{\cal V}^2}{\left( \hat{\cal V} - \hat{\xi} \right) \left( \hat{\xi} + 2\hat{\cal V} \right)} \] 

\[ = -2e^{K\hat{a}_s |A_s|} e^{-\hat{a}_s \hat{r}_s} \left( |W_{GWW}| - |A_s| e^{-\hat{a}_s \hat{r}_s} \right) \hat{r}_s \frac{4\hat{\xi}^2 + \hat{\xi} \hat{\cal V} + 4\hat{\cal V}^2}{\left( \hat{\cal V} - \hat{\xi} \right) \left( 2\hat{\cal V} + \hat{\xi} \right)} \]  

(3.96)

Furthermore, [28] shows that in the case of a single small blowup modulus, there will only be a large volume AdS minimum when an additional so-called “homogeneity condition”

\[ \kappa_{sst}\hat{t}^i \sim -c\sqrt{\hat{r}_s}, \quad c > 0 \]  

(3.97)
is satisfied\(^{11}\). Then, in the large volume limit we have \(e^K = \frac{g_s e^{K_{cs}}}{\hat{V}^2}\), and to leading order in each term

\[
V_{np1} = g_s e^{K_{cs}} \frac{2c \hat{a}_s^2 |A_s|^2 e^{-2 \hat{a}_s \hat{\tau}_s} \sqrt{\hat{\tau}_s}}{\hat{V}}
\]

(3.98)

\[
V_{np2} = -g_s e^{K_{cs}} \frac{2\hat{a}_s |A_s W_{GVW}| e^{-\hat{a}_s \hat{\tau}_s} \hat{\tau}_s}{\hat{V}^2}
\]

(3.99)

\[
V_{\alpha'} = g_s e^{K_{cs}} \frac{3 |W_{GVW}|^2 \hat{\xi}}{8\hat{V}^3}
\]

(3.100)

and we obtain the full potential

\[
V(\hat{\tau}_s, \hat{V}) = V_{tree} + V_{np1} + V_{np2} + V_{\alpha'}
\]

\[
= g_s e^{K_{cs}} \left[ \frac{2c \hat{a}_s^2 |A_s|^2 e^{-2 \hat{a}_s \hat{\tau}_s} \sqrt{\hat{\tau}_s}}{\hat{V}} - \frac{2\hat{a}_s |A_s W_{GVW}| e^{-\hat{a}_s \hat{\tau}_s} \hat{\tau}_s}{\hat{V}^2} + \frac{3 |W_{GVW}|^2 \hat{\xi}}{8\hat{V}^3} \right]
\]

(3.101)

We can now stabilize both the blowup modulus and the volume by finding a local minimum of the potential where \(\frac{\partial V}{\partial \hat{\tau}_s} = \frac{\partial V}{\partial \hat{V}} = 0\). Following the work of [44] and taking \(\hat{a}_s \hat{\tau}_s \sim \ln \hat{V} \gg 1\) in order to cut off higher instanton corrections, obtain the simple result

\[
\langle \hat{\tau}_s \rangle \simeq \left( \frac{3c \xi}{16} \right)^{2/3}
\]

and

\[
\langle \hat{V} \rangle \simeq \frac{|W_{GVW}|}{2c \hat{a}_s |A_s|} \sqrt{\hat{\tau}_s} e^{\hat{a}_s \hat{\tau}_s}
\]

(3.102)

\(^{11}\)The minus sign in Equation (3.97) originates from the fact that inside the Kähler cone \(\oint J > 0\), the Kähler metric must be positive definite.
Finally, we convert back to the string frame using the transformations $\hat{V} = g_s^{-3/2}V$, $\hat{\tau}_i = g_s^{-1}\tau_i$, $\hat{a}_i = g_s a_i$, and $\hat{\xi} = g_s^{-3/2} \xi = \frac{\chi(X)\zeta(3)}{2g_s^{3/2}}$. We find that

$$\langle \tau_s \rangle \simeq \frac{1}{4} \left( \frac{3\chi(X)\zeta(3)}{4} \right)^{2/3}$$  \hspace{1cm} (3.103)

$$\langle V \rangle \simeq \frac{|W_{GVW}|}{2ca_s |A_s|} \sqrt{\tau_s e^{a_s \tau_s}}$$  \hspace{1cm} (3.104)

### 3.5 Example Moduli Stabilization

#### 3.5.1 Toric Swiss Cheese with $N_L = N_S = 2$

We choose an example from our database at [www.rossealtman.com](http://www.rossealtman.com) with $h^{1,1}(X) = 4$, $h^{2,1}(X) = 94$, $\chi(X) = -180$ and database indexes

<table>
<thead>
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<th>Geometry ID</th>
</tr>
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<tr>
<td>1145</td>
<td>1</td>
</tr>
</tbody>
</table>

---

$^{12}$For a small number $h^{1,1}(X)$ of Kähler moduli relative to complex structure moduli $h^{2,1}(X)$, the Euler number will have a negative value $\chi(X) = 2 \left( h^{1,1}(X) - h^{2,1}(X) \right) < 0$. Thus, the volume $V > 0$. 

The intersection numbers and Kähler cone matrix in the original bases are given by

\[ I_3 = J_1^2 J_2 - 3J_1 J_2^2 - 9J_2^3 + 2J_1^2 J_3 + 6J_1 J_2 J_3 + 6J_1 J_3^2 
+ 18J_2 J_3^2 + 18J_3^3 + J_1^2 J_4 - 3J_1 J_2^2 + 9J_4^3 \] (3.105)

\[ K = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \] (3.106)

This Calabi-Yau geometry was found, as a result of our toric Swiss cheese scan presented in Section 3.2, to have a Swiss cheese solution with \( N_L = N_S = 2 \), with original basis, \( A \)-basis, and \( B \)-basis given by

\[ J_1 = D_3, \quad J_2 = D_6, \quad J_3 = D_7, \quad J_4 = D_8 \] (3.107)

\[ J_1^A = D_5, \quad J_2^A = D_7, \quad J_3^A = D_1, \quad J_4^A = D_4 \] (3.108)

\[ J_1^B = D_1, \quad J_2^B = D_5, \quad J_3^B = D_4, \quad J_4^B = D_8 \] (3.109)
The toric divisors have independent Hodge numbers $h^\bullet = \{h^{0,0}, h^{0,1}, h^{0,2}, h^{1,1}\}$ given by

\begin{align*}
h^\bullet(D_1) &= h^\bullet(D_2) = h^\bullet(D_3) = \{1, 0, 2, 30\} \\
h^\bullet(D_4) &= h^\bullet(D_8) = \{1, 0, 0, 1\} \\
h^\bullet(D_5) &= \{1, 0, 1, 20\} \\
h^\bullet(D_6) &= \{1, 0, 0, 19\} \\
h^\bullet(D_7) &= \{1, 0, 10, 92\}
\end{align*}

Since the $B$-basis separates 4-cycles into large and small volumes given by $\tau_i^B = \frac{1}{2\pi} \int_{J^B_i} J \wedge J$, this tells us immediately that the two small volume divisors $J^B_3$ and $J^B_4$ are both dP$_0$ blowup cycles, while $J^B_2$ is a K3 fiber. Then, $J^B_3$ and $J^B_4$ are precisely the divisors desired to host the non-perturbative contributions to the superpotential (due to E3-instantons or gaugino condensation on a stack of D7 branes) required to stabilize some of the Kähler moduli using the LVS prescription.

Using Equation (3.44) and the relations between toric divisors, we find the rotation matrices

\begin{align*}
T^A &= \begin{pmatrix}
-3 & 1 & 1 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
-3 & 0 & 1 & -1
\end{pmatrix} \\
\text{and} \\
T^B &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
-3 & 1 & 1 & -1 \\
-3 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{align*}

(3.111)
We can use these to rotate the intersection tensor into the $AAA$ and $BBA$ configurations

$$\kappa^{AAA}_{ijk} = (T^A)_i^r (T^A)_j^s (T^A)_j^t \kappa_{rst}$$

$$\kappa^{BBA}_{ijk} = (T^B)_i^r (T^B)_j^s (T^A)_j^t \kappa_{rst} \quad (3.112)$$

The intersection numbers in this configuration are given by

$$I^{AAA}_3 = 36 J_2^A J_3^A J_1^A + 54 (J_2^A)^2 J_1^A + 6 (J_3^A)^2 J_1^A + 18 (J_2^A)^2 J_3^A + 3 (J_3^A)^2 J_4^A$$

$$+ 18 (J_2^A)^3 + 9 (J_4^A)^3 + 6 J_2^A (J_3^A)^2 - 9 J_3^A (J_4^A)^2 \quad (3.113)$$

$$I^{BBA}_3 = 2 J_1^A (J_1^B)^2 + 4 J_3^A J_2^B J_1^B - 6 J_4^A J_3^B J_1^B + 12 J_2^A J_2^B J_1^B + 2 J_3^A J_3^B J_1^B + 2 J_3^A J_4^B J_1^B$$

$$+ 2 J_2^A (J_1^B)^2 + J_4^A (J_1^B)^2 + 9 J_4^A (J_3^B)^2 - 3 J_3^A (J_3^B)^2 - 3 J_3^A (J_4^B)^2 \quad (3.114)$$

We can then write out the $\tau_i^B$ in terms of the $t^A$ using

$$\tau_i^B = \frac{1}{2!} t^B j t^B k \kappa^{BBA}_{ijk} \quad (3.115)$$
and we get

\[ \tau^B_1 = 3 (t^{A2})^2 + 2t^{A2}t^{A3} + 2t^{A1} (3t^{A2} + t^{A3}) + t^{A3}t^{A4} - \frac{3}{2} (t^{A4})^2 \]

\[ \tau^B_2 = (3t^{A2} + t^{A3})^2 \]

\[ \tau^B_3 = \frac{1}{2} (t^{A3} - 3t^{A4})^2 \]

\[ \tau^B_4 = \frac{1}{2} (t^{A3})^2 \]  \hspace{1cm} (3.116)

Thus, we see that the \( B \)-basis is at least partially explicit. We can invert the perfect squares to get

\[ 3t^{A2} + t^{A3} = \pm \sqrt{\tau^B_2} \]

\[ \frac{1}{\sqrt{2}} (t^{A3} - 3t^{A4}) = \pm \sqrt{\tau^B_3} \]  \hspace{1cm} (3.117)

\[ \frac{t^{A3}}{\sqrt{2}} = \pm \sqrt{\tau^B_4} \]

We can fix the signs on the right hand side by computing the Kähler cone in the \( A \)-basis

\[ K^A = K (T^A)^T = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \]  \hspace{1cm} (3.118)
with \((K^A)_i^j t^A_j > 0\) so that

\[
t^{A4} < 0, \quad t^{A3} > 0, \quad t^{A1} + t^{A4} > 0, \quad t^{A2} + t^{A4} > 0
\] (3.119)

This fixes the signs in Equation (3.117) to be \((+, +, +)\). Solving the rest of Equation (3.116), we get the rather messy result

\[
\begin{align*}
t^{A}_1 &= \frac{1}{6\sqrt{\tau^B_2}} \left(3\tau^B_1 - \tau^B_2 + \tau^B_3 + \tau^B_4\right) \\
t^{A}_2 &= \frac{1}{3} \left(\sqrt{\tau^B_2} - \sqrt{2\tau^B_4}\right) \\
t^{A}_3 &= \sqrt{2\tau^B_4} \\
t^{A}_4 &= \frac{\sqrt{2}}{3} \left(\sqrt{\tau^B_4} - \sqrt{\tau^B_3}\right)
\end{align*}
\] (3.120)

Substituting these into the expression for volume, we get

\[
\mathcal{V} = \frac{1}{3!} t^{A_i} t^{A_j} t^{A_k} \kappa^{AAA}_{ijkl}
\]

\[
= \frac{1}{18} \left[9\tau^B_1 \sqrt{\tau^B_2} + 3\sqrt{\tau^B_2} (\tau^B_3 + \tau^B_4) - (\tau^B_2)^{3/2} - 2\sqrt{2} \left((\tau^B_3)^{3/2} + (\tau^B_4)^{3/2}\right)\right]
\] (3.121)

Thus, we have determined that this is an explicit and partially diagonal Swiss cheese solution. In order to stabilize the Kähler moduli, we must write down the effective potential and find a stable AdS minimum, which can later be uplifted. To find the form of the potential, we need to know the inverse Kähler metric. This is, to leading order in
$V^{-1}$ (see Appendix (3.4)), given by

$$(\mathcal{K}^{-1})_{ij} = -2\mathcal{V}\kappa_{ijk}^{BRA}t^{Ak}$$

$$= 2\sqrt{2}\mathcal{V} \left( \begin{array}{ccc} \frac{1}{3}\left(\tau_3^B + \tau_4^B\right) & \frac{1}{3}\sqrt{2}\left(\tau_3^B + \tau_4^B\right) & -\sqrt{2}\sqrt{\tau_3^B} & -\sqrt{\tau_3^B} & -\sqrt{\tau_4^B} \\ -\sqrt{2}\sqrt{\tau_3^B} & 0 & 0 & 0 \\ -\sqrt{\tau_3^B} & 0 & 3\sqrt{\tau_3^B} & 0 \\ -\sqrt{\tau_4^B} & 0 & 0 & 3\sqrt{\tau_4^B} \end{array} \right)$$

From this form of the inverse Kähler metric, the effective potential takes the form

$$V(V, \tau_3^B, \tau_4^B) = \frac{a_3^2|A_3|^2}{\mathcal{V}^2}(\mathcal{K}^{-1})_{33} e^{-2a_3\tau_3^B} + \frac{a_4^2|A_4|^2}{\mathcal{V}^2}(\mathcal{K}^{-1})_{44} e^{-2a_4\tau_4^B}$$

$$+ \frac{2a_3a_4|A_3A_4|}{\mathcal{V}^2}(\mathcal{K}^{-1})_{34} e^{-(a_3\tau_3^B + a_4\tau_4^B)} - \frac{2a_3|A_3W_{GVW}|\tau_3^Be^{-a_3\tau_3^B}}{\mathcal{V}^2}$$

$$- \frac{2a_4|A_4W_{GVW}|\tau_4^Be^{-a_4\tau_4^B}}{\mathcal{V}^2} + \frac{3\xi|W_{GVW}|^2}{8\mathcal{V}^3}$$

$$= \frac{6\sqrt{2}a_3^2|A_3|^2\sqrt{\tau_3^B} e^{-2a_3\tau_3^B}}{\mathcal{V}} + \frac{6\sqrt{2}a_4^2|A_4|^2\sqrt{\tau_4^B} e^{-2a_4\tau_4^B}}{\mathcal{V}} - \frac{2a_3|A_3W_{GVW}|\tau_3^Be^{-a_3\tau_3^B}}{\mathcal{V}^2}$$

$$- \frac{2a_4|A_4W_{GVW}|\tau_4^Be^{-a_4\tau_4^B}}{\mathcal{V}^2} + \frac{3\xi|W_{GVW}|^2}{8\mathcal{V}^3}$$

We attempt to plot $\ln V(\tau_3^B, \tau_4^B)$ using an estimate of $\langle\mathcal{V}\rangle \sim 10^9$ and reasonable values for $a_3 = a_4 = 2\pi$, $A_1 = A_2 = 1$, $W_{GVW} = 1$, and $\xi = -\frac{\chi(X)\zeta(3)}{2}$. We find Where the logarithm gets cut off, the potential has gone negative. This gives us an AdS minimum.

We notice that the potential is symmetric in $\tau_3^B$ and $\tau_4^B$, so we can choose the direction where they are equal, i.e. $\tau_3^B : \tau_3^B = \tau_4^B$. Then, we can plot the potential in terms of
Figure 3.1: Plot of $\ln V(\tau_3^B, \tau_4^B)$ with $\langle V \rangle \sim 10^9$ and reasonable values for $a_3 = a_4 = 2\pi$, $A_1 = A_2 = 1$, $W_{GVW} = 1$, and $\xi = -\frac{\chi(X)\zeta(3)}{2}$.

$\tau_s^B$ and $V$. We find again, we see the AdS minimum. With $\tau_3^B$ and $\tau_4^B$ identified, the potential takes the form

$$
V(V, \tau_s) = \frac{12 \sqrt{2} a_s^2 |A_s|^2}{\sqrt{\tau_s^B} e^{-a_s \tau_s^B}} - \frac{4 a_s |A_s| W_{GVW} |\tau_s^B e^{-a_s \tau_s^B}|}{V^2} + \frac{3 \xi W_{GVW}^2}{8 V^3}
$$

(3.124)

In fact, this is exactly the form of Equation (3.101), the potential for $N_S = 1$ with $A_s \rightarrow 2A_s$ and $c = \frac{3}{\sqrt{2}}$. Using Equations (3.102) and (3.103) to find the minima, we
Figure 3.2: Plot of $\ln V(\mathcal{V}, \tau_s^B)$ with $\tau_s^B := \tau_3^B = \tau_4^B$, and reasonable values for $a_3 = a_4 = 2\pi$, $A_1 = A_2 = 1$, $W_{GVW} = 1$, and $\xi = -\chi(X)\zeta(3)$. We arrive at

$$\langle \tau_s^B \rangle \simeq \frac{1}{4} \left( \frac{3c_\chi(X)\zeta(3)}{4} \right)^{2/3} \simeq 12.3$$  \hfill (3.125)$$

$$\langle \mathcal{V} \rangle \simeq \frac{|W_{GVW}|}{2ca_s|A_s|} \sqrt{\tau_s^B} e^{a_s\tau_s^B} \simeq 2.12 \times 10^{32}$$  \hfill (3.126)$$

Thus, we do indeed get a large volume solution. And finally, we notice that using this minimum, we can find a flat direction for the other two large 4-cycle volumes $\tau_1^B$ and $\tau_2^B$.

$$\tau_1^B = \frac{(\tau_2^B)^{3/2} - 6\tau_s^B \sqrt{\tau_2^B} + 4\sqrt{2} (\tau_s^B)^{3/2} + 18V}{9\sqrt{\tau_2^B}}$$  \hfill (3.127)$$
This kind of feature will be particular interesting in the context of fiber modulus mediated inflation. Recall that $J_2^B$ is, in fact, a $K3$ fiber in this case. We leave this topic to future work.

### 3.6 Results

The results of our scans for toric Swiss cheese solutions (as described in Section 3.2) and for geometries $X$ whose volume can be written explicitly in terms of the dynamical 4-cycle volumes (as described in Section 3.3) are available in our database through the search engine at [www.rossealtman.com](http://www.rossealtman.com). For convenience, some useful statistics for these results are given in Table 3.2. It is clear from these results that there is a scarcity of Calabi-Yau threefolds $X$, whose volumes can be made explicit at higher values of $h^{1,1}(X)$. Recall from Section 3.3 that this does not mean that we cannot ever write the 6-cycle volume of these manifolds in terms of 4-cycle volumes, but merely that to do so might involve a linear combination of arbitrary square roots, for which is difficult to scan.

We also see from Table 3.2 that there are few toric Swiss cheese manifolds $X$ at higher $h^{1,1}(X)$ with many large 4-cycles. Unfortunately, when $N_S = h^{1,1}(X) - N_L > 2$, it becomes prohibitively difficult to stabilize the axion component of the complexified 4-cycle moduli [28]. However, we see that there are still 18 cases in $h^{1,1}(X) = 4$ for which the Kähler moduli can still be explicitly stabilized through the LVS. We demonstrate an example of this in Section 3.5.1. We note in Section 3.2 that these results can be expanded by running this scan iteratively, each time with an arbitrary rotation of the toric intersection tensor. In addition, there is currently a more exhaustive scan being run...
Chapter 3. *Scanning for Toric and Explicit Swiss Cheese Solutions*

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Explicit and Toric Swiss Cheese Geometries

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<td>32 (22)</td>
<td>86 (84)</td>
<td>173 (171)</td>
<td>603 (577)</td>
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<td>-</td>
<td>23 (23)</td>
<td>17 (17)</td>
<td>12 (10)</td>
<td>17 (13)</td>
</tr>
<tr>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>1 (1)</td>
<td>0 (0)</td>
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<td>-</td>
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<td>$N_L = 5$</td>
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<td>-</td>
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</tbody>
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Table 3.2: Statistics for explicitness and toric Swiss cheese scans over favorable Calabi-Yau threefold geometries. (Note: the scan for explicitness has not completed at the time of writing.)

using the general case described in Section 3.1.4.

Finally, we notice that a large number of the toric Swiss cheese solutions satisfy the homogeneity condition of Section 3.1.3. This condition ensures that the effective potential contains terms with the correct order in $V^{-1}$ for a minimum to exist. It is not always necessary to achieve such a minimum when $N_S > 1$, but it greatly simplifies the minimization procedure [28].
Chapter 4

Divisor Involutions and Free Actions on Toric Calabi-Yau Orientifolds

4.1 Constructing Calabi-Yau Orientifolds

4.1.1 Polytopes, Geometries, and Triangulations

It is a well-known result of Batyrev [101] that Calabi-Yau threefolds can generically be obtained by taking the anticanonical hypersurface in an ambient 4-dimensional Gorenstein toric Fano variety. In a previous work [99], we lay out the procedure for computationally extracting the topology of such a toric variety, as well as its restriction to the anticanonical hypersurface, from combinatorical information encoded in a 4-dimensional reflexive lattice polytope $\Delta$. In fact, a complete enumeration of all 4-dimensional reflexive polytopes exists due to Kreuzer and Skarke [1].
In the context of toric geometry, when the toric divisor classes on the Calabi-Yau hypersurface $X$ are all descended from ambient space $\mathcal{A}$, we say that it is a favorable geometry (as discussed earlier in Section 2.6.4). Consider the short exact sequence and its dual sequence

\begin{equation}
0 \to TX \to T\mathcal{A}|_X \to N_{X/\mathcal{A}} \to 0, \quad (4.1)
\end{equation}

\begin{equation}
0 \to N_{X/\mathcal{A}}^\ast \to T^\ast \mathcal{A}|_X \to T^\ast X \to 0.
\end{equation}

This induces the long exact sequence in sheaf cohomology

\begin{equation}
\cdots \to H^1(X,N_{X/\mathcal{A}}^\ast) \to H^1(X,T^\ast \mathcal{A}|_X) \to H^1(X,T^\ast X) \to H^2(X,N_{X/\mathcal{A}}^\ast) \to H^2(X,T^\ast \mathcal{A}|_X) \to H^2(X,T^\ast X) \to \cdots
\end{equation}

(4.2)

By Dolbeault’s theorem, $H^1(X,T^\ast X) \cong H^{1,1}(X) \cong \text{coker}(\alpha) \oplus \ker(\beta)$. It has two contributions, one from the descent of the Kähler moduli on $\mathcal{A}$ to Kähler moduli on $X$, while another from the kernel part. If the kernel part is zero, we say the geometry is favorable and $h^{1,1}(X) = \dim(H^{1,1}(X)) \cong \dim(\text{Pic}(\mathcal{A}))$. Studying these unfavorable cases is a problem we leave for future work.

If we restrict ourselves to smooth manifolds, we must at least partially desingularize the ambient toric variety $\mathcal{A}$ by blowing up enough of its singular points that $X$ is generically smooth, but without adding any discrepancies to its cohomology class. A method for doing such a maximal projective crepant partial (MPCP) desingularization has been
worked out by Batyrev [101], which involves the triangulation of the polar dual reflexive polytope $\Delta^*$. 

There are generally many MPCP desingularization configurations possible, each of which sets different topological restrictions on how $X$ can be deformed within $\mathcal{A}$. These deformations are parameterized by Kähler moduli, and the different configurations serve to divide up the space of moduli into discontinuous chambers. When passing from one chamber to another, some singular points are blown up, while others are blown down, so that the singularities cannot be consistently resolved at the boundary. This exchange is called a flop. Each distinct chamber defines a unique resolved ambient space $\tilde{\mathcal{A}}$, and there are, in general, exponentially more of these than the original ambient toric varieties $\mathcal{A}$.

It often happens, however, that the smooth Calabi-Yau hypersurface $X$ does not intersect any of the singular points involved in a flop between chambers of the moduli space. If, in addition, the topology of $X$ is invariant under the flop, then the singularity can be neglected for our purposes, and the chambers can be effectively glued and associated with a single unique, smooth hypersurface $X$.

It is clear from the above discussion that obtaining an accurate count of unique, smooth Calabi-Yau geometries a priori is highly non-trivial. However, the calculations have been performed for geometries with Hodge number $h^{1,1}(X) \leq 6$, and as a result, we have generated a large (and growing) database of $> 10^5$ unique Calabi-Yau threefold geometries.
One can see that the number of triangulations considered in this thesis is larger than in previous classifications [24]. This is because in our previous paper we only considered full desingularizations of the ambient toric variety, rather than the less restrictive MPCP desingularizations that we use here.

### 4.1.2 Topology of Toric Divisors

We will assume the ambient space $\mathcal{A}$ to be a resolved Gorenstein toric Fano variety with dimension $n = 4$ whose anticanonical divisor $X = -K_\mathcal{A}$ is a Calabi-Yau threefold hypersurface. Denote $x_i$ as the weighted homogeneous coordinates used to define $X$ inside the ambient space $\mathcal{A}$. Then the divisor $D_i = \{x_i = 0\}$ defines a 4-cycle on $X$ which is dual to a 2-cycle $\omega_i$.

In the context of toric geometry, a pair of “Non-trivial Identical Divisors” refers to a pair of divisors with distinct charges under the torus action, but which intersect with the Calabi-Yau hypersurface to identical topological surfaces with the same Hodge number.

In order to determine the Hodge number of an individual divisor on the Calabi-Yau threefold, we use the Koszul extension to the cohomCalg package [102, 103] with the HodgeDiamond module to calculate these quantities. However, in many situations relevant to our purposes, this module cannot give an explicit result.

For an irreducible divisor $D$, the complex conjugation $h^{p,q}(D) = h^{q,p}(D)$ and Hodge star $h^{p,q}(D) = h^{2-p,2-q}(D)$ dualities constrain the independent Hodge numbers of $D$ down to only $h^{1,0}(D), h^{2,0}(D)$, and $h^{1,1}(D)$. As a result, when we encounter difficulties with cohomCalg, we first calculate the Euler number $\chi(D) = \int_D c_2(D)$ of the divisor on
the hypersurface, and then determine $h^{1,0}(D)$ and $h^{2,0}(D)$ by calculating the trivial line bundle cohomology of the divisor $h^*(D, \mathcal{O}_D) = \{h^{0,0}(D), h^{1,0}(D), h^{2,0}(D)\}$ by chasing the Koszul sequence:

$$0 \rightarrow \mathcal{O}_A(-X - D) \rightarrow \mathcal{O}_A(-X) \oplus \mathcal{O}_A(-D) \rightarrow \mathcal{O}_A \rightarrow \mathcal{O}_{D/X} \rightarrow 0 \quad (4.3)$$

Then, using the expression

$$\chi(D) = \sum_{i=0}^{2} (-1)^i \dim \left( H^0_{\text{DR}}(D) \right) = \sum_{p+q=0}^{2} (-1)^{p+q} \dim \left( H^q(D, \Omega^p) \right) \quad (4.4)$$

we can fix $h^{1,1}(D)$ and get the full Hodge diamond for any divisor. The internal topology of these divisors plays an important role in string compactification and moduli stabilization. In our procedure for scanning divisor involutions, several divisors classifications are of particular phenomenological interest.

**Completely rigid divisor:** The Hodge numbers of these divisors are characterized by

$h^*(D) = \{h^{0,0}(D), h^{0,1}(D), h^{0,2}(D), h^{1,1}(D)\} = \{1, 0, 0, h^{1,1}(D)\}$ such that $h^{1,1}(D) \neq 0$.

This class of divisors is further subdivided into either the del Pezzo surfaces $\{\mathbb{P}^2 \cong dP_0, dP_n, \text{ with } n = 1, \ldots, 8\}$ (which may be shrinkable or non-shrinkable depending on the diagonalizability of its intersection tensor) with $n = h^{1,1}(D) - 1$, and the “non-shrinkable rigid divisors” (NRD) with $h^{1,1}(D) > 9$. 
The del Pezzo divisors $D = dP_n$ are obtained by blowing up (a set of) generic points in a $\mathbb{P}^2$, and play a crucial role in string compactification.

- Because they always have arithmetic genus $\chi(D, O(D)) = 1$, the necessary condition for branes wrapping them to contribute non-trivially to the superpotential [104] is automatically satisfied.

- GUT models can be realized as compactifications of F-theory on Calabi-Yau four-folds where the gauge theory degrees of freedom of the GUT localize on the world-volume of a non-perturbative D7 brane. In order to decouple from gravity, these space-time filling D7 branes must wrap a shrinkable del Pezzo surface [48,105].

- del Pezzo surfaces play an important role in realizing the Swiss cheese structure in the LARGE volume scenario (LVS) of moduli stabilization and cosmological inflation.

"Wilson" divisor: The Hodge numbers of these divisors are characterized by $h^\bullet(D) = \{h^{0,0}(D), h^{0,1}(D), h^{0,2}(D), h^{1,1}(D)\} = \{1, h^{1,0}, 0, h^{1,1}\}$ with $h^{1,0}(D), h^{1,1}(D) \neq 0$. We will also further specify the "Exact-Wilson" divisor as $h^\bullet(D) = \{1, 1, 0, h^{1,1}\}$ with $h^{1,1}(D) \neq 0$.

The wrapping of a Euclidean D3-brane instanton, on an "Exact-Wilson" divisor with $h^{1,0}_+(D) = 1$ is sufficient to generate a so-called poly-instanton contribution to the superpotential\(^1\). An important feature of a poly-instanton is that its zero modes cannot be

---

\(^1\)A poly-instanton refers to the correction of a Euclidean D-brane instanton action by other D-brane instantons [106–111].
lifted by the background fluxes [112], and is therefore more robust with respect to quantum fluctuations. Such new non-perturbative effects result in a new variation of Kähler moduli inflation, called poly-instanton inflation, which treats the “Exact-Wilson” divisor volume (or its complexified analogue together with an axion part) as the inflaton [110,111].

**Deformation divisor:** These divisors are characterized simply by $h^{2,0}(D) \neq 0$. It has been proposed that turning on background magnetic fluxes, one can lift these (extra) deformation zero modes [113] while leaving the poly-instanton zero modes encoded in $h_{+}^{1,0}(D)$ unaffected. Since these fluxes can rigidify some deformation divisors, such circumstances facilitate the moduli stabilization process by introducing more terms for superpotential contributions. We mainly focus on three kinds of deformation divisors.

- A K3 divisor is a deformation divisor with Hodge numbers $h^\bullet(D) = \{1, 0, 1, 20\}$. A K3-fibration is useful for obtaining an anisotropic shape of the Calabi-Yau compactification. This leads to some LVS models with what is effectively two large extra dimensions of micron size and a fundamental gravity scale around $\sim 1$ TeV [27,114]. The property of spaces which contain both $K3$ and Wilson surface are also studied in [24,109,115].

- Another deformation divisor that appears often in our scan is similar to a K3 divisor, but with an extra $h^{1,1}$ deformation degree of freedom, i.e. $h^\bullet(D) = \{1, 0, 1, 21\}$. We will refer to this as a type-1 special deformation divisor, $SD1$. 
The last deformation divisor that appears in our scan has Hodge numbers $h^*(D) = \{1, 0, 2, 30\}$, which we refer to as $SD2$.

$SD1$ and $SD2$ appear to share similar properties, particularly in their volume forms. In our scan, we will first identify the Hodge numbers of all the toric divisors on the Calabi-Yau hypersurface $X$, which descend from the ambient space $\mathcal{A}$. This calculation will be performed for each Calabi-Yau geometry in our database www.rossealtman.com.

4.1.3 Holomorphic Divisor Exchange Involutions

The map $\sigma : x_i \leftrightarrow x_j$ exchanging two homogeneous coordinates of the ambient toric variety $\mathcal{A}$ can be pulled back to a holomorphic involution on the corresponding toric divisor cohomology classes $\sigma^* : D_i \leftrightarrow D_j$, which, on favorable manifolds, restricts in a straightforward way to the Calabi-Yau hypersurface $X$. We then define the even and odd parity eigendivisor classes $D_{\pm} \equiv D_i \pm D_j \in H^{1,1}_{\pm}(X/\sigma^*)$. In general, a given geometry will allow multiple disjoint involutions $\sigma_1, \sigma_2, \ldots, \sigma_n$. In this case, the full involution is given by $\sigma \equiv \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n$.

We consider two divisors $D_i, D_j$ to be identical if $h^*(D_i) \cong h^*(D_j)$, however they may still be cohomologically distinct on $X$. This can readily be determined in favorable geometries by inspecting their toric $\mathbb{C}^*$ weights. If an identical pair of divisors also share the same weights, then an involution $\sigma$ exchanging them will act trivially on the hypersurface polynomial and will leave $h^{-1,1}_{-}(X/\sigma^*) = 0$.

From a phenomenological point of view, there are a variety of reasons to work on a Calabi-Yau manifold with non-trivial odd parity $h^{-1,1}_{-}(X/\sigma^*) > 0$. In the Ansatz of KKLT and the
LARGE volume scenario (LVS), the non-perturbative corrections to the superpotential which allow for Kähler moduli stabilization arise due to E3-brane instantons (or gaugino condensation on stacks of D7 branes). However, if the divisor $D_{np}$ hosting these non-perturbative effects intersects with the D7 brane hosting the visible sector on $D_{vis}$, then the resulting charged fermion zero modes must be soaked up by including charged matter fields in the instanton superpotential [72]. These zero modes are counted by [116]

$$H^i(C; \mathcal{L} \otimes K_C^{1/2}), \quad i = 0, 1 \text{ with } C = D_{np} \cap D_{vis}$$

(4.5)

where $\mathcal{L}$ is a line bundle carried by the D7 brane. This instanton is not necessarily gauge invariant, and so cannot be used to stabilize all the Kähler moduli [27]. However, by choosing $D_{vis}$ to be a rigid divisor with odd parity under the orientifold involution, the vanishing of the Fayet-Iliopoulos parameter due to D-flatness implies that $D_{vis}$ shrinks to zero size, and therefore avoids any problematic intersections [72,78]. Another possibility arises in supporting the R-R and NS-NS 2-form potentials on the odd divisor 2-forms $\omega_a$ as $C_2 = c^a \omega_a, B_2 = b^a \omega_a$, where $a = 1, \cdots, h^{1,1}_-(X/\sigma^*)$. Then, these introduce extra degrees of freedom with which to make the instanton gauge invariant [74].

As a consequence, we consider only holomorphic divisor involutions between “Non-trivial Identical Divisor pairs” (NIDs) in this work. In particular, since we only consider favorable Calabi-Yau threefolds, we will not include coordinate reflections such as $\sigma : x_i \leftrightarrow -x_i$, as the corresponding divisor involution $\sigma^* : D_i \leftrightarrow D_i$ is manifestly trivial and will not contribute to $h^{1,1}_-(X/\sigma^*)$ [24].
After identifying the “Non-trivial Identical Divisors”, we will check which involutions between these divisors are consistent with the topology of Calabi-Yau manifold. Since the hypersurface is embedded in a desingularized ambient variety \( \mathcal{A} \), we will require our orientifold involution to be an automorphism of \( \mathcal{A} \), leaving invariant the exceptional divisors from resolved singularities. The information describing the desingularization is encoded in the Stanley-Reisner ideal \( I_{SR}(\mathcal{A}) \). As a consequence, the involution should be a symmetry of \( I_{SR}(\mathcal{A}) \).

We then further distinguish the “proper” involutions which are symmetries of the linear ideal \( I_{lin}(\mathcal{A}) \) encoding toric divisor redundancy. This ensures that the defining polynomial of the Calabi-Yau hypersurface generically remains homogeneous under the coordinate exchange without tuning any coefficients to zero. These involutions are then also symmetries of the graded Chow ring

\[
A^\bullet(\mathcal{A}) \cong \frac{\mathbb{Z}(D_1, \cdots, D_k)}{I_{lin}(\mathcal{A}) + I_{SR}(\mathcal{A})} \tag{4.6}
\]

Due to the favorability condition on the Calabi-Yau hypersurface we have

\[
A^1(\mathcal{A}) \cong H^{1,1}(\mathcal{A}) \cong \text{Pic}(\mathcal{A}) \cong \text{Pic}(X) \cong H^{1,1}(X) \cong A^1(X) \tag{4.7}
\]
Then the toric triple intersection tensor defined in the Chow ring $A^4(X)$ of Calabi-Yau threefolds by

$$d_{ijk} = \int_X D_i \wedge D_j \wedge D_k \equiv D_i \cdot D_j \cdot D_k \cdot X \quad \text{and} \quad X = -K_X = \sum_{i=1}^k D_i$$

(4.8)

will also remain invariant. If the involution involves non-rigid, deformation divisors, this condition will ensure that they can be exchanged in a consistent way.

Let us consider a simple example with $h^{1,1}(X) = 3$. The triple intersection tensor can be written in the basis of divisor class $\{J_1, J_2, J_3\} \in H^{1,1}(X; \mathbb{Z})$ as

$$\kappa_{ijk} = \int_X J_i \wedge J_j \wedge J_k \equiv J_i \cdot J_j \cdot J_k \cdot X$$

(4.9)

Suppose we have a proper involution $\sigma^*: J_2 \leftrightarrow J_3$. Then, under the basis change

$$\{J_1, J_2, J_3\} \mapsto \{J_0, J_+, J_-\} \quad \text{where} \quad J_\pm = J_2 \pm J_3$$

(4.10)

the intersection numbers with an odd number of minus indices ($\kappa_{00-}, \kappa_{++-}, \kappa_{0+-}, \kappa_{--}$) should vanish identically. In this basis, we can write down the Kähler form $J = t^0 J_0 + t^+ J_+ + t^- J_-$. For a consistent orientifold, we must have both

$$\sigma^* J = J \quad \text{and} \quad \sigma^* \Omega_3 = -\Omega_3$$

(4.11)

where $\Omega_3$ is the unique holomorphic $(3,0)$-form on $X$. The constraint on the Kähler
form given by Equation (4.11) can be used to show that the Kähler cone condition 
\( f_c J > 0 \) will be consistently satisfied on the orientifold.

The resulting orientifold projection that breaks \( \mathcal{N} = 2 \) SUGRA down to \( \mathcal{N} = 1 \) is given by \( \Omega \sigma (-1)^{F_L} \), where \( \Omega \) is the world-sheet parity transformation and \( F_L \) is the left-moving space-time fermion number.

The procedure, then, is to first scan the database of Calabi-Yau threefolds extracted from the Kreuzer-Skarke list up to \( h^{1,1}(X) = 6 \) and pick out the desingularized configurations (i.e., triangulations of reflexive 4D polytopes) that support an exchange of NIDs. We then identify all the “proper” involutions that allow for a consistent orientifold geometries. Our results are available via the database search engine at www.rossealtman.com and the statistics of this scan is summarized in Tables 4.1-4.8. In this table, we further classify the different involutions into exchanges that include completely rigid surfaces, Wilson surfaces, and some deformation surfaces. We also count the triangulations that exchange several different types of involutions simultaneously. These methods can and will be systemically generalized to higher \( h^{1,1}(X) \geq 7 \), and database will be updated accordingly.

4.1.4 Classifications of the Fixed Orientifold Planes

The next question to ask whether there exist any point-wise fixed loci for a given involution on the Calabi-Yau threefold. If there is a fixed codimension-\( n \) subvariety defined by the simultaneous vanishing of \( n \) polynomials, then \( X \) will define an orientifold with an \( Om \) plane, where \( m = 9 - 2n \). For a Calabi-Yau threefold, we must have \( 0 \leq n \leq 3 \),
and so we can only have O9, O7, O5, or O3 planes. If we have none of these, then the
involution is a free $\mathbb{Z}_2$ action on $X$.

We now present an algorithm for identifying and classifying fixed-point sets under an
orientifold of the type described in the previous section. We first define the MPCP-
desingularized ambient 4D toric variety

$$A = \frac{\mathbb{C}^k \setminus Z}{\mathbb{C}^*} \times G$$

(4.12)

where $Z$ is the locus of points in $\mathbb{C}^k$ ruled out by the Stanley-Reisner ideal $I_{SR}(A)$, and
$G$ is the stringy fundamental group (i.e. trivial in most cases we’re interested in). The
geometry on this toric variety can be described by the projective coordinates $\{x_1, ..., x_k\}$
and their toric $\mathbb{C}^*$ equivalence classes

$$(x_1, ..., x_k) \sim (\lambda^{w_{11}} x_1, ..., \lambda^{w_{kk}} x_k)$$

(4.13)

which define a projective weight matrix $W$. In addition, the image of the symplectic
moment map $\mu : A \to \mathbb{R}^4$ defines a convex Newton polytope by Atiyah and Guillemin-
Sternberg. In cases gleaned from the Kreuzer-Skarke list, as in our study, these are
reflexive lattice polytopes $\Delta = \mu(A)$ with polar dual $\Delta^*$.

4.1.4.1 Invariant Calabi-Yau Hypersurface Polynomial

Consider the smooth Calabi-Yau anticanonical hypersurface $X = -K_A$ defined by the
vanishing of a homogeneous polynomial $\{P = 0\}$. The polynomial $P$ can be expressed in
terms of the known vertices \( m \in M, n \in N \) of the Newton and dual polytopes, respectively

\[
P = \sum_{m \in \Delta} a_m M_m = 0, \quad \text{where} \quad M_m = \prod_{i=1}^{k} x_i^{(m,n_i)+1}
\]  

(4.14)

An involution of the type that we are considering will consist of a set of \( n \) fully disjoint coordinate exchanges, \( \sigma_s : x_i \leftrightarrow x_j, \quad s = 1, \ldots, n \) and \( \sigma = \sigma_1 \circ \cdots \circ \sigma_n \). In order for the Calabi-Yau hypersurface to be invariant under \( \sigma \), we must restrict to the subset of moduli space in which the defining polynomial is invariant. To check this, we define the set of monomials \( \mathcal{M} = \{ M_m | m \in \Delta \} \). Then for \( M_m, M_n \in \mathcal{M} \), we identify three cases

1. \( \sigma(M_m) = M_m \) \( \Rightarrow \) \( a_m \) is generic

2. \( \sigma(M_m) = M_n, \quad n \neq m \) \( \Rightarrow \) \( a_m = a_n \)

3. \( \sigma(M_m) \notin \mathcal{M} \) \( \Rightarrow \) \( a_m = 0 \)

Clearly this requires some tuning of the complex structure moduli space, but end result is that \( P \mapsto P_{\text{symm}} \) such that \( \sigma(P_{\text{symm}}) = P_{\text{symm}} \) in addition to \( \sigma^* J = J \). It is important to note that this tuning may introduce singularities into the CY hypersurface. The MPCP desingularization of the ambient space \( \mathcal{A} \) only guarantees resolution of codimension > 2 singularities, which can be generically avoided by the Calabi-Yau polynomial by Bertini’s theorem. However, now that we have forced some coefficients to zero, it is not necessarily the case that the hypersurface can avoid these singularities.
We can now search for the set of points fixed under $\sigma$. We first locate the fixed-point set in the ambient space, and only later restrict this set to the Calabi-Yau hypersurface$^2$.

4.1.4.2 Minimal generating set of homogeneous polynomials

We look for the minimal generating set $\mathcal{G}$ of homogeneous polynomials $y(x_1, ..., x_k)$ that are (anti-)invariant under $\sigma$. Each of these generators then has definite parity $\sigma(y) = \pm y$ for each $y \in \mathcal{G}$.

Clearly, the monomials in $\mathcal{M}$ satisfying case (1) above are included in $\mathcal{G}$. In addition, we define the subset $\mathcal{G}_0 \subset \{x_1, ..., x_k\}$ of the projective coordinates that are left unexchanged by $\sigma$. Then, for ease of notation, we can define the orthogonal decomposition

$$\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_+ \cup \mathcal{G}_- \quad (4.15)$$

In order to determine these generating polynomials, we note that because $\sigma^2 = id$, any (anti-)invariant polynomial can be written in terms of monomials $N_i$ as

$$Q_{0,+} = \sum_i c_i N_i + \sum_j d_j (N_j + \sigma(N_j))$$

$$Q_- = \sum_j d_j (N_j - \sigma(N_j)), \quad \text{where } \sigma(N_i) = N_i \text{ and } \sigma(N_j) \neq N_j \quad (4.16)$$

Thus, $\mathcal{G}_0 \cup \mathcal{G}_+$ will only contain monomial and binomial generators, and $\mathcal{G}_-$ only binomial generators.

---

$^2$This restriction can be done in a straightforward manner since we only consider favorable manifolds.
The unexchanged coordinates in $G_0$ are known a priori from our choice of involution, so we restrict our attention to finding the non-trivial even and odd parity generators in $G_+$ and $G_-$. To do this, we must consider not only $\sigma$, but the $2^n - 1$ non-trivial “sub-involutions” $\rho \subseteq \sigma$ given by the non-empty subsets of $\{\sigma_1, ..., \sigma_n\}$ of size $1 \leq m \leq n$.

For $m = 1$, $\rho \equiv \sigma_s : x_{i_s} \leftrightarrow x_{j_s}$, it is clear that $G_+ = \{x_{i_s} x_{j_s}\}$ and $G_- = \emptyset$. Since we are considering only exchanges of NIDs, we do not need to consider binomials (i.e. $x_{i_s} \pm x_{j_s}$) as these are not homogeneous.

For $m > 1$ sub-involutions, any invariant monomials are just products of the generators from $m = 1$ sub-involutions. Therefore, we will now turn our attention to binomial generators. In particular, we look for those of the form

$$y_\pm(a) = x_{1}^{a_1} x_{2}^{a_2} \cdots x_{m}^{a_m} \pm x_{1}^{a_1} x_{2}^{a_2} \cdots x_{m}^{a_m}$$

(4.17)

where $a = (a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m$ are such as to ensure that the binomial is homogeneous.

The condition for homogeneity, in terms of the columns $w_{i_s}$ and $w_{j_s}$ of the weight matrix $W$ is given by

$$a_1 w_{i_1} + a_2 w_{i_2} + \cdots + a_m w_{i_m} = a_1 w_{j_1} + a_2 w_{j_2} + \cdots + a_m w_{j_m}$$

(4.18)

which we rewrite as

$$a_1(w_{i_1} - w_{j_1}) + a_2(w_{i_2} - w_{j_2}) + \cdots + a_m(w_{i_m} - w_{j_m}) = 0$$

(4.19)
Let $D$ be the matrix whose columns are the difference vectors $d_s = w_i - w_j$. The above equality implies that $a$ lies in $\ker D \cap \mathbb{Z}^m$. Via the identity

$$y_\pm(a + b) = y_\pm(a)y_\pm(b) \mp y_\pm(|a_1 - b_1|, \ldots, |a_m - b_m|) \prod_{s=1}^m (x_i^s x_j^s)^\min(a_s, b_s) \tag{4.20}$$

one can see that the generators of $\ker D \cap \mathbb{Z}^m$ as a $\mathbb{Z}$-module give the exponents of binomial generators in $\mathcal{G}_\pm$. If a kernel generator contains negative entries, we simply multiply through by the necessary monomial factor to clear any denominators and obtain a proper binomial. We repeat this procedure for each sub-involution $\rho \subseteq \sigma$. The sets $\mathcal{G}_\pm$ are given by the union of the generators found for each of the sub-involutions, and each will contain the same number of elements.

### 4.1.4.3 Naive Fixed Point Loci

We first perform a version of the Segre embedding, transforming the projective coordinates into the (anti-)invariant generators $\{x_1, \ldots, x_k\} \mapsto \{y_1, \ldots, y_{k'}\} \equiv \mathcal{G}$. The cardinality $k' = |\mathcal{G}|$ may be less than, equal to, or greater than $k$. We construct the weight matrix $\tilde{W}$ for these generators by taking appropriate linear combinations of the original weight matrix columns $w_i$.

In order for a codimension-1 subvariety on $D \subset X$ to be point-wise fixed under the involution $\sigma^*$, the corresponding coordinate exchange must force its defining polynomial to vanish, i.e. $\sigma : y \mapsto -y$, so that $D = \{y = 0\}$ is fixed. This implies that the defining polynomial $y$ of every point-wise fixed codimension-1 subvariety must be generated by odd-parity generators in $\mathcal{G}_-$. In the case where $\mathcal{G}_-$ is empty, there is no non-trivial fixed
subvariety. Then, naively, the only locus in the ambient space (if it exists) fixed under $\sigma$ is the set

$$\{x_{i_1} = x_{i_2} = \cdots = x_{i_n} = x_{j_1} = x_{j_2} = \cdots = x_{j_n} = 0\}$$

which is trivially unaffected by $\sigma$. However, when $\mathcal{G}_-$ is non-empty, this trivial locus is immediately a subspace of any generator $y \in \mathcal{G}_-$, and is therefore redundant.

To determine the point-wise fixed loci for codimension $> 1$, we must check whether the involution forces a subset of generators $\mathcal{F} \subseteq \mathcal{G}$ to vanish simultaneously. The number of checks required can be reduced by noting that if a set is not point-wise fixed, then no set containing it will be either. For this reason, we consider only the subsets $\mathcal{F}$ where $\mathcal{F} \cap \mathcal{G}_- \neq \emptyset$.

It would seem, then, that the only point-wise fixed set would be $\{y_i = \cdots = y_r = 0\}$, $y_1, \ldots, y_r \in \mathcal{G}_-$ and all other fixed sets would be its proper subsets. However, it is important to note that the torus $\mathbb{C}^*$ actions provide $\text{rank}(\tilde{\mathcal{W}})$ additional degrees of freedom for the generators to avoid being forced to zero. That is, there may be a toric equivalence class that neutralizes the odd parity of some set of generators, while adding odd parity to another set. In each subset of generators $\mathcal{F}$, we check for this by solving the system of equations

$$\lambda_1^{\tilde{W}_{i_1}} \lambda_2^{\tilde{W}_{i_2}} \cdots \lambda_m^{\tilde{W}_{i_m}} = \sigma(y_i)/y_i, \quad i = 1, \ldots, k'$$  \hspace{1cm} (4.21)
By the construction of the generator $y_i$, the right-hand side is equal to $\pm 1$. The set is point-wise fixed if this equation is solvable in the $\lambda_i$.

Since the right-hand side of each of these equations has magnitude 1, we only need to look for solutions with $\lambda_i$ on the complex unit circle. As any element of the unit circle can be written $\lambda_i = e^{i\pi u_i}$ with $0 \leq u_i < 2$, each of the equations above can be rewritten as a linear congruence in the $x$ as

$$
\tilde{W}_1 u_1 + \cdots + \tilde{W}_m u_m \equiv \begin{cases} 
0 \pmod{2}, & y_i \in G_0 \cup G_+ \\
1 \pmod{2}, & y_i \in G_- 
\end{cases} \quad (4.22)
$$

We can rewrite this as a system of linear Diophantine equations

$$
\tilde{W}_1 u_1 + \cdots + \tilde{W}_m u_m - 2q_i \equiv \begin{cases} 
0, & y_i \in G_0 \cup G_+ \\
1, & y_i \in G_- 
\end{cases} \quad (4.23)
$$

for $q_i \in \mathbb{Z}$. The fact that $0 \leq u_i < 2$ means that $0 \leq q_i < \tilde{W}_1 + \cdots + \tilde{W}_m$, and thus there are only a finite number of vectors $q := (q_1, ..., q_{k'}) \in \mathbb{Z}^{k'}$ for which the solvability of this linear system needs to be checked. This can easily be done using standard matrix techniques. If the system has a solution $u := (u_1, ..., u_m) \in \mathbb{Q}^m$ for any of the allowed $q$ vectors, then the set is point-wise fixed under $\sigma$.

One may notice that the generators we have defined above are not all entirely independent.

For example, consider an involution $\sigma : x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_4$ such that we get generators $y_1 = x_1x_2, y_2 = x_3x_4, y_3 = x_1x_3 + x_2x_4, y_4 = x_1x_3 - x_2x_4$. Then the generators are related
via the consistency condition $y_3^2 = y_1^2 + 4y_1y_2$. This manifests in a constraint on the vanishing of the generators. For example, if $y_1 = y_3 = 0$, then $y_4$ is necessarily equal to zero. When scanning the subsets $\mathcal{F}$ of vanishing generators, we implement a consistency check to filter out any spurious sets that violate these restrictions.

Before moving forward, it is important to note that it is indeed sufficient to consider only the zero loci of the minimal generators, and not more general combinations. If we have $y_1, y_2 \in \mathcal{G}$ with zero loci $D_1 = \{y_1 = 0\}$ and $D_2 = \{y_2 = 0\}$, the vanishing of the product is simply the union of the two subvarieties $\{y_1y_2 = 0\} = D_1 \cup D_2$. Thus, $D_1 \cup D_2$ is only fixed when $D_1$ and $D_2$ are already fixed independently. We can also consider the sum or difference $y_1 + y_2$, such that both $y_1$ and $y_2$ have the same $\mathbb{C}^*$ weights and the sum has definite parity. If either one vanishes, then both do, and the resulting subvariety lies on $D_1 \cap D_2$, so that clearly both must be fixed independently for their intersection to be fixed. Then, finally, we assume that neither generator vanishes. Because $\sigma$ acts as a homomorphism on the generators, the definite parity condition ensures that $y_1$, $y_2$, and $y_1 + y_2$ must all have the same parity via

$$\sigma(y_1 + y_2) = \sigma(y_1) + \sigma(y_2) = \begin{cases} y_1 + y_2, & y_1, y_2 \in \mathcal{G}_0 \cup \mathcal{G}_+ \\ -(y_1 + y_2), & y_1, y_2 \in \mathcal{G}_- \end{cases}$$

We saw above that determining whether or not a set is point-wise fixed under $\sigma$ is dependent only on its weights and parity, so $y_1 + y_2$ defines a fixed locus only if $D_1$ and $D_2$ are fixed as well. The same argument applies to differences $y_1 - y_2$. 
4.1.4.4 Stanley-Reisner Ideal and Calabi-Yau Transversality

All of our calculations thus far have depended only on the toric weights, but Equation (4.12) tells us that the ambient space $\mathcal{A}$ has a set of points $Z$ ruled out by the Stanley-Reisnner ideal $I_{SR}$. Now that we have found all possible point-wise fixed loci for each subset of generators $\mathcal{F}$, we must check that none of these lie in $Z$. $I_{SR}$ is defined as the square-free ideal of non-intersections of toric divisors. For example, if $D_1 D_2 \in I_{SR}$, then $d_{ij} = D_1 \cdot D_2 \cdot D_i \cdot D_j = 0$ for all $i, j = 1, \ldots, k$. However, the concept works equally well with the projective coordinates. One can say that if $x_1 x_2$ is in the SR ideal, then the simultaneously vanishing set $\{x_1 = x_2 = 0\}$ will not exist in $\mathcal{A}$. As our fixed sets will often consist of the zero loci of binomials, compatibility with the SR ideal cannot be generally read off in this way. We check compatibility with the Stanley-Reisner ideal by expressing the ideal and the generators as Boolean expressions. For each generator $y \in \mathcal{G}$, we define a Boolean variable $Y$ whose value is $\text{True}$ when $y = 0$.

To make this more clear, we give a few examples. The monomial $y_1 y_2$, which is zero when $y_1 = 0$ or $y_2 = 0$, is assigned the Boolean expression $Y_1 \lor Y_2$. We build the expressions for binomials from those of their constituent parts. For example, consider $y_1 y_2 - y_3 y_4$. This expression is zero if both $y_1 y_2$ and $y_3 y_4$ are zero, or if neither of them are (which means that none of the variables is zero). We thus get the Boolean expression

$$y_1 y_2 - y_3 y_4 \rightarrow \left[ (Y_1 \lor Y_2) \land (Y_3 \lor Y_4) \right] \lor \left[ (\neg Y_1) \land (\neg Y_2) \land (\neg Y_3) \land (\neg Y_4) \right]$$
By this iterative method, we can create the Boolean expression for any polynomial. Although tedious, this process is easily automated. Using this method, we create the expressions for the SR ideal and the polynomial generators. If these Boolean expressions together form a contradiction, the fixed-point set does not intersect the Calabi-Yau and is thus discarded. Unfortunately, this Boolean approach will not catch every case. In the above example, if \( y_1 y_2 = y_3 y_4 \neq 0 \), this method is insufficient and we need something more robust. However, this first pass will rule out many spurious cases.

At least one coordinate in each element of the Stanley-Reisner ideal must be non-zero at each point in \( \mathcal{A} \). By finding the minimal generating sets of coordinates that, when set non-zero, satisfy this condition, we can split \( \mathcal{A} \) up into disjoint regions \( U_i \). In each of these sectors, we implement this non-zero condition by setting the coordinates equal to 1. By working in these sectors, we ensure that any remaining fixed sets that contradict the SR ideal will be filtered out.

We now check whether each set can be restricted to the Calabi-Yau hypersurface. For a given fixed set, we compute in each sector \( U_i \) the dimension of the ideal generated by the Calabi-Yau polynomial \( P_{\text{symm}} \) and the fixed set generators \( \{y_1, \ldots, y_r\} = \mathcal{F} \)

\[
\mathcal{I}_{\text{ir}}^{\text{fixed}} = \langle U_i, P_{\text{symm}}, y_1, \ldots, y_r \rangle
\]

If the dimension \( \dim \mathcal{I}_{\text{ir}}^{\text{fixed}} < 0 \) for all \( U_i \), then there is no consistent Groebner basis, and we conclude that this set does not intersect the CY hypersurface.

For each set that is not discarded, we repeat this calculation for the ideal with one fixed
set generator \( \dim I_{i1}^{\text{fixed}} \), and then two \( \dim I_{i2}^{\text{fixed}} \), etc. until \( \dim I_{i\ell}^{\text{fixed}} = \dim I_{ir}^{\text{fixed}} \) when adding more generators to the ideal no longer changes the dimension for any region \( U_i \).

Then, the intersection \( \{y_1 = \cdots = y_\ell = 0\} \) of these generators gives the final point-wise fixed locus, with redundancies eliminated. Since each generator has codimension 1 in \( X \), the length \( \ell \) of this set of generators is the codimension in \( X \) of the fixed point locus. An O3 plane corresponds to a codimension-3 point-wise fixed subvariey, an O5 plane has codimension-2, an O7 plane has codimension-1, and an O9 plane has codimension-0 (which means the entire CY is fixed under \( \sigma \)).

4.1.4.5 Smoothness

Finally, we must check whether the invariant Calabi-Yau hypersurface defined by \( P_{\text{symm}} \) is smooth. We do this by checking if there is any solution to the condition \( P_{\text{symm}} = dP_{\text{symm}} = 0 \) that is not ruled out by the Stanley-Reisner ideal. In practice, this is done by setting up the ideals

\[
I_i^{\text{smooth}} = \langle U_i, P_{\text{symm}}, \frac{\partial P_{\text{symm}}}{\partial x_1}, \ldots, \frac{\partial P_{\text{symm}}}{\partial x_k} \rangle
\]

for each region \( U_i \) allowed by the Stanley-Reisner ideal, and computing the dimension. If \( \dim I_i^{\text{smooth}} < 0 \) for all \( U_i \), then the invariant Calabi-Yau hypersurface is smooth.

If no \( \text{O}_m \) planes exist and the invariant Calabi-Yau hypersurface is smooth, then the involution defines a \( \mathbb{Z}_2 \) free action on \( X \) \[117\]. The classification of these orientifold involutions is summarized in Tables 4.9-4.16. In Section 4.2, we demonstrate this algorithm on a few explicit examples.
4.1.5 Hodge splitting on Calabi-Yau orientifold

The orientifold involution $\sigma$ exchanges two projective coordinates on $X$. The holomorphicity condition requires that the pullback $\sigma^*$ maps $(p,q)$-forms on $X$ to $(p,q)$-forms. This is also true at the level of cohomology as the Dolbeault operator $\bar{\partial}$ commutes with the pullback $\sigma^*$. This implies that

$$H^{p,q}(X/\sigma^*) = H^{p,q}_+(X/\sigma^*) \oplus H^{p,q}_-(X/\sigma^*)$$

(4.24)

However, only when $\sigma$ is a free action (i.e. $X/\sigma^*$ is smooth), does it make sense to describe these as the Hodge numbers of the orientifold. Nevertheless, we will still compute the dimensions of these split cohomology groups and refer them to as Hodge numbers as in the smooth case.

Because we work only with favorable geometries, the calculations are simplified by the fact that the toric classes of ambient space $\mathcal{A}$ always restrict in a straightforward way to the Calabi-Yau hypersurface, and the divisor classes are the same as shown in Equation (4.7). Then we can always expand the Kähler form in terms of these divisor classes.

To illustrate, we choose a toy example with $h^{1,1}(X) = 3$, admitting a proper orientifold involution $\sigma^*: D_2 \leftrightarrow D_3$. Suppose the divisor classes $\{D_1, D_2, D_3\}$ form a basis for $H^{1,1}(X;\mathbb{Z})$. Then, the Kähler form can be expanded as the linear combination

$$J = t_1 D_1 + t_2 D_2 + t_3 D_3$$
with $t_1, t_2, t_3 \in \mathbb{Z}$. But, the Kähler form must obey the constraint of even parity under the orientifold involution, and must therefore only have components in $H^{1,1}_+(X)$, so that

$$J = \sigma^*J = t_1\sigma^*D_1 + t_2\sigma^*D_2 + t_3\sigma^*D_3$$

$$= t_1D_1 + t_2D_3 + t_3D_2$$  \hspace{1cm} (4.25)

Then, comparing Equations (4.1.5) and (4.25), we note that we must have $t_2 = t_3 = t_+, \text{ for some } t_+ \in \mathbb{Z}$. Then defining the even and odd parity eigendivisor $D_\pm = D_2 \pm D_3$, we can write

$$J = t_1D_1 + t_+D_+$$  \hspace{1cm} (4.26)

Thus, on the orientifold $X/\sigma^*$, there are only two independent directions in the Kähler moduli space, $D_1$ and $D_+$, while $D_- \in H^{1,1}_-(X/\sigma^*)$ does not appear in the Kähler form. We therefore find the equivariant cohomology $h^{1,1}_+(X/\sigma^*) = 2$ and $h^{1,1}_-(X/\sigma^*) = 1$. It will be shown later that this can only be done in a consistent way when $\sigma$ is a proper\textsuperscript{3} involution respecting the linear ideal $I_{lin}$, which fixes the redundancy toric divisor classes. Since the involution exchanges pairs of NIDs, one can always expand the Kähler form in the orientifold invariant basis, depending on some pairs of the divisors involved in the involution.

In addition to finding the even and odd parity splitting of $h^{1,1}_+(X/\sigma^*)$, if the involution is a free action, we can further fix the splitting of $h^{2,1}_+(X\sigma^*)$ using the fact that the Euler

\textsuperscript{3}In fact, in the case of a proper involution, it is extremely rare that we can write down an involution exchanging only a single pair of NIDs. However, for clarity, we use this simple toy example only to illustrate the underlying concept.
Chapter 4. Divisor Involutions and Free Actions on Toric Calabi-Yau Orientifolds

characteristic of a free action quotient group obeys \( \chi(X/\sigma^*) = \chi(X)/|\sigma^*| \), where \(|\sigma^*|\) is the order of the group action defined by \( \sigma^* \cong \mathbb{Z}_2 \). Then, the following formula applies

\[
\chi(X/\sigma^*) = \frac{\chi(X)}{2} = h^{1,1}(X) - h^{2,1}(X)
\]

\[
= 2 \left( h^{1,1}(X/\sigma^*) - h^{2,1}(X/\sigma^*) \right) \tag{4.27}
\]

We then find \( h^{2,1}(X/\sigma^*) = \frac{\chi(X)}{4} - h^{1,1}(X/\sigma^*) \). Since this formula only holds in the case of a free action, we only determine the \( h^{1,1}(X/\sigma^*) \) splitting for singular Calabi-Yau orientifolds. These results are enumerated in Tables 4.17 and 4.18.

4.2 Examples

4.2.1 Non-trivial Action: 2 Coordinate Exchange

4.2.1.1 Orientifold Planes

In this section, we demonstrate an explicit example of finding and classifying the point-wise fixed sets of a Calabi-Yau orientifold. We have chosen an example from our database of Calabi-Yau threefolds at www.rossealtman.com with Hodge numbers \( h^{1,1}(X) = 4, \ h^{2,1}(X) = 34 \). It can be identified by its index in the database

<table>
<thead>
<tr>
<th>Polytope ID</th>
<th>Geometry ID</th>
<th>Triangulation ID</th>
</tr>
</thead>
<tbody>
<tr>
<td>288</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
This example defines an MPCP desingularized ambient toric variety with weight matrix \( W \) given by

\[
\begin{array}{cccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
  0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 \\
  0 & 0 & 2 & 1 & 2 & 4 & 3 & 0 \\
  0 & 1 & 1 & 1 & 1 & 2 & 0 & 0 \\
  1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

and Stanley-Reisner ideal

\[
I_{SR} = \langle x_2x_4, x_2x_8, x_3x_7, x_3x_8, x_4x_7, x_1x_5x_6 \rangle
\]

For this example, there is only one proper involution exchanging NIDs, which leaves the SR ideal and intersection numbers invariant

\[
\sigma : x_2 \leftrightarrow x_4, \ x_7 \leftrightarrow x_8
\]

Since the projective coordinates \( x_1, x_3, x_5, \) and \( x_6 \) are not affected by the involution, they are included in our list of (anti-)invariant polynomial generators

\[
G_0 = \{x_1, x_3, x_5, x_6\}
\]
If we define $\sigma_1 : x_2 \leftrightarrow x_4$ and $\sigma_2 : x_7 \leftrightarrow x_8$, such that $\sigma = \sigma_1 \circ \sigma_2$, then we have the 3 sub-involutions: $\sigma_1$, $\sigma_2$, and $\sigma$. We now explore these cases.

1. $\sigma_1 : x_2 \leftrightarrow x_4$

   Because we only consider NIDs, $x_2$ and $x_4$ have different weights and cannot be combined into a homogenous binomial. Thus, we are left only with the invariant monomial $x_2 x_4$.

   $\Rightarrow \mathcal{G}_+ = \{x_2 x_4\}, \; \mathcal{G}_- = \emptyset$

2. $\sigma_2 : x_7 \leftrightarrow x_8$

   For the same reason, we are left only with the invariant monomial $x_7 x_8$.

   $\Rightarrow \mathcal{G}_+ = \{x_7 x_8\}, \; \mathcal{G}_- = \emptyset$

3. $\sigma : x_2 \leftrightarrow x_4, \; x_7 \leftrightarrow x_8$

   Any invariant monomials in this case are just products of the ones we found above. However, we must now consider binomial generators of the form

   \[
   x_2^m x_7^n \pm x_4^m x_8^n
   \]

   for $m, n \in \mathbb{Z}$. The homogeneity of this binomial is determined by the following condition on the weights

   \[
   m(W_{i2} - W_{i4}) + n(W_{i7} - W_{i8}) = 0
   \]
Thus, the vector \((m, n) \in \mathbb{Z}^2\) lies in the kernel of the matrix of difference vectors

\[
\begin{array}{c|c|c|c}
W_{i2} - W_{i4} & W_{i7} - W_{i8} \\
\hline
0 & 0 \\
\hline
-1 & 3 \\
\hline
0 & 0 \\
\hline
0 & 0 \\
\end{array}
\]

The kernel is generated by the vector \((m, n) = (3, 1)\), so that our binomial generators\(^4\) are given by \(x_2^3x_7 \pm x_4^3x_8\).

\[
\Rightarrow \quad G_+ = \{x_2^3x_7 + x_4^3x_8\}, \quad G_- = \{x_2^3x_7 - x_4^3x_8\}
\]

Now, all the (anti-)invariant polynomial generations in \(G = G_0 \cup G_+ \cup G_-\) are given by

\[
y_1 = x_1, \quad y_2 = x_3, \quad y_3 = x_5, \quad y_4 = x_6, \quad y_5 = x_2x_4,
\]

\[
y_6 = x_7x_8, \quad y_7 = x_2^3x_7 + x_4^3x_8, \quad y_8 = x_2^3x_7 - x_4^3x_8
\]

\(^4\)Note that if the kernel was generated by \((m, n) = (3, -1)\), then the binomial generators would be given by \(x_2^3x_7^{-1} \pm x_4^3x_8^{-1}\). If we multiply through by \(x_7x_8\), we get \(x_2^3x_8 \pm x_4^3x_7\). This just implies that we assumed the wrong positions of \(x_7\) and \(x_8\), so they must be switched in the binomial.
This coordinate transformation defines the Segre embedding with consistency condition
\[ y_7^2 = y_8^2 + 4y_5^3y_6 \]
and new weight matrix

\[
\begin{array}{cccccccc}
  y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\
  0 & 1 & 1 & 2 & 0 & 2 & 1 & 1 (1) \\
  0 & 2 & 2 & 4 & 1 & 3 & 3 & 3 (2) \\
  0 & 1 & 1 & 2 & 2 & 0 & 3 & 3 (3) \\
  1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 (4)
\end{array}
\]

However, this matrix only has rank 3, and is redundant. We see that row (3) = 2 × (2) − 3 × (1), so we eliminate this row

\[
\begin{array}{cccccccc}
  y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\
  0 & 1 & 1 & 2 & 0 & 2 & 1 & 1 \lambda \\
  0 & 2 & 2 & 4 & 1 & 3 & 3 & 3 \rho \\
  1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \gamma
\end{array}
\]

Now, the involution \( \sigma \) can be rewritten simply in these coordinates as \( y_8 \mapsto -y_8 \). Thus, we see that \( F_1 = \{ y_8 = 0 \} \) is a point-wise fixed, codimension-1 subvariety. This defines an O7 plane on the orientifold of \( X \). By inspecting Equation (4.30), we may in addition, naively assume the fixed point set \( F_0 = \{ x_2 = x_4 = x_7 = x_8 = 0 \} \), which is trivially unaffected by the involution. However, \( F_0 \) is a subset of \( F_1 \), and is therefore redundant. The redundancy of this trivial fixed point set is a general feature of any involution which
admits odd parity binomial generators in $G_\ast$.

By taking advantage of the toric degrees of freedom, however, there may be other non-trivial fixed sets in addition to $F_1$. We therefore check whether any subset $\mathcal{F}$ of the generators can neutralize the odd parity of $y_8$, becoming fixed themselves in the process. This scan can be simplified by noting that if a set of points is not fixed, then neither is any set containing it. Thus, if the simultaneous vanishing of a set of generators is not fixed, then neither is the vanishing of any subset. We therefore begin our scan with the largest set of generators and work our way down. The largest set we can choose has 4 generators, since their simultaneous vanishing defines a set of isolated points on the ambient space $\mathcal{A}$.

Consider the subset $\{y_1, y_2, y_4, y_7\} \equiv \mathcal{F}_2 \subset \mathcal{G}$. In order for the locus $F_2 = \{y_1 = y_2 = y_4 = y_7 = 0\}$ to be fixed, we must use the 3 independent $\mathbb{C}^\ast$ actions to set $y_1$, $y_2$, $y_4$, and $y_7$ to odd parity, while setting $y_8$ to even parity and leaving everything else invariant. This constraint is defined by the toric equivalence class

\[
(y_1, y_2, y_3, y_4, y_5, y_6, y_7, -y_8) \sim (\gamma y_1, \lambda \rho^2 y_2, \lambda \rho^2 \gamma y_3, \lambda^2 \rho^4 \gamma y_4, \rho y_5, \lambda^2 \rho^3 y_6, \lambda \rho^3 y_7, -\lambda \rho^3 y_8)
\]

\[
= (-y_1, -y_2, y_3, -y_4, y_5, y_6, -y_7, y_8)
\]  

(4.34)
where $\lambda, \rho, \gamma \in \mathbb{C}^*$. This can be written more simply as the system of multiplicative equations

\begin{align*}
\gamma &= -1 \\
\lambda \rho^2 &= -1 \\
\lambda \rho^2 \gamma &= 1 \\
\lambda^2 \rho^4 \gamma &= -1 \\
\rho &= 1 \\
\lambda^2 \rho^3 &= 1 \\
\lambda \rho^3 &= -1 \\
-\lambda \rho^3 &= 1
\end{align*}  \hfill (4.35)

This system can clearly be solved visually with solution $(\lambda, \rho, \gamma) = (-1, 1, -1)$, but we will continue with our algorithm as the computer would not be able to obtain the solution this way.

By defining $\lambda = e^{i\pi u}$, $\rho = e^{i\pi v}$, $\gamma = e^{i\pi w}$ as explicit values on the complex unit circle with $u, v, w \in \mathbb{Q}$, and noting that $0 \leq u, v, w < 2$ is the primitive domain of unique values, we can replace the multiplicative equations with linear congruences

\begin{align*}
w &\equiv 1 \pmod{2} & u + 2v &\equiv 1 \pmod{2} \\
 u + 2v + w &\equiv 0 \pmod{2} & 2u + 4v + w &\equiv 1 \pmod{2} \\
v &\equiv 0 \pmod{2} & 2u + 3v &\equiv 0 \pmod{2} \hfill (4.36) \\
u + 3v &\equiv 1 \pmod{2} & u + 3v &\equiv 1 \pmod{2}
\end{align*}
Then finally, by adding in the auxiliary variables $q_1, ..., q_8 \in \mathbb{Z}$, we can write this as a linear system of Diophantine equalities

\begin{align*}
  w - 2q_1 &= 1 \\
  u + 2v + w - 2q_3 &= 0 \\
  v - 2q_5 &= 0 \\
  u + 3v - 2q_7 &= 1 \\
  u + 2v - 2q_2 &= 1 \\
  2u + 4v + w - 2q_4 &= 1 \\
  2u + 3v - 2q_6 &= 0 \\
  u + 3v - 2q_8 &= 1
\end{align*}

(4.37)

Since $0 \leq u, v, w < 2$, we notice that the vector $(q_1, ..., q_8) \in \mathbb{Z}^8$ is in the lattice $\Lambda \subset \mathbb{Z}^8$ defined by

\begin{align*}
  0 \leq q_1, q_5 < 1 \\
  0 \leq q_2 < 3 \\
  0 \leq q_3, q_7, q_8 < 4 \\
  0 \leq q_6 < 5 \\
  0 \leq q_4 < 7
\end{align*}

This gives us a finite range to scan over, and the search is relatively fast. For each point $(q_1, ..., q_8) \in \Lambda$, we search for a solution to Equation (4.37). If any solution is found for any lattice point, then the set of generators $\mathcal{F}$ has a point-wise fixed vanishing locus. As we saw earlier, this particular system can indeed be solved, and so $F_2 = \{y_1 = y_2 = y_4 = y_7 = 0\}$ defines a point-wise fixed set. Similarly, we find that the sets $F_3 = \{y_2 = y_3 = y_7 = 0\}$ and $F_4 = \{y_5 = y_6 = y_7 = 0\}$ are point-wise fixed as well.

For $F_4$, however, we notice that the consistency condition of the Segre map $y_7^2 = y_8^2 + 4y_5^3y_6$ requires that $y_8$ vanish as well. Thus, $F_4 = \{y_5 = y_6 = y_7 = y_8 = 0\}$, but this is just a subset of $F_1$ and is redundant.
We must now check that the fixed sets intersect the Calabi-Yau hypersurface transversally, so that they are not redundant. The orientifold-symmetric Calabi-Yau polynomial has the form

\[
P_{\text{symm}} = a_1x_1^3x_3^6 + a_2x_1x_2x_3x_4x_7x_8 + a_3x_1^3x_2^3x_7^2x_8 + a_4x_1^3x_2^3x_7x_8^2 + a_5x_1^2x_2^2x_3^2x_7^2x_8^2
\]

\[\quad + a_6x_1^2x_3x_4x_7^3x_8 + a_7x_1^2x_2x_3x_4x_7^2x_8 + a_8x_1x_2^2x_4x_7x_8 + a_9x_1^3x_2^3x_7x_8^2 + a_{10}x_1^3x_2^3x_7x_8^4
\]

\[\quad + a_11x_1^2x_3^5x_5 + a_{12}x_1^2x_3^4x_6 + a_{13}x_1^2x_2x_3^3x_4x_5x_7x_8 + a_{14}x_1^2x_2^2x_3^2x_5x_7x_8 + a_{15}x_1^2x_2^2x_3^2x_5x_7x_8^2
\]

\[\quad + a_{16}x_1x_2^2x_4^2x_6x_7x_8 + a_{17}x_1^2x_2^2x_3^2x_5x_7x_8^2 + a_{18}x_1^2x_2^2x_3x_6x_7^2x_8 + a_{19}x_1^2x_3x_6x_7x_8^2
\]

\[\quad + a_{20}x_1^2x_2^4x_5x_7x_8^2 + a_{21}x_1^2x_2^4x_5x_7x_8^3 + a_{22}x_1^2x_2^4x_6x_7x_8^2 + a_{23}x_1^2x_3^2x_5x_6 + a_{24}x_1^2x_3^2x_5x_6
\]

\[\quad + a_{25}x_1x_2^2x_4x_5^2x_7x_8 + a_{26}x_1x_3x_6^2 + a_{27}x_1x_2x_3x_5^2x_7^2x_8 + a_{28}x_1x_3x_3x_5^2x_7^2x_8^2
\]

\[\quad + a_{29}x_1x_2x_3x_4x_5x_6x_7x_8 + a_{30}x_1x_2x_3x_4x_5x_6x_7^2x_8 + a_{31}x_1x_2x_5x_6x_7^2x_8 + a_{32}x_1x_3x_5x_6x_7x_8^2
\]

\[\quad + a_{33}x_1x_2x_4x_6x_7x_8 + a_{34}x_1x_3^3x_6 + a_{35}x_1x_3^3x_5x_6 + a_{36}x_2x_3x_4x_5^2x_7x_8 + a_{37}x_3x_5x_6^2 + a_{38}x_3^2x_5^2x_7x_8
\]

\[\quad + a_{39}x_4^2x_5^2x_7x_8 + a_{40}x_2x_4^2x_6x_7x_8 + a_{41}x_6^3
\]

The fixed point set \( F_2 = \{y_1 = y_2 = y_4 = y_7 = 0\} \) can be written in terms of the original projective coordinates \( \{x_1 = x_3 = x_6 = 0\} \cap \{x_3^2x_8 = -x_2^3x_7\} \). If we make these substitutions in \( P_{\text{symm}} \), it reduces to

\[
P_{\text{symm}} = x_2^3x_7x_8(a_{38}x_2^3x_7 + a_{39}x_2^3x_8)
\]

\[= x_2^3x_7x_8(a_{38} - a_{39})
\]
But, the Stanley-Reisner ideal forbids \( \{ x_1 = x_5 = x_6 = 0 \} \), \( \{ x_3 = x_7 = 0 \} \), and \( \{ x_3 = x_8 = 0 \} \), so the only possibility for \( P_{\text{symm}} \) vanishing is \( a_{38} = a_{39} = a \), which allows us to rewrite the Calabi-Yau polynomial

\[
P_{\text{symm}} = ax_5^3x_7x_8(x_2^3x_7 + x_4^3x_8) = ax_5^3x_7x_8y_7
\]

Thus, \( P_{\text{symm}} = 0 \) implies \( y_7 = 0 \), and \( y_7 \) is redundant when restricting the fixed set to \( X \).

The reduced set is then \( F'_2 = \{ y_1 = y_2 = y_4 = 0 \} \).

We do a similar check for the other fixed sets \( F_1 = \{ y_8 = 0 \} \) and \( F_3 = \{ y_2 = y_3 = y_7 = 0 \} \), but we find that there is no redundancy in these cases.

In practice, we combine the transversality and SR ideal checks by performing Groebner basis calculations to check the dimension of the ideal

\[
\mathcal{I}_j^{\text{fixed}} = \langle U_i, P_{\text{symm}}, F_j \rangle
\]
for $U_i$ a region allowed by the Stanley-Reisner ideal. Given $\mathcal{I}_{SR}$, the allowed regions in this case are

\begin{align*}
U_1 &= \langle x_1 - t_1, x_2 - t_2, x_3 - t_3, x_4 - t_4 \rangle \\
U_2 &= \langle x_1 - t_1, x_2 - t_2, x_3 - t_3, x_7 - t_7 \rangle \\
U_3 &= \langle x_1 - t_1, x_2 - t_2, x_7 - t_7, x_8 - t_8 \rangle \\
U_4 &= \langle x_1 - t_1, x_3 - t_3, x_4 - t_4, x_8 - t_8 \rangle \\
U_5 &= \langle x_1 - t_1, x_4 - t_4, x_7 - t_7, x_8 - t_8 \rangle \\
U_6 &= \langle x_2 - t_2, x_3 - t_3, x_4 - t_4, x_5 - t_5 \rangle \\
U_7 &= \langle x_2 - t_2, x_3 - t_3, x_4 - t_4, x_6 - t_6 \rangle \\
U_8 &= \langle x_2 - t_2, x_3 - t_3, x_5 - t_5, x_7 - t_7 \rangle \\
U_9 &= \langle x_2 - t_2, x_3 - t_3, x_6 - t_6, x_7 - t_7 \rangle \\
U_{10} &= \langle x_2 - t_2, x_5 - t_5, x_7 - t_7, x_8 - t_8 \rangle \\
U_{11} &= \langle x_2 - t_2, x_6 - t_6, x_7 - t_7, x_8 - t_8 \rangle \\
U_{12} &= \langle x_3 - t_3, x_4 - t_4, x_5 - t_5, x_8 - t_8 \rangle \\
U_{13} &= \langle x_3 - t_3, x_4 - t_4, x_6 - t_6, x_8 - t_8 \rangle \\
U_{14} &= \langle x_4 - t_4, x_5 - t_5, x_7 - t_7, x_8 - t_8 \rangle \\
U_{15} &= \langle x_4 - t_4, x_6 - t_6, x_7 - t_7, x_8 - t_8 \rangle
\end{align*}

for auxiliary variables $t_1, \ldots, t_8 \in \mathbb{C}$. In order to simplify the calculation, however, we set $t_1 = \ldots = t_8 = 1$. If the dimension $\dim \mathcal{I} > 0$ and removing any generator from $F_i$ changes the dimension, then we know that $F_i$ intersects $X$ transversally and is allowed by the SR ideal.

Finally, we tell from the number of intersecting codimension-1 subvarieties in each of the fixed sets $F_1, F'_2$, and $F_3$ that they have complex codimension 1, 3, and 3 in $X$ respectively. This implies that $F_1$ is an O7 plane, while $F'_2$ and $F_3$ are both O3 planes.
4.2.1.2 Hodge Splitting

We now turn to the procedure for computing the splitting of the Hodge numbers on the Calabi-Yau orientifold. The linear ideal, which fixes toric divisor redundancies, is given by

$$\mathcal{I}_{\text{lin}} = \langle -D_1 - D_2 - D_3 - D_4 - D_5 + 2D_6 - D_7 - D_8, $$
$$ -D_1 - D_2 - D_3 + 2D_4 + 2D_5 - D_6 + 0 + D_8, $$
$$ 0 + 0 + D_3 + 0 + D_5 - D_6 + 0 + 0, $$
$$ D_1 + 2D_2 + D_3 - D_4 + 0 - D_6 + D_7 + 0 \rangle \quad (4.38)$$

and a basis in $H^{1,1}(X;\mathbb{Z})$ given by $J_1 = D_4, J_2 = D_5, J_3 = D_6, J_4 = D_8$. The Hodge numbers of the toric divisors are

$$h^\bullet(D_1) = \{1, 0, 0, 9\}$$
$$h^\bullet(D_2) = h^\bullet(D_3) = h^\bullet(D_4) = \{1, 0, 0, 10\}$$
$$h^\bullet(D_5) = \{1, 0, 1, 19\} \quad (4.39)$$
$$h^\bullet(D_6) = \{1, 0, 3, 33\}$$
$$h^\bullet(D_7) = h^\bullet(D_8) = \{1, 1, 0, 2\}$$
For this example, there is only one proper involution exchanging NIDs, which leaves the SR ideal and intersection numbers invariant

$$\sigma^*: D_2 \leftrightarrow D_4, D_7 \leftrightarrow D_8$$  \hspace{1cm} (4.40)

From Equation (4.67), we see that this is an exchange of two dP$_8$, del Pezzo divisors and two exact Wilson divisors.

Under the orientifold involution, the Kähler form is even and must therefore belong to $H^{1,1}(X/\sigma^*)$. In a case with favorable geometry, as in this example, the calculation is simplified by the fact that the toric divisor classes of ambient space $\mathcal{A}$ always restrict in a straightforward way to the Calabi-Yau hypersurface via

$$H^{1,1}(\mathcal{A}) \cong \text{Pic}(\mathcal{A}) \cong \text{Pic}(X) \cong H^{1,1}(X)$$  \hspace{1cm} (4.41)

We can now expand the Kähler form in terms of these divisor classes

$$J = t_1J_1 + t_2J_2 + t_3J_3 + t_4J_4 = t_1D_4 + t_2D_5 + t_3D_6 + t_4D_8$$  \hspace{1cm} (4.42)
with $t_1, t_2, t_3, t_4 \in \mathbb{Z}$. But, the Kähler form must obey the constraint of even parity under the orientifold involution, and must therefore only have components in $H^{1,1}_+(X)$, so that

$$J = \sigma^* J = t_1 \sigma^* D_4 + t_2 \sigma^* D_5 + t_3 \sigma^* D_6 + t_4 \sigma^* D_8$$

$$= t_1 D_2 + t_2 D_5 + t_3 D_6 + t_4 D_7$$

$$= t_1 D_2 + t_2 J_2 + t_3 J_3 + t_4 D_7$$

(4.43)

In order to relate Equations (4.70) and (4.71), we must be able to write $D_2$ and $D_7$ (restricted to the CY hypersurface $X = \sum_{i=1}^{8} D_i$) in terms of our chosen basis. This can be done by reducing $D_2$ and $D_7$ by the linear ideal $I_{lin}$ (also restricted to the CY hypersurface). We find that\(^5\) on $X$, $D_2$ and $D_7$ are uniquely given by

$$D_2 = 2J_1 + J_2 - J_3 + J_4 \quad \text{and} \quad D_7 = -3J_1 - 3J_2 + 3J_3 - 2J_4$$

(4.44)

Plugging these into Equation (4.71), we deduce

$$J = (2t_1 - 3t_4)J_1 + (t_1 + t_2 - 3t_4)J_2 + (-t_1 + t_3 + 3t_4)J_3 + (t_1 - 2t_4)J_4$$

(4.45)

\(^5\)We do this calculation using the symbolic algebraic geometry software packages **Singular** and **Sage**.
Now, comparing Equations (4.70) and (4.73), we obtain the following system of linear equations

\[
\begin{align*}
2t_1 - 3t_4 &= t_1 \\
t_1 + t_2 - 3t_4 &= t_2 \\
-t_1 + t_3 + 3t_4 &= t_3 \\
t_1 - 2t_4 &= t_4
\end{align*}
\] (4.46)

for which the only independent solution is \( t_1 = 3t_4 \). Thus, we see that only 3 directions in the Kähler moduli space are independent, and so \( h_{-1}^{1,1}(X/\sigma^*) = 3 \), and by extension \( h_{1}^{1,1}(X/\sigma^*) = 1 \). This will be the case for any choice of integral basis on the Kähler moduli space. In fact, choosing even and odd parity eigendivisors \( D_{\pm,1} = D_2 \pm D_4 \), \( D_{\pm,2} = D_7 \pm D_8 \), the Kähler form can be written

\[
J = t_2 J_2 + t_3 J_3 + \frac{1}{2} t_4 (3D_{+,1} + D_{+,2})
\]

\[= t_2 J_2 + t_3 J_3 + t_+ D_+\] (4.47)

where the latter equality makes the redefinitions \( t_+ = \frac{t_4}{2} \) and \( D_+ = 3D_{+,1} + D_{+,2} = 3(D_2 + D_4) + (D_7 + D_8) \). The even parity of the Kähler form is now manifest.

In this example, the involution \( \sigma^* \) induces an O7 plane and two O3 planes, and so is not a free action. For that reason, Equation (4.27) does not strictly apply, and we only record the splitting of \( h^{1,1}(X/\sigma^*) \).
Finally we check that the locus \( \{ P_{\text{symm}} = 0 \} \) is smooth by constructing the ideal

\[
I_{i}^{\text{smooth}} = \langle U_i, P_{\text{symm}}, \frac{\partial P_{\text{symm}}}{\partial x_1}, \ldots, \frac{\partial P_{\text{symm}}}{\partial x_k} \rangle
\]

and computing the dimension \( \dim I_{i}^{\text{smooth}} \) for each disjoint region \( U_i \) allowed by the Stanley-Reisner ideal. We find that the maximum dimension is \(-1\), so that \( \{ P_{\text{symm}} = 0 \} \).

This allows us to state that \( H^{1,1}_{X}(X/\sigma^*) \) are, in fact, the split Hodge numbers on the orientifold.

### 4.2.2 Possible Free Action: 1 Coordinate Exchange

#### 4.2.2.1 Orientifold Planes

In this section, we demonstrate an explicit example of a Calabi-Yau orientifold for which an involution with a single coordinate exchange is a free action. This turns out to be a general feature of single coordinate exchange involutions, and we will see why that is. We have chosen an example from our database of Calabi-Yau threefolds at www.rossealtman.com with Hodge numbers \( h^{1,1}(X) = 4, \ h^{2,1}(X) = 82 \). It can be identified by its index in the database

<table>
<thead>
<tr>
<th>Polytope ID</th>
<th>Geometry ID</th>
<th>Triangulation ID</th>
</tr>
</thead>
<tbody>
<tr>
<td>917</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
This example defines an MPCP desingularized ambient toric variety with weight matrix

\[ W \] given by

\[
\begin{array}{cccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
  0 & 0 & 0 & 1 & 0 & 1 & 3 & 1 \\
  0 & 0 & 1 & 1 & 1 & 0 & 3 & 0 \\
  0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 \\
  1 & 1 & 0 & 1 & 0 & 2 & 5 & 0 \\
\end{array}
\]  

(4.48)

and Stanley-Reisner ideal

\[ I_{SR} = \langle x_1x_3, x_1x_6, x_3x_4, x_6x_8, x_1x_2x_5, x_2x_5x_7, x_4x_7x_8 \rangle \]  

(4.49)

For this example, there is only one involution exchanging NIDs that leaves the SR ideal invariant, however, it is not “proper” in the sense that it also leaves the intersection numbers invariant. This appears to be a general property of the potential freely acting involutions at low \( h^{1,1}(X) \). In fact, we do not begin to see proper freely acting involutions until \( h^{1,1}(X) = 6 \) with upwards of four disjoint coordinate exchanges. In the interest of keeping this example simple, but not too simple, we will carry on with this choice of “improper” involution. This involution is given by

\[ \sigma : x_2 \leftrightarrow x_5 \]  

(4.50)
Since the projective coordinates $x_1, x_3, x_4, x_6, x_7,$ and $x_8$ are not affected by the involution, they are included in our list of (anti-)invariant polynomial generators

$$\mathcal{G}_0 = \{x_1, x_3, x_4, x_6, x_7, x_8\} \quad (4.51)$$

In this case, there is only one sub-involution, $\sigma$ itself. Because we only consider NIDs, $x_2$ and $x_5$ have different weights and cannot be combined into a homogenous binomial. Thus, we are left only with the invariant monomial $x_2 x_5$. This implies that the even and odd parity generator sets are $\mathcal{G}_+ = \{x_2 x_5\}$ and $\mathcal{G}_- = \emptyset$.

Now, all the (anti-)invariant polynomial generations in $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_+ \cup \mathcal{G}_-$ are given by

$$y_1 = x_1, \quad y_2 = x_3, \quad y_3 = x_4, \quad y_4 = x_6, \quad y_5 = x_7,$$

$$y_6 = x_8, \quad y_7 = x_2 x_5$$

There are no generators with odd parity under the involution which are manifestly fixed. It is for this reason that single coordinate exchange involutions are often free actions. Naively, however, we may still have the fixed set $F_0 = \{x_2 = x_5 = 0\}$. Note that this set is allowed by the Stanley-Reisner ideal and does, in fact, exist in the ambient space.
However, the orientifold-symmetric Calabi-Yau polynomial has the form

\[
P_{\text{symm}} = a_1 x_7^2 + a_1 x_1^2 x_2^2 x_4^2 x_5^2 x_6 x_8 + a_3 x_1^4 x_2^2 x_3 x_4 x_5^2 x_6 x_8 + a_4 x_1^2 x_2^2 x_3 x_4 x_5^2 x_6 x_8 + a_5 x_1 x_2 x_3 x_4 x_5^2 x_6 x_8
\]

\[
+ a_6 x_1^2 x_2 x_3^2 x_4 x_5^2 x_6 x_8 + a_7 x_1^2 x_2 x_3^2 x_4 x_5^2 x_6 x_8 + a_8 x_2 x_3 x_4 x_5^2 x_6 x_8 + a_9 x_2 x_4 x_5 x_6 x_8 + a_{10} x_1^2 x_2^2 x_4 x_5^2 x_6 x_8 + a_{11} x_1^3 x_2 x_3 x_4 x_5 x_7 x_8^2 + a_{12} x_1^2 x_2^2 x_3 x_4 x_5 x_7 x_8^2 + a_{13} x_1 x_2 x_3 x_4 x_5 x_7 x_8^2
\]

\[
+ a_{14} x_1^2 x_2 x_3^2 x_4 x_5^2 x_6 x_8 + a_{15} x_1^2 x_2^2 x_4 x_5^2 x_6 x_8 + a_{16} x_1^2 x_2 x_3 x_4 x_5 x_7 x_8 + a_{17} x_1 x_2 x_3 x_4 x_5 x_7 x_8 + a_{18} x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 + a_{19} x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 + a_{20} x_1 x_2^2 x_3 x_4 x_5^2 x_6 x_8 + a_{21} x_1 x_2^2 x_3 x_4 x_5 x_6 x_8 + a_{22} x_1 x_2 x_3 x_4 x_5^2 x_6 x_8 + a_{23} x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 + a_{24} x_1 x_2 x_3 x_4 x_5^2 x_6 x_8 + a_{25} x_1 x_2 x_3 x_4 x_5 x_6 x_8 + a_{26} x_1 x_2 x_3 x_4 x_5^2 x_6 x_8 + a_{27} x_1 x_2 x_3 x_4 x_5^2 x_6 x_8
\]

where \(a_i \in \mathbb{C}\) are arbitrary coefficients. On the ambient space fixed set, where \(\{x_2 = x_5 = 0\}\), this reduces simply to

\[
P_{\text{symm}} = a_1 x_7^2
\]

Thus, the vanishing with the Calabi-Yau polynomial on this set is equivalent to the vanishing of \(x_7\). However, as \(x_2 x_5 x_7\) is an element of the Stanley-Reisner ideal, \(x_7\) cannot vanish on this set. Hence, the region where the Calabi-Yau hypersurface intersects the fixed-point locus of the ambient space is ruled out, giving us what is potentially a free action on the orientifold, provided that it is also non-singular.
4.2.2.2 Hodge Splitting

We now turn to the procedure for computing the splitting of the Hodge numbers on the Calabi-Yau orientifold. The linear ideal, which fixes toric divisor redundancies, is given by

\[
I_{\text{lin}} = \langle -D_1 - D_2 - D_3 - D_4 - D_5 - D_6 + D_7 - D_8, \\
0 + 0 + 0 + D_4 + 2D_5 + 2D_6 - D_7 + 0, \\
-D_1 - D_2 + D_3 - D_4 + 3D_5 + 4D_6 - D_7 + 0, \\
0 + D_2 + 0 + D_4 - D_5 - D_6 + 0 + 0 \rangle
\]

and a basis in \( H^{1,1}(X; \mathbb{Z}) \) given by \( J_1 = D_1, \ J_2 = D_5, \ J_3 = D_7, \ J_4 = D_8 \). The Hodge numbers of the toric divisors are

\[
h^\bullet(D_1) = h^\bullet(D_3) = \{1, 0, 0, 10\} \\
h^\bullet(D_2) = h^\bullet(D_5) = \{1, 0, 0, 13\} \\
h^\bullet(D_4) = h^\bullet(D_6) = \{1, 0, 1, 20\} \quad (4.53) \\
h^\bullet(D_7) = \{1, 0, 19, 134\} \\
h^\bullet(D_8) = \{1, 1, 0, 2\}
\]

The involution acts on the divisor classes as

\[
\sigma^* : D_2 \leftrightarrow D_5 \quad (4.54)
\]
From Equation (4.67), we see that this is an exchange of two rigid divisors.

Under the orientifold involution, the Kähler form is even and must therefore belong to $H^{1,1}_+(X/\sigma^*)$. In a case with favorable geometry, as in this example, the calculation is simplified by the fact that the toric divisor classes of ambient space $\mathcal{A}$ always restrict in a straightforward way to the Calabi-Yau hypersurface via

$$H^{1,1}(\mathcal{A}) \cong \text{Pic}(\mathcal{A}) \cong \text{Pic}(X) \cong H^{1,1}(X)$$  \hspace{1cm} (4.55)

We can now expand the Kähler form in terms of these divisor classes

$$J = t_1 J_1 + t_2 J_2 + t_3 J_3 + t_4 J_4 = t_1 D_1 + t_2 D_5 + t_3 D_7 + t_4 D_8$$  \hspace{1cm} (4.56)

with $t_1, t_2, t_3, t_4 \in \mathbb{Z}$. But, the Kähler form must obey the constraint of even parity under the orientifold involution, and must therefore only have components in $H^{1,1}_+(X)$, so that

$$J = \sigma^* J = t_1 \sigma^* J_1 + t_2 \sigma^* J_2 + t_3 \sigma^* J_3 + t_4 \sigma^* J_4 = t_1 D_1 + t_2 D_2 + t_3 D_7 + t_4 D_8$$  \hspace{1cm} (4.57)

In order to relate Equations (4.70) and (4.71), we must be able to write $D_2$ (restricted to the CY hypersurface $X = \sum_{i=1}^8 D_i$) in terms of our chosen basis. This can be done by reducing $D_2$ by the linear ideal $I_{lin}$ (also restricted to the CY hypersurface). We find
that\textsuperscript{6} on $X$, $D_2$ is uniquely given by

$$D_2 = 6J_1 + 3J_2 - J_3 + 3J_4$$  \hfill (4.58)

Plugging this into Equation (4.71), we deduce

$$J = (t_1 + 6t_2)J_1 + 3t_2J_2 + (t_3 - t_2)J_3 + (3t_2 + t_4)J_4$$  \hfill (4.59)

Now, comparing Equations (4.70) and (4.73), we obtain the following system of linear equations

\begin{align*}
t_1 + 6t_2 &= t_1 \\
3t_2 &= t_2 \\
t_3 - t_2 &= t_3 \\
3t_2 + t_4 &= t_4
\end{align*}

(4.60)

for which the only independent solution is $t_2 = 0$. Thus, we see that only 3 directions in the Kähler moduli space are independent, and so $h^{1,1}_+(X/\sigma^*) = 3$, and by extension $h^{1,1}_-(X/\sigma^*) = 1$. This will be the case for any choice of integral basis on the Kähler moduli

\textsuperscript{6}We do this calculation using the symbolic algebraic geometry software packages \textbf{Singular} and \textbf{Sage}.\n
space. In fact, with $t_2 = 0$, the Kähler form can be written

$$J = t_1 J_1 + t_3 J_3 + t_4 J_4$$

Thus, the even parity of the Kähler form is manifest.

Finally we check that the locus $\{P_{symm} = 0\}$ is smooth by constructing the ideal

$$\mathcal{I}_{i}^{\text{smooth}} = \langle U_i, P_{symm}, \frac{\partial P_{symm}}{\partial x_1}, \ldots, \frac{\partial P_{symm}}{\partial x_k} \rangle$$

and computing the dimension $\dim \mathcal{I}_{i}^{\text{smooth}}$ for each disjoint region $U_i$ allowed by the Stanley-Reisner ideal. We find that the maximum dimension is 2, so that $\{P_{symm} = 0\}$ is not smooth. Therefore, the involution $\sigma^*$ is not actually a free action on $X$, and it is not entirely accurate to say that $H^{1,1}_\pm(X/\sigma^*)$ are, in fact, the split Hodge numbers on the orientifold.

**4.2.3 Possible Free Action: 2 Coordinate Exchanges**

**4.2.3.1 Orientifold Planes**

In this section, we demonstrate an explicit example of a Calabi-Yau orientifold for which the involution is a free action. We have chosen an example from our database of Calabi-Yau threefolds at [www.rossealtman.com](http://www.rossealtman.com) with Hodge numbers $h^{1,1}(X) = 5, \ h^{2,1}(X) =$
49. It can be identified by its index in the database

<table>
<thead>
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<th>Polytope ID</th>
<th>Geometry ID</th>
<th>Triangulation ID</th>
</tr>
</thead>
<tbody>
<tr>
<td>1726</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

This example defines an MPCP desingularized ambient toric variety with weight matrix $W$ given by

\[
\begin{array}{ccccccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
  0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 3 \\
  0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\
  0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
  1 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 \\
\end{array}
\] (4.62)

and Stanley-Reisner ideal

\[
\mathcal{I}_{SR} = \langle x_2 x_5, x_2 x_7, x_2 x_9, x_3 x_4, x_5 x_7, x_5 x_8, x_8 x_9, x_1 x_6 x_7, x_1 x_6 x_8, x_1 x_6 x_9 \rangle
\] (4.63)

For this example, there are three involutions exchanging NIDs that leave the SR ideal invariant, however, they are not “proper” in the sense that they also leave the intersection numbers invariant. This appears to be a general property of the potential freely acting involutions at low $h^{1,1}(X)$. In fact, we do not begin to see proper freely acting involutions until $h^{1,1}(X) = 6$ with upwards of four disjoint coordinate exchanges. In the interest of
keeping this example simple, but not too simple, we will carry on with this choice of “improper” involution. Of the three involutions exchanging NIDs, we choose the first

\[ \sigma : x_2 \leftrightarrow x_5, \ x_8 \leftrightarrow x_9 \]  

(4.64)

Since the projective coordinates \(x_1, x_3, x_4, x_6,\) and \(x_7\) are not affected by the involution, they are included in our list of (anti-)invariant polynomial generators

\[ G_0 = \{x_1, x_3, x_4, x_6, x_7\} \]  

(4.65)

If we define \(\sigma_1 : x_2 \leftrightarrow x_5\) and \(\sigma_2 : x_8 \leftrightarrow x_9,\) such that \(\sigma = \sigma_1 \circ \sigma_2,\) then we have the 3 sub-involutions: \(\sigma_1, \sigma_2,\) and \(\sigma.\) We now explore these cases

1. \(\sigma_1 : x_2 \leftrightarrow x_5\)

   Because we only consider NIDs, \(x_2\) and \(x_5\) have different weights and cannot be combined into a homogenous binomial. Thus, we are left only with the invariant monomial \(x_2x_5.\)

   \[ \Rightarrow \ G_+ = \{x_2x_5\}, \ G_- = \emptyset \]

2. \(\sigma_2 : x_8 \leftrightarrow x_9\)

   For the same reason, we are left only with the invariant monomial \(x_8x_9.\)

   \[ \Rightarrow \ G_+ = \{x_8x_9\}, \ G_- = \emptyset \]
3. \( \sigma : x_2 \leftrightarrow x_5, \; x_8 \leftrightarrow x_9 \)

Any invariant monomials in this case are just products of the ones we found above. However, we must now consider binomial generators of the form

\[ x_2^m x_8^n \pm x_5^m x_9^n \]

for \( m, n \in \mathbb{Z} \). The homogeneity of this binomial is determined by the following condition on the weights

\[ m(W_{i2} - W_{i5}) + n(W_{i8} - W_{i9}) = 0 \]

Thus, the vector \((m, n) \in \mathbb{Z}^2\) lies in the kernel of the matrix of difference vectors

\[
\begin{array}{c|c|c|c}
W_{i2} - W_{i5} & W_{i8} - W_{i9} \\
\hline
0 & 0 \\
0 & -3 \\
1 & -2 \\
0 & 0 \\
0 & 3 \\
\end{array}
\]

The kernel is trivial, so there are no binomial generators of this form.

\[ \Rightarrow \; G_+ = \emptyset, \; G_- = \emptyset \]
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Now, all the (anti-)invariant polynomial generations in $G = G_0 \cup G_+ \cup G_-$ are given by

$$y_1 = x_1, \quad y_2 = x_3, \quad y_3 = x_4, \quad y_4 = x_6, \quad y_5 = x_7,$$

$$y_6 = x_2 x_5, \quad y_7 = x_8 x_9$$

There are no generators with odd parity under the involution which are manifestly fixed, but naively, we may still have the fixed set $F_0 = \{x_2 = x_5 = x_8 = x_9 = 0\}$, which is trivially unaffected by the involution. However each of the loci $\{x_2 = x_5 = 0\}$, $\{x_2 = x_9 = 0\}$, $\{x_5 = x_8 = 0\}$, and $\{x_8 = x_9 = 0\}$ of which $F_0$ is a subspace, are forbidden by the Stanley-Reisner ideal and do not, in fact, exist on the ambient space $A$. Therefore, there will be no point-wise fixed sets, and $\sigma^*$ is then potentially a free action on $X$, provided that the orientifold is also non-singular.

4.2.3.2 Hodge Splitting

We now turn to the procedure for computing the splitting of the Hodge numbers on the Calabi-Yau orientifold. The linear ideal, which fixes toric divisor redundancies, is given by

$$I_{lin} = \langle -D_1 - D_2 - D_3 - D_4 - D_5 - D_6 + D_7 + 0 + 0, \quad -D_1 - D_2 - D_3 + 0 + 3D_5 + 6D_6 - D_7 - D_8 + D_9, \quad -D_1 + 3D_2 + 6D_3 - D_4 - D_5 + 0 - D_7 + D_8 - D_9, \quad D_1 + 0 - D_3 + D_4 + 0 - D_6 + 0 + 0 + 0 \rangle \quad (4.66)$$
and a basis in $H^{1,1}(X; \mathbb{Z})$ given by $J_1 = D_5$, $J_2 = D_6$, $J_3 = D_7$, $J_4 = D_8$, $J_5 = D_9$. The Hodge numbers of the toric divisors are
\begin{align*}
    h^*(D_1) &= h^*(D_3) = h^*(D_4) = h^*(D_6) = \{1, 0, 1, 20\} \\
    h^*(D_2) &= h^*(D_5) = h^*(D_8) = h^*(D_9) = \{1, 0, 0, 2\} \\
    h^*(D_7) &= \{1, 0, 9, 84\}
\end{align*}

For this example, there is only one proper involution exchanging NIDs, which leaves the SR ideal and intersection numbers invariant
\[ \sigma^* : D_2 \leftrightarrow D_5, \; D_8 \leftrightarrow D_9 \]  

From Equation (4.67), we see that this is an exchange of two sets of dP$_1$, del Pezzo divisors.

Under the orientifold involution, the Kähler form is even and must therefore belong to $H^{1,1}_+(X/\sigma^*)$. In a case with favorable geometry, as in this example, the calculation is simplified by the fact that the toric divisor classes of ambient space $\mathcal{A}$ always restrict in a straightforward way to the Calabi-Yau hypersurface via
\[ H^{1,1}(\mathcal{A}) \cong \text{Pic}(\mathcal{A}) \cong \text{Pic}(X) \cong H^{1,1}(X) \]  

(4.69)
We can now expand the Kähler form in terms of these divisor classes

\[ J = t_1 J_1 + t_2 J_2 + t_3 J_3 + t_4 J_4 + t_5 J_5 = t_1 D_5 + t_2 D_6 + t_3 D_7 + t_4 D_8 + t_5 D_9 \]  \hspace{1cm} (4.70)

with \( t_1, t_2, t_3, t_4, t_5 \in \mathbb{Z} \). But, the Kähler form must obey the constraint of even parity under the orientifold involution, and must therefore only have components in \( H^{1,1}_+(X) \), so that

\[ J = \sigma^* J = t_1 \sigma^* D_5 + t_2 \sigma^* D_6 + t_3 \sigma^* D_7 + t_4 \sigma^* D_8 + t_5 \sigma^* D_9 \]

\[ = t_1 D_2 + t_2 D_6 + t_3 D_7 + t_4 D_9 + t_5 D_8 \]

\[ = t_1 D_2 + t_2 J_2 + t_3 J_3 + t_4 J_5 + t_5 J_4 \]  \hspace{1cm} (4.71)

In order to relate Equations (4.70) and (4.71), we must be able to write \( D_2 \) (restricted to the CY hypersurface \( X = \sum_{i=1}^{8} D_i \)) in terms of our chosen basis. This can be done by reducing \( D_2 \) by the linear ideal \( I_{lin} \) (also restricted to the CY hypersurface). We find that\(^7\) on \( X \), \( D_2 \) is uniquely given by

\[ D_2 = 7J_1 + 12J_2 - 3J_3 - 2J_4 + 2J_5 \]  \hspace{1cm} (4.72)

\(^7\)We do this calculation using the symbolic algebraic geometry software packages \texttt{Singular} [118] and \texttt{Sage} [40].
Plugging this into Equation (4.71), we deduce

\[ J = 7t_1 J_1 + (12t_1 + t_2) J_2 + (t_3 - 3t_1) J_3 + (t_5 - 2t_1) J_4 + (2t_1 + t_4) J_5 \]  

(4.73)

Now, comparing Equations (4.70) and (4.73), we obtain the following system of linear equations

\[
\begin{align*}
7t_1 &= t_1 \\
12t_1 + t_2 &= t_2 \\
t_3 - 3t_1 &= t_3 \\
t_5 - 2t_1 &= t_4 \\
2t_1 + t_4 &= t_5
\end{align*}
\]  

(4.74)

for which the only independent solution is \( t_1 = 0, \ t_4 = t_5 \). Thus, we see that only 3 directions in the Kähler moduli space are independent, and so \( h^{1,1}_{+}(X/\sigma^*) = 3 \), and by extension \( h^{1,1}_{-}(X/\sigma^*) = 2 \). This will be the case for any choice of integral basis on the Kähler moduli space. In fact, setting \( t_1 = 0 \) and \( t_4 = t_5 = t_+ \), and choosing even and odd parity eigendivisors \( D_{\pm} = D_8 \pm D_9 = J_4 \pm J_5 \), the Kähler form can be written

\[ J = t_2 J_2 + t_3 J_3 + t_+ D_+ \]  

(4.75)

Thus, the even parity of the Kähler form is now manifest.
Finally we check that the locus \( \{ P_{\text{symm}} = 0 \} \) is smooth by constructing the ideal

\[
\mathcal{I}_{i}^{\text{smooth}} = \langle U_i, P_{\text{symm}}, \frac{\partial P_{\text{symm}}}{\partial x_1}, ..., \frac{\partial P_{\text{symm}}}{\partial x_k} \rangle
\]

and computing the dimension \( \dim \mathcal{I}_{i}^{\text{smooth}} \) for each disjoint region \( U_i \) allowed by the Stanley-Reisner ideal. We find that the maximum dimension is 2, so that \( \{ P_{\text{symm}} = 0 \} \) is not smooth. Therefore, the involution \( \sigma^* \) is not actually a free action on \( X \), and it is not entirely accurate to say that \( H^{1,1}_{\pm}(X/\sigma^*) \) are, in fact, the split Hodge numbers on the orientifold.
4.3 Results

4.3.1 NID Involutions

<table>
<thead>
<tr>
<th>$h^{1,1}(X)$</th>
<th>1</th>
<th>2</th>
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<table>
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<tr>
<th>Triangulation-wide Involutions with Particular NIDs</th>
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<tbody>
<tr>
<td>del Pezzo surface dP$_n$, $n \leq 8$</td>
</tr>
<tr>
<td>Non-Shrinkable Rigid surface dP$_n$, $n &gt; 8$</td>
</tr>
<tr>
<td>(Exact-)Wilson surface</td>
</tr>
<tr>
<td>K3 surface</td>
</tr>
<tr>
<td>Special Deformation SD1 surface</td>
</tr>
<tr>
<td>Special Deformation SD2 surface</td>
</tr>
<tr>
<td>del Pezzo and K3</td>
</tr>
<tr>
<td>del Pezzo and (Exact-)Wilson</td>
</tr>
<tr>
<td>K3 and (Exact-)Wilson</td>
</tr>
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<td>del Pezzo, K3 and (Exact-)Wilson</td>
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Table 4.1: Numbers of triangulation-wide NID exchange involutions, and combinations of particular toric divisors exchanged. (Note: the scan for NID involutions has not completed at the time of writing.)
### Table 4.2: Numbers of triangulations that admit at least one NID exchange involution, and combinations of particular toric divisors exchanged. (Note: the scan for NID involutions has not completed at the time of writing.)

<table>
<thead>
<tr>
<th>$h^{1,1}(X)$</th>
<th>1</th>
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<td>91075</td>
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<tr>
<td>(Exact-)Wilson surface</td>
</tr>
<tr>
<td>K3 surface</td>
</tr>
<tr>
<td>Special Deformation SD1 surface</td>
</tr>
<tr>
<td>Special Deformation SD2 surface</td>
</tr>
<tr>
<td>del Pezzo and K3</td>
</tr>
<tr>
<td>del Pezzo and (Exact-)Wilson</td>
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<td>K3 and (Exact-)Wilson</td>
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<td>del Pezzo, K3 and (Exact-)Wilson</td>
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<table>
<thead>
<tr>
<th>$h^{1,1}(X)$</th>
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<td>83170</td>
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<th>Non-Shrinkable Rigid surface dP$_n$, n &gt; 8</th>
<th>(Exact-)Wilson surface</th>
<th>K3 surface</th>
<th>Special Deformation SD1 surface</th>
<th>Special Deformation SD2 surface</th>
<th>del Pezzo and K3</th>
<th>del Pezzo and (Exact-)Wilson</th>
<th>K3 and (Exact-)Wilson</th>
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<td>16152</td>
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<td>0 (0)</td>
<td>17 (0)</td>
<td>3 (1)</td>
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Table 4.3: Numbers of triangulation-wide “proper” NID exchange involutions, and combinations of particular toric divisors exchanged. (Note: the scan for NID involutions has not completed at the time of writing.)
Chapter 4. *Divisor Involutions and Free Actions on Toric Calabi-Yau Orientifolds* 159

% of Favorable Triangulations Scanned

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# of Triangulations with “Proper” NID Involutions

<table>
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Triangulations that Admit a “Proper” Involution with Particular NIDs

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<th>5</th>
<th>6</th>
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<td>24</td>
<td>542</td>
<td>6570</td>
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<td>0 (0)</td>
<td>0 (0)</td>
<td>40 (0)</td>
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**Table 4.4:** Numbers of triangulations that admit at least one “proper” NID exchange involution, and combinations of particular toric divisors exchanged. (Note: the scan for NID involutions has not completed at the time of writing.)
<table>
<thead>
<tr>
<th>h^{1,1}(X)</th>
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<td>806</td>
<td>3189</td>
<td>10053</td>
</tr>
</tbody>
</table>

**Geometry-wide Involutions with Particular NIDs**

| Del Pezzo surface dP_n, n ≤ 8 | 0 | 0 | 12 | 205 | 1337 | 5765 |
| Non-Shrinkable Rigid surface dP_n, n > 8 | 0 | 0 | 14 | 556 | 4517 | 13718 |
| (Exact-)Wilson surface | 0 (0) | 0 (0) | 10 (0) | 48 (7) | 93 (33) | 329 (190) |
| K3 surface | 0 | 0 | 121 | 858 | 1098 | 2149 |
| Special Deformation SD1 surface | 0 | 0 | 9 | 43 | 222 | 644 |
| Special Deformation SD2 surface | 0 | 36 | 10 | 45 | 107 | 252 |
| Del Pezzo and K3 | 0 | 0 | 0 | 6 | 96 | 404 |
| Del Pezzo and (Exact-)Wilson | 0 (0) | 0 (0) | 1 (0) | 19 (0) | 69 (16) | 281 (124) |
| K3 and (Exact-)Wilson | 0 (0) | 0 (0) | 12 (0) | 36 (8) | 17 (4) | 38 (2) |
| Del Pezzo, K3 and (Exact-)Wilson | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 13 (0) | 36 (1) |

**Table 4.5:** Numbers of geometry-wide NID exchange involutions, and combinations of particular toric divisors exchanged. (Note: the scan for NID involutions has not completed at the time of writing.)
### Table 4.6: Numbers of geometries that admit at least one consistent NID exchange involution, and combinations of particular toric divisors exchanged. (Note: the scan for NID involutions has not completed at the time of writing.)

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<td>99.99</td>
<td>99.96</td>
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<td># of Geometries with NID Involutions</td>
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Geometries that Admit an Involution with Particular NIDs

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<td>1 (0)</td>
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<td>7 (0)</td>
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### Chapter 4. Divisor Involutions and Free Actions on Toric Calabi-Yau Orientifolds

#### Table 4.7: Numbers of geometry-wide “proper” NID exchange involutions, and combinations of particular toric divisors exchanged. (Note: the scan for NID involutions has not completed at the time of writing.)

<table>
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<tr>
<th>( h^{1,1}(X) )</th>
<th>1</th>
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</tr>
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<tbody>
<tr>
<td>% of Favorable Triangulations Scanned</td>
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<td>100</td>
<td>100</td>
<td>100</td>
<td>99.99</td>
<td>99.96</td>
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<tr>
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<td>49</td>
<td>425</td>
<td>2196</td>
<td>8097</td>
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<table>
<thead>
<tr>
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<th>del Pezzo surface ( dP_n ), ( n \leq 8 )</th>
<th>0</th>
<th>0</th>
<th>12</th>
<th>168</th>
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<td>7 (0)</td>
<td>33 (5)</td>
<td>91 (32)</td>
<td>324 (190)</td>
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<td>370</td>
<td>428</td>
<td>917</td>
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<tr>
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<td>9</td>
<td>39</td>
<td>204</td>
<td>630</td>
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<tr>
<td>Special Deformation SD2 surface</td>
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<td>68 (16)</td>
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<td>K3 and (Exact-)Wilson</td>
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<td>17 (4)</td>
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<td>0 (0)</td>
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### Geometries that Admit a “Proper” Involutions with Particular NIDs

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<td></td>
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<td>100</td>
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<td>99.96</td>
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<td># of Geometries with</td>
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<td></td>
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</tr>
<tr>
<td>Non-Shrinkable</td>
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<td>dP$_n$, $n &gt; 8$</td>
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<tr>
<td>(Exact-)Wilson surface</td>
<td>(0)</td>
<td>(0)</td>
<td>7 (0)</td>
<td>33 (5)</td>
<td>90 (32)</td>
<td>324 (190)</td>
</tr>
<tr>
<td>K3 surface</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Special Deformation</td>
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<td></td>
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</tr>
<tr>
<td>SD1 surface</td>
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<td>0</td>
<td>5</td>
<td>33</td>
<td>193</td>
<td>625</td>
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<tr>
<td>Special Deformation</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SD2 surface</td>
<td>0</td>
<td>12</td>
<td>8</td>
<td>30</td>
<td>84</td>
<td>234</td>
</tr>
<tr>
<td>del Pezzo and K3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Exact-)Wilson</td>
<td>(0)</td>
<td>(0)</td>
<td>1 (0)</td>
<td>19 (0)</td>
<td>67 (16)</td>
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<td>K3 and (Exact-)Wilson</td>
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<td>(0)</td>
<td>6 (0)</td>
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<td>9 (2)</td>
<td>31 (2)</td>
</tr>
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<td>del Pezzo, K3 Wilson</td>
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<td>(0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
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**Table 4.8:** Numbers of geometries that admit at least one consistent, “proper” NID exchange involution, and combinations of particular toric divisors exchanged. (Note: the scan for NID involutions has not completed at the time of writing.)
4.3.2 Orientifold Planes

<table>
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<tr>
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<tbody>
<tr>
<td>% of Favorable Triangulations Scanned</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>99.99</td>
<td>99.96</td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Fixed Loci</td>
<td>100</td>
<td>100</td>
<td>97.81</td>
<td>96.34</td>
<td>98.71</td>
<td>55.38</td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Smoothness</td>
<td>100</td>
<td>100</td>
<td>65.03</td>
<td>90.1</td>
<td>15.27</td>
<td>1.28</td>
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</tbody>
</table>

Orientifolds from Triangulation-wide NID Exchange Involutions

<table>
<thead>
<tr>
<th># with Fixed Loci</th>
<th>O3</th>
<th>O5</th>
<th>O7</th>
<th>O3 and O5</th>
<th>O3 and O7</th>
<th>O5 and O7</th>
<th>O3, O5, and O7</th>
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</thead>
<tbody>
<tr>
<td></td>
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<td>0</td>
<td>29</td>
<td>2</td>
<td>29</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>12</td>
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<td>34</td>
<td>771</td>
<td>1590</td>
<td>561</td>
</tr>
<tr>
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<td>0</td>
<td>145</td>
<td>1369</td>
<td>12441</td>
<td>56148</td>
<td>1590</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>34</td>
<td>199</td>
<td>561</td>
<td>1758</td>
</tr>
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<td>414</td>
<td>21654</td>
<td>119393</td>
<td>1758</td>
<td>512</td>
</tr>
<tr>
<td># without Fixed Loci</td>
<td>Singular X</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>64</td>
<td>240</td>
<td>512</td>
</tr>
<tr>
<td>Free Action on X</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.9: Numbers of orientifold point-wise fixed loci and free actions for triangulation-wide NID involutions. (Note: the scans for NID involutions and orientifold fixed loci have not completed at the time of writing.)
### h^{1,1}(X)

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<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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<tbody>
<tr>
<td>% of Favorable Triangulations Scanned</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>99.99</td>
<td>99.96</td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Fixed Loci</td>
<td>100</td>
<td>100</td>
<td>97.81</td>
<td>96.34</td>
<td>98.71</td>
<td>55.38</td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Smoothness</td>
<td>100</td>
<td>100</td>
<td>65.03</td>
<td>90.1</td>
<td>15.27</td>
<td>1.28</td>
</tr>
</tbody>
</table>

### Orientifolds from Triangulations that Admit an NID Exchange Involution

<table>
<thead>
<tr>
<th># with Fixed Loci</th>
<th>O3</th>
<th>O5</th>
<th>O7</th>
<th>O3 and O5</th>
<th>O3 and O7</th>
<th>O5 and O7</th>
<th>O3, O5, and O7</th>
</tr>
</thead>
<tbody>
<tr>
<td>O3</td>
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<td>0</td>
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<td>8686</td>
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<td>30</td>
<td>364</td>
<td>2871</td>
<td>5017</td>
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<tr>
<td>O7</td>
<td>0</td>
<td>0</td>
<td>145</td>
<td>1276</td>
<td>10818</td>
<td>45139</td>
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<tr>
<td>O3 and O5</td>
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<td>0</td>
<td>2</td>
<td>17</td>
<td>167</td>
<td>236</td>
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<td>O3 and O7</td>
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<td>0</td>
<td>27</td>
<td>770</td>
<td>8483</td>
<td>36611</td>
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<td>0</td>
<td>2</td>
<td>24</td>
<td>116</td>
<td>283</td>
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<tr>
<td>O3, O5, and O7</td>
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<td>2</td>
<td>17</td>
<td>92</td>
<td>234</td>
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### # without Fixed Loci

<table>
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<th>Free Action on X</th>
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<td>6</td>
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<td>0</td>
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<tr>
<td>0</td>
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</table>

**Table 4.10:** Numbers of orientifold point-wise fixed loci and free actions for triangulations that admit at least one NID involution. (Note: the scans for NID involutions and orientifold fixed loci have not completed at the time of writing.)
<table>
<thead>
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</tr>
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<tr>
<td>% of Favorable Triangulations Scanned</td>
<td>100</td>
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<td>100</td>
<td>100</td>
<td>99.99</td>
<td>99.96</td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Fixed Loci</td>
<td>100</td>
<td>100</td>
<td>97.81</td>
<td>96.34</td>
<td>98.71</td>
<td>55.38</td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Smoothness</td>
<td>100</td>
<td>100</td>
<td>65.03</td>
<td>90.1</td>
<td>15.27</td>
<td>1.28</td>
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</tbody>
</table>

**Orientifolds from Triangulation-wide “Proper” NID Exchange Involutions**

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<th>23</th>
<th>1203</th>
<th>19736</th>
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<td>861</td>
<td>9733</td>
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<td>0</td>
<td>33</td>
<td>0</td>
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<td>O3 and O7</td>
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<tr>
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<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>O3, O5, and O7</td>
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<td>0</td>
<td>0</td>
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<table>
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<th>0</th>
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Table 4.11: Numbers of orientifold point-wise fixed loci and free actions for triangulation-wide “proper” NID involutions. (Note: the scans for NID involutions and orientifold fixed loci have not completed at the time of writing.)
### Table 4.12:

Numbers of orientifold point-wise fixed loci and free actions for triangulations that admit at least one “proper” NID involution. (Note: the scans for NID involutions and orientifold fixed loci have not completed at the time of writing.)

<table>
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<th>3</th>
<th>4</th>
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<tbody>
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<td>% of Favorable Triangulations Scanned</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>99.99</td>
<td>99.96</td>
<td></td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Fixed Loci</td>
<td>100</td>
<td>100</td>
<td>97.81</td>
<td>96.34</td>
<td>98.71</td>
<td>55.38</td>
<td></td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Smoothness</td>
<td>100</td>
<td>100</td>
<td>65.03</td>
<td>90.1</td>
<td>15.27</td>
<td>1.28</td>
<td></td>
</tr>
<tr>
<td>Orientifolds from Triangulations that Admit a “Proper” NID Exchange Involution</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>220</td>
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<td>824</td>
<td>8937</td>
<td>40082</td>
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<tr>
<td></td>
<td>O3 and O5</td>
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<td>0</td>
<td>0</td>
<td>33</td>
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<tr>
<td></td>
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<td>23</td>
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<td>7697</td>
<td>33871</td>
</tr>
<tr>
<td></td>
<td>O5 and O7</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>O3, O5, and O7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td># without Fixed Loci</td>
<td>Singular X</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>42</td>
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<tr>
<td></td>
<td>Free Action on X</td>
<td>0</td>
<td>0</td>
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</tr>
</tbody>
</table>
## Chapter 4. Divisor Involutions and Free Actions on Toric Calabi-Yau Orientifolds

<table>
<thead>
<tr>
<th>h^{1,1}(X)</th>
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<th>3</th>
<th>4</th>
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<th>6</th>
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<tbody>
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<td>% of Favorable Triangulations Scanned</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>99.99</td>
<td>99.96</td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Fixed Loci</td>
<td>100</td>
<td>100</td>
<td>97.81</td>
<td>96.34</td>
<td>98.71</td>
<td>55.38</td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Smoothness</td>
<td>100</td>
<td>100</td>
<td>65.03</td>
<td>90.1</td>
<td>15.27</td>
<td>1.28</td>
</tr>
</tbody>
</table>

### Orientifolds from Geometry-wide NID Exchange Involutions

<table>
<thead>
<tr>
<th># with Fixed Loci</th>
<th>O3</th>
<th>O5</th>
<th>O7</th>
<th>O3 and O5</th>
<th>O3 and O7</th>
<th>O5 and O7</th>
<th>O3, O5, and O7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0</td>
<td>8</td>
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<td>0</td>
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<td>12</td>
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<td>433</td>
<td>307</td>
<td>22</td>
<td>14</td>
<td>2</td>
</tr>
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<td>0</td>
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<td>538</td>
<td>3215</td>
<td>16559</td>
<td>14</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># without Fixed Loci</th>
<th>Singular X</th>
<th>Free Action on X</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>0</td>
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<tr>
<td>0</td>
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<tr>
<td>0</td>
<td>62</td>
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<tr>
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<td>137</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.13: Numbers of orientifold point-wise fixed loci and free actions for geometry-wide NID involutions. (Note: the scans for NID involutions and orientifold fixed loci have not completed at the time of writing.)
% of Favorable Triangulations Scanned

<table>
<thead>
<tr>
<th>$h^{1,1}(X)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>99.99</td>
<td>99.96</td>
</tr>
</tbody>
</table>

% of NID Involutions Scanned for Fixed Loci

<table>
<thead>
<tr>
<th></th>
<th>100</th>
<th>100</th>
<th>97.81</th>
<th>96.34</th>
<th>98.71</th>
<th>55.38</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

% of NID Involutions Scanned for Smoothness

<table>
<thead>
<tr>
<th></th>
<th>100</th>
<th>100</th>
<th>65.03</th>
<th>90.1</th>
<th>15.27</th>
<th>1.28</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Orientifolds from Geometries that Admit an NID Exchange Involution

<table>
<thead>
<tr>
<th># with Fixed Loci</th>
<th>O3</th>
<th>O5</th>
<th>O7</th>
<th>O3 and O5</th>
<th>O3 and O7</th>
<th>O5 and O7</th>
<th>O3, O5, and O7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>12</td>
<td>21</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>61</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
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<td>0</td>
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<td></td>
<td>41</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>165</td>
<td>1282</td>
<td>4876</td>
</tr>
<tr>
<td></td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>2</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>0</td>
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<td>0</td>
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<td>2</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># without Fixed Loci</th>
<th>Singular X</th>
<th>Free Action on X</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.14: Numbers of orientifold point-wise fixed loci and free actions for geometries that admit at least one consistent NID involution. (Note: the scans for NID involutions and orientifold fixed loci have not completed at the time of writing.)
% of Favorable Triangulations Scanned  
100 100 100 100 99.99 99.96

% of NID Involutions Scanned for Fixed Loci  
100 100 97.81 96.34 98.71 55.38

% of NID Involutions Scanned for Smoothness  
100 100 65.03 90.1 15.27 1.28

<table>
<thead>
<tr>
<th># with Fixed Loci</th>
<th>O3</th>
<th>O5</th>
<th>O7</th>
<th>O3 and O5</th>
<th>O3 and O7</th>
<th>O5 and O7</th>
<th>O3, O5, and O7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>338</td>
<td>1740</td>
<td>221</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>12</td>
<td>38</td>
<td>229</td>
<td>7239</td>
<td>2886</td>
<td>14883</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>29</td>
<td>1740</td>
<td>2886</td>
<td>14883</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>338</td>
<td>1740</td>
<td>221</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>12</td>
<td>38</td>
<td>229</td>
<td>7239</td>
<td>2886</td>
<td>14883</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>29</td>
<td>1740</td>
<td>2886</td>
<td>14883</td>
<td>0</td>
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<tr>
<td></td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># without Fixed Loci</th>
<th>Singular X</th>
<th>Free Action on X</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 0 0 0 0 0</td>
<td>0 0 0 0 0 0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Orientifolds from Geometry-wide “Proper” NID Exchange Involutions</th>
</tr>
</thead>
</table>

Table 4.15: Numbers of orientifold point-wise fixed loci and free actions for geometry-wide “proper” NID involutions. (Note: the scans for NID involutions and orientifold fixed loci have not completed at the time of writing.)
## Chapter 4: Divisor Involutions and Free Actions on Toric Calabi-Yau Orientifolds

### Table 4.16: Numbers of orientifold point-wise fixed loci and free actions for geometries that admit at least one consistent, “proper” NID involution. (Note: the scans for NID involutions and orientifold fixed loci have not completed at the time of writing.)

<table>
<thead>
<tr>
<th>h^{1,1}(X)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>% of Favorable Triangulations Scanned</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>99.99</td>
<td>99.96</td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Fixed Loci</td>
<td>100</td>
<td>100</td>
<td>97.81</td>
<td>96.34</td>
<td>98.71</td>
<td>55.38</td>
</tr>
<tr>
<td>% of NID Involutions Scanned for Smoothness</td>
<td>100</td>
<td>100</td>
<td>65.03</td>
<td>90.1</td>
<td>15.27</td>
<td>1.28</td>
</tr>
</tbody>
</table>

| Orientifolds from Geometries that Admit a “Proper” NID Exchange Involution |
|-----------------------------|----------|----------|----------|----------|----------|
| # with Fixed Loci | O3 | 0 | 0 | 8 | 175 | 1179 | 4487 |
|                   | O5 | 0 | 12 | 19 | 108 | 669 | 808 |
|                   | O7 | 0 | 0 | 29 | 234 | 1463 | 5398 |
|                   | O3 and O5 | 0 | 0 | 0 | 0 | 1 | 0 |
|                   | O3 and O7 | 0 | 0 | 8 | 144 | 1147 | 4474 |
|                   | O5 and O7 | 0 | 0 | 0 | 0 | 0 | 0 |
|                   | O3, O5, and O7 | 0 | 0 | 0 | 0 | 0 | 0 |
| # without Fixed Loci | Singular X | 0 | 0 | 0 | 0 | 0 | 0 |
|                   | Free Action on X | 0 | 0 | 0 | 0 | 0 | 0 |
### 4.3.3 Hodge Splitting

<table>
<thead>
<tr>
<th>$h^{1,1}(X)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>% of Favorable Triangulations Scanned</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>99.99</td>
<td>99.96</td>
</tr>
<tr>
<td># of Triangulation-wide (“Proper”) NID Involutions</td>
<td>0 (0)</td>
<td>12 (12)</td>
<td>183 (125)</td>
<td>1778 (1142)</td>
<td>14123 (11630)</td>
<td>91075 (83170)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># with $h^{1,1}$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangulations that Admit (“Proper”) NID Involutions with Some Odd Parity</td>
<td>–</td>
<td>12 (12)</td>
<td>179 (123)</td>
<td>1613 (1043)</td>
<td>11662 (9688)</td>
<td>41057 (36129)</td>
</tr>
<tr>
<td>2</td>
<td>–</td>
<td>–</td>
<td>0 (0)</td>
<td>100 (34)</td>
<td>2229 (1766)</td>
<td>8858 (7850)</td>
</tr>
<tr>
<td>3</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0 (0)</td>
<td>50 (0)</td>
<td>525 (418)</td>
</tr>
<tr>
<td>4</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0 (0)</td>
<td>1 (0)</td>
</tr>
<tr>
<td>5</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0 (0)</td>
</tr>
</tbody>
</table>

**Table 4.17:** Numbers of triangulations that admit at least one (“proper”) NID orientifold involution with a particular $h^{1,1}(X/\sigma^*)$ splitting. (Note: the scan for NID involutions has not completed at the time of writing.)
### Chapter 4. Divisor Involutions and Free Actions on Toric Calabi-Yau Orientifolds

#### Table 4.18: Numbers of geometries that admit at least one consistent ("proper") NID orientifold involution with a particular $h^{1,1}(X/\sigma^*)$ splitting. (Note: the scan for NID involutions has not completed at the time of writing.)

<table>
<thead>
<tr>
<th>$h^{1,1}(X)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>% of Favorable Triangulations Scanned</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>99.99</td>
<td>99.96</td>
</tr>
<tr>
<td># of Geometry-wide (&quot;Proper&quot;) NID Involutions</td>
<td>0 (0)</td>
<td>12 (12)</td>
<td>86 (49)</td>
<td>806 (425)</td>
<td>3189 (2196)</td>
<td>10053 (8097)</td>
</tr>
<tr>
<td>Geometries that Admit (&quot;Proper&quot;) NID Involutions with Some Odd Parity</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># with $h^{1,1}$</td>
<td>1</td>
<td>–</td>
<td>12 (12)</td>
<td>85 (48)</td>
<td>705 (349)</td>
<td>2574 (1843)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>–</td>
<td>–</td>
<td>0 (0)</td>
<td>49 (24)</td>
<td>536 (320)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0 (0)</td>
<td>42 (0)</td>
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<tr>
<td></td>
<td>4</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0 (0)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
Chapter 5

Conclusion

The work described in this thesis was motivated primarily by the desire of the string theory and phenomenology communities for a large, diverse, and concrete sample of the string vacuum landscape in an easy-to-access form, as a laboratory in which to advance the current state of research towards more realistic and complete models of our universe.

The traditional tool, PALP (Package for Analyzing Lattice Polytopes), for approaching the largest such set Calabi-Yau string vacua, the Kreuzer-Skarke database [33], is a wonderful resource, but it only goes so far in providing physicists with the information they need.

As an example of this limitation, PALP is unable to compute a secondary polytope in more than three dimensions. That is, the triangulation algorithm is coded in such a way as to be limited to polytopes with no more than three points interior to the facets of $\Delta^*$. As a result, PALP’s ability to triangulate the Kreuzer-Skarke dataset is limited to only the smaller reflexive polytopes. In fact, 377 of the 4,990 reflexive polytopes with $h^{1,1} = 5$,
or 7.6%, could not even be triangulated with PALP. At \( h^{1,1} = 6 \) the fraction of polytopes which could not be triangulated grew to 23.4% (4,007 out of 17,101). No polytopes with \( h^{1,1} > 6 \), which can be triangulated using PALP, have been identified.

In addition, and perhaps more importantly, the derivation of the data that a physicist may want from the Kreuzer-Skarke list is a computationally intensive task. The computations in this work were performed on dual Intel E5 2650 CPUs with 128GB of RAM per node over the course of 6 years using resources at the Massachusetts Green High Performance Computing Center. It is a waste of resources for every group interested in such topics to be forced to recompute these results in isolation. To this end, the online repository located at www.rossealtman.com, compiled from the Kreuzer-Skarke database is an excellent resource, which I hope will continue to be utilized effectively, and contributed to, by the string theory community. With so many of us working to refine this data set from its toric geometry foundation toward high level phenomenology and low-energy dynamics, this sector of the string landscape may become an excellent lamppost under which to identify more general features of the space of vacua.

The methods described in Chapter 2 were applied to the 23,573 polytopes with \( h^{1,1} \leq 6 \). The number of independent triangulations and glued Calabi-Yau geometries obtained for each value of \( h^{1,1} \) is collected in Table 2.2, while statistics on the types of toric divisors on these geometries are are given in Tables 2.3-2.6.

The total number of triangulations performed was 653,062, resulting in 101,681 unique
CY threefolds. Of these, 100,368 are favorable cases, and are thus amenable to meaningful phenomenological study. Where results could be obtained using PALP 2.1, we find agreement between the Sage implementation and the output from PALP.

Work is currently underway to extend these results to $h^{1,1} = 7$ and beyond. As the database expands, the newly-computed Calabi-Yau data will be appended to that already hosted on the web-based repository [34].

In Chapter 3, the case was made for moduli stabilization in the context of the LARGE Volume Scenario (LVS), in particular for Calabi-Yau vacuum configurations satisfying the so-called Swiss cheese form of the volume. The special toric case, for which the problem of identifying such a solution becomes purely combinatorical, was presented in detail in Section 3.1.5 along with an explicit algorithm for performing this scan in Section 3.2.

In total, 2,313 toric Swiss cheese manifolds were identified, distributed as shown in Table 3.6. While these numbers represent only a subset of all the possible Swiss cheese manifolds that may exist in this dataset, they represent an ample starting point for phenomenological investigation with nearly 160 explicit examples of guaranteed moduli stabilization using the techniques of [28]. A concrete example of this was explored in Section 3.5.

More general techniques that do not rely on the ability to trivialize the unknown basis change (see Equation (3.40)), such as those employed in [67] are far more computationally expensive. Nevertheless, we expect such cases to form the majority of all manifolds which admit a large volume limit, particularly for higher values of $h^{1,1}$ and/or greater numbers
of large cycles. A full analysis, extending the results of [67], is currently underway.

The $\sim 2,000$ cases with $h^{1,1} > 4$ are, to our knowledge, unknown before now. Such cases were not approachable using the PALP software [39], and required the redesigned techniques described in Chapter 2. These high Picard number cases include 29 cases with two or more large cycles. These are of particular interest for the possibility of identifying flat directions which may be relevant for inflationary cosmology [119–122].

The second stage of the analysis involved identifying directly which favorable threefolds can be rotated into a basis for which the volume can be written explicitly in terms of 4-cycle volumes. Given the difficulty of studying the dynamics of Kähler moduli in the 4-dimensional effective supergravity theory when the Calabi-Yau volume cannot be put into a fully-explicit or (preferably) strong form, we also flag those cases for which a suitable basis exists, and provide the necessary rotation matrix ($T^C$ in Section 3.3). We expect these results to be expanded significantly in the future as the more general case is completed. While this brute force attempt was performed over the entire dataset of Calabi-Yau threefolds, the identification of the components of the basis matrix involves solving a large system of polynomials using Groebner basis techniques. As such, this stage is computationally expensive, and both the CPU time and physical memory required to find a solution for any given example can be hard to predict. The results of the second stage in the analysis are found in Table 3.6, and we find a total of $109$ usable cases in $h^{1,1} \leq 3$. 

In Chapter 4, we expanded upon the phenomenological study of our database of Calabi-Yau threefolds, by asking for the existence of a holomorphic $\mathbb{Z}_2$ orientifold involution $\sigma$, which, in conjunction with the world-sheet parity transformation, projects out a gravitino from the theory, thereby reducing the $\mathcal{N} = 2$ supergravity to $\mathcal{N} = 1$ in the low energy effective theory. By requiring that the pullback of $\sigma$ to cohomology classes exchange only toric divisors with identical surface topology, but separate cohomology on the Calabi-Yau, we ensure that the orientifold has non-trivial odd equivariant cohomology $h^{1,1}_-(X/\sigma^*) \neq 0$. By enforcing the requirement of such non-trivial, identical divisors (NIDs), we further ensure the existence of some mechanism for protecting a moduli-stabilizing non-perturbative superpotential (as discussed in Chapter 3) from being spoiled by charged zero modes originating from the intersection of D7 branes with the non-perturbative Euclidean D3-instanton branes. In Section 4.1.3, we outline the procedure for obtaining the desired NID involutions, and present the results of this scan in Tables 4.1-4.8, which include an analysis of particular classes of exchanged divisors. In total, we find that there are 107,171 such involutions present, 96,079 of which are “proper” in the sense that they preserve the intersection structure of $X$. We also find that 10,779 favorable CY geometries admit a consistent involution.

The fixed points of the involution correspond to orientifold planes (primarily O3 and O7 planes), which acquire charges that must be cancelled by appropriate configurations of D3 and D7 branes in order to avoid anomalies in our theory. We scan for these orientifold planes in Section 4.1.4 by seeking the fixed points in the ambient toric manifold
and restricting down to components transversal with the involution-invariant part of the Calabi-Yau hypersurface. The results of this scan can be found in Tables 4.9-4.16. Note than in all but one case, the fixed sets allowed by a “proper” involution across an entire consistent CY geometry are either individual O3, O5, or O7 planes, or combinations of O3 and O7 planes with a dimension 4 separation. Only in one case do we obtain an O3 and O5 with dimension 2 separation. In addition, we enumerate the 822 NID involutions for which there are no fixed point sets on the Calabi-Yau hypersurface. However, because none of these hypersurfaces appear to be smooth, the involution cannot be regarded simply as a freely acting $Z_2$.

Finally, in Section 4.1.5, we discuss the decomposition of the Kähler moduli space into odd and even parity equivariant cohomology $H^{1,1}(X/\sigma^*) = H^{1,1}_+(X/\sigma^*) \oplus H^{1,1}_-(X/\sigma^*)$. The constraint that the Kähler form must be invariant $\sigma^*J = J$ ensures that we can always find the dimension of the even parity space, and then by deduction, the dimension of the odd party space $h^{1,1}_-(X/\sigma^*)$, which as has been discussed, must be non-trivial in our case. The results of this Kähler moduli space splitting can be found in Tables 4.17 and 4.18. Several examples of the entire procedure are explicitly performed in Section 4.2.

The most significant result of this thesis, from a big picture point of view, is the systematic progression from the raw, abstract polyhedra of the Kreuzer-Skarke dataset to actual phenomenology with real physical implications. This procedure has led to the generation of a vast amount of valuable and informative data about each possible vacuum configuration at many different levels of refinement. This data, all stored and accessible
through a robust and user-friendly search engine at www.rossealtman.com, is open and available to all members of the string theory community in the hope that it will serve as a resource and a laboratory in which to help further elucidate the complexities of this ever-evolving discipline.
Appendix A

Mathematical Background

The mathematical tools needed to construct proper, smooth Calabi-Yau threefolds from the vertices of a reflexive polytope are varied, and some of the knowledge is both technical and quite specialized to the problem at hand. Many mathematical physicists will be familiar with at least some of the terms, concepts and procedures. A few will be expert in all of them. But a great many formal theorists and string phenomenologists will know almost none of this background material. A number of very good textbooks and review articles exist which cover many of these subjects in far greater detail – and with more mathematical rigor – than we will present here (see [79, 81, 92, 123–131]). But our own entry into this subject has revealed that no one review or text covers all of the process, from beginning to end. Nor do these sources uniformly provide recipes for actually computing certain quantities of interest. Hopefully the following will, at least partially, rectify that situation.

A.1 Why Calabi-Yau?

It is well known that in a superstring theory, Lorentz invariance of the massless string states demands a critical dimension of 10. However, direct observation tells us that a
physically consistent formulation must reduce to 4 dimensions in the low energy effective theory. In the spirit of Kaluza and Klein, we consider the naturally symmetry-breaking procedure of deforming the 6 extra dimensions into a compact Riemannian manifold $X$ with metric $g$. In this paradigm, the extra dimensions can only be probed directly at energies inversely proportional to the compactification radius, and perhaps not at all if the visible sector is localized on the compactification manifold. In this latter case, the extra dimensions may be taken to be large.

**A.1.1 Supersymmetric Constraints**

We must take care that 4-dimensional Poincaré symmetry is not one of those broken by compactification [123]. This implies the vanishing of all supersymmetric variations in the vacuum, which turns out to be a very stringent requirement. The vanishing of bosonic variations, which are directly proportional to fermion fields $\delta \phi \sim \psi$, is trivial by Poincaré symmetry, but the vanishing of fermionic variations imposes additional structure on $X$. A fermionic field on a 6-dimensional manifold $X$ can be thought of as a spinor in the fundamental (Weyl) representation $4$ of Spin(6) $\cong SU(4)$, and the fact that its supersymmetric variation vanishes implies that there exists some non-zero spinor $\psi \in 4$ on $X$ which is covariantly constant with respect to the Levi-Civita connection $\nabla$ on $X$. That is, $\nabla \psi = 0$.

**A.1.2 Complex Structure**

Consider the Noether current $I_{\mu}^{\nu} = i \psi \Gamma_{\mu}^{\nu} \psi$, where $\Gamma_{\mu}^{\nu} = \frac{1}{2} (\Gamma_{\mu}^{\nu} \Gamma^{\nu} - \Gamma^{\nu} \Gamma_{\mu})$ is the fully anti-symmetrized product of 6-dimensional Dirac gamma matrices. With some work, it can be shown that $I^2 = -1$ and therefore defines an *almost complex structure* on $X$. $I$ assigns to the tangent space $T_pX$ at a point $p \in X$ the structure of a complex vector space, with eigenvalues $\pm i$ and corresponding eigenspaces defined as the holomorphic and anti-holomorphic tangent spaces $T_p^{(1,0)}X$ and $T_p^{(0,1)}X$ respectively. We can summarize this
by decomposing $I$ into (anti-)holomorphic indices: $I^\nu_\mu = I_a^b + I_{\bar{a}}{}^\bar{b}$, where $I_a^b = i\delta_a^b$ and $I_{\bar{a}}{}^\bar{b} = -i\delta_{\bar{a}}{}^{\bar{b}}$. However, this (anti-)holomorphic structure does not necessarily extend to an open neighborhood around $p$, and in fact will only do so if $T_p^{(1,0)}X$ and $T_p^{(0,1)}X$ are each closed under the Lie brackets preserving their eigenstructures. This is equivalent to a vanishing anti-holomorphic projection of the holomorphic Lie bracket. This projection is called the Nijenhuis tensor and can be expressed as

$$N^\rho_\nu_\mu = I^\rho_\sigma \nabla_\sigma I^\nu_\mu - I^\rho_\nu \nabla_\nu I^\sigma_\mu - I^\nu_\sigma \nabla_\sigma I^\rho_\mu + I^\nu_\mu \nabla_\nu I^\rho_\sigma \quad (A.1)$$

Since $\nabla \psi = 0$, we deduce that $\nabla I = 0$ and therefore $N = 0$. Thus, we have shown that we can choose local holomorphic coordinates for each point in $X$, which can then be patched together into a holomorphic atlas for $X$. $I$ is said to be integrable and the almost complex structure of $X$ is promoted to a complex structure [123,124]. This structure is analogous to that imposed by the Cauchy-Riemann equations in the complex plane.

### A.1.3 Symplectic and Kähler Structure

By considering the relation between the metric $g_{\mu\nu}$, the Dirac gamma matrices $\Gamma_\mu$, and the complex structure tensor $I^\nu_\mu$, it can be shown that $I^\rho_\mu I^\nu_\sigma g_{\mu\nu} = g_{\rho\sigma}$ and therefore that $g_{\mu\nu}$ is Hermitian with respect to the complex structure on $X$. But, recalling that $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ is a bilinear form, we see that one argument must be holomorphic with eigenvalue $i$ and the other anti-holomorphic with eigenvalue $-i$ with respect to $I$. Again, we can summarize this by writing $g$ in (anti-)holomorphic coordinates $g_{\mu\nu} = g_{ab} + g_{\bar{a}\bar{b}}$ with $g_{ab} = g_{\bar{a}\bar{b}} = 0$.

Any complex manifold with Hermitian metric is also endowed with a (1,1)-form defined by $J_{\mu\nu} = I^e_\mu g_{e\nu}$. Decomposing into (anti-)holomorphic indices, we find $J_{\mu\nu} = I^e_a g_{eb} + I^e_{\bar{a}} g_{\bar{b}} = i(g_{ab} - g_{\bar{b}\bar{a}}) = J_{ab} + J_{\bar{a}\bar{b}}$. In our construction, both the complex structure tensor $I$ and the Hermitian metric $g$ are covariantly constant with respect to the Levi-Civita
Appendix A. Mathematical Background

connection. Therefore, $\nabla J = 0$, so that $J$ is a closed (1,1)-form and $X$ is shown to have symplectic structure as well. A manifold with consistent complex, Riemannian, and symplectic structure is called a Kähler manifold, with $J$ its Kähler (1,1)-form, and $g$ its associated Kähler metric [123,124].

A.1.4 Calabi-Yau Structure

The covariantly constant spinor $\psi$ also places requirements on the curvature of $X$. The Riemann curvature tensor is defined by

$$[\nabla_\mu, \nabla_\nu]v^\rho = R^\rho_{\mu\nu\sigma} v^\sigma \quad (A.2)$$

with $R^\rho_{\mu\nu\sigma}$ taking values in a vector representation of the Lie algebra of the Lorentz group, restricted to $X$. The Dirac spinor representation of the Lorentz group is related to this vector representation by $\Lambda^\rho_\sigma \mapsto \frac{1}{4} \Lambda^\rho_\sigma \Gamma^\sigma_\rho = \frac{1}{4} \Lambda^\rho_\sigma \Gamma^\rho_\sigma$. Then the spin curvature tensor $R_{\mu\nu}$ can be defined in terms of the Riemann curvature tensor and we find

$$[\nabla_\mu, \nabla_\nu] \psi = R_{\mu\nu} \psi = \frac{1}{4} R_{\mu\nu\rho\sigma} \Gamma^\rho_\sigma \psi = 0$$

with the second equality resulting from the fact that $\nabla \psi = 0$. Using the anticommutation relations of the 6-dimensional Dirac gamma matrices and the first Bianchi identity $(R_{\mu\nu\rho} + R_{\mu\sigma\nu} + R_{\mu\rho\sigma} = 0)$, we find that

$$0 = \frac{1}{4} R_{\mu\nu\rho\sigma} \Gamma^\rho_\sigma \Gamma^\nu_\sigma = \frac{1}{12} R_{\mu\rho} \Gamma^\rho_\sigma$$ \quad (A.3)$$

where $R_{\mu\rho}$ is the Ricci tensor on $X$. Thus, $X$ is a Ricci-flat Kähler manifold. We can decompose $R_{\mu\rho}$ into its (anti-)holomorphic components by considering the fact that since $\nabla I = 0$, we must have $[\nabla_\mu, \nabla_\nu](Iv)^\rho = I^\rho_\gamma [\nabla_\mu, \nabla_\nu]v^\gamma$. Therefore, $I$ commutes with the
Appendix A. Mathematical Background

Riemann curvature tensor. Making use of the interchange symmetry \( R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \), we find

\[
I^\sigma_{\tau} g^{\rho\alpha} R_{\mu\nu\alpha\sigma} = I^\rho_{\gamma} g^{\gamma\beta} R_{\mu\nu\beta\tau}
\]  
(A.4)

From this, we find that in (anti-)holomorphic indices, the Ricci tensor is \( R_{\mu\nu} = R_{ab} + R_{\bar{a}\bar{b}} \).

Switching gears for a moment, we define the \( k \)-th Chern form of a complex vector bundle \( V \) of rank \( n \) over \( X \) is defined by the characteristic polynomial

\[
\det \left( 1 + \frac{i}{2\pi} F t \right) = \sum_k c_k(V) t^k
\]  
(A.5)

where \( F \) is the curvature 2-form of a connection on \( V \). From this, we see that \( c_1(V) = \frac{i}{2\pi} \text{Tr}(F) \). However, since \( F \) is a 2-form, each \( c_k \in H^{2k}(X, \mathbb{R}) \). Therefore, \( c_1 \) is actually a representative of a Chern class \( c_1(V) = \left[ \frac{i}{2\pi} \text{Tr}(F) \right] \). Furthermore, the Ambrose–Singer theorem tells us that the curvature form \( F \) on a vector bundle can be exponentiated to an element of its holonomy group. Thus, the cohomology class of \( \frac{i}{2\pi} \text{Tr}(F) \) must be integer-valued, so \( c_1 \in H^2(X, \mathbb{Z}) \). The Levi-Civita connection \( \nabla \) on \( TX \) has a curvature 2-form given by the Ricci tensor. Then (using holomorphic indices), we find

\[
c_1(TX) = \left[ \frac{i}{2\pi} R_{ab} dz^a \wedge d\bar{z}^b \right]
\]

\[= 0\]

with the second equality resulting from Ricci-flatness. It is clear then that \( c_1(TX) = 0 \) is a necessary condition for Ricci-flatness, but is it also sufficient? Eugenio Calabi conjectured in 1957 that the answer was yes, and the proof was given 20 years later by Shing-Tung Yau. Given \( c_1(TX) = 0 \), then there is a unique Kähler form in the same class with vanishing Ricci curvature. Thus, \( c_1(TX) = 0 \) is known as the Calabi-Yau condition and \( X \) is a Calabi-Yau threefold [123,124].
A.1.5 Calabi-Yau Structure from Holonomy

We can also learn something about $X$ by looking at its holonomy group. The holonomy group $\text{Hol}_x(\nabla)$ generated by the Levi-Civita connection $\nabla$ is the group of all invertible linear maps $P_\gamma : T_x X \to T_x X$ that are equivalent to parallel transporting some tangent vector around a closed loop $\gamma$ beginning and ending at $x \in X$. Because these maps are linear and invertible, we have $\text{Hol}_x(\nabla) \subset \text{GL}(T_x X) = \text{GL}(6, \mathbb{R})$.

Because $\nabla$ is a metric connection on $X$, the holonomy group must preserve vector lengths. Furthermore, we have shown that $X$ is Ricci-flat, and therefore orientable, so $\text{Hol}_x(\nabla) \subset \text{SO}(6)$. Consider the Christoffel symbols defining the Levi-Civita connection $\nabla$

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} \sum_r g^{\mu\sigma} (\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\sigma\rho} + \partial_\sigma g_{\nu\rho}) \quad (A.6)$$

Because the metric $g$ is Hermitian, the Christoffel symbols have no mixed holomorphic/anti-holomorphic indices. This guarantees that a holomorphic vector will remain holomorphic under parallel transport, and therefore the complex structure is respected by the holonomy and we can restrict the holonomy group further: $\text{Hol}_x(\nabla) \subset \text{U}(3)$.

Now, we can write down an arbitrary tangent vector $V = V^\mu \partial_\mu \in T_x X$. We parallel transport $V$ around an infinitesimal parallelogram with edges parallel to $\partial_\mu$ and $\partial_\nu$ and area equal to $\delta a^{\mu\nu}$. A common result in differential geometry is the following

$$V^\rho \mapsto V'^\rho = (\delta^\rho_\sigma + \delta a^{\mu\nu} R_{\mu\nu \rho \sigma}) V^\sigma \quad (A.7)$$

From this, we can clearly see that $\delta a^{\mu\nu} R_{\mu\nu \rho \sigma}$ is a skew-Hermitian matrix in the Lie algebra $\mathfrak{u}(3)$. In a neighborhood of the identity, such a matrix can be decomposed into a traceless skew-Hermitian matrix and a pure imaginary number equal to the trace, or $\mathfrak{u}(3) \cong \mathfrak{su}(3) \oplus i \mathbb{R} = \mathfrak{su}(3) \oplus \mathfrak{u}(1)$. The trace is equal to $\delta a^{\mu\nu} R_{\mu\nu \rho} = -4 \delta a^{\mu\nu} R_{\mu\nu}$, where $R_{\mu\nu}$ is the
Ricci curvature tensor. We saw earlier that $X$ is Ricci-flat, and so the $u(1)$ component of $u(3)$ is trivial and the holonomy group of $X$ is reduced further to $\text{Hol}_x(\nabla) \subset \text{SU}(3)$. It turns out that the converse is true as well, and this fact can be used to determine whether a Kähler manifold is Ricci-flat, and therefore Calabi-Yau [123,124].

A.1.6 Unique Holomorphic Volume Form

The Lie algebra decomposition $u(3) \cong su(3) \oplus u(1)$ near the identity lifts to the decomposition of Lie groups $U(3) = SU(3) \times U(1)$. Since $u(1)$ has the form of a trace of $u(3)$, $U(1)$ must have the form of a determinant of $U(3)$. Vectors in the holomorphic cotangent bundle $T^{(1,0)} X$ transform under $U(3)$, so elements of the determinant bundle of $T^{(1,0)} X$ must transform under $U(1)$. The determinant bundle, or top exterior power of the holomorphic cotangent bundle $\omega_X = \Lambda^3 T^{(1,0)} X$, is called the canonical bundle and it is a line bundle (i.e. rank 1), which transforms as the determinant $dz^1 \wedge dz^2 \wedge dz^3$.

Because the $u(1)$ component of the Lie algebra is trivial for a Calabi-Yau manifold, the holonomy group restricted to $\omega_X$ is trivial. Thus, all holomorphic $(3,0)$-forms of $\omega_X$ must be parallel, and we can therefore normalize and write down a unique holomorphic $(3,0)$-form (or volume form) $\Omega^{(3,0)}_{\mu \nu \rho}$, which can also be interpreted as a unique, global holomorphic section of $\omega_X$. This means that $\Omega^{(3,0)}_{\mu \nu \rho}$ is covariantly constant with respect to the Levi-Civita connection $\nabla$. So, we can again use our covariantly constant spinor $\psi$ to define it

$$\Omega^{(3,0)}_{\mu \nu \rho} = \psi^\dagger \Gamma_{\mu \nu \rho} \psi$$ \hspace{1cm} (A.8)

where

$$\Gamma_{\mu_1 \mu_2 \mu_3} = \frac{1}{3!} \sum_{\sigma \in S_3} \epsilon^{\mu_\sigma(1) \mu_\sigma(2) \mu_\sigma(3)} \Gamma_{\mu_\sigma(1)} \Gamma_{\mu_\sigma(2)} \Gamma_{\mu_\sigma(3)}$$ \hspace{1cm} (A.9)

is the fully anti-symmetrized product of 6-dimensional Dirac gamma matrices [123,132].
A.2 Topological Structure

Because the holomorphic volume form is covariantly constant, it is a closed \((3,0)\)-form, and therefore \(\Omega_{\mu\nu\rho}^{(3,0)} \in H^{(3,0)}(X)\). We can begin to calculate the Hodge numbers on \(X\) by recognizing that since \(\Omega_{\mu\nu\rho}^{(3,0)}\) is unique, \(h^{(3,0)}(X) = \dim H^{(3,0)}(X) = 1\). We now consider various dualities that are active on a Calabi-Yau manifold \(X\):

- Conjugation duality defined on a complex manifold requires \(h^{(p,q)} = h^{(q,p)}\).
- Hodge star duality on a threefold tells us that \(h^{(p,q)} = h^{(3-q,3-p)}\).
- For \(\Omega := \Omega_{\mu\nu\rho}^{(3,0)} \in H^{(3,0)}(X)\) and given \([\alpha] \in H^{(0,q)}(X)\), there is a unique \([\beta] \in H^{(0,3-q)}(X)\) such that \(\int_X \alpha \wedge \beta \wedge \Omega = 1\).

Taking these dualities into account, we find the following Hodge diamond for \(X\):

\[
\begin{array}{cccc}
1 & & & \\
& 0 & 0 & \\
& 0 & h^{1,1} & 0 & \\
1 & h^{2,1} & h^{2,1} & 1 & \\
& 0 & h^{1,1} & 0 & \\
& 0 & 0 & & \\
1 & & & \\
\end{array}
\]

(A.10)

We recall that the Euler characteristic is given by the Hodge (or Betti) numbers

\[
\chi = \sum_{k=0}^{6} (-1)^k b_k = \sum_{k=0}^{6} (-1)^k \sum_{p+q=k} h^{p,q}
\]

\[
= 1 - 0 + h^{1,1} - (1 + 2h^{2,1} + 1) + h^{1,1} - 0 + 1
\]

\[
= 2 \left( h^{1,1} - h^{2,1} \right)
\]

(A.11)
From the Hodge diamond, we see that the only independent Hodge numbers are \( h^{1,1} \) and \( h^{2,1} \). We now have all the machinery we need for a discussion of moduli space.

### A.3 Kähler and Complex Structure Moduli

On any complex manifold \( X \), there are many possible choices of almost complex structure tensor \( I \); it is not unique. Furthermore, if that manifold is Kähler, there are many choices of the Kähler form \( J \) as well. We can account for this by introducing infinitesimal deformations \( I \mapsto I + \delta I \) and \( J \mapsto J + \delta J \).

In order to maintain the almost complex structure on \( X \), \( \delta I \) must satisfy \((I + \delta I)^2 = -1\). Expanding to leading order in \( \delta I \), we find that \( \delta I_{\mu}^{\nu} = \delta I_{\alpha}^{\beta} + \delta I_{\bar{\alpha}}^{\bar{\beta}} \). In addition, the deformation must preserve integrability of the almost complex structure, implying a vanishing Nijenhuis tensor and therefore \( \nabla \delta I = 0 \). We can contract \( \delta I \) with the constant holomorphic volume form \( \Omega^{(3,0)} \) to form the isomorphism \( \delta I_{\gamma}^{\rho} \mapsto u_{\gamma\mu\nu} = \delta I_{\gamma}^{\rho} \Omega_{\mu\nu}^{(3,0)} \), where \( u_{\gamma\mu\nu} = u_{\alpha\beta\gamma} \) in holomorphic indices. Because both \( \delta I \) and \( \Omega^{(3,0)} \) are covariantly constant, \( u_{\alpha\beta\gamma} \) is a closed \((2,1)\)-form. The \( u_{\alpha\beta\gamma} \in H^{2,1}(X) \) are called complex structure moduli, and they parameterize the various deformations of the complex structure on \( X \). The Hodge number \( h^{2,1}(X) \) is the dimension of the space of complex structure moduli.

We can also find a space of deformations \( \delta J \) of the Kähler form under which the Kähler structure is preserved. Applying the Hermitian metric condition to the definition of the Kähler form, we find that \( I_{\rho}^{\mu} I_{\alpha}^{\nu} J_{\mu\nu} = J_{\rho\sigma} \). Substituting \( I \mapsto I + \delta I \) and \( J \mapsto J + \delta J \) and expanding to linear order in \( \delta I \) and \( \delta J \), we find that \( \delta J \) is also Hermitian and therefore \( \delta J_{\mu\nu} = \delta J_{\bar{\alpha}\bar{\beta}} + \delta J_{\bar{\beta}\bar{\alpha}} \). Furthermore, the deformation must preserve closure of the Kähler form, so \( \nabla(J + \delta J) = 0 \) and \( \delta J \) is a closed \((1,1)\)-form. The \( \delta J \in H^{1,1}(X) \) are called Kähler moduli, and they parameterize the deformations of the Kähler form on \( X \). The Hodge number \( h^{1,1}(X) \) is the dimension of the space of Kähler moduli.
We may characterize the moduli in a more cohesive way by considering generic deformations of the metric $g_{\mu\nu} = I_\mu^\rho J_{\rho\nu}$. We apply the transformation $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu} = (I + \delta I)_\mu^\rho (J + \delta J)_{\rho\nu}$ and find that in (anti-)holomorphic indices

$$\delta g_{a\bar{b}} = -i \delta J_{a\bar{b}}$$
$$\delta g_{ab} = i \delta I^b_{\bar{c}} g_{a\bar{c}}$$  \hspace{1cm} (A.12)

Thus, $(1, 1)$-deformations of the metric correspond to Kähler moduli and $(2, 0)$- or $(0, 2)$-deformations correspond to complex structure moduli.

### A.4 Chow Ring

We define $\mathcal{A}$ to be an $n$-dimensional algebraic variety along with an open cover

$$\mathcal{U}(\mathcal{A}) = \{ U \subset \mathcal{A} \mid \mathcal{A} \cong \bigcup U \}.$$  \hspace{1cm} (A.13)

On each coordinate patch $U \in \mathcal{U}(\mathcal{A})$, we define a local ring of meromorphic functions $\mathcal{M}_U$, containing within it the subring of holomorphic functions $\mathcal{O}_U \subset \mathcal{M}_U$. These rings each glue together on all of $\mathcal{A}$ to form the sheaves of meromorphic and holomorphic functions, $\mathcal{M}_\mathcal{A}$ and $\mathcal{O}_\mathcal{A}$ respectively.

We define the group of cycles $Z(\mathcal{A})$ to be the free abelian group of closed cycles generated by subvarieties of $\mathcal{A}$. Then, $Z^k(\mathcal{A}) \subset Z(\mathcal{A})$ is the group of cycles generated by codimension-$k$ subvarieties.

Let $C$ be a rational curve, i.e. birationally equivalent to $\mathbb{P}^1$, and let $Y_1, Y_2 \in Z^k(\mathcal{A})$. We say that $Y_1$ and $Y_2$ are *rationally equivalent*\footnote{The definition of rational equivalence is slightly more rigorous than this, but for the sake of simplicity, we will content ourselves with this.}, i.e. $Y_1 \sim_{\text{rat}} Y_2$, if $Y_1$ and $Y_2$ are both fibers of some total space $E \subset \mathcal{A} \times C$ over $C$. The space of rational equivalence classes $[Y]_{\text{rat}}$
of $k$-cocycles is called the $k$th Chow group $A^k(\mathcal{A}) = Z^k(\mathcal{A})/\sim_{\text{rat}}$. The Chow groups sum to form the graded Chow ring

$$A^i(\mathcal{A}) = \bigoplus_{k=0}^{\dim(\mathcal{A})} A^k(\mathcal{A})$$

Similarly, we say that $Y_1, Y_2 \in Z^k(\mathcal{A})$ are numerically equivalent, i.e. $Y_1 \sim_{\text{num}} Y_2$, if

$$\int_{\mathcal{A}} Y_1 \wedge Q = \int_{\mathcal{A}} Y_2 \wedge Q, \quad \forall Q \in Z^{n-k}(\mathcal{A}).$$

The space of numerical equivalence classes $[Y]_{\text{num}}$ of $k$-cocycles is denoted $N^k(\mathcal{A}) = Z^k(\mathcal{A})/\sim_{\text{num}}$.

### A.5 Divisors

We now consider the special case of hypersurfaces in $\mathcal{A}$, or subvarieties of codimension 1. A Weil divisor $W$ is a 1-cocycle, which can be described by a linear combination of irreducible codimension 1 subvarieties on $\mathcal{A}$, i.e.

$$W = \sum_{\text{codim}(Y) = 1, Y \text{ irreducible}} v_Y \cdot Y.$$  \hfill (A.15)

The Weil divisors form an additive, free abelian group $\mathcal{W}(\mathcal{A}) \cong Z^1(\mathcal{A})$. If all the coefficients $v_Y \geq 0$, then $W$ is said to be effective.

Weil divisors do not always behave well when treated locally, however, and so we therefore define the more refined group of Cartier divisors $\mathcal{C}(\mathcal{A}) \subset \mathcal{W}(\mathcal{A})$, which are defined locally on each coordinate patch $U \in \mathcal{U}(\mathcal{A})$ by a single, non-zero\footnote{The starred ring $\mathcal{M}_U^*$ refers to the subset of meromorphic functions which are non-zero. This notion can be formalized by considering the exponential map, $\exp : \mathcal{M}_U \to \mathcal{M}_U^*$. There is a corresponding map for $\mathcal{O}_U \subset \mathcal{M}_U$.} meromorphic function $f_U \in \mathcal{M}_U^*$. That is, each zero or pole of $f_U$ is an irreducible codimension 1 subvariety $Y$ of order $v_Y$, and their formal sum has the form of Equation (A.15). Then, locally on $U$,
the Cartier divisor $D \in \mathcal{C}(\mathcal{A})$ can be written
\[
D|_U = \sum_{i=1}^{k} v_i Y_i \quad \text{with} \quad f_U(z) = (z - Y_1)^{v_1} \cdots (z - Y_k)^{v_k} g_U(z), \quad g_U \in \mathcal{O}_U^*.
\] (A.16)

Furthermore, the local restrictions $D|_U$ must glue together in a piecewise fashion such that $f_{U_1}(z) = f_{U_2}(z)$ for $z \in U_1 \cap U_2 \subset \mathcal{A}$. An effective Cartier divisor is one which is defined by a purely holomorphic function. A Cartier divisor which is defined on the whole of $\mathcal{A}$ by a single function $f \in \mathcal{O}_A^*$ is called a principal divisor, and is denoted $(f)$.

The non-zero meromorphic functions $M^*_\mathcal{U}$ can be parameterized by a rational curve $C$. Then, because a Cartier divisor $D$ is locally defined by a single function $f_U \in M^*_\mathcal{U}$, it can be treated as a fiber over $C$. We can define the rational equivalence class $[D]_{\text{rat}}$ to be the set of fibers that can be smoothly deformed into $D$, i.e. described by functions in the orbit $f_U \cdot \mathcal{O}_A^*$ which only differ multiplicatively by a non-zero holomorphic function on all of $\mathcal{A}$. Equivalently, the divisors themselves will only differ additively by a principal divisor, i.e. $D_1 = D_2 + (f)$. Because the rational equivalence of divisors is given by a linear relation between functions, it is sometimes called linear equivalence.

When each coordinate patch $U \in \mathcal{U}(\mathcal{A})$ is a unique factorization domain, any Weil divisor can be expressed uniquely by a Cartier divisor, and we say that $\mathcal{A}$ is factorial. Then, the first Chow group can be written $A^1(\mathcal{A}) = \mathcal{C}(\mathcal{A})/\sim_{\text{lin}}$. By this construction, linear equivalence classes of divisors $[D]_{\text{lin}}$ are locally generated by a single holomorphic function. The effective divisors in each class form a linear system, with a base locus defined by $\text{Bl}(D) = \bigcap_{E_{\text{eff}} \in [D]_{\text{lin}}} E_{\text{eff}}$. If $\text{Bl}(D) = \emptyset$, then $[D]_{\text{lin}}$ is said to be base point free.

### A.6 Line Bundles

Given the locally ringed structure of $\mathcal{A}$ on the open cover $\mathcal{U}(\mathcal{A})$, $\mathcal{A}$ always admits holomorphic rank $r$ vector bundles with sections generated locally on coordinate patch
$U$ by non-zero holomorphic functions $g_1, ..., g_r \in \mathcal{O}_U^*$. The holomorphic line bundles, i.e. rank 1 vector bundles, form a group up to isomorphism, called the Picard group $	ext{Pic}(\mathcal{A}) := H^1(\mathcal{A}, \mathcal{O}_\mathcal{A}^*)$.

The isomorphism class of holomorphic line bundles $\mathcal{L} \in \text{Pic}(\mathcal{A})$ has sections $s \in \Gamma(\mathcal{L})$ which are locally generated on $U$ by the orbit of a single non-zero holomorphic function $f \in \mathcal{O}_U^*$ such that

$$\Gamma(\mathcal{L}|_U) = f \cdot \mathcal{O}_\mathcal{A}^* \subset \mathcal{O}_U^*.$$  \hspace{1cm} (A.17)

However, we have seen that a non-zero holomorphic function $f \in \mathcal{O}_U^*$ also locally generates the linear equivalence class of effective Cartier divisors $[D]_{\text{lin}} \in A^1(\mathcal{A})$. Thus, there is a one-to-one correspondence between line bundles $\mathcal{L} \in \text{Pic}(\mathcal{A})$ and divisor classes $[D]_{\text{lin}} \in A^1(\mathcal{A})$ (i.e. $\text{Pic}(\mathcal{A}) \cong A^1(\mathcal{A})$). We can write the space of sections $\Gamma(\mathcal{L}) = \mathcal{O}_\mathcal{A}(D)$, i.e. sections in $\mathcal{O}_\mathcal{A}$ of a line bundle corresponding to $[D]_{\text{lin}}$. By a common abuse of notation, we denote the line bundle itself this way as well $\mathcal{O}_\mathcal{A}(D) := \mathcal{L}$. Then, the base locus is the intersection of the support of $\mathcal{O}_\mathcal{A}(D)$, and so base point freeness occurs when $\mathcal{O}_\mathcal{A}(D)$ is globally generated by sections $g_1, ..., g_r$ whose orbits do not overlap.

Furthermore, because the bundle $\mathcal{O}_\mathcal{A}(D)$ over $\mathcal{A}$ is supported by $[D]_{\text{lin}}$, the restriction $\mathcal{O}_\mathcal{A}(D)|_D$ can be viewed as the orthogonal complement of the manifestly differentiable tangent bundle $TD$ in $T\mathcal{A}|_D$. This complement is called the normal bundle $\mathcal{N}_{D/\mathcal{A}}$ of $D$ in $\mathcal{A}$, and we have the short exact sequence

$$0 \rightarrow TD \rightarrow T\mathcal{A}|_D \rightarrow \mathcal{N}_{D/\mathcal{A}} \equiv \mathcal{O}_\mathcal{A}(D)|_D \rightarrow 0$$  \hspace{1cm} (A.18)

and its dual

$$0 \rightarrow \mathcal{N}_{D/\mathcal{A}}^* \equiv \mathcal{O}_\mathcal{A}(-D)|_D \rightarrow T^*\mathcal{A}|_D \rightarrow T^*D \rightarrow 0,$$  \hspace{1cm} (A.19)

where $\mathcal{O}_\mathcal{A}(-D)$ is the dual bundle corresponding to the support divisor $-D$ such that
$-D|_U = -(f) = (f^{-1})$. $\mathcal{O}_A(-D)$ is called the ideal sheaf $\mathcal{I}_{D/A}$ of $D$ in $A$, and its restriction to $D$ is called the conormal bundle

$$\mathcal{N}_{D/A}^* = \mathcal{I}_{D/A}|_D \equiv \mathcal{O}_A(-D)|_D \quad (A.20)$$

### A.7 Triality of Vector Spaces

The Lefschetz theorem on (1,1)-classes tells us that when $A$ is compact and Kähler, the first Chern class defines a bijection from the Picard group onto the integer-valued Kähler moduli space $H^{1,1}(A) \cap H^2(A, \mathbb{Z})$. Furthermore, the Kodaira embedding theorem tells us that when the Kähler moduli space is entirely integer-valued, i.e. $H^{1,1}(A) \subseteq H^2(A, \mathbb{Z})$, then $A$ is a projective variety, and can therefore be embedded in projective space $A \subset \mathbb{P}^{k-1}$. Thus, for projective varieties, we have the map

$$c_1 : \text{Pic}(A) \xrightarrow{\sim} H^{1,1}(A) \subseteq H^2(A, \mathbb{Z}), \quad (A.21)$$

so that $c_1(\mathcal{O}_A(D)) = [J]$, where $[J]$ is the cohomology class of the Kähler form.

But, because $\text{Pic}(A) \cong A^1(A)$, we can equivalently write $c_1(\mathcal{O}_A(D)) = \gamma(D)$ where $\gamma$ represents the Poincaré duality map $[D]|_{\text{lin}} \xrightarrow{\sim} [J]$. The main result, then, of these last three sections is that when $A$ is projective, we have a triality between the vector spaces $A^1(A)$, $\text{Pic}(A)$, and $H^{1,1}(A)$, which can be summarized by the commutative diagram

$$
\begin{array}{ccc}
A^1(A) & \xrightarrow{\sim} & H^{1,1}(A) \\
\mathcal{O}_A \xrightarrow{\sim} & \gamma & \\
\text{Pic}(A) \xrightarrow{c_1} & & \\
\end{array}
$$

(A.22)
A.8 Adjunction

Recall the short exact sequence

\[ 0 \rightarrow TD \rightarrow T\mathcal{A}|_D \rightarrow N_{D/A}|_D \cong \mathcal{O}_A(D)|_D \rightarrow 0. \tag{A.23} \]

By the definition of the Chern class, we can write

\[ c_1(T\mathcal{A}|_D) = c_1(TD) + c_1(\mathcal{O}_A(D)|_D) = c_1(TD) + \gamma(D|_D). \tag{A.24} \]

\[ c_1(T\mathcal{A}|_D) = c_1(TD) + \gamma(D|_D). \tag{A.25} \]

Assuming smoothness, the tangent bundle \( T\mathcal{A} \) is a rank \( n \) vector bundle and \( TD \) is rank \( n - 1 \), but we can consider the special case where they each decompose into line bundles

\[ T\mathcal{A} = \bigoplus_{i=1}^n L_i^A \quad \text{and} \quad TD = \bigoplus_{i=1}^{n-1} L_i^D. \]

Then, the top exterior powers are line bundles

\[ \bigwedge^n T\mathcal{A} = \bigotimes_{i=1}^n L_i^A \quad \text{and} \quad \bigwedge^{n-1} TD = \bigotimes_{i=1}^{n-1} L_i^D \] with first Chern class

\[ c_1\left(\bigwedge^n T\mathcal{A}\right) = \sum_{i=1}^n c_1(L_i^A) = c_1(T\mathcal{A}) \]

\[ c_1\left(\bigwedge^{n-1} TD\right) = \sum_{i=1}^{n-1} c_1(L_i^D) = c_1(TD). \tag{A.26} \]

The splitting principle ensures that this can be done even when \( T\mathcal{A} \) and \( TD \) do not decompose. The top exterior power of the cotangent bundle \( T^*\mathcal{A} \) is called the canonical bundle, which corresponds via \( \text{Pic}(\mathcal{A}) \cong A^1(\mathcal{A}) \) to a canonical divisor \( K_{\mathcal{A}} \) such that

\[ \bigwedge^n T^*\mathcal{A} = \mathcal{O}_A(K_{\mathcal{A}}). \]

Dual to this is the top exterior power of the tangent bundle, called the anticanonical bundle \( \bigwedge^n T\mathcal{A} = \mathcal{O}_A(-K_{\mathcal{A}}) \) with anticanonical divisor \(-K_{\mathcal{A}}\).
Then, equation (A.24) becomes
\[ K_D = (K_A + D)|_D. \] (A.27)

This result is called the \textit{adjunction formula}.

\section*{A.9 Fano Varieties}

Consider the hypersurface \( X \subset A \). We want to know how to choose \( X \) so that it is Calabi-Yau. The Calabi-Yau condition tells us that \( c_1(TX) = \gamma(-K_X) = 0 \), and considering \( X \) to be a Cartier divisor, equation (A.27) tells us that
\[ X = -K_A. \] (A.28)

Furthermore, we want \( X \) to be smooth. Bertini’s theorem asserts that because \( A \) is a projective variety, the linear system \([X]_{\text{lin}}\) will be smooth at points away from its base locus \( \text{Bl}(X) \), up to intersections with singular points of \( A \). More importantly, it will be smooth everywhere when \([X]_{\text{lin}}\) is base point free. In this case, the line bundle \( O_A(X) \) associated to \([X]_{\text{lin}}\) is globally generated, and its sections define an embedding into \( \mathbb{P}^{k-1} \) such that \( O_A(X) = O_{\mathbb{P}^{k-1}}(1)|_{A} \). Such a line bundle (or divisor) is said to be \textit{ample}. When the anticanonical divisor \(-K_A\) is ample, the variety \( A \) is called a \textit{Fano variety}.

\section*{A.10 Mori and Kähler Cones}

In order to present a clearer picture, we must first consider a condition slightly weaker than base point freeness. We say that a linear system \([D]_{\text{lin}}\) is \textit{eventually free} when \([\ell D]_{\text{lin}}\) is base point free for some \( \ell \in \mathbb{Z}_{>0} \). The property of base point freeness requires that \( C \cdot D_2 = \int_C \gamma(D_2) = 0 \) for all \( D_1, D_2 \in [D]_{\text{lin}} \) and all curves \( C \subset D_1 \). However, if \([D]_{\text{lin}}\) is only eventually base point free, then we have the less strict requirement that
\[ \int_C \gamma(D_2) \geq 0 \] for all curves \( C \subset D_1 \).

If two divisors \( D_1, D_2 \in \mathcal{C}(\mathcal{A}) \) have the same intersection \( \int_C \gamma(D_1) = \int_C \gamma(D_2) \) for all curves \( C \subset \mathcal{A} \), then they are said to be numerically equivalent, i.e. \( D_1 \sim_{\text{num}} D_2 \). The quotient subgroup of Cartier divisors \( N^1(\mathcal{A}) \cong \mathcal{C}(\mathcal{A})/\sim_{\text{num}} \) assigns all numerically equivalent divisors \( D \) an equivalence class \([D]_{\text{num}}\). For a projective variety, this is just the Chow group modulo torsion, i.e. \( N^1(\mathcal{A}) \cong A^1(\mathcal{A})/A^1_{\text{tor}}(\mathcal{A}) \). If a numerical equivalence class \([D]_{\text{num}}\) contains an eventually base point free linear system, then the divisors \( D \in [D]_{\text{num}} \) are said to be nef (or numerically eventually free).

Dual to the space of 1-cocycles modulo numerical equivalence \( N^1(\mathcal{A}) \) is the space \( N_1(\mathcal{A}) \) of 1-cycles (or curves) modulo numerical equivalence. Then, given an independent set of irreducible curves \( C_i \subset \mathcal{A} \), the set of positive linear combinations

\[
\text{NE}(\mathcal{A}) = \left\{ \sum_i a_i[C_i]_{\text{num}} \in N_1(\tilde{\mathcal{A}}) \mid a_i \in \mathbb{R}_{>0} \right\}
\]  (A.29)

forms a convex cone known as the Mori cone (or the cone of (numerically effective) curves). It can be shown that the nef divisors \( D \) form a cone in \( N^1(\mathcal{A}) \) dual to \( \text{NE}(\mathcal{A}) \subset N_1(\mathcal{A}) \) defined by

\[
\text{Nef}(\mathcal{A}) = \left\{ D \in N^1(\mathcal{A}) \mid \int_{[C]_{\text{num}}} \gamma(D) \geq 0, \ [C]_{\text{num}} \in \text{NE}(\mathcal{A}) \right\}.
\]  (A.30)

Furthermore, all divisors classes \([D]_{\text{num}}\) in the interior of \( \text{Nef}(\mathcal{A}) \) are ample. This ample cone also forms a cone in the Chow group of linear equivalence classes \( A^1(\mathcal{A}) \). And finally, via the triality of vector spaces \( A^1(\mathcal{A}), \text{Pic}(\mathcal{A}), \) and \( H^{1,1}(\mathcal{A}) \), we also obtain a cone of ample holomorphic line bundles, and a cone of allowed Kähler forms

\[
\mathcal{K}(\mathcal{A}) = \left\{ J \in H^{1,1}(\tilde{\mathcal{A}}) \mid \int_{[C]_{\text{num}}} J > 0, \ [C]_{\text{num}} \in \text{NE}(\tilde{\mathcal{A}}) \right\}.
\]  (A.31)
The latter is called the Kähler cone, and is what the physicists among us will find most interesting.

**A.11 Gorenstein Toric Fano Varieties and Reflexive Polytopes**

In the preceding sections, we have been building up to a method of constructing a Calabi-Yau threefold $X$ as the anticanonical divisor of a Fano variety $A$. In order for this construction to have any real computational power, however, we must be able to determine its properties using only combinatorics and linear algebra. It turns out that when we choose $A$ to be a toric variety as well, it can be described combinatorially by the fan of a $n$-dimensional convex polytope. Furthermore, when $A$ is also chosen to be Gorenstein, then it is described specifically by an $n$-dimensional reflexive polytope. For $n = 4$, all such reflexive polytopes have been painstakingly classified by Kreuzer and Skarke [1,33], and so this class of Calabi-Yau threefolds can be computed in its entirety, which would mean an obvious boon for scan-based phenomenology.

In the following sections, we will lay out the construction of Gorenstein toric Fano varieties from reflexive polytopes in a relatively complete, if somewhat technical manner. This construction will culminate in Section A.11.2 with a dictionary of terminology on the polytope side of the story, and in Section A.11.4 with the relationship between the Gorenstein toric Fano variety $A$ and the reflexive polytope $\Delta$.

**A.11.1 $\mathbb{Q}$-Gorenstein Fano Varieties**

Because we have chosen the variety $A$ to be Fano, it is necessarily projective, and as a result, there exists a morphism into projective space $\phi : A \rightarrow \mathbb{P}^{k-1}$. Furthermore, there must exist a very ample, effective Cartier divisor $E \subset A$ whose corresponding line bundle is the pullback of the hyperplane bundle on $\mathbb{P}^{k-1}$, i.e. $O_A(E) = \phi^*(O_{\mathbb{P}^{k-1}}(1))$. 
Then, Chow’s theorem tells us that the divisor $E$ can be defined as the zero-locus of a holomorphic, homogenous polynomial $P$ in the homogenous coordinates $z_1, ..., z_k$ of $\mathbb{P}^{k-1}$.

We choose an open cover $U(\mathcal{A}) = \{ U \subset \mathcal{A} \mid \mathcal{A} \cong \bigcup U \}$ such that the Cartier divisor $E$ can be defined locally as the zero-locus of a holomorphic, homogenous monomial $P_U$ on each patch $U \in U(\mathcal{A})$. Without loss of generality, we write this monomial

$$P_U = \prod_{\rho=1}^{k} z^\chi_{U,\rho}, \quad \chi_{U,\rho} \in \mathbb{Z}_{\geq 0}.$$  \hfill (A.32)

It is clear that the loci $\{ z_\rho = 0 \}_{\rho=1,...,k}$ define hypersurfaces in $\mathbb{P}^{k-1}$ and therefore define Cartier divisors $D_\rho$ whose corresponding line bundles $\mathcal{O}_\mathcal{A}(D_\rho)$ also descend from the hyperplane bundle $\mathcal{O}_{\mathbb{P}^{k-1}}(1)$. As a result, we can write $E$ as the locally principal divisor

$$E|_U = (P_U)|_U = \left( \prod_{\rho=1}^{k} z^\chi_{U,\rho} \right)|_U = \sum_{\rho=1}^{k} \chi_{U,\rho} \cdot (z_\rho)|_U$$

$$= \sum_{\rho=1}^{k} \chi_{U,\rho} \cdot D_\rho|_U. \hfill (A.33)$$

In addition, $E$ must be consistent on the intersection of any pair of patches $U_1, U_2 \in U(\mathcal{A})$ so that $\chi_{U_1 \cap U_2,\rho}$ is well-defined for all $\rho = 1, ..., k$.

By choosing $\mathcal{A}$ Fano, we require that the anticanonical divisor $-K_\mathcal{A}$ be ample, and therefore that some positive integer multiple $-\ell K_\mathcal{A}$ be very ample for $\ell \in \mathbb{Z}_{>0}$. Without loss of generality, let us choose $-\ell U K_\mathcal{A}|_U = E|_U$ so that $-\ell K_\mathcal{A}$ is very ample when $\ell = \text{LCM}(\{ \ell_U \}_{U \in U(\mathcal{A})})$. Then, we define $\psi_{U,\rho} = \frac{1}{\ell_U} \cdot \chi_{U,\rho}$ and write

$$-\ell U K_\mathcal{A}|_U = E|_U = \ell_U \sum_{\rho=1}^{k} \psi_{U,\rho} \cdot D_\rho|_U \hfill (A.34)$$
with defining monomial

\[ P_U = \prod_{\rho=1}^{k} z_\rho^\psi_{U,\rho}, \quad \psi_{U,\rho} \in \frac{1}{l_U} \mathbb{Z}_{\geq 0}. \]  

(A.35)

A divisor is \( \mathbb{Q} \)-Cartier when some integer multiple is Cartier and therefore defines a line bundle. When this is true of the anticanonical divisor, we say that \( \mathcal{A} \) is \( \mathbb{Q} \)-Gorenstein.

### A.11.2 \( \mathbb{Q} \)-Gorenstein Toric Fano Varieties and Polytopes

Recall that a toric variety \( \mathcal{A} \) is defined as an algebraic variety containing a torus \( T \) as a dense open subset such that the action of \( T \) on itself extends to all of \( \mathcal{A} \), i.e. \( T \times \mathcal{A} \rightarrow \mathcal{A} \).

If we choose an appropriate variety \( V \subset \mathbb{C}^k \), then we can write

\[ \mathcal{A} \cong V/T \]  

(A.36)

In general, the torus \( T \) can not be split entirely into a product of copies of the 1-torus \( T^1 = \mathbb{R}/\mathbb{Z} \cong \mathbb{C}^* \). However, in these cases when \( T \) is non-split, we can always perform a Galois transformation into a field extension in which \( T \) can be so split. Then, we can write the toric variety

\[ \mathcal{A} \cong V/((\mathbb{C}^*)^{k-n} \times G), \]  

(A.37)

where \( G \) is the group of orbifold automorphisms taking \( \mathcal{A} \) to itself. In the case of \( T \) already split, \( G \) is trivial.

Given a representation \( \gamma : T \rightarrow GL(\mathcal{A}) \), the multiplicative character of the toric variety is given by

\[ \chi^\gamma \equiv \text{Tr} \circ \gamma : \quad T \rightarrow \mathbb{C}^* \]

\[ (t_1, ..., t_n) \mapsto t_1^{\gamma_1} ... t_n^{\gamma_n}. \]  

(A.38)
The characters form a \textit{weight lattice} \( M = \text{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^n \ni \gamma = (\gamma_1, ..., \gamma_n) \) parameterized by the representations \( \gamma \). In addition, we can define a \textit{coweight lattice} \( N = \text{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^n \) such that there is a bilinear form \( \langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z} \).

This procedure can be modified slightly by defining tori over the \( \ell \)-th roots of unity; that is, we redefine the 1-torus to be \( T_1 = \mathbb{R}/\ell \mathbb{Z} \). Then, the character map \( \chi_{\gamma} \) changes to
\[
(t_1, ..., t_n) \mapsto t_1^{\frac{\gamma_1}{\ell}} ... t_n^{\frac{\gamma_n}{\ell}} \quad (A.39)
\]
and the weight and coweight lattices are scaled so that \( M \mapsto M_{\ell} \cong \left( \frac{1}{\ell} \mathbb{Z} \right)^n \) and \( N \mapsto N_{\ell} \cong \left( \frac{1}{\ell} \mathbb{Z} \right)^n \). Considering arbitrary \( \ell \in \mathbb{Z}_{>0} \), we can extend \( M \) and \( N \) to the \textit{rational extensions} \( M_{\mathbb{Q}} \cong M \otimes \mathbb{Z} \mathbb{Q} \) and \( N_{\mathbb{Q}} \cong N \otimes \mathbb{Z} \mathbb{Q} \) with bilinear form \( \langle \cdot, \cdot \rangle : M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q} \).

It is worth noting that \( \mathcal{A} \) can be equivalently described as a weighted projective space with respect to each 1-torus \( T^1 \). If the torus is already split, then the Picard group is parameterized by \( k - n \) hyperplane line bundles such that \( \text{Pic}(\mathcal{A}) \cong \mathbb{Z}^{k-n} \). If the torus is not split, and we must perform a Galois transformation, then the Picard group has the form \( \text{Pic}(\mathcal{A}) \cong \mathbb{Q}^{k-n} \).

Because \( \mathcal{A} \) is both \( \mathbb{Q} \)-Gorenstein and toric, we have a natural candidate for the coefficients \( \psi_{U, \rho} \) in Equation (A.34) by choosing dual sets \( \mathcal{V}_M \subset M_{\mathbb{Q}} \) and \( \mathcal{V}_N \subset N_{\mathbb{Q}} \), and integers \( \ell_U \in \mathbb{Z}_{>0} \) for \( U \in \mathcal{U}(\mathcal{A}) \), such that the identification \( \psi_{U, \rho} = \langle m_U, n_{\rho} \rangle \) holds for all \( m_U \in \mathcal{V}_M \) and \( n_{\rho} \in \mathcal{V}_N \) and such that \( \ell_U \cdot \psi_{U, \rho} \in \mathbb{Z}_{>0} \). We can define the linear functions
\[
\psi_U(n) = \langle m_U, n \rangle. \quad (A.40)
\]

In fact, because \( -K_{\mathcal{A}} \) is an ample divisor, each inner product must satisfy \( \langle m_U, n_{\rho} \rangle = \psi_{U, \rho} \geq 0 \), and as a result, the sets \( \mathcal{V}_M \) and \( \mathcal{V}_N \) must be convex. Then, it makes sense to say that the convex hulls of \( \mathcal{V}_M \) and \( \mathcal{V}_N \) form dual (or \textit{polar}) polytopes \( \Delta = \text{conv}(\mathcal{V}_M) \subset M_{\mathbb{Q}} \).
Appendix A. Mathematical Background

and $\Delta^* = \text{conv}(\mathcal{V}_N) \subset \mathbb{N}_Q$ with vertices given by $\mathcal{V}(\Delta) = \mathcal{V}_M$ and $\mathcal{V}(\Delta^*) = \mathcal{V}_N$. Because each vertex of $\Delta$, $m_U \in \mathcal{V}(\Delta)$ corresponds to an exponent in the monomial $P_U$, we sometimes refer to $\Delta$ as the Newton polytope corresponding to the variety $\mathcal{A}$.

Notice, however, from equation (A.35) that the $\psi_{U,\rho}$ can never be equal to zero for all $\rho = 1, \ldots, k$ if we want the monomial $P_U$ to have a zero-locus. Then, the origin $0 \notin \Delta, \Delta^*$. Unfortunately, this makes certain combinatorial computations too unwieldy. We can correct this with the translation $\psi_U(n) \mapsto \psi_U(n) - \frac{1}{\ell_U}$. Then equation (A.35) becomes

$$P_U = \prod_{\rho=1}^{k} z_\rho^{\ell_U \psi_U(n_\rho)+1}, \quad \psi_U(n_\rho) \in \frac{1}{\ell_U} \mathbb{Z}_{\geq -1} \subset \mathbb{Q}_{\geq -1} \quad (A.41)$$

and equation (A.34) becomes

$$- \ell_U K_{\mathcal{A}|U} = \sum_{\rho=1}^{k} (\ell_U \cdot \psi_U(n_\rho) + 1) \cdot D_\rho|U, \quad \psi_U(n_\rho) \in \frac{1}{\ell_U} \mathbb{Z}_{\geq -1} \subset \mathbb{Q}_{\geq -1}, \quad (A.42)$$

where the divisors $D_\rho$ are referred to as toric divisors, and clearly have a one-to-one correspondence with the vertices of the dual polytope $n_\rho \in \mathcal{V}(\Delta^*)$ via the bijection $D_\rho \to n_\rho$ and with the homogenous coordinates $z_1, \ldots, z_k$ on $\mathcal{A}$ via $D_\rho = (z_\rho)|_{\mathcal{A}}$.

We can demonstrate the duality between $\Delta$ and $\Delta^*$ explicitly by expressing one in terms of the other via

$$\Delta^* = \left\{ n \in \mathbb{N}_Q \left\vert \psi_U(n) \geq -\frac{1}{\ell_U}, \forall U \in \mathcal{U}(\mathcal{A}) \right. \right\}. \quad (A.43)$$

We can also define the supporting hyperplanes of $\Delta^*$ by

$$H_U = \left\{ n \in \mathbb{N}_Q \left\vert \psi_U(n) = -\frac{1}{\ell_U} \right. \right\}. \quad (A.44)$$

As a result, the linear function $\psi_U$ is said to be the support function on $\Delta^*$ corresponding
to the coordinate patch $U$. Furthermore, because $\psi_U(n) \geq -\frac{1}{\ell_U}$ for all $n \in \Delta^*$, it is a convex function on $\Delta^*$. We see that this is, again, a direct result of the fact that $-K_A$ is an ample divisor.

The truncated hyperplanes $F_U = H_U \cap \Delta^*$ are called the facets (or $(n-1)$-faces) of $\Delta^*$. The intersections of the facets form lower-dimensional faces, and we can consider the set containing the facets and all their intersections $F(\Delta^*)$. We denote the subset of $d$-dimensional faces $F_d(\Delta^*)$, and the union of these is called the $d$-skeleton $\Lambda_d(\Delta^*) = \bigcup_{F \in F_d(\Delta^*)} F$.

Each facet $F \in F_{n-1}(\Delta^*)$ can be extended to an $n$-dimensional (or maximal) convex rational polyhedral cone $\sigma = \text{cone}(F)$ through the origin. The intersections of these cones form lower dimensional cones, and we can consider the set $\Sigma(\Delta^*)$ containing the maximal cones and all their intersections. $\Sigma(\Delta^*)$ is called the fan of the dual polytope $\Delta^*$ (sometimes also referred to as the normal fan of the Newton polytope $\Delta$). The subset of $d$-dimensional cones in $\Sigma(\Delta^*)$ is denoted $\Sigma_d(\Delta^*)$, and the union of all the cones is denoted $|\Sigma(\Delta^*)| = \bigcup_{\sigma \in \Sigma(\Delta^*)} \sigma$.

The above description suggests that for each coordinate patch $U \in \mathcal{U}(\mathcal{A})$, there is a corresponding vertex $m \in \mathcal{V}(\Delta)$, a corresponding facet $F \in F_{n-1}(\Delta^*)$, and a corresponding maximal cone $\sigma \in \Sigma_n(\Delta^*)$. As such, any of these objects can be used to describe local properties of the toric variety $\mathcal{A}$, and each version can be very useful for different purposes.

**A.11.3 $\ell$-Gorenstein Toric Fano Varieties and $\ell$-Reflexive Polytopes**

Recall from Section A.11.1 that when $\ell = \text{LCM}(\{\ell_U\}_{U \in \mathcal{U}(\mathcal{A})})$, then the divisor $-\ell K_A$ is very ample. In fact $-\ell K_A$ must then be a Cartier divisor, so we say that $-K_A$ is $\ell$-Gorenstein. Now, consider the case in which $\ell_U = \ell$, $\forall U \in \mathcal{U}(\mathcal{A})$. Then, clearly $-K_A$
is $\ell$-Gorenstein. In addition, we may restrict the rational extension $N_\mathbb{Q}$ to $N_\ell \cong (\frac{1}{\ell}\mathbb{Z})^n$. In this special case, the supporting hyperplanes $H_U$ defined in Equation (A.44) can be expressed

$$H_U = \left\{ n \in N_\ell \left| \langle m_U, n \rangle = -\frac{1}{\ell} \right. \right\} \quad (A.45)$$

such that each facet $F_U = H_U \cap \Delta^*$ has the same lattice distance $\ell$. When this is true, we say that $\Delta^*$ is $\ell$-reflexive.

In the $\ell$-reflexive case, $\psi_{U_1, \rho} = \psi_{U_2, \rho}$ on $U_1 \cap U_2$ for $U_1, U_2 \in \mathcal{U}(\mathcal{A})$, and we can therefore write a single piecewise linear support function $\psi : N_\ell \to \frac{1}{\ell}\mathbb{Z}$, which is defined in the usual way when restricted to a cone

$$\psi|_{\sigma_U} = \psi_U \cdot (A.46)$$

Because ampleness of $-K_{\mathcal{A}}$ requires convexity of the $\psi_U$, the union $\psi$ is also convex. If the convexity of $\psi$ is strict, i.e. $\psi(n_1) + \psi(n_2) > \psi(n_1 + n_2)$ with $n_1 \in \sigma_1 \neq \sigma_2 \ni n_2$, then it can be used to distinguish maximal cones, and the map $\sigma_U \to U$ is well-defined. Then we say that the union of cones $|\Sigma(\Delta^*)|$ supports $\Sigma(\Delta^*)$. Because $\mathcal{A}$ is compact, the strict convexity of $\psi$ implies that $|\Sigma(\Delta^*)| = N_\ell$, and we say that $\Sigma(\Delta^*)$ is complete [126].

### A.11.4 Gorenstein Toric Fano Varieties and Reflexive Polytopes

Restricting even further to the special case in which $\ell = 1$, $-K_{\mathcal{A}}$ itself is very ample. It is therefore Cartier, and we therefore say that it is Gorenstein. Furthermore, each facet $F_U = H_U \cap \Delta^*$ has uniform lattice distance 1, and so $\Delta^*$ is reflexive. Because there is only one way to choose LCM $\{\ell_U\}_{U \in \mathcal{U}(\mathcal{A})} = 1$, we find that there is a one-to-one correspondence between Gorenstein toric Fano varieties $\mathcal{A}$ and reflexive polytopes $\Delta$ (or $\Delta^*$).

Notice also that when $\ell = 1$, both $\Delta$ and $\Delta^*$ are lattice polytopes with vertices $\mathcal{V}(\Delta) \subset M$. 
and $\mathcal{V}(\Delta^*) \subset N$. In addition, because the lattice distance of its facets is 1, it is impossible for $\Delta^*$ to contain any lattice points in its interior save for the origin. Due to the duality corresponding to the bilinear form $\langle \mathbf{m}_U, \mathbf{n}_\rho \rangle$, the same must be true for $\Delta$ as well. Then, a reflexive polytope $\Delta$ can be defined equivalently as a lattice polytope containing only the origin in its interior, whose dual polytope is also a lattice polytope containing only the origin in its interior.

When $\ell = 1$, the anticanonical divisor can be written

$$-K_A|_U = \sum_{\rho=1}^{k} \left( \langle \mathbf{m}_U, \mathbf{n}_\rho \rangle + 1 \right) \cdot D_{\rho|_U}, \quad \langle \mathbf{m}_U, \mathbf{n}_\rho \rangle \in \mathbb{Z}_{\geq -1} \quad (A.47)$$

and its defining equation as

$$P_U = \prod_{\rho=1}^{k} z_{\rho}^{\langle \mathbf{m}_U, \mathbf{n}_\rho \rangle + 1}, \quad \langle \mathbf{m}_U, \mathbf{n}_\rho \rangle \in \mathbb{Z}_{\geq -1}. \quad (A.48)$$
Appendix B

Identities for Deriving the Inverse Kähler Metric

For a given Calabi-Yau threefold $X$, the volume form is given by

$$\mathcal{V} = \frac{1}{3!} \kappa_{ijk} t^i t^j t^k$$  \hspace{1cm} (B.1)

From this, we can derive the volume of the $i^{th}$ 4-cycle divisor

$$\tau_a = \frac{\partial \mathcal{V}}{\partial t^a} = \frac{1}{3!} \kappa_{ijk} \left( \frac{\partial t^i}{\partial t^a} t^j t^k + t^i \frac{\partial t^j}{\partial t^a} t^k + t^i t^j \frac{\partial t^k}{\partial t^a} \right)$$

$$= \frac{1}{3!} \kappa_{ijk} \left( \delta^i_i \delta^j_j \delta^k_k + t^i \delta^j_j t^k + t^i t^j \delta^k_k \right)$$

$$= \frac{1}{2} \kappa_{ajk} t^j t^k$$  \hspace{1cm} (B.2)
For ease of computation, we will also derive the following two relations involving derivatives of 2-cycle divisor volumes

\[
\delta^a_b = \frac{\partial \tau_b}{\partial \tau_a} = \frac{\partial}{\partial \tau_a} \left(\frac{1}{2} \kappa_{bijk} t^j t^k \right) \\
= \frac{1}{2} \kappa_{bijk} \left( \frac{\partial t^j}{\partial \tau_a} t^k + t^j \frac{\partial t^k}{\partial \tau_a} \right) \\
= \kappa_{bijk} t^k \frac{\partial t^j}{\partial \tau_a}
\]  
(B.3)

\[
0 = \frac{\partial \delta^i_b}{\partial \tau_c} = \kappa_{bijk} \left( \frac{\partial t^k}{\partial \tau_c} \frac{\partial t^i}{\partial \tau_a} + t^k \frac{\partial^2 t^i}{\partial \tau_a \partial \tau_c} \right) \\
\Rightarrow \kappa_{bijk} t^k \frac{\partial^2 t^i}{\partial \tau_c \partial \tau_a} = -\kappa_{bijk} \frac{\partial t^k}{\partial \tau_c} \frac{\partial t^i}{\partial \tau_a}
\]  
(B.4)

Using Equation (B.3), we now compute the first derivative of the volume form

\[
\gamma^a = \frac{\partial V}{\partial \tau_a} = \frac{1}{3!} \kappa_{ijk} \left( \frac{\partial t^i}{\partial \tau_a} t^j t^k + t^i \frac{\partial t^j}{\partial \tau_a} t^k + t^i t^j \frac{\partial t^k}{\partial \tau_a} \right) \\
= \frac{1}{2} \frac{\partial t^i}{\partial \tau_a} \kappa_{ijk} l^i l^j l^k = \frac{1}{2} \delta^a_k l^k = \frac{1}{2} t^a
\]  
(B.5)

and using Equation (B.4), we compute the second derivative

\[
\gamma^{ab} = \frac{\partial \gamma^a}{\partial \tau_b} = \frac{1}{2} \kappa_{ijk} \left( \frac{\partial^2 t^i}{\partial \tau_b \partial \tau_a} t^j t^k + \frac{\partial t^i}{\partial \tau_a} \frac{\partial t^j}{\partial \tau_b} t^k + \frac{\partial t^i}{\partial \tau_a} t^j \frac{\partial t^k}{\partial \tau_b} \right) \\
= \frac{1}{2} \kappa_{ijk} \left( -\frac{\partial t^k}{\partial \tau_b} \frac{\partial t^i}{\partial \tau_a} l^j + \frac{\partial t^i}{\partial \tau_a} \frac{\partial t^j}{\partial \tau_b} t^k + \frac{\partial t^i}{\partial \tau_a} t^j \frac{\partial t^k}{\partial \tau_b} \right) \\
= \frac{1}{2} \delta^b_i \frac{\partial t^i}{\partial \tau_a} = \frac{1}{2} \frac{\partial t^b}{\partial \tau_a}
\]  
(B.6)
Again using Equation (B.3), we can then derive the inverse of the second derivative of the volume form

\[ \delta^a_b = \kappa_{bijk} t^k \frac{\partial t^j}{\partial \tau_a} = 2 \kappa_{bijk} t^k \nu^j \]

\[ \Rightarrow (\nu^{-1})_{jb} = 2 \kappa_{jblk} t^k \] (B.7)

We now derive the following three relations, which will be useful in computing the inverse Kähler metric

\[ \nu^{ab} (\nu^{-1})_{bc} = \left( \frac{1}{2} \frac{\partial t^b}{\partial \tau_a} \right) (2 \kappa_{bck} t^k) = \delta^a_c \] (B.8)

\[ \nu^a (\nu^{-1})_{ab} = \left( \frac{1}{2} t^a \right) (2 \kappa_{abk} t^k) = 2 \tau_b \] (B.9)

\[ \nu^a (\nu^{-1})_{ab} \nu^b = \left( \frac{1}{2} t^a \right) (2 \kappa_{abk} t^k) \left( \frac{1}{2} t^b \right) = 3 \nu \] (B.10)
References


References


References


