Gauge Theory
and
Self-Linking of Legendrian Knots

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Abstract of Dissertation

This thesis presents the perturbative contact Chern-Simons gauge field theory on 3-dimensional manifold. We identify the linking number and self-linking number of Legendrian knots as the first perturbative estimate of the expectation value of the Wilson loop operator with respect to the Legendrian knots and links of Legendrian knots.

In chapter 2, we briefly discuss the usual 3 dimensional perturbative Chern-Simons theory. We write down the propagators and set up the Feynman rules for this theory. We also discuss the linking number of links and self-linking of knots. We interpret the linking integral and self-linking integral as the first term in the perturbative expansion of the expectation value of the Wilson loop operator. We examine two different approaches to define the self-linking number of a knot. One approach is framing and the other one is adding a counterpart to the naive Gauss self-linking integral given in Pohl’s work in 1967.

In chapter 3, we promote the Yang-Mills Chern-Simons theory to a supersymmetric theory. We break the Yang-Mills action into magnetic and electric components after introducing a contact structure. We show that the contact Chern-Simons action is a infrared limit of the supersymmetric Yang-Mills theory.

In chapter 4, we study the perturbative theory of the contact Chern-Simons theory. We take the expansion of the contact Chern-Simons theory and derive the contact Hessian and contact Laplacian in the quadratic components of the contact Chern-Simons theory on $\mathbb{R}^3$. We note that the contact Hessian and Laplacian are invariant under the Heisenberg group structure on $\mathbb{R}^3$. We express the contact Hessian and Laplacian in terms of an operator studied by Folland and Stein in 1974 and write the Green’s functions of the two operators explicitly. The contact Hessian and Laplacian
are closely related to contact complex introduced by Rumin in 1994. We study the homology properties of the differential operator in the contact complex. A new concept called relative contact cohomology is introduced. We set up the Feynman rule for the contact Chern-Simons theory.

In chapter 5, we study the perturbative expansion of the expectation value of the Wilson loop operator with respect to the Legendrian knots. First we review some of the basics of Legendrian knots and then we define the linking number in this theory for a multi-component Legendrian link, and show that it agrees with the usual Gauss linking number. Then, we introduce perturbative localization for the contact Chern-Simons theory, and consider the asymptotic behavior of the propagator of the contact Chern-Simons theory.

In the last chapter, we discuss the analytic notions of the self-linking for Legendrian knots in $\mathbb{R}^3$. The definition of self-linking in this chapter is based on the perturbative localization of the propagator of the contact Chern-Simons theory. We discover the Thurston-Bennequin invariant as a special case of the self-linking.
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Introduction

The basic content of this thesis is studying the perturbative expansion of the expectation value of Wilson loop operator in contact Chern-Simons theory and identifying the first perturbative correction in the expansion as topological invariants of Legendrian knots. In the following four sections of this introduction, we give the basic ideas and present the main results in this thesis.

0.1 Contact Chern-Simons Functional and Its Propagators

Let $M$ be an oriented 3-dimensional contact manifold with contact 1-form $\kappa$, the action of the contact Chern-Simons theory can be given as

$$I_r(A) = rS(A) + CS(A) \quad (0.1.1)$$

where $r \in \mathbb{C}$ is a parameter,

$$S(A) = \int_M \kappa \wedge d\kappa \text{Tr} \left[ \left( \frac{\kappa \wedge F_A}{\kappa \wedge d\kappa} \right)^2 \right] \quad (0.1.2)$$

and

$$CS(A) = \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (0.1.3)$$

is the Chern-Simons action, $A \in \Omega^1(M, \mathfrak{g})$ (g is the Lie algebra of a Lie group $G$) is a connection with value in $\mathfrak{g}$ over $M$, $F_A = dA + A \wedge A$ is the curvature of $A$. Note that, since the top form $\kappa \wedge d\kappa \neq 0$, $\frac{\kappa \wedge F_A}{\kappa \wedge d\kappa}$ is well defined. This action was first given and
studied in [12] and further analyzed in [13, 14, 41, 42, 51]. The origin of this action is explained as a supersymmetric Yang-Mills theory in the contact limit in 2.1. An important property of the contact Chern-Simons action is when \( r = i \), \( I_r \) is invariant under the following shift:

\[
A \rightarrow A + f \kappa
\]  

\( \forall f \in C^\infty(M, g). \)

We take the expansion of \( I_r(A) \) at a fixed flat connection \( A_0 \in \Omega^1(M, g) \), write \( A = A_0 + B \), we get

\[
I_r(A_0 + B) = CS(A_0 + B) + i r S(A_0 + B),
\]

\[
= CS(A_0) + i r \int_M \text{Tr} (B \wedge D_r B) + P_r(B),
\]

where the integral in the last line above is the quadratic component in \( I_r(A) \) and \( P_r(B) \) includes all the terms with order 3 or higher. To perform perturbative Feynman diagram calculations, we need to take the inverse of \( D_r \). However \( D_r \) is not invertible for its kernel is too large. So we adapt the usual BRST procedure by introducing ghost fields \( c \in \Omega^0(M, g), b \in \Omega^3(M, g) \) and an auxiliary commuting scalar field \( \phi \in \Omega^3(M, g) \). The action takes the form:

\[
S_r(A) = CS(A_0) - \frac{1}{4\pi} (B, D_r B) + \frac{ir}{2\pi} (b, \triangle_H c) + O_3,
\]

where, the second and the third terms on the RHS of the above equation are the quadratic components of \( S_r(A) \) and \( O_3 \) contains all order 3 and higher terms. Here \( \triangle_H \) is a contact version of the scalar Laplacian, \( D_r \) can be regarded as a contact version of Heisenberg operator which is called contact Hessian, and the parentheses appear in the quadratic terms are \( L^2 \) inner products. The specific forms of the operators \( D_r \) and \( \triangle_H \) are given in Chapter 2.

We write

\[
G = \triangle_H^{-1}, \quad K_r = D_r^{-1}.
\]
These two operators $G$ and $K_r$ are the propagators for the contact Chern-Simons theory. Both of these two operators can be written as integral operators:

$$(G c)^{a_1} (x) = \int_{M_y} \kappa_y \wedge d\kappa_y \, G(x, y)^{a_1}_{a_2} c^{a_2}(y), \quad (x, y) \in M \times M, \quad (0.1.8)$$

and

$$(K_r \Phi)^{a_1} (x) = \int_{M_y} K_{r,1}^{1,2}(x, y)^{a_1}_{a_2} \wedge \Phi(y)^{a_2}, \quad (x, y) \in M \times M, \quad (0.1.9)$$

where $\Phi \in \Omega^1(H, g)$ is a horizontal form on $M$. The indices $a_1$ and $a_2$ are indices of the Lie algebra $g$ w.r.t. a basis. $K_{r,1}^{1,2}(x, y)$ is a 1-form in $x$ and 2-form in $y$. We dualize the kernel $K_{r,1}^{1,2}$ on the right to obtain the more symmetric form,

$$K_{r,1}^{1,1}(x, y)^{a_1}_{a_2} = \star_y K_{r,1}^{1,2}(x, y)^{a_1}_{a_2} \in \Omega^{1,1}(H \times H, \text{End}(g)). \quad (0.1.10)$$

Note that the kernel $K_{r,1}^{1,1}$ is a horizontal $\text{End}(g)$-valued $(1, 1)$-form on $M \times M$. We may find the kernel $G(x, y)^{a_1}_{a_2} \in \Omega^{0,0}(M \times M, \text{End}(g))$ by solving the equation,

$$\triangle_{H, x} G(x, y)^{a_1}_{a_2} = \delta_{\triangle_H}(x, y) \cdot 1^{a_1}_{a_2}, \quad a_1, a_2 = 1, \ldots, \text{dim } g. \quad (0.1.11)$$

We may also find the kernel $K_{r,1}^{1,1}(x, y)^{a_1}_{a_2} \in \Omega^{1,1}(H \times H, \text{End}(g))$, by solving the equation

$$D_{r,x} K_{r,1}^{1,1}(x, y)^{a_1}_{a_2} = \delta_D(x, y) \cdot 1^{a_1}_{a_2}, \quad a_1, a_2 = 1, \ldots, \text{dim } g, \quad (0.1.12)$$

The delta functional $\delta_D$ is naturally viewed as distributional horizontal $(1, 1)$-form. The specific forms of $\delta_{\triangle_H}$ and $\delta_D$ are given in Chapter 2.

\footnote{\[1\Omega^1(H, g)\] is the space of horizontal 1-form valued in $g$ given by $\{ \beta \in \Omega^1(M, g) | \beta(R) = 0 \}$. $R$ is the Reeb field corresponding to $\kappa$: $\kappa(R) = 1$, $d\kappa(R, \bullet) = 0$.}
0.2 Wilson Loop Operator and Legendrian Knots

Knots are described by Wilson loop operator in gauge theory. Given a knot $C$ in a manifold $M$ with connection $A \in \Omega^1(M, g)$, the Wilson loop operator $W$ is defined as

$$W_R(C) = \text{Tr}_R \text{Pexp} \left( - \oint_C A \right) \equiv \text{Tr}_R \text{Hol}_C(A), \quad (0.2.1)$$

where $R$ is a representation of $g$, $P$ is the path-ordering operator, and $\text{Hol}_C(A)$ is the holonomy of $A$ around the knot $C$.

One can take the expansion of $W_R(C)$ as

$$W_R(C) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \oint_{C^n} P \left[ A(s_1) \cdots A(s_n) \right], \quad C^n = C \times \cdots \times C. \quad (0.2.2)$$

Note that the operator $P$ requires that $s_1, \ldots, s_n$ are in the clockwise order. If we take an orthonormal basis $\{T_a\}$ of the Lie algebra $g$ and write

$$A = \sum_{a=1}^{\dim g} A^a T_a, \quad A^a \in \Omega^1(M) \quad (0.2.3)$$

Then

$$W_R(C) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} I_{a_1 \cdots a_n}^R \oint_{C^n} P \left[ A^{a_1}(s_1) \cdots A^{a_n}(s_n) \right], \quad (0.2.4)$$

where

$$I_{a_1 \cdots a_n}^R = \text{Tr}_R (T_{a_1} \cdots T_{a_n}) \in (g^\vee)^\otimes n. \quad (0.2.5)$$

Our interest is in the study of the perturbative expansion of the expectation value of the Wilson loop operator in the contact Chern-Simons theory:

$$\langle W_R(C) \rangle = \frac{Z(k; C, R)}{Z(k)}. \quad (0.2.6)$$

where

$$Z(k) = \frac{1}{\text{Vol}(g)} \int_A \mathcal{D}A \exp[i k I_r(A)]. \quad (0.2.7)$$
and
\[ Z(k; C, R) = \frac{1}{\text{Vol}(G)} \int_A DA W_R(C) \exp[i k \mathbf{I}_r(A)], \tag{0.2.8} \]
are two path integrals over the space of connections. By 0.2.4, the expectation value of the Wilson loop operator can be written as
\[ \langle W_R(C) \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} I^R_{a_1 \ldots a_n} \oint_{C^n} \langle P[A^{a_1}(s_1) \ldots A^{a_n}(s_n)] \rangle. \tag{0.2.9} \]
where the integrand is given by
\[ \langle P[A^{a_1}(s_1) \ldots A^{a_n}(s_n)] \rangle = \frac{1}{Z(k)} \frac{1}{\text{Vol}(G)} \int_A DA P[A^{a_1}(s_1) \ldots A^{a_n}(s_n)] \exp[i k \mathbf{I}_r(A)]. \tag{0.2.10} \]

We set up certain Feynman rules for the integral above in Chapter 1 and the expectation value of the Wilson loop operator can be written as
\[ \langle W_R(C) \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \oint_{C^n} P \left[ \begin{array} \text{central blob} \end{array} \right]. \tag{0.2.11} \]

In the diagrammatic shorthand, each diagram inside the circle represents the asymptotic expansion of the expectation value \( \langle B^{a_1}(x_1) \ldots B^{a_n}(x_n) \rangle \) for \( (x_1, \ldots, x_n) \in C^n \), where the central blob represents all connected ways to tie together the \( n \) external gauge field propagators using the propagators \( G \) and \( K^{1,1} \) and related vertices. The whole wheel represents the path-ordered integrand for \( (x_1, \ldots, x_n) \in C^n \) with \( n \) spokes. At least to leading order, the graphical expansion in (0.2.11) is quite simple,
\[ \langle W_R(C) \rangle = \dim R + \frac{1}{2} \oint_{C^2} P \left[ \begin{array} \text{central blob} \end{array} \right] + \mathcal{O}(g^4). \tag{0.2.12} \]
where $g^2 = \frac{2\pi}{k}$.

For the invariance of $I_r(A)$ under the shift (0.1.4), we require that the Wilson loop operator is also invariant under the shift. That is to say

$$\oint_C f\kappa = 0, \quad \forall f \in C^\infty(M, g) \tag{0.2.13}$$

This implies that $\kappa|_C = 0$. By definition, $C$ is a Legendrian knot. In this thesis, $M$ is taken to be $\mathbb{R}^3$ and the contact form is taken to be $\kappa = dz + \frac{1}{2}(xdy - ydx)$ \footnote{In the last two chapters, the coefficient $\frac{1}{2}$ will be suppressed. However, the absence of the $\frac{1}{2}$ will not change the result of any computation in those two chapters.}. In this case, Legendrian knots follow the differential equation:

$$\frac{dz}{d\theta} = \frac{1}{2}(y\frac{dx}{d\theta} - x\frac{dy}{d\theta}) \tag{0.2.14}$$

where $\theta$ is the parameter of the Legendrian knot.

### 0.3 Linking Number and Self-Linking Number

The first correction term in the expansion (0.2.12) can be written as

$$\oint_C \mathbf{P} = \lim_{\epsilon \to 0} \int_{C \times C \setminus \Delta\epsilon} K^{1,1}_r(x, y), \tag{0.3.1}$$

Here we follow the standard way to excise a neighborhood of the diagonal $\Delta\epsilon := \{(x, x) | x \in C\}$ with width $\epsilon$ in the integral domain and take the limit $\epsilon \to 0$ to define this improper integral.

Let $X : S^1 \to \mathbb{R}^3$ be the map giving the Legendrian knot $C$,

$$\psi = \frac{1}{8\pi} \frac{\epsilon_{\mu\nu\rho} x^\mu dx^\nu \wedge dx^\rho}{||x||^3}, \quad x \neq 0 \in \mathbb{R}^3. \tag{0.3.2}$$
be the angular form on the punctured space $\mathbb{R}^3 - \{0\}$, and
\[
\Gamma : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \Gamma(x, y) = x - y, \tag{0.3.3}
\]
be the difference map. If we replace the integrand $K$ in 0.3.1 by $(-1) \cdot (X \times X)^* \Gamma^* \psi$, we get the Gauss self-linking integral:
\[
L(C) = \lim_{\epsilon \rightarrow 0^+} \int_{S^1 \times S^1 - \Delta(\epsilon)} (-1) \cdot (X \times X)^* \Gamma^* \psi. \tag{0.3.4}
\]
This self-linking integral is neither gauge invariant nor homotopic invariant. In fact, we show that the variation of $L(C)$ is given by
\[
\delta L(C) = -\frac{1}{2\pi} \oint_{S^1} d\varphi \, \epsilon_{\mu\nu\rho} \dot{X}^\mu \delta X^\nu \dot{X}^\rho, \quad ||\dot{X}(\varphi)||^2 = 1. \tag{0.3.5}
\]
where $\varphi$ is the arc-length parameter of $C$.

There are two approaches for us to find a homotopic invariant quantity based on the Gauss self-linking integral. Both of these two ways involve renormalization of the Wilson loop operator. One approach is subtracting a suitable counter term to cancel the variation. Famously, the necessary counterterm is the total torsion measured with respect to arc-length,
\[
T(C) = \frac{1}{2\pi} \oint_{S^1} d\varphi \, \tau, \tag{0.3.6}
\]
where $\tau$ is the torsion of $C$. So if we define
\[
L_\tau(C) = L(C) + T(C), \tag{0.3.7}
\]
then $L_\tau$ is invariant under arbitrary deformation of $C$.

In Chapter 1, we show that $L_\tau$ is an integer and understand it as an outcome of the renormalization of the Wilson loop operator as follows:
\[
W_q(C)_\tau \equiv W_q(C) \cdot \exp \left[ \frac{i q^2}{2k} \oint_{S^1} d\varphi \, \tau \right]. \tag{0.3.8}
\]
The other approach to construct a homotopic invariant is framing. A framing is a trivialization of the normal bundle for $C \subset \mathbb{R}^3$, up to homotopy. Such a trivialization is specified by a pair $\{n_1, n_2\}$ of linearly-independent, normal vector fields along $C$. If we denote the unit tangent vector of $C$ as $t$ and further assume that $\{t, n_1, n_2\}$ is an oriented orthonormal basis at each point of $C$, then $n_2$ is determined by $\{t, n_1\}$. So a framing of $C$ is given by a single, nowhere-vanishing normal vector $n := n_1$ along $C$. Up to homotopy, there are countably many ways to fame a knot, but if the curvature $\kappa > 0$ everywhere along $C$, as a result in elementary differential geometry, a canonical normal vector $n$ can be given by

$$\ddot{X} = \kappa n, \quad ||n||^2 = 1,$$

We can renormalize the Wilson loop operator by point splitting, i.e., constructing a new knot $\tilde{C}$ by shifting the knot $C$ in the direction of $n$ in a short distance. We can replace the naive Gauss self-linking integral $L(C)$ by the framed self-linking integral:

$$L_f(C) = L(C, \tilde{C}) := \int_{S^1 \times S^1} (-1) \cdot (X \times \tilde{X})^\ast \Gamma^\ast \psi$$

In the end of Chapter 1, we give an argument to show that

$$L_f(C) = L_r(C)$$

In Chapter 4 and 5, we study the following two integrals:

$$I(C_1, C_2) = \int_{C_1 \times C_2} K^{1,1}_{r=1}(x, y),$$

and

$$I(C) = \lim_{\epsilon \to 0} \int_{C \times C \setminus \Delta} K^{1,1}_{r<1}(x, y),$$

where $C_1$, $C_2$, and $C$ are Legendrian knots. The results turn out to be invariant under Legendrian isotopies.
0.4 Main Results

Our first main result concerns the propagators $G$ and $K$ of the contact Chern-Simons theory. The manifold we are working on is $M = \mathbb{R}^3$. It has a Heisenberg group structure:

\[
x * y = (u_x + u_y, v_x + v_y, t_x + t_y - u_x v_y + u_y v_x), \quad (0.4.1)
\]

\[
= (\zeta_x + \zeta_y, t_x + t_y + \text{Im}(\zeta_x \zeta_y)).
\]

where

\[
x = (u_x, v_x, t_x)
\]

\[
y = (u_y, v_y, t_y) \quad (0.4.2)
\]

\[
\zeta = u + iv
\]

The operators $\Delta_H$ and $D_r$ are left invariant with respect to this structure and they can be written as

\[
\Delta_H(f) = \mathfrak{L}_0 f,
\]

\[
D_r(fd\zeta) = [(-ir \cdot \mathfrak{L}_{\lambda_r})f]d\zeta,
\]

\[
D_r(fd\bar{\zeta}) = [(-ir \cdot \mathfrak{L}_{-\lambda_r})f]d\bar{\zeta},
\]

here

\[
\mathfrak{L}_\lambda = ZZ + ZZ - i\lambda \partial_t. \quad (0.4.3)
\]

\[
\lambda_r = \frac{i}{r} + 1 \quad (0.4.4)
\]

\[
Z = \frac{1}{2} (U - i V) \quad (0.4.5)
\]

and

\[
U = \frac{\partial}{\partial u} + v \frac{\partial}{\partial t},
\]

\[
V = \frac{\partial}{\partial v} - u \frac{\partial}{\partial t},
\]
The operator $\mathcal{L}_\lambda$ was studied in [30], and the fundamental solution of the operator $\mathcal{L}_\lambda$ is given by

$$\phi_\lambda = c_\lambda \left( t + i \frac{|\zeta|^2}{2} \right)^{-\frac{1+i\lambda}{2}} \cdot \left( t - i \frac{|\zeta|^2}{2} \right)^{-\frac{1-i\lambda}{2}}, \quad (0.4.6)$$

where,

$$c_\lambda = - \frac{\Gamma(\frac{1}{2}(1+\lambda))\Gamma(\frac{1}{2}(1-\lambda))}{i^\lambda \pi^{1/2}}.$$

With the help of the above result, we have

**Theorem 0.4.1.** For $x, y \in \mathbb{R}^3$, the kernels $G(x, y)^{a_1}_{a_2}$ and $K(x, y)^{a_1}_{a_2}$ are given by,

$$G(x, y)^{a_1}_{a_2} = \phi_0(y^{-1} \ast x) \cdot T^{a_1}_{a_2}, \quad (0.4.7)$$

and,

$$K^{1,1}_{r}(x, y)^{a_1}_{a_2} = \frac{i}{2r} \left[ \phi_\lambda_r(y^{-1} \ast x) d\zeta_x \wedge d\zeta_y + \phi_{-\lambda_r}(y^{-1} \ast x) d\zeta_x \wedge d\zeta_y \right] \cdot T^{a_1}_{a_2}. \quad (0.4.8)$$

where $y^{-1}$ is the inverse of $y$ in the Heisenberg structure of $\mathbb{R}^3$.

In the second main result, we consider the following integral:

$$I(C_1, C_2) = \int_{C_1 \times C_2} K^{1,1}_{r}(x, y), \quad (0.4.9)$$

where $C_1$ and $C_2$ are Legendrian knots. Apparently, this integral is an analogy of the Gauss linking integral. In fact we prove in Chapter 4

**Theorem 0.4.2.** Suppose $C_{1,2} \subset \mathbb{R}^3$ are disjoint oriented Legendrian curves. Then

$$I(C_1, C_2) = \int_{C_1 \times C_2} K^{1,1}_{r}(x, y) = L(C_1, C_2), \quad (0.4.10)$$

is the linking number of the Legendrian link $(C_1, C_2)$.

We study the asymptotic behavior of the propagator $K^{1,1}_{r}$ when $r \ll 1$ and show

**Theorem 0.4.3.** When $r \ll 1$, the leading order asymptotic estimate of $K^{1,1}_{r}$ is given
by

\[ K^{1,1}_r(x, y) \sim -\frac{2}{\pi r \Delta t} \exp \left( -\frac{\left|\Delta \zeta\right|^2}{2r \Delta t} \right) d\Delta u \wedge d\Delta v, \tag{0.4.11} \]

where

\[ x^{-1} \ast y = (\Delta u, \Delta v, \hat{\Delta} t) \]
\[ \Delta \zeta = \zeta_x - \zeta_y \tag{0.4.12} \]

The leading order asymptotic estimate is in the form of heat kernel. This kernel has nice behavior at short distance, i.e., in a small neighborhood of the diagonal \{\((x, x)\mid x \in C\}\}. We study the following integral

\[ I(C) = \lim_{\epsilon \to 0} \int_{C \times C \setminus \Delta} \frac{2}{\pi r \Delta t} \exp \left( -\frac{\left|\Delta \zeta\right|^2}{2r \Delta t} \right) d\Delta u \wedge d\Delta v, \tag{0.4.13} \]

for \(\forall r > 0\) and prove

**Theorem 0.4.4.**

\[ I(C) = \text{tb}(C), \tag{0.4.14} \]

where \(\text{tb}(C)\) is the Thurston-Bennequin invariant of \(C\).

The critical technique in the proof of the above theorem is perturbative localization. The basic idea of this technique is when \(r \ll 1\), the integrand of 0.4.13 is negligible unless \(\left|\Delta \zeta\right|\) is small. We prove that the integral 0.4.13 is independent with the value of \(r > 0\). So, to evaluate the integral, we just need to take \(r \ll 1\) and focus on the pairs of points \((x, y) \in C \times C\) for which \(\left|\Delta \zeta\right|\) is small. These kinds of pairs of points can be divided into two classes: points with close values of parameter and points whose Lagrangian projections are close to a crossing point in the Lagrangian projection of the knot \(C\). In Chapter 5, we give a very detailed analysis.
Chapter 1

Perturbative Chern-Simons Theory on One Leg

In preparation for our analysis of Legendrian Wilson loops in contact Chern-Simons theory, we begin by reviewing the corresponding computations for topological Wilson loops in ordinary Chern-Simons theory. In Section 1.1 we establish basic gauge theory conventions, followed in Section 1.2 by a sketch of the main ideas underlying Chern-Simons perturbation theory. This material is all very well-known. Important early work on Chern-Simons perturbation theory includes [7, 8, 9, 10, 11, 36, 37], from which we especially recommend Bar-Natan’s Princeton thesis [9] as a source which will not soon go out of style. A recent overview of the subject, suitable for mathematicians, can be found in [61].

Along the way, we highlight in Section 1.3 some well-known geometric features of the self-linking integrals that arise in Chern-Simons perturbation theory. Our focus lies on the topological anomaly in the naive Gauss self-linking integral, discussed originally in [21, 54] and later rediscovered [55, 69] from a more physical perspective. Though this anomaly can be characterized systematically [2, 20, 63] to all orders in perturbation theory (and cancelled after framing the knot), it represents a significant complication for the analysis of perturbative knot invariants. Among the more beautiful results in this paper, we eliminate the anomaly entirely when we pass to the Legendrian world.
Finally, Appendices 5.3.3 and 5.3.3 contain two computations related to the topological anomaly. These results are classics dating to [21, 54] but simple, direct computations seem to be scarce in the literature, so we have included our own.

1.1 Preliminary Remarks

We first fix conventions. Throughout, the gauge group $G$ will be a compact, connected, simply-connected, and simple Lie group, for instance $G = SU(N)$. To leading order in perturbation theory, there is little difference between abelian and non-abelian gauge theory, so we also consider $G = U(1)$ as a degenerate case.

The Lie algebra $\mathfrak{g}$ of $G$ is equipped with an invariant, negative-definite form $\text{Tr}$, which enters the Chern-Simons functional

$$
\text{CS}(A) = \frac{1}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).
$$

(1.1.1)

The gauge field $A$ is a connection on a principal $G$-bundle over an oriented three-manifold $M$. Working in perturbation theory, we always assume that the $G$-bundle is both trivial and trivialized, so that $A \in \Omega^1(M; \mathfrak{g})$ can be regarded as a Lie algebra-valued one-form on $M$. Explicitly, $A$ admits an expansion

$$
A = \sum_{a=1}^{\dim \mathfrak{g}} A^a T_a, \quad A^a \in \Omega^1(M), \quad T_a \in \mathfrak{g},
$$

(1.1.2)

where we choose a set of orthonormal generators $T_a$ for the Lie algebra, such that $\text{Tr}(T_a T_b) = -\delta_{ab}$ for $a, b = 1, \ldots, \dim \mathfrak{g}$. Here $\delta_{ab}$ is the Kronecker-delta. As standard, both the generators $T_a$ and the gauge field $A$ are represented by anti-hermitian matrices. For $G = U(1)$, this convention means that $A$ is purely imaginary.

After the gauge field $A$ is expanded in terms of the Lie algebra generators in (1.1.2), all perturbative dependence upon the choice of the gauge group $G$ occurs through the structure constants $f^c_{ab}$ of the Lie algebra,

$$
[T_a, T_b] = \sum_{c=1}^{\dim \mathfrak{g}} f^c_{ab} T_c, \quad f^c_{ab} \in \mathbb{R}.
$$

(1.1.3)
Manifestly, since $\text{Tr}(A^0 A^1 A^2) = \frac{1}{2} \text{Tr}([A, A] \wedge A) = -\frac{1}{2} f_{abc} A^a \wedge A^b \wedge A^c$, the Chern-Simons functional can be expressed solely in terms of the collection of one-forms $A^a$ and the structure constants $f_{abc} = f_{eab} \delta_{ec}$.

$$\text{CS}(A) = -\frac{1}{4\pi} \int_M \left( \delta_{ab} A^a \wedge dA^b + \frac{1}{3} f_{abc} A^a \wedge A^b \wedge A^c \right), \quad a, b, c = 1, \ldots, \dim \mathfrak{g}. \quad (1.1.4)$$

We employ the Einstein summation convention for the Lie algebra indices in (1.1.4) and henceforth.

Given our assumptions on $G$, the invariant form $\text{Tr}$ is unique up to normalization. For non-abelian $G$, we fix the normalization so that the value of the Chern-Simons functional $\text{CS}(A)$ in (1.1.1) is well-defined as an element of $\mathbb{R}/2\pi\mathbb{Z}$ if $M$ is a closed three-manifold. With this choice, the Chern-Simons level $k \in \mathbb{Z}$ obeys the familiar integral quantization when it appears in the formal path integral over the affine space $\mathcal{A}$ of connections on $M$,

$$Z(k) = \frac{1}{\text{Vol}(G)} \int_\mathcal{A} \mathcal{D}A \exp[i k \text{CS}(A)]. \quad (1.1.5)$$

As usual, we divide the path integral by the formal volume of the group $G \simeq \text{Map}(M, G)$ of gauge transformations. For $G = SU(N)$, our normalization convention means that $\text{Tr}$ is the matrix trace in the fundamental representation.

Otherwise, when $G = U(1)$ we simply set

$$\text{CS}(A) = \frac{1}{4\pi} \int_M A \wedge dA, \quad (1.1.6)$$

with $A$ normalized so that the cohomology class of the curvature satisfies the integrality condition $[F_A]/2\pi i \in H^2(M; \mathbb{Z})$. Gauge-invariance of the integrand in (1.1.5) requires the Chern-Simons level $k \in 2\mathbb{Z}$ to be an even integer in the abelian case. When $M$ is endowed with a spin structure [17, 27], abelian Chern-Simons theory becomes sensible for all integers $k \in \mathbb{Z}$, though we will not see this refinement in our perturbative computations.

When the orientation of $M$ is reversed, the Chern-Simons functional in (1.1.1)
changes sign, so without loss we take $k > 0$ to be positive.

**Knots and Wilson Loops.** In the language of gauge theory, knots are described by Wilson loop operators

$$ W_R(C) = \text{Tr}_R \text{Pexp} \left( - \oint_C A \right) \equiv \text{Tr}_R \text{Hol}_C(A), \quad (1.1.7) $$

where $C \subset M$ is a smoothly embedded, oriented\(^1\) curve, decorated with an irreducible representation $R$ of the gauge group $G$. According to the usual notation in physics, the path-ordered exponential ‘Pexp’ in (1.1.7) is shorthand for the holonomy of $A$ around the curve $C$, and ‘Tr\(_R\)’ indicates the trace in the representation $R$. Because we work with the covariant derivative $d_A = d + A$, a minus sign occurs naturally in the argument of the exponential.

Concretely, path-ordering is applied to the Taylor expansion of the exponential in (1.1.7), so that in perturbation theory

$$ W_R(C) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \oint_{C^n} \text{P} \left[ A(s_1) \cdots A(s_n) \right], \quad C^n = C \times \cdots \times C. \quad (1.1.8) $$

Equivalently, in terms of the Lie algebra expansion (1.1.2) for $A$,

$$ W_R(C) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} l^R_{a_1 \cdots a_n} \oint_{C^n} \text{P} \left[ A^{a_1}(s_1) \cdots A^{a_n}(s_n) \right], \quad (1.1.9) $$

for which all dependence on the representation $R$ is encoded by the invariant tensors

$$ l^R_{a_1 \cdots a_n} = \text{Tr}_R(T_{a_1} \cdots T_{a_n}) \in (g^\vee)^{\otimes n}. \quad (1.1.10) $$

To make the structure of the multiple integrals in (1.1.8) and (1.1.9) clear, we introduce parameters $(s_1, \ldots, s_n)$ for each copy of $C$ in the $n$-fold product $C^n$, with $s_j \in [0, 2\pi)$ for each $j$. The path-ordering symbol $\text{P}$ then indicates that the factors of

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\(^1\) As a functional of the gauge field, the Wilson loop operator is invariant under orientation-reversal of $C$ accompanied by an exchange of $R$ with the dual representation $R^\vee$. So if $R \simeq R^\vee$ is a real or pseudoreal representation, as for instance always the case when $G = SU(2)$, the Wilson loop does not depend upon the choice of orientation for $C$. 

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A in the multiple integrals are to be ordered so that \( s_1 \geq s_2 \geq \cdots \geq s_n \), up to cyclic permutations. Because the tensor \( I^R_{a_1 \cdots a_n} \) is defined by the trace in the representation \( R \), it is trivially preserved under corresponding cyclic permutations of the Lie algebra indices, so only the cyclic ordering of \((s_1, \ldots, s_n)\) around \( C \) actually matters. To indicate these features, we adopt the diagrammatic notation for \( I^R_{a_1 \cdots a_n} \) in Figure 1-1.

![Figure 1-1: Diagram for the invariant tensor \( I^R_{a_1 \cdots a_n} \).](image)

For the record, if the group \( G \) is simple,

\[
I^R_0 = \dim R, \quad I^R_{a_1} = 0, \quad I^R_{a_1 a_2} = -c_2(R) \cdot \delta_{a_1 a_2}, \quad (1.1.11)
\]

where \( c_2(R) \) is the quadratic Casimir of the representation. With our normalization, the quadratic Casimir of the fundamental representation of \( SU(N) \) is \( c_2(N) = 1 \).

When \( G = U(1) \) is abelian, path-ordering is no longer necessary for the classical Wilson loop. We just write

\[
W_q(C) = \exp \left[ -q \oint_C A \right], \quad q \in \mathbb{Z}, \quad (1.1.12)
\]

where \( q \) is the charge of the \( U(1) \) representation.

In any discussion of Wilson loop operators in Chern-Simons theory, one immediately faces the question of whether to consider the absolutely-normalized Wilson loop path integral or the Wilson loop expectation value. The Wilson loop path integral is
given by the generalization of (1.1.5) to include $W_R(C)$,

$$Z(k; C, R) = \frac{1}{\text{Vol}(\mathcal{G})} \int_{\mathcal{A}} \mathcal{D}A \ W_R(C) \exp[i \ k \ \text{CS}(A)], \quad (1.1.13)$$

whereas the Wilson loop expectation value is given by the ratio

$$\langle W_R(C) \rangle = \frac{Z(k; C, R)}{Z(k)}. \quad (1.1.14)$$

When the Wilson loop is studied semi-classically using localization, as for instance in [13, 43, 45], the absolutely-normalized Wilson loop path integral $Z(k; C, R)$ is by far the easier quantity to consider. On the other hand, since $W_R(C)$ reduces to the identity operator when $R = 1$ is the trivial representation, the expectation value obeys

$$\langle W_{R=1}(C) \rangle = 1. \quad (1.1.15)$$

In perturbation theory, the normalization condition in (1.1.15) implies that disconnected “vacuum-bubble” Feynman diagrams do not contribute to the Wilson loop expectation value. As a result, the Wilson loop expectation value $\langle W_R(C) \rangle$ is the more convenient observable for a perturbative analysis, our present focus.

Finally, if $L = C_1 \cup \cdots \cup C_m$ is a link with $m$ components, each decorated with a representation $R_1, \ldots, R_m$, the Wilson loop path integral in (1.1.13) generalizes in the obvious way,

$$Z\left(k; (C_1, R_1), \cdots, (C_m, R_m)\right) = \frac{1}{\text{Vol}(\mathcal{G})} \int_{\mathcal{A}} \mathcal{D}A \ W_{R_1}(C_1) \cdots W_{R_m}(C_m) \exp[i \ k \ \text{CS}(A)], \quad (1.1.16)$$

with link expectation value

$$\langle W_{R_1}(C_1) \cdots W_{R_m}(C_m) \rangle = \frac{Z\left(k; (C_1, R_1), \cdots, (C_m, R_m)\right)}{Z(k)}. \quad (1.1.17)$$

If $L$ is a trivial link, meaning that each component $C_j$ for $j = 1, \ldots, m$ can be isotoped to lie in a three-ball $B^3 \subset M$ which is disjoint from the other components, the link
expectation value factorizes,

\[ \langle W_{R_1}(C_1) \cdots W_{R_m}(C_m) \rangle = \langle W_{R_1}(C_1) \rangle \cdots \langle W_{R_m}(C_m) \rangle, \quad [L \text{ trivial}] \]

but not generally otherwise.

1.2 Gauge Fixing, Feynman Rules, and All That

Before tackling the novelties of contact Chern-Simons theory, let us recall the basic setup for the perturbative analysis of ordinary Chern-Simons theory.

To perform computations in perturbation theory, one must always make additional choices. The first choice is simply that of a classical, background solution to the equation of motion

\[ F_A = dA + A \wedge A = 0, \]

which characterizes a critical point of the Chern-Simons functional in (1.1.1). A solution to (1.2.1) is a flat connection \( A_0 \) on \( M \). For simplicity, we take \( A_0 \) to be isolated and irreducible. Otherwise, one must integrate over the component of the moduli space of flat connections which contains \( A_0 \), and one must account for the subgroup \( G_0 \subset G \) of gauge transformations which fix \( A_0 \), respectively.

Given the flat connection \( A_0 \), we write \( A = A_0 + B \) and expand the Chern-Simons functional in terms of the fluctuating field \( B \in \Omega^1(M, g) \),

\[ \text{CS}(A) = \text{CS}(A_0) + \frac{1}{4\pi} \int_M \text{Tr} \left( B \wedge d_{A_0} B + \frac{2}{3} B \wedge B \wedge B \right), \]

where \( d_{A_0} B = dB + [A_0, B] \) is the covariant derivative determined by \( A_0 \). Since \( A_0 \) is a critical point of the Chern-Simons functional, no term linear in \( B \) appears in the expansion (1.2.2), and the expansion stops at third-order for obvious reasons!

Of particular interest is the quadratic term in (1.2.2). This term describes the Hessian of the Chern-Simons functional at the point \( [A_0] \in \mathcal{A} \) in the affine space of connections. Explicitly, the Hessian \( \mathcal{H} : T_{[A_0]} \mathcal{A} \otimes T_{[A_0]} \mathcal{A} \to \mathbb{R} \) is given by the sym-
metric pairing

\[ \mathcal{H}(\eta, \xi) = \frac{1}{4\pi} \int_M \text{Tr}(\eta \wedge d_{A_0} \xi), \quad \eta, \xi \in T_{[A_0]}\mathcal{A} \simeq \Omega^1(M; \mathfrak{g}), \quad (1.2.3) \]

where we identify tangent vectors to \( \mathcal{A} \) with one-forms on \( M \). To avoid cluttering the notation, the dependence of \( \mathcal{H} \) on the flat connection \( A_0 \) has been suppressed.

Once we specify a metric \((\cdot, \cdot)\) on \( \mathcal{A} \), the Hessian dually determines a linear operator \( D \) on \( \Omega^1(M; \mathfrak{g}) \) via the identity

\[ \mathcal{H}(\eta, \xi) = -\frac{1}{4\pi} (\eta, D \xi), \quad D : \Omega^1(M; \mathfrak{g}) \rightarrow \Omega^1(M; \mathfrak{g}). \quad (1.2.4) \]

Such a metric on \( \mathcal{A} \) is induced from a Riemannian metric \( g \) on \( M \),

\[ (\eta, \xi) = -\int_M \text{Tr}(\eta \wedge \star \xi), \quad (1.2.5) \]

where \( \star \) is the Hodge star associated to the metric on \( M \). The minus sign in (1.2.5) arises from our convention that the quadratic form ‘\( \text{Tr} \)’ on \( \mathfrak{g} \) is negative-definite. Evidently from (1.2.3), (1.2.4), and (1.2.5),

\[ D = \star d_{A_0}, \quad (1.2.6) \]

since \( \star^2 = 1 \) when acting on \( \Omega^1(M) \).

All subtleties of Chern-Simons perturbation theory arise from the fact that \( D \) is highly degenerate, with a very large kernel. Because \( A_0 \) is flat, \( d_{A_0}^2 = F_{A_0} = 0 \) by definition, so \( D \) automatically annihilates the image of \( d_{A_0} \) acting on \( \Omega^0(M; \mathfrak{g}) \),

\[ D\big|_{\text{Im}(d_{A_0})} = 0. \quad (1.2.7) \]

Hence the kernel of \( D \) has infinite dimension.

This degeneracy is no accident, as it is guaranteed by the gauge-invariance of the Chern-Simons functional. Under an infinitesimal gauge transformation generated by an element \( \varphi \in \text{Lie}(\mathcal{G}) \simeq \Omega^0(M; \mathfrak{g}) \), the gauge field \( A \) transforms by \( \delta A = -d_A \varphi \). The
one-form $d_{A_0} \varphi$ therefore describes the tangent vector to the gauge orbit generated by $\varphi$ at the point $[A_0] \in \mathcal{A}$, and the Hessian of any gauge-invariant functional must vanish on the tangent vectors to gauge orbits.

**BRST Procedure.** To perform perturbative Feynman diagram calculations, we are instructed to invert $D$ to determine the propagator $K = D^{-1}$ for the Chern-Simons gauge field. By the preceding discussion, $K$ can only be defined after we fix the gauge symmetry, effectively replacing the naive version of $D$ in (1.2.6) with an operator which is invertible, at least away from a finite-dimensional kernel.

We follow the usual BRST procedure, which involves the introduction of an anti-commuting ghost field $c \in \Omega^0(M; \mathfrak{g})$ valued in the Lie algebra of $G$. The BRST operator $Q$ acts on the pair $(A, c)$ via

$$
\delta A = -d_A c, \quad \delta c = \frac{1}{2} [c, c],
$$

or in terms of the generators $T^a$ for $\mathfrak{g}$,

$$
\delta A^a = -\left( dc^a + f_{bc}^a A^b c^c \right), \quad \delta c^a = \frac{1}{2} f_{bc}^a c^b c^c.
$$

The Jacobi identity for the Lie bracket ensures that $Q^2 = 0$ in (1.2.8). We also introduce a second anti-commuting ghost field $b \in \Omega^3(M; \mathfrak{g})$, along with an auxiliary commuting scalar field $\phi \in \Omega^3(M; \mathfrak{g})$. The BRST operator acts on the pair $(b, \phi)$ by

$$
\delta b = \phi, \quad \delta \phi = 0.
$$

Though we will not take the following route, see the work [7, 8] of Axelrod and Singer for a very elegant superspace formalism which treats the BRST multiplet $(A, \phi, b, c)$ in a unified fashion.

2For any field $\Phi$, the symbol $\delta \Phi$ is shorthand for the graded commutator $\{Q, \Phi\}$. If $\Phi$ is commuting, then $\{Q, \Phi\} \equiv [Q, \Phi]$ is the usual commutator. Otherwise if $\Phi$ is anti-commuting, $\{Q, \Phi\} \equiv \{Q, \Phi\}$ is the anti-commutator.
Next, we replace the Chern-Simons functional in (1.1.5) by the gauge-fixed action

\[ S(A, \phi, b, c) = \text{CS}(A) + \delta V(A, \phi, b, c), \]  

(1.2.11)

where \( V \) is any functional of \((A, \phi, b, c)\) which ensures that the Hessian of the new action in (1.2.11) be non-degenerate. The BRST prepotential \( V \) itself cannot be gauge-invariant, and \( V \) is allowed to depend upon the choice of the background flat connection \( A_0 \) about which we expand in perturbation theory.

In Chern-Simons theory on \( M \) there is a standard choice for \( V \), but there is no canonical choice. A canonical choice, if one existed, would preserve all symmetries in the problem, including the diffeomorphism invariance of the Chern-Simons action in (1.1.1). Yet the BRST prepotential \( V \) inevitably involves the choice of a Riemannian metric \( g \) on \( M \), as appears already in the operator \( D \), and any particular \( g \) is only preserved by those diffeomorphisms which act by isometries. Ultimately, the need to specify a metric on \( M \) through the gauge-fixing term \( \delta V \) leads to the rather subtle quantum dependence on framing [6, 69] in the knot and three-manifold invariants derived from Chern-Simons gauge theory. For the case of knots, we will have a great deal more to say about framing issues in Section 1.3.

These caveats aside, a standard choice for \( V \) is given by

\[ V = \frac{1}{2\pi} \int_M \text{Tr}(b \wedge \star d_{A_0} \star B), \]  

(1.2.12)

for which the gauge-fixed action in (1.2.11) becomes\(^3\)

\[ S = \text{CS}(A) + \frac{1}{2\pi} \int_M \text{Tr}(\phi \wedge \star d_{A_0} \star B + b \wedge \star d_{A_0} \star d_{A_c}). \]  

(1.2.13)

\(^3\)Since \( A_0 \) is a fixed background connection, \( \delta A_0 = 0 \) under the BRST symmetry. Hence the BRST operator acts by \( \delta B = -d_{A_0} c = -d_{A_0} c - [B, c] \).
Expanding further about $A_0$,

$$S = CS(A_0) + \frac{1}{4\pi} \int_M \text{Tr}(B \wedge d_{A_0} B + 2 \phi \wedge *d_{A_0}*B - 2 b \wedge \Delta c) +$$

$$+ \frac{1}{4\pi} \int_M \text{Tr}\left(\frac{2}{3} B \wedge B \wedge B + 2 d_{A_0}^\dagger b \wedge [B, c]\right).$$

(1.2.14)

Here $\Delta = d_{A_0}^\dagger d_{A_0}$ is the covariant Laplacian acting on sections of $\Omega^0(M; g)$, and $d_{A_0}^\dagger = -* d_{A_0}*$ is the $L^2$-adjoint of $d_{A_0}$. Our sign conventions ensure that $\Delta \geq 0$ is a positive operator. Since $\phi$ enters the action $S$ linearly, with no derivatives, the classical equation of motion for $\phi$ enforces the harmonic gauge condition

$$d_{A_0} * B = 0.$$  

(1.2.15)

The fields $B$ and $\phi$ can be naturally grouped together as a section $\Phi \equiv (B, \phi)$ of the direct sum

$$\Omega^-(M; g) = \Omega^1(M; g) \oplus \Omega^3(M; g),$$

(1.2.16)

where the minus sign indicates the parity of the degree as a differential form on $M$. Similarly, for future reference,

$$\Omega^+(M; g) = \Omega^0(M; g) \oplus \Omega^2(M; g).$$

(1.2.17)

In terms of $\Phi \in \Omega^-(M; g)$, the quadratic piece of the gauge-fixed action in (1.2.14) takes the form

$$S = CS(A_0) - \frac{1}{4\pi} (\Phi, L \Phi) + \frac{1}{2\pi} (b, * \Delta c) + \cdots.$$  

(1.2.18)

Here $(\cdot, \cdot)$ is again the $L^2$-inner product from (1.2.5), and `$\cdots$' indicates the cubic interaction terms for $B$ and the $bc$-ghosts in the second line of (1.2.14). Notably, the quadratic term for $\Phi$ involves the self-adjoint elliptic operator $L : \Omega^-(M; g) \to \Omega^-(M; g)$ whose signature is the APS eta-invariant [4],

$$L = (* d_{A_0} + d_{A_0} *) \circ 1,$$  

(1.2.19)
where \( l \) acts on \( \Omega^\bullet(M; \mathfrak{g}) \) by the respective signs

\[
I = +1 \quad \text{on} \quad \Omega^{1,2}(M; \mathfrak{g}), \quad I = -1 \quad \text{on} \quad \Omega^{0,3}(M; \mathfrak{g}). \tag{1.2.20}
\]

**Euclidean Propagator.** In any quantum field theory, the short-distance behavior of individual Feynman diagrams is controlled by the corresponding short-distance behavior of the various propagators in the theory. For Chern-Simons theory with the gauge-fixed action in (1.2.18), we have formal propagators

\[
G = \triangle^{-1}, \quad K = L^{-1}. \tag{1.2.21}
\]

For the present discussion, we suppress miscellaneous factors of 2, \( \pi \), and \( k \) which would otherwise multiply the propagators.

Both \( G \) and \( K \) are integral operators, and the kernel for the \( bc \)-ghost propagator \( G \) is simply the Green’s function for the covariant scalar Laplacian \( \triangle \) on \( M \),

\[
(G c)^{a_1}(x) = \int_M d^3y \sqrt{g} \, G(x, y)^{a_1}_{a_2} c^{a_2}(y), \quad (x, y) \in M \times M, \tag{1.2.22}
\]

where

\[
\triangle_l G(x, y)^{a_1}_{a_2} = \delta_\Delta(x, y) \cdot 1^{a_1}_{a_2}, \quad a_1, a_2 = 1, \ldots, \dim \mathfrak{g}. \tag{1.2.23}
\]

In (1.2.22) the convolution integral over \( y \in M \) is performed using the Riemannian measure on \( M \) determined by the metric \( g \) previously used to fix harmonic gauge. Likewise, the subscript on \( \triangle_l \) in (1.2.23) indicates that the scalar Laplacian acts on the left argument \( x \) of the Green’s function \( G(x, y)^{a_1}_{a_2} \), with \( y \) fixed. Finally, \( \delta_\Delta(x, y) \) is a delta-function supported along the diagonal inside \( M \times M \) (and defined with respect to the Riemannian measure), and \( 1 \in \text{End}(\mathfrak{g}) \) is the identity. Since we take the background flat connection \( A_0 \) to be irreducible, \( \triangle > 0 \) is strictly positive, with no kernel, and the existence of a smooth Green’s function on \( M \times M \) follows from basic Hodge theory.
More important to us today will be the propagator $K = L^{-1}$ for the gauge field. Like the Dirac operator, the operator $L$ provides a square-root for the Laplacian on $M$. By a brief calculation,

$$L^2|_{\Omega^{-}(M;g)} = \star d_A \star d_A - d_A \star d_A \star = \triangle|_{\Omega^{-}(M;g)}.$$  \hspace{1cm} (1.2.24)

Our assumptions that $A_0$ is irreducible and isolated imply that $\triangle > 0$ is strictly positive on $\Omega^{-}(M;g)$. Hence the elliptic operator $L$ has trivial kernel and will also be invertible.

As an integral operator on $\Omega^{-}(M;g)$, we write

$$\left(K \Phi\right)^{a_1}(x) = \int_{\{x\} \times M} K(x, y)^{a_1} \wedge \Phi(y)^{a_2}, \hspace{1cm} (x, y) \in M \times M ,$$  \hspace{1cm} (1.2.25)

where the kernel $K(x, y)^{a_1}_{a_2}$ is a differential form on $M \times M$ of mixed type $(-, +)$,

$$K(x, y)^{a_1}_{a_2} \in \Omega^{-+}(M \times M, \mathrm{End}(g)),$$  \hspace{1cm} (1.2.26)

relative to the inclusion

$$\Omega^{-}(M) \otimes \Omega^{+}(M) \hookrightarrow \Omega^{\ast}(M \times M).$$  \hspace{1cm} (1.2.27)

In (1.2.25) $K$ acts by convolution with $\Phi \in \Omega^{-}(M;g)$, pulled back to a differential form on the product $M \times M$ and then pushed down by fiberwise integration over the second copy of $M$, as in the push-pull diagram

$$\int_{\{x\} \times M} \xrightarrow{\pi^*_{\nu}} M \times M \xleftarrow{\pi^*_{\nu}} M.$$  \hspace{1cm} (1.2.28)

Although the convolution per se in (1.2.25) involves no explicit metric dependence, unlike the volume integral in (1.2.22), the kernel $K(x, y)^{a_1}_{a_2}$ certainly does depend upon the metric on $M$. 

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With respect to the direct sums in (1.2.16) and (1.2.17), we naturally decompose $K$ into blocks

$$K = \begin{pmatrix} K^{1,2} & K^{1,0} \\ K^{3,2} & K^{3,0} \end{pmatrix}, \quad (1.2.29)$$

where $K^{i,j}(x, y)^{a_1}_{a_2} \in \Omega^{i,j}(M \times M; \text{End}(g))$ has the indicated bi-degree as a differential form on $M \times M$. In our conventions, $K^{1,2}$ is then the propagator for the fluctuating one-form $B$, and the other components of $K$ describe the mixed $B$-$\phi$ and $\phi$-$\phi$ propagators.

By assumption the propagator satisfies the tensorial Green’s equation

$$L_\ell K(x, y)^{a_1}_{a_2} = \delta^{-,+}_\Delta(x, y) \cdot 1^{a_1}_{a_2}, \quad a_1, a_2 = 1, \ldots, \text{dim } g. \quad (1.2.30)$$

Again, the subscript ‘$\ell$’ indicates that the operator $L_\ell$ acts on the left argument of $K(x, y)^{a_1}_{a_2}$, with $y$ fixed. The right-hand side of (1.2.30) takes the same form as the scalar Green’s equation in (1.2.23), but the geometric interpretation is different. Now $\delta^{-,+}_\Delta(x, y)$ is a differential form with delta-function support which represents the $(-, +)$-component of the Poincaré dual to the diagonal $\Delta \subset M \times M$. Explicitly, in a local model for which $M = \mathbb{R}^3$ with the standard orientation and Euclidean metric, the Poincaré dual of the diagonal is given by the distributional three-form

$$\delta_\Delta(x, y) = \delta^{(3)}(x - y) \cdot d(x^1 - y^1) \wedge d(x^2 - y^2) \wedge d(x^3 - y^3), \quad (1.2.31)$$

where $\delta^{(3)}(x)$ is the usual delta-function on $\mathbb{R}^3$. Thus $\delta^{-,+}_\Delta(x, y)$ has components

$$\delta^{-,+}_\Delta(x, y) = \delta^{(3)}(x - y) \cdot \left[ dx^1 \wedge dy^1 \wedge dy^3 + dx^2 \wedge dy^2 \wedge dy^1 + dx^3 \wedge dy^3 \wedge dy^1 \right] + \delta^{(3)}(x - y) \cdot dx^1 \wedge dx^2 \wedge dx^3. \quad (1.2.32)$$

Because $L^2 = \Delta$ when acting on $\Omega^-(M; g)$, the propagator $K$ can be expressed in terms of the inverse $G^{-,+} = \Delta^{-1}$ for the vector Laplacian,

$$\left(G^{-,+} \Phi \right)^{a_1}(x) = \int_{M_y} G^{-,+}(x, y)^{a_1}_{a_2} \wedge \Phi(y)^{a_2}, \quad (x, y) \in M \times M, \quad (1.2.33)$$
whose kernel satisfies the analogue of (1.2.30),

$$
\Delta_\ell G^{-,+}(x, y)^{a_1}_{a_2} = \delta^{-,+}(x, y) \cdot 1^{a_1}_{a_2}, \quad a_1, a_2 = 1, \ldots, \dim g. \quad (1.2.34)
$$

Clearly,

$$
K = L_\ell \circ G^{-,+}. \quad (1.2.35)
$$

Solving for $G^{-,+}$ is slightly easier than solving for $K$, for the simple reason that $\Delta$ preserves the degree of a differential form and so acts in a ‘block-diagonal’ fashion on $\Omega^-(M; g) = \Omega^1(M; g) \oplus \Omega^3(M; g)$. In this regard, note that the top and bottom lines on the right in (1.2.32) describe the respective sources for $\Omega^1(M)$ and $\Omega^3(M)$ in (1.2.34).

When $M = \mathbb{R}^3$ with the Euclidean metric, the kernels for both $G^{-,+}$ and $K$ have elementary analytic expressions which can be used for perturbative calculations. On $\mathbb{R}^3$ we take the background flat connection $A_0 = 0$ to be trivial, so $d_{A_0} \equiv d$ becomes the ordinary de Rham operator. Because $\mathbb{R}^3$ is not compact, we must also specify boundary conditions at infinity for the fields and propagators. We do so by requiring all quantities to extend in a regular fashion over the conformal compactification to $S^3 = \mathbb{R}^3 \cup \{\infty\}$. For the gauge field, this condition requires $A$ to approach $A_0$ at least as fast as $O\left(1/||x||^2\right)$ for $||x||^2 \to \infty$.\(^4\) Likewise, the gauge field propagator must vanish as $O\left(1/||x||^2\right)$. This condition will allow us to determine the propagator uniquely.

Contrary to our simplifying assumption, $A_0 = 0$ is a reducible connection, fixed by the constant gauge transformations on $\mathbb{R}^3$. Equivalently, the scalar Laplacian $\Delta$ on $\mathbb{R}^3$ has a non-trivial kernel, so the $bc$-ghost fields in the gauge-fixed action (1.2.14) have zero-modes. These zero-modes must be treated with care when one attempts to evaluate the Wilson loop path integral [13, 43, 45] exactly. For our computations in perturbation theory, however, the ghost zero-modes will not play a role and can safely be ignored. Formally, ignoring the ghost zero-modes amounts to working with the group $G_\infty$ of based gauge transformations. Elements of $G_\infty$ are gauge transformations

\(^4\)As usual, $||x||$ is the Euclidean norm of $x \in \mathbb{R}^3$.\]
on $S^3 = \mathbb{R}^3 \cup \{\infty\}$ which restrict to the identity at the point $\{\infty\}$. The only constant gauge transformation to satisfy this boundary condition on $\mathbb{R}^3$ is the identity in $G$.

As well-known, for the scalar Green’s equation in (1.2.23) the solution on $\mathbb{R}^3$ is given by

$$G(x, y)^{a_1}_{a_2} = \frac{1}{4\pi ||x-y||} \cdot 1^{a_1}_{a_2}, \quad x, y \in \mathbb{R}^3. \quad (1.2.36)$$

With respect to the orthonormal basis $\{dx^1, dx^2, dx^3, dx^1 \wedge dx^2 \wedge dx^3\}$ of $\Omega^-(M)$, the vector Laplacian acts componentwise just like the scalar Laplacian, so the solution of (1.2.34) is given similarly by

$$G^{-+}(x, y)^{a_1}_{a_2} = \frac{1}{4\pi ||x-y||} \left[ dx^1 \wedge dy^2 \wedge dy^3 + dx^2 \wedge dy^3 \wedge dy^1 + dx^3 \wedge dy^1 \wedge dy^2 \right] \cdot 1^{a_1}_{a_2} + \frac{1}{4\pi ||x-y||} dx^1 \wedge dx^2 \wedge dx^3 \cdot 1^{a_1}_{a_2}. \quad (1.2.37)$$

Acting with $L = (\ast d + d \ast) \circ 1$ on the expression in (1.2.37) for fixed $y$, we then determine $K$. Of the four components in (1.2.29), the most important for us will be the one-form propagator $K^{1,2} \in \Omega^{1,2}(M \times M, \text{End}(g))$. By a brief calculation,

$$K^{1,2}(x, y)^{a_1}_{a_2} = \frac{\epsilon_{\mu\nu\rho}(x - y)\mu dx^\nu \wedge \ast dy^\rho}{4\pi ||x-y||^3} \cdot 1^{a_1}_{a_2}, \quad \mu, \nu, \rho = 1, 2, 3, \quad (1.2.38)$$

where $\epsilon_{\mu\nu\rho}$ is the fully anti-symmetric tensor, satisfying $\epsilon_{123} = +1$. Clearly $K^{1,2}$ vanishes as $O(1/||x||^2)$ for $||x||^2 \to \infty$ and so obeys the asymptotic boundary condition on $\mathbb{R}^3$.

We shall also have occasion to work with a small variant of $K^{1,2}$. On the product $M \times M$, we have individual left/right Hodge operators

$$\ast_{\ell} : \Omega^{i,j}(M \times M) \longrightarrow \Omega^{3-i-j}(M \times M),$$

$$\ast_{r} : \Omega^{i,j}(M \times M) \longrightarrow \Omega^{i,j}(M \times M), \quad (1.2.39)$$

defined with respect to the dense decomposition $\Omega^\bullet(M \times M) \simeq \Omega^\bullet(M) \otimes \Omega^\bullet(M)$ and satisfying $\ast_{\ell}^2 = \ast_r^2 = 1$. We then dualize $K^{1,2}$ on the right to obtain a more symmetric
presentation of the propagator as a $(1,1)$-form on $M \times M$,

$$K^{1,1}(x,y)^{a_1}_{a_2} = \ast, K^{1,2}(x,y)^{a_1}_{a_2} \in \Omega^{1,1}(M \times M; \text{End}(g)). \quad (1.2.40)$$

For the Euclidean propagator in (1.2.38),

$$K^{1,1}(x,y)^{a_1}_{a_2} = \frac{\epsilon_{\mu\nu}(x - y)^{\mu} dx^{\nu} \wedge dy^{\nu}}{4\pi ||x - y||^3} \cdot 1^{a_1}_{a_2}. \quad (1.2.41)$$

Evidently under the exchange $x \leftrightarrow y$, the propagator $K^{1,1}$ transforms by a sign,$^5$

$$K^{1,1}(y,x)^{a_1}_{a_2} = -K^{1,1}(x,y)^{a_1}_{a_2}. \quad (1.2.42)$$

**Feynman Diagrams.** From the perspective of the present paper, the most important feature of (bosonic) Chern-Simons perturbation theory is simply the analytic behavior as $||x - y||^2 \to 0$ of the ghost and gauge field propagators in (1.2.36) and (1.2.38). Though we derived this short-distance behavior for the special case $M = \mathbb{R}^3$, the leading divergence is a universal property which holds for all Riemannian three-manifolds and is manifested topologically by the need to frame ‘bare’ Chern-Simons observables such as (1.1.14). Later in Section 1.3, we review the basic example of this phenomenon.

Beyond its analytic features, Chern-Simons perturbation theory famously exhibits a combinatoric aspect as well, associated to the Feynman diagram expansion of each observable in powers of the gauge coupling$^6$

$$g^2 = \frac{2\pi}{k} \ll 1. \quad (1.2.43)$$

Though our interest here is mostly analytic, not combinatoric, we pause to sketch the

$^5$The anti-symmetry of $K^{1,1}$ in (1.2.42) may be disconcerting to those physicists who expect to find a symmetric tensor $K^{1,1}_{\mu\nu}(x,y)^{a_1}_{a_2} = K^{1,1}_{\nu\mu}(y,x)^{a_1}_{a_2}$ on the basis of Bose statistics. The tensor $K^{1,1}_{\mu\nu}(x,y)^{a_1}_{a_2} = \epsilon_{\mu\nu}(x - y)^{\mu} dx^{\nu} \wedge dy^{\nu} / 4\pi ||x - y||^3$ does obey the conventional Bose symmetry, but for geometric reasons we have promoted this symmetric tensor to the $(1,1)$-form $K^{1,1}_{\mu\nu}(x,y)^{a_1}_{a_2} dx^{\mu} \wedge dy^{\nu}$, an object more naturally integrated on $M \times M$.

$^6$The appearance of $g^2$ as opposed to $g$ in (1.2.43) is a standard convention, motivated by positivity of $k$. The factor of $2\pi$ eliminates similar factors elsewhere.
Feynman rules for perturbation theory about the background flat connection $A_0$ in the gauge-fixed action (1.2.14).

For the Wilson loop expectation value in (1.1.14), two expansions are relevant. First, we have the classical expansion (1.1.9) of $W_R(C)$ in powers of $A$, so that

$$
\langle W_R(C) \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} I^R_{a_1 \cdots a_n} \oint_{C^n} \langle P[A^{a_1}(s_1) \cdots A^{a_n}(s_n)] \rangle.
$$

(1.2.44)

Then for each multiple integral over $C^n$ in (1.2.44), the integrand itself is given by

$$
\langle P[A^{a_1}(s_1) \cdots A^{a_n}(s_n)] \rangle = \frac{1}{Z(k) \text{Vol}(G)} \int_A \mathcal{D}A \ P[A^{a_1}(s_1) \cdots A^{a_n}(s_n)] \exp[i k \text{CS}(A)].
$$

(1.2.45)

To make sense of (1.2.45) for $k \gg 1$, we apply a path integral version of the stationary-phase approximation about the flat connection $A_0$ to obtain an asymptotic expansion of the right-hand side in the coupling $g^2$.

The two series expansions of $\langle W_R(C) \rangle$ in powers of $A$ and $g^2$ overlap, insofar as terms with distinct values for $n$ in (1.2.44) may contribute at identical orders in $g^2$ for the expansion of (1.2.45). Nonetheless, these expansions rest on very different theoretical footings. Though the Taylor series expansion in (1.2.44) provides a helpful organizational scheme for calculations, this expansion does not respect the gauge symmetry. Only the overall asymptotic expansion of $\langle W_R(C) \rangle$ in powers of $g^2$ respects the gauge symmetry and so leads, term-by-term, to a series of perturbative invariants for framed knots.

Briefly, the various terms in the asymptotic expansion of (1.2.45) are depicted by connected trivalent Feynman diagrams assembled from the following ingredients.

The ghost and gauge field propagators are represented by the dashed and dotted lines in Figure 1-2. The ends of each line are labelled by distinct points $x, y \in M$ and Lie algebra indices $a_{1,2} = 1, \ldots, \text{dim } g$. As indicated, both propagators are ori-

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7But at any given order in $g^2$, terms for only finitely-many values of $n$ in (1.2.44) contribute.

8Because the auxiliary boson $\phi$ does not interact with the gauge field or ghosts in (1.2.14), $\phi$ can be ignored in perturbative computations of the Wilson loop expectation value.
The ghost propagator describes the correlator
\[ \langle c_{a_1}(x) b_{a_2}(y) \rangle = i g^2 \star_r G(x, y)^{a_1 a_2} \in \Omega^{0,3}(M \times M, g^\otimes 2) \],
\noindent (1.2.46)

where the prefactor \( i g^2 \) arises from the overall \( i k \) which multiplies the Chern-Simons action \( \text{CS}(A) \) in (1.2.45).\(^9\) Due to the anti-commuting nature of the ghosts, reversing the order of \( b \) and \( c \) on the left in (1.2.46) is accompanied by a minus sign on the right, so the ghost propagator in Figure 1-2 must be oriented. Similarly, the gauge field propagator describes the leading-order contribution to the correlator
\[ \langle B^{a_1}(x) B^{a_2}(y) \rangle = i g^2 K^{1,1}(x, y)^{a_1 a_2} \in \Omega^{1,1}(M \times M, g^\otimes 2) \].
\noindent (1.2.47)

Here \( K^{1,1} = \star_r K^{1,2} \) is the (1,1)-form version of the propagator appearing in (1.2.40). The orientation of the gauge field propagator in Figure 1-2 is necessary to keep track of the minus sign in (1.2.42). Relative to expressions such as (1.2.36) and (1.2.41), the Lie algebra index \( a_2 \) has been raised using the invariant metric on \( g \), so the Euclidean propagators on \( \mathbb{R}^3 \) are both proportional to \( \delta^{a_1 a_2} \).

In addition to the propagators, the pair of cubic interaction terms in the second line of the gauge-fixed action (1.2.14) are represented by the trivalent vertices in Figure 1-3. The endpoints of each line in the figure carry a Lie algebra index \( a_{1,2,3} = 1, \ldots, \dim g \), and the vertex itself is labelled by a point \( z \in M \). Again, the factor \( i k \) in (1.2.45) leads to a corresponding factor \( -i/g^2 \) for the vertices, and the

---

\(^9\) If desired, one can rescale the fluctuating fields \( (B, b, c) \) so that no factor of \( g^2 \) multiplies the propagators in Figure 1-2. Though convenient for some purposes, such a rescaling causes extraneous factors of \( g^2 \) to appear in the Wilson loop operator, so we prefer to use instead the canonical scaling which follows from (1.2.45).
Relative factor of $1/6$ which multiplies the gauge interaction vertex can be interpreted as a symmetry factor for permutations of the three legs.

Both vertices in Figure 1-3 involve an integral over the point $z \in M$, so the integrand must be a three-form on $M$. For the gauge interaction vertex, the required three-form is proportional to the wedge product $K^{1,1}(z, \cdot)^{a_1 a_4} \wedge K^{1,1}(z, \cdot)^{a_2 a_5} \wedge K^{1,1}(z, \cdot)^{a_3 a_6}$ of three external gauge field propagators attached to the vertex. For the ghost interaction, the three-form is proportional to $\star_r d_r G(\cdot, z)^{a_1 a_4} \wedge K^{1,1}(z, \cdot)^{a_2 a_5} G(z, \cdot)^{a_3 a_6}$, where $\star_r d_r$ acts on the right, with respect to the argument $z$. These structures are indicated schematically in the formulas below each vertex in the figure. Via (1.2.42), the orientation on any gauge field propagator attached to either vertex in Figure 1-3 can be reversed at the cost of a minus sign.

Feynman diagrams representing terms in the asymptotic expansion of (1.2.45) are given by connected graphs, each with $n$ endpoints, which are constructed by gluing together tinkertoy-fashion copies of the vertices in Figure 1-3 with the propagators in Figure 1-2. Two diagrams relevant for $n = 2$ appear in Figure 1-4. Both diagrams contribute to $\langle B^{a_1}(x) B^{a_2}(y) \rangle$ at order $g^4$.

For sake of illustration, let us evaluate these loop diagrams. Each diagram has a pair of vertices so represents an expression which is quadratic in the Jacobi tensor $f_{a_1 a_2 a_3}$ and involves an integral over the points $(w, z) \in M \times M$. For first diagram in Figure 1-4, the integrand is constructed from four copies of $K^{1,1}$ (for the four solid
Figure 1-4: Diagrams contributing to $\langle B^{a_1}(x) B^{a_2}(y) \rangle$ at $\mathcal{O}(g^4)$.

The two on the right of (1.2.48) is a symmetry factor, necessary to account for the similar diagram in which the arrows on the internal lines are reversed.

Due to the singularity in $K_{1,1}^{1,1}$ at coincident points, we are careful to excise from the naive integration domain in (1.2.48) a neighborhood $\Delta(\varepsilon) \subset M \times M$ of radius $\varepsilon > 0$ about the points where $w = z$ (the diagonal) as well as $z = x$ and $w = y$. Hence the integral is manifestly finite, though its value generally depends upon the small parameter $\varepsilon$, which serves as a short-distance regulator. A major theme of our work will be the careful analysis of the limit $\varepsilon \to 0$ in related configuration space integrals.

In the case $M = \mathbb{R}^3$ of main interest, the analytic expression for $K_{1,1}^{1,1}$ in (1.2.41) can be used to make the integrand of (1.2.48) quite explicit. That exercise turns out to be academic, however, because the contribution from the second loop diagram in Figure 1-4 exactly cancels the contribution from the first.

The second loop diagram involves two copies of the ghost propagator $\ast_r G$ as well as two copies of the gauge field propagator $K_{1,1}^{1,1}$, again arranged to yield a top-form
on $M \times M$ valued in $\Omega^1(M, g)^{\otimes 2}$. With the same regularization as (1.2.48),

$$
\rightarrow \quad \circ \quad \rightarrow = g^4 \int_{M \times M - \Delta(\varepsilon)} f_{a_3 a_5} f_{a_6 a_8} \times \\
\times K^{1,1}(w, y)^{a_3 a_2} \wedge \star_r d_r G(w, z)^{a_4 a_7} \wedge \star_r d_r G(z, w)^{a_6 a_5} \wedge K^{1,1}(x, z)^{a_1 a_8}.
$$

(1.2.49)

Because of the asymmetry between the $b$ and $c$ ghosts, no combinatoric factor of two appears on the right in (1.2.49). The relative signs in (1.2.48) and (1.2.49) are also delicate. Beyond the various factors of 'i' in the propagators and vertices, the signs depend upon the arrangement of Lie algebra indices on the propagators and upon the anti-commuting nature of the $bc$-ghosts. Nonetheless, computing $\star_r d_r G$ from (1.2.36), one can verify that the integrand of (1.2.49) is exactly twice the integrand of (1.2.48), but with the opposite sign. So for all values of $\varepsilon > 0$,

$$
\rightarrow \quad \circ \quad \rightarrow + \quad \rightarrow \quad \circ \quad \rightarrow = 0.
$$

(1.2.50)

The one-loop cancellation in (1.2.50) was observed early in the study of Chern-Simons theory [36]. From a physical perspective, the sum of loop diagrams is necessarily finite and independent of the cutoff parameter $\varepsilon$ due to gauge invariance, as manifested by the integrality of the level $k \in \mathbb{Z}$. Because $k$ is quantized, the Chern-Simons coupling $g^2 = 2\pi/k$ cannot vary under renormalization group flow, so the beta-function for $g^2$ must vanish. As standard, that beta-function is determined by the divergence as $\varepsilon \to 0$ in the configuration space integrals contributing to $\langle B^{a_1}(x) B^{a_2}(y) \rangle$, and vanishing of the beta-function is tantamount to finiteness of the sum of loop diagrams in (1.2.50).

1.3 Linking and Self-Linking for Framed Knots

In the diagrammatic shorthand, each term in the asymptotic expansion of the expectation value $\langle B^{a_1}(x_1) \cdots B^{a_n}(x_n) \rangle$ for $x_1, \ldots, x_n \in M$ has the structure in Figure 1-5, where the central blob represents all connected ways to tie together the $n$ external
gauge field propagators using the propagators and vertices in Figures 1-2 and 1-3.

![Figure 1-5: Expectation value \( \langle B^{a_1}(x_1) \cdots B^{a_n}(x_n) \rangle \) for \( x_1, \ldots, x_n \in M \).](image)

To evaluate the Wilson loop expectation value (1.2.44), expanded perturbatively about the trivial connection \( A_0 = 0 \) on \( M = \mathbb{R}^3 \), we pair the diagram in Figure 1-5 with the invariant tensor \( I^{R|a_1\cdots a_n} \) in Figure 1-1 to write schematically

\[
\langle W_R(C) \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \oint_{C^n} P \left[ \right].
\]  

(1.3.1)

Here the wheel representing the path-ordered integrand for \( (x_1, \ldots, x_n) \in C^n \) has exactly \( n \) spokes.

At least to leading order, the graphical expansion in (1.3.1) is quite simple,

\[
\langle W_R(C) \rangle = \dim R + \frac{1}{2} \oint_{C^2} P \left[ \right] + \mathcal{O}(g^4).
\]  

(1.3.2)

According to the description of \( I^{R|a_1\cdots a_n} \) in (1.1.11), the term for \( n = 0 \) in (1.3.1) is just the dimension of \( R \), and no term for \( n = 1 \) appears. For \( n = 2 \), the dominant contribution occurs at order \( g^2 \) and is given by the single diagram in (1.3.2). Otherwise,
all other diagrams with \( n \geq 2 \) contribute at order \( g^4 \) and higher. Explicitly,

\[
\begin{align*}
&= i g^2 \mathfrak{I}^R_{a_1 a_2} K^{1,1}(x, y)^{a_1 a_2} , \quad x, y \in C \subset \mathbb{R}^3 , \\
&= -i g^2 \epsilon_{\mu \nu \rho} (x - y)^\mu d^\nu y^\rho \frac{1}{4\pi ||x - y||^3} c_2(R) \dim R ,
\end{align*}
\]

where we recall that \( \mathfrak{I}^R_{a_1 a_2} = -c_2(R) \delta_{a_1 a_2} \), and we sum over the Lie algebra indices \( a_1, a_2 \) in passing to the second line. Geometrically, the diagram in (1.3.3) thus represents a \((1, 1)\)-form to be integrated over \( C^2 = C \times C \).

To interpret the integrand topologically, let us introduce the angular form \( \psi \) on the punctured space \( \mathbb{R}^3 - \{0\} \),

\[
\psi = \frac{1}{8\pi} \epsilon_{\mu \nu \rho} x^\mu d^\nu x^\rho \frac{1}{||x||^3} , \quad x \neq 0 \in \mathbb{R}^3.
\]

After summing repeated indices \( \mu, \nu, \rho = 1, 2, 3 \),

\[
\psi = \frac{1}{4\pi} \frac{x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2}{||x||^3} .
\]

As discussed for instance in Ch I.6 in [19], \( \psi \) is the pullback of the volume form on the unit sphere \( S^2 \subset \mathbb{R}^3 \) under the retraction

\[
\varrho : \mathbb{R}^3 - \{0\} \rightarrow S^2 , \quad \varrho(x) = \frac{x}{||x||}.
\]

Hence \( \psi \) is a closed, invariant generator for \( H^2(\mathbb{R}^3 - \{0\}; \mathbb{R}) \simeq \mathbb{R} \), with normalization

\[
\int_{S^2} \psi = 1 .
\]

The Euclidean Chern-Simons propagator \( K^{1,1} \) bears a close relation to the angular form \( \psi \). Comparing the formulas in (1.2.41) and (5.1.2), we see that \( K^{1,1} \) is the pullback

\[
K^{1,1}(x, y)^{a_1}_{a_2} = -[\Gamma^* \psi]^{1,1}(x, y) \cdot \mathfrak{I}^R_{a_1 a_2} ,
\]
where $\Gamma$ is the difference map

$$
\Gamma : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \Gamma(x, y) = x - y, \quad (1.3.9)
$$

and $[\cdot]^{1,1}$ indicates the component which has type $(1, 1)$ on $\mathbb{R}^3 \times \mathbb{R}^3$. Since $\Gamma$ is odd under the exchange $x \leftrightarrow y$, the presentation of $K^{1,1}$ in (1.3.8) makes the anti-symmetry in (1.2.42) manifest.

To discuss the integral over $C^2$ in (1.3.2), we switch perspectives slightly and consider the knot $C \subset \mathbb{R}^3$ as a parametrized curve, with smooth parametrization

$$
X : S^1 \rightarrow \mathbb{R}^3, \quad C = \text{Im}(X). \quad (1.3.10)
$$

The integral over $C^2$ then pulls back under the product map $X \times X$ to an integral over the torus $S^1 \times S^1$, in terms of which

$$
\frac{1}{2} \oint_{C^2} P \left[ \begin{array}{c}
\vdots \\
\end{array} \right] = -\frac{ig^2}{4} c_2(R) \dim R \cdot L(C), \quad (1.3.11)
$$

where $L(C)$ is the Gauss self-linking integral

$$
L(C) = \lim_{\epsilon \to 0^+} \int_{S^1 \times S^1 - \Delta(\epsilon)} (-1) \cdot (X \times X)^* \Gamma^* \psi. \quad (1.3.12)
$$

Several comments about $L(C)$ are in order.

First, due to the path-ordering $P$ on the left in (1.3.11), the naive domain of integration consists of pairs $(s_1, s_2) \in S^1 \times S^1$ for which $s_1 \geq s_2$, as depicted in Figure 1-6. However, given the symmetry of the integrand under exchange, we are free to unfold the domain of integration to the full $S^1 \times S^1$, provided that we divide by two on the right in (1.3.11). The other numerical factors appear already in (1.3.3). The minus sign in the integrand of (1.3.12) is the same minus sign in (1.3.8), necessary to ensure that our conventions for linking and self-linking agree with the standard ones.
More significantly, because $\psi$ is singular at the origin, we are careful to excise from the integration domain in (1.3.12) a neighborhood of radius $\varepsilon$ about the diagonal $\Delta \subset S^1 \times S^1$, depicted by the shaded region in Figure 1-7. Topologically, the resulting integration domain is a cylinder,

$$S^1 \times S^1 - \Delta(\varepsilon) \simeq [\varepsilon, 2\pi - \varepsilon] \times S^1,$$

(1.3.13)

with a pair of circle boundaries where $s_1 = s_2 \pm \varepsilon \mod 2\pi$.

The small parameter $\varepsilon > 0$ serves as a short-distance cutoff, and the integral over the cylinder in (1.3.13) generally depends on $\varepsilon$. For composite operators in the typical quantum field theory, one is usually plagued as $\varepsilon \to 0$ by divergences, which must be then swept under the rug by operator renormalization. Chern-Simons gauge theory is not a typical quantum field theory, though (eg. due to its classical diffeomorphism invariance), and for the self-linking integral in (1.3.12), the situation is much simpler. Namely, the singularity along the diagonal $\Delta \subset S^1 \times S^1$ in the pullback $(X \times X)^* \Gamma^* \psi$
is integrable. So the naive, unrenormalized limit $\varepsilon \to 0$ exists and can be used to define $L(C)$ independently of any cutoff.

We will later be concerned with similar statements about the Legendrian variant of (1.3.12), so let us briefly analyze the behavior of the integrand in the vicinity of the diagonal $\Delta \subset S^1 \times S^1$. We set

$$s_1 = \varphi + \eta, \quad s_2 = \varphi, \quad \eta \in [-\varepsilon, \varepsilon], \quad (1.3.14)$$

in terms of which we expand the difference

$$X^\mu(s_1) - X^\mu(s_2) = \eta X^\mu(\varphi) + \frac{\eta^2}{2} \dddot{X}^\mu(\varphi) + \frac{\eta^3}{6} \dddot{X}^\mu(\varphi) + \mathcal{O}(\varepsilon^4). \quad (1.3.15)$$

Here we abbreviate $\dddot{X}^\mu \equiv dX^\mu/d\varphi$, and similarly for higher derivatives with respect to the parameter $\varphi$ along the curve $C$. For convenience, we assume that the parametrization is regular with unit speed,

$$||\dot{X}(\varphi)||^2 = 1. \quad (1.3.16)$$

Finally, $S^1 \times S^1$ is oriented according to $ds_1 \wedge ds_2 = d\eta \wedge d\varphi$.

In terms of $X : S^1 \to \mathbb{R}^3$, the integrand in (1.3.12) is given concretely by

$$(X \times X)^* \Gamma^\* \psi = \frac{1}{8\pi} \epsilon_{\mu\nu\rho} \Delta X^\mu d(\Delta X^\nu) \wedge d(\Delta X^\rho) \left/ ||\Delta X||^3 \right., \quad (1.3.17)$$

$$\Delta X^\mu(s_1, s_2) \equiv X^\mu(s_1) - X^\mu(s_2).$$

After a small calculation, the expansion in (1.3.15) implies that the numerator vanishes quartically along the diagonal $\eta = 0$,

$$\epsilon_{\mu\nu\rho} \Delta X^\mu d(\Delta X^\nu) \wedge d(\Delta X^\rho) = -\frac{1}{6} \left( \epsilon_{\mu\nu\rho} \dddot{X}^\mu \dddot{X}^\nu \dddot{X}^\rho \right) \eta^4 d\eta \wedge d\varphi + \mathcal{O}(\varepsilon^5), \quad (1.3.18)$$

whereas the denominator vanishes cubically,

$$||\Delta X||^3 = \eta^3 \left( 1 + \mathcal{O}(\varepsilon^2) \right). \quad (1.3.19)$$
Here we use the unit-speed assumption in (1.3.16). Hence for \( \varepsilon \ll 1 \),

\[
(X \times X)^* \Gamma^* \psi = -\frac{1}{48\pi} \left( \epsilon_{\mu
u\rho} \dot{X}^\mu \dot{X}^\nu \dot{X}^\rho \right) \eta d\eta \wedge d\varphi + \mathcal{O}(\varepsilon^2).
\] (1.3.20)

As a result, the potentially divergent contribution to the self-linking integral from the region \( \Delta(\varepsilon) \subset S^1 \times S^1 \) in Figure 1-7 vanishes for \( \varepsilon \to 0 \).

**The Topological Anomaly.** Although the Gauss self-linking integral \( L(C) \in \mathbb{R} \) is well-defined as a real number, the naive result in (1.3.11) for the leading perturbative contribution to the Wilson loop expectation value \( \langle W_R(C) \rangle \) respects neither topological nor gauge invariance. As usual with quantum anomalies, the trouble in either case is caused by the short-distance behavior of the Euclidean propagator \( K^{1,1}(x, y)_{a_1} \) for nearby points \( x, y \in \mathbb{R}^3 \). The gauge and topological anomalies are intimately related, but the failure of gauge invariance is the more fundamental and perhaps less appreciated, so we begin with it.

With no essential loss of generality and with considerable gain in transparency, we temporarily specialize to the abelian case \( G = U(1) \), for which the level \( k \in 2\mathbb{Z} \) must be an even integer in the conventions from Section 1.1. Because the abelian Chern-Simons functional (1.1.6) is quadratic in \( A \), the expectation value for any \( m \)-component Wilson link \( L = C_1 \cup \cdots \cup C_m \) can be evaluated exactly in terms of the propagator \( K^{1,1} \) alone, by summing diagrams of the form in Figure 1-8 (where \( m = 2 \) for simplicity). Note that \( C_1 \) and \( C_2 \) in Figure 1-8 may be non-trivially linked in \( \mathbb{R}^3 \), since the diagram itself does not carry information about the respective embeddings \( X_1, X_2 : S^1 \to \mathbb{R}^3 \), just the combinatorial structure of an integral over \( C_1 \times C_2 \) in this example.

Beyond the charges \( q_1, \ldots, q_m \in \mathbb{Z} \) which decorate each component of the link, the abelian expectation value \( \langle W_{q_1}(C_1) \cdots W_{q_m}(C_m) \rangle \) can depend only upon the pairwise linking numbers

\[
L(C_j, C_\ell) = \int_{S^1 \times S^1} (-1) \cdot (X_j \times X_\ell)^* \Gamma^* \psi, \quad j \neq \ell, \quad j, \ell = 1, \ldots, m,
\] (1.3.21)
Figure 1-8: An $\mathcal{O}(g^6)$ contribution to $\langle W_{q_1}(C_1) W_{q_2}(C_2) \rangle$.

as well as the Gauss self-linking constants $L(C_j)$ in (1.3.12). Because $C_j$ and $C_\ell$ are distinct, non-intersecting curves in (1.3.21), the domain of integration runs over the full torus $S^1 \times S^1$, with no need to excise the diagonal. As $\psi$ is the pullback from a cohomology generator on $S^2$, $L(C_j, C_\ell)$ is the degree of the induced map from $S^1 \times S^1$ to $S^2$. Of course, $L(C_j, C_\ell) \in \mathbb{Z}$ is both integral and a topological invariant.

The diagram in Figure 1-8 is proportional to the product of linking and self-linking numbers $L(C_1) \cdot L(C_1, C_2) \cdot L(C_2)$, a statement which generalizes for all other diagrams in the abelian theory. With care for the combinatorics, these diagrams can be summed to yield the exact result

$$
\langle W_{q_1}(C_1) \cdots W_{q_m}(C_m) \rangle = \exp \left[ i g^2 \sum_{j \neq \ell} q_j q_\ell L(C_j, C_\ell) + \frac{i g^2}{2} \sum_j q_j^2 L(C_j) \right].
$$

(1.3.22)

The factors of ‘$i$’ in (1.3.22) derive from the same factor in the propagator $K^{1,1}$, and the relative factor of two between the linking and self-linking terms arises from the symmetry under the exchange $j \leftrightarrow \ell$. The result in (1.3.22) is very well-known and can also be derived by formal Gaussian integration or from the relation [69] to abelian current algebra. See Exercise 10.1 in [53] for a thorough discussion from the latter perspective, but be aware that the conventions in [53] are such that $k_{\text{there}} = 2k_{\text{here}}$.

Recall that $g^2$ is related to the level $k \in 2\mathbb{Z}$ by $g^2 = 2\pi/k$. If the self-linking constant $L(C_j) \in \mathbb{R}$ on the right in (1.3.22) is assumed to be an integer, the Wilson
link expectation value is invariant under shifts of the individual charges\(^\text{10}\)

\[ q_j \mapsto q_j + k, \quad j = 1, \ldots, m. \quad (1.3.23) \]

Since the quantum symmetry in (1.3.23) holds for arbitrary Wilson link expectation values, it is equivalent to the operator identity

\[ W_q(C) \sim W_{q+k}(C), \quad (1.3.24) \]

which asserts that \( W_q(C) \) depends only on the value of \( q \) mod \( k \). As well-known, in the non-abelian counterpart of (1.3.23), the Weyl action on the weight lattice of \( G \) is promoted to an affine Weyl action, described for instance in Ch 5.1 of [57].

From any number of perspectives, the operator identity in (1.3.24) is a fundamental feature of \( U(1) \) Chern-Simons theory at level \( k \). At its heart, this identity is a consequence of gauge symmetry [52]. To explain very briefly the role of gauge symmetry, we consider the Wilson loop path integral

\[ Z(k; C, q) = \frac{1}{\text{Vol}(G)} \int_A \mathcal{D} A \; W_q(C) \exp \left[ \frac{i k}{4\pi} \int_M A \wedge dA \right]. \quad (1.3.25) \]

The integrand in (1.3.25) is invariant under non-singular gauge transformations on \( M \), but in the presence of the Wilson loop operator, we can also consider gauge transformations which are singular along the curve \( C \subset M \).

To describe such a gauge transformation, let \((r, \theta, \varphi)\) be local coordinates on a tubular neighborhood of \( C \). Here \( \varphi \) is a coordinate along \( C \), and \((r, \theta)\) are polar coordinates on the plane transverse to \( C \), which passes through the origin where \( r = 0 \). For any winding-number \( n \), the following gauge transformation \( g \) is well-defined in the tubular neighborhood where \( r \neq 0 \),

\[ g(r, \theta, \varphi) = \exp(-i n \theta), \quad n \in \mathbb{Z}. \quad (1.3.26) \]

\( ^{10} \)Note that invariance of (1.3.22) also requires \( k \) to be an even integer, due to the appearance of \( q_j^2/2 \) in the self-linking term.
Of course, \( n \) must be an integer in (1.3.26) so that \( g \) determines a sensible map to \( U(1) \). By standard patching arguments, \( g \) can be extended in a non-singular fashion to the whole of the complement \( M - C \). Because the integrand of (1.3.25) is invariant under non-singular gauge transformations, the choice of extension for \( g \) away from \( C \) will not matter.

Under the action by \( g \), the abelian gauge field \( A \) transforms to

\[
A^g = A - g^{-1}dg = A + i n d\theta .
\]  

(1.3.27)

This gauge transformation does not change the Wilson loop operator \( W_q(C) \) appearing in the integrand of (1.3.25), but it does alter the classical action via

\[
\frac{i k}{4\pi} \int_M A^g \wedge dA^g = \frac{i k}{4\pi} \int_M A \wedge dA - n k \oint_C A .
\]  

(1.3.28)

Here we use the distributional identity \( d(d\theta) = 2\pi\delta_C \), where \( \delta_C \) is a two-form with delta-function support which represents the Poincaré dual of \( C \). We also note a factor of two which arises from the quadratic dependence on \( A \) in the classical action. When exponentiated, the second term on the right in (1.3.28) induces the replacement \( W_q(C) \rightarrow W_{q+nk}(C) \) in the Wilson loop path integral. Thus the mod \( k \) equivalence in (1.3.24) is properly interpreted as a kind of gauge equivalence in \( U(1) \) Chern-Simons theory at level \( k \).

As a gauge equivalence, the invariance under shifts \( q_j \rightarrow q_j + k \) in the Wilson link expectation value is sacrosanct. However, we have already noted that this invariance only holds for the expression in (1.3.22) if each self-linking constant \( L(C_j) \in \mathbb{R} \) is actually an integer. Integrality is a very strong property, and with the naive definition in (1.3.12), there is absolutely no reason for integrality to hold,

\[
L(C_j) \notin \mathbb{Z} , \quad C_j \text{ generic} .
\]  

(1.3.29)

The naive definition of the Gauss self-linking integral is therefore not compatible with gauge invariance!
Nor is the naive self-linking constant $L(C)$ in (1.3.12) a deformation invariant of the embedding $X : S^1 \to \mathbb{R}^3$. Specifically, the variation of $L(C)$ with respect to the map $X$ is given by the integral

$$
\delta L(C) = -\frac{1}{2\pi} \int_{S^1} d\varphi \, \epsilon_{\mu\nu\rho} \dot{X}^\mu \delta X^\nu \dot{X}^\rho, \quad ||\dot{X}(\varphi)||^2 = 1. \quad (1.3.30)
$$

Without loss, $X$ is assumed to be a regular unit-speed parametrization, so that $d\varphi$ is the arc-length measure along $C \subset \mathbb{R}^3$. Also, $\delta X$ is a smooth vector field along $C$ describing an arbitrary first-order deformation of the curve. As a small consistency check, note that the integrand in (1.3.30) vanishes identically at points where $\delta X$ is tangent to $C$, due to the anti-symmetrization with $\dot{X}^\mu$. If $\delta X$ is tangent to $C$, the curve $C \subset \mathbb{R}^3$ is not changed under the variation by $\delta X$ (only its parametrization changes), and $L(C)$ is defined in a reparametrization-invariant fashion.

Generically $\delta L(C) \neq 0$, so $L(C)$ can be neither an integer nor a topological invariant of the curve $C$. Hence the formula in (1.3.30) can be interpreted as a kind of ‘topological anomaly’ in the naive Gauss self-linking integral.

Though the result in (1.3.30) is simple, the computation leading to this formula is involved, so we have relegated it to Appendix 5.3.3. The computation there is instructive however, since it highlights the role played by the short-distance behavior of the Euclidean propagator $K^{1,1}$ in determining the topological anomaly. Very briefly, for any fixed $\varepsilon > 0$, the variation of the self-linking integrand in (1.3.12) is a total derivative on the cylinder $S^1 \times S^1 - \Delta(\varepsilon)$ depicted in Figure 1-7. As such, the variation is given entirely by contributions from the two boundaries, where $s_1 = s_2 + \varepsilon$ in the configuration space. For small $\varepsilon \ll 1$, these contributions can be evaluated by Taylor expansion of $X$, and crucially, the result in (1.3.30) is non-zero even in the limit $\varepsilon \to 0$.

Later, we will repeat a similar analysis for the Legendrian self-linking integral, with an altogether more pleasing outcome.

**The Role of Framing.** Due to the violation of both gauge and topological invariance, the definition of $L(C)$ via the naive Gauss self-linking integral in (1.3.12) is not
acceptable. We now review two approaches to give a definition of self-linking which is acceptable, in the sense that $L(C)$ will be an integer which is invariant under smooth deformations of $C$. Physically, both definitions can be interpreted as renormalizations of the Wilson loop operator $W_R(C)$.

We begin with the most straightforward way to make sense of $L(C)$, which requires the choice of a framing for $C$. By definition, a framing is a trivialization of the normal bundle for $C \subset \mathbb{R}^3$, up to homotopy. Such a trivialization is specified by a pair $\{n_1, n_2\}$ of linearly-independent, normal vector fields along $C$. Letting $t$ be the unit tangent vector to $C$, we further assume that $\{t, n_1, n_2\}$ is an oriented, orthonormal basis at each point of $C$. Since $n_2$ is determined by $t$ and $n_1$ in this case, a framing of $C$ amounts to the choice of the single, nowhere-vanishing normal vector field $n_1 \equiv n$.

Given any two framings, with unit normal vectors $n$ and $n'$, the difference between $n$ and $n'$ is measured by a map from $C$ to $SO(2)$. Up to homotopy, such maps are classified by the winding-number, so $C$ admits countably-many choices of framing.

Given a non-vanishing normal vector field $n$ along $C$, the Wilson loop operator $W_R(C)$ can be renormalized by point-splitting. For the self-linking number, point-splitting just means that we consider a new curve $\tilde{C} \subset \mathbb{R}^3$ obtained after displacing $C$ by a small amount in the direction of $n$ at each point. See Figure 1-9 for an example. The self-linking number of $C$ as a framed knot is then defined to be the usual linking number of $C$ with $\tilde{C}$,

$$L_f(C) = L(C, \tilde{C}) \in \mathbb{Z}.$$  (1.3.31)
With this definition, \( L_f(C) \) is clearly an integer and an invariant of the pair \((C, n)\). However, \( L_f(C) \) also contains no topological information about the knot \( C \) itself, since the self-linking number takes all integer values as the winding of the normal vector field \( n \) is shifted relative to any fixed framing.

We cannot do better than the crude definition (1.3.31) via point-splitting if we demand \( L(C) \) to be invariant under smooth isotopies of the arbitrary space curve. But we can definitely do better if we allow ourselves to impose additional geometric constraints on \( C \), as well as to restrict the kinds of deformations under which we require invariance of \( L(C) \). This statement neatly captures the essential philosophy of our work.

As one illustration dating back to [21, 54], let us suppose that \( C \subset \mathbb{R}^3 \) is a space curve with everywhere non-zero curvature \( \kappa > 0 \). In terms of a unit-speed parametrization \( X(\varphi) \), the curvature is given by the second-derivative

\[
\dddot{X} = \kappa n, \quad ||n||^2 = 1, \quad (1.3.32)
\]

where \( n \) is the unit normal to the curve. Thus \( \kappa(\varphi) > 0 \) is strictly-positive and \( n \) is globally-defined so long as \( \dddot{X}(\varphi) \neq 0 \) for all \( \varphi \). Moreover, \( C \) is canonically framed by \( n \) as well as the unit binormal vector\(^{11}\)

\[
\begin{align*}
b &= t \times n, \\
t &= \dot{X}.
\end{align*}
\]

The condition that \( \kappa \) be non-zero at all points on \( C \) is an open condition which holds generically, eg. for the circular unknot in Figure 1-9, so we do not lose much by the assumption. In recompense, we gain a way to cancel the variation of the naive Gauss self-linking integral by subtracting a suitable counterterm.

Famously, the necessary counterterm is the total torsion measured with respect to arc-length,

\[
T(C) = \frac{1}{2\pi} \oint_{S^1} d\varphi \, \tau, \quad (1.3.34)
\]

\(^{11}\)As standard, ‘\( \times \)’ indicates the cross-product of vectors in \( \mathbb{R}^3 \).
where the local torsion $\tau(\varphi)$ is defined in a unit-speed parametrization by the derivative of the binormal,
\[
\dot{b} = -\tau n.
\] (1.3.35)

With the sign convention in (1.3.35), $\tau$ is positive when the Frenet-Serret frame $\{t, n, b\}$ rotates in a right-handed fashion as $C$ is traversed in the positive direction. Note that some authors, including that of the classic text [22], follow the opposite sign convention for $\tau$. In terms of any regular parametrization $X(s)$, not necessarily unit-speed,
\[
\tau = \frac{\epsilon_{\mu\nu\rho} \dot{X}^\mu \ddot{X}^\nu \dddot{X}^\rho}{||\dot{X} \times \ddot{X}||^2}, \quad \dddot{X}(s) \neq 0.
\] (1.3.36)

Again, we require the condition $\dddot{X}(s) \neq 0$ to ensure that $\tau$ is everywhere well-defined. See Problem 12 in §1.5 of [22] for a derivation of (1.3.36).

We have seen the triple-product $\epsilon_{\mu\nu\rho} \dot{X}^\mu \ddot{X}^\nu \dddot{X}^\rho$ in (1.3.36) before, when we analyzed the behavior of the Gauss self-linking integrand (1.3.20) in the vicinity of the diagonal. This observation hints at the following miracle: the variation of the total torsion $T(C)$ is precisely opposite the variation (1.3.30) of the Gauss self-linking integral,
\[
\delta T(C) = \frac{1}{2\pi} \int_{S^1} d\varphi \epsilon_{\mu\nu\rho} \dot{X}^\mu \delta X^\nu \dddot{X}^\rho, \quad ||\dot{X}(\varphi)||^2 = 1,
\] (1.3.37)

Consequently, if we consider the ‘renormalized’ self-linking number
\[
L_r(C) = L(C) + T(C),
\] (1.3.38)

then $L_r(C)$ is invariant under arbitrary deformations of $C$,
\[
\delta L_r(C) = 0.
\] (1.3.39)

Globally, the renormalized self-linking number $L_r(C)$ is invariant under smooth isotopies of $C$ which preserve the non-degeneracy condition $\kappa > 0$, necessary to define $T(C)$ in the first place.
A very elegant, indirect proof of (1.3.39) is given by Pohl in [54]. In keeping with the style of the present paper, we supply in Appendix 5.3.3 an alternative proof by directly evaluating the variation of the torsion in (1.3.37). Despite the complicated dependence of \( \tau \) on \( X \) in (1.3.36), this calculation turns out to be surprisingly easy when one exploits appropriately the Frenet-Serret apparatus.

At least for abelian Chern-Simons theory, the torsion counterterm in (1.3.38) deserves its name, since its appearance amounts to a multiplicative redefinition of the Wilson loop operator of charge \( q \) by a phase

\[
W_q(C)_r \equiv W_q(C) \cdot \exp \left[ \frac{i q^2}{2k} \oint_{S^1} d\varphi \tau \right].
\]

The renormalized operator \( W_q(C)_r \) depends explicitly on the geometry of \( C \subset \mathbb{R}^3 \) as an embedded space curve. However, due to the additional phase in (1.3.40), the link expectation value now depends upon the renormalized (as opposed to naive) self-linking number,

\[
\langle W_{q_1}(C_1)_r \cdots W_{q_m}(C_m)_r \rangle = \exp \left[ i g^2 \sum_{j \neq \ell} q_j q_\ell L(C_j, C_\ell) + \frac{i g^2}{2} \sum_j q_j^2 L_r(C_j) \right],
\]

and according to (1.3.39), the link expectation value is a deformation-invariant of the curves \( C_1, \ldots, C_m \), as we originally hoped. To leading order, the same discussion applies to the Wilson loop operator \( W_R(C) \) in non-abelian Chern-Simons theory, where the role of \( q^2 \) in (1.3.40) is played by the quadratic Casimir \( c_2(R) \) appearing in (1.3.11).

The renormalization of the Wilson loop operator in (1.3.40) should be compared to the renormalization of the Chern-Simons action itself on a framed three-manifold \( M \). As explained originally in [69], to maintain diffeomorphism invariance of the partition function, one must correct the naive Chern-Simons action by a one-loop counterterm (a gravitational Chern-Simons term) which depends explicitly upon the metric on \( M \).

For consistency with gauge invariance, the renormalized self-linking number \( L_r(C) \)
must be more than a deformation-invariant of $C$. It must be an integer as well,

$$L_t(C) \in \mathbb{Z}. \quad (1.3.42)$$

We demonstrate the integrality of $L_t(C)$ by computing the invariant for any curve which satisfies the non-degeneracy condition $\kappa > 0$. This computation can be done in many ways. We will follow a strategy which generalizes, in a more sophisticated fashion, to the analysis of Legendrian knots in Section 5.3.

A standard way to present a knot in $\mathbb{R}^3$ is by projecting the image of the knot onto a chosen plane, so as to obtain a planar diagram with double-point singularities at over- and under-crossings, as for instance in Figure 1-10. To compute the integrals which comprise the renormalized self-linking number $L_t(C)$, we will go one step further by actually flattening the knot (as opposed to merely projecting its image) into a plane.

In terms of coordinates $(x^1, x^2, x^3) \in \mathbb{R}^3$, we consider a smooth isotopy $F_\Lambda$ with real parameter $\Lambda > 0$,

$$F_\Lambda : \mathbb{R}^3 \to \mathbb{R}^3, \quad F_\Lambda(x^1, x^2, x^3) = \left(x^1, x^2, \frac{x^3}{\Lambda}\right). \quad (1.3.43)$$

The isotopy $F_\Lambda$ interpolates in a simple way from the identity map for $\Lambda = 1$ to the projection onto the plane $x^3 = 0$ as $\Lambda$ increases to infinity. Thus, if we consider the
one-parameter family \( X_\Lambda = F_\Lambda \circ X : S^1 \to \mathbb{R}^3 \) of embeddings, this family describes a compression of the knot described by \( X \) into the plane \( x^3 = 0 \) as \( \Lambda \) increases to infinity. Without loss, we assume that the projection of the knot onto the plane \( x^3 = 0 \) is generic, in the sense of having only double-point (as opposed to triple-point and higher) singularities as occur in Figure 1-10. Otherwise, if the projection is non-generic, one can perform a small, rigid rotation of the knot in \( \mathbb{R}^3 \) to achieve a generic projection.

For any finite value of \( \Lambda \), the embedding \( X_\Lambda \) satisfies the non-degeneracy condition \( \kappa > 0 \), or equivalently \( \tilde{X}_\Lambda(s) \neq 0 \), if the original embedding \( X_{\Lambda=1} \) does. Thus, by isotopy invariance, the value of the renormalized self-linking number \( L_r(C) \) does not depend upon \( \Lambda \). We will compute \( L_r(C) \) by considering the asymptotic regime \( \Lambda \gg 1 \), in which the knot is nearly planar.

Recall that \( L_r(C) \) consists of two integrals,

\[
L_r(C) = L(C) + T(C),
\]

where

\[
L(C) = \lim_{\epsilon \to 0^+} \int_{S^1 \times S^1 - \Delta(\epsilon)} \frac{\epsilon_{\mu\nu\rho} \Delta X^\mu(s_1) \tilde{X}^\nu(s_1) \tilde{X}^\rho(s_2) ds_1 \wedge ds_2}{4\pi \|\Delta X\|^3},
\]

\[
\Delta X^\mu(s_1, s_2) = X^\mu(s_1) - X^\mu(s_2),
\]

and

\[
T(C) = \frac{1}{2\pi} \int_{S^1} d\phi \, \tau.
\]

These two integrals behave very differently in the regime of large \( \Lambda \). The torsion \( \tau \) is defined for any immersed (not necessarily embedded) curve with \( \kappa > 0 \), so \( \tau \) is insensitive to the double-point singularities which appear in the limit \( \Lambda \to \infty \). Because \( \tau \) measures the rotation of the Frenet-Serret frame in space, \( \tau \) vanishes identically for a plane curve. So immediately,

\[
\lim_{\Lambda \to \infty} T(C) = 0.
\]
We are left to evaluate the (unrenormalized) self-linking integral $L(C)$ when $\Lambda \gg 1$. We do so by dividing the domain of integration in (1.3.45) into three regions. First, for the generic point $(s_1, s_2) \in S^1 \times S^1$ on the torus, the denominator $||\Delta X||^3$ in the integrand is bounded away from zero for all values of $\Lambda$. Equivalently, $X(s_1)$ and $X(s_2)$ span a non-trivial chord on the knot in the planar limit. Such points make a negligible contribution to the integral in (1.3.45) when $\Lambda$ is large, because the triple-product of the vectors $\Delta X(s_1, s_2)$, $\dot{X}(s_1)$, and $\ddot{X}(s_2)$ in the numerator of (1.3.45) vanishes in the planar limit.

We next consider the other two regions on the torus, where the denominator $||\Delta X||^3$ is small for large $\Lambda$. Of course, for all values of $\Lambda$ this denominator is small if $(s_1, s_2)$ lies near the diagonal $\Delta \subset S^1 \times S^1$. We have already analyzed the self-linking integrand in a neighborhood of the diagonal and found a non-singular result (1.3.20), proportional to the triple-product of $\dot{X}$, $\ddot{X}$, and $\dddot{X}$. Again, this triple-product vanishes in the planar limit, so the contribution from a neighborhood of $\Delta$ is also negligible for large $\Lambda$.

Otherwise, $||\Delta X||^3$ becomes small for $\Lambda \gg 1$ when the points $X(s_1)$ and $X(s_2)$ lie near a crossing in the plane projection of the knot. For this case, the contribution to the self-linking integral is non-vanishing but can be explicitly evaluated.

In a small neighborhood of the crossing, the curve $C$ can be approximated to first-order by its tangent lines, and by isotopy invariance we might as well assume those tangent lines to be perpendicular, as in Figure 1-11. We assume that the two strands in the figure are separated by a vertical distance $1/\Lambda \ll 1$. Each strand can

Figure 1-11: Writhe at right- and left-handed crossings.
parametrized as

\[ X(s_1) = (s_1, 0, \pm 1/\Lambda), \quad X(s_2) = (0, s_2, 0), \quad (1.3.48) \]

where the sign in \( X(s_1) \) distinguishes the right- and left-handed crossings in Figure 1-11. We then evaluate the contribution to the self-linking integral (1.3.45) from the crossing as

\[
\begin{align*}
L(C)\bigg|_{\text{right/left}} &= \pm \frac{1}{4\pi \Lambda} \int_{\mathbb{R}^2} ds_1 ds_2 \frac{1}{s_1^2 + s_2^2 + (1/\Lambda^2)^{3/2}} + (s_1 \leftrightarrow s_2), \\
&= \pm 1.
\end{align*}
\]

(1.3.49)

In the first line of (1.3.49), we indicate an identical contribution, and hence an overall factor of two, which arises when the roles of the parameters \( s_1 \) and \( s_2 \) in (1.3.48) are interchanged.

Each crossing in the planar diagram thus contributes \( \pm 1 \) to the self-linking number. The sum of the signed contributions from all crossings is a well-known quantity in knot theory. By definition, this sum is the writhe \( w(C) \) of the planar diagram for the knot,

\[
L_W(C) = \sum_{\text{crossings } j} w_j, \quad (1.3.50)
\]

At the \( j \)-th crossing, \( w_j = \pm 1 \) according to Figure 1-11. Note immediately that \( w(C) \) does not depend upon the orientation of \( C \), since reversing the direction on both strands in Figure 1-11 preserves each diagram, after a rotation in the plane. As a consistency check, the integrals defining \( L(C) \) and \( T(C) \) in (1.3.45) and (1.3.46) are also invariant under orientation-reversal of \( C \). Moreover, the writhe in (1.3.50) is trivially an integer, so the renormalized self-linking number satisfies the necessary condition for gauge invariance.

The renormalized self-linking number \( L_W(C) \) is not invariant under arbitrary smooth isotopies of \( C \), as illustrated by the unknot diagrams with different writhes in Figure 1-12. Since the self-linking number is invariant under those isotopies which preserve
the non-degeneracy condition $\kappa > 0$, any isotopy that relates the distinct versions of the unknot in Figure 1-12 must evidently pass through a configuration which has vanishing curvature $\kappa = 0$ at some point along the knot. This statement is not altogether obvious, so it demonstrates that $L_r(C)$ indeed encodes non-trivial geometric information about the space curve $C \subset \mathbb{R}^3$.

Finally, let us mention the relation between the ‘renormalized’ self-linking number $L_r(C)$ and the framed self-linking number $L_f(C)$, both of which correspond to different ways to regulate the short-distance behavior of the Wilson loop operator $W_R(C)$. The essential clue is the non-degeneracy condition $\kappa > 0$ required to define $L_r(C)$. Under this condition, $C$ is equipped with a distinguished Frenet-Serret framing by the unit normal vector $n$ in (5.1.20), or equivalently by the unit binormal vector $b$ in (1.3.33). Hence one might naturally guess that $L_r(C) = L_f(C)$, where the framed self-linking number is defined with respect to the Frenet-Serret normal $n$.

This guess is correct, as can be seen by considering the behavior of the Frenet-Serret frame $\{t, n, b\}$ in the planar limit $\Lambda \to \infty$. Very briefly, when the curve $C$ is planar, the Frenet-Serret normal $n$ lies everywhere in the plane $x^3 = 0$ of the curve. By the same token, the binormal $b$ points everywhere perpendicular to the plane of the curve, so this framing is commonly called the vertical framing. After displacing $C$ by $n$ to obtain $\tilde{C}$, one finds a ribbon graph, depicted in a neighborhood of the positive crossing in Figure 1-13. Each self-crossing of $C$ contributes $\pm 1$ to the planar writhe, but such a crossing also contributes the same amount to the linking number $L(C, \tilde{C})$, computable as a signed count of crossings of $C$ by $\tilde{C}$. Equivalently, the ribbon can be straightened to the twisted band on the right of Figure 1-13, with one unit of winding.

Figure 1-12: Writhe for a selection of unknots.
for $n$.

An important goal in the remainder of our work will be to explain a third way to define the self-linking number, again by regulating the behavior of the Wilson loop operator $W_R(C)$ at short-distances. Our regularization will be more delicate by far than either a crude point-splitting or the addition of explicit counterterms à la Călugăreanu [21] and Pohl [54]. Rather, we shall promote $W_R(C)$ to a supersymmetric operator in what is effectively a supersymmetric version of Chern-Simons theory.
Chapter 2

Contact Chern-Simons Theory

The supersymmetric version of Chern-Simons theory relevant to us will be the ‘contact’ Chern-Simons theory introduced in [12] and further analyzed in [13, 14, 41, 42, 51]. Even as a classical gauge theory, contact Chern-Simons theory possesses several exotic features. Our goal in this Chapter is both to review these features as well as to introduce several essential notions from contact geometry and topology.

Contact Chern-Simons theory is a bosonic quantum field theory, involving only a gauge field $A$ on a contact three-manifold $M$, so the role of supersymmetry is not manifest in the naive formulation of this theory. Yet at least for certain classes of observables, many instances are known in which a supersymmetric quantum field theory is secretly equivalent to a bosonic quantum field theory. Examples include the relation between $\mathcal{N} = (2, 2)$ supersymmetric and physical Yang-Mills theory in two dimensions [70], as well as the more recently exploited relation [32, 71] between the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on the half-space $\mathbb{R}^4_+$, with suitable boundary conditions, and the usual Chern-Simons theory on $\mathbb{R}^3$.

For contact Chern-Simons theory, its counterpart in the superworld is the $\mathcal{N} = 2$ supersymmetric Yang-Mills-Chern-Simons (YM-CS) theory, without chiral matter. The latter equivalence underlies the localization computations in [12, 13, 43] and has been explained from a formal, algebraic perspective by Källén and Zabzine in §2.2 of [44]. Mostly out of a desire to streamline the exposition, we do not employ supersymmetric technology here. Nonetheless, as a means to motivate the classical Lagrangian
for contact Chern-Simons theory, we will sketch yet another, more geometric way to understand the relation to $\mathcal{N} = 2$ supersymmetric YM-CS theory.

### 2.1 Supersymmetric Yang-Mills Theory in the Contact Limit

Let us first consider the classical Lagrangian for purely bosonic (i.e., non-supersymmetric) YM-CS theory on a Riemannian three-manifold $M$,

$$I(A) = -\frac{1}{8\pi e^2} \int_M \text{Tr}(F_A \wedge \star F_A) + i k \text{CS}(A), \quad (2.1.1)$$

where $F_A = dA + A \wedge A$ is the curvature, and $\text{CS}(A)$ is the Chern-Simons functional with the same normalization in Section 1.1,

$$\text{CS}(A) = \frac{1}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (2.1.2)$$

Beyond the level $k \in \mathbb{Z}$, the classical Lagrangian in (2.1.1) depends upon the Yang-Mills coupling $e^2$ as well as the Riemannian metric $g$, which determines the Hodge operator $\star$ on $M$. Because the signature of the metric on $M$ is Euclidean, the Yang-Mills path integral takes the exponentially-damped\(^1\) form

$$Z(k, e^2, g) = \frac{1}{\text{Vol}(g)} \int_A \mathcal{D}A \exp[-I(A)], \quad (2.1.3)$$

and the relative factor of ‘$i$’ in the Euclidean action (2.1.1) is required for gauge-invariance of the path integrand. In particular, $I(A)$ is complex-valued, with positive real part. This innocuous feature will become crucial for the analysis in Section 5.3.

Without supersymmetry, $Z$ depends not only upon the Chern-Simons level $k$ but also on the Yang-Mills coupling $e^2$ and the metric $g$. The latter dependence is not arbitrary, for reasons of dimensional analysis. Fix a fiducial metric $g_0$ on $M$, and consider the one-parameter family of rescaled metrics

$$g_\sigma = \exp(2\sigma) \cdot g_0, \quad \sigma \in \mathbb{R}. \quad (2.1.4)$$

\(^1\)Recall that ‘$\text{Tr}$’ is by convention a negative-definite form on the Lie algebra $\mathfrak{g}$.  

55
The Chern-Simons term in $\mathbf{I}(A)$ does not depend upon the parameter $\sigma$, or upon $g_0$ for that matter, whilst the Yang-Mills term depends upon $\sigma$ as

$$
\int_M \text{Tr}(F_A \wedge *_\sigma F_A) = \exp(-\sigma) \cdot \int_M \text{Tr}(F_A \wedge *_0 F_A). 
$$

(2.1.5)

Here $*_\sigma$ and $*_{0}$ are the respective Hodge operators for the metrics $g_{\sigma}$ and $g_{0}$. Evidently, the Lagrangian in (2.1.1) is invariant under the simultaneous scaling of the metric $g$ and the coupling $e^2$ by

$$
g \mapsto \exp(2\sigma) \cdot g, \quad e^2 \mapsto \exp(-\sigma) \cdot e^2.
$$

(2.1.6)

With the transformation in (2.1.6), the value of $e^2$ can be set to unity, and all dependence in $Z(k, e^2, g)$ on the Yang-Mills coupling is absorbed into a dependence on the overall scale of the metric on $M$.

Accordingly, the parameter regimes of weak or strong Yang-Mills coupling become equivalent to geometric regimes in which the volume of $M$ is respectively small or large. When $e^2 \gg 1$, or equivalently when the volume of $M$ is large, the Chern-Simons term dominates the YM-CS Lagrangian in (2.1.1), and $Z$ reduces to the partition function (1.1.5) for pure Chern-Simons theory on $M$. Conversely when $e^2 \ll 1$, or the volume of $M$ is small, the Yang-Mills term dominates the Lagrangian in (2.1.1). In three dimensions, Yang-Mills theory is super-renormalizable, meaning that the typical Feynman diagram is convergent at short-distances. The Yang-Mills term in (2.1.1) thus serves as an implicit short-distance regulator for the pure Chern-Simons theory, at the cost of an explicit metric dependence in the YM-CS Lagrangian.

**A Little Contact Geometry.** We next promote the bosonic YM-CS theory to a supersymmetric theory on $M$. To obtain a representation of the $\mathcal{N} = 2$ supersymmetry algebra, one must consider extra quantum fields beyond the gauge field $A$, and one must add terms to $\mathbf{I}(A)$ which are related to the Yang-Mills and Chern-Simons terms by the action of supersymmetry. We will not explain those details here, as the supersymmetric gauge theory *per se* is not our focus. See [1, 25, 26, 38, 39, 40, 43, 46]
for a variety of recent works concerning ‘curved’ supersymmetry on three-manifolds.

We require only a few broad statements about supersymmetric YM-CS theory on $M$.

First, to preserve a twisted version of $\mathcal{N} = 2$ supersymmetry, the Riemannian metric on $M$ must take a special form compatible with a reduction in the structure group of the tangent bundle $TM$ from $\text{Spin}(3) \simeq SU(2)$ to $\text{Spin}(2) \simeq U(1)$.\footnote{Because the $\mathcal{N} = 2$ supersymmetry algebra involves spinors, the spin groups of $TM$ are the relevant ones for twisting. The tangent bundle $TM$ of any compact, orientable three-manifold is topologically trivial, so a lift of the Riemannian structure group $SO(3)$ to $\text{Spin}(3)$ always exists. See for instance Theorem 4.2.1 in [34] for three separate proofs of the triviality of $TM$.} After the reduction in the structure group, one can twist the $\mathcal{N} = 2$ supersymmetric YM-CS in a standard fashion [68] by means of its global $U(1)_R$ symmetry. This beautiful observation first appeared in [43], though its mathematical antecedents go back at least to [48].

In each tangent fiber, the reduction in structure group from $\text{Spin}(3)$ to $\text{Spin}(2)$ amounts to an isomorphism $\mathbb{R}^3 \simeq \mathbb{R} \oplus \mathbb{C}$, from which we obtain globally a smooth family of two-planes $H \subset TM$ on $M$. Here $H$ should be read as “horizontal.” Locally in an open patch $\mathcal{U} \subset M$, $H$ can be specified as the kernel of a non-vanishing one-form $\kappa \neq 0$,

$$H = \ker \kappa \subset T\mathcal{U} \quad \kappa \in \Omega^1(\mathcal{U}). \quad (2.1.7)$$

Conversely, $\kappa$ is determined by $H$ up to multiplication by a non-vanishing function $f \neq 0$,

$$\kappa \sim f \cdot \kappa, \quad f \in \Omega^0(\mathcal{U}). \quad (2.1.8)$$

Depending on the topology of $H$, the local one-form $\kappa$ may or may not exist globally, due to the non-trivial equivalence relation in (2.1.8). At issue is the sign of the function $f$ in a consistent patching to define $\kappa$ everywhere on $M$. Briefly, if $\kappa \in \Omega^1(M)$ exists globally, then $\kappa \neq 0$ trivializes the dual of the real line-bundle $TM/H$. Since $TM$ is orientable by assumption, triviality of $TM/H \simeq \mathbb{R}$ is equivalent to orientability of $H$, which is the necessary and sufficient condition for $\kappa$ to exist globally as an honest one-form on $M$. For convenience, we assume throughout that
$H \subset TM$ is orientable and hence that a representative one-form $\kappa \in \Omega^1(M)$ does exist on $M$.\footnote{The orientability assumption on $H$ is also typical throughout the subject.}

In order to obtain a Riemannian metric compatible with $H$, we must make further topological assumptions about the family of two-planes on $M$. For instance, if $\kappa \wedge d\kappa = 0$ identically on $M$, then $H$ is a codimension-one foliation. In this case, each plane fiber of $H$ can be locally modelled on the tangent space of an embedded surface $\Sigma \subset M$. The simplest example occurs when $M = S^1 \times \Sigma$ and $H = T\Sigma$ is the product foliation. A representative one-form for this foliation is just the angular form $\kappa = d\theta/2\pi$ on the circle. To reduce the structure group of $TM$ compatibly with $H$, we consider a product metric

$$ds_M^2 = \frac{1}{2} \omega(\cdot, J \cdot) + \kappa \otimes \kappa, \quad M = S^1 \times \Sigma, \quad (2.1.9)$$

where $\omega$ and $J$ are any compatible Kähler form and complex structure on $\Sigma$. For the product manifold with the product metric in (2.1.9), the twist of the $N = 2$ supersymmetry algebra to preserve two supercharges is well-known.

Our interest today lies at the opposite extreme, for which the three-form $\kappa \wedge d\kappa \neq 0$ is everywhere non-vanishing on $M$. By definition, the plane field $H$ is then a contact structure on $M$. Equivalently, $d\kappa|_H \neq 0$, so $d\kappa$ restricts to symplectic form on each two-plane fiber of $H$. In this sense, contact structures [18, 28, 34, 35] are a generalization of symplectic structures to manifolds of odd-dimension. Contact structures in dimension three are plentiful. By fundamental results of Lutz [49] and Martinet [50], each homotopy class of orientable plane fields on a compact, orientable three-manifold $M$ admits a contact structure. Note that orientability of both $H$ and $M$ follows from the existence of the contact form $\kappa$, since $d\kappa|_H \neq 0$ orients $H$, and $\kappa \wedge d\kappa \neq 0$ orients $M$.

For the remainder, we assume that the orientation of $M$ is compatible with the...
orientation induced by the contact form, in the sense that
\[ \int_M \kappa \wedge d\kappa > 0. \tag{2.1.10} \]
This condition is invariant under the local rescaling of \( \kappa \) by the non-vanishing function \( f \neq 0 \) in (2.1.8), since \( \kappa \wedge d\kappa \) scales quadratically with \( f \),
\[ \kappa \wedge d\kappa \mapsto f^2 \cdot \kappa \wedge d\kappa. \tag{2.1.11} \]
In particular, derivatives of \( f \) do not enter (2.1.11), and the sign of \( \kappa \wedge d\kappa \) does not depend upon the sign of \( f \). Both the contact condition \( \kappa \wedge d\kappa \neq 0 \) and the positivity condition in (2.1.10) are therefore intrinsic properties of the underlying plane field \( H \subset TM \).

Associated to any contact form is a Reeb vector field \( R \), determined by
\[ \iota_R \kappa = 1, \quad \iota_R d\kappa = 0, \tag{2.1.12} \]
where \( \iota_R \) indicates the interior product (or contraction) with \( R \). Since the Lie derivative \( \mathcal{L}_R \) along the Reeb field is given by the anti-commutator
\[ \mathcal{L}_R = \{ d, \iota_R \} = d \circ \iota_R + \iota_R \circ d, \tag{2.1.13} \]
the conditions in (2.1.12) imply
\[ \mathcal{L}_R \kappa = 0. \tag{2.1.14} \]
The Reeb flow thus preserves the contact structure \( H \) on \( M \). Since \( [\mathcal{L}_R, d] = 0 \), we automatically have \( \mathcal{L}_R d\kappa = 0 \) as well. Note that the Reeb field \( R \) depends not only upon \( H \) but also upon the representative contact form \( \kappa \).

Along with being plentiful, contact structures in dimension three are tame, in the sense of admitting metrics which realize the reduction [23] in structure group to \( U(1) \). To write such a metric, we require a compatible complex structure tensor \( J \in \text{End}(TM) \). The notion of a complex structure on a three-manifold \( M \) may
seem exotic at first glance, but $J$ will simply provide a way to specify simultaneously a complex structure on each two-plane in $H$, as implicit in the local isomorphism $\mathbb{R}^3 \simeq \mathbb{R} \oplus \mathbb{C}$.

By definition, the complex structure tensor $J (\equiv J^\mu_\nu$ in local coordinates) satisfies

$$J^2 = -1 + R \otimes \kappa, \quad J \in \text{End}(TM). \quad (2.1.15)$$

Immediately, $J$ restricts to a complex structure tensor on $H$,

$$J^2 |_H = -1, \quad (2.1.16)$$

which we assume to be compatible with the symplectic form $d\kappa |_H$ in the usual sense that

$$d\kappa(J \cdot, J \cdot) = d\kappa(\cdot, \cdot), \quad d\kappa(\cdot, J \cdot) |_H > 0. \quad (2.1.17)$$

By (2.1.15) the Reeb field $R$ lies in the kernel of $J$,

$$J(R) = 0. \quad (2.1.18)$$

Hence a local frame for $TM$ exists in which $J$ takes the canonical form

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.1.19)$$

Because we consider complex structures on two-planes, any integrability condition for $J$ would be vacuous, and a complex structure tensor satisfying the conditions in (2.1.15) and (2.1.17) always exists. See Theorem 4.4 in [18] for a proof of this statement, along with many other useful facts about Riemannian contact geometry.

With the contact form $\kappa$ and the complex structure $J$ in hand, we can immediately
produce an associated contact metric

\[ ds_M^2 = \frac{1}{2} d\kappa(\cdot, J \cdot) + \kappa \otimes \kappa. \]  \hspace{1cm} (2.1.20)

With respect to the contact metric, the Reeb vector field has unit length, \(||R||^2 = 1\), and it is orthogonal to each contact plane in \(H\). Also, given the relative factor of one-half in (2.1.20),

\[ d\kappa = 2 \star \kappa, \quad \text{vol}_M = \star 1 = \frac{1}{2} \kappa \wedge d\kappa. \]  \hspace{1cm} (2.1.21)

Like the product metric in (2.1.9), the contact metric in (2.1.20) manifests the reduction in structure group to \(U(1)\), with \(d\kappa\) now playing the role of the Kähler form \(\omega\) on \(\Sigma\).

The product metric on \(M = S^1 \times \Sigma\) admits an isometry by rotations in the circle direction. In order to preserve two supercharges \([43]\) from the \(\mathcal{N} = 2\) supersymmetry algebra on the contact manifold \(M\), we must similarly demand that the Reeb vector field \(R\) generate an isometry of the contact metric in (2.1.20). Because \(\kappa\) and \(d\kappa\) are invariant under the Reeb flow by definition, we require

\[ \mathcal{L}_R g = 0 \iff \mathcal{L}_R J = 0, \]  \hspace{1cm} (2.1.22)

which is the condition that \(R\) be Killing with respect to the contact metric \(g\) in (2.1.20).

The condition \(\mathcal{L}_R J = 0\) imposes very stringent constraints on the topology of \(M\). According to the classification \([33]\) by Geiges, \(M\) must be a Seifert-fibered three-manifold. Such a three-manifold can be described succinctly as the total space of a

\footnote{Our convention for the contact metric follows Taubes \([62]\) and is natural for a variety of reasons, some of which will be mentioned shortly.}
nontrivial circle bundle over a Riemann surface $\Sigma$,

$$S^1 \rightarrow^\pi M \quad , \quad \Sigma$$

(2.1.23)

where $\Sigma$ is generally allowed to have orbifold points and the circle bundle is allowed to be a corresponding orbifold bundle. See for instance §3.2 in [12] for more about the orbifold case and the topological classification of Seifert three-manifolds. We take the contact form $\kappa$ to be a $U(1)$-connection on $M$, regarded as the total space of a principal $U(1)$-bundle, with curvature

$$d\kappa = n \pi^* \omega, \quad n \neq 0 .$$

(2.1.24)

Here $\omega$ is a unit-area symplectic form on $\Sigma$. So long as the degree $n$ of the bundle is non-zero, $\kappa \wedge d\kappa \neq 0$ by definition of the connection. The Reeb field $R$ then generates rotations in the circle fiber of $M$, and an invariant complex structure $J$ can be lifted from $\Sigma$.

The basic example of this construction occurs when $M$ is the three-sphere $S^3$ and the contact form $\kappa$ is a connection on the degree-one Hopf fibration over $\mathbb{C}P^1$. If $\kappa$ and $J$ are taken to be left-invariant under the identification $S^3 \simeq SU(2)$, the contact metric in (2.1.20) just becomes the round metric on $S^3$. Without the relative factor of two, the contact metric associated to a left-invariant contact form on $S^3$ would have been squashed, which is one motivation for our metric convention.

**Into the Anisotropic Infrared.** When we specialize to the case (2.1.23) that $M$ is a Seifert-fibered three-manifold with a Seifert contact structure, the supersymmetric Yang-Mills-Chern-Simons action takes the familiar form

$$I_{SUSY} = -\frac{1}{8\pi e^2} \int_M Tr\left( F_A \wedge *F_A \right) + ik \text{CS}(A) + \cdots ,$$

(2.1.25)
where the ‘⋯’ indicate all the other terms required to preserve a twisted form of \( \mathcal{N} = 2 \) supersymmetry on \( M \). For our analysis here, the precise nature of the omitted terms is irrelevant.

By arguments dating to [68], the expectation value of any supersymmetric operator in the twisted \( \mathcal{N} = 2 \) YM-CS theory is independent of the Yang-Mills coupling \( e^2 \). With the simultaneous transformation in (2.1.6), the expectation value is equally-well invariant under an overall scaling of the contact metric \( g \) on the Seifert manifold \( M \). Moreover, in the infrared limit \( e^2 \to \infty \), the supersymmetric YM-CS theory reduces to the bare Chern-Simons theory in Section 1.1.\(^5\) Consequently, supersymmetric observables in the twisted YM-CS theory on the Seifert manifold \( M \) agree with corresponding non-supersymmetric observables in the bare Chern-Simons theory.

Supersymmetric observables in the twisted YM-CS theory can be studied in other limits. An immediate alternative is the ultraviolet limit \( e^2 \to 0 \), in which the supersymmetric action in (2.1.25) becomes weakly-coupled. In this limit, the supersymmetric YM-CS path integral can be evaluated semiclassically by localization [43, 45], though practical computations are only feasible when \( M \) is a relatively simple Seifert manifold, such as a lens space, with finite fundamental group. One can also apply localization techniques to compute the expectation values of supersymmetric Wilson loops which wrap the Seifert fibers of \( M \). Examples of the latter include all torus knots in \( S^3 \).

If \( M \) is a Riemannian three-manifold with a generic metric \( g \), then \( M \) is best pictured as a blob, with no structure. The only geometric limits for a blob are the limits of large and small volume, corresponding to the preceding limits \( e^2 \to \infty \) and \( e^2 \to 0 \). However, as soon as we endow \( M \) with the specific contact metric in (2.1.20), additional geometric limits become possible.

From the perspective of supersymmetry, nothing is special about the normalization of the contact metric in the directions of \( H \) versus the direction \( R \), so the contact metric

\(^5\)In the limit \( e^2 \to \infty \) with \( k \neq 0 \), all component fields in the \( \mathcal{N} = 2 \) vector multiplet other than the gauge field \( A \) have large masses and can be eliminated by Gaussian integration.
metric in (2.1.20) is best considered as a single member of the two-parameter family

\[ ds^2_M = \frac{t}{2r} d\kappa(\cdot, J\cdot) + t^2 \kappa \otimes \kappa, \quad r, t \in \mathbb{R}_+. \]  

(2.1.26)

All metrics in (2.1.26) are compatible with a reduction in the structure group of $TM$ to $U(1)$. For $S^3$ with the standard, left-invariant contact form, these metrics are squashed metrics with an $SU(2) \times U(1)$ isometry.

For any values $r, t > 0$, a twisted version of $\mathcal{N} = 2$ supersymmetry can be preserved on the Seifert manifold $M$ with the contact metric in (2.1.26), and the expectation values of supersymmetric operators do not depend upon the parameters $(r, t)$. This statement is hardly a surprise and follows from the preceding observation that supersymmetric expectation values are independent of the Yang-Mills coupling $e^2$. For in the limit $e^2 \to \infty$ with $(r, t)$ fixed, the supersymmetric YM-CS theory reduces to the bare, bosonic Chern-Simons theory, with no metric dependence.

The two-dimensional family of contact metrics in (2.1.26) can be parametrized in many ways. With our choice, the dependence on $t$ can always be absorbed by a rescaling of the contact form $\kappa$ itself, as a special case of the transformation in (2.1.8). Nonetheless we assume $\kappa$ and the Reeb field $R$ to be fixed, and we leave $t$ as a metric parameter. Otherwise, the parameter $r$ controls the duality relation

\[ d\kappa = 2r \star \kappa. \]  

(2.1.27)

For economy of notation, we suppress the dependence of the Hodge operator $\star$ on the parameters $(r, t)$. Also, the volume form on $M$ now depends upon $r$ and $t$ as

\[ \text{vol}_M = \frac{t^2}{2r} \kappa \wedge d\kappa. \]  

(2.1.28)

Evidently, the conformal scaling of the metric by $\exp(2\sigma)$ in (2.1.6) corresponds to

\[ t \mapsto \exp(\sigma) \cdot t, \quad r \mapsto \exp(-\sigma) \cdot r. \]  

(2.1.29)
In terms of the parametrization (2.1.26), the naive, isotropic large-volume limit is then
\[ t \to \infty, \quad r \to 0, \quad rt \text{ fixed}. \] (2.1.30)

In this paper we are interested in a different large-volume limit, for which
\[ t \to \infty, \quad r \text{ fixed}. \] (2.1.31)

This limit is anisotropic, since the two terms in the contact metric depend differently on the parameter \( t \). According to (2.1.26), the length of the Reeb field \( R \) scales linearly with \( t \), whereas the lengths of tangent vectors in the contact plane \( H \) scale only with the square-root \( t^{1/2} \).

Globally in terms of the Seifert fibration \( S^1 \to M \to \Sigma \), the limit in (2.1.31) describes a process by which both the circle fiber and the base \( \Sigma \) of \( M \) become large, but the radius of the fiber grows much faster than the radius of the base. Up to a conformal rescaling of the metric, the limit in (2.1.31) is the ‘diabatic’ limit [16] of the Seifert fibration, with large fiber and small base. The diabatic limit is so-named as the geometric limit which is opposite to the adiabatic limit, with small fiber and large base. For the special case \( M = S^3 \), the limit in (2.1.31) corresponds to an infrared limit of large squashing.

To see why the anisotropic large-volume limit (2.1.31) is interesting, we need only consider how the Yang-Mills action depends upon the parameters \( r \) and \( t \).

By the small computations in Appendix 5.3.3, the Yang-Mills action on any contact three-manifold \( M \) with the contact metric in (2.1.26) can be rewritten in terms of contact form \( \kappa \), the Reeb field \( R \), and the complex structure \( J \) as
\[
\int_M \text{Tr}(F_A \wedge \ast F_A) = 2r \int_M \kappa \wedge d\kappa \text{Tr}
\left[\frac{(\kappa \wedge F_A)^2}{\kappa \wedge d\kappa}\right] - \frac{1}{t} \int_M \kappa \wedge \text{Tr}
\left[\iota_R F_A \wedge J \circ \iota_R F_A\right].
\] magnetic electric (2.1.32)

As in [12, 13], we abuse notation somewhat by writing “\( 1/\kappa \wedge d\kappa \)” in the first term on the right in (2.1.32). More precisely, because \( \kappa \wedge d\kappa \) is everywhere non-zero, the
three-form $\kappa \wedge F_A$ satisfies

$$\kappa \wedge F_A = f_A \cdot \kappa \wedge d\kappa, \quad f_A \in \Omega^0(M; g), \quad (2.1.33)$$

for some adjoint-valued function $f_A$ on $M$. By definition, $\kappa \wedge F_A / \kappa \wedge d\kappa \equiv f_A$, in terms of which the first term on the right in (2.1.32) becomes $2 r f_M \kappa \wedge d\kappa \text{Tr}(f_A^2)$.

Otherwise, the second term on the right in (2.1.32) is proportional to the norm-square of the one-form

$$\iota_R F_A \in \Omega^1(H; g). \quad (2.1.34)$$

Here $\Omega^1(H)$ indicates the subspace of one-forms annihilated by contraction with the Reeb field $R$,

$$\Omega^1(H) = \ker(\iota_R) \subset \Omega^1(M). \quad (2.1.35)$$

Elements of $\Omega^1(H)$ said to be ‘horizontal’ with respect to the contact structure on $M$. Because $\iota_R^2 = 0$, the one-form $\iota_R F_A$ is trivially horizontal. The horizontal one-forms will play a special role in Sections 3 and 5.3.

The geometric formula in (2.1.32) is valid for an arbitrary contact three-manifold $M$, whether Seifert or no. Especially in the Seifert case, though, one is naturally tempted to interpret the Reeb direction $R$ as a kind of periodic Euclidean time, after which the orthogonal plane $H$ spans the spatial directions on $M$. Under this interpretation, the horizontal one-form $\iota_R F_A$ is the electric component of the Yang-Mills field, and likewise the scalar $f_A$ in (2.1.33) is the magnetic component. Hence from the physical perspective, we have simply decomposed the Yang-Mills action in (2.1.32) into respective magnetic and electric terms on each contact two-plane $H \subset TM$.

The essential feature of the electric-magnetic decomposition for the Yang-Mills action on $M$ is the dependence on $r$ and $t$. As a check, note that the dependence on the right in (2.1.32) is consistent with the homogeneous scaling (2.1.5) of the Yang-Mills action under the isotropic scaling for $r$ and $t$ in (2.1.29). In the isotropic limit $t \to \infty$ with $rt$ fixed, both the electric and magnetic terms of the Yang-Mills action
scale to zero at the same rate, for any fixed configuration of $A$.

By contrast, in the anisotropic limit $t \to \infty$ with $r$ fixed, the electric term in the Yang-Mills action vanishes, but the magnetic term survives to the infrared. Applied to the supersymmetric YM-CS action in (2.1.25), the anisotropic infrared limit is

$$\lim_{t \to \infty} I_{\text{SUSY}} = \frac{k}{4\pi} S_E(A), \quad r \text{ fixed,}$$

(2.1.36)

where

$$S_E(A) = -\frac{r}{e^2 k} \int_M \kappa \wedge d\kappa \text{Tr} \left[ \left( \frac{\kappa \wedge F_A}{\kappa \wedge d\kappa} \right)^2 \right] - i \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

(2.1.37)

Again, all components of the $\mathcal{N} = 2$ vector multiplet other than the gauge field $A$ develop large masses when $t \to \infty$, so these components can be eliminated trivially from the supersymmetric action on $M$. On the right-hand side of (2.1.36), we are left with a classical action $S_E(A)$ which depends only on the gauge field. As promised, we have converted the supersymmetric YM-CS theory into a purely bosonic field theory on $M$.

### 2.2 Properties of the Contact Functional

In the infrared limit $t \to \infty$, most geometric data on $M$ is washed away from the Yang-Mills action. Only the contact form $\kappa$ and the parameter $r > 0$ appear in the limiting functional $S_E(A)$ in (2.1.37), and even this dependence is less than it seems.

As already mentioned, under a local rescaling of the contact form by a non-vanishing function $\lambda \neq 0$,

$$\kappa \mapsto \lambda \cdot \kappa, \quad \lambda \in \Omega^0(M),$$

(2.2.1)

the non-vanishing three-form $\kappa \wedge d\kappa$ scales quadratically,

$$\kappa \wedge d\kappa \mapsto \lambda^2 \cdot \kappa \wedge d\kappa.$$

(2.2.2)

From this observation, one can easily check that $S_E(A)$ is invariant under the local
scaling of $\kappa$ in (2.2.1). Hence $S_E(A)$ depends only upon the geometry of the contact plane field $H \subset TM$, not the choice of representative contact form $\kappa$.

Second, we have already noted that supersymmetric observables are independent of the values of the real parameters $e^2, r, \text{ and } t$. This statement remains true in the limit $t \to \infty$, so we can set the dimensionless coefficient $r/e^2k$ in front of the first term in (2.1.37) to any convenient value. For reasons to be explained momentarily, the most convenient value is

$$\frac{r}{e^2k} = 1,$$

after which

$$S_E(A) = -\int_M \kappa \wedge d\kappa \text{Tr} \left( \frac{\kappa \wedge F_A}{\kappa \wedge d\kappa} \right)^2 + i \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

(2.2.4)

By contrast, the supersymmetric YM-CS theory depends upon the value of the level $k \in \mathbb{Z}$. For this reason, we extract the coefficient $k$ from the limiting action in (2.1.36), appropriate when we compute perturbatively in the regime $k \gg 1$.

**A Word about the Signature.** The limiting functional $S_E(A)$ in (2.2.4) is derived from the supersymmetric YM-CS action for a metric of Euclidean signature $(+ + +)$; hence the subscript. Normally this feature would not be worth comment, but here we are in an unusual situation, insofar as we can also consider the gauge theory in Lorentz signature $(− + +)$ on the compact three-manifold $M$.

Given the Euclidean contact metric in (2.1.20), we obtain a Lorentzian contact metric by a sign flip,

$$ds^2_M = \frac{1}{2} d\kappa(\cdot, J \cdot) - \kappa \otimes \kappa.$$  

(2.2.5)

With respect to the Lorentz metric, the Reeb direction $R$ is timelike, $||R||^2 = −1$, and the contact plane $H$ is spacelike. Since all compact orientable three-manifolds admit contact structures, they also all admit Lorentz metrics, which is not the case for compact manifolds in higher dimensions.

Especially when $M = \mathbb{R}^3$, working with a Lorentz metric is perfectly natural. Only in Lorentz signature do we obtain a physically-sensible, unitary quantum field theory.
Additionally, in Lorentz signature the YM-CS action is real as opposed to complex,

\[ I(A) = -\frac{1}{8\pi e^2} \int_M \text{Tr}(F_A \wedge \star F_A) + k \text{CS}(A), \quad (2.2.6) \]

as befits the oscillatory path integral

\[ Z(k, e^2, g) = \frac{1}{\text{Vol}(G)} \int_A \mathcal{D}A \exp\left[-i I(A)\right]. \quad (2.2.7) \]

The pair of minus signs in (2.2.6) and (2.2.7) ensure that the kinetic (electric) term in the Yang-Mills action is positive. We can again promote \( I(A) \) to a supersymmetric action on the Lorentzian contact manifold \( M \), and in the infrared limit \( t \to \infty \) with \( r/e^2k = 1 \) we obtain the real version of (2.2.4),

\[ S(A) = -\int_M \kappa \wedge d\kappa \text{Tr} \left[ \frac{(\kappa \wedge F_A)^2}{\kappa \wedge d\kappa} \right] + \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (2.2.8) \]

The real functional \( S(A) \) is precisely the contact Chern-Simons functional introduced in [12], cf. equation (3.9) therein. See §3.1 of [12] for an alternative derivation of (2.2.8), proceeding from bosonic Chern-Simons theory on \( M \).

Because the real functional \( S(A) \) and the complex functional \( S_E(A) \) arise in the same limit of supersymmetric YM-CS theory, the only difference being the metric signature, both deserve to be treated on the same footing. For the remainder of Sections 4 and 3, we will focus on the real contact functional \( S(A) \), which describes a consistent, unitary quantum theory on \( \mathbb{R}^3 \). However, at a judicious moment in Section 5.3, we will perform an analytic continuation to \( S_E(A) \) to take special advantage of the decay of the Euclidean propagator.
Chapter 3

A First Look at Contact Perturbation Theory

3.1 The Contact Hessian

Consider the following action with respect to the variable $r = \frac{1}{\sqrt{2}k}$ and a contact form $\kappa \in \Omega^1(M)$,

$$S(A, r) = i r \int_M \kappa \wedge d\kappa \operatorname{Tr} \left( \left( \frac{\kappa \wedge F_A}{\kappa \wedge d\kappa} \right)^2 \right) + \int_M \operatorname{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (3.1.1)$$

In this section we will derive the Hessian for the action in (3.1) for general $r$. Note that the Euclidean and Lorentzian theories are related by analytic continuation and setting $r = 1$ yields the Euclidean theory with action $i S(A, r = 1) = S_E(A)$ as in (2.2.4), and taking $r = i$ corresponds to the Lorentzian theory with action $S(A, r = i) = S(A)$ as in (2.2.8). Note that the action is multiplied by an overall factor of $\left( \frac{-i k}{4\pi} \right)$ in the path integral and we therefore require that $r > 0$ in order to ensure that the Yang-Mills part of the action is properly damped.

We begin by defining the duality operators, $\star : \Omega^3(M, \mathfrak{g}) \rightarrow \Omega^0(M, \mathfrak{g})$,

$$\star \left( \frac{1}{2} \kappa \wedge d\kappa \right) = 1,$$
and, $\ast_H : \Omega^2(M, \mathfrak{g}) \to \Omega^0(M, \mathfrak{g})$ as,

$$\ast_H \beta := \ast(\kappa \wedge \beta).$$

The equation of motion for the action $S(A, r)$ may be written as,

$$F_A + \frac{ir}{2}(\ast_H F_A) \wedge d\kappa + \frac{ir}{2} \kappa \wedge d_A \ast_H F_A = 0. \quad (3.1.2)$$

We claim that the critical points of this action are precisely the flat connections. Note that there are two cases to consider here:

- **Degenerate case:** $r = i$,

- **Generic case:** $r \neq i$.

In the degenerate case the critical locus is enlarged precisely by the action of the shift symmetry $\delta A = \sigma \kappa$, where $\sigma \in \Omega^0(M, \mathfrak{g})$ [12, Eq. 3.3]. As is shown in [12], the critical points may be represented by flat connections within their shift class after choosing the shift gauge $\kappa \wedge F_A = 0$. In the generic case, first consider wedging (3.1.2) by $\kappa$ so that,

$$\kappa \wedge F_A + \frac{ir}{2}(\ast_H F_A) \kappa \wedge d\kappa = 0. \quad (3.1.3)$$

Note that by definition of $\ast_H$ we have $\kappa \wedge F_A = \frac{1}{2}(\ast_H F_A) \kappa \wedge d\kappa$. Hence,

$$(1 + ir)\kappa \wedge F_A = 0. \quad (3.1.4)$$

Since $r \neq i$, we have $\kappa \wedge F_A = 0$, and therefore $\ast(\kappa \wedge F_A) = \ast_H F_A = 0$. Plugging $\ast_H F_A = 0$ back into (3.1.2), we deduce that $F_A = 0$. Write a general connection as $A = A_0 + B$, where $A_0$ is flat and $B \in \Omega^1(M, \mathfrak{g})$. Let

$$S_0(A) = \int_M \kappa \wedge d\kappa \text{Tr} \left[ \left( \frac{\kappa \wedge F_A}{\kappa \wedge d\kappa} \right)^2 \right]. \quad (3.1.5)$$

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Expanding $S_0(A)$ about $A_0$ we have,

\[ S_0(A_0 + B) = \frac{1}{2} \int_M \kappa \wedge \text{Tr} \left[ d_A^H B \wedge *_H d_A^H B \right] + \tilde{P}(B), \quad (3.1.6) \]

where,

\[ \tilde{P}(B) = \frac{1}{2} \int_M \kappa \wedge \text{Tr} \left[ d_A^H B \wedge *_H [B, B] \right] + \frac{1}{8} \int_M \kappa \wedge \text{Tr} \left[ [B, B] \wedge *_H [B, B] \right]. \quad (3.1.7) \]

By Stokes' theorem we have,

\[ S_0(A_0 + B) = \frac{1}{2} \int_M \kappa \wedge \text{Tr} \left( B \wedge [d_A^H *_H d_A^H] B \right) + \frac{1}{2} \int_M \text{Tr} \left( B \wedge [d_\kappa \wedge *_H d_A^H] B \right) + \tilde{P}(B). \quad (3.1.8) \]

Computing further we find,

\[ S(A_0 + B, r) = irS_0(A_0 + B) + 4\pi CS(A_0 + B), \quad (3.1.9) \]

\[ = \int_M \text{Tr} (B \wedge \mathcal{D}_r B) + \mathcal{P}_r(B), \]

where for $R \in \Gamma(TM)$ the Reeb vector field,

\[ \mathcal{D}_r = \kappa \wedge \left\{ (\mathcal{L}_R - d \circ \iota_R + \iota_R A_0, \cdot) - \frac{ir}{2} d_A^H *_H d_A^H \right\}. \quad (3.1.10) \]

and,

\[ \mathcal{P}_r(B) = ir\tilde{P}(B) + \int_M \text{Tr} \left( \frac{2}{3} B \wedge B \wedge B \right). \]

In the case of interest $M = \mathbb{R}^3$, $A_0 = 0$, and we impose the gauge condition,

\[ \iota_R B = 0. \quad (3.1.11) \]

In this gauge $S(A_0 + B, r)$ may be expressed in terms of the horizontal forms $\Omega^1(H, g) = \{ \alpha \in \Omega^1(M, g) | \iota_R \alpha = 0 \}$. Note that for a horizontal form $B \in \Omega^1(H, g)$ the cubic
Chern-Simons term in $\mathcal{P}(B)$ vanishes and $\mathcal{P}_r(B) = ir\tilde{\mathcal{P}}(B)$. We also have,

$$\mathcal{D}_r = \kappa \wedge \left( \mathcal{L}_R - \frac{ir}{2} d_H *_{H} d_H \right), \quad (3.1.12)$$

when acting on horizontal forms. Now choose a contact metric $g$,

$$g = \frac{1}{2} d\kappa \circ (\mathbb{I} \otimes J) + \kappa \otimes \kappa,$$

where $J \in \text{End}(H)$ is an almost complex structure on the contact distribution $H$, $J^2 = -\mathbb{I}$. We extend $J$ to $TM$ by setting $J(R) = 0$. Note that the dependence of our theory on the signature of the metric coming from the original Chern-Simons-Yang-Mills action is completely encoded in the variable $r$, and we are free to choose any contact compatible metric at this point. The compatibility of the metric with the contact structure is required in order to ensure that the Hodge star for the associated metric $g$ induces the duality operators $\ast$, $\ast_H$ defined above. Using the contact metric $g$ on $M$, the associated $L^2$ metric $(\cdot, \cdot)$ on $\Omega^1(M, g)$ defines a linear operator,

$$\mathcal{D}_r = \ast \mathcal{D}_r = \ast_H \mathcal{L}_R - \frac{ir}{2} d_H^\dagger d_H,$$

where the formal adjoint is given by $d_H^\dagger = \ast_H d_H \ast_H$.

Now we seek to invert the contact operator $\mathcal{D}_r$ to determine the propagator $K_r = \mathcal{D}_r^{-1}$ for the contact Chern-Simons gauge field. As in ordinary Chern-Simons theory, $K_r$ can only be defined after we fix the gauge symmetry. For us, we must adapt the gauge fixing procedure to our contact geometric set up. Given $\varphi \in \text{Lie}(G) \simeq \Omega^0(M, \mathfrak{g})$, observe that the corresponding infinitesimal gauge transformation acts on the reduced space of gauge fields $A \in \mathcal{A}$ by $\delta A = d^H_A \varphi$. This will then allow us to modify the non-invertible operator $\mathcal{D}_r$ with an operator which is invertible, at least away from a finite-dimensional kernel.

Following the usual BRST procedure let $c \in \Omega^0(M, \mathfrak{g})$ be an anti-commuting ghost.
field valued in the Lie algebra of $G$. The contact BRST operator $Q$ acts on the pair $(A, c)$ via
\[
\delta A = -d_A^H c, \quad \delta c = \frac{1}{2} [c, c]. \tag{3.1.14}
\]
The Jacobi identity for the Lie bracket ensures that $\delta^2 = 0$ in (3.1.14). We also introduce a second anti-commuting ghost field $b \in \Omega^3(M, g)$, along with an auxiliary commuting scalar field $\phi \in \Omega^3(M, g)$. The BRST operator acts on the pair $(b, \phi)$ by
\[
\delta b = \phi, \quad \delta \phi = 0. \tag{3.1.15}
\]
Next we replace the contact Chern-Simons functional in (3.1.9) by the gauge-fixed action
\[
S_{\text{gauge}}(r) = S(A, r) + \delta V_r(A, \phi, b, c), \tag{3.1.16}
\]
where $V_r$ is any functional of $(A, \phi, b, c)$ which ensures that the Hessian of the new action in (1.2.11) is non-degenerate. The BRST prepotential $V_r$ itself cannot be gauge-invariant, and $V_r$ is allowed to depend upon the choice of the background flat connection $A_0$ about which we perturb.

We choose $V_r$ in an analogous fashion to the usual choice of prepotential in ordinary Chern-Simons theory. Although there is no canonical choice for the prepotential, the associated Riemannian metric $g$ for the contact structure at least provides a natural metric to use in its construction. Let
\[
V_r = -\frac{i r}{4\pi} \int_M \text{Tr} \left( b \wedge \left[ *_H d_H *_H B + \frac{1}{2} *\phi \right] \right), \tag{3.1.17}
\]
for which the gauge-fixed action becomes

\[ S_{\text{gauge}}(r) = S(A, r) - \frac{ir}{4\pi} \int_M \text{Tr} \left( \phi \wedge \star_H d_H \star_H B + \frac{1}{2} \phi \wedge \star \phi - 2 b \wedge \Delta_H c - \star_H d_H (\star b) \wedge [B, c] \right), \]

\[ = CS(A_0) + \frac{1}{4\pi} \int_M \text{Tr} \left( B \wedge \mathcal{D}_r B - \frac{ir}{2} \kappa \wedge \star_H d_H \star_H \star_H d_H \star_H B + 2i r b \wedge \Delta_H c - \frac{ir}{2} \phi \wedge \star \phi \right) \]

\[ + \frac{ir}{4\pi} \int_M \kappa \wedge \text{Tr} \left( \star_H d_H (\star b) \wedge [B, c] + \frac{1}{2} [d_H B \wedge \star_H [B, B]] + \frac{1}{8} [[B, B] \wedge \star_H [B, B]] \right). \]  

(3.1.18)

Here \( \Delta_H = \frac{1}{2} d_H^\dagger d_H \) is the covariant horizontal scalar Laplacian acting on \( \Omega^0(M, g) \).

Since \( \phi \) enters the action \( S_{\text{gauge}}(r) \) quadratically, we perform the Gaussian path integral over \( \phi \) and obtain the effective gauge fixed action

\[ S_{\text{gauge}}(r) = CS(A_0) + \frac{1}{4\pi} \int_M \text{Tr} \left( B \wedge \mathcal{D}_r B - \frac{ir}{2} \kappa \wedge B \wedge \star_H d_H \star_H \star_H d_H \star_H B + 2i r b \wedge \Delta_H c \right) \]

\[ + \frac{ir}{4\pi} \int_M \kappa \wedge \text{Tr} \left( \star_H d_H (\star b) \wedge [B, c] + \frac{1}{2} [d_H B \wedge \star_H [B, B]] + \frac{1}{8} [[B, B] \wedge \star_H [B, B]] \right). \]  

(3.1.19)

The gauge-fixed action \( S_{\text{gauge}}(r) \) in (3.1.19) may be expressed in terms of the \( L^2 \) inner product \((\cdot, \cdot)\) as

\[ S_{\text{gauge}}(r) = CS(A_0) - \frac{1}{4\pi} (B, \mathcal{D}_r B) + \frac{ir}{2\pi} (\star b, \Delta_H c) + O_3, \]  

(3.1.20)

where,

\[ \mathcal{D}_r = \star_H \mathcal{L}_R - \frac{ir}{2} \left( d_H^\dagger d_H + d_H d_H^\dagger \right). \]  

(3.1.21)

Consider the formal propagators

\[ G = \Delta_H^{-1}, \quad K_r = \mathcal{D}_r^{-1}. \]  

(3.1.22)

Use the subscripts \( x \) and \( y \) to denote quantities defined on \( M_x \) and \( M_y \) in the product \( M \times M = M_x \times M_y \). Suppressing factors of \( 2, \pi, i \) and \( k \) which would otherwise multiply the propagators, write \( G \) and \( K_r \) as integral operators,

\[ (G c)^a_1 (x) = \int_{M_y} \kappa_y \wedge d\kappa_y \ G(x, y)^a_1 c^{a_2} (y), \quad (x, y) \in M \times M, \]  

(3.1.23)
and
\[
(K_r, \Phi)^{a_1}(x) = \int_{M_y} K_r^{1,2}(x, y)^{a_1} \wedge \Phi(y)^{a_2}, \quad (x, y) \in M \times M, \tag{3.1.24}
\]
where \( \Phi \in \Omega^1(H, \mathfrak{g}) \) is a horizontal form on \( M \). As in (1.2.40), we dualize the kernel \( K_r^{1,2} \) on the right to obtain the more symmetric form,
\[
K_r^{1,1}(x, y)^{a_1} = *_y K_r^{1,2}(x, y)^{a_1} \in \Omega^{1,1}(H \times H, \text{End}(\mathfrak{g})). \tag{3.1.25}
\]
Note that the kernel \( K_r^{1,1} \) is a horizontal \( \text{End}(\mathfrak{g}) \)-valued \( (1, 1) \)-form on \( M \times M \). We may find the kernel \( G(x, y)^{a_1}_{a_2} \in \Omega^{0,0}(M \times M, \text{End}(\mathfrak{g})) \) in (3.1.23) by solving the equation,
\[
\triangle_{H,x} G(x, y)^{a_1}_{a_2} = \delta_{H}(x, y) \cdot 1^{a_1}_{a_2}, \quad a_1, a_2 = 1, \ldots, \text{dim } \mathfrak{g}. \tag{3.1.26}
\]
We may also find the kernel \( K_r^{1,1}(x, y)(x, y)^{a_1}_{a_2} \in \Omega^{1,1}(H \times H, \text{End}(\mathfrak{g})) \), by solving the equation
\[
D_{r,x} K_r^{1,1}(x, y)^{a_1}_{a_2} = \delta_{D}(x, y) \cdot 1^{a_1}_{a_2}, \quad a_1, a_2 = 1, \ldots, \text{dim } \mathfrak{g}, \tag{3.1.27}
\]
The delta functional \( \delta_D \) is naturally viewed as distributional horizontal \( (1, 1) \)-form which we will write down explicitly after establishing some notation.

### 3.2 The Heisenberg Group and the Contact Operator

In this section we establish some basic definitions and facts about the three-dimensional Heisenberg group and the standard contact structure on \( \mathbb{R}^3 \). We then study the contact operator (3.1.21) and express it in terms of the well known Folland-Stein operators [30]. Let \( x = (u, v, t) \) denote coordinates on \( \mathbb{R}^3 \). To distinguish \( x \in \mathbb{R}^3 \) we also write \( x = (u_x, v_x, t_x) \). Make the identification \( \mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R} \) and define \( \zeta = u + iv, \) \( \overline{\zeta} = u - iv \) and \( x = (\zeta_x, t_x) = (u_x + iv_x, t_x) \). Recall that the three-dimensional Heisenberg group is given by \( \mathbb{H} = (\mathbb{R}^3, *) \) where the group multiplication \( * : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \)
is defined for \( x, y \in \mathbb{R}^3 \) by

\[
\begin{aligned}
x \ast y &= (u_x + u_y, v_x + v_y, t_x + t_y - u_x v_y + u_y v_x), \\
&= (\zeta_x + \zeta_y, t_x + t_y + \text{Im}(\zeta_x \overline{\zeta_y})).
\end{aligned}
\]  

(3.2.1)

The origin remains the identity in \( \mathbb{H} \), and the Heisenberg inverse is \( x^{-1} = (-u_x, -v_x, -t_x) \).

A spanning set for the left invariant vector fields on the Heisenberg group is given by

\[
\begin{aligned}
U &= \frac{\partial}{\partial u} + v \frac{\partial}{\partial t}, \\
V &= \frac{\partial}{\partial v} - u \frac{\partial}{\partial t}, \\
R &= \frac{\partial}{\partial t}.
\end{aligned}
\]

Define

\[
\begin{aligned}
Z &= \frac{1}{2} (U - i V) = \frac{\partial}{\partial \zeta} + \frac{i}{2} \zeta \frac{\partial}{\partial t}, \\
\overline{Z} &= \frac{1}{2} (U + i V) = \frac{\partial}{\partial \overline{\zeta}} - \frac{i}{2} \zeta \frac{\partial}{\partial t}.
\end{aligned}
\]

Let \( \kappa = dt + u \, dv - v \, du = dt + \frac{i}{2} \left( \zeta \, d\overline{\zeta} - \overline{\zeta} \, d\zeta \right) \) represent the standard contact structure on \( \mathbb{R}^3 \), for which the top-form \( \kappa \wedge d\kappa > 0 \) is positive with respect to the standard orientation on \( \mathbb{R}^3 \). This choice of contact form respects many symmetries, which are important here. Manifestly, \( \kappa \) is preserved under translations generated by the Reeb field \( R = \partial/\partial t \) as well as rotations in the \( uv \)-plane. Though \( \kappa \) is not preserved by translations in the \( uv \)-plane, \( \kappa \) is preserved by the left-action of the Heisenberg Lie group on itself. Finally, \( \kappa \) is homogeneous of degree two under the parabolic scaling

\[
(u, v, t) \mapsto (\lambda u, \lambda v, \lambda^2 t), \quad \lambda \in \mathbb{R}_+,
\]

(3.2.2)

which commutes with Heisenberg multiplication. As a result, the parabolic scaling fixes each contact plane \( H \subset \mathbb{R}^3 \) in the kernel of \( \kappa \). A picture of the family of contact planes \( H = \ker \kappa \) appears in Figure 3-1. The contact planes are approximately hori-
horizontal near \( u = v = 0 \), but they twist vertically as one moves outward from the origin in the \( uv \)-plane. Recall the horizontal Laplacian \( \triangle_H = \frac{1}{2} d_H^1 d_H : \Omega^0(M, g) \rightarrow \Omega^0(M, g) \).

![Figure 3-1: The standard radially-symmetric contact structure on \( \mathbb{R}^3 \). (Courtesy of the Mathematica routine ‘CSPlotter’ by Matias Dahl.)](image)

and the contact operator \( D_r : \Omega^1(H, g) \rightarrow \Omega^1(H, g) \) is defined by

\[
D_r = \star_H \mathcal{L}_R - \frac{ir}{2} \left( d_H^1 d_H + d_H d_H^1 \right).
\]

Our next goal is to express these operators in terms of the natural vector fields on the Heisenberg group. We begin with the following

**Lemma 3.2.1.** \( d_H f = Z f \, d\zeta + \overline{Z} f \, d\overline{\zeta} \).

**Proof.** By definition \( d_H = \Pi_H \circ d \) where for \( \alpha \in \Omega(M, g) \), \( \Pi_H(\alpha) = \alpha - \kappa \wedge \iota_R \alpha \). Given \( df = \partial_u f \, du + \partial_v f \, dv + \partial_t f \, dt \), \( \kappa = dt + udv - vdu \), and \( R = \partial_t \), then

\[
d_H f = \Pi_H \circ d f,
\]

\[
= df - \kappa \wedge \iota_R df,
\]

\[
= (\partial_u f \, du + \partial_v f \, dv + \partial_t f \, dt) - (dt + udv - vdu) \wedge \partial_t f,
\]

\[
= (\partial_u + v \partial_t) f \, du + (\partial_v - u \partial_t) f \, dv,
\]

\[
= (U f) du + (V f) dv.
\]
Using $d\zeta = du + idv$, $d\bar{\zeta} = du - idv$, $Z = \frac{1}{2} (U - iV)$, and $\bar{Z} = \frac{1}{2} (U + iV)$, we have

\[
d_H f = (U f) du + (V f) dv,
\]
\[
= (U f) \frac{1}{2}(d\zeta + d\bar{\zeta}) + (V f) \frac{1}{2i}(d\zeta - d\bar{\zeta}),
\]
\[
= \frac{1}{2} (U - iV) f \, d\zeta + \frac{1}{2} (U + iV) f \, d\bar{\zeta},
\]
\[
= Zf \, d\zeta + \bar{Z}f \, d\bar{\zeta}.
\]

Lemma 3.2.2. $\Delta_H f = [Z\bar{Z} + \bar{Z}Z] f$.

Proof. Some basic properties of the Hodge star are given as follows,

\[
\begin{align*}
\star 1 &= d \text{Vol}, \\
\star d \text{Vol} &= 1, \\
\star \kappa &= \frac{1}{2} d \kappa, \\
\star d \kappa &= 2 \kappa, \\
\star_H \kappa &= 0, \\
\star_H d \kappa &= 2.
\end{align*}
\]

Given that $\star_H = \star \circ (\kappa \wedge)$ and using (3.2.3) we have,

\[
\begin{align*}
\star_H du &= dv, \\
\star_H dv &= -du, \\
\star_H d\zeta &= -id\zeta, \\
\star_H d\bar{\zeta} &= id\bar{\zeta}, \\
\star_H d\zeta \wedge d\bar{\zeta} &= -2i, \\
\star_H^2 &= -\mathbb{I},
\end{align*}
\]
where the last equation is valid on $\Omega^1(H)$. Using the above relations, we compute

\[
\Delta_H f = \frac{1}{2} d_H^\dagger d_H f, \\
= \frac{1}{2} \ast_H d_H \ast_H d_H f, \\
= \frac{1}{2} \ast_H d_H \ast_H [Z f d\zeta + Z f d\bar{\zeta}], \\
= \frac{i}{2} \ast_H d_H [-Z f d\zeta + Z f d\bar{\zeta}], \\
= \frac{i}{2} \ast_H [(Z Z + Z Z) f] d\zeta \wedge d\bar{\zeta}, \\
\]

□

**Lemma 3.2.3.** For $f d\zeta, f d\bar{\zeta} \in \Omega(\mathbb{R}^3, \mathbb{C})$,

\[
D_r(f d\zeta) = [-i r (Z Z + Z Z) - (r + i) \partial_t] f d\zeta, \quad \text{(3.2.5)}
\]
and,

\[
D_r(f d\bar{\zeta}) = [-i r (Z Z + Z Z) + (r + i) \partial_t] f d\bar{\zeta}. \quad \text{(3.2.6)}
\]

**Proof.** Consider,

\[
\ast_H \mathcal{L}_R f d\zeta = \ast_H (d_R + \iota_R d) f d\zeta, \\
= \ast_H \iota_R \partial_t f dt \wedge d\zeta, \\
= \ast_H \partial_t f d\zeta, \\
= -i \partial_t f d\zeta.
\]
\[ d_H d_H^d f d\zeta = d_H \ast_H d_H \ast_H f d\zeta, \]
\[ = -i d_H \ast_H d_H f d\zeta, \]
\[ = i d_H \ast_H (Z f d\zeta \wedge d\zeta), \]
\[ = 2d_H Z f, \]
\[ = 2(Z \overline{Z} f d\zeta + \overline{Z}^2 f d\zeta). \]

\[ d_H^d d_H f d\zeta = \ast_H d_H \ast_H d_H f d\zeta, \]
\[ = -\ast_H d_H \ast_H (Z f d\zeta \wedge d\zeta), \]
\[ = 2i \ast_H d_H Z f, \]
\[ = 2i \ast_H (Z \overline{Z} f d\zeta + \overline{Z}^2 f d\zeta), \]
\[ = 2(Z \overline{Z} f d\zeta - \overline{Z}^2 f d\zeta). \]

Thus, we have,
\[
\mathbf{D}_r f d\zeta = -i\partial_t f d\zeta - ir \left[ (Z \overline{Z} f d\zeta + \overline{Z}^2 f d\zeta) + (Z \overline{Z} f d\zeta - \overline{Z}^2 f d\zeta) \right],
\]
\[
= \left[ (-2ir Z \overline{Z} - i\partial_t) f \right] d\zeta.
\]

Using the definition of \( Z = \frac{1}{2} (U - i V) \) and \( \overline{Z} = \frac{1}{2} (U + i V) \), we see that \( Z \overline{Z} = \frac{1}{4} (U^2 + V^2 + i[U, V]) \), and we also compute \([U, V] = -2\partial_t\). We then have \( Z \overline{Z} = \frac{1}{4} (U^2 + V^2 - 2i\partial_t) = \frac{1}{2} (Z \overline{Z} + \overline{Z} Z - i\partial_t) \) and so
\[
\mathbf{D}_r f d\zeta = \left[ (-2ir Z \overline{Z} - i\partial_t) f \right] d\zeta,
\]
\[
= \left[ (-ir [Z \overline{Z} + \overline{Z} Z] - (r + i)\partial_t) f \right] d\zeta.
\]

A similar computation yields
\[
\mathbf{D}_r (f d\zeta) = \left[ (-ir [Z \overline{Z} + \overline{Z} Z] + (r + i)\partial_t) f \right] d\zeta. \tag{3.2.7}
\]

\[ \square \]
For $\lambda \in \mathbb{C}$, define the operators,

$$
\mathcal{L}_\lambda = ZZ + ZZ - i\lambda \partial_t. \tag{3.2.8}
$$

After a coordinate change $(u, v) \rightarrow \sqrt{2}(u, v)$, the operators $\mathcal{L}_\lambda$ are precisely those studied by Folland and Stein [30]. Lemmas 3.2.2 and 3.2.3 show that our operators are related to the Folland-Stein operators,

$$
\Delta_H(f) = \mathcal{L}_0 f,
$$

$$
D_r(f \, d\zeta) = \left[(-ir \cdot \mathcal{L}_\lambda r) f \right] d\zeta,
$$

$$
D_r(f \, d\bar{\zeta}) = \left[(-ir \cdot \mathcal{L}_{-\lambda} r) f \right] d\bar{\zeta},
$$

where $\lambda_r = \frac{i}{r} + 1$. By [30, Theorem 1], a fundamental solution to the equation,

$$
\mathcal{L}_\lambda \phi = \delta, \tag{3.2.9}
$$

where $\delta$ is the standard delta functional, is given by,

$$
\phi = c_\lambda \left( t + i \frac{|\zeta|^2}{2} \right)^{-\frac{1+\lambda}{2}} \cdot \left( t - i \frac{|\zeta|^2}{2} \right)^{-\frac{1-\lambda}{2}}, \tag{3.2.10}
$$

where,

$$
c_\lambda = -\frac{\Gamma(\frac{1}{2}(1 + \lambda))\Gamma(\frac{1}{2}(1 - \lambda))}{i^{-\lambda}\pi^2}. \tag{3.2.11}
$$

Since $\Delta_H, D_r$ are left-invariant operators under the Heisenberg group, we deduce that the kernels $G(x, y)_{a_2}$ and $K(x, y)_{a_2}$ are given by,

$$
G(x, y)_{a_2} = \phi_0(y^{-1} * x) \cdot 1_{a_2}^{a_1}, \tag{3.2.12}
$$

and,

$$
K^{1,1}_r(x, y)_{a_2} = \frac{i}{2r} \left[ \phi_{\lambda_r}(y^{-1} * x) d\zeta_x \wedge d\zeta_y + \phi_{-\lambda_r}(y^{-1} * x) d\bar{\zeta}_x \wedge d\bar{\zeta}_y \right] \cdot 1_{a_2}^{a_1}. \tag{3.2.13}
$$
By construction the expression given for $G(x, y)^{a_1}_{a_2}$ in (3.2.11) satisfies (3.1.26)

$$\Delta_{H, x} G(x, y)^{a_1}_{a_2} = \delta_{\Delta_H} (x, y) \cdot 1^{a_1}_{a_2}, \quad a_1, a_2 = 1, \ldots, \dim g,$$

where $\delta_{\Delta_H} (x, y) = \delta(y^{-1} \ast x)$ for the ordinary delta distribution $\delta$ on $\mathbb{R}^3$. Also, by construction the expression given for $K_r(x, y)^{a_1}_{a_2}$ in (3.2.12) satisfies (3.1.27)

$$D_{r, x} K_r(x, y)^{a_1}_{a_2} = \delta_D (x, y) \cdot 1^{a_1}_{a_2}, \quad a_1, a_2 = 1, \ldots, \dim g,$$

where $\delta_D$ is the distributional $(1, 1)$-form given by

$$\delta_D (x, y) := \frac{1}{2} \delta(y^{-1} \ast x) \left[ d\zeta_x \wedge d\zeta_y + d\zeta_x \wedge d\zeta_y \right]. \quad (3.2.13)$$

### 3.3 The Contact Complex

In this section we review the contact complex that was introduced and studied by M. Rumin [58]. This complex captures several key properties of the operator,

$$D = \kappa \wedge \left( \mathcal{L}_R + [i_R A_0, \cdot] + \frac{1}{2} d^H_H \ast_H d^H_H \right), \quad (3.3.1)$$

that occurs in the contact Hessian above (3.1.10) in the degenerate case when $r = i$.

We work with a general contact structure $H$ on a three-manifold $M$, and choose a contact form $\kappa \in \Omega^1(M)$ (i.e. $\kappa \wedge d\kappa \neq 0$ and $H = \ker \kappa$) with Reeb vector field $R$. Recall that $R$ is defined uniquely by the conditions $i_R \kappa = 1$ and $i_R d\kappa = 0$. For $j = 1, 2$, the horizontal forms are defined as $\Omega^j(H, g) = \{ \alpha \in \Omega^j(M, g) \mid i_R \alpha = 0 \}$, and the vertical forms are defined by $\Omega^j(V, g) := \{ \beta \in \Omega^j(M, g) \mid \beta = \kappa \wedge \alpha, \text{ for some } \alpha \in \Omega^j(M, g) \}$. $D$ is an operator on horizontal forms $D : \Omega^1(H, g) \to \Omega^2(V, g)$. Recall also that we take $g$ to denote the Euclidean signature metric,

$$g = \frac{1}{2} d\kappa \circ (\mathbb{I} \otimes J) + \kappa \otimes \kappa,$$
where \( J \in \text{End}(H) \) is an almost complex structure on the contact distribution \( H \), \( J^2 = -I \), and \( \star \) denotes the Hodge-star for \( g \), and \( \star_H = \star \circ (\kappa \wedge) \) the horizontal Hodge-star. Let \( \text{dVol} = \frac{1}{2}\kappa \wedge d\kappa \) denote the Riemannian volume form for the metric \( g \). Let \( \alpha, \beta \in \Omega^j(M) \). We define the Hodge-star by the condition,

\[
\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \text{dVol},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the induced metric on forms. The contact complex \( (\mathcal{E}(M, g), \mathcal{D}) \) is defined as,

\[
C^\infty(M, g) \xrightarrow{\mathcal{D}} \Omega^1(H, g) \xrightarrow{\mathcal{D}} \Omega^2(V, g) \xrightarrow{\mathcal{D}} \Omega^3(M, g), \tag{3.3.2}
\]

where \( C^\infty(M, g) \xrightarrow{\mathcal{D}} \Omega^1(H, g) \) is defined by,

\[
\mathcal{D} := \Pi_H \circ d_{A_0},
\]

and \( \Omega^2(V, g) \xrightarrow{\mathcal{D}} \Omega^3(M, g) \) is defined by,

\[
\mathcal{D} := d_{A_0}.
\]

Note that we suppress the dependence of \( \mathcal{D} \) on \( A_0 \) in our notation. We call the second order operator \( \mathcal{D} \) the contact operator. Note that \( (\mathcal{E}(M, g), \mathcal{D}) \) is defined for any orientable three-manifold with a choice of contact form \( \kappa \in \Omega^1(M) \) and indeed does define a complex, and \( \mathcal{D}^2 = 0 \) for \( A_0 \) flat. For completeness, we have provided a proof of this below in the trivial case where \( A_0 = 0 \). Note that it is precisely in the degenerate case where \( r = i \) that we obtain a differential complex and this is the key to several of our constructions below. We also note that it is possible to define the complex \( (\mathcal{E}(M, g), \mathcal{D}) \) without specifying a choice of contact form and only a contact structure \( H \subset TM \) need be specified as in [58], [60]. One may also define an analogous contact complex on contact manifolds of arbitrary odd dimension [60]. Of course, once a contact structure has been specified and one also assumes that it is coorientable then it is possible to choose a global form \( \kappa \) such that \( H = \ker \kappa \).
In general, not every contact structure need be coorientable and in this case the contact structure is only defined locally by forms \( \kappa_i \in \Omega^1(U_i) \) say, for \( U_i \subset M \) open. A theorem of J. Martinet [50] says, however, that every compact orientable three-manifold admits a coorientable contact structure and is therefore defined by a global one form \( \kappa \in \Omega^1(M) \). It is also possible to show that the cohomology of \( (\mathcal{E}(M, \mathfrak{g}), \mathcal{D}) \) is the same as the usual twisted de Rham complex. We have provided a proof of this below.

**Remark 3.3.1.** For the sake of simplicity, we now take \( A_0 = 0 \) and use the notation \( \Omega^\bullet(\ast, \mathfrak{g}) = \Omega^\bullet(\ast) \), and \( d_{A_0=0} = d_H \). The contact operator is then given by,

\[
\mathcal{D} = \kappa \wedge \left( \mathcal{L}_R + \frac{1}{2} d_H \ast_H d_H \right). \tag{3.3.3}
\]

The operator \( \mathcal{D} \) is closely related to the ordinary de Rham operator. To see this, we need the following,

**Lemma 3.3.2.** Let \((M, H)\) be a contact three manifold with a given choice of contact form \( \kappa \in \Omega^1(M) \). For any \( \alpha \in \Omega^1(H) \) there exists a unique \( \tilde{\alpha} \in \Omega^1(M) \) such that \( \alpha - \tilde{\alpha} \in \Omega^1(V) \) and \( d\tilde{\alpha} \in \Omega^2(V) \).

**Proof.** Notice that the operator \( d_H \) induces an isomorphism,

\[
d_H : \Omega^1(V) \rightarrow \Omega^2(H), \text{ with } d_H(f\kappa) = f \cdot d\kappa,
\]

where \( f \in C^\infty(M) \). Given \( \alpha \in \Omega^1(H) \), then using the isomorphism induced by \( d_H \) let \( \alpha_0 \in \Omega^1(V) \) such that \( d_H \alpha_0 = d_H \alpha \). Define \( \tilde{\alpha} := \alpha - \alpha_0 \). Clearly, \( \alpha - \tilde{\alpha} = \alpha_0 \in \Omega^1(V) \) and,

\[
d\tilde{\alpha} = d\alpha - d\alpha_0 = d_H \alpha + \kappa \wedge \mathcal{L}_R \alpha - d_H \alpha = \kappa \wedge \mathcal{L}_R \alpha \in \Omega^2(V).
\]

To see that \( \tilde{\alpha} \) is the unique form satisfying these properties observe that if \( \beta \in \Omega^1(V) \) is another form such that,

\[
\alpha - \beta \in \Omega^1(V),
\]
and,
\[ d\beta \in \Omega^2(V), \]
then \( \tilde{\alpha} - \beta = h \cdot \kappa \in \Omega^1(V) \) for some \( h \in C^\infty(M) \) such that \( d(h \cdot \kappa) = 0 \). But,
\[ d(h \cdot \kappa) = dh \wedge \kappa + h \cdot d\kappa = 0, \]
implies that \( h \cdot d\kappa \in \Omega^2(H) \cap \Omega^2(V) = \{0\} \) and therefore \( h = 0 \), giving us \( \beta = \tilde{\alpha} \) as desired. \( \square \)

**Proposition 3.3.3.** The contact operator \( \mathcal{D} : \Omega^1(H) \to \Omega^2(V) \) defined for \( \alpha \in \Omega^1(H) \) as,
\[ \mathcal{D}\alpha = \kappa \wedge (\mathcal{L}_R + \frac{1}{2} d_H \ast_H d_H) \alpha. \] (3.3.4)
may also be expressed as,
\[ \mathcal{D}\alpha = d\tilde{\alpha}, \] (3.3.5)
where \( \tilde{\alpha} \in \Omega^1(M) \) is unique form defined in Lemma (3.3.2).

**Proof.** First we claim that if \( \beta = \alpha - \frac{1}{2}(\ast_H d_H \alpha) \kappa \in \Omega^1(M) \), then \( \beta = \tilde{\alpha} \), where \( \tilde{\alpha} \) is the unique form in Lemma (3.3.2). Clearly, \( \alpha - \beta \in \Omega^1(V) \) by definition. To see that \( d\beta \in \Omega^2(V) \), consider,
\[ d\beta = d\alpha - d[\frac{1}{2}(\ast_H d_H \alpha) \kappa], \]
\[ = d\alpha - \frac{1}{2} d(\ast_H d_H \alpha) \wedge \kappa - \frac{1}{2} (\ast_H d_H \alpha) \wedge d\kappa, \]
\[ = [d_H \alpha - \frac{1}{2}(\ast_H d_H \alpha) \wedge d\kappa] + [\kappa \wedge \mathcal{L}_R \alpha - \frac{1}{2} d(\ast_H d_H \alpha) \wedge \kappa]. \]
We claim that \( [d_H \alpha - \frac{1}{2}(\ast_H d_H \alpha) \wedge d\kappa] = 0 \), which will then imply that \( d\beta \in \Omega^2(V) \). To see this, write,
\[ d_H \alpha = h \cdot d\kappa, \]
for some \( h \in C^\infty(M) \), and observe that,
\[ \ast_H d_H \alpha = 2h. \]
Thus, \( d_H \alpha = h \cdot dk = \frac{1}{2}(\star_H d_H \alpha) \land dk \). Since \( \beta \) satisfies the main properties of Lemma (3.3.2), we must have \( \beta = \bar{\alpha} \). In order to prove the proposition we compute,

\[
\begin{align*}
\mathcal{D} \alpha &= d\bar{\alpha} = d\beta, \\
&= \kappa \land \mathcal{L}_R \alpha - \frac{1}{2}d(\star_H d_H \alpha) \land \kappa, \\
&= \kappa \land [\mathcal{L}_R + \frac{1}{2}d_H \star_H d_H] \alpha,
\end{align*}
\]

where the last line follows since the vertical part of \( d(\star_H d_H \alpha) \) will vanish in the wedge product with \( \kappa \).

\[
\square
\]

**Proposition 3.3.4.** Let \((M, H)\) be a contact three-manifold with a given choice of contact form \( \kappa \in \Omega^1(M) \). Then \((\mathcal{E}, \mathcal{D})\) is a complex; i.e. \( \mathcal{D} \circ \mathcal{D} = 0 \).

**Proof.** First consider \( \mathcal{D} \circ \mathcal{D} \phi \) when \( \phi \in \Omega^0(M) \). By definition of \( \mathcal{D} \) we have,

\[
\mathcal{D} \phi = d_H \phi = d\phi - \kappa \land (i_R d\phi), \tag{3.3.6}
\]

and,

\[
\mathcal{D} \circ \mathcal{D} \phi = d(\mathcal{D} \phi), \tag{3.3.7}
\]

where \( \mathcal{D} \phi \in \Omega^1(M) \) is the unique form defined in Lemma (3.3.2). As seen in the proof of Prop. (3.3.3), we know that,

\[
\mathcal{D} \phi = d_H \phi = d_H \phi - \frac{1}{2}(\star_H d_H d_H \phi) \kappa. \tag{3.3.8}
\]

First we compute the term \( \star_H d_H d_H \phi \) in (3.3.8),

\[
\begin{align*}
d_H d_H \phi &= d_H (d\phi - \kappa \land (i_R d\phi)), \\
&= d(d\phi - \kappa \land (i_R d\phi)) - \kappa \land [i_R (d\phi - \kappa \land (i_R d\phi))], \\
&= -d[\kappa \land (i_R d\phi)], \\
&= -(i_R d\phi) dk - \kappa \land (d i_R d\phi).
\end{align*}
\]

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Now,
\[ \star_H[(\iota_R d\phi) d\kappa] := \star(\kappa \land [(\iota_R d\phi) d\kappa]) = 2\iota_R d\phi, \]
since \( \star(\kappa \land d\kappa) = 2 \) and,
\[ \star_H[\kappa \land (d\iota_R d\phi)] = \star(\kappa \land [\kappa \land (d\iota_R d\phi)]) = 0. \]
Thus,
\[ \frac{1}{2} \star_H d_H d_H \phi = -\iota_R d\phi. \quad (3.3.9) \]
By (3.3.8) and (3.3.9) we have,
\[ \widetilde{\mathcal{D}} \phi = d_H \phi + (\iota_R d\phi)\kappa. \quad (3.3.10) \]
Plugging (3.3.10) into (3.3.7) we compute,
\[
\begin{align*}
\mathcal{D} \circ \mathcal{D} \phi &= d(\widetilde{\mathcal{D}}\phi), \\
&= d(d_H \phi + (\iota_R d\phi)\kappa), \\
&= d[d\phi - \kappa \land (\iota_R d\phi)] + d[(\iota_R d\phi)\kappa], \\
&= -d[(\iota_R d\phi)\kappa] + d[(\iota_R d\phi)\kappa], \\
&= 0.
\end{align*}
\]
Thus, \( \mathcal{D} \circ \mathcal{D} = 0 \) on \( \Omega^0(M) \). Next consider \( \mathcal{D} \circ \mathcal{D} \alpha \) when \( \alpha \in \Omega^1(H) \). By definition of \( \mathcal{D} \),
\[
\mathcal{D} \circ \mathcal{D} \alpha = \mathcal{D}(d\tilde{\alpha}), \text{ with } \tilde{\alpha} \text{ as in Lemma (3.3.2)},
\]
\[
= d(d\tilde{\alpha}), \text{ by definition of } \mathcal{D} \text{ on } \Omega^2(V),
\]
\[
= 0, \text{ since } d^2 = 0 \text{ for the ordinary exterior derivative.}
\]
Hence, \( \mathcal{D} \circ \mathcal{D} = 0 \) on \( \Omega^1(H) \) as well. This completes the proof. \( \square \)
Definition 3.3.5. We define the contact cohomology of $M$ as:

$$H^\ast(\mathcal{E}(M)) = \text{Ker}D/\text{Im}D$$  \hspace{1cm} (3.3.11)

Proposition 3.3.6. The following diagram is commutative:

$$
\begin{array}{ccccccc}
C^\infty(M) & \overset{d}{\longrightarrow} & \Omega^1(M) & \overset{d}{\longrightarrow} & \Omega^2(M) & \overset{d}{\longrightarrow} & \Omega^3(M) \\
\downarrow{l_0} & & \downarrow{l_1} & & \downarrow{l_2} & & \downarrow{l_3} \\
C^\infty(M) & \overset{\mathcal{D}}{\longrightarrow} & \Omega^1(H) & \overset{\mathcal{D}}{\longrightarrow} & \Omega^2(V) & \overset{\mathcal{D}}{\longrightarrow} & \Omega^3(M)
\end{array}
$$

where, $l_0$ and $l_3$ are identity maps, $l_1$ is given by

$$l_1\alpha = \alpha - \kappa \wedge \iota_R\alpha$$  \hspace{1cm} (3.3.12)

and $l_2$ is given by

$$l_2\beta = \beta - d\star(\beta - \kappa \wedge \iota_R\beta)$$  \hspace{1cm} (3.3.13)

Proof. For $f \in C^\infty(M)$,

$$l_1df = df - \kappa \wedge \iota_Rdf = \mathcal{D}f$$  \hspace{1cm} (3.3.14)

For $\alpha \in \Omega^1(M)$,

$$\mathcal{D}l_1\alpha = d(l_1\alpha - \star d_H l_1\alpha)$$

$$= d(\alpha - \kappa \wedge \iota_R\alpha - \star d_H(\alpha - \kappa \wedge \iota_R\alpha))$$

$$= d(\alpha - \star d_H\alpha - \kappa \wedge \iota_R\alpha + \star d_H(\kappa \wedge \iota_R\alpha))$$  \hspace{1cm} (3.3.15)

$$= d(\alpha - \star d_H\alpha)$$

$$= l_2d\alpha$$

For $\alpha \in \Omega^2(M)$,

$$\mathcal{D}l_2\alpha = d(\alpha - d\star(\alpha - \kappa \wedge \iota_R\alpha)) = d\alpha$$  \hspace{1cm} (3.3.16)
This shows that $l_i$ induces a map in cohomology:

$$L_i : H^i(M) \to H^i(\mathcal{E}(M))$$

$$[\alpha] \mapsto [l_i \alpha]_{\mathcal{E}} \quad (3.3.17)$$

**Proposition 3.3.7.** The map $L_i : H^i(M) \to H^i(\mathcal{E}(M))$ is an isomorphism for $\forall i$.

**Proof.** 1. The injectivity of $L_0$ is trivial. The surjectivity of $L_0$: for $[f]_{\mathcal{E}} \in H^0(\mathcal{E}(M))$, we have

$$df - \kappa \wedge \nu_R df = 0 \quad (3.3.18)$$

This implies that

$$0 = d(\kappa \wedge \nu_R df) = d\kappa \wedge \nu_R df - \kappa \wedge d\nu_R df \quad (3.3.19)$$

so

$$\kappa \wedge d\kappa \nu_R df = \kappa \wedge \kappa \wedge d\nu_R df = 0 \quad (3.3.20)$$

Since

$$\kappa \wedge d\kappa \neq 0 \quad (3.3.21)$$

we have

$$\nu_R df = 0 \quad (3.3.22)$$

By (3.3.18), we have $df = 0$. Thus,

$$L_0([f]) = [f]_{\mathcal{E}} \quad (3.3.23)$$

2. Suppose $[\alpha]_{\mathcal{E}} \in H^1(\mathcal{E}(M))$, then

$$\mathcal{D}\alpha = d(\alpha - *d_H \alpha) = 0 \quad (3.3.24)$$
On the other hand,

\[ l_1(\alpha - \star d_H \alpha) = \alpha - \star d_H \alpha - \kappa \wedge i_R(\alpha - \star d_H \alpha) = \alpha \]  

(3.3.25)

so

\[ L_1([\alpha - \star d_H \alpha]) = [\alpha]_\mathcal{E} \]  

(3.3.26)

This shows the surjectivity of \( L_1 \).

Suppose \( L_1([\alpha]) = 0 \), then

\[ l_1 \alpha = \mathcal{D}f \]  

(3.3.27)

where \( f \in C^\infty(M) \). Thus

\[ l_1 \alpha = l_1 df \]  

(3.3.28)

i.e.,

\[ \alpha - df - \kappa \wedge i_R(\alpha - df) = 0 \]  

(3.3.29)

Let \( g = i_R(\alpha - df) \), then

\[ \alpha = df + g\kappa \]  

(3.3.30)

Since

\[ 0 = d\alpha = dg \wedge \kappa + g\kappa \]  

(3.3.31)

we have

\[ 0 = dg \wedge \kappa \wedge \kappa + g\kappa \wedge \kappa = g\kappa \wedge \kappa \]  

(3.3.32)

so \( g = 0 \), and \( \alpha = df \), i.e., \([\alpha] = 0\). This shows the injectivity of \( L_1 \).

3. Suppose \([\alpha]_\mathcal{E} \in H^2(\mathcal{E}(M))\), then \( d\alpha = 0 \). Write \( \alpha = \kappa \wedge \beta \), then

\[ \kappa \wedge i_R \alpha = \alpha \]  

(3.3.33)

Thus

\[ l_2 \alpha = \alpha - d \star (\alpha - \kappa \wedge i_R \alpha) = \alpha \]  

(3.3.34)
Hence
\[ L_2([\alpha]) = [\alpha]_\varepsilon \quad (3.3.35) \]

This shows the surjectivity of \( L_2 \).

Suppose \( L_2[\alpha] = 0 \), then
\[ l_2\alpha = D\beta \quad (3.3.36) \]

for some \( \beta \in \Omega^1(H) \), i.e.,
\[ \alpha - d \star (\alpha - \kappa \wedge \iota_R \alpha) = d(\beta - \star d_H \beta) \quad (3.3.37) \]

Thus \([\alpha] = 0\). This shows the injectivity of \( L_2 \).

4. For \( L_3 \), everything is trivial.
\[ \square \]

Stokes’ theorem for de Rham \( d \) says that,
\[ \int_M d\alpha = \int_{\partial M} \alpha. \quad (3.3.38) \]

We also have an analogue of Stokes’ theorem for \( D \). It turns out that we need to put some constraints on the manifolds or their boundaries.

**Proposition 3.3.8.** Let \( M \) be a contact manifold with contact form \( \kappa \), and let \( C : [0,1] \to M \) be a Legendrian curve. Then
\[ \int_C Df = f(C(1)) - f(C(0)) \quad (3.3.39) \]

**Proof.** We just need to note that
\[ \int_C Df = \int_C df - \kappa \wedge \iota_R df \quad (3.3.40) \]

and \( \kappa|_C = 0 \). \( \square \)

**Proposition 3.3.9.** Let \( N \) be an oriented two-dimensional sub-manifold of a contact
three-manifold $M$, where $\partial N$ is a closed Legendrian link. Then,

$$\int_N \mathcal{D} \alpha = \int_{\partial N} \alpha, \quad \forall \alpha \in \Omega^1(H). \quad (3.3.41)$$

Proof. By proposition 1.1, we have

$$\int_N \mathcal{D} \alpha = \int_N d(\alpha - \star d_H \alpha)$$

$$= \int_{\partial N} \alpha - \star d_H \alpha \quad (3.3.42)$$

Since $\star d_H \alpha$ is a vertical 1-form (i.e. proportional to $\kappa$), then $\int_{\partial N} \star d_H \alpha = 0$. $\square$

A special case is when two closed Legendrian knots $X_0, X_1$ are equivalent, so there exist a $C^\infty$ map

$$\Gamma : [0, 1] \times [0, 1] \to M \quad (3.3.43)$$

such that

$$\Gamma(0, t) = X_0(t)$$

$$\Gamma(1, t) = X_1(t) \quad (3.3.44)$$

and $\Gamma(s, t)$ is Legendrian knot for $\forall s$.

Proposition 4.2 tells us that for $\forall \alpha \in \Omega^1(H)$,

$$\int_{[0, 1] \times [0, 1]} \Gamma^* \mathcal{D} \alpha = \int_{[0, 1]} X_2^* \alpha - \int_{[0, 1]} X_1^* \alpha. \quad (3.3.45)$$

### 3.4 Contact Relative Cohomology

In this section, we define contact relative cohomology. This concept can be regarded as an analogy of the regular relative cohomology. Let $f : S^1 \to \mathbb{R}^3 \setminus \{0\}$, and

$$\Omega^p(f) = \Omega^p(\mathcal{E}(\mathbb{R}^3 \setminus \{0\})) \oplus \Omega^{p-1}(S^1) \quad (3.4.1)$$
Then define

\[ \tilde{\partial} : \Omega^p(f) \to \Omega^{p+1}(f) \]

\[ (\omega, \theta) \mapsto (\mathcal{D}\omega, f^\ast \omega - d\theta) \] (3.4.2)

The square of \( \tilde{\partial} \) is given by

\[ \tilde{\partial}^2(\omega, \theta) = \tilde{\partial}(\mathcal{D}\omega, f^\ast \omega - d\theta) \]

\[ = (0, f^\ast \mathcal{D}\omega - d(f^\ast \omega - d\theta)) \] (3.4.3)

\[ = (0, f^\ast \mathcal{D}\omega - df^\ast \omega) \]

The following proposition shows that a version of cohomology relative to \( \tilde{\partial} \) can be given for Legendrian knots:

**Proposition 3.4.1.** We have \( \tilde{\partial}^2 = 0 \), if \( f^\ast \kappa = 0 \), i.e., \( f : S^1 \to \mathbb{R}^3 \setminus \{0\} \) is a Legendrian knot.

**Proof.** The proof is straight forward.

Recall that when \( \omega \in C^\infty(\mathbb{R}^3 \setminus \{0\}) \), we have

\[ \mathcal{D}\omega = d\omega - \kappa \wedge \nu d\omega \] (3.4.4)

When \( \omega \in \Omega^1(H) \),

\[ \mathcal{D}\omega = d\omega - d \ast d_H\omega \]

\[ = d\omega - d(g\kappa), \] (3.4.5)

for some \( g \in C^\infty(\mathbb{R}^3 \setminus \{0\}) \).

When \( \omega \in \Omega^2(V) \), we have

\[ \mathcal{D}\omega = d\omega \] (3.4.6)

In each case, \( \tilde{\partial}^2 = 0 \). \qed
Define

\[ H^p(f) = \frac{\text{Ker } \tilde{d}}{\text{Im } d} \]  

(3.4.7)

We have a short exact sequence:

\[ 0 \to \Omega^*(S^1) \to \Omega^*(f) \to \Omega^*(\mathcal{E}(\mathbb{R}^3 \setminus \{0\})) \to 0 \]  

(3.4.8)

where

\[ \phi_1(\theta) = (0, \theta) \]

\[ \phi_2(\omega, \theta) = \omega \]  

(3.4.9)

Obviously, \( \phi_2 \) is a chain map, and \( \phi_1 \) is an “anti” chain map, i.e., \( \tilde{d}\phi_1 = -\phi_1 d \). These lead to a long exact sequence:

\[
\begin{array}{c}
0 \longrightarrow H^0(S^1) \xrightarrow{\phi_1^*} H^1(f) \xrightarrow{\phi_2^*} H^1(\mathcal{E}(\mathbb{R}^3 \setminus \{0\})) \longrightarrow \\
\xrightarrow{\phi_1^*} H^1(S^1) \xrightarrow{\phi_2^*} H^2(f) \xrightarrow{\phi_2^*} H^2(\mathcal{E}(\mathbb{R}^3 \setminus \{0\})) \longrightarrow \ldots
\end{array}
\]

Since \( H^0(S^1) = \mathbb{R} \), and \( H^1(\mathcal{E}(\mathbb{R}^3 \setminus \{0\})) \simeq H^1(\mathbb{R}^3 \setminus \{0\}) = 0 \), so we have

\[ H^1(f) = \mathbb{R} \]  

(3.4.10)

So we have a generator \((\omega, \theta)\) for \( H^1(f) \) such that \( f^*\omega \) is exact, and \( D\omega = 0 \).

### 3.5 Feynman Rules

In this section we present the Feynman rules for contact Chern-Simons theory. We follow the same notation as in Chapter 1. There are two types of propagators corresponding to the gauge and ghost terms in the contact Chern-Simons action. As before, both propagators are oriented and, to leading-order in the stationary-phase
approximation, the gauge propagator describes the correlator,

\[ \langle B^{a_1}(x) B^{a_2}(y) \rangle = i g^2 K(x, y)^{a_1 a_2} \in \Omega_H^{1,1}(M \times M, g^{\otimes 2}). \]  \hspace{1cm} (3.5.1)

Similarly, the ghost field propagator describes the leading-order contribution to the correlator,

\[ \langle c^{a_1}(x) b^{a_2}(y) \rangle = i g^2 \star_x G(x, y)^{a_1 a_2} \in \Omega^{0,3}(M \times M, g^{\otimes 2}). \]  \hspace{1cm} (3.5.2)

The gauge and ghost field propagators are expressed diagrammatically as in Figure 3-2

In addition, there are cubic and quartic interaction terms in the last line of the gauge-fixed action (3.1.19). The first two of these are represented in Figure 3-3.

\[ i g^2 \star_x G(x, y)^{a_1 a_2} \]

Figure 3-2: Ghost and gauge field propagators.

gauge-fixed action (3.1.19). The first two of these are represented in Figure 3-3.

The last interaction vertex is new and is represented by the quartic diagram in Figure 3-4. The vertices in Figure 3-3 involve an integral over the point \( z \in M \), so the integrand must be a three-form on \( M \). For the first diagram in Figure 3-3, the cubic gauge interaction vertex, the required three-form is proportional to the wedge product of three external gauge field propagators attached to the vertex,

\[ \kappa_z \wedge \star_{H, z} d_{H, z} K(z, \cdot)^{a_1 a_3} \wedge [K(z, \cdot)^{a_2 a_5} \wedge K(z, \cdot)^{a_3 a_6}]. \]
For the second diagram in Figure 3-3, the ghost interaction vertex, the three-form is proportional to,

\[ \kappa_z \wedge \star_{H,z} d_{H,z} G(\cdot, z)^{a_1a_4} \wedge K(z, \cdot)^{a_2a_4} G(z, \cdot)^{a_3a_6}. \]

For the quartic gauge interaction vertex in Figure 3-4 the required three-form is proportional to the wedge product,

\[ \kappa_z \wedge K(z, \cdot)^{a_1a_5} \wedge K(z, \cdot)^{a_2a_6} \wedge \star_{H,z} [K(z, \cdot)^{a_3a_7} \wedge K(z, \cdot)^{a_4a_8}]. \]

There are three main diagram contributions at order \( g^4 \). The first two of these diagrams are given in Figure 3-5 and contain only cubic interaction vertices.

The last diagram is given in Figure 3-6 and contains a single quartic vertex. Consider the two loop diagrams in Figure 3-5. We shall neglect the loop diagram in Figure 3-6 for now because this requires additional regularization due to the coincident point \( z \) for multiple propagators. Each diagram in Figure 3-5 has a pair of vertices and represents an expression which is quadratic in the Jacobi tensor \( f_{a_1a_2a_3} \) and involves
an integral over the points \((w, z) \in M_w \times M_z\). The first diagram in Figure 3-5 is constructed from four copies of \(K\) (for the four solid lines), wedged together to make a top-form on \(M \times M\) valued in \(\Omega^1_H(M, g) \otimes^2\),

\[
2g^4 \int_{M \times M - \Delta(\epsilon)} f_{a_3 a_4 a_5} f_{a_6 a_7 a_8} \kappa_w \wedge \kappa_z \wedge K(w, y)^{a_3 a_2} \wedge \star_{H, w} d_{H, w} K(w, z)^{a_4 a_7} \wedge \star_{H, z} d_{H, z} K(z, w)^{a_6 a_5} \wedge K(x, z)^{a_1 a_8}.
\]

(3.5.3)

Due to the singularity in \(K\) at coincident points, we are careful to excise from the naive integration domain in (3.5.3) a neighborhood \(\Delta(\epsilon) \subset M \times M\) of radius \(\epsilon > 0\) about the points where \(w = z\) (the diagonal) as well as \(z = x\) and \(w = y\). The factor of 2 on the right of (3.5.3) is a symmetry factor, necessary to account for the similar diagram in which the arrows on the internal lines are reversed. The integrand in the second diagram in Figure 3-5 contains two copies of the ghost propagator \(\star_{H, g} G\) as well as two copies of the gauge field propagator \(K\), arranged to yield a top-form on \(M \times M\) valued in \(\Omega^1_H(M, g) \otimes^2\). Using the same \(\epsilon\)-regularization as in (3.5.3),

\[
ge^4 \int_{M \times M - \Delta(\epsilon)} f_{a_3 a_4 a_5} f_{a_6 a_7 a_8} \kappa_w \wedge \kappa_z \wedge K(w, y)^{a_3 a_2} \wedge \star_{H, w} d_{H, w} G(w, z)^{a_4 a_7} \wedge \star_{H, z} d_{H, z} G(z, w)^{a_6 a_5} \wedge K(x, z)^{a_1 a_8}.
\]

(3.5.4)

We conjecture that the contributions from these two diagrams combined with the quartic diagram cancel exactly, as is what happens for the analogous diagram contri-
butions in the Bosonic theory. So for all values of $\varepsilon > 0$,

\[ = 0 \quad (3.5.5) \]
Chapter 4

Loop Calculations

In this section we establish some results for the 1-loop invariants of contact Chern-Simons theory. First we review some of the basics of Legendrian knots and then we define the linking number in this theory for a multi-component Legendrian link, and show that it agrees with the usual Gauss linking number. Then, we introduce perturbative localization for the contact Chern-Simons theory, and consider the asymptotic behavior of the propagator of the contact Chern-Simons theory. The asymptotic estimate of the propagator is closely related to the self-linking given in the next chapter.

4.1 Legendrian Wilson Loops

We begin by reviewing some facts about Legendrian knots. As usual, we use the notation \((u, v, t)\) to denote the standard coordinates on \(\mathbb{R}^3\). By definition, \(C \subset \mathbb{R}^3\) is Legendrian when the tangent line \(T_pC\) at any point \(p \in C\) lies in the contact plane \(H_p\) at that point. Equivalently, the pullback of \(\kappa\) to \(C\) vanishes,

\[
\kappa \bigg|_C = 0 \iff C \text{ is Legendrian}, \quad (4.1.1)
\]

or in terms of a parametrization \(X : S^1 \to \mathbb{R}^3\),

\[
\frac{dt}{d\theta} = v \frac{du}{d\theta} - u \frac{dv}{d\theta}, \quad X(\theta) \equiv (u(\theta), v(\theta), t(\theta)). \quad (4.1.2)
\]
Any smooth knot admits a Legendrian representative, so the theory of Legendrian knots is extremely rich. Moreover, equivalence by Legendrian isotopy, i.e. continuous isotopy through a family of Legendrian knots, strictly refines the usual topological equivalence. A given topological knot may admit infinitely-many inequivalent Legendrian representatives.

Unlike topological knots, Legendrian knots have canonical plane projections. For this reason, Legendrian knots behave in many ways like plane curves. Our interest lies in the so-called Lagrangian projection to the $uv$-plane, 

$$\Pi : \mathbb{R}^3 \to \mathbb{R}^2, \quad \Pi(u, v, t) = (u, v), \quad (4.1.3)$$

for which the image $\Pi(C)$ of a Legendrian knot $C \subset \mathbb{R}^3$ is a smoothly immersed curve. The smoothness of $\Pi(C)$ is already a non-trivial feature of the Legendrian condition (4.1.2), since this condition implies that $\dot{t} = 0$ at any point on $C$ where $\dot{u} = \dot{v} = 0$. Hence if $X$ is a regular parametrization of $C$, then $\Pi \circ X$ is a regular parametrization of $\Pi(C)$. Trivially, $\Pi(C)$ is a Lagrangian submanifold of $\mathbb{R}^2$ with respect to the symplectic form $dk = 2du \wedge dv$, whence the name.

In Figure 4-1 we display the Lagrangian projection of a Legendrian trefoil knot. To guide the eye, we indicate over- and under-crossings in the figure. Unlike for topological knot diagrams, the crossing information for a Legendrian knot is redundant, since the spatial configuration of $C \subset \mathbb{R}^3$ can be completely recovered from the plane.
curve $\Pi(C)$ by integrating the contact relation in (4.1.2),

$$t(\theta) = t_0 + \int_0^\theta d\theta' [v(\theta') \dot{u}(\theta') - u(\theta') \dot{v}(\theta')].$$  

(4.1.4)

Here $t_0$ is the height of $C$ at the basepoint corresponding to $\theta = 0$. Due the symmetry of $\kappa$ under translations in $t$, this constant is both arbitrary and irrelevant.

Not every immersed curve can be the plane projection of a Legendrian knot. For instance, if we take the parameter $\theta$ in (4.1.4) to have periodicity $2\pi$, then

$$0 = t(2\pi) - t(0) = \int_0^{2\pi} d\theta' [v(\theta') \dot{u}(\theta') - u(\theta') \dot{v}(\theta')].$$  

(4.1.5)

Equivalently by Stokes’ Theorem, the plane region $D$ enclosed by $\Pi(C)$ must have zero symplectic area,

$$\int_D du \wedge dv = 0,$$  

(4.1.6)

where each component of $D$ is oriented consistently with $\partial D = C$. Also, if $\theta_1 \neq \theta_2$ are distinct parameter values for which $u(\theta_1) = u(\theta_2)$ and $v(\theta_1) = v(\theta_2)$, corresponding to the location of a crossing in $\Pi(C)$, then

$$0 \neq t(\theta_2) - t(\theta_1) = \int_{\theta_1}^{\theta_2} d\theta' [v(\theta') \dot{u}(\theta') - u(\theta') \dot{v}(\theta')].$$  

(4.1.7)

The necessary conditions in (4.1.5) and (4.1.7) are sufficient for the immersed plane curve to lift to an embedded Legendrian knot. These conditions depend upon the signed areas of the regions enclosed by $\Pi(C)$, so the Lagrangian projection cannot be manipulated in a wholly topological fashion á la Reidemeister.

Legendrian knots always carry a canonical framing by the Reeb vector field $R = \partial/\partial t$. Concretely from (4.1.2), a Legendrian curve $C \subset \mathbb{R}^3$ cannot have a vertical tangent, where $\dot{u} = \dot{v} = 0$ but $\dot{t} \neq 0$. With the choice $n = R$, the framed self-linking number $\text{slk}_f(C, n)$ can then be converted into a Legendrian invariant

$$\text{tb}(C) := \text{slk}_f(C, R) \in \mathbb{Z},$$  

(4.1.8)
a kind of self-linking number for $C$.

The Thurston-Bennequin invariant $tb(C)$ can be easily computed from the Lagrangian projection of $C$. After a rigid rotation, the vertical framing by the Reeb field $R$ becomes equivalent to the planar, blackboard framing of $\Pi(C)$. But again by Figure 1-13, the self-linking number in the blackboard framing is exactly the writhe of the knot diagram. Thus,

\[ tb(C) = w(\Pi(C)) \quad (4.1.9) \]

Since the writhe is fixed under orientation-reversal, so too is

\[ tb(-C) = tb(C) \quad (4.1.10) \]

The Thurston-Bennequin invariant is one of a pair of classical Legendrian invariants. To state the Main Theorem, we also need the other.

Because $C$ is determined by its Lagrangian projection $\Pi(C)$, any isotopy invariant of immersed plane curves yields a Legendrian invariant of $C$. According to the Whitney-Graustein Theorem [66], the unique such invariant of an immersion $\gamma : S^1 \rightarrow \mathbb{R}^2$ is the rotation number

\[ \text{rot}(\gamma) = \deg \dot{\gamma}, \quad \dot{\gamma} : S^1 \rightarrow \mathbb{R}^2 - \{0\}, \quad (4.1.11) \]

defined as the topological degree of the derivative $\dot{\gamma}$. Equivalently, $\text{rot}(\gamma)$ is the total (signed) curvature of the immersed plane curve,

\[ \text{rot}(\gamma) = \frac{1}{2\pi} \int_{S^1} d\theta \frac{\dot{\gamma} \times \ddot{\gamma}}{||\dot{\gamma}||^2}, \quad (4.1.12) \]

where we use the shorthand ‘$\times$’ for the scalar cross-product,

\[ \dot{\gamma} \times \ddot{\gamma}(\theta) \equiv \dot{u}(\theta) \ddot{v}(\theta) - \ddot{u}(\theta) \dot{v}(\theta), \quad \gamma(\theta) = (u(\theta), v(\theta)) \in \mathbb{R}^2. \quad (4.1.13) \]

As will be essential later, the formula for $\text{rot}(\gamma)$ in (5.1.9) presents the rotation number as a local invariant, in the sense of being the integral of a locally-defined geometric
quantity along the curve. With our conventions, \( \text{rot}(\gamma) = 1 \) when \( \gamma \) is a circle traversed in the counterclockwise direction.

For the Legendrian knot \( C \subset \mathbb{R}^3 \), we set

\[
\text{rot}(C) := \text{rot}(\Pi(C)) \in \mathbb{Z}.
\] (4.1.14)

See Definition 3.5.12 in [35] for an intrinsically three-dimensional characterization of \( \text{rot}(C) \). In terms of the diagram for \( \Pi(C) \), the rotation number can be computed as a signed count of upwards vertical tangencies, as in Figure 5-1. For the Legendrian trefoil in Figure 4-1, two upwards vertical tangencies occur, but they do so with opposite signs, so \( \text{rot}(C) = 0 \).

Note that the rotation number depends upon the orientation of the curve, and under a reversal of orientation, the rotation number changes sign. So in contrast to the behavior (5.1.7) of the Thurston-Bennequin invariant,

\[
\text{rot}(-C) = -\text{rot}(C).
\] (4.1.15)

We have not specified an orientation for the trefoil knot in Figure 4-1, but because \( \text{rot}(C) = 0 \), the orientation does not matter.

4.2 The Linking Number

Recall that

\[
D_r = \star_H L_R - \frac{ir}{2} \left( d_H^t d_H + d_H d_H^t \right),
\] (4.2.1)
denotes the gauge fixed contact operator. The symmetrized kernel for \( D_r \) is given by

\[
K^{1,1}_r(x, y) = \frac{i}{2r} \left[ \phi_{\lambda_r}(y^{-1} \ast x) d\zeta_x \wedge d\zeta_y + \phi_{-\lambda_r}(y^{-1} \ast x) d\zeta_x \wedge d\zeta_y \right],
\]

(4.2.2)

\textbf{Remark 4.2.1.} At this point it will be useful to distinguish the degenerate case \( r = i \) in our notation. Our convention will be to remove the \( r \)-subscript when we set \( r = i \).

For example, \( K^{1,1}(x, y) = K^{1,1}_{i}(x, y) \), \( D = D_{r=i}, D = D_{r=i}, \) etc.

\textbf{Proposition 4.2.2} (Legendrian isotopic invariance). Suppose \( C_{1,2} \subset \mathbb{R}^3 \) are disjoint oriented Legendrian curves. Then

\[
I(C_1, C_2) = \int_{C_1 \times C_2} K^{1,1}(x, y),
\]

(4.2.3)

is invariant under isotopies with Legendrian boundary.

\textit{Proof.} We first show that \( I \) is invariant under an isotopy of \( C_1 \). Let \( \tilde{C}_1 \) be a Legendrian curve that is isotopic to \( C_1 \) such that the isotopy \( \Psi : [0, 1] \times S^1 \to \mathbb{R}^3 \) does not intersect \( C_2 \). Let \( \Sigma = \Psi([0, 1] \times S^1) \subset \mathbb{R}^3 \). Let \( D_x, D_x, d_{H,x} \) denote the operators \( D, D, d_H \) acting with respect to the first factor of the product \( \mathbb{R}^3 \times \mathbb{R}^3 \). Since \( K^{1,1} \) is the kernel for \( D \), then \( D_x K^{1,1} = 0 \) away from the diagonal in \( \mathbb{R}^3 \times \mathbb{R}^3 \). Since (3.3.2) defines a complex, \( D^2 = D^\dagger D + \frac{1}{4}(d_H d_{H}^\dagger)^2 \), and it follows that \( D_x K^{1,1} = 0 \) (i.e. the result follows by taking the \( L^2 \) inner product \( (K^{1,1}, (D_x^\dagger D_x + \frac{1}{4}(d_{H,x} d_{H,x}^\dagger)^2) K^{1,1}) = (D_x K^{1,1}, D_x K^{1,1}) + \frac{1}{4}(d_{H,x} d_{H,x}^\dagger K^{1,1}, d_{H,x} d_{H,x}^\dagger K^{1,1}) = 0) \). By Lemma 3.3.2, \( D_x K^{1,1} = d_x \tilde{K}^{1,1} \) where \( \tilde{K}^{1,1} \equiv K^{1,1} \mod \kappa_x \), where \( \kappa_x = \pi_x^* \kappa \) is the pullback of \( \kappa \) under the projection map \( \pi_x : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \) to the first factor. Write \( \tilde{K}^{1,1} = K^{1,1} + (g \circ \pi_x) \kappa_x \).
where $g \in \Omega^0(\mathbb{R}^3)$. We have,

$$
I(C_1, C_2) = \int_{C_1 \times C_2} K^{1,1}(x, y),
$$

$$
= \int_{C_1 \times C_2} \left( K^{1,1}(x, y) + (g \circ \pi_x)\kappa_x \right), \ C_1 \text{ is Legendrian},
$$

$$
= \int_{\Sigma \times C_2} d_x \left( K^{1,1}(x, y) + (g \circ \pi_x)\kappa_x \right) + I(\tilde{C}_1, C_2), \ \text{Stokes’ theorem, } \tilde{C}_1 \text{ is Legendrian},
$$

$$
= \int_{\Sigma \times C_2} D_x K^{1,1}(x, y) + I(\tilde{C}_1, C_2),
$$

$$
= I(\tilde{C}_1, C_2), \ D_x K^{1,1} = 0.
$$

Invariance of $I$ under an isotopy of $C_2$ follows by the symmetry $K^{1,1}(x, y) = K^{1,1}(y, x)$. \hfill \Box

**Remark 4.2.3.** Consider,

$$
I_r(C_1, C_2) = \int_{C_1 \times C_2} K^{1,1}_r(x, y). \tag{4.2.4}
$$

Proposition 5.1.1 shows that $I_{r=i}(C_1, C_2)$ is deformation invariant in the degenerate case. Deformation invariance for general $r$ will follow if analyticity of $I_r(C_1, C_2)$ in the variable $r$ is established along with $\frac{d}{dr} I_r(C_1, C_2) = 0$. 

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Define,

\[ \alpha = t_x - t_y - \text{Im}(\zeta_y \cdot \bar{\zeta}_x) + \frac{i}{2} |\zeta_x - \zeta_y|^2 \]  
(4.2.5)

\[ \beta = \frac{\bar{\alpha}}{\alpha} \]  
(4.2.7)

\[ \Theta = (\alpha \cdot \bar{\alpha})^{-1/2} = ||\alpha||^{-1}, \]  
(4.2.8)

\[ \phi = \beta \cdot \Theta, \quad \bar{\phi} = \bar{\beta} \Theta. \]  
(4.2.9)

\[ U = \frac{\partial}{\partial u} + v \frac{\partial}{\partial t}, \]  
(4.2.10)

\[ V = \frac{\partial}{\partial v} - u \frac{\partial}{\partial t}, \]  
(4.2.11)

\[ R = \frac{\partial}{\partial t}, \]  
(4.2.12)

\[ Z = \frac{1}{2} (U - i V) = \frac{\partial}{\partial \zeta} + \frac{i}{2} \frac{\partial}{\partial t}, \]  
(4.2.13)

\[ \bar{Z} = \frac{1}{2} (U + i V) = \frac{\partial}{\partial \bar{\zeta}} - \frac{i}{2} \frac{\partial}{\partial t}. \]  
(4.2.14)

Lemma 4.2.4.

\[ Z_x \phi = -Z_y \Theta, \]  
(4.2.15)

\[ Z_x \bar{\phi} = -Z_y \Theta. \]  
(4.2.16)

Proof. Observe that,

\[ Z_x \alpha = Z_y \bar{\alpha} = i(\zeta_x - \zeta_y), \]  
(4.2.17)

\[ \bar{Z}_y \alpha = \bar{Z}_x \bar{\alpha} = -i(\zeta_x - \zeta_y), \]  
(4.2.18)

\[ Z_x \bar{\alpha} = Z_y \alpha = \bar{Z}_y \bar{\alpha} = \bar{Z}_x \alpha = 0. \]  
(4.2.19)

Consider,

\[ \bar{Z}_x \phi = \bar{Z}_x (\beta \cdot \Theta) = Z_x (\beta) \cdot \Theta + \beta \cdot \bar{Z}_x \Theta. \]  
(4.2.20)
Using (4.2.7), (4.2.18) and (4.2.19) we find,

\[ Z_x \beta = \frac{Z_y \alpha}{\alpha}, \]  
(4.2.21)

Using (4.2.8) and (4.2.19) we also find,

\[ \frac{Z_y \alpha}{Z_y \Theta} = -\frac{2}{\Theta^3 \pi}, \]  
(4.2.22)

and,

\[ \frac{Z_x \Theta}{Z_y \Theta} = \beta. \]  
(4.2.23)

From (4.2.20) we find,

\[
Z_x \phi = Z_x (\beta) \cdot \Theta + \beta \cdot Z_x \Theta,
\]
\[
= \frac{Z_y \alpha}{\alpha} \cdot \Theta + \beta \cdot Z_x \Theta, \text{ by (4.2.21),}
\]
\[
= -\frac{2 Z_y \Theta}{\Theta^3 (\pi \alpha)} \cdot \Theta + \beta \cdot Z_x \Theta, \text{ by (4.2.22),}
\]
\[
= -2 Z_y \Theta + \beta \cdot Z_x \Theta, \text{ by (4.2.8),}
\]
\[
= -2 Z_y \Theta + (\beta \beta) Z_y \Theta, \text{ by (4.2.23),}
\]
\[
= -Z_y \Theta, \text{ by (4.2.7).}
\]

Thus, we have shown that

\[ Z_x \phi = -Z_y \Theta. \]  
(4.2.24)

Taking the complex conjugate of (4.2.24) yields,

\[ Z_x \bar{\phi} = -Z_y \Theta. \]  
(4.2.25)

\[ \square \]

**Lemma 4.2.5.**

\[ d_{H,x} \ast_{H,x} K_{1,1} = d_{H,y} F, \]
(4.2.26)

where \( F = -\frac{i}{2} \Theta d \xi_x \wedge d \bar{\xi}_x \in \Omega^{(2,0)}(\mathbb{R}^3 \times \mathbb{R}^3). \)
Proof. The proof is a straightforward computation,

\[
d_{H,x} \ast_{H,x} K^{1,1} = \frac{1}{2} d_{H,x} \ast_{H,x} \left[ \phi \, d\zeta_x \wedge d\zeta_y + \overline{\phi} \, d\overline{\zeta}_x \wedge d\overline{\zeta}_y \right],
\]

(4.2.27)

\[
= \frac{i}{2} d_{H,x} \left[ -\phi \, d\zeta_x \wedge d\zeta_y + \overline{\phi} \, d\overline{\zeta}_x \wedge d\overline{\zeta}_y \right],
\]

(4.2.28)

\[
= \frac{i}{2} \left[ Z_x \phi \, d\zeta_x \wedge d\zeta_y + Z_x \overline{\phi} \, d\overline{\zeta}_x \wedge d\overline{\zeta}_y \right],
\]

(4.2.29)

\[
= -\frac{i}{2} \left[ Z_y \Theta \, d\zeta_x \wedge d\zeta_y + Z_y \Theta \, d\zeta_x \wedge d\zeta_y \right],
\]

(4.2.30)

\[
= d_{H,y} \left( -\frac{i}{2} \Theta \, d\zeta_x \wedge d\zeta_y \right),
\]

(4.2.31)

\[
= d_{H,y} F.
\]

(4.2.32)

Let \( \sigma \subset \mathbb{R}^3 \) be a Seifert surface for \( C_1 \). Consider,

\[
\int_{\sigma \times C_2} \ast_x d_{H,x} (d_{H,x})^\dagger K^{1,1} = \int_{\sigma \times C_2} \ast_x d_{H,x} \ast_{H,x} \left( d_{H,x} \ast_{H,x} K^{1,1} \right),
\]

(4.2.26)

\[
= \int_{\sigma \times C_2} \ast_x d_{H,x} \ast_{H,x} \left( d_{H,y} F \right), \text{ by (4.2.26)},
\]

\[
= \int_{\sigma} \int_{C_2} d_{H,y} \left( \ast_x d_{H,x} \ast_{H,x} F \right),
\]

\[
= \int_{\sigma} \int_{C_2} d_y \left( \ast_x d_{H,x} \ast_{H,x} F \right), \text{ \( C_2 \) Legendrian}
\]

\[
= 0, \text{ Stokes' theorem}.
\]

We therefore have shown,

\[
\int_{\sigma \times C_2} \ast_x d_{H,x} (d_{H,x})^\dagger K^{1,1} = 0.
\]

(4.2.33)

Thus, we have

\[
\int_{\sigma \times C_2} \ast_x D_x K^{1,1} = \int_{\sigma \times C_2} \left( D_x + \ast_x d_{H,x} (d_{H,x})^\dagger \right) K^{1,1},
\]

(4.2.34)

\[
= \int_{\sigma \times C_2} D_x K^{1,1}, \text{ by (4.2.33)}.
\]
Recall that the kernel $K^{1,1}$ satisfies,

$$D_x K(x, y) = \delta_D(x, y),$$

where $\delta_D$ is the distributional $(1, 1)$-form given by

$$\delta_D(x, y) := \frac{1}{2} \delta(y^{-1} \ast x) \left[ d\zeta_x \land d\zeta_y + d\zeta_x \land d\zeta_y \right]. \quad (4.2.35)$$

**Proposition 4.2.6 (Heisenberg Linking = Gauss Linking).** Suppose $C_{1,2} \subset \mathbb{R}^3$ are disjoint oriented Legendrian curves. Then

$$\text{lk}(C_1, C_2) = I(C_1, C_2). \quad (4.2.36)$$

**Proof.** Consider,

$$I(C_1, C_2) = \int_{C_1 \times C_2} K^{1,1},$$

$$= \int_{C_1 \times C_2} \left( K^{1,1} + (g \circ \pi_x) \kappa_x \right), \ C_1 \text{ Legendrian},$$

$$= \int_{\sigma \times C_2} d_x \left( K^{1,1} + (g \circ \pi_x) \kappa_x \right), \ \text{Stokes’ theorem},$$

$$= \int_{\sigma \times C_2} D_x K^{1,1}, \ \text{by Lemma 3.3.2},$$

$$= \int_{\sigma \times C_2} \ast_x D_x K^{1,1}, \ \text{by (4.2.34)},$$

$$= \int_{\sigma \times C_2} \ast_x \delta_D(x, y),$$

$$= \text{lk}(C_1, C_2),$$

where the last equality follows by definition of $\text{lk}(C_1, C_2).$ \qed

### 4.3 Perturbative Localization

Perturbative localization refers to the study of our theory in the limit $r \to 0^+$. This limit leads to some useful simplifications that are not possible in ordinary Bosonic
Chern-Simons theory. We first consider the kernel

\[ K_{r,1}^{1,1}(x, y) = \frac{i}{2r} \left[ \phi_{\lambda_r}(y^{-1} \ast x) d\zeta_x \wedge d\bar{\zeta}_y + \phi_{-\lambda_r}(y^{-1} \ast x) d\zeta_x \wedge d\zeta_y \right], \tag{4.3.1} \]

in the limit as \( r \to 0^+ \). Recall that \( \lambda_r = \frac{i}{r} + 1 \) and

\[ \phi_{\lambda_r} = c_{\lambda_r} \left( t + i \frac{|\zeta|^2}{2} \right)^{-\frac{1+i\lambda_r}{2}} \cdot \left( t - i \frac{|\zeta|^2}{2} \right)^{-\frac{1-i\lambda_r}{2}}, \tag{4.3.2} \]

where,

\[ c_{\lambda_r} = -\frac{\Gamma\left(\frac{1}{2}(1 + \lambda_r)\right) \Gamma\left(\frac{1}{2}(1 - \lambda_r)\right)}{i^{-\lambda_r} \pi^2}. \]

By Euler’s reflection formula, we have

\[ c_{\lambda_r} = -\frac{\Gamma\left(\frac{1}{2}(1 + \lambda_r)\right) \Gamma\left(\frac{1}{2}(1 - \lambda_r)\right)}{i^{-\lambda_r} \pi^2} = -\frac{1}{i^{-\lambda_r} \pi \sin\left(\frac{\pi}{2}(1 - \lambda_r)\right)}. \tag{4.3.3} \]

Since,

\[ \sin\left(\frac{\pi}{2}(1 - \lambda_r)\right) = \frac{e^{i\frac{\pi}{2}(1-\lambda_r)} - e^{-i\frac{\pi}{2}(1-\lambda_r)}}{2i} = \frac{e^{-\frac{i}{2} \lambda_r} + e^{\frac{i}{2} \lambda_r}}{2}, \tag{4.3.4} \]

we have,

\[ i^{-\lambda_r} \pi \sin\left(\frac{\pi}{2}(1 - \lambda_r)\right) = \pi e^{-\frac{i}{2} \lambda_r} \frac{e^{-\frac{i}{2} \lambda_r} + e^{\frac{i}{2} \lambda_r}}{2} = \pi e^{-\frac{i}{2} \lambda_r} + 1. \tag{4.3.5} \]

Thus,

\[ c_{\lambda_r} = -\frac{2}{\pi(e^{-\pi i \lambda_r} + 1)} = \frac{2}{\pi(e^{\pi/r} - 1)}, \tag{4.3.6} \]

and,

\[ c_{-\lambda_r} = \frac{2}{\pi(e^{-\pi/r} - 1)}. \tag{4.3.7} \]
We now consider the leading order of the asymptotic estimate of the kernel $K^{1,1}_r$,

$$K^{1,1}_r(x, y) \sim_{r \to 0} K^{\text{asym}}_r(x, y). \quad (4.3.8)$$

Given the formal asymptotic estimate of the coefficients in (4.3.6) and (4.3.7), we expect to see that $K^{\text{asym}}_r(x, y)$ represents an operator acting on anti-holomorphic 1-forms $f d\zeta \in \Omega^1(H)$. Let

$$\Delta_+ = dv_x \wedge du_y - du_x \wedge dv_y, \quad (4.3.9)$$
$$\Delta_- = du_x \wedge du_y + dv_x \wedge dv_y. \quad (4.3.10)$$

By direct calculation we see that,

$$d\zeta_x \wedge d\zeta_y = -i\Delta_+ + \Delta_-., \quad (4.3.11)$$

and,

$$\Delta_+ \wedge d\zeta_y = i\Delta_- \wedge d\zeta_y. \quad (4.3.12)$$

Thus, in the expression for the anti-holomorphic part of the integral kernel we may replace $\Delta_-$ by $-i\Delta_+$ and write,

$$\frac{i}{2r} \phi_{-\lambda_r} d\zeta_x \wedge d\zeta_y = \frac{i}{2r} \phi_{-\lambda_r} (-i\Delta_+ + \Delta_-),$$

$$= \frac{1}{r} \phi_{-\lambda_r} \Delta_+.$$

Let $\Delta u = u_x - u_y$ and $\Delta v = v_x - v_y$, and observe that $\Delta_+ = d\Delta u \wedge d\Delta v$. Thus, the integral kernel acting on anti-holomorphic forms is then given by,

$$\frac{1}{r} \phi_{-\lambda_r} d\Delta u \wedge d\Delta v. \quad (4.3.13)$$
Let the leading order of the asymptotic estimate of \( \phi - \lambda r \) be denoted by \( \chi_r(x, y) \).

Given,

\[
\phi - \lambda r(\zeta, t) = c_{-\lambda r} \left( t + i \frac{|\zeta|^2}{2} \right)^{-1 + i \lambda r} \left( t - i \frac{|\zeta|^2}{2} \right)^{-1 - i \lambda r} = c_{-\lambda r} \left( t - i \frac{|\zeta|^2}{2} \right)^{-1} \left( t + i \frac{|\zeta|^2}{2} \right)^{\frac{i}{2\pi}} ,
\]

write,

\[
t + i \frac{|\zeta|^2}{2} = ae^{i\theta} . \tag{4.3.15}
\]

Then,

\[
\left( t + i \frac{|\zeta|^2}{2} \right)^{\frac{i}{2\pi}} \left( t - i \frac{|\zeta|^2}{2} \right)^{-\theta/r} = e^{-\theta/r} \tag{4.3.16}
\]

The above equation tells us that when \( r \ll 1 \),

\[
\left( t + i \frac{|\zeta|^2}{2} \right)^{\frac{i}{2\pi}} \left( t - i \frac{|\zeta|^2}{2} \right)
\]

will be negligible unless \( \theta \) is small.

Write,

\[
\theta = \arctan \left( \frac{|\zeta|^2}{2t} \right) . \tag{4.3.17}
\]

When \( \theta \ll 1 \), we have

\[
\arctan \left( \frac{|\zeta|^2}{2t} \right) \sim \frac{|\zeta|^2}{2t} \tag{4.3.18}
\]

The leading order of the asymptotic estimate of the right hand side of (4.3.14) is then given by,

\[
- \frac{2}{\pi} \left( t - i \frac{|\zeta|^2}{2} \right)^{-1} \exp \left( - \frac{|\zeta|^2}{2tr} \right) \tag{4.3.19}
\]

For \( \tan \theta = \frac{|\zeta|^2}{2t} \ll 1 \), we also have \( \left( t - i \frac{|\zeta|^2}{2} \right)^{-1} \sim \frac{1}{t} \). Let \( \Delta t = t_x - t_y + (u_y v_x - u_x v_y) \), \( \Delta \zeta = \zeta_x - \zeta_y \). Hence, the leading order of the asymptotic estimate of the kernel \( K^{1,1}_r \) is given by

\[
\chi_r(x, y) d\Delta u \wedge d\Delta v = - \frac{2}{\pi r \Delta t} \exp \left( - \frac{\Delta \zeta^2}{2r \Delta t} \right) d\Delta u \wedge d\Delta v . \tag{4.3.20}
\]
This asymptotic estimate is in the form of heat kernel. In the next chapter, we will consider the self-linking integral given by the asymptotic estimate above and show that this self-linking integral is in fact invariant under Legendrian homotopy.
Chapter 5

Legendrian Self-Linking

5.1 Introduction

In the previous chapter, we already considered the following new form of linking integral:

\[ I_r(C_1, C_2) = \int_{C_1 \times C_2} K^{1,1}_r(x, y). \]  

(5.1.1)

The value of this integral is the usual linking number when \( C_1 \) and \( C_2 \) are both Legendrian knots. The purpose of this chapter is studying the self-linking counterpart of the above integral which one would like to make sense of the linking number of \( C_1 \) and \( C_2 \) as a knot invariant in the degenerate case \( C_1 = C_2 \). We will not use the integrand \( K^{1,1}_r \) directly. Instead, we follow the approach we construct integrand of the Gauss linking integral to construct our heat kernel version integrand. Recall that to construct the Gauss linking integral, we start with a generator \([\psi]\) of \( \Omega^2(S^2) \):

\[ \psi = \frac{1}{8\pi} \epsilon_{\mu\nu\rho} x^\mu \, dx^\nu \wedge dx^\rho, \quad x \neq 0 \in \mathbb{R}^3. \]  

(5.1.2)

Then we take the pullback of \( \psi \) under the Gauss map and the Euclidean difference map in succession:

\[ \varrho : \mathbb{R}^3 \setminus \{0\} \to S^2 \]

\[ x \mapsto \frac{x}{||x||}. \]  

(5.1.3)
\[ \Gamma : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \]
\[ (x, y) \mapsto y - x \]  

(5.1.4)

Our theory is supersymmetric and this leads to a modification of the the \( \psi \) enters in the Gauss integral. Because the Legendrian condition on \( C \subset \mathbb{R}^3 \) does not respect the Euclidean action by \( SO(3) \), we forgo the seemingly-natural requirement that \( \psi \) itself be \( SO(3) \)-invariant. In recompense, the supersymmetric version of \( \psi \) will enjoy superior analytic behavior near the origin in \( \mathbb{R}^3 \).

As heuristic motivation for the following, one should imagine that we alter the angular form \( \psi \) in (5.1.2) by concentrating the support for the generator of \( H^2(S^2; \mathbb{Z}) \) at the North Pole on the sphere. This trick is well-known to aficionados of Chern-Simons perturbation theory, but here we must take care to preserve the underlying symmetries of the contact structure on \( \mathbb{R}^3 \).

5.1.1 Thurston-Bennequin Number and Rotation Number

We will need two basic concepts before we state our main theorem in this chapter. The first one is Thurston-Bennequin number. Legendrian knots always carry a canonical framing by the Reeb vector field \( R = \partial / \partial z \). Concretely from (4.1.2), a Legendrian curve \( C \subset \mathbb{R}^3 \) cannot have a vertical tangent, where \( \dot{x} = \dot{y} = 0 \) but \( \dot{z} \neq 0 \). With the choice \( n = R \), the framed self-linking number \( \text{slk}_f(C, n) \) can then be converted into a Legendrian invariant

\[ \text{tb}(C) := \text{slk}_f(C, R) \in \mathbb{Z}, \]  

(5.1.5)

a kind of self-linking number for \( C \) known as Thurston-Bennequin number.

The Thurston-Bennequin invariant \( \text{tb}(C) \) can be easily computed from the Lagrangian projection of \( C \). After a rigid rotation, the vertical framing by the Reeb field \( R \) becomes equivalent to the planar, blackboard framing of \( \Pi(C) \). But again by Figure 1-13, the self-linking number in the blackboard framing is exactly the writhe of the knot diagram. Thus,

\[ \text{tb}(C) = w(\Pi(C)). \]  

(5.1.6)
Since the writhe is fixed under orientation-reversal, so too is
\[ \text{tb}(-C) = \text{tb}(C). \] (5.1.7)

The Thurston-Bennequin invariant is one of a pair of classical Legendrian invariants. To state the Main Theorem, we also need the other.

Because \( C \) is determined by its Lagrangian projection \( \Pi(C) \), any isotopy invariant of immersed plane curves yields a Legendrian invariant of \( C \). According to the Whitney-Graustein Theorem [66], the unique such invariant of an immersion \( \gamma : S^1 \to \mathbb{R}^2 \) is the rotation number
\[ \text{rot}(\gamma) = \deg \dot{\gamma}, \quad \dot{\gamma} : S^1 \to \mathbb{R}^2 - \{0\}, \] (5.1.8)
defined as the topological degree of the derivative \( \gamma \). Equivalently, \( \text{rot}(\gamma) \) is the total (signed) curvature of the immersed plane curve,
\[ \text{rot}(\gamma) = \frac{1}{2\pi} \oint_{S^1} d\theta \frac{\dot{\gamma} \times \ddot{\gamma}}{||\dot{\gamma}||^2}, \] (5.1.9)
where we use the shorthand ‘\( \times \)’ for the scalar cross-product,
\[ \dot{\gamma} \times \ddot{\gamma}(\theta) \equiv \dot{x}(\theta) \ddot{y}(\theta) - \dot{y}(\theta) \ddot{x}(\theta), \quad \gamma(\theta) = (x(\theta), y(\theta)) \in \mathbb{R}^2. \] (5.1.10)

As will be essential later, the formula for \( \text{rot}(\gamma) \) in (5.1.9) presents the rotation number as a local invariant, in the sense of being the integral of a locally-defined geometric quantity along the curve. With our conventions, \( \text{rot}(\gamma) = 1 \) when \( \gamma \) is a circle traversed in the counterclockwise direction.

For the Legendrian knot \( C \subset \mathbb{R}^3 \), we set
\[ \text{rot}(C) := \text{rot}(\Pi(C)) \in \mathbb{Z}. \] (5.1.11)

See Definition 3.5.12 in [35] for an intrinsically three-dimensional characterization of \( \text{rot}(C) \). In terms of the diagram for \( \Pi(C) \), the rotation number can be computed as
Figure 5-1: Rotation number at upwards tangencies.

a signed count of upwards vertical tangencies, as in Figure 5-1. For the Legendrian trefoil in Figure 4-1, two upwards vertical tangencies occur, but they do so with opposite signs, so $\text{rot}(C) = 0$.

Note that the rotation number depends upon the orientation of the curve, and under a reversal of orientation, the rotation number changes sign. So in contrast to the behavior (5.1.7) of the Thurston-Bennequin invariant,

$$\text{rot}(-C) = -\text{rot}(C) .$$

(5.1.12)

We have not specified an orientation for the trefoil knot in Figure 4-1, but because $\text{rot}(C) = 0$, the orientation does not matter.

5.1.2 Main theorem

We introduce the Gaussian two-form $\omega_\Lambda \in \Omega^2(\mathbb{R}^2)$ on the $xy$-plane,

$$\omega_\Lambda = \frac{\Lambda}{2\pi} e^{-\Lambda(x^2+y^2)/2} \, dx \wedge dy , \quad \Lambda > 0 .$$

(5.1.13)

Here $\Lambda$ is a positive real parameter which sets the width of the Gaussian, and in the limit $\Lambda \to \infty$, the Gaussian becomes a delta-function concentrated at the origin in $\mathbb{R}^2$. Clearly $\omega_\Lambda$ is invariant under rotations, and $\omega_\Lambda$ is normalized so that for all $\Lambda$,

$$\int_{\mathbb{R}^2} \omega_\Lambda = 1 .$$

(5.1.14)
The various factors of two in (5.1.13) are standard and could be absorbed into \( \Lambda \) if desired.

Though the support of \( \omega_\Lambda \) is not compact, \( \omega_\Lambda \) decays very rapidly at infinity. At least morally, \( \omega_\Lambda \) should be regarded as a generator for the compactly-supported cohomology \( H^2_c(\mathbb{R}^2; \mathbb{Z}) \simeq \mathbb{Z} \) of the plane, on the same footing as the unit area form on the sphere. Because the normalization condition in (5.1.14) does not depend on \( \Lambda \), neither does the cohomology class of \( \omega_\Lambda \). Explicitly, a small computation shows

\[
\frac{\partial \omega_\Lambda}{\partial \Lambda} = \frac{1}{2\pi} \left[ 1 - \frac{\Lambda(x^2 + y^2)}{2} \right] e^{-\Lambda(x^2+y^2)/2} dx \wedge dy = d\alpha_\Lambda, \tag{5.1.15}
\]

where

\[
\alpha_\Lambda = \frac{1}{4\pi} e^{-\Lambda(x^2+y^2)/2} (x\,dy - y\,dx) \in \Omega^1(\mathbb{R}^2). \tag{5.1.16}
\]

The transgression form \( \alpha_\Lambda \) will reappear in the proof of our Fundamental Lemma. In the meantime, note that \( \alpha_\Lambda \) is also \( SO(2) \)-invariant, as required by the relation to \( \omega_\Lambda \) in (5.1.15).

We next introduce the planar analogue for the retraction onto \( S^2 \) in (??). To preserve the parabolic scaling in (3.2.2), we consider the map \( \varphi_+: \mathbb{R}^3_+ \to \mathbb{R}^2 \) defined on the upper half-space \( \mathbb{R}^3_+ \) by

\[
\varphi_+(x, y, z) = \left( \frac{x}{\sqrt{z}}, \frac{y}{\sqrt{z}} \right), \quad z > 0. \tag{5.1.17}
\]

Trivially, the image of \( \varphi_+ \) is preserved under the scaling for which \( x \) and \( y \) have weight one and \( z \) has weight two.

Using the planar retraction in (5.1.17), we pull the Gaussian form \( \omega_\Lambda \in \Omega^2(\mathbb{R}^2) \) back to a new two-form

\[
\chi_\Lambda = \varphi_+^* \omega_\Lambda, \\
= \frac{\Lambda}{2\pi z} e^{-\Lambda(x^2+y^2)/2z} \left[ dx \wedge dy + \frac{1}{2} (x\,dy - y\,dx) \wedge \frac{dz}{z} \right], \quad z > 0. \tag{5.1.18}
\]

By construction, \( \chi_\Lambda \) is invariant under the parabolic scaling and the action of \( SO(2) \),
but not \( SO(3) \).

Of course, \( \chi_\Lambda \) strongly resembles the heat kernel for the Laplacian in two dimensions. As with the heat kernel, so long as \( x^2 + y^2 \neq 0 \), the expression in (5.1.18) vanishes smoothly as \( z \to 0 \) from above. To define \( \chi_\Lambda \) on the entire punctured space \( \mathbb{R}^3 - \{0\} \), we simply extend by zero,

\[
\chi_\Lambda = 0, \quad z \leq 0. \tag{5.1.19}
\]

With this choice, \( \chi_\Lambda \in \Omega^2(\mathbb{R}^3 - \{0\}) \) is automatically closed away from \( \{0\} \) and generates the cohomology \( H^2(\mathbb{R}^3 - \{0\}; \mathbb{Z}) \). For instance, over the unit sphere \( S^2 \subset \mathbb{R}^3 \),

\[
\int_{S^2} \chi_\Lambda = \int_{S^2 \cap \mathbb{R}^3_+} \chi_\Lambda = \int_{\mathbb{R}^2} \omega_\Lambda = 1. \tag{5.1.20}
\]

We have yet to incorporate the Heisenberg symmetry of the contact structure. In the elementary Gauss linking integral, the abelian structure of \( \mathbb{R}^3 \) as a vector space enters implicitly through the definition of the difference map \( \Gamma \) in (1.3.11). To preserve instead the non-abelian symmetry by left-translation in \( \mathbb{H} \simeq \mathbb{R}^3 \), we consider a Heisenberg difference map \( \hat{\Gamma} : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \), given by

\[
\hat{\Gamma}(X_1, X_2) = \mu(X_1^{-1}, X_2) = \mu(-X_1, X_2), \quad X_{1,2} \in \mathbb{H}, \tag{5.1.21}
\]

\[
= (x_2 - x_1, y_2 - y_1, z_2 - z_1 + x_1 y_2 - x_2 y_1). \]

Since \( \mu \) is the Heisenberg multiplication, \( \hat{\Gamma}(X_1, X_2) = X_1^{-1} \cdot X_2 \) in the usual shorthand. This combination of \( X_1 \) and \( X_2 \) is invariant under simultaneous left-multiplication,

\[
\hat{\Gamma}(g \cdot X_1, g \cdot X_2) = \hat{\Gamma}(X_1, X_2), \quad g \in \mathbb{H}, \tag{5.1.22}
\]

and we have selected the relative signs of \( X_1 \) and \( X_2 \) in (5.1.21) to agree with the convention for the abelian difference in (1.3.11).

**Proposition 5.1.1** (Heisenberg Linking). Suppose \( C_{1,2} \subset \mathbb{R}^3 \) are disjoint oriented curves, not necessarily Legendrian, with respective parametrizations \( X_{1,2} : S^1 \to \mathbb{R}^3 \).
Then
\[
\text{lk}(C_1, C_2) = \int_{T^2} (X_1 \times X_2)^* \tilde{\Gamma}^* \chi_\Lambda, \quad \Lambda > 0. \quad (5.1.23)
\]

This proposition follows from the fact that the heat form \(\chi_\Lambda\) is equivalent in cohomology to the global angular form \(\psi\),
\[
[\chi_\Lambda] = [\psi] \in H^2(\mathbb{R}^3 - \{0\}; \mathbb{Z}). \quad (5.1.24)
\]

Also, the Heisenberg difference \(\tilde{\Gamma}\) in (5.1.21) is homotopic to the abelian difference \(\Gamma\). To see this, set
\[
\mu_\hbar(X_1, X_2) = \left( x_1 + x_2, y_1 + y_2, z_1 + z_2 - \hbar (x_1 y_2 - x_2 y_1) \right), \quad \hbar \in [0, 1]. \quad (5.1.25)
\]
Then \(\tilde{\Gamma}_\hbar = \mu_\hbar(X_1^{-1}, X_2)\) smoothly interpolates from the abelian to the Heisenberg difference as the Planck constant \(\hbar\) ranges over the interval from \(\hbar = 0\) to \(\hbar = 1\).

Though the angular form \(\psi\) and the heat form \(\chi_\Lambda\) agree in cohomology, they behave very differently near the origin in \(\mathbb{R}^3\). This analytic distinction matters crucially for approaches to self-linking.

Let \(C \subset \mathbb{R}^3\) be an oriented Legendrian curve, with regular parametrization \(X : S^1 \to \mathbb{R}^3\). By analogy to the naive Gauss self-linking integral in (1.3.12), we consider a new Heisenberg self-linking integral
\[
\text{slk}_\hbar(C) := \lim_{\varepsilon \to 0} \int_{T^2 - \Delta(\varepsilon)} (X \times X)^* \tilde{\Gamma}_\hbar^* \chi_\Lambda, \quad \hbar \in [0, 1], \quad \Lambda > 0. \quad (5.1.26)
\]

The remainder of this chapter is devoted to the proof of the following Main Theorem.

**Theorem 5.1.2 (Legendrian Self-Linking).** The limit defining \(\text{slk}_\hbar(C)\) exists. The value of \(\text{slk}_\hbar(C)\) is independent of \(\Lambda\) and depends only upon the Legendrian isotopy class of \(C\). In terms of the Thurston-Bennequin invariant \(tb(C)\) and the rotation
number \ rot(C),

\[ \text{slk}_\alpha(C) = \begin{cases} 
\text{tb}(C) - \text{rot}(C), & h \neq 1, \\
\text{tb}(C), & h = 1. 
\end{cases} \]  

(5.1.27)

Informally, the Main Theorem states that the framing anomaly for knots in bosonic Chern-Simons theory is absent in supersymmetric Chern-Simons theory. The corresponding statement for Seifert-fibered three-manifolds was observed previously in [12]. The Main Theorem is also consistent with results of Fuchs and Tabachnikov [31] identifying \( \text{tb}(C) \) and \( \text{rot}(C) \) as the only order \( \leq 1 \) finite-type invariants of Legendrian knots.

The strategy of proof for Theorem 5.1.2 is straightforward. We first demonstrate that \( \text{slk}_\alpha(C) \) is independent of the parameter \( \Lambda \) which sets the width of the Gaussian in \( \chi_\Lambda \). For any \( \varepsilon > 0 \), the derivative of \( \text{slk}_\alpha(C) \) with respect to \( \Lambda \) is given in terms of a boundary integral of the transgression form \( \alpha_\Lambda \) in (5.1.16). The content of our Fundamental Lemma in Section 5.2 is to demonstrate that the potentially anomalous boundary contribution from \( \alpha_\Lambda \) vanishes in the limit \( \varepsilon \to 0 \), due to the rapid decay of the heat kernel away from the diagonal.

In Section 5.1.2 we directly evaluate the self-linking integral in the limit \( \Lambda \to \infty \). In this regime, the self-linking integrand is non-negligible only in the neighborhood of points on \( T^2 \) which map under the product \( X \times X \) to pairs of points \( p, q \in C \) that are coincident under the Lagrangian projection to the \( xy \)-plane, ie. \( \Pi(p) = \Pi(q) \). Such points on \( T^2 \) correspond either to the preimage of crossings in \( \Pi(C) \), or more trivially, to points on the diagonal \( \Delta \subset T^2 \). The local contribution from the crossings leads to the appearance of the Thurston-Bennequin invariant \( \text{tb}(C) \), while the local contribution from the diagonal \( \Delta \) leads to the appearance of the rotation number \( \text{rot}(C) \) for \( h \neq 1 \). At the special value \( h = 1 \), the Heisenberg symmetry of the integrand is restored, and the anomalous contribution from \( \Delta \) vanishes. The Legendrian condition is used crucially throughout the local analysis, and invariance under Legendrian isotopy follows a postiori from the formula in (5.1.27).\(^1\)

\(^1\)A direct computation of the Legendrian variation \( \delta \text{slk}_\alpha(C) \), achieved for \( \delta \text{slk}_0(C) \) in (1.3.30), involves some formidable differential algebra and did not appear practical to the author.
Let us emphasize one striking feature of Theorem 5.1.2, which is perhaps best appreciated after one works through the localization computation in Section 5.1.2. Namely, since the coefficients of \( \text{tb}(C) \) and \( \text{rot}(C) \) in (5.1.27) are integers, so is the value of \( \text{slk}_\kappa(C) \)! The coefficient of \( \text{tb}(C) \) is directly related to our normalization condition (5.1.20) on the heat form \( \chi_\Lambda \), required to recover the usual linking number in Proposition 5.1.1. Thus the coefficient of \( \text{tb}(C) \) is fixed by fiat to unity. In contrast, the coefficient of \( \text{rot}(C) \) is determined by a delicate calculation near the diagonal \( \Delta \subset T^2 \), so its integrality for all \( \hbar \) is a non-trivial feature of the Legendrian self-linking integral.

From the physical perspective, integrality of \( \text{slk}_\kappa(C) \) is necessary for gauge invariance as well as consistency with standard lore about non-renormalization and the infrared behavior of supersymmetric Yang-Mills-Chern-Simons theory. We already have a discussion of this statement in Chapter 2. However, on the principle that no integer appears by chance, a simple topological explanation for the integrality of \( \text{slk}_\kappa(C) \in \mathbb{Z} \) would be nice to have.

For instance, the difference of classical Legendrian invariants \( \text{tb}(C) - \text{rot}(C) \) in (5.1.27) occurs naturally in contact topology as the transverse self-linking invariant \( \text{slk}(C_+) \) of the canonical positive transverse push-off \( C_+ \) of \( C \) (Proposition 3.5.36 in [35]). The transverse invariant \( \text{slk}(C_+) \) can be interpreted in terms of a relative Euler class on a Seifert surface for \( C_+ \), so its appearance here is surely not a coincidence.

5.1.3 Notation and conventions

For the convenience of the reader, we summarize the notation and conventions used in the rest of the paper.

- \( \mathbb{R}^3 \) has Euclidean coordinates \((x, y, z)\) and is oriented by \( dx \wedge dy \wedge dz \).

- The map \( \Pi : \mathbb{R}^3 \to \mathbb{R}^2 \) is the projection onto the \( xy \)-plane.

- \( \kappa = dz + x dy - y dx \) is the standard radially-symmetric contact form, positive with respect to the orientation on \( \mathbb{R}^3 \).

- \( C \subset \mathbb{R}^3 \) is an oriented Legendrian knot, with regular parametrization \( X : S^1 \to \mathbb{R}^3 \).
• \( \theta \sim \theta + 2\pi \) is an angular coordinate on \( S^1 \), compatible under \( X \) with the given orientation on \( C \). We abbreviate \( dX/d\theta \equiv \dot{X}(\theta) \), and so on.

• The torus \( T^2 = S^1 \times S^1 \) has angular coordinates \( (\theta_1, \theta_2) \) and is oriented by \( d\theta_1 \wedge d\theta_2 \). The diagonal \( \Delta \subset T^2 \) is the subset where \( \theta_1 = \theta_2 \).

• For \( \varepsilon > 0 \), a tubular neighborhood \( \Delta(\varepsilon) \) of the diagonal \( \Delta \subset T^2 \) is parametrized by \( \theta_1 = \phi \) and \( \theta_2 = \phi + \eta \) for \( |\eta| < \varepsilon \). The cylinder \( T^2 - \Delta(\varepsilon) \) has boundary circles \( S^1_{\pm} \) on which \( \eta = \pm \varepsilon \), respectively.

• \( \gamma : S^1 \to \mathbb{R}^2 \) is an immersed plane curve which is the Lagrangian projection of \( C \), ie. \( \gamma = \Pi \circ X \).

• We use the abbreviation ‘\( \times \)’ for the scalar cross-product on the plane, as in

\[
\gamma \times \dot{\gamma}(\theta) \equiv x(\theta) \dot{y}(\theta) - y(\theta) \dot{x}(\theta), \quad \gamma(\theta) = (x(\theta), y(\theta)) \in \mathbb{R}^2.
\]

• The contact form \( \kappa \) is left-invariant with respect to the Heisenberg multiplication

\[
\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{H}, \quad \mathbb{H} \simeq \mathbb{R}^3, \quad [h = 1]
\]

\[
\mu(X_1, X_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 - x_1y_2 + x_2y_1).
\]

• The left-invariant Heisenberg difference \( \hat{\Gamma}_h : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \) for \( h \in [0, 1] \) is given by

\[
\hat{\Gamma}_h(X_1, X_2) = \mu_h(X_1^{-1}, X_2),
\]

\[
= (x_2 - x_1, y_2 - y_1, z_2 - z_1 + h(x_1y_2 - x_2y_1)).
\]

• The Gaussian fundamental class \( \omega_{\Lambda} \in \Omega^2(\mathbb{R}^2) \) of the \( xy \)-plane is given by

\[
\omega_{\Lambda} = \frac{\Lambda}{2\pi} e^{-\Lambda(x^2 + y^2)/2} dx \wedge dy, \quad \Lambda > 0.
\]
• The transgression form $\alpha_\Lambda \in \Omega^1(\mathbb{R}^2)$ satisfies $\partial \omega / \partial \Lambda = d\alpha_\Lambda$ and is given by

$$\alpha_\Lambda = \frac{1}{4\pi} e^{-\Lambda(x^2+y^2)/2} (x \, dy - y \, dx).$$

• The parabolic scaling $\varrho_+: \mathbb{R}^3_+ \to \mathbb{R}^2$ is defined on the upper half-space $\mathbb{R}^3_+$ by

$$\varrho_+(x, y, z) = \left( \frac{x}{\sqrt{z}}, \frac{y}{\sqrt{z}} \right), \quad z > 0.$$

• The heat form $\chi_\Lambda \in \Omega^2(\mathbb{R}^3 - \{0\})$ is the pullback

$$\chi_\Lambda = \begin{cases} 
\varrho_+^* \omega_\Lambda, & z > 0, \\
0, & z \leq 0.
\end{cases}$$

Explicitly,

$$\varrho_+^* \omega_\Lambda = \frac{\Lambda}{2\pi z} e^{-\Lambda(x^2+y^2)/2z} \left[ dx \wedge dy + \frac{1}{2} (x \, dy - y \, dx) \wedge \frac{dz}{z} \right], \quad z > 0.$$

### 5.2 Fundamental Lemma

We first demonstrate that the value of the Legendrian self-linking integral $\text{slk}_\hbar(C)$ does not depend upon the parameter $\Lambda > 0$ which sets the width of the Gaussian in the heat form $\chi_\Lambda$.

**Lemma 5.2.1** (Fundamental Lemma). *The limit which defines the Legendrian self-linking integral $\text{slk}_\hbar(C)$ exists,*

$$\text{slk}_\hbar(C) = \lim_{\varepsilon \to 0} \int_{T^2 - \Delta(\varepsilon)} (X \times X)^* \hat{\Gamma}_h \chi_\Lambda, \quad \hbar \in [0, 1], \quad \Lambda > 0, \quad (5.2.1)$$

*and the value of $\text{slk}_\hbar(C) \in \mathbb{R}$ is independent of the parameter $\Lambda$.*

Precisely at the special value $\hbar = 1$, the self-linking integrand is Heisenberg-invariant. For this reason, the analysis to prove Lemma 5.2.1 will differ slightly depending on whether $\hbar \neq 1$ or $\hbar = 1$. 

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Throughout, we make two extra topological assumptions about the Legendrian knot $C$. Both assumptions hold generically and help to simplify the proofs.

1. The Lagrangian projection $\Pi(C)$ is an immersed plane curve $\gamma$ with only simple double-point singularities, as in Figure 4-1.

2. The height function $z(\theta)$ on $C \subset \mathbb{R}^3$ is Morse, with isolated non-degenerate critical points. That is, $\dot{z}$ vanishes only at isolated points $p \in C$, at which $\ddot{z} \neq 0$. Because $C$ is Legendrian, the Morse condition on $z(\theta)$ is equivalent by (4.1.2) to the condition that the function $\gamma \times \dot{\gamma}$ vanish only at isolated points $\theta \in S^1$, at which $\gamma \times \ddot{\gamma} \neq 0$.

The first assumption is standard. Otherwise, the Morse condition is used only in the proof of the Fundamental Lemma for the case $\hbar \neq 1$ and could possibly be relaxed, at the cost of further effort.

Before embarking on the proof of our Fundamental Lemma, let us mention an easy corollary which is handy for the gauge theory analysis in Chapter 4. Let $t > 0$ be a positive scaling parameter. As in the remarks following Proposition 5.1.1, we consider a version of the Heisenberg difference $\hat{\Gamma}_{th}$ with rescaled Planck constant $th$ for fixed $h \in [0, 1]$,

$$\hat{\Gamma}_{th}(X_1, X_2) = (x_2 - x_1, y_2 - y_1, z_2 - z_1 + th(x_1 y_2 - x_2 y_1)).$$

(5.2.2)

For each value of $t$, we associate the contact form

$$\kappa_t = t^{-1/2} dz + t^{1/2} (x dy - y dx).$$

(5.2.3)

The relative power of $t$ between the two terms in $\kappa_t$ ensures that the contact form is left-invariant under the Heisenberg multiplication $\mu_{h=t}$ in (5.1.25). The overall power of $t$ ensures that the contact condition $\kappa_t \wedge d\kappa_t = 2 dx \wedge dy \wedge dz \neq 0$ is satisfied trivially for all values $t > 0$.

Finally, suppose that $C \subset \mathbb{R}^3$ is a Legendrian knot with respect to the standard contact form $\kappa_{t=1}$. Just as we consider the family of isotopic contact forms in (5.2.3),
we would like to consider a family of isotopic knots $C_t \subset \mathbb{R}^3$, each of which is Legendrian with respect to $\kappa_t$ for the given value of $t$. The family of knots is determined if we simply require the Lagrangian projection of $C_t$ to coincide with that of $C$,

$$\Pi(C_t) = \Pi(C), \quad t > 0.$$ (5.2.4)

Either by integrating the contact condition as in (4.1.4) or just by scaling, the embedding map $X_t : S^1 \to \mathbb{R}^3$ for $C_t$ is then related to the original embedding $X$ for $C$ via

$$X_t(\theta) \equiv (x_t(\theta), y_t(\theta), z_t(\theta)) = (x(\theta), y(\theta), t z(\theta)).$$ (5.2.5)

In particular, the abelian limit $t \to 0$ of the Heisenberg structure corresponds to a limit in which $C_t$ flattens to a curve in the $xy$-plane, and the Lagrangian projection $\Pi(C)$ is realized geometrically.

Given the family of curves $C_t$, we consider the three-parameter self-linking integral

$$\text{slk}_\kappa(C_t) = \lim_{\varepsilon \to 0} \int_{T^2 - \Delta(\varepsilon)} (X_t \times X_t)^* \hat{\Gamma}_{th}^* \chi_\Lambda, \quad h \in [0, 1], \quad t, \Lambda > 0.$$ (5.2.6)

Precisely for $h = 1$, the self-linking integrand is invariant under the Heisenberg symmetry with multiplication map $\mu_t$.

The $t$-dependence of the integrand in (5.2.6) is very simple. In terms of the finite differences

$$\Delta x = x_2 - x_1, \quad \Delta y = y_2 - y_1, \quad \Delta z = z_2 - z_1 + t (x_1 y_2 - x_2 y_1),$$ (5.2.7)

the formula for the heat form $\chi_\Lambda$ in (5.1.18) implies

$$(X_t \times X_t)^* \hat{\Gamma}_{th}^* \chi_\Lambda =$$

$$\frac{\Lambda}{2\pi t \Delta z} e^{-\Lambda(\Delta x^2 + \Delta y^2)/2t \Delta z} \left[ d\Delta x \wedge d\Delta y + \frac{1}{2} (\Delta x d\Delta y - \Delta y d\Delta x) \wedge \frac{d\Delta z}{\Delta z} \right],$$ (5.2.8)

provided $\Delta z > 0$. Otherwise, the pullback of $\chi_\Lambda$ vanishes. Evidently, $t$ just multiplies

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\[ \hat{\Delta} \hat{z} \text{ in } (5.2.8), \text{ and all dependence on } t \text{ can be absorbed by rescaling the Gaussian parameter } \Lambda. \]

**Remark 5.2.2 (Scaling Identity).** For all \( t, \Lambda > 0, \)

\[
(X_t \times X_t)^*\hat{\Gamma}_t^*\chi_\Lambda = (X \times X)^*\hat{\Gamma}_\Lambda^*\chi_{\Lambda/t}. \tag{5.2.9}
\]

By the Scaling Identity, the behavior of the self-linking integrand in limit \( \Lambda \to \infty \) with fixed \( t \) is the same as the behavior in the limit \( t \to 0 \) with fixed \( \Lambda \). Because \( C_t \) flattens to a plane curve in the latter limit, \( \Lambda \) plays a similar role to the parameter of the same name in (1.3.43).

**Corollary 5.2.3.** The value of \( \text{slk}_\kappa(C_t) \) is independent of both \( t \) and \( \Lambda \).

This corollary follows immediately from Lemma 5.2.1 and the Scaling Identity in (5.2.9).  □

### 5.2.1 Analysis near the diagonal

The non-trivial aspect of our work concerns the local analysis of the self-linking integrand in the vicinity of the diagonal \( \Delta \subset T^2 \). In practice, this analysis amounts to the Taylor expansion of expressions such as occur in (5.2.8). Rather than scatter such expansions willy-nilly throughout the paper, we collect here the basic ingredients to be used again and again.

Let \((\theta_1, \theta_2)\) be angular coordinates on \( T^2 \). To parametrize the tubular neighborhood \( \Delta(\varepsilon) \subset T^2 \) of the diagonal, we set

\[
\theta_1 = \phi, \quad \theta_2 = \phi + \eta. \tag{5.2.10}
\]

Here \( \phi \) is an angular coordinate along the diagonal, and \( \Delta(\varepsilon) \) is the subset where \(|\eta| < \varepsilon\). Our local expansions near the diagonal will then be Taylor expansions in \( \eta \), appropriate for the regime \( \varepsilon \ll 1 \). We must be careful about orientations. In terms of the coordinates \((\phi, \eta)\), the orientation form on \( T^2 \) is given by \( d\phi \land d\theta_2 = d\phi \land d\eta \).

Topologically, the configuration space \( T^2 - \Delta(\varepsilon) \) is a cylinder with oriented boundary
circles

\[ S^1_{\pm} : \eta = \pm \epsilon \mod 2\pi. \]  \hfill (5.2.11)

As shown in Figure A.1, the boundary orientation of \( S^1_{+} \) is positive with respect to the direction of increasing \( \phi \) and the orientation of \( S^1_{-} \) is negative, so

\[ \partial \left( T^2 - \Delta(\epsilon) \right) = S^1_{+} - S^1_{-}. \]  \hfill (5.2.12)

We now expand the various terms appearing in the self-linking integrand \( (X \times X)^* \tilde{\Gamma}_h \chi_\Lambda \), given by the expression in (5.2.8) for \( t = 1 \). We start with

\[ \Delta x = x(\theta_2) - x(\theta_1) = x(\phi + \eta) - x(\phi) = \eta \dot{x}(\phi) + O(\epsilon^2), \]  \hfill (5.2.13)

and similarly for \( \Delta y \). Hence

\[ \Delta x^2 + \Delta y^2 = \eta^2 \left( \dot{x}^2 + \dot{y}^2 \right) + O(\epsilon^3) = \eta^2 ||\dot{\gamma}||^2 + O(\epsilon^3), \]  \hfill (5.2.14)

where \( \gamma(\phi) = (x(\phi), y(\phi)) \) is the Lagrangian immersion \( \gamma = \Pi \circ X \) as before. For the
related two-form, we can write

\[ d\Delta x \wedge d\Delta y = \frac{1}{2} d\Delta \gamma \times d\Delta \gamma, \quad \Delta \gamma \equiv (\Delta x, \Delta y), \]

\[ = \frac{1}{2} d(\eta \dot{\gamma}) \times d(\eta \dot{\gamma}) + O(\varepsilon^3), \quad (5.2.15) \]

\[ = \frac{1}{2} (\dot{\gamma} d\eta + \eta \ddot{\gamma} d\phi) \times (\dot{\gamma} d\eta + \eta \ddot{\gamma} d\phi) + O(\varepsilon^3). \]

After collecting terms in the product,

\[ d\Delta x \wedge d\Delta y = -\eta (\dot{\gamma} \times \ddot{\gamma}) d\phi \wedge d\eta + O(\varepsilon^2) d\phi \wedge d\eta. \quad (5.2.16) \]

Recall the definition

\[ \hat{\Delta} z = z_2 - z_1 + \hbar (x_1 y_2 - x_2 y_1). \quad (5.2.17) \]

The Legendrian condition on \( C \) is absolutely critical for our results, because it implies that the local behavior of \( \hat{\Delta} z \) near the diagonal is controlled by the Lagrangian immersion \( \gamma \). Moreover, for \( \hat{\Delta} z \) the Heisenberg group rears its head!

The expansion of the abelian difference \( \Delta z = z_2 - z_1 \) is fixed by the Legendrian condition

\[ \ddot{z} = y \dot{x} - x \dot{y} = -\gamma \times \dot{\gamma}. \quad (5.2.18) \]

Thus

\[ z_2 - z_1 = z(\phi + \eta) - z(\phi) = \eta \ddot{z}(\phi) + \frac{1}{2} \eta^2 \dddot{z}(\phi) + \frac{1}{6} \eta^3 \dddot{\dddot{z}}(\phi) + O(\varepsilon^4), \]

\[ = -\eta (\gamma \times \dot{\gamma}) - \frac{1}{2} \eta^2 (\gamma \times \ddot{\gamma}) - \frac{1}{6} \eta^3 (\dot{\gamma} \times \ddot{\gamma} + \gamma \times \dddot{\gamma}) + O(\varepsilon^4). \quad (5.2.19) \]

In passing to the second line of (5.2.19), we repeatedly differentiate the Legendrian condition on \( \dot{z} \) in (5.2.18).

The attentive reader may wonder why we have expanded \( \Delta z = z_2 - z_1 \) all the way to cubic order in \( \eta \). The question answers itself once we expand the remaining
quadratic terms in the Heisenberg difference $\hat{\Delta} z$, 

\[
x_1 y_2 - x_2 y_1 = \gamma(\phi) \times \gamma(\phi + \eta),
\]

\[
= \eta (\gamma \times \dot{\gamma}) + \frac{1}{2} \eta^2 (\gamma \times \ddot{\gamma}) + \frac{1}{6} \eta^3 (\gamma \times \dddot{\gamma}) + \mathcal{O}(\varepsilon^4).
\]

(5.2.20)

So long as $h \neq 1$, the expansion of $\hat{\Delta} z$ begins at linear order in $\eta$,

\[
\hat{\Delta} z \overset{h \neq 1}{=} -\eta (1 - h) (\gamma \times \dot{\gamma}) + \mathcal{O}(\varepsilon^2),
\]

(5.2.21)

as one naively expects. But precisely at $h = 1$, cancellations occur in the sum of (5.2.19) and (5.2.20), and the leading term in the expansion of $\hat{\Delta} z$ near the diagonal begins at cubic order,

\[
\hat{\Delta} z \overset{h = 1}{=} -\frac{1}{6} \eta^3 (\dot{\gamma} \times \ddot{\gamma}) + \mathcal{O}(\varepsilon^4).
\]

(5.2.22)

The cancellation in (5.2.22) is forced by the Heisenberg symmetry at $h = 1$. Clearly, the quantity $\gamma \times \dot{\gamma}$ in (5.2.21) is not invariant under translations $\gamma \mapsto \gamma + \gamma_0$ for constant $\gamma_0 \in \mathbb{R}^2$. Upon projection to the $xy$-plane, such translations are generated by the Heisenberg action, so $\gamma \times \dot{\gamma}$ is forbidden to appear at $h = 1$.

Combining the expansions in (5.2.14), (5.2.21), and (5.2.22), we see that the argument of the heat kernel is given in the neighborhood $\Delta(\varepsilon)$ by

\[
\frac{\Delta x^2 + \Delta y^2}{2\hat{\Delta} z} = \begin{cases} 
-\frac{||\dot{\gamma}||^2 \eta}{2 (1 - h) (\gamma \times \dot{\gamma})} + \mathcal{O}(\varepsilon^2), & [h \neq 1] \\
-\frac{3 ||\dot{\gamma}||^2}{(\dot{\gamma} \times \ddot{\gamma}) \eta} + \mathcal{O}(1). & [h = 1]
\end{cases}
\]

(5.2.23)

For generic $h \neq 1$, the argument of the heat kernel in (5.2.23) vanishes linearly near the diagonal. However, at the symmetric point $h = 1$, the argument instead diverges as $\eta \to 0$. In Section 5.1.2, this difference will ultimately lead to the discontinuity in the value of $\text{slk}_k(C)$ at $h = 1$.

Remark 5.2.4 (Local Positivity). The pullback of the heat form $\chi_A$ vanishes identically unless $\hat{\Delta} z > 0$. On the neighborhood $\Delta(\varepsilon)$, positivity of $\hat{\Delta} z$ becomes equivalent via
(5.2.21) and (5.2.22) to the local sign condition

\[ \frac{\Delta z}{\Delta(\varepsilon)} > 0 \iff \begin{cases} \eta (1 - h) (\gamma \times \dot{\gamma}) < 0, & [h \neq 1], \\ \eta (\dot{\gamma} \times \ddot{\gamma}) < 0. & [h = 1]. \end{cases} \]  \hspace{1cm} (5.2.24)

Again, the nature of the positivity condition depends upon whether or not \( h = 1 \). For generic values of \( h \), the sign of \( \eta \) is determined by the sign of \( \gamma \times \dot{\gamma} \) and hence the sign of the derivative \( \dot{z} \). For \( h = 1 \), the sign of \( \eta \) is instead fixed by the sign of \( \dot{\gamma} \times \ddot{\gamma} \), proportional to the plane curvature of \( \gamma \).

Since the local positivity condition depends upon the sign of \( \eta \), the self-linking integrand always vanishes in the neighborhood of one or the other of the boundary circles \( S^1_\pm \) in Figure A.1, and the geometry of \( \Gamma \) at any given point determines on which boundary circle the integrand vanishes. See Figure 5-3 for a geometric illustration of the local positivity condition \( \Delta z > 0 \) in the case \( h = 1 \). For clarity, we exaggerate the small separation between the points \( \gamma(\theta_1) \) and \( \gamma(\theta_2) \) in the figure. In both cases, the positivity condition is sensitive to the orientation of \( \Gamma \), as a reversal of orientation flips the signs of \( \gamma \times \dot{\gamma} \) and \( \dot{\gamma} \times \ddot{\gamma} \) in (5.2.24).

![Figure 5-3: Local positivity condition \( \Delta z > 0 \) for \( h = 1 \).](image)

Figure 5-3: Local positivity condition \( \Delta z > 0 \) for \( h = 1 \).

Let us complete the expansion for small \( \eta \) of the self-linking integrand. The second bracketed term in (5.2.8) involves the angular one-form

\[ \Delta x d\Delta y - \Delta y d\Delta x = \Delta \gamma \times d\Delta \gamma, \]

\[ = (\eta \dot{\gamma}) \times d(\eta \dot{\gamma}) + O(\varepsilon^3), \]  \hspace{1cm} (5.2.25)

\[ = \eta^2 (\dot{\gamma} \times \ddot{\gamma}) d\phi + O(\varepsilon^3). \]
From the expansions of $\hat{\Delta}z$ in (5.2.21) and (5.2.22),

\[
(\Delta x d\Delta y - \Delta y d\Delta x) \wedge d\hat{\Delta}z = \begin{cases}
- (1 - h) (\gamma \times \dot{\gamma}) (\dot{\gamma} \times \ddot{\gamma}) \eta^2 d\phi \wedge d\eta, & [h \neq 1] \\
- \frac{1}{2} (\dot{\gamma} \times \ddot{\gamma})^2 \eta^4 d\phi \wedge d\eta, & [h = 1]
\end{cases}
\]

(5.2.26)

to leading order, so

\[
(\Delta x d\Delta y - \Delta y d\Delta x) \wedge d\hat{\Delta}z = \begin{cases}
\eta (\dot{\gamma} \times \ddot{\gamma}) d\phi \wedge d\eta + \mathcal{O}(\varepsilon^2) d\phi \wedge d\eta, & [h \neq 1] \\
3 \eta (\dot{\gamma} \times \ddot{\gamma}) d\phi \wedge d\eta + \mathcal{O}(\varepsilon^2) d\phi \wedge d\eta. & [h = 1]
\end{cases}
\]

(5.2.27)

After a bit of algebra, one finds that the pullback (5.2.8) of $\chi_\Lambda$ behaves near the diagonal $\Delta \subset T^2$ for $h \neq 1$ as

\[
(X \times X)^* \hat{\Gamma}_h^* \chi_\Lambda \bigg|_{\Delta = \Delta(\varepsilon)} \equiv - \text{sgn}(\eta) \frac{\Lambda (\dot{\gamma} \times \ddot{\gamma})}{4\pi |1 - h| |\gamma \times \dot{\gamma}|} \exp \left[ - \frac{\Lambda ||\dot{\gamma}||^2 |\eta|}{2 |1 - h| |\gamma \times \dot{\gamma}|} \right] d\phi \wedge d\eta + \cdots,
\]

(5.2.28)

assuming the sign condition $\eta (1 - h) (\gamma \times \dot{\gamma}) < 0$ in (5.2.24) holds. Otherwise, the pullback is equal to zero. The sign condition ensures that the exponential in (5.2.28) is always decaying, and it leads to a non-analytic dependence on the sign of $\eta$. The ellipses in (5.2.28) indicate subleading terms which vanish as $\eta \to 0$.

By contrast, at the symmetric value $h = 1$,

\[
(X \times X)^* \hat{\Gamma}_1^* \chi_\Lambda \bigg|_{\Delta = \Delta(\varepsilon)} \equiv - \frac{3 \Lambda}{2\pi \eta^2} \exp \left[ - \frac{3 \Lambda ||\dot{\gamma}||^2}{|\gamma \times \dot{\gamma}|} \frac{1}{|\eta|} \right] d\phi \wedge d\eta + \cdots,
\]

(5.2.29)

under the sign condition $\eta (\dot{\gamma} \times \ddot{\gamma}) < 0$ in (5.2.24). In passing to the second line, we make the substitution $\nu = 1/\eta$ for clarity. In this case, the pullback of the heat form $\chi_\Lambda$ vanishes exponentially as $|\nu| \to \infty$, or equivalently $|\eta| \to 0$.

---

\(^2\)As usual, $\text{sgn}(\eta) = +1$ for $\eta > 0$, and $\text{sgn}(\eta) = -1$ for $\eta < 0$. 

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5.2.2 Proof of the fundamental lemma

The proof of our Fundamental Lemma 5.2.1 is now an exercise in calculus.

As a brief formality, we first establish that the singularity in the pullback of the heat form \( \chi_\Lambda \) is integrable, so that the defining limit \( \varepsilon \to 0 \) in (5.2.1) does exist. By assumption, \( ||\dot{\gamma}||^2 > 0 \) is everywhere non-vanishing, and the functions \( |\gamma \times \dot{\gamma}| \) and \( |\dot{\gamma} \times \ddot{\gamma}| \) are bounded from above on \( C \). Integrability in the region of small \( |\eta| < \varepsilon \) follows immediately from the local expressions in (5.2.28) and (5.2.29), both of which remain finite as \( \eta \to 0 \).

Otherwise, we must check that the value of \( \text{slk}_\kappa(C) \) does not depend upon the parameter \( \Lambda > 0 \). This argument will be equally straightforward but the result is significant; the analogue for the naive Gauss self-linking integral \( \text{slk}_0(C) \) is simply false. Our strategy will be to show that the derivative of \( \text{slk}_\kappa(C) \) with respect to \( \Lambda \) vanishes for all values of \( \Lambda > 0 \). The details differ somewhat depending upon whether \( h \neq 1 \) or \( h = 1 \), but the main idea is the same in both cases.

We compute

\[
\frac{d \text{slk}_\kappa(C)}{d\Lambda} = \lim_{\varepsilon \to 0} \int_{T^2 - \Delta(\varepsilon)_+} (X \times X)^* \hat{\Gamma}_\hbar^* \left( \frac{\partial \chi_\Lambda}{\partial \Lambda} \right),
\]

\[
= \lim_{\varepsilon \to 0} \int_{[T^2 - \Delta(\varepsilon)]_+^\circ} (X \times X)^* \hat{\Gamma}_\hbar^* \left( \frac{\partial \omega_\Lambda}{\partial \Lambda} \right). \quad (5.2.30)
\]

Here \( [T^2 - \Delta(\varepsilon)]_+^\circ \) indicates the closed subset of the cylinder where \( \hat{\Delta}z \geq 0 \) is positive,

\[
[T^2 - \Delta(\varepsilon)]_+^\circ = \left\{ (\theta_1, \theta_2) \left| \hat{\Delta}z(\theta_1, \theta_2) \geq 0 \right. \right\}, \quad (5.2.31)
\]

on which the pullback of the heat form \( \chi_\Lambda \) is non-vanishing. Of course, the positive subset in (5.2.31) depends upon the Legendrian embedding \( X \). We omit this dependence from the notation as \( X \) is fixed throughout.

By the calculation in (5.1.15),

\[
\frac{\partial \omega_\Lambda}{\partial \Lambda} = d\alpha_\Lambda, \quad (5.2.32)
\]
for the transgression one-form

\[
\alpha_\Lambda = \frac{1}{4\pi} e^{-\Lambda(x^2+y^2)/2} (x \, dy - y \, dx) \in \Omega^1(\mathbb{R}^2).
\] (5.2.33)

We apply the commutativity of the de Rham operator with pullback, followed by Stokes’ Theorem, to reduce the bulk integral in the second line of (5.2.30) to a boundary integral,

\[
\int_{[T^2-\Delta(\epsilon)]_+} (X \times X)^* \tilde{\Gamma}_h^* \varrho_+^* \frac{\partial \omega_\Lambda}{\partial \Lambda} = \int_{\partial[T^2-\Delta(\epsilon)]_+} (X \times X)^* \tilde{\Gamma}_h^* \varrho_+^* \alpha_\Lambda.
\] (5.2.34)

Explicitly, the boundary integrand in (5.2.34) is given where non-zero by

\[
(X \times X)^* \tilde{\Gamma}_h^* \varrho_+^* \alpha_\Lambda = \frac{1}{4\pi \Delta z} e^{-\Lambda(\Delta x^2+\Delta y^2)/2\Delta z} \left(\Delta x \, d\Delta y - \Delta y \, d\Delta x\right).
\] (5.2.35)

This expression vanishes smoothly whenever \(\Delta z \to 0^+\) with \(\Delta x^2 + \Delta y^2 \neq 0\).

The boundary of the positive subset \(\partial[T^2-\Delta(\epsilon)]_+\) in (5.2.34) includes those curves where \(\Delta z = 0\) as well as the intersection of \([T^2-\Delta(\epsilon)]_+\) with the boundary circles \(S^1_\pm\) themselves. Recall that points on \(S^1_\pm\) satisfy \(\eta = \pm \varepsilon\), respectively. By the preceding, only the boundary integral over the intersection \(S^1_\pm \cap [T^2-\Delta(\epsilon)]_+\) is relevant, because the boundary integrand in (5.2.35) vanishes on the locus where \(\Delta z = 0\).

Altogether, in terms of the boundary integral on the right in (5.2.34),

\[
\frac{d\text{slk}_\kappa(C)}{d\Lambda} = \lim_{\varepsilon \to 0} \left[ \int_{S^1_\pm \cap [T^2-\Delta(\epsilon)]_+} (X \times X)^* \tilde{\Gamma}_h^* \varrho_+^* \alpha_\Lambda - \int_{S^1_\pm \cap [T^2-\Delta(\epsilon)]_+} (X \times X)^* \tilde{\Gamma}_h^* \varrho_+^* \alpha_\Lambda \right]
\] (5.2.36)

The minus sign for the boundary integral over \(S^1_-\) accounts for the relative orientation in Figure A.1.

Despite the minus sign, no possibility exists for a trivial cancellation between the two boundary integrals in (5.2.36) for any fixed \(\varepsilon > 0\). According to the local
positivity condition in (5.2.24) for respectively \( h \neq 1 \) or \( h = 1 \),

\[
\begin{align*}
&\begin{cases}
(1 - h) (\gamma \times \dot{\gamma}) \text{ or } \dot{\gamma} \times \ddot{\gamma} \leq 0 \quad \text{on } S^1_+ \cap [T^2 - \Delta(\varepsilon)]_+ , \\
(1 - h) (\gamma \times \dot{\gamma}) \text{ or } \dot{\gamma} \times \ddot{\gamma} \geq 0 \quad \text{on } S^1_+ \cap [T^2 - \Delta(\varepsilon)]_+ .
\end{cases}
\end{align*}
\tag{5.2.37}
\]

The domains of integration over the two boundary circles \( S^1_\pm \) in (5.2.36) are therefore disjoint away from the degeneracy locus where \( (1 - h) (\gamma \times \dot{\gamma}) \) or \( \dot{\gamma} \times \ddot{\gamma} = 0 \), so no cancellation can occur. Generically, the degeneracy locus consists of a finite set of isolated inflection points on the curve.

Let us examine the behavior of the boundary integrand (5.2.35) via the expansion near the diagonal from Section 5.2.1.

**Symmetric case** \( h = 1 \)

We initially consider the Heisenberg-symmetric case \( h = 1 \). Similar to the bulk integrand in (5.2.29), the boundary integrand behaves to leading-order at \( \eta = \pm \varepsilon \) as

\[
(X \times X)^* \tilde{\Gamma}_h^* g_+^* \alpha_\Lambda \bigg|_{S^1_\pm} \overset{h=1}{=} \mp \frac{3}{2\pi \varepsilon} \exp \left[ -\frac{3\Lambda \|\dot{\gamma}\|^2}{\varepsilon |\dot{\gamma} \times \ddot{\gamma}|} \right] d\phi + \cdots ,
\tag{5.2.38}
\]

where the omitted terms vanish more rapidly as \( \varepsilon \to 0 \). By a conspiracy of signs, the difference on the right of (5.2.36) can be rewritten as the single integral

\[
d \text{slk}_\kappa (C) = \overset{h=1}{\text{lim}} \left[ \frac{3}{\pi \varepsilon} \int_{S^1} d\phi \exp \left( -\frac{3\Lambda \|\dot{\gamma}\|^2}{\varepsilon |\dot{\gamma} \times \ddot{\gamma}|} \right) \right] = 0 .
\tag{5.2.39}
\]

To deduce the vanishing of the limit \( \varepsilon \to 0 \), we note that the ratio \( \|\dot{\gamma}\|^2 / |\dot{\gamma} \times \ddot{\gamma}| \geq m \) is everywhere bounded from below on \( S^1 \) by a positive constant \( m > 0 \), so the integrand in (5.2.39) is dominated by the exponentially-small constant

\[
\exp \left( -\frac{3\Lambda m}{\varepsilon} \right) \leq \exp \left( -\frac{3\Lambda m}{\varepsilon} \right).
\tag{5.2.40}
\]

Since \( \Lambda > 0 \) has been arbitrary throughout, \( \text{slk}_\kappa (C) \) is independent of \( \Lambda \) for \( h = 1 \).

\( \square \)

**Generic case** \( h \neq 1 \)

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The analysis for generic $\hbar \neq 1$ is slightly more delicate. Here

\[
(X \times X)^* \hat{\Gamma}_h \circ \alpha_{\Lambda}|_{S^1_{\pm}} \xrightarrow{\hbar \neq 1} \frac{\varepsilon (\gamma \times \dot{\gamma})}{4\pi |1-\hbar||\gamma \times \dot{\gamma}|} \exp \left[ -\frac{\Lambda||\dot{\gamma}||^2 \varepsilon}{2|1-\hbar||\gamma \times \dot{\gamma}|} \right] d\phi + \cdots ,
\]
so the derivative becomes

\[
\frac{d\text{slk}_\kappa(C)}{d\Lambda} \xrightarrow{\hbar \neq 1} \lim_{\varepsilon \to 0} \left[ I_+(\varepsilon) - I_- (\varepsilon) \right],
\]
with

\[
I_\pm (\varepsilon) = \frac{\varepsilon}{4\pi |1-\hbar|} \int_{S^1_{\pm} \cap [T^2-\Delta(\zeta)]_{\pm}} d\phi \frac{\dot{\gamma} \times \ddot{\gamma}}{|\gamma \times \dot{\gamma}|} \exp \left( -\frac{\Lambda||\dot{\gamma}||^2 \varepsilon}{2|1-\hbar||\gamma \times \dot{\gamma}|} \right).
\]

The functions $I_\pm (\varepsilon)$ differ only in the domain of integration over $S^1$, and our goal will be to show individually

\[
\lim_{\varepsilon \to 0} I_\pm (\varepsilon) = 0.
\]

Were the function $\gamma \times \dot{\gamma}$ to be everywhere non-zero on $S^1$, the conclusion in (5.2.44) would be immediate, as we would know the integral in (5.2.43) to be bounded in magnitude even for $\varepsilon = 0$. The explicit prefactor of $\varepsilon$ then ensures the vanishing of $I_\pm (\varepsilon)$ in the limit $\varepsilon \to 0$. However, $\dot{z} = -\gamma \times \dot{\gamma}$ always vanishes for at least two points (the highest and the lowest) on the knot $C \subset \mathbb{R}^3$, and we must worry about what happens to the integral in (5.2.43) near a zero of $\gamma \times \dot{\gamma}$, when $\varepsilon$ is very small.

Let us make an elementary simplification. Since $|\dot{\gamma} \times \ddot{\gamma}|$ is bounded from above and $||\dot{\gamma}||^2 > 0$ is bounded from below on $S^1$,

\[
|I_\pm (\varepsilon)| \leq J_\pm (\varepsilon) = \int_{S^1_{\pm} \cap [T^2-\Delta(\zeta)]_{\pm}} d\phi \frac{A \varepsilon}{|\gamma \times \dot{\gamma}|} \exp \left( -\frac{B \varepsilon}{|\gamma \times \dot{\gamma}|} \right), \quad A, B > 0,
\]
for some positive constants $A$ and $B$, into which we also absorb the dependence on $\Lambda$ and $\hbar$ and the various other numerical factors in (5.2.43). To deduce the limit (5.2.44) for $I_\pm (\varepsilon)$, we show that $J_\pm (\varepsilon)$ vanishes in the same limit.

By assumption, the height function $z(\phi)$ is Morse, with isolated non-degenerate critical points. Equivalently, the function $(\gamma \times \dot{\gamma})(\phi)$ vanishes non-degenerately at an
isolated set of points on $S^1$. By the criteria in (5.2.37), these points are precisely the endpoints of the intervals which compose each integration domain $S^1_\pm \cap [T^2 - \Delta(\varepsilon)]_+$. Locally near such an endpoint $\phi = \phi_0$,

$$ (\gamma \times \dot{\gamma}) (\phi) = c_0 (\phi - \phi_0) + \mathcal{O}(|\phi - \phi_0|^2), \quad c_0 \neq 0. \quad (5.2.46) $$

When we examine $J_\pm(\varepsilon)$ in the limit $\varepsilon \to 0$, only the contribution to the integral from a (one-sided) neighborhood of $\phi_0$ can be non-zero, so we simplify further by replacing $J_\pm(\varepsilon)$ by the model

$$ K(\varepsilon) = \int_{\phi_0}^{\phi_1} d\phi \frac{\varepsilon}{f(\phi)} \exp \left( -\frac{\varepsilon}{f(\phi)} \right). \quad (5.2.47) $$

Here $\phi_1$ is an arbitrary upper cutoff, and $f(\phi) (= |\gamma \times \dot{\gamma}|/B)$ is now any continuous function defined on the interval $[\phi_0, \phi_1]$ such that

$$ f(\phi) > 0 \text{ for } \phi > \phi_0, \quad f(\phi_0) = 0, \quad \text{and} \quad \lim_{\phi \to \phi_0} \left[ \frac{\phi - \phi_0}{f(\phi)} \right] > 0 \text{ exists.} \quad (5.2.48) $$

For all $\varepsilon > 0$, the integral defining $K(\varepsilon)$ exists, since the integrand vanishes at the endpoint $\phi = \phi_0$. For convenience, we take $\phi_0 = 0$ and $\phi_1 = 1$ by a suitable choice of parameter. The proof of the Fundamental Lemma 5.2.1 for generic $\hbar \neq 1$ reduces to the following claim.
Lemma 5.2.5. Let $K(\varepsilon)$ and $f(\phi)$ be defined as in (5.2.47) and (5.2.48). Then

$$\lim_{\varepsilon \to 0} K(\varepsilon) = \lim_{\varepsilon \to 0} \left[ \int_0^1 \frac{\varepsilon}{f(\phi)} \exp \left( -\frac{\varepsilon}{f(\phi)} \right) \right] = 0. \quad (5.2.49)$$

Proof. We consider a succession of three cases.

(i) We start with the basic example $f(\phi) = \phi$, so that

$$K(\varepsilon) = \varepsilon \int_0^1 \frac{d\phi}{\phi} \exp \left( -\frac{\varepsilon}{\phi} \right). \quad (5.2.50)$$

After the substitution $x = \varepsilon/\phi$,

$$K(\varepsilon) = \varepsilon \int_{\varepsilon}^{\infty} \frac{dx}{x} e^{-x} \leq \varepsilon \int_{1}^{\infty} \frac{dx}{x} + \varepsilon \int_{1}^{\infty} dx e^{-x} = \varepsilon |\ln \varepsilon| + \varepsilon e^{-1}, \quad (5.2.51)$$

from which the limit follows.

(ii) Next, let $g(\phi)$ and $h(\phi)$ be continuous functions on the interval $[0, 1]$ obeying bounds

$$0 < m \leq g(\phi), \quad |h(\phi)| \leq M, \quad (5.2.52)$$

for some constants $m$ and $M$. Set

$$K(\varepsilon) = \varepsilon \int_0^1 \frac{d\phi}{\phi} h(\phi) \exp \left( -\frac{\varepsilon}{\phi} \right). \quad (5.2.53)$$

Then

$$K(\varepsilon) \leq M \varepsilon \int_0^1 \frac{d\phi}{\phi} \exp \left( -\frac{m}{\phi} \right) = M \varepsilon \int_0^{1/m} \frac{d\phi}{\phi} \exp \left( -\frac{\varepsilon}{\phi} \right). \quad (5.2.54)$$

The function $K(\varepsilon)$ vanishes as $\varepsilon \to 0$ by (i).

(iii) In the general case of interest,

$$K(\varepsilon) = \int_0^1 \frac{d\phi}{f(\phi)} \frac{\varepsilon}{f(\phi)} \exp \left( -\frac{\varepsilon}{f(\phi)} \right) = \varepsilon \int_0^1 \frac{d\phi}{\phi} \left( \frac{\phi}{f(\phi)} \right) \exp \left( -\frac{\varepsilon}{f(\phi)} \right). \quad (5.2.55)$$

Because $f(\phi) > 0$ for $\phi > 0$ by assumption, the function $g(\phi) = h(\phi) = \phi/ f(\phi)$ is con-
tinuous and positive for all $\phi > 0$. Since the limit $\lim_{\phi \to 0} [\phi / f(\phi)] > 0$ is also assumed to exist and be non-zero, $g(\phi) > 0$ is continuous and non-vanishing throughout the unit interval. Hence $0 < m \leq g(\phi) \leq M$ for some constants $m$ and $M$, and the general case follows from (ii).

\[ 5.3 \text{ Planar Limit} \]

According to the Fundamental Lemma 5.2.1, the value of the self-linking integral $\text{slk}_\kappa(C)$ does not depend upon the positive parameter $\Lambda > 0$ which sets the width of the Gaussian in the heat form $\chi_\Lambda$. To evaluate $\text{slk}_\kappa(C)$, and in the process to show that $\text{slk}_\kappa(C)$ is invariant under Legendrian isotopy, we now analyze the self-linking integral (5.1.26) in the limit $\Lambda \to \infty$. The Legendrian knot $C \subset \mathbb{R}^3$ and its regular embedding $X : S^1 \to \mathbb{R}^3$ remain fixed throughout.

The limit $\Lambda \to \infty$ has several interpretations.

In terms of the heat kernel, this limit is the short-time limit, in which the Gaussian generator $\omega_\Lambda$ for $H^2_c(\mathbb{R}^2; \mathbb{Z})$ concentrates to a two-form with delta-function support at the origin. More geometrically, by the Scaling Identity in (5.2.9), the limit $\Lambda \to \infty$ is equivalent to the limit $t \to 0$ in which the contact planes represented by $\kappa_t$ in (5.2.3) and the Legendrian knot $C_t$ in (5.2.5) flatten to the $xy$-plane. Simultaneously, the Planck constant $\hbar$ in the Heisenberg multiplication scales to zero, and the abelian structure of $\mathbb{R}^3$ is restored. For this reason, we refer to the limit $\Lambda \to \infty$ as the planar limit.

In the planar limit, the Legendrian self-linking integral simplifies immensely, as can be understood from the formula for the integrand (where non-zero)

\[
(X \times X)^* \hat{\Gamma}_h \chi_\Lambda = \frac{\Lambda}{2\pi \Delta z} e^{-\Lambda(\Delta x^2 + \Delta y^2)/2\Delta z} \times
\]

\[
\times \left[ d\Delta x \wedge d\Delta y + \frac{1}{2} (\Delta x d\Delta y - \Delta y d\Delta x) \wedge \frac{d\Delta z}{\Delta z} \right], \quad \Delta z > 0.
\]

(5.3.1)

Intuitively, the behavior of the integrand is controlled by the exponential factor in the first line of (5.3.1). When $\Lambda$ is sufficiently large, the integrand is negligible away
from the locus where
\[
\left[ \Delta x^2 + \Delta y^2 \right]_{(\theta_1, \theta_2)} \ll \frac{1}{\Lambda}, \quad (\theta_1, \theta_2) \in T^2 - \Delta(\varepsilon). \tag{5.3.2}
\]

Because \( \Delta x \) and \( \Delta y \) are given by the differences
\[
\Delta x = x(\theta_2) - x(\theta_1), \quad \Delta y = y(\theta_2) - y(\theta_1), \tag{5.3.3}
\]
the asymptotic condition in (5.3.2) means that the pair \( \theta_1, \theta_2 \) map under the embedding \( X : S^1 \to \mathbb{R}^3 \) to points \( p, q \in C \) which are nearly coincident under the Lagrangian projection to the \( xy \)-plane. Thus the point \((\theta_1, \theta_2)\) either lies near the preimage of a crossing (aka double-point) on the Lagrangian projection \( \Pi(C) \), or \((\theta_1, \theta_2)\) lies near the diagonal \( \Delta \) itself, in the boundary region that we previously analyzed in Section 5.2.1.

With this observation, our proof of the Main Theorem proceeds in three steps.

1. Estimate the contribution to \( \text{slk}_k(C) \) from each crossing of \( \Pi(C) \) when \( \Lambda \) is large.

2. Estimate the contribution to \( \text{slk}_k(C) \) from the diagonal \( \Delta \subset T^2 \) when \( \Lambda \) is large.

3. Bound the contributions from elsewhere on the integration domain, as well as the errors in the preceding estimates, by a quantity \( \delta \) which can be made arbitrarily small as \( \Lambda \to \infty \).

Conceptually, the local estimates in the first two steps are most important, because these estimates explain why the Thurston-Bennequin invariant \( \text{tb}(C) \) and the rotation number \( \text{rot}(C) \) appear in the formula (5.1.27) for \( \text{slk}_k(C) \). We therefore begin in Section 5.3.1 with simple, informal computations for the first two steps in the proof.

The technical heart of the proof resides in the third step, when we carefully bound the errors in the preceding local computations. This step is required for a rigorous analysis, but the ideas are standard and offer no surprises. For this reason, Sections 5.3.2 and 5.3.3 could be omitted on a first reading of the paper. In Section 5.3.2 we
introduce various geometric quantities to be used in the error analysis, and in Section 5.3.3 we make the necessary bounds.

### 5.3.1 Local computations

We first compute the local contribution to $\text{slk}_\kappa(C)$ from a right-handed crossing in the Lagrangian projection. We depict such a crossing on the left in Figure 5-5. We shall proceed softly, reserving precise inequalities for Section 5.3.3.

![Figure 5-5: Neighborhoods of right- and left-handed crossings of $\Pi(C)$.](image)

To first-order over the double-point, the curve $C$ is approximated by a pair of straight lines. For our local computation, we take the lines to be parametrized by maps $X^\pm : \mathbb{R} \to \mathbb{R}^3$, where $X^+$ passes over $X^-$ by convention. As in the figure, we take $X^\pm$ to describe lines which are perpendicular and lie in parallel planes,

$$X^-(\theta_1) = (0, \theta_1, 0), \quad X^+(\theta_2) = (\theta_2, 0, \Delta z), \quad \theta_{1,2} \in \mathbb{R}. \quad (5.3.4)$$

Here $\Delta z > 0$ is a positive constant, the height of the overpass. With this choice, both $X^\pm$ satisfy the Legendrian condition (4.1.2) and so describe a Legendrian crossing. Because we have yet to establish isotopy-invariance of any kind, our assumptions about even the first-order geometry of $C$ require justification. A significant portion of the analysis in Section 5.3.3 will be devoted exactly to this issue.

The local contribution to $\text{slk}_\kappa(C)$ from the right-handed crossing at $\{0\} \in \mathbb{R}^2$ is now given by

$$\text{slk}_\kappa(C)|_{\{0\}} = \int_{\mathbb{R}^2} \left( X^- \times X^+ \right)^* \tilde{\Gamma}_h \chi_\Lambda, \quad \Lambda \gg 1, \quad (5.3.5)$$
where we integrate over all \((\theta_1, \theta_2) \in \mathbb{R}^2\), with the standard orientation \(d\theta_1 \wedge d\theta_2\). Informally, the error which we make when we extend the range of integration from a small region on \(T^2\) to the entire plane \(\mathbb{R}^2\) vanishes exponentially as \(\Lambda \to \infty\), due to the rapid decay of the heat form \(\chi_\Lambda\) away from the origin. By extending over all of \(\mathbb{R}^2\), we will be able to evaluate the integral (5.3.5) explicitly in closed form.

For the perpendicular lines \(X^\pm\) in (5.3.4), the differences \(\Delta x, \Delta y,\) and \(\hat{\Delta} z\) in (5.3.1) become

\[
\Delta x = x^+ - x^- = \theta_2, \quad \Delta y = y^+ - y^- = -\theta_1, \quad (5.3.6)
\]

and

\[
\hat{\Delta} z = z^+ - z^- + \hbar (x^- y^+ - x^+ y^-) = \Delta z - \hbar \theta_1 \theta_2. \quad (5.3.7)
\]

After a small calculation, one finds for the self-linking integrand

\[
(\mathbf{X}^- \times \mathbf{X}^+)^* \hat{\Gamma}_h^* \chi_\Lambda = \frac{\Lambda}{2\pi \Delta z (1 - \hbar \theta_1 \theta_2 / \Delta z)^2} \exp \left[ -\frac{\Lambda (\theta_1^2 + \theta_2^2)}{2 (\Delta z - \hbar \theta_1 \theta_2)} \right] d\theta_1 \wedge d\theta_2, \quad (5.3.8)
\]

assuming the positivity condition \(\hat{\Delta} z > 0 \Leftrightarrow \Delta z > \hbar \theta_1 \theta_2\) (else the integrand vanishes).

Thus,

\[
\text{slk}_\kappa(C)|_{\{0\}} = \int_{\Delta z > \hbar \theta_1 \theta_2} d\theta_1 d\theta_2 \frac{\Lambda}{2\pi \Delta z (1 - \hbar \theta_1 \theta_2 / \Delta z)^2} \exp \left[ -\frac{\Lambda (\theta_1^2 + \theta_2^2)}{2 (\Delta z - \hbar \theta_1 \theta_2)} \right]. \quad (5.3.9)
\]

Since \(\Lambda \gg 1\) is large, let us rescale the integration variables to eliminate the overall factor of \(\Lambda\) from the argument of the exponential,

\[
\text{slk}_\kappa(C)|_{\{0\}} = \int_{\Lambda \Delta z > \hbar \theta_1 \theta_2} d\theta_1 d\theta_2 \frac{1}{2\pi \Delta z (1 - \hbar \theta_1 \theta_2 / \Lambda \Delta z)^2} \exp \left[ -\frac{\theta_1^2 + \theta_2^2}{2 \Delta z (1 - \hbar \theta_1 \theta_2 / \Lambda \Delta z)} \right]. \quad (5.3.10)
\]

After we expand the integrand of (5.3.10) asymptotically in \(1/\Lambda\), the local contribution to \(\text{slk}_\kappa(C)\) from the right-handed crossing can be evaluated as a Gaussian integral.
over $\mathbb{R}^2$,

$$\text{slk}_\kappa(C)|_{\{0\}} = \int_{\mathbb{R}^2} d\theta_1 d\theta_2 \frac{1}{2\pi \Delta z} \exp \left[ -\frac{\theta_1^2 + \theta_2^2}{2\Delta z} \right] + \mathcal{O}\left(1/\Lambda\right),$$

$$= 1 + \mathcal{O}\left(1/\Lambda\right).$$

Note that all dependence on the homotopy parameter $\hbar$ disappears as soon as we perform the asymptotic expansion in $\Lambda$.

In principle, the contribution from the right-handed crossing in Figure 5-5 also includes the portion of the integration domain $T^2 - \Delta(\varepsilon)$ where the roles of $\theta_1$ and $\theta_2$ are swapped in (5.3.4), with $X^+ \equiv X^+(\theta_1)$ and $X^- \equiv X^-(\theta_2)$. In this case, $\hat{\Delta}z = -\Delta z + \hbar \theta_1 \theta_2 < 0$ is negative near the origin, and the self-linking integrand vanishes identically by the definition of the heat form $\chi_\Lambda$.

Finally, to evaluate the local contribution from the left-handed crossing in Figure 5-5, we simply swap the roles of $X^+$ and $X^-$. Apparently from (5.3.5), this swap is equivalent to an orientation-reversal on $\mathbb{R}^2$, so the sign of $\text{slk}_\kappa(C)|_{\{0\}}$ is reversed. Comparing to our conventions for the writhe in Figure 1-11, we conclude that $\text{slk}_\kappa(C)|_{\{0\}}$ is the local writhe of the given crossing in $\Pi(C)$. In total, the local contribution to $\text{slk}_\kappa(C)$ from the crossings is precisely the Thurston-Bennequin invariant of $C$,

$$\sum_{a \in I} \text{slk}_\kappa(C)|_a = w\left(\Pi(C)\right) = \text{tb}(C),$$

(5.3.12)

where $I$ indexes the set of all crossings in the Lagrangian projection.

The localization computation is also consistent with Proposition 5.1.1 regarding Heisenberg linking, together with the diagrammatic formula for $\text{lk}(C_1, C_2)$ in (1.3.50).

More interesting is the local contribution to $\text{slk}_\kappa(C)$ from the diagonal $\Delta \subset T^2$. This contribution does depend (weakly) on the value of $\hbar$, a small remnant of the topological anomaly. Integrating over a neighborhood of the diagonal really means integrating over the two boundary regions on the cylinder $T^2 - \Delta(\varepsilon)$, as we have already considered in our proof of the Fundamental Lemma in Section 5.2. So we do not need to perform any new computations to evaluate the contribution from the
We begin with the generic case \( h \neq 1 \), for which the local expression for the self-linking integrand appears in (5.2.28). Directly for \( \Lambda \gg 1 \),

\[
\begin{align*}
\text{slk}_κ(C)_Δ \Big|_{h \neq 1} &= - \int_{S^1_+ \cap [(1-h)(γ×\dot{γ})<0]} dφ \int_0^∞ dη \frac{Λ (\dot{γ} × \ddot{γ})}{4π |1-h| |γ × \dot{γ}|} \exp \left(- \frac{1}{2} \frac{Λ ||\dot{γ}||^2 η}{|1-h||γ × \dot{γ}|} \right) \\
&- \int_{S^1_+ \cap [(1-h)(γ×\dot{γ})<0]} dφ \int_0^∞ dη \frac{Λ (\dot{γ} × \ddot{γ})}{4π |1-h| |γ × \dot{γ}|} \exp \left(- \frac{1}{2} \frac{Λ ||\dot{γ}||^2 η}{|1-h||γ × \dot{γ}|} \right) .
\end{align*}
\]

The integrals in the two lines of (5.3.13) describe the respective local contributions to \( \text{slk}_κ(C) \) from collar neighborhoods of the boundary circles \( S^1_\pm \) on the cylinder in Figure A.1. Both integrals appear with identical signs, after one takes into account the relative boundary orientations on \( S^1_\pm \) and the explicit dependence of the integrand on \( \text{sgn}(η) \) in (5.2.28).\(^3\) By the positivity condition in (5.2.24), the integral over \( S^1_+ \) runs over the subset where \( (1-h)(γ × \dot{γ}) < 0 \), and the integral over \( S^1_- \) runs over the complement. Finally, we extend the integration range over the normal coordinate \( η \) to infinity at the cost of an exponentially small error for large \( \Lambda \), and we set \( ε = 0 \) at the lower limit of integration.

After integrating over \( η \) in (5.3.13),

\[
\begin{align*}
\text{slk}_κ(C)_Δ \Big|_{h \neq 1} &= - \frac{1}{2π} \int_{S^1_+ \cap [(1-h)(γ×\dot{γ})<0]} dφ \frac{\dot{γ} × \ddot{γ}}{||\dot{γ}||^2} - \frac{1}{2π} \int_{S^1_+ \cap [(1-h)(γ×\dot{γ})>0]} dφ \frac{\dot{γ} × \ddot{γ}}{||\dot{γ}||^2} ,
\end{align*}
\]

or put more succinctly,

\[
\text{slk}_κ(C)_Δ \Big|_{h \neq 1} = - \frac{1}{2π} \int_{S^3} dφ \frac{\dot{γ} × \ddot{γ}}{||\dot{γ}||^2} .
\]

(5.3.15)

So long as \( h \neq 1 \), for which the expressions in (5.3.13) are obviously inadequate, all dependence on \( h \) disappears. Comparing to the geometric expression for the rotation

---

\(^3\)We make a trivial change of variables so that the integral in the neighborhood of \( S^1_- \) also runs over positive, as opposed to negative, values of \( η \).
number in (5.1.9), we deduce

\[ \text{slk}_\kappa(C) \bigg|_{\Delta} \overset{h \neq 1}{=} \text{rot}(C). \]  

(5.3.16)

On general grounds, the appearance of the rotation number in this calculation is not so surprising, as one was guaranteed to find some local geometric invariant of \( C \). However, the integrality of the result (5.3.16) comes as a minor miracle, which is far from obvious from the definition of the self-linking integral in (5.1.26). Remember, the naive Gauss self-linking integral \( \text{slk}_0(C) \) is not even a deformation-invariant!

We return to our formula in (5.2.29) to evaluate the local self-linking contribution from the diagonal when \( h = 1 \),

\[ (X \times X)^* \tilde{\Gamma}_h^* \chi_{\Lambda} \bigg|_{\Delta(\varepsilon)} \overset{h = 1}{=} \frac{-3 \Lambda}{2\pi \eta^2} \exp \left[ -\frac{3 \Lambda \norm{\dot{\gamma}}^2}{\norm{\dot{\gamma} \times \ddot{\gamma}} \norm{\eta}} \right] d\phi \wedge d\eta + \cdots. \]  

(5.3.17)

Unlike the expressions in (5.3.13), which are non-zero for \( \eta = 0 \), the self-linking integrand in (5.3.17) vanishes exponentially as \( \eta \to 0 \) for any \( \Lambda > 0 \). By inspection we conclude

\[ \text{slk}_\kappa(C) \bigg|_{\Delta} \overset{h \equiv 1}{=} 0. \]  

(5.3.18)

If the Heisenberg symmetry is preserved, the diagonal does not contribute to the Legendrian self-linking integral.

At least informally, modulo precise control of the error terms, we obtain from these local computations the statement in the Main Theorem,

\[ \text{slk}_\kappa(C) = \lim_{\Lambda \to \infty} \text{slk}_\kappa(C) = \sum_{a \in I} \text{slk}_\kappa(C) \bigg|_a + \text{slk}_\kappa(C) \bigg|_{\Delta}, \]

\[ = \begin{cases} \text{tb}(C) - \text{rot}(C), & h \neq 1, \\ \text{tb}(C), & h = 1. \end{cases} \]  

(5.3.19)

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5.3.2 Some preliminaries

The informal localization computation in Section 5.3.1 is useful for developing geometric intuition about the behavior of the Legendrian self-linking integral. To prove the Main Theorem, we retrace the same route philosophically, but we exercise greater care in analyzing the error terms and their dependence on $\Lambda$, at least when $\Lambda$ is large.

Before we establish precise inequalities in Section 5.3.3, we need to introduce a bevy of constants related to the geometry of the knot $C \subset \mathbb{R}^3$ and its Lagrangian projection $\Pi(C)$. These constants are important, as the required bounds fall out naturally from them.

Local Neighborhoods of Crossings

The notion of a collar neighborhood for the boundary of $T^2 - \Delta(\varepsilon)$ is unambiguous, but we also need a proper notion for the neighborhood of each crossing in $\Pi(C)$. The trick will be to choose these neighborhoods to be small enough so that the geometry of $C$ is controlled over each neighborhood. Informally, we treated this issue by linearizing $C$ in Figure 5-5, but now we work nonlinearly.

As before, $\gamma : S^1 \to \mathbb{R}^2$ is the regular immersed plane curve which is the Lagrangian projection of the embedding $X : S^1 \to \mathbb{R}^3$,

$$\gamma = \Pi \circ X, \quad \gamma(\theta) \equiv (x(\theta), y(\theta)).$$  \hfill (5.3.20)

By assumption, $\gamma$ has only a finite number $n$ of simple, double-point singularities, which are located at positions

$$\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{R}^2. \hfill (5.3.21)$$

See Figure 4-1 for our canonical trefoil example, with $n = 5$. Each crossing $\gamma_a$ for $a = 1, \ldots, n$ lies under a pair of corresponding points $(p_a, q_a)$ on the knot $C$,

$$\gamma_a = \Pi(p_a) = \Pi(q_a), \quad p_a, q_a \in C. \hfill (5.3.22)$$
Figure 5-6: Two points $p_a$ and $q_a$ on $C$ with coincident Lagrangian projections.

Let $z_a$ and $z'_a$ be the respective heights of $p_a$ and $q_a$, so that these points have coordinates in $\mathbb{R}^3$ given by

$$p_a = (\gamma_a, z_a), \quad q_a = (\gamma_a, z'_a). \quad (5.3.23)$$

As in the informal computation, an important geometric quantity is the absolute difference $\Delta z_a$ in the heights of $p_a$ and $q_a$ over the crossing,

$$\Delta z_a = |z_a - z'_a| > 0. \quad (5.3.24)$$

See Figure 5-6 for a local (nonlinear) picture of $C$ near the points $p_a$ and $q_a$.

Let $D(\gamma_a; h) \equiv D_a(h) \subset \mathbb{R}^2$ be the open disc of radius $h > 0$ centered at the location $\gamma_a$ of a given crossing in the plane. The union $\cup D_a(h)$ of these discs, each with the same radius $h$, provides an open neighborhood for all crossings in $\Pi(C)$. We now choose $h > 0$ to be sufficiently small so that the following statements are true at each crossing. By continuity of $X$ and compactness of the closure $\overline{D_a(h)}$, such a choice is always possible.

For ease of notation, we suppress the crossing index ‘a’ below.

1. The disc $D(h)$ intersects the immersed plane curve $\gamma$ in two arcs, as shown in Figure 5-7. We denote these arcs by $\gamma^+$ and $\gamma^-$, where ‘$\pm$’ indicate the respective upper and lower strands at the crossing. Over the disc, the embedding $X$ restricts to a pair of maps $X^-(\theta_1) = (\gamma^-(\theta_1), z^-(\theta_1))$ and $X^+(\theta_2) = (\gamma^+(\theta_2), z^+(\theta_2))$, with $z^+ > z^-$. Here $X^\pm$ are nonlinear analogues of the expressions in (5.3.4).

2. With the same arcs $\gamma^\pm$ in mind, let $\Gamma_{2d} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be the two-dimensional
difference map
\[ \Gamma_{2d}(u, v) = v - u. \quad (5.3.25) \]

Consider the composition
\[ \varphi = \Gamma_{2d} \circ (\gamma^- \times \gamma^+) = \gamma^+ - \gamma^-, \quad (5.3.26) \]

which maps the region on \( T^2 \) where \( \gamma^- \times \gamma^+ \) is locally defined to another region on the \( uv \)-plane. Then \( \varphi \equiv (u(\theta_1, \theta_2), v(\theta_1, \theta_2)) \) is a diffeomorphism to a curvy quadrilateral region \( Q \) around the origin in the \( uv \)-plane, as in Figure 5-8.

Eventually, we will use \( \varphi \) to make a change-of-variables to simplify the self-linking integrand in the neighborhood of the crossing.

3. Let \( J_\varphi \) be the Jacobian for the change-of-variables induced by \( \varphi \) from \((\theta_1, \theta_2)\) to \((u, v)\),
\[ J_\varphi = \left| \frac{d\gamma^+}{d\theta_2} \times \frac{d\gamma^-}{d\theta_1} \right|. \quad (5.3.27) \]

Since \( \varphi \) is a diffeomorphism, \( J_\varphi \neq 0 \) is non-vanishing throughout the domain of \( \varphi \), as illustrated in Figure 5-9. We go slightly further and assume that \( J_\varphi \) is
uniformly bounded from below by a positive constant

\[ 0 < J_0 < J_\varphi. \quad (5.3.28) \]

Figure 5-9: The Jacobian \( J_\varphi \neq 0 \) of \( \varphi \).

4. For the crossing labelled by ‘\( a \)’, consider all pairs of points on \( C \subset \mathbb{R}^3 \) which lie in the image of \( X^+_a \times X^-_a \) over the disc \( D_a(h) \). Then the difference in heights \( |z^+_a - z^-_a| \) for all such pairs lies in the range

\[ \left( 1 - \frac{c}{2} \right) \Delta z_a < |z^+_a - z^-_a| < \left( 1 + \frac{c}{2} \right) \Delta z_a. \quad (5.3.29) \]

Here \( c \) is a positive constant, independent of \( a \), bounded by

\[ 0 < c < \frac{1}{2}. \quad (5.3.30) \]

This assumption in (5.3.29) implies that the difference \( |z^+_a - z^-_a| \) for all pairs of points on \( C \) projecting to \( D_a(h) \) obeys\(^4\)

\[ 1 - c < \frac{\Delta z_a}{|z^+_a - z^-_a|} < 1 + c, \quad (5.3.31) \]

for \( c \) in the given range. Informally, the constant \( c \) controls the variation in the vertical separation between the two strands of \( C \) over the disc \( D_a(h) \), relative to the separation \( \Delta z_a \) over the crossing itself. The constant ‘1/2’ in the bound (5.3.30) is a convenient choice related to other inequalities later.

**Bounds on the Integrand**

\(^4\)The bound in (5.3.31) is not sharp but will suffice for us.
Our assumption about the radius \( h \) of the discs \( D_a(h) \) gives us adequate control on the local geometry of \( C \) above each crossing \( \gamma_a \). We now introduce further constants related to the magnitude of the self-linking integrand itself.

First, let \( Z > 0 \) be the height of the knot \( C \subset \mathbb{R}^3 \). More formally, \( Z \) is the maximum vertical separation between any pair of points on \( C \),

\[
Z = \max_{(\theta_1, \theta_2) \in T^2} \left| z(\theta_2) - z(\theta_1) \right|.
\]  

We now fix a small positive constant \( \delta > 0 \) which will control the errors. In reference to the dependence of the heat form \( \chi_{\Lambda} \) on \( r^2 = x^2 + y^2 \) and \( z \) in (5.1.18), we note the following lemma.

**Lemma 5.3.1.** Given \( \delta > 0 \) and sufficiently large \( \Lambda \), there exists a constant \( r_\Lambda > 0 \), depending on \( \Lambda \), so that for all \( r > r_\Lambda \),

\[
\sup_{0 < z \leq Z} \left[ \frac{\Lambda}{2\pi z} \exp \left( - \frac{\Lambda r^2}{2z} \right) \right] < \delta,
\]

and

\[
\sup_{0 < z \leq Z} \left[ \frac{\Lambda}{4\pi z^2} \exp \left( - \frac{\Lambda r^2}{2z} \right) \right] < \delta.
\]

**Proof.** The proof of the lemma is elementary, but we wish to gain precise knowledge about how \( r_\Lambda \) must depend upon \( Z \) and \( \Lambda \) for the bounds to hold. The bounds in (5.3.33) and (5.3.34) can be treated similarly; we start with the bound in (5.3.33).

We differentiate the function in (5.3.33) with respect to \( z \),

\[
\frac{\partial}{\partial z} \left[ \frac{\Lambda}{2\pi z} \exp \left( - \frac{\Lambda r^2}{2z} \right) \right] = \frac{\Lambda}{2\pi z^2} \left( \frac{r^2}{2z} - 1 \right) \exp \left( - \frac{\Lambda r^2}{2z} \right). \tag{5.3.35}
\]

The derivative in (5.3.35) is positive so long as

\[
\frac{\Lambda r^2}{2z} > 1, \tag{5.3.36}
\]
which in turn is equivalent to
\[ r^2 > \frac{2Z}{\Lambda}. \]  

(5.3.37)

We will actually require a stronger bound on \( r \) in regard to its dependence on \( \Lambda \).

We set
\[ r_\Lambda^2 = \frac{2Z}{\sqrt{\Lambda}}. \]  

(5.3.38)

So long as \( \Lambda > 1 \), the condition \( r > r_\Lambda \) implies the bound in (5.3.37), so that
\[ \frac{\partial}{\partial z} \left[ \frac{\Lambda}{2\pi z} \exp \left( - \frac{\Lambda r^2}{2z} \right) \right] > 0. \]  

(5.3.39)

The function in (5.3.33) therefore increases with \( z \) for all \( r > r_\Lambda \), and the supremum is achieved at the value \( z = Z \),
\[ \sup_{0 < z \leq Z} \left\{ \frac{\Lambda}{2\pi z} \exp \left( - \frac{\Lambda r^2}{2z} \right) \right\} = \frac{\Lambda}{2\pi Z} \exp \left( - \frac{\Lambda r_\Lambda^2}{2Z} \right), \quad r > r_\Lambda. \]  

(5.3.40)

For \( r > r_\Lambda \), the argument of the exponential in (5.3.40) satisfies
\[ \frac{\Lambda r^2}{2Z} > \frac{\Lambda r_\Lambda^2}{2Z} = \sqrt{\Lambda}. \]  

(5.3.41)

Had we imposed the weak inequality in (5.3.37), the quantity \( \Lambda r^2/2Z \) would have been bounded from below only by a constant, independent of \( \Lambda \), and we would have no chance to achieve the bound by \( \delta \) in (5.3.33). Instead, with the strong inequality \( r > r_\Lambda \) in (5.3.38),
\[ \frac{\Lambda}{2\pi Z} \exp \left( - \frac{\Lambda r^2}{2Z} \right) < \frac{\Lambda}{2\pi Z} \exp \left( -\sqrt{\Lambda} \right). \]  

(5.3.42)

For any positive \( \delta > 0 \), we now take \( \Lambda \) sufficiently large so that
\[ \frac{\Lambda}{2\pi Z} \exp \left( -\sqrt{\Lambda} \right) < \delta, \]  

(5.3.43)

implying via (5.3.40) and (5.3.42) the desired inequality in (5.3.33).

The analysis of the function in (5.3.34) is identical, up to a factor of 2, due to appearance of the same Gaussian factor. In this case, we take \( r_\Lambda^2 = 4Z/\sqrt{\Lambda} \). To treat
both cases of Lemma 5.3.1 simultaneously, we set

\[ r_\Lambda^2 = \max \left\{ \frac{2Z}{\sqrt{\Lambda}}, \frac{4Z}{\sqrt{\Lambda}} \right\} = \frac{4Z}{\sqrt{\Lambda}}. \]  
(5.3.44)

With this choice for \( r_\Lambda \), the lemma follows. \( \square \)

We require one additional quantity to make our bounds on the error. This quantity will be a function of \( \Lambda \) which arises in reference to the magnitude of the Gaussian integrand (5.3.11) at a crossing. Briefly, for given \( \delta \) and crossing index ‘a’, we let \( R_a(\Lambda) > 0 \) be the solution to

\[ \frac{\Lambda}{2\pi\Delta z_a} \exp \left[ -\Lambda R_a(\Lambda)^2 \right] = \delta. \]  
(5.3.45)

Explicitly,

\[ R_a(\Lambda)^2 = \frac{2\Delta z_a}{\Lambda} \ln \left( \frac{\Lambda}{2\pi\Delta z_a \delta} \right). \]  
(5.3.46)

As a function of \( \Lambda \), \( R_a(\Lambda) \) is decreasing for sufficiently large \( \Lambda \) and vanishes in the limit \( \Lambda \to \infty \).

For the convenience of the reader, we summarize all these constants in Table 5.1.

### 5.3.3 Bounds on the error

We now establish the necessary bounds to prove the Main Theorem.

The most fundamental bound emerges trivially from the definition of the constant \( r_\Lambda \) in Lemma 5.3.1. Let \( \mathcal{U} \) and \( \mathcal{V} \) be the subsets of the cylinder \( T^2 - \Delta(\varepsilon) \) defined by

\[ \mathcal{U} = \left\{ (\theta_1, \theta_2) \in T^2 - \Delta(\varepsilon) \left| \left[ \Delta x^2 + \Delta y^2 \right]_{(\theta_1, \theta_2)} \leq r_\Lambda^2 \right. \right\}, \]  
(5.3.47)

and

\[ \mathcal{V} = \left\{ (\theta_1, \theta_2) \in T^2 - \Delta(\varepsilon) \left| \left[ \Delta x^2 + \Delta y^2 \right]_{(\theta_1, \theta_2)} > r_\Lambda^2 \right. \right\}. \]  
(5.3.48)

That is, \( \mathcal{U} \) consists of those pairs of points \( (\theta_1, \theta_2) \) whose images in the \( xy \)-plane under the map \( \Pi \circ (X \times X) \) lie within the critical distance \( r_\Lambda \), and \( \mathcal{V} \) consists of those pairs
Width of Gaussian in \( \chi_\Lambda \).

Any positive constant which only depends upon \( C \subset \mathbb{R}^3 \).

The values of \( M \) and \( m \) may differ at various places in the text.

A fixed, small positive constant. Any error less than \( M \delta \) is negligible.

Vertical displacement of \( C \) over each crossing \( \gamma_a \in \mathbb{R}^2 \).

Radius of disc \( D_a(h) \subset \mathbb{R}^2 \) centered at each crossing.

Lower bound \( 0 < J_0 < J_\varphi \) for the Jacobian of \( \varphi \).

Positive constant \(< 1/2\) for which \( 1 - c < \Delta z_a / |z^+_a - z^-_a| < 1 + c \).

Total height of \( C \subset \mathbb{R}^3 \).

A positive number given by \( r^2_\Lambda = 4Z / \sqrt{\Lambda} \). For sufficiently large \( \Lambda \) and \( r > r_\Lambda \), the inequalities in Lemma 5.3.1 are true.

Positive solution to \((\Lambda / 2\pi \Delta z_a) \exp[-\Lambda R_a(\Lambda)^2 / 2\Delta z_a] = \delta.\)

Table 5.1: List of Constants.

Separated in the \( xy \)-plane by a distance greater than \( r_\Lambda \).

Since \( \mathcal{U} \) and \( \mathcal{V} \) are complementary subsets of the cylinder, the self-linking integral can be written as the sum

\[
\text{slk}_\kappa(C) = \int_{\mathcal{U}} (X \times X)^* \hat{\Gamma}_h^* \chi_\Lambda + \int_{\mathcal{V}} (X \times X)^* \hat{\Gamma}_h^* \chi_\Lambda. \tag{5.3.49}
\]

By the very definition of \( r_\Lambda \) in Lemma 5.3.1, the magnitude of the self-linking integrand (5.3.1) on \( \mathcal{V} \) is everywhere bounded by \( \delta \). Automatically,

\[
\left| \text{slk}_\kappa(C) - \int_{\mathcal{U}} (X \times X)^* \hat{\Gamma}_h^* \chi_\Lambda \right| = \left| \int_{\mathcal{V}} (X \times X)^* \hat{\Gamma}_h^* \chi_\Lambda \right| < M \delta. \tag{5.3.50}
\]

Here \( M \) is a constant, depending on the geometry of the knot \( C \subset \mathbb{R}^3 \) but independent of \( \Lambda \). For small \( \delta \), the value of \( \text{slk}_\kappa(C) \) is well-approximated by the integral over the subset \( \mathcal{U} \).

Our next task is to characterize the points in the domain of integration that lie in \( \mathcal{U} \). If we formally set \( r_\Lambda = 0 \) in (5.3.47), then \( \mathcal{U} \) consists of those pairs \((\theta_1, \theta_2)\) which become coincident after projection to the \( xy \)-plane. Such pairs either lie along the
diagonal $\Delta \subset T^2$, or they lie in the preimage of a crossing in the Lagrangian projection of $C$. When $r_\Lambda > 0$ is positive but sufficiently small, these closed sets fatten, and $U$ is contained within the disjoint union of a tubular neighborhood $N_\Delta(w)$ of the diagonal and a collection of open balls $B_b(h) \subset T^2$ associated to the crossings,

$$U \subset N_\Delta(w) \cup \bigcup_{b=1}^{2n} B_b(h).$$

See Figure 5-10 for a schematic picture of the open set containing $U$ for sufficiently small $r_\Lambda$. As in the picture, each crossing in $\Pi(C)$ has two preimages on $T^2$, which are exchanged when the roles of $\theta_1$ and $\theta_2$ swap. Thus if $\Pi(C)$ has $n$ crossings, the index ‘$b$’ on the balls $B_b(h)$ runs to $2n$.

In writing $B_b(h)$ for the open ball in $T^2$, we abuse notation somewhat. By assumption, the radius of $B_b(h)$ is fixed so that this ball lies in the preimage of the corresponding disc $D_a(h) \subset \mathbb{R}^2$ under the map $\gamma_a^- \times \gamma_a^+$ in Figure 5-7,

$$\big(\gamma_a^- \times \gamma_a^+(h)\big) \subset D_a(h), \quad a \equiv b \mod n.$$

Therefore the radius of $B_b(h)$ is not necessarily equal to $h$, but it is determined by $h$ independently of $\Lambda$. Once $h$ is fixed in terms of the geometry of $C$, we can always take $\Lambda$ sufficiently large and $r_\Lambda \sim \Lambda^{-1/4}$ sufficiently small so that $B_b(h)$ contains the relevant portion of $U$.

Similarly, the tubular neighborhood $N_\Delta(w)$ of the diagonal has width $w > 0$, meaning that points in $N_\Delta(w)$ satisfy $|\theta_2 - \theta_1| < w$ for the parameters in Figure 5-10. A
crucial step will be to fix the value of \( w \), which must be large enough so that \( N_\Delta(w) \) contains the piece of \( \mathcal{U} \) near the diagonal, but also small enough so that the local analysis from Section 5.2.1 is applicable everywhere in \( N_\Delta(w) \).

According to the next lemma, both conditions on \( N_\Delta(w) \) can be simultaneously satisfied once we set

\[
w = \frac{r_\Lambda \sqrt{2}}{m},
\]

with a positive constant \( m > 0 \) defined geometrically by

\[
m^2 = \min_{\theta \in S^1} \left[ |\dot{\gamma}(\theta)|^2 \right].
\]

Since \( \gamma = \Pi \circ X \) is an immersion, the minimum speed in (5.3.54) is bounded away from zero, which is essential. The constant \( \sqrt{2} \) is inessential and could be absorbed into the definition of \( m \).

**Lemma 5.3.2.** For sufficiently large \( \Lambda \) and the given value of \( w \), the tubular neighborhood \( N_\Delta(w) \) contains the diagonal component of \( \mathcal{U} \).

**Proof.** The lemma says that, away from crossings, all points on the projection \( \Pi(C) \) which lie within the distance \( r_\Lambda \) of a given point \( \gamma(\theta) \) are contained within the image of the interval \([\theta - r_\Lambda \sqrt{2}/m, \theta + r_\Lambda \sqrt{2}/m]\) under \( \gamma \), for all values of \( \theta \). See Figure 5-11 for an illustration of this claim. The minimum speed along \( \gamma \) naturally sets the scale of the required interval, which decreases with increasing \( m \).

![Figure 5-11: Points within the distance \( r_\Lambda \) of \( \gamma(\theta) \).](image)

We prove the lemma using a bit of geometry for plane curves. Let \( t \) be the unit tangent vector for \( \gamma \),

\[
t = \frac{\dot{\gamma}}{|\dot{\gamma}|},
\]

(5.3.55)
and let $n$ be the unit normal,

$$\gamma' = Kn, \quad K = \frac{\gamma' \times \gamma'}{||\gamma'||^3},$$

with plane curvature $K$.

For any fixed $\theta_0 \in S^1$, define the function

$$F(\theta) = \frac{d}{d\theta} ||\gamma(\theta) - \gamma(\theta_0)||^2 = 2 \left(\gamma(\theta) - \gamma(\theta_0)\right) \cdot \gamma'(\theta).$$

Then $F(\theta_0) = 0$, and

$$\frac{dF}{d\theta}(\theta) = 2 ||\gamma'||^2 + 2 \left(\gamma(\theta) - \gamma(\theta_0)\right) \cdot \gamma'(\theta),$$

from the definition of the unit normal. Here $||\gamma'|| \geq m$ is bounded below, $|K| < \infty$ is bounded above, and $||n|| = 1$. So if $||\gamma(\theta) - \gamma(\theta_0)|| \leq r_\Lambda$ for sufficiently large $\Lambda$ and small $r\Lambda$, then

$$\frac{dF}{d\theta}(\theta) > m^2.$$

Integrating the inequality in (5.3.59), we obtain

$$F(\theta) > m^2 |\theta - \theta_0|.$$

Integrating once more from the definition (5.3.57) of $F(\theta)$,

$$||\gamma(\theta) - \gamma(\theta_0)||^2 > \frac{1}{2} m^2 |\theta - \theta_0|^2,$$

or

$$||\gamma(\theta) - \gamma(\theta_0)|| > \frac{m}{\sqrt{2}} |\theta - \theta_0|.$$

Thus if $||\gamma(\theta) - \gamma(\theta_0)|| = r_\Lambda$, then $|\theta - \theta_0| < r_\Lambda \sqrt{2}/m$, which is the required bound. \hfill \square
Because $\mathcal{U}$ is contained in the union of $N_\Delta(w)$ and $\bigcup B_b(h)$, we have a relation between the corresponding self-linking integrals,

$$\left| \int_\mathcal{U} (X \times X)^* \hat{\Gamma}_h^* \chi_\Lambda \right| - \int_{N_\Delta(w)} (X \times X)^* \hat{\Gamma}_h^* \chi_\Lambda - \sum_{b=1}^{2n} \int_{B_b(h)} (X \times X)^* \hat{\Gamma}_h^* \chi_\Lambda < M \delta .$$  

(5.3.63)

This inequality again follows from Lemma 5.3.1, because points contained in either $N_\Delta(w)$ or $B_b(h)$ but not in $\mathcal{U}$ lie in $\mathcal{V}$, where the magnitude of the self-linking integrand is bounded by $\delta$. So for sufficiently large $\Lambda$, we just need to evaluate the self-linking integral over the balls $B_b(h)$ and the tubular neighborhood $N_\Delta(w)$ in Figure 5-10.

**Error analysis at a crossing**

We first evaluate the self-linking integral over the ball $B_b(h)$. By the positivity condition on the heat form $\chi_\Lambda$, the self-linking integrand vanishes in exactly half the balls. Reshuffling indices as necessary, we consider only those $B_a(h)$ for $a = 1, \ldots, n$ on which $\hat{\Delta}z > 0$ and the integrand is non-zero.

With malice aforethought, we have arranged that the image of $B_a(h)$ under the product map $\gamma_a^{-} \times \gamma_a^{+}$ lies in the disc $D_a(h) \subset \mathbb{R}^2$, where we have control over the geometry of $C$. In particular, the map $\varphi_a$ in Figure 5-8 restricts to a diffeomorphism from $B_a(h)$ to a curvy quadrilateral region $Q$ about the origin in the $uv$-plane,

$$\varphi_a = \gamma_a^{+}(\theta_2) - \gamma_a^{-}(\theta_1) \equiv (u(\theta_1, \theta_2), v(\theta_1, \theta_2)) .$$  

(5.3.64)

Our analysis will be performed using the $uv$-coordinates because in these coordinates the self-linking integrand (5.3.1) simplifies,

$$\frac{\Lambda}{2\pi \hat{\Delta}z(u,v)} e^{-\Lambda(u^2 + v^2)/(2\hat{\Delta}z)} \left[ d\Delta x \wedge d\Delta y + \frac{1}{2} (\Delta x d\Delta y - \Delta y d\Delta x) \wedge \frac{d\hat{\Delta}z}{\hat{\Delta}z} \right]$$

$$= \frac{\Lambda}{2\pi \hat{\Delta}z(u,v)} e^{-\Lambda(u^2 + v^2)/(2\hat{\Delta}z(u,v))} \left[ du \wedge dv + \frac{1}{2} (u dv - v du) \wedge \frac{d\hat{\Delta}z(u,v)}{\hat{\Delta}z(u,v)} \right].$$  

(5.3.65)

Via (5.3.64), $\Delta x$ and $\Delta y$ are identified with the Cartesian coordinates $u$ and $v$, and $\hat{\Delta}z$ is considered to be a function of $(u,v)$. All unknown functional dependence of the self-linking integrand in (5.3.65) is absorbed into $\hat{\Delta}z(u,v)$.

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The integrand in (5.3.65) is a sum of the two terms,

$$\Psi = \frac{\Lambda}{2\pi \Delta z(u, v)} e^{-\Lambda(u^2 + v^2)/2\Delta z(u, v)} du \wedge dv,$$

(5.3.66)

and

$$\Xi = \frac{\Lambda}{4\pi \Delta z(u, v)} e^{-\Lambda(u^2 + v^2)/2\Delta z(u, v)} (u du - v dv) \wedge \frac{d\Delta z(u, v)}{\Delta z(u, v)}.$$

(5.3.67)

Hence after making the change-of-variables in \(Q\),

$$\int_{B_a(h)} (X \times X) \hat{\Gamma}_+ \chi_{\Lambda} = \int_Q \Psi + \int_Q \Xi.$$

(5.3.68)

The analysis of the two terms on the right in (5.3.68) is different. Morally, \(\Xi\) is higher-order in \(u\) and \(v\) so will be irrelevant when \(\Lambda\) is large and \(u, v \ll 1/\Lambda\). By contrast, \(\Psi\) is always relevant. We analyze the integrals of \(\Psi\) and \(\Xi\) over \(Q\) in turn.

Explicitly, the integral of \(\Psi\) is given by a kind of nonlinear Gaussian,

$$\int_Q \Psi = \deg(\varphi_a) \int_Q \frac{\Lambda}{2\pi \Delta z(u, v)} \exp \left[ -\frac{\Lambda (u^2 + v^2)}{2 \Delta z(u, v)} \right] du \wedge dv, \quad \Delta z(u, v) > 0.$$

(5.3.69)

Here \(\deg(\varphi_a) = \pm 1\) depending upon whether the diffeomorphism \(\varphi_a : B_a(h) \to Q\) preserves or reverses orientation. Equivalently, from the expression in (5.3.64), the sign is determined by the Jacobian in the expansion

$$du \wedge dv = -\left( \frac{d\gamma^-}{d\theta_1} \times \frac{d\gamma^+}{d\theta_2} \right) d\theta_1 \wedge d\theta_2.$$  

(5.3.70)

By inspection of Figure 5-12, \(\deg(\varphi_a)\) is exactly the local writhe at the given crossing,

$$\deg(\varphi_a) = w_a.$$  

(5.3.71)

To suppress pernicious signs for the remainder, we assume \(w_a = +1\).

For large \(\Lambda\), we evaluate the integral of \(\Psi\) in two steps.

1. We replace the unknown function \(\Delta z(u, v)\) by the constant displacement \(\Delta z_a\)
at the crossing, with error
\[
\left| \int_{\mathcal{Q}} \frac{\Lambda}{2\pi \Delta z(u,v)} \exp \left[ -\frac{\Lambda (u^2 + v^2)}{2 \Delta z(u,v)} \right] du \, dv - \int_{\mathcal{Q}} \frac{\Lambda}{2\pi \Delta z_a} \exp \left[ -\frac{\Lambda (u^2 + v^2)}{2 \Delta z_a} \right] du \, dv \right| < M\delta. \tag{5.3.72}
\]

2. We extend the range of Gaussian integration from $\mathcal{Q}$ to $\mathbb{R}^2$ so that the Gaussian integral can be performed analytically, with error
\[
\left| \int_{\mathcal{Q}} \frac{\Lambda}{2\pi \Delta z_a} \exp \left[ -\frac{\Lambda (u^2 + v^2)}{2 \Delta z_a} \right] du \, dv - \int_{\mathbb{R}^2} \frac{\Lambda}{2\pi \Delta z_a} \exp \left[ -\frac{\Lambda (u^2 + v^2)}{2 \Delta z_a} \right] du \, dv \right| < \delta. \tag{5.3.73}
\]

Of these steps, only the first is non-trivial. For the second, because the Gaussian integral over $\mathbb{R}^2$ is normalized to unity independent of $\Lambda$, we can always choose $\Lambda$ sufficiently large so that
\[
\int_{\mathbb{R}^2-\mathcal{Q}} \frac{\Lambda}{2\pi \Delta z_a} \exp \left[ -\frac{\Lambda (u^2 + v^2)}{2 \Delta z_a} \right] du \, dv < \delta. \tag{5.3.74}
\]
Informally, we performed both these steps in arriving at (5.3.11).

For the first step, we use heavily the positive function $R_a(\Lambda)$ which satisfies
\[
\frac{\Lambda}{2\pi \Delta z_a} \exp \left[ -\frac{\Lambda R_a(\Lambda)^2}{2 \Delta z_a} \right] = \delta, \tag{5.3.75}
\]
and vanishes monotonically as $\Lambda \to \infty$. The function $R_a(\Lambda)$ sets the minimum distance from the origin for which the Gaussian integrand in (5.3.72) becomes negligible.

To be on the safe side in our bounds, we will have to work with a slightly larger
distance $R_a(\Lambda/2) > R_a(\Lambda)$. Let $B_0 \equiv B_0(R_a(\Lambda/2))$ be the ball of radius $R_a(\Lambda/2)$ which is centered at the origin in $Q$,

$$B_0 : \quad u^2 + v^2 < R_a^2(\Lambda/2). \quad (5.3.76)$$

We shall prove that when $(u, v)$ lies in $B_0$, the difference between the nonlinear and the usual Gaussian in (5.3.72) is small,

$$\left| \int_{B_0} \frac{\Lambda}{2\pi\Delta z(u,v)} \exp \left[ -\frac{\Lambda(u^2 + v^2)}{2\Delta z(u,v)} \right] du \, dv - \int_{B_0} \frac{\Lambda}{2\pi\Delta z_a} \exp \left[ -\frac{\Lambda(u^2 + v^2)}{2\Delta z_a} \right] du \, dv \right| < \delta. \quad (5.3.77)$$

Otherwise, when $(u, v)$ lies outside $B_0$ in $Q$, we show that both integrals are separately small, with

$$\int_{Q-B_0} \frac{\Lambda}{2\pi\Delta z(u,v)} \exp \left[ \frac{\Lambda(u^2 + v^2)}{2\Delta z(u,v)} \right] du \, dv < M \delta, \quad (5.3.78)$$

and

$$\int_{Q-B_0} \frac{\Lambda}{2\pi\Delta z_a} \exp \left[ -\frac{\Lambda(u^2 + v^2)}{2\Delta z_a} \right] du \, dv < M \delta. \quad (5.3.79)$$

Thus the difference must also be small in $Q - B_0$. This trick is the engine of asymptotic analysis. See Ch. 6 in [15] for further background on this idea.

We begin by establishing some easy bounds when $(u, v)$ lies inside the ball $B_0 \subset Q$. From the definition of $R_a(\Lambda/2)$,

$$u^2 + v^2 < R_a^2(\Lambda/2) = \frac{4\Delta z_a}{\Lambda} \ln \frac{\Lambda}{4\pi\Delta z_a \delta}, \quad (5.3.80)$$

so the argument of the Gaussian is bounded by

$$\frac{\Lambda(u^2 + v^2)}{2\Delta z_a} < 2 \ln \frac{\Lambda}{4\pi\Delta z_a \delta}. \quad (5.3.81)$$

On the other hand, consider the difference $\Delta z(u,v) - \Delta z_a$. As a function of $(u,v)$, the difference vanishes at $u = v = 0$ and is differentiable there, so

$$\left| \Delta z(u,v) - \Delta z_a \right| < M \sqrt{u^2 + v^2} < M R_a(\Lambda/2), \quad (5.3.82)$$
for some constant $M$ depending as usual on $C$. Immediately, since $R^2_a(\Lambda/2)$ scales like $\ln \Lambda/\Lambda$, the relative fluctuations in height about $\Delta z_a$ satisfy

$$\frac{|\hat{z}(u,v) - \Delta z_a|}{\Delta z_a} < M \frac{R_a(\Lambda/2)}{\Delta z_a} \sim \left( \frac{\ln \Lambda}{\Lambda} \right)^{1/2}. \quad (5.3.83)$$

Directly from (5.3.81) and (5.3.83),

$$\frac{\Lambda(u^2 + v^2)}{2 \Delta z_a} \cdot \frac{|\hat{z}(u,v) - \Delta z_a|}{\Delta z_a} < M \frac{(\ln \Lambda)^{3/2}}{\Lambda^{1/2}}. \quad (5.3.84)$$

The constant $M$ in (5.3.84) is not necessarily the same as the constant $M$ in (5.3.83)!

Given the relative similarity between the integrands in (5.3.77), we consider their ratio

$$q = \frac{\Lambda}{2\pi \Delta z(u,v)} \exp \left[ -\frac{\Lambda (u^2 + v^2)}{2\Delta z(u,v)} \right] / \frac{\Lambda}{2\pi \Delta z_a} \exp \left[ -\frac{\Lambda (u^2 + v^2)}{2\Delta z_a} \right]. \quad (5.3.85)$$

With some algebra, this ratio can be recast as

$$q = \frac{\Delta z_a}{\Delta z(u,v)} \exp \left[ \frac{\Lambda (u^2 + v^2)}{2} \cdot \frac{\hat{z}(u,v) - \Delta z_a}{\Delta z(u,v) \Delta z_a} \right],$$

$$= \left[ 1 + \frac{\Delta z(u,v) - \Delta z_a}{\Delta z_a} \right]^{-1} \cdot \exp \left[ \frac{\Lambda (u^2 + v^2)}{2\Delta z_a} \cdot \frac{\hat{z}(u,v) - \Delta z_a}{\Delta z_a} \cdot \frac{\Delta z_a}{\Delta z(u,v)} \right]. \quad (5.3.86)$$

By the estimates in (5.3.83) and (5.3.84), the prefactor in $q$ approaches unity and the argument of the exponential vanishes as $\Lambda \to \infty$. Hence we can choose $\Lambda$ sufficiently large so that

$$|q - 1| < \delta. \quad (5.3.87)$$

With this control over the fractional error, the difference between the nonlinear and
the usual Gaussian in (5.3.77) is bounded by

$$\int_{B_0} \left| \frac{\Lambda}{2\pi \Delta z(u,v)} \exp \left[ -\frac{\Lambda (u^2 + v^2)}{2\Delta z(u,v)} \right] \right| du dv \leq \int_{B_0} \frac{\Lambda |q - 1|}{2\pi \Delta z_a} \exp \left[ -\frac{\Lambda (u^2 + v^2)}{2\Delta z_a} \right] du dv \tag{5.3.88}$$

We are left to examine what happens when \((u, v)\) lies outside the ball \(B_0 \subset Q\) of radius \(R_a(\Lambda/2)\), meaning

$$u^2 + v^2 \geq R_a^2(\Lambda/2). \tag{5.3.89}$$

To start, the bound on the Gaussian in (5.3.79) is trivial because the integrand is bounded by \(\delta\) for all points \((u, v)\) outside the ball of radius \(R_a(\Lambda)\), and \(R_a(\Lambda/2) > R_a(\Lambda)\). So the real task is to establish the bound for the nonlinear Gaussian in (5.3.78).

Now consider the following function on \(Q\),

$$\tilde{\Lambda}(u, v) = \Lambda \cdot \frac{\Delta z_a}{\Delta z(u,v)}. \tag{5.3.90}$$

Conceptually, we interpret \(\tilde{\Lambda}\) as a fluctuating, position-dependent version of the parameter \(\Lambda\), so that the width of the nonlinear Gaussian varies from point-to-point on \(Q\). By the estimate in (5.3.31), the relative fluctuation factor is bounded from below everywhere on \(Q\) by

$$\frac{1}{2} < 1 - c < \frac{\Delta z_a}{\Delta z(u,v)}. \tag{5.3.91}$$

Consequently, by the definition of \(\tilde{\Lambda}\),

$$\frac{\Lambda}{2} < \tilde{\Lambda}(u, v). \tag{5.3.92}$$

Associated to the local parameter \(\tilde{\Lambda}(u, v)\) we have a local scale \(R_a(\tilde{\Lambda}(u,v))\), also function of \(u\) and \(v\). Since \(R_a\) is monotonically decreasing for large \(\Lambda\), the lower bound
on $\hat{\Lambda}$ in (5.3.92) means that
\[ R_2^2(\hat{\Lambda}(u, v)) < R_2^2(\Lambda/2) < u^2 + v^2. \quad (5.3.93) \]

Thus, again by the definition of $R_a(\Lambda)$ in (5.3.75),
\[ \frac{\hat{\Lambda}}{2\pi \Delta z_a} \exp \left[ -\frac{\hat{\Lambda}(u^2 + v^2)}{2 \Delta z_a} \right] < \delta, \quad (5.3.94) \]
or by substitution from (5.3.90),
\[ \frac{\Lambda}{2\pi \Delta z(u, v)} \exp \left[ -\frac{\Lambda(u^2 + v^2)}{2 \Delta z(u, v)} \right] < \delta. \quad (5.3.95) \]

The bound on the nonlinear Gaussian in (5.3.95) is exactly what we need to control
the integral over $Q - B_0$, so that
\[ \left| \int_{Q - B_0} \Psi - \deg(\varphi_a) \right| \int_{\mathbb{R}^2} \frac{\Lambda}{2\pi \Delta z_a} \exp \left[ -\frac{\Lambda(u^2 + v^2)}{2 \Delta z_a} \right] du \, dv < M \delta. \quad (5.3.96) \]

Combining with the trivial bound in (5.3.73), we deduce that the desired integral of
$\Psi$ over $Q$ can be well-approximated for large $\Lambda$ by the naive Gaussian integral,
\[ \left| \int_{Q} \Psi - \deg(\varphi_a) \int_{\mathbb{R}^2} \frac{\Lambda}{2\pi \Delta z_a} \exp \left[ -\frac{\Lambda(u^2 + v^2)}{2 \Delta z_a} \right] du \, dv \right| < M \delta. \quad (5.3.97) \]

We are not finished with our error analysis at the crossing, because we still must
consider the integral of $\Xi$ over $Q$ in (5.3.68). We will show that the contribution of
$\Xi$ is negligible for large $\Lambda$,
\[ \left| \int_{Q} \Xi \right| < M \delta. \quad (5.3.98) \]

Explicitly, from the formula in (5.3.67), the integral of $\Xi$ is given in the $(u, v)$-
coordinates by
\[ \int_{Q} \Xi = \int_{Q} \frac{\Lambda}{4\pi \Delta z(u, v)^2} e^{-\Lambda(u^2 + v^2)/2\Delta z(u, v)} \left( u \frac{\partial \Delta z}{\partial u} + v \frac{\partial \Delta z}{\partial v} \right) du \, dv. \quad (5.3.99) \]
Again, we consider the cases that \((u, v)\) lies inside the ball \(B_0\) and outside the ball \(B_0\) separately. When \((u, v)\) lies outside the ball \(B_0 \subset \mathcal{Q}\), then by the definition of \(R_a(\Lambda)\) in (5.3.75),

\[
\left| \frac{\Lambda}{4\pi \hat{\Delta} z(u, v)^2} e^{-\Lambda(u^2 + v^2)/2\hat{\Delta} z(u, v)} \left( u \frac{\partial \hat{\Delta} z}{\partial u} + v \frac{\partial \hat{\Delta} z}{\partial v} \right) \right| < M \delta. \tag{5.3.100}
\]

Here we note that the extra factors of \(1/\hat{\Delta} z\) and \((u \partial/\partial u + v \partial/\partial v) \hat{\Delta} z\) in (5.3.100) are smooth functions bounded independently of \(\Lambda\) on \(\mathcal{Q}\).

Otherwise, for points inside \(B_0\), we have a bound

\[
\int_{B_0} \frac{\Lambda}{4\pi \hat{\Delta} z(u, v)^2} \exp \left[ -\frac{\Lambda (u^2 + v^2)}{2\hat{\Delta} z(u, v)} \right] du dv < M, \tag{5.3.101}
\]

which follows by the same arguments used to produce the estimate in (5.3.88). Also, since \((u \partial/\partial u + v \partial/\partial v) \hat{\Delta} z\) is bounded in \(B_0\) and \(|u|, |v| \leq R_a(\Lambda/2) \to 0\) as \(\Lambda \to \infty\), we can always choose \(\Lambda\) so that

\[
\left| u \frac{\partial \hat{\Delta} z}{\partial u} + v \frac{\partial \hat{\Delta} z}{\partial v} \right| < \delta, \tag{5.3.102}
\]

for all \((u, v)\) in \(B_0\). Combining the bounds in (5.3.100), (5.3.101), and (5.3.102) for outside and inside \(B_0\), we obtain the conclusion in (5.3.98).

In total, these bounds establish the informal localization formula in (5.3.12) for any crossing of \(\Pi(C)\). \square

**Error analysis near the diagonal**

Our final goal is to evaluate the self-linking integral over the tubular neighborhood \(N_\Delta(w)\) of the diagonal \(\Delta \subset T^2\), where \(w \sim \Lambda^{-1/4}\) is the width set in Lemma 5.3.2. For the informal localization computation in (5.3.13), we used the leading term in the Taylor expansion of the self-linking integrand near the diagonal to approximate the integral. Depending upon the value of the parameter \(\hbar\), we abbreviate this leading
term by

\[ \Phi_h \overset{\hbar \neq 1}{=} - \text{sgn}(\eta) \frac{\Lambda(\dot{\gamma} \times \ddot{\gamma})}{4\pi|1 - \hbar||\gamma \times \dot{\gamma}|} \exp \left[ - \frac{\Lambda||\dot{\gamma}||^2|\eta|}{2|1 - \hbar||\gamma \times \dot{\gamma}|} \right] d\phi \wedge d\eta, \quad (5.3.103) \]

or

\[ \Phi_h \overset{\hbar = 1}{=} - \frac{3\Lambda}{2\pi\eta^2} \exp \left[ - \frac{3\Lambda||\dot{\gamma}||^2}{|\gamma \times \dot{\gamma}|} \right] d\phi \wedge d\eta. \quad (5.3.104) \]

In both cases we assume that the local positivity condition \( \delta z > 0 \) is satisfied, as in (5.2.24). Otherwise, \( \Phi_h \equiv 0 \).

To justify our localization computation, we must demonstrate for sufficiently large \( \Lambda \) the bound

\[ \left| \int_{N_\Delta(w)} (X \times X)^* \hat{\Gamma}_h^* \chi_\Lambda - \int_{N_\Delta(w)} \Phi_h \right| < M \delta. \quad (5.3.105) \]

The behavior of \( \Phi_h \) for small \( \eta \) depends very much on whether \( \hbar \) is equal to one or not, so we treat the cases in (5.3.103) and (5.3.104) separately. Because the generic case \( \hbar \neq 1 \) is the more involved, and the more interesting, we begin with it.

**Generic case \( \hbar \neq 1 \)**

In principle, the error in the leading approximation to \( (X \times X)^* \hat{\Gamma}_h^* \chi_\Lambda \) for small \( \eta \) is controlled by the magnitude of the next-order term in the Taylor expansion. We will need this correction term for our analysis. Briefly, by the same computations leading to (5.2.23) in Section 5.2.1, the argument of the heat kernel admits the second-order expansion

\[ \frac{\Delta x^2 + \Delta y^2}{2 \Delta z} \overset{\hbar \neq 1}{=} - \frac{||\dot{\gamma}||^2 \eta}{2(1 - \hbar)(\gamma \times \dot{\gamma})} \left[ 1 + \left( \frac{\dot{\gamma} \cdot \ddot{\gamma}}{||\dot{\gamma}||^2} - \frac{1}{2} \frac{\gamma \times \dot{\gamma}}{\gamma \times \dot{\gamma}} \right) \eta + O(\eta^2) \right], \quad (5.3.106) \]

where \( \dot{\gamma} \cdot \ddot{\gamma} \equiv \dot{x}\ddot{x} + \dot{y}\ddot{y} \). Similarly for the pullback of the heat form \( \chi_\Lambda \) itself,

\[ (X \times X)^* \hat{\Gamma}_h^* \chi_\Lambda \overset{\hbar \neq 1}{=} \]

\[ - \frac{\Lambda \text{sgn}(\eta)}{4\pi|1 - \hbar||\gamma \times \dot{\gamma}|} \left[ \dot{\gamma} \times \ddot{\gamma} \right] + \frac{1}{2} \left( \dot{\gamma} \times \ddot{\gamma} \right) \eta - \left( \frac{\gamma \times \dot{\gamma}}{\gamma \times \dot{\gamma}} \right) \eta + O(\eta^2) \times \]

\[ \times \exp \left[ - \frac{\Lambda||\dot{\gamma}||^2|\eta|}{2|1 - \hbar||\gamma \times \dot{\gamma}|} \left( 1 + \left( \frac{\dot{\gamma} \cdot \ddot{\gamma}}{||\dot{\gamma}||^2} - \frac{1}{2} \frac{\gamma \times \dot{\gamma}}{\gamma \times \dot{\gamma}} \right) \eta + O(\eta^2) \right) \right]. \quad (5.3.107) \]
We omit the computation leading to (5.3.107), since the details of this formula will not be so important. The expansion merely confirms that both the prefactor and the argument of the exponential for $\Phi_h$ in (5.3.103) receive further corrections at the next order in $\eta$, as determined by the geometry of the projection $\Pi(C)$.

Validity of the leading approximation $\Phi_h$ requires that the correction terms in (5.3.107) be small. At least informally, for the argument of the heat kernel in (5.3.106) we require

$$
\left| \left( \frac{\dot{\gamma} \cdot \ddot{\gamma}}{||\ddot{\gamma}||^2} - \frac{1}{2} \frac{\gamma \times \ddot{\gamma}}{\gamma \times \dot{\gamma}} \right) \eta \right| \ll 1.
$$

(5.3.108)

By assumption, $|\eta| < w \sim \Lambda^{-1/4}$ is always small on $N_\Delta(w)$, and $||\ddot{\gamma}||^2 > 0$ is bounded from below. The condition in (5.3.108) is therefore only violated at points where $\gamma \times \dot{\gamma} = 0$. At these points, the Taylor expansion in (5.3.107) breaks down.

Despite the failure of the Taylor expansion at points where $\gamma \times \dot{\gamma} = 0$, these points cause no difficulties. Recall that points where $\gamma \times \dot{\gamma} = 0$ correspond to critical points of the height function $z$ on $C$. By the Morse assumption which follows Lemma 5.2.1, the function $(\gamma \times \dot{\gamma})(\phi)$ vanishes at only a finite number of isolated critical points $\{\phi_1, \ldots, \phi_{2k}\}$ on $S^1$. According to the analysis at the end of Section 5.2.2, these points are precisely the endpoints of the disjoint collection of intervals $S^1_\pm \cap [T^2 - \Delta(\varepsilon)]$ in (5.2.37). At the endpoints, $\Phi_h$ vanishes, and the exact integrand $(X \times X)^*\Gamma_h^* \chi_\Lambda$ in (5.3.1) is exponentially small (since $\Delta z$ is small). Consequently, the troublesome points for the Taylor expansion can just be removed from the domain of integration, with negligible error.

Technically, about each critical point $\phi_c$,

$$
(\gamma \times \dot{\gamma})(\phi_c) = 0, \quad c = 1, \ldots, 2k,
$$

(5.3.109)

we consider a small neighborhood $(\phi_c - \ell, \phi_c + \ell)$ with fixed width $\ell > 0$. Let $I_c(\ell) \subset N_\Delta(w)$ be the corresponding closed strip

$$
I_c(\ell) = \left\{ (\phi, \eta) \mid \phi_c - \ell \leq \phi \leq \phi_c + \ell, \ 0 \leq |\eta| \leq w \right\},
$$

(5.3.110)
and set
\[ N_\Delta(w; \ell) = N_\Delta(w) - \bigcup_{c=1}^{2k} I_c(\ell). \] (5.3.111)

See Figure 5-13 for a sketch of \( N_\Delta(w; \ell) \) near one boundary of the cylinder \( T^2 - \Delta(\varepsilon) \). The shaded regions indicate the strips of width \( \ell \) which have been excised about a pair of zeroes \( \phi_1 \) and \( \phi_2 \) of the function \( \gamma \times \dot{\gamma} \).

We choose the width \( \ell > 0 \) of each strip small enough so that
\[ \left| \sum_{c=1}^{2k} \int_{I_c(\ell)} (X \times X)^* \hat{\Gamma}_h \chi_\Lambda \right| < \delta, \quad \left| \sum_{c=1}^{2k} \int_{I_c(\ell)} \Phi_h \right| < \delta. \] (5.3.112)

Both integrands in (5.3.112) are bounded (and indeed vanish) at the points \( \phi_c \), so the integrals over \( I_c(\ell) \) can be made as small as desired by the choice of \( \ell \). Moreover, both integrands decrease monotonically as functions of \( \Lambda \) for sufficiently large \( \Lambda \), so the width \( \ell \) can be chosen independently of \( \Lambda \), our crucial requirement.

By definition, the function \( (\gamma \times \dot{\gamma})(\phi) \) is now bounded away from zero everywhere on the new domain \( N_\Delta(w; \ell) \),
\[ |\gamma \times \dot{\gamma}| > m > 0 \quad \text{on} \quad N_\Delta(w; \ell). \] (5.3.113)

Here \( m \) is a constant which depends upon the curve \( C \) and the parameter \( \ell \), but not
on $\Lambda$. Because the respective contributions (5.3.112) from the excised strips are small by assumption, we are free to replace $N_\Delta(w)$ by $N_\Delta(w;\ell)$ in the inequality (5.3.105) to be proven. On the other hand, due to the lower bound in (5.3.113), we will also have uniform control of error terms such as (5.3.108) in the Taylor approximation on $N_\Delta(w;\ell)$.

The remainder of the discussion proceeds in rough correspondence to the asymptotic analysis near a crossing. By analogy to the ball $B_0$ in (5.3.76), we introduce a smaller tubular neighborhood $N_0 \subset N_\Delta(w;\ell)$ defined by

$$N_0 : |\eta| < \frac{w}{\sqrt{\Lambda}} \sim \Lambda^{-3/4}.$$  

(5.3.114)

We indicate the coaxial configuration schematically in Figure 5-13. For points inside the small tube $N_0$, we will show that the difference between the integrals of $(X \times X)^* \hat{\gamma}_h \chi_\Lambda$ and $\Phi_h$ is small,

$$\left| \int_{N_0} (X \times X)^* \hat{\gamma}_h \chi_\Lambda - \int_{N_0} \Phi_h \right| < M \delta.$$  

(5.3.115)

For points outside $N_0$ but inside $N_\Delta(w;\ell)$, we will show that both integrals are separately small, with

$$\left| \int_{N_\Delta(w;\ell)-N_0} (X \times X)^* \hat{\gamma}_h \chi_\Lambda \right| < M \delta,$$  

(5.3.116)

and

$$\left| \int_{N_\Delta(w;\ell)-N_0} \Phi_h \right| < M \delta.$$  

(5.3.117)

The extra factor of $1/\sqrt{\Lambda}$ in the definition of $N_0$ is simply what is needed to ensure the inequality in (5.3.115).

We first consider the points inside $N_0$. From the second-order expansion in
(5.3.107), the ratio of the self-linking integrand to its approximation $\Phi_h$ satisfies

$$q = \frac{(X \times X)^* \Gamma_h^* \chi}{\Phi_h} = \left( 1 + \frac{1}{2} \frac{\gamma \times \ddot{\gamma}}{\dot{\gamma} \times \dot{\gamma}} \right) \eta - \frac{\gamma \times \ddot{\gamma}}{\dot{\gamma} \times \dot{\gamma}} \eta + O(\eta^2) \times$$

$$\exp \left[ - \frac{\Lambda \norm{\dot{\gamma}}}{2 |1 - \hbar| |\gamma \times \dot{\gamma}|} \left( \frac{\dot{\gamma} \cdot \ddot{\gamma}}{\norm{\dot{\gamma}}^2} - \frac{1}{2} \frac{\gamma \times \ddot{\gamma}}{\gamma \times \dot{\gamma}} \right) \eta + O(\eta^3) \right].$$

Again, to deal with the denominator in the prefactor, we assume that any zeroes of $\gamma \times \ddot{\gamma}$ are isolated, and we remove small neighborhoods as necessary about those zeroes so that the functions which multiply $\eta$ in both the prefactor and the argument of the exponential in (5.3.118) are bounded, independently of $\Lambda$.

For any point in the small tube $N_0$, the argument of the exponential in (5.3.118) is bounded in magnitude by

$$\frac{\Lambda \norm{\dot{\gamma}}^2 \eta^2}{2 |1 - \hbar| |\gamma \times \dot{\gamma}|} \cdot \left| \frac{\dot{\gamma} \cdot \ddot{\gamma}}{\norm{\dot{\gamma}}^2} - \frac{1}{2} \frac{\gamma \times \ddot{\gamma}}{\gamma \times \dot{\gamma}} \right| < \Lambda M \eta^2 < M w^2,$$

where we apply the conditions $|\gamma \times \dot{\gamma}| > m > 0$ as well as $|\eta| < w/\sqrt{\Lambda}$ in $N_0$. Because $w \sim \Lambda^{-1/4}$, this inequality means that the argument of the exponential in $q$ vanishes in the limit $\Lambda \to \infty$, and the prefactor approaches unity. Thus for sufficiently large $\Lambda$, the fractional error is small,

$$|q - 1| < \delta.$$  (5.3.120)

By the same idea in (5.3.88),

$$\left| \int_{N_0} (X \times X)^* \Gamma_h^* \chi \right| \leq |q - 1| \cdot \left| \int_{N_0} \Phi_h \right| < M \delta,$$  (5.3.121)

since we already know the integral of $\Phi_h$ to be finite and independent of $\Lambda$ by the local computation in Section 5.3.1.

The inequalities for points outside $N_0$ are even easier.

From the explicit expression for $\Phi_h$ in (5.3.103),

$$|\Phi_h| < A \Lambda \exp \left[ - B \Lambda |\eta| \right], \quad A, B > 0,$$  (5.3.122)
for some positive constants $A$ and $B$. So in the allowed range $\Lambda^{-1/2}w \leq |\eta| \leq w$ on the complement of $N_0$,

\[
|\Phi_h| < A \Lambda \exp\left[ -B \Lambda^{1/2}w \right], \quad w = m \Lambda^{-1/4},
\]

\[
= A \Lambda \exp\left[ -mB \Lambda^{1/4} \right].
\]

By taking $\Lambda$ sufficiently large, we can make the magnitude of $\Phi_h$ as small as desired on the complement of $N_0$ inside $N_{\Delta}(w; \ell)$, from which the bound in (5.3.117) follows.

To establish a similar bound for the pullback of $\chi_{\Lambda}$ in (5.3.107), observe that

\[
\frac{|||\dot{\gamma}|^2 |\eta|}{2\sqrt{1 - h^2} |\gamma \times \dot{\gamma}|} \left[ 1 + \left( \frac{\dot{\gamma} \cdot \ddot{\gamma}}{||\gamma||^2} - \frac{1}{2} \frac{\gamma \times \ddot{\gamma}}{\gamma \times \dot{\gamma}} \right) \eta \right] > B |\eta|, \quad B > 0,
\]

provided that $\Lambda$ is sufficiently large and $\eta$ sufficiently small. Here $B > 0$ is a suitable positive constant. Then according to the expansion in (5.3.107),

\[
\left| (X \times X)^* \hat{T}_h^* \chi_{\Lambda} \right| < A \Lambda \exp\left[ -B \Lambda |\eta| \right], \quad A, B > 0,
\]

exactly as for the preceding bound on $\Phi_h$ in (5.3.122). The claim in (5.3.116) now follows by an identical argument.

In total, the three inequalities in (5.3.115), (5.3.116), and (5.3.117) finish the proof of the localization formula for $\text{slk}_m(C)|_{\Delta}$ in the generic case $h \neq 1$. \[\square\]

**Symmetric case** $h = 1$

For the Heisenberg-symmetric value $h = 1$, the localization formula from Section 5.3.1 states $\text{slk}_m(C)|_{\Delta} = 0$. Consistent with this result, we establish the basic bound in (5.3.105) by showing individually

\[
\left| \int_{N_{\Delta}(w)} (X \times X)^* \hat{T}_h^* \chi_{\Lambda} \right| < \delta, \quad [h = 1]
\]

and

\[
\left| \int_{N_{\Delta}(w)} \Phi_h \right| < \delta. \quad [h = 1]
\]

Our workhorse is the next-order expansion of the self-linking integrand, which
behaves differently for $\hbar = 1$. For the argument of the heat kernel, the calculations in Section 5.2.1 yield

$$\frac{\Delta x^2 + \Delta y^2}{2 \Delta z} \equiv \frac{3}{\eta} \left[ \frac{||\dot{\gamma}||^2 + (\dot{\gamma} \cdot \ddot{\gamma}) \eta + \mathcal{O}(\eta^2)}{(\dot{\gamma} \times \ddot{\gamma}) + \frac{1}{2} (\dot{\gamma} \times \ddot{\gamma}) \eta + \mathcal{O}(\eta^2)} \right]. \quad (5.3.128)$$

Similarly,

$$(X \times X)^* \bar{\Gamma}_h \lambda \equiv \frac{3 \Lambda d\phi \wedge d\eta}{2\pi \eta^2} [1 + \mathcal{O}(\eta^2)] \times \exp \left[ -\frac{3 \Lambda}{|\eta|} \cdot \frac{||\dot{\gamma}||^2 + (\dot{\gamma} \cdot \ddot{\gamma}) \eta + \mathcal{O}(\eta^2)}{(\dot{\gamma} \times \ddot{\gamma}) + \frac{1}{2} (\dot{\gamma} \times \ddot{\gamma}) \eta + \mathcal{O}(\eta^2)} \right]. \quad (5.3.129)$$

For sake of brevity, we omit the calculation leading to (5.3.129). The details of this formula are not important.\footnote{Curiously, the order-$\eta$ correction to the prefactor in (5.3.129) vanishes when $\hbar = 1$.}

For the approximation $\Phi_h$, recall the formula

$$\Phi_h \equiv \frac{3 \Lambda}{2\pi \eta^2} \exp \left[ -\frac{3 \Lambda \eta^2}{|\dot{\gamma} \times \ddot{\gamma}|} \right] d\phi \wedge d\eta. \quad (5.3.130)$$

Then $\Phi_h$ vanishes smoothly for $\eta = 0$ and otherwise satisfies

$$|\Phi_h| < \frac{M \Lambda}{\eta^2} \exp \left[ -\frac{A \Lambda}{|\eta|} \right], \quad A, M > 0, \quad (5.3.131)$$

for some positive constants $A$ and $M$. For $\Lambda$ sufficiently large, $\Phi_h$ can be made as small as desired everywhere on $N_\Delta(w)$, and the inequality in (5.3.127) holds.

To treat the pullback of $\chi_\Lambda$ in the same fashion, note that the denominator in (5.3.128) obeys

$$|\dot{\gamma} \times \ddot{\gamma}| \eta + \frac{1}{2} (\dot{\gamma} \times \ddot{\gamma}) \eta^2 < A |\eta| + B |\eta|^2 < 2 A |\eta|, \quad (5.3.132)$$

provided $|\eta|$ is sufficiently small (with $B |\eta| < A$), as holds when $\Lambda$ is sufficiently large.
Also in this regime, the numerator in (5.3.128) is bounded from below by

$$
\left| ||\dot{\gamma}||^2 + (\dot{\gamma} \cdot \ddot{\gamma}) \eta \right| > m > 0.
$$

(5.3.133)

Hence on the tubular neighborhood $N_{\Delta}(w)$,

$$
\left| (X \times X)^* \hat{\Gamma}^*_h \chi \Lambda \right| < \frac{M \Lambda}{\eta^2} \exp \left[ -\frac{3m \Lambda}{2 \Lambda |\eta|} \right].
$$

(5.3.134)

This inequality has the same shape as that for $\Phi_h$ in (5.3.131), from which we reach the conclusion in (5.3.126).

The proof of our Main Theorem is complete. $\square$
Bibliography


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Appendix A: The Topological Anomaly

In this appendix, we provide a derivation of the formula for the topological anomaly in the Gauss self-linking integral

$$L(C) = \lim_{\varepsilon \to 0^+} \int_{S^1 \times S^1 - \Delta(\varepsilon)} (-1) \cdot (X \times X)^* \Gamma^* \psi.$$  \hspace{1cm} (A.1)

Here $X: S^1 \to \mathbb{R}^3$ is a regular, unit-speed parametrization of the space curve $C$. We introduce periodic coordinates $(s_1, s_2)$ on the torus $S^1 \times S^1$, in terms of which

$$\Delta X^\mu(s_1, s_2) \equiv X^\mu(s_1) - X^\mu(s_2).$$  \hspace{1cm} (A.2)

Our goal is to determine the first-order change $\delta L(C)$ in the self-linking integral (A.1) under a general variation $\delta X$ of the map $X$.

The topological anomaly is the statement that $\delta L(C) \neq 0$. Before we analyze a quantity which is non-zero, though, let us first compute a simpler quantity which is zero. For the latter, we consider the Gauss linking (as opposed to self-linking) integral

$$L(C_1, C_2) = \int_{S^1 \times S^1} (-1) \cdot (X_1 \times X_2)^* \Gamma^* \psi \in \mathbb{Z},$$  \hspace{1cm} (A.3)

where $C_{1,2}$ are distinct, non-intersecting curves with respective parametrizations $X_{1,2}$.  

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The integrand takes the same form in (A.2), subject to the replacement
\[ \Delta X^\mu(s_1, s_2) \equiv X_1^\mu(s_1) - X_2^\mu(s_2). \] (A.4)

The value of \( L(C_1, C_2) \) depends continuously on \( X_1, X_2 \) and is quantized for topological reasons. So for any variation of \( X_1 \) or \( X_2 \), automatically
\[ \delta L(C_1, C_2) = 0. \] (A.5)

For application to the self-linking integral in (A.1), which lacks an obvious topological meaning, we wish to understand the vanishing in (A.5) more directly. The variation of the integrand in (A.3) with respect to \( X_1 \) and \( X_2 \) is straightforward, if somewhat laborious, to compute. We find
\[ \delta (X_1 \times X_2)^* \Gamma^* \psi = -\frac{1}{4\pi} \frac{\epsilon_{\mu\nu\rho} T^{\mu\nu\rho}}{||\Delta X||^3} ds_1 \wedge ds_2, \] (A.6)

where the tensor \( T^{\mu\nu\rho} \) is given by the soup of terms
\[ T^{\mu\nu\rho} = -\frac{3 (\Delta X \cdot \delta X_1)}{||\Delta X||^3} \Delta X^\mu \dot{X}_1^\nu \dot{X}_2^\rho + \delta X_1^\mu \dot{X}_1^\nu \dot{X}_2^\rho + \Delta X^\mu \delta \dot{X}_1^\nu \dot{X}_2^\rho + \frac{3 (\Delta X \cdot \delta X_2)}{||\Delta X||^3} \Delta X^\mu \dot{X}_1^\nu \dot{X}_2^\rho - \delta X_2^\mu \dot{X}_1^\nu \dot{X}_2^\rho + \Delta X^\mu \dot{X}_1^\nu \delta \dot{X}_2^\rho. \] (A.7)

Here we abuse notation slightly to abbreviate \( \dot{X}_1^\mu \equiv dX_1^\mu/ds_1 \) and \( \dot{X}_2^\mu \equiv dX_2^\mu/ds_2 \), and we employ the standard notation for the Euclidean inner product, eg. \( \Delta X \cdot \delta X_1 = \Delta X^\mu \delta X_1^\mu \). The factors in \( T^{\mu\nu\rho} \) proportional to \( \Delta X \cdot \delta X_{1,2} \) arise from the variation of \( 1/||\Delta X||^3 \) in (A.2), and the other factors arise from the variation of the numerator in (A.2).

Although the integrand for \( L(C_1, C_2) \) in (A.3) itself varies non-trivially with \( X_1 \) and \( X_2 \), the cohomology class of the integrand on \( S^1 \times S^1 \) does not vary, according to (A.5). Hence it must be possible to write the expression in (A.6) as
\[ \delta (X_1 \times X_2)^* \Gamma^* \psi = d\omega_{12}, \quad \omega_{12} \in \Omega^1(S^1 \times S^1; \mathbb{R}). \] (A.8)
Here \( \omega_{12} \) is a smooth one-form on the torus which depends upon the embeddings \( X_{1,2} \) as well as the vector fields \( \delta X_{1,2} \) which describe the variation of each map. Clearly, \( \omega_{12} \) is defined by (A.8) up to the addition of a closed form on \( S^1 \times S^1 \).

The one-form \( \omega_{12} \) will be the essential ingredient for computing the topological anomaly, and determining \( \omega_{12} \) from (A.6) and (A.7) effectively becomes an exercise in integration-by-parts. Briefly, we claim that a representative for \( \omega_{12} \) is given by

\[
\omega_{12} = -\frac{1}{4\pi} \epsilon_{\mu\nu\rho} \frac{\Delta X^\mu}{||\Delta X||^3} \left( \delta X_1^\nu - \delta X_2^\nu \right) \left( \dot{X}_2^\rho \, ds_2 - \dot{X}_1^\rho \, ds_1 \right). \tag{A.9}
\]

As a small check, note that both \( \omega_{12} \) and \( \delta(X_1 \times X_2)^* \Gamma^* \psi \) are odd under the exchange \( X_1 \leftrightarrow X_2 \) and \( s_1 \leftrightarrow s_2 \). The general form of \( \omega_{12} \) in (A.9) is also easy to guess, since \( \omega_{12} \) must depend linearly on \( \delta X_{1,2} \), involve a single copy of the anti-symmetric tensor \( \epsilon_{\mu\nu\rho} \), and be proportional to \( \dot{X}_{1,2}^\mu \, ds_{1,2} \) for reparametrization invariance.

Otherwise, the claim in (A.8) can be verified from (A.6), (A.7), and (A.9) by direct calculation, which we spare the reader. In the process, one must use the following tensor identity. Let \( x, y, z \) be vectors in \( \mathbb{R}^3 \), and let \( u \in \mathbb{R}^3 \) be a unit vector, \( ||u||^2 = 1 \). Introduce the trilinear form

\[
Q(x, y, z) = \epsilon_{\mu\nu\rho} \left[ (u \cdot x) y^\mu z^\nu u^\rho - (u \cdot y) x^\mu z^\nu u^\rho + (u \cdot z) x^\mu y^\nu u^\rho \right]. \tag{A.10}
\]

Then \( Q(x, y, z) \) is independent of \( u \) and given alternatively by

\[
Q(x, y, z) = \epsilon_{\mu\nu\rho} x^\mu y^\nu z^\rho. \tag{A.11}
\]

To prove the identity for \( Q \) in (A.11), note that the expression in (A.10) is invariant under the action by \( SO(3) \) on \( \mathbb{R}^3 \). With this action we can always fix \( u = (1, 0, 0) \) say, so \( Q \) cannot depend on \( u \). Otherwise, one can readily check that the right-hand side of (A.10) is fully anti-symmetric under the exchanges of \( x, y, \) and \( z \), so \( Q(x, y, z) \) must be proportional to \( \epsilon_{\mu\nu\rho} x^\mu y^\nu z^\rho \). The constant of proportionality is fixed to unity in (A.11) by evaluating \( Q(x, y, z) \) on any convenient set of vectors, for instance an orthonormal frame.
From the result in (A.8), the variation of $L(C_1, C_2)$ vanishes by Stokes’ Theorem,

$$
\delta L(C_1, C_2) = \int_{S^1 \times S^1} (-1) \cdot \delta(X_1 \times X_2)^* \Gamma^* \psi = -\int_{S^1 \times S^1} d\omega_{12} = 0. \quad (A.12)
$$

Having successfully computed zero, we now apply a similar logic to the Gauss self-linking integral. By exactly the same computations, the variation of the self-linking integrand is trivial in cohomology on the cylinder $S^1 \times S^1 - \Delta(\varepsilon)$,

$$
\delta(X \times X)^* \Gamma^* \psi = d\omega, \quad \omega \in \Omega^1(S^1 \times S^1 - \Delta(\varepsilon); \mathbb{R}), \quad (A.13)
$$

where we make the obvious modifications to (A.9),

$$
\omega = -\frac{1}{4\pi} \frac{\epsilon_{\mu
u\rho} \Delta X^\mu}{||\Delta X||^3} \left[ \delta X(s_1)^\nu - \delta X(s_2)^\nu \right] \left[ \dot{X}(s_2)^\rho ds_2 - \dot{X}(s_1)^\rho ds_1 \right]. \quad (A.14)
$$

Via Stokes’ Theorem, the variation of $L(C)$ with $X$ is thus

$$
\delta L(C) = \lim_{\varepsilon \to 0^+} \int_{S^1 \times S^1 - \Delta(\varepsilon)} (-1) \cdot \delta(X \times X)^* \Gamma^* \psi = (-1) \cdot \lim_{\varepsilon \to 0^+} \int_{S^1 \times S^1 - \Delta(\varepsilon)} d\omega,
$$

$$
= (-1) \cdot \lim_{\varepsilon \to 0^+} \left[ \oint_{S^1_+} \omega - \oint_{S^1_-} \omega \right]. \quad (A.15)
$$

In particular, the integral of $d\omega$ over the cylinder does not necessarily vanish but reduces to a contribution from the oriented boundary circles,

$$
\partial \left( S^1 \times S^1 - \Delta(\varepsilon) \right) = S^1_+ - S^1_-, \quad (A.16)
$$

where

$$
S^1_{\pm} : \quad s_1 = s_2 \pm \varepsilon. \quad (A.17)
$$

As in Section 1.3, we set $s_1 = \varphi + \eta$ and $s_2 = \varphi$, with orientation $ds_1 \wedge ds_2 = d\eta \wedge d\varphi$ on the cylinder. The signs in (A.16) are explained most succinctly by Figure A.1.

Because the variation of the Gauss self-linking integral $L(C)$ is entirely determined by the integral of $\omega$ over the respective boundary circles $S^1_{\pm}$ where $\eta = \pm \varepsilon$, we see that the topological anomaly is a short-distance phenomenon, sensitive to the divergence
of the Euclidean propagator $K^{1,1}$ near the diagonal. This feature is characteristic of all anomalies in quantum field theory.

To evaluate $\delta L(C)$ in the last line of (A.15), we expand the difference

$$X^\mu(\varphi \pm \epsilon) - X^\mu(\varphi) = \pm \epsilon \dot{X}^\mu(\varphi) + \frac{\epsilon^2}{2} \ddot{X}^\mu(\varphi) \pm \frac{\epsilon^3}{6} \dddot{X}^\mu(\varphi) + \mathcal{O}(\epsilon^4).$$  (A.18)

Restricted to either boundary, the anomaly form $\omega$ in (A.14) becomes

$$\omega \big|_{S_{\pm}^1} = \pm \frac{1}{4\pi} \epsilon_{\mu\nu\rho} \dot{X}^\mu \delta \dot{X}^\nu \ddot{X}^\rho d\varphi + \mathcal{O}(\epsilon), \quad ||\dot{X}(\varphi)||^2 = 1,$$  (A.19)

where we assume without loss that $X$ is a unit-speed parametrization, and $d\varphi$ is the arc-length measure. Otherwise, a term proportional to $\dot{X} \cdot \ddot{X}$ also contributes to the anomaly. According to the formula for $\omega \big|_{S_{\pm}^1}$, the limit as $\epsilon$ goes to zero in (A.15) is non-vanishing, with

$$\delta L(C) = \frac{1}{2\pi} \oint_{S^1} d\varphi \epsilon_{\mu\nu\rho} \dot{X}^\mu \delta \dot{X}^\nu \ddot{X}^\rho, \quad ||\dot{X}(\varphi)||^2 = 1,$$  (A.20)

For the latter equality, we integrate by parts on $S^1$ to recover the expression in (1.3.30).

This computation is important because we will follow essentially the same strategy
to compute the anomaly in the Legendrian self-linking integral. In that case, though, the anomaly vanishes.
Appendix B: The Torsion Counterterm

In this appendix we compute the variation of the total torsion

\[ T(C) = \frac{1}{2\pi} \oint_{S^1} d\varphi \tau, \]  

(B.1)

where \( d\varphi \) is the arc-length measure on \( C \), and

\[ \tau = \frac{\epsilon_{\mu\nu\rho} \dot{X}^\mu \ddot{X}^\nu \dddot{X}^\rho}{||X \times \dot{X}||^2}, \quad \dddot{X} \neq 0. \]  

(B.2)

Crucially, this formula for \( \tau \) is valid for any regular parametrization with \( \dddot{X}(s) \neq 0 \) at all points along \( C \).

Both \( d\varphi \) and \( \tau \) depend upon the map \( X : S^1 \to \mathbb{R}^3 \), so the variation of \( T(C) \) with respect to \( X \) has two terms,

\[ \delta T(C) = \frac{1}{2\pi} \oint_{S^1} [\delta(d\varphi) \tau + d\varphi \delta \tau]. \]  

(B.3)

The variation of the measure \( d\varphi \) is easy to evaluate. For any regular parametrization \( X(s) \) of \( C \),

\[ d\varphi = \sqrt{||\dot{X}||^2} \, ds, \]  

(B.4)

so

\[ \delta(d\varphi) = \frac{\dot{X} \cdot \delta \dot{X}}{\sqrt{||\dot{X}||^2}} \, ds. \]  

(B.5)
To simplify the expression in (B.5) further, we take the parameter $s = \varphi$ to be unit-speed, after which
\[
\delta(d\varphi) = t \cdot \delta\dot{X} \, d\varphi, \quad ||\dot{X}||^2 = 1. \tag{B.6}
\]

As before, $t = \dot{X}$ is the unit tangent vector along $C$.

To calculate the variation of $\tau$ requires additional cleverness. Directly from (B.2),
\[
\delta\tau = -\frac{2 \epsilon_{\mu\nu\rho} \dot{X}^\mu \dot{X}^\nu \ddot{X}^\rho}{||\dot{X} \times \ddot{X}||^4} \left[ \left( \delta \dot{X} \times \ddot{X} \right) \cdot \left( \dot{X} \times \ddot{X} \right) + \left( \dot{X} \times \delta \ddot{X} \right) \cdot \left( \dot{X} \times \ddot{X} \right) \right] + \frac{1}{||\dot{X} \times \ddot{X}||^2} \epsilon_{\mu\nu\rho} \left[ \delta \dot{X}^\mu \dot{X}^\nu \ddot{X}^\rho + \dot{X}^\mu \delta \dot{X}^\nu \ddot{X}^\rho + \ddot{X}^\mu \ddot{X}^\nu \delta \ddot{X}^\rho \right]. \tag{B.7}
\]

To simplify this result, we again pass to a unit-speed parametrization. Recall that the unit normal vector $n$ to $C$ is given for $\kappa > 0$ by the double-derivative
\[
\ddot{X} = \kappa n, \quad ||\dot{X}||^2 = 1, \tag{B.8}
\]
and the unit binormal vector is
\[
b = t \times n. \tag{B.9}
\]

Together, the Frenet-Serret vectors \{t, n, b\} satisfy the structure equation
\[
\begin{pmatrix}
t \\
\dot{n} \\
\dot{b}
\end{pmatrix} =
\begin{pmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix}
\begin{pmatrix}
t \\
n \\
b
\end{pmatrix}, \tag{B.10}
\]
derived for instance in §1.5 of [22], from which we deduce the triple-derivative
\[
\dot{X} = \kappa n + \kappa \dot{n} = -\kappa^2 t + \kappa n + \kappa \tau b, \quad ||\dot{X}||^2 = 1. \tag{B.11}
\]

Thus for instance in terms of the Frenet-Serret data,
\[
||\dot{X} \times \ddot{X}||^2 = \kappa^2, \quad \epsilon_{\mu\nu\rho} \dddot{X}^\mu \dddot{X}^\nu \dddot{X}^\rho = \kappa^2 \tau. \tag{B.12}
\]
consistent with the expression for $\tau$ in (B.2). Substituting similarly for $\dot{X}$, $\ddot{X}$, and $\dddot{X}$ in (B.7), we find after a bit of vector algebra

$$
\delta \tau = -\frac{2\tau}{\kappa^2} \left[ \kappa^2 (t \cdot \delta \dot{X}) + \kappa (n \cdot \delta \dot{X}) \right] +
+ \frac{1}{\kappa^2} \left[ \kappa^3 (b \cdot \delta \dot{X}) + \kappa^2 \tau (t \cdot \delta \dot{X}) - \dot{\kappa} (b \cdot \delta \ddot{X}) + \kappa \tau (n \cdot \delta \ddot{X}) + \kappa (b \cdot \delta \dddot{X}) \right],
$$

or more succinctly,

$$
\delta \tau = -\tau t \cdot \delta \dot{X} + \kappa b \cdot \delta \dot{X} - \left( \frac{\tau}{\kappa} \right) n \cdot \delta \ddot{X} - \left( \frac{\dot{\kappa}}{\kappa^2} \right) b \cdot \delta \ddot{X} + \left( \frac{1}{\kappa} \right) b \cdot \delta \dddot{X}. \quad (B.13)
$$

In terms of the respective variations for $d\varphi$ and $\tau$ in (B.6) and (B.14), the variation of the total torsion in (B.3) becomes

$$
\delta T(C) = \frac{1}{2\pi} \int_{S^1} d\varphi \left[ \kappa b \cdot \delta \dot{X} - \left( \frac{\tau}{\kappa} \right) n \cdot \delta \ddot{X} - \left( \frac{\dot{\kappa}}{\kappa^2} \right) b \cdot \delta \ddot{X} + \left( \frac{1}{\kappa} \right) b \cdot \delta \dddot{X} \right]. \quad (B.15)
$$

In (B.15), the term in $\delta \tau$ proportional to $t \cdot \delta \dot{X}$ is exactly cancelled by the variation of the measure $d\varphi$, as required by the underlying reparametrization invariance of the integrand for $T(C)$.

To simplify the formula in (B.15) further, we apply integration-by-parts to the final term, for which

$$
\left( \frac{1}{\kappa} \right) b \cdot \delta \dddot{X} = \frac{d}{d\varphi} \left[ \left( \frac{1}{\kappa} \right) b \cdot \delta \dddot{X} \right] + \left( \frac{\dot{\kappa}}{\kappa^2} \right) b \cdot \delta \ddot{X} - \left( \frac{1}{\kappa} \right) b \cdot \delta \dot{X},
$$

$$
= \frac{d}{d\varphi} \left[ \left( \frac{1}{\kappa} \right) b \cdot \delta \dddot{X} \right] + \left( \frac{\dot{\kappa}}{\kappa^2} \right) b \cdot \delta \ddot{X} + \left( \frac{\tau}{\kappa} \right) n \cdot \delta \dddot{X}. \quad (B.16)
$$

In passing to the second line of (B.16), we recall that $\dot{b} = -\tau n$. Comparing (B.15) to (B.16), we see that the terms involving $\delta \dddot{X}$ precisely cancel those involving $\delta \ddot{X}$, up to a total derivative which integrates to zero on $S^1$.  

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We thus obtain the extremely simple result

\[
\delta T(C) = \frac{1}{2\pi} \oint_{S^1} d\phi \kappa b \cdot \delta \dot{X}, \quad ||\dot{X}||^2 = 1,
\]

\[
= \frac{1}{2\pi} \oint_{S^1} d\phi \epsilon_{\mu\nu\rho} \dot{X}^\mu \ddot{X}^\nu \delta \dot{X}^\rho,
\]

\[
= \frac{1}{2\pi} \oint_{S^1} d\phi \epsilon_{\mu\nu\rho} \dot{X}^\mu \delta X^\nu \ddot{X}^\rho,
\]

as claimed in (1.3.37).
Appendix C: Metric Manifesto

In this appendix we collect some elementary geometric facts related to the Yang-Mills action on the contact three-manifold \((M, H)\), with the contact metric

\[
d s^2_M = \frac{t}{2r} d\kappa(\cdot, J \cdot) + t^2 \kappa \otimes \kappa, \quad r, t \in \mathbb{R}^+. \tag{C.1}
\]

In particular, we wish to check the parameter dependence on \(r\) and \(t\) in the electric-magnetic decomposition (2.1.32) for the three-dimensional Yang-Mills action,

\[
\int_M \text{Tr}(F_A \wedge \star F_A) = 2r \int_M \kappa \wedge d\kappa \text{Tr} \left( \frac{(\kappa \wedge F_A)^2}{\kappa \wedge d\kappa} \right) - \frac{1}{t} \int_M \kappa \wedge \text{Tr} \left[ t_R F_A \wedge J \circ t_R F_A \right]. \tag{C.2}
\]

As the decomposition is a local statement, we do not require \(M\) to be Seifert, nor do we make any assumption about the contact structure \(H \subset TM\) beyond orientability.

For convenience we work dually with a one-form \(\alpha \in \Omega^1(M)\), for which the counterpart of (C.2) is a decomposition for the \(L^2\)-norm \(||\alpha||^2\) with respect to the contact data. Without loss, we work locally on \(M\) with an orthogonal coframe

\[
\{\varphi^1, \varphi^2, \kappa\}. \tag{C.3}
\]

Here \(\varphi^{1,2} \in \Omega^1(H)\) are horizontal one-forms, ie. \(t_R \varphi^{1,2} = 0\), which we choose to satisfy

\[
\varphi^2 = -J(\varphi^1), \tag{C.4}
\]

and

\[
d\kappa = \varphi^1 \wedge \varphi^2. \tag{C.5}
\]
The sign convention in (C.4) ensures that $J$ and $d\kappa$ are compatible. Otherwise, $\varphi^2$ is uniquely determined by $\varphi^1$ in (C.4), and we locally scale $\varphi^1$ so that the relation in (C.5) is obeyed.

In terms of the coframe, the contact metric is given by

$$ds^2_M = \frac{t}{2r} \varphi^1 \otimes \varphi^1 + \frac{t}{2r} \varphi^2 \otimes \varphi^2 + t^2 \kappa \otimes \kappa. \quad (C.6)$$

Evidently the coframe is orthogonal but not generally orthonormal. An orthonormal coframe $\{e_1, e_2, e_3\}$ is easy enough to produce by further scaling,

$$e^1 = \left(\frac{t}{2r}\right)^{1/2} \varphi^1, \quad e^2 = \left(\frac{t}{2r}\right)^{1/2} \varphi^2, \quad e^3 = t \kappa. \quad (C.7)$$

From these expressions we immediately deduce the statement in (2.1.27),

$$\ast \kappa = \frac{1}{t} \ast e^3 = \frac{1}{t} e^1 \wedge e^2 = \frac{1}{2r} d\kappa, \quad (C.8)$$

as well as that in (2.1.28),

$$\text{vol}_M = e^1 \wedge e^2 \wedge e^3 = \frac{t^2}{2r} \kappa \wedge d\kappa. \quad (C.9)$$

Any one-form $\alpha \in \Omega^1(M)$ can be expanded in the orthonormal coframe as

$$\alpha = \alpha_1 e^1 + \alpha_2 e^2 + \alpha_3 e^3, \quad \alpha_{1,2,3} \in \Omega^0(M), \quad (C.10)$$

for some coefficient functions $\alpha_{1,2,3}$. In terms of these, the $L^2$-norm is by definition

$$\int_M \alpha \wedge \ast \alpha = \int_M \left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2\right) \cdot \text{vol}_M. \quad (C.11)$$

Since

$$J \circ \alpha = -\alpha_1 e^2 + \alpha_2 e^1, \quad (C.12)$$
the expression on the right of (C.11) can be compared to the hermitian pairing

\[
\int_M \kappa \wedge \alpha \wedge J \circ \alpha = \int_M \kappa \wedge (\alpha_1 e^1 + \alpha_2 e^2) \wedge (-\alpha_1 e^2 + \alpha_2 e^1),
\]

\[
= \frac{1}{t} \int_M (\alpha_1^2 + \alpha_2^2) \cdot \text{vol}_M.
\]  

(C.13)

From the relation in (C.9) and the contraction \( \iota_R e^3 = t \), also

\[
\int_M \kappa \wedge d\kappa (\iota_R \alpha)^2 = 2r \int_M \alpha_3^2 \cdot \text{vol}_M.
\]  

(C.14)

Combining the results in (C.11), (C.13), and (C.14), we obtain an expression for the \( L^2 \)-norm of \( \alpha \) in terms of the contact data,

\[
\int_M \alpha \wedge \star \alpha = \frac{1}{2r} \int_M \kappa \wedge \kappa \wedge (\iota_R \alpha)^2 - t \int_M \kappa \wedge \alpha \wedge J \circ \alpha, \quad \alpha \in \Omega^1(M).
\]  

(C.15)

Dually, if \( \beta \in \Omega^2(M) \) is any smooth two-form, the \( L^2 \)-norm of \( \beta \) with respect to the contact metric in (C.1) is given by

\[
\int_M \beta \wedge \star \beta = 2r \int_M \kappa \wedge d\kappa \left[ \left( \frac{\kappa \wedge \beta}{\kappa \wedge d\kappa} \right)^2 \right] - \frac{1}{t} \int_M \kappa \wedge \iota_R \beta \wedge J \circ \iota_R \beta, \quad \beta \in \Omega^2(M).
\]  

(C.16)

The formula in (C.16) can be obtained in exactly the same way as (C.15), so we omit the details. Applied to the Yang-Mills curvature \( F_A \), the right-hand side of (C.16) expresses the electric-magnetic decomposition with respect to the contact plane \( H \).