Two explorations in symplectic geometry: I. Moduli spaces of parabolic vector bundles over curves II. Characters of quantisations of Hamiltonian actions of compact Lie groups on symplectic manifolds

by Elisheva Adina Gamse

B.A., Cambridge University
M.Math., Cambridge University

A dissertation submitted to

The Faculty of
the College of Science of
Northeastern University
in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

April 5, 2016

Dissertation directed by

Jonathan Weitsman
Professor of Mathematics
Dedication

To My Family
Acknowledgments

I would like to thank Jonathan Weitsman for suggesting all the problems and for his advice and encouragement throughout. I am also grateful to the anonymous reviewers for making valuable corrections and suggestions, and to Nate Bade, Barbara Bolognese, Victor Guillemin, Alina Marian, and Ryan Mickler for helpful comments and discussions.

Elisheva Adina Gamse

Northeastern University

April 2016
Abstract of Dissertation

In Part I we study the moduli space of holomorphic parabolic vector bundles over a curve, using combinatorial techniques to obtain information about the structure of the cohomology ring. We consider the ring generated by the Chern classes of tautological line bundles on the moduli space of parabolic bundles of arbitrary rank on a Riemann surface. We show the Poincaré duals to these Chern classes have simple geometric representatives, and use this construction to show that the ring generated by these Chern classes vanishes below the dimension of the moduli space, in analogy with the Newstead-Ramanan conjecture for stable bundles.

In Part II we study the geometric quantisation of symplectic manifolds with Lie group actions, and the characters of the resulting virtual representations. In particular, let $K \subset G$ be compact connected Lie groups with common maximal torus $T$. Let $(M, \omega)$ be a prequantisable compact connected symplectic manifold with a Hamiltonian $G$-action. Geometric quantisation gives a virtual representation of $G$; we give a formula for the character $\chi$ of this virtual representation as a quotient of virtual characters of $K$. When $M$ is a generic coadjoint orbit our formula agrees with the Gross-Kostant-Ramond-Sternberg formula. We then derive a generalisation of the Guillemin-Prato multiplicity formula which, for $\lambda$ a dominant integral weight of $K$, gives the multiplicity in $\chi$ of the irreducible representation of $K$ of highest weight $\lambda$. 
# Table of Contents

Acknowledgments .................................................. iii  

Abstract of Dissertation .......................................... iv  

Table of Contents .................................................. v  

List of Figures ..................................................... vii  

Disclaimer ........................................................... viii  

Chapter 1  Introduction ........................................... 1  

I  Part I: Moduli spaces of parabolic vector bundles over curves 7  

Chapter 2  $G = SU(n)$ ............................................ 9  
   2.1 Introduction .................................................. 9  
   2.2 Combinatorial Preliminaries .................................. 12  
   2.3 Algebraic Preliminaries ...................................... 14  
   2.4 Proof of the main theorem .................................... 19  

Chapter 3  $G = SO(2n + 1)$ ..................................... 23  
   3.1 Introduction .................................................. 23  
   3.2 Identifying the desirable subgroups ......................... 26
3.3 Topology: Sections and particular vanishing classes .................................. 28
3.4 The main argument .................................................................................. 32
  3.4.1 Combinatorial Preliminaries ............................................................... 32
  3.4.2 Algebraic Preliminaries ...................................................................... 33
  3.4.3 Relations and equivalences ................................................................. 36
3.5 Proof of the main theorem ....................................................................... 51

II Part II: Characters of quantisations of Hamiltonian actions
of compact Lie groups on symplectic manifolds ........................................ 53

Chapter 4 Characters of quantisations of Hamiltonian actions of compact
Lie groups on symplectic manifolds ........................................................ 55
  4.1 Character Formula .............................................................................. 55
    4.1.1 Review of Equivariant Cohomology .............................................. 56
    4.1.2 Actions of the Torus and of a Subtorus ......................................... 58
    4.1.3 Some Notation ............................................................................. 60
    4.1.4 The Main Theorem ..................................................................... 61
    4.1.5 Example ....................................................................................... 66
    4.1.6 Relation to the GKRS formula .................................................... 67
  4.2 Multiplicity Formula ........................................................................... 69
    4.2.1 Derivation of the Multiplicity Formula ......................................... 70
    4.2.2 Examples ..................................................................................... 73

Bibliography ................................................................................................. 77
List of Figures

3.1 The inductive hypothesis gave us a block of zeros in $B[X \setminus \{z\}]$, using a
monomial of degree $4gwh$ ................................. 40

3.2 The inductive hypothesis gave us a block of zeros in $B[X \setminus \{z\}]$, using a
monomial of degree $4gwh$ ................................. 41

3.3 We can apply the inductive hypothesis to one of these triangles $T_h, T_w$ ...... 41

3.4 Applying the inductive hypothesis to $T_h$, we find either a block of zeros, or a
block of $R_-$ with the rest $R_+$ ................................. 42

3.5 We can apply either Corollary 3.4.14 to $\overline{T}_h$ (left), or the inductive hypothesis
to $T_w$ ................................................................. 43

3.6 Applying Corollary 3.4.14 to $\overline{T}_h$ gives a block $E$ of $R_+$ or $R_-$ as shown in $\overline{T}_h$,
which combines with the $R_+$ and $R_-$ in $T_h$ to get zeros everywhere in $E$. The
union $C \cup E$ contains a full block $B_{CE} \in B[X]$ ................................. 44

3.7 Applying the inductive hypothesis in $T_w$ results in one of these two situations 45

3.8 The inductive hypothesis gave us a block $C \in B[X \setminus \{z\}]$ of $R_-$, and $R_+$
everywhere else above the diagonal and away from $z$ ................................. 46

3.9 Lemma 3.4.15 puts us in one of these situations ................................. 46

3.10 Combining the block $G$ (Figure 3.9b) with $R_-$ everywhere in $C$ and $R_+$ else-
where away from $z$ (Figure 3.8) gives a block of zeros in $B[X \setminus \{z\}]$, reducing
us to the earlier case (Figure 3.2) ................................. 47
4.1 The circled points are the only points in $\mathcal{W}_K$ with a non-zero net multiplicity, showing that they are the highest weights of the four characters of $K$ that appear in $Q(M, \omega, \nabla, L; J)$ with multiplicity one.
Disclaimer

I hereby declare that the work in this thesis is that of the candidate alone, except where indicated in the text, and as described below.


Chapter 4 has been published as *Character and Multiplicity Formulas for Compact Hamiltonian G-spaces* in the Journal of Geometry and Physics and is available at arXiv:1411.0345.

Chapter 1 is partially based on the introductions to each of the aforementioned papers.
Chapter 1

Introduction

The first part of this thesis studies the moduli space of parabolic vector bundles over a curve.

Let $G$ be a compact Lie group with maximal torus $T$, let $\Sigma^g$ be a compact, connected, oriented 2-manifold of genus $g$, and let $p \in \Sigma^g$. For $t \in T$, let

$$R_g(t) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} \mid \prod_{i=1}^{g}[A_i, B_i] \sim t\}$$

(where $\sim$ denotes conjugacy in $G$). The fundamental group $\pi_1(\Sigma^g \setminus \{p\})$ can be presented by generators $a_1, \ldots, a_g, b_1, \ldots, b_g, c$ with the relation $\prod_{i=1}^{g}[a_i, b_i] = c$, where $c$ can be thought of as representing the boundary curve of a small disc containing $p$; we choose such a set of generators. Then

$$R_g(t) = \{\rho \in \text{Hom}(\pi_1(\Sigma^g \setminus \{p\}), G) \mid \rho(c) \sim t\},$$

$G$ acts on $R_g(t)$ by conjugation, and $S_g(t) = R_g(t)/G$ is the space of characters of the fundamental group of $\Sigma^g \setminus \{p\}$ in $G$ where the conjugacy class of the image of $c$ is fixed.

When $t = e$, we get the moduli space $\bar{S}_g = \text{Hom}(\pi_1(\Sigma^g), G)/G$ of flat connections on $\Sigma^g$.

Take $t = \xi \in Z(G)$ to be in the centre of $G$. In particular, take $G = SU(N)$ and $\xi = e^{2\pi ik/N}I \in Z(G)$, where $(k, N) = 1$. Then the space

$$R_g(\xi) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} \mid \prod_{i=1}^{g}[A_i, B_i] = \xi\}$$
is the moduli space of flat connections on a principal $G$-bundle over $\Sigma^g$. If we equip $\Sigma^g$ with a conformal structure, then $S_g(\xi)$ acquires a Kähler structure as a moduli space of stable rank $N$ vector bundles, of degree $k$ and fixed determinant, over the corresponding Riemann surface. In this case, the following generalisations of the Newstead-Ramanan conjecture have been established:

**Theorem 1.0.1** ([42]). The Chern classes of $S_g(\xi)$ vanish above degree $N(N - 1)(g - 1)$.

**Theorem 1.0.2** ([8]). The ring generated by the Chern classes of the vector bundle associated to $R_g(\xi) \rightarrow S_g(\xi)$ via the standard representation of $SU(N)$ on $\mathbb{C}^N$ vanishes in dimension strictly greater than $2N(N - 1)(g - 1)$.

Let $a_2, \ldots, a_r$ be the Chern classes of the vector bundle associated to $R_g(\xi)$ via the standard representation of $SU(N)$ on $\mathbb{C}^N$. Earl shows in [7] that the Pontryagin ring of $S_g(\xi)$ is contained in the subring of $H^*(S_g(\xi))$ generated by the $a_i$, and so Theorem 1.0.2 implies

**Theorem 1.0.3** ([8]). The Pontryagin ring of $S_g(\xi)$ vanishes in dimension strictly greater than $2N(N - 1)(g - 1)$.

These results are generalisations of a conjecture of Newstead published in [37]. For some other references on this subject, see [1, 2, 4, 5, 6, 9, 12, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 33, 34, 35, 36, 38, 39, 40, 41, 43, 45, 46, 47].

We now take $t$ to be a generic element of $T$:

**Definition 1.0.4.** Let $t_1, \ldots, t_N \in (0, 1)$ be such that $t_i \neq t_j$ for $i \neq j$ and $t_1 + \cdots + t_N \in \mathbb{Z}$, but the sum of any proper nonempty subset of the $t_j$ is not an integer, and let $t = \text{Diag}(e^{2\pi it_1}, \ldots, e^{2\pi it_N}) \in T$.

Consider

$$R_g(t) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} | \prod_{i=1}^{g} [A_i, B_i] \sim t\}.$$
Again, $G$ acts on $R_g(t)$ by conjugation; in this case, $S_g(t) = R_g(t)/G$ is a moduli space of rank $N$ vector bundles over $\Sigma^g$ with parabolic structure at the marked point $p$.

Consider the torus bundle $V_g(t) \to S_g(t)$ given by

$$V_g(t) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} | \prod_{i=1}^g [A_i, B_i] = t\}$$

$$= \{\rho \in \text{Hom}(\pi_1(\Sigma^g \setminus \{p\}), SU(N)) \mid \rho(c) = t\};$$

then $S_g(t) = R_g(t)/G = V_g(t)/T$. For $1 \leq i, j \leq N$ with $i \neq j$, let $L_{ij} \to S_g(t)$ be the line bundle associated to $V_g(t)$ by the representation

$$T \times \mathbb{C} \to \mathbb{C}$$

$$(\text{Diag}(e^{\sqrt{-1}\theta_1}, \ldots, e^{\sqrt{-1}\theta_N}, z) \mapsto e^{\sqrt{-1}(\theta_i - \theta_j)} z.$$ (1.1)

We will denote this representation of $T$ by $C_{(ij)}$, and its weight by $\chi_{ij}$. Note that since $L_{ij} \otimes L_{jk} = L_{ik}$ and $L_{ij} = L_{ji}^*$, for $i, j, k$ all distinct, we have $c_1(L_{ij}) + c_1(L_{jk}) = c_1(L_{ik})$ and $c_1(L_{ij}) = -c_1(L_{ji})$. We are interested in the subring of $H^*(S_g(t); \mathbb{Q})$ generated by the $c_1(L_{ij})$. When $G = SU(2)$, $V_g(t)$ is a circle bundle, and

**Theorem 1.0.5** ([44]). $c_1(L_{12})^{2g} = 0$.

In chapter 2, we extend this result to $G = SU(n)$; this chapter is based on the paper *Geometry of the intersection ring and vanishing relations in the cohomology of the moduli space of parabolic bundles on a curve* to be published in the Journal of Differential Geometry.

In chapter 3, we extend this result to $G = SO(2n + 1)$.

The second part of this thesis is concerned with quantisations of Hamiltonian $G$-actions on symplectic manifolds. Let $(M, \omega)$ be a compact connected symplectic manifold, and let $G$ be a compact connected Lie group acting on $M$ in a Hamiltonian fashion with moment map $\mu : M \to g^*$. Assume that the equivariant cohomology class $[\omega + \mu]$ is integral. Let $L \to M$ be a complex Hermitian line bundle and let $\nabla$ be a Hermitian connection on $L$ whose equivariant curvature form is the equivariant symplectic form $\omega + \mu$. Suppose that
the $G$-action on $M$ lifts to a $G$-action on $L$ which preserves $\nabla$. Let $J$ be a $G$-equivariant almost complex structure on $M$. This almost complex structure gives us a totally complex distribution $\Delta$ in the complexified tangent bundle to $M$, such that $TM \otimes \mathbb{C} = \Delta \oplus \overline{\Delta}$. This gives us a splitting $\Lambda^k(TM \otimes \mathbb{C}) = \sum_{i+j=k} \Lambda^i(\Delta) \otimes \Lambda^j(\overline{\Delta})$, and thus a bigrading on the space of $L$-valued differential forms. We will take $J$ to be compatible with $\omega$; that is, $\omega$ is a $(1,1)$-form and $\omega(v, Jv) > 0$ for all $p \in M$ and $v \in T_p M$.

Let $D : \Omega^k(M; L) \to \Omega^{k+1}(M; L)$ be the operator $D(s \otimes \alpha) = \nabla s \otimes \alpha + s \otimes d\alpha$, where $s \in \Gamma(L)$ and $\alpha \in \Omega^k(M)$. Define $\overline{\partial}$ to be the $(0, k+1)$ component of $D$. Fix a Hermitian metric on $M$. Together with the Hermitian inner product on $L$, this gives a Hermitian inner product on $L \otimes \Lambda^{0,k}TM$. Define operators $\overline{\partial}^j : \Omega^{0,k}(M; L) \to \Omega^{0,k-1}(M; L)$ which are $L^2$-adjoint to $\overline{\partial}$. Then the operator $\overline{\partial} + \overline{\partial}^j : \Omega^{0,\text{even}}(M; L) \to \Omega^{0,\text{odd}}(M; L)$ is elliptic; the quantisation $Q(M, \omega, \nabla, L; J)$ is defined as the virtual $G$-representation on $\ker(\overline{\partial} + \overline{\partial}^j) \ominus \text{coker}(\overline{\partial} + \overline{\partial}^j)$; that is, the equivariant index of the $\overline{\partial} + \overline{\partial}^j$ operator on $L$. (See chapter 6 of Ginzburg, Guillemin and Karshon’s book [15].)

In their paper [16], Guillemin and Prato prove a formula for the multiplicity with which each irreducible character of $G$ appears in the character of this representation $Q(M, \omega, L, \nabla; J)$. In the special case of a torus $T$ acting on a coadjoint orbit $G/T$ by left multiplication, their formula becomes the Kostant multiplicity formula that Kostant obtained in [31].

Let $G$ be semisimple, let $K \subset G$ be a Lie subgroup of equal rank, choose a common maximal torus $T \subset K \subset G$, let $\lambda$ be a dominant integral weight for $G$, and let $M$ be the coadjoint orbit $G \cdot \lambda$. The choice of positive roots for $G$ determines a complex structure on $G \cdot \lambda$; take $J$ to be the corresponding almost complex structure on $G \cdot \lambda$. Let $L_\lambda = G \times_T \mathbb{C}_{(\lambda)}$ (where $T$ acts on the line $\mathbb{C}_{(\lambda)}$ with weight $\lambda$). Then the quantisation $Q(M, \omega, L_\lambda, \nabla; J)$ is a $G$-representation on the space of holomorphic sections of $L_\lambda$ (see [15] for details). The Borel-Weil theorem tells us that the quantisation of $(M, \omega, L_\lambda, \nabla, J)$ is the irreducible representation of $G$ of highest weight $\lambda$, and that all irreducible representations arise in this way,
as described by Bott in [3]. In this case, Gross, Kostant, Ramond and Sternberg provided in [14] a formula for the character of this $G$-representation as a quotient of the alternating sum of a multiplet of $K$-characters. Their formula has its origins in String Theory and is the motivation for our work which provides a generalisation. In the special case where $K = T$, their formula becomes the Weyl character formula.

In this thesis, we extend the result of Gross, Kostant, Ramond and Sternberg by replacing the coadjoint orbit $G \cdot \lambda$ with any compact connected symplectic Hamiltonian $G$-manifold $M$, and relate the resulting character formula to the Guillemin-Prato multiplicity formula. In section two we obtain, for arbitrary compact connected symplectic manifolds $M$ with Hamiltonian $G$-actions, a formula for the character of $Q(M, \omega, \nabla, L; J)$ as a quotient of $K$-characters. In section three, we derive from our formula a generalisation of the Guillemin-Prato multiplicity formula. This chapter is based on the paper *Character and Multiplicity Formulas for Compact Hamiltonian $G$-spaces* published in the Journal of Geometry and Physics ([10]).
Part I

Part I: Moduli spaces of parabolic vector bundles over curves
Chapter 2

\[ G = SU(n) \]

2.1 Introduction

Let \( G \) be a compact Lie group with maximal torus \( T \), let \( \Sigma^g \) be a compact, connected, oriented 2-manifold of genus \( g \), and let \( p \in \Sigma^g \). For \( t \in T \), let

\[ R_g(t) = \{ (A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} | \prod_{i=1}^{g} [A_i, B_i] \sim t \} \]

(where \( \sim \) denotes conjugacy in \( G \)). The fundamental group \( \pi_1(\Sigma^g \setminus \{p\}) \) can be presented by generators \( a_1, \ldots, a_g, b_1, \ldots, b_g, c \) with the relation \( \prod_{i=1}^{g} [a_i, b_i] = c \), where \( c \) can be thought of as representing the boundary curve of a small disc containing \( p \); we choose such a set of generators. Then

\[ R_g(t) = \{ \rho \in Hom(\pi_1(\Sigma^g \setminus \{p\}), G) | \rho(c) \sim t \} \],

\( G \) acts on \( R_g(t) \) by conjugation, and \( S_g(t) = R_g(t)/G \) is the space of characters of the fundamental group of \( \Sigma^g \setminus \{p\} \) in \( G \) where the conjugacy class of the image of \( c \) is fixed. We take \( t \) to be a generic element of \( T \):

**Definition 2.1.1.** Let \( t_1, \ldots, t_N \in (0, 1) \) be such that \( t_i \neq t_j \) for \( i \neq j \) and \( t_1 + \cdots + t_N \in \mathbb{Z} \), but the sum of any proper nonempty subset of the \( t_j \) is not an integer, and let \( t = \text{Diag}(e^{2\pi it_1}, \ldots, e^{2\pi it_N}) \in T \).
Consider
\[ R_g(t) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} \mid \prod_{i=1}^g [A_i, B_i] \sim t\}. \]

Again, \( G \) acts on \( R_g(t) \) by conjugation; in this case, \( S_g(t) = R_g(t)/G \) is a moduli space of rank \( N \) vector bundles over \( \Sigma^g \) with parabolic structure at the marked point \( p \).

Consider the torus bundle \( V_g(t) \to S_g(t) \) given by
\[ V_g(t) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} \mid \prod_{i=1}^g [A_i, B_i] = t\} = \{\rho \in Hom(\pi_1(\Sigma^g \setminus \{p\}), SU(N)) \mid \rho(c) = t\}; \]
then \( S_g(t) = R_g(t)/G = V_g(t)/T \). For \( 1 \leq i, j \leq N \) with \( i \neq j \), let \( L_{ij} \to S_g(t) \) be the line bundle associated to \( V_g(t) \) by the representation
\[
T \times \mathbb{C} \to \mathbb{C} \\
(\text{Diag}(e^{\sqrt{-1} \theta_1}, \ldots, e^{\sqrt{-1} \theta_N}), z) \mapsto e^{\sqrt{-1} (\theta_i - \theta_j)} z.
\]
(2.1)

We will denote this representation of \( T \) by \( C_{(ij)} \), and its weight by \( \chi_{ij} \). Note that since \( L_{ij} \otimes L_{jk} = L_{ik} \) and \( L_{ij} = L_{ji}^* \), for \( i, j, k \) all distinct, we have \( c_1(L_{ij}) + c_1(L_{jk}) = c_1(L_{ik}) \) and \( c_1(L_{ij}) = -c_1(L_{ji}) \). We are interested in the subring of \( H^*(S_g(t); \mathbb{Q}) \) generated by the \( c_1(L_{ij}) \). When \( G = SU(2) \), \( V_g(t) \) is a circle bundle, and

**Theorem 2.1.2** ([44]). \( c_1(L_{12})^{2g} = 0 \).

We will sketch here the proof of Theorem 2.1.2 found in [44], as the purpose of this chapter is to extend this technique to arbitrary rank. The idea is to find explicit geometric cycles Poincaré dual to the Chern class \( c_1(V_g(t)) \). For \( 1 \leq i \leq g \), consider the sections \( s_{A_i} \) of \( L_{12} \to S_g(t) \) induced by the equivariant maps
\[
V_g(t) \to C_{(12)} \\
(A_1, \ldots, A_g, B_1, \ldots, B_g) \mapsto (A_i)_{12},
\]

10
where the subscript 12 denotes the \((1, 2)\) matrix entry. These sections \(s_{A_i}\) vanish on the cycles

\[ D(A_i) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in V_g(t) | [A_i, t] = 1 \}/T. \]

Define sections \(s_{B_i}\) and cycles \(D(B_i)\) similarly, and consider the intersection

\[ D := D(A_1) \cap \cdots \cap D(A_g) \cap D(B_1) \cap \cdots \cap D(B_g). \]

This is the image in \(S_g(t)\) of the subset of \(V_g(t)\) consisting of elements \((A_1, \ldots, A_g, B_1, \ldots, B_g)\) where the matrices \(A_1, \ldots, A_g, B_1, \ldots, B_g\) are all diagonal, and so \(\prod_{i=1}^{g} [A_i, B_i] = 1\). Since \(t \neq 1\) by Definition 2.1.1, \(D = \emptyset\), and so \(c_1(L_{12})^{2g} = 0\).

The purpose of this chapter is to generalise this result to \(G = SU(N)\). Our theorem is

**Theorem 2.1.3.** For \(1 \leq i, j \leq N\) and \(i \neq j\), let \(k_{ij}\) be nonnegative integers. Then the cohomology class

\[ \prod_{1 \leq i, j \leq N \atop i \neq j} c_1(L_{ij})^{k_{ij}} \in H^*(S_g(t); \mathbb{Q}) \]

vanishes whenever

\[ \sum_{1 \leq i, j \leq N \atop i \neq j} k_{ij} \geq N(N - 1)g - N + 2. \]

**Remark 2.1.4.** The dimension of \(S_g(t)\) is given by

\[ \dim(S_g(t)) = 2g \dim SU(N) - \dim T - \dim SU(N) \]

\[ = 2g(N^2 - 1) - (N - 1) - (N^2 - 1) \]

\[ = 2g(N^2 - 1) - N^2 - N + 2. \]

Theorem 2.1.3 above says that monomials in the \(c_1(L_{ij})\) vanish in degree \(r\) for \(r \geq 2gN(N - 1) - 2N + 4\), which is well below the dimension of \(S_g(t)\).

The proof uses the same technique as [44] but the combinatorics of the intersections is much more complicated; we illustrate it here for the case \(G = SU(3)\).
For $1 \leq i, j \leq 3$ with $i \neq j$, consider the line bundles $L_{ij} \to S_g(t)$ associated to $V_g(t)$ as above. Their Chern classes $c_{ij} = c_1(L_{ij})$ satisfy

$$c_{ij} + c_{jk} + c_{ki} = 0$$

$$c_{ij} + c_{ji} = 0.$$  \hfill (2.2)

For $1 \leq m \leq g$, we can as before find sections $s_{A_m}^{ij}$ of $L_{ij} \to S_g(t)$ that are zero on

$$D_{ij}(A_m) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in V_g(t)|(A_m)_{ij} = 0\}/T,$$

and similarly sections $s_{B_m}^{ij}$ with zero locus $D_{ij}(B_m)$. The intersection

$$D := \bigcap_{m=1}^{g} D_{12}(A_m) \cap D_{13}(B_m) \cap D_{13}(A_m) \cap D_{13}(B_m)$$

is the image in $S_g(t)$ of the set of elements of $V_g(t)$ where $(A_m)_{12} = (A_m)_{13} = 0$ and $(B_m)_{12} = (B_m)_{13} = 0$ for all $m$. Hence

$$\left(\prod_{m=1}^{g} [A_m, B_m]\right)_{11} = 1,$$

so again if $t$ is generic as in Definition 2.1.1, then $D = \emptyset$. Hence the monomial $c_{12}^{2g}c_{13}^{2g}$ vanishes, and similarly, so do $c_{23}^{2g}c_{21}^{2g}$ and $c_{31}^{2g}c_{32}^{2g}$. But any monomial in the $c_{ij}$ of degree at least $6g - 1$ may be written using the relations (2.2) as a sum of monomials containing at least one such factor, and thus

**Proposition 2.1.5.** Let $G = SU(3)$. Then any monomial in the Chern classes of the $L_{ij}$ of degree at least $6g - 1$ vanishes.

The generalisation of this argument to higher rank requires more careful attention to the combinatorics and algebra of possible monomials.

### 2.2 Combinatorial Preliminaries

We fix an integer $g \geq 2$; in the geometric application this is the genus of the 2-manifold $\Sigma^g$. 
Notation. For any set $X$, we will denote by $X^{(2)}$ the set of 2-element subsets of $X$.

Lemma 2.2.1. Let $X$ and $Y$ be finite sets with $|X| = n \geq 3$ and $|Y| \geq n(n-1)g - n + 2$, and let $f : Y \rightarrow X^{(2)}$ be a function. Then there exists $z \in X$ such that $|f^{-1}((X \setminus \{z\})^{(2)})| \geq (n-1)(n-2)g - n + 3$.

Proof. For $x \in X$, let $Q_x$ denote the set $(X \setminus \{x\})^{(2)}$. Suppose there is no $z$ for which $|f^{-1}(Q_x)| \geq (n-1)(n-2)g - n + 3$. So for each $x \in X$, $|f^{-1}(Q_x)| \leq (n-1)(n-2)g - n + 2$, and thus

$$ \sum_{x \in X} |f^{-1}(Q_x)| \leq n(n-1)(n-2)g - n(n-2). $$

But each element $\{u,v\} \in X^{(2)}$ is contained in exactly $n-2$ of the sets $Q_x$. So

$$ \sum_{x \in X} |f^{-1}(Q_x)| = (n-2)|Y| $$

. Hence

$$ |Y| \leq \frac{n(n-1)(n-2)g - n(n-2)}{n-2} $$

$$ = n(n-1)g - n $$

$$ < |Y|. $$

This is a contradiction, so such a $z$ must exist. \hfill \Box

Definition 2.2.2. Let $X$ be a finite set. A block in $X \times X$ is a subset $B \subset X \times X$ of the form $V \times (X \setminus V)$, where $V \subset X$ is a proper nonempty subset. If $|V| = h$ we call $B$ an $h$-by-$\left(|X| - h\right)$ block.

Definition 2.2.3. For a finite set $X$, let $\mathcal{B}[X]$ be the set of blocks in $X \times X$.

Notation. If $n \in \mathbb{N}$, we will write $[n]$ for the set $\{1, \ldots, n\}$.

Remark 2.2.4. Suppose $A \in SU(N)$ and $1 \leq h \leq N$, let $V \subset [N]$ be a subset with $|V| = h$, and consider the block $B = V \times ([N] \setminus V) \subset [N] \times [N]$. If $A_{ij} = 0$ for all $(i, j) \in B$, then
there is some ordering of basis elements for which $A$ is upper block diagonal. More precisely, if $\sigma$ is a permutation of $[N]$ such that $\sigma(V) = [h]$, then the matrix $(C_{ij}) = (A_{\sigma^{-1}(i)\sigma^{-1}(j)})$ has the form

$$
\begin{pmatrix}
A_{\sigma^{-1}(1)\sigma^{-1}(1)} & \cdots & A_{\sigma^{-1}(1)\sigma^{-1}(h)} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{\sigma^{-1}(N-h)\sigma^{-1}(1)} & \cdots & A_{\sigma^{-1}(N-h)\sigma^{-1}(h)} & 0 & \cdots & 0 \\
| & & & | & & \\
u_1 & \cdots & u_h & v_1 & \cdots & v_{N-h}
\end{pmatrix}.
$$

Since $A$, and hence $C$, are unitary, the vectors $v_i$ form a basis for $\mathbb{C}^{N-h}$, and $u_i \cdot v_j = 0$ for all $i, j$. Hence each of the $u_i$ must be zero, so the matrix $C$ is block diagonal. Hence $A_{ji} = 0$ for all $(i,j) \in B$. Thus the condition $A_{ij} = 0$ for all $(i,j) \in B$ implies that $A$ is block diagonal, with blocks of size $h$ and $N - h$, up to reordering of basis elements.

### 2.3 Algebraic Preliminaries

Let $X$ be a finite set.

**Definition 2.3.1.** Let $\mathbb{Q}[x_{ij}]$ be the ring $\mathbb{Q}\{x_{ij} | 1 \leq i, j \leq |X|, i \neq j\}$, where we adjoin variables $x_{ij}$ for all ordered pairs $(i,j)$ with $i,j \in X$ and $i \neq j$.

We will need the following lemmas.

**Lemma 2.3.2.** Suppose $|X| = n \geq 3$, and let $p \in \mathbb{Q}[x_{ij}]$ be a monomial of degree at least $n(n-1)g - n + 2$. Then there exists some $z \in X$ such that if we factor $p$ as $p = qr$, where $q \in \mathbb{Q}\{x_{iz}, x_{zi} | i \in X \setminus \{z\}\}$ and $r \in \mathbb{Q}\{x_{ij} | i, j \in X \setminus \{z\}, i \neq j\}$ are monomials, then $r$ has degree at least $(n-1)(n-2)g - n + 3$.

**Proof.** Write $p = \lambda \prod_{i,j} x_{ij}^{d_{ij}}$. Let $Y_{ij}$ be disjoint sets with $|Y_{ij}| = d_{ij}$, for each pair $(i,j)$ with
\(i, j \in X\) and \(i \neq j\). Let \(Y = \bigcup Y_{ij}\). We have

\[
|Y| = \sum_{i,j} d_{ij} \geq n(n-1)g - n + 2.
\]

Consider the function

\[
f : Y \to X^{(2)}
\]

given by

\[
f|_{Y_{ij}} = \{i, j\}.
\]

Then

\[
|f^{-1}(\{i, j\})| = d_{ij} + d_{ji}
\]

for all \(\{i, j\} \in X^{(2)}\). By Lemma 2.2.1, there exists \(z \in X\) with

\[
|f^{-1}((X \setminus \{z\})^{(2)})| \geq (n-1)(n-2)g - n + 3.
\]

But

\[
|f^{-1}((X \setminus \{z\})^{(2)})| = \sum_{(i,j) \in (X \setminus \{z\})^{(2)}} d_{ij} + d_{ji} = \deg(r),
\]

and so \(\deg(r) \geq (n-1)(n-2)g - n + 3\).

**Lemma 2.3.3.** Let \(z \in X\) and let \(w, h \in \mathbb{N}\) with \(w + h = |X| - 1\). Let \(\eta \in \mathbb{Q}[\{x_{zi} | i \in X \setminus \{z\}\}]\) be a monomial of degree at least \(|X|(|X|-1)g - |X| + 2 - 2gh\). Given a partition \(X \setminus \{z\} = \{e_1, \ldots, e_h\} \sqcup \{f_1, \ldots, f_w\}\), factor \(\eta\) as \(\eta = \eta_h \eta_w\), with \(\eta_h \in \mathbb{Q}[\{x_{ze_i} | 1 \leq i \leq h\}]\) and \(\eta_w \in \mathbb{Q}[\{x_{zf_i} | 1 \leq i \leq w\}]\). Then either the degree of \(\eta_h\) is at least \(gh(h + 1) - h + 1\), or the degree of \(\eta_w\) is at least \(gw(w + 1) - w + 1\).
Proof. Suppose \(\deg(\eta_h) < gh(h + 1) - h + 1\). Then

\[
\deg(\eta_w) = \deg(\eta) - \deg(\eta_h) \\
\geq |X|(|X| - 1)g - |X| + 2 - 2gwh - gh(h + 1) + h \\
= (w + h + 1)(w + h)g - (w + h + 1) + 2 - 2gwh - gh(h + 1) + h \\
= g(w^2 + 2hw + h^2 + w + h - 2wh - h^2 - h) - w - h + 1 + h \\
= gw(w + 1) - w + 1.
\]

\[\square\]

**Definition 2.3.4.** Let \(I \subset \mathbb{Q}[x_{ij}]\) be the ideal generated by \(x_{ij} + x_{ji}\) and \(x_{ij} + x_{jk} + x_{ki}\), for all triples of distinct elements \(i, j, k \in X\). Let \(R = \mathbb{Q}[x_{ij}]/I\) be the quotient of \(\mathbb{Q}[x_{ij}]\) by this ideal.

Note that the quotient preserves the grading by degree. If \(\zeta \in \mathbb{Q}[x_{ij}]\) we will write \([\zeta]\) for its image in \(R\).

**Lemma 2.3.5.** Let \(\xi \in \mathbb{Q}[x_{ij}]\) be a monomial, and let \(z \in X\). Then there exists a homogeneous polynomial \(\eta \in \mathbb{Q}[\{x_{zj}|j \in X \setminus \{z\}\}]\) of the same degree as \(\xi\) such that \([\xi] = [\eta] \in R\).

**Proof.** If \(\xi = \lambda \prod_{i,j} x_{ij}^{d_{ij}}\) where \(\lambda \in \mathbb{Q}\), let

\[
\eta = \lambda \prod_{i,j \neq z} (-x_{zi} + x_{zj})^{d_{ij}} \prod_{i \neq z} (-1)^{d_{iz}} x_{2i}^{d_{iz}} \prod_{j \neq z} x_{2j}^{d_{iz}}.
\]

Then \([\xi] = [\eta]\), and the degree of each term of \(\eta\) is equal to the degree of \(\xi\). \[\square\]

**Proposition 2.3.6.** Let \(X\) be a finite set with \(|X| \geq 2\). Let \(R = \mathbb{Q}[x_{ij}]/I\) as before. Let \(\zeta \in \mathbb{Q}[x_{ij}]\) be a monomial of degree at least \(|X|(|X| - 1)g - |X| + 2\). Then for each block \(B \subset X \times X\) we can find a monomial \(\psi_B\) in \(\mathbb{Q}[x_{ij}]\) such that

\[
[\zeta] = \left[ \sum_{B \in \mathcal{B}[X]} \psi_B \prod_{(i,j) \in B} x_{ij}^{2g} \right].
\]

**Proof.** By induction on \(|X|\).
If $|X| = 2$, take $X = \{1, 2\}$. The monomials in $\mathbb{Q}[x_{ij}]$ of degree at least $2g$ are of the form $\lambda x_{12}^a x_{21}^b$ where $a + b \geq 2g$, $\lambda \in \mathbb{Q}$. The set $\{(1, 2)\}$ is a 1-by-1 block in $X \times X$. Since $[x_{12}] = [-x_{21}]$ in $R$, the class $[\lambda x_{12}^a x_{21}^b] = [\lambda (-1)^b x_{12}^{a+b-2g} x_{12}^{2g}] \in R$ is of the desired form.

Now suppose $|X| = n \geq 3$. Let $\zeta \in \mathbb{Q}[x_{ij}]$ be a monomial of degree $d \geq n(n-1)g - n + 2$. By Lemma 2.3.2, there exists $z \in X$ such that if we factor $\zeta$ as $\zeta = qr$, where $q \in \mathbb{Q}[\{x_{ij} | i \in X \setminus \{z\}\}]$ and $r \in \mathbb{Q}[\{x_{ij} | i, j \in X \setminus \{z\}, i \neq j\}]$ are monomials, then the degree of $r$ is at least $(n-1)(n-2)g - n + 3$.

By the inductive hypothesis, for each block $C \subset (X \setminus \{z\}) \times (X \setminus \{z\})$ we can find a monomial $\theta_C \in \mathbb{Q}[\{x_{ij} | i, j \in X \setminus \{z\}, i \neq j\}]$ such that

$$[r] = \left[ \sum_{C \in \mathcal{B}[X \setminus \{z\}]} \theta_C \prod_{(i,j) \in C} x_{ij}^{2g} \right],$$

and so

$$[\zeta] = [qr] = \left[ q \sum_{C \in \mathcal{B}[X \setminus \{z\}]} \theta_C \prod_{(i,j) \in C} x_{ij}^{2g} \right].$$

It suffices to show that each nonzero monomial in the sum can be written as a sum of terms having the desired form. For each $C$ with $\theta_C \neq 0$, consider

$$\left[ q \theta_C \prod_{(i,j) \in C} x_{ij}^{2g} \right].$$

This is a monomial of degree $d \geq n(n-1)g - n + 2$. Suppose the block $C$ is given by

$$C = \{e_1, \ldots, e_h\} \times \{f_1, \ldots, f_w\},$$

where $h, w \geq 1$ and $X \setminus \{z\}$ is the disjoint union $X \setminus \{z\} = \{e_1, \ldots, e_h\} \sqcup \{f_1, \ldots, f_w\}$ (so $w + h = n - 1$). By Lemma 2.3.5, we can find a homogeneous polynomial $p_1 + \cdots + p_m$, where $p_1, \ldots, p_m$ are monomials in $\mathbb{Q}[\{x_{ij} | j \in X \setminus \{z\}\}]$, such that

$$\left[ q \theta_C \prod_{(i,j) \in C} x_{ij}^{2g} \right] = \left[ (p_1 + \cdots + p_m) \prod_{(i,j) \in C} x_{ij}^{2g} \right].$$
Again, it suffices to show that each monomial in the sum can be written as a sum of terms having the desired form, so consider
\[
\left[ p \prod_{(i,j) \in C} x_{ij}^{2g} \right],
\]
where \( p \in \{p_1, \ldots, p_m\} \). Note that
\[
\deg(p) = d - 2gwh \geq n(n - 1)g - n + 2 - 2gwh.
\]

Factor \( p \) as \( p = p_hp_w \) where \( p_h \in \mathbb{Q}[[x_{zei} \mid 1 \leq i \leq h}] \) and \( p_w \in \mathbb{Q}[[x_zf_j \mid 1 \leq j \leq w}] \) are monomials. By Lemma 2.3.3, either \( \deg(p_h) \geq gh(h+1) - h + 1 \) or \( \deg(p_w) \geq gw(w+1) - w + 1 \); without loss of generality we assume the former.

By the inductive hypothesis, for each block \( D \subset \{e_1, \ldots, e_h, z\} \times \{e_1, \ldots, e_h, z\} \) we can find a monomial \( \phi_D \in \mathbb{Q}[[x_{ij} \mid i, j \in \{e_1, \ldots, e_h, z\}, i \neq j]] \) such that
\[
\left[ p_h \right] = \left[ \sum_{D \in \mathcal{B}[[e_1, \ldots, e_h, z]]} \phi_D \prod_{(i,j) \in D} x_{ij}^{2g} \right],
\]
and so
\[
\left[ p \prod_{(i,j) \in C} x_{ij}^{2g} \right] = \left[ p_w \sum_{D \in \mathcal{B}[[e_1, \ldots, e_h, z]]} \phi_D \prod_{(i,j) \in D} x_{ij}^{2g} \prod_{(i,j) \in C} x_{ij}^{2g} \right].
\]

For each \( D \), consider the monomial
\[
\left[ p_w \phi_D \prod_{(i,j) \in D} x_{ij}^{2g} \prod_{(i,j) \in C} x_{ij}^{2g} \right].
\]
Observe that \( C \cap D = \emptyset \), and so
\[
\left[ p_w \phi_D \prod_{(i,j) \in D} x_{ij}^{2g} \prod_{(i,j) \in C} x_{ij}^{2g} \right] = \left[ p_w \phi_D \prod_{(i,j) \in C \cup D} x_{ij}^{2g} \right].
\]
We may assume
\[
D = \{e_{\sigma(1)}, \ldots, e_{\sigma(d)}\} \times \{e_{\sigma(d+1)}, \ldots, e_{\sigma(h)}, z\},
\]
for some permutation \( \sigma \) of \([h]\) and \( 1 \leq d \leq h \), since \( x_{ij} - x_{ji} \in I \). But
\[
C \cup D = \{e_1, \ldots, e_h\} \times \{f_1, \ldots, f_w\} \cup \{e_{\sigma(1)}, \ldots, e_{\sigma(d)}\} \times \{e_{\sigma(d+1)}, \ldots, e_{\sigma(h)}, z\}
\]
contains
\[ E := \{e_{\sigma(1)}, \ldots, e_{\sigma(d)}\} \times \{e_{\sigma(d+1)}, \ldots, e_{\sigma(h)}, f_1, \ldots, f_w, z\} \in \mathcal{B}[X]. \]

So
\[
\begin{bmatrix}
p_w \phi_D \prod_{(i,j) \in D} x_{ij}^{2g} \prod_{(i,j) \in C} x_{ij}^{2g} \\
p_w \phi_D \prod_{(i,j) \in D} x_{ij}^{2g} \prod_{(i,j) \in C} x_{ij}^{2g}
\end{bmatrix} = \begin{bmatrix}
\psi_E \prod_{(i,j) \in E} x_{ij}^{2g} \\
\psi_E \prod_{(i,j) \in E} x_{ij}^{2g}
\end{bmatrix}
\]
for some \( \psi_E \in \mathbb{Q}[x_{ij}] \). We have shown that \([\zeta]\) has a representative in \( \mathbb{Q}[x_{ij}] \) that is a sum of monomials of this form, i.e.

\[
[\zeta] = \left[ \sum_{B \in \mathcal{B}[X]} \psi_B \prod_{(i,j) \in B} x_{ij}^{2g} \right],
\]
for some monomials \( \psi_B \) in \( \mathbb{Q}[x_{ij}] \).

\[ \Box \]

### 2.4 Proof of the main theorem

**Definition 2.4.1.** Suppose \( u \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\} \) is one of the chosen generators of \( \pi_1(\Sigma_g \setminus \{p\}) \), and define maps

\[
f_u^{ij} : V_g(t) \rightarrow \mathbb{C}_{(ij)}
\]

\[
\rho \mapsto (\rho(u))_{ij}
\]

for each pair \((i,j)\) with \(1 \leq i, j \leq N\) and \(i \neq j\). These maps \(f_u^{ij}\) are \(T\)-equivariant since \(T\) acts on the matrix entry \((\rho(u))_{ij}\) with weight \(\chi_{ij}\). These maps then induce sections

\[
s_u^{ij} : S_g(t) \rightarrow V_g(t) \times_T \mathbb{C}_{(ij)}
\]

\[
[\rho] \mapsto (\rho, f_u^{ij}(\rho))
\]

of the line bundles \(L_{ij}\).

Let \(D_u^{ij}\) be the image in \(S_g(t)\) of the subspace \(\{\rho \in V_g(t) | (\rho(u))_{ij} = 0\}\). Then the section \(s_u^{ij}\) is nonzero on the complement of \(D_u^{ij}\).
To prove Theorem 2.1.3, we will show that intersections of certain sets of these subspaces \(D_{ij}^u\) are empty, and conclude that the corresponding polynomials in the Chern classes \(c_1(L_{ij})\) are zero. This is the same technique that was used in [44].

**Lemma 2.4.2.** Let \(M\) be a manifold. Let \(\mathcal{L}_i \rightarrow M, i = 1, \ldots, m,\) be complex line bundles with sections \(s_i : M \rightarrow \mathcal{L}_i.\) If these sections have no common zeros, i.e. \(s_1^{-1}(0) \cap \cdots \cap s_m^{-1}(0) = \emptyset,\) then 
\[c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_m) = 0.\]

**Proof.** Consider the vector bundle \(E := \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m \rightarrow M\) with the section 
\[\sigma := (s_1, \cdots, s_m) : M \rightarrow E.\]

The section \(\sigma\) is nowhere zero, so the Euler class \(e(E) = 0.\) Since \(c_m(E) = e(E),\) the top Chern class \(c_m(E) = c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_m)\) is equal to zero. \(\square\)

**Lemma 2.4.3.** Let \(0 < h < N,\) let \(B \subset [N] \times [N]\) be an \(h\)-by-\(N - h\) block, and let 
\[\zeta = \prod_{(i,j) \in B} c_1(L_{ij})^{2g} \in H^{4gh(N-h)}(S_g(t)).\]

Then \(\zeta = 0.\)

**Proof.** Consider the sections \(s_{ij}^u : S_g(t) \rightarrow L_{ij}\) (as defined in 2.4.1), for \((i, j) \in B\) and generators \(u \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}\) of \(\pi_1(S_g \setminus \{p\}).\) We have \((s_{ij}^u)^{-1}(0) = D_{ij}^u.\) By Lemma 2.4.2, it suffices to show that 
\[D := \bigcap_{(i,j) \in B} (D_{a_1}^{ij} \cap \cdots \cap D_{a_g}^{ij} \cap D_{b_1}^{ij} \cap \cdots \cap D_{b_g}^{ij}) = \emptyset.\]

By definition, \(D\) is the image in \(S_g(t)\) of the set of homomorphisms \(\rho \in V_g(t)\) such that the \((i,j)^{th}\) entry of \(\rho(u)\) is zero for all \((i, j) \in B\) and all \(u \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}.\) Suppose \(\rho \in D.\) By Remark 2.2.4, we can find a permutation \(\sigma\) of \([N]\) such that the matrices 
\[(\Phi(u))_{ij} := (\rho(u))_{\sigma^{-1}(i)\sigma^{-1}(j)},\]

20
for \( u \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\} \) are all block diagonal with blocks of size \( h \) and \( N-h \). So \( \prod_{i=1}^g [\Phi(a_i), \Phi(b_i)] \) is also block diagonal with blocks of size \( h \) and \( N-h \), and each block has determinant equal to 1. Let \( E \in SU(N) \) be a product of elementary matrices representing this permutation \( \sigma \) of basis elements, so that

\[
\rho(u) = E^\dagger \Phi(u) E \quad \text{for all} \quad u \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}.
\]

In particular, the matrix \( EtE^\dagger \) is obtained from \( t \in T \) by permuting the diagonal entries. Then

\[
t = \rho(c) = \prod_{i=1}^g [\rho(a_i), \rho(b_i)]
\]

\[
= \prod_{i=1}^g [E^\dagger \Phi(a_i)E, E^\dagger \Phi(b_i)E]
\]

\[
= \prod_{i=1}^g E^\dagger [\Phi(a_i), \Phi(b_i)]E
\]

\[
= E^\dagger \left( \prod_{i=1}^g [\Phi(a_i), \Phi(b_i)] \right) E.
\]

Thus \( E\rho(c)E^\dagger = EtE^\dagger = \prod_{i=1}^g [\Phi(a_i), \Phi(b_i)] \) is a diagonal matrix where the first \( h \) diagonal entries have product equal to 1. But this is impossible because for \( \rho \in V_g(t) \) we chose \( t = \rho(c) \) such that no \( h \) of its diagonal entries could have product equal to 1 (see Definition 2.1.1). Hence the set \( D \) of such \( \rho \) is empty. \( \square \)

**Proof of Theorem 2.1.3.** Let \( X = [N] \) and consider the rings \( \mathbb{Q}[x_{ij}] \) and \( R \) as in Definitions 2.3.1 and 2.3.4. Let \( J \subset H^*(S_g(t)) \) be the subring generated by the \( c_1(L_{ij}) \) for \( 1 \leq i, j \leq N \) and \( i \neq j \). Since \( c_1(L_{ij}) = -c_1(L_{ji}) \) and \( c_1(L_{ij}) + c_1(L_{jk}) = c_1(L_{ik}) \), the map

\[
\pi : R \rightarrow J
\]

\[
[x_{ij}] \mapsto c_1(L_{ij})
\]

defines a ring homomorphism. Consider the element

\[
\prod_{1 \leq i, j \leq N \atop i \neq j} c_1(L_{ij})^{k_{ij}} \in J.
\]
It has a representative

\[
\left[ \prod_{1 \leq i, j \leq N} x_{ij}^{k_{ij}} \right]
\]

in \( \mathbb{R} \). Suppose

\[
\sum_{\substack{1 \leq i, j \leq N \\ i \neq j}} k_{ij} \geq N(N - 1)g - N + 2.
\]

Then by Proposition 2.3.6, for each block \( B \subset [N] \times [N] \) we can find a monomial \( \theta_B \) in \( \mathbb{Q}[x_{ij}] \) such that

\[
\left[ \prod_{1 \leq i, j \leq N} x_{ij}^{k_{ij}} \right] = \left[ \sum_{B \in \mathcal{B}[X]} \theta_B \prod_{(i, j) \in B} x_{ij}^{2g} \right].
\]

So

\[
\prod_{1 \leq i, j \leq N} c_1(L_{ij})^{k_{ij}} = \pi \left( \sum_{B \in \mathcal{B}[X]} \theta_B \prod_{(i, j) \in B} x_{ij}^{2g} \right)
\]

\[
= \sum_{B \in \mathcal{B}[X]} \pi(\theta_B) \prod_{(i, j) \in B} c_1(L_{ij})^{2g},
\]

which vanishes by Lemma 2.4.3. \( \square \)
Chapter 3

\[ G = SO(2n + 1) \]

3.1 Introduction

Let \( g \geq 2 \) and let \( \Sigma \) be a Riemann surface of genus \( g \). Mark a point \( p \) on \( \Sigma \) and choose generators \( a_1, \ldots, a_g, b_1, \ldots, b_g, c \) for the fundamental group \( \pi_1(\Sigma \setminus \{p\}) \), such that \( c \) represents the boundary of \( \Sigma \setminus \{p\} \) and \( \prod_{i=1}^g [a_i, b_i] = c \). Let \( g = SO(2n + 1) \) and pick the maximal torus

\[
T = \left\{ \begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1 \\
\vdots \\
\cos \theta_n & -\sin \theta_n \\
\sin \theta_n & \cos \theta_n \\
1
\end{pmatrix} \mid \theta_1, \ldots, \theta_n \in [0, 2\pi) \right\} \subset G.
\]

For ease of notation we will denote these torus elements by \((\theta_1, \ldots, \theta_n)\).

**Definition 3.1.1.** Call an element \( \theta = (\theta_1, \ldots, \theta_n) \in T \) *generic* if it satisfies the following conditions:

- \( Stab\theta = T \)
• if \( \lambda_1, \ldots, \lambda_n \in \{1, -1, 0\} \) and \( \sum_{i=1}^{n} \lambda_i \eta_i \in 2\pi \mathbb{Z} \) then \( \lambda_1 = \cdots = \lambda_n = 0. \)

Fix a generic element \( t \in T \) and let

\[
S_g(t) := \{ \rho \in \text{Hom}(\pi_1(\Sigma \setminus \{p\}), G) \mid \rho(c) \sim t \}/G,
\]

where \( \sim \) denotes conjugacy in \( G \). Consider the torus bundle \( V_g(t) \to S_g(t) \) given by

\[
V_g(t) := \{ \rho \in \text{Hom}(\pi_1(\Sigma \setminus \{p\}), G) \mid \rho(c) = t \}.
\]

We associate line bundles to this torus bundle line bundles to this torus bundle via roots of the Lie algebra \( g \) of \( G \) as follows: Let \( \Phi(G) \) denote the set of roots of \( g \). If \( \phi \) is a root of \( g \), let \( \mathbb{C}(\phi) \) denote the corresponding one dimensional representation of \( T \), and form the line bundle \( L_{\phi} := (V_g(t) \times \mathbb{C}(\phi))/T \to V_g(t)/T = S_g(t) \), where the quotient is by the diagonal action of \( T \).

**Theorem 3.1.2.** For each root \( \phi \in \Phi(G) \), let \( k_{\phi} \) be a nonnegative integer. Then the cohomology class

\[
\prod_{\phi \in \Phi(G)} (c_1(L_{\phi}))^{k_{\phi}} \in H^2 \sum_{\phi \in \Phi(G)} k_{\phi}(S_g(t); \mathbb{Q})
\]

vanishes whenever

\[
\sum_{\phi \in \Phi(G)} k_{\phi} \geq 2gn^2 + \frac{1}{2}(n-1)(n-2)
\]

> **Remark 3.1.3.** The dimension of \( S_g(t) \) is \( 2gn(2n+1) - 2n(n+1) \).

This is a generalisation of the Newstead conjecture (see [37]). An analogous version was proved for \( SU(2) \) in [44] and for \( SU(n) \) in [11]; see the previous chapter of this thesis. The method of proof of Theorem 3.1.2 is similar to that in [11]; we illustrate it here for \( G = SO(3) \).
Proposition 3.1.4. Let \( G = \text{SO}(3) \). Let \( L \to S_g(t) \) be the line bundle associated to \( V_g(t) \) by

the representation

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
1 & 0
\end{pmatrix}
\cdot z = e^{2i\theta} z.
\]

Then \((c_1(L))^{2g} = 0\) in \( H_{4g}(S_g(t); \mathbb{Q})\).

Proof. For \( x \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\} \), consider the section \( s_x \) of \( L \) induced by the \( T \)-equivariant map

\[
f_x : V_g(t) \to \mathbb{C}
\]

\[
\rho \mapsto (\rho(x))_{11} - (\rho(x))_{22} + i((\rho(x))_{21} + (\rho(x))_{12})
\]

(where the subscript \( ij \) denotes the \((i,j)\)th matrix entry). If \( s_x = 0 \), then

\[
\rho(x) = \begin{pmatrix}
a & b & c \\
b & a & d \\
e & f & g
\end{pmatrix} \in \text{SO}(3)
\]

But since \( \rho(x) \in \text{SO}(3) \), we also have \(|c| = |d|\) and \(cd = 0\), thus \(c = d = 0\). Similarly \(e = f = 0\), and so \(a^2 + b^2 = 1\); that is, \( \rho(x) \in T \). Suppose \( \rho(x) = 0 \) for all \( x \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\} \). Then \( \rho(a_1), \ldots, \rho(a_g), \rho(b_1), \ldots, \rho(b_g) \in T \). But then \( \rho(c) = \prod_{i=1}^{g} [\rho(a_i), \rho(b_i)] = 1 \neq t \), so in fact there are no such \( \rho \) in \( V_g(t) \). Hence the section \((s_{a_1}, \ldots, s_{a_g}, s_{b_1}, \ldots, s_{b_g})\) of \( L^{\oplus 2g} \) is nowhere zero, so \( c_{2g}(L^{\oplus 2g}) = (c_1(L))^{2g} = 0 \).

For \( n > 1 \), the combinatorics of the intersections of the vanishing loci of the relevant sections is more complicated.

The outline of the rest of this chapter is as follows. We first, in Section 3.2, identify further subgroups \( H \subset G \) such that if \( \rho(a_i), \rho(b_i) \in H \) for all \( i \), then \( \prod_{i=1}^{g} [\rho(a_i), \rho(b_i)] \) cannot equal the generic torus element \( t \). Next in Section 3.3 we identify sections of the line bundles \( L_\phi \) which vanish exactly when the \( \rho(x) \) are elements of these subgroups \( H \), and thus exhibit some particular monomials in the \( c_1(L_\phi) \) which vanish. Finally, Section 3.4 which makes up the bulk of this chapter is devoted to showing that any monomial in the \( c_1(L_\phi) \) of high
enough degree can be expressed as a combination of these particular vanishing monomials, and hence also vanishes.

### 3.2 Identifying the desirable subgroups

**Lemma 3.2.1.** Consider the inclusion \( \iota : SU(n) \to SO(2n) \) defined by

\[
\iota : \begin{pmatrix}
    a_{11} & \cdots & \cdots & a_{ij} \\
    \vdots & \ddots & \ddots & \vdots \\
    \cdots & \cdots & a_{nn}
\end{pmatrix}
\mapsto
\begin{pmatrix}
    \Re a_{11} & -\Im a_{11} & \cdots & \cdots & : \\
    \Im a_{11} & \Re a_{11} & \cdots & : & : \\
    \vdots & \Re a_{ij} & -\Im a_{ij} & \cdots & : \\
    \Im a_{ij} & \Re a_{ij} & \cdots & \Re a_{nn} & -\Im a_{nn} \\
    \vdots & \cdots & \Im a_{ij} & \Re a_{nn}
\end{pmatrix}
\]

Let \( \kappa : SO(2n) \to SO(2n) \times \{1\} \subseteq SO(2n+1) \) be the map that adds a 1 to the bottom right corner, and zeros elsewhere in the bottom row and right column. Suppose \( A_1, \ldots, A_g, B_1, \ldots, B_g \in \kappa \circ \iota(SU(n)) \) and \( \prod_{i=1}^g [A_i, B_i] \in T \). Then \( \prod_{i=1}^g [A_i, B_i] \) cannot be generic in the sense of Definition 3.1.1.

**Proof.** We first observe that \( \iota \) is a homomorphism: Take \( A = (a_{ij}), B = (b_{ij}) \in SU(n) \). Then

\[
\iota(A)\iota(B) = \begin{pmatrix}
    \cdots & \cdots \\
    \sum_l (\Re a_{il} b_{lj} - \Im a_{il} b_{lj}) & -\sum_l (\Re a_{il} b_{lj} + \Im a_{il} b_{lj}) & \cdots \\
    \sum_l (\Re a_{il} b_{lj} + \Im a_{il} b_{lj}) & \sum_l (\Re a_{il} b_{lj} - \Im a_{il} b_{lj}) & \cdots \\
    \vdots & \cdots & \vdots 
\end{pmatrix}
\]

and

\[
\iota(AB) = \begin{pmatrix}
    \cdots & \Re(\sum_l a_{il} b_{lj}) & -\Im(\sum_l a_{il} b_{lj}) & \cdots \\
    \Im(\sum_l a_{il} b_{lj}) & \Re(\sum_l a_{il} b_{lj}) & \cdots \\
    \vdots & \cdots & \vdots 
\end{pmatrix}.
\]
But $\Re(\sum_i a_i b_{ij}) = \sum_i \Re(a_i b_{ij}) = \sum_i (\Re a_i \Re b_{ij} - \Im a_i \Im b_{ij})$, and $\Im(\sum_i a_i b_{ij}) = \sum_i \Im(a_i b_{ij}) = \sum_i (\Re a_i \Im b_{ij} + \Im a_i \Re b_{ij})$, so $i$ is indeed a homomorphism. Observe also that $\kappa \circ i$ is an isomorphism onto its image, and that for $A \in SU(n)$, $\kappa \circ i(A) \in T$ if and only if $A$ is diagonal.

Suppose that $A_i = \kappa \circ i(C_i)$ and $B_i = \kappa \circ i(D_i)$ for some $C_i, D_i \in SU(n)$ for each $1 \leq i \leq g$, and that $\prod_{i=1}^g [A_i, B_i] \in T$. Then $\prod_{i=1}^g [\kappa \circ i(C_i), \kappa \circ i(D_i)] = \kappa \circ i \prod_{i=1}^g [C_i, D_i] \in T$, so $\prod_{i=1}^g [C_i, D_i] \in SU(n)$ is diagonal. Thus $\prod_{i=1}^g [C_i, D_i] = \begin{pmatrix} e^{i\theta_1} & & & \\ & \ddots & & \\ & & \ddots & \cos \theta_n - \sin \theta_n \\ & & & \sin \theta_n \cos \theta_n \end{pmatrix}$, for some $\theta_j$ with $\theta_1 + \cdots + \theta_n \in 2\pi \mathbb{Z}$ (since $\prod_{i=1}^g [C_i, D_i] \in SU(n)$). So

$$\prod_{i=1}^g [A_i, B_i] = \kappa \circ i \begin{pmatrix} e^{i\theta_1} & & & \\ & \ddots & & \\ & & \ddots & \cos \theta_n - \sin \theta_n \\ & & & \sin \theta_n \cos \theta_n \end{pmatrix}$$

is not generic. \hfill \Box

**Lemma 3.2.2.** Let $1 \leq l \leq n$, and let $E_l$ be the matrix $E_l = (0, \ldots, 0, \frac{\pi}{2}, 0, \ldots, 0) \in T$, where the $\frac{\pi}{2}$ is in the $l$th position (so conjugation by $E_l$ switches rows $2l - 1$ and $2l$ and columns $2l - 1$ and $2l$). For a subset $X$ of $\{1, \ldots, n\}$, consider the product $E = \prod_{l \in X} E_l$. If $A_1, \ldots, A_g, B_1, \ldots, B_g \in E \kappa \circ i(SU(n)) E^{-1}$, then $\prod_{i=1}^g [A_i, B_i]$ is not a generic torus element.

**Proof.** Suppose $A_1, \ldots, A_g, B_1, \ldots, B_g \in E \kappa \circ i(SU(n)) E^{-1}$, and $\prod_{i=1}^g [A_i, B_i] = h \in T$. Then $E^{-1}hE = \prod_{i=1}^g [E^{-1}A_i E, E^{-1}B_i E]$, which cannot be generic by Lemma 3.2.1. Note that $E^{-1}(\theta_1, \ldots, \theta_n)E_l = (\theta_1, \ldots, -\theta_l, \ldots, \theta_n)$; thus $E^{-1}hE$ not being generic implies $h$ cannot be generic. \hfill \Box
Corollary 3.2.3. Suppose $X$ is a subset of $\{1, \ldots, h\}$, and let $E = \prod_{l \in X} E_l$. Let $M \in SO(2n)$ represent a permutation $\tilde{\sigma}$ of the form

$$\tilde{\sigma}(m) = \begin{cases} 
\sigma(m/2) & m \text{ even} \\
\sigma(\frac{m+1}{2}) & m \text{ odd}
\end{cases},$$

where $\sigma$ is a permutation of $[n]$. Suppose $A_1, \ldots, A_g, B_1, \ldots, B_g \in \kappa(ME_i(SU(h))E^{-1} \times SO(2m)M^{-1}) \subset SO(2h + 2m + 1)$ are block diagonal, with upper block in the same form as in Lemma 3.2.2. Then $\prod_{i=1}^{g}[A_i, B_i]$ cannot be a generic torus element. \hfill \Box

Remark 3.2.4. Let $V \subseteq \{1, \ldots, n\}$ and let $E = \prod_{l \in V} E_l$. Then $A \in E\kappa \circ \iota(SU(n))E^{-1}$ if and only if $A$ takes the form

$$\begin{pmatrix} R_{11} & \cdots & R_{1n} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
R_{n1} & \cdots & R_{nn} & 0 \\
0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix},$$

where $R_{ij}$ is a 2 by 2 matrix of the form

$$\begin{cases}
x & y \\
y & -x
\end{cases}$$

everywhere exactly one of $i, j \in V$

$$\begin{cases}
x & -y \\
y & x
\end{cases}$$

otherwise

3.3 Topology: Sections and particular vanishing classes

In the proof of Theorem 3.1.2 we will make use of the following lemma:

Lemma 3.3.1. Consider a manifold $M$ together with $m$ complex line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_m \rightarrow M$ and sections $s_i : M \rightarrow \mathcal{L}_i$ for each $1 \leq i \leq m$. If $s_1^{-1}(0) \cap \cdots \cap s_m^{-1}(0) = \emptyset$, then $c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_m) = 0$.

Proof. Suppose $s_1, \ldots, s_m$ have no common zeros. Consider the vector bundle $E = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m \rightarrow M$. The section $(s_1, \ldots, s_m) : M \rightarrow E$ is nowhere zero, so the Euler class $e(E) = 0$. 28
Hence $c_1(L_1) \ldots c_1(L_m) = c_m(E) = e(E) = 0$.

Our technique in this chapter will be to exhibit sections of the $L_\phi$ with no common zeros, and deduce that the corresponding monomials in the $c_1(L_\phi)$ vanish.

Recall that the roots of $g$ are $\pm(\epsilon_i + \epsilon_j)$ for $1 \leq i \leq j \leq n$ and $\pm(\epsilon_i - \epsilon_j)$ for $1 \leq i < j \leq n$, and the corresponding torus representations are given by

\[
T \times \mathbb{C}(\epsilon_i + \epsilon_j) \to \mathbb{C}(\epsilon_i + \epsilon_j)
\]

\[
((\alpha_1, \ldots, \alpha_n), z) \mapsto e^{\sqrt{-1}(\alpha_i + \alpha_j)}z
\]

and

\[
T \times \mathbb{C}(\epsilon_i - \epsilon_j) \to \mathbb{C}(\epsilon_i - \epsilon_j)
\]

\[
((\alpha_1, \ldots, \alpha_n), z) \mapsto e^{\sqrt{-1}(\alpha_i - \alpha_j)}z
\]

respectively. A $T$-equivariant map $V_g(t) \to \mathbb{C}(\phi)$ induces a section of $(V_g(t) \times \mathbb{C}(\phi))/T \to V_g(t)/T = S_g(t)$. Let $x \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a generator of the fundamental group $\pi_1(\Sigma \setminus \{p\})$ other than $c$, and consider the $T$-equivariant map

\[
f_{i,j}^\pm(x) : V_g(t) \to \mathbb{C}(\epsilon_i \pm \epsilon_j)
\]

\[
\rho \mapsto (\rho(x))_{2i-1,2j-1} \mp (\rho(x))_{2i,2j} + \sqrt{-1}((\rho(x))_{2i,2j-1} \pm (\rho(x))_{2i-1,2j}).
\]

These maps induce sections $s_{i,j}^\pm(x)$ of $L_{\epsilon_i \pm \epsilon_j}$.

**Lemma 3.3.2.** Let $V$ be a subset of $X = \{1, \ldots, n\}$, and let $B = V \times V^c$. Then

\[
M_B^X := \prod_{i=1}^n c_1(L_{2\epsilon_i})^{2g} \prod_{i \in V} c_1(L_{\epsilon_i - \epsilon_j})^{2g} \prod_{i, j \notin V} c_1(L_{\epsilon_i + \epsilon_j})^{2g} = 0
\]

in $H^{4gm^2}(S_g(t); \mathbb{Q})$.

**Proof.** Consider the sections $s_{i,j}^+(x)$ for each $i, j$ with either both $i, j \in V$ or both $i, j \notin V$, and each $x \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}$, together with $s_{i,j}^-(x)$ for each pair $i, j$ where exactly
one of $i, j$ is in $V$, and each $x \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}$. These sections have common zeros when $\rho(x) \in E \kappa \circ \iota SU(n) E^{-1}$ for each $x$, where $E = \prod_{t \in V} E_t$. Thus $\rho(c) = \prod_{i=1}^g [\rho(a_i), \rho(b_i)]$ cannot be a generic torus element by Lemma 3.2.2, and in particular $\rho(c) \neq t$. Thus the set of such $\rho$ is empty, so these sections have no common zeros. So by Lemma 3.3.1, the monomial in the $c_1(L_\phi)$ vanishes.

Before we continue it will help to set some notation. We will denote by $[n]$ the set $\{1, \ldots, n\}$.

**Definition 3.3.3.** Let $X$ be a finite set. A **block** $B$ in $X \times X$ is a subset of $X \times X$ of the form $V \times V^c$, where $\emptyset \subseteq V \subseteq X$ is a proper nonempty subset of $X$. We denote the set of all blocks in $X \times X$ by $\mathcal{B}[X]$. If $B = V \times V^c$, let $\bar{B} = V^c \times V$. If $i, j \in X$, let

$$
\epsilon_B(i, j) = \begin{cases} 
1 & (i, j) \in B \cup \bar{B} \\
-1 & \text{otherwise}
\end{cases}
$$

**Lemma 3.3.4.** Let $A \in (\mathcal{B}[X])^{2g}$ and $B = V \times V^c \in \mathcal{B}[X] \cup \{\emptyset\}$ (so we are allowing $V$ to be the empty set here). Then the cohomology class

$$
N_{B,A}^X := \prod_{i \in X} c_1(L_{2e_i})^{2g} \prod_{i<j} c_1(L_{e_i - \epsilon_B(i,j)e_j})^{2g} \prod_{l=1}^{2g} \prod_{i<j} c_1(L_{e_i - \epsilon_A \Delta_B(i,j)e_j})
$$

vanishes in $H^*(S_g(t))$.

**Proof.** We again exhibit sections of each line bundle appearing in $N_{B,A}^X$ which have common zeros precisely if all $\rho(x)$ lie in $E \kappa \circ \iota(SU(n)) E^{-1}$ for $E = \prod_{t \in V} E_t$. Consider the following sections:

- For $1 \leq l \leq g$:
  - $s_{i,i}^+(a_i)$ for all $i \in X$
  - $s_{i,j}^+(a_i)$ and $s_{i,j}^-(a_i)$ for all $(i, j) \in A_l$
  - $s_{i,j}^-(a_i)$ for all $i \neq j$ with $(i, j) \notin A_l \cup \bar{A}_l$ and $(i, j) \in B \cup \bar{B}$
- $s_{i,j}^+(a_l)$ for all $i \neq j$ with $(i, j) \notin A_l \cup \tilde{A}_l \cup B \cup \tilde{B}$

- and for $g + 1 \leq l \leq 2g$:

  - $s_{i,i}^+(b_l - g)$ for all $i \in X$
  - $s_{i,j}^+(b_l - g)$ and $s_{i,j}^-(b_l - g)$ for all $(i, j) \in A_l$
  - $s_{i,j}^-(b_l - g)$ for all $i \neq j$ with $(i, j) \notin A_l \cup \tilde{A}_l$ and $(i, j) \in B \cup \tilde{B}$
  - $s_{i,j}^+(b_l - g)$ for all $i \neq j$ with $(i, j) \notin A_l \cup \tilde{A}_l \cup B \cup \tilde{B}$

Fix $l$, and suppose without loss of generality that $l \leq g$. If the relevant sections all vanish, then $(\rho(a_l))_{2i,2j} = (\rho(a_l))_{2i-1,2j} = (\rho(a_l))_{2i,2j-1} = (\rho(a_l))_{2i-1,2j-1} = 0$ for all $(i, j) \in A_l$. Pick a permutation $\sigma$ of $[n]$ such that $A_l = \{1, \ldots, h\} \times \{h+1, \ldots, n\}$ and let $\tilde{\sigma}$ be the permutation of $[2n]$ given by

$$\tilde{\sigma}(m) = \begin{cases} 
\sigma(m/2) & m \text{ even} \\
\sigma((m+1)/2) & m \text{ odd}
\end{cases}.$$ 

Under the reordering of basis elements given by $\tilde{\sigma}$, $\rho(a_l)$ has the following form:

- $(\rho(a_l))_{ij} = 0$ for $1 \leq i \leq 2h$ and $2h + 1 \leq j \leq 2n$

- the square $1 \leq i, j \leq 2h$ breaks up into 2-by-2 matrices of the form

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$ 

In particular, for each $1 \leq i \leq h$,

$$\sum_{j=1}^{2h} (\rho(a_l))_{2i,j}^2 = \sum_{j=1}^{2h} (\rho(a_l))_{2i-1,j}^2,$$

and

$$\sum_{j=1}^{2h} (\rho(a_l))_{2i-1,j}(\rho(a_l))_{2i,j} = 0.$$

Thus since $\rho(a_l) \in SO(2n + 1)$, we have $(\rho(a_l))_{i,2n+1} = 0$ for $1 \leq i \leq 2h$. 31
• Thus \( \rho(a_l) \) is block diagonal.

• The rest of the top 2n-by-2n square bit of \( \rho(a_l) \) also breaks up into 2-by-2 matrices of the form
  \[
  \begin{pmatrix}
  x & y \\
  y & -x
  \end{pmatrix}
  \]
  for \( (i, j) \in B \cup \bar{B} \) and
  \[
  \begin{pmatrix}
  x & y \\
  -y & x
  \end{pmatrix}
  \]
  otherwise.

In particular, \( \rho(a_l) \in E \kappa \circ \iota(SU(n))E^{-1} \), where \( E = \prod_{v \in V} E_v \). Note this is independent of \( A_l \), so the full collection of sections above vanishes when each \( \rho(x) \) is in \( E \kappa \circ \iota SU(n)E^{-1} \), i.e., again, this intersection is empty so \( N^{X}_{B,A} = 0 \).

Lemma 3.3.5. Let \( X = [n] \) and let \( B \in \mathcal{B}[X] \), with \( B = V \times V^c \). If \( D \in \mathcal{B}[V] \), \( A \in (\mathcal{B}[V])^{2g} \), and \( \Psi = M^{Y}_{D} \) or \( \Psi = N^{Y}_{D,A} \), then the monomial \( \prod_{(i,j) \in B}(c_1(L_{\epsilon_i+\epsilon_j})c_1(L_{\epsilon_i-\epsilon_j}))^{2g}\Psi \) vanishes in \( H^*(S_t) \).

Proof. Consider the sections \( s^+_{i,j}(x) \) and \( s^-_{i,j}(x) \) for all \( (i, j) \in B \) and \( x \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\} \), together with the appropriate sections for \( \Psi \) described in the proofs of Lemmas 3.3.2 and 3.3.4 respectively. If these sections all vanish, then \( \rho(x) \in E \kappa \circ \iota (SU(|V|)) \times SO(2|V^c|)E^{-1} \) for each \( x \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\} \), where \( E = \prod_{l \in V} E_l \). By Corollary 3.2.3, \( \prod_{i=1}^{g}[\rho(a_i), \rho(b_i)] \neq t \), so again these sections have no common zeros.

### 3.4 The main argument

The rest of this chapter is devoted to showing that any monomial in the \( c_1(L_{\phi}) \) of degree at least \( 2gn^2 + \frac{1}{2}(n-1)(n-2) \) can be expressed as a combination of those appearing in Lemmas 3.3.2, 3.3.4 and 3.3.5, and hence also vanishes.

#### 3.4.1 Combinatorial Preliminaries

We begin with some combinatorial lemmas.
Lemma 3.4.1. Suppose \( X = \{e_1, \ldots, e_h, f_1, \ldots, f_w, z\} \). Let \( B \in \mathcal{B}[X \setminus \{z\}] \) be the block \( B = \{e_1, \ldots, e_h\} \times \{f_1, \ldots, f_w\} \), and let \( C \) be a block in \( \mathcal{B}[[f_1, \ldots, f_w, z]] \). Then the union \( B \cup C \cup \bar{C} \) contains a block \( D \in \mathcal{B}[X] \).

Proof. Suppose without loss of generality \( C = \{f_1, \ldots, f_d\} \times \{f_{d+1}, \ldots, f_w, z\} \), for some \( 1 \leq d \leq w \). Then \( D = \{e_1, \ldots, e_h, f_1, \ldots, f_d\} \times \{f_{d+1}, \ldots, f_w, z\} \subset B \cup C \cup \bar{C} \).

Lemma 3.4.2. Again, suppose \( X = \{e_1, \ldots, e_h, f_1, \ldots, f_w, z\} \). Let \( B = \{e_1, \ldots, e_h\} \times \{f_1, \ldots, f_w\} \in \mathcal{B}[X \setminus \{z\}] \). Let \( C \in \mathcal{B}[\{e_1, \ldots, e_h, z\}] \) and \( E \in \mathcal{B}[\{f_1, \ldots, f_w, z\}] \). Then there exists a block \( A_{CE} \in \mathcal{B}[X] \) such that either \( C \cup E \subset A_{CE} \subset C \cup E \cup B \cup \bar{B} \), or \( C \cup \bar{E} \subset A_{CE} \subset C \cup \bar{E} \cup B \cup \bar{B} \).

Proof. Suppose without loss of generality \( C = \{e_1, \ldots, e_k\} \times \{e_{k+1}, \ldots, e_h, z\} \), for some \( 1 \leq k \leq h \). Either \( E \) or \( \bar{E} \) has the form \( \{f_1, \ldots, f_l\} \times \{f_{l+1}, \ldots, f_w, z\} \) (up to relabelling), so take \( A_{CE} = \{e_1, \ldots, e_k, f_1, \ldots, f_l\} \times \{e_{k+1}, \ldots, e_h, f_{l+1}, \ldots, f_w, z\} \).

Lemma 3.4.3 (The symmetric difference of two blocks is a block). Let \( B, C \in \mathcal{B}[X] \). Then there exists a block \( D \in \mathcal{B}[X] \) such that \( (B \cup \bar{B}) \triangle (C \cup \bar{C}) = D \cup \bar{D} \).

Proof. Suppose \( B = \{e_1, \ldots, e_h\} \times \{f_1, \ldots, f_w\} \), and

\[
C = \{e_1, \ldots, e_s, f_1, \ldots, f_{t+1}\} \times \{e_{s+1}, \ldots, e_h, f_{t+1}, \ldots, f_w\}.
\]

Then take \( D = \{e_1, \ldots, e_s, f_{t+1}, \ldots, f_w\} \times \{e_{s+1}, \ldots, e_h, f_1, \ldots, f_t\} \).

Remark 3.4.4. Let \( B \in \mathcal{B}[X] \) and \( z \in X \). Then \( B|_{X \setminus \{z\}} := B \cap ((X \setminus \{z\}) \times (X \setminus \{z\})) \in \mathcal{B}[X \setminus \{z\}] \).

3.4.2 Algebraic Preliminaries

We continue with some algebraic discussion.

Definition 3.4.5. Let \( X \) be a finite subset of \( \mathbb{N} \). Define the associated auxiliary set \( Y(X) = \{y_{ij}^{\pm}, y_{ij} \mid i < j, i, j \in X\} \cup \{y_{ii}^{\pm} \mid i \in X\} \). We also define the following useful subsets of \( Y(X) \):
• for $z \in X$: $Y_z(X) = \{y^+_ij, y^-ij \mid i < j, i = z \text{ or } j = z\} \cup \{y^+_zz\}$

• for $B \in \mathcal{B}[X] \sqcup \{\emptyset\}$:
  - $Y^+_B(X) = \{y^+_ij \mid i < j \in X, (i, j) \notin B \cup \bar{B}\} \cup \{y^+ij \mid (i, j) \in B \cup \bar{B}\}$
  - $Y^-_B(X) = \{y^-ij \mid i < j \in X, (i, j) \notin B \cup \bar{B}\} \cup \{y^-ij \mid (i, j) \in B \cup \bar{B}\}$

We need a few results about the polynomial ring $\mathbb{Q}[Y(X)]$ formed by adjoining the elements of $Y(X)$ to $\mathbb{Q}$.

**Lemma 3.4.6.** Let $X$ be a finite subset of $\mathbb{N}$ with $|X| = m > 3$, and let $p \in \mathbb{Q}[Y(X)]$ be a monomial of degree at least $2gm(m - 1) - m + 1$. Then there exists some $z \in X$ such that if we factorise $p$ as $p = q_z r_z$, where $q_z \in \mathbb{Q}[Y(X \setminus \{z\})]$ and $r_z \in \mathbb{Q}[Y_z(X)]$ are monomials, then $q_z$ has degree at least $2g(m - 1)(m - 2) - m + 2$.

**Proof.** Write

$$p = \lambda \prod_{\substack{i,j \in X \\text{i} \leq j}} (y^+_ij)^{d^+_ij} (y^-ij)^{d^-ij}$$

(where we take $d^-ii = 0$ for all $i$). Given $z \in X$,

$$q_z = \prod_{\substack{i,j \in X \setminus \{z\} \\text{i} \leq j}} (y^+_ij)^{d^+_ij} (y^-ij)^{d^-ij}.$$  

Note that each factor $y^+_ij$ of $p$ appears in $q_z$ precisely when $i, j \neq z$, and thus appears in at least $m - 2$ of the $q_z$ as $z$ ranges over $X$ (exactly $m - 2$ except for the $y^+_ii$). Hence

$$\prod_{z=1}^m a_z = p^{m-2} \prod_{i \in X \setminus \{z\}} (y^+_ii)^{d^+_ii}$$

has degree $\geq (m - 2)(2gm(m - 1) - m + 1)$. Suppose by way of contradiction that each $q_z$ has degree at most $2g(m - 1)(m - 2) - m + 1$. Then $\prod_{z=1}^m q_z$ has degree at most

$$m(2g(m - 1)(m - 2) - m + 1) = 2gm(m - 1)(m - 2) - m + 1$$

$$< (m - 2)2gm(m - 1) - (m - 2)(m - 1)$$

$$= (m - 2)(2g(m - 1) - m + 1).$$
This is a contradiction, so the desired $z$ must exist.

Lemma 3.4.7. Let $X = \{e_1, \ldots, e_h, f_1, \ldots, f_w, z\}$ be a subset of $\mathbb{N}$ with $|X| = m$. Let $p \in \mathbb{Q}[Y_z(X)]$ be a monomial of degree at least $2gm(m-1) - m + 1 - 4gwh$ that factorises as $p = p_np_w$, where $p_h \in \mathbb{Q}[Y(\{e_1, \ldots, e_h, z\})]$ and $p_w \in \mathbb{Q}[Y(\{f_1, \ldots, f_w, z\})]$ are monomials. Then either $\deg p_h \geq 2gh(h+1) - h$, or $\deg p_w \geq 2gw(w+1) - w$.

Proof. Suppose $\deg p_h \leq 2gh(h+1) - h - 1$. Then

$$
\deg p_w \geq 2gm(m-1) - m + 1 - 4gwh - 2gh(h+1) + h + 1
$$

$$
= 2gm(w+h) - (w+h) - 4gwh - 2gh(m-w) + (m-w)
$$

$$
= 2gw(m+h) - 4gwh - w - h + m - w
$$

$$
= 2gw(w+h+1+h) - 4gwh - w + 1
$$

$$
= 2gw(w+1) - w + 1 > 2gw(w+1) - w.
$$

Lemma 3.4.8. Let $X = H \sqcup W$ be a finite subset of $\mathbb{N}$ with $|H| = h$, $|W| = w$, and $|X| = w + h = n$. Let $p \in \mathbb{Q}[Y(X)]$ be a monomial of degree at least $2gn^2 + \frac{1}{2}(n-1)(n-2) - 4gwh$ that factorises as $p = p_wp_h$, where $p_w \in \mathbb{Q}[Y(W)]$ and $p_h \in \mathbb{Q}[Y(H)]$. Then either $\deg p_w \geq 2gw^2 + \frac{1}{2}(w-1)(w-2)$ or $\deg p_h \geq 2gh^2 + \frac{1}{2}(h-1)(h-2)$.

Proof. Suppose $\deg p_h \leq 2gh^2 + \frac{1}{2}(h-1)(h-2) - 1$. Then

$$
\deg p_w \geq 2gn^2 + \frac{1}{2}(n-1)(n-2) - 4gwh - 2gh^2 - \frac{1}{2}(h-1)(h-2) + 1
$$

$$
= 2g(h^2 + 2wh + w^2) + \frac{1}{2}(w + h - 1)(w + h - 2) - 4gwh - 2gh^2 - \frac{1}{2}(h-1)(h-2) + 1
$$

$$
= 2gw^2 + \frac{1}{2}(h-1)(h-2) + \frac{1}{2}w(h-2) + \frac{1}{2}w(h-1) - \frac{w^2}{2} - \frac{1}{2}(h-1)(h-2) + 1
$$

$$
= 2gw^2 + \frac{1}{2}(w^2 - 2w - w + 2) + wh
$$

$$
= 2gw^2 + \frac{1}{2}(w-1)(w-2) + wh > 2gw^2 + \frac{1}{2}(w-1)(w-2).
$$

Lemma 3.4.9. Let $p = qr$ be a monomial of degree at least $2gn^2 + \frac{1}{2}(n-1)(n-2) - n(n-1)g - 2g(n-1)$. Then either $\deg q \geq 2g$ or $\deg r \geq n(n-1)g - n + 2$. 

35
Proof. Suppose deg \( r \leq n(n-1)g - n + 1 \). Then

\[
\begin{align*}
\deg q & \geq 2gn^2 + \frac{1}{2}(n-1)(n-2) - n(n-1)g - 2g(n-1) - n(n-1)g + n - 1 \\
& = 2gn^2 - 2gn(n-1) - 2g(n-1) + \frac{1}{2}(n-1)(n-2) + n - 1 \\
& = 2g + \frac{1}{2}n(n-1) \geq 2g.
\end{align*}
\]

\[\square\]

Lemma 3.4.10. Let \( w, h \in \mathbb{N} \) with \( w + h = m - 1 \), and let \( p = qr \) be a monomial of degree at least \( 2gm(m-1) - m + 1 - 4gwh - h(h+1)g \). Then either \( \deg q \geq 2gw(w+1) - w \) or \( \deg r \geq h(h+1)g - (h+1) + 2 \).

Proof. Suppose \( \deg r \leq h(h+1)g - (h+1) + 1 \). Then

\[
\begin{align*}
\deg q & \geq 2gm(m-1) - m + 1 - 4gwh - h(h+1)g - h(h+1)g + (h+1) - 1 \\
& = 2g(w+h+1)(w+h) - (w+h) - 4gwh - 2gh(h+1) + h \\
& = 2gw^2 + 2gh(h+1) + 2gw - w - 2gh(h+1) \\
& = 2gw(w+1) - w.
\end{align*}
\]

\[\square\]

### 3.4.3 Relations and equivalences

Let \( \phi, \psi, \chi \in \Phi(G) \) be roots of \( g \) with \( \phi + \psi = \chi \). Then \( L_\phi \otimes L_\psi \cong L_\chi \), and so

\[
c_1(L_\phi) + c_1(L_\psi) = c_1(L_\chi).
\]  

(3.1)

It is the relations (3.1) that will enable us to express any monomial in the \( c_1(L_\phi) \) of high enough degree in terms of those monomials which we have already shown to vanish.

Definition 3.4.11. Let \( I \subset \mathbb{Q}[Y(X)] \) be the ideal generated by the elements

- \( y_{ij}^- + y_{ji}^- \)
- \( y_{ij}^- + y_{jk}^- + y_{ki}^- \)
- \( y_{ij}^+ - y_{jk}^+ + y_{ki}^- \)
for all triples of elements $i, j, k \in X$, where $y_{ii} = 0$ for all $i$. Let $R = \mathbb{Q}[Y(X)]/I$ be the quotient of $\mathbb{Q}[Y(X)]$ by this ideal.

Note that this quotient preserves the grading by degree. If $p \in \mathbb{Q}[Y(X)]$ we will denote by $[p]$ its image in $R$.

**Lemma 3.4.12.** Let $p \in \mathbb{Q}[Y^{-}(X)]$ be a monomial of degree at least $\frac{1}{2}|X|(|X| - 1)a - |X| + 2$, for some $a \in \mathbb{N}$. Then for each $B \in \mathcal{B}[X]$ we can find a monomial $\phi_B \in \mathbb{Q}[Y^{-}(X)]$ such that

$$[p] = \left[ \sum_{B \in \mathcal{B}[X]} \phi_B \prod_{(i,j) \in B} (y_{ij}^{-})^a \right].$$

**Proof.** This follows from replacing $2g$ by $a$ in the proof of Proposition 3.6 in [11], which is Proposition 2.3.6 in Chapter 2 of this thesis. \hfill \Box

**Remark 3.4.13.** Let $V \subset X$. The map on $\mathbb{Q}[Y(X)]$ given by

$$f_B : y_{ij}^{\pm} \mapsto \begin{cases} -y_{ij}^{\pm} & i, j \in V \\ -y_{ij}^{\pm} & i \in V, j \notin V \\ y_{ij}^{\pm} & i \notin V, j \in V \\ y_{ij}^{\pm} & i, j \notin V \end{cases}$$

descends to the quotient $R = \mathbb{Q}[Y(X)]/I$.

**Corollary 3.4.14.** Let $X$ be a finite set, let $B = V \times V^c \in \mathcal{B}[X]$ be a block, let $a \in \mathbb{N}$, and let $p \in \mathbb{Q}[Y_B^{-}(X)]$ be a monomial of degree at least $\frac{1}{2}|X|(|X| - 1)a - |X| + 2$. Then for each $C \in \mathcal{B}[X]$ we can find a monomial $\phi_C \in \mathbb{Q}[Y_B^{-}(X)]$ such that

$$[p] = \left[ \sum_{C \in \mathcal{B}[X]} \phi_C \prod_{(i,j) \in C} (y_{ij}^{-} \epsilon_B(i,j))^a \right].$$

**Proof.** Observe that $f_B : \mathbb{Q}[Y^{-}(X)] \to \mathbb{Q}[Y_B^{-}(X)]$ is an isomorphism. Applying Lemma 3.4.12 to $f_B^{-1}(p)$, we can write

$$[f_B^{-1}(p)] = \left[ \sum_{C \in \mathcal{B}[X]} \phi_C \prod_{(i,j) \in C} (y_{ij}^{-})^a \right].$$
Thus
\[ [p] = \left[ \sum_{C \in \mathcal{B}[X]} f_B(\phi_C) \prod_{(i,j) \in C} (-1)^{\lambda_C} (y_{ij}^{-\epsilon_v(i,j)})^a \right], \]
where \( \lambda_C = |\{(i, j) \in C \mid i \in V\}|. \)

\[ \square \]

Lemma 3.4.15. Let \( X \) be a finite subset of \( \mathbb{N} \), and let \( B = V \times V^c \in \mathcal{B}[X] \). Let \( z \in V^c \).

Let \( p \in \mathbb{Q}[Y(X)] \) be a homogeneous polynomial of degree at least \( 2gm(m - 1) - m + 1 - (m - 1)(m - 2)g \). Then \([p]\) has a representative in \( \mathbb{Q}[Y_B^-(X) \cup \{y_{iz}^{-\epsilon_v(i)} \mid i \in X \setminus \{z\}\}] \); furthermore this representative can be chosen to be a sum of monomials, each of which, when factorised as \( qr \) with \( q \in \mathbb{Q}[Y_B^- (X)] \) and \( r \in \mathbb{Q}[\{y_{iz}^{-\epsilon_v(i)} \mid i \in X \setminus \{z\}\}] \), either satisfies \( \deg q \geq m(m - 1)g - m + 2 \), or \( \prod_{i \in X \setminus \{z\}} (y_{iz}^{-\epsilon_v(i)})^{2g} \) divides \( r \).

Proof. We first observe that any \( y_{iz}^{-\epsilon_v(i)} \), together with \( Y_B^-(X) \), generates \( \mathbb{Q}[Y(X)] \) over \( \mathbb{Q} \).

This is because for each \( (i, j) \) we have either \( y_{ij}^+ \in Y_B^+(X) \) or \( y_{ij}^- \in Y_B^-(X) \), and for fixed \( i \), \( y_{iz}^+ \in Y_B^+(X) \iff \epsilon_v(i) = \pm 1 \). So both \( y_{iz}^+ \) and \( y_{iz}^- \) are in \( Y_B^-(X) \cup \{y_{iz}^{-\epsilon_v(i)}\} \). We need to check that

- if \( (i, j) \in B \), then \( y_{ij}^- \in \mathbb{Q}[Y_B^- (X) \cup \{y_{iz}^{-\epsilon_v(i)}\}] \)

- if \( (i, j) \notin B \), then \( y_{ij}^+ \in \mathbb{Q}[Y_B^- (X) \cup \{y_{iz}^{-\epsilon_v(i)}\}] \)

. For \( (i, j) \in B \): \( y_{ij}^- = -y_{ij}^+ + y_{iz}^+ + y_{iz}^- \). For \( (i, j) \notin B \): \( y_{ij}^+ = -y_{ij}^- + y_{iz}^+ + y_{iz}^- \). \( \checkmark \)

Suppose \( X = [m] \), and without loss of generality take \( z = m \). First factorise each term of \([p]\) as \([q_1(y_{iz}^{-\epsilon_v(1)})^{d_1}]\), with \( q_1 \in \mathbb{Q}[Y_B^-(X)] \). If \( \deg q_1 \geq m(m - 1)g - m + 2 \), this term has the desired form; else

\[ d_1 \geq 2gm(m - 1) - m + 1 - (m - 1)(m - 2)g - m(m - 1)g + m - 1 \]
\[ = g(m - 1)(2m - m - 2 - m) \]
\[ = 2g(m - 1) > 2g. \]

In this case, factorise \([q_1(y_{iz}^{-\epsilon_v(1)})^{d_1}]\) as \((y_{iz}^{-\epsilon_v(i)})^{2g} q_2(y_{iz}^{-\epsilon_v(2)})^{d_2}\), where \( q_2 \in \mathbb{Q}[Y_B^-(X)] \). Any term where \( \deg q_2 \geq m(m - 1)g - m + 2 \) now has the desired form; else \( d_2 \geq 2g(m - 1) - 2g = \]

38
2g(m - 2) \geq 2g. Repeat this process: for each term, either \( \deg q_i \geq m(m - 1)g - m + 2 \) or \( d_i \geq 2g(m - i) \). Thus any term where \( \deg q_{m-1} < m(m - 1)g - m + 2 \) has \( d_i \geq 2g \) for all \( 1 \leq i \leq m - 1 \), and so contains the factors \((y_i^{-(i)})^{2g}\) for all \( i \in X \setminus \{z\}\). \qed

**Lemma 3.4.16.** Let \( X \) be a finite subset of \( \mathbb{N} \) with \(|X| \geq 2\), and suppose \( p \in \mathbb{Q}[Y(X)] \) is a monomial of degree at least \( 2g|X|(|X| - 1) - |X| + 1 \). Then we can find a homogeneous polynomial \( \chi_{\emptyset} \in \mathbb{Q}[Y(X)] \), together with polynomials \( \psi_B, \chi_B \in \mathbb{Q}[Y(X)] \) for each block \( B \in \mathcal{B}[X] \), such that

\[
[p] = \left[ \sum_{B \in \mathcal{B}[X]} \sum_{(i,j) \in B} \psi_B \prod_{(i,j) \in B} (y_{ij}^+ y_{ij}^-)^{2g} + \sum_{B \in \mathcal{B}[X] \cup \{\emptyset\}} \chi_B \prod_{y \in Y_B^+} y^{2g} \right].
\]

**Illustration.** Recall that our goal is to use the relations (3.1) to express any monomial in the \( c_1(L_{\emptyset}) \) of large enough degree as a combination of the special monomials which are shown to vanish in Lemmas 3.3.2, 3.3.4 and 3.3.5. The elements \( y_{ij}^\pm \) correspond to \( c_1(L_{e_i \pm e_j}) \), and we will depict these monomials by considering the intersection of the vanishing loci of the appropriate sections. Although these vanishing loci lie in \((SO(2n + 1))^{2g}\), we will illustrate them as \( n \times n \) grids, representing the \( 2 \times 2 \) blocks \( R_+ \) by \( \Box \), the \( 2 \times 2 \) blocks \( R_- \) by \( \Box \), and the \( 2 \times 2 \) blocks of zeros (in the vanishing loci of \((s_{ij}^+(x), s_{ij}^-(x))\)) by \( \bigcirc \). Lemma 3.4.16 says we can express monomials of high enough degree as a sum of terms whose corresponding sections have vanishing loci taking one of the forms shown in Figure 3.1.

**Proof.** By induction on \(|X|\).

For \(|X| = 2\), without loss of generality take \( X = \{1, 2\} \). Then

\[
\mathbb{Q}[Y(X)] = \mathbb{Q}[y_{12}^+, y_{12}^-, y_{21}^+, y_{21}^-]
\]. Since \([y_{12}^+] = [y_{21}^+] \) and \([y_{12}^-] = [-y_{21}^-] \) in \( R \), any monomial \( p \in \mathbb{Q}[Y(X)] \) of degree at least \( 4g - 1 \) is equivalent in \( R \) to \( \lambda(y_{12}^+)^a(y_{12}^-)^b \), where \( \lambda \in \mathbb{Q} \) and \( a + b \geq 4g - 1 \). So either \( a \geq 2g \) or \( b \geq 2g \). If \( a \geq 2g \), take \( \chi_{\emptyset} = (y_{12}^+)^{a-2g}(y_{12}^-)^b \), and \( \psi_B = \chi_B = 0 \) for all \( B \in \mathcal{B}[X] \). If \( b \geq 2g \), take \( \chi_{(1,2)} = (y_{12}^+)^a(y_{12}^-)^{b-2g} \), and \( \psi_B = \chi_{\emptyset} = 0 \) for all \( B \in \mathcal{B}[X] \). (\( \checkmark \))
(a) A block of $R^-$, otherwise $R^+$, above the diagonal. (Or all $R^+$, if $B = \emptyset$)

(b) A block of zeros. Note that this alone does not make $\rho(x) \in SO(2n + 1)$ block diagonal, since this is only a $2w$-by-$2(n-h)$ block of zeros.

Figure 3.1

Now suppose $|X| = m \geq 3$. By Lemma 3.4.6, there is some $z \in X$ for which when we factorise $p$ as $q_z r_z$, with monomials $q_z \in \mathbb{Q}[Y(X \setminus \{z\})]$ and $r_z \in \mathbb{Q}[Y_z(X)]$, the degree of $q_z$ is at least $2g(m-1)(m-2) - m + 2$. By the inductive hypothesis, there exist homogeneous polynomials $\tilde{\psi}_C, \tilde{\chi}_C, \tilde{\chi}_\emptyset \in \mathbb{Q}[Y(X \setminus \{z\})]$, for all $C \in \mathcal{B}[X \setminus \{z\}]$, such that

$$[q_z] = \left[ \sum_{C \in \mathcal{B}[X \setminus \{z\}]} \tilde{\psi}_C \prod_{(i,j) \in C} (y_{ij}^+ y_{ij}^-)^{2g} + \sum_{C \in \mathcal{B}[X \setminus \{z\}] \cup \{\emptyset\}} \tilde{\chi}_C \prod_{y \in y_C^+ (X \setminus \{z\})} y^{2g} \right].$$

We will consider these terms separately, as it suffices to show each term has the desired form.

Fix a block $C \in \mathcal{B}[X \setminus \{z\}]$, say $C = \{e_1, \ldots, e_h\} \times \{f_1, \ldots, f_w\}$ (so $h + w = m - 1$). Consider the term

$$[r_z \tilde{\psi}_C \prod_{(i,j) \in C} (y_{ij}^+ y_{ij}^-)^{2g}]$$

in $[p]$ (see Figure 3.2). Observe that $\deg r_z \tilde{\psi}_C \geq 2gm(m-1) - m + 1 - 4gw$. Since $[y_{ij}^+] = [y_{ij}^+ + y_{ij}^-]$ and $[y_{ij}^-] = [y_{ij}^- + y_{ij}^+]$, each term of $r_z \tilde{\psi}_C$ can be expressed as a sum of terms of the form $[p_h p_w]$, where $p_h \in \mathbb{Q}[Y(\{e_1, \ldots, e_h, z\})]$ and $p_y \in \mathbb{Q}[Y(\{f_1, \ldots, f_w, z\})]$ are monomials. By Lemma 3.4.7, either $\deg p_h \geq 2gh(h+1) - h$, or $\deg p_w \geq 2gw(w+1) - w$ (see figure 3.3). Consider one such monomial in the sum, and without loss of generality assume
Figure 3.2: The inductive hypothesis gave us a block of zeros in $B[X \setminus \{z\}]$, using a monomial of degree $4gwh$.

Figure 3.3: We can apply the inductive hypothesis to one of these triangles $T_h, T_w$. 
(a) A block $D$ of zeros in $B\{e_1, \ldots, e_h, z\}$.

Observe that the union $C \cup D$ contains a full block $B_{CD} \in B[X]$

Figure 3.4: Applying the inductive hypothesis to $T_h$, we find either a block of zeros, or a block of $R_-$ with the rest $R_+$.

the former. By the inductive hypothesis, we can find homogeneous polynomials $\psi_D, \chi_D$ for each $D \in B\{e_1, \ldots, e_h, z\}$, together with $\chi_{\emptyset}$, with $\psi_D, \chi_D, \chi_{\emptyset} \in \mathbb{Q}[Y\{e_1, \ldots, e_h, z\}]$, such that

$$[p_h] = \left[ \sum_{D \in B\{e_1, \ldots, e_h, z\}} \psi_D \prod_{(i,j) \in D} (y_{ij}^+ y_{ij}^-)^{2g} + \sum_{D \in B\{e_1, \ldots, e_h, z\} \cup \emptyset} \chi_D \prod_{y \in Y_D^+ \{e_1, \ldots, e_h, z\}} y^{2g} \right]$$

(see figure 3.4).

Fix a block $D = \{e_1, \ldots, e_d\} \times \{e_{d+1}, \ldots, e_h, z\}$ (some $1 \leq d \leq h$), and consider the term

$$p_w \psi_D \prod_{(i,j) \in D} (y_{ij}^+ y_{ij}^-)^{2g} \prod_{(i,j) \in C} (y_{ij}^+ y_{ij}^-)^{2g}$$

of $[p]$. By Lemma 3.4.1, there is a block $B_{CD} \in B[X]$ with $B_{CD} \subset C \cup D \cup \bar{D}$, so our term is a multiple of $\prod_{(i,j) \in B_{CD}} (y_{ij}^+ y_{ij}^-)^{2g}$ and hence has the desired form (see Figure 3.4a).

Now consider the term

$$p_w \chi_D \prod_{y \in Y_{D}^+ \{e_1, \ldots, e_h, z\}} y^{2g} \prod_{(i,j) \in C} (y_{ij}^+ y_{ij}^-)^{2g}$$
Figure 3.5: We can apply either Corollary 3.4.14 to $\bar{T}_h$ (left), or the inductive hypothesis to $T_w$ of $[p]$ (see Figure 3.4b). Factorise $[p_w\chi_D]$ as a sum of terms of the form $[\alpha_w\alpha_h]$, where $\alpha_w \in \mathbb{Q}[Y([f_1, \ldots, f_w, z])]$ and $\alpha_h \in \mathbb{Q}[Y^D_D(\{e_1, \ldots, e_h, z\})]$. By Lemma 3.4.10, either $\deg \alpha_w \geq 2gw(w + 1) - w$ or $\deg \alpha_h \geq h(h + 1)g - (h + 1) + 2$ (see figure 3.5). If $\deg \alpha_h \geq h(h + 1)g - (h + 1) + 2$, then by Corollary 3.4.14, for each block $E \in B[e_1, \ldots, e_h, z]$ we can find a monomial $\beta_E$ such that

$$[\alpha_h] = \left[ \sum_{E \in B[H \cup \{z\}]} \beta_E \prod_{(i,j) \in E} (y_{ij}^{e(i,j)}2g) \right].$$

Then for fixed $E$, the term

$$\left[ \alpha_w\beta_E \prod_{(i,j) \in E} (y_{ij}^{e(i,j)}2g) \prod_{y \in Y^D_D(\{e_1, \ldots, e_h, z\})} y^{2g} \prod_{(i,j) \in C} (y_{ij}^+y_{ij}^-2g) \right]$$

of $[p]$ contains the factor

$$\prod_{(i,j) \in E} (y_{ij}^+y_{ij}^-2g) \prod_{(i,j) \in C} (y_{ij}^+y_{ij}^-2g).$$

By Lemma 3.4.1, $C \cup E \cup \bar{E}$ contains a block $B_{CE} \in B[X]$, so this term has the desired form (see Figure 3.6).
Figure 3.6: Applying Corollary 3.4.14 to $\overline{T_h}$ gives a block $E$ of $R_+$ or $R_-$ as shown in $\overline{T_h}$, which combines with the $R_+$ and $R_-$ in $T_h$ to get zeros everywhere in $E$. The union $C \cup E$ contains a full block $B_{CE} \in B[X]$

Suppose instead that $\deg \alpha_w \geq 2gw(w + 1) - w$. Then by the inductive hypothesis,

$$[\alpha_w] = \left[ \sum_{F \in B([\{f_1, \ldots, f_w, z\}])} \theta_F \prod_{(i,j) \in F} (y_{ij}^+y_{ij}^-)^{2g} + \sum_{F \in B([\{f_1, \ldots, f_w, z\}]) \cup \{\emptyset\}} \phi_F \prod_{y \in Y_F^+([\{f_1, \ldots, f_w, z\}])} y^{2g} \right],$$

for some homogeneous polynomials $\theta_F, \phi_F \in \mathbb{Q}[Y([\{f_1, \ldots, f_w, z\}])]$.

Fix a block $F \in B([\{f_1, \ldots, f_w, z\}])$ and consider each term separately. By Lemma 3.4.1, $C \cup F \cup \overline{F}$ contains a block $B_{CF} \in B[X]$, and so the term

$$\left[ \alpha_h \theta_F \prod_{(i,j) \in F} (y_{ij}^+y_{ij}^-)^{2g} \prod_{y \in Y_F^+([\{f_1, \ldots, f_w, z\}])} y^{2g} \prod_{(i,j) \in C} (y_{ij}^+y_{ij}^-)^{2g} \right]$$

of $[p]$ contains the factor $\prod_{(i,j) \in B_{CF}} (y_{ij}^+y_{ij}^-)^{2g}$ and hence is of the desired form (see Figure 3.7a).

By Lemma 3.4.2, there exists a block $B_{DF} \in B[X]$ with either $D \cup F \subset B_{DF} \subset D \cup F \cup C \cup \overline{C}$, or $\overline{D} \cup F \subset B_{DF} \subset \overline{D} \cup F \cup C \cup \overline{C}$, and thus the term

$$\left[ \alpha_h \phi_F \prod_{y \in Y_F^+[\{f_1, \ldots, f_w, z\}])} y^{2g} \prod_{y \in Y_F^+[\{f_1, \ldots, f_w, z\}])} y^{2g} \prod_{(i,j) \in C} (y_{ij}^+y_{ij}^-)^{2g} \right]$$

of $[p]$ contains the factor $\prod_{y \in Y_{B_{DF}(X)}^+} y^{2g}$, and hence is of the desired form (see Figure 3.7b).
(a) A block $F$ of zeros in $T_w$; the union $C \cup E$ contains a full block $B_{CF} \in \mathcal{B}[X]$

(b) A block of $R_-$, the rest $R_+$, in $T_w$. There is now a full block $B_{DF} \in \mathcal{B}[X]$ with $R_-$ (or zeros) in every position inside and $R_+$ (or zero) in every position outside

Figure 3.7: Applying the inductive hypothesis in $T_w$ results in one of these two situations

Hence each term $\left[ r_z \psi C \prod_{(i,j) \in C} (y_i y_j)^{2g} \right]$ has a representative in $\mathbb{Q}[Y(X)]$ of the desired form.

It remains to show that

$$\left[ r_z \sum_{C \in \mathcal{B}[Y \setminus \{z\}] \cup \{\emptyset\}} \bar{X}_C \prod_{y \in Y_C^e (X \setminus \{z\})} y^{2g} \right]$$

(Figure 3.8) has a representative in $\mathbb{Q}[Y(X)]$ of the desired form; again it suffices to show this for each term separately. Fix $C = V \times (X \setminus \{z\}) \setminus V \in \mathcal{B}[X \setminus \{z\}]$ and consider $r_z \bar{X}_C \prod_{y \in Y_C^e (X \setminus \{z\})} y^{2g}$. Observe that $\deg r_z \bar{X}_C \geq 2gm(m-1) - m + 1 - g(m-1)(m-2)$. By Lemma 3.4.15, $[r_z \bar{X}_C]$ has a representative in $\mathbb{Q}[Y(X)]$ that is a sum of terms of the form $q \prod_{i \in X \setminus \{z\}} (y_i y_i)^{d_i}$, where either $\deg q \geq m(m-1)g - m + 2$, or $d_i \geq 2g \forall i \in X \setminus \{z\}$ (Figure 3.9).
Figure 3.8: The inductive hypothesis gave us a block $C \in \mathcal{B}[X \setminus \{z\}]$ of $R_-$, and $R_+$ everywhere else above the diagonal and away from $z$

<table>
<thead>
<tr>
<th></th>
<th>$z$</th>
<th>$f_1$</th>
<th>$f_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$e_h$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

(a) We can either get the appropriate pattern of $R_+$ and $R_-$ along $z$ to extend $C$ to a full block

(b) ... or we can apply Corollary 3.4.14 to find a full block in the subring $\mathbb{Q}[Y_B^{-1}(X)]$, where $B$ is the extension of the block $C$ as shown above

Figure 3.9: Lemma 3.4.15 puts us in one of these situations
Figure 3.10: Combining the block $G$ (Figure 3.9b) with $R_-$ everywhere in $C$ and $R_+$ elsewhere away from $z$ (Figure 3.8) gives a block of zeros in $\mathcal{B}[X \setminus \{z\}]$, reducing us to the earlier case (Figure 3.2).

In the latter case, the corresponding terms of $[p]$ have the factor

$$\prod_{y \in Y_C^+(X \setminus \{z\})} y^{2g} \prod_{i \in X \setminus \{z\}} (y_{iz} - e^{(i)})^{2g} = \prod_{y \in Y_C^+(X)} y^{2g},$$

where $\tilde{C} = V \times (X \setminus V)$ is the extension of the block $C$ obtained by adding $z$ to the second factor. Hence these terms have the desired form (see Figure 3.9a).

If a term has $\text{deg } q \geq m(m-1)g - m + 2$, then by Corollary 3.4.14 there exist homogeneous polynomials $\gamma_G$, for each $G \in \mathcal{B}[X]$, such that

$$[q] = \left[ \sum_{G \in \mathcal{B}[X]} \gamma_G \prod_{(i,j) \in G} (y_{ij}^c)^{(i,j)2g} \right]$$

(see Figure 3.9b), where $C' = (V \cup Z) \times (X \setminus V)$. This contains the factor $\prod_{(i,j) \in G \setminus X \setminus \{z\}} (y_{ij}^+ y_{ij}^-)^{2g}$. By Remark 3.4.4, $G|_{X \setminus \{z\}} \in \mathcal{B}[X \setminus \{z\}]$, and thus we are reduced to a monomial of the form described in case 1 above (Figure 3.10), which we have already shown how to put in the desired form.

Thus we have found a representative for $[p]$ in $\mathbb{Q}[Y(X)]$ that has the desired form. □
Lemma 3.4.17. Let $B \in B[[n]]$ be a block. Let $p \in \mathbb{Q}[Y(X)]$ be a monomial, and fix $1 \leq i \leq n$. Then $[p]$ is equivalent in $R$ to a sum of terms of the form $[qr]$, where $q \in \mathbb{Q}[y_i^+]$ and $r \in \mathbb{Q}[Y_B^-(X)]$. Such polynomials also exist if we take $B = \emptyset$ (so $r \in \mathbb{Q}[Y^{-}(X)]$).

Proof. We check that the elements of $Y_B^-(X)$ together with the $y_i^+$ generate all of $R = \mathbb{Q}[Y(X)]/I$.

- $[y_{jj}^+] = [y_{ii}^+ - y_{ij}^+ - y_{ij}^-] \text{ for } i < j, (i, j) \notin B \cup \bar{B}$
- $[y_{jj}^+] = [y_{ii}^+ + y_{ji}^+ + y_{ji}^-] \text{ for } i > j, (i, j) \notin B \cup \bar{B}$
- $[y_{jj}^-] = [y_{ij}^+ + y_{ij}^- - y_{ii}^+] \text{ for } i > j, (i, j) \in B \cup \bar{B}$
- $[y_{jj}^+] = [y_{ij}^+ + y_{ij}^- - y_{ii}^+] \text{ for } i < j, (i, j) \in B \cup \bar{B}$
- $[y_{jk}^+] = \frac{1}{2}[y_{jj}^+ + y_{kk}^+] \text{ for all } j \neq k$
- $[y_{jk}^-] = \frac{1}{2}[y_{jj}^+ - y_{kk}^+] \text{ for all } j \neq k$

Lemma 3.4.18. Let $X = [n]$ and let $p \in \mathbb{Q}[Y(X)]$ be a monomial of degree at least $2gn^2 + \frac{1}{2}(n-1)(n-2) - n(n-1)g - 2gn$. Let $C \in B[X]$ be a fixed block. Then we can find homogeneous polynomials $\theta_B$ and $\phi_A \in \mathbb{Q}[Y(X)]$, for each $B \in B[X]$ and $A \in (B[X])^{2g}$, such that

$$[p] = \left[ \sum_{B \in B[X]} \theta_B \prod_{(i,j) \in B} (y_{ij}^{-C(i,j)})^{2g} + \sum_{A \in (B[X])^{2g}} \phi_A \prod_{l=1}^{2g} \prod_{y \in Y_A^-(X)} y \right].$$

Proof. Factorise $p$ as $p_+ p_-$, where $p_+ \in \mathbb{Q}[Y_C^+(X)]$ and $p_- \in \mathbb{Q}[Y_C^-(X)]$. If $\deg p_- \geq n(n-1)g - n + 2$, then we are done by Corollary 3.4.14. Otherwise, let $a$ be the largest integer such that $\deg p_- \geq \frac{1}{2}n(n-1)a - n + 2$, and write

$$[p_-] = \left[ \sum_{B \in B[X]} \psi_B \prod_{(i,j) \in B} (y_{ij}^{-C(i,j)})^{a} \right],$$
where $\psi_B \in \mathbb{Q}[Y(X)]$ are homogeneous polynomials of degree at most $\frac{1}{2}n(n - 1) - n + 1$. Fix $B$, and consider the monomial

$$p_+ \prod_{(i, j) \in B} \left( y_{ij}^{\epsilon C(i, j)} \right)^a = \lambda \prod_{i, j \in X} (y_{ij}^+)^{d_{ij}} \prod_{i, j \in X, i \neq j} (y_{ij}^-)^{d_{ij}}$$

for some $d_{ij}^+ \in \mathbb{N}$. For $i < j \in X$, let $d_{ij} := d_{ij}^+ + d_{ij}^- + d_{ji}^+ + d_{ji}^-$, and let $d_{ii} = d_{ii}^+$. If $d_{ij} \geq 2g$ for all $i, j \in X$ with $i < j$, then this monomial contains the factor

$$\prod_{(i, j) \in B} \left( y_{ij}^{\epsilon C(i, j)} \right)^a \prod_{(i, j) \in B \cup \bar{B}, i \neq j} \left( y_{ij}^{\epsilon C(i, j)} \right)^{2g - d_{ij}}$$

which is in the required form. So it suffices to show that any monomial has a representative in $\mathbb{Q}[Y(X)]$ where in each term, either degree $p_- \geq n(n - 1)g - n + 2$, or $d_{ij} \geq 2g \forall i < j$. We show this by giving a procedure which when applied to a monomial either increases $\deg p_-$ or decreases the quantity $\sum_{d_{ij} < 2g} 2g - d_{ij}$. The procedure is as follows: Suppose $\deg p_- \leq n(n - 1)g - n + 1$. Then

$$\deg p_+ \geq 2gn^2 + \frac{1}{2}(n - 1)(n - 2) - n(n - 1)g - 2gn$$

$$> (2g)\frac{1}{2}n(n - 1) = n(n - 1)g.$$ 

Suppose there's some $i < j$ with $d_{ij} < 2g$. Under the map

$$[y_{ki}^{\epsilon C(k, i)}] \mapsto [y_{ki}^{\epsilon C(k, i)} + y_{ij}^{\epsilon C(i, j)} + y_{jl}^{\epsilon C(j, l)}],$$

in each term of the resulting expression, either the degree of the factor in $\mathbb{Q}[Y(X)]$ has increased, or $2g - d_{ij}$ has decreased. Applying this procedure finitely many times to each monomial puts $[p]$ in the required form.

\[\square\]

**Proposition 3.4.19.** Let $X \subseteq \mathbb{N}$ with $|X| = n$. Let $p \in \mathbb{Q}[Y(X)]$ be a monomial of degree

$$\geq 2gn^2 + \frac{1}{2}(n - 1)(n - 2).$$

Then for each $B = V \times V^c \subseteq \mathcal{B}[X]$ we can find homogeneous


polynomials $\theta_B, \phi_B$ and $\chi_B \in \mathbb{Q}[Y(X)]$ such that

$$[p] = \left[ \sum_{B \in \mathcal{B}(X \cup \{\emptyset\})} \prod_{B \in \mathcal{B}(X \cup \{\emptyset\})} (y_{ij}^+ y_{ij}^-)^{2g} (\theta_B p_{\emptyset} + \phi_B p_{\emptyset}) + \sum_{B \in \mathcal{B}(X \cup \{\emptyset\})} \psi_B \prod_{y \in Y_B^+(X)} y^{2g} \left( \sum_{A \in \mathcal{B}(X \cup \{\emptyset\})} \prod_{y \in Y_B^+(X)} y^{2g} \right) \right], \quad (3.2)$$

where

- $p_{\emptyset} = 0$
- $p_{\{z\}} = (y_{zz}^+)^{2g}$ for $z \in X$
- for $W \subset X$ with $|W| \geq 2$, $p_W$ is defined recursively as an element of $\mathbb{Q}[Y(W)]$ of the form (3.2).

**Proof.**

For $n = 1$, if $p \in \mathbb{Q}[y_{11}^+]$ is a monomial of degree $\geq 2g$, it is in the desired form. ($\checkmark$)

Suppose $n \geq 2$. By Lemma 3.4.16, we can write

$$[p] = \left[ \sum_{B \in \mathcal{B}(X \cup \{\emptyset\})} \psi_B \prod_{B \in \mathcal{B}(X \cup \{\emptyset\})} (y_{ij}^+ y_{ij}^-)^{2g} + \sum_{B \in \mathcal{B}(X \cup \{\emptyset\})} \chi_B \prod_{y \in Y_B^+(X)} y^{2g} \right].$$

As usual, we treat each term separately. Fix a block $B \in \mathcal{B}(X)$ and consider the monomial $\psi_B \prod_{B \in \mathcal{B}(X \cup \{\emptyset\})} (y_{ij}^+ y_{ij}^-)^{2g}$. Suppose $B = \{e_1, \ldots, e_h\} \times \{f_1, \ldots, f_w\}$; then $\deg \psi_B \geq 2gn^2 + \frac{1}{2}(n-1)(n-2)-4gwh$. Factorise $[\psi_B]$ as $[\sum_i \psi_i^h \psi_i^w]$, where for each $i \in X$, $\psi_i^h \in \mathbb{Q}[Y(\{e_1, \ldots, e_h\})]$ and $\psi_i^w \in \mathbb{Q}[Y(\{f_1, \ldots, f_w\})]$ are homogeneous polynomials. By Lemma 3.4.8, for each $i$ either $\deg \psi_i^h \geq 2gh^2 + \frac{1}{2}(h-1)(h-2)$ or $\deg \psi_i^w \geq 2gw^2 + \frac{1}{2}(w-1)(w-2)$; thus by the inductive hypothesis we can either write $[\psi_i^h] = [p_{\{e_1, \ldots, e_h\}}]$ or $[\psi_i^w] = [p_{\{f_1, \ldots, f_w\}}]$ in the form (3.2). Hence

$$\left[ \psi_B \prod_{B \in \mathcal{B}(X \cup \{\emptyset\})} (y_{ij}^+ y_{ij}^-)^{2g} \right] = \left[ (\theta_B p_{\emptyset} + \phi_B p_{\emptyset}) \prod_{B \in \mathcal{B}(X \cup \{\emptyset\})} (y_{ij}^+ y_{ij}^-)^{2g} \right]$$

has the desired form.
Now fix $B \in \mathcal{B}[X] \cup \{\emptyset\}$ and consider the term $\chi_B \prod_{y \in Y_B^+(X)} y^{2g}$. Observe that $\deg \chi_B \geq 2gn^2 + \frac{1}{2}(n-1)(n-2) - n(n-1)g$. By repeated application of Lemmas 3.4.9 and 3.4.17, we can write $[\chi_B] = \prod_{i \in X} (y_{ii}^+)^{2q} \prod_{y \in Y_B^+(X)} y^{2g}$, where $r \in \mathbb{Q}[Y_B^+(X)]$ is a homogeneous polynomial with $\deg r \geq n(n-1)g - n + 2$, and $\theta, \phi \in \mathbb{Q}[Y(X)]$. By Corollary 3.4.14, $[r] = \left[ \sum_{C \in \mathbb{B}[X]} \theta_C \prod_{(i,j) \in C} (y_{ij}^{c(i,j)})^{2g} \right]$, for homogeneous polynomials $\theta_C \in \mathbb{Q}[Y(X)]$. Thus

$$\left[ \phi \prod_{y \in Y_B^+(X)} y^{2g} \right] = \left[ \sum_{C \in \mathbb{B}[X]} \psi_C \prod_{(i,j) \in C} (y_{ij}^+ y_{ij}^-)^{2g} \right]$$

(for some $\phi_C \in \mathbb{Q}[Y(X)]$), which is of the form considered above.

Finally, consider the monomial $\prod_{i \in X} (y_{ii}^+)^{2g} \prod_{y \in Y_B^+(X)} y^{2g}$. Note that $\deg \theta \geq 2gn^2 + \frac{1}{2}(n-1)(n-2) - \frac{1}{2}n(n-1)g - 2gn$. Applying Lemma 3.4.18,

$$[\theta] = \left[ \sum_{D \in \mathbb{B}[X]} \phi_D \prod_{(i,j) \in D} (y_{ij}^{c(i,j)})^{2g} + \sum_{A \in \mathbb{B}[X]^{2g}} \psi_A \prod_{l=1}^{2g} \prod_{y \in Y_{A_l}^+(X)} y \right] .$$

As usual, consider each monomial individually. Fix $D$, and consider

$$\prod_{i \in X} (y_{ii}^+)^{2g} \prod_{y \in Y_B^+(X)} y^{2g} \phi_B \prod_{(i,j) \in D} (y_{ij}^{c(i,j)})^{2g} .$$

This contains the factor $\prod_{(i,j) \in D} (y_{ij}^+ y_{ij}^-)^{2g}$, and so is of the form discussed above. The remaining term

$$\prod_{i \in X} (y_{ii}^+)^{2g} \prod_{y \in Y_B^+(X)} y^{2g} \left( \sum_{A \in \mathbb{B}[X]^{2g}} \psi_A \prod_{l=1}^{2g} \prod_{y \in Y_{A_l}^+(X)} y \right)$$

is already in the form (3.2).

\[ \square \]

### 3.5 Proof of the main theorem

**Theorem 3.5.1.** For each $\phi \in \Phi(G)$, let $k_{\phi}$ be a nonnegative integer. Then the cohomology class $\prod_{\phi \in \Phi(G)} (c_1(L_\phi))^{k_{\phi}} \in H^2 \sum_{\phi} k_{\phi}(S_\phi(t); \mathbb{Q})$ vanishes whenever $\sum_{\phi \in \Phi(G)} k_{\phi} \geq 2gn^2 + \frac{1}{2}(n-1)(n-2)$. 

51
Proof. Let $X = [n]$ and consider the rings $\mathbb{Q}[Y(X)]$ and $R = \mathbb{Q}[Y(X)]/I$. Let $J \subset H^*(S_g(t); \mathbb{Q})$ be the subring generated by the $c_1(L_\phi)$ for $\phi \in \Phi(G)$. Since the relations (3.1) hold in $J$, the map

$$
\pi : R \rightarrow J
$$

$$
[y_{ij}^\pm] \mapsto c_1(L_{\varepsilon^i \varepsilon^j})
$$
defines a ring homomorphism. Consider the element $\prod_{\phi \in \Phi(G)} c_1(L_\phi)^{k_\phi} \in J$. It has a representative $[\prod (y_{ij}^\pm)^{d_{ij}^\kappa}]$ in $R$. Suppose $\sum_\phi k_\phi \geq 2gn^2 + \frac{1}{2}(n-1)(n-2)$. Then by Proposition 3.4.19, $[p]$ is equivalent in $R$ to an expression of the form (3.2). So $\pi(p)$ vanishes in $H^*(S_g(t); \mathbb{Q})$ by Lemma 3.3.5. $\square$
Part II

Part II: Characters of quantisations of Hamiltonian actions of compact Lie groups on symplectic manifolds
Chapter 4

Characters of quantisations of Hamiltonian actions of compact Lie groups on symplectic manifolds

4.1 Character Formula

Let $G$ and $K$ be compact connected Lie groups of equal rank with $K \subset G$, and choose a common maximal torus $T \subset K \subset G$. Write $\mathfrak{t}$, $\mathfrak{k}$ and $\mathfrak{g}$ for the Lie algebras of $T$, $K$, and $G$ respectively. Let $\mathcal{N}_G(T)$ denote the normaliser in $G$ of $T$ and let $W(G) = \mathcal{N}_G(T)/T$ be the Weyl group of $G$. Choose a set $\Phi^+(G)$ of positive roots of $G$, and let $W_G \subset \mathfrak{t}^*$ be the positive Weyl chamber for $G$. Let $\Lambda \subset \mathfrak{t}^*$ denote the weight lattice. For $\phi \in \Phi^+(G)$, let $H_\phi$ denote the hyperplane orthogonal to $\phi$ in $\mathfrak{t}^*$, and let $w_\phi \in W(G)$ be the reflection in this hyperplane. Let $(M, \omega)$ be a compact connected symplectic manifold, and let $G$ act on $(M, \omega)$ in a Hamiltonian manner. Then $T$ acts on $(M, \omega)$ in a Hamiltonian fashion with $T$-equivariant moment map $\mu : M \to \mathfrak{t}^*$; we will assume that the fixed points of this torus action are isolated. Suppose the equivariant cohomology class $[\omega + \mu]$ is integral, choose a prequantisation line bundle $(L, \nabla)$, and let $J$ be a $G$-equivariant almost complex structure.
on $M$ that is compatible with $\omega$. Let $Q(M, \omega, \nabla, L; J) = ind(\bar{\partial} + \bar{\partial}')$ be the quantisation of $(M, \omega, L, \nabla; J)$, and let $\chi$ denote its character. In this section we will give an expression for the character $\chi$, as a sum of quotients of (virtual) $K$-characters. We begin by setting up the equivariant cohomology we need, and by recalling the equivariant index theorem and the localisation theorem which will be our main tools.

### 4.1.1 Review of Equivariant Cohomology

Let $G$ be a compact Lie group acting on a manifold $M$. Let $EG$ be a contractible space on which $G$ acts freely, so that $M \times EG \simeq M$ and the diagonal action of $G$ on $M \times EG$ is free. Form the homotopy quotient $M_G := (M \times EG)/G$.

**Definition 4.1.1.** The equivariant cohomology ring $H^*_G(M)$ is the ordinary cohomology ring $H^*(M_G)$.

Let $H \subset G$ be a subgroup. Then $H$ also acts freely on $EG$, so we can take $EH = EG$ and thus $M_G = M_H/G$. If $p : M_H \to M_G$ denotes the projection, we can pull back classes in $H^*_G(M)$ along $p$ to $H^*_H(M)$.

An alternative approach to defining equivariant cohomology is known as the Cartan model. Define an equivariant differential form to be a $G$-equivariant polynomial on $\mathfrak{g}$ taking values in $\Omega^*(M)$. More precisely, the equivariant differential $k$-forms are elements of $\Omega^k_G(M) = \bigoplus_{k=2i+j}(S^i(\mathfrak{g}^*) \otimes \Omega^j(M))^G$. The equivariant exterior differential $d_G : \Omega^k_G(M) \to \Omega^{k+1}_G(M)$ is given by

$$(d_G \alpha)(X) = d(\alpha(X)) - i_{X_M} \alpha(X),$$

where $\alpha \in \Omega^k_G(M)$, $X \in \mathfrak{g}$, and $X_M$ is the vector field defined by the infinitesimal action of $X$ on $M$. Note that $d_G^2 = 0$ by invariance.

Hence, an equivariant form $\alpha \in \Omega^*_G(M)$ is closed if $d(\alpha(X)) - i_{X_M} \alpha(X) = 0$ for all $X \in \mathfrak{g}$.

**Example 4.1.2 (Equivariant symplectic form).** Let $\omega + \mu$ be the equivariant symplectic form. Then $d_G(\omega + \mu)(X) = d(\omega + \mu)(X) - i_{X_M}(\omega + \mu)(X)$. But $d\omega = 0$ since $\omega$ is the non-equivariant
symplectic form, and \( i_{X_M}(\mu) = 0 \) since \( \mu \) is a 0-form. So \( d_G(\omega + \mu)(X) = d\mu(X) - i_{X_M}\omega(X) \), which is zero by definition of the moment map \( \mu \). So the equivariant symplectic form is equivariantly closed.

**Theorem 4.1.3** (Equivariant de Rham Theorem). Let \( G \) be a compact connected Lie group acting on a manifold \( M \). Then the equivariant cohomology is given by

\[
H^*_G(M) = \frac{\ker d_G}{\text{im} d_G}.
\]

Suppose \( S \subset T \) is a subtorus with Lie algebra \( \mathfrak{s} \). A \( T \)-equivariant form \( \alpha \) is an \( \Omega(M) \)-valued polynomial on \( \mathfrak{t} \); by restriction we can view this as an \( \Omega(M) \)-valued polynomial on \( \mathfrak{s} \) and hence as an \( S \)-equivariant form. If \( \alpha \) is \( T \)-equivariantly closed then \( d(\alpha(X)) - i_{X_M}\alpha(X) = 0 \) for all \( X \in \mathfrak{t} \), in which case we certainly have \( d(\alpha(X)) - i_{X_M}\alpha(X) = 0 \) for all \( X \in \mathfrak{s} \) and so \( \alpha \) is also \( S \)-equivariantly closed.

Let \( c_{1,T}, e_T \) and \( Td_T \) denote the \( T \)-equivariant first Chern, Euler and Todd classes respectively. For a detailed discussion of these equivariant characteristic classes, see [17] or [15].

We are now ready to state the following two theorems, which are the key components of our proof. For more details about these theorems we refer the reader to [15].

**Theorem 4.1.4** (Equivariant Index Theorem). Let \((M, \omega, L, \nabla; J)\) be a \( G \)-manifold, and let \( \chi \) denote the character of the quantisation \( Q(M, \omega, \nabla, L; J) \). Then, in a neighbourhood of \( 0 \in \mathfrak{g} \),

\[
\chi \circ \exp = \int_M e^{c_{1,T}(L)}Td_T(M).
\]

**Theorem 4.1.5** (Atiyah-Bott-Berline-Vergne Localisation Formula). Let \( M \) be a compact, oriented manifold and let the torus \( T \) act on \( M \) with fixed point set \( F \), and let \( \iota : F \to M \) denote the inclusion. Let \( \alpha \) be an equivariantly closed form on \( M \). Let \( NF \) denote the normal bundle to \( F \) in \( M \). Then

\[
\int_M \alpha = \int_F \frac{\iota^*_T \alpha}{e_T(NF)}.
\]
In particular, when $F$ is a finite set of isolated points,

$$
\int_M \alpha = \sum_{p \in F} \frac{\alpha|_p}{e_T(T_pM)}.
$$

### 4.1.2 Actions of the Torus and of a Subtorus

The Equivariant Index Theorem and the Localisation Theorem together give an expression for the character of $Q(M, \omega, \nabla, L; J)$ as a sum of contributions from each fixed point of the torus action on $M$. The fixed point data involved is the $T$-equivariant Chern classes of the line bundle $L$, and the equivariant Todd and Euler classes of the tangent bundle $TM$, all restricted to each fixed point $p$. These classes are all determined by the weights of the torus action on the fibres of the vector bundles above $p$. So we now discuss what the fixed points are and how the torus acts on the fibres above them.

**Lemma 4.1.6.** The equivariant first Chern class $c_{1,T}(L)|_p$ of the fibre of the line bundle $L$ above the point $p \in M$ is the moment map image $\mu(p)$.

**Proof.** We chose our prequantisation line bundle $L \to M$ and connection $\nabla$ such that its equivariant curvature form was equal to the equivariant symplectic form $\omega + \mu$. Restricted to a point $p$ this becomes the image $\mu(p)$. \qed

**Lemma 4.1.7.** The Weyl group $W(G)$ of $G$ acts on the set of fixed points of the $T$ action on $M$.

**Proof.** Let $p \in M$ be a fixed point of the torus action, and let $wT \in W(G)$. We define the action of $W(G)$ on the fixed point set by $wT \cdot p = w \cdot p$. Since the torus action fixes $p$, this doesn’t depend on the coset representative $w$, and so the action is well-defined. We would like to show that $wT \cdot p$ is a fixed point of the torus action. By definition of the Weyl group, if $wT \in W(G)$ and $t \in T$, then $tw = wt'$ for some $t' \in T$. So $t \cdot (w \cdot p) = tw \cdot p = wt' \cdot p = w \cdot (t' \cdot p) = w \cdot p$, so $w \cdot p$ is a fixed point as claimed. \qed
Corollary 4.1.8. The fixed points of the torus action can be partitioned into $W(G)$ orbits.

Lemma 4.1.9. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the weights of the representation of $T$ on the tangent space $T_pM$ to the fixed point $p$, and let $w \in N_G(T)$ be a representative for $wT \in W(G)$. Then the weights of the $T$-representation on $T_{w \cdot p}M$ are $w \cdot \alpha_1, \ldots, w \cdot \alpha_n$.

Proof. Let $\alpha$ be a weight of the torus action on the tangent space $T_pM$, and let $V_\alpha \subset T_pM$ be the weight space of $\alpha$. Let $v \in V_\alpha \subset T_pM$, and let $X \in t$ with $\exp X = t$. By definition this means $t \cdot v = e^{\langle \alpha, X \rangle} v$. We wish to know how $T$ acts on $w \cdot v$. Since $w \in N_G(T)$, we know $w^{-1}tw = t'$ for some $t' \in T$. Suppose $t' = \exp Y$. So $t \cdot (w \cdot v) = (tw) \cdot v = wt' \cdot v = w \cdot (t' \cdot v) = w \cdot (e^{\langle \alpha, Y \rangle} v) = e^{\langle \alpha, Y \rangle} w \cdot v$. But $\exp Y = w^{-1} \exp Xw$, so $Y = w^{-1}Xw$ since the exponential map commutes with the adjoint action. So $\langle \alpha, Y \rangle = \langle \alpha, w^{-1}Xw \rangle = \langle w \cdot \alpha, X \rangle$, so $t \cdot w \cdot v = e^{\langle w \cdot \alpha, X \rangle} w \cdot v$. That is, $w \cdot \alpha$ is a weight of the $T$ action on the tangent space $T_{w \cdot p}M$, as required.

If we apply localisation to the equivariant index theorem, we get a formula for the character of the $G$-representation in terms of torus characters. We combine these torus characters into $K$-characters. Geometrically, this arises as follows.

Let $s \subseteq t$ be the algebra $\{ \zeta \in t | \langle \alpha, \zeta \rangle = 0 \forall \alpha \in \Phi(K) \}$, and let $S = \exp(s)$. Then $S \subseteq T$ is the maximal torus of the centraliser of $K$ in $G$.

Lemma 4.1.10. The fixed manifolds of the $S$-action are preserved by the $K$-action.

Proof. Suppose $x \in M$ is fixed by the $S$ action. We claim that $k \cdot x$ is also fixed by $S$, for all $k \in K$. But $S$ is contained in the centraliser of $K$, so indeed $s \cdot k \cdot x = k \cdot s \cdot x = k \cdot x$. Since $K$ is connected, the $K$-action preserves connected components of the fixed set of $S$.

Corollary 4.1.11. Each fixed manifold $F$ of the $S$-action is a Hamiltonian $K$-space. In particular, $W(K)$ acts on the set of fixed points $p_i$ of the torus action that are contained in $F$, and on the weights of the torus action on $T_{p_i}F$. 

59
Let $p \in M$ be a fixed point of the torus action, and consider the action of $T$ on the tangent space $T_p M$. The tangent space can be broken up into a direct sum of 2-dimensional weight spaces. The point $p$ is contained in some manifold $F$ fixed by the $S$-action. Denote by $NF$ the normal bundle to $F$ in $M$; since the torus action preserves $F$, the decomposition $T_p M = T_p F \oplus N_p F$ respects the weight decomposition.

Recall that an element $\xi$ of a Lie algebra $\mathfrak{k}$ is regular if the dimension of its centraliser is minimal among all centralisers of elements of $\mathfrak{k}$. For Lie algebras of compact Lie groups this means that the centraliser of $\xi$ is a maximal torus of $\mathfrak{k}$. Write $\mathfrak{k}^*_{\text{reg}}$ for the set of regular elements of $\mathfrak{k}^*$, and $t^*_{\text{reg}}$ for the intersection of $\mathfrak{k}^*_{\text{reg}}$ with $t^*$.

Since the action of $K$ preserves $F$, if $\mu(p)$ is in $t^*_{\text{reg}}$ then $T_p F$ contains a copy of $\mathfrak{k}/t$; in particular the set of weights of the representation on $T_p F$ contains either $\phi$ or $-\phi$ for all positive roots $\phi$ of $K$. Note however that unlike Guillemin and Prato in [16] we are not assuming all of the $\mu(p)$ are regular; this allows us to consider actions on non-generic coadjoint orbits. If $\mu(p)$ is not in $t^*_{\text{reg}}$ then it’s fixed by some of the $w_\alpha$, where $w_\alpha \in W(K)$ is the element of the Weyl group that acts on $t^*$ by reflection in the hyperplane orthogonal to $\alpha$. These $\alpha$ are then not necessarily weights of the torus action on $T_p F$.

### 4.1.3 Some Notation

Let $P = \{p_1, \ldots, p_n\}$ be the set of fixed points of the torus action on $M$. For the fixed point $p_i$, let $A_i$ be the set $\{\alpha \in \Phi^+(K) | \langle \alpha, \mu(p_i) \rangle = 0 \}$ of roots $\alpha$ of $K$ such that $\mu(p_i)$ lies on the hyperplane orthogonal to $\alpha$ (and so $w_\alpha$ fixes $\mu(p_i)$). Let $A_i$ be the group generated by the $w_\alpha$, for $\alpha \in A_i$, and let $U_i$ be the set of left cosets of $A_i$ in $W(G)$. If $X$ is an $A_i$-invariant set then we define the action of $U_i$ on $X$ by $[w] \cdot x = w \cdot x$, where $w \in W(G)$, the coset of $A_i$ containing $w$ is denoted $[w]$, and $x \in X$; this action does not depend on the choice of coset representative and so is well defined. For ease of notation we will often drop the square brackets and refer to elements $w \in U_i$. 

60
If $w \cdot p_i = p_j$, we will also write $w \cdot i = j$. Let $P^+$ be the set of fixed points which map into the closed positive Weyl chamber $\overline{W_K}$ under the moment map.

Given a fixed point $p_i \in M$, let $F$ be the fixed manifold of the $S$ action containing $p_i$. Let $\{\alpha_i\}_{j=1}^n$ denote the weights of the torus action on the tangent space to the $K$-orbit of $p_i$ (these form a subset of $\Phi(K)$), let $\{\beta_{ij}\}_{j=1}^{f_i}$ denote the remaining weights of $T_{p_i}F$, and let $\{\beta_{ij}\}_{j=f_i+1}^{n_i}$ denote the weights of $(NF)_{p_i}$. We write $B_i = \{\beta_{ij} | 1 \leq j \leq n_i\}$. We will polarise the weights as follows. Choose a $\xi \in \mathfrak{t}$ such that $\langle \alpha, \xi \rangle = 0$ for all $\alpha \in W(K)$, and $\langle \lambda, \xi \rangle \neq 0$ for all other weights $\lambda$ of the torus action on tangent spaces to fixed points. So $W(K)$ preserves $\xi$. Note that there may be some weights $\beta_{ij}$ of the torus action on the normal bundle to the $K$-orbit of $p_i$ which also happen to be roots of $K$; such $\beta_{ij}$ will have $\langle \beta_{ij}, \xi \rangle = 0$. Define polarised weights

$$\beta^+_{ij} = \begin{cases} \beta_{ij}, & \beta_{ij}(\xi) \geq 0 \\ -\beta_{ij}, & \beta_{ij}(\xi) < 0 \end{cases},$$

and for fixed $i$, let $s_i$ be the number of $j$ with $\beta_{ij}(\xi) < 0$. Notice that we've chosen $\xi$ such that $s_i = s_{w \cdot i}$ for all $w \in W(K)$.

For the fixed point $p_i$, write $\beta_i = \frac{1}{2} \sum_j \beta_{ij}$. Let $\mathcal{B}_i$ be the set $\{w \cdot \beta_{ij} | w \in W(K), \beta_{ij} \in B_i\}$, and let $\mathcal{B}_i^+$ be the set of polarised weights $\{\beta^+ | \beta \in \mathcal{B}_i\}$. Let $\overline{\beta}_i = \sum_{\beta \in \mathcal{B}_i} \beta = -\beta_i$. Let $C_i$ be the set of weights $\beta \in \mathcal{B}_i^+$ that are not equal to $\beta^+_{ij}$ for any $\beta_{ij} \in B_i$; that is, polarisations of all weights in the orbit except for those at the point $p_i$. Let $m_i(\eta)$ be the multiplicity of $e^\eta$ in $\prod_{\gamma \in C_i} (1 - e^{-\gamma})$.

### 4.1.4 The Main Theorem

**Theorem 4.1.12.** Let $G$ be a compact Lie group with maximal torus $T$. Let $(M, \omega, L, \nabla; J)$ be a Hamiltonian $G$-space such that the $T$-action has isolated fixed points. Let $K \subset G$ be a closed connected subgroup with maximal torus $T$. Write $V^K_\lambda$ for the irreducible $K$-
representation of highest weight $\lambda$. Then the character $\chi$ of the quantisation $Q(M, \omega, L, \nabla; J)$ is given by

$$\chi(Q(M, \omega, \nabla, L; J)) = \sum_{F \in \pi_0(M^S)} \sum_{p_i \in F} \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{|F|}} \frac{\sum_{\eta \in \Lambda} (-1)^s m_i(\eta) \chi(V^K_{\rho(p_i)+\bar{\eta}+\eta})}{\prod_{\gamma \in B_+} (1 - e^{-\gamma})}.$$  

(4.1)

**Proof.** The equivariant index theorem (Theorem 4.1.4) tells us that $\chi \circ \exp = \int_M e^{c_1(T)T}d_T(M)$. By localisation with respect to the $T$ action we get

$$\chi \circ \exp = \sum_{p_i \in M_T} \left. \frac{(e^{c_1(T)L}T)(M))}{e_T(T(p_i)M)} \right|_{p_i}.$$  

(4.2)

Let $M^S$ denote the fixed set of the $S$ action on $M$. Since every fixed point $p_i \in M^T$ is contained in $M^S$, we can write $M^T$ as the disjoint union over all $F \in \pi_0(M^S)$ of the fixed sets $F^T$ of the $T$ action on $F$. So

$$\chi \circ \exp = \sum_{F \in \pi_0(M^S)} \sum_{p_i \in F^T} \left. \frac{(e^{c_1(T)L}T)(M))}{e_T(T(p_i)M)} \right|_{p_i}.$$  

(4.3)

At a fixed point $p_i \in F$, we can write $TM|_{p_i}$ as $T_{p_i}F \oplus (NF)_{p_i}$. Hence $Td_T(TM)_{p_i} = Td_T(T_{p_i}F)Td_T((NF)_{p_i})$, and $e_T(TM)_{p_i} = e_T(T_{p_i}F)e_T((NF)_{p_i})$. So

$$\chi \circ \exp = \sum_{F \in \pi_0(M^S)} \sum_{p_i \in F^T} \frac{e^{(c_1(T)L)p_i}Td_T(T_{p_i}F)Td_T((NF)_{p_i})}{e_T((NF)_{p_i})e_T(T_{p_i}F)}.$$  

(4.4)

Recalling the definitions of the equivariant characteristic classes and lemma 4.1.9, we can write

$$e_T(T_{p_i}F) = \prod_j \alpha_{ij} \prod_{k=1}^{f_i} \beta_{ik};$$  

(4.5)

$$Td_T(T_{p_i}F) = \frac{e_T(T_{p_i}F)}{\prod_j (1 - e^{-\alpha_{ij}}) \prod_{k=1}^{f_i} (1 - e^{-\beta_{ik}})};$$  

(4.6)

$$e_T((NF)_{p_i}) = \prod_{l=f_i+1}^{m_i} \beta_l;$$  

(4.7)

$$Td_T((NF)_{p_i}) = \frac{\prod_{l=f_i+1}^{m_i} \beta_l}{\prod_{l=f_i+1}^{m_i} (1 - e^{-\beta_l})};$$  

(4.8)
\[ e^{(c_1\tau(L)|_F)_{p_i}} = e^{\mu(p_i)}. \]

(4.9)

And so our formula becomes

\[
\chi \circ \exp = \sum_{F \in \pi_0(M_S)} \sum_{p_i \in F} e^{\mu(p_i)} \prod_j \left( 1 - e^{-\alpha_{ij}} \right) \prod_{k=1}^{m_i} \left( 1 - e^{-\beta_{ik}} \right).
\]

(4.10)

We now organise this formula into characters of representations of \( K \). By Lemma 4.1.7, \( W(K) \) acts on the fixed points of the fixed manifolds \( F \). This action partitions the fixed points contained in \( F \) into \( W(K) \)-orbits; each orbit contains \( |U_i| \) points, and has a unique representative \( p_i \) whose image \( \mu(p_i) \) lies in the closed positive Weyl chamber \( \overline{W_K} \). For given \( F \), the set of such orbit representatives is \( F \cap P^+ \).

Recall from Lemma 4.1.9 that the action of \( w \in W(K) \) takes weights of the tangent space at \( p_i \) to weights of the tangent space at \( w \cdot p_i \). So we can write our formula more suggestively as

\[
\chi \circ \exp = \sum_{F \in \pi_0(M_S)} \sum_{p_i \in F \cap P^+} \sum_{w \in U_i} e^{w \cdot \mu(p_i)} \prod_j \left( 1 - e^{-w \cdot \alpha_{ij}} \right) \prod_{\beta \in B_i} \left( 1 - e^{-w \cdot \beta} \right).
\]

(4.11)

Recall that an irreducible \( K \)-character has the form \( \sum_{w \in U_i} e^{w \cdot \lambda} \prod_{\alpha \in \Phi^+(K)} \left( 1 - e^{-w \cdot \alpha} \right) \), where the \( \alpha \) are the positive roots of \( K \) whose inner product with \( \lambda \) is non-zero. (When \( \lambda \) is regular, this is all the positive roots and this becomes the more familiar \( \sum_{w \in W(K)} e^{w \cdot \lambda} \prod_{\phi \in \Phi^+(K)} \left( 1 - e^{-w \cdot \phi} \right) \).) So to get irreducible \( K \)-characters, we would like to rewrite

\[
\sum_{w \in U_i} \frac{e^{w \cdot \mu(p_i)}}{\prod_j \left( 1 - e^{-w \cdot \alpha_{ij}} \right) \prod_{\beta \in B_i} \left( 1 - e^{-w \cdot \beta} \right)}
\]

(4.12)

as

\[
\sum_{w \in U_i} \frac{e^{w \cdot \mu(p_i)}w \cdot R}{\prod_j \left( 1 - e^{-w \cdot \alpha_{ij}} \right) \prod_{\beta \in B_i} \left( 1 - e^{-w \cdot \beta} \right)}
\]

(4.13)

where \( R = \sum_{\tau \in E} e^{\tau} \), for \( E \) a finite subset of the integral weight lattice, and \( c_\tau \in \mathbb{Z} \). Equivalently, we wish to rewrite \( \left( \prod_{\beta \in B_i} (1 - e^{-\beta}) \right)^{-1} \) as \( R/H \) with \( H \) required to be \( W(K) \)-invariant. One \( W(K) \)-invariant set we have is the set \( B_i \) of all weights of the torus action on the spaces \( T_{w \cdot p_i} M/T_{p_i}(K \cdot p_i) \) for \( w \) ranging over \( W(K) \). In fact, since we chose \( \xi \) to be fixed
by $W(K)$, we know $\langle w \cdot \beta, \xi \rangle = \langle \beta, \xi \rangle$ and so $W(K)$ also preserves the set $\mathcal{B}_i^+$ of polarised weights. Let us therefore take

$$H = \prod_{\gamma \in \mathcal{B}_i^+} (1 - e^{-\gamma}). \quad (4.14)$$

Thus we find that

$$R = \frac{\prod_{\gamma \in \mathcal{B}_i^+} (1 - e^{-\gamma})}{\prod_{\beta \in \mathcal{B}_i} (1 - e^{-\beta})}. \quad (4.15)$$

Consider $\beta \in B_i$. If $\beta = \beta^+$, then $\beta \in \mathcal{B}_i^+$, so the corresponding terms cancel. If $\beta = -\beta^+$, then $-\beta \in \mathcal{B}_i^+$, so we can replace the corresponding terms with a $-e^\beta$. By definition the number of $\beta \in B_i$ with $\beta = -\beta^+$ is $s_i$, and so we are left with

$$R = (-1)^{s_i} e^{\sum_{\beta \in B_i; \beta = -\beta^+} \beta} \prod_{\gamma \in C_i} (1 - e^{-\gamma}). \quad (4.16)$$

where $C_i$ is the set of all polarised weights $\gamma \in \mathcal{B}_i^+$ of the torus action on the tangent spaces to the whole orbit of fixed points such that neither $\gamma$ nor $-\gamma$ is in $B_i$; that is, neither $\gamma$ nor $-\gamma$ is a weight of the torus action on the tangent space $T_{p_i} M$ at the particular fixed point $p_i$. Write $\overline{\beta}$ for $\sum_{\beta \in B_i; \beta = -\beta^+}$. So we have rewritten (4.12) as

$$\sum_{w \in U_i} w \cdot \left( \frac{e^{\mu(p_i)(-1)^{s_i} e^\overline{\beta} \prod_{\gamma \in C_i} (1 - e^{-\gamma})}}{\prod_{\gamma \in \mathcal{B}_i^+} (1 - e^{-\gamma})} \right). \quad (4.16)$$

**Lemma 4.1.13.** Let $A_i \subset \Phi^+(K)$ be the set $\{ \alpha \in \Phi^+(K) | \langle \alpha, \mu(p_i) \rangle = 0 \}$, and let $A_i$ be the subgroup of $W(K)$ generated by the reflections $w_\alpha$ for $\alpha \in A_i$. Then

$$\sum_{w \in A_i} \frac{1}{\prod_{\alpha \in A_i} (1 - e^{-w \cdot \alpha})} = 1. \quad (4.16)$$

**Proof.** Let $s_w$ denote the number of $\alpha$ in $A_i$ such that $w \cdot \alpha$ is a negative root of $K$, and let $\rho_{A_i}$ be the sum of the elements of $A_i$. Then

$$\sum_{w \in A_i} \frac{1}{\prod_{\alpha \in A_i} (1 - e^{-w \cdot \alpha})} = \frac{\sum_{w \in A_i} (-1)^{s_w} e^{w \cdot \rho_{A_i}}}{\prod_{\alpha \in A_i} (e^{\alpha/2} - e^{-\alpha/2})}. \quad (4.16)$$

We claim that this is equal to 1. Let $\Psi \subset A_i$ be a set of simple roots (so that each $\alpha \in A_i$ can be expressed uniquely as a non-negative integer combination of the $\psi \in \Psi$). Let
\( x = (x_1, \ldots, x_{|A_i|}) \in \{-1, 1\}^{A_i} \), and consider the term \( X = e^{\sum x_i \alpha_i/2} \) in the expansion of the product in the denominator.

If for all \( i, j, k \) for which \( \alpha_i = \alpha_j + \alpha_k \) and \( x_j = x_k \) we also have \( x_i = x_j \), then \( \sum x_i \alpha_i/2 = w \cdot \rho_{A_i} \) for some \( w \in A_i \), and the coefficient of \( X \) in the denominator is \( \prod x_i = (-1)^{s_w} \). Every \( e^{w \cdot \rho_{A_i}} \) appears once in the denominator.

Otherwise, suppose there are \( i, j, k \) with \( \alpha_i = \alpha_j + \alpha_k \) and \( x_j = x_k \) but \( x_i = -x_j \). Then there’s another term \( \bar{X} = e^{\sum \bar{x}_i \alpha_i/2} \) where \( \bar{x}_l = -x_l \) for \( l = i, j, k \) and \( \bar{x}_l = x_l \) otherwise. Then \( X \) and \( \bar{X} \) both appear once in the expansion of the denominator, each with opposite sign and so they cancel.

Thus we are left with \( \prod_{\alpha} (e^{\alpha/2} - e^{-\alpha/2}) = \sum_w (-1)^{s_w} e^{w \cdot \rho_{A_i}} \), which proves the lemma.

By multiplying (4.16) by \( \sum_{w \in A_i} (\prod_{\alpha \in A_i} (1 - e^{-w \cdot \alpha}))^{-1} \), we get
\[
\sum_{w \in W(K)} w \cdot \left( \frac{e^{u(p_i)(-1)^{s_i} e^\bar{\gamma}_i \prod_{\gamma \in C_i} (1 - e^{-\gamma})}}{\prod_{\phi \in \Phi^+(K)} (1 - e^{-\gamma})} \right)_{\prod_{\gamma \in B_i^+(1 - e^{-\gamma})},}
\]
which is a quotient of \( K \)-characters, where the \( K \)-character of highest weight \( \mu(p_i) + \bar{\gamma}_i + \lambda \) appears with multiplicity equal to the multiplicity of \( e^\lambda \) in \( (-1)^{s_i} \prod_{\gamma \in C_i} (1 - e^{-\gamma}) \). Write \( V^K_{\lambda} \) for the irreducible \( K \)-representation of highest weight \( \lambda \), and \( m_i(\eta) \) for the multiplicity of \( e^\eta \) in \( \prod_{\gamma \in C_i} (1 - e^{-\gamma}) \). Then summing over all orbits we get
\[
\chi(Q(M, \omega, \nabla, L; J)) = \sum_{F \in \pi_0(M^S)} \sum_{p_i \in F \cap P^+} \sum_{\eta \in A} (-1)^{s_i} m_i(\eta) \chi(V^K_{\mu(p_i)+\bar{\gamma}_i+\eta})_{\prod_{\gamma \in B_i^+(1 - e^{-\gamma})},}
\]
which is equation (4.1), as claimed.

\( \square \)

**Remark 4.1.14.** There are at most \( 2^{|C_i|} \) points \( \eta \in \Lambda \) where \( m_i(\eta) \) is non zero.

**Remark 4.1.15.** Although the proof of our main theorem only requires the \( T \) action, considering the \( S \) action as well helps with the geometric intuition. Consider one connected component \( F \in \pi_0(M^S) \) of the fixed set of the \( S \)-action. By Corollary 4.1.11 \( F \) is a Hamiltonian \( K \)-space, so the character of the quantisation of the restriction of \( L \) to \( F \) is given
by $\int_F e^{\gamma(L)} Td_T(F)$. This is an integral of $S$-equivariant forms, so we can apply localisation with respect to the $S$ action on $F$ to get $\sum_{p \in F} \frac{e^{\gamma(L)} Td_T(T_pF)}{e_T(T_pF)}$. These terms, for each $F \in \pi_0(M^S)$, appear in the formula (4.4) for the full character $\chi \circ \exp$, each multiplied by $Td_T(NF)_p/e_T(NF)_p$. Geometrically, this tells us that the way a few $K$-characters combine to form a $G$-character is a consequence of the way that the fixed manifolds of the $S$-action – each of which is a $K$-space – are embedded in the $G$-space.

4.1.5 Example

Example 4.1.16. Let $G = SU(3)$ and let $K \subset G$ be $S(U(2) \times U(1))$. Denote by $\alpha, \beta$ and $\gamma$ the positive roots of $G$, where $\beta = \alpha + \gamma$, and let $\alpha$ be the positive root of $K$. Choose $\xi$ such that the polarised roots are the positive roots. Let $\nu = \beta + \gamma$, and let $M$ be the coadjoint orbit $G.\nu \cong G/K$. In this case the moment map is the inclusion $i : G \cdot \nu \hookrightarrow g^*$ composed with the projection $\pi : g^* \to t^*$.

The centraliser of $K$ in $G$ is the subtorus $S$ consisting of matrices of the form

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix},$$

for $\theta \in [0, 2\pi)$. The $T$ action on $G/K$ has three fixed points: their moment map images are $\nu, w \cdot \nu$, and $w^2 \cdot \nu$ where $w$ is one of the rotations in $W(G)$. Let’s assume $w \cdot \nu$ is in the dominant Weyl chamber $W_K$. These fixed points lie in the two fixed manifolds of the $S$-action: the point $\nu$, and the copy of $K/T \cong \mathbb{P}^1$ which maps to the edge between $w \cdot \nu$ and $w^2 \cdot \nu$ under the moment map. Each fixed manifold contains just one $W(K)$-orbit of fixed points.

For the fixed manifold $F \cong \mathbb{P}^1$, the weights of tangent spaces are $-\gamma$ (at $T_{w\cdot\nu}G/K$) and $-\beta$ (at $T_{w^2\cdot\nu}G/K$). So the denominator will be $(1 - e^{-\beta})(1 - e^{-\gamma})$. Each fixed point in this $W(K)$-orbit has one negative weight on its tangent space that’s not a root of $K$, so $s_i = 1$. 66
Here $\overline{\beta}_i$ is the sum of the negative weights at $w \cdot \gamma$, which is $-\gamma$. Recall that $C_i$ is the set of weights $\delta$ of the $T$ action on tangent spaces to $M$ at fixed points of the orbit of $p_i$ such that neither $\delta$ nor $-\delta$ is a weight of the tangent space at $p_i$, where $p_i$ is the fixed point in the dominant Weyl chamber for $K$; in this case $p_i$ is $w \cdot \nu$ and $C_i$ is $-\beta$. Thus $m_i(\eta)$ is the multiplicity of $e^\eta$ in $1 - e^{-\beta}$, so

$$m_i(\eta) = \begin{cases} 1, & \eta = 0 \\ -1, & \eta = -\beta \\ 0, & \text{otherwise.} \end{cases}$$

At the point $p$, the weights of the tangent space are $\beta$ and $\gamma$, so the denominator is again $(1 - e^{-\beta})(1 - e^{-\gamma})$. Both weights are positive so $s_i = 0$ and $\overline{\beta}_i = 0$. This time there are no weights on tangent spaces in that orbit that are not weights of the tangent space at that point, so $C_i$ is empty and $m_i(\eta)$ is non-zero (and equal to one) only when $\eta = 0$.

So our theorem tells us that

$$\chi(V_G^\nu) = \frac{\chi(V^K_\nu) - \left(\chi(V^K_{w \cdot \nu - \gamma}) - \chi(V^K_{w \cdot \nu - \beta - \gamma})\right)}{(1 - e^{-\beta})(1 - e^{-\gamma})}.$$  

### 4.1.6 Relation to the GKRS formula

In their paper [14], Gross, Kostant, Ramond and Sternberg consider equal rank Lie algebras $\mathfrak{k} \subset \mathfrak{g}$ with $\mathfrak{g}$ semisimple and $\mathfrak{k}$ reductive, and give a formula for the characters of irreducible $\mathfrak{g}$-representations in terms of certain irreducible $\mathfrak{k}$-characters. Since representations of Lie groups are also representations of their Lie algebras, in the case where $G$ is semisimple and the quantisation of the $G$-action on $M$ is an irreducible $G$-representation, we expect our formula to agree with the GKRS formula. We now show that the two formulae do indeed coincide.

At the end of the proof of Theorem 4.1.12, we had expressed the contribution from one
orbit of fixed points as

$$\sum_{w \in W(K)} w \cdot \left( \frac{e^{\mu(p_i)}(-1)^{\delta_i}e^{\beta_i} \prod_{\gamma \in C_i}(1-e^{-\gamma})}{\prod_{\delta \in \Phi^+ (K)}(1-e^{-\delta})} \right) \prod_{\gamma \in B_+} (1-e^{-\gamma}).$$

(4.18)

We may write $B_+^i = \{ \beta \in B_i | \beta = \beta^+ \} \sqcup \{ -\beta | \beta \in B_i, \beta = -\beta^+ \} \sqcup C_i$. So multiplying numerator and denominator of (4.18) by $\prod_{\gamma \in B_+} e^{\gamma/2}$ (which is $W(K)$-invariant), we get

$$\sum_{w \in W(K)} w \cdot \left( \frac{e^{\mu(p_i)}(-1)^{\delta_i}e^{\beta_i} \prod_{\gamma \in C_i}(1-e^{-\gamma})}{\prod_{\delta \in \Phi^+ (K)}(1-e^{-\delta})} \right) \prod_{\gamma \in B_+} (1-e^{-\gamma}).$$

(4.19)

Since $\gamma/2$ is not necessarily a weight of $T$, the numerator of (4.19) is a character not of $K$ but of a covering of $K$; it is however a character of the Lie algebra $\mathfrak{t}$. Thus if we work at the Lie algebra level we have the alternative expression

$$\chi(Q(M, \omega, \nabla, L; J)) = \sum_{F \in \pi_0(M^S)} \sum_{p_i \in F \cap P^+} \frac{\sum_{\eta \in \Lambda} (-1)^{s_i} \tilde{m_i}(\eta) \chi(V^r_{\mu(p_i)+\beta_i+\eta})}{\prod_{\gamma \in B_+^i}(e^{\gamma/2} - e^{-\gamma/2})},$$

(4.20)

where $\tilde{m_i}(\lambda)$ is the multiplicity of $e^\lambda$ in $\prod_{\gamma \in C_i}(e^{\gamma/2} - e^{-\gamma/2})$.

Let $K \subset G$ be compact connected Lie groups of equal rank, and let $G$ act on $G/T$ (by left multiplication). Choose positive roots for $G$, let $\rho_G$ be half their sum, and let $\rho_K$ be half the sum of the positive roots of $K$. Recall that the choice of positive roots determines a complex structure on $G/T$; let $J$ be the associated almost complex structure. Choose $\xi$ such that the polarised roots are the positive roots. Let $\lambda$ be a dominant integral weight, and let $L_\lambda = G \times_T C(\lambda) \to G/T$ be the prequantisation line bundle, where $T$ acts on $C(\lambda)$ with weight $\lambda$ (as in [3]). In this case the quantisation is the irreducible representation of $G$ of highest weight $\lambda$. The images of the fixed points of the $T$ action are the $w \cdot \lambda$, for $w \in W(G)$.

Write $p_w$ for the fixed point whose image $\mu(p_w)$ is $w \cdot \lambda$. Let $\mathcal{W}_G$, $\mathcal{W}_K$ denote the positive Weyl chambers of $G$, $K$ respectively (so $\mathcal{W}_G \subset \mathcal{W}_K$). There is one fixed manifold of the $S$ action containing each of the fixed points whose images lie in the positive Weyl chamber $\mathcal{W}_K$, and each of those fixed manifolds contains a single $W(K)$-orbit of fixed points. As in
Gross, Kostant, Ramond and Sternberg’s paper [14], let \( C \) be the subset of \( W(G) \) mapping \( \mathcal{W}_G \) into \( \mathcal{W}_K \); the fixed manifolds are indexed by \( C \). The weights of the torus action on \( T_p.w \) are \( \{ w \cdot \phi | \phi \in \Phi^+(G) \} \). The orbit \( K \cdot \lambda \) is a generic coadjoint orbit (of \( K \)) and so the set of weights of the \( T \) action on the tangent space to the orbit is \( \{ w \cdot \phi | \phi \in \Phi^+(G) \} \cap \Phi(K) \), while the remaining weights make up the \( \beta_w \). For every fixed manifold \( F, \mathcal{B}^+_i \) is exactly the set \( \Phi^+(G/K) \) of positive roots of \( G \) that are not roots of \( K \). For each weight \( \beta \in \mathcal{B}_w \) of the torus action on the tangent space at a fixed point, either \( \beta \) or \(-\beta\) appears as a weight of the torus action on the tangent space at each other fixed point in the \( W(K) \)-orbit; thus \( C_i \) is empty, and so \( m_i(\eta) \) is non-zero only when \( \eta = 0 \) in which case \( m_i(\eta) = 1 \). By definition \( s_c \) is the number of positive roots changed into negative roots under \( c \); that is, \((-1)^{s_c} = \epsilon(c)\).

The weights at \( T_p.G/T \) are \( \{ c \cdot \phi | \phi \in \Phi^+(G) \} \). Note that if \( c \cdot \phi \in \Phi(K) \) then \( c \cdot \phi \in \Phi^+(K) \) since \( C \) maps \( \mathcal{W}_G \) into \( \mathcal{W}_K \). So \( \beta_c = \frac{\sum_{\phi \in \Phi^+(G)} c \cdot \phi - \sum_{\alpha \in \Phi^+(K)} \alpha}{2} = c \cdot \rho_G - \rho_K \). Substituting this all into equation (4.20), we get

\[
\chi(V) = \sum_{c \in C} \epsilon(c) \chi(V_c(\lambda + \rho_G - \rho_K)) \prod_{\phi \in \Phi^+(G/K)} (e^{\phi/2} - e^{-\phi/2})^\epsilon(c),
\]

which is exactly the GKRS formula (equation 5 in [14]).

An alternative approach to obtaining a GKRS-like formula that applies to Lie groups is discussed in Landwebber and Sjamaar’s paper [32].

### 4.2 Multiplicity Formula

In this section, we will give a formula for the multiplicity in \( Q(M, \omega, \nabla, L; J) \) of \( V^K_\lambda \), the irreducible representation of \( K \) of highest weight \( \lambda \). We will compare our formula with the Guillemin-Prato formula from [16].
4.2.1 Derivation of the Multiplicity Formula

Our starting point is the equation

$$\chi \circ \exp = \sum_{F \in \pi_0(M^S)} \sum_{P \in \mathcal{P}} \frac{e^{w(p_i)}}{\prod_{j(1-e^{-w\cdot\beta_j})}} \sum_{p_i \in F \cap P} \prod_{\beta \in B_i} (1-e^{-w\cdot\beta}) \cdot (4.21)$$

from the previous section. Let us start by only considering fixed points belonging to one Weyl orbit. The contribution to formula (4.21) from such a set of points is

$$\sum_{w \in U_i} e^{w \cdot (\mu(p_i) - \beta_i - \beta_i)} \prod_{j} (1-e^{-w\cdot\alpha_{ij}}) \prod_{k} (1-e^{-w\cdot\beta_k}) \cdot (4.22)$$

We polarise the $\beta_{ij}$ as above. If $\beta_{ij} = \beta_{ij}^+$, we have $(1-e^{-\beta_{ik}})^{-1} = (1-e^{-\beta_{ik}^+})^{-1}$; if $\beta_{ij} = -\beta_{ij}^+$ then rewrite $(1-e^{-\beta_{ik}})^{-1}$ as $((1)^{s_i}e^{-\beta_{ik}^+}(1-e^{-\beta_{ik}^+}))^{-1}$. By definition the number of $\beta_{ij}$ with $\beta_{ij} = -\beta_{ij}^+$ is $s_i$; thus $((1)^{s_i}e^{-\beta_{ik}^+}(1-e^{-\beta_{ik}^+}))^{-1}$ becomes $(((1)^{s_i}e^{-\beta_{ik}^+}(1-e^{-\beta_{ik}^+}))^{-1}$. As discussed earlier, we have $s_{w(i)} = s_i$, and $w \cdot (\beta_{i}^+ - \beta_i) = \beta_{w(i)}^+ - \beta_{w(i)}$. Thus we can rewrite (4.22) as

$$\sum_{w \in U_i} (1)^{s_i} e^{-w \cdot (\mu(p_i) - \beta_i + \beta_i)} \prod_{j} (1-e^{-w\cdot\alpha_{ij}}) \prod_{k} (1-e^{-w\cdot\beta_k}) \cdot (4.23)$$

Expanding $(1-x)^{-1}$ as $1 + x + x^2 + \cdots$, this gives us

$$\sum_{w \in U_i} (1)^{s_i} e^{-w \cdot (\mu(p_i) - \beta_i + \beta_i)} \prod_{j} (1-e^{-w\cdot\alpha_{ij}}) \prod_{k} (1+ e^{-w\cdot\beta_k^+} + e^{-2w\cdot\beta_k^+} + e^{-3w\cdot\beta_k^+} + \cdots) \cdot (4.24)$$

For $\zeta \in \mathfrak{t}^*$, write $P_i(\zeta)$ for the number of ways to write $\zeta$ as a sum $\sum_j c_j \beta_{ij}^+$, where the $c_j$ are non-negative integers. So the contribution of this $W(K)$-orbit of fixed points is

$$(-1)^{s_i} \sum_{\zeta} P_i(\zeta) \sum_{w \in U_i} w \cdot (e^{\mu(p_i) - \beta_i + \beta_i}) \prod_{j} (1-e^{-w\cdot\alpha_{ij}}) \cdot (4.25)$$
Recall that an irreducible character of $K$ of highest weight $\lambda$ takes the form
\[
\sum_{w \in W(K)} e^{w \cdot \lambda} \prod_{\alpha \in \Phi^+(K)} \left(1 - e^{-w \cdot \alpha}\right).
\]

Note that if $\mu(p_i)$ is regular and $w^{-1} \mu(p_i)$ is in the positive Weyl chamber $W_K$, then the $\alpha_{ij}$ are exactly the $w \cdot \alpha$, for $\alpha \in \Phi^+(K)$. If $\mu(p_i)$ is not regular and lies on a hyperplane orthogonal to the root $\alpha$ of $K$, then neither $\alpha$ nor $-\alpha$ will be a weight on $T_{p_i}(K \cdot p_i)$. As before let $A_i$ be the set $\{\alpha \in \Phi^+(K) \mid \langle \alpha, \mu(p_i) \rangle = 0\}$; notice that this is exactly the set of positive roots $\alpha$ of $K$ such that neither $\alpha$ nor $-\alpha$ is a weight of the torus action on $T_{p_i}(K \cdot p_i)$. Hence $\Phi^+(K) = \{\alpha_{ij}\} \sqcup A_i$. By lemma 4.1.9 we know that $A_{w \cdot i} = \{w \cdot \alpha \mid \alpha \in A_i\}$ for $w \in W(K)$.

Let $A_i < W(K)$ be the subgroup generated by the reflections $w_\alpha$ for $\alpha \in A_i$. Note that $\prod_{\alpha \in A_i}((1 - e^{-\alpha})^{-1} + (1 - e^{\alpha})^{-1})$ is $A_i$-invariant, so the action of $U_i$ on this product is well defined.

Using Lemma 4.1.13 we can rewrite (4.25) as
\[
(-1)^{s} \sum_{\zeta} P_i(\zeta) \sum_{w \in U_i} \left(\frac{w \cdot (e^{\mu(p_i)} - \beta_i^+ + \beta_i^- - \zeta)}{\prod_{k}(1 - e^{-w \cdot \alpha_{ik}})} \sum_{v \in A_i} \frac{1}{\prod_{\alpha \in A_i}(1 - e^{-v \cdot \alpha})}\right).
\]

But since $\Phi^+(K) = \{\alpha_{ik}\} \sqcup A_i$, this can be simplified to
\[
(-1)^{s} \sum_{\nu} P_i(\nu) \sum_{w \in W(K)} \frac{w \cdot (e^{\mu(p_i)} - \beta_i^+ + \beta_i^- - \nu)}{\prod_{\alpha \in \Phi^+(K)}(1 - e^{-w \cdot \alpha})}.
\]

**Remark 4.2.1.** In the previous formula (4.26), the sum over $U_i$ corresponded to a sum over the fixed points in one $W(K)$-orbit. In the formula (4.27), we instead sum over the set of pairs $(p, w)$ where $p = w \cdot p_i$ for $w \in W(K)$. Geometrically, what we have done is replace each point $p$ with several copies of itself, one for each Weyl chamber on whose boundary $\mu(p)$ lies.

Let $Z_i$ denote the set of pairs $(p, w)$ where $p = w \cdot p_i$ for $w \in W(K)$. Note that $|Z_i| = |W(K)|$. We let $Z$ be the union over all $p_i \in P^+$ of the $Z_i$; that is, let $Z$ be the set of pairs $(p, w)$ where $p$ is a fixed point of the $T$ action on $M$ and $w \in W(K)$ is such that
$w^{-1}(\mu(p))$ is in the closed positive Weyl chamber $\overline{W_K}$. (So each regular point appears once; the non-regular points appear once for each Weyl chamber in whose boundaries they lie.)

Equation (4.27) expresses the contribution of this orbit as a sum of irreducible $K$-characters; we would now like to know the multiplicity with which the $K$-character of highest weight $\lambda$ occurs, for each $\lambda \in \mathfrak{t}^*$. Let $\rho_K$ denote half the sum of the positive roots of $K$. We can rewrite (4.27) as

$$(-1)^s_i \sum_{\zeta} P_i(\zeta) \sum_{(p_i,w) \in \mathbb{Z}_i} \epsilon(w) e^{\mu(p_i) - \beta_i^+ + \beta_i - \zeta - \rho_K + w \cdot \rho_K} \prod_{\alpha \in \Phi^+(K)} (1 - e^{-\alpha})$$

(4.28)

The contribution of one $W(K)$-orbit to the multiplicity of $\lambda$ is given by the multiplicity in (4.28) of terms that look like $e^\lambda \prod_{\alpha \in \Phi^+(K)} (1 - e^{-\alpha})$. So write $\lambda = \mu(p_i) - \beta_i^+ + \beta_i - \zeta - \rho_K + w \cdot \rho_K$. Then we can rewrite the contribution from one $W(K)$ orbit as

$$(-1)^s_i \sum_{\lambda} \sum_{(p_i,w) \in \mathbb{Z}_i} \epsilon(w) P_i(-\lambda + \mu(p_i) - \beta_i^+ + \beta_i - \rho_K + w \cdot \rho_K) \frac{e^\lambda}{\prod_{\alpha \in \Phi^+(K)} (1 - e^{-\alpha})}$$

(4.29)

and hence, summing over all Weyl orbits, we get the following formula:

**Theorem 4.2.2.** Let $K \subset G$ be compact Lie groups of equal rank with a common maximal torus $T$. Let $(M,\omega,L,\nabla;J)$ be a compact Hamiltonian $G$-space, and let $Q(M,\omega,\nabla,L;J)$ be its quantisation. Let $\lambda$ be a dominant weight for $K$. Then the multiplicity of the irreducible $K$-representation of highest weight $\lambda$ in the $G$-representation $Q(M,\omega,\nabla,L;J)$ is given by

$$\#(\lambda,Q(M,\omega,\nabla,L;J)) = \sum_{(p_i,w) \in \mathbb{Z}} (-1)^s_i \epsilon(w) P_i(\mu(p_i) + w \cdot \rho_K + \beta_i - \lambda - \rho_K - \beta_i^+) \cdot$$

**Remark 4.2.3.** This is almost the Guillemin-Prato multiplicity formula from [16], generalised to allow fixed points whose images under $\mu$ are not regular. The two formulae do not quite agree; the formula appearing in [16], in our notation, is the following:

$$\#(\lambda,Q(M,\omega,\nabla,L;J)) = \sum_{(p_i,w) \in \mathbb{Z}} (-1)^s_i \epsilon(w) P_i(\lambda + w \cdot \rho_K + \beta_i^+ - \beta_i - \rho_K - \mu(p_i))$$
4.2.2 Examples

We will illustrate our formula using the same examples as in the previous section.

Example 4.2.4 (Generic coadjoint orbits). Suppose $M = G/T$, and denote by $\nu$ the moment map image of the fixed point $p$ that lands in the dominant Weyl chamber. So we can identify $G/T$ with the coadjoint orbit $G \cdot \nu$. Then the set of fixed points is $\{u \cdot p | u \in W(G)\}$, and their moment map images are the $u \cdot \nu$. Write $p_u = u \cdot p$ for $u \in W(G)$. So the set $Z$ is $\{(p_u, w) | u \in W(G), w \in W(K), w^{-1}(u \cdot \nu) \in W_K\}$. The weights of the $T$ action on the spaces $T_p M / T_p(K \cdot p)$ are the roots of $G$ that are not roots of $K$. So $\{\beta_{u_j} \} = \Phi^+(G/K)$, and at the point $p_u$ the $\beta_{u_j}$ are $\{u \cdot \phi | \phi \in \Phi^+(G/K), u \cdot \phi \notin \Phi(K)\}$. For each $u$, $\beta_u = u \cdot \rho_G - w \cdot \rho_K$, and $\beta_u^+ = \rho_G - \rho_K$. Since the $\beta_{u_j}^+$ are the same at each fixed point, we will write $P(\lambda)$ for the number of ways to express $\lambda$ as a sum $\sum_j c_j \beta_j$, where the $\beta_j$ are the positive roots of $G$ that are not roots of $K$, and the $c_j$ are non-negative integers. Finally, $(-1)^s_u \epsilon(w) = (-1)^n$, where $n$ is the number of roots of $G$ that change sign under $u$, and so $(-1)^s_u \epsilon(w) = \epsilon(u)$. So in this case, our formula becomes

$$
\#(\lambda, Q(M, \omega, \nabla, L; J)) = \sum_{u \in W(G)} \epsilon(u) P(u \cdot \nu + u \cdot \rho_G - \lambda - \rho_G).
$$

Remark 4.2.5. Goodman and Wallach proved a branching formula which gives the multiplicity of the irreducible representation of $H$ of highest weight $\lambda$ in the irreducible representation of $G$ of highest weight $\nu$ (Theorem 8.2.1 in [13]). They do not require $G$ and $H$ to be of equal rank, only that the chosen maximal torus of $H$ is contained in that of $G$, but in the case where the two groups do have a common maximal torus their formula coincides with our (4.30) above.

Example 4.2.6. Consider the same non-generic coadjoint orbit we discussed in the previous section. That is, take $G = SU(3)$ and $K = S(U(2) \times U(1))$. Let $\alpha, \beta$ and $\gamma$ be the positive roots of $G$ (with $\alpha + \gamma = \beta$), and take $\alpha$ to be the positive root of $K$. Let $\nu = \beta + \gamma$, and let $M$ be the coadjoint orbit $G \cdot \nu \cong G/K$. Then the torus action has three fixed points: $\nu,$
$w_+\nu$ and $w_-\nu$ (where $w_+$ and $w_-$ are the rotations in $W(G)$ that map $\nu$ into the chambers $W_K$ and $-W_K$ respectively).

Let $u$ be the non trivial element in $W(K)$. Since $w_+\nu$ is in the positive chamber $W_K$, $w_-\nu$ is in the other chamber and $\nu$ is on the boundary wall, the set $Z$ consists of the four elements $(w_+\nu, e), (w_-\nu, u), (\nu, e)$, and $(\nu, u)$. The data at each of these points is as follows:

<table>
<thead>
<tr>
<th>$(p, w) \in Z$</th>
<th>Weights of $T$ action on $T_pM/T_p(K \cdot p)$</th>
<th>$s_i$</th>
<th>$\epsilon(w)$</th>
<th>$\beta_i$</th>
<th>$\beta_i^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(w_+\nu, e)$</td>
<td>$-\gamma$</td>
<td>1</td>
<td>1</td>
<td>$-\gamma/2$</td>
<td>$\gamma/2$</td>
</tr>
<tr>
<td>$(w_-\nu, u)$</td>
<td>$-\beta$</td>
<td>1</td>
<td>-1</td>
<td>$-\beta/2$</td>
<td>$\beta/2$</td>
</tr>
<tr>
<td>$(\nu, e)$</td>
<td>$\beta, \gamma$</td>
<td>0</td>
<td>1</td>
<td>$(\beta + \gamma)/2$</td>
<td>$(\beta + \gamma)/2$</td>
</tr>
<tr>
<td>$(\nu, u)$</td>
<td>$\beta, \gamma$</td>
<td>0</td>
<td>-1</td>
<td>$(\beta + \gamma)/2$</td>
<td>$(\beta + \gamma)/2$</td>
</tr>
</tbody>
</table>

In this case $\rho_K = \alpha/2$. Putting this into our formula, we get

$$-P_\gamma(w_+\nu - \lambda - \gamma) + P_\beta(w_-\nu - \beta - \alpha - \lambda) + P_{\beta, \gamma}(\nu - \lambda) - P_{\beta, \gamma}(\nu - \alpha - \lambda),$$

where $P_A(\psi)$ denotes the number of ways to write $\psi$ as a sum $\sum_{a \in A} c_a a$ where the $c_a$ are in $\mathbb{N}$. Notice that in this example, each $A$ is a linearly independent set of roots, so $P_A$ will always be one or zero. For each point $(p_i, w) \in Z$, we will shade the region of $t^*$ containing those $\lambda$ where $P_i(\lambda + \beta_i^w - \beta_i + \rho_K - w \cdot \rho_K - \mu(p_i)) = 1$ with red lines of positive slope if $(-1)^{s_i}\epsilon(w)$ is $+1$ or blue lines of negative slope if $(-1)^{s_i}\epsilon(w)$ is $-1$. (Thus the regions where
Figure 4.1: The circled points are the only points in $\overline{W_K}$ with a non-zero net multiplicity, showing that they are the highest weights of the four characters of $K$ that appear in $Q(M,\omega,\nabla,L;J)$ with multiplicity one.

cancellations occur will appear hashed in red and blue.) The point $(w_+\nu,e)$ contributes a multiplicity of $-1$ for every $\lambda$ that can be written as $w_+\nu - \gamma - c\gamma$ for $c \in \mathbb{N}$, so we illustrate this with a blue dashed halfline starting at $w_+\nu - \gamma$ and travelling in the $-\gamma$ direction. The point $(\nu,e)$ contributes a multiplicity of $+1$ for every $\lambda$ that can be written as $\nu - b\beta - c\gamma$ for $b,c \in \mathbb{N}$, so this is illustrated with a red shaded cone with vertex $\nu$. The contributions from the other points are shaded in the same way.

The portion of the picture we are interested in is the top half, corresponding to the closed positive Weyl chamber $\overline{W_K}$, since our formula tells us, for an element $\lambda \in W_K$, the multiplicity of the irreducible $K$ representation of highest weight $\lambda$. We can see that most of this region is either unshaded or hashed in red and blue; in fact the only integral weights in $W_K$ that lie in the positive region are the four lying along the top edge of the triangle. Thus we see that our $G$ character breaks up into the four $K$ characters which have the circled weights as their highest weights (see figure 4.1).
Bibliography


