Generalized Optical Theorem Detection in Random and Complex Media

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I dedicate my dissertation to my family.

I express a special feeling of gratitude to my loving parents, Dr. Hanjun Tu and Mrs. Jun Lv, whose consistent support and encouragement makes me who I am today. I am truly thankful having such unconditional love from you.

I dedicate this work to my dear fiancee, Miss Ting Shen. It’s a miracle that we met each other and fell in love in Boston. I feel grateful for the happiness and support you give to me.

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List of Acronyms

2D  Two-Dimensional.
CCD  Charge-Coupled Device.
cdf  cumulative distribution function.
ED  Energy Detector.
FDTD  Finite Difference Time Domain.
GLRT  Generalized Likelihood Ratio Test.
LTI  Linear Time Invariant.
OT  Optical Theorem.
pdf  probability distribution function.
PEC  Perfect Electric Conductor.
PML  Perfect Matched Layer.
ROC  Receiver Operator Characteristic.
ROI  Region Of Interest.
SLM  Space-Light Modulator.
SNR  Signal-to-Noise Ratio.
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Abstract of the Dissertation

Generalized Optical Theorem Detection in Random and Complex Media

by

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Northeastern University, April 2016
Dr. Edwin A. Marengo, Advisor

The problem of detecting changes of a medium or environment based on active, transmit-plus-receive wave sensor data is at the heart of many important applications including radar, surveillance, remote sensing, nondestructive testing, and cancer detection. This is a challenging problem because both the change or target and the surrounding background medium are in general unknown and can be quite complex. This Ph.D. dissertation presents a new wave physics-based approach for the detection of targets or changes in rather arbitrary backgrounds. The proposed methodology is rooted on a fundamental result of wave theory called the optical theorem, which gives real physical energy meaning to the statistics used for detection.

This dissertation is composed of two main parts. The first part significantly expands the theory and understanding of the optical theorem for arbitrary probing fields and arbitrary media including nonreciprocal media, active media, as well as time-varying and nonlinear scatterers. The proposed formalism addresses both scalar and full vector electromagnetic fields. The second contribution of this dissertation is the application of the optical theorem to change detection with particular emphasis on random, complex, and active media, including single frequency probing fields and broadband probing fields.

The first part of this work focuses on the generalization of the existing theoretical repertoire and interpretation of the scalar and electromagnetic optical theorem. Several fundamental generalizations of the optical theorem are developed. A new theory is developed for the optical theorem for scalar fields in nonhomogeneous media which can be bounded or unbounded. The bounded media context is essential for applications such as intrusion detection and surveillance in enclosed environments such as indoor facilities, caves, tunnels, as well as for nondestructive testing and communication systems based on wave-guiding structures. The developed scalar optical theorem theory applies to arbitrary lossless backgrounds and quite general probing fields including...
near fields which play a key role in super-resolution imaging. The derived formulation holds for arbitrary passive scatterers, which can be dissipative, as well as for the more general class of active scatterers which are composed of a (passive) scatterer component and an active, radiating (antenna) component. Furthermore, the generalization of the optical theorem to active scatterers is relevant to many applications such as surveillance of active targets including certain cloaks, invisible scatterers, and wireless communications. The latter developments have important military applications. The derived theoretical framework includes the familiar real power optical theorem describing power extinction due to both dissipation and scattering as well as a reactive optical theorem related to the reactive power changes. Meanwhile, the developed approach naturally leads to three optical theorem indicators or statistics, which can be used to detect changes or targets in unknown complex media. In addition, the optical theorem theory is generalized in the time domain so that it applies to arbitrary full vector fields, and arbitrary media including anisotropic media, nonreciprocal media, active media, as well as time-varying and nonlinear scatterers.

The second component of this Ph.D. research program focuses on the application of the optical theorem to change detection. Three different forms of indicators or statistics are developed for change detection in unknown background media: a real power optical theorem detector, a reactive power optical theorem detector, and a total apparent power optical theorem detector. No prior knowledge is required of the background or the change or target. The performance of the three proposed optical theorem detectors is compared with the classical energy detector approach for change detection. The latter uses a mathematical or functional energy while the optical theorem detectors are based on real physical energy. For reference, the optical theorem detectors are also compared with the matched filter approach which (unlike the optical theorem detectors) assumes perfect target and medium information. The practical implementation of the optical theorem detectors is based for certain random and complex media on the exploitation of time reversal focusing ideas developed in the past 20 years in electromagnetics and acoustics. In the final part of the dissertation, we also discuss the implementation of the optical theorem sensors for one-dimensional propagation systems such as transmission lines. We also present a new generalized likelihood ratio test for detection that exploits a prior data constraint based on the optical theorem. Finally, we also address the practical implementation of the optical theorem sensors for optical imaging systems, by means of holography. The later is the first holographic implementation the optical theorem for arbitrary scenes and targets.
Chapter 1

Introduction

The problem of detecting changes of a medium or environment based on active, transmit-plus-receive wave sensor data is at the heart of many important applications including radar, surveillance, remote sensing, nondestructive testing, and cancer detection. This is a challenging problem because both the change or target and the surrounding background medium are in general unknown and can be quite complex. This Ph.D. dissertation develops a new wave physics-based approach for the detection of targets or changes in rather arbitrary backgrounds. The developed approach is rooted on a fundamental result of wave theory called the optical theorem, which gives real physical energy meaning to the statistics used for detection.

The dissertation is composed of two main parts. The first part (consisting of Chapter 2, Chapter 3, and Chapter 5) significantly expands the theory and understanding of the optical theorem for arbitrary probing fields and arbitrary media including nonreciprocal media, active scatterers, as well as time-varying and nonlinear scatterers. The developed formalism addresses both scalar and full vector electromagnetic fields. The second part (consisting of Chapter 4, Chapter 6, Chapter 7, and Chapter 8) explores applications of the optical theorem, with emphasis on change detection in random and complex media. Both single frequency as well as multifrequency and broadband fields are considered. Receiver operator characteristic (ROC) curves are provided for many examples which consistently illustrate the effectiveness of the proposed optical theorem detectors versus alternative approaches, such as the classical energy detector and the ideal matched filter approach (which relies on more prior information about the target and the background). The second part of the thesis also addresses the application of the optical theorem to detect changes in transmission line systems. In addition, it also presents the holographic implementation of the optical theorem for optical systems in which one measures only field intensity. Thus this dissertation presents a new theory of the optical
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Theorem along with applications of this fundamental result of wave scattering theory. The application areas emphasized are the detection of unknown targets in unknown background media, the detection of changes in high frequency circuits such as transmission lines, and the use of holography to optically measure the power extinction of targets in complex backgrounds.

The main contributions of this dissertation can be summarized as follows:

1. This dissertation develops the most general theory of the optical theorem for scalar fields under arbitrary probing fields, nonhomogeneous media, and active scatterers that have a self-emitting component, including the scalar form of the reactive optical theorem. The derived formulation holds for arbitrary passive scatterers, which can be dissipative, as well as for the more general class of active scatterers which are composed of a (passive) scatterer component and an active, radiating (antenna) component.

2. This effort develops the most general theory of the generalized optical theorem of electromagnetics directly in the time domain. This contribution holds for arbitrary fields and media, and arbitrary scatterers including those made of nonlinear and time-varying media. It includes the time domain form of the reactive optical theorem. Both background and scatterers can be anisotropic and nonhomogeneous.

3. This dissertation demonstrates that one can design a new class of optical theorem detectors that are based on the optical theorem. They are quite effective in change detection of unknown targets in unknown media. The results are obtained for both the single frequency and multifrequency or broadband cases, and applied to general transmit-receive array systems. Several variants of the optical theorem detector are considered, being of particular interest optical theorem detectors linked to the real, reactive, and apparent extinguished power of the scattering phenomenon. The new detectors are consistently found to perform better than alternative approaches such as the energy detector. They perform close to the matched filter even though they do not rely (unlike the matched filter) on prior information about the target or the background. In addition, in the final part of the thesis we present a new generalized likelihood ratio test (GLRT) detector that exploits an optical theorem data constraint.

4. In addition, this effort provides the detailed theory and implications of the optical theorem for transmission lines or one-dimensional propagation systems. This establishes, as a corollary, a theoretical basis to compare the apparent power optical theorem detector and the conventional energy detector. It also provides the starting point of a more general theory applicable to
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arbitrary high frequency circuits and networks, with applications to change detection of civil structures, electronic systems, and power systems.

5. The thesis also addresses the practical implementation of the optical theorem via holography, which permits the intensity-only measurement of the extinguished power by a scatterer that is probed by an arbitrary probing field and is embedded in an arbitrary medium. Prior work had considered only free space. This is the first optical theorem holography for more arbitrary media or scenes.

The rest of the dissertation is organized as follows. This chapter reviews the state of the art on the research topics tackled in the dissertation. The novelty of the proposed work is summarized in the context of the existing literature. In Chapter 2, the optical theorem is developed for scalar fields in general nonhomogeneous media. Extension to active scatterers having a self-emitting component is given in Chapter 3. Chapter 4 focuses on demonstrating the application to detection in random and complex media. Many numerical results (ROC curves) are given which show the three variants of the optical theorem detector for the detection of unknown targets in unknown media. In Chapter 5, the optical theorem is developed directly in the time domain. In that chapter, we consider full vector electromagnetic fields in quite general anisotropic media. In Chapter 6, we present the optical theorem for transmission lines. In Chapter 7, we develop a detection-theoretic framework, based on the generalized likelihood ratio test (GLRT), which sheds further insight on the interpretation and exploitation of prior information extracted from the optical theorem. Finally, in Chapter 8, we present the holographic implementation of the optical theorem. Appendix A provides a review of theoretical results for scattered fields used in the example in Chapter 8.

1.1 Review of the Optical Theorem

The optical theorem is a basic result of wave scattering theory, which applies to acoustic, electromagnetic, and quantum-mechanical scattering. In its most familiar form, which is applicable for plane wave excitation and free space, it relates the total extinction cross section of the scatterer due to both dissipation and scattering (to all directions) to the imaginary part of the forward scattering amplitude [1, 2]. For scalar waves, relevant to acoustics and quantum mechanics, this well known result can be written in the form [1, 3]:

\[
\sigma_{tot} = \frac{4\pi}{k} \Im \left( f(s_0, s_0) \right) \quad (1.1)
\]
where \( f(s_0, s_0) \) is the (forward) scattering amplitude, corresponding to the direction of incidence defined by the unit vector \( s_0 \), \( k \) is the wavenumber of the field, and \( \sigma_{tot} \) is (apart from a multiplicative factor) equal to the total extinction cross section of the scatterer including elastic and inelastic scattering. A similar result applies to electromagnetic waves. The extinction cross-section of the scatterer is a measure of its rate of uptake of energy from the incident plane wave, irrespective of whether the energy is permanently absorbed or subsequently re-radiated (scattered) into the surrounding space. The beauty of the optical theorem is in the simplicity and generality of the relation between the extinction cross-section and the forward scattering amplitude, regardless of the complexity of the interaction between the scatterer and the incident wave. The optical theorem is a consequence of energy conservation in electromagnetics and acoustics or of conservation of probability in quantum mechanics.

Originally, the optical theorem was obtained independently by Wolfgang Von Sellmeier and Lord Rayleigh almost simultaneously in 1871. Lord Rayleigh applied this result in the study of the color and polarization of the sky. The equation was later extended to quantum scattering theory by several researchers such as Wheeler [4] and Bohr et al. [5], and was known after the publication [5] as “the Bohr-Peierls-Placzek relation”. It was first referred to as “the optical theorem” in a book published by Bethe, De Hoffman and Schweber in 1955 [6].

In 1943, Heisenberg employed the S matrix in his investigations on the quantum scattering of particles. Heisenberg proved the unitarity of the S matrix and, as an observable consequence of this abstract property, derived the generalized optical theorem [7]. This more general result, known as the generalized optical theorem, puts constraints on the entries of the scattering amplitude \( f(s', s) \) (for scattering direction \( s' \) and incidence direction \( s \)), and is usually stated as

\[
\Im f(s', s) = \frac{k}{4\pi} \int f(s', s'') f^*(s'', s) ds''.
\] (1.2)

A special case occurs when \( s' = s = s_0 \). This corresponds to the result in equation [1.1] the ordinary optical theorem. As World War II ended, there came up a explosion of research on scattering theory, with further usage of the optical theorem. Representative work was that of Wick [8], Lax [9, 10], and Glauber together with Schomaker [11]. At this time, it was still called the Bohr-Peierls-Placzek relation in many authoritative books at that time, such as Messiah [12], Gottfried [13], and Sakurai [14]. Meanwhile, physicists applied this theorem more often. In 1952, this theorem was first called “a well-known theorem of optics” in a paper published by Rohrlich and Gluckstern [15]. After that, it became customary for physicists to call it simply “the optical theorem”.

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In the context of electromagnetic theory, van de Hulst [16] rediscovered the optical theorem in 1949, unaware that it was well known in optics and in quantum scattering theory. Nevertheless, the first edition of the optics treatise by Born and Wolf [1] credited van de Hulst with the first derivation of the theorem in the domain of classical optics. In addition, the optical theorem was derived without using phase shift method (S matrix) and proved to be valid for all the scattering phenomena by Levine and Schwinger [17]. For a more detailed discussion of the history and significance of the theorem, see [18], which reproduced van de Hulst’s scalar derivation as a very nice and intuitive exposition of the theorem from a physical point of view, albeit one that lacked the generality and accuracy of the more general derivation. Several years after World War II, the optical theorem was regarded as an effective tool to calculate the total extinction cross section instead of old-fasion direct (far field) integration. Nevertheless, if that were the only use of this theorem, its value would be limited. Fortunately, throughout the years, many new applications had emerged. Hanbury-Brown and Twiss discovered the optical-theorem-based method to measure the phase of intensity fluctuation correlations effect between two different and independent incident fields scattered by the same scatterers [19]. Particle beam scattering applications were addressed in papers by Goldberger, Lewis and Watson [20, 21]. An important use of the generalized optical theorem was to regard equation 1.2 as a nonlinear integral equation of the phase of the scattering amplitude if the differential cross section was given as a function of the scattering angle. This conclusion was obtained by different scientists independently including Puzikov et al. [22], Newton [23] and Martin [24]. What is more, an interesting application of the optical theorem was found in an article by Carney et al. [25], where it was generalized to scatterers whose properties are known only in a statistical sense. The optical theorem was there shown to be a powerful tool for inverse scattering applications.

Recent works address other applications, such as testing of algorithms for the computation of scattered wave fields [26, 27, 28]. Another area is the estimation of backscattering from measurements of the scattered wave field taken at other angles [29]. Chadan and Sabatier use the optical theorem in the formulation of inverse problems in quantum scattering theory [30]. In seismology, De Hoop and Snieder discuss the acoustic optical theorem [31] and a special version of the theorem for surface waves and its application to the understanding of propagation attenuation (see [32]). In addition, by using a statistical approach it may be possible to infer the structure of the scattering media. Representative contributions in this area include the unified Green’s operator theory in [33] and the works in imaging [34, 35, 36, 37] showing connections to the Green’s function extraction from field correlations. Snieder et al. and Wapenaar et al. have recognized that the Green’s function representation used in seismic interferometry resembles the generalized optical theorem.
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They showed that the generalized optical theorem can be obtained as a special case from the Green’s function representation for interferometry. The derivation is similar to that in [38, 39] in the sense that it is based on substituting far-field expressions for direct and scattered waves in the Green’s function representation. However, instead of using stationary phase analysis, they use reciprocity theory to analyze this representation, term by term. By comparing the final result with the original Green’s function representation, the generalized optical theorem follows straightforwardly.

Another set of important developments is the generalization to nonhomogeneous media shown in [40, 41, 42, 43]. Further formulations for specialized sensing geometries and coordinate systems are given in [44, 45, 46]. The optical theorem has also found application in lasers [47], meta-materials [48], sensors [49, 50], and other areas [51].

1.2 Dissertation Contributions, Relevance, and Novelty

Of particular relevance for this Ph.D. dissertation are the previous efforts to generalize the ordinary optical theorem to arbitrary fields and media. This was done by Carney et al. in the paper [40] for scalar fields. And this is expanded in a subsequent paper by the same group [41] for vector electromagnetic fields with emphasis on including evanescent fields. On the other hand, the work in those papers suggests only precursor relations for the optical theorem. But it does not conclusively prove the conditions under which they are equivalent to the theorem. In addition, only the ordinary optical theorem is considered. The more general form of this result, called the generalized optical theorem, is not discussed in those previous papers. The vector work in [41] considers only isotropic media. Also relevant to this thesis is the expanded generalization by Marengo [52] applicable to both the ordinary and the generalized optical theorem. There the generalized optical theorem is generalized for the full vector electromagnetic fields including arbitrary fields and media including anisotropic materials. The same paper also proposes a new reactive optical theorem and demonstrates the role of time reversal in the practical implementation of the theorem. The results in [52] correspond to the frequency domain optical theorem for linear time-invariant media. The time-domain version of the optical theorem is less known than its frequency domain counterpart. The time domain optical theorem was studied for the usual special cases of plane wave excitation and free space in [53, 54, 55, 56, 57]. Some of the motivation is the validation of computational methods. More recent work by Stumpf and Lager [58] has addressed time-domain antenna applications but again the focus is only on plane waves and free space.

Despite the existing literature, a number of pending fundamental questions remain to be
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addressed in this area. For example, prior to the effort leading to the present dissertation, the optical theorem had not been generalized for active scatterers having a self-emitting component. This extension is relevant to novel cloaking and invisibility devices. In this dissertation, a new theory is developed (in Chapter 2 and Chapter 3) for the optical theorem for scalar fields in nonhomogeneous media which can be bounded or unbounded. The bounded media context is essential for applications such as intrusion detection and surveillance in enclosed environments. The developed scalar optical theorem theory applies to arbitrary lossless backgrounds and quite general probing fields including near fields which play a key role in super-resolution imaging. The derived formulation holds for arbitrary passive scatterers, which can be dissipative, as well as for the more general class of active scatterers which are composed of a (passive) scatterer component and an active, radiating (antenna) component. The generalization of the optical theorem to active scatterers has important military applications. The derived theoretical framework includes the familiar real power optical theorem describing power extinction due to both dissipation and scattering as well as a reactive optical theorem related to the reactive power changes. Meanwhile, the developed approach naturally leads to three optical theorem indicators or statistics, presented and validated in Chapter 4, which can be used to detect changes or targets in unknown complex media. In summary, this dissertation develops (in Chapter 2 and Chapter 3) a general scalar theory of the optical theorem for arbitrary probing fields and backgrounds, which applies to conventional scatterers as well as scatterers having an active self-emitting component. This dissertation also demonstrates (in Chapter 4) the application of the resulting optical theorem theory to detection of unknown targets or medium changes in unknown media. Furthermore, in Chapter 4, the dissertation also illustrates optical theorem detectors for multifrequency and broadband data.

In addition, there is a vast amount of literature associated with the frequency-domain optical theorem. On the contrary, existing works on the optical theorem in the time domain are very limited (see [53, 54, 55, 56, 57, 58]). However, those developments hold only for plane waves and free space. They apply only to the ordinary optical theorem and not the generalized optical theorem. It is important to extend this result to arbitrary transient fields and media. In addition, it is important to develop the reactive power form of the optical theorem directly in the time domain. Importantly, this fundamental generalization enables applicability to scatterers that can be nonlinear and time-varying. Consequently, this dissertation tackles (in Chapter 5) this fundamental open problem. This effort leads to the development of the electromagnetic form of the optical theorem theory directly in the time domain. This theoretical development significantly expands the state of the art in this area: it applies to arbitrary full vector fields, and arbitrary media including anisotropic media, nonreciprocal...
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media, active media, as well as time-varying and nonlinear scatterers. Envisioned applications include novel imaging methods, tools to test computational methods, and the development of novel detection methods for transient and broadband data.

This dissertation also contains novel results on the application of the optical theorem for transmission lines. This simple system is relevant to a broad range of electromagnetic wave propagation and scattering systems such as microwave circuits, subsurface sensing of the stratified media, models of sonar, lidar and radar, and other applications. Understanding of the optical theorem for transmission lines is essential to understand the optical theorem in higher dimensions. In addition, this contribution provides, among other results, the proof that for single-sensor systems the apparent power form of the optical theorem detector is essentially equivalent to the classical energy detector approach. On the other hand, we find that this is not the case for multiple receivers or frequencies, in which the optical theorem detectors consistently perform better than the energy detector. Our published work [59] on this fundamental problem, developed during the course of the present dissertation research, was the first to address this basic problem of wave theory. A summary of the main results in this basic topic is given in Chapter 6. Moreover, in Chapter 7 the dissertation presents a new detection approach that is based on a generalized likelihood ratio test with optical theorem constraint. The constraint is very general. It can be applied to simple systems such as transmission lines. We present the theoretical derivation of this new detection approach. The successful detection performance of this method is validated for several transmission line examples.

In addition, even though [52] addresses the practical implementation of the optical theorem in complex media, by means of time reversal, the proposed approach holds only for acoustics and electromagnetics. In optics the available data is “intensity only”, so the question is how to implement this result in optics. In the final part of this dissertation, we address the design of holography-based optical theorem sensors. They can be used in practice for the measurement of the power extinguished by a scatterer that is interrogated by an arbitrary probing field and is embedded in an arbitrary medium. In this regard, there is very limited literature associated with the holographic implementation of the optical theorem. [60] discusses the nonuniqueness of optical theorem detectors, and sheds light on optical implementation in free space. The work of Berg et al. [61] illustrates an approach to measure power extinction using holography. But this approach holds only for the typical plane wave and free space scenario. Our contribution is more general. It holds for arbitrary backgrounds and fields. We develop several practical configurations based on lens imaging combined with holography, with particular emphasis on the classical Leith-Upatnieks holography. Motivation for this research is provided by the possibility of developing novel sensors of constitutive properties of materials, as well
as novel approaches for short-range optical communication systems, information storage devices, biometric readers, and scanners. Representative efforts in this area are described in [62, 63, 64], which address holographic fingerprinting sensors, in [65, 66, 67], which address hologram-based identification cards, and in [68, 69, 70] which discuss holographic scanning systems.

A major application of this dissertation is the detection of unknown targets or media changes in unknown background media. This is a very general problem that has many important practical applications. To facilitate the subsequent discussion of change detection applications of the optical theorem, we conclude this introduction chapter with an outline of the basic theory and relevant literature pertinent to this fundamental problem.

Detecting changes of a medium or environment based on active, transmit-plus-receive wave sensor data is a big problem that plays a key role in many important applications including radar, surveillance, remote sensing, nondestructive testing, and cancer detection. This is a challenging problem because both the change or target and the surrounding background medium are in general unknown and can be quite complex. In the previous literature, a classical detection method of unknown signals, which is called the energy detector, is usually adopted [71]. In addition, in another related area, time reversal focusing has been a successful concept in acoustics and electromagnetics. It is used to refocus waves in highly scattering and multipath channels, see, e.g., the seminal papers [72, 73, 74, 75, 76, 77, 78, 79, 80]. Other representative works in this area include those in [81, 82, 83, 84, 85, 86, 87, 88, 89, 90]. Application of the time reversal principle to the detection problem has led to a change detection approach known as the time reversal detector which has been the topic of well-known papers by Moura and co-workers [91, 92, 93, 94, 95]. Nehorai and co-workers have considered a related method in [96, 97]. Their studies address the binary hypothesis test of detecting the presence or absence of a target in a highly cluttered environment [98, 99] by using time reversal. This method is adaptive. In addition, it relies on using not only the baseline scattering data of the probing array, but also additional data associated to a second round of experiments using time-reversal signals. The latter lead to enhanced focusing on the target and can thus enhance the signal-to-noise ratio (SNR). This enhances the detection performance relative to the use of a single set of (generally not-time-reversed) experiments.

Likewise, the optical theorem motivates new methods for detection. Three different forms of indicators or statistics are developed for change detection in unknown background media: a real power optical theorem detector, a reactive power optical theorem detector, and a total apparent power optical theorem detector. Marengo and Gruber in [100] have obtained encouraging preliminary results suggesting the effectiveness of the apparent power form of the optical theorem detector.
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This dissertation expands the understanding and characterization of the optical theorem approach to change detection including active targets and broadband fields that were not addressed in [100]. Furthermore, detailed work is shown to fully characterize the properties of the optical theorem detector approach, including its relevance for active scatterers. No prior knowledge is required of the background or the change or target. In this work the performance of the three proposed optical theorem detectors is compared with the classical energy detector approach for change detection. The derived theory also sheds insight on the similarities and differences between the proposed optical theorem approach to detection and the existing time-reversal detector.

In summary, this dissertation demonstrates the broad generality of the optical theorem for arbitrary probing conditions and media. It exploits the derived results for the design of novel optical theorem approaches for change detection in random, complex, and active media. This work is the first to generalize the optical theorem to active scatterers, the latter being an important generalization for cloaking and invisibility systems and imaging applications. This work provides the most general theory of the electromagnetic full vector form of the optical theorem in the time domain, addressing arbitrary anisotropic, nonreciprocal, nonhomogeneous media, as well as nonlinear and time-varying scatterers. It leads to novel optical theorem detectors for both passive and active scatterers or medium changes, in rather arbitrary media including both bounded and unbounded media. The methods hold for both single and multiple frequency conditions or broadband fields. This work comparatively addressed the effectiveness of the proposed optical theorem detector approaches versus conventional approaches such as the energy detector. In addition, this effort provides the particular theory and applications of the optical theorem for transmission lines, as well as new holographic-based implementations of optical-theorem-based sensors for optical applications.
Chapter 2

Scalar Optical Theorem in Nonhomogeneous Media

2.1 Background

In this chapter, we develop a new theory of the optical theorem for scalar fields in nonhomogeneous media. The derived formulation holds for arbitrary scatterers (possibly lossy) and applies to arbitrary lossless backgrounds and general incident fields. The derived optical theorems can be implemented in practice via standard wave time reversal and phase conjugation techniques developed in recent decades in the fields of acoustics, electromagnetic and optics. In particular, we derive the theoretical framework of the real power optical theorem associated with the dissipated and scattered power, and a novel reactive power optical theorem corresponding to the energy storage of the scattering phenomenon. Moreover, these results are pertinent for the design of novel energy sensors. The results also lead to three new and physically well-motivated detection algorithms that are studied in Chapter 4. Meanwhile, we also give a brief derivation of a related scattering theorem, which is linked to the well-known approach called energy detector. These results hold for quite arbitrary lossless background media (bounded or unbounded) embedding the arbitrary scatterer.

2.2 Scattering Theory Review

In this chapter, radiation and scattering are described in the context the scalar Helmholtz equation. In the basic free space case, this means that radiating sources $\rho$ and their fields $\psi$ are
related in the frequency domain by

\[ \nabla^2 + k_0^2(\omega)\psi(r,\omega) = \rho(r, \omega) \]  

(2.1)

where \( k_0 = \omega/c \), where \( \omega \) is the angular oscillation frequency and \( c \) is the speed of light. For simplicity, next we suppress the frequency dependence, with the understanding that all quantities depend on the frequency. Thus (2.1) becomes

\[ (\nabla^2 + k_0^2)\psi(r) = \rho(r). \]  

(2.2)

The field \( \psi \) due to source \( \rho \) of compact spatial support \( \tau \) obeys suitable boundary conditions.

In this chapter we focus on unbounded media in which the fields obey the radiation condition at infinity. Here it is important to point out that the main results developed in this thesis apply also to bounded media, as we show in our key paper [101]. The extension to bounded media is important since it enables application of the principles developed in this thesis to enclosed environments such as caves, tunnels, indoor environments, cavities, waveguides, and so on. For brevity, we do not dwell next on the theoretical formalities behind this important extension. Instead, the interested reader is referred to our key paper [101] which presents the pertinent theoretical account.

Under the classical radiation condition the relevant field is given by

\[ \psi(r) = \int_\tau dr' \rho(r')G_0(r, r') \]  

(2.3)

where \( G_0 \) is the free space Green’s function which is given by

\[ G_0(r, r') = -\frac{e^{ik_0|r-r'|}}{4\pi|r-r'|}. \]  

(2.4)

It has the far-zone behavior

\[ G_0(r, r') \sim -\frac{e^{ik_0r}}{4\pi r}e^{-ik_0\hat{r}\cdot r'} \text{ as } k_0r \to \infty \]  

(2.5)

where \( r \equiv |r| \) and \( \hat{r} \equiv r/r \). Using this in (2.3) gives the far-zone behavior of the field,

\[ \psi(r\hat{r}) \sim f(\hat{r})e^{ik_0r}/r, \]  

(2.6)

where

\[ f(\hat{r}) = -\frac{1}{4\pi} \int_\tau dr' \rho(r')e^{-ik_0\hat{r}\cdot r'} \]  

(2.7)

is the far-field radiation pattern of the field.
CHAPTER 2. SCALAR OPTICAL THEOREM IN NONHOMOGENEOUS MEDIA

2.3 Nonhomogeneous Media

The previous results are expanded next to nonhomogeneous media. In this more general
case
\[
[\nabla^2 + k^2(\mathbf{r})] \psi(\mathbf{r}) = \rho(\mathbf{r}),
\]
(2.8)
where \( k(\mathbf{r}) \) denotes the space-dependent wavenumber. We focus on lossless or non-dissipative media
so that \( k(\mathbf{r}) \) (which corresponds next to the background medium) is real-valued. This assumption
preserves both reciprocity and time-reversal invariance. We assume that the medium is formed by
material of finite extent that is embedded in unbounded free space so that \( k(\mathbf{r}) \to k_0 \) as \( r \equiv |\mathbf{r}| \to \infty \). Thus the radiated field \( \psi \) obeys the radiation condition at infinity.

In particular, the field \( \psi \) radiated by a source \( \rho \) of support \( \tau \) can be expressed in terms of
the background Green’s function \( G(\mathbf{r}, \mathbf{r}') \) as
\[
\psi(\mathbf{r}) = \int_{\tau} d\mathbf{r}' \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}')
\]
(2.9)
where it can be shown that \( G(\mathbf{r}, \mathbf{r}') \) is related to the free space Green’s function \( G_0(\mathbf{r}, \mathbf{r}') \) via
\[
G(\mathbf{r}, \mathbf{r}') = G_0(\mathbf{r}, \mathbf{r}') + \int d\mathbf{r}'' G_0(\mathbf{r}, \mathbf{r}'') v(\mathbf{r}'') G(\mathbf{r}'', \mathbf{r}')
\]
(2.10)
where
\[
v(\mathbf{r}) = k_0^2 - k^2(\mathbf{r}).
\]
(2.11)
According to (2.5, 2.9, 2.10) the field has the far-zone behavior in (2.6) with
\[
f(\hat{\mathbf{r}}) = \int_{\tau} d\mathbf{r}' \rho(\mathbf{r}') g(\hat{\mathbf{r}}, \mathbf{r}')
\]
(2.12)
where
\[
g(\hat{\mathbf{r}}, \mathbf{r}') = e^{-ik_0 \hat{\mathbf{r}} \cdot \mathbf{r}'} + \int_{\tau} d\mathbf{r}'' e^{-ik_0 \hat{\mathbf{r}} \cdot \mathbf{r}''} v(\mathbf{r}'') G(\mathbf{r}'', \mathbf{r}').
\]
(2.13)

2.4 Scattered Fields

We formulate scattering in the familiar context in which the medium inside the region of
interest \( U' \) is probed by an arbitrary incident field obeying the homogeneous form of (2.8) in \( U' \), thus
\[
[\nabla^2 + k^2(\mathbf{r})] \psi_i(\mathbf{r}) = 0 \quad \mathbf{r} \in U'.
\]
(2.14)
When inhomogeneities or perturbations (scatterers) are added to the background medium considered above, the wavenumber changes from the background wavenumber \( k(\mathbf{r}) \) to the total wavenumber \( K(\mathbf{r}) \) so that the equation for the total field becomes
\[
[\nabla^2 + K^2(\mathbf{r})] \psi(\mathbf{r}) = 0.
\] (2.15)

The scatterer is made of an arbitrary material, and thus the value of the wavenumber in the scatterer’s support is an arbitrary complex number. In this chapter we consider only the material part of the scatterer. It can be a dissipative material, or even an active medium. In the next chapter we develop a generalization to scatterers that have both the material constituent plus an additional (possibly independent) self-emitting component. We call those more general scatterers as “active scatterers” to differentiate them from the more typical scatterers considered in this chapter.

It follows from (2.8, 2.15) that
\[
[\nabla^2 + k^2(\mathbf{r})] \psi(\mathbf{r}) = \rho_{\text{ind}}(\mathbf{r})
\] (2.16)

where we have introduced the induced source
\[
\rho_{\text{ind}}(\mathbf{r}) = \mathcal{V}(\mathbf{r}) \psi(\mathbf{r})
\] (2.17)

where the scattering potential function
\[
\mathcal{V}(\mathbf{r}) = k^2(\mathbf{r}) - K^2(\mathbf{r}).
\] (2.18)

We assume \( \mathcal{V} \) to be of compact support \( \tau \). Writing the total field as the sum of the incident field \( \psi_i \) plus a perturbation component called the scattered field \( \psi_s \) we get
\[
\psi(\mathbf{r}) = \psi_i(\mathbf{r}) + \psi_s(\mathbf{r}).
\] (2.19)

From this expression and (2.14, 2.16) we get
\[
[\nabla^2 + k^2(\mathbf{r})] \psi_s(\mathbf{r}) = \rho_{\text{ind}}.
\] (2.20)

Using (2.9, 2.10) we get that the scattered field is given by
\[
\psi_s(\mathbf{r}) = \int_{\tau} d\mathbf{r}' \rho_{\text{ind}}(\mathbf{r}') G(\mathbf{r}, \mathbf{r}').
\] (2.21)

We outline next the key principles of reciprocity and interaction, which enable us to sense extinction power of the scattering phenomenon via scattered field measurements outside a certain scattering region of interest, \( \tau \subseteq U' \), where the scatterer of support \( \tau \) resides.
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Figure 2.1: The reaction $<1,2>$ of field 2 on source 1, given by $<1,2>=\int d\rho_1 \psi_2$, is equal to the reaction $<2,1>$ of field 1 on source 2 (see eq. 2.23).

2.5 Reciprocity and Interaction Relations

To arrive at the optical theorem for arbitrary fields and media, we need two key principles of wave theory. The first principle is the reciprocity theorem, which corresponds more generally to what is called reciprocity relation of the convolution type. The second principle characterizes the source-field power or energy interaction between a source and a field, and is a corollary of a more general relation called reciprocity of the correlation type. These connections become obvious in chapter 5 where we present the electromagnetic formulation of the generalized optical theorem in the time domain. Next we outline the basic reciprocity and interaction relations as they apply to scalar fields.

Consider two sources and their fields, $\rho_1, \psi_1$ and $\rho_2, \psi_2$ as illustrated in Figure 2.1 which obey the Helmholtz equation, i.e.,

$$[\nabla^2 + k^2(r)]\psi_j(r) = \rho_j(r) \quad j = 1, 2$$  \hspace{1cm} (2.22)

plus the radiation condition at infinity. It is not hard to show that the following condition, called reciprocity, holds:

$$\int d\rho_1(r)\psi_2(r) = \int d\rho_2(r)\psi_1(r).$$  \hspace{1cm} (2.23)

This result is key in the formulation of measurements of the field. For example, consider a receiver characterized by receiver mode $R(r)$. Then the output $a_r$ of the measurement of the field $\psi_1$ at this
receiver is equal to the projection of the field onto this mode, in particular,

\[ a_r = \int d\mathbf{r} R(\mathbf{r}) \psi_1(\mathbf{r}). \]  

(2.24)

The receiver mode \( R \) also has a transmit counterpart, in which this mode corresponds to a radiation mode (antenna) whose generated field is given by

\[ \psi_R(\mathbf{r}) = \int d\mathbf{r}' R(\mathbf{r}') G(\mathbf{r}, \mathbf{r}'). \]  

(2.25)

It follows from (2.25) that the measurement \( a_r \) defined in (2.24) is also equal to

\[ a_r = \int d\mathbf{r} \rho_1(\mathbf{r}) \psi_R(\mathbf{r}). \]  

(2.26)

Thus reciprocity is key for the measurement of the field. In electromagnetics, it can be thought as the key result to measure forces. Next we outline another key relation, pertinent to the measurement of the exchange energy, power or interaction, between a source and a field.

Consider a source \( \rho \) of support \( \tau \) whose generated field is \( \psi \). Using (2.8) and the identity

\[ \nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \nabla^2 v \]  

(2.27)

one finds that

\[ \psi^* (\nabla^2 + k^2) \psi = \rho \psi^* \]

\[ \nabla \cdot (\psi^* \nabla \psi) - |\nabla \psi|^2 + k^2 |\psi|^2 = \rho \psi^*. \]  

(2.28)

Here we conveniently introduce the quantity

\[ F = \psi^* \nabla \psi \]  

(2.29)

which is analogous to the Poynting vector of electromagnetic theory. It is also called the energy flux vector. Substituting (2.29) in (2.28) we get

\[ \Im \nabla \cdot F + \Im(k^2)|\psi|^2 = \Im(\rho \psi^*) \]  

(2.30)

and

\[ \Re \nabla \cdot F - |\nabla \psi|^2 + \Re(k^2)|\psi|^2 = \Re(\rho \psi^*) \]  

(2.31)

where \( \Re \) and \( \Im \) denote the real and imaginary parts, respectively. For real \( k^2 \), \( \Im(k^2) = 0 \). Thus (2.30) becomes

\[ \Im \nabla \cdot F = \Im(\rho \psi^*). \]  

(2.32)
Here it is convenient to introduce the following quantity \( I \), which we call the source-field interaction integral:

\[
I = \int \! d\mathbf{r} \rho(\mathbf{r}) \psi^*(\mathbf{r}). \tag{2.33}
\]

By integrating equation (2.32) over a sufficiently large ball such that the far-field approximation \( (2.6) \) holds over the surface of the ball, and employing the divergence theorem, one obtains the following relation which characterizes the real-power budget:

\[
P = \Im I = k_0 \int_{4\pi} \! d\hat{\mathbf{r}} |f(\hat{\mathbf{r}})|^2 \tag{2.34}
\]

where \( f(\hat{\mathbf{r}}) \) is the corresponding far-field radiation pattern. The quantity \( P \) is, apart from a multiplicative factor that depends on the physical nature of the radiation under consideration, equal to the real, average power radiated away from the source. Thus the imaginary part \( \Im I \) of the source-field interaction integral \( I \) represents the real power radiated by the source in the lossless embedding medium under consideration.

In addition, it is not hard to show that in the far zone, the relevant \( O(r^{-2}) \) contributing term of \( F \) is purely imaginary, so that by integrating \( (2.6) \) over all space and using the divergence theorem one obtains yet another key result, which defines the reactive power associated with the storage of wave-field energy everywhere in space:

\[
Q = \Re I = \int \! d\mathbf{r} \left[ k^2(\mathbf{r}) |\psi(\mathbf{r})|^2 - |\nabla \psi(\mathbf{r})|^2 \right]. \tag{2.35}
\]

Thus the real part \( \Re I \) of the interaction integral \( I \) corresponds to the reactive power of the source.

The preceding results describe self-field powers, put by a source to its own field. More generally, we can consider cross-field powers as well, put by a source to an arbitrary field. Based on the geometry shown in Figure 2.1, this follows readily by considering two pairs of sources and fields, say \( \rho_1, \psi_1 \) and \( \rho_2, \psi_2 \). The interaction integral of the form \( (2.33) \) for the resulting total source \( \rho_1 + \rho_2 \) and total field \( \psi_1 + \psi_2 \) is given by the sum of two self-field terms plus two cross-field terms. The latter correspond physically to the sought-after cross-field power perturbations. Thus in this broader picture the power delivered by a source \( \rho_1 \) to an arbitrary field \( \psi_2 \) is governed by the interaction integral

\[
J = \int \! d\mathbf{r} \rho_1(\mathbf{r}) \psi_2^*(\mathbf{r}) \tag{2.36}
\]

and, in particular, the real power put by source \( \rho_1 \) into field \( \psi_2 \) is given by

\[
P_{1 \rightarrow 2} = \Im J \tag{2.37}
\]
while the reactive power put by source \( \rho_1 \) into field \( \psi_2 \) is given (apart from a multiplicative factor) by
\[
Q_{1 \rightarrow 2} = \Re J.
\] (2.38)

Conversely, the real and reactive powers taken away from field \( \psi_2 \) by source \( \rho_1 \) are \( P_{1 \leftarrow 2} = -\Im J \) and \( Q_{1 \leftarrow 2} = -\Re J \), respectively. These results are used next to obtain the sought-after real power and reactive power optical theorems for scalar fields.

## 2.6 Optical Theorem in Nonhomogeneous Media

In this section we discuss the optical theorem for conventional scatterers. They have only the material component and lack the separate self-emitting component which is treated in the next chapter. We develop two optical theorems, one for the real extinguished power and another for the reactive extinguished power. These results give physical meaning to the optical theorem detectors used in this dissertation. They suggest a real-power optical theorem detector, a reactive-power optical theorem detector, as well as a total (apparent) power optical theorem detector. In addition, in this section we also outline other related findings established in our key paper [101] which assign physical meaning to the more conventional energy detector approach to change detection.

### 2.6.1 Derivation of the Optical Theorem

Consider a lossless background medium having space-dependent wavenumber \( k(r) \). Let \( \psi_i(r) \) be an arbitrary incident field used to probe a scatterer characterized by scattering potential \( \mathcal{V}(r) \) of support \( \tau \subseteq U' \).

It follows from (2.20) and (2.21), that the scattered field \( \psi_s \) is the field generated by the induced source \( \rho_{ind} \). Thus the real scattered power \( P_s \) is given from (2.33) and (2.34) by
\[
P_s = \Im \int_{U'} d\mathbf{r} \rho_{ind}(\mathbf{r}) \psi_s^*(\mathbf{r}).
\] (2.39)

From this, plus equations (2.17) and (2.18) and the fact that the background \( k^2(\mathbf{r}) \) is real-valued, one obtains
\[
P_s = \int_{U'} d\mathbf{r} \Im \mathcal{V}(\mathbf{r}) |\psi(\mathbf{r})|^2 - \Im \int_{U'} d\mathbf{r} \mathcal{V}(\mathbf{r}) \psi(\mathbf{r}) \psi_i^*(\mathbf{r})
\-
\int_{U'} d\mathbf{r} \Im K^2(\mathbf{r}) |\psi(\mathbf{r})|^2 - \Im \int_{U'} d\mathbf{r} \rho_{ind}(\mathbf{r}) \psi_i^*(\mathbf{r}).
\] (2.40)
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It follows from an analysis similar to that in (2.27-2.32) that the power dissipated inside the scatterer, to be denoted as \( P_{\text{loss}} \), is equal to

\[
P_{\text{loss}} = \int \Im K^2(r) |\psi(r)|^2.
\]  

(2.41)

By using this, (2.40) can be rewritten as

\[
P_s + P_{\text{loss}} = -\Im \int \rho_{\text{ind}}(r) \psi_i^*(r).
\]  

(2.42)

Now, the total extinguished power, to be denoted as \( P_e \), is given by the sum of the scattered power \( P_s \) and dissipation power \( P_{\text{loss}} \), in particular,

\[
P_e = P_s + P_{\text{loss}}.
\]  

(2.43)

This means that the right side of expression (2.42) corresponds to the total extinguished power, i.e.,

\[
P_e = -\Im \int \rho_{\text{ind}}(r) \psi_i^*(r).
\]  

(2.44)

Now, it follows from the reciprocity principle that one can measure this extinguished power in the form of a field measurement \( R \), so long as the receiver \( R \) creates the complex conjugate field \( \psi_i^*(r) \) in the region of interest containing the scatterer. In particular, let the receiver be such that when it operates as a source it radiates \( \psi_i^*(r) \) in the region of interest. Then from reciprocity

\[
\int \rho_{\text{ind}}(r) \psi_i^*(r) = \int d\mathbf{r} R(\mathbf{r}) \psi_s(\mathbf{r}).
\]  

(2.45)

Then the average, real power \( P_e \) extinguished by this scatterer via re-radiation in all directions (scattering) and dissipation inside the scatterer can be sensed outside the region of interest \( U' \) by means of a projective measurement of the form

\[
w = \int d\mathbf{r} R(\mathbf{r}) \psi_s(\mathbf{r})
\]  

(2.46)

such that

\[
P_e = -\Im w
\]  

(2.47)

corresponds to the extinguished real power and where \( R(\mathbf{r}) \) is a receiver mode which in its transmit mode counterpart radiates the complex conjugate field \( \psi_i^*(\mathbf{r}) \) in the region of interest \( U' \), i.e.,

\[
\int d\mathbf{r}' R(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') = \psi_i^*(\mathbf{r}) \quad \mathbf{r} \in U'.
\]  

(2.48)
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This is a complete statement of the optical theorem for an arbitrary probing field interrogating a scatterer in an arbitrary lossless background medium. It defines a field sensor $R$ that can measure directly the real power that is extinguished by the scatterer. Here it is interesting to note that, as has been discussed in detail in the recent paper by Marengo [60], such sensor is inherently nonunique. One possible realization is via Huygens sources which is the theoretical foundation of the so-called time-reversal cavities and mirrors in time-reversal acoustics and electromagnetics. We outline this later on in this thesis when we present the application to change detection.

2.6.2 The Reactive Optical Theorem

Consider a general background medium of space-dependent $k(r)$, which can be regarded as an arbitrary lossless medium. Let $\psi_i(r)$ be an incident field, which can be described by an arbitrary probing incident field, used to probe a passive scatterer characterized by scattering potential $V(r)$ of support $\tau \subseteq U'$. 

The scattered reactive power put by source $\rho_{ind}$ into field $\psi_s$ given according to (2.38) by

$$Q_s = \Re \int_{U'} dr \rho_{ind}(r) \psi_s^*(r).$$

(2.49)

From (2.17) and (2.18) plus the fact that the background is lossless so $k^2(r)$ is real-valued, we can rewrite (2.49) as

$$Q_s = \int_{U'} dr k^2(r) |\psi(r)|^2 - \int_{U'} dr \Re K^2(r) |\psi(r)|^2 - \Re \int_{U'} dr \rho_{ind}(r) \psi_i^*(r).$$

(2.50)

It is not hard to show via an analysis similar to the derivation process shown in (2.27-2.32) that the rate of the stored energy change associated with the difference between the two media can be quantified as

$$\Delta Q = \Re \int_{U'} dr K^2(r) |\psi(r)|^2 - \int_{U'} dr k^2(r) |\psi(r)|^2.$$ 

(2.51)

By using this, (2.50) can be rewritten as

$$Q_s + \Delta Q = -\Re \int_{U'} dr \rho_{ind}(r) \psi_i^*(r).$$

(2.52)

Then, the total extinguished reactive power

$$Q_e = Q_s + \Delta Q$$

(2.53)
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in particular, is given by

$$Q_e = -\Re \int_{U'} d\mathbf{r} \rho_{\text{ind}}(\mathbf{r}) \psi^*_i(\mathbf{r}).$$  (2.54)

To obtain the reactive optical theorem, we recall the reciprocity theorem. We assume the receiver mode $R$ creates the field of the complex conjugate field $\psi^*_i(\mathbf{r})$. Then from reciprocity

$$\Re \int_{U'} d\mathbf{r} \rho_{\text{ind}}(\mathbf{r}) \psi^*_i(\mathbf{r}) = \Re \int d\mathbf{r} R(\mathbf{r}) \psi_s(\mathbf{r})$$  (2.55)

the reactive power $Q_e$ extinguished by this scatterer can be sensed outside the region of interest $U'$ by means of the same projective measurement $w$ of the form in (2.46) such that

$$Q_e = -\Re w$$  (2.56)

corresponds to the extinguished reactive power associated to the scattering phenomenon and where $R(\mathbf{r})$ is a receiver mode which in transmit mode radiates the complex conjugate field $\psi^*_i(\mathbf{r})$ in the region of interest $U'$, i.e., eq.(2.48) holds.

2.6.3 Scattering Theorem

We conclude with another result that sheds insight on the physical justification of the conventional energy detector. Consider a general background medium of space-dependent $k(\mathbf{r})$, which can be regarded as an arbitrary lossless medium. Let $\psi_i(\mathbf{r})$ be an incident field, which can be described by an arbitrary probing incident field, used to probe a passive scatterer characterized by scattering potential $V(\mathbf{r})$ of support $\tau \subseteq U'$.

From (2.34), the scattered power can be regarded as the field produced by the induced source, in particular,

$$P_s = \int_{U'} d\mathbf{r} \rho_{\text{ind}}(\mathbf{r}) \psi^*_s(\mathbf{r}) = k_0 \int_4 d\hat{r} |f_s(\hat{r})|^2$$  (2.57)

where $f_s(\hat{r})$ is the far field scattering amplitude. To generate a field measurement $R(\mathbf{r})$ to calculate the real scattered power, we can use reciprocity theorem. In particular, we recall the reciprocity relation

$$\int_{U'} d\mathbf{r} \rho_{\text{ind}}(\mathbf{r}) \psi_s(\mathbf{r}) = \int d\mathbf{r} R(\mathbf{r}) \psi_s(\mathbf{r})$$  (2.58)

where $\psi_s(\mathbf{r})$ is the field generated by $R(R)$. Comparing (2.58) and (2.57), we can see that $R(\mathbf{r})$ must generate complex conjugate scattered field $\psi^*_s(\mathbf{r})$ in the scattering region. The required receiver $R(\mathbf{r})$ is realizable via standard time reversal or phase conjugation methods. Perhaps the simplest realization is a time reversal cavity over a far-zone surface enclosing the region of interest and having source excitations equal to $R(\mathbf{r}) = k_0 \psi^*_s(\mathbf{r})$ at each point over that surface.
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2.7 Conclusion

In this chapter, we derived the optical theorem in nonhomogeneous media in the framework of the scalar Helmholtz equation. We obtain the real power optical theorem associated with the dissipated and scattered power, and a novel reactive power optical theorem corresponding to the energy storage of the scattering. The derived optical theorem results are relevant for the sensing of power associated to the scattering, dissipation, and field energy storage arising from the presence of a target or the occurrence of a change in the wave propagation environment. As such, these results are pertinent for the design of novel energy sensors. The theorem also leads to three new and physically well-motivated detection algorithms that are studied in Chapter 4. Meanwhile, we also give reviewed a scattering-only theorem, which leads to the well-known approach called conventional energy detector which is considered also in Chapter 4. Given the broad generality of the derived results, which hold for arbitrary probing fields and media, the obtained results can be useful for many applications including nondestructive testing, radar, lidar, communications, and surveillance in complex environments. In the next chapter, we generalize the optical theorem results to the more general active scatterers that can carry a self-emitting component, which we call active scatterers.
Chapter 3

Scalar Optical Theorem For Active Scatterers

3.1 Background

In this chapter, we extend the optical theorem to active scatterers which carry an additional radiating source. In fact, any physical radiator or antenna is made of material media, and it is thus both a source and a scatterer. Furthermore, it can be embedded in a vehicle which makes the compound object a combination of a scatterer and a source that can be dependent or independent of the probing field. Among other applications, we are motivated by the use of active sources for cloaking and invisibility. In fact, artificial materials such as metamaterials can be built using active circuit components so that their constitutive parameters contain an active filtering component that is not present in natural passive media. The developed results are relevant to applications such as nondestructive diagnostics of active materials and devices, radar, lidar, wireless communications in complex environments, and security and military scenarios in which one may need to detect an intruder or target, which is a scattering entity to the externally applied probing fields, but which can also carry wave transmitting equipment of its own.

3.2 Generalized Scattered Fields

In this chapter, we focus on an active scatterer of compact support \( \tau \) whose scattering characteristics are governed by scattering potential \( \mathcal{V}(r) \) and whose active radiation nature is modeled by a source or antenna mode \( \rho_a(r) \). We clarify that what we call “active scatterer” here may differ
from other terminology used in the literature. In particular, the active scatterers under consideration are not what are usually termed “active media”. In fact, the material part (scattering potential) may correspond to either passive or active media. Under excitation by probing field $\psi_i$ produced by sources outside the region of interest $U'$ containing $\tau$, the total field $\tilde{\psi}$ is given by the sum of this probing field plus the generalized scattered field $\tilde{\psi}_s$ generated by this active scatterer which is characterized as follows.

In this chapter we focus on unbounded media in which the fields obey the radiation condition at infinity. Here it is important to point out that the main results for the active scatterers developed in this dissertation apply also to bounded media, as we show in our key paper [101]. The extension to bounded media, such as the homogeneous Dirichlet boundary condition scenario, is important since it enables application of the principles developed in this thesis to enclosed environments such as caves, tunnels, indoor environments, cavities, waveguides, and so on. For brevity, we do not dwell next on the theoretical formalities behind this important extension. Instead, the interested reader is referred to our key paper [101] which presents the theoretical generalization of these key results. In the unbounded media case, we can formulate scattering in the familiar context in which the medium inside the region of interest $U'$ is probed by an arbitrary incident field obeying the homogeneous form of (2.8) and (2.14) in $U'$. The total field $\tilde{\psi}(r)$ obeys

$$\left[\nabla^2 + k^2(r)\right]\tilde{\psi}(r) = \rho_{ind,a}(r) \quad (3.1)$$

where we introduce a generalized induced source $\rho_{ind,a}(r)$, which is defined as the combination of the induced source due to the scatterer component plus the source or antenna mode $\rho_a(r)$, i.e.,

$$\rho_{ind,a}(r) = \tilde{\rho}_{ind}(r) + \rho_a(r) \quad (3.2)$$

where we know

$$\tilde{\rho}_{ind}(r) = \mathcal{V}(r)\tilde{\psi}(r). \quad (3.3)$$

Meanwhile, the definition of the scattering potential $\mathcal{V}$ is as in (2.18). Consequently from (2.14, 3.1) the generalized scattered field $\tilde{\psi}_s$ obeys in $U'$

$$\left[\nabla^2 + k^2(r)\right]\tilde{\psi}_s(r) = \rho_{ind,a}(r). \quad (3.4)$$

Furthermore, since $\tau \subseteq U'$, it follows that in the unbounded media case the same equation (3.4) holds everywhere in space and the scattered field obeys the radiation condition at infinity. Using (2.10), the generalized scattered field is given by

$$\tilde{\psi}_s(r) = \int_{\tau} d\mathbf{r}' \rho_{ind,a}(r') G(r, r'). \quad (3.5)$$
CHAPTER 3. SCALAR OPTICAL THEOREM FOR ACTIVE SCATTERERS

3.3 Optical Theorem For Active Scatterers

In this section, we extend the optical theorem for active scatterers. We develop two optical theorems, one for the real extinguished power and another for the reactive extinguished power. These results give physical meaning to the optical theorem detectors for active scatterers used in this dissertation. They suggest a real-power optical theorem detector, a reactive-power optical theorem detector, as well as a total (apparent) power optical theorem detector. In addition, in this section we also outline other related findings established in our key paper [101] which assign physical meaning to the more conventional energy detector approach to change detection.

![Figure 3.1: The geometry and power budget for an active scatterer.](image)

3.3.1 Total Extinction Optical Theorem For Active Scatterers

Consider a general background medium of space-dependent $k(r)$ used to probe the active scatterer characterized by scattering potential $V(r)$ of support $\tau \subseteq U'$ and a radiating source component $\rho_a(r)$ of the same support $\tau$.

It follows from (3.4) and (3.5), that the scattered field $\tilde{\psi}_s$ is the field generated by the induced source $\rho_{\text{ind},a}$. Thus the real scattered power $\tilde{P}_s$ is given from (2.33) and (2.34) by

$$\tilde{P}_s = \Im \int_{U'} d\mathbf{r} \rho_{\text{ind},a}(\mathbf{r}) \tilde{\psi}_s^*(\mathbf{r})$$

(3.6)
CHAPTER 3. SCALAR OPTICAL THEOREM FOR ACTIVE SCATTERERS

From this, plus equations (2.17) and (2.18) and the fact that the background $k^2(r)$ is real-valued, one obtains

\[ \tilde{P}_s = \int_{U'} d\mathbf{r} k^2(|\tilde{\psi}(\mathbf{r})|^2 + \Im \int_{U'} d\mathbf{r} \rho_a(\mathbf{r}) \tilde{\psi}(\mathbf{r}) - \Im \int_{U'} d\mathbf{r} \nabla(\mathbf{r}) \tilde{\psi}(\mathbf{r}) \psi^*(\mathbf{r}) \] 

It follows from an analysis similar to that in (2.27–2.32) that the power dissipated inside the scatterer, to be denoted as $\tilde{P}_{\text{loss}}$, is equal to

\[ \tilde{P}_{\text{loss}} = \int_{U'} d\mathbf{r} k^2(|\tilde{\psi}(\mathbf{r})|^2) \tag{3.8} \]

Meanwhile, the power taken away or put by the active scatterers can be described as $P_a$, which is corresponding to the interaction between the active source $\rho_a$ and total field $\tilde{\psi}$, or

\[ P_a = -\Im \int_{U'} d\mathbf{r} \rho_a(\mathbf{r}) \tilde{\psi}^*(\mathbf{r}). \tag{3.9} \]

The sum of scattered power due to scattering $\tilde{P}_s$, and power $P_a$ taken away as the result of radiation corresponding to the active component $\rho_a(\mathbf{r})$ and dissipation $\tilde{P}_{\text{loss}}$ should be equal to the total extinguished power $\tilde{P}_e$, in particular,

\[ \tilde{P}_e = \tilde{P}_s + \tilde{P}_{\text{loss}} + P_a. \tag{3.10} \]

Using (3.7) and (3.10), we get the expression of the total extinguished power as

\[ \tilde{P}_e = -\Im \int_{U'} d\mathbf{r} \rho_{\text{ind},a}(\mathbf{r}) \psi^*_i(\mathbf{r}). \tag{3.11} \]

Now, it follows from the reciprocity principle that one can measure this extinguished power in the form of a field measurement $R$, so long as the receiver $R$ creates the complex conjugate field $\psi^*_i(\mathbf{r})$ in the region of interest containing the scatterer. In particular, let the receiver be such that when it operates as a source it radiates $\psi^*_i(\mathbf{r})$ in the region of interest. Then from reciprocity

\[ \int_{U'} d\mathbf{r} \rho_{\text{ind},a}(\mathbf{r}) \psi^*_i(\mathbf{r}) = \int d\mathbf{r} R(\mathbf{r}) \tilde{\psi}_s(\mathbf{r}). \tag{3.12} \]

Then the average real power $\tilde{P}_e$ extinguished by this scatterer via radiation of $\rho_a(\mathbf{r})$ and re-radiation in all directions (scattering) and dissipation inside the scatterer can be sensed outside the region of interest $U'$ by means of a projective measurement of the form

\[ \tilde{w} = \int d\mathbf{r} R(\mathbf{r}) \tilde{\psi}_s(\mathbf{r}) \tag{3.13} \]
such that
\[ \tilde{P}_e = -\Im \tilde{w} \] (3.14)
corresponds to the extinguished real power and where \( R(r) \) is a receiver mode which in its transmit mode counterpart radiates the complex conjugate field \( \psi_i^* (r) \) in the region of interest \( U' \) (see eq. (2.48)).

### 3.3.2 The Reactive Optical Theorem For Active Scatterers

Consider a general background medium of space-dependent \( k(r) \) used to probe the active scatterer characterized by scattering potential \( \mathcal{V}(r) \) of support \( \tau \subseteq U' \) and a radiating source component \( \rho_a (r) \) of the same support \( \tau \).

The scattered reactive power put by source \( \rho_{\text{ind},a} \) into field \( \tilde{\psi}_s \) is given according to (2.38) by
\[ \tilde{Q}_s = \Re \int_{U'} dr \rho_{\text{ind},a}(r) \tilde{\psi}_s^*(r). \] (3.15)

From (2.17) and (2.18) plus the fact that the background is lossless so \( k^2(r) \) is real-valued, we can rewrite (3.15) as
\[ \tilde{Q}_s = \int_{U'} dr k^2(r) |\tilde{\psi}(r)|^2 - \int_{U'} dr \mathcal{V}(r) \tilde{\psi}(r) \psi_i^* (r) \]
\[ = \int_{U'} dr k^2(r) |\tilde{\psi}(r)|^2 - \int_{U'} dr K^2(r) |\tilde{\psi}(r)|^2 \]
\[ + \Re \int_{U'} dr \rho_a (r) \tilde{\psi}(r) - \Re \int_{U'} dr \rho_{\text{ind},a}(r) \psi_i^* (r). \] (3.16)

Now via an analysis similar to the derivation process shown in (2.27)-(2.32), we find that the rate of the stored energy change associated with the difference between the two media can be quantified as
\[ \Delta \tilde{Q} = \Re \int_{U'} dr k^2(r) |\tilde{\psi}(r)|^2 - \int_{U'} dr k^2(r) |\tilde{\psi}(r)|^2. \] (3.17)

Combining (3.16) and (3.17), one gets
\[ \tilde{Q}_s + \Delta \tilde{Q} = \Re \int_{U'} dr \rho_a (r) \tilde{\psi}(r) - \Re \int_{U'} dr \rho_{\text{ind},a}(r) \psi_i^* (r). \] (3.18)

Meanwhile, the reactive power put by field \( \tilde{\psi} \) into source \( \rho_a \) can be described as \( Q_a \)
\[ Q_a = -\Re \int_{U'} dr \rho_a (r) \tilde{\psi}^*(r). \] (3.19)
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By using (3.17) and (3.19), (3.16) can be rewritten as

\[ \tilde{Q}_s + \Delta \tilde{Q} + Q_a = -\Re \int_{U'} d\mathbf{r} \rho_{\text{ind}}(\mathbf{r}) \psi_s^*(\mathbf{r}). \]  

(3.20)

Then, the total extinguished reactive power defined by

\[ \tilde{Q}_e = \tilde{Q}_s + \Delta \tilde{Q} + Q_a \]  

(3.21)

in particular, is given by

\[ \tilde{Q}_e = -\Re \int_{U'} d\mathbf{r} \rho_{\text{ind},a}(\mathbf{r}) \psi_i^*(\mathbf{r}). \]  

(3.22)

To obtain the reactive optical theorem, we recall the reciprocity theorem. We assume the receiver mode \( R \) creates the field of the complex conjugate field \( \psi_i^*(\mathbf{r}) \). Then from reciprocity

\[ \Re \int_{U'} d\mathbf{r} \rho_{\text{ind},a}(\mathbf{r}) \psi_i^*(\mathbf{r}) = \Re \int_{U'} d\mathbf{r} R(\mathbf{r}) \tilde{\psi}_s(\mathbf{r}) \]  

(3.23)

the reactive power \( \tilde{Q}_e \) extinguished by this scatterer can be sensed outside the region of interest \( U' \) by means of the same projective measurement \( \tilde{w} \) of the form in (3.13) such that

\[ \tilde{Q}_e = -\Re \tilde{w} \]  

(3.24)

corresponds to the extinguished reactive power associated to the scattering phenomenon and where \( R(\mathbf{r}) \) is a receiver mode which in transmit mode radiates the complex conjugate field \( \psi_i^*(\mathbf{r}) \) in the region of interest \( U' \) (see eq. (2.48)).

3.3.3 Scattering Theorem For Active Scatterers

Similarly, we conclude with another result that sheds insight on the physical justification of the conventional energy detector. Consider a general background medium of space-dependent \( k(\mathbf{r}) \) used to probe the active scatterer characterized by scattering potential \( \mathcal{V}(\mathbf{r}) \) of support \( \tau \subseteq U' \) and a radiating source component \( \rho_a(\mathbf{r}) \) of the same support \( \tau \).

From (2.34), the scattered power can be regarded as the field produced by the induced source, in particular,

\[ \tilde{P}_s = \int_{U'} d\mathbf{r} \rho_{\text{ind},a}(\mathbf{r}) \tilde{\psi}_s^*(\mathbf{r}) = k_0 \int_{4\pi} d\hat{\mathbf{r}} |\tilde{f}_s(\hat{\mathbf{r}})|^2 \]  

(3.25)

where \( \tilde{f}_s(\hat{\mathbf{r}}) \) is the far field scattering amplitude. To generate a field measurement \( \tilde{R}(\mathbf{r}) \) to calculate the real scattered power, we can use reciprocity. In particular, we recall the reciprocity relation

\[ \int_{U'} d\mathbf{r} \rho_{\text{ind},a}(\mathbf{r}) \psi R(\mathbf{r}) = \int d\mathbf{r} \tilde{R}(\mathbf{r}) \tilde{\psi}_s(\mathbf{r}) \]  

(3.26)
where $\psi_R(r)$ is the field generated by $R(r)$. Comparing (3.26) and (3.25), we see that $R(r)$ must generate the complex conjugate scattered field $\tilde{\psi}_s^*(r)$ in the scattering region. The required receiver $R(r)$ is realizable via standard time reversal or phase conjugation methods. Perhaps the simplest realization is a time reversal cavity over a far-zone surface enclosing the region of interest and having source excitations equal to $R(r) = k_0 \tilde{\psi}_s^*(r)$ at each point over that surface.

3.4 Conclusion

This chapter extended the optical theorem to the class of active scatterers. This is an important extension since it generalizes the optical theorem theory to arbitrary probing field, and arbitrary media, including scatterers that have both the conventional material response components as well as an additional active, self-emitting component. The derived optical theorem results are relevant for the sensing of power associated to the scattering, dissipation, and field energy storage arising from the presence of a target or the occurrence of a change in the wave propagation environment. Up to this point, we have considered only the frequency domain theory, which applies to linear time-invariant media. In Chapter 5, we develop the time domain generalization relevant to more arbitrary media, including nonlinear and time-varying media.
Chapter 4

Change Detection Based on the Optical Theorem

4.1 Background

In this chapter, we investigate the change detection application of the derived optical theorem detectors in Chapter 2 and Chapter 3. Detecting changes of a medium or environment based on active, transmit-plus-receive wave sensor data plays a key role in many important applications including radar, surveillance, remote sensing, nondestructive testing, and cancer detection. This is a challenging problem because both the change or target and the surrounding background medium are in general unknown and can be quite complex. The optical theorem motivates new methods for detection. In this chapter, we focus on three different forms of indicators or statistics that are inspired by the optical theorem detectors: a real power optical theorem detector, a reactive power optical theorem detector, and a total apparent power optical theorem detector. Marengo and Gruber in [100] have obtained encouraging preliminary results suggesting the effectiveness of the apparent power form of the optical theorem detector. This chapter expands the understanding and characterization of the optical theorem approach to change detection including active targets. The performance is studied with the help of ROC curves, but theoretical results are also developed which lead to a new form of generalized likelihood ratio test with an optical-theorem-based constraint. For reference, the performance of the developed detectors is compared with that of the classical energy detector as well as the ideal matched filter.
4.2 Optical-Theorem-Based Indicators

In this chapter we consider the $N$-sensor array realizations of the optical theorem detectors defined in Chapters 2 and 3. We describe the detectors using the notation used in section 2.6 of Chapter 2, which applies to the conventional (“passive”) scatterers. However, the same results apply to the more general active scatterers considered in Chapter 3.

Let $X_1, X_2, \cdots, X_N$ denote the sensor positions. We denote the incident field in the background or clutters as $\psi_i$. The total field in the medium including the scatterer is denoted as $\psi$. The scattered field is denoted as $\psi_s$. The respective incident, total, and scattered field vector signals at the $N$-sensor array are given by

$$v_i = [\psi_i(X_1), \ldots, \psi_i(X_N)]^T, \quad (4.1)$$
$$v_t = [\psi(X_1), \ldots, \psi(X_N)]^T, \quad (4.2)$$

and

$$v_s = [\psi_s(X_1), \ldots, \psi_s(X_N)]^T = v_t - v_i, \quad (4.3)$$

where $(\cdot)^T$ stands for matrix transpose. The respective array realization of the optical theorem receiver mode $R(r)$ is

$$R(r) = 2ik_0 \sum_{n=1}^{N} \psi_n^*(X) \delta(r - X_n). \quad (4.4)$$

In many complex and highly reverberating media, the field produced by the transmit counterpart of this receiver is approximately equal to the complex conjugate field $\psi_i^*$ in the scattering region $U'$, as desired.

Now, we denote as $\hat{\psi}_s$ the noisy version of the scattered field $\psi_s$. Estimates of the real, reactive, and apparent extinguished power of the scatterer can be obtained for realistic, finite-sized apertures, via the projective measurement $v$ of this noisy field involving the projection onto the receiver mode $R$ in (4.4). In particular,

$$v = 2k_0 \sum_{n=1}^{N} \psi_n^*(X_n) \hat{\psi}_s(X_n) \psi_i(X_n), \quad (4.5)$$

which corresponds, apart from a factor, to the cross-correlation of the noisy scattered field $\hat{\psi}_s$ with the probing field $\psi_i$, as measured at the sensor array. The corresponding estimates of the real, reactive,
and apparent extinguished powers are given by
\[
\hat{P}_e = -\Re v, \\
\hat{Q}_e = \Im v, \\
\hat{S}_e = |v|
\] (4.6)

where the apparent extinguished power can be rewritten in terms of \( P_e \) and \( Q_e \) as
\[
S_e = \sqrt{\hat{P}_e^2 + \hat{Q}_e^2}. 
\] (4.7)

In this chapter we study the adoption of the quantities in (4.6) as statistical indicators to detect the presence of a target or medium change.

Meanwhile, we also recall the scattering theorem shown in Chapter 2. As shown in (2.57), the quantity \( P_s \) is proportional to the square of the 2 norm of the scattered signal. This provides a physical justification for the basic energy detector approach to change detection which uses the 2 norm of the measured scattered signal as the indicator function[71]. Under the \( N \)-sensor array realization considered in this chapter, the relevant energy detector indicator function is
\[
E = k_0 \sum_{n=1}^{N} |\hat{\psi}_s(\mathbf{X}_n)|^2. 
\] (4.8)

### 4.3 Statistical Characterization of the Optical Theorem Detectors

In this section we review the statistical behavior of the optical theorem detectors under circularly symmetric Gaussian noise (see [100] for more details). We introduce the vector of total field with noise \( v_{tn} \) which is given by
\[
v_{tn} = v_t + n
\] (4.9)

where \( n \) represents additive white complex Gaussian noise with zero mean and \( \sigma^2 \) variance. Similarly, we represent the measured scattered field signal with noise as \( v_{sn} \). It is defined as
\[
v_{sn} = v_s + n.
\] (4.10)

Vector signal \( v_{sn} \) has a Gaussian distribution with a mean vector \( v_s \) and variance vector \( \sigma^2 \textbf{I} \). Consequently, the conditional probability distribution functions (pdfs) for the two hypotheses of target present \( (H_1) \) and no target (null hypothesis \( H_0 \)) are:
\[
P(v_{sn}|H_0) = \frac{1}{(\pi \sigma^2)^N} \exp\left(-\frac{||v_{sn}||^2}{\sigma^2}\right) 
\] (4.11)
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and

\[ P(v_{sn}|H_1) = \frac{1}{(\pi\sigma^2)^N} \exp\left(-\frac{||v_{sn} - v_s||^2}{\sigma^2}\right) \]  

(4.12)

where hypotheses $H_0$ represents “no target presence”, while $H_1$ corresponds to “target presence”. In this section we obtain the probability of detection, $P_d = P(\text{“target presence”}|H_1)$, and the probability of false alarm, $P_{fa} = P(\text{“target presence”}|H_0)$, for the different detectors, illustrated in Figure 4.1.

Figure 4.1: Illustration of the two hypotheses.

We focus on the following detectors: real and apparent power OT (Optical Theorem) detectors, and ED (Energy Detector). The matched filter is used in standard signal processing but it assumes prior knowledge of the scattering signal. It is useful as reference performance bound for the other detectors which do not assume prior knowledge of the scattering signal.

1) Energy Detector: In the detection problem, we seek to detect an unknown scattered signal contaminated with noise. The energy detector uses as test statistic the mathematical signal energy or $L^2$ norm of the measured signal. In the present scatterer detection context the relevant signal is the measured scattered field vector $v_{sn}$, the relevant signal energy is

\[ E_{ED} = ||v_{sn}||^2 = \sum_{m=1}^{M} |v_{sn}(m)|^2. \]  

(4.13)

We apply the following normalization:

\[ \hat{E}_{ED} = \frac{2E_{ED}}{\sigma^2}. \]  

(4.14)

This is the indicator function adopted next for the discussion of the energy detector. The detection rule is as follows: if $\hat{E}_{ED} > \eta_{ED}$, where $\eta_{ED}$ is a threshold, we decide target presence ($H_1$); if $\hat{E}_{ED} \leq \eta_{ED}$, we decide no target presence ($H_0$).
Under the null hypothesis, the measured scattered signal has the circularly symmetric complex Gaussian distribution, \( (v_{sn}|H_0) \sim \mathcal{CN}(0, \sigma^2 I) \). Then the conditional probability distribution function (pdf) of the test statistic is
\[
\hat{E}_{ED}(v_{sn}|H_0) \sim \chi^2_{2M}(0). \tag{4.15}
\]
Under the alternative hypothesis, the distribution is \( (v_{sn}|H_1) \sim \mathcal{CN}(v_{sn}, \sigma^2 I) \). Then
\[
\hat{E}_{ED}(v_{sn}|H_1) \sim \chi^2_{2M}(\mu) \tag{4.16}
\]
where
\[
\mu = \frac{2||v_{sn}||^2}{\sigma^2}. \tag{4.17}
\]
\( \chi^2_{2M}(\nu) \) represents a noncentral \( \chi \)-square function with \( 2M \) freedom and noncentrality parameter \( \nu \).

In the follow part, we will also use such definition. Here we use \( \Psi_{2M,\nu}(x) \) to donate the cumulative distribution function (cdf) of a noncentral \( \chi \)-square function with \( 2M \) freedom and noncentrality parameter \( \nu \). Consequently, we can obtain the result of the probability of false alarm and probability of detection
\[
P_{fa} = 1 - \Psi_{2M,0}(\eta_{ED}) \tag{4.18}
\]
\[
P_d = 1 - \Psi_{2M,\mu}(\eta_{ED}) \tag{4.19}
\]
where \( \eta_{ED} \) represents the threshold chosen for the detection.

2): Optical Theorem Detector: Based on the discussion in (4.6), we consider the quantity
\[
E_{OT} = v_i^H v_{sn} \tag{4.20}
\]
where \( (\cdot)^H \) denotes complex conjugate transpose (Hermitian transpose) and carries information about the total scattered power due to the target. The real part of this quantity is related to the real extinguished power. The magnitude is related to the total apparent power of the scattering phenomenon. Therefore it is logical to formulate the scatterer detection problem in complex media using as test statistic the amplitude
\[
E_{OTR} = \Re(v_i^H v_{sn}) \tag{4.21}
\]
or the magnitude
\[
E_{OTM} = |v_i^H v_{sn}|. \tag{4.22}
\]
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These quantities represent the real and apparent optical theorem statistics. It’s easy to show that

\[ E_{OTR}(v_{sn}|H_0) \sim \mathcal{N}(0, \frac{\sigma^2||v_i||^2}{2}) \]  

while

\[ E_{OTR}(v_{sn}|H_1) \sim \mathcal{N}[\Re(v_i^H v_s), \frac{\sigma^2||v_i||^2}{2}] \].

Then the probability of detection and false alarm for the real power optical theorem detector are given by

\[ P_a = \frac{1}{2} - \operatorname{erf}\left(\frac{\eta_{OTR}}{||v_i||\sigma}\right) \]  

\[ P_d = \frac{1}{2} - \operatorname{erf}\left[\frac{\eta_{OTR} - \Re(v_i^H v_s)}{||v_i||\sigma}\right] \]

where \( \eta_{OTR} \) represents the threshold chosen for the detection, and \( \operatorname{erf}(\cdot) \) represents the error function.

Moreover, the statistical behavior of the apparent power optical theorem detector is described as follows. We have

\[ E_{OTM}(v_{sn}|H_0) \sim \text{Rayleigh}\left(\frac{\sigma||v_i||}{\sqrt{2}}\right) \]  

while

\[ E_{OTM}(v_{sn}|H_1) \sim \text{Rice}(||v_i^H v_s||, \frac{\sigma||v_i||}{\sqrt{2}}) \]  

where Rayleigh(\cdot) and Rice(\cdot) are the Rayleigh and Rice distribution, respectively. It can be shown that the probability of detection and false alarm are given by

\[ P_a = \exp\left(-\frac{\eta_{OTM}^2}{2\sigma^2||v_i||^2}\right) \]  

\[ P_d = Q_1\left(\frac{\sqrt{2}||v_i^H v_s||}{\sigma||v_i||}, \frac{\sqrt{2}\eta_{OTM}}{\sigma||v_i||}\right) \]

where \( \eta_{OTM} \) represents the threshold adopted in the detection rule and \( Q_1(\cdot) \) represents the first order of the Marcum Q-function.

4.4 Numerical Simulation Results

We illustrate in this section the performance of the optical theorem detectors. The main goal is to illustrate, through the comparison of typical receiver operator characteristic (ROC) curves applicable to random and complex media, the general detection performance associated to the energy detector and the optical theorem detectors. For reference, we also provide the corresponding results.
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for the ideal matched filter detection approach which, unlike the detectors above, relies on prior information about the noise-free scattering response of the target in the background medium. We consider both conventional, “passive” target, and “active” scatterers having a self-emitting parts.

### 4.4.1 Ten Clutter Inhomogeneities Detection

As a first example, we considered, in two-dimensional (2D) space, the complex background medium illustrated in Figure 4.2, which is formed by 10 point-like scatterers. In all the simulations associated to this example, we assumed that the incident field was generated by a point source located at the origin \((x = 0, y = 0)\). The field data were gathered at an array of \(N\) receivers having positions \(X_n = (d - n\lambda/2, 0), n = 0, 1, \cdots, N - 1\), where \(\lambda\) is the free space wavelength of the field and \(d = 100\lambda\). In the following, we provide the results for the special cases of a single receiver \((N = 1)\) and 10 receivers \((N = 10)\). The target was placed at the center of the region of interest, corresponding to position \(R_{ROI} = (d/2, \sqrt{3}d/2)\). Radiation and scattering was computed numerically using the well-known Foldy-Lax multiple scattering model (see, e.g., [102], pages 246-248). The target was modelled as being of scattering strength \(\tau_t = 1\), while the 10 clutters had scattering strengths \(\tau_1 = \tau_3 = \tau_4 = \tau_5 = \tau_6 = \tau_8 = \tau_9 = \tau_{10} = 1\) and \(\tau_2 = 0.5 + i\) and positions \(R_1 = R_{ROI} + (-2\lambda, -2\lambda), R_2 = R_{ROI} + (0.5\lambda, -\lambda), R_3 = R_{ROI} + (0.8\lambda, -\lambda), R_4 = R_{ROI} + (-\lambda, 0.8\lambda), R_5 = R_{ROI} + (0.4\lambda, 0.8\lambda), R_6 = R_{ROI} + (-\lambda, -1.6\lambda), R_7 = R_{ROI} + (-2.4\lambda, 0.8\lambda), R_8 = R_{ROI} + (-3.2\lambda, 2.4\lambda), R_9 = R_{ROI} + (-0.4\lambda, 2\lambda), R_{10} = R_{ROI} + (\lambda, -4\lambda)\).

In the following simulations, the scattered field data are corrupted by circularly symmetric complex Gaussian noise of variance \(\sigma^2 \sim \mathcal{CN}(0, \sigma^2 I_N)\), where \(I_N\) is the \(N \times N\) identity matrix. The post-processing SNR, defined as

\[
\text{SNR} = \frac{\sum_{n=1}^{N} |\psi_s(X_n)|^2}{\sigma^2},
\]

is equal to 5 (6.99 dB) in all the results. All the ROC curves presented next were obtained numerically using 10000 noise realizations.

We consider first the case of a purely passive scatterer which corresponds to \(\rho_a = 0\). Figure 4.3 shows the ROC plots of the probability of detection \(P_d\) versus the probability of false alarm \(P_{fa}\) corresponding to the single receiver case. These results illustrate the fact, demonstrated under a basic transmission line model in [59], that for scalar data associated to passive scatterers, the conventional energy detector and the apparent power optical theorem detector are essentially
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Figure 4.2: Illustration of the background medium used in the example, which consists of 10 point-like clutter inhomogeneities (indicated with circles). The target is indicated with a larger circle.

equivalent. For the particular scattering parameters and sensing geometry considered in the example, they outperform both the real power and reactive power optical theorem detectors. For reference, we also show the ROC for the optimal matched filter, which unlike the previous detectors, assumes perfect information about the target response in the given clutter medium.

Figure 4.4 and Figure 4.5 show other sets of ROC curves corresponding to the case of 5 and 10 receivers, respectively. In these cases, the apparent power optical theorem detector is clearly the best among the studied detectors that do not assume prior information about the target response, and its performance is close to that of the matched filter which is given for reference. Furthermore, for these multiple receivers cases, the three considered optical theorem detectors clearly outperform the conventional energy detector.

We consider next the case of an active scatterer that is predominantly active, meaning that its combined scattered plus radiated signal is due mostly to its primary source constituent. The scattering geometry is the same one in Figure 4.2, the active source is located at \( \mathbf{R}_a = \mathbf{R}_{ROI} + (2\lambda, 2\lambda) \). Figure 4.6 shows the ROC plots for a single receiver. The apparent power optical theorem detector and the energy detector have essentially the same performance, but the best detector among those that do not rely on prior target information is the reactive power optical theorem detector. Figure 4.7
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Figure 4.3: ROC curves for the detection of a passive scatterer in 10 clutters medium using a single receiver and a single frequency. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the optimal matched filter which, unlike the four previous detectors, assumes perfect information about the target response in the clutter medium.

Figure 4.4: ROC curves corresponding to the detection of a passive scatterer in 10 clutters medium using 5 receivers and a single frequency. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
Figure 4.5: ROC curves corresponding to the detection of a passive scatterer in 10 clutters medium using 10 receivers and a single frequency. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

and Figure 4.8 show the results for 5 and 10 receivers, respectively. The apparent power and reactive power optical theorem detectors perform very well. The energy detector performs better than the real power optical theorem detector.

To conclude the discussion of this example, we consider the case in which the active scatterer has comparable active and passive parts. In particular, we provide ROC curves for different values of the ratio \( \alpha = ||\rho_a||/||\rho_{ind}|| \) of the 2 norms of the corresponding primary source \( \rho_a \) and the induced source \( \rho_{ind} \) associated to scattering. We illustrate next for the case \( \alpha = 0.99 \) in which the norms of the primary and induced sources are almost identical, and the cases \( \alpha = 0.1 \) and \( \alpha = 0.01 \) in which the norm of the induced source is 10 times and 100 times larger than that of the primary source component, respectively.

The results in Figure 4.9 for \( \alpha = 0.99 \) and a single receiver, are similar to those in Figure 4.6 for the predominantly active target and a single receiver. Figure 4.10 shows the ROC curves for \( \alpha = 0.1 \) and a single receiver. Figure 4.11 shows the ROC curves for \( \alpha = 0.01 \) and a single receiver. In this case, the real power optical theorem detector performs slightly better than the apparent power optical theorem detector and the energy detector, whose performances are essentially identical. Put together, the ROC curves for a single receiver, Figs. 4.3, 4.6, 4.9, 4.10, 4.11, clearly suggest that for a single receiver the apparent power optical theorem detector performs essentially like the energy detector, with a detection performance that is consistently good. In certain cases the real
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Figure 4.6: ROC curves for the detection of a predominantly active scatterer using a single receiver. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.7: ROC curves for the detection of a predominantly active scatterer using 5 receivers. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
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Figure 4.8: ROC curves for the detection of a predominantly active scatterer using 10 receivers. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

power detector or the reactive power detector performs better. On the other hand, the performance of the real power and reactive power optical theorem detectors fluctuates dramatically from a case to another, suggesting that these detectors are very sensitive to the target and clutter characteristics and the sensing setup.

Figure 4.12 shows the ROC curves for $\alpha = 0.99$ and 10 receivers, which have characteristics similar to those in Figure 4.8 corresponding to the predominantly active scatterer case. Figure 4.13 shows the corresponding results for $\alpha = 0.1$ and 10 receivers. Figure 4.14 shows the corresponding results for $\alpha = 0.01$ and 10 receivers. In this case the real power optical theorem detector performs slightly better than the apparent power optical theorem detector. Both perform much better than the energy detector. The provided ROC curves for 10 receivers, Figs. 4.5, 4.8, 4.12, 4.13, 4.14 consistently reveal the tendency of the apparent power optical theorem detector to outperform the conventional energy detector. Furthermore, the apparent power optical theorem detector performs consistently well, sometimes approaching the ideal matched filter, unlike the real power and reactive power optical theorem detectors which perform very well under certain conditions but behave quite poorly in other cases.
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Figure 4.9: ROC curves for the detection with a single receiver of a scatterer with comparable active and passive parts ($\alpha = 99\%$). Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.10: ROC curves for the detection with a single receiver of a mildly active scatterer having $\alpha = 10\%$. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
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Figure 4.11: ROC curves for the detection with a single receiver of a mildly active scatterer having $\alpha = 1\%$. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.12: ROC curves for the detection with 10 receivers of a scatterer with comparable active and passive parts ($\alpha = 99\%$). Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
Figure 4.13: ROC curves for the detection with 10 receivers of a mildly active scatterer having $\alpha = 10\%$. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.14: ROC curves for the detection with 10 receivers of a mildly active scatterer having $\alpha = 1\%$. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
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We conclude that the apparent power optical theorem detector works consistently well for arbitrary value of $\alpha$. This conclusion holds for single frequency data only. It does not extend to the multiple frequencies cases as we briefly illustrate later in this section.

### 4.4.2 Special Cases For A Thousand Random-positioned Clutters Detection

![Figure 4.15: Illustration of the background medium region used in the example. It has 1000 point-like random-positioned clutter inhomogeneities.](image)

In this example the background medium is composed of 1000 point-like inhomogeneities that are randomly positioned within a $10\lambda \times 10\lambda$, $30\lambda \times 30\lambda$, $50\lambda \times 50\lambda$ and $70\lambda \times 70\lambda$ square region centered around the target which is located at $R_{ROI}$. We used a scattering strength of 1.5 for the target and of 0.1 for all the clutter inhomogeneities. We used $N = 5$ receivers having positions $X_n = (n\lambda/2, 0)$, $n = 0, 1, \cdots, N - 1$. 

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Figure 4.16: ROC curves for the detection with 5 receivers of a purely passive scatterer surrounded by 1000 randomly positioned clutter inhomogeneities within a $10\lambda \times 10\lambda$ square region centered around the target. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.17: ROC curves for the detection with 5 receivers of a predominantly active scatterer surrounded by 1000 randomly positioned clutter inhomogeneities within a $10\lambda \times 10\lambda$ square region centered around the target. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
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Figure 4.18: ROC curves for the detection with 5 receivers of a purely passive scatterer surrounded by 1000 randomly positioned clutter inhomogeneities within a $30\lambda \times 30\lambda$ square region centered around the target. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.19: ROC curves for the detection with 5 receivers of a predominantly active scatterer surrounded by 1000 randomly positioned clutter inhomogeneities within a $30\lambda \times 30\lambda$ square region centered around the target. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
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Figure 4.20: ROC curves for the detection with 5 receivers of a purely passive scatterer surrounded by 1000 randomly positioned clutter inhomogeneities within a $50\lambda \times 50\lambda$ square region centered around the target. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.21: ROC curves for the detection with 5 receivers of a predominantly active scatterer surrounded by 1000 randomly positioned clutter inhomogeneities within a $50\lambda \times 50\lambda$ square region centered around the target. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
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Figure 4.22: ROC curves for the detection with 5 receivers of a purely passive scatterer surrounded by 1000 randomly positioned clutter inhomogeneities within a $70\lambda \times 70\lambda$ square region centered around the target. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.23: ROC curves for the detection with 5 receivers of a predominantly active scatterer surrounded by 1000 randomly positioned clutter inhomogeneities within a $70\lambda \times 70\lambda$ square region centered around the target. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
Figure 4.24: ROC curves for the detection with 5 receivers of a purely passive scatterer surrounded by 1000 randomly positioned clutter inhomogeneities within a $80\lambda \times 80\lambda$ square region centered around the target. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.25: ROC curves for the detection with 5 receivers of a predominantly active scatterer surrounded by 1000 randomly positioned clutter inhomogeneities within a $80\lambda \times 80\lambda$ square region centered around the target. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
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These ROC plots illustrate the following tendencies. For the passive scatterers, the apparent power optical theorem detector is consistently superior. The behavior for active scatterers depends on the size of the clutter region. For small clutter region, the apparent power optical theorem detector performs better than the energy detector. However, as the clutter region becomes larger, the energy detector performs better and can outperform the optical theorem detector. These results also illustrate the observed greater stability of the apparent power optical theorem detector versus that of the real power and reactive power optical theorem detectors.

4.4.3 Broadband Signal Results

We illustrate next the use of the optical theorem detectors for multi-frequency data. The following results apply to passive scatterers only. The scattering geometry is the one shown in Figure 4.2.

![ROC curves](image)

Figure 4.26: ROC curves for detection in 10 clutters medium with 10 frequencies in the (narrow) band of 2% bandwidth and one receiver. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
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Figure 4.27: ROC curves for detection in 10 clutters medium with 10 frequencies in the band of 4% bandwidth and one receiver. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.28: ROC curves for detection in 10 clutters medium with 10 frequencies in the (narrow) band of 2% bandwidth and 5 receivers. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
Figure 4.29: ROC curves for detection in 10 clutters medium with 10 frequencies in the band of 4% bandwidth and 5 receivers. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.30: ROC curves for detection in 10 clutters medium with 10 frequencies in the (narrow) band of 2% bandwidth and 10 receivers. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
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Figure 4.31: ROC curves for detection in 10 clutters medium with 10 frequencies in the band of 4% bandwidth and 10 receivers. Results for the real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 4.26, Figure 4.27, Figure 4.28, Figure 4.29, Figure 4.30, and Figure 4.31 show the ROC curves for different bandwidth and numbers of receivers. These results consistently show the effectiveness of the apparent power optical theorem detectors, whose performance is close to that of the matched filter.

4.5 Conclusion

In this chapter, we presented the detection application of the optical theorem detectors of Chapter 2 and Chapter 3. The results apply to both passive and active targets. The derived optical theorems shown in the previous two chapters naturally lead to three different kinds of optical theorem detectors. Two of them are associated with the real and reactive power budgets of the scattering phenomenon. The other one is linked to a measurement of the total power (real plus reactive), also called apparent power. Typical ROC curves corresponding to the Foldy-Lax-based multiple scattering in complex and random media were demonstrated which show the practical feasibility of at least one of these three optical-theorem-based approaches. We found that the performance of the apparent power optical theorem detector is consistently very good for passive scatterers. For single frequency and single receiver, it performs essentially as the energy detector. However, for the multiple receivers, multiple frequencies, it is clearly superior to the energy detector. Meanwhile, the real power optical
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Theorem detector is also better than the energy detector in many cases. The reactive power optical theorem detector performs well in some of cases pertinent to active scatterers. For pure active scatterers, we see that optical theorem detectors are not necessarily better than the conventional energy detector. For the random media used in the simulation, it was found that the performance results depend on the clutter region size. Finally, we also illustrated the multi-frequency performance for passive scatterers. The apparent power optical theorem detector was found to perform very well. We conclude that the apparent power optical theorem detector performs very well for both single frequency and multiple frequencies. However, for active targets, the conventional energy detector can perform better.
Chapter 5

Optical Theorem in the Time Domain

5.1 Background

This chapter expands the theoretical repertoire on the optical theorem to the time domain. We cover aspects of the theory not covered in the previous key papers [53, 54, 55, 58]. The results in this chapter are a summary of those in our key paper [103]. We consider not only the ordinary optical theorem but also the most general form of the optical theorem known as the generalized optical theorem (see [1], p. 723, and [52]). This is the first formulation of the generalized optical theorem in the time domain. In this chapter, we also provide the first time domain formulation of the reactive optical theorem. Furthermore, the focus of past time-domain formulations has been on homogeneous backgrounds such as free space, as well as probing of the scatterer by homogeneous plane waves. We generalize the time-domain optical theorem to more general nonhomogeneous background media that are LTI and lossless, as well as to arbitrary probing fields, including near fields which are relevant such as super-resolution imaging. The lossless nature of the background permits the measurement (in a field) of the electromagnetic interaction as reaction and LTI nature allows us to invoke the reciprocity theorem of the convolution type of the optical theorem in time domain [104]. Moreover, the background can be anisotropic and nonreciprocal. Importantly, as in the previous key paper [58], the scatterer itself can be very general. For instance, it can be electromagnetically nonlinear, time-varying, and lossy. In fact, the main appeal of the time-domain optical theorem is its applicability to arbitrary targets that can be time-varying and nonlinear, for which the frequency domain theory of most previous work (relevant only to LTI scatterers) is not applicable.
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5.2 Notation

We denote time and position vector in three-dimensional space as \( t \in \mathbb{R} \) and \( r \in \mathbb{R}^3 \), respectively. For simplicity, we do not use the familiar boldface font for vector fields, for example, the electric field is denoted simply as \( E(r, t) \). In addition, we introduce the convolution inner product \( \odot \) defined for any vector fields \( F \) and \( G \) as

\[
(F \odot G)(r, t) \equiv \int_{-\infty}^{\infty} d\tau F(r, \tau) \cdot G(r, t - \tau).
\] (5.1)

It follows from a well-known convolution property that

\[
F \odot \frac{\partial}{\partial t} G = \frac{\partial}{\partial t} F \odot G = \frac{\partial}{\partial t} (F \odot G).
\] (5.2)

We also introduce the convolution cross product \( \otimes \) defined as

\[
(F \otimes G)(r, t) \equiv \int_{-\infty}^{\infty} d\tau F(r, \tau) \times G(r, t - \tau).
\] (5.3)

We consider the time reversal operation \( \bar{\cdot} \) defined as

\[
\bar{F}(r, t) \equiv F(r, -t)
\] (5.4)

and introduce the correlation inner product \( \diamond \) defined as

\[
(F \diamond G)(r, t) \equiv (\bar{F} \odot G)(r, t)
= \int_{-\infty}^{\infty} d\tau F(r, \tau) \cdot G(r, t + \tau)
\] (5.5)

as well as the correlation cross product \( \star \) defined as

\[
(F \star G)(r, t) \equiv (\bar{F} \otimes G)(r, t)
= \int_{-\infty}^{\infty} d\tau F(r, \tau) \times G(r, t + \tau).
\] (5.6)

See that \( F \odot G = G \odot F \) and \( F \otimes G = -G \otimes F \) while \( (F \diamond G)(t) = (G \diamond F)(-t) \) and \( (F \star G)(t) = -(G \star F)(-t) \).

5.3 Problem Formulation, Reciprocity, and Interaction Relations in the Time Domain

We consider an arbitrary scatterer embedded in a general locally-reacting, LTI, lossless, anisotropic background medium. The background is in general nonhomogeneous and can be
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nonreciprocal. It is assumed that the background medium is made of material of finite spatial extent so that the background’s constitutive properties are equal to those of free space at infinity. The fields generated by sources and scatterers behave as outgoing waves at infinity. The scatterer is assumed to be of compact spatial support $V_0$ but is otherwise arbitrary.

In the background medium, Maxwell’s equations are

$$\nabla \times E(r, t) = -M(r, t) - \frac{\partial}{\partial t} B(r, t)$$
$$\nabla \times H(r, t) = J(r, t) + \frac{\partial}{\partial t} D(r, t)$$

(5.7)

where $E, H, D, B, J,$ and $M$ denote, respectively, the electric field, the magnetic field, the electric flux density, the magnetic flux density, the impressed electric current density, and the impressed magnetic current density. In the background, the electric flux density $D$ is given by

$$D(r, t) = \varepsilon(r, t) \odot E(r, t)$$

(5.8)

where $\varepsilon$ is the background medium’s permittivity tensor. The magnetic flux density $B$ is given by

$$B(r, t) = \underline{\mu}(r, t) \odot H(r, t)$$

(5.9)

where $\underline{\mu}$ is the permeability tensor of the background.

To develop the optical theorems in a framework applicable to both reciprocal and nonreciprocal media, we consider the complementary medium in which the permittivity and permeability are given by

$$\varepsilon^C(r, t) = \varepsilon^T(r, t)$$
$$\underline{\mu}^C(r, t) = \underline{\mu}^T(r, t)$$

(5.10)

where $(\cdot)^T$ denotes the transpose. If $\varepsilon^T = \varepsilon$ and $\underline{\mu}^T = \underline{\mu}$ the medium is reciprocal. If these conditions do not hold the medium is nonreciprocal.

Consider any two pairs of sources $J_1, M_1$ and $J_2, M_2$ having respective supports $V_1$ and $V_2$. Sources “1” generate fields $E_1, H_1$ in the background medium. The same sources generate fields $E_1^C, H_1^C$ in the complementary medium. Sources “2” generate fields $E_2, H_2$ in the background. The same sources generate fields $E_2^C, H_2^C$ in the complementary medium. We develop next the fundamental reciprocity relations describing the reaction and interaction of these two sets of sources and their fields.
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It is well known (see [105] and [106]) that the following modified reciprocity theorem holds:

\[
\int_{V_1} dV[(E_2 \odot J_1)(\mathbf{r}, t) - (H_2 \odot M_1)(\mathbf{r}, t)]
= \int_{V_2} dV[(E_1 \odot J_2)(\mathbf{r}, t) - (H_1 \odot M_2)(\mathbf{r}, t)].
\] (5.11)

In the special case in which the background is reciprocal then \(E_2 = E_2^C, H_2 = H_2^C\) so that this reduces to the familiar Lorentz reciprocity theorem.

We develop next two fundamental correlation-type reciprocity relations. The starting point is provided by the background medium Maxwell equations corresponding to sources and fields “1”, which we write as

\[
\nabla \times E_1(\mathbf{r}, \tau) = -M_1(\mathbf{r}, \tau) - \frac{\partial}{\partial \tau} B_1(\mathbf{r}, \tau)
\]
\[
\nabla \times H_1(\mathbf{r}, \tau) = J_1(\mathbf{r}, \tau) + \frac{\partial}{\partial \tau} D_1(\mathbf{r}, \tau).
\] (5.12)

The corresponding equations for sources and fields “2” are

\[
\nabla \times E_2(\mathbf{r}, \tau) = -M_2(\mathbf{r}, \tau) - \frac{\partial}{\partial \tau} B_2(\mathbf{r}, \tau)
\]
\[
\nabla \times H_2(\mathbf{r}, \tau) = J_2(\mathbf{r}, \tau) + \frac{\partial}{\partial \tau} D_2(\mathbf{r}, \tau).
\] (5.13)

Substituting \(\tau\) by \(t + \tau\) in the above result we obtain

\[
\nabla \times E_2(\mathbf{r}, t + \tau) = -M_2(\mathbf{r}, t + \tau) - \frac{\partial}{\partial \tau} B_2(\mathbf{r}, t + \tau)
\]
\[
\nabla \times H_2(\mathbf{r}, t + \tau) = J_2(\mathbf{r}, t + \tau) + \frac{\partial}{\partial \tau} D_2(\mathbf{r}, t + \tau).
\] (5.14)

Multiplying the first of eqs.(5.12) by \(H_2(\mathbf{r}, t + \tau)\) (i.e., applying \(H_2(\mathbf{r}, t + \tau) \cdot \cdot\)) one obtains

\[
H_2(\mathbf{r}, t + \tau) \cdot [\nabla \times E_1(\mathbf{r}, \tau)] = -H_2(\mathbf{r}, t + \tau) \cdot M_1(\mathbf{r}, \tau) - H_2(\mathbf{r}, t + \tau) \cdot \frac{\partial}{\partial \tau} B_1(\mathbf{r}, \tau).
\] (5.15)

Multiplying the last of eqs.(5.14) by \(E_1(\mathbf{r}, \tau)\) we get

\[
E_1(\mathbf{r}, \tau) \cdot [\nabla \times H_2(\mathbf{r}, t + \tau)] = E_1(\mathbf{r}, \tau) \cdot J_2(\mathbf{r}, t + \tau) + E_1(\mathbf{r}, \tau) \cdot \frac{\partial}{\partial \tau} D_2(\mathbf{r}, t + \tau).
\] (5.16)

It follows from (5.15) (5.16) and the identity

\[
\nabla \cdot (A \times B) = B \cdot \nabla \times A - A \cdot \nabla \times B
\] (5.17)
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that

\[
\nabla \cdot \left[ E_1(r, \tau) \times H_2(r, t + \tau) \right] = -H_2(r, t + \tau) \cdot M_1(r, \tau) - E_1(r, \tau) \cdot J_2(r, t + \tau) - H_2(r, t + \tau) \cdot \frac{\partial}{\partial \tau} B_1(r, \tau) - E_1(r, \tau) \cdot \frac{\partial}{\partial \tau} D_2(r, t + \tau).
\]

(5.18)

Similarly, from the last of eqs. (5.12) and the first of eqs. (5.14) one also obtains

\[
\nabla \cdot \left[ E_2(r, t + \tau) \times H_1(r, \tau) \right] = -H_1(r, \tau) \cdot M_2(r, t + \tau) - E_2(r, t + \tau) \cdot J_1(r, \tau) - H_1(r, \tau) \cdot \frac{\partial}{\partial \tau} B_2(r, t + \tau) - E_2(r, t + \tau) \cdot \frac{\partial}{\partial \tau} D_1(r, \tau).
\]

(5.19)

For the special case in which sources and fields “1” are identical to sources and fields “2”, and for \( t = 0 \), the above results, eqs. (5.18) and (5.19), reduce to the differential form of Poynting’s theorem:

\[
\nabla \cdot \left[ E_1 \times H_1 \right] = -M_1 \cdot B_2 - H_1 \cdot J_2 - E_1 \cdot D_2 - \frac{\partial}{\partial t} B_1 \cdot H_2 - \frac{\partial}{\partial t} B_2 \cdot E_1 - \frac{\partial}{\partial t} D_1 \cdot E_2.
\]

(5.20)

This reveals that the correlation reciprocity relations to be developed next are a generalization of the electromagnetic energy conservation relation based on Poynting’s theorem. The left side term in (5.20) is related to the flux density of power exiting the differential volume. The first two source-field interaction terms in the right side of (5.20) define the density of power put by the sources. Finally, the last two terms in the right side of the same equation define the time rate at which energy is stored in the differential volume, which is usually termed the reactive power density.

Adding eqs. (5.18) and (5.19) and integrating the resulting expression over \( \tau \) we get

\[
\nabla \cdot \left[ E_1 \star H_2 - H_1 \star E_2 \right] = -M_1 \cdot H_2 - H_1 \cdot M_2 - E_1 \cdot J_2 - J_1 \cdot E_2 - \frac{\partial}{\partial t} B_1 \cdot H_2 - \frac{\partial}{\partial t} B_2 \cdot E_1 - \frac{\partial}{\partial t} D_1 \cdot E_2.
\]

(5.21)

where we conveniently suppress the space-time dependence \((r, t)\) with the understanding that all the quantities appearing in the formulation depend on time and position. Also, subtracting eqs. (5.18)
and (5.19) and integrating over \( \tau \) we get
\[
\nabla \cdot \left[ E_1 \ast H_2 + H_1 \ast E_2 \right] = -M_1 \circ H_2 + H_1 \circ M_2 - E_1 \circ J_2 + J_1 \circ E_2 \\
- \frac{\partial}{\partial t} B_1 \circ H_2 + H_1 \circ \frac{\partial}{\partial t} B_2 - E_1 \circ \frac{\partial}{\partial t} D_2 \\
+ \frac{\partial}{\partial t} D_1 \circ E_2.
\]
(5.22)

In view of (5.2, 5.5, 5.8, 5.9) the result (5.21) can be written as (see also [105]):
\[
\nabla \cdot \left[ E_1 \ast H_2 - H_1 \ast E_2 \right] = -M_1 \circ H_2 - H_1 \circ M_2 - E_1 \circ J_2 - J_1 \circ E_2 \\
+ \frac{\partial}{\partial t} \{ \bar{H}_1 \circ ([\mu^T - \mu] \circ H_2) \} \\
+ \frac{\partial}{\partial t} \{ \bar{E}_1 \circ ([\varepsilon^T - \varepsilon] \circ E_2) \} 
\]
(5.23)

which further reduces to
\[
\nabla \cdot \left[ E_1 \ast H_2 - H_1 \ast E_2 \right] = -M_1 \circ H_2 - H_1 \circ M_2 - E_1 \circ J_2 - J_1 \circ E_2 
\]
(5.24)

if the condition
\[
\xi^T(r,-t) = \xi(r,t) \\
\mu^T(r,-t) = \mu(r,t)
\]
(5.25)

holds. We assume this condition corresponding to a lossless background. For the special case of a nondispersive medium for which the permittivity and permeability are of the form
\[
\xi(r,t) = \xi_0(r)\delta(t) \\
\mu(r,t) = \mu_0(r)\delta(t)
\]
(5.26)

where \( \delta(\cdot) \) is Dirac’s delta function and where \( \xi_0 \) and \( \mu_0 \) are space-dependent dyadics, this further implies that the medium is reciprocal, in particular, \( \xi^T_0 = \xi_0 \) and \( \mu^T_0 = \mu_0 \).

Similarly, in view of (5.2, 5.5, 5.8, 5.9) the result (5.22) reduces to
\[
\nabla \cdot \left[ E_1 \ast H_2 + H_1 \ast E_2 \right] = -M_1 \circ H_2 + H_1 \circ M_2 - E_1 \circ J_2 + J_1 \circ E_2 \\
+ 2 \frac{\partial}{\partial t} \{ \bar{H}_1 \circ (\mu \circ H_2) \} \\
+ 2 \frac{\partial}{\partial t} \{ \bar{E}_1 \circ (\varepsilon \circ E_2) \}.
\]
(5.27)

The correlation reciprocity relation (5.27) has linkages to the reactive energy and appears to be new.
The integral form of (5.24) follows from the divergence theorem and can be written as

$$
\int_{\partial V} dS \hat{n} \cdot (E_1 \star H_2 - H_1 \star E_2) = - \int_{V_1} dV (M_1 \circ H_2 + J_1 \circ E_2) - \int_{V_2} dV (H_1 \circ M_2 + E_1 \circ J_2)
$$

(5.28)

where \( V \) is a volume containing \( V_1 \) and \( V_2 \) (\( V_1 \subseteq V \) and \( V_2 \subseteq V \)), \( \partial V \) is the boundary of \( V \), \( dS \) is surface differential element, and \( \hat{n} \) is the unit vector in the outward-normal direction associated to the differential element. Similarly, the integral form of (5.27) is

$$
\int_{\partial V} dS \hat{n} \cdot (E_1 \star H_2 + H_1 \star E_2) = \int_{V_1} dV (-M_1 \circ H_2 + J_1 \circ E_2) + \int_{V_2} dV (H_1 \circ M_2 - E_1 \circ J_2) + 2 \frac{\partial}{\partial t} \int_V dV \{ \bar{H}_1 \circ [\mu \circ H_2] + \bar{E}_1 \circ [\varepsilon \circ E_2]\}.
$$

(5.29)

Two classes of time-domain optical theorems follow from the key results (5.28, 5.29). The first class is the most general and will be referred to as “correlation-type optical theorems”. The second class is an important special case of the former, corresponding to “autocorrelations” as opposed to the more general “cross-correlations” of the general theory. The autocorrelations in question correspond to the physical electromagnetic energies. We shall term this special class “autocorrelation-type optical theorems” or simply “energy-type optical theorems”.

We conclude this section with the associated precursor relations for the ordinary and reactive optical theorems. In particular, for the special case in which the sources and fields “1” and “2” are identical the key result (5.28) takes the form

$$
\int_{\partial V} dS \hat{n} \cdot (E_1 \star H_1 - H_1 \star E_1) = - \int_{V_1} dV (M_1 \circ H_1 + J_1 \circ E_1 + H_1 \circ M_1 + E_1 \circ J_1).
$$

(5.30)

This is the precursor of the ordinary optical theorems derived next. The correlation-type ordinary optical theorem follows from the general expression (5.30). The more specialized energy-type version is derived by evaluating (5.30) for \( t = 0 \), which gives

$$
\int_{-\infty}^{\infty} d\tau \int_{\partial V} dS \hat{n} \cdot [E_1(r, \tau) \times H_1(r, \tau)] = - \int_{-\infty}^{\infty} d\tau \int_{V_1} dV [M_1(r, \tau) \cdot H_1(r, \tau) + J_1(r, \tau) \cdot E_1(r, \tau)].
$$

(5.31)
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The term in the left side of this equation is the total radiated energy. It is equal to the time-integrated sum of the source-field interaction integrals in the right side of the equation. Appendix in our key work [103] provides further interpretation.

Similarly, for the special case in which the sources and fields “1” and “2” are identical, the key result (5.29) becomes

\[
\int_{\partial V} dS \hat{n} \cdot (E_1 \times H_1 + H_1 \times E_1) = \int_{V} dV (-M_1 \circ H_1 + J_1 \circ E_1 + \frac{\partial}{\partial t}(H_1 \circ \epsilon \circ E_1) + \epsilon \circ \frac{\partial}{\partial t} E_1 + (\mu \circ H_1) + \frac{\partial}{\partial t} H_1 \circ \epsilon \circ E_1)).
\]  
(5.32)

Evaluating this for \( t = 0 \) we get that the time-integrated sum of the electric and magnetic reactive powers is equal to zero, i.e.,

\[
\int_{-\infty}^{\infty} d\tau \int_{V} dV \left[ H_1(r, \tau) \cdot \frac{\partial}{\partial \tau} B_1(r, \tau) + E_1(r, \tau) \cdot \frac{\partial}{\partial \tau} D_1(r, \tau) \right] = 0.
\]  
(5.33)

For \( t \neq 0 \) the general expression (5.32) may be exploited to gain information about the reactive energy, near field dynamics, which motivates the new correlation-type reactive optical theorem.

5.4 Scattering Formulation in the Time Domain

Let \( V_s \) be the scattering region where the scatterer of support \( V_0 \subseteq V_s \) is contained. The scatterer is embedded in the background medium having permittivity and permeability dyadics \( \epsilon \) and \( \mu \), respectively. It follows from (5.7) that when the scatterer is interrogated by arbitrary sources labelled “\( n \)” which are located outside the volume \( V_s \) the corresponding probing fields \( E_1^{(n)} \), \( H_1^{(n)} \) due to those sources obey in \( V_s \)

\[
\nabla \times E_i^{(n)}(r, t) = -\frac{\partial}{\partial t} B_i^{(n)}(r, t)
\]

\[
\nabla \times H_i^{(n)}(r, t) = \frac{\partial}{\partial t} D_i^{(n)}(r, t)
\]  
(5.34)

where \( D_i^{(n)} \) and \( B_i^{(n)} \) are the electric and magnetic flux densities associated to the electric and magnetic fields \( E_i^{(n)} \) and \( H_i^{(n)} \), in particular,

\[
\nabla \times E_i^{(n)}(r, t) = -\frac{\partial}{\partial t}(\mu \circ H_i^{(n)})(r, t)
\]

\[
\nabla \times H_i^{(n)}(r, t) = \frac{\partial}{\partial t}(\epsilon \circ E_i^{(n)})(r, t).
\]  
(5.35)
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Let $\varepsilon_t$ and $\mu_t$ be, respectively, the permittivity and permeability of the total medium composed by the background plus the scatterer. It follows that, relative to the background, the scatterer’s constitutive properties are $\delta\varepsilon = \varepsilon_t - \varepsilon$ and $\delta\mu = \mu_t - \mu$. The total fields $E_{t}^{(n)}, H_{t}^{(n)}$ in the total medium composed by the background plus the scatterer upon the excitation by sources “$n$” located outside $V_s$ obey in $V_s$

$$\nabla \times E_{t}^{(n)}(r, t) = -\frac{\partial}{\partial t} B_{t}^{(n)}(r, t)$$
$$\nabla \times H_{t}^{(n)}(r, t) = \frac{\partial}{\partial t} D_{t}^{(n)}(r, t).$$

(5.36)

This can be written in terms of the fields as

$$\nabla \times E_{t}^{(n)}(r, t) = -\frac{\partial}{\partial t} (\delta\mu \odot H_{t}^{(n)})(r, t) - \frac{\partial}{\partial t} (\mu \odot H_{t}^{(n)})(r, t)$$
$$\nabla \times H_{t}^{(n)}(r, t) = \frac{\partial}{\partial t} (\delta\varepsilon \odot E_{t}^{(n)})(r, t) + \frac{\partial}{\partial t}(\varepsilon \odot E_{t}^{(n)})(r, t).$$

(5.37)

It follows from (5.35, 5.37) that the scattered fields

$$E_{s}^{(n)}(r, t) = E_{t}^{(n)}(r, t) - E_{i}^{(n)}(r, t)$$
$$H_{s}^{(n)}(r, t) = H_{t}^{(n)}(r, t) - H_{i}^{(n)}(r, t)$$

(5.38)

obey

$$\nabla \times E_{s}^{(n)}(r, t) = -M_{s}^{(n)}(r, t) - \frac{\partial}{\partial t}(\mu \odot H_{s}^{(n)})(r, t)$$
$$\nabla \times H_{s}^{(n)}(r, t) = J_{s}^{(n)}(r, t) + \frac{\partial}{\partial t}(\varepsilon \odot E_{s}^{(n)})(r, t)$$

(5.39)

where we have conveniently introduced the electric and magnetic sources, $J_{s}^{(n)}$ and $M_{s}^{(n)}$, respectively, that are induced in the scatterer upon excitation by the sources labelled “$n$” located outside the scattering region $V_s$. They are given by

$$J_{s}^{(n)}(r, t) = \frac{\partial}{\partial t} (\delta\varepsilon \odot E_{t}^{(n)})(r, t)$$
$$M_{s}^{(n)}(r, t) = \frac{\partial}{\partial t} (\delta\mu \odot H_{t}^{(n)})(r, t).$$

(5.40)

The scattered fields $E_{s}^{(n)}, H_{s}^{(n)}$ are uniquely defined by (5.39) and the radiation condition. Expressions (5.35, 5.36) hold for arbitrary scatterers. The particular expressions (5.40) for the induced sources $J_{s}^{(n)}, M_{s}^{(n)}$ hold only for LTI scatterers.
5.5 Field Measurements

To formulate the optical theorems, we need to define how to implement scattered field measurements. It follows from antenna theory that the most general measurement of the scattered fields $E_s^{(n)}, H_s^{(n)}$ is of the form

$$V^{(m,n)}(t) = \int_{V_r} dV \left[ (E_s^{(n)} \odot h_e^{(m)})(\mathbf{r},t) - (H_s^{(n)} \odot h_m^{(m)})(\mathbf{r},t) \right]$$  \hspace{1cm} (5.41)

where $V_r$ represents the probe region of localization, $V^{(m,n)}$ represents the output voltage due to the sensing of fields $E_s^{(n)}, H_s^{(n)}$ with electric and magnetic probe modes labeled “$m$” and characterized by local impulse responses $h_e^{(m)}$ and $h_m^{(m)}$, respectively. In view of the reciprocity theorem eq.(5.11) the value of the voltage $V^{(m,n)}$ is given in terms of the induced sources $J_s^{(n)}, M_s^{(n)}$ by

$$V^{(m,n)}(t) = \int_{V_s} dV \left[ (E^{C(m)}_h \odot J_s^{(n)})(\mathbf{r},t) - (H^{C(m)}_h \odot M_s^{(n)})(\mathbf{r},t) \right]$$  \hspace{1cm} (5.42)

where $E^{C(m)}_h, H^{C(m)}_h$ are the electric and magnetic fields generated by electric and magnetic sources $h_e^{(m)}, h_m^{(m)}$ which obey Maxwell’s equations in the complementary medium, namely,

$$\nabla \times E^{C(m)}_h(\mathbf{r},t) = -h_m^{(m)}(\mathbf{r},t) - \frac{\partial}{\partial t} (\mu^C \odot H^{C(m)}_h)(\mathbf{r},t)$$

$$\nabla \times H^{C(m)}_h(\mathbf{r},t) = h_e^{(m)}(\mathbf{r},t) + \frac{\partial}{\partial t} (\epsilon^C \odot E^{C(m)}_h)(\mathbf{r},t)$$  \hspace{1cm} (5.43)

plus the radiation condition.

5.6 The Ordinary Optical Theorem in the Time Domain

The correlation-type form of the ordinary optical theorem follows from (5.31) with the substitutions of $E_1, H_1$ for the scattered fields $E_s^{(n)}, H_s^{(n)}$, and of $J_1, M_1$ for the induced sources $J_s^{(n)}, M_s^{(n)}$. In addition, we substitute $V_1$ for $V_s$ and $\partial V$ for the boundary $\partial V_s$ of $V_s$. We also borrow from (5.38). We obtain the following correlation-type optical theorem:

$$U^{(n)}(t) = S_{e}^{(n,n)}(t) + S_{d}^{(n,n)}(t)$$  \hspace{1cm} (5.44)

where

$$U^{(n)}(t) = \frac{1}{2} \left[ U^{(n)}(t) + U^{(n)}(-t) \right],$$  \hspace{1cm} (5.45)
where \( U^{(n)} \) is the quantity defined by

\[
U^{(n)} = \int_{V_s} dV \left( H_i^{(n)} \odot M_i^{(n)} + E_i^{(n)} \odot J_i^{(n)} \right)
\]

\[
= \int_{V_s} dV \left( \bar{H}_i^{(n)} \odot M_i^{(n)} + \bar{E}_i^{(n)} \odot J_i^{(n)} \right),
\] (5.46)

and where

\[
S^{(n,n)}_e(t) = \frac{1}{2} \left[ S^{(n,n)}_e(t) + S^{(n,n)}_e(-t) \right]
\] (5.47)

where

\[
S^{(n,n)}_e = \int_{\partial V_s} dS \hat{n} \cdot (E_s^{(n)} \ast H_s^{(n)}),
\] (5.48)

while

\[
S^{(n,n)}_d(t) = \frac{1}{2} \left[ S^{(n,n)}_d(t) + S^{(n,n)}_d(-t) \right]
\] (5.49)

where

\[
S^{(n,n)}_d = \int_{V_s} dV \left( H_t^{(n)} \odot M_t^{(n)} + E_t^{(n)} \odot J_t^{(n)} \right).
\] (5.50)

It is necessary to specify how to implement the key measurement (5.46) using probes located outside the scattering region \( V_s \). It follows from (5.41, 5.42, 5.46) that \( U^{(n)}(t) \) can be measured in the form of a scattered field measurement \( V^{(n,n)}(t) \) as defined in (5.41), i.e., \( U^{(n)}(t) = V^{(n,n)}(t) \), where according to (5.42) the required electric and magnetic field probes characterized by the local impulse response functions \( h^{(n)}_e, h^{(n)}_m \) are such that they generate, when operating as transmitters, the time-reversed incident fields \( \bar{E}_i^{(n)}, \bar{H}_i^{(n)} \) for \( \vec{r} \in V_s \) in the complementary medium defined in (5.10), in particular,

\[
E_h^{C(n)}(\vec{r}, t) = E_i^{(n)}(\vec{r}, -t) \quad \vec{r} \in V_s
\]

\[
H_h^{C(n)}(\vec{r}, t) = -H_i^{(n)}(\vec{r}, -t) \quad \vec{r} \in V_s.
\] (5.51)

Furthermore, for the case of an incident electromagnetic pulse passing by the scattering region \( V_s \) in a finite, time interval, say, \([0, T]\), it suffices that (5.51) be obeyed within that interval. It is not hard to show from Maxwell’s equations for the complementary medium plus the vector analogues of Green’s theorem and Kirchhoff’s integral formula ([2], sec. 10.6) that these fields are realizable using sources located outside \( V_s \), thus the optical theorem measurement is realizable with probes located outside \( V_s \). Thus for probes \( (h^{(n)}_e, h^{(n)}_m) \) having these characteristics, the optical theorem expression (5.44) can be written in terms of the scattering measurement \( V^{(n,n)}(t) \) as

\[
V^{(n,n)}(t) = U^{(n)}(t) = S^{(n,n)}_e(t) + S^{(n,n)}_d(t)
\] (5.52)
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where

\[ V^{(n,n)+}(t) = \frac{1}{2} \left[ V^{(n,n)}(t) + V^{(n,n)}(-t) \right]. \] (5.53)

Here it is very important to point out that the actual probes \((h_e^{(n)}, h_m^{(n)})\) used to implement in practice the optical theorem (5.52) are nonunique \([60]\).

The energy-type form of the ordinary optical theorem follows from (5.31) via the same substitutions or, alternatively, by evaluating the expression for the correlation-type optical theorem (5.52) for \(t = 0\). In particular, the extinct energy, taken away from the probing beam by the scatterer, is equal to \(U^{(n)+}(0) = U^{(n)}(0)\), which is given from (5.45 5.46) by

\[ U^{(n)+}(0) = \int_{-\infty}^{\infty} d\tau \int_{V_s} dV \left[ H_i^{(n)}(r, \tau) \cdot M_s^{(n)}(r, \tau) + E_s^{(n)}(r, \tau) \cdot J_s^{(n)}(r, \tau) \right]. \] (5.54)

As expected, it is given by the integral of the interaction of the incident field \((E_i^{(n)}, H_i^{(n)})\) with the source \((J_s^{(n)}, M_s^{(n)})\) that is induced in the scatterer. Note that if the incident electromagnetic pulse passes by the scattering region \(V_s\) in a finite time window, say \([0, T]\), then the only relevant value of the induced source is the one corresponding to that particular interval. Furthermore, it follows from (5.41) that the extinct energy can be measured with probes \(h_e^{(n)}, h_m^{(n)}\) located outside the scattering region \(V_s\) via an optical theorem measurement of the form

\[ V^{(n,n)+}(0) = \int_{-\infty}^{\infty} d\tau \int_{V_s} dV \left[ E_s^{(n)}(r, \tau) \cdot h_e^{(n)}(r, -\tau) - H_s^{(n)}(r, \tau) \cdot h_m^{(n)}(r, -\tau) \right]. \] (5.55)

where the probes \((h_e^{(n)}(r, t), h_m^{(n)}(r, t))\) obey the required conditions in (5.51). In addition, the source-field correlation function \(S_e^{(n,n)+}(t)\) reduces for \(t = 0\) to the total (over all time) scattered energy,

\[ S_e^{(n,n)+}(0) = \text{Energy}_{s}^{(n)} = \int_{-\infty}^{\infty} d\tau \int_{\partial V_s} dS \mathbf{n} \cdot [E_s^{(n)}(r, \tau) \times H_s^{(n)}(r, \tau)]. \] (5.56)

Also, the source-field correlation function \(S_d^{(n,n)+}(t)\) reduces for \(t = 0\) to the total energy that is dissipated in the scatterer,

\[ S_d^{(n,n)+}(0) = \text{Energy}_{d}^{(n)} = \int_{-\infty}^{\infty} d\tau \int_{V_s} dV [M_i^{(n)}(r, \tau) \cdot H_i^{(n)}(r, \tau) + J_s^{(n)}(r, \tau) \cdot E_i^{(n)}(r, \tau)]. \] (5.57)

In summary, evaluating (5.52) for \(t = 0\) and using (5.56 5.57) we obtain the energy-type ordinary optical theorem:

\[ V^{(n,n)}(0) = U^{(n)}(0) = \text{Energy}^{(n)}_{s} + \text{Energy}^{(n)}_{d}. \] (5.58)

Likewise, it is shown in our paper \([103]\) that there is also a correlation-type reactive optical theorem which follows from (5.32), and generalizes the frequency domain results in \([52]\), sec. V.
Let $V^{(m,n)}$ be the scattered field measurement defined in (5.41) where the probes characterized by the local impulse response functions $h_e^{(m)}$, $h_e^{(m)}$ are such that they generate, in transmit mode, the fields given by (5.51), with $n$ substituted by $m$, in the complementary medium defined in (5.10). Then from (5.42)

$$V^{(m,n)} = \int_{V_s} dV \left( H^{(n)}_s \circ h^{(m)}_m + E^{(n)}_s \circ h^{(m)}_e \right) = \int_{V_s} dV \left( \bar{H}^{(m)}_i \circ M^{(n)}_s + \bar{E}^{(m)}_i \circ J^{(n)}_s \right).$$

(5.59)

The correlation-type generalized optical theorem follows from (5.28) after the substitutions $E_1, H_1 \rightarrow E^{(m)}_s, H^{(m)}_s, J_1, M_1 \rightarrow J^{(m)}_s, M^{(m)}_s, V_1 \rightarrow V_s, \partial V \rightarrow \partial V_s,$ and $E_2, H_2 \rightarrow E^{(n)}_s, H^{(n)}_s, J_2, M_2 \rightarrow J^{(n)}_s, M^{(n)}_s,$ and $V_2 \rightarrow V_s$. In view of (5.51,5.59), it can be conveniently stated directly in terms of the required generalized optical theorem measurement $V^{(m,n)}$ as

$$V^{(m,n)}(t) = S^{(m,n)}_e(t) + S^{(m,n)}_d(t)$$

(5.60)

where

$$V^{(m,n)}(t) = \frac{1}{2} \left[ V^{(m,n)}(t) + V^{(n,m)}(-t) \right],$$

(5.61)

$$S^{(m,n)}_e(t) = \frac{1}{2} \left[ S^{(m,n)}_e(t) + S^{(n,m)}_e(-t) \right]$$

(5.62)

where

$$S^{(m,n)}_e = \int_{\partial V_s} dS \hat{n} \cdot \left( E^{(m)}_s \ast H^{(n)}_s \right),$$

(5.63)

and

$$S^{(m,n)}_d(t) = \frac{1}{2} \left[ S^{(m,n)}_d(t) + S^{(n,m)}_d(-t) \right]$$

(5.64)

where

$$S^{(m,n)}_d = \int_{V_s} dV \left( H^{(m)}_i \circ M^{(n)}_s + E^{(m)}_i \circ J^{(n)}_s \right).$$

(5.65)

Evaluating (5.60) for $t = 0$ gives the energy-type generalized optical theorem. The latter is an important special case since its ordinary counterpart corresponding to $m = n$ characterizes the energy extinction of the scattering by probing field mode "n", as explained earlier.

We also obtain a complementary generalized optical theorem based on (5.29) via the same procedure used to obtain (5.60). For reasons already explained in connection with the discussion of
we assume next nonmagnetic scatterers. Similar results hold for nondielectric scatterers. We get
\[ V^{(m,n)}(t) = S^{(m,n)}_{r}(t) + S^{(m,n)}_{r}(t) + \delta S^{(m,n)}_{r} \]
\[ (5.66) \]
where
\[ V^{(m,n)}(t) = \frac{1}{2} \left[ V^{(m,n)}(t) - V^{(m,n)}(-t) \right], \]
\[ (5.67) \]
\[ S^{(m,n)}_{r}(t) = \frac{1}{2} \left[ S^{(m,n)}_{r}(t) - S^{(m,n)}_{r}(-t) \right], \]
\[ (5.68) \]
\[ S^{(m,n)}_{r}(t) = \frac{1}{2} \left[ S^{(m,n)}_{r}(t) - S^{(m,n)}_{r}(-t) \right]. \]
\[ (5.69) \]
where
\[ S^{(m,n)}_{r} = \int_{V_{s}} dV \left( E^{(m)}_{t} \odot J^{(n)}_{s} \right), \]
\[ (5.70) \]
and
\[ \delta S^{(m,n)}_{r} = - \frac{\partial}{\partial t} \int_{V_{s}} dV \left[ H^{(m)}_{s} \odot (\mu \odot H^{(n)}_{s}) + \right. \]
\[ \left. E^{(m)}_{s} \odot (\epsilon \odot E^{(n)}_{s}) \right]. \]
\[ (5.71) \]

5.8 The Special Case of Free Space and Plane Waves

In this section we show that the optical theorem results derived in this paper are consistent with those of prior work, particularly the pioneering work of Karlsson [53, 54, 55] which focuses on homogeneous media such as free space and plane wave excitation. In particular, we show that the energy-type ordinary optical theorem (5.58) reduces for free space and plane wave excitation to the results eq.(2.5) in [53] and eq.(15) in [55].

Consider the particular case in which the scatterer is probed with a plane wave traveling in the direction of the unit vector \( s_{0} \). The respective incident electric field \( (E^{(n)}_{i}) \) is given by \( E_{0}(t - s_{0} \cdot r/c) \) where \( s_{0} \cdot E_{0} = 0 \). We denote the respective scattered electric field as \( E_{s}(r, t; s_{0}) \), and similarly the sources induced in the scatterer denoted as \( J_{s}(r, t; s_{0}) \), \( M_{s}(r, t; s_{0}) \).

It is not hard to show that the (scattered) electric field \( (E_{s}) \) that is generated by sources \( (J_{s}, M_{s}) \) induced in a scatterer that is embedded in free space behaves in the far zone as
\[ E_{s}(r \hat{r}, t; s_{0}) \sim \frac{F_{s}(\tilde{r}, t - r/c; s_{0})}{r} \]
\[ (5.72) \]
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where \( F_s \) is the far-field scattering amplitude, which is given in terms of the sources by (107), eqs. 46-49

\[
F_s(\hat{r}, t; s_0) = -\frac{\mu_0}{4\pi} \frac{\partial}{\partial t} (1 - \hat{r}\hat{r} \cdot) \left[ \hat{J}_s(\hat{r}, t; s_0) - \hat{r} \times \hat{M}_s(\hat{r}, t; s_0)/\eta \right] \tag{5.73}
\]

where \( \hat{J}_s \) and \( \hat{M}_s \) are the slant-stack transforms of \( J_s \) and \( M_s \), respectively, which are given by

\[
\hat{J}_s(\hat{r}, t; s_0) = \int_{V_s} dV J_s(r, t + \hat{r} \cdot r/c; s_0)
\]
\[
\hat{M}_s(\hat{r}, t; s_0) = \int_{V_s} dV M_s(r, t + \hat{r} \cdot r/c; s_0). \tag{5.74}
\]

Now, it follows from (5.73) with the substitution \( \hat{r} = s_0 \) and the fact that \( s_0 \cdot E_0 = 0 \) that

\[
E_0(t) \cdot \left[ \hat{J}_s(s_0, t; s_0) - s_0 \times \hat{M}_s(s_0, t; s_0)/\eta \right] = -\frac{4\pi}{\mu_0} E_0(t) \cdot \int_{-\infty}^{t} dt' F(s_0, t'). \tag{5.75}
\]

It follows from (5.58, 5.54) that the extinct energy under this plane wave excitation case is given by

\[
U = \int_{V_s} dV \int_{-\infty}^{\infty} dt E_0(t - s_0 \cdot r/c) \cdot J_s(r, t; s_0) + s_0 \times H_0(t - s_0 \cdot r/c) \cdot M_s(r, t; s_0)/\eta
\]
\[
= \int_{V_s} dV \int_{-\infty}^{\infty} dt' E_0(t') \cdot J_s(r, t' + s_0 \cdot r/c; s_0) + s_0 \times E_0(t') \cdot M_s(r, t' + s_0 \cdot r/c; s_0)/\eta
\]
\[
= \int_{-\infty}^{\infty} dt' E_0(t') \left[ \hat{J}_s(s_0, t'; s_0) - s_0 \times \hat{M}_s(s_0, t'; s_0)/\eta \right]
\]
\[
= -\frac{4\pi}{\mu_0} \int_{-\infty}^{\infty} dt' E_0(t') \cdot \int_{-\infty}^{t'} dt'' F(s_0, t''; s_0), \tag{5.76}
\]

where we used the fact that the incident magnetic field is

\[
H_i(r, t; s_0) = s_0 \times E_0(t - s_0 \cdot r/c)/\eta,
\]

followed by the change of variable \( t \to t' + s_0 \cdot r/c \), followed by the identity \((A \cdot (B \times C)) = C \cdot (A \times B))\), followed by eq. (5.74), and finally by (5.75). The last equation in (5.76) is identical to the statement of the optical theorem for free space and plane waves as derived by Karlsson in [53, 54, 55], as we wanted to demonstrate.

5.9 Numerical Illustrations

In this section we consider in two-dimensional space the scattering of an electromagnetic pulse by a uniform circular cylinder of radius \( R \). The scattering cylinder is surrounded by a
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background medium consisting of two perfect electric conductor (PEC) plates that act as a corner reflector. The corresponding propagation and scattering is simulated computationally with the FDTD method, which is well-known and effective method for the implementation of computational electromagnetic (see [108, 109]). For all examples illustrated in this section, we consider a 9 m x 9 m scattering region or region of interest (ROI). The boundaries located under and to the right of the ROI are assumed to be open. Computationally, this is modeled via perfectly matched layer (PML) absorbing boundary conditions. The depth of boundary for each side is 0.5 m. The probing field is generated by a point source ($J_z$). Its time-dependence is that of the modulated Gaussian pulse shown in Fig. 5.1, which has 2.5 GHz center frequency and fractional bandwidth of 2. In the FDTD simulations, we used a computational grid having space step $\Delta x = \Delta y = \Delta = 0.02$ m, and time step $\Delta t = \frac{\Delta}{\sqrt{2c}}$ where $c$ is the free space speed of light.

Simulation 1:

In this example, the background is the free space medium including the corner reflector. The relevant scattering geometry is illustrated in Figure 5.2. The origin of the coordinates is the center of the computational domain, as shown in Figure 5.2 and Figure 5.3. The probing field is generated by a point source located at $(x, y) = (0 \text{ m}, -4.48 \text{ m})$ as shown in the figures. Meanwhile, we set the scatterer at the center of the ROI ($(x, y) = (0, 0)$). The scatterer has radius $R$. The scatterer and the PEC reflectors are shown in Figure 5.3.

The results of a typical FDTD simulation are given next. In the simulation, we consider a lossless nonmagnetic scatterer having permittivity perturbation $\delta \varepsilon = 0.1 \varepsilon_0$ where $\varepsilon_0$ is the free space...
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Figure 5.2: Scattering geometry, showing the probing source, one scatterer, and the corner reflector which acts as the background. PEC boundary conditions are applied to the top and left sides (this is the corner reflector), while PML boundary conditions are implemented at the bottom and right sides of the FDTD computational grid.

Figure 5.3: Numerical geometry of example 1, showing the scatterer and PEC in white.
permittivity. The scatterer has radius $R = 0.4$ m. The corresponding incident electric field used to probe the ROI is illustrated in Figure 5.4. In the presence of the scatterer in the ROI, this field is perturbed. Figure 5.5 illustrates the total electric field after the scatterer is added. Figure 5.6 shows the corresponding scattered electric field.

To validate the ordinary optical theorem eq. (5.58), we computed numerically the key terms appearing in that theorem, namely, the total extinct energy $V = U$, the scattered energy $S_e = E_{nergy_s}$, and the dissipated energy $S_d = E_{nergy_d}$, versus different scattering parameters. Figure 5.7 shows, for fixed scatterer radius $R = 0.4$ m, the variation of $U$ and $S_e$ versus permittivity perturbation $\delta \varepsilon$ ranging from $0.01\varepsilon_0$ to $\varepsilon_0$. In this example the scatterer is lossless so the total extinct energy must be equal to the scattered energy. The plots of $U$ and $S_e$ are, indeed, very similar, as
Figure 5.5: Snapshots of the total electric field after adding a scatterer in front of the corner reflector. The scatterer has radius $R = 0.4$ m and permittivity $\delta \epsilon = 0.1 \epsilon_0$. 

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Figure 5.6: Snapshots of the scattered electric field. The scatterer has radius $R = 0.4$ m and permittivity $\delta \epsilon = 0.1 \epsilon_0$. 
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expected.

Figure 5.7: Extinct energy $U$ and scattered energy $S_e$ versus permittivity perturbation $\delta \epsilon$. The dashed line represents $U$ while the solid line corresponds to $S_e$.

Figure 5.8 shows the corresponding results for a lossy scatterer having conductivity $\sigma = 1$ S/m. Other properties of the geometry do not change. As expected in this case, the dissipated energy is not equal to zero any more. The computed values of the total extinct energy $U$ are seen to coincide with the sum of the scattered energy $S_e$ and dissipated energy $S_d$.

In another set of plots, Figure 5.9, Figure 5.10, and Figure 5.11 show $U$ and $S_e$ as functions of the normalized scatterer radius $r_0 = R/\Delta$, which was chosen to vary in the range $[1, 25]$. In these simulations, we considered scatterer radii ($R$) in the range $[0.02$ m, $0.5$ m]. In these simulations we considered a lossless nonmagnetic scatterer having $\delta \epsilon = 0.01 \epsilon_0$ (in Figure 5.9), $\delta \epsilon = 0.1 \epsilon_0$ (in Figure 5.10) and $\delta \epsilon = 0.2 \epsilon_0$ (in Figure 5.11). Meanwhile, we keep other properties of the geometry same. Again we find that, as expected, the plots of the extinct and scattered energies based on the results of section 4 are very similar. The scattered energy $S_e$ is almost equal to the extinguished power $U$. The minor differences between the corresponding plots are attributed to the numerical truncation errors of the FDTD method.
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Figure 5.8: Extinct energy $U$, scattered energy $S_e$, and dissipated energy $S_d$, for a lossy scatterer having $\sigma = 1 \text{ S/m}$. The dashed line represents $U$, the solid line corresponds to $S_e$, and the dotted line refers to $S_d$.

Figure 5.9: Extinct energy $U$ and scattered energy $S_e$ for fixed $\delta \epsilon = 0.01\epsilon_0$ versus the normalized radius $r_0 = R/\Delta$. The dashed line represents $U$ while the solid line corresponds to $S_e$. 

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Figure 5.10: Extinct energy $U$ and scattered energy $S_e$ for fixed $\delta\epsilon = 0.1\epsilon_0$ versus the normalized radius $r_0 = R/\Delta$. The dashed line represents $U$ while the solid line corresponds to $S_e$.

Figure 5.11: Extinct energy $U$ and scattered energy $S_e$ for fixed $\delta\epsilon = 0.2\epsilon_0$ versus the normalized radius $r_0 = R/\Delta$. The dashed line represents $U$ while the solid line corresponds to $S_e$. 
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Simulation 2:

We continue with another example, which is inspired by an intriguing implication of the time-domain optical theorem discussed in [53, 54, 55]. In particular, the energy extinguished by two scatterers is not, in general, equal to the sum of the individual extinct energies of the scatterers. However, the extinction is governed by the interaction of the incident field with the induced source in the scatterer only in the time window in which the probing field passes by the scatterer’s support. Therefore, if two scatterers are sufficiently far so that the induced source (in each scatterer), in said time window, is due only to the incident field excitation, and not due to the field scattered by the neighboring scatterer, then the extinction resulting from each scatterer remains identical to that which would be obtained if that scatterer is alone. Thus for such well-separated scatterers the total extinct energy is equal to the sum of the individual extinct energies of the scatterers. This was shown in [53, 54, 55] for the case of an unbounded homogeneous background probed by plane waves. The same principle applies to arbitrary backgrounds and probing fields, as can be easily shown from the results in section 6 (see the discussion in eqs. (5.51, 5.54)). On the other hand, for media that are highly reverberating, such as the corner reflector background considered in this section, the probing field has two contributions: the initial wave launched by the probing source plus the waves bounced by the reflecting boundaries. The latter components arrive late and may significantly lengthen the passing time, by the scattering region, of the entire incident field. For such media, one expects that only for very specialized conditions, e.g., weak scattering (Born approximation), can the total extinction be equal to the sum of the individual extinguished energies. In the next set of figures we illustrate these ideas for the corner reflector background medium.

Figure 5.12 and Figure 5.13 show the geometry for two well-separated circular scatterers of radius \( R = 0.4 \) m. In this case the scatterers are centered at positions \((x, y) = (-1 \text{ m}, 0)\) (left scatterer) and \((1 \text{ m}, 0)\) (right scatterer). In this configuration the initial or early contribution to the incident pulse arrives at the same time at both scatterers. The time window associated to the passing of this early probing pulse by each scatterer is approximately equal to \(2R/c\). The distance \(2R = 0.8 \) m is shorter than the separation distance of the two scatterers (equal to 1.2 m in this example). Thus the multiple scattering interaction between the scatterers happens only after the early part of the probing pulse has passed their supports. In addition to this early probing pulse contribution, there is a late contribution to the probing pulse due to the reverberations at the reflector. Figure 5.14 shows that for small values of the scatterer permittivity (weak scattering regime) the extinct energy is equal to the sum of the individual extinct energies, as expected since in this limit the scatterers do not interact. The same figure also shows that the effect of the late probing pulse contribution becomes
Figure 5.12: Scattering geometry, showing the probing source, two scatterers having $R = 0.4$ m, and centered at $(x, y) = (-1, 0)$ and $(1, 0)$ and the corner reflector which acts as the background. PEC boundary conditions are applied to the top and left sides (this is the corner reflector), while PML boundary conditions are implemented at the bottom and right sides of the FDTD computational grid.

Figure 5.13: Simulation geometry showing the two scatterers having $R = 0.4$ m, and centered at $(x, y) = (-1, 0)$ and $(1, 0)$. 
Figure 5.14: Computed extinct energy versus $\delta\epsilon$ for two scatterers having $R = 0.4$ m, and centered at $(x, y) = (-1 \text{ m}, 0)$ and $(1 \text{ m}, 0)$. Bold dashed line: right scatterer. Bold solid line: left scatterer. Thin solid line: sum of the individual extinct energies. Dashed line: extinct energy computed for the two scatterers together.

more noticeable as the scatterer permittivity increases, as expected. In particular, for sufficiently large scatterer permittivity ($\delta\epsilon > 0.4\varepsilon_0$), the extinct energy of the two-scatterer compound is no longer equal to the sum of the extinct energies of the individual scatterers.
Simulation 3:

In the next example, the two scatterers are quite close to each other. Figure 5.15 and Figure 5.13 show the corresponding simulation geometry. In this case the scatterers are in close proximity, so that the extinct energy is approximately given by the sum of the individual extinct energies only for small $\delta \epsilon$ (corresponding to weak multiple scattering). Figure 5.17 shows the corresponding result.

Figure 5.15: Scattering geometry, showing the probing source, two scatterers having $R = 0.4$ m, and centered at $(x, y) = (-1$ m, 0) and $(-0.2$ m, 0) and the corner reflector which acts as the background. PEC boundary conditions are applied to the top and left sides (this is the corner reflector), while PML boundary conditions are implemented at the bottom and right sides of the FDTD computational grid.
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Figure 5.16: Numerical geometry showing two scatterers having $R = 0.4$ m, and centered at $(x, y) = (-1 \text{ m}, 0)$ and $(-0.2 \text{ m}, 0)$.

Figure 5.17: Computed extinct energy versus $\delta \epsilon$ for two scatterers next to each other having $R = 0.4$ m, and centered at $(x, y) = (-1 \text{ m}, 0)$ and $(-0.2 \text{ m}, 0)$. The solid line is the sum of the individual extinct energies. The dashed line corresponds to the extinct energy computed for the two scatterers together.
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Simulation 4:

Figure 5.18 and Figure 5.19 show the corresponding results for well-separated scatterers that are placed near the top PEC boundary, at positions (−1 m, 4 m) and (1 m, 4 m). In this case the late reverberating contribution of the probing field passes by the scatterer’s support shortly after the early pulse, and the multiple scattering interaction is insignificant within the passing window of the entire probing pulse. Consequently, the extinct energy is given by the sum of the individual extinct energies. Figure 5.20 shows the corresponding results.

Figure 5.18: Scattering geometry, showing the probing source, two scatterers having $R = 0.4$ m, and centered at $(x, y) = (−1$ m, 4 m) and (1 m, 4 m) and the corner reflector which acts as the background. PEC boundary conditions are applied to the top and left sides (this is the corner reflector), while PML boundary conditions are implemented at the bottom and right sides of the FDTD computational grid.
Figure 5.19: Simulation geometry showing the two scatterers having $R = 0.4$ m, and centered at $(x, y) = (-1$, $4)$ m and $(1$, $4)$ m.

Figure 5.20: Computed extinct energy versus $\delta\epsilon$ for two scatterers having $R = 0.4$ m, and centered at $(x, y) = (-1$, $4)$ m and $(1$, $4)$ m. The bold dashed line applies to the right scatterer only. The bold solid line applies to the left scatterer only. The thin solid line is the sum of the individual extinct energies. The dashed line corresponds to the extinct energy computed for the two scatterers together.
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Simulation 5:

Figure 5.21 and Figure 5.22 illustrate the corresponding geometry when the two scatterers are placed next to each other, at positions (−1 m, 4 m) and (−0.2 m, 4 m). As expected, in this case the total extinct energy is in general different from the sum of the individual extinct energies. These two quantities are equal only for small values of the scatterer permittivity $\delta\epsilon$, since then multiple scattering becomes negligible. Figure 5.23 illustrates the corresponding result.

Figure 5.21: Scattering geometry, showing the probing source, two scatterers having $R = 0.4$ m, and centered at $(x, y) = (−1$ m, $4$ m) and $(−0.2$ m, $4$ m) and the corner reflector which acts as the background. PEC boundary conditions are applied to the top and left sides (this is the corner reflector), while PML boundary conditions are implemented at the bottom and right sides of the FDTD computational grid.
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Figure 5.22: Simulation geometry showing the two scatterers having $R = 0.4$ m, and centered at $(x, y) = (-1 \text{ m}, 4 \text{ m})$ and $(-0.2 \text{ m}, 4 \text{ m})$.

Figure 5.23: Computed extinct energy versus $\delta \varepsilon$ for two scatterers having $R = 0.4$ m, and centered at $(x, y) = (-1 \text{ m}, 4 \text{ m})$ and $(-0.2 \text{ m}, 4 \text{ m})$. Bold dashed line: right scatterer. Bold solid line: left scatterer. Thin solid line: sum of the individual extinct energies. Dashed line: extinct energy computed for the two scatterers together.
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5.10 Conclusion

This chapter presented a very general formulation of the optical theorem in the time domain that applies to arbitrary probing fields and media. The derived theoretical framework for the optical theorem in the time domain is applicable to the most general scatterer, which can be time-varying and nonlinear, and has important envisioned applications such as the validation of computational electromagnetics codes, the fast computation of energy and field correlations, and the design of broadband detectors based on the time domain optical theorem. The derived formulation covers not only the ordinary optical theorem, but also the most general form of this result, known as the generalized optical theorem. Furthermore, two classes of time-domain optical theorems were developed: correlation-type optical theorems, which are the most general relations, as well as more specialized versions called autocorrelation- or energy-type optical theorems. In the discussion of the practical implementation of the optical theorem to measure the extinguished energy using external electromagnetic probes or sensors, it was mentioned that the sensors in question are inherently nonunique. Thus even though the statements of the optical theorems presented here are universal, their implementations can take an infinitude of alternative forms (see [60]). To demonstrate the link between our formulation and prior work in this area, we showed that the ordinary form of the time-domain optical theorem presented in this work renders the particular time-domain optical theorem for free space and plane wave excitation derived in previous papers. The derived ordinary optical theorem results were illustrated with the help of numerical examples in which the background medium is a corner reflector. The numerical results confirmed the validity of the formulation and shed insight on important implications of the time-domain optical theorem.
Chapter 6

Optical Theorem For Transmission Lines

6.1 Background

In this chapter we consider the theory and application of the optical theorem for the important special case of transmission line systems. This simple system is relevant on its own as well as a model for a broad range of electromagnetic wave propagation and scattering systems that can be modeled as one-dimensional, transmission-line-like systems. Such systems are relevant to microwave circuits, power transmission and distribution lines, simplified models of radar, sonar, and lidar, subsurface sensing of stratified media, and other applications. The focus on the transmission line case allows us to obtain analytical results which shed physical and insight on the feasibility of using the optical theorem as the basis of the new envisaged sensors. Methodologically, we reinterpret and implement, for one-dimensional electromagnetic systems such as transmission lines, the recently derived general version of the optical theorem reported in [52] which applies to arbitrary probing fields and lossless media. The results reveal previously unknown connections between the optical theorem predictions and standard results in transmission line theory, which enhances understanding of both the optical theorem and its application to transmission line problems. The insight gained by exploring this, simplest of the electromagnetic wave propagation systems from the point of view of the optical theorem is important for understanding power budget of electromagnetic scattering due to the presence of targets in a medium, and of changes of loads due to parasitics, faults, switching, and other reasons, in transmission lines, with applications to quality control in manufacturing, self-
monitoring of microwave circuits, and the detection of load changes and faults in power transmission and distribution systems. In addition, the results of this chapter corroborate an important optical theorem data constraint that is the basis of a new kind of detector considered in the next chapter.

6.2 Review of the Transmission Line Relations

In this section, we summarize the transmission line model of one-dimensional electromagnetic propagation, restricting attention to the description in terms of phasors, or equivalently the frequency domain picture in terms of the temporal Fourier transform of the electromagnetic signals. In this section we provide the transmission line relations for the baseline system in Figure 6.1 without the perturbations or changes whose consideration is left for the following sections. Figure 6.1 shows the circuit diagram of the baseline system under consideration. The transmitter, used to probe the one-dimensional propagation or transmission line system, is composed of signal generator and impedance. We denote the generator voltage phasor and the generator impedance as \( V_g \) and \( Z_g \), respectively. The transmission line has characteristic impedance \( Z_0 \) and length \( l \). Application of the optical theorem to the analysis of scattering phenomena due to load or line perturbations or changes requires that the transmission line be lossless, thus we assume that \( Z_0 \) is real. Also, the baseline transmission line system (without changes or perturbations) is ended by a passive, lossless, purely reactive load having impedance \( Z_L = jX_L \) where \( X_L \) is the load reactance. The voltage signal in

Figure 6.1: Circuit diagram showing the transmitter and the transmission line system with load.
the transmission line is given by \((110),\) ch.2

\[ V(z) = V_0^+ [\exp(-j\beta z) + \Gamma_L \exp(j\beta z)] \]  
(6.1)

where

\[ V_0^+ = \frac{Z_{in} V_g}{Z_{in} + Z_g} \frac{1}{1 + \Gamma_L} \]  
(6.2)

where the input impedance at the transmitter output terminals

\[ Z_{in} = Z_0 \left( \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right). \]  
(6.3)

The current signal is given by

\[ I(z) = \frac{V_0^+}{Z_0} [\exp(-j\beta z) - \Gamma_L \exp(j\beta z)]. \]  
(6.4)

In the following we shall work mostly with the normalized voltage signal

\[ v(z) \equiv \frac{V(z)}{V_0^+} = \exp(-j\beta z) + \Gamma_L \exp(j\beta z) \]  
(6.5)

which is equivalent to setting the incident wave amplitude to unity \(V_0^+ = 1.\) It is well known \((110),\) ch.2 that the time-average power of the incident and reflected waves, \(P_{av}^i\) and \(P_{av}^r ,\) respectively, are given by

\[ P_{av}^i = \frac{1}{2Z_0} \]

\[ P_{av}^r = -\frac{|\Gamma_L|^2}{2Z_0} \]  
(6.6)

and

\[ P_{av} = P_{av}^i + P_{av}^r = \frac{(1 - |\Gamma_L|^2)}{2Z_0}. \]  
(6.7)

### 6.3 The Scattered Signal

We consider the situation in which the transmission line system changes due to thermal changes, spurious or parasitic loads, or as part of switching operations in the subsystem modeled as a load above, such as for communications, signal processing, etc. Figure 6.2 illustrates two scenarios of interest in the following section. For the baseline system the normalized reflected signal is, according to (6.5), equal to

\[ v_r(z) = \Gamma_L \exp(j\beta z) \]  
(6.8)
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Figure 6.2: (a) Impedance change of the load circuit, modelled as a new load $Z_C$. (b) Parasitic effect or fault along the line, modelled as an impedance $Z_P$ in parallel.

while for the perturbed system it is given by

$$\hat{v}_r(z) = \Gamma_C \exp(j\beta z).$$

(6.9)

The normalized scattered voltage signal is the difference between the baseline and perturbed reflected signals, thus from (6.8, 6.9)

$$v^{(s)}(z) = \hat{v}_r(z) - v_r(z) = (\Gamma_C - \Gamma_L) \exp(j\beta z).$$

(6.10)

We assume next that the datum available for signal processing and detection is the value of the reflected signal at $z = 0$, the transmitter output terminals. Thus the measured reflected signal under the baseline system is

$$v^{(b)}_{rec} \equiv v_r(0) = \Gamma_L$$

(6.11)

and the scattering datum is

$$v_{scat} \equiv v^{(s)}(0) = \Gamma_C - \Gamma_L.$$ 

(6.12)

6.4 Optical Theorem Applied to Transmission Line Scattering

We apply next the optical theorem of electromagnetics to the study of wave scattering resulting from perturbations or changes to the original transmission line and its load (see [52] for the details).
The complex conjugate (or time-reversed) version of the voltage signal in (6.5) is
\[ v^*(z) = \exp(j\beta z) + \Gamma_L^* \exp(-j\beta z). \] (6.13)

The excitation \( V_0^+ \) that physically produces this voltage signal along the line, by radiation and propagation, is simply \( V_0^+ = \Gamma_L^* \) which is the complex conjugate of the measured reflected signal datum \( v_{rec}^{(b)} \) in (6.11), and this was expected from time-reversal electromagnetics.

According to the optical theorem, if we measure the value of the scattered signal \( v^{(s)} \) at \( z = 0 \), that is, \( v_{scat} \equiv v^{(s)}(0) \) (see eq. (6.12)), and filter this signal as
\[ f = v_{rec}^{(b)*} v_{scat}, \] (6.14)
then the real part of this projection
\[ \Re(f) = \Re(v_{rec}^{(b)*} v_{scat}) \propto P^{(s)} + P^{(loss)} \] (6.15)
where \( P^{(s)} \) is the time-average scattered power, corresponding to the scattered voltage signal, and \( P^{(loss)} \) is the power dissipated at the load \( Z_C \). Carrying out the computation in (6.14) with the help of (6.11, 6.12) we get
\[ f = \Gamma_L^* \Gamma_C - 1 \] (6.16)
so that the real part
\[ \Re(f) = \Re(\Gamma_L^* \Gamma_C) - 1. \] (6.17)

To see how this follows from familiar transmission line results we compute the average power of the scattered signal
\[ P^{(s)} = \frac{|\Gamma_C - \Gamma_L|^2}{2Z_0} = 1 - 2\Re(\Gamma_L^* \Gamma_C) + |\Gamma_C|^2 \] (6.18)
In addition, the power dissipated at the load is given by
\[ P^{(loss)} = \frac{1}{2Z_0} (1 - |\Gamma_C|^2) \] (6.19)
so that the sum
\[ P^{(s)} + P^{(loss)} = \frac{1}{Z_0} [1 - \Re(\Gamma_L^* \Gamma_C)]. \] (6.20)
In arriving at this result we used \( V_0^+ = 1 \). For general \( V_0^+ \) we obtain the general expression
\[ P^{(s)} + P^{(loss)} = -\frac{|V_0^+|^2}{Z_0} \Re(f) = -\frac{|V_0^+|^2}{Z_0} \Re(v_{rec}^{(b)*} v_{scat}). \] (6.21)
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We let

\[ \Gamma_L = |\Gamma_L| \exp(j\alpha) \]
\[ \Gamma_C = |\Gamma_C| \exp(j\gamma). \]  

(6.22)

Then

\[ \Re(\Gamma_L^* \Gamma_C) = |\Gamma_C| \cos(\gamma - \alpha) \]
\[ \Im(\Gamma_L^* \Gamma_C) = |\Gamma_C| \sin(\gamma - \alpha) \]  

(6.23)

so that from (6.17) \( \Re(f) \), which is a measure of the scattering real power (due to the scattered wave plus dissipation at the load), is given by

\[ \Re(f) = |\Gamma_C| \cos(\gamma - \alpha) - 1 \]  

(6.24)

and this has the Smith chart representation in Figure 6.3. Figure 6.3 shows the geometric, Smith chart representation of \( |\Im(f)| \). The imaginary part \( \Im(f) \) of \( f \) is a measure of the reactive power of the scattering phenomenon. We explain this with reference to the Smith chart as follows. Thus the respective change of reactive power due to the load change is given by

\[ \Delta P_{\text{react}} = P_{\text{react}}^C - P_{\text{react}}^L = \frac{|V_0^z|^2}{Z_0} \Im(\Gamma_C \Gamma_L^*) \propto \Im(f) \]  

(6.25)

as desired, where \( P_{\text{react}}^L \) is corresponding to the reference reactive load the reactive power going into the load at position \( z \), while \( P_{\text{react}}^C \) represents the changed load having reflection coefficient \( \Gamma_C \) the reactive power at the same reference.
Figure 6.3 also shows that $|f|$ is in fact equal to the magnitude of the scattered signal, $|\Gamma_L - \Gamma_C|$:

$$
\frac{|f|}{Z_0} = \frac{|v_{rec}v_{scat}|}{Z_0} = P_{ap} = \sqrt{(P(s) + P(\text{loss}))^2 + (\Delta P_{\text{react}})^2} = \frac{|\Gamma_C - \Gamma_L|}{Z_0}
$$

(6.26)

where $P_{ap}$ denotes the scattering apparent power.

These general results yield some interesting special cases. We assume for simplicity $V_0^+ = 1$. Note for instance that if the change or perturbation represented by the new load $Z_C$ is purely reactive or nondissipative, then knowledge of $|f|$ suffices to define the magnitude of both scattering real and reactive power, in particular,

$$
\Re(\Gamma_L^* \Gamma_C) + (\Im(\Gamma_L^* \Gamma_C))^2 = 1 \\
(\Re(\Gamma_L^* \Gamma_C) - 1)^2 + (\Im(\Gamma_L^* \Gamma_C))^2 = |f|^2
$$

(6.27)

so that

$$
\Re(f) = \Re(\Gamma_L^* \Gamma_C) - 1 = -|f|^2/2 \\
|\Im(f)| = |\Im(\Gamma_L^* \Gamma_C)| = |f|\sqrt{1 - |f|^2/4}
$$

(6.28)

which in view of the discussion in (6.21,6.25,6.26) implies

$$
P(s) = \frac{|f|^2|V_0^+|^2}{2Z_0} \\
|\Delta P_{\text{react}}| = \frac{|f|\sqrt{1 - |f|^2/4}|V_0^+|^2}{Z_0} \\
P_{ap} = \frac{|f||V_0^+|^2}{Z_0}
$$

(6.29)

so that the magnitude of both real and reactive power associated to the load change can be deduced from the magnitude of $f$ alone, which is very interesting and also quite relevant to high frequencies such as the optical regime in which one measures directly only field intensities. Note, however, that while the real power is positive, the reactive power can be positive or negative, and this sign still remains undetermined from the knowledge of $|f|$ alone. This implies that for nondissipative changes, all the physical power information, except only the sign of the reactive power, is carried by the magnitude of the change signal, which corresponds to the mathematical signal energy.

These results also allow us to establish some bounds for the real, reactive, and apparent power that is extinguished due to the scattering, which can be used if only partial information is available about the received signal or the associated reflection coefficient (see [59] for the details).
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6.5 Numerical Examples

We illustrate the preceding optical theorem developments with two transmission line examples which can be used, among other applications, for communications based on the change of a load (switching) at the end of a transmission line. We exploit the well known fact that many problems of electromagnetic wave propagation are analogous to the propagation of voltage and current signals in a transmission line, with the plane wave parameters of electric and magnetic field intensity, wavenumber, and wave impedance playing the role of the transmission line parameters of voltage and current, wavenumber, and line or load impedance, respectively.

Here we consider the case that the transmission line (of impedance $Z_0$ and wavenumber $\beta$) is ended by a segment of length $D$ constituted by another transmission line of impedance $Z_1$ that is ended by a short circuit load shown in Figure 6.4. The segment of transmission line has wavenumber $\beta_1$. This transmission line segment is assumed to be lossless so that both $Z_1$ and $\beta_1$ are real-valued. This corresponds to the original condition of the system, which can change at subsequent measurements. For example, the length ($D$) and impedance ($Z_1$) of the transmission line segment can change, taking possibly different values $D + \delta$ and $Z_1^C$ as illustrated in the bottom part of Figure 6.4. In addition, the wavenumber of the segment of transmission line can also change, from its original value of $\beta_1$ to a new value of $\beta_1^C$. Again, for simplicity we assume that the transmission line remains lossless so that $Z_1^C$ and $\beta_1^C$ are both real. The corresponding electromagnetic plane
wave propagation analogy is illustrated in Figure 6.5. A transmitter sends in free space a normally-

incident wave
reflected wave
original dielectric slab

Figure 6.5: Electromagnetic plane wave propagation analog of the system in Figure 6.4. Probing is done using a normally incident plane wave. The reflected wave is measured and the reflection coefficient is used as the key signal. The background system contains an infinite slab of dielectric material of width $D$ that is bounded by a PEC wall. The possible changes that can occur to this system are shown in the bottom portion of the figure, including change of the slab width (from its original value of $D$ to a new value of $D + \delta$) or change of the constitutive properties of the dielectric material.

incident plane wave to an infinite slab that is made of a dielectric material and is bounded below by a perfect electric conductor (PEC) or ground plane. The original slab material is assumed to be lossless and of thickness $D$. The reflected wave corresponding to this background medium acts as the reference signal. Later a change to this material can occur, e.g., the thickness can change from $D$ to $D + \delta$, or the slab material permittivity can change. For example, the slab material can correspond to paint or a protective sealant applied to the surface of a metallic object, and the system can be designed to detect significant variations of the layer thickness, e.g., regions of insufficient or too much material being sprayed or applied, or changes to the applied protective material due to exposure to the environment, e.g., corrosion, or thermal changes. In another, remote sensing scenario, the slab can model vegetation, earth, or water, whose changes one may wish to detect remotely. In this electromagnetic analogy, the role of the transmission line impedance $Z_0$ is taken by the electromagnetic free space impedance $\eta = \sqrt{\mu_0/\epsilon_0}$ where $\epsilon_0$ and $\mu_0$ are the free space permittivity
and permeability, respectively. The corresponding value of the wavenumber is \( \beta = \omega \sqrt{\mu_0 \varepsilon_0} \). The original slab material has permittivity and permeability \( \varepsilon_1 \) and \( \mu_1 \), respectively, which corresponds to impedance \( \eta_1 = \sqrt{\mu_1 / \varepsilon_1} \) and wavenumber \( \beta_1 = \omega \sqrt{\mu_1 \varepsilon_1} \). In the electromagnetic analogy, \( \eta_1 \) corresponds to \( Z_1 \). Finally, the changed slab material has permittivity and permeability \( \varepsilon'_1 \) and \( \mu'_1 \), respectively, which corresponds to impedance \( \eta'_1 = \sqrt{\mu'_1 / \varepsilon'_1} \) and wavenumber \( \beta'_1 = \omega \sqrt{\mu'_1 \varepsilon'_1} \). In the electromagnetic analogy, \( \eta'_1 \) corresponds to \( Z'_1 \).

We use a well-known result to compute the equivalent impedance of the short circuit at that point:

\[
Z'_L = jZ_1 \tan(\beta_1 D).
\]

(6.30)

Then the reflection coefficient at the interface of the two transmission line segments of impedance \( Z_0 \) and \( Z_1 \) is equal to

\[
\Gamma'_L = \frac{jZ_1 \tan(\beta_1 D) - Z_0}{jZ_1 \tan(\beta_1 D) + Z_0}.
\]

(6.31)

Similarly, for the changed system, the equivalent impedance at the interface of the two transmission line segments is equal to

\[
Z'_C = jZ'_1 \tan[\beta'_1 (D + \delta)].
\]

(6.32)

The respective reflection coefficient is

\[
\Gamma'_C = \frac{jZ'_1 \tan[\beta'_1 (D + \delta)] - Z_0}{jZ'_1 \tan[\beta'_1 (D + \delta)] + Z_0}.
\]

(6.33)

Here we note that to apply our results, the reflection coefficients \( \Gamma'_L \) and \( \Gamma'_C \) must correspond to the same position in the line. We use as reference point the interface of the transmission line segments of the original, background system. We translate the reflection coefficient in \(6.33\) to this point, obtaining the corrected value

\[
\Gamma'_C = \frac{jZ'_1 \tan[\beta'_1 (D + \delta)] - Z_0}{jZ'_1 \tan[\beta'_1 (D + \delta)] + Z_0} \exp(2j\delta) \exp(2j\beta\delta).
\]

(6.34)

Using these results we get

\[
f = \frac{[\exp(2j\delta) - 1]F_1 - j[\exp(2j\beta\delta) + 1]F_2}{F_1 + jF_2},
\]

(6.35)

where

\[
F_1 = z'_1 z_1 \tan(\beta_1 D) \tan[\beta'_1 (D + \delta)] + 1
\]

\[
F_2 = z'_1 \tan[\beta'_1 (D + \delta)] - z_1 \tan(\beta_1 D).
\]

(6.36)
where we have introduced the normalized impedances

\[ z_1 \equiv \frac{Z_1}{Z_0} \]
\[ z_C^1 \equiv \frac{Z_C^1}{Z_0} \]

This gives

\[
\begin{align*}
(F_1^2 + F_2^2)\Re(f) &= (F_1^2 - F_2^2) \cos(\beta \delta) - (F_1^2 + F_2^2) + 2F_1F_2 \sin(\beta \delta) \\
(F_1^2 + F_2^2)\Im(f) &= (F_1^2 - F_2^2) \sin(\beta \delta) - 2F_1F_2 \cos(\beta \delta).
\end{align*}
\] (6.38)

For the special case in which \( Z_C^1 = Z_1 \) and \( \beta_C^1 = \beta_1 \) (the properties of the transmission line segment remain the same) the results in (6.36) take the special form

\[
\begin{align*}
F_1 &= z_1^2 \tan(\beta_1 D) \tan[\beta_1(D + \delta)] + 1 \\
F_2 &= z_1[\tan[\beta_1(D + \delta)] - \tan(\beta_1 D)].
\end{align*}
\] (6.39)

Figure 6.6: Normalized real, reactive, and apparent power versus \( \delta/\lambda \) for \( D = \lambda/8 \). Results for \( Z_1 = Z_0/2 \) and \( \beta_1 = 2\beta \). The solid line corresponds to the real power. The dashed line represents the reactive power. The dashdot line corresponds to the apparent power.

Figure 6.6 and Figure 6.7 illustrate the dependence of the real, reactive, and apparent powers on the slab width change \( \delta \), for the special case associated to eq. (6.39). In this computer illustration we adopt the following numerical values: \( |V_0^+|^2/Z_0 = 1 \) (which conveniently normalizes the power...
Figure 6.7: Normalized real, reactive, and apparent power versus $\delta/\lambda$ for $D = \lambda/4$. Results for $Z_1 = Z_0/2$ and $\beta_1 = 2\beta$. The solid line corresponds to the real power. The dashed line represents the reactive power. The dashdot line corresponds to the apparent power.

values); $Z_1 = Z_0/2$, which gives $z_1 = 1/2$; $\beta_1 = 2\beta$; $\delta \in [-\lambda/4, \lambda/4]$ where the wavelength $\lambda = 2\pi/\beta$. We plot the results versus $\delta/\lambda$ for the following values of $D$: $D = \lambda/8$ (Figure 6.6) and $\lambda/4$ (Figure 6.7). It follows from the discussion given after eq.(6.38) that the dependence of $f$ on $\delta$ is periodic with period bounded by the biggest between $\lambda/2$ and $\lambda_1 = 2\pi/\beta_1 = \lambda/2$. In this example the power varies periodically with $\delta$ with period $\lambda/2$ which indicates the detectability of spatial changes of the load associated to the half-wavelength range only. This is very interesting since in the present example, the change in question is not only a displacement of the reactive load (as in the short circuit case). Also the load itself as perceived at the interface between the two transmission line segments changes. Still, we obtain again the same detectability range, but the actual dependence on the distance in question (in the present example, the slab width change $\delta$) is not the sinusoidal function encountered in the short circuit case (or any other reactive load as we explained in that example). The variation depends on the background load itself as we see in these plots corresponding to two different values of $D$. As another example we consider the values: $|V_0^+|^2/Z_0 = 1$; $Z_1 = Z_0/2$; $\beta_1 = \beta/2$; $\delta \in [-\lambda/4, \lambda/4]$ where the wavelength $\lambda = 2\pi/\beta$. Figure 6.8 shows the variation of $f$ with $\delta/\lambda$ for $D = \lambda/8$. In this case the power varies periodically with $\delta$ with period $\lambda$, as expected from the discussion given after eq.(6.38) since here $\lambda_1 = 2\lambda$. 

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Figure 6.8: Normalized real, reactive, and apparent power versus $\delta/\lambda$ for $D = \lambda/4$. Results for $Z_1 = Z_0/2$ and $\beta_1 = \beta/2$. The solid line corresponds to the real power. The dashed line represents the reactive power. The dashdot line corresponds to the apparent power.

6.6 Conclusion

In this chapter, we demonstrated the application of the optical theorem to the simplest of the electromagnetic wave propagation environments: the one-dimensional, transmission line system. The optical theorem for transmission lines is fundamental for our theoretical understanding of the optical theorem, in both its classical form related to the real extinction power as well as its recently developed reactive power format [52]. This insight paves the way for a myriad of applications in sensing and detection both of transmission line systems such as power distribution and microwave circuits as well as more complex systems that can be modeled using transmission line concepts. For instance, the optical theorem data can be used as statistics or indicators for the detection of changes in transmission-line-like systems with applications in remote sensing and communications. We rigorously validated the optical theorem predictions as they apply to transmission lines, and showed their connections to standard transmission line theory concepts such as the Smith chart. The derived results were illustrated with numerical examples motivated by practical contexts in which optical theorem indicators can be implemented as a physically-motivated concept for sensing and change detection. In particular, the optical theorem provides a way to directly measure power associated to the scattering, which can be used to measure constitutive properties.
Chapter 7

Generalized Likelihood Ratio Test With Constraint

7.1 Background

This chapter presents a new detection approach that is based on a generalized likelihood ratio test (GLRT) with optical theorem constraint. The constraint is very general. It applies to complex media that are highly reverberating so the conventional time reversal focusing applies. It also applies to simple systems such as transmission lines. The chapter presents the theoretical derivation of this new detection approach. The successful detection performance of this method is validated for several transmission line examples.

7.2 Optical Theorem Constraint

We consider a scatterer or medium perturbation of support \( \tau \) that is inside the ROI \( U' \) \( (\tau \in U') \). As shown in Chapter 2, the extinguished power is given by

\[
P_e = -3 \int_{U'} d\mathbf{r} \rho_{ind}(\mathbf{r}) \psi_i^*(\mathbf{r}) \tag{7.1}
\]

where \( \rho_{ind}(\mathbf{r}) \) is the source induced in the scatterer upon the incidence of probing field \( \psi_i(\mathbf{r}) \). In Chapter 2 we showed that in view of reciprocity,

\[
\int_{U'} d\mathbf{r} \rho_{ind}(\mathbf{r}) \psi_i^*(\mathbf{r}) = \int d\mathbf{r} R(\mathbf{r}) \psi_s(\mathbf{r}), \tag{7.2}
\]
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\[ \int dr' R(r')G(r, r') = \psi_i^*(r) \quad r \in U', \quad (7.3) \]

where \( G \) is Green’s function in the background. Therefore, if (7.3) holds,

\[ P_e = -\Im \int dr R(r)\psi_s(r) \quad (7.4) \]

where \( R(r) \) is an optical theorem receiver or sensor mode. Clearly there are infinite optical theorem receivers \( R(r) \) obeying this requirement [60]. In many media, we can use a realization based on Huygens’ sources. It is of the general form

\[ R(r) = 2ik_0\psi_i^*(r). \quad (7.5) \]

Then

\[ P_e = -2k_0\Re \int dr \psi_i^*(r)\psi_s(r). \quad (7.6) \]

In the realistic case in which \( \psi_i(r) \) is due to a finite support source, this can be conveniently written as

\[ P_e = -2k_0\Re \int_{4\pi} d\hat{r} f_s^*(\hat{r})f_s(\hat{r}) \quad (7.7) \]

where \( f_s \) is the far field scattering amplitude and \( f_i \) is the far field radiation pattern of the source.

Meanwhile, the scattered power

\[ P_s = k_0 \int_{4\pi} d\hat{r} |f_s(\hat{r})|^2 = k_0 ||f_s||^2. \quad (7.8) \]

For a finite size aperture, this can be approximated in many highly reverberating media as

\[ P_s = \frac{k_0}{C} \int_A dr |\psi_s(r)|^2 \quad (7.9) \]

where \( A \) is the support of the sensing aperture and \( C \) is a constant that takes into account the fact that the aperture captures only a fraction of the total scattered field (into all directions).

Furthermore, we also know that

\[ P_e = P_s + P_{\text{loss}} \quad (7.10) \]

which indicates that \( P_e \) is the extinguished power due to both scattering (\( P_s \) term) and dissipation (\( P_{\text{loss}} \) term). For scatterers made of passive media, the dissipation power \( P_{\text{loss}} \geq 0 \). Then from (7.10)

\[ P_e - P_s \geq 0. \quad (7.11) \]
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For lossless scatterers, \( P_e - P_s = 0 \).

From (7.11), we obtain a useful relationship that represents a prior physical constraint of the noise-free part of the data signals. In particular, using (7.6) and (7.9), we get

\[
P_e = -2C \int_A \text{d}r \Re[\psi_s(r)\psi_i^*(r)] \geq \int_A \text{d}r |\psi_s(r)|^2 = P_s \tag{7.12}
\]

To simplify the derivation in the next section, we set \( U(r) = C\psi_i(r) \). Because of the real nature of the constant \( C \), (7.12) can be rewritten as

\[
\int_A \text{d}r |\psi_s(r)|^2 + 2 \int_A \text{d}r \Re[\psi_s(r)U^*(r)] \leq 0. \tag{7.13}
\]

For the special case of a finite support probing source, we can also use (7.7) and (7.8), we get the following expression in terms of far field radiation patterns \( f_s \) and \( f_i \):

\[
P_e = -2\Re \int_{4\pi} \text{d}\hat{r} f_i^*(\hat{r}) f_s(\hat{r}) \geq \int_{4\pi} \text{d}\hat{r} |f_s(\hat{r})|^2 = P_s. \tag{7.14}
\]

Similar results apply for transmission line systems. From (6.18), (6.20), and (6.21) in Chapter 6, we find that

\[
P_e = -\frac{|V_o^+|^2\Re(\Gamma_L^*\Gamma_S)}{Z_0} \geq \frac{|V_o^+|^2|\Gamma_S|^2}{2Z_0} = P_s \tag{7.15}
\]

where \( \Gamma_S = \Gamma_C - \Gamma_L \). Here \( \Gamma_L \) is the background response (reflection coefficient), while \( \Gamma_C \) is the changed medium response, and \( \Gamma_S \) is the scattered signal. We can write this compactly as

\[
-2\Re(\Gamma_L^*\Gamma_S) \geq |\Gamma_S|^2 \tag{7.16}
\]

which is the one-dimensional version of the results (7.13) and (7.14), as desired. The results of (7.13), (7.14), and (7.16), all have the same general form, with \( U = C\psi_i \) in (7.13) and \( U = f_i \) in (7.14) and \( U = \Gamma_L \) in (7.16), respectively.

In summary, for scatterers made of passive media, an optical theorem constraint of the general form in (7.13, 7.14, 7.16) holds, which relates the noise-free scattering signal and the known background signal relevant to the problem at hand (\( \psi_i, f_i, \) or \( \Gamma_L \), in the situations corresponding to (7.13, 7.14, 7.16)).

7.3 Detection Theory

In the rest of the chapter, we focus on the realistic case in which scattering data are gathered at an array of \( N \) receivers. For the special case of transmission line systems, we focus on the reflective geometry studied in Chapter 6, in which it suffices to use a single receiver (N=1).
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Let the \( N \) receivers be located at positions \( X_n, n = 1, \ldots, N \). The respective incident field vector is defined by

\[
v_i = [\psi_i(X_1), \ldots, \psi_i(X_N)]^T
\]

(7.17)

where \((\cdot)^T\) stands for matrix transpose. For the generality of the following development, we consider the modified incident field vector

\[
u = [U(X_1), \ldots, U(X_N)]^T = Cv_i
\]

(7.18)

where \( C \) is a constant that is equal to 1 in many sensing scenarios (see (7.14) and (7.16)). Similarly, we use \( v_t \) to represent the total field vector at the receiver array, in particular,

\[
v_t = [\psi(X_1), \ldots, \psi(X_N)]^T.
\]

(7.19)

Also,

\[
v_s = v_t - v_i = [\psi_s(X_1), \ldots, \psi_s(X_N)]^T.
\]

(7.20)

Now, in this framework, the optical theorem constraint in (7.13, 7.14, 7.16) becomes

\[-2\Re(u^Hv_s) \geq ||v_s||^2.
\]

(7.21)

This is the form of the constraint for the general case of scatterers made of passive media, including lossy scatterers. If one knows a prior that the scatterer is passive, then one can use this prior constraint. In the more specialized case in which the scatterer is lossless, and it is known that the scatterer is lossless then on can use a more restricted constraint for lossless scatterers:

\[-2\Re(u^Hv_s) = ||v_s||^2.
\]

(7.22)

In this chapter, we demonstrate the application of both types of constraints to detection.

In practice, measured fields have noise. We consider additive noise so that the noisy field vector

\[
v_{tn} = v_t + n,
\]

(7.23)

where \( n \) is circular complex Gaussian noise defined by

\[
n \sim CN(0, \sigma^2 I_N),
\]

(7.24)

where \( I_N \) is the \( N \times N \) identity matrix. Similarly, the noise-added scattered field vector is

\[
v_{sn} = v_s + n.
\]

(7.25)
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The detection problem involves the two hypotheses:

\[ H_1 : v_{sn} = v_s + n \]
\[ H_0 : v_{sn} = n \]  \hspace{1cm} (7.26)

where \( v_s \) is unknown. Typically, the prior knowledge of the background field \( (v_i) \) is used only for background suppression. On the other hand, the physics tells us that the knowledge of the background field can be exploited via the prior optical theorem constraint in (7.13). Thus we formulate the detection problem as the binary hypotheses testing problem in (7.26) with the physics-based, optical theorem constraint in (7.13).

The relevant conditional pdfs are well known to be\[ p(v_{sn}|H_1) = \frac{1}{\pi N} \frac{1}{N} \exp\left(-\frac{|v_{sn} - v_s|^2}{\sigma^2}\right) \]
\[ p(v_{sn}|H_0) = \frac{1}{\pi N} \frac{1}{N} \exp\left(-\frac{|v_s|^2}{\sigma^2}\right) \]  \hspace{1cm} (7.27)

where \(|\cdot|\) represents the 2 norm. Then the corresponding likelihood ratio test statistic

\[ l(v_{sn}) = \frac{p(v_{sn}|H_1)}{p(v_{sn}|H_0)} \]  \hspace{1cm} (7.28)

where \( p(v_{sn}|H_1) \) and \( p(v_{sn}|H_0) \) are the probability density functions of the data conditioned on \( H_1 \) and \( H_0 \) defined in (7.27). Then from (7.27) and (7.28),

\[ l(v_{sn}) = \exp\left(\frac{|v_{sn}|^2 - |v_{sn} - v_s|^2}{\sigma^2}\right) \]  \hspace{1cm} (7.29)

Taking the logarithm of both sides of (7.29), and neglecting the constants, we obtain the logarithm likelihood ratio test statistic given by

\[ \hat{l}(v_{sn}) = |v_{sn}|^2 - |v_{sn} - v_s|^2 \]
\[ = 2\Re(v_s^H v_{sn}) - |v_s|^2 \]  \hspace{1cm} (7.30)

where \((\cdot)^H\) represents the Hermitian transpose of the vector.

If \( v_s \) is known, then this statistic corresponds to the matched filter detector. On the other hand, if \( v_s \) is not known, one can consider the GLRT defined by

\[ \hat{l}_{ed}(v_{sn}) \equiv \max_{v_s} [\hat{l}(v_{sn})] \]  \hspace{1cm} (7.31)

which can be shown to reduce to the energy detector

\[ \hat{l}_{ed}(v_{sn}) = |v_{sn}|^2. \]  \hspace{1cm} (7.32)
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Our goal is to maximize (7.30) as the function of the unknown data $v_s$, which obeys the constraint in (7.21) for passive scatterers or (7.22) for lossless scatterers. The constraint in (7.22) corresponds to the sphere in N-dimensional space, $||v_s + u||^2 = ||u||^2$. For further analysis, the constraint in (7.21) and (7.22) can be simplified as

$$||v_s + u||^2 \leq ||u||^2,$$

which corresponds to a ball in the N-dimensional data space.

We solve first for general passive scatterers using (7.21). Here we should note that there are two different situations based on the difference of the noise-added data signal position. If data signal (with noise) is inside the circle shown in (7.32), the constraint is not binding in this case. So that, the maximum value of likelihood ratio test in (7.30) is corresponding to the universal maximum value without any constraint. In another word, the maximum occurs if we pick $v_s = v_{sn}$. In particular, from (7.30),

$$\hat{l}_{\text{max}}(v_{sn}) = ||v_{sn}||^2,$$

if $||v_{sn} + u||^2 \leq ||U||^2$, We can pick the absolute best which is the GLRT statistics of the energy detector under this circumstance.

However, if the noise-added data signal is outside the ball shown in (7.32), then we cannot pick $v_s = v_{sn}$ because this does not obey the lossy scatterer condition. In this case, the constraint is binding.

In this case we apply the Lagrange Multipliers method [112] to find the maximum value of (7.30). The modified Lagrange multiplier operator function is given by

$$L(v_s, \lambda) = 2\Re(v_s^H v_{sn}) - ||v_s||^2 + \lambda[||v_s||^2 + 2\Re(u^H v_s)]$$

where $\lambda$ is a real coefficient. In order to get the maximum value of the corresponding function, we can let the gradient of the function above equal zero, in particular,

$$\nabla_{v_s,\lambda}L = 0.$$

Then we obtain

$$v_s = \frac{v_{sn} + \lambda u}{1 - \lambda}$$

$$||v_s||^2 + 2\Re(u^H v_s) = 0.$$

Substituting (7.37) into (7.38), we get the value of $\lambda$, in particular,

$$\lambda = 1 \pm \frac{||u + v_{sn}||}{||u||}.$$
Meanwhile, using \((7.37)\), we simplify the target function \((7.30)\), i.e.,
\[
\hat{l}(v_{sn}) = 2\Re(\frac{v_{sn}^H + \lambda u^H}{1 - \lambda} \cdot v_{sn}) - \| v_{sn} + \lambda u \|^2
\]
\[
= \| v_{sn} \|^2 - \left( \frac{\lambda}{1 - \lambda} \right)^2 \| v_{sn} + u \|^2. \tag{7.40}
\]
Here we recall that we want to get the maximum value of \(\hat{l}(v_{sn})\). We pay attention that based on \((7.40)\), \(\hat{l}(v_{sn})\) increases while \(\lambda\) decreases. So that, we pick the smaller value of \(\lambda\) out of the two possible values shown in \((7.39)\), in particular,
\[
\lambda = 1 - \frac{\| u + v_{sn} \|}{\| u \|}. \tag{7.41}
\]
Substituting \((7.41)\) into \((7.40)\), \((7.40)\) can be rewritten as
\[
\hat{l}_{\text{max}}(v_{sn}) = \| v_{sn} \|^2 - (\| u \| - \| v_{sn} + u \|)^2
\]
\[
= -2\Re(u^H v_{sn}) + 2\| v_{sn} + u \|\| u \| - 2\| u \|^2, \tag{7.42}
\]
if \(\| v_{sn} + u \|^2 \geq \| u \|^2\). \((7.34)\) and \((7.42)\) define the new GLRT detector with optical theorem constraint.

Finally, for the special case of purely lossless scatterers, the constraint in \((7.22)\) is always binding, so the respective detector is the one in \((7.42)\).

### 7.4 Numerical Simulation Results

Next we illustrate, through ROC curves, the performance of the new GLRT detector of this section. For reference, we also provide the corresponding results for the classical energy detector, the other optical theorem detectors, and the matched filter. In all the following examples, no prior knowledge of the nature of the medium change, or the scattering signal, is assumed. Only the matched filter detector employs such prior knowledge, and this detector is considered only for reference.

#### 7.4.1 Purely Lossless Scatterers

In this subsection we consider purely lossless scatterers. We use the same examples in the examples section in [59]. Figure 7.1 shows the first transmission line system to be considered, which consists of a transmission line ended by a short circuit. The above transmission line system is also a simple model for an electromagnetic sensor, possibly a radar system, to sense changes in the position
Figure 7.1: Transmission line system with switching load. The reference load is equivalent to a short circuit at a given point in the line (top). The changed load state is equivalent to a short circuit at a shifted position in the line (bottom).

of a target exhibiting high electric conductivity, e.g., a person, a vehicle, machinery in an industrial facility, etc. It can thus be used to detect motion or the proximity of a target of interest.

In this example the load is \(Z_L = 0\) which means that the reflection coefficient at the original load position is \(\Gamma_L = -1\). If the load position changes by a distance \(\delta\), moving into the transmitter, as is shown in Fig. 7.1, then the respective scattering signal is given by

\[
\Gamma_S = \Gamma_L \exp(j2\beta\delta) - \Gamma_L = 1 - \exp(j2\beta\delta)
\]

(7.43)

where \(\beta = 2\pi/\lambda\).

In the following simulations, the scattered field data are corrupted by circularly symmetric complex Gaussian noise of variance \(\sigma^2\) \((\sim CN(0, \sigma^2I_N))\) where \(I_N\) is the \(N \times N\) identity matrix). The post-processing SNR, defined as

\[
\text{SNR} = \frac{\sum_{n=1}^{N} |\psi_s(X_n)|^2}{\sigma^2},
\]

(7.44)

is equal to 5 (6.99 dB) in all the results. All the ROC curves presented next were obtained numerically using 10000 noise realizations.

Figure 7.2 and Figure 7.3 show the ROC plots of the probability of detection \(P_d\) versus the probability of false alarm \(P_{fa}\) corresponding to \(\delta = \lambda/10\) and \(\delta = \lambda/8\), respectively. These results illustrate the fact, demonstrated under a basic transmission line model in the previous chapter [59], that the conventional energy detector and the apparent power optical theorem detector are essentially
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Figure 7.2: ROC curves corresponding to the detection in lossless transmission lines with $\delta = \lambda/10$. Results for the new GLRT detector (squared line), real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 7.3: ROC curves corresponding to the detection in lossless transmission lines with $\delta = \lambda/8$. Results for the new GLRT detector (squared line), real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
equivalent. For the particular scattering parameters and transmission lines geometry in the example, they outperform both the real power and reactive power optical theorem detectors. Furthermore, the new detector (drawn with squared line) performs better than energy detector and apparent power of optical theorem detector. For reference, we also show the ROC for the optimal matched filter, which unlike the previous detectors, assumes perfect information about the target response in the given background medium.

As a second example, we assume a transmitter sends in free space a normally-incident plane wave to an infinite slab that is made of a dielectric material and is bounded below by a perfect electric conductor (PEC) or ground plane. The original slab material is assumed to be lossless and of thickness \(D\). The reflected wave corresponding to this background medium acts as the reference signal. Later a change to this material can occur, e.g., the thickness can change from \(D\) to \(D + \delta\), or the slab material permittivity can changes shown in Fig. 7.1. In this electromagnetic analogy, the role of the transmission line impedance \(Z_0\) is taken by the electromagnetic free space impedance \(\eta = \sqrt{\mu_0/\epsilon_0}\) where \(\epsilon_0\) and \(\mu_0\) are the free space permittivity and permeability, respectively. The corresponding value of the wavenumber is \(\beta = \omega \sqrt{\mu_0 \epsilon_0}\). This transmission line segment is assumed to be lossless so that both \(Z_1\) and \(\beta_1\) are real-valued. For the special case, the properties of the
transmission line segment remain the same. We obtain the relative statistics
\[
\Gamma_L = \frac{jZ_1 \tan(\beta_1 D)}{jZ_1 \tan(\beta_1 D) + Z_0},
\]
(7.45)
\[
\Gamma_S = \frac{\left[ jZ_1 \tan(\beta_1 (D + \delta)) - Z_0 \right]}{\left[ jZ_1 \tan(\beta_1 (D + \delta)) + Z_0 \right]} \exp(2j\beta\delta) - \frac{jZ_1 \tan(\beta_1 D) - Z_0}{jZ_1 \tan(\beta_1 D) + Z_0}
\]
(7.46)

Figure 7.5: ROC curves corresponding to the detection in lossless transmission line system for 
\(\delta = \lambda/8, D = \lambda/8, Z_1 = Z_0/2\) and \(\beta_1 = 2\beta\). Results for the new GLRT detector (squared line),
real power optical theorem detector (solid line), reactive power optical theorem detector (circle),
apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star
line corresponds to the matched filter.

Figure 7.5, Figure 7.6 and Figure 7.7 show the ROC plots of the probability of detection 
\(P_d\) versus the probability of false alarm \(P_{fa}\) corresponding to three different situations. Figure 7.5
shows the results for \(\delta = \lambda/8, D = \lambda/8, Z_1 = Z_0/2\) and \(\beta_1 = 2\beta\), and Figure 7.6 shows the results
for \(\delta = \lambda/8, D = \lambda/4, Z_1 = Z_0/2\) and \(\beta_1 = 2\beta\), and Figure 7.7 shows the results for \(\delta = \lambda/8, 
D = \lambda/4, Z_1 = Z_0/2\) and \(\beta_1 = \beta/2\). For the particular scattering parameters and transmission lines
geometry in the example, they outperform both the real power and reactive power optical theorem
detectors. Furthermore, the new detector (drawn with squared line) performs better than energy
detector and apparent power optical theorem detector. For reference, we also show the ROC for the
optimal matched filter, which unlike the previous detectors, assumes perfect information about the
target response in the given background medium.
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Figure 7.6: ROC curves corresponding to the detection in lossless transmission line system for $\delta = \lambda/8$, $D = \lambda/4$, $Z_1 = Z_0/2$ and $\beta_1 = 2\beta$. Results for the new GLRT detector (squared line), real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 7.7: ROC curves corresponding to the detection in lossless transmission line system for $\delta = \lambda/8$, $D = \lambda/4$, $Z_1 = Z_0/2$ and $\beta_1 = \beta/2$. Results for the new GLRT detector (squared line), real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
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7.4.2 Lossy Scatterers

In this example the load is $Z_L = 0$ which means that the reflection coefficient at the original load position is $\Gamma_L = -1$. We use $|\Gamma_C| < 1$ corresponding to a lossy scatterer. In the following examples, we choose $\Gamma_C = 0.25(1 + j)$ and $\Gamma_C = 0.25(1 - j)$. Then the respective scattering signal is $\Gamma_S = 1 - 0.25(1 + j)$ and $\Gamma_S = 1 - 0.25(1 + j)$. The post-processing SNR in (7.44) is equal to 5 (6.99 dB). All the ROC curves presented next were obtained numerically using 10000 noise realizations. We apply the detector shown in (7.34) and (7.42), we get the ROC curves illustrated in Figure 7.8 and Figure 7.9. In the next set of plots (Figure 7.10 and Figure 7.11), we add more noise to the signal for detection. We change the post-processing SNR to 2.5 (3.99 dB). The other parameters are the same as in the previous examples. We get the ROC curves illustrated in Figure 7.10 and Figure 7.11. As expected, the GLRT detector performs very well. In all those results (Figure 7.8 to Figure 7.11), the new GLRT detector with constraint is clearly better than the energy detector, and the apparent power optical theorem detector.

![Figure 7.8: ROC curves corresponding to the detection of a lossy scatterer with $\Gamma_C = 0.25(1 + j)$ and $SNR = 5$ (6.99 dB). Results for the new GLRT detector (squared line), real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.](image_url)
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Figure 7.9: ROC curves corresponding to the detection of a lossy scatterer with $\Gamma_C = 0.25(1 - j)$ and $SNR = 5$ (6.99 dB). Results for the new GLRT detector (squared line), real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

Figure 7.10: ROC curves corresponding to the detection of a lossy scatterer with $\Gamma_C = 0.25(1 + j)$ and $SNR = 2.5$ (3.99 dB). Results for the new GLRT detector (squared line), real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.
Figure 7.11: ROC curves corresponding to the detection of a lossy scatterer with $\Gamma_C = 0.25(1 - j)$ and $SNR = 2.5$ (3.99 dB). Results for the new GLRT detector (squared line), real power optical theorem detector (solid line), reactive power optical theorem detector (circle), apparent power optical theorem detector (dashed line), and the energy detector (dotted line). The star line corresponds to the matched filter.

7.5 Conclusion

This chapter demonstrated a new GLRT detector that exploits prior information about the unknown noise-free scattering signal. The exploited constraint is based on the optical theorem. It leads to better performance than the conventional energy detector. It also performed better than the other optical theorem detectors.
Chapter 8

Holographic Implementation of the Optical Theorem

8.1 Background

In this chapter, we develop the holographic implementation of the optical theorem for the measurement of the power extinguished by a scatterer that is interrogated by an arbitrary probing field and is embedded in an arbitrary medium or scene. A general recipe has been proposed in [52, 100] for the design of optical theorem sensors that measure the extinguished power for arbitrary fields and media. However, this particular approach relies on the use of time reversal mirrors or cavities which is suitable for acoustics and electromagnetics but not for optics. In particular, in the optical regime one measures only field intensities. As a consequence, the corresponding optical implementation, be it analogically or digitally, of the required time reversal mirror is less obvious. In this chapter, we develop a conceptually simple and highly practical holography-based approach which allows one to measure both $P_e$ and $P_r$ (real and reactive extinguished power, respectively) with a single optical sensor (a single-pixel camera) in combination with a lens and a hologram of the background scene. This holographic realization of the optical theorem can pave the way for novel sensors of constitutive properties of materials as well as novel approaches for short-range optical scanners and communication systems, and information storage devices.
8.2 Lens-based Imaging Approach

In order to develop the novel optical theorem holography, firstly, we consider the basic problem of designing an optical theorem detector that is based on conventional lens-based imaging. The developed system applies to the probing with arbitrary fields of a scatterer that is embedded in free space where the wavenumber \( k_0 = 2\pi/\lambda \) where \( \lambda \) is the wavelength. Figure 8.1 shows the geometry of the system. Here \( \frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{F} \), where \( F \) is the focal length. The field used to probe the ROI where the scatterer may be located is generated by passing an incident plane wave (laser beam) through a space-light modulator (SLM) which can operate in either transmissive (as shown in the figure) or reflective modality. We denote the field at the object plane immediately behind the SLM as \( \psi_O \). It is assumed next that the ROI is a few wavelengths removed from the object plane so that the probing field passing by the ROI is approximately purely propagating, which allows us to describe propagation and backpropagation or time reversal in the canonical framework of the Fresnel approximation. Using well-known results of Fourier optics (\[113\], secs. 5.3.2 and 5.3.3) we find that the field \( \psi_I \) at the image plane is given by

\[
\psi_I(u, v) = h(u, v) \otimes \hat{\psi}_I(u, v)
\]

where \( \otimes \) denotes convolution, \( h \) is the Fraunhofer diffraction pattern of the lens pupil and where

\[
\hat{\psi}_I(u, v) = \frac{1}{|M|} \psi_O \left( \frac{u}{M}, \frac{v}{M} \right),
\]
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where the magnification \( M = -z_2/z_1 \). Furthermore, it can be shown [114] that if one feeds complex conjugate field \( \hat{\psi}_I^* \) at the image plane, the respective field at objet plane is (apart from diffraction effect) precisely the complex conjugate probing field \( \psi_s^* \). Then, at least approximately, it is possible to generate the complex conjugate probing field in the ROI from a source at the image plane. It follows from basic Rayleigh-Sommerfeld diffraction theory ([115], ch. 2) that an equivalent source at the original image plane is

\[
R(u, v) = 2ik_0 \psi_I^*(u, v). \tag{8.3}
\]

This corresponds to a viable realization of receiver mode. Substituting this in (2.46, 2.47, 2.54 and 2.56) we obtain the extincted powers

\[
P_e = -2Re \int \int_A dudv \psi_I^*(u, v) \psi_s(u, v) \]
\[
P_r = -2Im \int \int_A dudv \psi_I^*(u, v) \psi_s(u, v) \tag{8.4}
\]

where integration is over the available sensing aperture \( A \) (e.g., the CCD camera area).

In practice one employs a finite number of sensors (CCD detectors) or pixels as shown in Figure 8.1 in which case the result (8.4) needs to be approximated as

\[
P_e \simeq -2KRe \sum_{n=1}^{N} \psi_I^*(X_n) \psi_s(X_n) \]
\[
P_r \simeq -2KIm \sum_{n=1}^{N} \psi_I^*(X_n) \psi_s(X_n) \tag{8.5}
\]

where \( X_n, n = 1, \cdots, N \) denotes the sensor positions and \( K \) is a calibration constant representing the effective sensor area.

In the absence of the scatterer, the field intensity \( I_i(X_n) \) measured at the \( n \)th detector is (apart from noise) equal to

\[
I_i(X_n) = |\psi_I(X_n)|^2. \tag{8.6}
\]

In the presence of the scatterer, the output of the \( n \)th detector is the total intensity

\[
I_t(X_n) = |\psi_I(X_n) + \psi_s(X_n)|^2 \simeq |\psi_I(X_n)|^2 + 2Re[\psi_I^*(X_n)\psi_s(X_n)] \tag{8.7}
\]

where in the last statement we assume that the scatterer is weak so that \( |\psi_s| << |\psi_I| \). If this approximation holds, then it follows from (8.5) that the value of the quantity

\[
u_{OT} = K \sum_{n=1}^{N} |I_i(X_n) - I_t(X_n)| \simeq P_e \tag{8.8}
\]
so that the real power \( P_e \) extinguished by the scatterer upon excitation by the given probing field can be estimated directly from the intensity-only data. In contrast, the measurement of \( P_r \) via (8.5) is based on the same setup but requires an additional interferometer.

### 8.3 Optical Theorem Holography

In this section, we develop a new holographic method to measure the total extinguished power using a single sensor such as a single-pixel camera. Figure 8.2 illustrates the general system

![Figure 8.2: Interrogation by a general probing wave of a scatterer that is embedded in a complex, possibly unknown background medium or scene.](image)

under consideration, in which a scatterer that is embedded in a nonhomogeneous background or clutter is interrogated with an arbitrary probing field. Let \( \psi_p \) be the field produced by the source in the free space medium without the background scene. As illustrated in the figure, this probing field can be quite general. The field in the presence of the background is termed the incident field \( \psi_i \) and is given by

\[
\psi_i = \psi_p + \psi_c \tag{8.9}
\]

where \( \psi_c \) is field scattered by the background. Finally, in the presence of both the background and the scatterer, the total field \( \psi \) is given by

\[
\psi = \psi_p + \psi_c + \psi_s = \psi_i + \psi_s \tag{8.10}
\]

where \( \psi_s \) denotes the scattered field. The essence of the proposed optical theorem holography is
Figure 8.3: (a) Recording of the hologram of the background scene. (b) Synthesis of the complex conjugate probing field in the ROI using a point source or pinhole at the spot where the lens-based-imaging system focuses the reference wave due to the source and reflecting mirror in part (a).

explained next for the special case of a probing beam generated by a point source, as shown in Figure 8.3a. As shown in the figure, part of the probing beam can be relayed into a mirror for the generation of a reference wave $\psi_r$. As shown in Figure 8.3a, a recording medium such as a photographic plate can be used to capture the intensity of the sum of the incident field $\psi_i$ plus the reference wave $\psi_r$. Under the reflective geometry assumed in the figure, $\psi_i = \psi_c$, so that the recording medium captures over its sensing aperture region $A$ the intensity

$$I_{c+r}(r) = |\psi_c(r) + \psi_r(r)|^2$$

$$= |\psi_c(r)|^2 + \psi_c^*(r)\psi_r(r) + \psi_c(r)\psi_r^*(r) + |\psi_r(r)|^2$$  \hspace{1cm} (8.11)

of $\psi_c + \psi_r$. In particular, in the following we represent the hologram by a transparency function $t$ over the hologram aperture region $A$ that is equal to the intensity $I_{c+r}$ in (8.11) multiplied by a measurement constant $C$, i.e.,

$$t(r) = CI_{c+r}(r) \quad r \in A.$$  \hspace{1cm} (8.12)

Now, if the thus recorded hologram is excited by the complex conjugate reference field $\psi_r^*$, then it radiates back into the medium the complex conjugate background field $\psi_c^*$ (this is the real image field of the hologram), as shown in Figure 8.3b. In practice, the generated field is approximately equal to the complex conjugate background field $\psi_c^*$ in the relevant ROI containing the scatterers or medium.
perturbations of interest. This procedure completes the definition of the sought-after holographic optical theorem detector for any scatterer in the ROI. The obtained holographic detector is shown in Figure 8.4. Importantly, this holographic detector is nonadaptive or universal in the sense that it applies to any scatterer appearing subsequently in the ROI.

As an important special case, we discuss next the case of a plane reference wave. The general results presented next remain valid for diverging spherical waves, as can be shown, e.g., via standard Fourier optics ([113], p. 119-121). Assume that both the background scene and the ROI are several wavelengths apart from the hologram aperture region \( A \). Then it follows from basic Rayleigh-Sommerfeld diffraction theory that an approximate version of the complex conjugate field \( \psi_c^* \) can be generated in the ROI using a source \( 2ik_0\psi_c^* \) supported in the hologram aperture area \( A \).

The extinguished power \( P_e \) is given by

\[
P_e = -2\text{Re} \int_A dxdy \psi_c^*(x,y)\psi_s(x,y)
\]

where here \( x, y \) denote the coordinates of point \( r \) in the area \( A \) of the hologram (see also the related result eq.(8.4)). If the reference wave arriving at the photographic plate is a plane wave \( \psi_r(r) = A' e^{ik_0s_0 \cdot r} \) traveling in the direction of the unit vector \( s_0 \) then from (8.11, 8.12) and assuming \( C = 1 \) the hologram transmissivity

\[
t(x,y) = |\psi_c(x,y)|^2 + A' e^{ik_0s_0 \cdot r} \psi_c^*(x,y) + A'^* e^{-ik_0s_0 \cdot r} \psi_c(x,y) + |A'|^2 \quad r \in A.
\]
In order to implement the optical theorem, we need to isolate the real image component in (8.14) (the second term) so as to be able to generate the complex conjugate probing field in the ROI. This can be done via the Leith-Upatnieks approach ([113], sec. 9.4). In particular, referring to Fig. 8.5, if the reference wave angle $\theta_0 \geq \sin^{-1}(3B\lambda)$ where $B$ is the bandwidth of the background, then the field produced by exciting the hologram with the complex conjugate reference field $\psi_r^*$ is approximately equal to the complex conjugate probing field $\psi_c^*$ near the background. Under this condition, the fields due to the other three components of $t$ in (8.14) (the first, third, and fourth terms), focus away from the ROI. Thus as a radiator, the Leith-Upatnieks hologram aperture excited by the complex conjugate field $\psi_r^*$ effectively radiates the complex conjugate probing field $\psi_c^*$ near the background, as desired. The receiver counterpart of this “hologram plus complex conjugate field $\psi_r^*$” system is the sought-after optical theorem detector based on Leith-Upatnieks holography. Under these conditions, when the background medium is probed with the same probing field used to generate the hologram, the corresponding field intensity $I_c$ measured at the detector is given by

$$\sqrt{I_c} = \frac{|A'|}{AF} \int \int_A dx dy |\psi_c(x, y)|^2$$

(8.15)

where $F$ is the lens focal length and the integration is over the area $A$ of the hologram.

When there is a scatterer in the ROI the total field arriving at the hologram is the sum of the background field $\psi_c$ plus the scattered field $\psi_s$, and the respective intensity $I_{c+s}$ measured at the detector is given by

$$I_{c+s} = |\sqrt{I_c} + V|^2$$

(8.16)
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where

\[ V = \frac{|A'|}{\lambda F} \oint_A dx dy \psi_c^*(x, y) \psi_s(x, y) \]  

(8.17)

so that if the scattered field is weak relative to the background field then from (8.15, 8.16, 8.17)

\[ I_{c+s} - I_c \simeq 2 \left( \frac{|A'|}{\lambda F} \right)^2 \text{Re} \oint_A dx dy \psi_c^*(x, y) \psi_s(x, y) . \]  

(8.18)

Finally, comparing (8.18) with (8.13) we conclude that the extinguished power \( P_e \) can be estimated from the intensity-only data captured by the single sensor of the optical theorem detector which is located at the focal spot of the reference wave imaging system (shown in Figure 8.5) via the formula

\[ P_e = - \left( \frac{\lambda F}{|A'|} \right)^2 (I_{c+s} - I_c) . \]  

(8.19)

8.4 Illustration

We illustrate the previous developments for the system in two-dimensional space shown in Figure 8.6. In the geometry considered, a point source is used to probe the background which is modeled as a perfectly reflecting flat surface or mirror. As shown in the figure, the background field \( \psi_c \) consists of the superposition of the original probing field plus the reflected field which can be readily computed via the method of images. In addition, the reference field \( \psi_r \) arriving at the hologram aperture is provided by the same point source used to probe the scene. In the

Figure 8.6: System considered in the example.
following numerical results we considered a hologram aperture of length \( L = 400\lambda \) centered at the origin of coordinates \((x = 0, y = 0)\). The mirror was located at \( y = -1500\lambda \). The probing source was located at \((x = -400\lambda, y = -500\lambda)\). The ROI was the rectangular region \( \tau = \{ x \in \mathbb{R}, y \in \mathbb{R}, |x + 240\lambda| \leq R, y - 1500\lambda \geq R \} \) where \( R \) characterizes the size of the ROI. In the simulations we used \( R = 80\lambda \). In the scattering computations we considered a scatterer for which an exact analytical solution of the scattering problem is available. In particular, we used a perfectly reflecting semicircle of radius \( a < R \) centered at \((x = -240\lambda, y = -1500\lambda)\). This scatterer is illustrated in Figure 8.6 and the corresponding exact scattering computations for a unit-amplitude point source are provided in Appendix A.

Figure 8.7 shows plots of the real part of the reflective component of the background field in the boundary \( y = -1500\lambda \) of the ROI (solid line), and of the real part of the incident component of the field generated in the same boundary by the hologram aperture upon excitation by the complex conjugate reference wave (dotted line). The two plots are very similar, indicating that the hologram successfully generates the complex conjugate background field in the ROI, as desired.

Figure 8.8 provides plots of the imaginary part of the complex conjugate version of the reflective component of the background field in the boundary \( y = -1500\lambda \) of the ROI (solid line), and of the imaginary part of the incident component of the field generated in the same boundary by the hologram aperture upon excitation by the complex conjugate reference wave (dotted line). In the latter plot, the field produced by the hologram is multiplied by the same corrective factor \((1/|A'|^2)\) involved in the companion (real part) plot in Figure 8.7. Figure 8.7 and Figure 8.8 convincingly show
that the hologram, when acting as a source, effectively generates the complex conjugate background field in the ROI, as is required in order for this holographic sensor to perform the optical theorem operation of measuring the extinguished power.

Figure 8.9 shows the effective isolation in the ROI of the field produced by the real image component of the hologram. In particular, the figure provides plots of the amplitude of the incident fields generated in the boundary \( y = -1500\lambda \) of the ROI corresponding to three components of the hologram aperture upon excitation by the complex conjugate reference field: The solid line plot corresponds to the amplitude of the real image field (second term in (8.14)). The dashed line plot corresponds to the virtual image field (third term in (8.14)). The dotted line corresponds to the DC or constant term associated to the intensity of the reference field alone (fourth term in (8.14)). The amplitude of the key real image field is significantly larger than that of the two other components, as desired. The field amplitude of the first term in (8.14) is not shown, but its value is comparable in this example to that of the virtual image. These results illustrate the successful isolation, in the ROI, of the key real image field of the hologram. This is accomplished in this example by putting the reference beam source away from the main optical axis, in analogy with the classical Leith-Upatnieks holography.

The solid line in Figure 8.10 illustrates the variation of the scattered power \( P_s \) versus the scatterer radius \( a \) in the \( 0 - 4\lambda \) range. The scattered power \( P_s \) was evaluated using (A.8). The same figure shows the plot corresponding to the extinguished power as measured by the holographic optical theorem detector. Both plots are almost identical, as expected from Figure 8.7.

Figure 8.8: Plot of the imaginary part of the complex conjugate version of the reflective component of the background field in the boundary \( y = -1500\lambda \) of the ROI (solid line), and of the imaginary part of the incident component of the field generated in the same boundary by the hologram aperture upon excitation by the complex conjugate reference wave (dotted line).
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Figure 8.9: Base-10 semilog plots of the amplitude of the incident fields generated in the boundary $y = -1500\lambda$ of the ROI by three components of the hologram aperture upon excitation by the complex conjugate reference field: Amplitude of the real image (solid line), of the virtual image (dashed line), and of the DC or constant term associated to the intensity of the reference field alone (dotted line).

Figure 8.10: Dotted line: Plot of the scattered power $P_s$ based on the analytical expression (A.8) versus the scatterer radius $a$. Solid line: Plot of the extinguished power measured by the holographic optical theorem detector versus $a$. 

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Figure 8.9 and the optical theorem holography theory presented in the chapter.

![Figure 8.9](image)

Figure 8.11: Percentage of error of the holographic optical theorem detector versus the scatterer radius $a$.

Figure 8.11 shows the percentage of error of the holographic optical theorem detector versus the scatterer radius $a$ in the larger $0 - 160\lambda$ range. For radius smaller than $80\lambda$, corresponding to a scatterer confined within the ROI $\tau$, the error percentage remains below 5%. This is consistent with the plots in Figure 8.7 and Figure 8.8, which show that the complex conjugate background field is well approximated in this region by the transmit form of the optical theorem hologram. Thus this hologram obeys the basic requirement that allows it to function as extinction power sensor for scatterers residing in this ROI. On the other hand, as the scatterer radius increases beyond the critical radius of $80\lambda$ the sensing error grows quickly. This is expected from the loss of imaging capability, due to diffraction, of the finite-sized hologram aperture in the correspondingly larger scattering region, which implies that the hologram can no longer function (for $a > 80\lambda$) as extinction power sensor.

8.5 Conclusion

In this chapter, we showed how to design optical theorem power sensors that are based on classical holography. The derived optical theorem holography broadens the perspectives on the practical implementation of the optical theorem. It further facilitates the development of optical theorem sensors using optical fields and quantum-mechanical matter waves in which one measures directly only field intensities. Possible applications include novel holography-based power sensors, detectors, scanners, communication systems, and information storage devices based on photonics and quantum electronics. We derived the lens-imaging-based implementation of the optical theorem in
free space. This included the general case using arbitrary propagating fields as well as a special case using point source excitation in which the optical theorem sensor can be implemented with a single-detector (a single-pixel camera). These results on the free-space lens-imaging-based implementation were applied later to the development of the more general optical theorem holography which applies to arbitrary scenes or backgrounds. In the derived optical theorem holography, information about the background (without the scatterer) is captured in the optical theorem hologram which is created by the interference of the background field with a known reference wave. The resulting hologram plus lens-based detector system is the sought-after holography-based optical theorem sensor. It is a nonadaptive sensor that measures the power extinguished by any scatterer appearing in the neighborhood of the background. We also showed a canonical example based on a basic reflective geometry relevant for certain applications such as optical scanners, biometric readers, and synthetic aperture radar systems, in which one is interested in detecting irregularities or changes over a surface.
Bibliography


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Appendix A

Useful Relations For Scattered Fields

Next we derive key results used in the example in Chapter 8. Let \((\rho', \phi')\) and \((\rho, \phi)\) be the cylindrical coordinates of the point source and the observation point, respectively. Here the origin is the center of the semicircular scatterer of radius \(a\). It is not hard to show using standard Green’s function techniques that the relevant background Green’s function \(G_b\) for the background medium bounded by the mirror, represented mathematically by homogeneous Dirichlet boundary condition for the field at the plane \(y = 0\), is

\[
G_b(\rho, \phi; \rho', \phi') = i \sum_{m=1}^{\infty} H_m^{(1)}(k_0 \rho) J_m(k_0 \rho) \sin(m\phi) \sin(m\phi')
\]

(A.1)

where \(J_m\) is the Bessel function of order \(m\), \(H_m^{(1)}\) is the Hankel function of the first kind of order \(m\), \(\rho = \max(\rho, \rho')\), and \(\rho = \min(\rho, \rho')\). This Green’s function defines the probing field in the region under investigation. On the other hand, total Green’s function \(G_t\) which corresponds to the total (incident plus scattered) field in the same region can be shown to be given by

\[
G_t(\rho, \phi; \rho', \phi') = i \sum_{m=1}^{\infty} \left[ J_m(k_0 \rho) - J_m(k_0 a) H_m^{(1)}(k_0 a) \right] \frac{J_m(k_0 a)}{H_m^{(1)}(k_0 a)} H_m^{(1)}(k_0 \rho) \sin(m\phi) \sin(m\phi').
\]

(A.2)

Thus from (A.1,A.2) the scattering Green’s function \(G_s\) corresponding to the scattered field is given by

\[
G_s(\rho, \phi; \rho', \phi') =
-i \sum_{m=1}^{\infty} H_m^{(1)}(k_0 \rho') H_m^{(1)}(k_0 \rho) \frac{J_m(k_0 a)}{H_m^{(1)}(k_0 a)} \sin(m\phi) \sin(m\phi')
\]

(A.3)
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It follows from (A.3) and the asymptotic behavior

\[ H_m^{(1)}(k_0 \rho) \sim \sqrt{\frac{2}{\pi k_0 \rho}} e^{ik_0 \rho} e^{-im \pi/2} e^{-i \pi/4} \quad k_0 \rho \to \infty \]  \hspace{1cm} (A.4)

that for far zone observation points

\[ G_s(\rho, \phi; \rho', \phi') \sim -i C_0 \sum_{m=1}^{\infty} (-i)^m H_m^{(1)}(k_0 \rho') \frac{J_m(k_0 a)}{H_m^{(1)}(k_0 a)} \sin(m \phi) \sin(m \phi') \]  \hspace{1cm} (A.5)

where

\[ C_0 = \sqrt{\frac{2}{\pi k_0 \rho}} e^{ik_0 \rho} e^{-i \pi/4}. \]  \hspace{1cm} (A.6)

Using (A.5) we can evaluate the far-zone scattered power

\[ P_s = \int_0^{\pi} \rho d\phi |G_s(\rho, \phi; \rho', \phi')|^2 \]
\[ = k_0^{-1} \sum_{m=1}^{\infty} |H_m^{(1)}(k_0 \rho')|^2 \frac{|J_m(k_0 a)|^2}{|H_m^{(1)}(k_0 a)|^2} \sin^2(m \phi'). \]  \hspace{1cm} (A.7)

Finally, the Bessel functions \( J_m(x) \) decay exponentially fast for \( m > x \), so that the series expansion in (A.7) can be effectively truncated as

\[ P_s \simeq k_0^{-1} \sum_{m=1}^{M} |H_m^{(1)}(k_0 \rho')|^2 \frac{|J_m(k_0 a)|^2}{|H_m^{(1)}(k_0 a)|^2} \sin^2(m \phi'), \]  \hspace{1cm} (A.8)

where \( M = [k_0 a] \), which is the form used in the computer simulations.