ESTIMATION OF ZEROS FROM RESPONSE MEASUREMENTS

A Dissertation Presented

by

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ABSTRACT

This study examines conditions where one can estimate the zeros of transfer functions from response measurements. It is shown in the literature that the cepstrum can be exploited to estimate the poles and zeros of a system from its response measurements provided that the system is excited at a single point with an input which has a flat log-spectrum and the location of the input is known. Poles and zeros are estimated by curve-fitting an analytical expression of the transfer function to the region of the cepstrum where the contribution of the input is minimal. Since the source and input information cannot be separated completely anywhere in the cepstrum, the approach is always approximate even in the case of noise-free data. This dissertation improves the cepstrum technique such that the requirement on the characteristics of the input is completely eliminated provided that the number of measurements is more than one. Although the improved technique cannot estimate the poles, it can estimate the zeros exactly in the case of noise-free data.

The main shortcoming of the cepstrum approach to the estimation of zeros in the unknown input case is the requirement of a single input. In many civil engineering applications systems are constantly excited by unknown forces such as wind, traffic and passengers that act at many locations so the cepstrum technique is not applicable. A second objective of the dissertation is to devise a technique which can estimate the zeros from response measurements in the case of multiple inputs. The technique that has been developed is for systems whose mass matrices can be approximated as diagonal. Assuming the measured responses are due to collocated forces, the technique makes use of state-space matrices \( A \) and \( C \) which are obtained from stochastic system identification and estimates a surrogate matrix for the missing input matrix \( B \) connected with a collocated distribution of forces. The approach works by forcing two constraints: (1) the off-diagonal terms of the mass matrix are zero, (2) the fact that there is no direct transmission term relating forces to velocity or displacement measurements, namely \( (CA^T)B = 0 \). The technique yields exact results provided that mass matrix is diagonal and the system matrices \( A \) and \( C \) are not truncated and the number of sensors is larger than a certain threshold. Otherwise, the technique is approximate.
ACKNOWLEDGEMENTS

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1 INTRODUCTION

1.1 Research Context

Engineers and scientists, on the way to understand the behavior of physical systems, need representation of the fundamental dynamics in the form of analytical expressions, i.e., they need models. While the form of these expressions can often be obtained directly from physical laws, parameters sometimes need to be obtained by physical observations or by interpreting response measurements taken on the physical system when the system is exposed to some form of a disturbance which may or may not be measured.

In many engineering fields including civil engineering, the dynamics of the systems encountered may be best represented by the system resonances, also known as modes. Each mode can be fully defined by its modal parameters, i.e., the natural frequencies, damping values, mode shapes and modal scaling factors (or modal participation factors). One method that has been widely used in practice to estimate the modal parameters of physical systems is the so called experimental modal analysis (EMA). In EMA technique, the system is artificially excited from particular locations and the applied input(s) and the resulting response which is measured at a number of geometrical locations on the system are processed in certain ways such that the desired modal parameters can be extracted.

Provided that the test structure is linear and its dynamical properties do not change with time (time-invariant), the input \( u \) can be related to the measured output \( y \) through the transfer function matrix, \( G \). A transfer function matrix is a linear mapping in the Laplace domain between inputs and outputs which can be represented as shown in Figure 1-1. An inherent assumption in this definition is that the system is at rest at time zero. In other words, the output does not contain any contribution from the initial condition of the system.

\[
\begin{array}{ccc}
  u(s) & \rightarrow & G(s) \\
  & \rightarrow & y(s)
\end{array}
\]

Figure 1-1 Transfer function matrix, Input-Output Map in Laplace Domain

Each entry of the matrix \( G(s) \), can be written as the ratio of two polynomials in \( s \), with the denominator polynomial being common at each entry. The roots of the denominator polynomial are the poles of the transfer function and the roots of the numerator polynomial, which is unique for each entry, are the zeros of the entry that relates a particular input coordinate to a particular output coordinate. While each entry of the transfer function matrix has its own set of zeros, the transfer function matrix itself or any other partition of the matrix may have zeros too. These zeros are defined as the complex values at which the matrix considered loses its normal rank. Apart from the poles and the zeros, another important parameter that is needed to fully define a transfer function is
the gain. When the common denominator polynomial is written in monic form (the coefficient of the highest order term is one), the coefficient of the highest order term in the numerator polynomial is then the gain.

Once a transfer function model is extracted from measured input and output data, the natural frequencies and associated damping values can be obtained directly from the poles whereas the mode shapes and modal scaling factors can be retrieved from the zeros and the gains. It is opportune to note that when the Laplace variable \( s \) is replaced by \( i\omega \) where \( \omega \) is the frequency variable, the resulting transfer function matrix, \( G(\omega) \), is called frequency response function (FRF), which is just a cut of the transfer function matrix along the imaginary axis in the complex plane.

The relationship between the measured inputs and the corresponding outputs can also be extracted in the time-domain. Provided that the structure is linear, time-invariant and finite dimensional, the input-to-state-to-output connection may be established through the well-known state-space representation which can be characterized by the quadruple matrices \( \{A_c, B_c, C_c, D_c\} \). The matrix \( A_c \) is the state transition matrix, \( B_c \) is the input-to-state matrix, \( C_c \) is the output matrix and \( D_c \) is the direct transmission term. The connection with the transfer function matrix is given by \( G(s) = C_c(sI-A_c)^{-1}B_c + D_c \).

While the poles of the transfer matrix are simply equivalent to a subset of the eigenvalues of \( A_c \), the zeros are a function of the state quadruple.

It should be clear by now that when one has the measured input and output data, it is possible to express the information on the system dynamics in the form of a transfer function matrix in the frequency domain or in the form of a state-space model in the time domain. In some instances such as in civil engineering systems, however, the test structure is naturally excited by ambient vibration and it is difficult and sometimes impossible to measure the excitation. In this particular case, due to the missing information on the input, only part of the modal model can be acquired from the observations. Operational Modal Analysis (OMA) is the term used to refer to EMA in cases where the input is not known deterministically.

In OMA case, one cannot obtain the transfer function matrix in the frequency domain. What can be obtained, however, are the auto- and cross- spectral density functions of the measurements which generally allow one to estimate the poles as well as the arbitrarily scaled mode shapes. From a state-space perspective, one can obtain only the matrices \( \{A_c, C_c\} \) but not \( B_c \) or \( D_c \).

While a transfer function is not available in an OMA, an interesting question is whether one can still extract the zeros. Apparently, if there is no information other than \( \{A_c, C_c\} \), the answer to the question appears to be “NO”. However, if some information on the physical system or on the source of excitation is available, some opportunities appear. For example, if it is known that there is a single input acting on the system and its location is also known, then the measured response may be processed in a particular way which finally enables one to extract the zeros. Or, even when there is no information
available on the excitation, the knowledge of some physical constraints may be exploited, as will be described later, to estimate the zeros.

1.2 Motivation

The reason why we may want to get zeros is because, if we could, then they can be used in a variety of applications such as structural health monitoring, finite element model updating, model validation and control. The use of transfer function zeros in damage identification has been discussed by many researchers. Recently, the work done by Dilena & Morassi (2004) and Reich & Park (2000) indicated that there is significant potential in the zeros as successful system parameters that can be effectively related to damage. A damage localization technique, DDLV, has been recently shown to operate optimally at the zeros of the transfer function of the delta system which relates the input(s) to the difference of the measurements taken on the system before and after damage takes place (Bernal 2007b).

In model updating field, the most widely used feature has been the natural frequencies of the system. The second, but less frequently used feature has been the mode shapes. The use of zeros in model updating, in place of mode shapes, has been investigated by many researchers especially in the last decade and it has been concluded that zeros can be useful alternatives to mode shapes (Rade & Lallement, 1998; Gordis 1999; Jones and Turcotte, 2002; D’ambrogio & Fregolent, 2000, 2003; Lammens et. al, 1993; Mottershead 1998; Mottershead 2001; Nam et. al, 2005; Hanson et al., 2007b).

Another interesting area where the zeros of transfer functions can prove useful is the estimation of modal scaling factors within an OMA framework. As it is well-known, in OMA, one can extract only the arbitrarily scaled mode shapes and misses the modal scaling factors. Given the fact that many available techniques in the structural health monitoring frequently require appropriately scaled mode shapes, the lack of modal scaling factors constitutes a significant limitation in some applications. In order to alleviate this problem, there have been a number of studies aiming at recovering the missing modal scaling factors (Bernal 2004; Parloo et. al, 2001, 2002; Schwarz and Richardson 2003). These studies mainly require additional tests to be performed after changing the structure with a known modification such as adding masses or stiffnesses.

If the zeros of a collocated transfer function are available in addition to the poles and the arbitrarily scaled mode shapes, it can be shown that the modal scaling factors can actually be obtained to within an unknown scalar. In other words, it becomes possible to scale the mode shapes with respect to each other without having to perform additional tests.

The information connected with the poles and the arbitrarily scaled mode shapes are stored in the state-space matrices \( A_c \) and \( C_c \). The information on the input (or modal scaling factors), however, is carried by the input matrix \( B_c \). In other words, if one can obtain \( B_c \), then it is a simple matter to obtain the modal scaling factors. This last point can be illustrated as follows:
Let the state space matrices $A_c \in \mathbb{R}^{2nx2n}$, $B_c \in \mathbb{R}^{2nxr}$, $C_c \in \mathbb{R}^{mx2n}$ are available for an $n$-DOF system. Let also the unscaled and scaled (mass normalized) mode shapes be represented by vectors $(\varphi_i)$ and $(\phi_i)$ for the $i^{th}$ mode. Then, the relationship between the two mode shapes can be accomplished through the $i^{th}$ modal scaling factor $(\alpha_i)$ as,

$$\phi_i = \varphi_i \alpha_i$$  \hspace{1cm} (1.1)

The well-known relationship between the mass normalized mode shapes and the inverse of the mass matrix is

$$\sum_{i=1}^{n} \phi_i \phi_i^T = M^{-1}$$  \hspace{1cm} (1.2)

As explained in detail in Chapter 5, the inverse of the mass matrix is equivalent to the direct transmission term and can be related to the state-space matrices as

$$M^{-1} = C_c A^{-p}_c B_c$$  \hspace{1cm} (1.3)

where $p$ is 0, 1 or 2 depending on whether the measured output is displacement, velocity or acceleration. The solution for the modal scaling factors can be obtained after expressing the state-space matrices in modal coordinates (see Chapter 2) and combining Eqs. (1.1) through (1.3) as shown in the following.

Let $a$, $b$ and $c$ be the state space matrices $A_c$, $B_c$ and $C_c$ expressed in modal coordinates and partitioned such that the complex conjugate entries are separated as indicated below.

$$a = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^* \end{bmatrix}, \quad b = \begin{bmatrix} \Gamma \\ \Gamma^* \end{bmatrix}, \quad c = \begin{bmatrix} \Upsilon & \Upsilon^* \end{bmatrix}$$

Then, Eq. (1.3) can be written as

$$M^{-1} = \sum_{i=1}^{n} \left( \frac{c_{(i,j)} b_{(i,:)}}{a_{p,i}^{p-1}} + \frac{c_{(i,j)}^* b_{(i,:)}}{a_{p,i}^{p-1*}} \right)$$  \hspace{1cm} (1.4)

where subscripts $(;i)$ and $(i,:)$ refer to the $i^{th}$ column or $i^{th}$ row respectively. A comparison of Eqs. (1.1), (1.2) and (1.4) reveals that the solution for the $i^{th}$ modal scaling factor can be realized as

$$\frac{c_{(i,j)} b_{(i,:)}}{a_{p,i}^{p-1}} + \frac{c_{(i,j)}^* b_{(i,:)}}{a_{p,i}^{p-1*}} = \alpha_i \varphi_i \phi_i^T$$  \hspace{1cm} (1.5)
It is shown that once the matrix $B_c$ is available besides the matrices $A_c$ and $C_c$, it is possible to compute the modal scaling factors. If, however, the matrix $B_c$ is available only to within an unknown scalar, then one can only compute the modal scaling factors to within a scalar. The following example is intended to provide more insight to this point and to exemplify the motivation.

**Example:**

Consider an undamped 3-DOF shear frame whose story stiffnesses and story masses are given in consistent units as $\{800, 700, 1000\}$ and $\{1.25, 1.10, 1.35\}$ respectively. It is assumed that there are two displacement sensors located at the first two stories. The poles, the arbitrarily scaled mode shapes defined at the sensor coordinates and the zeros of the collocated transfer function defined at the second mass are assumed to be available as presented in Table 1-1 and Table 1-2.

<table>
<thead>
<tr>
<th>Table 1-1 The poles and the zeros of $G_{22}$</th>
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<tbody>
<tr>
<td>Poles</td>
</tr>
<tr>
<td>$\pm 11.1120i$</td>
</tr>
<tr>
<td>$\pm 32.5867i$</td>
</tr>
<tr>
<td>$\pm 47.9669i$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 1-2 Arbitrarily scaled mode shapes</th>
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</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
</tr>
<tr>
<td>Sensor 1</td>
</tr>
<tr>
<td>Sensor 2</td>
</tr>
</tbody>
</table>

The key idea in estimating the modal scaling factors within to a scalar is to estimate the matrix $B_c$ within to a scalar. This may be achieved by making use of the zeros of the collocated transfer function through the relation

$$C_{c,2} \left( z_i I - A_c \right)^{-1} B_{c,2} = 0, \quad i = 1, 2, ..., 2n - 2$$

(1.6)

where $C_{c,2}$ represents the row of $C_c$ associated with the second mass, $z_i$ is the $i^{th}$ zero of the collocated transfer function and $B_{c,2}$ is the required column of the matrix $B_c$ associated with the second mass. Since there are $2n-2$ zeros ($2 \times 3 - 2 = 4$ zeros are shown in Table 1-2) in the collocated transfer function, Eq. (1.6) produces $2n-2$ equations. The number of unknowns in $B_{c,2}$, however, is $2n$ and at least an additional equation is needed to be able to estimate $B_{c,2}$ to within a scalar. This additional equation is derived from the physics of the problem and is given as
The details and the derivation of Eq. (1.7) can be found in Chapter 5. It should be noted that the total number of equations available is $2n-1$ whereas the number of unknowns (entries of $B_{c,2}$) is equal to $2n$, which makes it possible to compute $B_{c,2}$ to within a scalar. For this example, the estimated $B_{c,2}$ and the true $B_{c,2}$ are given in Table 1-3. It is clear from the table that the solution obtained misses a scalar to the true solution.

Table 1-3 The true and estimated column of the matrix $B_c$

<table>
<thead>
<tr>
<th>True $B_{c,2}$</th>
<th>Estimated $B_{c,2}$</th>
<th>$(\text{True } B_{c,2})/ (\text{Estimated } B_{c,2})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1501</td>
<td>-0.1017 - 0.4209i</td>
<td>0.0814 + 0.3369i</td>
</tr>
<tr>
<td>0.1518</td>
<td>-0.1029 - 0.4258i</td>
<td>0.0814 + 0.3369i</td>
</tr>
<tr>
<td>0.1962</td>
<td>-0.133 - 0.5503i</td>
<td>0.0814 + 0.3369i</td>
</tr>
<tr>
<td>0.1675</td>
<td>-0.1135 - 0.4696i</td>
<td>0.0814 + 0.3369i</td>
</tr>
<tr>
<td>-0.042</td>
<td>0.0284 + 0.1177i</td>
<td>0.0814 + 0.3369i</td>
</tr>
<tr>
<td>0.079</td>
<td>-0.0535 - 0.2214i</td>
<td>0.0814 + 0.3369i</td>
</tr>
</tbody>
</table>

Next, the estimated column, $B_{c,2}$, is used to calculate the modal scaling factors. For that, first, we need to expand $B_{c,2}$ to $B_c$, in other words, the columns of $B_c$ associated with the other measured coordinates has to be obtained. This can be accomplished in several different ways. The approach proposed by Bernal and Tigli (2006), which expands the transfer function matrix defined between a set of input and output channels to a square transfer matrix relating all the measured coordinates has been followed here. The approach uses reciprocity enforced at a number of values of the Laplace variable $s$ to form a set of linear equations from where one solves for additional columns of the matrix $B_c$ corresponding to unloaded coordinates. Once the expansion is performed, one obtains the matrix $B_c$ to within an unknown scalar. Then, it becomes possible to calculate the scaling factors following the formulation given earlier. For the example considered, the true and estimated modal scaling factors are given in Table 1-4.
Table 1-4 Exact and estimated (mass normalized) scaling factors

<table>
<thead>
<tr>
<th></th>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\phi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>True $\alpha$</td>
<td>-0.5437</td>
<td>0.7551</td>
<td>0.7608</td>
</tr>
<tr>
<td>Estimated $\alpha$</td>
<td>0.4477i</td>
<td>0.6217i</td>
<td>0.6263i</td>
</tr>
<tr>
<td>$(\text{True } \alpha)/(\text{Estimated } \alpha)$</td>
<td>1.2145i</td>
<td>1.2145i</td>
<td>1.2145i</td>
</tr>
</tbody>
</table>

1.3 Objectives

This dissertation studies the conditions where one is able to estimate the zeros of a transfer matrix from output measurements only. The problem has been examined in two phases: (1) Single-input multiple-output systems – SIMO-, (2) Multiple-input multiple output systems – MIMO-.

In the first case, the only available technique present in the literature intended to estimate the zeros in the absence of input measurements, cepstral analysis, has been investigated and the main limitations are clarified. It is shown that the technique is successful in extracting the zeros as well as the poles from response measurements provided that there is a single input with a flat and smooth log spectrum and the location of the input is known (Randall and Gao, 1994). The technique employs a nonlinear curve-fitting approach to extract the system parameters from the cepstrum of the response measurements. An important difficulty in the cepstrum technique lies in the fact that an initial set of estimates of the poles and the zeros need to be specified and the segment along the quefrency axis to which the mathematical model is curve-fitted has to be selected. The cepstrum on that segment is desired to have minimal contribution from the input and maximal contribution from the system. Since there is always some contribution from input, the poles and zeros extracted are never exact even when the data is noise-free.

In order to improve the accuracy of the zero estimates obtained in the cepstrum technique the dissertation proposes a simple modification which enables one to extract the zeros exactly (when the data is noise-free) with no restriction on the type of the input provided that more than one coordinate is measured. The modified cepstrum technique proposed in the dissertation can estimate only the zeros of the transfer functions unlike the original cepstrum technique which can also estimate the poles. However, in the proposed modification, the contribution of the any type of input to the cepstrum is completely avoided which eliminates the need to look for the best location in the quefrency axis for curve-fitting and the exact estimates becomes feasible provided that the measured outputs are noise-free.

In the second phase, the zeros of transfer function matrices that relate a set of input coordinate(s) to a set of output coordinate(s) exclusively from response measurements has been investigated. Note that this proposition may appear ill-posed since without an input set the concept of zeros has no meaning. What is intended is to estimate the zeros of
a “collocated” set of input-output coordinates which are (of course) defined by the outputs.

Since civil engineering structures are constantly excited by many unknown forces such as wind, traffic, passengers etc. and their locations are not available; the applicability of the cepstrum technique to which as noted limited to a single input is restricted. The dissertation proposes a technique that can treat multiple input case, even though it is applicable only when the mass matrices can be approximated as diagonal. The technique makes use of state-space matrices $A_c$ and $C_c$ which are obtained from stochastic system identification (Di Ruscio 1996, Van Overschee and Moor 1996) and estimates a surrogate matrix for the missing input to state matrix $B_c$ connected with a collocated distribution of forces. The approach works by forcing two constraints (Bernal, 2006): (1) the off-diagonal terms of the mass matrix are zero, (2) there is no direct transmission term relating forces to velocity or displacement measurements. The technique yields exact results provided that mass matrix is diagonal, the system matrices $A_c$ and $C_c$ are not truncated and the number of sensors is larger than a certain threshold. Otherwise the results obtained are approximate.

1.4 Contributions of the Dissertation

There are mainly two contributions in the dissertation. The first contribution is an enhancement to the cepstrum technique which enables one to extract the zeros exactly (when the data is noise-free) with no restriction on the type of the input provided that there are more than one measurement coordinates.

The second contribution is an approximate method intended to estimate the zeros of transfer function matrices from output-only measurements. The technique, unlike the cepstral analysis which is intended only to SIMO systems, can also estimate the zeros of MIMO systems from output measurements. The zeros that can be estimated are associated with input coordinates that are collocated.

1.5 Organization of the Dissertation

Chapter 1 introduces the research context and the motivation of the dissertation and highlights the objectives and the original contributions of the work presented.

Chapter 2 provides background information that is frequently used throughout the dissertation. It also describes the theory of the zeros for both discrete and continuous time systems and presents the way the zeros are computed for systems that can be defined by a state-space representation.

Chapter 3 presents the cepstrum technique and clarifies its limitations.


Chapter 4 introduces a modification to the cepstrum technique, which enables one to extract the zeros exactly (when the data is noise-free) with no restriction on the type of the input provided that there are more than one measurement coordinates.

Chapter 5 introduces a technique which can estimate the zeros of transfer functions from response measurements in the case of multiple inputs for systems whose mass matrices can be assumed to be diagonal.

Chapter 6 summarizes the conclusions of the dissertation.
2 PRELIMINARIES

The Chapter presents basic mathematical tools and theoretical descriptions used frequently in the dissertation, including state-space and transfer function representations of both continuous- and discrete-time systems. The theory of the zeros of multivariable systems is given for continuous- and discrete-time systems. The chapter concludes by discussing the zeros of sampled discrete-time systems and their connection to the true zeros of the original continuous-time system. In this regard, emphasis is placed on discrete-time systems that are obtained by sampling continuous-time input and output functions where the input(s) is delivered following the zero-order-hold condition.

Although in theory, it is possible to retrieve the continuous time system from a discrete time realization provided that the input is delivered based on a certain hold and both input and output functions are measured, the situation becomes complicated when only the output functions are measured and it becomes no longer possible to obtain the continuous time system completely from the discrete time realization (In fact, the discrete time realization will not be complete anyways). As it is shown in Chapters 3 and 4, under certain conditions, one can estimate the zeros of the discrete time system directly from measured output data. However, there is still a need to transfer the estimated discrete-time zeros to continuous-time zeros, hence the connection between the two sets of zeros needs to be further elaborated as presented at the end of the Chapter.

2.1 Theoretical Background

This section presents the two well-known system descriptions in continuous and discrete time: (1) State-space description, (2) Transfer function description.

2.1.1 Continuous-Time State-Space Description

The state-space description assumes that the structure can be represented by a set of finite number of parameters, which is, strictly speaking, not true for many Structural Health monitoring applications where the true behavior of the structures of interest requires infinitely many parameters. After noting this inherent assumption, a linear finite dimensional structural system subjected to a time varying excitation \( u(t) \), can be described by the following ordinary linear differential equation

\[
M\dot{w}(t) + \zeta \ddot{w}(t) + K w(t) = b_2 u(t)
\]  

(2.1)

where the dot represents differentiation with respect to time, \( w \in \mathbb{R}^{nx1} \) is the displacement vector at the degrees of freedom, \( M \), \( \zeta \) and \( K \) are the mass, damping and stiffness matrices respectively and \( b_2 \in \mathbb{R}^{nxr} \) is a vector describing the spatial distribution of the excitation \( u(t) \in \mathbb{R}^{rx1} \) and \( n \) and \( r \) are the number of degrees of freedom and input excitations respectively. In order to eliminate clutter, we drop explicit reference to time variable \( t \) in the sequel. Taking
\[
\begin{bmatrix}
  x_1 \\
  x_2 
\end{bmatrix} = \begin{bmatrix}
  w \\
  \dot{w}
\end{bmatrix} \quad (2.2)
\]

and substituting Eq. (2.2) into Eq. (2.1) one gets
\[
\begin{bmatrix}
  I & 0 \\
  0 & M
\end{bmatrix} \begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
\end{bmatrix} = \begin{bmatrix}
  0 & I \\
  -K & -\zeta
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} + \begin{bmatrix}
  0 \\
  b_2
\end{bmatrix} u \quad (2.3)
\]

which can conveniently be written as;
\[
Ex = Fx + Gu \quad (2.4)
\]

where \( E, F \in \mathbb{R}^{2nx2n} \) are the continuous-time system matrices, \( G \in \mathbb{R}^{2nxr} \) is the continuous-time state-to-input matrix, \( x = [x_1^T \ x_2^T]^T \in \mathbb{R}^{2nx1} \) is the state vector and integer \( 2n \) is the order of the system. Assuming that the available measurements are linear combinations of the state one can write:
\[
y = C_c x + D_c u \quad (2.5)
\]

where matrix \( D_c \) is zero, unless acceleration is measured at a collocated coordinate, \( C_c \in \mathbb{R}^{mx2n} \) is the state to output matrix and integer \( m \) is the number of measurements. The linear system represented by Eqs.(2.4) and (2.5) is called regular if the matrix \( E \) is nonsingular, and it is called singular or descriptor system if \( E \) is singular. While Eq. (2.4) represents ordinary differential equations in the case of a nonsingular \( E \), it represents differential algebraic equations when \( E \) is singular. For a detailed discussion of differential algebraic equations and their solutions, the reader is referred to Campbell (1980). In this dissertation, we limit examination to cases where the matrix \( E \) becomes non-singular. In these cases, Eq. (2.3) can be written as
\[
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
\end{bmatrix} = \begin{bmatrix}
  0 & I \\
  -M^{-1}K & -M^{-1}\zeta
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} + \begin{bmatrix}
  0 \\
  M^{-1}b_2
\end{bmatrix} u \quad (2.6)
\]

which can also be written as
\[
\dot{x} = A_c x + B_c u \quad (2.7)
\]

where \( A_c \in \mathbb{R}^{mx2n} \) is the continuous-time system matrix, \( B_c \in \mathbb{R}^{2nxr} \) is the continuous-time state-to-input matrix. The expressions given in Eqs. (2.7) and (2.5) constitute the continuous-time state-space description of a linear time invariant system. The triplet \( \{A_c, B_c, C_c\} \) depends on the selection of the state-space variables. Specifically, taking \( x = T x' \), where \( T \) is any invertible matrix with appropriate size, Eqs. (2.7) and (2.5) can also be written as
\[
\dot{x}' = T^{-1}A_cTx' + T^{-1}B_c u \\
y = C_c Tx + D_c u
\] (2.8)

Note that Eqs. (2.8) and (2.9) provide the same input-output map given by Eqs. (2.7) and (2.5).

### 2.1.1.1 Decoupling and the Canonical Forms

One of the widely used canonical forms that produce an uncoupled formulation for Eq. (2.8) can be realized when the matrix \( A_c \) is non-degenerate (degeneracy is defined shortly) by expressing the system matrix as

\[
\Lambda = \Psi^\dagger A_c \Psi
\] (2.10)

where \( \Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n) \), \( \lambda_j \) is the \( j \)th eigenvalue of \( A_c \) \((j = 1,2,...,n)\), \( (*) \) denotes complex conjugation and \( \Psi \) is the matrix collecting all eigenvectors of \( A_c \). Corresponding state-space and output equations can be obtained from Eqs. (2.8) and (2.9) simply by replacing \( T \) with \( \Psi \) as

\[
\dot{Y} = \Lambda Y + \Psi^\dagger B_c u
\] (2.11)

\[
y = C_c \Psi Y + D_c u
\] (2.12)

Eqs. (2.11) and (2.12) are referred to as state-space description in modal coordinates. In this form, the system matrix becomes diagonal composed of poles of the system and the output matrix contains the mode shapes in physical coordinates.

When the matrix \( A_c \) is degenerate, however, it cannot be diagonalized as given in Eq. (2.10). The closest form to diagonalization can be obtained through Jordan blocks. Before moving to Jordan blocks, the definition for the degeneracy is given. Let the characteristic polynomial of the matrix \( A_c \) is given by

\[
\Delta(A_c) = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n
\] (2.13)

which can also be written as

\[
\Delta(A_c) = (\lambda - \hat{\lambda}_1)^{q_1}(\lambda - \hat{\lambda}_2)^{q_2}...(\lambda - \hat{\lambda}_N)^{q_N}
\] (2.14)

where \( \hat{\lambda}_j \) is the \( j \)th eigenvalue of \( A_c \). The exponent \( q_j \) is known as the algebraic multiplicity of the \( j \)th eigenvalue. When \( q_j \) is greater than one, it means that \( j \)th eigenvalue is a repeated eigenvalue. When the \( j \)th root that has an algebraic multiplicity of \( q_j \) is substituted to the following generalized eigenvalue problem
\[ [A_c - \lambda I] \phi = 0 \]  

(2.15)

one may find up to \( q_j \) number of linearly independent vectors \( \phi \) that satisfy Eq. (2.15). The number of vectors, \( \phi_i \), that satisfy Eq. (2.15) is known as the geometric multiplicity of the \( j \)th eigenvalue. The geometric multiplicity is never larger than the algebraic multiplicity but it may be smaller. When the geometric multiplicity of an eigenvalue is smaller than the algebraic multiplicity then there are not enough eigenvectors to form a square eigenvector matrix and thus the spectral decomposition of \( A_c \) as given in Eq. (2.10) is not possible and such systems are known as defective systems while such a matrix \( A_c \) is called degenerate.

The closest thing to diagonalization in the case of defective matrices is the Jordan Canonical Form which is given by

\[
W A_c W^{-1} = \begin{bmatrix}
J_1 & & \\
& \ddots & \\
& & J_m
\end{bmatrix}
\]

(2.16)

where \( J_i \) is known as a Jordan block and it has the form;

\[
J_i = \begin{bmatrix}
\lambda_i & 1 & & \\
& \lambda_i & \ddots & \\
& & \ddots & 1 \\
& & & \lambda_i
\end{bmatrix}
\]

(2.17)

The number of Jordan blocks associated with a given eigenvalue is equal to the geometric multiplicity of the eigenvalue. In other words, there is one Jordan block for each eigenvector. The algebraic multiplicity of the eigenvalues is equal to the sum of the dimensions of the Jordan blocks for that eigenvalue. The size of each Jordan block is equal to one plus the number of generalized eigenvectors that are associated with the selected eigenvector. If an eigenvalue has algebraic multiplicity \( q \) and geometric multiplicity \( p \) one can find \( q-p \) generalized eigenvectors. For each one of the \( p \) eigenvectors there may be none, one, or more generalized eigenvectors. The generalized eigenvector of order \( i \), \( \phi^{(i)} \) for eigenvector \( \phi \) is a vector that satisfies;

\[
[A_c - \lambda I] \phi^{(i)} = \phi^{(i-1)}
\]

(2.18)

where \( \phi^{(0)} = \phi \). The generalized eigenvector is not unique (not even to within a scalar) because the coefficient matrix is rank deficient so there are infinite solutions – in practice one often uses the one with minimum norm.
Since the dissertation deals with non-degenerate matrices and the study of degenerate matrices can take us far from the main path of the work presented here, we close the discussion of degenerate matrices at this point.

### 2.1.2 Discrete-Time State-Space Description

The solution to Eq. (2.7) is given by the well known expression

$$x(t) = e^{A_c t} x_0 + \int_0^t e^{A_c (t-\tau)} B_c u(\tau) d\tau$$  \hspace{1cm} (2.19)

when Eq. (2.19) is substituted into Eq. (2.5), we get

$$y(t) = C_c e^{A_c t} x_0 + C_c \int_0^t e^{A_c (t-\tau)} B_c u(\tau) d\tau + D_c u(t)$$  \hspace{1cm} (2.20)

where $x_0$ is the initial state. Since experimental data from structural vibration tests are usually obtained in digital form what is fitted in practice to the data is a realization in discrete time given by

$$x_{k+1} = A_d x_k + B_d u_k$$  \hspace{1cm} (2.21)

$$y_k = C_d x_k + D_d u_k$$  \hspace{1cm} (2.22)

The relation between the discrete time matrices $A_d$ and $C_d$ and their continuous time counterparts $A_c$ and $C_c$ follow by inspection of Eqs. (2.19)-(2.22) as,

$$A_d = e^{A_c \Delta t}$$  \hspace{1cm} (2.23)

$$C_d = C_c$$  \hspace{1cm} (2.24)

The connections between the discrete-time matrices $B_d$ and $D_d$ and the continuous-time matrices $B_c$ and $D_c$ depend on the inter-sample behavior of the input. For the common zero-order-hold assumption (input is assumed constant between time steps), one has

$$B_d = (A_d - I) A_c^{-1} B_c$$  \hspace{1cm} (2.25)

$$D_d = D_c$$  \hspace{1cm} (2.26)

Other relations, corresponding to a half-step forward shift of zero-order-hold (SZOH), to a linear variation of the input (FOH), to a band limited hold (BLH) where the input is assumed to be sampled as modulated train of Dirac impulses, as derived in Bernal (2007a), are shown in Table 2-1.
Table 2-1 Summary of discrete to continuous transfer relationships

<table>
<thead>
<tr>
<th>Inter-sample Assumption</th>
<th>Matrices in continuous time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$B_c$</td>
</tr>
<tr>
<td>SZOH</td>
<td>$A_c(A_d - I)^{-1}A_d^{-0.5}B_d$</td>
</tr>
<tr>
<td>FOH</td>
<td>$A_c^2(A_d - I)^{-2}B_d\Delta t$</td>
</tr>
<tr>
<td>BLH</td>
<td>$A_d^{-1}B_d\Delta t^{-1}$</td>
</tr>
</tbody>
</table>

By explicitly carrying out the recursion in Eqs. (2.21) and (2.22), one finds that the input – output relationship in discrete time can be written as:

$$y_k = C_dA_d^kx_0 + \sum_{j=1}^{k} C_dA_d^{k-j}B_du_j$$

(2.27)

where the terms $C_dA_d^kB_d$ are known as the discrete time Markov parameters. As can be seen, the discrete time Markov parameters are a function of the inter-sample behavior of the input. The collection of the Markov parameters for increasing $k$ computed with the ZOH premise constitutes the pulse response of the system.

2.1.3 Transfer Function

The transfer function description of a continuous-time state-space system can be readily obtained by taking a Laplace transform of Eq. (2.7) which yields,

$$sx(s) - x_0 = A_c x(s) + B_c u(s)$$

(2.28)

solving for the state vector, one gets

$$x(s) = (Is - A_c)^{-1}x_0 + (Is - A_c)^{-1}B_c u(s)$$

(2.29)

Similarly, taking a Laplace transform of the output equation (Eq. (2.5)) yields

$$y(s) = C_c x(s) + D_c u(s)$$

(2.30)

The input-output map in the Laplace domain, then, can be written as

$$y(s) = C_c (Is - A_c)^{-1}x_0 + \left( C_c (Is - A_c)^{-1}B_c + D_c \right) u(s)$$

(2.31)

The expression
\[
G(s) = C_c (I_s - A_c)^{-1} B_c + D_c
\]  

(2.32)

is called the transfer-function matrix and, as can be seen, it gives the relation between the input and output in the Laplace domain for the case \( x_0 = 0 \) (or after steady state conditions are realized).

### 2.1.4 Transfer Function in Discrete Systems

Since the definition of the transfer function for discrete-time systems involves \( z \)-transforms, an introduction to \( z \)-transforms follows first.

#### 2.1.4.1 Z-Transform

The \( z \)-transform, \( Y(z) \) of a sequence \( y(n) \) is defined as

\[
Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}
\]  

(2.33)

where \( z \) is a complex variable. The inverse \( z \)-transform relation is given by the following contour integral

\[
y(n) = \frac{1}{2\pi i} \oint_{C} Y(z) z^{n-1} dz
\]  

(2.34)

where \( C \) is a counterclockwise closed contour in the region of convergence of \( Y(z) \) and encircling the origin of the \( z \)-plane. The region of convergence is the set of \( z \) values for which \( Y(z) \) converges. A powerful observation is that \( Y(z) \) is analytic at every \( z \) within the region of convergence. The connection between the \( z \)-transform and the discrete Fourier transform can be established by expressing the complex variable \( z \) in polar form, \( z=re^{j\omega} \), in Eq. (2.33) which produces

\[
Y(re^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n) r^{-n} e^{-j\omega n}
\]  

(2.35)

Eq. (2.35) indicates that the \( z \)-transform of \( y(n) \) is equal to the discrete Fourier transform of \( y(n)r^{-n} \). Therefore, for \( |z| = 1 \) (on the unit circle), the \( z \)-transform is identical to the discrete Fourier transform. For finite-length sequences, the discrete Fourier transform is given by

\[
Y(k) = Y\left(e^{j\omega}\right) = \sum_{n=0}^{N-1} y(n) e^{-j(2\pi/N)kn}
\]  

(2.36)
where \( N \) is the length of the data sequence and \( Y(k) \) is the discrete Fourier transform of the sequence \( y(n) \). One of the properties of the \( z \)-transforms that is used in the derivation of the pulse transfer function is the shift property, namely

\[
Z \left[ y(n + \Delta n) \right] = z^n Y(z)
\]

(2.37)

where \( Z \) denotes the \( z \)-transform. Another interesting property of the \( z \)-transforms which has been utilized in this dissertation is the derivative property, namely

\[
Z[ny(n)] = -z \frac{dY(z)}{dz}
\]

(2.38)

### 2.1.4.2 Pulse Transfer Function

Structural systems that are of main interest in this dissertation are continuous-time systems, so are the inputs and the resulting responses. However, when the inputs and outputs that are analog originally are measured, they are passed through an analog to digital converter and a sampled discrete version of the analog signal is obtained. Therefore, the information on the input and the output becomes available only at discrete time stations. If, however, the input is delivered based on a hold, then its behavior between the time stations is also available. The relation between the discrete input and output data can be established either in the form of a discrete-time state-space description as outlined in Section 2.1.2 or more directly through a transfer function which relates the \( z \)-transform of the discrete output data to the \( z \)-transform of the discrete input data. The transfer function defined that way is known as pulse-transfer function.

The poles of the pulse-transfer function are related to the poles of the original continuous-time system through the standard mapping used to connect the Laplace domain (variable \( s \)) to \( z \)-domain, namely, \( z = e^{s \Delta t} \) where \( \Delta t \) is the sampling period, the time duration between the two consecutive time stations of the measurements. However, the same mapping cannot be used to relate the zeros of the two systems. In fact, there is no closed form equation to relate the zeros of the sampled system to that of the continuous time directly (Astrom et.al, 1984; Hagiwara et.al, 1993). Besides, the discrete-time system and the associated continuous time systems might have different number of zeros.

At this point, it can be argued that if a discrete-time system realization \( \{A_d, B_d, C_d, D_d\} \) can be obtained from the measured input and output data, one can actually transfer the discrete-time realization to continuous-time and obtain the matrices \( \{A_c, B_c, C_c, D_c\} \) through the formulations given in Section 2.1.2 provided that the input satisfies a certain hold that is known by the user. Then, the zeros of the continuous time system can be computed in a standard way as it is outlined in Section 2.2. Although this is true, the real situation we are interested in the dissertation is the case where the input is not measured. In that case, one can neither estimate a discrete-time system realization \( \{A_d, B_d, C_d, D_d\} \)
nor calculate pulse-transfer function due to the missing information on the input. However, as it is shown in Chapters 3 and 4, it may be possible to estimate the poles and the zeros of the pulse-transfer from the cepstrum of the measured output data. Therefore, it proves useful to examine the form of the pulse transfer function and the connection of its zeros to the zeros of the continuous-time transfer function.

It has been shown that the location of the zeros of the discrete-time system depends on the inter-sample behavior of the input, the sampling period $\Delta t$, and the relative degree of the continuous time transfer function $G(s)$ (Astrom et.al, 1984). We close the section by noting that a detailed discussion regarding the connection between the zeros of the continuous- and discrete-time systems is given in Section 2.2.6.

2.1.4.3 Pulse Transfer Function from State-Space Matrices

The pulse transfer function can be derived by taking the z-transform of the discrete time state space equations, (Eqs. (2.21) and (2.22)) as

$$z x(z) = A_d x(z) + B_d u(z) \quad (2.39)$$
$$y(z) = C_d x(z) + D_d u(z) \quad (2.40)$$

combining Eqs. (2.39) and (2.40), one obtains the following relationship between the z-transforms of the input and output sequences

$$y(z) = (C_d(zI - A_d)B_d + D_d)u(z) \quad (2.41)$$

then, the pulse transfer function is obtained as

$$H(z) = C_d(zI - A_d)^{-1}B_d + D_d \quad (2.42)$$

Note that $H(z)$ is the general form of the pulse transfer function in terms of the discrete time state-space matrices. The type of hold the input satisfies changes the form of the pulse-transfer function through the terms $B_d$ and $D_d$.

2.1.4.4 Pulse Transfer Function Sampled from Continuous Transfer Function

The form of the pulse transfer function primarily depends on the discretization process. For example, if the input satisfies ZOH, then the pulse transfer function is such that its step response is equivalent to the sampled version of the step response of the continuous-time system. Because of this property, the conversion between the discrete and continuous time system representations are also known as step-invariant conversion in the literature and it can be represented by (Kollar et.al, 1996)
where $Z$ and $L^{-1}$ represent z- and inverse Laplace transforms respectively. If the transfer function $G(s)$ is strictly proper (the number of poles is greater than the number of zeros) then Eq. (2.43) can also be written as (Astrom et.al, 1984)

$$H(z) = \left(\frac{z-1}{z}\right) \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{z\Delta t}}{z-e^{s\Delta t}} \frac{G(s)}{s} ds$$ \hspace{1cm} (2.44)

where $\gamma$ is a real number such that all poles of $G(s)/s$ have real parts less than $\gamma$. Provided that $G(s)$ is finite when $s \to \infty$, the integration path in Eq. (2.44) can be closed and residue calculus can be used to evaluate the integral. Closing the path with a large semicircle on the left gives

$$H(z) = \left(\frac{z-1}{z}\right) \sum_{\text{poles of } G(s)/s} \text{Residues of } \left(\frac{e^{z\Delta t}}{z-e^{s\Delta t}} \frac{G(s)}{s}\right)$$ \hspace{1cm} (2.45)

where the residue for the $i^{\text{th}}$ pole of $G(s)/s$ that is repeated $k$ times can be computed from

$$\lim_{s \to p_i} \frac{d^{k-1}}{ds^{k-1}} \left( (s - p_i)^k \frac{e^{s\Delta t}}{z-e^{s\Delta t}} \frac{G(s)}{s} \right)$$ \hspace{1cm} (2.46)

If the input satisfies first order hold (FOH) the pulse transfer function is such that its ramp response is equivalent to the sampled version of the ramp response of the continuous-time system. This type of transfer is also known as ramp-invariant conversion in the literature and it can be represented by

$$H(z) = \left(\frac{z-1}{z}\right) \frac{1}{\Delta t z} \left[ L^{-1} \left\{ \frac{G(s)}{s^2} \right\} \right]$$ \hspace{1cm} (2.47)

In this dissertation, we operate under the ZOH premise.

2.2 Zeros of Multivariable Systems

In literature, the zeros of multivariable systems have been defined in a number of ways depending on whether the system considered is defined through a state-space or a transfer function representation (Macfarlane and Karcamias 1976, Schrader and Sain 1989; Rosenbrock 1973; Wolovich 1973). As it will be apparent in the following subsections, since the information carried by a state-space representation of a system is more than that present in a transfer function representation of the same system, the zeros defined through
state-space matrices constitute a larger set. Moreover, the physical interpretations of the zeros defined accordingly differ from one another slightly although both definitions are related with zero output response of the system. In particular, the zeros of transfer functions are associated with zero steady state response, those defined through state-space matrices are connected with zero output both in the initial and the steady states.

In this dissertation, the definitions connected with a state-space representation are examined in detail. In general, the whole set of zeros that can be obtained from a state-space representation can be classified into four categories: (1) Transmission zeros, (2) Decoupling zeros, (3) Invariant zeros, (4) System zeros. The relationship between each set of zeros is presented in Figure 2-1. Namely, the system zeros contain any set of zeros defined in the literature and it can be represented as the union of the invariant and the decoupling zeros. Moreover, the invariant zeros contain all of the transmission zeros and some of the decoupling zeros.

![Diagram showing the connection between the set of zeros defined in the literature](image)

**Figure 2-1** The connection between the set of zeros defined in the literature

### 2.2.1 Invariant Zeros

Invariant zeros are physically associated with zero-output behavior of a system (both free and forced vibrations are blocked) which is the situation where the system has an identically zero output while the states and inputs are not identically zero. Let the system represented by a continuous-time state-space representation where the input and output coordinates are collected in sets $I$ and $a$ respectively is given by

\[
\begin{align*}
\dot{x} &= A_c x + B_{c,1} u_I, \quad (2.48) \\
y_a &= C_{c,a} x + D_{c,a1} u_I, \quad (2.49)
\end{align*}
\]

where $B_{c,1}$ is the columns of the matrix $B_c$ associated with the input coordinates collected in set $I$, $C_{c,a}$ are the rows of the matrix $C_c$ corresponding to the output coordinates...
contained in set \(a\) and \(D_{c,a1}\) is the associated partition of the matrix \(D_c\). Suppose that the system is driven by an input in the form,

\[ u(t) = ge^{zt \tau} , \quad t \geq 0 \tag{2.50} \]

where \(g\) is some spatial distribution of the loading on set \(I\). The invariant zeros are the complex numbers, \(z_0\), for which the input in the form of Eq. \((2.50)\) results in no motion at the sensor coordinates (set \(a\)) with a non-zero initial condition \((x_0 \neq 0)\), namely,

\[ y_a(t) = 0 \quad \text{for} \quad t \geq 0 \tag{2.51} \]

The computation of such complex numbers can be cast into matrix pencil form, as it is detailed next. Since \(y_a(t_0) = 0\), Eq. \((2.49)\) requires,

\[ C_{c,a} x_0 + D_{c,a1} g = 0 \tag{2.52} \]

where \(x_0\) is the initial state vector. When \(t > 0\), \(y_a(t) = 0 \implies y_a(s) = 0\), after plugging Eq. \((2.29)\) into Eq. \((2.30)\) by considering only the partitions involving coordinates collected in sets \(I\) and \(a\) and realizing that \(u_1(s) = g/(s-z_0)\), and pre-multiplying both sides of the resulting equation by \((s-z_0)\), one gets,

\[ \begin{bmatrix} 0 \\ C_{c,a} (I \cdot s - A_c) \end{bmatrix}^{-1} \begin{bmatrix} (I \cdot s - I \cdot z_0) x_0 + B_{c,1} g \\ D_{c,a1} g \end{bmatrix} + D_{c,a1} g \]

which can also be written as,

\[ \begin{bmatrix} C_{c,a} (I \cdot s - A_c) \end{bmatrix}^{-1} \begin{bmatrix} (I \cdot s - I \cdot z_0) x_0 + B_{c,1} g \\ D_{c,a1} g \end{bmatrix} + D_{c,a1} g \]

or,

\[ \begin{bmatrix} 0 \\ C_{c,a} x_0 + D_{c,a1} g + C_{c,a} (I \cdot s - A_c) \end{bmatrix}^{-1} \begin{bmatrix} (A_c - I \cdot z_0) x_0 + B_{c,1} g \\ D_{c,a1} g \end{bmatrix} \]

which is satisfied at all \(s\) provided that \((A_c-I.z)x_0 + B_c g\) is equal to zero (given that this term is not a function of \(s\)). From this result and the requirement of no motion at set \(a\), Eq. \((2.52)\), one gets

\[ \begin{bmatrix} A_c \\ B_{c,1} \\ C_{c,a} \\ D_{c,a1} \end{bmatrix} \begin{bmatrix} x_0 \\ g \end{bmatrix} = z_0 \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x_0 \\ g \end{bmatrix} \tag{2.56} \]

or,

\[ \begin{bmatrix} 1 \cdot z_0 - A_c \\ -B_{c,1} \\ C_{c,a} \\ D_{c,a1} \end{bmatrix} \begin{bmatrix} x_0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.57} \]
which becomes a necessary condition for the existence of a \( z_0 \), \( x_0 \) and \( g \) such that \( y_a(s) = 0 \).

It is opportune to note that the first matrix on the left-hand side of Eq. (2.57) is referred to as system matrix (Rosenbrock, 1970) and it is essential for the definition of the invariant zeros. As it should be clear by now, an invariant zero is associated with the complex number \( z_0 \), which makes the system matrix loose its normal rank, a nonzero initial state \( x_0 \), which is needed to make the free response at the coordinates defined by the set \( a \) to be zero, and the vector \( g \), which is the spatial distribution of the exponential loading given in Eq. (2.50). Note that Eq. (2.57) can be solved as an eigenvalue problem provided that the number of input and output coordinates collected in sets \( I \) and \( a \) are equal. Otherwise, Eq. (2.57) provides a matrix pencil and its solution requires a special treatment which has been studied extensively in the control field (Enami-Naeini and Van Dooren 1982; Moylan 1977; Davison and Wang 1978; Laub and Moore 1978). Once an invariant zero is computed, the state vector, \( x(s) \), can be obtained from Eq. (2.29) by introducing the expression present in the upper partition of Eq. (2.57), which results

\[
x(s) = \frac{1}{s - z_0} x_0
\]

and the associated state-space motion becomes,

\[
x(t) = x_0 e^{z_0 t}, \quad t \geq 0
\]

plugging Eqs. (2.50) and (2.59) into Eq. (2.49) and realizing (2.52), one gets,

\[
y_a(t) = (C_{c,a} x_0 + D_{c,a} g) e^{z_0 t} = 0, \quad t \geq 0
\]

### 2.2.1.1 A Comment on the Complex Input and Initial Condition

Note that although both input \( ge^{z_0 t} \) and the initial condition \( x_0 \) associated with the invariant zero \( (z_0) \) are complex, one can easily note that there are two real valued input and initial condition pairs for a given invariant zero which produces zero output at the sensor coordinates. These two cases are simply the real (Re) and imaginary (Im) parts of the input and the initial state. In other words, a real valued initial condition given by Re(\( x_0 \)) together with the real valued input given by Re(\( ge^{z_0 t} \)) produce identically zero output. Similarly, a real valued initial condition given by Im(\( x_0 \)) together with the real valued input given by Im(\( ge^{z_0 t} \)) satisfy zero output condition. In order to illustrate this, consider the system depicted in Figure 2-2.

![Figure 2-2](imageurl) Two-DOF system (\( m_1 = 2 \), \( m_2 = 1 \), \( k_1 = 150 \), \( k_2 = 100 \) in consistent units)
Suppose that it is of interest to zero the output at the first mass by applying an input of the form \( ge^{at} \) at the same coordinate. The state-space matrices corresponding to the system with the collocated input-output coordinate are given by,

\[
A_c = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-125 & 50 & 0 & 0 \\
100 & -100 & 0 & 0 \\
\end{bmatrix}, \quad B_{c,i} = \begin{bmatrix}
0 \\
0 \\
0.5 \\
0 \\
\end{bmatrix}, \quad C_{c,i} = [1 \ 0 \ 0 \ 0], \quad D_{c,i} = 0
\]

From Eq. (2.56), one gets \( z_0 = -10i \), \( g = -i \) and \( x_0 = [0 \ 0.01i \ 0 \ 0.1]^T \). So, the input required is found to be complex valued as,

\[ u_1(t) = ge^{at} = -\sin(10t) - \cos(10t)i \]

However, when only the real parts of the input and the initial condition is considered, one has \( u_1(t) = -\sin(10t) \) and \( x_0 = [0 \ 0 \ 0.1]^T \). The resulting output can be easily obtained in the Laplace domain following Eq. (2.31) and realizing \( u_1(s) = -10/(s^2+100) \) and

\[
\begin{pmatrix}
1 \\
100s \\
-125 \left( s^2 + 60 \right) \\
100s^2 \\
\end{pmatrix}
\begin{pmatrix}
s \left( s^2 + 100 \right) \\
50s \\
s \left( s^2 + 125 \right) \\
100s \\
\end{pmatrix}
\begin{pmatrix}
s \left( s^2 + 100 \right) \\
50s \\
s \left( s^2 + 125 \right) \\
100s \\
\end{pmatrix}
\]

as

\[ y_2(s) = C_{c,i} \left( I_s - A_c \right)^{-1} x_0 + \left( C_{c,i} \left( I_s - A_c \right)^{-1} B_{c,i} + D_{c,i} \right) u_1(s) = 0 \]

It is straightforward to see that when only the imaginary parts of the input and initial condition are considered, one has \( u_1(t) = -\cos(10t) \) and \( x_0 = [0 \ 0.01 \ 0 \ 0]^T \), and realizing \( u_1(s) = -s/(s^2+100) \), the output can be found to be identically equal to zero.

### 2.2.1.2 Invariant Transformations

The set of invariant zeros is invariant with respect to the following set of transformations (Rosenbrock, 1970):

- Nonsingular coordinate transformations in the state space,
- Nonsingular transformations of the inputs,
- Nonsingular transformations of the outputs,
• State feedback to the inputs
• Output feedback to the inputs

A coordinate transformation in the state space, \( x = Tx' \) where \( T \) is a nonsingular matrix with appropriate size, transforms Eq. (2.56) to

\[
\begin{bmatrix}
T^{-1}A & T^{-1}B_x \\
C_{x,a} & D_{x,a,1}
\end{bmatrix}
\begin{bmatrix}
x'_0 \\
g
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x'_0 \\
g
\end{bmatrix}
\tag{2.61}
\]

realizing \( x'_0 = T^{-1}x_0 \) and the fact that pre-multiplication of Eq. (2.61) by a matrix, \( Q \), of the form

\[
Q = \begin{bmatrix}
T & 0 \\
0 & I_{m'}
\end{bmatrix}
\tag{2.62}
\]

where \( m' \) denotes the number of output coordinates in set \( a \), does not change the equality in Eq. (2.61), proves the invariance property of the invariant zeros to nonsingular state space transformations.

A nonsingular transformation of the inputs, \( u = Vu' \), where \( V \) is a nonsingular matrix with appropriate size, transforms Eq. (2.56) to

\[
\begin{bmatrix}
A_x & B_{x,1}V \\
C_{x,a} & D_{x,a,1}V
\end{bmatrix}
\begin{bmatrix}
x_0 \\
g'
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_0 \\
g'
\end{bmatrix}
\tag{2.63}
\]

realizing \( g' = V^{-1}g \), shows that invariant zeros are invariant to nonsingular input transformations.

A nonsingular transformation of the outputs, \( y_a = Uy_a' \), where \( U \) is a nonsingular matrix with appropriate size, transforms Eq. (2.56) to

\[
\begin{bmatrix}
A_x & B_{x,1} & U^{-1}C_{x,a} \\
U^{-1}C_{x,a} & D_{x,a,1} & U^{-1}D_{x,a,1}
\end{bmatrix}
\begin{bmatrix}
x_0 \\
g
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_0 \\
g
\end{bmatrix}
\tag{2.64}
\]

realizing the fact that pre-multiplication of Eq. (2.64) by a matrix, \( Q \), of the form

\[
Q = \begin{bmatrix}
I_{2a} & 0 \\
0 & U
\end{bmatrix}
\tag{2.65}
\]

does not change the equality in Eq. (2.64), proves the invariance property of the invariant zeros to nonsingular output transformations.
Introducing the control law, \( \mathbf{u}_1 = \mathbf{u}_1' + \mathbf{Kx} \), where \( \mathbf{K} \) is a real matrix with appropriate size, transforms Eq. (2.56) to

\[
\begin{bmatrix}
\mathbf{A}_{c,e} + \mathbf{B}_{c,e} \mathbf{K} & \mathbf{B}_{c,e} \\
\mathbf{C}_{c,a} + \mathbf{D}_{c,a} \mathbf{K} & \mathbf{D}_{c,a}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_0 \\
\mathbf{g}'
\end{bmatrix}
= \begin{bmatrix}
\mathbf{I} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_0 \\
\mathbf{g}'
\end{bmatrix}
\]  

(2.66)

realizing \( \mathbf{g}' = \mathbf{g} - \mathbf{Kx}_0 \), shows that invariant zeros are invariant with respect to the state feedback to the inputs.

Introducing the control law, \( \mathbf{u}_1 = \mathbf{u}_1' + \mathbf{Ty}_a \), where \( \mathbf{T} \) is a real matrix with appropriate size, such that the matrix \( (\mathbf{I}_m - \mathbf{DT}) \) is invertible, transforms Eq. (2.56) to

\[
\begin{bmatrix}
\mathbf{A}_{c,e} + \mathbf{B}_{c,e} (\mathbf{I}_m - \mathbf{D}_{c,a} \mathbf{T})^{-1} \mathbf{C}_{c,a} & \mathbf{B}_{c,e} + \mathbf{B}_{c,e} (\mathbf{I}_m - \mathbf{D}_{c,a} \mathbf{T})^{-1} \mathbf{D}_{c,a} \\
(\mathbf{I}_m - \mathbf{D}_{c,a} \mathbf{T})^{-1} \mathbf{C}_{c,a} & (\mathbf{I}_m - \mathbf{D}_{c,a} \mathbf{T})^{-1} \mathbf{D}_{c,a}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_0 \\
\mathbf{g}'
\end{bmatrix}
= \begin{bmatrix}
\mathbf{I} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_0 \\
\mathbf{g}'
\end{bmatrix}
\]  

(2.67)

realizing \( \mathbf{y}_a = \mathbf{0} \), hence, \( \mathbf{g} = \mathbf{g}' \) and \( \mathbf{C}_{c,a} \mathbf{x}_0 = -\mathbf{D}_{c,a} \mathbf{g}' \) together with the fact pre-multiplication of Eq. (2.67) by a matrix, \( \mathbf{Q} \), of the form

\[
\mathbf{Q} = \begin{bmatrix}
\mathbf{I}_{2n} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}_m - \mathbf{D}_{c,a} \mathbf{T}
\end{bmatrix}
\]  

(2.68)

does not change the equality in Eq. (2.67), proves the invariance property of the invariant zeros to output feedback to the inputs.

2.2.2 Transmission Zeros

Transmission zeros are defined via the transfer function matrix \( \mathbf{G}(s) \) and they are associated with transmission-blocking properties of a system (forced vibrations are blocked). Transmission zeros are the complex values at which the transfer function matrix \( \mathbf{G}(s) \) looses its normal rank. Transmission zeros can be obtained as a subset of invariant zeros. Strictly speaking, the invariant zeros that are associated with transmission zeros are not identical to transmission zeros. Because, an invariant zero is defined by a complex number \( (\mathbf{z}_0) \) and its direction which contains the spatial distribution \( (\mathbf{g}) \) of the input as well as the initial condition \( (\mathbf{x}_0) \) needed to make the free response zero. However, the transmission zero is connected only with the complex number \( \mathbf{z}_0 \) by definition.

2.2.3 Decoupling Zeros

Decoupling zeros are the first zeros introduced by Rosenbrock (1970) and are associated with the modal motion of the system state that is uncoupled from the system’s input and/or output.
2.2.3.1 Input Decoupling Zeros

These zeros are actually the poles associated with uncontrollable modes. The rank deficiency definition states that the complex numbers, $z_0$, that satisfy

$$\text{rank} \left( \left[ z_0 I - A_c \quad B_{c,1} \right] \right) < 2n$$

are called input decoupling zeros. Input decoupling zeros are associated with loss of row-rank of the matrix

$$P_i(s) = \left[ sI - A_c \quad B_{c,1} \right]$$

The number of input decoupling zeros is shown (Rosenbrock, 1970) to be the rank defect of the controllability matrix

$$\left[ B_{c,1} \quad A_c B_{c,1} \quad A_c^2 B_{c,1} \ldots A_c^{2n-1} B_{c,1} \right]$$

Another way of visualizing input decoupling zeros may be established by considering the matrices $A_c$ and $B_{c,1}$ expressed in modal coordinates. Note that if there is a mode that is non-controllable from the input coordinates considered in $B_{c,1}$, the row of $B_{c,1}$ corresponding to that mode takes values of zero. At the same time, matrix $A_c$ which is diagonal in modal coordinates, contain the value of the pole corresponding to that non-controllable mode at the diagonal term associated with the row of $B_{c,1}$ that has zeros. It is, then, straightforward to see that when the matrix $P_i(s)$ evaluated at $s$ equals to the pole of the non-controllable mode, the row corresponding to that mode becomes full of zeros, hence the normal rank of the matrix $P_i(s)$ is reduced.

2.2.3.2 Output Decoupling Zeros

These zeros are the poles corresponding to unobservable modes. The rank deficiency definition states that the complex numbers, $z_0$, that satisfy

$$\text{rank} \left( \left[ z_0 I - A_c \quad C_{c,a} \right] \right) < 2n$$

are called output decoupling zeros. Output decoupling zeros are associated with loss of column rank of the matrix

$$P_o(s) = \left[ sI - A_c \quad C_{c,a} \right]$$

The number of output decoupling zeros are shown (Rosenbrock, 1970) to be the rank defect of the observability matrix
Another way of visualizing output decoupling zeros may be established by considering the matrices $A_c$ and $C_{c,a}$ expressed in modal coordinates. Note that if there is a mode that is non-observable from the output coordinates considered in $C_{c,a}$, the column of $C_{c,a}$ corresponding to that mode takes values of zeros while the matrix $A_c$, which is diagonal, contain the value of the pole corresponding to that non-observable mode at the diagonal term associated with the column of $C_{c,a}$ that has zeros. It is, then, straightforward to see that when the matrix $P_o(s)$ evaluated at $s$ equals to the pole of the non-observable mode, the column corresponding to that mode becomes full of zeros, hence a drop in the normal rank of the matrix $P_o(s)$ is observed.

2.2.3.3 Input-Output Decoupling Zeros

These zeros are the poles associated with both uncontrollable and unobservable modes. One obvious way to obtain them is just to look at the intersection of the input and output decoupling zeros.

2.2.4 Relationship between Invariant, Transmission and Decoupling Zeros

The only difference between the set of invariant zeros and the set of transmission zeros is that while the invariant zeros contain the transmission zeros they constitute a larger set. The additional zeros that may exist in the set of invariant zeros are some or all of the decoupling zeros (see Figure 2-1). The subset of decoupling zeros that belong to the set of invariant zeros is determined by the number of input and output coordinates present in sets $I$ and $a$.

Realizing the fact that normal rank of the system matrix is equal to $2n + min(m, r)$ (provided that the matrix $A_c$ is full rank) where $m$ and $r$ are the number of sensors in sets $a$ and $I$ respectively, and that the invariant zeros are connected with the complex values at which the system matrix reduces its normal rank, following three conditions can be observed: (1) If the number of outputs is less than the number of inputs ($m < r$), all of the output decoupling zeros are invariant zeros. (2) If the number of inputs is less than the number of outputs ($r < m$), then all of the input decoupling zeros are invariant zeros. (3) If there are equal number of inputs and outputs ($r = m$), then both input decoupling zeros and output decoupling zeros as well as the input output decoupling zeros are in the set of invariant zeros.

The fact that the invariant zeros contain the transmission zeros and they become the same set if and only if the system realization $\{A_c, B_{c,1}, C_{c,a}, D_{c,a}\}$ is of the minimal order and the relationships between the decoupling and invariant zeros noted above can be easily verified by writing the system matrix for the case where $r > m$ as

$$\left[ C_{c,a}^T A_{c,a}^T C_{c,a}^T (A_{c,a}^T)^2 C_{c,a}^T \ldots (A_{c,a}^T)^{2n-1} C_{c,a}^T \right]^T$$

(2.74)
where it is clear that the normal rank of the system matrix given on the lhs of Eq. (2.75) is \(2n+m\) and it depends on the row rank of the two matrices: (1) \(G(z_0)\), (2) \(\begin{bmatrix} I \cdot z_0 - A_c & -B_{c,1} \\ C_{c,a} & D_{c,a,1} \end{bmatrix}\). Therefore, any complex number \(z_0\) that reduces the row rank of the transfer function \(G(s)\), transmission zero, is an invariant zero. Similarly, any complex number that reduces the row rank of the matrix \(\begin{bmatrix} I \cdot s - A_c & -B_{c,1} \end{bmatrix}\), input decoupling zero, is an invariant zero. Note that when the system \(\{A_c, B_{c,1}, C_{c,a}, D_{c,a,1}\}\) is completely controllable, then there will be no input decoupling zero, but there may be output decoupling zeros some of which may also be an invariant zero.

When \(m > r\), the system matrix can be written as

\[
\begin{bmatrix} I \cdot z_0 - A_c & -B_{c,1} \\ C_{c,a} & D_{c,a,1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_{c,a}(I \cdot z_0 - A_c)^{-1} & I^T \\ 0 & G(z_0) \end{bmatrix} \begin{bmatrix} I \cdot z_0 - A_c & -B_{c,1} \end{bmatrix}^{-1}
\]

(2.76)

where it is clear that the normal rank of the system matrix given on the lhs of Eq. (2.76) is \(2n+r\) and it depends on the column rank of the two matrices: (1) \(G(z_0)\), (2) \(\begin{bmatrix} (I \cdot z_0 - A_c)^T \\ C_{c,a}^T \end{bmatrix}\). Therefore, any complex number \(z_0\) that reduces the column rank of the transfer function \(G(s)\), transmission zero, is an invariant zero. Similarly, any complex number that reduces the column rank of the matrix \(\begin{bmatrix} (I \cdot z_0 - A_c)^T & C_{c,a}^T \end{bmatrix}\), output decoupling zero, is an invariant zero. Note that when the system \(\{A_c, B_{c,1}, C_{c,a}, D_{c,a,1}\}\) is completely observable, then there will be no output decoupling zero, but there may be input decoupling zeros some of which may also be an invariant zero.

Note that when \(m = r\), the normal rank of the system matrix is \(2n + m = 2n + r\), hence any complex value that reduces the column or row rank of the system matrix will be considered as an invariant zero. Therefore, if the system is not minimal, all of the decoupling zeros (input, output and input-output decoupling zeros) will be in the set of invariant zeros. If, however, the system \(\{A_c, B_{c,1}, C_{c,a}, D_{c,a,1}\}\) is minimal (both observable and controllable), then there will be no decoupling zeros and the set of invariant zeros and the set of transmission zeros will be the same set.

**Example:**

In order to illustrate the relationships between the invariant, transmission and decoupling zeros, consider the following state space representation of order 4, with \(m = 2\) and \(r = 1\).
\[
A_c = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -5 & 0 \\
0 & 0 & 0 & 7
\end{bmatrix}, \quad B_{c,1} = \begin{bmatrix}
0 \\
-1 \\
-1 \\
-1
\end{bmatrix}, \quad C_{c,a} = \begin{bmatrix}
1 & 0 & 2 & 1 \\
0 & 0 & 2 & 1
\end{bmatrix}, \quad D_{c,a1} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

The system matrix associated with the given state-space matrices can be obtained as

\[
\begin{bmatrix}
z_0 - 1 & 0 & 0 & 0 & 0 \\
0 & z_0 + 1 & 0 & 0 & 1 \\
0 & 0 & z_0 + 5 & 0 & 1 \\
0 & 0 & 0 & z_0 - 7 & 1 \\
1 & 0 & 2 & 1 & 0 \\
0 & 0 & 2 & 1 & 0
\end{bmatrix}
\]

since the normal rank of the system matrix is 5 \((2n+r)\), the invariant zeros are defined as the complex values \(z_0\) that reduces the column rank of the system matrix. The invariant zeros that are computed from MATLAB using the \textit{tzero} command are found to be \{-1 and 3\}. Based on the discussion given in the previous section, if the system given is not completely observable, it is expected to have all of the output decoupling zeros to be in the set of invariant zeros. In order to check whether this is the case, we explicitly calculate the output decoupling zeros, again using the \textit{tzero} command in MATLAB but this time the matrices \(B_{c,1}\) and \(D_{c,a1}\) are taken to be empty, in other words, the zeros of Eq. (2.73) are computed. The set of output decoupling zeros is found to be \{-1\} which verifies the expectation that the invariant zeros contain the output decoupling zeros when \(m>r\). Next, we compute the input decoupling zeros by forming Eq. (2.70), which again can be computed conveniently from \textit{tzero} command where the matrix \(C_{c,a}\) and \(D_{c,a1}\) are taken to be empty. The set of input decoupling zeros are found to be \{1\}. Clearly, in this particular case, the set of invariant zeros does not contain the input decoupling zeros. It is worth noting that the invariant zero at \(s=3\) is neither an input decoupling zero nor an output decoupling zero. Therefore, it has to be a transmission zero. As a summary, one can combine the results in the following sketch.
2.2.5 Invariant Zeros of Discrete-Time Systems

While the invariant zeros are defined for continuous-time systems so far, a similar discussion can be made for discrete-time systems and parallel expressions for the invariant zeros of discrete-time systems can be obtained. For example, the invariant zeros of a discrete-time system that relates the input coordinates collected in set $I$ to output coordinates in set $a$ can be computed from

$$
\begin{bmatrix}
I - \bar{z}_0 A_d & -B_{d,1} \\
C_{d,a} & D_{d,a1}
\end{bmatrix}
\begin{bmatrix}
x_0 \\
g
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

(2.77)

where $B_{d,1}$ is the columns of the discrete-time input-to-state matrix $B_d$ associated with the input coordinates collected in set $I$, $C_{d,a}$ are the rows of the discrete-time output matrix $C_d$ corresponding to the output coordinates contained in set $a$, $D_{d,a1}$ is the associated partition of the matrix $D_d$ and $\bar{z}_0$ is the zero of the discrete-time system.

Similarly, if the discrete-time system realization is of minimal order, the zeros computed from Eq. (2.77) correspond to the transmission zeros at which the pulse transfer function matrix $H(z)$ looses its normal rank. If the realization is not minimal, then only some of the zeros obtained from Eq. (2.77) correspond to the transmission zeros of the associated pulse transfer function.

2.2.6 Connection between the Continuous- and Discrete- Time Zeros

It is well-known in the control community that the zeros of the sampled systems (discrete-time systems) differ from the zeros of the original continuous time system depending on the discretization process being employed. In the literature, the zeros of the sampled systems for sufficiently small sampling periods are classified into two categories. The first one consists of the zeros corresponding to the original continuous-time system
which are known as *intrinsic* zeros and the second one contains the zeros generated due to the sampling process and they are known as *discretization* zeros.

Therefore, it is not uncommon to have different number of zeros in discrete and continuous-time systems associated with the same physical structure. Moreover, even if the original continuous time system is known to be minimum-phase, i.e., all the zeros are located on the left-half plane in the Laplace domain, the sampled system might have zeros outside the unit circle, which map to the right-half plane in the Laplace domain. It is also well-known that the intrinsic zeros cannot be related to the zeros of the continuous time system through any closed form equation (Astrom 1984; Hagiwara 1996). The standard mapping between the Laplace and z-planes provides just an approximation while the degree of the approximation declines with higher sampling rates.

Astrom (1984) investigated the limiting behavior of the zeros of the sampled systems satisfying ZOH condition with sufficiently small and large sampling periods. He showed that the pulse transfer function of these systems which is given in Eq. (2.43) converges to

$$
\lim_{A \Delta t \to 0} H(z) = \frac{K}{(n-m)!} \left(\frac{(z-1)^m B_{n-m}(z)}{(z-1)^n}\right)
$$

when the sampling period goes to zero. The pulse transfer function has generically $n-1$ zeros while $m$ of these zeros (intrinsic zeros) approach $z = 1$, the remaining $n-m-1$ zeros approach the zeros of the polynomial $B_{n-m}(z)$ which is shown to be a function of the relative degree $(n-m)$ of the transfer function, $G(s)$, where $n$ and $m$ are the number of poles and zeros of the transfer function respectively. The coefficients of the polynomial $B_{n-m}(z)$ can be computed recursively from

$$
b_k^n = k b_k^{n-1} + (n-k+1) b_{k-1}^{n-1} \quad k = 2, \ldots, n-1
$$

$$
b_1^n = b_1^n = 1
$$

The first few samples of the polynomial $B_k(z)$ are listed below for future references (general formula for the polynomial $B_k(z)$ can be found in Astrom (1984)).

$$
B_1(z) = 1
$$

$$
B_2(z) = z + 1
$$

$$
B_3(z) = z^2 + 4z + 1
$$

$$
B_4(z) = z^3 + 11z^2 + 11z + 1
$$

The results of Astrom’s work can be summarized as follows: for a strictly proper continuous time transfer function which does not have a zero at $s = 0$, if $n - m = 1$, there is no discretization zeros since there are $n - 1$ intrinsic zeros as suggested by Eq. (2.80). If $n - m = 2$, there is only one discretization zero, which approaches to $z = -1$, which is the root of the polynomial given in Eq. (2.81). If $n - m$ is an odd number greater than or equal
to 3, the discretization zeros are always located in pairs where each pair consists of a root inside the unit circle and its reciprocal outside. For example, when \( n - m = 3 \), the discretization zeros approach to the roots of the polynomial given in Eq. (2.82), which are -0.268 and -3.732. If \( n - m \) is even, then \((n - m - 2)/2\) pairs of the roots which have the same property as above, and the additional root approaches to \( z = -1 \). For example, the roots of the polynomial given in Eq. (2.83) are -0.101, -1, -9.899. Note that the discrete time transfer functions sampled from continuous time transfer functions that satisfy the condition \( n-m>2 \) always have zeros that are outside the unit circle, regardless of whether the continuous time system is of minimum-phase or not.

In civil or mechanical engineering applications, one of the cases where these findings directly apply is the transfer function which relates forces to displacements. In particular, let \( G^d(s) \) be a collocated transfer function relating forces to displacements,

\[
G^d(s) = K \frac{(s-z_1)(s-z_2) \ldots (s-z_m)}{(s-p_1)(s-p_2) \ldots (s-p_n)}
\]

(2.84)

where \( m = n - 2 \). Note that \( G^d(s)/s \) can be written in pole-residue form as

\[
\frac{G^d(s)}{s} = K \left( \frac{R_0^d}{s} + \frac{R_1^d}{s-p_1} + \ldots + \frac{R_n^d}{s-p_n} \right)
\]

(2.85)

it is a simple matter to see that the sum of the residues is equal to zero (the sum of the residues is the coefficient of the term \( s^n \) in the numerator which has to be equal to zero in this case). Then, the pulse transfer function can be obtained from Eq. (2.44) as

\[
H^d(z) = \left( \frac{z-1}{z} \right) K \left( \frac{z}{z-1} \right) q_0 z^{-1} + q_1 z^{-2} + \ldots + q_n + q_0 z^{-1} \right)
\]

\[
\frac{\left( z^{-1} \right)}{\left( z-e^{\rho_i \Delta t} \right) \ldots \left( z-e^{\rho_n \Delta t} \right)}
\]

(2.86)

where \( q_0 = (-1)^n \sum_{i=0}^{n} R_i^d \) and \( q_n = R_0^d + \sum_{i=1}^{n} R_i^d e^{\rho_i \Delta t} \). Note that \( q_0 = 0 \) due to the last term (sum of the residues is zero). For any non-zero sampling period \( \Delta t \), there is no reason to expect any other coefficient to be equal to zero including the first one, \( q_n \). Therefore, one can easily conclude that the pulse transfer function \( H^d(z) \) has \( n - 1 \) zeros. While it may appear from Eq. (2.44) that the pulse transfer function has a zero at \( z = 1 \) and a pole at \( z = 0 \), Eq. (2.86) clearly indicates that they are actually cancelled. The fact that \( H(z) \) does not have a pole at \( z = 0 \) has been mentioned by Astrom (1984) who stated that “it can be shown through a state space representation that \( H(0) \) is actually finite”. Eq. (2.86) is powerful in the sense that there is no need to get a state-space representation to show this fact, besides it also clearly shows that \( H^d(z) \) does not have a zero at \( z = 1 \).

If the continuous time transfer function has actually a zero at \( s = 0 \), then the pulse transfer function has always an intrinsic zero at \( z = 1 \) regardless of the sampling period. This is
the only case where a discrete time zero can be mapped to its continuous time counterpart through the mapping 

\[ z = e^{s \Delta t} \]

exactly. This condition corresponds to continuous time transfer matrices which relate forces to velocity measurements. The ultimate condition where \( n - m = 1 \) is realized in the collocated transfer functions. The fact that there is always a zero at \( z = 1 \) can be shown as follows. Let \( G^t(s) \) be the continuous time collocated transfer function relating forces to velocity measurements, which can be written as

\[
G^t(s) = K \frac{s(s-z_1)(s-z_2)\ldots(s-z_{m-1})}{(s-p_1)(s-p_2)\ldots(s-p_n)} \tag{2.87}
\]

and \( G^v(s)/s \) can be expressed in the pole-residue form as

\[
\frac{G^t(s)}{s} = K \left( \frac{R_1^v}{s-p_1} + \ldots + \frac{R_n^v}{s-p_n} \right) \tag{2.88}
\]

since \( m = n - 1 \) for \( G^v(s) \), it is a simple matter to see that the sum of the residues is equal to zero (the sum of the residues is the coefficient of the term \( s^{n-1} \) in the numerator which has to be equal to zero since \( G^v(s)/s \) does not have a term \( s^{n-1} \) in the numerator). Then, the pulse transfer function can be obtained from Eq. (2.44) as

\[
H^v(z) = \left( \frac{z-1}{z} \right) \left( K \left( \frac{q_{n-1}z^{n-2} + q_{n-2}z^{n-3} + \ldots + q_1 + q_0 z^{-1}}{z - e^{p_1 \Delta t}} \ldots (z - e^{p_n \Delta t}) \right) \right) \tag{2.89}
\]

where \( q_0 = (-1)^{n-1} e^{(p_1 + \ldots + p_n) \Delta t} \sum R_i^v \) and \( q_{n-1} = \sum R_i^v e^{p_i \Delta t} \). Note that \( q_0 = 0 \) due to the last term and it can easily be concluded that there are \( n - 1 \) zeros, one of which is at \( z = 1 \) regardless of the sampling period. Eq. (2.89) also clearly indicates the fact that \( H^v(z) \) does not have a pole at \( z = 0 \).

An interesting case which is frequently encountered in many civil and mechanical engineering applications is that of collocated transfer functions which relate forces to acceleration measurements, \( G^a(s) \). In this particular case, the continuous time transfer function has a zero relative degree, i.e \( m = n \), and has two zeros at \( s = 0 \). Since there are equal number of poles and zeros one can write the transfer function after long division as,

\[
G^a(s) = D + K \frac{(s-z_1)(s-z_2)\ldots(s-z_{m-1})}{(s-p_1)(s-p_2)\ldots(s-p_n)} \tag{2.90}
\]

and \( G^a(s)/s \) can be written as
\[
G^a(s) = \frac{D}{s} + K \left( \frac{R_0^a}{s} + \frac{R_1^a}{s-p_1} + \ldots + \frac{R_n^a}{s-p_n} \right) \tag{2.91}
\]

Note the second term on the right hand side of Eq. (2.91) cannot have a term of \( s^n \) in the numerator so the coefficient of that term, which is equal to the sum of the residues have to be equal to zero. When Eq. (2.91) plugged back into Eq. (2.44) one obtains

\[
H(z) = D + \left( \frac{z-1}{z} \right) \left( K \left( \frac{z}{z-1} \right) \frac{q_0 z^{n-1} + q_{n-1} z^{n-2} + \ldots + q_1 + q_0 z^{-1}}{(z - e^{\eta \Delta t}) \ldots (z - e^{\rho_i \Delta t})} \right) \tag{2.92}
\]

where \( q_0 = (-1) e^{(p_1 + \ldots + p_n) \Delta t} \sum_{i=0}^{n} R_i^a \) and \( q_n = R_0^a + \sum_{i=1}^{n} R_i^a e^{p_i \Delta t} \). Note that since \( \sum_{i=0}^{n} R_i^a = 0 \), then \( q_0 \) is zero and the pole at \( z = 0 \) cancels as well as the terms \( z - 1 \) in the second part of Eq. (2.92). It is interesting to see that even though there are two zeros at \( s = 0 \) in the continuous-time transfer function, Eq. (2.92) does not guarantee the placement of even a single zero at \( z = 1 \) in the pulse transfer function. Limited numerical runs verified this statement although usually one of the zeros is found to be almost equal to one but not exactly. Moreover its location is found to be dependent on the sampling period. This observation is quite important in the sense that the connectivity between the continuous time and discrete time transfer functions is broke down for proper transfer functions (\( n = m \)), which happens to be an important case (a collocated transfer function relating forces to accelerations) in many applications in the civil or mechanical engineering. Aforementioned inconsistency between the two transfer functions reflects itself as very large approximations when the zeros of the pulse transfer function are mapped to the Laplace plane through the mapping \( z = e^{s \Delta t} \). In order to illustrate the main points presented in this section, consider the following example.

**Example:**

A 5 story shear frame with story stiffnesses of \{800, 700, 600, 650, 500\} and story masses of \{1.20, 1.15, 1.05, 1.25, 1.30\} with 1 \% classical damping at each mode is considered. While the largest frequency of the structure is 7.25 Hz, a sampling rate of 0.04 seconds has been selected, which correspond to a Nyquist frequency of 12.5 Hz. The zeros of the continuous time and discrete time collocated transfer functions defined at the second mass have been computed for three cases where the collocated forces are related to displacement, velocity and acceleration measurements (see Table 2-2, Table 2-3 and Table 2-4 respectively). In particular, the zeros of the pulse transfer function associated with collocated displacement measurements contain a single zero approaching to \( z = -1 \) which is the single discretization zero since \( n - m = 1 \) in this case. When the intrinsic zeros are mapped to the Laplace plane it is clear that they do not correspond to the continuous time zeros exactly but the level of accuracy can be acceptable for particular applications. The zeros of the pulse transfer function associated with the velocity measurement do not contain any discretization zeros, while one of the intrinsic zeros is located at \( z = 1 \) which correspond to the zero at \( s = 0 \) in Laplace plane exactly. The
locations of the rest of the zeros when mapped to the Laplace plane can be considered to be close to their continuous time counterparts. The zeros of the pulse transfer function associated with the acceleration measurements when mapped to the Laplace domain are found to be quite approximate especially in their real parts. The reason for that can be attributed to the fact that one of the zeros which is supposed to be located at \( z = 1 \) is at \( z = 0.57 \) which maps to \( s = -14.18 \) on the Laplace plane. Therefore, it can be concluded that the error associated with that zero affects the location of the other zeros. In fact, when a higher sampling rate is used, it has been observed that that zero got closer to \( z = 1 \) and the approximation observed in the rest of the zeros has also been reduced.

Aforementioned issue can also be observed in the FRF plots which are given in Figure 2-4, Figure 2-5 and Figure 2-6. Each figure shows the magnitude of the FRF of the original continuous-time system and the discrete-time system for each type of measurement. While the two curves are quite consistent for displacement and velocity measurements, the distortion in the real parts of the zeros in the acceleration measurement case is quite visible.

Table 2-2 Zeros of continuous and discrete-time systems associated with displacement measurement

<table>
<thead>
<tr>
<th>True Zeros</th>
<th>Zeros of Discrete System</th>
<th>Discrete Zeros mapped to Laplace domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.085 ± 9.745i</td>
<td>0.922 ± 0.379i</td>
<td>-0.085 ± 9.744i</td>
</tr>
<tr>
<td>-0.260 ± 26.707i</td>
<td>0.477 ± 0.867i</td>
<td>-0.259 ± 26.688i</td>
</tr>
<tr>
<td>-0.340 ± 35.354i</td>
<td>0.157 ± 0.974i</td>
<td>-0.337 ± 35.267i</td>
</tr>
<tr>
<td>-0.406 ± 41.068i</td>
<td>-0.069 ± 0.982i</td>
<td>-0.405 ± 41.02i</td>
</tr>
<tr>
<td>-0</td>
<td>-0.990</td>
<td>-0.252 ± 78.54i</td>
</tr>
</tbody>
</table>

Table 2-3 Zeros of continuous and discrete-time systems associated with velocity measurement

<table>
<thead>
<tr>
<th>True Zeros</th>
<th>Zeros of Discrete System</th>
<th>Discrete Zeros mapped to Laplace domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>-0.085 ± 9.745i</td>
<td>0.919 ± 0.386i</td>
<td>-0.088 ± 9.939i</td>
</tr>
<tr>
<td>-0.260 ± 26.707i</td>
<td>0.467 ± 0.872i</td>
<td>-0.266 ± 26.991i</td>
</tr>
<tr>
<td>-0.340 ± 35.354i</td>
<td>0.125 ± 0.978i</td>
<td>-0.357 ± 36.083i</td>
</tr>
<tr>
<td>-0.406 ± 41.068i</td>
<td>-0.084 ± 0.98i</td>
<td>-0.413 ± 41.404i</td>
</tr>
</tbody>
</table>
Table 2-4 Zeros of continuous and discrete-time systems associated with acceleration measurement

<table>
<thead>
<tr>
<th>True Zeros</th>
<th>Zeros of Discrete System</th>
<th>Discrete Zeros mapped to Laplace domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0.567</td>
<td>-14.183</td>
</tr>
<tr>
<td>-0.085 ± 9.745i</td>
<td>0.880 ± 0.289i</td>
<td>-1.905 ± 7.938i</td>
</tr>
<tr>
<td>-0.260 ± 26.707i</td>
<td>0.431 ± 0.814i</td>
<td>-2.060 ± 27.084i</td>
</tr>
<tr>
<td>-0.340 ± 35.354i</td>
<td>0.137 ± 0.824i</td>
<td>-4.507 ± 35.144i</td>
</tr>
<tr>
<td>-0.406 ± 41.068i</td>
<td>-0.047 ± 0.949i</td>
<td>-1.270 ± 40.511i</td>
</tr>
</tbody>
</table>

Figure 2-4 Magnitude plots of the continuous and discrete time collocated FRFs associated with displacement measurement
Figure 2-5 Magnitude plots of the continuous and discrete time collocated FRFs associated with velocity measurement

Figure 2-6 Magnitude plots of the continuous and discrete time collocated FRFs associated with acceleration measurement
2.3 Summary

This Chapter, first, presented the two widely used system representations, namely, the transfer function and the state-space representations for both continuous-time and discrete-time systems.

Then, the theory for the zeros of multivariable systems has been presented. In this regard, the invariant zeros, the transmission zeros and the decoupling zeros are defined and their connections have been identified. In particular, it is shown that the invariant zeros always contain the transmission zeros and the decoupling zeros exist only when the system is not both observable and controllable. If they exist, some or all of the decoupling zeros constitute a subset of the invariant zeros depending on the relative value of the number of input and output coordinates. If the number of input and output coordinates is equal, then all of the decoupling zeros are invariant zeros. If the number of inputs is larger than the number of outputs, then every input decoupling zero is an invariant zero and some of the output decoupling zeros may or may not be an invariant zero. If the number of outputs is larger than the number of inputs, then every output decoupling zero is an invariant zero and some of the input decoupling zeros may or may not be an invariant zero.

In the last part, the Chapter discusses the zeros of sampled systems in discrete-time and their connection to the zeros of the original continuous time systems. It is concluded that there is no closed form formulation that directly connects the zeros of the discrete-time systems to their continuous time counterparts. In fact, the set of zeros in the discrete-time may contain additional zeros which are not associated with any zeros of the original continuous time system and are referred to as discretization zeros whose locations are determined as a function of sampling period, the hold being used to deliver the input and the relative degree of the continuous time transfer function. It is shown through a numerical example that if the sampling period is not properly selected, the location of the zeros of a sampled system that is associated with a continuous-time system with zero relative degree can significantly be distorted if the approximate relation \( z = e^{\Delta t} \) is used to connect the two sets of zeros.
This Chapter is on the use of the cepstrum technique to estimate the zeros of transfer functions of a single-input multiple-output (SIMO) system from its response measurements. As it may be apparent, the zeros of interest are associated with transfer functions that relate the input to a particular output coordinate. In other words, the zeros of transfer function matrices that relate multiple input coordinates to multiple output coordinates are outside the scope of the cepstrum technique. The estimation of the zeros of MIMO systems is examined in Chapter 6.

In this Chapter, first, the theoretical basis of the cepstrum technique is presented by emphasizing the two most widely used cepstra: complex cepstrum and power cepstrum. The computation of these cepstra from measured data sequences is presented next. Then, the connection between the response cepstrum and the poles and zeros of the pulse transfer function is established. Two approaches developed for the extraction of the system parameters from the response cepstra are explained and some issues that arise when it comes to the actual implementation of the cepstrum technique with measured data are addressed.

3.1 Introduction

The two frequently utilized and closely related cepstra are the power cepstrum and the complex cepstrum. Power cepstrum is defined as the inverse Fourier transform of the logarithm of the magnitude of the Fourier transform of a signal. Similarly, complex cepstrum is defined as the inverse Fourier transform of the logarithm of the Fourier transform of a signal. The power cepstrum was first introduced by Bogert et al. (1963) to find echoes in seismic signals from earthquakes so that the depth of the seismic source can be determined from echo delay. Bogert et al. (1963) also introduced the following terms by following a syllabic interchange rule:

- frequency…………….quefrency
- spectrum…………….cepstrum
- phase……………….. saphe
- amplitude……………gamnitude
- filtering…………….. liftering
- harmonic…………… rahmonic
- period………………. repiod

The complex cepstrum was developed by Oppenheim (1965) and can be seen to be a more general form of the power cepstrum. The cepstral analysis has been traditionally used in the general problem of the deconvolution of two or more signals (Childers et al., 1977). The cepstrum technique has also been applied in a variety of areas including audio processing, speech processing, geophysics, radar, medical imaging, fault detection, parameter identification etc (Oppenheim and Schafer, 2004). The type of application that
is of interest here is the possibility of recovering system information in the form of poles and zeros directly from response measurements.

One of the earliest applications of the cepstrum concept to the extraction of system information is the work done by Derby (1978) who estimated the pole and zero locations of a discrete-time system by minimizing the error between predicted and measured cepstrum sequences in a least square sense. A similar approach has been proposed by Randall (1984) and applied to the diagnosis of gearbox faults. Later on, Randall and Gao (1994) recovered poles and zeros of frequency response functions (FRFs) of a free-free beam from measured response autospectra by curve-fitting analytical expressions to the selected regions of the response power cepstra which are expected to be dominated by the path effect (FRF). The separation of the input and the path effects in the cepstrum is realized under the premise that the structure is excited from a single input source with a flat and smooth log spectrum such that the contribution of the input mainly concentrates along the earlier part of the quefrency axis. In order to establish the physical interpretation of the recovered zeros, the point of application of the single load has to be known.

With regards to the curve-fitting step of the approach, the number of poles and zeros to be extracted from the data has to be known a-priori and initial trial values of these parameters, which should presumably be close to their true values need to be specified. Moreover, the part of the cepstrum that is mainly dominated by system parameters has to be determined so that the analytical expressions representing the system can be curve-fitted to these regions. It is worth noting that the contribution of the unmeasured input to the cepstrum will be present everywhere along the quefrency axis even when the input has a flat log spectrum, hence the recovered system parameters will always be approximate due to the inevitable contamination coming from the input.

In 1996, Gao and Randall adapted the Ibrahim time domain (ITD) method to extract poles and zeros of transfer functions from the differential cepstrum of the measured response. The differential cepstrum of a sequence is defined as the inverse z-transform of the ratio of the z-transform of a sequence that is obtained by multiplying each entry of the original sequence by its index to the z-transform of the sequence itself (see Section 3.3.2 for details). The advantage of using differential cepstrum with the ITD method over the nonlinear curve-fitting algorithm is that the number of the poles and zeros together with their initial values need not to be specified a priori anymore. The downside of the ITD method, however, is that one has to discriminate system related parameters (poles and zeros) from the computational and/or noise related parameters since ITD method, like any other system identification technique, identifies an oversized parameter space. Moreover, the differential cepstrum of measured output data is more sensitive to aliasing than power or complex cepstra which further restricts the range of application areas of the technique.

In order to extend the cepstrum technique to multiple input case, Randall et. al, (1998) made use of singular value decomposition of a square response spectral matrix. It has been concluded that as long as there is a single dominant excitation, the approach may be
extended to systems with multiple inputs. Obviously, the idea is to extract the part of the response associated with the dominant excitation, hence, reducing the system from a multiple input case to a single input case in an approximate way.

More recently, the cepstrum analysis has been extended to multiple-input multiple-output (MIMO) systems excited by at least one cyclostationary input with a unique cyclic frequency (Hanson et al. 2007a). Cyclostationary signals are a special class of nonstationary signals which exhibit periodicity in their statistics. It has been shown that the power spectral density (PSD) may be evaluated at a particular frequency shift, corresponding to the unique statistical periodicity in the cyclostationary signal (Antoni et al., 2002). At this frequency shift, the only contribution to the PSD will come from that cyclostationary signal. The system is thereby reduced from a MIMO scenario to a SIMO scenario. Then, the cepstrum technique is used in the usual way. Obviously, the requirement on the input having a flat and smooth logarithmic spectrum is always necessary to separate the contribution of the input from that of the transfer function in the cepstrum.

3.2 Theoretical Background

In this Section, the theoretical background for the two widely-used cepstra is presented: (1) Complex cepstrum, (2) Power cepstrum.

3.2.1 Complex Cepstrum

In its basic form, the complex cepstrum can be defined as the inverse Fourier transform of the logarithm of the Fourier transform of a time signal. If the time signal of interest is the response, \( y(t) \), of a linear time-invariant system to an excitation, \( u(t) \), it can be represented by the following convolution integral,

\[
y(t) = \int_{0}^{t} h(t-\tau)u(\tau)d\tau
\]

(3.1)

where \( h(t) \) is the impulse response function. Following the definition, we first take Fourier transform of Eq (3.1), which translate the convolution integral to a multiplication in the frequency domain;

\[
Y(\omega) = G(\omega)U(\omega)
\]

(3.2)

where \( \omega \) is frequency variable and \( G \) is the transfer function. Next, we take the logarithm of Eq. (3.2) and get

\[
\log[Y(\omega)] = \log[G(\omega)] + \log[U(\omega)]
\]

(3.3)
The last step in the cepstrum analysis is to take an inverse Fourier transform of Eq. (3.3) which yield the complex cepstrum of the response,

\[ \mathcal{Y}_c(\xi) = \mathcal{F}^{-1} \log[Y(\omega)] = \overline{h}_c(\xi) + \overline{u}_c(\xi) \]  

(3.4)

where \( \mathcal{Y}_c(\xi) \) is the complex cepstrum of the response, \( \mathcal{F}^{-1} \) is the inverse Fourier transform, \( \overline{h}_c(\xi) = \mathcal{F}^{-1} \log[G(\omega)] \) is the complex cepstrum of the impulse response function which contains the information on the transfer function (path effect), \( \overline{u}_c(\xi) = \mathcal{F}^{-1} \log[U(\omega)] \) is the complex cepstrum of the input (source effect) and the subscript \( c \) is selected to refer to complex cepstrum. The variable of the cepstrum is called quefrency and has units of time. For the sake of clarity, we present a very simple example next.

**Example:**

A 5 story shear frame with story stiffnesses of \{800, 700, 600, 650, 500\} and story masses of \{1.20, 1.15, 1.05, 1.25, 1.30\} with a 1\% classical damping in each mode is considered. The structure is excited from the second mass by a burst random excitation and the acceleration response of the same mass is measured as output. The data is sampled at 50 Hz. The magnitude spectrum of the input is given in Figure 3-1. The complex cepstrum of the input (source effect), the impulse response function (path effect) and the measured response are plotted in Figure 3-2 and Figure 3-3 for the quefrency ranges of 0 – 1 and 1- 3 seconds respectively. Note that the sum of the source and path effects is identical to the complex cepstrum of the measured response. For the purposes of extracting system information from measured response, it is necessary to utilize the regions of the cepstrum where the cepstrum of the measured response can be approximated as the contribution of the path effect only. It is clear from the figures that the correspondence gets better and better when one gets away from the origin. The correspondence can be better observed in Figure 3-3 than Figure 3-2.
Figure 3-1 Magnitude of the Fourier transform of the burst random input

Figure 3-2 Complex cepstra of the input, the impulse response function and the measured response in the range of 0 - 1 sec
3.2.2 Power Cepstrum

The power cepstrum of a time signal can be defined as the inverse Fourier transform of the logarithm of the magnitude of the Fourier transform of the time signal. Following the same notations, one has the magnitude of the Fourier transform of $y(t)$ as,

$$ |Y(\omega)| = |G(\omega)||U(\omega)| $$  \hspace{1cm} (3.5)

then, the logarithm of Eq. (3.5) is

$$ \log[|Y(\omega)|] = \log[|G(\omega)|] + \log[|U(\omega)|] $$  \hspace{1cm} (3.6)

and inverse Fourier transform of Eq. (3.6) yields

$$ \bar{y}_p(\xi) = \mathcal{F}^{-1}\log[|Y(\omega)|] = \bar{h}_p(\xi) + \bar{u}_p(\xi) $$  \hspace{1cm} (3.7)

where $\bar{y}_p(\xi)$ is the power cepstrum of the response, $\bar{h}_p(\xi) = \mathcal{F}^{-1}\log[|G(\omega)|]$ is the power cepstrum of the impulse response function, $\bar{u}_p(\xi) = \mathcal{F}^{-1}\log[|U(\omega)|]$ is the power cepstrum of the input and the subscript $p$ refers to the power cepstrum.
Another definition of the power cepstrum operates on the autocorrelation function \( R_{yy} \) of the time signal of interest. Autocorrelation function for a stationary process, \( y(t) \), can be defined as (Bendat and Piersol, 2000)

\[
R_{yy}(\tau) = E[y(t)y(t+\tau)]
\]  

(3.8)

where \( E \) is the expectation operator. Following the general steps of the cepstrum technique, first we take Fourier transform of Eq. (3.8) which yields auto-spectral density function \( S_{yy} \), which can be written in terms of the transfer function and the spectral density of the input \( S_{uu} \) as (Wiener-Khinchine relations)

\[
S_{yy}(\omega) = |G(\omega)|^2 S_{uu}(\omega)
\]  

(3.9)

taking the logarithm of both sides of Eq. (3.9) gives

\[
\log[S_{yy}(\omega)] = 2\log|G(\omega)| + \log[S_{uu}(\omega)]
\]  

(3.10)

then, as a last step, taking an inverse Fourier transform of Eq. (3.10), one gets the power cepstrum, \( \overline{y}_{po}(\xi) \),

\[
\overline{y}_{po}(\xi) = \mathcal{F}^{-1}\log[S_{yy}(\omega)] = \overline{h}_{po}(\xi) + \overline{u}_{po}(\xi)
\]  

(3.11)

where \( \overline{h}_{po}(\xi) = 2\mathcal{F}^{-1}\log|G(\omega)| \) is the transfer function contribution to the power cepstrum, and \( \overline{u}_{po}(\xi) = \mathcal{F}^{-1}\log[S_{uu}(\omega)] \) is the power cepstrum of the autocorrelation function of the input. Note that the relationship between the two power cepstra definitions given above with regards to the transfer function contributions can be expressed as

\[
\overline{h}_{po}(\xi) = 2\overline{h}_p(\xi)
\]  

(3.12)

**Example:**

The same structure presented in Section 3.2.1 is also considered here. The power cepstrum of the input (source effect), the impulse response function (path effect) and the measured response are plotted in Figure 3-4 and Figure 3-5 for the quefrency ranges of 0 – 1 and 1- 3 seconds respectively. Note that the sum of the source and path effects is identical to the power cepstrum of the measured response. For the purposes of extracting system information from measured response, it is necessary to find the regions of the quefrency axis where \( \overline{y}_{po}(\xi) \approx \overline{h}_{po}(\xi) \). As it is clear from the figures, the equality assumption can be better satisfied when one gets away from the origin.
Figure 3-4 Power cepstra of the input, the impulse response function and the measured response in the range of 0 - 1 sec

Figure 3-5 Power cepstra of the input, the impulse response function and the measured response in the range of 1 - 3 sec
3.2.3 The Connection between Complex and Power Cepstra

The relationship between the complex and power cepstra can be established by considering the following equality.

\[
|Y(\omega)|^2 = Y(\omega)Y^*(\omega)
\]  

(3.13)

where superscript (*) refers to complex conjugation. Taking the logarithm of Eq. (3.13) yields

\[
2\log|Y(\omega)| = \log|Y(\omega)| + \log|Y^*(\omega)|
\]  

(3.14)

since \(Y^*(\omega) = Y(-\omega)\), by taking inverse Fourier transform of the both sides of Eq. (3.14) one gets,

\[
\bar{y}_p(\xi) = \frac{\bar{y}_c(\xi) + \bar{y}_c(-\xi)}{2}
\]  

(3.15)

Eq. (3.15) shows that power cepstrum is equal to the even part of the complex cepstrum. One of the fundamental differences between complex and power cepstra is that the computation of the complex cepstrum is complicated due to the fact that complex logarithm is multi-valued; hence a phase unwrapping step needs to be included just after the complex logarithm is computed. The power cepstrum, on the other hand, involves the logarithm of the magnitude of the Fourier transform of the time signal, therefore, contains no phase information and the problem of phase unwrapping is not an issue. Although the phase information is lost in the power cepstrum, when it comes to stable and minimum-phase systems, there is no discrepancy between the power and complex cepstra.

This follows from the fact that the phase information can be recovered from the log magnitude of the transfer function since they are Hilbert transform pairs provided that the system considered is stable and minimum-phase, which also means that the impulse response is casual. Similarly, the contribution of a stable and minimum-phase system to the complex cepstrum (\(\bar{h}_c(\xi)\)) is right-sided (casual), namely \(\bar{h}_c(\xi) = 0\) when \(\xi < 0\), and it can be related to the contribution of the same system to the power cepstra with the help of Eq. (3.15) as \(\bar{h}_{po}(\xi) = 2\bar{h}_p(\xi) = \bar{h}_c(\xi)\). In fact, the comparison of the power and complex cepstra plotted in Figure 3-2 and Figure 3-3 with Figure 3-4 and Figure 3-5 reveals that the power and complex cepstra of the path effect are identical.

3.3 Practical Implementation of the Cepstrum Concept

The cepstra definitions given in Section 3.2 involve Fourier transforms, hence valid only for continuous time functions. However, in reality, what is available is just a sampled version of these continuous functions. So, more realistic definitions involving discrete
Fourier transforms or z-transforms of available data sequences are needed. In that case, the system information present in the cepstrum will be associated with the pulse transfer function instead of the transfer function of the continuous-time system. Therefore, the poles and the zeros that may be recovered from the cepstrum will be the poles and the zeros of the pulse-transfer function rather than that of the continuous system’s transfer function. As explained in Chapter 2, while there is a one-to-one correspondence between the poles of the pulse-transfer function and the transfer function of the continuous system, the same is not true for the zeros due to the additional zeros (discretization zeros) present in the pulse-transfer function. Therefore, there are basically two issues one has to be careful about. The first one is whether one has to account for all or some of the contribution of the discretization zeros to the cepstrum while extracting the poles and the zeros of the system. And, the second issue is how well one can estimate the zeros of the continuous system’s transfer function from the zeros of the pulse transfer function of the sampled system, given that there are no closed form equation for the transfer (Astrom, 1984). While the first issue is addressed in Section 3.3.4.1, the second issue has been investigated in a more general context at the end of the second Chapter.

### 3.3.1 Complex and Power Cepstra of Sequences

The complex cepstrum of a sequence can be defined as the inverse z-transform of the logarithm of the z-transform of the sequence. If the sequence of interest is the sampled response, \( y(n) \), of a linear time invariant system to an input sequence, \( u(n) \), it can be expressed by the following convolution sum

\[
y(n) = \sum_{j=-\infty}^{\infty} h(j)u(n-j)
\]

where \( h(j) \) is the sequence composed of the impulse response samples of the system. Following the definition of the complex cepstrum, one obtains

\[
Y(z) = H(z)U(z)
\]

\[
\hat{Y}(z) = \log[Y(z)] = \log[H(z)] + \log[U(z)]
\]

\[
\hat{y}_c(n) = Z^{-1}\left[\hat{Y}(z)\right] = \hat{h}_c(n) + \hat{u}_c(n)
\]

where \( H(z) \) is the transfer function (usually referred to as pulse transfer function) representing the sampled continuous-time system transfer function and \( Z^{-1} \) refers to the inverse z-transform. Similarly, power cepstrum definition can be given by the following set of equations.

\[
|Y(z)| = |H(z)||U(z)|
\]

\[
\hat{Y}_p(z) = \log[|Y(z)|] = \log[|H(z)|] + \log[|U(z)|]
\]

\[
\hat{y}_p(n) = Z^{-1}\left[\hat{Y}_p(z)\right] = \hat{h}_p(n) + \hat{u}_p(n)
\]
The definition of the power cepstrum involving spectral density functions can also be represented for data sequences as,

\[ S_{yy}(z) = |H(z)|^2 S_{uu}(z) \]  
(3.23)

\[ \log[S_{yy}(z)] = 2 \log[|H(z)|] + \log[S_{uu}(z)] \]  
(3.24)

\[ \hat{y}_{po}(n) = Z^{-1} \log[S_{yy}(z)] = \hat{h}_{po}(n) + \hat{u}_{po}(n) \]  
(3.25)

and the contribution of the transfer function to the two power cepstra are related to each other through

\[ \hat{h}_{po}(n) = 2 \hat{h}_p(n) \]  
(3.26)

### 3.3.2 Differential Cepstrum

The differential cepstrum of a sequence is defined as the inverse z-transform of the ratio of the z-transform of the sequence that is multiplied by its index to the z-transform of the sequence (Polydoros & Fam, 1981). Namely,

\[ \hat{y}_{dc} = Z^{-1} \left[ \frac{Z[n.y(n)]}{Z[y(n)]} \right] \]  
(3.27)

The connection between differential cepstrum and the complex cepstrum can be established by making use of the derivative property of the z-transform, which enables one to write Eq. (3.27) as

\[ \hat{y}_{dc} = Z^{-1} \left[ -z \frac{dY(z)}{dz} \frac{dY(z)}{dz} \right] \]  
(3.28)

Note that z-transform of the complex cepstrum is given by

\[ Z[\hat{y}_c] = \log[Y(z)] \]  
(3.29)

and the z-transform of the complex cepstrum which is multiplied by its index can be written as

\[ Z[n\hat{y}_c(n)] = -z \frac{d \left( Z[\hat{y}_c(n)] \right)}{dz} = -z \frac{d \left( \log[Y(z)] \right)}{dz} \]  
(3.30)
A comparison of Eqs. (3.30) and (3.28) reveals the connection between the differential and complex cepstra, namely

\[ \tilde{y}_{dc}(n) = n\tilde{y}_c(n) \quad \text{for } n \neq 0 \quad (3.31) \]

Eq. (3.31) clearly shows that if the differential cepstrum is available, one can calculate the complex cepstrum directly from the differential cepstrum. The main advantage of computing the complex cepstrum from the differential cepstrum is that the calculation of the complex logarithm which requires phase unwrapping can be avoided. While that is in fact the case, it has been demonstrated that the down-side of the differential cepstrum lies in the fact that it is more prone to aliasing problem than the complex cepstrum when discrete Fourier transforms are used (Reddy & Rao, 1987). The reason lies in the periodic nature of the discrete Fourier transforms. For example the complex cepstrum computed with a discrete Fourier transform of length \( N \) will be periodic and its relation to the true complex cepstrum, \( \overline{y}_c(n) \), can be expressed as

\[ \tilde{y}_c(n) = \sum_{k=-\infty}^{\infty} \overline{y}_c(n + kN) \quad (3.32) \]

since \( \tilde{y}_c(n) \) is decaying at least as fast as \( 1/n \), the aliasing can be reduced simply by increasing \( N \) which can be done by appending zeros to the original sequence. When the complex cepstrum is to be computed from differential cepstrum, one has

\[ \tilde{y}_c(n) = \frac{1}{n} \sum_{k=-\infty}^{\infty} (n + kN)\overline{y}_c(n + kN) \quad (3.33) \]

which clearly shows that the complex cepstrum computed from differential cepstrum will always be more aliased than the complex cepstrum computed directly from data by computing the complex logarithm with a phase unwrapping step. While appending zeros may help alleviate aliasing in general, it does not solve the problem completely. An interesting approach to the aliasing problem has been developed by Zazula (1998) who showed that the aliasing problem can be asymptotically avoided by using discrete Fourier transforms with interpolation in the frequency domain.

3.3.3 Pulse Transfer function and its Cepstrum

Since the main objective of the dissertation is to estimate poles and zeros from response measurements, it is essential to represent the contribution of the path to the cepstrum. For that purpose, let's consider the pulse transfer function, \( H(z) \) expressed in the following rational polynomial form (Oppenheim and Schafer, 1989)
where $A$ is the gain, $r$ is the delay term and $|a_k|, |b_k|, |c_k|, |d_k|$ are all less than unity such that $a_k$ and $c_k$ are the zeros and poles, respectively, inside the unit circle, and the $b_k$ and $d_k$ are the zeros and poles outside the unit circle with integers $m_o, p_o, m_o$ and $p_o$ being their numbers respectively. The logarithm of Eq. (3.34) can be written as,

$$
\log\left[H(z)\right] = \log[A] + \log\left[z^r\right] + \sum_{k=1}^{m_o} \log\left[1 - a_k z^{-1}\right] + \sum_{k=1}^{m_o} \log\left[1 - b_k z\right]
- \sum_{k=1}^{p_o} \log\left[1 - c_k z^{-1}\right] - \sum_{k=1}^{p_o} \log\left[1 - d_k z\right]
$$

(3.35)

Since $A$ is real, $\log[A]$ exists only if $A>0$ and it contributes only to the first term of the cepstrum. If $A<0$, it is more difficult to determine the contribution. In order to see this, recognize the fact that a negative $A$ can expressed as $A = |A|e^{i\pi}$ and its logarithm is $\log[A]=\log(|A|) + i\pi$ and it contributes only to the first term in the cepstrum but this time the contribution is complex and it is equal to $\log(|A|) + i\pi$. Realizing the fact that cepstrum is defined to be a real sequence, it can not take a value like $\log(|A|)+i\pi$, hence the ambiguity prevails in the case of negative $A$. The term $z^r$ corresponds to a delay or advance of the sequence $h(n)$. When $r=0$, this term vanishes from Eq.(3.35), otherwise it will contribute to the cepstrum. In practice, if the sign of $A$ and the value of $r$ can be measured, then the problem can be avoided simply by altering the input such that $A$ becomes positive and $r$ goes to zero. However, in civil or mechanical engineering applications where only the response measurements are available, the information on $A$ and $r$ is already convolved with the unmeasured input, hence the proposed solution cannot be pursued. Bearing in mind these difficulties in the complex cepstrum, we move forward with the case where $r = 0$ and $A$ is positive and look at the remaining terms in Eq. (3.35) which are of the form $\log \left[1 - \alpha z^{-1}\right]$ and $\log \left[1 - \beta z\right]$. By noting the fact that the convergence region of the $z$-transform must include the unit circle, the logarithms can be represented by the following power-series expansions

$$
\log\left[1 - \alpha z^{-1}\right] = -\sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^{-n}, \quad |z| > |\alpha|
$$

(3.36)

$$
\log\left[1 - \beta z\right] = -\sum_{n=1}^{\infty} \frac{\beta^n}{n} z^n, \quad |z| < |\beta^{-1}|
$$

(3.37)

Using these expressions, the complex cepstrum of the pulse transfer function given in Eq.(3.34) with a positive $A$ and $r = 0$ can be expressed as,
\[
\tilde{h}_c(n) = \begin{cases} 
\log[|A|] & n = 0 \\
-\sum_{k=1}^{m_i} a_k^n / n + \sum_{k=1}^{p_i} c_k^n / n & n > 0 \\
\sum_{k=1}^{m_o} b_k^{-n} / n - \sum_{k=1}^{p_o} d_k^{-n} / n & n < 0
\end{cases}
\tag{3.38}
\]

Eq. (3.38) shows that if the system is stable and minimum-phase there will be no poles and zeros outside the unit circle, so \( \tilde{h}_c(n) = 0 \) when \( n < 0 \). In this particular case, with the help of Eq. (3.15), it becomes clear that complex cepstrum is related to the power cepstrum through

\[
\tilde{h}_c(n) = \begin{cases} 
\tilde{h}_p(n) & n = 0 \\
2\tilde{h}_p(n) & n > 0 \\
0 & n < 0
\end{cases}
\tag{3.39}
\]

and,

\[
\tilde{h}_c(n) = \begin{cases} 
\tilde{h}_{po}(n) / 2 & n = 0 \\
\tilde{h}_{po}(n) & n > 0 \\
0 & n < 0
\end{cases}
\tag{3.40}
\]

where Eq. (3.40) follows from Eq.(3.26). If the system is non-stable and maximum phase (i.e., all poles and zeros are outside the unit circle), a parallel discussion yields

\[
\tilde{h}_c(n) = \begin{cases} 
\tilde{h}_p(n) & n = 0 \\
0 & n > 0 \\
2\tilde{h}_p(n) & n < 0
\end{cases}
\tag{3.41}
\]

and

\[
\tilde{h}_c(n) = \begin{cases} 
\tilde{h}_{po}(n) / 2 & n = 0 \\
0 & n > 0 \\
\tilde{h}_{po}(n) & n < 0
\end{cases}
\tag{3.42}
\]

If the system is of mixed type, that is there are poles and zeros both inside and outside the unit circle, we cannot establish a one-to-one relationship between complex and power cepstra. In that case, power cepstra contain the information regarding the poles and zeros inside the unit circle plus the poles and zeros outside the unit circle that are mapped to
inside the unit circle to the points that correspond to their reciprocals. This follows from a
careful consideration of Eqs. (3.15) and (3.38). It is worth noting that an interesting
application of the complex cepstrum is to obtain a stable and minimum-phase (causal)
sequence from a non-minimum phase one. This can be easily achieved by forcing \( \tilde{h}_c(n) = 0 \) when \( n<0 \), then taking an inverse cepstrum of the resulting sequence. Note that this
application cannot be pursued with power cepstrum since the contributions of the poles
and zeros that are inside and outside of the unit circle are combined in the power
cepstrum.

3.3.4 Curve-fitting the Cepstrum

Observing the fact that Eq. (3.38) presents the analytical expressions for the contribution
of a pulse transfer function to the complex cepstrum in terms of its poles and zeros, it has
been suggested to curve-fit this analytical structure to the part of the measured cepstrum
(\( \tilde{y}_c(n) \) or \( \tilde{y}_p(n) \) or \( \tilde{y}_{ps}(n) \)) where the contribution of the input is minimal in relative
terms. In order to curve-fit the measured cepstrum data to the analytical expression of Eq.
(3.38), a non-linear optimization has to be carried out such that the discrepancies between
the measured cepstrum and the analytical estimate, which is obtained from a set of trial
poles and zeros, is minimized by adjusting the values of the poles and zeros. At this point,
it should be noted that the number of poles and zeros in the frequency band of interest
should be known as a priori knowledge from a finite element model or inspection of the
auto-spectrum of the response or any other means. For a stable minimum-phase system,
the complex cepstrum becomes right-sided and the analytical expression reduces to

\[
\tilde{h}_c(n) = \frac{1}{n} \sum_{k=1}^{m} a_k^n + \sum_{k=-1}^{p} c_k^n, \quad n > 0 \quad (3.43)
\]

Although Eq. (3.43) can be utilized in the curve-fitting process directly, a more
convenient form which allows certain physical constraints to be enforced within the
optimization algorithm follows next. Once the poles and zeros with their complex
conjugate pairs (which are denoted by \( * \)) are expressed in exponential form as,

\[
a_k = A_{ak} e^{i\omega_{ak}} \quad a^*_k = A_{ak} e^{-i\omega_{ak}} \quad (3.44)
\]
\[
c_k = A_{ck} e^{i\omega_{ck}} \quad c^*_k = A_{ck} e^{-i\omega_{ck}} \quad (3.45)
\]

where \( A_{ak} \) and \( A_{ck} \) are the magnitudes, and \( \omega_{ak} \) and \( \omega_{ck} \) are the phase angles of the \( k^{th} \) zero
and pole, respectively, it is a simple matter to see that Eq. (3.43) can also be written as

\[
\tilde{h}_c(n) = 2 \sum_{k=1}^{\frac{m}{2}} \frac{1}{n} A_{ck}^n \cos(n\omega_{ck}) - 2 \sum_{k=1}^{\frac{p}{2}} \frac{1}{n} A_{ak}^n \cos(n\omega_{ak}), \quad n > 0 \quad (3.46)
\]

The set of parameters that are updated within the optimization framework can be
combined in the following vector,
One advantage of exploiting the analytical structure given in Eq. (3.46) over Eq. (3.43) is that the following set of constraint conditions can be enforced within the optimization framework

\[ \beta = \{ A_{c1}, \omega_{c1}, \ldots , A_{rpi/2}, \omega_{rpi/2}, A_{a1}, \omega_{a1}, \ldots , A_{ami/2}, \omega_{ami/2} \}^T \]  

(3.47)

\[ 0 < |A_{ck}| < 1 \quad -\pi < \omega_{ck} < \pi \]  

(3.48)

\[ 0 < |A_{ak}| < 1 \quad -\pi < \omega_{ak} < \pi \]  

(3.49)

It is opportune to note that one of the fundamental difficulties associated with the curve-fitting is that one has to decide on the size of the vector \( \beta \) and select a set of initial values for the poles and zeros before starting the optimization process. As is the case with any non-convex optimization problem, success depends on conditions such as how close the initial trial values are to the correct values of the poles and the zeros and whether the correct number of parameters is being optimized.

Several numerical simulations performed in order to assess how these conditions are important for the particular application here revealed that when the number of parameters selected to be curve-fitted is larger than what is really needed, the optimization was able to converge to the correct poles and zeros, while the extra poles and zeros were simply pushed toward the origin in the z-plane so that their contribution to the cepstrum were insignificant. On the other hand, when the parameter set selected to be curve-fitted contained less number of poles and zeros than the correct number of poles and zeros, the optimization produced poor results for the parameters curve-fitted, while of course, the ones that are skipped could not be extracted. It has also been observed that the distortion introduced to the estimated parameters in the second case were found to be stronger in the poles and zeros that are located close to the ones that are skipped. So, it can be concluded that when the number of poles and zeros is not clearly available, it is preferable to overspecify the order.

On the other hand, the success of the optimization depends on how close the initial trial values selected are to the true values. Numerical simulations indicated that when the number of poles and zeros selected is correct, the optimization sometimes converged at a local minimum. However, the value of the cost function at these local minima was found to be larger than the value of the cost function that is found at the true minimum. In other words, it is possible to run a number of cases in which a different set of initial values are selected and the best solution set can be selected as the ones with the least value of the cost function. When the number of poles and zeros selected is not correct, based on the previous paragraph, it is always better to overspecify the order. Numerical simulations suggested that when the order is overspecified with an incorrect set of initial values of the poles and the zeros, it has been found that the optimization sometimes converged to a local minimum. However, similar to the case where the true order is selected, the convergence attained with the least value of the cost function corresponded to the best solution set that is possible with the given quality of the data.
Another difficulty one faces when it comes to the curve-fitting in the cepstrum domain is that the user has to decide on the region of the cepstrum to which the analytical expressions are to be curve-fitted. The basic criterion is to select the parts of the cepstrum where the contribution of the input is minimal with respect to the contribution of the system parameters. This requires a priori knowledge on the characteristics of the excitation source. Besides, the contribution of the input is never zero anywhere in the cepstrum, which makes the estimates of the system parameters always approximate. In order to illustrate main points made above, the following example has been considered.

Example:

A 5 story shear frame with story stiffnesses of \{800, 700, 600, 650, 500\} and story masses of \{1.20, 1.15, 1.05, 1.25, 1.30\} with a 1% classical damping in each mode is considered. The natural frequencies of the structure range between 1 Hz and 7.25 Hz. The response has been measured with a sampling frequency of 50 Hz.

Part I (Over- vs Under- specified number of parameters)

In this part of the example, the effect of curve-fitting with an over- or under specified number of parameters to the accuracy of the estimates has been examined. For that purpose, the power cepstrum of the displacement response of the second mass to a pulse applied at the same mass has been used. It is expected to have 10 poles and 9 zeros in the pulse-transfer function where one of the zeros should be located around $z = -1$ (discretization zero). Two cases are simulated: (1) The number of poles and zeros are overspecified by 6 and 2 respectively; (2) both the number of poles and the zeros are underspecified by 2.

The true poles and zeros of the collocated pulse transfer function together with the trial and estimated poles and zeros for Case 1 are listed in Table 3-1. Note that additional poles and zeros are presented in italic. As it is clear from the table, the poles and zeros estimated are exact at least within the digits presented. Some of the additional poles and zeros are pushed to the same location in the complex plane such that their contributions to the cepstrum are cancelled. The rest of the additional poles are pushed near the origin in the $z$-plane so that their contribution to the cepstrum becomes negligible.

Similarly, the true poles and zeros together with the trial and estimated poles and zeros for Case 2 are listed in Table 3-2. A comparison of Table 3-1 and Table 3-2 reveals that the estimates obtained in the second case are less accurate and the deviations are more significant for the poles and zeros located near the ones that are skipped.
Table 3-1 True, trial and estimated poles and zeros for Case 1

<table>
<thead>
<tr>
<th>True Poles</th>
<th>True Zeros</th>
<th>Trial Poles</th>
<th>Trial Zeros</th>
<th>Estimated Poles</th>
<th>Estimated Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.989 ± 0.135i</td>
<td>0.979 ± 0.193i</td>
<td>0.993 ± 0.110i</td>
<td>0.980 ± 0.191i</td>
<td>0.989 ± 0.135i</td>
<td>0.979 ± 0.193i</td>
</tr>
<tr>
<td>0.923 ± 0.375i</td>
<td>0.856 ± 0.506i</td>
<td>0.929 ± 0.361i</td>
<td>0.891 ± 0.444i</td>
<td>0.923 ± 0.375i</td>
<td>0.856 ± 0.506i</td>
</tr>
<tr>
<td>0.829 ± 0.549i</td>
<td>0.755 ± 0.645i</td>
<td>0.849 ± 0.518i</td>
<td>0.829 ± 0.548i</td>
<td>0.755 ± 0.645i</td>
<td></td>
</tr>
<tr>
<td>0.704 ± 0.699i</td>
<td>0.676 ± 0.726i</td>
<td>0.797 ± 0.593i</td>
<td>0.740 ± 0.662i</td>
<td>0.704 ± 0.699i</td>
<td>0.676 ± 0.726i</td>
</tr>
<tr>
<td>0.608 ± 0.783i</td>
<td>-0.996</td>
<td>0.734 ± 0.668i</td>
<td>-0.999</td>
<td>0.608 ± 0.783i</td>
<td>-0.996</td>
</tr>
</tbody>
</table>

Table 3-2 True, trial and estimated poles and zeros for Case 2

<table>
<thead>
<tr>
<th>True Poles</th>
<th>True Zeros</th>
<th>Trial Poles</th>
<th>Trial Zeros</th>
<th>Estimated Poles</th>
<th>Estimated Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.989 ± 0.135i</td>
<td>0.979 ± 0.193i</td>
<td>0.99 ± 0.134i</td>
<td>0.903 ± 0.421i</td>
<td>0.990 ± 0.135i</td>
<td>0.979 ± 0.193i</td>
</tr>
<tr>
<td>0.923 ± 0.375i</td>
<td>0.856 ± 0.506i</td>
<td>0.936 ± 0.342i</td>
<td>0.760 ± 0.640i</td>
<td>0.924 ± 0.375i</td>
<td>0.854 ± 0.506i</td>
</tr>
<tr>
<td>0.829 ± 0.549i</td>
<td>0.755 ± 0.645i</td>
<td>0.83 ± 0.547i</td>
<td>0.692 ± 0.711i</td>
<td>0.831 ± 0.550i</td>
<td>0.743 ± 0.636i</td>
</tr>
<tr>
<td>0.704 ± 0.699i</td>
<td>0.676 ± 0.726i</td>
<td>0.765 ± 0.634i</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.608 ± 0.783i</td>
<td>-0.996</td>
<td>-</td>
<td>-0.999</td>
<td>0.601 ± 0.775i</td>
<td>-0.996</td>
</tr>
</tbody>
</table>

**Part II (The distance between the initial trial values and the correct values of the poles and the zeros):**

In the second part of the example, the effect of having incorrect initial estimates to the accuracy of the estimated parameters that are obtained by curve-fitting has been examined. The structure is excited by a burst random excitation at the second mass. The loading is simulated by a 1000-point long white noise followed by a 9000-point long zero padding. The displacement response has been measured at the same mass and then contaminated by an additive white noise that has an RMS equal to the 3% of the RMS of the response. The power cepstrum of the resulting noisy data has been used for the extraction of the poles and the zeros of the collocated transfer function.

Four different sets of initial values for the poles and zeros to be estimated are prepared and the curve-fitting algorithm has been used for each of these sets separately. The initial and the estimated values of the poles and the zeros in each of these cases together with the value of the cost function when the convergence is attained are presented in Table 3-3,
Table 3-4, Table 3-5 and Table 3-6 (True values of the poles and zeros can be obtained from the first part of the example).

In particular, Table 3-3 presents the results for the case where the initial set is selected to be the true values and the converged results correspond to the minimum value of the cost function (0.3244). Table 3-4 presents the results for a run where the initial values are not the true values, yet the optimization was successful in converging to the true minimum of the cost function, (i.e. 0.3244). Table 3-5 and Table 3-6 present the cases where the optimization converged to local minima with the given initial selection of the poles and zeros. Note that the value of the cost function at these local minima (0.4704 and 0.5854) is more than the value of the cost function at the true minimum, which indicates that the true minimum can be reached by trying different set of initial values and checking the value of the cost function.

### Table 3-3 Trial and estimated poles and zeros

<table>
<thead>
<tr>
<th>Trial Poles</th>
<th>Estimated Poles</th>
<th>Trial Zeros</th>
<th>Estimated Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.068 ± 6.780i</td>
<td>0.000 ± 6.753i</td>
<td>-0.085 ± 9.745i</td>
<td>-0.259 ± 9.868i</td>
</tr>
<tr>
<td>-0.193 ± 19.302i</td>
<td>-0.080 ± 19.444i</td>
<td>-0.260 ± 26.706i</td>
<td>-0.296 ± 26.522i</td>
</tr>
<tr>
<td>-0.292 ± 29.221i</td>
<td>-0.284 ± 29.125i</td>
<td>-0.340 ± 35.349i</td>
<td>-0.676 ± 35.402i</td>
</tr>
<tr>
<td>-0.391 ± 39.097i</td>
<td>-0.503 ± 38.821i</td>
<td>-0.406 ± 41.066i</td>
<td>-0.494 ± 41.417i</td>
</tr>
<tr>
<td>-0.455 ± 45.542i</td>
<td>-0.359 ± 45.412i</td>
<td>-0.999</td>
<td>0.151</td>
</tr>
</tbody>
</table>

Norm of the residual vector : 0.3244

### Table 3-4 Trial and estimated poles and zeros

<table>
<thead>
<tr>
<th>Trial Poles</th>
<th>Estimated Poles</th>
<th>Trial Zeros</th>
<th>Estimated Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ± 20i</td>
<td>0.000 ± 6.753i</td>
<td>0 ± 21i</td>
<td>-0.259 ± 9.868i</td>
</tr>
<tr>
<td>0 ± 22i</td>
<td>-0.080 ± 19.444i</td>
<td>0 ± 23i</td>
<td>-0.295 ± 26.522i</td>
</tr>
<tr>
<td>0 ± 24i</td>
<td>-0.284 ± 29.125i</td>
<td>0 ± 25i</td>
<td>-0.674 ± 35.400i</td>
</tr>
<tr>
<td>0 ± 26i</td>
<td>-0.506 ± 38.819i</td>
<td>0 ± 27i</td>
<td>-0.497 ± 41.422i</td>
</tr>
<tr>
<td>0 ± 28i</td>
<td>-0.357 ± 45.413i</td>
<td>-0.999</td>
<td>0.151</td>
</tr>
</tbody>
</table>

Norm of the residual vector : 0.3244
Table 3-5 Trial and estimated poles and zeros

<table>
<thead>
<tr>
<th>Trial Poles</th>
<th>Estimated Poles</th>
<th>Trial Zeros</th>
<th>Estimated Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ± 20i</td>
<td>0.000 ± 6.732i</td>
<td>0 ± 20.5i</td>
<td>-0.281 ± 9.939i</td>
</tr>
<tr>
<td>0 ± 21i</td>
<td>-0.706 ± 18.852i</td>
<td>0 ± 21.5i</td>
<td>-0.814 ± 19.922i</td>
</tr>
<tr>
<td>0 ± 22i</td>
<td>0.000 ± 19.729i</td>
<td>0 ± 22.5i</td>
<td>-0.291 ± 26.672i</td>
</tr>
<tr>
<td>0 ± 23i</td>
<td>-0.300 ± 28.915i</td>
<td>0 ± 23.5i</td>
<td>-4.129 ± 39.107i</td>
</tr>
<tr>
<td>0 ± 24i</td>
<td>-0.428 ± 45.667i</td>
<td>-0.999</td>
<td>0.285</td>
</tr>
</tbody>
</table>

Norm of the residual vector : 0.4704

Table 3-6 Trial and estimated poles and zeros

<table>
<thead>
<tr>
<th>Trial Poles</th>
<th>Estimated Poles</th>
<th>Trial Zeros</th>
<th>Estimated Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ± 10i</td>
<td>-0.079 ± 6.750i</td>
<td>0 ± 10.5i</td>
<td>0.000 ± 9.644i</td>
</tr>
<tr>
<td>0 ± 11i</td>
<td>-0.673 ± 10.738i</td>
<td>0 ± 11.5i</td>
<td>0.000 ± 10.911i</td>
</tr>
<tr>
<td>0 ± 12i</td>
<td>-2.797 ± 11.387i</td>
<td>0 ± 12.5i</td>
<td>-6.100 ± 14.340i</td>
</tr>
<tr>
<td>0 ± 13i</td>
<td>-0.058 ± 19.424i</td>
<td>0 ± 13.5i</td>
<td>-7.480 ± 39.572i</td>
</tr>
<tr>
<td>0 ± 14i</td>
<td>-0.407 ± 45.772i</td>
<td>-0.999</td>
<td>0.544</td>
</tr>
</tbody>
</table>

Norm of the residual vector : 0.5854

3.3.4.1 The Contribution of Discretization Zeros to the Cepstrum

The poles and zeros extracted through curve-fitting in the cepstrum are associated with the discrete-time system which is characterized by the pulse transfer function \( H(z) \). While a subset of the discrete-time zeros are associated with the zeros of the continuous system, \textit{intrinsic zeros}, the rest is an outcome of the discretization process, \textit{discretization zeros}, and they are not associated with any physical meaning. While the connection between the poles of the discrete time and that of the continuous time can be easily established through the mapping between the Laplace and z-planes, there is no closed form equation to relate the zeros of the sampled system to that of the continuous time (Astrom et.al, 1984).

The matter of interest here, is to elaborate the effect of the discretization zeros to the estimation of the system parameters through curve-fitting in the cepstrum domain. In particular, the question whether the contribution of the discretization zeros to the cepstrum is significant enough to be accounted for in the curve-fitting process or not is
addressed. As it is the case throughout the dissertation, the discretization process that is connected with an input satisfying ZOH condition has been considered.

Although the location of the discretization zeros in the \( z \)-plane can vary with the sampling period, for sufficiently small sampling periods, it has been shown that they converge to the roots of a particular polynomial defined by the excess number of poles of the continuous time transfer function (Astrom, 1984). Based on the derived formula the discretization zeros are located on the negative real axis in the \( z \)-plane between 0 and -1, if any, and they are accompanied with the discretization zeros that are located at their reciprocals outside the unit circle. As far as the contribution of the discretization zeros to the cepstrum is concerned, it should be clear from Eq. (3.38) that the discretization zeros located away from the point \( z = -1 \) are expected to have negligible contribution with respect to a zero at \( z = -1 \) or with respect to poles and intrinsic zeros that are usually located inside the unit circle close to |\( z \)| = 1. From this perspective, it can be concluded that it is very difficult, if not impossible, to extract the discretization zeros especially the ones located away from \( z = -1 \) by curve-fitting in the quefrency domain due to the fact that their contribution is negligible. However, it is suggested that one should account for the discretization zeros located close to \( z = -1 \) or \( z = 1 \), if any, since their contribution to the cepstrum is comparable to the contribution of any other intrinsic zeros. These points are highlighted through the following numerical example.

**Example:**

The same example that is considered in Section 3.3.4 is also considered here. The power cepstrum of the displacement response of the structure at the second mass due to a pulse applied at the same mass has been used for the extraction of the poles and the zeros of the collocated transfer function. Since the measurement is at a collocated coordinate, we expect to have 10 poles, and 8 zeros in the continuous time transfer function and 10 poles and 9 zeros in the pulse-transfer function where the additional zero should be located at around \( z = -1 \).

The power cepstrum computed from pulse response history does not contain any contribution from the input; hence an exact correspondence can be established in the cepstrum provided that the correct number of poles and zeros are to be curve-fitted. Two curve-fitting analyses have been carried out where in the first one, 10 poles and 9 zeros (the last zero is taken to be real) and in the second one, 10 poles and 8 zeros (the discretization zero is omitted) are considered. The set of initial values are selected such that the real and imaginary parts of the true values (in Laplace plane) are perturbed by adding a randomly selected perturbation which varies between -30% and +30% of the original value. Then, the perturbed values are transformed to \( z \)-plane and used as initial values to the curve-fitting algorithm. The region of the power cepstrum used in the curve-fitting algorithm can be selected arbitrarily since there is no contribution of the input. Therefore, the region defined between the first point and the 200th one has been used for that purpose. The true poles and zeros of the associated pulse transfer function are presented in Table 3-7 and the poles and zeros that are extracted in the two cases are
presented in Table 3-8 which clearly indicates that the effect of the discretization zeros located close to \( z = -1 \) can be quite significant and has to be accounted for.

<table>
<thead>
<tr>
<th>True Poles</th>
<th>True Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.989 ± 0.135i</td>
<td>0.979 ± 0.193i</td>
</tr>
<tr>
<td>0.923 ± 0.375i</td>
<td>0.856 ± 0.506i</td>
</tr>
<tr>
<td>0.829 ± 0.549i</td>
<td>0.755 ± 0.645i</td>
</tr>
<tr>
<td>0.704 ± 0.699i</td>
<td>0.676 ± 0.726i</td>
</tr>
<tr>
<td>0.608 ± 0.783i</td>
<td>-0.996</td>
</tr>
</tbody>
</table>

Table 3-7 True poles and zeros

Table 3-8 The effect of discretization zeros to the extracted discrete-time poles and zeros

<table>
<thead>
<tr>
<th>Estimated Poles</th>
<th>Estimated Zeros</th>
<th>Estimated Poles</th>
<th>Estimated Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.989 ± 0.135i</td>
<td>0.979 ± 0.193i</td>
<td>0.991 ± 0.135i</td>
<td>0.977 ± 0.193i</td>
</tr>
<tr>
<td>0.923 ± 0.375i</td>
<td>0.856 ± 0.506i</td>
<td>0.925 ± 0.374i</td>
<td>0.851 ± 0.508i</td>
</tr>
<tr>
<td>0.829 ± 0.549i</td>
<td>0.755 ± 0.645i</td>
<td>0.833 ± 0.548i</td>
<td>0.719 ± 0.629i</td>
</tr>
<tr>
<td>0.704 ± 0.699i</td>
<td>0.676 ± 0.726i</td>
<td>-0.445 &amp; -0.445</td>
<td>-</td>
</tr>
<tr>
<td>0.608 ± 0.783i</td>
<td>-0.998</td>
<td>0.604 ± 0.780i</td>
<td>-0.892</td>
</tr>
</tbody>
</table>

### 3.3.4.2 Which Cepstrum is More Convenient For Curve-Fitting?

As one can gather from the preceding discussions, a natural question that can be asked is whether the power or complex cepstrum is more advantageous for the purposes of estimating the poles and the zeros of pulse transfer functions from response measurements. While one can look at the question from different perspectives, maybe, one of the most important requirements that need to be satisfied before starting the curve-fitting procedure is that the contribution of the system to the cepstrum has to be separated from that of the input.

From the separation perspective, let’s consider a case where the measured response is given in the form of Eq. (3.17) and let the transfer function and the input in the frequency domain be expressed in the following pole-zero forms

\[
H(z) = \frac{A \prod_{s}(z-z_s)}{\prod_{s}(z-p_s)}
\]  

(3.50)
\[ U(z) = \frac{B \prod_{u}(z - z_u)}{\prod_{u}(z - p_u)} \]  

where \( S \) and \( U \) refer to the system and the input respectively while \( A \) and \( B \) are the gains corresponding to \( H(z) \) and \( U(z) \) respectively. The logarithm of Eq. (3.17) yields

\[
\log[Y(z)] = \log[AB] + \sum_{S} \log[z - z_s] - \sum_{S} \log[z - p_s] + \sum_{U} \log[z - z_u] - \sum_{U} \log[z - p_u]
\]  

and the discrete inverse Fourier transform of Eq. (3.52) produces the complex cepstrum of the measured response

\[
\hat{y}_c(n) = \sum_{SI} \left( \frac{z_{si}}{n} \right)^n - \sum_{SI} \left( \frac{p_{si}}{n} \right)^n + \sum_{UI} \left( \frac{z_{ui}}{n} \right)^n - \sum_{UI} \left( \frac{p_{ui}}{n} \right)^n \quad n > 0
\]  

\[
\hat{y}_c(n) = -\sum_{SO} \left( \frac{z_{so}}{n} \right)^n + \sum_{SO} \left( \frac{p_{so}}{n} \right)^n - \sum_{UO} \left( \frac{z_{uo}}{n} \right)^n + \sum_{UO} \left( \frac{p_{uo}}{n} \right)^n \quad n < 0
\]  

where \( SI \) and \( UI \) refer to the inside of the unit circle associated with the system and the input, \( SO \) and \( UO \) denote the outside of the unit circle again associated with the system and the input respectively. Also, \( |z_{si}|, |p_{si}|, |z_{ui}|, |p_{ui}| \) are all less than unity while \( |z_{so}|, |p_{so}|, |z_{uo}|, |p_{uo}| \) are all greater than unity. Eqs. (3.53) and (3.54) suggest that for a specific region of the cepstrum to have minimal contribution from the input and having maximal contribution from the system, one of the following conditions should hold

- The poles and the zeros of the input located inside the unit circle should have smaller values (closer to the origin) than that of the system so that the contribution of the input to the complex cepstrum would be minimal compared to that of the system whose poles and zeros are usually located close to the unit circle.
- The same argument holds for the negative part of the cepstrum but here we need to have the poles and zeros of the input to be larger than that of the system.

While the comments made above are correct, it might be difficult to come across the kind of inputs that satisfies these requirements in many practical situations. Without going into further discussion on this topic let’s also look at the form of the power cepstrum. With the help of Eq. (3.15), it is not difficult to see that the power cepstrum of the measured response can be written as
Note that Eq. (3.55) offers an additional option for the separation of the input from the system which cannot be realized in the complex cepstrum:

- If the input is such that the terms representing the contribution of the input to the power cepstrum cancels each other out, then a successful separation can be obtained.

Above statement can be summarized as

\[
2\hat{y}_p(n) = \sum_{SI} (z_{w})^n - \sum_{SI} (p_{w})^n + \sum_{UI} (z_{w})^n - \sum_{UI} (p_{w})^n + \sum_{SO} (z_{wo})^{-n} - \sum_{SO} (p_{wo})^{-n} + \sum_{UO} (z_{wo})^{-n} - \sum_{UO} (p_{wo})^{-n} \quad n > 0
\]  \( (3.55) \)

One interpretation of Eq. (3.56) is as follows: If the pole-zero form of the input is such that the zeros that are outside the unit circle are located at the reciprocals of the poles that are inside the unit circle and the poles that are outside the unit circle are located at the reciprocals of the zeros that are inside the unit circle, the power cepstrum of the input is exactly equal to zero since perfect pole-zero cancellations are realized. While this condition looks too good to be true for any particular input that can be observed frequently in reality, it is interesting to see that it is actually approximately satisfied by inputs that have relatively flat spectrum such as white-noise sequences.

Note that if the measured response is due to a load which satisfies the above requirements, then the power cepstrum of the measured response will contain much less contribution from the input with respect to the contribution of the same input to the complex cepstrum. When it comes down to the contribution of the system to the cepstrum, it usually does not differ much between the complex and power cepstra, due to the fact that the systems to be identified are often stable and minimum-phase. Therefore, in such cases the power cepstrum can be seen to be much more preferable than the complex cepstrum for parameter estimation purposes. The main points made in this Section are illustrated in the following example.

Example:

Consider the white noise sequence given in Figure 3-6 which is sampled at 10 Hz for a duration of 200 seconds. When the negative quefrency part of the cepstrum is forced to be zero, the cepstrum contains the information on the poles and zeros that are inside the unit circle in z-plane (left half plane in s-plane). An inverse cepstrum of the resulting sequence provides the data in the time domain whose frequency response function is plotted in Figure 3-7 as solid line. Similarly, in a second attempt, the positive quefrency
part of the cepstrum is forced to be zero and the frequency response function of the resulting sequence is plotted in Figure 3-7 as dotted line. As the figure suggests, the poles and the zeros are almost symmetrically located with respect to the imaginary axis so as to cancel each others’ effects on the frequency axis. Furthermore, the complex and power cepstra of the considered white-noise sequence are plotted in Figure 3-8 which clearly indicates the superior property of the power cepstrum when it comes down to the separation of the input effects from that of the system.

Figure 3-6 Time-history and the magnitude of the frequency response function of a white noise sequence
Figure 3-7 Magnitude of the frequency response functions of the positive (solid line) and negative (dotted line) quefrency components

Figure 3-8 Complex and power cepstra of the white-noise sequence
Going back to Eq. (3.55), one can conclude that if the input is in the form that satisfies Eq. (3.56) even approximately, then a successful curve-fitting may be realized. The approximation is due to the fact that the poles and the zeros do not cancel each other exactly. It is worth noting that with the same measured data, one may not be able to separate the system information from the effect of the input if the complex cepstrum is used. The last statement is made assuming the user who performs the curve-fitting to the complex cepstrum uses the analytical expression given in Eq. (3.43) which corresponds only to the contribution of the system which is given as

$$\hat{h}_{e}(n) = \sum_{s} \frac{(z_{si})^{n}}{n} - \sum_{s} \frac{(p_{si})^{n}}{n} \quad n > 0$$

(3.57)

In that particular case, the user assumes the contribution of the input which is given as

$$\hat{u}_{c} = \sum_{u} \frac{(z_{ui})^{n}}{n} - \sum_{u} \frac{(p_{ui})^{n}}{n}$$

(3.58)

to be zero, which is not correct. If however, the curve-fitting was to be performed by accounting for the contribution of the input as well, which requires a larger set of parameters to be used in the curve-fitting, then one would still be able to estimate the poles and the zeros of the system although the set of poles and zeros estimated in this case contain the poles and the zeros of the input as well. The problem, then, would be to separate the system poles and zeros from that of the input. While the statement made above can be verified through numerical examples, due to the difficulty with regards to the selection of the right order (correct number of poles and zeros associated with both system and the input) and high number of parameters to be curve-fitted, it has been postponed to Section 3.3.4.3 where all of these difficulties can be avoided by using an alternative technique which exploits the differential cepstrum and does not require the expected number of poles and zeros as an input and allows one to estimate all of the poles and the zeros (system and the input together).

The approach taken in this Section clearly explains why the power cepstrum, not the complex cepstrum, of a measured response to an input with a flat-log spectrum has been found to be useful in the literature to estimate the poles and zeros of a transfer function from response measurements (Randall and Gao1994). It is interesting to see that there is no explanation made in the literature as to why the power cepstrum is more appealing for parameter estimation than complex cepstrum and why an input with relatively flat spectrum is needed. It is hoped that the explanation given in this Section addresses these two points.

3.3.4.3 Parameter Estimation from Differential Cepstrum

One of the main difficulties of the nonlinear curve-fitting methodology presented in Section 3.3.4 is that the user has to decide on the number of poles and zeros to be
extracted from the cepstrum. In order to alleviate this difficulty, Gao and Randall (1996) studied the use of differential cepstrum instead of the power cepstra. The benefit of the differential cepstrum over the power or complex cepstra is that the differential cepstrum coefficients resemble the free decay response data and they can be used in any time-domain identification algorithm which uses the free decay data such as Ibrahim time-domain method (Ibrahim and Mikulcik, 1977) or eigensystem realization algorithm, ERA, (Juang and Pappa, 1985; Juang 1994). It is straightforward to show that the part of the differential cepstrum that is associated with the transfer function has the following form

\[
\hat{h}_{dc}(n) = \begin{cases} 
  r & n = 0 \\
  \sum_{k=1}^{m_d} a_k^n + \sum_{k=1}^{p_d} c_k^n & n > 0 \\
  \sum_{k=1}^{m_u} b_k^{-n} - \sum_{k=1}^{p_u} a_k^{-n} & n < 0
\end{cases}
\]  

(3.59)

With the same terminology used in Section 3.3.4.2, the general expression for the differential cepstrum of a measured response can be written in the following form

\[
\tilde{y}_{dc}(n) = \sum_{SI} (z_s)^n - \sum_{SI} (p_{sl})^n + \sum_{UJ} (z_{ul})^n - \sum_{UJ} (p_{ul})^n & n > 0 \\
\tilde{y}_{dc}(n) = -\sum_{SO} (z_{so})^n + \sum_{SO} (p_{so})^n - \sum_{CO} (z_{co})^n + \sum_{CO} (p_{co})^n & n < 0
\]  

(3.60)

(3.61)

Note that for the systems that are stable and minimum phase which is usually the case in many civil or mechanical applications, then the poles and zeros of the system will be located inside the unit circle, hence one need to use the differential cepstrum along the positive quefrency axis as given in Eq. (3.60). When these differential cepstrum coefficients are treated as Markov parameters and used in ERA, for example, one can identify the poles and the zeros of the system together with the poles and the zeros of the input that are located inside the unit circle.

While it alleviates the difficulty with regards to the determination of the correct order of the system, the use of differential cepstrum faces a number of challenges. The first one is the high sensitivity of the differential cepstrum to aliasing. From that perspective, unless the measured time-domain data is decaying fast enough so that the sequence by itself goes almost to zero long before the end of the measurement time segment, the associated differential cepstrum can be distorted so much that it may not carry any useful information.

A second issue which is actually one of the classical problems in system identification field is the discrimination of the true system modes from those that are associated with noise. It is a common practice in system identification field to extract a high order model by taking into consideration the contribution of the noise, which in turn allows more accurate estimates of the true system parameters. Once a high order model is extracted,
the challenge, of course, is to pick the ones associated with the true system. For that purpose, a number of guidelines and techniques have been developed such as modal confidence factor, stabilization diagrams etc. (Auweraer and Peeters, 2004; Pappa et. al, 1998). Most of these guidelines make use of the mode shape information which can only be available if the number of measurements is more than one.

It is worth noting that although in the regular system identification problems the true system modes are associated with the poles, when differential cepstrum data is treated as free decay response the true modes are associated with both poles and zeros. Therefore, the third issue is to decide which of the identified modes correspond to the poles and which of them correspond to the zeros. Gao and Randall (1996) were able to separate the poles from the zeros by checking the values of the mode shapes associated with each mode identified. Obviously, the fundamental requirement for this is, again, to have multiple measurements.

A careful consideration of the form of the differential cepstrum, however, reveals that one can actually separate the poles from the zeros even in the case of a single measurement. The key idea is to realize that the contribution of the poles to the differential cepstrum goes with a negative sign, while that of the zeros goes with a positive sign. Within the ERA context, first the connection between the differential cepstrum coefficients and the Markov parameters has to be established, which can be expressed as

\[
\hat{y}_{dk}(n) = C_d A_d^n B_d \quad n > 0
\]  

where \( C_d, A_d \) and \( B_d \) are the discrete-time output, system and input-to-state matrices respectively. It is straightforward to see that one possible form of these matrices is

\[
C_d = \begin{bmatrix} 1 & \ldots & 1 & -1 & \ldots & -1 \end{bmatrix}
\]  

\[
A_d = \text{diag}\left(p_1, \ldots, p_l, z_1, \ldots, z_k\right)
\]  

\[
B_d = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T
\]

where \( k \) is the number of zeros and \( l \) is the number of poles, and \( \text{diag()} \) represents a diagonal matrix with the values being in the parenthesis. Note that the first Markov parameter \( C_d B_d \) (zeroth entry of the differential cepstrum) corresponds to the number of excess zeros of the transfer function. Due to the contribution of the input, however, the zeroth term of the differential cepstrum does not contain any valuable information with regards to the excess number of zeros in the transfer function hence it can be selected by the user as any real number. It has been found that the algorithm adjust the number of excess zeros to the value entered (the first Markov parameter) by creating artificial zeros at \( z = 0 \) that do not contribute to the differential cepstrum.

As is always the case, when the differential cepstrum coefficients are treated as Markov parameters in ERA, a system realization in the form of \( \{T^{-1}A_d T, T^{-1}B_d, C_d T\} \) is obtained where \( T \) is an unknown non-singular matrix with appropriate size. However, the
realization obtained can be transformed back to the modal coordinates where the system matrix $A_d$ is diagonal, which actually corresponds to the form given in Eq. (3.64). However, the matrices $C_d$ and $B_d$ in modal coordinates will be in the form of $\{Z^{-1}B_d, C_dZ\}$ where $Z$ is an unknown diagonal matrix. Note that although $C_dZ$ is not equivalent to $C_d$ in Eq. (3.63) which would have enabled one to separate the poles from the zeros, $(C_dZ)(Z^{-1}B_d)$ where $(.)$ refers to a dot multiplication of the two vectors is free from the unknown matrix $Z$, hence contains information regarding the separation of the poles from the zeros. Then, one can actually check the sign of the real part of $(C_dZ)(Z^{-1}B_d)$ and distinguish the diagonal entries of $A_d$ that are associated with a positive real part in the vector $(C_dZ)(Z^{-1}B_d)$ as poles and those with a negative real part as zeros.

Although the idea has been presented for the case of a single measurement, the expansion to multiple measurements can easily be done. In the multiple measurement case, one needs to expand the size of the matrix $A_d$ such that it accounts for the zeros of the transfer functions relating the input coordinate to the additional output coordinates as well. Obviously, the number of columns and rows in the matrices $C_d$ and $B_d$ has to be enlarged while considering the fact that each row of the matrix $C_d$ is associated with a particular measurement coordinate. In the multiple measurement case, the separation of the zeros associated with a particular measurement coordinate than the rest of the zeros can easily be accomplished by checking the mode shape of the zero (the column of the matrix $C_dZ$ corresponding to the zero in question). It is expected that the mode shape of the zero should have a significant value only at the coordinate to which it corresponds to. This idea has successfully been applied by Gao and Randall (1996) to match the zeros to the coordinates that they are associated with. In order to illustrate the main points discussed above consider the following simple example.

**Example:**

The same example that was introduced in Section 3.3.4 has also been considered here. The structure has been excited by a burst random excitation at the second mass and the acceleration response at the same mass is recorded. Both input and the output is sampled at 25 Hz. The input is composed of a random noise of length 0.2 seconds followed by zeros of length 199.8 seconds. The first 600 samples of the differential cepstrum has been taken as free decay response and used in ERA. The first Markov parameter has been selected to be equal to one. The singular value plot of the Hankel matrix assembled from the differential cepstrum coefficients is given in Figure 3-9 where one can easily see that the order of the system is 23. Identified modes have been separated as poles and zeros based on the sign criteria mentioned earlier. The resulting poles and zeros are presented in Table 3-9. As it is clear from the table, there are one pole and two zeros that are not associated with the system but the input and there is no clear way of discriminating the ones that are related with the system than those of the input. If there were multiple response measurements, mode shape information may have been utilized and the system parameters would have been discriminated. Another solution would be possible if the response of the system to a different loading was available. In that case, the poles and zeros of the input would probably be different while the poles and the zeros of the system
would be the same. So, the intersection of the poles and zeros identified from the two loading conditions would give the desired system parameters.

Note that the number of non-zero values of the loading used in this example is just five and the associated pole-zero form has only one pole and two zeros. If, however, a different loading is applied with a larger set of poles and zeros, the extraction of the system parameters would require a higher order to be selected. In order to see this, now let’s consider a different burst random excitation which is composed of a random noise of length 4 seconds followed by zeros of length 196 seconds. Here, we again use 600 samples of the differential cepstrum in ERA and obtain the singular value plot given in Figure 3-10. It is clear that the order that needs to be identified has grown with the type of the input being used and it is 76 now. After the poles and zeros separated according to the sign criteria, it has been found that there are 4 poles and 52 zeros in addition to the 10 poles and 10 zeros that are associated with the system. This observation while confirming the expectations that were stated in the previous section, namely, one can still estimate the true parameters from complex cepstrum provided that the poles and the zeros of the input are also considered, it also indicates that the use of differential cepstrum as a free decay data to estimate poles and zeros of transfer function might be an efficient tool only for systems that are excited by inputs that can be represented by a relatively low number of poles and zeros, i.e, impulse-like inputs. Otherwise, the order of the system to be extracted grows very fast which may require formation of huge Hankel matrices. Besides, the discrimination of the system related parameters from that of the input remains to be a challenge.

Figure 3-9 Singular value plot of the Hankel matrix for the case where input has non-zero values for 0.2 seconds
Table 3-9 True poles and zeros of the system and identified poles and zeros

<table>
<thead>
<tr>
<th>True System Poles</th>
<th>Poles Identified</th>
<th>True System Zeros</th>
<th>Zeros Identified</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.244 ± 0.951i</td>
<td>-0.244 ± 0.951i</td>
<td>-0.047 ± 0.949i</td>
<td>-0.047 ± 0.949i</td>
</tr>
<tr>
<td>0.007 ± 0.984i</td>
<td>0.007 ± 0.984i</td>
<td>0.137 ± 0.824i</td>
<td>0.137 ± 0.824i</td>
</tr>
<tr>
<td>0.387 ± 0.910i</td>
<td>0.387 ± 0.910i</td>
<td>0.431 ± 0.814i</td>
<td>0.431 ± 0.814i</td>
</tr>
<tr>
<td>0.711 ± 0.692i</td>
<td>0.711 ± 0.692i</td>
<td>0.880 ± 0.289i</td>
<td>0.880 ± 0.289i</td>
</tr>
<tr>
<td>0.961 ± 0.267i</td>
<td>0.961 ± 0.267i</td>
<td>0.567</td>
<td>0.567</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.948</td>
<td>0.683</td>
</tr>
</tbody>
</table>

Figure 3-10 Singular value plot of the Hankel matrix for the case where input has non-zero values for 4 seconds

3.3.5 The Cepstrum of Noisy Measurements

Since the measured data in reality is always noisy, it is essential to see how the performance of the techniques presented so far compares when the measured data is contaminated with noise. From a conceptual perspective, the noisy measurements can be expressed as

\[ \hat{y}(n) = y(n) + w(n) \]  \hspace{1cm} (3.66)
where \( \hat{y}(n) \) is the measured response, \( y(n) \) is the true response and \( w(n) \) is the noise. Z transform of Eq. (3.66) may be written as

\[
\hat{Y}(z) = H(z)U(z) + W(z)
\]  

(3.67)

Expressing Eq. (3.67) in pole-zero form, one gets

\[
\hat{Y}(z) = \frac{A}{\prod_s (z-p_s)} \frac{B}{\prod_u (z-p_u)} + \frac{C}{\prod_w (z-p_w)}
\]  

(3.68)

which can also be written as

\[
\hat{Y}(z) = \frac{AB}{\prod_s (z-z_s)} \frac{B}{\prod_u (z-z_u)} \frac{C}{\prod_w (z-w_w)} \frac{C}{\prod_w (z-p_w)}
\]  

(3.69)

While the complex cepstrum is defined as the inverse discrete Fourier transform of the logarithm of Eq. (3.69), one can make a number of important observations regarding the noise related distortion on the performance of the cepstrum based parameter identification. First of all, it is clear from Eq. (3.69) that the poles of the system, the input and the noise are still in the denominator with no apparent distortion introduced. In other words, the noise does not change the location of the poles of the system, but the number of poles that contribute to the cepstrum of the measured response has been increased. The situation for the zeros is worse. Because, not only the number of zeros present in the data has changed, but also their locations have been distorted to the new locations corresponding to the zeros of the numerator term in Eq. (3.69) which contains the convolved information of the zeros and the poles of the system, the input and the noise. Obviously, one way to reduce the distortions incurred on the zeros is to process the measured raw data before computing the cepstrum in such a way that the measurement noise in the data is reduced. In practice, the common approach used for noise reduction is averaging which can be done in different forms such as the averaging of the spectral density functions computed at different sections of the measured data. While the averaging of the measured data can be quite useful for noise reduction, we will focus on other alternatives to better handle the noise within the cepstrum concept. In this regard, one question that needs to be answered is whether there is any advantage of using the power or the complex cepstrum over the other one when it comes to the estimation of system poles and zeros by curve-fitting to the cepstrum of noisy measurements. For that purpose, let’s rewrite Eq. (3.68) as

\[
\hat{Y}(z) = \hat{H}(z)\hat{U}(z)
\]  

(3.70)

where
\[\hat{H}(z) = H(z)U(z)W(z) = \frac{ABC \prod_s (z-z_s) \prod_u (z-z_u) \prod_w (z-z_w)}{\prod_s (z-p_s) \prod_u (z-p_u) \prod_w (z-p_w)}\] (3.71)

\[\hat{U}(z) = \frac{1}{W(z)} + \frac{1}{Y(z)} = \frac{\prod_w (z-p_w)}{C \prod_w (z-z_w)} + \frac{\prod_s (z-p_s) \prod_u (z-p_u)}{AB \prod_s (z-z_s) \prod_u (z-z_u)}\] (3.72)

Note that Eq. (3.70) simply states that the noisy measurements can be taken as the response of an artificial system \(\hat{H}(z)\) to an artificial input whose z-transform is denoted by \(\hat{U}(z)\). Note that the zeros of the artificial system correspond to the union of the zeros of the physical system, the input and the noise and similarly the poles of the artificial system are associated with the union of the set of poles of the physical system, the input and the noise. The logarithm of Eq. (3.70) can be written as

\[\log\left[\hat{Y}(z)\right] = \log\left[\prod_s (z-z_s)\right] + \log\left[\prod_u (z-z_u)\right] + \log\left[\prod_w (z-z_w)\right] + \log\left[\hat{U}(z)\right] - \log\left[\prod_s (z-p_s)\right] - \log\left[\prod_u (z-p_u)\right] - \log\left[\prod_w (z-p_w)\right] + \log[ABC]\] (3.73)

The inverse discrete Fourier transform of Eq. (3.73) yields the complex cepstrum. It is a simple matter to write the terms that constitute the positive and negative quefrency axis of the complex cepstrum directly from Eq. (3.73) by separating the poles and the zeros that are located inside and outside the unit circle such that a form similar to Eq. (3.38) is obtained. However, for the purposes of this Section, we do not pursue that and draw our conclusions directly from Eq. (3.73) as it turns out to be more useful to see the general problem in a more compact form.

The power cepstrum of the noisy measurements, on the other hand, can be computed from the complex cepstrum through the relation given in Eq. (3.15). The term that needs special consideration in Eq. (3.73) is \(\log\left[\hat{U}(z)\right]\). In this regard, we are mainly interested in the form of the log spectrum of the artificial input. Because, it has been shown previously that if the input has a relatively flat spectrum, its contribution to the power cepstrum will concentrate on the early parts hence a reasonable separation of the path and source effects can be achieved, which is of great importance for the estimation of the system parameters.

As Eq. (3.72) clearly indicates, the spectrum of the artificial input has two components. The first one is the inverse of the spectrum of the noise and the second one is the inverse of the spectrum of the true response. Note that when the measurement noise is white, the spectrum of the artificial input will be the sum of two spectra, one is white and the other one is colored. The final form of the sum, however, depends on the relative magnitudes

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of the two spectra added, namely the relative values of the gains $AB$ and $C$. Although there is no information with regards to the relative values of $AB$ and $C$, it may be claimed that they can assume values which can make the contribution of the white-noise noticeable in relative terms to that of the inverse spectrum of the noise-free response. If that happens, although the artificial noise has a colored nature, it may be assumed to be white.

The reason why we are interested in obtaining an artificial input with a white-noise characteristics is because it has been shown in Section 3.3.4.2 that power cepstrum has an advantage over the complex cepstrum when it comes down to the separation of the path effects from that of the source provided the input has a dynamical structure which can be represented by a set of poles and zeros inside the unit circle and another set outside the unit circle such that the poles outside the unit circle are located at the close proximity of the reciprocals of the zeros that are inside the unit circle and vice versa. In this particular case, the contribution of the input to the positive and negative regions of the complex cepstrum tends to be approximately the same except the sign while the contribution of such an input to the power cepstrum based on Eq. (3.15) becomes the sum of these terms that have approximately the same value with an opposite sign, meaning the contribution tends to go to zero. Therefore, there is a good potential that while the contribution of the term $\log \hat{U}(z)$ to the complex cepstrum can be significant, it can be very low for the power cepstrum and the power cepstrum can be preferred over the complex cepstrum when it comes down to the estimation of system parameters from the cepstrum of the noisy measurements. These expectations are actually verified in the following numerical example.

**Example:**

The same system presented in Section 3.3.4 has also been considered in this example. The structure has been excited from the second mass by a white noise sequence of length 655.36 seconds and the resulting acceleration response at the second and fifth masses are measured. The data has been sampled with a sampling period of 0.04 seconds. Two cases are considered: (1) Noise-free data, (2) Noisy data which is obtained by adding a white noise sequence with an RMS of 5% of that of the output measured on the second mass. In order to illustrate the form of the artificial input defined above, first the inverse of the discrete Fourier transform of the noise-free response, $1/Y(z)$, (Figure 3-11) and the inverse of the discrete Fourier transform of the additive noise, $1/W(z)$, are computed (Figure 3-11). The spectrum of the artificial input then is just the sum of the two spectra as shown in Figure 3-11 for the response measured at the second mass. As it can be seen, the artificial input has a relatively flat spectrum which is expected to have less contribution to the power cepstrum than to the complex cepstrum. The artificial input that is connected with the measurement on the fifth mass is also computed in the same way and the resulting spectra are plotted in Figure 3-12. As it is clear, the same behavior is observed, i.e, the contribution of the noise is found to be more significant, hence the artificial input has a relatively flat log spectrum.
The complex and power cepstra of the noisy measurements are compared with that of the noise-free measurements in Figure 3-13 and Figure 3-14 for the measurement on the second mass and in Figure 3-15 and Figure 3-16 for the measurement on the fifth mass. The results clearly indicate that the power cepstrum of the noisy measurement closely follows the power cepstrum of the noise-free measurement whereas the discrepancy between the complex cepstra of the noisy and that of the noise-free measurements are quite noticeable. Therefore, it can be suggested to use power cepstrum instead of complex cepstrum for the purposes of estimating poles and zeros of the transfer functions from noisy measurements in the cepstrum domain.

Figure 3-11 Spectrum of the artificial input together with its components associated with the response measured at the second mass.
Figure 3-12 Spectrum of the artificial input together with its components associated with the response measured at the fifth mass

Figure 3-13 Power cepstra of the noisy and noise-free measurements on the second mass
Figure 3-14 Complex cepstra of the noisy and noise-free measurements on the second mass

Figure 3-15 Power cepstra of the noisy and noise-free measurements on the fifth mass
3.3.5.1 Parameter Estimation from Cepstrum of the Noisy Measurements

In the previous section, it has been shown that when the data is noisy, the best noise performance can be obtained if the power cepstrum of the measurements are used. When it comes to the use of the cepstrum to extract system poles and zeros from the data, there are basically two approaches one can follow. The first one is to use the nonlinear curve-fitting algorithm explained in Section 3.3.4 and the second one is to use the differential cepstrum of the response in conjunction with a time-domain system identification algorithm that uses free decay responses. Differential cepstrum is known to be sensitive to aliasing, therefore, its use can be restricted only to the response measurements that are impulse-like responses. When it comes to noisy measurements, the complications of the differential cepstrum multiply. Because the differential cepstrum is closely connected with complex cepstrum and the performance of the complex cepstrum is found to be poor for parameter estimation.

The advantage of the differential cepstrum however lies in the fact that the user does not need to know the number of poles and zeros of the transfer function a priori in order to start the estimation process. Considering the fact that when the data is noisy, the number of poles and zeros that contribute to the cepstrum of the measurements is higher than the noise-free case (extra parameters are the poles and zeros of the noise component), the differential cepstrum concept, if it can be adapted to the noisy case, becomes more appealing. Because, a high order model which may account for the total number of poles

![Complex cepstra of the noisy and noise-free measurements on the fifth mass](image)
and zeros associated with the system and the noise can be extracted, which in turn may improve the accuracy of the estimated system parameters.

Based on the potential improvements attained by the differential cepstrum concept, we define differential power cepstrum, which may be obtained directly from the power cepstrum by simply multiplying each coefficient of the power cepstrum with its index, namely

\[
\hat{y}_{dpc}(n) = n\hat{y}_{po}(n)
\]  

(3.74)

where \(\hat{y}_{dpc}(n)\) is differential power cepstrum, \(n\) is the index of the cepstrum and \(\hat{y}_{po}(n)\) is the power cepstrum. It can also be verified that the differential power cepstrum can be computed from auto correlation functions of the measured output sequences as

\[
\hat{y}_{dpc}(n) = Z^{-1} \left[ \frac{Z(\tau_{yy}(k))}{Z(\tau_{yy}(k))} \right]
\]  

(3.75)

where \(\tau_{yy}(k)\) is the auto-correlation function of the measured response, \(Z\) and \(Z^{-1}\) denote the z-transform and inverse z-transform respectively. It should be apparent that the computation of the differential power cepstrum from Eq. (3.75) is not advised due to the high sensitivity to aliasing unless the data is noise-free and impulse-response like. Therefore, it is much more advisable to compute the differential power cepstrum directly from power cepstrum following Eq. (3.74).

Example:

The same system presented in Section 3.3.4 has also been considered in this example. The structure has been excited by a burst random excitation at the second mass and the displacement response at the same mass is recorded. The input is composed of a random noise of length 0.2 seconds followed by zeros of length 199.8 seconds. The data has been sampled with a sampling period of 0.04 seconds. Two cases are considered: (1) Noise-free data, (2) Noisy data which is generated by adding a white noise sequence with an RMS of 5% of that of the output measured on the second mass to the measured response. First, both differential complex cepstrum and differential power cepstrum of the measured responses are computed directly from noise-free and noisy-data following Eqs. (3.27) and (3.75) respectively. The resulting cepstra are plotted in Figure 3-17 and Figure 3-18. As it is evident, the differential cepstra computed from noisy measurements are useless, which is mainly due to the fact that they are very sensitive to the aliasing problem. Next, both differential complex and power cepstra are computed from complex and power cepstra based on the relationships given in Eqs. (3.31) and (3.74) and plotted in Figure 3-19 and Figure 3-20. As it is expected, the differential complex cepstrum is found to be more sensitive to noise than differential power cepstrum, which directly follows from the fact that power cepstrum has a superior performance to noise than complex cepstrum.
Figure 3-17 Differential complex cepstra calculated directly from noisy and noise-free data

Figure 3-18 Differential power cepstra calculated directly from noisy and noise-free data
Figure 3-19 Differential complex cepstra calculated directly from complex cepstra for noisy and noise-free data.

Figure 3-20 Differential power cepstra calculated directly from power cepstra for noisy and noise-free data.
Next, the poles and zeros of the collocated transfer function is estimated from the noisy measurements by following two approaches: (1) Curve-fitting to the power cepstrum of the measurements, (2) Using ERA with the differential power cepstrum computed directly from power cepstrum. In the curve-fitting approach, the trial set of poles and zeros are obtained by perturbing the true values in the Laplace plane. The perturbation that is introduced to the real and imaginary parts separately has been computed as random values changing between ± 10% of the original values. Then, the trial poles and zeros are obtained by transforming the perturbed values to the z-plane. The discretization zero located at \( z = -1 \), has also been accounted for by an additional zero at \( z = -0.999 \). The region of the power cepstrum where the curve-fitting is applied is selected to be the range between the 25\(^{\text{th}}\) and 200\(^{\text{th}}\) samples of the power cepstrum. In the second approach, the first 100 differential power cepstrum coefficients have been employed to form Hankel matrices of size 50x50. From the singular values of the Hankel matrices, a model of order 30 has been selected. The poles and the zeros are separated based on the sign criterion mentioned in Section 3.3.4.3. Most of the modes that are not associated with the physical structure are found to be concentrated around the Nyquist frequency region. Other computational modes are found to have very high damping values, hence easily discarded. The estimated parameters from both approaches are listed in Table 3-11 and Table 3-12 while the true values of the poles and zeros are given in Table 3-10 (true values correspond to the poles and zeros of the pulse transfer function that are mapped to the Laplace plane). Note that although the estimated parameters are in the z-domain, what is presented in the tables are the values obtained after mapping the original values to Laplace domain through the standard mapping. The results indicate that the two approaches produce similar results in this particular case.

<table>
<thead>
<tr>
<th>Table 3-10 Poles and zeros of the sampled physical structure</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>True Poles</strong></td>
</tr>
<tr>
<td>(-0.011 \pm 1.079i)</td>
</tr>
<tr>
<td>(-0.031 \pm 3.072i)</td>
</tr>
<tr>
<td>(-0.047 \pm 4.651i)</td>
</tr>
<tr>
<td>(-0.062 \pm 6.222i)</td>
</tr>
<tr>
<td>(-0.072 \pm 7.248i)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3-11 Poles and zeros estimated based on the power cepstrum concept</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Est. Poles</strong></td>
</tr>
<tr>
<td>(-0.010 \pm 1.078i)</td>
</tr>
<tr>
<td>(-0.033 \pm 3.063i)</td>
</tr>
<tr>
<td>(-0.050 \pm 4.648i)</td>
</tr>
<tr>
<td>(-0.066 \pm 6.215i)</td>
</tr>
<tr>
<td>(-0.069 \pm 7.253i)</td>
</tr>
</tbody>
</table>
Table 3-12 Poles and zeros estimated based on the differential power cepstrum concept

<table>
<thead>
<tr>
<th>Est. Poles</th>
<th>Est. Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.008 ± 1.082i</td>
<td>-0.066 ± 1.535i</td>
</tr>
<tr>
<td>-0.024 ± 3.095i</td>
<td>-0.104 ± 4.207i</td>
</tr>
<tr>
<td>-0.072 ± 4.637i</td>
<td>-0.047 ± 5.618i</td>
</tr>
<tr>
<td>-0.089 ± 6.235i</td>
<td>-0.098 ± 6.553i</td>
</tr>
<tr>
<td>-0.074 ± 7.248i</td>
<td>-1.265 +12.500i</td>
</tr>
</tbody>
</table>

3.4 Summary

This Chapter examined the use of the cepstrum technique for estimating the zeros of transfer functions from response measurements in SIMO systems. The Chapter first introduces the cepstra that have been implemented in the literature for parameter estimation, namely, power, complex and differential cepstra. Then, the question of which cepstra is more advantageous to use for estimating the zeros has been explored.

In particular, it has been shown that provided the input has a flat log spectrum, the power cepstrum is more successful than complex cepstrum in separating the system information from that of the input. This is explained by the fact that the complex cepstrum of the stable and non-stable parts of the input turns out to be approximately equal to each other in value with an opposite sign. Since the power cepstrum is the sum of these two cepstra, it tends to be almost negligible compared to the contribution of the system information (poles and zeros) to the power cepstrum. Hence, the separation of the system information from that of the input is achieved naturally. It is opportune to note the question mentioned above has never been addressed in the literature, although it has been concluded that the technique has been found to be working only with the inputs having flat-log spectra.

Next, the Chapter studies the characteristics of the estimated zeros. It is worth noting that since measured data is of discrete nature in reality, the zeros estimated from the cepstrum technique are actually the zeros of the discrete-time system, which is obtained by sampling the original continuous time system. While a subset of the zeros of the discrete-time system (intrinsic zeros) is connected to the zeros of the continuous time system, the remaining zeros (discretization zeros) are artificially generated and are not associated with the physics of the continuous-time system.

Since the cepstrum of a measured response contains information on the poles and the zeros of the discrete-time system, it is important to account for the contribution of some of the discretization zeros too, in addition to the intrinsic zeros, to the cepstrum in order to estimate the zeros with acceptable accuracy. It has been concluded that it is sufficient to consider only the discretization zeros that are located close to the unit circle in the parameter estimation since the contribution of a zero to the cepstrum goes to zero as the zero gets closer to the origin.
Next, the Chapter examines the effect of noise present in the measured data to the accuracy of the zeros estimated by the cepstrum technique. The problem is approached by treating the noisy measurement as being the response of an artificial system to an artificial input. The dynamics of the artificial system is defined by the combination of the dynamics of the original system, the noise and the input being applied. It has been concluded that provided that the original input has a flat-log spectrum and the noise is of white nature, then the power cepstrum can be effectively used to estimate the zeros. On the other hand, the complex cepstrum has been found to be more sensitive to noise than power cepstrum, hence not recommended.

One of the difficulties associated with the use of power or complex cepstra for parameter estimation is the need for specifying the number and the trial values of the poles and the zeros a priori for the curve-fitting algorithm. The differential cepstrum has been proposed for parameter estimation due to the fact that differential cepstrum coefficients can be treated as free decay data and the parameters can be estimated by any system identification algorithm which uses free-decay data. The advantage is clear; the need for specifying the number and values of the poles and zeros a priori is avoided. However, the downside of the differential cepstrum is that it is very sensitive to aliasing. Therefore, unless the measured response is almost an impulse response and the measurements go to almost zero long before the end of the measurement duration, the use of differential cepstrum for parameter estimation is not recommended.
One of the limitations of the cepstrum technique with regards to the estimation of poles and zeros directly from response measurements is the restrictions placed on the characteristics of the input. In particular, the input is required to be in such a form that its contribution to the cepstrum is significantly smaller than the contribution of the system at the locations where the system information is concentrated. There are basically two difficulties associated with this requirement: (1) Since the input is not measured, there is no deterministic information to be used to find out where exactly the input contribution is minimal compared to the contribution of the system. Therefore, the user has to use his or her own judgment to decide on the particular regions of the cepstrum along which the curve-fitting is performed. (2) Since the contribution of the input is never zero anywhere in the cepstrum, even if the best place is found and used in the curve-fitting, the estimates are always approximate. This Chapter presents a modification to the cepstrum technique which eliminates or significantly alleviates the limitations listed above in the estimation of the zeros.

4.1 Theoretical Background

In this Section, a simple technique which eliminates the contribution of any type of input to the cepstrum is proposed. The fundamental requirement of the proposed technique is that the number of measurements has to be equal to or larger than two. Let $y_i$ and $y_j$ be the two measurements taken at coordinate $i$ and $j$, due to the input, $u_l$, applied at coordinate $l$. Then, the complex cepstrum of the measurements can be expressed as

$$
\tilde{y}_{c,j} = \tilde{h}_{c,j} + \tilde{u}_{c,j} \tag{4.1}
$$

$$
\tilde{y}_{c,j} = \tilde{h}_{c,j} + \tilde{u}_{c,j} \tag{4.2}
$$

by taking the difference of the Eqs. (4.1) and (4.2), one obtains

$$
\tilde{y}_{c,j} - \tilde{y}_{c,j} = \tilde{h}_{c,j} - \tilde{h}_{c,j} \tag{4.3}
$$

As it is apparent from Eq. (4.3), the contribution of the input to the complex cepstrum is completely eliminated. With respect to the system information present in the difference of the complex cepstra, consider the transfer functions given in the following forms

$$
H_i(z) = \frac{\left| A \prod_{k=1}^{m_l} \left( 1 - a_{i,k}z^{-1} \right) \prod_{k=1}^{m_k} \left( 1 - h_{i,k}z \right) \right|}{\prod_{k=1}^{p_l} \left( 1 - c_{i,k}z^{-1} \right) \prod_{k=1}^{p_k} \left( 1 - d_{i,k}z \right)} \tag{4.4}
$$
\[
H_j(z) = \frac{\prod_{k=1}^{m_j} (1-a_{j,k}z^{-1}) \prod_{k=1}^{p_j} (1-b_{j,k}z^{-1})}{\prod_{k=1}^{m_j} (1-c_{j,k}z^{-1}) \prod_{k=1}^{p_j} (1-d_{j,k}z^{-1})}
\]  
(4.5)

where the parameters \(a, b, c\) and \(d\) are as defined just after Eq. (3.34), except the sub-indices \(i\) and \(j\), which are introduced to the poles and the zeros of the transfer functions in order to differentiate them from one another based on the coordinates that they are associated with. The contribution of the transfer functions given in Eqs. (4.4) and (4.5) to the complex cepstrum can be written as

\[
\tilde{h}_{c,i}(n) = \begin{cases} 
\log[|A|] & n = 0 \\
-\sum_{k=1}^{m_i} a_{i,k}^n / n + \sum_{k=1}^{p_i} c_{i,k}^n / n & n > 0 \\
\sum_{k=1}^{m_i} b_{i,k}^n / n - \sum_{k=1}^{p_i} d_{i,k}^n / n & n < 0 
\end{cases}
\]  
(4.6)

\[
\tilde{h}_{c,j}(n) = \begin{cases} 
\log[|B|] & n = 0 \\
-\sum_{k=1}^{m_j} a_{j,k}^n / n + \sum_{k=1}^{p_j} c_{j,k}^n / n & n > 0 \\
\sum_{k=1}^{m_j} b_{j,k}^n / n - \sum_{k=1}^{p_j} d_{j,k}^n / n & n < 0 
\end{cases}
\]  
(4.7)

for the general case where the poles of the two transfer functions are equal in number, it is a simple matter to see that Eq. (4.3) is equivalent to

\[
\tilde{h}_{c,i}(n) - \tilde{h}_{c,j}(n) = \begin{cases} 
\log \left[ \frac{|A|}{|B|} \right] & n = 0 \\
-\sum_{k=1}^{m_i} a_{i,k}^n / n + \sum_{k=1}^{m_j} a_{j,k}^n / n & n > 0 \\
\sum_{k=1}^{m_i} b_{i,k}^n / n - \sum_{k=1}^{m_j} b_{j,k}^n / n & n < 0 
\end{cases}
\]  
(4.8)

In order to connect the result presented in Eq. (4.8) with the theory already present in the literature, it is worth noting that the result of the subtraction in the complex cepstrum is actually equivalent to the inverse discrete Fourier transform of the logarithm of the transmissibility function which is defined by the ratios of the transfer functions in frequency domain as
\[ T_{ij}(z) = \frac{H_i(z)}{H_j(z)} \]  

(4.9)

where \( T_{ij} \) is the transmissibility function associated with the coordinates \( i \) and \( j \).

Going back to the Eq. (4.8), it is clear that \( \tilde{h}_{c,i}(n) - \tilde{h}_{c,j}(n) \) contains only one part of the system information, the zeros of the transfer functions \( H_i(z) \) and \( H_j(z) \). This follows from the fact that the contribution of the poles to the cepstrum is cancelled as a result of the subtraction (the special case where the two transfer functions have different number of poles is considered separately in Section 4.2.1). It is opportune to note that the zeros of the two transfer functions can be easily differentiated from one another based on the sign of their contributions to the difference of the cepstra. In particular, it can be seen from Eq. (4.8) that when the difference of the cepstra is treated as the contribution of a single artificial transfer function to the complex cepstrum, then the zeros of the transfer functions \( H_i(z) \) and \( H_j(z) \) will be associated with the zeros and the poles of the artificial transfer function, hence it will be a simple matter to associate estimated parameters to the transfer functions they belong to.

Although the formulations presented above are associated with the complex cepstrum, the extension to the power and/or differential cepstra is straightforward based on the relationships given in Chapter 3 (Eq. (3.15) and (3.31) respectively). Therefore, any curve-fitting algorithm used with complex or power cepstra or any time-domain identification technique adapted to be used with differential cepstrum data that have been used for parameter extraction can also be used with the subtracted cepstral coefficients. The only difference will be in the interpretation of the parameters extracted. In particular, the parameters identified as poles will correspond to the zeros of the transfer function associated with the coordinate whose contribution is subtracted (coordinate \( j \) in this case) and similarly the parameters identified as zeros will correspond to the zeros of the transfer function associated with the coordinate \( i \).

It should be clear by now that the proposed technique enables one to estimate the zeros of the transfer functions exactly (when the measured data is noise-free) while placing no restrictions on the input provided that there are at least two measurements. Moreover, the user does not need to look for the optimum region in the cepstrum where the input contribution is minimal, simply because the input contribution is cancelled completely. Therefore, any section of the cepstrum can be used in the curve-fitting approach. The downside of the technique compared to the original cepstrum approach is that since poles are cancelled they cannot be identified. If the poles are of great importance besides the zeros for a particular application, then one can estimate the poles in a number of ways. For example, one can use the original cepstrum technique just after the zeros are estimated from the proposed technique. The advantage of this is that one can actually adjust the curve-fitting algorithm in such a way that the curve-fitter takes the already estimated zeros to be the true values and optimize only the locations of the poles, which might also improve the quality of the estimated poles. In some cases, however, it might
also be preferable to use other techniques such as stochastic system identification algorithms to estimate the poles directly from measurements.

4.2 Implementation Issues

While the proposed technique offers exact estimates of the zeros in the case of noise-free data, there are a number of theoretical and practical issues when it comes down to actual implementation of the technique. These issues are addressed in this Section.

4.2.1 De-coupling Zeros

Note that in the derivation of the proposed technique, it is assumed that the number of poles of the transfer functions $H_i(z)$ and $H_j(z)$ are the same. However, it is possible to have transfer functions with different number of poles associated with the same physical system. The reason why a transfer function of a system might have less number of poles than another transfer function defined on the same system can be explained by the phenomenon known as pole-zero cancellation. As the name implies, what happens is that sometimes a subset of the zeros of a particular transfer function may be located at the exact locations in the complex plane where a subset of the poles are located. In this special case, the poles and the zeros located at the same point cancel each other; hence produce transfer functions with reduced number of poles. The pole-zero cancellation is intimately connected with any of the three physical situations: (1) There is a pole for which the associated mode shape has a node (zero amplitude) at the output coordinate where the transfer function is related to, (2) There is a pole for which the associated mode shape has a node at the input coordinate where the transfer function is related to, (3) There is a pole for which the associated mode shape has nodes both at the input and the output coordinates where the transfer function is related to. The poles that are cancelled based on the three physical conditions mentioned above are called non-observable, non-controllable, and both non-controllable and non-observable poles respectively. Similarly, the zeros that are cancelled are referred to as output-decoupling, input-decoupling and input-output decoupling zeros.

The contribution of the excess poles can be shown to be in the same form of the contribution of the zeros of the transfer function with the less number of poles. It should be clear after the discussion given above; these poles are actually the de-coupling zeros of the transfer function with the less number of poles. Therefore, when the proposed technique is to be used with curve-fitting approach, the user has to account for the decoupling zeros as well while deciding on the order of the parameters to be estimated.

4.2.2 Zero-Zero Cancellation

One special case that might prevent the estimation of the zeros from the difference of the cepstra can occur when the transfer functions $H_i(z)$ and $H_j(z)$ have zeros that are at the
same location in the complex plane. As it is apparent from Eq. (4.8) that the contribution of these zeros cancels each other and makes it impossible to estimate these zeros from the difference of the cepstra, a phenomenon referred to here as zero-zero cancellation. Obviously, if there are additional measurements available, one would have the option of estimating the zeros of a particular transfer function from the differences of the cepstra associated with each available measurement, hence the probability of missing zeros due to zero-zero cancellation is greatly reduced.

In practice, it may be very difficult to find these special cases where an exact zero-zero cancellation occurs. Rather, it is much more probable to find cases where the zeros of the transfer functions are located close to each other. In that case, if the separation of the zeros is less than a certain threshold, then the identification of these zeros may become an issue. A similar problem constitutes a classical problem in system identification and it is usually referred to as close eigenvalue problem. Similar to the exact zero-zero cancellation, if there are more than two measurements available, then the probability of missing an existing zero can be greatly reduced by processing the difference of the cepstra for each measurement separately. The numerical example presented at the end of the Chapter further clarifies the comments made here.

4.2.3 Marginal Pole-Pole Cancellation

In the derivation of the proposed technique, it has been stated that the contribution of the poles associated with the transfer functions $H_i(z)$ and $H_j(z)$ to the cepstrum are cancelled after the two cepstra is subtracted. While this is exactly true in theory, when it comes down to the measurements, it is well-known that these poles will never be located at the same exact locations for two different transfer functions on the same system, although they may be located very closely. Therefore, in reality, the poles do not cancel completely, but most of their contributions to the cepstrum will be cancelled, a situation referred to, here, as marginal pole-pole cancellation. In this case, Eq. (4.8) can be written as

$$
\begin{align*}
\log \left[ \frac{|A|}{|B|} \right] & = 0 \\
\tilde{h}_{c,j}(n) - \tilde{h}_{c,j}(n) : & = -\sum_{k=1}^{m} a_{i,k}^n / n + \sum_{k=1}^{m} a_{i,k}^n / n + \sum_{k=1}^{m} (c_{i,k}^n - c_{j,k}^n) / n \\
& + \sum_{k=1}^{m} b_{j,k}^n / n - \sum_{k=1}^{m} b_{j,k}^n / n + \sum_{k=1}^{m} (d_{i,k}^n + d_{j,k}^n) / n \\
& \quad n > 0
\end{align*}
$$

(4.10)

Expressing $c_{j,k} = c_{i,k} + e_k$, and $d_{j,k} = d_{i,k} + f_k$, where $e_k$ and $f_k$ are the distances between $k^{th}$ pole of the transfer functions $H_i(z)$ and $H_j(z)$ in $z$-plane, the error terms in Eq. (4.10) can be expressed as
\[ \sum_{k=1}^{p_i} (c_{i,k}^n - (c_{i,k} + e_k)^n) / n = \sum_{k=1}^{p_i} (-nc_{i,k}^{n-1}e_k + O(e_k^2)) / n \quad n > 0 \quad (4.11) \]
\[ \sum_{k=1}^{p_j} (d_{j,k}^n - (d_{j,k} + f_k)^n) / n = \sum_{k=1}^{p_j} (-nd_{j,k}^{n-1}f_k + O(f_k^2)) / n \quad n > 0 \quad (4.12) \]

Given the fact that the magnitude of the zeros (|a_k|, |b_k|) are close to unity and that the deviations of the poles between the two transfer functions are expected to be much more smaller than one (|e_k|<<1, |f_k|<<1), the remainder of the marginal pole-pole cancellation can be considered to be insignificant with respect to the contribution of the zeros.

4.2.4 The effect of Noise to the Proposed Technique

The effect of noise to the cepstrum has been investigated in Section 3.3.5. In this section, we follow a similar methodology and investigate the effect of noise to the proposed technique. Let’s express the measured noisy responses \( i \) and \( j \) as
\[ \hat{y}_i(n) = y_i(n) + w(n) \quad (4.13) \]
\[ \hat{y}_j(n) = y_j(n) + v(n) \quad (4.14) \]

where \( y_i \) and \( y_j \) are the noise-free responses at coordinates \( i \) and \( j \) and \( w \) and \( v \) represent the measurement noises at the same coordinates respectively. Inverse discrete Fourier transforms of Eqs. (4.13) and (4.14) can be expressed as
\[ \hat{Y}_i(z) = H_i(z)U_i(z) + W(z) \quad (4.15) \]
\[ \hat{Y}_j(z) = H_j(z)U_j(z) + V(z) \quad (4.16) \]

Note that Eqs. (4.15) and (4.16) can also be written as,
\[ \hat{Y}_i(z) = \hat{H}_i(z)\hat{U}_i(z) \quad (4.17) \]
\[ \hat{Y}_j(z) = \hat{H}_j(z)\hat{U}_j(z) \quad (4.18) \]

where \( \hat{H}_i(z) \) and \( \hat{H}_j(z) \) are called artificial transfer functions, \( \hat{U}_i(z) \) and \( \hat{U}_j(z) \) as artificial inputs associated with coordinates \( i \) and \( j \) respectively, which are given as
\[ \hat{H}_i(z) = H_i(z)U(z)W(z) \quad (4.19) \]
\[ \hat{H}_j(z) = H_j(z)U(z)V(z) \quad (4.20) \]
\[ \hat{U}_i(z) = \frac{1}{W(z)} + \frac{1}{H_i(z)U(z)} \quad (4.21) \]
\[
\dot{U}_j(z) = \frac{1}{V(z)} + \frac{1}{H_j(z)U(z)}
\] (4.22)

Let’s express each term on the r.h.s of the Eqs. (4.19) and (4.20) in the pole-zero form as

\[
H_i(z) = \frac{A \prod_{s_i}(z - z_{s_i})}{\prod_{s_i}(z - p_{s_i})}
\] (4.23)

\[
U(z) = \frac{B \prod_{u}(z - z_u)}{\prod_{u}(z - p_u)}
\] (4.24)

\[
W(z) = \frac{C \prod_{w}(z - z_w)}{\prod_{w}(z - p_w)}
\] (4.25)

\[
H_j(z) = \frac{D \prod_{s_j}(z - z_{s_j})}{\prod_{s_j}(z - p_{s_j})}
\] (4.26)

\[
V(z) = \frac{E \prod_{v}(z - z_v)}{\prod_{v}(z - p_v)}
\] (4.27)

where \(z_{s_i}\) and \(p_{s_i}\) are the zeros and poles of the transfer function \(H_i(z)\), \(z_{s_i}\) and \(p_{s_i}\) are the zeros and poles of the transfer function \(H_j(z)\), \(z_u\) and \(p_u\) are the zeros and poles of the discrete Fourier transform of the applied input \(u\), \(z_w\) and \(p_w\) are the zeros and poles of the discrete Fourier transform of the measurement noise \(w\), \(z_v\) and \(p_v\) are the zeros and poles of the discrete Fourier transform of the measurement noise \(v\), \(A\), \(B\), \(C\), \(D\) and \(E\) are the gains of the associated z-transforms. It should be stated that the forms assumed in Eqs. (4.23) to (4.27) can be missing some of the terms such as a phase term. However, for the purposes of this Section and for conciseness, we do not complicate the expressions by adding these terms. The logarithm of the discrete Fourier transforms of the measured noisy data can be expressed as,

\[
\log\left[\hat{Y}_i(z)\right] = \log\left[\prod_{s_i}(z - z_{s_i})\right] + \log\left[\prod_{u}(z - z_u)\right] + \log\left[\prod_{w}(z - z_w)\right] + \log\left[\dot{U}_i(z)\right] - \log\left[\prod_{s_i}(z - p_{s_i})\right] - \log\left[\prod_{u}(z - p_u)\right] - \log\left[\prod_{w}(z - p_w)\right] + \log[ABC]
\] (4.28)
\[ \log[\hat{Y}_j(z)] = \log\left[ \prod_{sl}(z - z_{sl}) \right] + \log\left[ \prod_{u}(z - z_u) \right] + \log\left[ \prod_{v}(z - z_v) \right] + \log\left[ \hat{U}_j(z) \right] \]
\[ - \log\left[ \prod_{sl}(z - p_{sl}) \right] - \log\left[ \prod_{u}(z - p_u) \right] - \log\left[ \prod_{v}(z - p_v) \right] + \log[DBE] \] (4.29)

Note that the complex cepstrum of the measurements \(\hat{y}_{c,j}\) and \(\tilde{y}_{c,i}\) can be obtained by taking inverse discrete Fourier transform of the expressions given in Eqs. (4.28) and (4.29). Although the proposed technique operates on the difference of \(\hat{y}_{c,j}\) and \(\tilde{y}_{c,i}\), due to the linearity property of the inverse discrete Fourier transform, one can actually calculate the difference of the Eqs. (4.28) and (4.29) and then compute the inverse discrete Fourier transform of the result, which will be identical to Eq. (4.3). The difference of the Eqs. (4.28) and (4.29) can be expressed as

\[ \log\left[ \frac{\hat{Y}_j(z)}{\hat{Y}_j(z)} \right] = \log\left[ \frac{\prod_{sl}(z - z_{sl})}{\prod_{sl}(z - z_{sl})} \right] + \log\left[ \frac{\prod_{v}(z - z_{w})}{\prod_{v}(z - z_{w})} \right] \]
\[ + \log\left[ \frac{\hat{U}_j(z)}{\hat{U}_j(z)} \right] + \log\left[ \frac{ABC}{DBE} \right] \] (4.30)

Note that if the data was noise-free, Eq. (4.30) would reduce to the first term on the rhs plus the logarithm of the ratio of the gains associated with the transfer functions \(H_i(z)\) and \(H_j(z)\). In the noisy data case, however, there are basically two additional contributions to the difference of the cepstrum. The first one is the contribution of the noises in the measurements (second term on the rhs of Eq. (4.30)) and the second one is the contribution of the artificial inputs associated with the measurement coordinates (third term on the rhs of Eq. (4.30)). Note that in the case where the measurement noises \(v\) and \(w\) are white, the second term on the rhs of Eq. (4.30) will be white. Moreover, the third term which represents the contribution of the artificial inputs is also expected to be approximately white which follows from the discussions given in Section 3.3.5. Therefore, the difference of the two cepstra calculated from noisy measurements can be seen to be composed of mainly two parts: (1) the contribution of the zeros of the transfer functions \(H_i(z)\) and \(H_j(z)\), (2) the contribution of white noise sequences (provided that the measurement noise is white). For the purposes of comparing the proposed technique with the original cepstrum approach, we can also express Eq. (4.28) in a similar form to the Eq. (4.30) as

\[ \log[\hat{Y}_i(z)] = \log\left[ \prod_{sl}(z - z_{sl}) \right] + \log\left[ \prod_{u}(z - z_u) \right] + \log\left[ \prod_{v}(z - z_{w}) \right] \]
\[ + \log\left[ \hat{U}_i(z) \right] + \log[ABC] \] (4.31)
Note that Eq. (4.31) contains the contribution of the input directly, in addition to the contribution of the noise and the artificial input. Following the same discussion given above, we can conclude that if the noise is white, Eq. (4.31) can be seen to be composed of three parts: (1) the contribution of the zeros and poles of the transfer function $H_i(z)$, (2) the contribution of input, (3) the contribution of white noise sequences (provided that the measurement noise is white). Therefore, the proposed technique has the advantage of eliminating the direct input contribution. When the input is white, that would not be an advantage anymore because the contribution of the input in Eq. (4.31) can be seen to be equivalent to the contribution of the noise $v$ in Eq. (4.30), hence there would not be any significant advantage between the two forms. However, when input is colored, then the proposed technique can become more advantageous than the original technique which has actually been confirmed by the following numerical example.

**Example:**

The same system presented in Section 3.3.4 has also been considered in this example. The structure has been excited at the first mass and the displacement responses at the first, third and the fourth masses are taken as output. Both input and output data has been sampled with a sampling period of 0.04 seconds. Then, the measurements are contaminated by adding white noise sequences with an RMS of 3% of that of the output measured on the first mass. The true poles and zeros of the pulse transfer functions associated with the measurement coordinates are given in Table 4-1. Note that all poles and zeros are transformed to Laplace plane from their original locations in z-plane. A closer look to the zeros given in the table reveals that one of the zeros associated with the measurements taken on the first and third masses is located at $-0.329 \pm 33.557i$ and $-0.337 \pm 33.572i$ respectively, indicating that a zero-zero cancellation in a marginal sense can occur if the difference of the cepstra of these two coordinates to be used for parameter estimation. However, this particular zero can also be estimated by subtracting the cepstra of the measurements taken on the first and third or second and third masses.

<table>
<thead>
<tr>
<th>True Poles</th>
<th>True Zeros ($H_1$)</th>
<th>True Zeros ($H_3$)</th>
<th>True Zeros ($H_4$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.068 \pm 6.780i$</td>
<td>$-0.073 \pm 7.918i$</td>
<td>$-0.125 \pm 13.320i$</td>
<td>$-0.196 \pm 19.611i$</td>
</tr>
<tr>
<td>$-0.193 \pm 19.302i$</td>
<td>$-0.219 \pm 22.593i$</td>
<td>$-0.337 \pm 33.572i$</td>
<td></td>
</tr>
<tr>
<td>$-0.292 \pm 29.221i$</td>
<td>$-0.329 \pm 33.557i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.391 \pm 39.097i$</td>
<td>$-0.436 \pm 43.82i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.455 \pm 45.542i$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4-1 True poles and zeros of the discrete-time system

Since it has been shown that the power cepstrum has a superior performance in the case of noisy measurements than the complex cepstrum, the power cepstra of the measurements with the curve-fitting algorithm has been used for parameter extraction. The trial set of poles and zeros entered to the curve-fitting algorithm are listed in Table 4-2. In addition to the poles and zeros presented in the table, one of the discretization
zeros which is located at \( z = -j \) has also been considered by choosing a trial value of -0.999 for each of the measurements. The curve-fitting is performed on the region of the power cepstrum defined between the 10\textsuperscript{th} and 300\textsuperscript{th} points.

<table>
<thead>
<tr>
<th>Trial Poles</th>
<th>Trial Zeros (H\textsubscript{1})</th>
<th>Trial Zeros (H\textsubscript{3})</th>
<th>Trial Zeros (H\textsubscript{4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.056 ± 5.593i</td>
<td>-0.060 ± 6.540i</td>
<td>-0.111 ± 11.834i</td>
<td>-0.187 ± 18.655i</td>
</tr>
<tr>
<td>-0.181 ± 18.074i</td>
<td>-0.211 ± 21.817i</td>
<td>-0.280 ± 27.912i</td>
<td></td>
</tr>
<tr>
<td>-0.276 ± 27.624i</td>
<td>-0.294 ± 30.006i</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.338 ± 33.807i</td>
<td>-0.418 ± 42.097i</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.449 ± 44.894i</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Two cases are examined: (1) The input which has a spectra with frequent jumps obtained by passing a white noise sequence through a FIR filter which is defined by a series of frequency bands and associated relative amplitudes, which are 0 – 1.0 Hz with relative amplitude 0.4, 1.5 – 2.3 Hz with relative amplitude 1, 3.0 – 4.7 with relative amplitude 0.4 and for the frequencies above 6 Hz with the relative amplitude of 1 and with a linear transition region between the bands, (2) The excitation is a white-noise sequence. The spectra of the two types of inputs are presented in Figure 4-1 and Figure 4-2 respectively.

Figure 4-1 Magnitude spectrum of the input considered in Case I
Case I:

The true and estimated zeros from the original and the proposed techniques are plotted in Figure 4-3, Figure 4-4 and Figure 4-5 where the true zeros and the zeros estimated from the original technique are denoted by (*) and (+) respectively, and the zeros obtained from the proposed technique are represented by (□) and (◊) depending on the available subtraction options as referred to within the legends of the figures. Note that only the zeros located on the upper half plane are plotted due to the fact the rest of the zeros are just the complex conjugates of what is plotted. In particular, Figure 4-3 presents the zeros of the pulse transfer function related with the measurement on the first floor. Note that the error in the estimates of the original technique is most noticeable for the zero located at $-0.329 \pm 33.557i$, which can be attributed to the fact that the input has a dip at around $\pm 30.4i$ and the curve-fitting algorithm might have actually estimated a zero associated with the input instead of the system. Whereas, the proposed technique which uses the power cepstra difference of the measurements at the first and the fourth masses does not have any difficulty in estimating this particular zero. The reason why the estimate of this particular zero obtained from the proposed technique which uses the power cepstra difference of the measurements at the first and third masses is poor is not because of the contribution of the input but because of the marginal zero-zero cancellation noted previously. Note that the zeros associated with the first and third masses around this frequency are $-0.329 \pm 33.557i$ and $-0.337 \pm 33.572i$ respectively. An interesting property that should be noted is that although the contribution of the canceling zeros to the difference of the cepstra is very small, the curve-fitting algorithm was able to estimate the canceling zeros at a frequency close to its true value (although the real part is quite off) such that almost exactly the same value was determined as a zero for the pulse transfer.
functions whose effects are cancelled (the same zero at around -1.8 ±30i is estimated in Figure 4-3 and Figure 4-4). This observation can actually be used as a warning to the user such that if almost the same zero is identified as a zero for the two measurements subtracted, there is a good potential that these zeros are associated with cancelled zeros. When it comes to the zero at -0.219 ± 22.593i, although the estimates obtained from both techniques are accurate in the imaginary part, the real parts are in error. This error can be partly attributed to the pole of the system located at -0.193 ± 19.302i which is quite close to the zero to be estimated. In the regular approach, the error can be associated with a marginal pole-zero cancellation, and in the proposed technique with a marginal pole-pole cancellation.

The dip observed in the spectrum of the input, which is located around ±36.0i is found to have an adverse effect to the curve-fitting process when the original technique is used on the measurements taken at the third mass. In particular, Figure 4-4 clearly indicates how the original technique suffers from the contribution of the input in estimating the zero located at -0.337 ± 33.572i while the proposed technique can accurately estimate the zero when the difference of the power cepstra of measurements at the third and fourth masses is used. Note that the reason why the estimate of the proposed technique for this particular zero is in error when the power cepstra of the measurements at mass one and three is used has already been explained as a zero-zero cancellation. Lastly, the zero associated with the measurement at the fourth floor has been identified more accurately in the proposed approach than the standard approach as shown in Figure 4-5. The reason for the poor performance of the original technique can be attributed to the fact that the zero to be estimated is very close to one of the poles of the system (marginal pole-zero cancellation).

**Case II:**

In this case, the input spectrum does not have any dips, hence it is not expected the curve-fitting algorithm to converge to any zeros that might be associated with the input when the original cepstrum technique is employed. Hence, from a theoretical perspective, it is not expected to obtain any significant advantage on the proposed approach over the regular cepstrum technique. The zeros estimated for the transfer functions associated with the first, third and fourth masses are presented in Figure 4-6, Figure 4-7 and Figure 4-8. As the figures indicate, there is not any significant difference in the accuracy of the estimated zeros obtained from the original or proposed cepstrum techniques in general, although there are a couple of zeros for which the proposed cepstrum technique perform slightly better than the original technique such as the last zero in Figure 4-6, the second zero in Figure 4-7.
Figure 4-3 True and estimated zeros of the transfer function corresponding to measurement at the first mass (Case I)

Figure 4-4 True and estimated zeros of the transfer function corresponding to measurement at the third mass (Case I)
Figure 4-5 True and estimated zeros of the transfer function corresponding to measurement at the fourth mass (Case I)

Figure 4-6 True and estimated zeros of the transfer function corresponding to measurement at the first mass (Case II)
Figure 4-7 True and estimated zeros of the transfer function corresponding to measurement at the third mass (Case II)

Figure 4-8 True and estimated zeros of the transfer function corresponding to measurement at the fourth mass (Case II)
4.3 Summary

A modified cepstrum technique which eliminates or significantly alleviates the limitations that are set by the original cepstrum technique on the input has been proposed. The key idea is to eliminate the contribution of the input to the cepstrum of the response by subtracting the cepstra of any two outputs from each other, hence canceling the part associated with the input.

In particular, the proposed technique enables one to estimate the zeros exactly in the case of noise-free data with no restriction on the type of input whatsoever. It is worth noting that the original technique as it is presented in the literature estimates the zeros approximately even the measurements are noise-free provided that the input applied has a flat-log spectrum.

When the data is noisy, the performance of the proposed technique with respect to the original technique depends on the type of input applied to the system. Specifically, if the input has a flat log spectrum, the performance of the two techniques has been found comparable. On the other hand, if the input is such that its spectrum has frequent jumps, then the proposed technique has been found to yield more accurate estimates of the zeros than the original technique. This follows from the fact that the proposed technique cancels the contribution of the input to the cepstrum used in the curve-fitting step.

In the proposed technique, due to the subtraction involved, one loses the information connected with the poles, hence poles cannot be identified. This follows from the fact that the poles of any two transfer function of the same system are the same. Conversely, the technique enables one to estimate the zeros of the two transfer functions that are associated with the measured responses whose cepstra are subtracted. This follows from the fact that the zeros of each transfer function of a system is unique. One difficulty stems from the subtraction step is that whenever any zero of the two transfer functions are located very close to each other, there will be a marginal zero-zero cancellation in the cepstrum domain and the curve-fitting algorithm will not be able to estimate this zero. The solution proposed is to make use of all available measurements and try all different combinations of subtractions so as to minimize the probability of a marginal zero-zero cancellation.
5 ESTIMATION OF ZEROS FROM RESPONSE MEASUREMENTS OF MIMO SYSTEMS

The issue of estimating the zeros of transfer functions from response measurements has been investigated in Chapters 3 and 4 for SIMO systems within the cepstrum framework. In this Chapter, the estimation of the zeros of transfer function matrices that relate a set of input coordinate(s) to a set of output coordinate(s) exclusively from response measurements has been investigated. Note that this proposition may appear ill-posed since without an input set the concept of zeros has no meaning. What we intend to estimate are the zeros of a “collocated” set of input-output coordinates which are (of course) defined by the outputs. It is a simple matter to realize that SIMO systems addressed in Chapters 3 and 4 constitute just a special case of the treatment examined in this Chapter.

In essence, the Chapter proposes a technique which is basically intended for systems whose mass matrices can be approximated as diagonal. The technique makes use of state-space matrices $A_c$ and $C_c$ which are obtained from stochastic system identification (Di Ruscio 1996, Van Overschee and Moor 1996) to estimate a surrogate matrix for the missing input to state matrix $B_c$ connected with a collocated distribution of forces. The approach works by forcing two constraints: (1) the off-diagonal terms of the mass matrix are zero, (2) direct transmission term relating forces to velocity or displacement measurements is zero. The technique yields exact results provided that mass matrix is diagonal, the identified system and output matrices, $A_c$ and $C_c$, contain the full modal bases and the number of sensors is larger than a certain threshold. Otherwise, the technique produces approximate results.

The Chapter first introduces the constraint equations exploited to estimate the surrogate matrix and then derives the proposed formulation. The approximations present in the formulation are outlined and the performance of the technique under various conditions is examined through numerical simulations.

5.1 Constraint Equations

Bernal (2006) derived two constraint equations involving the continuous time state-space quadruple \{$A_c$, $B_c$, $C_c$, $D_c$\} and used these equations effectively to obtain an expression relating the input to state matrix $B_c$ to the triple \{$A_c$, $C_c$, $D_c$\}. Derivation starts by writing the output equation for displacements as,

$$y = C^{dis}x + D^{dis}u$$

(5.1)

where superscript $dis$ denotes ‘displacement’. It should be noted that the direct transmission term given in Eq. (5.1), $D^{dis} = 0$ due to the fact that there cannot be a direct transmission term between forces and displacements. Differentiating Eq. (5.1) with respect to time and then substituting the necessary expressions given for the derivative of the state vector, one obtains
\[
\dot{y} = C^{\text{dis}}_c A_c x + C^{\text{dis}}_c B_c u 
\]
(5.2)

taking an additional differentiation of Eq. (5.2) yields
\[
\ddot{y} = C^{\text{dis}}_c A_c^2 x + C^{\text{dis}}_c A_c B_c u + C^{\text{dis}}_c B_c \dot{u} 
\]
(5.3)
since there is no direct transmission term for velocity measurements, Eq. (5.2) requires
\[
C^{\text{dis}}_c B_c = D^{\text{vel}} = 0 
\]
(5.4)
For acceleration measurements, however, there is a direct transmission term and as Eq. (5.3) shows, it is given by
\[
D^{\text{acc}}_c = C^{\text{dis}}_c A_c B_c 
\]
(5.5)
Two other relationships that follow from the inspection of Eqs. (5.1) to (5.3) are,
\[
C^{\text{vel}}_c = C^{\text{dis}}_c A_c 
\]
(5.6)
\[
C^{\text{acc}}_c = C^{\text{vel}}_c A_c 
\]
(5.7)
Introducing Eqs. (5.6) and (5.7) into Eqs. (5.4) and (5.5), one obtains
\[
C_c A_c^{1-p} B_c = D^{\text{acc}}_c 
\]
(5.8)
\[
C_c A_c^{-p} B_c = 0 
\]
(5.9)
where \( p = 0, 1, \) or 2 depending on whether the measured output is displacement, velocity or acceleration and \( C_c \) is the corresponding state to output matrix. In the remainder of the Chapter, the non-zero direct transmission term \( D^{\text{acc}}_c \) is simply referred to as \( D_c \). Note that Eqs. (5.8) and (5.9) can be combined and written as
\[
H_p B_c = L D_c 
\]
(5.10)
where \( H_p \in \mathbb{R}^{2m \times 2n} \) and \( L \in \mathbb{R}^{2m \times m} \) (\( m \) is the number of sensors available, \( 2n \) is the order) are given by
\[
H_p = \begin{bmatrix} C_c A_c^{1-p} \\ C_c^{-p} A_c^{-p} \end{bmatrix} 
\]
(5.11)
\[
L = \begin{bmatrix} I \\ 0 \end{bmatrix} 
\]
(5.12)
As it was mentioned in the introduction, we let $B_c$ to be associated with the collocated coordinates regardless of whether the actual loads on the system that produces the measured response are located at these coordinates or not. This follows from the fact that any response measured at the sensors could have been generated by artificial inputs that are applied at the sensor coordinates.

## 5.2 Proposed Formulation

An expression for $B_c$ can be obtained from Eq. (5.10) as

$$B_c = H_p^\dagger LD_c + N(H_p)r$$

(5.13)

where $(\dagger)$ refers to the Moore-Penrose pseudo-inverse, $N(\cdot)$ denotes a basis for the null space of the matrix in the parenthesis and $r$ is an undetermined matrix of appropriate dimension. Note that if $2m \geq 2n$, then the second term on the rhs of Eq. (5.13) does not exist. For the sake of completeness, however, we will carry that term throughout the derivation. The column partition of $B_c$ associated with an arbitrary input set, $I$ can be obtained as

$$B_{c,i} = H_p^\dagger LD_c v + N(H_p)rv$$

(5.14)

where $v \in \mathbb{R}^{mc}$ ($mc$ is the number of inputs in set $I$) is Boolean such that it selects these columns of $B_c$ that are associated with coordinates collected in set $I$. Expressing the selected columns of the direct transmission term as

$$D_c v = vE + \varepsilon$$

(5.15)

where $E \in \mathbb{R}^{mc \times mc}$ and $\varepsilon \in \mathbb{R}^{mc \times mc}$, plugging Eq. (5.15) into Eq. (5.14) and following simple algebra, one has

$$B_{c,i} = \left(H_p^\dagger Lv + \chi\right)E$$

(5.16)

where

$$\chi = \left(N(H_p)rv + H_p^\dagger L \varepsilon\right)E^{-1}$$

(5.17)

Finally, substituting Eq. (5.16) into standard formulation for the computation of zeros from state-space matrices (Eq. (2.56)) and recognizing that $E$ can be placed as a pre-multiplier of $g$ one gets

$$\begin{bmatrix} A_c & H_p^\dagger Lv + \chi \\ C_{c,i} & D_{c,i} \end{bmatrix}\begin{bmatrix} x_0 \\ Eg \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ Eg \end{bmatrix}$$

(5.18)
Considering the diversity of the type of measured output in Structural Health Monitoring applications a more general form of Eq. (5.18) can be written as

\[
\begin{bmatrix}
A_c & H_p^t L v + \chi
\end{bmatrix}
\begin{bmatrix}
x_0
\end{bmatrix} =
\begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
x_0
\end{bmatrix} +
\begin{bmatrix}
0 & 0
\end{bmatrix}
\begin{bmatrix}
E_g
\end{bmatrix}
\]  

(5.19)

where \( p \) is 0, 1 or 2 depending on whether the measured output is displacement, velocity or acceleration as noted before. Although \( \chi \) is not known, an approximate solution can be obtained from Eq. (5.19) by taking \( \chi = 0 \). In cases where \( \chi = 0 \), the proposed approach estimates the zeros exactly. Therefore, it is essential to investigate under which conditions the term \( \chi \) is zero or close to zero. As it is clear from Eq. (5.17), there are two contributions to \( \chi \), one is associated with the null space of \( H_p \) (when \( H_p \) is a wide matrix) and the other one is related with the non-zero off-diagonal terms of the matrix \( D_c \). Therefore, if \( H_p \) is square or tall and the matrix \( D_c \) is diagonal the approach yields exact results.

### 5.2.1 A Comment on Equation (5.8)

It should be noted that the direct transmission term \( D_c \) is physically associated with the mass matrix of the system. In fact, it is equivalent to the partition of the inverse of the mass matrix associated with the input and output coordinates considered. Most of the systems that are of interest in civil or mechanical engineering can be represented by a diagonal mass matrix. Since the proposed technique estimates the columns of the matrix \( B_c \) that are associated with collocated coordinates, the corresponding direct transmission term, as Eq. (5.8) suggests, becomes diagonal provided that the matrices \( A_c \) and \( C_c \) contain full modal basis. When the identified state-space matrices are modally truncated, however, due to the missing terms, the direct transmission term that is calculated from the truncated matrices based on Eq. (5.8) ceases to become diagonal anymore. Even so, the matrix can be a close approximation to a diagonal matrix; hence the off-diagonal terms can still be treated as zero and an approximate solution can be obtained from Eq. (5.19).

### 5.2.2 A Comment on Equation (5.9)

Bernal (2006) showed that the validity of Eq. (5.9) is dependent on the type of damping present in the system and whether the matrices \( A_c \) and \( C_c \) are modally truncated or not. In particular, if the matrices \( A_c \) and \( C_c \) represent the full modal basis, Eq. (5.9) is valid regardless of the damping. If, however, the matrices are modally truncated, then Eq. (5.9) is still valid provided that the damping is of classical nature. A formal proof of this last statement can be found in Bernal (2006). If the damping is not classical, Eq. (5.9) does not hold exactly and further approximations are introduced to the zeros estimated from Eq. (5.19). It can, however, be argued that in many civil and mechanical engineering applications, most of the systems are lightly damped in which case Eq. (5.9) produces
very small values, if not zero, even in the case of truncation of a non-classically damped system, hence Eq. (5.9) although not exact, provides reasonable approximations.

Before moving to the numerical simulations, it is important to see how independent the information that is carried by each row of Eq. (5.9) is in order to estimate the column rank of the matrix \( H_p \). Since the validity of Eq. (5.9) depends on the type of damping present in a structure in the truncated case, we will look at the problem separately for classically and non-classically damped systems.

### 5.2.2.1 Classically Damped Systems

For classically damped systems, Eq. (5.9) produces \( n \) independent equations for each column of \( B_c \). Note that the number of rows in \( B_c \) is equal to the order (\( 2n \)). As it will be clear in the following, it becomes possible to get half of the unknowns directly from Eq. (5.9) once the matrices are expressed in modal coordinates. The number of sensors (rows of \( C_c \)) needed for this is just one and the information does not multiply with more sensors. As a matter of fact, any row of the matrix \( C_c \) contains the same information. In order to see this, let’s look at Eq. (5.9) \((p = 0)\) written in modal coordinates

\[
(C_c \Psi)(\Psi^{-1} B_c) = 0
\]  

(5.20)

Since Eq. (5.9) is valid for all levels of truncation, consider the simplest case where the system is truncated to a single mode, \( j \), then, Eq. (5.20) can be expressed as

\[
\begin{bmatrix}
c_{ij} & c_{1j}^* \\
c_{2j} & c_{2j}^* \\
.. & .. \\
c_{mj} & c_{mj}^*
\end{bmatrix}
\begin{bmatrix}
b_{ij} \\
b_{ij}^*
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  

(5.21)

where \( c_{ij} \) and \( b_{ij} \) represent entries of \( C_c \Psi \) and \( \Psi^{-1} B_c \), and the subscripts identify the location of the sensor and the mode considered respectively. A simple examination of the real part of Eq. (5.21) reveals that the following has to be satisfied for any sensor, \( i \) and any mode \( j \),

\[
\frac{\text{Im}(c_{ij})}{\text{Re}(c_{ij})} = \frac{\text{Re}(b_{ij})}{\text{Im}(b_{ij})}
\]  

(5.22)

where \( \text{Re}(\cdot) \) and \( \text{Im}(\cdot) \) refer to the real and imaginary parts of the complex numbers located inside the parentheses. It is a well-known fact that the mode shapes of a classically damped system can be made real with appropriate scaling. Bernal (2006) showed that when the mode shapes are made real, the associated modal participation factors, the elements of the corresponding matrix \( B_c \), become purely imaginary. It is
straightforward to confirm the same observation from Eq. (5.22) by simply looking at the limits, namely, when \( \text{Im}(c_{ij}) \to 0, \text{Re}(b_{ij}) \to 0 \).

Note that Eq. (5.22) simply states that the phase angles \( \phi_{b,ij} \) of the entries of the input matrix in modal coordinates, \( \Psi^T B_c \), is readily available from the phase angles \( \phi_{c,ij} \) of the associated entries of the output matrix in modal coordinates, \( C_c \Psi \). Therefore, employing all the sensors or only a single sensor in Eq. (5.9) introduces one and the same information. In fact, the phase angles are related by

\[
\phi_{b,ij} = \text{acot} \left( \frac{\tan(\phi_{c,ij})}{1} \right)
\]  

As it is clear from Eq. (5.23), half of the unknowns (phase angles) in any column of \( B_c \) can be obtained directly from \( C_c \). However, that information is not sufficient to find a reasonable estimate that can be used as a surrogate for \( B_c \), since the magnitudes of the elements of \( B_c \) are not available.

### 5.2.2.2 Non-classically Damped Systems

For non-classically damped systems, Eq. (5.9) is satisfied only when the full modal model is available. In that case, as opposed to classically damped systems, each sensor contains independent information. When the modal model is truncated, Eq. (5.9) becomes no longer exact, but deviations from zero are typically small and it can give reasonable estimates in many cases as shown in Section 5.3.

### 5.3 Summary and Further Clarifications

The performance of the proposed formulation differs whether the identified matrices \( A_c \) and \( C_c \) represent the full modal space or a truncated modal space. Moreover, the quality of the identified matrices \( A_c \) and \( C_c \) (identification error) plays an important role for an accurate estimation of the zeros. While the latter issue is examined through Monte-Carlo simulations in Section 5.4, the former issue is examined for each case separately in this Section.

**Case I : Full order model (\( A_c \) and \( C_c \) are not truncated)**

In this case, the direct transmission term \( D_c \) is diagonal; hence the part of the error in \( \chi \), denoted by \( \varepsilon \) disappears. If the number of sensors available is equal or more than half of the order, then the matrix \( H_p \) becomes full column rank and the formulation yields exact results. In other words, one can estimate the zeros of the true system exactly from Eq. (5.19). If however, the number of sensors is less than the half of the order, then the zeros estimated accordingly become approximate due to the error term that is connected with the null space of \( H_p \). In order to illustrate these points consider the following example:
Example:

A 3-story shear frame with story stiffnesses of \{800, 700, 1000\} and story masses of \{1.25, 1.10, 1.35\} which are in consistent units is considered. The structure is taken to be classically damped with a 2% of critical damping at each mode. The full order matrices $A_c$ and $C_c$ are assumed to be available. Two cases are considered: (1) all of the masses are measured ($m = 3$), (2) Only the second mass is measured ($m = 1$).

In the first case, since the number of sensors is equal to the half of the order ($6 / 2 = 3$), the proposed formulation produces exact results. For illustration purposes, the zeros of the collocated transfer function matrix ($G_{12,12}$) that relates the inputs at the first and second masses to the outputs at the same locations and the zeros of the collocated transfer function ($G_{22}$) defined at the second mass are estimated. The proposed formulation is executed in the following order. First, the surrogate input to state matrix is computed as $H_p^{-1}L$ where $H_p \in \mathbb{R}^{6 \times 6}$, $L \in \mathbb{R}^{6 \times 3}$. Then, the first two columns and then the second column of the surrogate matrix are treated as the unknown matrix $B_c$ and the zeros of the transfer function matrix ($G_{12,12}$) and the transfer function ($G_{22}$) are estimated respectively in the standard way. The estimated zeros and the true zeros are presented in Table 5-1 and Table 5-2. As it is clear, the estimated zeros are exact in both cases.

Table 5-1 The zeros of the collocated transfer function matrix defined at the first two masses ($m = 3$)

<table>
<thead>
<tr>
<th>Estimated Zeros ($m=3$)</th>
<th>True Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4591 ± 27.2130i</td>
<td>-0.4591 ± 27.2130i</td>
</tr>
</tbody>
</table>

In the second case where only the sensor at the second mass is available ($m = 1$), since the number of sensors is less than the half of the order, an exact solution is no longer available due to the fact that the matrix $H_p$ is wide. For illustration purposes, the zeros of the collocated transfer function defined at the second mass estimated by the proposed formulation are presented in Table 5-2. As it is clear from the table, only one of the zeros of the true system is approximately estimated.

Table 5-2 The zeros of the collocated transfer function defined at the second mass ($m = 1$)

<table>
<thead>
<tr>
<th>Estimated Zeros ($m=3$)</th>
<th>Estimated Zeros ($m=1$)</th>
<th>True Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4591 ± 27.213i</td>
<td>1.3351 &amp; 404.800</td>
<td>-0.4591 ± 27.213i</td>
</tr>
<tr>
<td>-0.6664 ± 34.634i</td>
<td>-0.7699 ± 34.2300i</td>
<td>-0.6664 ± 34.634i</td>
</tr>
</tbody>
</table>
**Case II: Truncated model (A\textsubscript{c} and C\textsubscript{c} are modally truncated)**

It is worth noting from the outset that in this case, even if there was no error associated with the formulation ($\chi = 0$), one cannot estimate the zeros of the true system, but the zeros of the truncated system defined by the truncated matrices A\textsubscript{c} and C\textsubscript{c}. This follows from the fact that the matrices A\textsubscript{c} and C\textsubscript{c} do not contain the full information of the true physical system instead a modally truncated version of the true system.

Moreover, the matrix D\textsubscript{c} is not diagonal anymore which gives rise to the error term $\varepsilon$ in $\chi$. If the number of sensors available is equal or more than half of the order, then the matrix H\textsubscript{p} becomes full column rank and the error term associated with the null space of H\textsubscript{p} disappears. However, due to the non-diagonal nature of D\textsubscript{c}, an exact solution which would produce the zeros of the truncated system cannot be obtained. If the number of sensors is less than the half of the order, then H\textsubscript{p} becomes wide and further approximations are introduced due to the null space of H\textsubscript{p}. All of these points are illustrated in the following example.

**Example:**

The same system defined in Case I has also been considered here. The matrices A\textsubscript{c} and C\textsubscript{c} are truncated to the first two modes (order is 4) in this case. Similar to Case I, two conditions are examined: (1) First and second masses are measured ($m = 2$), (2) Only the second mass is measured ($m = 1$). Since the system is truncated, the direct transmission matrix is not diagonal anymore; hence approximations in the estimated zeros are inevitable.

In the first case where the number of sensors is equal to the half of the order, the matrix H\textsubscript{p} is square hence the computation of the surrogate matrix is free from the error associated with the null space of H\textsubscript{p}. However, when it comes to the estimation of the zeros of the collocated transfer function defined at the second mass, one needs the second column of the matrix B\textsubscript{c}, or a column vector that is within to a scalar to the second column of the matrix B\textsubscript{c}. The second column of the surrogate matrix (H\textsubscript{p}\textsuperscript{-1}L) is not within to a scalar to the second column of the matrix B\textsubscript{c} as it was the case in the full order system. This follows from the fact that the matrix B\textsubscript{c} is equal to H\textsubscript{p}\textsuperscript{-1}LD\textsubscript{c} and D\textsubscript{c} is no longer diagonal. Therefore, one cannot take any column of the surrogate matrix (H\textsubscript{p}\textsuperscript{-1}L) and treat it as the respective column of B\textsubscript{c} without accepting the approximations introduced. Here, we proceed by accepting the approximations and treat the second column of the surrogate matrix as if it is within to a scalar to the second column of B\textsubscript{c}. In other words, we assume matrix D\textsubscript{c} to be diagonal. The zeros estimated accordingly and the true zeros of the truncated system are presented in Table 5-3. As it is clear, the estimated zero is no longer exact but the deviations can be acceptable depending on the type of application. It is also worth noting that the true zeros of the full and truncated systems are different both in number and in value.

In the second case, only one sensor is employed, hence the matrix H\textsubscript{p} is no longer square and the estimated surrogate matrix is perturbed due to the missing contribution of the null
space of $H_m$. However, the matrix $D_c$ turns out to be a scalar in this case, hence the error connected with $D_c$ having non-zero off-diagonal terms is completely avoided. The zeros of the collocated transfer function estimated by the proposed formulation are given in Table 5-3. As it is clear, the estimated zeros are much closer to the true zeros of the truncated system when a single sensor is available, which can be attributed to the fact that error term $\varepsilon$ is completely avoided.

Table 5-3 The estimated and the true zeros of the collocated transfer function defined at the second mass (truncated system)

<table>
<thead>
<tr>
<th>Estimated Zeros (m=2)</th>
<th>Estimated Zeros (m=1)</th>
<th>True Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.7149 ± 34.6336i</td>
<td>-0.5936 ± 28.9836i</td>
<td>-0.6066 ± 31.0323i</td>
</tr>
</tbody>
</table>

Lastly, the effect of non-classical damping to the quality of the estimated zeros in the case of modally truncated systems is illustrated within the context of the sample system considered herein. For this purpose, the damping matrix of the system is taken as the square of the stiffness matrix multiplied by $3.0965e-007$. Only the sensor on the second mass is considered ($m = 1$). The estimated zeros together with the true zeros of the collocated transfer function defined at the second mass of the truncated system are presented in Table 5-4. As it is clear, the distortion observed in the estimated zero is similar to that observed in the classically damped system, supporting the fact that the error associated with Eq. (5.9) is not significant for lightly damped systems.

Table 5-4 The estimated and the true zeros of the collocated transfer function defined at the second mass (truncated system)

<table>
<thead>
<tr>
<th>Estimated Zeros</th>
<th>True Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1920 ± 28.9921i</td>
<td>-0.1993 ± 31.0376i</td>
</tr>
</tbody>
</table>

5.4 Identification Error in the Identified Matrices $A_c$ and $C_c$

Since the proposed formulation employs the system matrix, $A_c$, and the output matrix, $C_c$, for the estimation of the system zeros, they need to be extracted directly from response measurements in practice. Therefore, there is a need to estimate whether the available techniques that can be used to extract these matrices directly from output measurements produce system realizations with decent quality such that the zeros can be estimated with no significant deviations. In order to examine this issue, the following numerical study has been carried out.
Example:

A 5 story shear frame with story stiffnesses of \{800, 700, 600, 650, 500\} and story masses of \{1.20, 1.15, 1.05, 1.25, 1.30\} with a 1 % classical damping at each mode is considered. The natural frequencies of the structure lie in the range of 1.08 Hz – 7.25 Hz. The structure is excited by random white noise sequences at levels 1, 3 and 4 and the resulting acceleration responses at all floors are taken as output. While the inputs are not measured, the response measurements are sampled with a sampling frequency of 40 Hz. In order to simulate noisy conditions, the measurements are contaminated by additive white noise sequences with an RMS value of 1%, 3%, 5% and 10% of the measured output at the first mass. Stochastic subspace identification algorithm has been used to obtain a realization in the form of \{A_c, C_c\} for each data set. Then, the zeros of the transfer function defined at the first mass are estimated based on the proposed formulation. Since the number of sensors is equal to the half of the order and the system identified is not truncated, the error in the estimated zeros is solely due to the error in the identified matrices \(A_c\) and \(C_c\). In order to better evaluate the effect of the identification error to the accuracy of the estimated zeros, a Monte Carlo analysis consisting of 100 simulations for each noise level is conducted. In each simulation, all the steps mentioned above are repeated for a different loading and noise condition. The normalized histogram of the real and imaginary parts of the first zero estimated for each level of noise considered are given in Figure 5-1 to Figure 5-4. The mean values and the standard deviations of the real and imaginary parts of the estimated zero are also provided in the figures. As it is clear from the figures, the identification error in the matrices \(A_c\) and \(C_c\) does not distort the locations of the estimated zeros significantly. In particular, the mean values of the real and imaginary parts of the estimates are quite close to the exact values for all practical purposes.
Figure 5-1 Normalized histogram of the real and imaginary parts of the first zero estimated (1% noise)

Figure 5-2 Normalized histogram of the real and imaginary parts of the first zero estimated (3% noise)
Figure 5-3 Normalized histogram of the real and imaginary parts of the first zero estimated (5% noise)

Figure 5-4 Normalized histogram of the real and imaginary parts of the first zero estimated (10% noise)
5.5 Summary

A technique which enables one to estimate the zeros of MIMO systems whose mass matrices can be approximated as diagonal has been proposed. The technique produces exact results provided that the mass matrix is diagonal (or its inverse, or any partition of its inverse, namely matrix \( D_c \)), the system matrices \( A_c \) and \( C_c \) are not truncated the number of measured coordinates is equal or larger than the order of the system matrices identified.

The technique yields approximate results if the number of sensors is less than the half of the order and/or the mass matrix is not diagonal, which is the case observed whenever the matrices \( A_c \) and \( C_c \) represent a modally truncated version of the original system. Numerical simulations suggested that the approximations in the estimated zeros are acceptable for many applications provided that the mass matrix associated with a truncated system has a form close to be diagonal.

Since the proposed technique makes use of the matrices \( A_c \) and \( C_c \) and they need to be extracted from measurements, a Monte Carlo simulation has been performed in order to study the effect of noise to the accuracy of the estimated matrices. The results indicate that the matrices extracted from noisy measurements can be effectively used to estimate zeros with reasonable accuracy.
A zero, $z_0$, is defined through a particular set of input and output coordinates such that if an input in the form of $ge^{z_0t}$ is applied to the input coordinates, the response at the output coordinates vanishes. Clearly, zeros may exist for transfer functions relating a single input to a single output coordinate (SISO systems) as well as for transfer function matrices relating multiple input and multiple output coordinates (MIMO systems).

The computation of zeros from a transfer function matrix or equivalently from a state-space representation of a system is a standard technique in control theory. The problem of estimating the zeros from measured input-output data has been studied by many researchers (Lallement and Cogan, 1992; Bernal and Tigli, 2007c). These works mainly focused on processing the measured data and then either obtaining frequency response functions (FRFs) in the frequency domain or extracting a system realization in the time domain. Once the data is converted into these forms, the computation of zeros follows standard techniques available in the literature.

A more challenging situation is the estimation of zeros from output measurements only. At first, the problem set forth may seem contradictory to the definition of the zeros, since zeros are defined through a particular set of input and output coordinates and if there is no information whatsoever on the inputs, the estimation of zeros might be impossible. If, however, some additional information on the physical system or on the source of excitation becomes available, some opportunities may be offered.

The reason why we are interested in estimating zeros from data is because, if we could, then they can be used in a variety of applications in the fields of structural health monitoring (Dilena and Morassi, 2004; Reich and Park, 2000; Bernal 2007b), finite element model updating (Rade and Lallement, 1998; D’ambrogio and Fregolent, 2000, 2003), model validation and control (Mcever and Leo, 2001).

Even though the zeros can be used effectively in many applications, the literature on the estimation of zeros from output measurements is quite limited. In fact, the single technique exploited for this purpose is the cepstrum approach and the published literature on the topic is only by Prof. Randall and his PhD students (Randall and Gao, 1994; Hanson et. al, 2007a and 2007b). The cepstrum technique can only be applied to SIMO systems where input has to satisfy certain requirements, namely, it is required to have a flat-log spectrum. In addition, the location of the input has to be known to interpret the estimated zeros.

The cepstrum of a response measurement is defined as the inverse Fourier transform of the logarithm of the Fourier transform of the measurement. While the preceding definition is referred to as complex cepstrum, the power cepstrum is defined as the inverse Fourier transform of the logarithm of the magnitude of the Fourier transform of
the measurement. These two types of cepstra are the ones that have been exploited most in the literature.

One of the challenges of the cepstrum technique is that since the cepstrum of the measurements contain both the information on the system (poles and zeros) and the information on the input combined, one has to look for ways which enable one to separate the two contributions and then to concentrate on the system information. In order to address this issue, the work presented in the literature puts constraints on the type of input as having flat-log spectrum so that the contribution of such an input mainly concentrates on the earlier part of the frequency axis in the cepstrum domain and one approximates the remaining part of the cepstrum to be composed of solely the contribution of the system. It is worth noting that there is no comment in the literature as to whether the trick mentioned above is equally applicable to the power and complex cepstra, although it has been used mainly with the power cepstra of the measurements. This ambiguity is clarified in this dissertation and it has been shown that the trick mentioned above is only valid for power cepstra. The clarification is based on the fact that the complex cepstrum of the stable and non-stable parts of an input with flat-log spectrum turns out to be approximately equal to each other in value with an opposite sign. Since the power cepstrum is the sum of these two cepstra, it tends to be almost negligible compared to the contribution of the system information (poles and zeros) to the power cepstrum. Hence, the separation of the system information from that of the input is achieved naturally.

One of the issues that is not addressed explicitly in the literature is how the cepstrum technique perform when the measured data is noisy. The problem is examined in this dissertation by treating the noisy measurements as being the response of an artificial system to an artificial input. The dynamics of the artificial system is defined by the combination of the dynamics of the original system, the noise and the input being applied. It has been concluded that provided that the original input has a flat-log spectrum and the noise is of white nature, then the power cepstrum can be effectively used to estimate the zeros. Moreover, the complex cepstrum has been found to be more sensitive to noise than power cepstrum, hence not recommended.

One of the difficulties associated with the use of power or complex cepstra for parameter estimation is the need for specifying the number and the trial values of the poles and the zeros a priori for the curve-fitting algorithm. The differential cepstrum has been proposed for parameter estimation due to the fact that differential cepstrum coefficients can be treated as free decay data and the parameters can be estimated by any system identification algorithm which uses free-decay data (Gao and Randall, 1996). The advantage is clear; the need for specifying the number and values of the poles and zeros a priori is avoided. However, the downside of the differential cepstrum is that it is very sensitive to aliasing. Therefore, unless the measured response is almost an impulse response and the measurements goes to almost zero long before the end of the measurement duration, the use of differential cepstrum for parameter estimation is not recommended.
One of the contributions of this dissertation is a simple modification to the cepstrum technique which enables one to remove the contribution of the input completely from the power or complex cepstrum of the measurements; hence the restrictions on the type of input present in the original cepstrum technique are not required in the modified version. The proposed technique enables one to estimate the zeros exactly in the case of noise-free data with no restriction on the type of input whatsoever. It is worth noting that the original technique as it is presented in the literature estimates the zeros approximately even the measurements are noise-free provided that the input applied has a flat-log spectrum.

When the data is noisy, the performance of the proposed technique with respect to the original technique depends on the type of input applied to the system. Specifically, if the input has a flat log spectrum, the performance of the two techniques has been found comparable. On the other hand, if the input is such that its spectrum has frequent jumps, then the proposed technique has been found to yield more accurate estimates of the zeros than the original technique. This follows from the fact that the proposed technique cancels the contribution of the input to the cepstrum used in the curve-fitting step.

In the proposed technique, due to the subtraction involved, one looses the information connected with the poles, hence poles cannot be identified. This follows from the fact that the poles of any two transfer function of the same system are the same. Conversely, the technique enables one to estimate the zeros of the two transfer functions that are associated with the measured responses whose cepstra are subtracted. This follows from the fact that the zeros of each transfer function of a system is unique. One difficulty stems from the subtraction step is that whenever any zero of the two transfer functions are located very close to each other, there will be a marginal zero-zero cancellation in the cepstrum domain and the curve-fitting algorithm will not be able to estimate this zero. The solution proposed is to make use of all available measurements and try all different combinations of subtractions so as to minimize the probability of a marginal zero-zero cancellation.

While the modified cepstrum technique eliminates all the restrictions on the input, it still requires there be a single input as the original cepstrum technique does. However, there are many cases especially in civil engineering field that the number of inputs is more than one and for the reasons of practicality and many others, they cannot be measured. The estimation of zeros in these MIMO systems has not been addressed completely in the literature. Available publications involve attempts to extend the cepstrum technique to MIMO systems provided that certain conditions regarding the input is satisfied such as the magnitude of one of the inputs being significantly larger than others or one of the inputs having cyclostationary characteristics.

In order to address the issue of estimating zeros in MIMO systems, the dissertation introduces a technique which is intended for systems whose mass matrices can be approximated as diagonal. Assuming the measured responses are due to collocated forces, the technique makes use of state-space matrices $A_c$ and $C_c$ which are obtained from stochastic system identification and estimates a surrogate matrix for the missing input
matrix \((B_c)\) connected with a collocated distribution of forces. The approach works by forcing two constraints: (1) the off-diagonal terms of the mass matrix are zero, (2) the direct transmission term relating forces to velocity or displacement measurements is zero.

The technique yields exact results provided that mass matrix is diagonal and the system matrices \(A_c\) and \(C_c\) are not truncated and the number of sensors is equal or larger than the order of the system matrices identified. Otherwise, the technique yields approximate results. Numerical simulations suggested that the approximations in the estimated zeros are acceptable for many applications provided that the mass matrix associated with a truncated system has a form close to be diagonal.

Since the proposed technique makes use of the matrices \(A_c\) and \(C_c\) and they need to be extracted from measurements, a Monte Carlo simulation has been performed in order to study the effect of noise to the accuracy of the estimated matrices. The results indicate that the matrices extracted from noisy measurements can be effectively used to estimate zeros with reasonable accuracy.
7 REFERENCES


