POSITIVE CONTROL WITH MAXIMUM STABILITY RADIUS
FOR CONTINUOUS-TIME DYNAMIC SYSTEMS

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Abstract

Positive systems have attracted much attention nowadays due to their numerous applications in modeling and control of physical, biological and economical systems. The state trajectory of such system remains in the nonnegative quadrant of the state space for any given nonnegative initial condition. This class of systems have nice stability and robustness properties. One can take advantage of these interesting properties to robustly stabilize general dynamic systems such that the closed-loop system becomes positive. One of the most important measures in robust control analysis is stability radius. This measure provides the amount of uncertainty that system can cope with before it becomes unstable. There are two types of stability radius defined; complex and real stability radius. Computation of real stability radius is more involved than its complex counterpart. Although the complex and real stability radius are different for a general LTI system, it has been found that they are equal for the class of positive system. In fact, a closed form expression can be obtained to find the stability radius of positive system. In this thesis, we try to positively stabilize a general uncertain system with the constraint of maximizing stability radius by using a state feedback control law.

First, some standard theorems and definitions on positive systems are discussed along with providing some preliminary results. The necessary and sufficient conditions for the existence of controllers satisfying the positivity constraints are provided. This constrained stabilization problem will be formulated and solved using linear programming (LP) and linear matrix inequality (LMI). With the aid of bounded real lemma, the major contribution of this thesis is to solve the constrained positive stabilization with maximum stability radius for both regular and time-delay systems.
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Introduction

Dynamical system theory has played a central role in the understanding of many biological, ecological and physiological processes. The dynamical models of many such systems involves state variables whose values are nonnegative. Hence, it follows from physical considerations that the state trajectory of such states remains in nonnegative orthant of state space for any given nonnegative initial condition. Such systems are commonly referred to nonnegative or positive dynamical systems in the literature. Roughly speaking, positive systems are systems whose inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear system behavior can be found in engineering, management service, economics, social sciences, biology and medicine, etc. An overview of state the art in positive system is provided in [1], [2].

The class of continuous-time positive systems is also known as Metzlerian systems due to the fact that the associated system matrix is a Metzler matrix and the input output coefficients matrices are nonnegative. A Metzler matrix has positive off-diagonal entries and in the strict sense its diagonal elements are negative. On the other hand, the class of discrete-time positive system has coefficient matrices which are element-wise nonnegative. The nonnegative and Metzlerian matrices are well-studied topics in matrix analysis and have found utilities in dynamical systems involving positivity as pointed out above. This class of matrices and their associated nonnegative and Metzlerian systems have also been examined in the context of stability analysis and robustness [3], [4].

Concentrating on the stable continuous-time linear systems with uncertainty, an interesting problem is to compute the stability radius. For complex-valued perturbation the stability radius which may be conveniently termed complex stability radius can be readily computed with respect to variety of norms of interest. This computation is facilitated by tools developed in $H_{\infty}$ analysis and those for computing the structured signal value; the latter, by definition, is the reciprocal of the stability radius [5]. When the perturbations are real-valued for which the stability radius is called real stability radius accordingly, the problem of computing the stability radius proved to be far more difficult. Nevertheless, a formula for computation of the real stability radius have been found [6] which requires the solution of an iterative global optimization problem. However, for the class of positive system the complex and real stability radii
coincide and can be computed with closed form expression for both continuous- and discrete-time cases [7], [8].

Recently efforts were devoted to solve the constrained stabilization problems based on structural characteristics of a special class of systems. The main idea behind this approach is to consider special properties of a certain class of systems and design controllers for general systems such that the closed-loop systems are stabilized and at the same time maintains those desirable properties. Here the special class of system considered for this purpose is linear continuous-time positive (Metzlerian) systems. The positive stabilization, and robust stabilization with non-negativity constraints have been tackled for both conventional and delay dynamical systems by a number of researchers [9] - [10]. The solution for this category of problem can be obtained using linear programming (LP) or linear matrix inequality (LMI) [11], [10].

In this thesis we consider the Metzlerian stabilization problem for general linear dynamical system with the aim of maximizing real stability radius for the closed-loop systems. Using the fundamental relation between real and complex stability radius for the class of Metzlerian systems, we are able to formulate and solve the problem in terms of an additional LMI to the original constrained stabilization problem. A major tool that makes the solution of this problem possible, is the bounded real lemma (BRL). As BRL has proven advantages in different arenas in connection to $H_\infty$ control theory, it plays a crucial role in establishing our main result in this thesis. In fact this is the major contribution of this thesis which relates BRL to compute maximum stability radius in connection to positive stabilization.

The key components of each chapter are highlighted in the following paragraphs. Chapter 1 reviews the essential matrix analysis background for the purpose of studying positive systems. In Chapter 2, positive systems are defined and several application examples representing positive system are provided. Exciting stability properties of positive systems are introduced in Chapter 3. Among these nice properties the stability radius, which is a robust stability measure, is defined and elaborated further since it plays a key role in the contribution of this thesis. Chapter 4 derives both linear programming (LP) and Linear Matrix Inequality (LMI) approaches for solving problem of constrained stabilization with maximum stability radius for positive systems. The proper controller to achieve this purpose is provided for any given uncertain system. Chapter 5 considers the same problem for the time-delay case. A generalization of the LMI based approach of Chapter 4 is introduced in Chapter 5 to obtain the proper controller for positive stabilization of any given time-delay uncertain system with maximum stability radius. The generalization should be done with extra care by proper adjustment of the theoretical development discussed in Chapter 5.
Chapter 6 includes several numerical example to support the main results of the thesis followed by Chapter 7 in which the concluding remarks are made.
Chapter 1 – Mathematical Background and Literature Review

In this chapter, we are going to discuss the essential mathematical tools needed to work with positive systems. Following definitions and lemmas are standard and can be found in [4]. There are various applications for this class of systems as discussed in introduction. Hence, analyzing them is a matter of interest. Note that analyzing any kind of system is prior to controlling them. In next chapters, we are going to analyze and study these class of systems. In this chapter, we will mainly focus on matrices and special class of them with known properties that can be utilized in next chapters for analyzing and controlling positive systems. We start with defining special class of matrices. These matrices and the mathematical background corresponding to them is needed for next chapters discussion.

1.1. Positive Matrices

Let $\mathbb{R}^{n \times m}$ be the set of $n \times m$ matrices with entries from the real field $\mathbb{R}$.

Definition 1.1. A matrix is called the monomial matrix (or generalized permutation matrix) if its every row and its every column contains only one positive entry and the remaining entries are zero.

The permutation matrix is a special case of monomial matrix. Every row and every column of the permutation matrix has only one entry equal to 1 and the remaining entries are zero. A monomial matrix is the product of permutation matrix and a nonsingular diagonal matrix.

The inverse matrix of the monomial matrix is also the monomial matrix. The inverse matrix of a permutation matrix $P$ is equal to the transpose matrix $P^T$, i.e. $P^{-1} = P^T$. The inverse matrix $A^{-1}$ of a monomial matrix $A$ is equal to the transpose matrix in which every nonzero entry is replaced by its inverse.

For example, the inverse matrix $A^{-1}$ of the monomial matrix

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 5 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

has the form
\[
A^{-1} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & \frac{1}{2} \\
\frac{1}{3} & 0 & 0
\end{bmatrix}
\] (1.2)

**Definition 1.2.** A matrix \( A \in \mathbb{R}^{n \times m} \) is called nonnegative if its entries \( a_{ij} \) are nonnegative \((a_{ij} \geq 0)\).

The nonnegative matrix \( A \) will be denoted by \( A \geq 0 \) and the set of \( n \times m \) nonnegative matrices by \( \mathbb{R}^{n \times m}_+ \).

In the particular case that the nonnegative matrix has all zero entries, i.e. it is known as a zero matrix.

**Definition 1.3.** A nonnegative matrix \( A \in \mathbb{R}^{n \times m}_+ \) is called positive if at least one of its entries is positive.

The positive matrix \( A \) will be denoted by \( A > 0 \) and the generalized permutation matrix is an example of a positive matrix.

**Definition 1.4.** A matrix \( A \in \mathbb{R}^{n \times m}_+ \) is called strictly positive if all its entries are positive. The strictly positive matrix \( A \) will denoted by \( A \gg 0 \).

**Theorem 1.1.** The inverse matrix of a positive matrix \( A \in \mathbb{R}^{n \times n}_+ \) is the positive matrix if and only if \( A \) is monomial matrix.

**Proof.** Sufficiency follows from the inverse matrix of a monomial matrix is also a monomial matrix which is the positive matrix. To show the necessity we assume that the inverse matrix \( A^{-1} = [b_{ij}] \) of a positive matrix \( A = [a_{ij}] \) is a nonnegative matrix. From the equation \( AA^{-1} = I \) we have

\[
\sum a_{ik}b_{kj} = \delta_{ij} = \begin{cases}
1 & \text{for } i = j \\
0 & \text{for } i \neq j
\end{cases} \quad i, j = 1, 2, \ldots, n
\] (1.3)

If the \( i \) th row of \( A \) has \( p \) positive entries \( a_{it} \) for \( t = 1, 2, \ldots, p \) and \( i \neq j \), then from equation (1.3) it follows that \( b_{ij} = 0 \) for \( k = j, t = 1, 2, \ldots, p \). In this case the matrix \( A^{-1} \) contains the \( p \times (n-1) \) zero submatrix. If \( p > 1 \) then \( \det A^{-1} = 0 \). Hence \( k = 1 \), since by assumption \( \det A \neq 0 \). The matrix \( A \) has only one positive entry in every row and every column. \[\blacksquare\]

**Theorem 1.2.** Let \( P = [p_{ij}] \in \mathbb{R}^{n \times n} \) be a monomial matrix. Then
(1) The matrix $B = P^{-1}AP$ is a positive matrix ($B > 0$) for every positive matrix $A > 0$.
(2) The matrices $A$ and $B$ have the same spectrum (the set of eigenvalues).
(3) The trace of matrix $A$ (sum of entries of the main diagonal) is equal to the trace of the matrix $B$, i.e. $\text{tr } A = \text{tr } B$.

**Proof.** The inverse matrix $P^{-1}$ of the monomial matrix $P$ is also the monomial matrix $P^{-1} > 0$. The matrix $B$ is the product of three matrices. We may write

$$\det[\text{Is } - P^{-1}AP] = \det[P^{-1}(\text{Is } - A)P] = \det P^{-1} \det[\text{Is } - A] \det P = \det[\text{Is } - A]$$

since $\det P^{-1} = (\det P)^{-1}$. The matrices $A$ and $B$ have the same characteristic polynomial and the same spectrum. Let $B = [b_{ij}], P = [p_{ij}]$ and $P^{-1} = [\bar{p}_{ij}]$ then we have

$$\text{tr } B = \sum_{i=1}^{n} \sum_{k=1}^{n} \bar{p}_{ik} a_{kj} p_{ji} = \sum_{k=1}^{n} a_{kk} = \text{tr } A \quad (1.4)$$

since $P$ and $P^{-1}$ are the monomial matrices and $\bar{p}_{ik} = \frac{1}{p_{ki}}$.

1.1.1. **Maximal Eigenvalue of a Nonnegative Matrix**

**Definition 1.5.** A matrix $A \in \mathbb{R}_{+}^{n \times n}$, $n \geq 2$ is called reducible if there exists a permutation matrix $P$ such that

$$P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \quad \text{or} \quad P^T AP = \begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix} \quad (1.5)$$

where $B$ and $D$ are nonzero square matrices. Otherwise the matrix is called irreducible.

**Theorem 1.3.** The matrix $A \in \mathbb{R}_{+}^{n \times n}$ is irreducible if and only if

1) The matrix $(I + A)^{n-1}$ is strictly positive

$$(I + A)^{n-1} \gg 0 \quad (1.6)$$

2) or equivalently if

$$I + A + \cdots + A^{n-1} \gg 0 \quad (1.7)$$
Proof. For every vector \( x > 0 \)
\[
(I + A)^{n-1} x \gg 0
\]  \hspace{1cm} (1.8)
holds if the matrix \( A \in \mathbb{R}_+^{n \times n} \) is irreducible. Let \( x = e_i \), where \( e_i \) is the \( i \) th column, \( i = 1, 2, \ldots, n \) of the \( n \times n \) identity matrix \( I \). From equation (1.8) we have \( (I + A)^{n-1} e_i \gg 0 \) for \( i = 1, 2, \ldots, n \), i.e. the columns of the matrix \( (I + A)^{n-1} \) are strictly positive.

If matrix \( A \) is reducible then equation (1.5) holds. Then the matrix \( (I + A)^{n-1} \) is also reducible since

\[
(I + A)^{n-1} = \begin{bmatrix} (B + I)^{n-1} & \tilde{C} \\ 0 & (D + I)^{n-1} \end{bmatrix}
\]
and the condition in equation (1.6) is not satisfied. The equivalence of the conditions in equations (1.6) and (1.7) follows from the relation

\[
(I + A)^{n-1} = I + C_1^{n-1} A + C_2^{n-1} A^2 + \cdots + C_{n-2}^{n-1} A^{n-2} + A^{n-1}
\]  \hspace{1cm} (1.9)
since \( C_k^{n-1} = \frac{(n-1)!}{k!(n-k-1)!} \) are positive coefficients.

Let \( \lambda \) be an eigenvalue of \( A \in \mathbb{R}_+^{n \times n} \) and \( x \) be its corresponding eigenvector

\[
Ax = \lambda x
\]  \hspace{1cm} (1.10)

**Lemma 1.1.** A positive eigenvector \( x \) of irreducible matrix \( A > 0 \) is strictly positive \( x \gg 0 \).

**Proof.** From equation (1.10) it follows that if \( A > 0 \) and \( x > 0 \) then \( \lambda \geq 0 \) and

\[
(I + A)x = (1 + \lambda)x
\]  \hspace{1cm} (1.11)
Let us presume that the vector \( x > 0 \) has \( k \), \( 1 \leq k \leq n \) zero components. Then the vector \( (1 + \lambda)x \) has also \( k \) zero components. It is clear that the vector \( (I + A)x \) has the less than \( k \) zero components. Therefore, we obtain the contradiction and \( x \gg 0 \).

The following theorem is the most important part of Perron–Frobenius theory [12].

**Theorem 1.4.** An irreducible nonnegative matrix \( A \) has a real positive eigenvalue \( r \) such that
\[ r \geq |\lambda_i|, \quad i = 1, 2, \ldots, n-1 \] (1.12)

for any eigenvalue \( \lambda_i \) of \( A \). Furthermore, there is a positive eigenvector corresponding to \( r \).

The eigenvalue \( r \) is called the maximal eigenvalue of the matrix \( A \) and the vector \( x_0 \) is called its maximal eigenvector. The strictly positive matrix is irreducible. From Theorem 1.4 the following corollary follows.

**Corollary 1.1.** The strictly positive matrix \( A \in \mathbb{R}^{n \times n}_+ \) has exactly one real eigenvalue \( r \) such that

\[ r \geq |\lambda_i|, \quad i = 1, 2, \ldots, n-1 \]

to which corresponds a strictly positive eigenvector \( x \), where \( \lambda_1, \lambda_2, \ldots, \lambda_{n-1} \) are eigenvalues of \( A \).

A submatrix obtained from \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) by deleting the rows \( R_i = [a_{i1}, a_{i2}, \ldots, a_{in}] \) and the columns \( C_j = [a_{1j}, a_{2j}, \ldots, a_{nj}]^T \) for \( i = 1, \ldots, i_p \) \((p < n)\) is called the principal submatrix of \( A \).

**Theorem 1.5.** The maximal eigenvalue of an irreducible positive matrix is larger than the maximal eigenvalue of its principal submatrices.

**Theorem 1.6.** A positive matrix \( A \) with maximal eigenvalue \( r \) is reducible if and only if \( r \) is an eigenvalue of a principal submatrix of \( A \).

**Proof.** Let \( A \) be reducible. Then by definition 1.5 there exists a permutation matrix \( P \) such that

\[ P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \] (1.13)

where \( B \) and \( D \) are square submatrices. From equation (1.13) it follows that the spectrum of \( A \) consists of the eigenvalues of \( B \) together with those of \( D \). Hence \( r \) must be either an eigenvalue of the maximal submatrix \( B \) or of the principal submatrix. From Theorem 1.5 it follows that the maximal eigenvalue \( r \) of \( A \) is an eigenvalue of its submatrix only if the matrix \( A \) is reducible. \( \square \)

### 1.1.2. Bounds on Maximal Eigenvalue of a Positive Matrix

Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n}_+ \) be positive reducible or irreducible matrix. Denote by

\[ r_i = \sum_{j=1}^{n} a_{ij}, \quad c_j = \sum_{i=1}^{n} a_{ij} \] (1.14)
the $i$ th row sum and the $j$ th column sum of $A$ respectively.

**Theorem 1.7.** If $r$ is a maximal eigenvalue of $A$ then

$$\min_i r_i \leq r \leq \max_i r_i \quad \text{and} \quad \min_j c_j \leq r \leq \max_j c_j \quad (1.15)$$

If $A$ is irreducible then equality can hold on either side of equations (1.15) if and only if $r_1 = r_2 = \cdots = r_n$ and $c_1 = c_2 = \cdots = c_n$ respectively.

**Proof.** Let $s$ be an eigenvalue of $A$ (and $A^T$), and $x = [x_1, x_2, \ldots, x_n]^T$, $y = [y_1, y_2, \ldots, y_n]^T$ be eigenvectors of $A^T$ and $A$, corresponding to $s$, respectively. From $A^T x = sx$ we have

$$\sum_{i=1}^{n} a_{ij} x_i = sx_j \quad \text{for} \quad j = 1, 2, \ldots, n$$

and

$$s \sum_{j=1}^{n} x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} x_i r_i \quad (1.16)$$

In a similar way, using $Ay = \lambda y$ it can be shown that $s \sum_{j=1}^{n} y_j = \sum_{i=1}^{n} y_i c_i$. Let $s = r$ and $x > 0$ such that $\sum_{i=1}^{n} x_i = 1$. In this case, from equation (1.16) we obtain

$$r = \sum_{i=1}^{n} x_i r_i \quad (1.17)$$

and the desired inequality of equation (1.15). If the matrix $A$ is irreducible then $x \gg 0$. In this case, from equation (1.17) it follows that the equality of equation (1.15) holds if and only if $r_1 = r_2 = \cdots = r_n$. In a similar way, equation (1.15) for $c_j$ can be shown.

**Theorem 1.8.** If for positive matrix $A = [a_{ij}]$

$$r_i = \sum_{j=1}^{n} a_{ij} > 0 \quad \text{for} \quad i = 1, 2, \ldots, n \quad (1.18)$$

then its maximal eigenvalue $r$ satisfies the inequality
\[
\min_i \left( \frac{1}{r_j} \sum_{j=1}^{n} a_{ij} r_j \right) \leq r \leq \max_i \left( \frac{1}{r_j} \sum_{j=1}^{n} a_{ij} r_j \right)
\]

(1.19)

**Proof.** Let \( D = \text{diag}[r_1, r_2, \ldots, r_n] \). The existence of the inverse matrix \( D^{-1} \) follows from equation (1.18).

It is easy to check that the \( i \) th row sum of \( D^{-1}AD \) is equal to \( \frac{1}{r_i} \sum_{j=1}^{n} a_{ij} r_j \). Using equation (1.15) for \( D^{-1}AD \) we obtain the inequality in equation (1.19). \( \square \)

**Theorem 1.9.** Let \( x = [x_1, x_2, \ldots, x_n]^T \) be the maximal eigenvector of a strictly positive matrix \( A = [a_{ij}] \) and \( \alpha = \max_{i,j} \frac{x_i}{x_j} \). Then

\[
\sqrt{\frac{\max_i r_i}{\min_i r_i}} \leq \alpha \leq \max_{i,j,k} \frac{a_{ij}}{a_{kj}}
\]

(1.20)

The equality holds on the left side of equation (1.20) if and only if \( r_1 = r_2 = \cdots = r_n \) and on the right hand side if and only if the \( p \) th row of \( A \) is proportional to its \( q \) th row and

\[
\frac{a_{ph}}{a_{qh}} = \max_{i,j,k} \frac{a_{ij}}{a_{kj}} \quad \text{for } h = 1, 2, \ldots, n
\]

(1.21)

**Example 1.1.** Find the bounds for \( \alpha \) of the matrix \( A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \)

**Solution.** Using equation (1.20) we obtain

\[
r_1 = 6, \; r_2 = r_3 = 5, \quad \sqrt{\frac{\max_i r_i}{\min_i r_i}} = \sqrt{\frac{6}{5}}, \quad \max_{i,j,k} \frac{a_{ij}}{a_{kj}} = 3 \quad \Rightarrow \quad \sqrt{\frac{6}{5}} \leq \alpha \leq 3
\]

1.2. Metzler Matrices

**Definition 1.6.** A matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) is called the Metzler matrix if its off-diagonal entries are nonnegative, \( a_{ij} \geq 0 \) for \( i \neq j ; i, j = 1, 2, \ldots, n \).
Theorem 1.10. Let $A \in \mathbb{R}^{n \times n}$. Then
\[ e^{At} > 0 \quad \text{for } t \geq 0 \quad (1.22) \]
if and only if $A$ is a Metzler matrix.

**Proof.** Necessity: From the expansion
\[ e^{At} = I + At + \frac{A^2 t^2}{2!} + \cdots \]
it follows that equation (1.22) holds for small $t > 0$ only if $A$ is the Metzler matrix.

Sufficiency: Let $A$ be the Metzler matrix. The scalar $\lambda > 0$ is chosen so that $A + \lambda I > 0$. Taking into account that
\[ (A + \lambda I)(-\lambda I) = (\lambda I)(A + \lambda I) \]
we obtain
\[ e^{At} = e^{(A + \lambda I)t - \lambda t} = e^{(A + \lambda I)t} e^{-\lambda t} > 0 \]
since $e^{(A + \lambda I)t} > 0$ and $e^{-\lambda t} > 0$.

**Remark 1.1.** Every Metzler matrix $A \in \mathbb{R}^{n \times n}$ has a real eigenvalue $\alpha = \max_i \text{Re } s_i$ and $\text{Re } s_i < 0$ for $i = 1, \ldots, n$ if $\alpha < 0$, where $s_i = s_i(A)$, $i = 1, \ldots, n$ are the eigenvalues of $A$.

For every Metzler matrix $A$ there exists a real number $\lambda$ such that $B = I \lambda + A \in \mathbb{R}_{+}^{n \times n}$. By Theorem 1.4 the matrix $B$ has a real eigenvalue equal to its spectral radius $\rho(B) = \max_i |s_i(B)|$. Hence the matrix $A$ has the real eigenvalue $\rho(B) - \lambda = \alpha$ and $\text{Re } s_i < 0$ for $i = 1, \ldots, n$ if $\alpha < 0$.

1.3. M-Matrices

**Definition 1.7.** A matrix $A \in \mathbb{R}^{n \times n}$ is called an M-matrix if (1) its entries of the main diagonal are nonnegative and its off-diagonal entries are nonpositive and (2) there exist a positive matrix $B \in \mathbb{R}_{+}^{n \times n}$ with maximal eigenvalue $r$ such that
\[ A = cI - B \quad (1.23) \]
where \( c \geq r \).

The set of M-matrices of dimension \( n \times n \) will be denoted by \( M_n \) and \( F_n \) denotes the set of \( n \times n \) matrices with nonpositive off-diagonal entries. Note that from equation (1.23) it follows that if \( A \) is an M-matrix then \(-A\) is the Metzler matrix. From Theorem 1.10 it follows that for every matrix \( A \) being the M-matrix it holds that

\[
e^{-At} > 0 \quad \text{for } t \geq 0
\]  
(1.24)

**Theorem 1.11.** A matrix \( A \in F_n \) is an M-matrix if and only if all its eigenvalues have nonnegative real parts.

**Proof.** Let a matrix \( A = [a_{ij}] \in F_n \) have all eigenvalues with negative real parts and \( a_{mm} = \max_i a_{ii} \). Then \( B \triangleq a_{mm} I - A \in \mathbb{R}_+^{n \times n} \). Let \( r \) be the maximal eigenvalue of the matrix \( B \). Then \( a_{mm} - r \) is a real nonnegative eigenvalue of the matrix \( A = a_{mm} I - B \) or \( a_{mm} \geq r \). Therefore, \( A = a_{mm} I - B \) is the M-matrix. Now let us assume that \( A = cI - B \) is an M-matrix and \( r \) is the maximal eigenvalue of the matrix \( B \). Hence \( c \geq r \).

Let \( \lambda_k \) be an eigenvalue of the matrix \( A \) and \( \text{Re}(\lambda_k) \) be its negative real part. Then

\[
0 = \det [I \lambda_k - A] = \det [I \lambda_k - cI + B] = \det [I (c - \lambda_k) - B]
\]  
(1.25)

From equation (1.25) it follows that \( c - \lambda_k \) is an eigenvalue of the matrix \( B \). But \( c \geq 0 \) and \( -\text{Re}(\lambda_k) > 0 \). Therefore, \( |c - \lambda_k| \geq c - \text{Re}(\lambda_k) > c \geq r \), which contradicts the assumption that \( r \) is the maximal eigenvalue of \( B \).

**Theorem 1.12.** A matrix \( A \in F_n \) is an M-matrix if and only if all its real eigenvalues are nonnegative.

**Proof.** The necessity follows immediately from Theorem 1.11. Let us assume that all real eigenvalues of the matrix \( A \) are nonnegative. Let \( B \triangleq Ia_{mm} - A \) where \( a_{mm} = \max_i a_{ii} \). Hence \( B \in \mathbb{R}_+^{n \times n} \). Let \( r \) be the maximal eigenvalue of \( B \). Then \( a_{mm} - r \) is a real nonnegative eigenvalue of the matrix \( A = Ia_{mm} - B \) or \( a_{mm} \geq r \). Therefore, \( A \) is the M-matrix.
A submatrix obtained from a square matrix by deleting the rows and columns denoted by the same numbers is called the principal submatrix. For example, the submatrix

\[
\begin{bmatrix}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{bmatrix}
\]

is the principal submatrix of the matrix

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\]

obtained by deleting the first row and first column and the fourth row and the fourth column.

**Theorem 1.13.** A principal submatrix of an M-matrix is an M-matrix.

**Proof.** Let \( A \) be an M-matrix and \( A = cI - B \), where \( B \in \mathbb{R}^{n \times n}_+ \) has the maximal eigenvalue \( r \) satisfying the condition \( c \geq r \). Let \( B_1 \in \mathbb{R}^{m_1 \times m_1}_+ \) be a principle submatrix of \( B \) and \( r_1 \) the maximal eigenvalue of \( B_1 \). The maximal eigenvalue \( r_1 \) of \( B_1 \) may not be larger than the maximal eigenvalue \( r \) of \( B \) or \( r_1 \leq r \). Hence \( c \geq r_1 \) and the principal submatrix \( A_1 = cI_{m_1} - B_1 \) is an M-matrix. \( \blacksquare \)

**Theorem 1.14.** A matrix \( A \in F_n \) is an M-matrix if and only if its principal minors are nonnegative.

**Proof.** First we shall show that the determinant of an M-matrix is nonnegative. By Theorem 1.12 the real roots of an M-matrix are nonnegative. Since the matrix is real, all its complex eigenvalues occur in conjugate pairs. Therefore, the determinant of an M-matrix which is product of its eigenvalues is nonnegative. Now suppose that all principal minors of \( A \in F_n \) are nonnegative. Let \( a_{nn} = \max a_{ii} \) and \( B = a_{nn} I - A \). Then, where \( B \in \mathbb{R}^{n \times n}_+ \) has the maximal eigenvalue \( r \). If all principal minors of \( A \) are zero then all its eigenvalues are zero and \( A = -B \) since \( a_{nn} = 0 \). In this case \( r = 0 \) and \( a_{nn} \geq r = 0 \) and \( A \) is an M-matrix. We assume now that all principal minors of \( A \) are nonnegative but not all are zero. Let \( m_i(A) \) be the sum of all its principal minors of order \( i \) of \( A \). Then \( m_i(A) \geq 0 \) for all \( i \) and the inequality is strict for at least one \( i \). If \( p \) is a positive number then

\[
\det [(p + a_{nn})I - B] = \det [pI + (a_{nn} I - B)] = \det [pI + A] = \sum_{k=0}^{n} p^{n-k} m_{n-k}(A) > 0
\]
The number \( p + a_{\text{nm}} \) is not an eigenvalue of \( B \) for every \( p > 0 \). Hence \( a_{\text{nm}} \geq r \) and \( A = a_{\text{nm}} I - B \) is an M-matrix.

**Theorem 1.15.** A nonsingular matrix \( A \in F^n \) is an M-matrix if and only if \( A^{-1} \in \mathbb{R}^{n \times n}_+ \).

**1.4. Totally nonnegative (positive) matrices**

In this section we shall consider nonnegative matrices with all their minors of all orders being nonnegative.

**Definition 1.8.** A matrix \( A \in \mathbb{R}^{m \times n}_+ \) is called totally nonnegative (positive) if and only if all its sub-determinants of all orders are nonnegative (positive).

The Vandermonde matrix

\[
V = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
a_1 & a_2 & \cdots & a_{n-1} & a_n \\
a_1^2 & a_2^2 & \cdots & a_{n-1}^2 & a_n^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
 a_1^{n-1} & a_2^{n-1} & \cdots & a_{n-1}^{n-1} & a_n^{n-1}
\end{bmatrix}
\]

is an example of a square totally positive matrix if \( 0 < a_1 < a_2 < \cdots < a_n \), since the matrix has positive determinant and all of its submatrices has positive determinants, too.

**Lemma 1.2.** A square matrix \( A \in \mathbb{R}^{n \times n}_+ \) is called totally positive (strictly Metzler) matrix if all of its diagonal entries are negative and all of its off-diagonal elements are nonnegative, i.e. \( a_{ii} < 0 \), \( a_{ij} < 0 \), \( \forall i \neq j \), \( i, j = 1, 2, \ldots, n \).

Note that the Metzler matrix is conventionally defined as a matrix with nonnegative off diagonal elements. Here, we define it in a strict sense to satisfy the necessary condition of stability, namely \( a_{ii} < 0 \). Metzler matrices are closely related to the class of M-matrices. An M-matrix has positive diagonal entries and negative off-diagonal entries. Thus, if \( A \) is a Metzler matrix, then \( -A \) is an M-matrix. Furthermore an M-matrix is called a nonsingular M-matrix if \( M^{-1} \geq 0 \). The nonsingular M-matrix has several nice properties.

One can deduce that stable Metzler matrices admit similar properties, i.e., if \( A \) is a stable Metzler matrix then \( -A \) is a nonsingular M-matrix. The underlying theory of such matrices stems from the theory of nonnegative (positive) matrices based on Frobenius Perron Theorem.
The spectral radius of an irreducible non-negative matrix $N$, denoted by $\rho(N)$, is positive and real. An irreducible Metzler matrix can be written as $A = N - \alpha I$ for some nonsingular matrix $N$ and a scalar $\alpha$. Thus $A$ is Hurwitz stable if and only if $\alpha > \rho(N)$, and its largest eigenvalue $\mu(A) = \rho(N) - \alpha$. Note that every Metzler matrix $A$ has a real eigenvalue $\lambda = \max \Re(\lambda_i)$ and if $\mu < 0$, then $\Re(\lambda_i) < 0$ for $i = 1, 2, \ldots, n$, where $\lambda_i$'s are the eigenvalues of $A$.

Due to the connection of these matrices with the corresponding models of continuous-time and discrete-time systems, one can similarly define Metzlerian and non-negative (positive) systems. Here we only concentrate on continuous-time Metzlerian systems which are going to be introduced and discussed in next chapter.
Chapter 2 – Continuous-time Positive Systems

2.1. Externally Positive Systems

Consider the linear continuous-time system described by the equations

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\] (2.1)

where \( x(t) \in \mathbb{R}^n \) is the state vector at the instant \( t \), \( u(t) \in \mathbb{R}^m \) is the input vector, \( y(t) \in \mathbb{R}^p \) is the output vector, \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \), \( D \in \mathbb{R}^{p \times m} \). Let \( \mathbb{R}_{n \times m}^+ \) be the set of \( n \times m \) matrices with nonnegative entries and \( \mathbb{R}_+^n = \mathbb{R}_+^{n \times 1} \).

Definition 2.1. The system (2.1) is called externally positive if and only if for every input \( u \in \mathbb{R}_+^m \) and \( x_0 = 0 \) the output \( y \in \mathbb{R}_+^p \) for all \( t \geq 0 \).

The impulse response \( g(t) \) of single-input single-output system is called the output of the system for the input equal to the Dirac impulse \( \delta(t) \) with zero initial conditions. In a similar way assuming successively that only one input is equal to \( \delta(t) \) and the remaining inputs are zero, we may define the matrix of impulse responses \( g(t) \in \mathbb{R}_{p \times m}^+ \) of a system with \( m \)-inputs and \( p \)-outputs.

Theorem 2.1. The system (2.1) is externally positive if and only if its matrix of impulse responses is nonnegative, i.e.

\[
g(t) = \mathbb{R}_{p \times m}^+ \quad \text{for all} \quad t \geq 0
\] (2.2)

Proof. The necessity of the condition in (2.2) follows immediately from definition 2.1. The output of the system in (2.1) with zero initial conditions for any input \( u(t) \) is given by the formula

\[
y(t) = \int_0^t g(t - \tau)u(\tau) \, d\tau
\] (2.3)

If the condition in (2.2) is satisfied and \( u(t) \in \mathbb{R}_+^m \), then from (2.3) we have \( y \in \mathbb{R}_+^p \) for \( t \geq 0 \). □

Theorem 2.2. The continuous-time system with the transfer function
is externally positive if $a_i \leq 0$ and $b_i \geq 0$ for $i = 1, 2, \ldots, n$.

**Proof.** We shall show that if the conditions are satisfied then $g(t) \in \mathbb{R}_+$ for $t \geq 0$. The transfer function can be expanded in the series

$$G(s) = g_1s^{-1} + g_2s^{-2} + \cdots$$  \hspace{1cm} (2.4)

From comparison of the right hand side of transfer function and (2.4) we have

$$b_{n-i}s^{n-i} + b_{n-i-2}s^{n-i-2} + \cdots + b_1s + b_0 = (s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0)(g_1s^{-1} + g_2s^{-2} + \cdots)$$  \hspace{1cm} (2.5)

Comparing the coefficients at the same powers of $s$ of the equality (2.5) we obtain

$$g_1 = b_{n-i}, \quad g_2 = b_{n-i-2} - a_{n-i}g_1, \quad \ldots, \quad g_k = b_{n-k} - a_{n-i}g_{k-1} - a_{n-k-2}g_{k-2} - \cdots - a_{n-k+i}g_1$$  \hspace{1cm} (2.6)

From equation (2.6) it follows that if the Theorem conditions are satisfied then $g_k \in \mathbb{R}_+$ for $k = 1, 2, \ldots$.

It is well-known that the impulse response $g(t)$ is the original of the transfer function $g(t) = L^{-1}[G(s)]$, where $L^{-1}$ is the inverse Laplace operator. From (2.4) we have $g(t) = g_1 + g_2t + g_3\frac{t^2}{2!} + \cdots$. Hence, if the conditions are satisfied then $g(t) \in \mathbb{R}_+$ for $t \geq 0$ and the system described by the transfer function is externally positive.

### 2.2. Internally Positive Systems

Consider the continuous-time system described by (2.1).

**Definition 2.2.** The system (2.1) is called internally positive (shortened to positive or Metzlerian) if and only if for any $x_0 \in \mathbb{R}_n^+$ and every $u \in \mathbb{R}_m^+$ we have $x \in \mathbb{R}_n^+$ and $y \in \mathbb{R}_r^+$ for all $t \geq 0$.

From definition 2.2 it follows that the system (2.1) is internally positive only if its matrix of impulse responses is nonnegative i.e. the condition in (2.2) is satisfied. This condition in general case is not
sufficient for the internal positivity of the system in (2.1). From definition 1.6, the matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a Metzler matrix if $a_{ij} \geq 0$ for $i \neq j$; $i, j = 1, 2, \ldots, n$.

**Theorem 2.3.** The continuous-time system (2.1) is internally positive if and only if the matrix $A$ is a Metzler matrix and $B \in \mathbb{R}^{n \times m}_+$, $C \in \mathbb{R}^{p \times n}_+$ and $D \in \mathbb{R}^{p \times m}_+$.

**Proof.**

Sufficiency: The solution of state equation in (2.1) has form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}B u(\tau) d\tau \quad (2.7)$$

By Theorem 1.10 the matrix $e^{At} \in \mathbb{R}^{n \times n}_+$ if and only if $A$ is Metzler matrix. If $A$ is the Metzler matrix and $B \in \mathbb{R}^{n \times m}_+$, $x_0 \in \mathbb{R}^n_+$, $u(t) \in \mathbb{R}^m_+$ for $t \geq 0$, then from (2.7) we obtain $x(t) \in \mathbb{R}^n_+$ for $t \geq 0$ and from equation (2.1) $y(t) \in \mathbb{R}^p_+$ since $C \in \mathbb{R}^{p \times n}_+$ and $D \in \mathbb{R}^{p \times m}_+$.

Necessity: Let $u(t) = 0$ for $t \geq 0$ and $x_0 = e_i$ (the $i$ th column of $I_n$). The trajectory does not leave the quarter $\mathbb{R}^n_+$ only if $\dot{x}(0) = Ae_i \geq 0$, which implies $a_{ji} \geq 0$ for $i \neq j$. The matrix $A$ has to be the Metzler matrix. For the same reasons, for $x_0 = 0$ we have $\dot{x}(0) = Bu(0) \geq 0$ which implies $B \in \mathbb{R}^{n \times m}_+$ since $u(0) \in \mathbb{R}^m_+$ may be arbitrary.

From equation (2.1) for $u(0) = 0$ we have $y(0) = Cx_0 \geq 0$ and $C \in \mathbb{R}^{p \times n}_+$, since $x_0 \in \mathbb{R}^n_+$ may be arbitrary. In a similar way, assuming $x_0 = 0$ we get $y(0) = Du(0) \geq 0$ and $D \in \mathbb{R}^{p \times m}_+$, since $u(0) \in \mathbb{R}^m_+$ may be arbitrary.

The matrix of impulse responses of the system in (2.1) is given by

$$g(t) = Ce^{At}B + D\delta(t) \quad \text{for } t \geq 0 \quad (2.8)$$

This formula may be obtained by substitution of equation (2.7) into the output equation of (2.1) and taking into account that for $x_0 = 0$ and $u(t) = \delta(t)$, $y(t) = g(t)$.

If $A$ is the Metzler matrix and $B \in \mathbb{R}^{n \times m}_+$, $C \in \mathbb{R}^{p \times n}_+$, $D \in \mathbb{R}^{p \times m}_+$, then from (2.8) it follows that $g(t) \in \mathbb{R}^{p \times m}_+$ for all $t \geq 0$. We have two important corollaries.
Corollary 2.1. The matrix of impulse responses of the internally positive system in (2.1) satisfies the condition in (2.2).

Corollary 2.2. Every continuous-time internally positive system is also externally positive.

Note that the internally positive continuous-time system, also known as Metzlerian system, is denoted as positive system from now on in this thesis. Here we provide some example of positive systems.

Example 2.1. [4] Given the circuit shown in Figure 2.1 with known resistances $R_1, R_2, R_3,$ inductances $L_1, L_2$ and source voltages $e_1 = e_1(t), e_2 = e_2(t).$ The currents $i_1 = i_1(t), i_2 = i_2(t)$ in the inductances are chosen as state variables and $y = y(t) = \begin{bmatrix} R_1i_1 \\ R_2i_2 \end{bmatrix}$ is chosen as the output.

![Figure 2.1 – A circuit as an Example of positive system](image)

Using the Kirchhoff law we may write the equations in the following state space format

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

(2.9)

$$y = C \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

(2.10)

where

$$A = \begin{bmatrix} \frac{R_2 + R_1}{L_1} & \frac{R_2}{L_1} \\ \frac{R_3}{L_2} & \frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}, \quad C = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

(2.11)
From equation (2.11) it follows that $A$ is the Metzler matrix and $B$ and $C$ have nonnegative entries. The circuit is an example of continuous-time positive system.

**Example 2.2.** Another example of positive system is the two nation model of Richardson’s Theory of arms force [13]. In this model two competing nations (or perhaps two competing coalitions of nations) are denoted $X$ and $Y$. The variables $x(t)$ and $y(t)$ represents, respectively, the armament levels of the nations $X$ and $Y$ at time $t$. The general form of the model is

$$
\begin{align*}
\dot{x}(t) &= ky(t) - \alpha x(t) + g \\
\dot{y}(t) &= lx(t) - \beta y(t) + h
\end{align*}
$$

(2.12)

In this model, the terms $g$ and $h$ are called “grievances”. They encompass the wide assortment of psychological and strategic motivations for changing armament levels, which are independent of existing levels of either nation. Roughly speaking, they are motives of revenge or dissatisfaction, and they may be due to dissatisfaction with treaties or other past political negotiations. The terms $k$ and $l$ are called “defense” coefficients. They are nonnegative constants that reflect the intensity of reaction by one nation to the current armament level of rivalry that can cause the exponential growth of armaments commonly associated with arms race. Finally, $\alpha$ and $\beta$ are called “fatigue” coefficients. They are nonnegative constants that represent the fatigue and expense of the effect of causing a nation to retard the growth of its own armament level; the retardation effect increasing as the level increase.

The system matrix is

$$
A = \begin{bmatrix}
-\alpha & k \\
l & -\beta
\end{bmatrix}
$$

(2.13)

which is a Metzler matrix.

**Example 2.3.** In chemical plants, it is often necessary to maintain the levels of liquids. A simplified model of a connection of two tanks can be described as follow

$$
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix}
\frac{1}{A_2 R_2} & \frac{1}{A_1 R_1} \\
\frac{1}{A_2 R_1} & -\frac{1}{A_2 R_1 + A_2 R_2}
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix}
\frac{1}{A_1} \\ 0
\end{bmatrix} u
$$

(2.14)
\[ y = \begin{bmatrix} 0 & \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \] \hspace{1cm} (2.15)

In this model, \( u \) is the inflow perturbation of the first tank which will cause variations in liquid level \( x_1 \) that will indirectly cause variations in liquid level \( x_2 \) and outflow variation \( y \) in the second tank. \( R_i \)'s are the flow resistances that can be controlled by valves and \( A_i \)'s are the cross section of tanks for \( i = 1, 2 \). Since all \( R_i \)'s and \( A_i \)'s are positive it can be seen that the system matrix is a Metzler matrix and therefore this chemical plant is also a positive system.

**Example 2.4. [14] (Bone Scanning)** The procedure for taking a scintigram is the following: the patient receives an injection of a radionuclide, which, transported by the blood, collects in the bones. More of it tends to collect in so-called “hot spots”, areas where there is increased metabolic activity (which in simple terms means that the bone is breaking down, or repairing itself). The gamma rays generated by the radionuclide are captured by a specific camera that provides the image.

Deciding when is the adequate time-after-injection for the scan is tricky: From a purely imaging point of view, the optimal time is the instant when the maximum contrast in the image between the ‘hot spots’ and the background is obtained. Unfortunately, the evolution of the contrast of the scintigram varies intra-patient. Therefore estimating this contrast is crucial for clinicians. Based on clinical measurements, the portion of the administered dose of this radionuclide in some compartments of the human body was determined to be quite precisely given by the following dynamical model

\[
\begin{bmatrix}
\frac{dx_1(t)}{dt} \\
\frac{dx_2(t)}{dt} \\
\frac{dx_3(t)}{dt} \\
\frac{dx_4(t)}{dt} \\
\frac{dx_5(t)}{dt}
\end{bmatrix} =
\begin{bmatrix}
-k_{21} - k_{41} - k_{51} & k_{12} & 0 & k_{14} & k_{15} \\
 k_{21} & -k_{21} - k_{32} & k_{23} & 0 & 0 \\
 0 & k_{32} & -k_{23} & 0 & 0 \\
 k_{41} & 0 & 0 & -k_{14} & 0 \\
 k_{51} & 0 & 0 & 0 & -k_{15} - k_{35}
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
x_4(t) \\
x_5(t)
\end{bmatrix}
\] \hspace{1cm} (2.16)

\[
y(t) = \begin{bmatrix} c & 0 & 0 & 0 & 0 \end{bmatrix} x(t) \] \hspace{1cm} (2.17)

where the states \( x_i(t) \)'s correspond to the portion of the dose of Tc-MDP (Tc-99m(Sn)Methylene Diphosphonate, i.e. a chemical affecting the contrast) in the different compartments: \( x_1(t) \) is the portion of the dose in the blood, \( x_2(t) \) in the extracellular fluid of the bone, \( x_3(t) \) in cellular bone, \( x_4(t) \) in the
tubular urine and $x_i(t)$ in the rest of the body. Some values for the parameters of the model were obtained from physiological data. Based on clinical measurements, the following parameters for the compartmental model were obtained, with some uncertainty that represents inter-patient variations:

$$
\begin{align*}
  k_{12} &= 0.540 \pm 0.038, \quad k_{21} = 0.095 \pm 0.003, \quad k_{14} = 0.277 \pm 0.007, \quad k_{41} = 0.431 \pm 0.011 \\
  k_{15} &= 0.233, \quad k_{51} = 0.024, \quad k_{35} = 0.749, \quad k_{23} = 0.049 \pm 0.001, \quad k_{32} = 1.055 \pm 0.0037
\end{align*}
$$

(2.18)

(2.19)

It can be seen that this system was also a positive system with Metzlerian matrix system and positive input output matrices.
Chapter 3 – Stability of Positive Systems

3.1. Asymptotic Stability

Consider a continuous-time internally positive system described by the equation

\[ \dot{x}(t) = Ax(t), \quad x(0) = x_0 \]  \hspace{1cm} (3.1)

where \( A \in \mathbb{R}^{n \times n} \) is the Metzler matrix. The solution of equation (3.1) has the form

\[ x(t) = e^{At}x_0 \]  \hspace{1cm} (3.2)

**Definition 3.1.** The internally positive system in (3.1) is called asymptotically stable if and only if the solution in (3.2) satisfies the condition

\[ \lim_{t \to \infty} x(t) = 0 \quad \text{for every} \quad x_0 \in \mathbb{R}^n_+ \]  \hspace{1cm} (3.3)

The roots \( s_1, s_2, \ldots, s_n \) of the equation \( \det[Is - A] = 0 \) are called the eigenvalues of the matrix \( A \) and their set is called the spectrum of \( A \).

**Theorem 3.1.** The internally positive system in (3.1) is called asymptotically stable if and only if all eigenvalues \( s_1, s_2, \ldots, s_n \) of the Metzler matrix \( A \) have negative real parts.

**Proof.** Proof can be found in [4].

**Lemma 3.1.** Let \( p = \max_i |s_i| \) be the spectral radius of a matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n}_+ \). Then a real number \( \lambda > p \) if and only if all principal minors \( M_j[I\lambda - A] \) of the matrix \( [I\lambda - A] \) are positive, i.e.

\[ M_1[I\lambda - A] = \lambda - a_{11} > 0, \quad M_2[I\lambda - A] = \begin{vmatrix} \lambda - a_{11} & a_{12} \\ a_{21} & \lambda - a_{22} \end{vmatrix} > 0, \quad \ldots, \quad M_n[I\lambda - A] = \det[I\lambda - A] > 0 \]
Theorem 3.2. The internally positive system in (3.1) is asymptotically stable if and only if all coefficients $a_i$ ($i = 0,1,\ldots,n-1$) of the characteristic polynomial

$$w_A(s) = \det[Is - A] = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0$$

are positive ($a_i > 0$).

Proof. Necessity: The eigenvalues $s_1, s_2,\ldots, s_n$ of $A$ are real or complex conjugate since the coefficients $a_i$ of $w_A(s)$ are real. Hence if $\Re s_i < 0$, $i = 0,1,\ldots,n-1$ then all coefficients of the polynomial $w_A(s) = (s-s_1)(s-s_2)\cdots(s-s_n)$ are positive, $a_i > 0$ for $i = 0,1,\ldots,n-1$.

Sufficiency: This will be proved by contradiction. If $A$ is Metzler matrix then by Remark 1.1 $\alpha = \max_i \Re s_i$ is its eigenvalue and $\Re s_i < 0$ if $\alpha < 0$. For $a_i > 0$ for $i = 0,1,\ldots,n-1$ and real $s$ we have $w_A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 > 0$ and $A$ has no real nonnegative eigenvalue. Thus, we get the contradiction and $\alpha < 0$.

To test the asymptotic stability of the system in (3.1) we do not need to know the characteristic polynomial (3.4) and we may use the following theorem.

Theorem 3.3. The internally positive system in (3.1) is asymptotically stable if and only if all principal minors $n$ of the matrix $-A$ are positive, i.e.

$$\begin{vmatrix} -a_{11} \end{vmatrix} > 0, \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \begin{vmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{vmatrix} > 0, \ldots, \det[-A] > 0$$

(3.5)

Proof. Note that the characteristic polynomial (3.4) maybe written as

$$w_A(s) = \det[Is - A] = \det[se_1 - a_1, se_2 - a_2,\ldots, se_n - a_n]$$

where $a_i$ and $e_i$ are the $i$th columns of $A$ and the $n\times n$ identity matrix $I$ respectively.
The decomposition of the determinant on the sum of $2^n$ yields determinants whose columns are $a_1, a_2, \ldots, a_n$ or $se_1, se_2, \ldots, se_n$. Among them we have $\frac{n!}{(n-i)!i!}$ determinants, which contains $i$ columns of the form $se_i$, $i \in \{1, 2, \ldots, n\}$. Every such determinant is equal to the principal minor of the $(n-i)$-th order of the matrix $A$. The sum of those determinants is equal to the term $a_is^i$, $i = 0, 1, \ldots, n-1$ of $w_A(s)$. From Koteljanski theorem in [15] it follows that if the conditions in (3.5) are satisfied then all principal minors are positive, since the matrix $-A$ has all nonpositive off-diagonal entries for the Metzler matrix $A$. Therefore all coefficients of $w_A(s)$ are positive if and only if the conditions in (3.5) are satisfied.

**Remark 3.1.** Theorem 3.3 follows from Lemma 3.1 for $\lambda = 0$.

In many cases the stability of the internally positive system in (3.1) can be tested by the use of the following instability sufficient condition.

**Theorem 3.4.** The internally positive system (3.1) is unstable if at least one diagonal entry of the matrix $A$ is positive, i.e. $a_{ii} > 0$ for some $i \in \{1, 2, \ldots, n\}$.

**Proof.** If $a_{ii} > 0$ for some $i \in \{1, 2, \ldots, n\}$ then $\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j > 0$ and for every $x_0 \in \mathbb{R}_+^n$ the $i$ th component of $x = x(t)$ tends to infinity for $t \to \infty$. In this case the internally positive system in (3.1) is unstable.

**3.2. Bounded-Input Bounded-Output (BIBO) Stability**

Consider an internally positive continuous-time single-input single-output system described by the equations

$$
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0
$$

$$
y(t) = Cx(t) + du(t)
$$

(3.6)

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ are the state, input and output vectors respectively and $A \in \mathbb{R}^{n \times n}$ is the Metzler matrix and $B \in \mathbb{R}_+^n$, $C \in \mathbb{R}_{+}^{1 \times n}$, $d \in \mathbb{R}_+$. 
A signal (input, output) $s(t)$ is called bounded if and only if its value (or the norm $\|s\|$) is bounded for all $t \in [0, +\infty)$.

**Definition 3.2.** The internally positive system in (3.6) is called BIBO stable if and only if its output is bounded for any bounded input and all $t \in [0, +\infty)$.

Let $g(t)$ be the impulse response of the system (3.6) that is the output of the system with zero initial conditions ($x_0 = 0$) for Dirac impulse $\delta(t)$ input.

The output $y(t)$ of the system (3.6) with zero initial conditions for any input $u(t)$ is given by

$$y(t) = \int_0^t g(t-\tau)u(\tau)\,d\tau = \int_0^t g(\tau)u(t-\tau)\,d\tau$$

(3.7)

**Theorem 3.5.** The internally positive system (3.6) is BIBO stable if and only if

$$\int_0^t g(\tau)d\tau < \infty \quad \text{for all } t \in [0, +\infty)$$

(3.8)

**Proof.** Taking into account that the impulse response $g(t)$ of internally positive system is nonnegative from (3.7), for a bounded $u(t) \in \mathbb{R}$, we obtain

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau \leq \int_0^t g(\tau)d\tau \bar{u}$$

(3.9)

where $\bar{u} \geq u(t)$ for $t \in [0, +\infty)$.

From (3.9) it follows that the output $y(t)$ is bounded for any bounded input $u(t)$ and all $t \in [0, +\infty)$ if and only if the condition in (3.8) is satisfied. $lacksquare$

Let $h(t)$ be the unit response of the (3.6) system that is the output of the system with zero initial conditions for the unit step

$$u(t) = \begin{cases} 
1 & \text{for } t > 0 \\
0 & \text{for } t < 0
\end{cases}$$

The unit response $h(t)$ and the impulse response $g(t)$ of the (3.9) system are related by the following formula.
\[ g(t) = \frac{dh(t)}{dt}, \quad h(0) = 0 \quad (3.10) \]

or

\[ h(t) = \int_0^t g(\tau)d\tau \quad (3.11) \]

Using equation (3.11) we may reformulate the Theorem 3.5 as follows.

**Theorem 3.6.** The internally positive system in (3.6) is BIBO stable if and only if its unit response is bounded for all \( t \in [0, +\infty) \).

Let \( G(s) \) be the transfer function of the system in (3.6)

\[ G(s) = C(1s - A)^{-1}B + d = \frac{L(s)}{M(s)} \quad (3.12) \]

where

\[ L(s) = b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 \]

\[ M(s) = s^{n'} + a_{n'-1} s^{n'-2} + \cdots + a_1 s + a_0 \]

where \( n' \leq n \) (the equality holds if the system does not have decoupling zeros).

It is assumed that the zeros \( z_1, z_2, \ldots, z_{n'} \) (the roots of \( L(s) = 0 \)) are different from the poles \( s_1, s_2, \ldots, s_{n'} \) (the roots of \( M(s) = 0 \)) of the transfer function in (3.12).

The impulse response \( g(t) \) is the original of \( G(s) \), i.e.

\[ g(t) = L^{-1}[G(s)] = C e^{At} B + d\delta(t) \quad (3.13) \]

where \( L^{-1} \) is the inverse Laplace transform operator. Without loss of generality we may assume that the poles \( s_1, s_2, \ldots, s_{n'} \) of transfer function in equation (3.12) are distinct (\( s_i \neq s_j \) for \( i \neq j \)). Then using (3.13) we obtain

\[ g(t) = \sum_{k=1}^{n'} A_k e^{s_k t} \quad (3.14) \]
where \( A_k = \frac{L(s_k)}{M'(s_k)}, \quad M'(s_k) = (s_k - s_1) \cdots (s_k - s_{k-1})(s_k - s_{k+1}) \cdots (s_k - s_{n'}) \).

From equation (3.14) it follows that the condition in (3.8) is satisfied if and only if \( \Re s_k < 0 \) for \( k = 1, 2, \ldots, n' \). By Theorem 3.1 the condition in equation (3.8) is satisfied if and only if the denominator \( M(s) \) of the transfer function in (3.12) has all positive coefficients. Therefore the following theorem has been proved.

**Theorem 3.7.** The internally positive system described in (3.6) is BIBO stable if and only if the denominator \( M(s) \) of the transfer function (3.12) has all positive coefficients, i.e. \( a_i > 0 \) for \( i = 0, 1, \ldots, n - 1 \).

The question then arises as to what the relationship is between the BIBO stability and the asymptotic stability of the internally positive system in (3.6). The answer is given by the following theorem.

**Theorem 3.8.** If the internally positive system in (3.6) is asymptotically stable then it is also BIBO stable.

**Proof.** It is well-known the set of poles \( s_1, s_2, \ldots, s_{n'} \) is a subset of the eigenvalues of the matrix \( A \) (the roots of the equation \( \det [Is - A] = 0 \)). The set of zeros of the minimal polynomial \( \Psi(s) \) of \( A \) contains all poles of the transfer function (3.12). Therefore, the system (3.6) is asymptotically stable only if the system is BIBO stable.

The considerations can be easily extended for systems with \( m \) inputs and \( p \) outputs by considering in turn the \( mp \) suitable single-input single-output subsystems. For multi-input multi-output systems the impulse response and the unit response in Theorem 3.5 and 3.6 should be replaced by the matrix of impulse responses and the matrix of unit responses, respectively, and the transfer function in the Theorem 3.7 should be replaced by the transfer matrix.

To sum up this chapter, we summarize all the above theorems in one Lemma. Then we are going to use this Lemma in next chapters as the main stability Lemma for positive continuous-time systems for the purpose of stabilizing.

**Lemma 3.2.** Let the system (3.6) be a positive continuous-time system (Metzlerian System). Then the system (3.6) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All eigenvalues of \( A \) have negative real parts.
2. All coefficients of the characteristic equation
\[ \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \]

are positive, i.e. \( a_i > 0 \) for \( i = 0, 1, 2, \ldots, n \).

iii. All principal minors of the matrix \(-A\) are positive.

iv. The matrix \( A \) is nonsingular and \(-A^{-1}\).

v. There exist a positive definite (possibly diagonal) matrix \( P \) such that \( A^T P + PA < 0 \).

vi. There exists a positive vector \( v \in \mathbb{R}^n \) such that \( Av < 0 \).

### 3.3. Asymptotic Stability using Lyapunov Equation

In the stability analysis of Metzlerian systems, it is of interest to find conditions under which the solution of the Lyapunov equation

\[ A^T P + PA = -Q \]  \hspace{1cm} (3.15)

is a positive matrix \( P > 0 \) in addition to its positive definiteness, i.e. \( P > 0 \). The following Lemma is useful for our main result.

**Lemma 3.3.** If a matrix \( A \) is Metzler and stable, then for any positive and positive definite symmetric matrix \( Q \), there is a positive and positive definite symmetric matrix \( P \) as a solution of Lyapunov equation (3.15).

**Proof.** The Lyapunov matrix equation (3.15) can be rewritten as a linear matrix equation

\[ Mp = -q \]  \hspace{1cm} (3.16)

where \( p \) and \( q \) are vectors whose elements are constructed from the components \( p_{ij} \) and \( q_{ij} \) of \( P \) and \( Q \), and

\[ M = A^T \otimes I + I \otimes A^T \]  \hspace{1cm} (3.17)

is an \( n^2 \times n^2 \) matrix with \( \otimes \) denoting the Kronecker product. The matrix \( M \) is stable and by construction it is also Metzlerian. Since for any Metzler matrix \(-M^{-1} > 0\), we conclude that for any \( q > 0 \) we have \( p > 0 \). The positive definiteness of \( P \) follows directly from stability result of the Lyapunov matrix equation.
Corollary 3.1. Let the matrix \( A = [a_{ij}] \) be any stable Metzler matrix. Then the following statements are equivalent:

i. There exists a positive diagonal matrix \( D \) such that \( A^T D + DA \) (or alternatively \( AD + DA^T \) ) is negative definite, i.e., \( A^T D + DA < 0 \).

ii. There exists a positive diagonal matrix \( D \) such that \( x^T A^T D x < 0 \) (or alternatively \( x^T A D x < 0 \) ) for all \( x \neq 0 \).

Furthermore, if \( B = [b_{ij}] \) is a matrix with negative diagonal elements \( b_{ii} \) with \( b_{ii} \leq a_{ii} \) and \( |b_{ij}| \leq a_{ij} \). Then \( B^T D + DB < 0 \). A direct consequence of Lemma 3.3 and Corollary 3.1 is the fact that it is always possible to find a diagonal positive matrix \( P = D \) for any Hurwitz stable Metzler matrix.

3.4. Robust Stability of Perturbed Systems

3.4.1. General Uncertain Systems

Consider the general continuous-time system with uncertainty structures defined by

\[
\dot{x}(t) = (A + \Delta A(t)) x(t) + (B + \Delta B(t)) u(t)
\]  

(3.18)

where

\[
\Delta A(t) = E\Delta(t) F_1
\]
\[
\Delta B(t) = E\Delta(t) F_2
\]  

(3.19)

with \( E \in \mathbb{R}^{m \times n} \), \( F_1 \in \mathbb{R}^{n \times m} \), \( F_2 \in \mathbb{R}^{n \times m} \) and

\[
\Delta(t) \in \Delta = \left\{ \Delta(t) \in \mathbb{R}^{ex} : \|\Delta(t)\| \leq 1 \right\}
\]  

(3.20)

Note that the elements of \( \Delta(t) \) are Lebesgue measurable and admissible uncertainties are such that \( \Delta^T(t) \Delta(t) \leq I \).

Now, for the purpose of defining positive uncertain system let the Metzler matrix \( A \) be associated with the system \( \dot{x}(t) = Ax(t) \). Let the perturbed system be defined by affine perturbation of the from shown in equations (3.18) and (3.19) but for simplicity let assume \( F_1 = F_2 \). Then the perturbed system will look as follow.
\[
\dot{x}(t) = (A + E\Delta F)x(t)
\]  
(3.21)

where \( E \in \mathbb{R}^{n \times e} \), \( F \in \mathbb{R}^{f \times n} \) represent the structure of uncertainties and \( \Delta \in \mathbb{R}^{e \times f} \) is an unknown uncertainty matrix. The following results provide robustness measures that will be used in the robustness analysis of the optimal constrained stabilization.

**Lemma 3.4.** Assume that the matrix \( A \) is Hurwitz stable and for any \( Q > 0 \), the solution \( A^TP + PA = -Q \) is given by \( P > 0 \). Then the perturbed system

\[
\dot{x}(t) = (A + H)x(t)
\]

is asymptotically stable provided that

\[
\|H\| < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}
\]  
(3.23)

Furthermore, if \( A \) is a Metzler stable matrix, the matrix \( P \) in (3.23) can be replaced by a diagonal positive matrix \( D \) obtained from the solution of \( A^TD + DA = -Q \).

Note that the bound (3.23) is well-known in the literature (see for example [16]). A more general measure of stability robustness is the stability radius defined in the next section.

### 3.4.2. Uncertain Interval Systems

Let the uncertain matrix \( A(\delta) \) be associated with the system

\[
\dot{x}(t) = A(\delta)x(t) + Bu(t)
\]  
(3.24)

defined by affine multiple parameter perturbations of the form

\[
A(\delta) = A + \sum_{r=1}^{q} \delta_r E_r, \quad r = 1, \ldots, q
\]  
(3.25)

where \( \delta_r \) are unknown scalars and \( E_r \) are given matrices specifying the structure of the perturbations, and \( \delta = \left[ \delta_1 \quad \delta_2 \quad \cdots \quad \delta_q \right]^T \) is the vector of uncertain parameters \( \delta_r, \ r = 1, \ldots, q \) confined within a prescribed set of interest \( \Omega \), i.e. \( \delta \in \Omega \). When \( r = 1 \), we have an affine single perturbation structure and \( A(\delta) \) reduces to \( A(\delta) = A + \delta E \) where \( \delta_1, E_1 \) are redefined as \( \delta, E \) respectively. Thus, for single and
multiple perturbations, \( \delta \) can be represented as the vector of uncertain parameters confined within a prescribed set of interest \( \Omega \), i.e. \( \delta \in \Omega \).

Since the interval matrices comprise of a closed and bounded convex set, they can be represented as a convex hull of its vertex matrices. Therefore, one can relate interval and affine uncertainties. For Metzlerian delay system with affine uncertainty structure and non-negative perturbation matrices \( E_r \in \mathbb{R}^{m \times n} \) we have \( \delta, E_r \in \left[ \underline{\delta}, E_r, \overline{\delta}, E_r \right] \) for all fixed \( \delta_r \in \left[ \underline{\delta_r}, \overline{\delta_r} \right] \). This indicates that \( A(\delta) \in \left[ A(\delta), \overline{A}(\delta) \right] \) with \( A(\delta) = A + \sum_{r=1}^{q} \delta E_r \) and \( \overline{A}(\delta) = A + \sum_{r=1}^{q} \overline{\delta E_r} \) where \( A \) is the nominal Metzler matrix and \( E_r \)'s represent the non-negative perturbation matrices. So, if \( A(\delta) \) is represented as the interval uncertainty through the affine perturbation structure discussed above leading to interval Metzlerian system or the uncertain Metzlerian system is defined as interval Metzlerian system, we can write (3.24) as

\[
\dot{x}(t) = \left[ A, \overline{A} \right] x(t) + Bu(t)
\]  

(3.26)

where \( A = A(\delta) \) and \( \overline{A} = \overline{A}(\delta) \). Then the following result for the uncertain interval Metzlerian system can be stated

**Theorem 3.9.** The Metzlerian interval system (3.26) is robustly stable if and only if the following single Metzlerian system

\[
\dot{x}(t) = \overline{A} x(t) + Bu(t)
\]

(3.27)

is asymptotically stable, and the stability can be verified by checking the positivity of leading principal minors of \( -\overline{A} \).

**Proof.** It has been established in [3] that the robust stability of Metzlerian interval systems \( \dot{x}(t) = \left[ A, \overline{A} \right] x(t) + Bu(t) \) is equivalent to the asymptotic stability of the system with the upper interval, namely \( \dot{x}(t) = \overline{A} x(t) + Bu(t) \).

**3.5. Stability Radius**

The stability radius can be defined for any objects such as a system, a function or a matrix. Stability radius at a given point is the radius of the largest ball, centered at the nominal point, all of whose elements
satisfy pre-determined stability conditions. Stability radius is a more general measure of stability robustness [6], [5].

We suppose the perturbed system can be described in the form of equation (3.21). Therefore, \( A \) is the nominal system matrix which under perturbation will be \( A+E\Delta F \) where \( E \) and \( F \) are given perturbation structure matrices and \( \Delta \) is unknown uncertainty which is allowed to be real or complex. In each case we try to find the smallest matrix \( \Delta \) that makes the system unstable. In the literature, it is common to measure the size of matrix \( \Delta \) by its norm. The system equations will be

\[
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) \\
u(t) = \Delta y(t)
\]

(3.28)

In this model signals \( u \) and \( y \) are fictitiously introduced and are not necessarily input or output of the system.

We measure size of \( \Delta \) using the following norm

\[
\|\Delta\| = \sup \{\|\Delta y\|_{\mathbb{F}} : y \in \mathbb{F}^f, \|y\|_{\mathbb{F}} \leq 1\}
\]

(3.29)

where \( \mathbb{F} \) could be either field of real or complex numbers and \( e \) and \( f \) are size of signals \( u \) and \( y \), respectively.

Let \( \mathcal{S} \) denotes the stability region in the complex plane and \( \lambda(M) \) denotes the eigenvalue of matrix \( M \). Since we assumed that matrix \( A \) is stable we can write \( \lambda(A) \subset \mathcal{S} \).

**Definition 3.3.** The stability radius, in the field \( \mathbb{F} \), of \( A \) with respect to the perturbation structure \( (E, F) \) is defined as

\[
r = \inf \{\|\Delta\| : \Delta \in \mathbb{F}^{n \times f}, \lambda(A+E\Delta F) \cap \mathbb{U} \neq \emptyset\}
\]

(3.30)

The operator norm of \( \Delta \) is most often measured by its maximum singular value, i.e., \( \|\Delta\| = \sigma(\Delta) \). In this case it can easily be established by continuity of the eigenvalues of \( A+E\Delta F \) on \( \Delta \), and the stability of \( A \), that
For a fixed $s \in \partial \mathcal{S}$, write $G(s) = M$. Then the calculation above reduces to solving the optimization problem

$$\inf \left\{ \bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{exf}, \det(I - \Delta G(s)) = 0 \right\}$$

When $\Delta$ is a complex matrix, the solution for this optimization problem is obtained as

$$\bar{\sigma}(\Delta) = \left[ \bar{\sigma}(M) \right]^{-1}$$

When $\Delta$ is constrained to be a real matrix the solution is much more complicated due to the fact that $M$ is a complex matrix.

### 3.5.1. Complex Stability Radius

Consider the case where all the system matrices are complex. Let us also assume that the unknown uncertainty is complex, too, i.e. $\Delta \in \mathbb{C}$. In this case we denote $r$ as $r_c$ and call it complex stability radius.

The following theorem enable us to compute $r_c$ assuming that the transfer function associated with perturbed triple $(A, E, F)$ is

$$G(s) = F(sI - A)^{-1}E$$

**Theorem 3.10.** If $A$ is stable with respect to $\mathcal{S}$, then

$$r_c = \frac{1}{\sup_{s \in \mathcal{S}} \|G(s)\|}$$

where $\|G(s)\|$ denotes the operator norm of $G(s)$ and by definition $0^{-1} = \infty$.

**Proof.** The proof is an immediate consequence of equations (3.31)-(3.33). □

When $\|\Delta\| = \bar{\sigma}(\Delta)$, the complex stability radius is obtained as
\[ r_c = \frac{1}{\sup_{s \in \mathbb{C}} \overline{\sigma} (G(s))} \]  

(3.35)

By choosing \( \mathbf{E} = \mathbf{F} = \mathbf{I} \) we can find the unstructured stability radius which for the case of Hurwitz stability can be found as follow.

\[ r_c = \frac{1}{\|G(s)\|_\infty} \]  

(3.36)

### 3.5.2. Real Stability Radius

Consider the case where all the system matrices are real. Let us also assume that the unknown uncertainty is constrained to be real, too, i.e. \( \Delta \in \mathbb{R} \). In this case we denote \( r \) as \( r_r \) and call it real stability radius.

To compute \( r_r \) we need to solve a two parameter optimization problem as discussed in the following theorem.

**Theorem 3.11.** The real stability radius is given by

\[ r_r = \inf_{s \in \mathbb{C}} \inf_{\gamma \in (0,1]} \sigma_2 \left[ \begin{array}{cc} \text{Re} G(s) & -\gamma \text{Im} G(s) \\ \gamma^{-1} \text{Im} G(s) & \text{Re} G(s) \end{array} \right] \]  

(3.37)

An important feature of this formula is the fact that the function

\[ \sigma_2 \left[ \begin{array}{cc} \text{Re} G(s) & -\gamma \text{Im} G(s) \\ \gamma^{-1} \text{Im} G(s) & \text{Re} G(s) \end{array} \right] \]

is unimodal over \( \gamma \in (0,1] \).

The computation of stability radius requires the solution of an iterative global optimization problem. Note that, computing the real stability radii is more complicated comparing to the complex stability radii and there is no closed form for none of them for the general systems. However for the class of positive systems the complex and real stability radii coincide and can be computed by closed form expressions for both continuous-time and discrete-time cases [7], [8]. For this class of system one can employ the Perron-Frobenius Theorem to derive Lemma 3.5.

**Theorem 3.12. (Perron-Frobenius)** If the matrix \( \mathbf{A} \) is nonnegative, then

i. \( \mathbf{A} \) has a positive eigenvalue \( r \) equal to the spectral radius of \( \mathbf{A} \).

ii. There is a positive eigenvector associated with the eigenvalue \( r \).
iii. The eigenvalue $r$ has algebraic multiplicity 1.

The eigenvalue $r$ will be called Perron-Frobenius eigenvalue.

The following lemma sums up all of our discussion about robust stability of positive systems. This Lemma is an example of nice stability properties that holds only for positive system. This is going to be a keystone for our controller design in the following chapters.

**Lemma 3.5.** [7] Let the Metzler matrix $A$ associated with the perturbed system (3.21) be Hurwitz stable. Then, the real and complex stability radii of the uncertain Metzlerian system $\dot{x}(t) = (A + E\Delta F)x(t)$ coincide and given by the following formulas depending on the characterization of $\Delta$,

i. Let $\|\cdot\|$ denotes the Euclidean norm in characterization of $\Delta$, then

$$r_c = r_R = \frac{1}{\|FA^{-1}E\|}$$

(3.38)

ii. Let $\Delta$ be defined by the set $\Delta = \{S \circ \Delta : S_{ij} \geq 0\}$ with $\|\Delta\| = \max \{|\delta_{ij}| : \delta_{ij} \neq 0\}$ where $[S \circ \Delta]_{ij} = S_{ij} \delta_{ij}$ represents the Schur product, then

$$r_c = r_R = \frac{1}{\rho\left(FA^{-1}ES\right)}$$

(3.39)

where $\rho(\cdot)$ denotes the spectral radius of a matrix. Furthermore, if the affine uncertainty structure is defined by $A(\delta) = A + \sum_{r=1}^{q} \delta_r E_r$, where $\delta = \begin{bmatrix} \delta_1 & \delta_2 & \cdots & \delta_q \end{bmatrix}^T$ is a vector of uncertain parameters confined within a prescribed set of interest $\Omega$, i.e. $\delta \in \Omega$. Then the real and complex stability radii of uncertain Metzlerian system $\dot{x}(t) = \left(A + \sum_{r=1}^{q} \delta_r E_r\right)x(t)$ coincide and it is given by

$$r_c = r_R = \frac{1}{\rho\left(-A^{-1}\sum_{r=1}^{q} E_r\right)}$$

(3.40)
Chapter 4 – Positive Stabilization with Maximum Stability Radius

4.1. Positive (Metzlerian) Stabilization

Consider the general unstable continuous-time linear system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]  
(4.1)

Let the state feedback control law of the form

\[ u(t) = v(t) + Kx(t) \]  
(4.2)

be applied to the system (4.1). Then we get the following closed-loop system

\[ \dot{x}(t) = (A + BK)x(t) + Bv(t) \]  
(4.3)

Thus, in the first stage of our design procedure we need to find \( K \in \mathbb{R}^{m \times n} \) such that \( A + BK \) is a Metzler matrix and \( A + BK \) is a Hurwitz stable matrix. There are many ways to achieve this goal by using the equivalent conditions of Lemma 3.2. For example, using the property 3 that the leading principal minors of \( M = -(A + BK) \) to be positive i.e. \( |M(\alpha)| > 0 \) where \( \alpha \in N \) defines the order of the minors and \( M_{ii} > 0, M_{ij} \leq 0 \), one can find the gain matrix \( K \) through a linear programming (LP) set-up [9]. Alternatively, one can construct an LP by using the property 6 of Lemma 3.2 applied to \( A + BK \) as outlined in the following theorem, which is more compact [11], [17].

**Theorem 4.1.** There exist a state feedback control law (4.2) for the system (4.1) such that the closed-loop system (4.3) becomes strictly Metzlerian stable if the following LP has a feasible solution with respect to the variables \( w = [w_1, w_2, \ldots, w_n]^T \in \mathbb{R}^n \) and \( z_i \in \mathbb{R}^m, \forall i = 1, \ldots, n \).

\[ Aw + B \sum_{i=1}^{n} z_i < 0, \quad w > 0 \]  
(4.4)

\[ a_{ij}w_j + b_{ij}z_j \geq 0 \quad \text{for } i \neq j \]  
(4.5)

\[ a_{ij}w_j + b_{ij}z_j < 0 \]  
(4.6)
with \( A = [a_{ij}] \) and \( B = [b_1 \ b_2 \ \cdots \ b_n]^T \). Furthermore, the gain matrix \( K \) is obtained from

\[
K = \begin{bmatrix}
\frac{z_1}{w_1} & \frac{z_2}{w_2} & \cdots & \frac{z_n}{w_n}
\end{bmatrix}
\]

\((4.7)\)

**Proof.** Imposing the structural constraint of Metzler matrix for \( A + BK \) implies that for \( i \neq j \) we have

\[
(A + BK)_{ij} = a_{ij} + b_j K_j = a_{ij} + b_j \frac{z_j}{w_j} \geq 0 \\
\text{and since } w_j > 0, \text{it leads to (4.5). Similarly, one can construct}
\]

the strict Metzlerian diagonal condition (4.6). The equivalent stability condition \((A + BK)w < 0\) for a positive vector \( w > 0 \) can be written as \( Aw + BKw < 0 \) or \( Aw + B \sum_{i=1}^{n} z_i < 0 \) with the aid of (4.7), which is (4.4).

The following theorem uses condition 5 of Lemma 2 along with structural constraint of Metzler matrix.

**Theorem 4.2.** [10] There exist a state feedback control law (4.2) for the system (4.1) such that the closed-loop system (4.3) becomes strictly Metzlerian stable if the following LMI has a feasible solution with respect to the variables \( Y \) and \( Z \)

\[
ZA^T + YT B^T + AZ + BY < 0 \\
(AZ + BY)_{ij} \geq 0 \quad \text{for } i \neq j \\
(AZ + BY)_{ii} < 0
\]

where \( Y \) and \( Z \) is a diagonal positive definite matrix.

**Proof.** Let \( K \) yet to be determined such that \( A + BK \) is a Metzler and stable matrix. Then using Corollary 3.1 the Lyapunov inequality

\[
Z(A + BK)^T + (A + BK)Z < 0
\]

\((4.11)\)

must have a positive definite diagonal solution for \( Z \) and the matrix \( K \) such that (4.11) is satisfied with its off-diagonal elements being non-negative. Since \( Z > 0 \) is diagonal, this condition holds if and only if all the off-diagonal entries of \((A + BK)Z\) are non-negative. Therefore, with the change of variable \( Y = KZ \), the asymptotic stability of the closed-loop Metzlerian system is equivalent to the LMI (4.8) along with
the structural constraints (4.9) and (4.10), which are similarly obtained from \((A+BK)_{ii} \geq 0\) and 
\((A+BK)_{ij} \geq 0\) by multiplying both sides with \(Z\) and using the same change of variable \(Y = KZ\).

The above procedure is applied to the following simple system to illustrate the Metzlerian stabilization.

Consider the unstable system with state space matrices

\[
A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}
\]

If we apply the Metzlerian stabilization based on LMI technique, then we obtain the feedback gain

\[
K = \begin{bmatrix} -1.5503 & 19.5143 & 0.0127 \\ 0.2356 & -19.2338 & 0.0118 \end{bmatrix}
\]

and the associated closed-loop system matrix becomes Metzlerian

\[
A_c = A + BK = \begin{bmatrix} -3.3146 & 0.2805 & 1.0245 \\ 1.2356 & -18.2338 & 1.0118 \\ 0.1351 & 19.7947 & -1.9628 \end{bmatrix}
\]

Furthermore, the system response with the initial condition \(x_0 = [1 \ 1 \ 1]^T\) is depicted below.

![State of closed-loop System](image)

Figure 4.1 – Closed-loop System State Response
The above example illustrates that a general linear time invariant system can be positively (Metzlerian) stabilized with state responses of the closed-loop system all lying in positive quadrant. However, the goal of the following section is to achieve Metzlerian stabilization with maximum stability radius.

4.2. Maximizing Stability Radius by State Feedback

In this section we show how to use the Lemma 3.5 in maximizing the stability radius by state feedback. Let the (4.3) closed-loop system be written as

\[ \dot{x}(t) = \left( A + BK + E \Delta F \right) x(t) \]  

(4.12)

Then, we seek to find a feedback controller such that the stability radius of the closed-loop system is maximized. Applying Lemma 3.5 for the closed-loop system (4.12) we need to solve the following problem

\[ \max_r \| F(A+BK)^{-1}E \| \]  

subject to the LMI constraint (4.8)-(4.10). However, it is not convenient to solve the above optimization problem. Since the real and complex stability radii of Metzlerian systems coincide, we avoid using the expression derived for real stability radius in the above optimization problem. Instead, we use the complex stability radius, which can conveniently be reformulated in terms of LMI with the aid of Bounded Real Lemma [18]. Consider the complex stability radius is computed by using Theorem 3.10 for the triple \((A_c, E, F)\). Then \( G(s) = F(sI - A_c)^{-1}E \) is the perturbed transfer function.

**Lemma 4.1. (Bounded Real Lemma)** Consider the transfer function \( G(s) = F(sI - A_c)^{-1}E \) with the state space parameters \( \{A_c, E, F\} \). Then the following are equivalent

i. \( \| F(sI - A_c)^{-1}E \|_{\infty} < \gamma \) and \( A_c \) is stable, i.e. \( \text{Re} \lambda_c(A_c) < 0 \).

ii. There exists a symmetric positive definite solution \( P_c \) to the LMI

\[
\begin{bmatrix}
A_{c}^{T}P_c + P_c A_{c} & P_c E & F^{T} \\
E^{T}P_c & -\gamma I & 0 \\
F & 0 & -\gamma I
\end{bmatrix} < 0
\]  

(4.14)
Note that the Bounded Real Lemma which is employed here for (4.12) can be regarded as a fictitious system with state spaces parameters \( \{A_c, E, F\} \) and stable \( A_c \). Thus, the equivalency of \( \left\| F(sI - A_c)^{-1} E \right\|_\infty < \gamma \) and LMI (4.14) is evident. Furthermore, using the Schur complement lemma one can rewrite the LMI (4.14) as the following Riccati equation

\[
A_c^T P_c + P_c A_c + \gamma^{-1} \left( F^T F + P_c E E^T P_c \right) < 0, \quad \gamma > 0
\] (4.15)

**Lemma 4.2. (Schur Complement Lemma)** The block matrix

\[
\begin{bmatrix}
P & M \\
M^T & Q
\end{bmatrix} < 0
\] (4.16)

is negative definite if and only if the following conditions are satisfied.

\[
Q < 0, \quad P - MQ^{-1} M^T < 0
\] (4.17)

With the aid of bounded real Lemma, one can obtain the maximum complex stability radius of the controlled system which is the inverse of the \( H_\infty \) norm of \( G(s) \) and can be recast as the following constrained optimization problem.

\[
\min \gamma
\] (4.18)

Subject to the LMI (4.14) with variables \( P_c = P_c^T > 0 \), \( K \) and \( \gamma \) where \( A_c = A + BK \).

In order to guarantee that the above optimization problem is formulated in terms of an LMI, we use the usual congruent transformation and pre- and post- multiply the inequality (4.18) by \( \text{diag} \{Q_c, I, I\} \), \( Q_c = P_c^{-1} \) and changing the variable \( Y_c = KQ_c \).

Thus, the problem can alternatively be formulated in terms of the following LMI with respect to the variables \( Q_c \) and \( Y_c \).

\[
\min \gamma
\] (4.19)

subject to
where \( W_c = Q_c A^T + Y_c^T B^T + A Q_c + B Y_c \) and the controller gain can be obtained by \( K = Y_c Q_c^{-1} \).

The above development can be summarized in the following theorem.

**Theorem 4.3.** There exist a state feedback control law (4.2) for the system (4.1) such that the closed-loop system (4.3) becomes strictly Metzlerian stable with maximum stability radius if the LMI (4.20) along with the structural constraints

\[
(AQ_c + BY_c)_{ij} \geq 0 \quad i \neq j \tag{4.21}
\]

\[
(AQ_c + BY_c)_{ii} < 0 \tag{4.22}
\]

has a feasible solution with respect to the variable \( Y_c \) and \( Q_c \). Furthermore, the feedback gain is obtained by \( K = Y_c Q_c^{-1} \).
Chapter 5 – Positive Time Delay System Stabilization with Maximum Stability Radius

5.1. Positive (Metzlerian) Stabilization

Consider the general unstable continuous-time linear system with delays

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{l} A_i x(t - \tau_i) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]  

(5.1)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( y \in \mathbb{R}^p \) are the state, input and output vectors, respectively and \( A_i \in \mathbb{R}^{n \times n} \), \( i = 0,1,\ldots,l \); \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \), \( D \in \mathbb{R}^{p \times m} \) and \( \tau_i > 0 \), \( i = 1,\ldots,l \) represent delays.

We assume that \( \tau_i = i\tau \) and \( i = 1,\ldots,l \) with initial condition \( x(t) = \varphi(t) \) for \( t \in [-d,0] \), \( d = l\tau \) and \( u(t), \ t \geq 0 \).

Lemma 5.1. The system (5.1) is internally positive if and only if \( A_0 \) is a Metzler matrix and the matrices \( A_i \); \( i = 1,\ldots,l \), \( B, C, D \) have nonnegative entries.

At this point, it is important to distinguish two terminologies. Let (5.1) be an internally positive delay system, then in the absent of delay state we have a delay free system, i.e. \( A_i = 0 \) for \( i = 1,2,\ldots,l \). On the other hand, we use the notion of internally positive delay system without delay or simply “Metzlerian system without delay” when \( \tau_i = 0 \) as it is formally defined below.

Definition 5.1. The continuous-time internally positive system (5.1) without delay i.e \( \tau_i = 0 \) is called a Metzlerian system without delay represented by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]  

(5.2)

where \( A = \sum_{i=0}^{l} A_i \) is assumed to be a Metzler matrix and \( B, C, D \) are positive matrices.
Theorem 5.1. [19], [20] The Metzlerian delay system (5.1) is asymptotically stable if and only if there exists a strictly positive vector \( v \in \mathbb{R}_+^n \) satisfying the inequality

\[ A v < 0 \quad \text{where} \quad A = \sum_{i=0}^{l} A_i \]

(5.3)

Corollary 5.1. The Metzlerian delay system (5.1) is asymptotically stable if and only if the Metzlerian system without delay

\[ \dot{x}(t) = Ax(t) + Bu(t) \quad \text{where} \quad A = \sum_{i=0}^{l} A_i \]

(5.4)

is asymptotically stable. Furthermore, the Metzlerian system is unstable for any \( A_i \), \( i = 1, 2, \ldots, l \) if the Metzlerian system \( \dot{x}(t) = A_ix(t) \) is unstable.

Note that the continuous-time linear delay system (5.1) is asymptotically stable if and only if

\[ \det \left( sI - A_0 - \sum_{i=1}^{l} A_i s^{-x_i} \right) \neq 0, \quad \forall \text{ Re } s \geq 0 \]

However, based on the Corollary 5.1, the above stability condition reduces to

\[ \det (sI - A) \neq 0, \quad \forall \text{ Re } s \geq 0, \quad A = \sum_{i=0}^{l} A_i \]

The problem of constrained positive stabilization for delay-free systems was introduced in [9] and later studied for delay systems in [21]. Here we further elaborate on it.

Lemma 5.2. Let the system (5.1) be Metzlerian delay system, i.e. \( A_0 \) is a Metzler matrix and \( A_i > 0 \), \( \forall i = 1, 2, \ldots, n \) with \( A = \sum_{i=0}^{l} A_i \), then the Metzlerian delay system is asymptotically stable if and only if any one of the equivalent conditions of Lemma 3.2 is satisfied.

Proof. Using Corollary 5.1, we know that the asymptotic stability of Metzlerian delay systems is equivalent to the asymptotic stability of Metzlerian system without delay. Thus, the equivalence of i - vi of Lemma 3.2 is obvious from the properties of positive systems and Theorem 5.1.
The above Lemma enables one to consider the constrained stabilization problem for a general time-delay systems. We consider two scenarios for it.

5.1.1. Case 1

Let the state feedback control law of the form

\[ u(t) = K_0 x(t) + \sum_{i=1}^{l} K_i x(t - \tau_i), \quad t \geq 0 \quad (5.5) \]

be applied to the system (5.1). Then we get the following dynamical equation for the closed-loop system

\[ \dot{x}(t) = (A_0 + BK_0) x(t) + \sum_{i=1}^{l} (A_i + BK_i) x(t - \tau_i), \quad t \geq 0, \quad x(t) = \varphi(t), \quad t \in [-l\tau, 0] \quad (5.6) \]

Lemma 5.3. Consider the closed-loop delay system (5.6). Then the closed-loop system without delay is asymptotically stable Metzlerian system if the following LMI has a feasible solution with respect to the variables \( Q \) and \( Y \)

\[
\begin{cases}
Q > 0, \quad (AQ+BY)_{ij} \geq 0, \quad i \neq j \\
QA^T + Y^TB^T + AQ + BY \prec 0
\end{cases}
\quad (5.7)
\]

where \( A = \sum_{i=0}^{l} A_i \) and the feedback gain \( K = \sum_{i=0}^{l} K_i \) is obtained from \( K = YQ^{-1} \).

Proof. Assuming that the Metzlerian feedback system (5.6) is asymptotically stable, then the equivalent conditions of Lemma 3.2 are satisfied with respect to the Metzlerian feedback systems without delay.

Now using the LMI formulation in [10], we seek a diagonal matrix \( Q > 0 \) and the matrix \( K = \sum_{i=0}^{l} K_i \) such that \( Q(A + BK)^T + (A + BK)Q \prec 0 \), with off-diagonal being non-negative. Since \( Q > 0 \) is diagonal, the last condition holds if and only if all the off-diagonal entries of \( (A + BK)Q \) are non-negative. Therefore, with the change of variable \( Y = KQ \), the asymptotic stability of the closed-loop Metzlerian system without delay is equivalent to the LMI (5.7) with the variables \( Q \) and \( Y \).
The above Lemma provides a solution to the constrained Metzlerian stabilization without delay. Unfortunately, the feedback gain $K$ does not provide a solution to $K_i$’s. However with the aid of the above Lemma one can state the following Theorem whereby $K_i$’s are exactly determined.

**Theorem 5.2.** Let the feedback control law (5.5) be applied to the time-delay system (5.1). Then the closed-loop system (5.6) is Metzlerian stable if and only if $A_0 + BK_0$ is a Metzler matrix, $A_i + BK_i \geq 0$, for $i = 1, 2, \ldots, l$ and one of the following equivalent conditions is satisfied:

i. $\sum_{i=0}^{l} (A_i + BK_i)$ is a stable Metzler matrix.

ii. There exists a vector $w \in \mathbb{R}^n$ and the gain matrices $K_i$’s such that $\left[ \sum_{i=0}^{l} (A_i + BK_i) \right] w < 0$.

iii. There exists a positive definite diagonal matrix $P$ such that $\left[ \sum_{i=0}^{l} (A_i + BK_i)^T \right] P + P \left[ \sum_{i=0}^{l} (A_i + BK_i) \right] < 0$.

Furthermore, the feedback control law (5.5) is determined if the following LMI has a feasible solution for $Q = P^{-1} \in \mathbb{R}^{m \times n}$ and $G_i \in \mathbb{R}^{m \times n}$

\[
\begin{align*}
\left( \sum_{i=0}^{l} A_i \right) Q + Q \left( \sum_{i=0}^{l} A_i \right)^T + B \left( \sum_{i=0}^{l} G_i \right) + \left( \sum_{i=0}^{l} G_i \right)^T B^T \prec 0
\end{align*}
\]

(5.8)

\[
\begin{align*}
(A_i Q + BG_i)_{jr} & \geq 0, \quad \forall \ j, r = 1, \ldots, n, \ i = 1, \ldots, n
\end{align*}
\]

(5.9)

\[
\begin{align*}
(A_0 Q + BG_0)_{jr} & \geq 0, \quad \forall \ j \neq r, \ j, r = 1, \ldots, n
\end{align*}
\]

(5.10)

\[
\begin{align*}
(A_0 Q + BG_0)_{jj} & < 0, \quad \forall \ j = 1, \ldots, n
\end{align*}
\]

(5.11)

where $(M)_{jr}$ means $jr$ element of a matrix $M$, and the feedback gain matrices $K_i, i = 0, 1, \ldots, l$ are obtained by $K_i = G_i Q^{-1}$.

**Proof.** Assuming (5.6) is Metzlerian, the equivalent conditions of Theorem 5.2 are obvious from Lemma 5.2. Taking advantage of LMI formulation of Lemma 5.3 and the equivalent conditions of Theorem 5.2, it
is sufficient to fulfill condition iii along with $A_0 + BK_0$, to be Metzler and $A_i + BK_i \geq 0$ for $i = 1, \ldots, l$.

Condition iii can alternatively be written as

$$Q \left[ \sum_{i=0}^{l} (A_i + BK_i) \right]^T + \left[ \sum_{i=0}^{l} (A_i + BK_i) \right] Q < 0$$

using congruent transformation with $Q = P^{-1}$ and by simple manipulation one can obtain (5.8).

Furthermore, if we multiply $A_i + BK_i \geq 0$ from right by $Q$ and use $G_i = K_i Q$ the constraint inequalities (5.9) and (5.10) regarding the positivity and Metzlerian constraints can be established. Note that, although condition (5.11) is not necessary, it is written to emphasize strict Metzlerian structure.

Remark 5.1. It is interesting to point out that the $\sum K_i$ obtained from LMI of Theorem 5.2 satisfies the LMI condition of Lemma 5.3.

Note also that it is possible to use condition ii along with the Metzlerian constraints for $A_0 + BK_0$ and $A_i + BK_i \geq 0$, $i = 1, \ldots, l$ to develop an equivalent LP solution. However, we skip this development and briefly elaborate on a special class of delay system in which $A_i$'s, $i = 1, \ldots, l$ are assumed to be positive.

In this case, the feedback control law $u(t) = K_0 x(t)$ is applied to delay system (5.1) with $A_i > 0$ for $i = 1, \ldots, l$ and $A = \sum_{i=1}^{l} A_i$, which results in closed-loop system with

$$\dot{x}(t) = (A_0 + BK_0)x(t) + \sum_{i=1}^{l} A_i x(t - \tau_i) .$$

Then Lemma 5.3 can be implemented with the LMI constraints

$$(A_0 Q + BY)_0 \geq 0 , \ A_0^T + Y^T B^T + AQ + BY < 0$$

whereby the feasibility of the solution with respect to the diagonal matrix $Q \succ 0$ and $Y \in \mathbb{R}^{mn}$ leads to $K_0 = YQ^{-1}$. This is evident due to the fact that if $A_0 + BK_0$ is a Metzler matrix and $A_i > 0$, the matrix $A_0 + BK_0 + \sum_{i=1}^{l} A_i$ becomes Metzler. Then by using the property v of Lemma 3.2 for stable Metzlerian matrices one can establish the result. In a similar fashion a simplified LP solution can also be obtained as reported in [22].

5.1.2. Case 2

Now let us assume the system is represented as follows
\[ \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{l} A_i x(t - \tau_i) + B u(t - \tau_i) \]  
(5.12)

and the feedback control law
\[ u(t) = K x(t), \quad t \geq 0 \]  
(5.13)

is applied to (5.12) such that the closed-loop system
\[ \dot{x}(t) = \sum_{i=0}^{l} (A_i + B_i K)x(t - \tau_i) \]  
(5.14)

is Metzlerian stable. It is clear from the previous development that Metzlerian system with delays (5.14) is asymptotically stable if and only if the Metzlerian system without delays, i.e.
\[ \dot{x}(t) = (A + BK) x(t) \]  
(5.15)

where \( A = \sum_{i=0}^{l} A_i \) and \( B = \sum_{i=0}^{l} B_i \) is Metzlerian stable. Thus, we have the following result.

**Theorem 5.3.** Let the feedback control law (5.13) be applied to the time-delay system (5.12). Then the closed-loop system (5.14) is Metzlerian stable if and only if there exists a diagonal matrix \( W = diag \{ w_1, w_2, \ldots, w_n \} \) with \( w_k > 0, \forall k = 1, \ldots, n \) and a real matrix \( Z \in \mathbb{R}^{m \times n} \) such that the following conditions are satisfied
\[ AW + BZ \text{ is Metzler} \]  
(5.16)
\[ (AW + BZ)J < 0 \]  
(5.17)

where \( J = [1 \ 1 \ \cdots \ 1]^T \) or equivalently the following LP has a feasible solution with respect to the variables \( w = [w_1 \ w_2 \ \cdots \ w_n]^T \in \mathbb{R}^n \) and \( z_i \in \mathbb{R}^m, \forall i = 1, \ldots, n. \)
\[ Aw + B \sum_{i=1}^{n} z_i < 0, \quad w > 0 \]  
(5.18)
\[ a_{ij} w_j + b_i z_j \geq 0 \text{ \ for } i \neq j \]  
(5.19)
\[ a_{ij} w_j + b_i z_j < 0 \]  
(5.20)
with \( A = [a_{ij}] \) and \( B = [b_1 \ b_2 \ \cdots \ b_n]^T \). Furthermore, the gain matrix \( K \) is obtained from

\[
K = ZW^{-1} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \\ w_1 & w_2 & \cdots & w_n \end{bmatrix}
\]  

(5.21)

Proof: Using (5.14), (5.15) and (5.21) we obtain

\[
\sum_{i=0}^{l} (A_i + BK_i) = A + BK = A + BZW^{-1} = (AW + BZ)W^{-1}
\]

which is Metzler if and only if (5.16) is satisfied i.e. \( AW + BZ \) is a Metzler matrix. Next, with the aid of \( KWJ = ZW^{-1}WJ = ZJ \) and \( WJ = [w_1 \ \cdots \ w_n]^T \), and using the property vi of Lemma 3.2 for stable Metzler matrices we have \((A + BK)w = (A + BK)WJ = (AW + BZ)J < 0\) which establishes the condition (5.17). Next, we show the validity of equivalent condition (5.18) - (5.20). Imposing the structural constraint of Metzler matrix for \( A + BK \) implies that for \( i \neq j \) we have \((A + BK)_{ij} = a_{ij} + b_j K_j = a_{ij} + b_j \frac{z_j}{w_j} \geq 0 \) and since \( w_j > 0 \) , it leads to (5.19). Similarly, one can add the strict Metzlerian diagonal condition (5.20) even though it is automatically satisfied as a result of stability condition (5.18). The equivalent stability condition \( (A + BK)w < 0 \) for a positive vector \( w > 0 \) can be written as \( Aw + Bkw < 0 \) or \( Aw + B \sum_{i=1}^{n} z_i < 0 \) with the aid of (5.21), which is (5.18).

It should be pointed out that an equivalent LMI can be written as in (5.7) of Lemma 5.3 with \( A = \sum_{i=0}^{l} A_i \) and \( B = \sum_{i=0}^{l} B_i \). Then the associated feedback gain is obtained from \( K = YQ^{-1} \).

5.2. Maximizing Stability Radius by State Feedback

Consider the general continuous-time delay system (5.1) with a general type of uncertainty structure described by

\[
\dot{x}(t) = \sum_{i=0}^{l} A_i(x(t - \tau_i) + Bu(t)) \\
y(t) = Cx(t) + Du(t)
\]  

(5.22)
where $A_i$'s are subjected to affine perturbations of the form

$$A_i(\Delta_i) = A_i + E_i \Delta_i F_i$$  \hspace{1cm} (5.23)$$

where $E_i \in \mathbb{R}^{n \times d_i}$, $F_i \in \mathbb{R}^{e \times n}$ represent the structure of the uncertainties and the matrices represent the structure of the uncertainties and the matrices $\Delta_i \in \mathbb{R}^{d_i \times n}$ are unknown uncertainty matrices. The robust stability results for (5.22), (5.22) are not trivial when general type of continuous-time delay systems is considered. There are several robust stability results are available for general time-delay systems using LMI conditions [23], [24], [25]. However, simple results can be obtained when (5.22) is assumed to be Metzlerian delay systems.

So, let us impose the constraint that the uncertain delay system (5.22) is Metzlerian delay system i.e. $A_i(\Delta_i)$ is a Metzler matrix and the matrices $A_i(\Delta_i) \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, l$ are non-negative for all $\Delta_i$, and $B \in \mathbb{R}^{n \times m}$. Then we have the following result, which can directly be stated with the aid of Corollary 5.1.

**Theorem 5.4.** Let the system (5.22) be uncertain Metzlerian delay system. Then it is robustly stable independent of delays if and only if the corresponding system without delays, i.e.

$$\dot{x}(t) = A(\Delta_i)x(t) + Bu, \quad \Delta_i \in \Omega$$  \hspace{1cm} (5.24)$$

is robustly stable, where $A(\Delta_i) = \sum_{i=0}^{l} A_i(\Delta_i)$ is a Metzler matrix for all $\Delta_i \in \Omega$.

Now, by using the relation between $r_C$ and $r_R$ for positive systems and using Lemma 3.5 we can obtain the following results.

**Theorem 5.5.** Let the system (5.22) with uncertainty structure (5.23) be an uncertain Metzlerian delay system and assume that it is asymptotically stable independent of delay without uncertainty. Then the real and complex stability radii of the uncertain Metzlerian system (5.23) coincide and it is given by the following formula if $E_i = E_j \in \mathbb{R}^{n \times d_i}$ or $F_i = F_j \in \mathbb{R}^{e \times n}$ for all $i, j = 0, 1, \ldots, l$

$$r_C = r_R = \frac{1}{\max_i \| F_i \left( -\sum_{i=0}^{l} A_i \right)^{-1} E_i \|}$$  \hspace{1cm} (5.25)$$
**Theorem 5.6.** Let the system (5.22) be uncertain Metzlerian delay system with uncertainty structure captured only on $A_i$, and let it be asymptotically stable independent of delay without uncertainty. Then, the real and complex stability radii of the uncertain Metzlerian delay system

$$
\dot{x}(t) = (A_0 + E_0 \Delta F_0)x(t) + \sum_{i=1}^{l} A_i x(t - \tau_i)
$$

coincide and it is given by following formulas depending on the characterization of $\Delta$

i. Let $\| \cdot \|$ denotes the Euclidean norm in characterization of $\Delta$, then

$$
r_c = r_k = \frac{1}{\left\| F_0 \left( \sum_{i=0}^{l} A_i \right)^{-1} E_0 \right\|} 
$$

(5.26)

ii. Let $\Delta$ be defined by the set $\Delta = \{ S \circ \Delta : S_y \geq 0 \}$ with $\| \Delta \| = \max \left\{ \| S_y \| : S_y \neq 0 \right\}$ where $[S \circ \Delta]_{ij} = S_y^{ij}$ represents the Schur product, then

$$
r_c = r_k = \frac{1}{\rho \left( F_0 \left( \sum_{i=0}^{l} A_i \right)^{-1} E_0 S \right)} 
$$

(5.27)

where $\rho(\cdot)$ denotes the spectral radius of a matrix.

Now, we show how to use Theorem 5.5 or Theorem 5.6 for maximizing the stability radius by state feedback. Let the uncertain closed-loop system for case 1 be written as

$$
\dot{x}(t) = \left( A + BK + E \Delta F \right) x(t)
$$

(5.28)

where $A = \sum_{i=0}^{l} A_i$, and with the assumption $F_i = F_j = F$ and $E_i = E_j = E$ for all $i, j$. 

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Then we seek to find a state feedback control gain $K = \sum_{i=0}^{l} K_i$ such that the stability radius of the closed-loop system is maximized. Applying Theorem 5.5 to the closed-loop system (5.28) we need to solve the following problem

$$\max_k r = \frac{1}{\|F(A + BK)^{-1}E\|}$$

(5.29)

Subject to the following LMI constraint

$$ZA^T + YT B^T + AZ + BY < 0$$

(5.30)

$$\left( AZ + BY \right)_{ij} \geq 0 \quad \text{for} \quad i \neq j$$

(5.31)

$$\left( AZ + BY \right)_{ii} < 0$$

(5.32)

where $Y$ and $Z > 0$ is a diagonal positive definite matrix.

It is evident that the above optimization problem is exactly the same one which we derived for the delay-free case in the previous section. Therefore, using the same procedure we can maximize the stability radius of the time-delay case. Note that here $A = \sum_{i=0}^{l} A_i$ and we have assumed that $F_i = F_j = F$ and $E_i = E_j = E$ for all $i, j$.

We sum up this section with the following theorem which enable us to positively stabilize a given general time delay system with maximum stability radius.

**Theorem 5.7.** There exist a state feedback control law (5.5) for the system (5.1) such that the closed-loop system (5.6) becomes strictly Metzlerian stable with maximum stability radius if the LMI

$$\begin{bmatrix}
W_c & E & Q_c F^T \\
E^T & -\gamma I & 0 \\
FQ_c & 0 & -\gamma I
\end{bmatrix} < 0$$

(5.33)

where $W_c = Q_c A^T + G_i^T B^T + AQ_c + BG_i$

along with the structural constraints


\[(A_i Q_c + B G_i)_{j'} \geq 0 \quad j \neq r \quad (5.34)\]

\[(A_0 Q_c + B G_0)_{j'} \geq 0 \quad j \neq r \quad (5.35)\]

\[(A_0 Q_c + B G_0)_{ij} < 0 \quad (5.36)\]

has a feasible solution with respect to the variable $G_i$ and $Q_c$. Furthermore, the feedback gain is obtained by $K_i = G_i Q_c^{-1}$.

Although Theorem 5.3 can be integrated with the objective function of maximizing the stability radius, it is computationally not attractive. Therefore, one can develop similar procedure for case 2 based on LMI which is implemented in example 6.5 of the next chapter.
Chapter 6 – Illustrative Examples

6.1. Delay-Free Positive Stabilization with Maximum Stability Radius

Example 6.1. [24] Consider the following for a perturbed system with unstructured unknown uncertainty

\[
A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E = F = I
\] (6.1)

Note that the system itself is a positive stable system. Therefore from Lemma 3.5 the stability radius is obtained as

\[
r_c = r_r = \frac{1}{\|A^{-1}\|} = 0.8740
\]

Now by using Theorem 4.3 and utilizing the CVX [26] we found the closed-loop system with the feedback gain

\[
K = \begin{bmatrix} -1 & -1 \end{bmatrix}
\]

which leads to a closed-loop Metzlerian system with maximum stability radius

\[
r_{c_{max}} = r_{r_{max}} = 2
\]

Example 6.2. Consider the following unstable MIMO perturbed system with the eigenvalues located at \( \{3.7427, -1.8713 \pm j0.7112\} \).

\[
A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 1 & 3 \\ 2 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}
\] (6.2)

Assuming \( E_i \) and \( F_i \) are structure matrices defining perturbation given by

\[
E_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad F_i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\]
Using the result of Theorem 4.3 and using the CVX software for the constrained LMI optimization problem [26], the state feedback gain $K_1$ which maximizes the stability radius of the closed-loop system is obtained as

$$K_1 = \begin{bmatrix} -1.5606 & -0.5262 & 0.0006 \\ -0.4394 & -1.4744 & -2.9994 \end{bmatrix}$$

and the close-loop system matrix $A_{c1}$ becomes a Metzlerian stable with the eigenvalues $\{-3.6445, -1.9988, -0.3905\}$, achieving the maximum stability radius of $r_{\text{max}1} = 3$.

$$A_{c1} = A + BK_1 = \begin{bmatrix} -3.5606 & 0.4744 & 0.0006 \\ 0.5606 & -0.4744 & 0.0006 \\ 0 & 0 & -1.9988 \end{bmatrix}$$

Next, we consider the same system with different structure matrices $E_2$ and $F_2$ given by

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which leads to

$$K_2 = \begin{bmatrix} -1.2328 & 0.4764 & 0.0007 \\ -0.7672 & -2.4764 & -2.9993 \end{bmatrix}$$

with $r_{\text{max}2} = 2.1213$.

Finally, considering structure matrices $E_3 = F_3 = I$, the unstructured stability radius is obtained as $r_{\text{max}3} = 2$ with the corresponding state feedback gain

$$K_3 = \begin{bmatrix} -1.2639 & 9.9100 & 0 \\ -0.7360 & -11.9095 & -3 \end{bmatrix}$$

Example 6.3. Consider the following example taken from [27]
with \( E = F = I \). Applying the two step procedure of LQR-based Metzlerian stabilization provided in [27] one can easily compute the feedback gain

\[
K_{LQR} = \begin{bmatrix} -3.9256 & -6.3936 & -8.8368 \end{bmatrix}
\]

leading to the Metzlerian stable matrix \( A + BK \). The associated robust stability radius is obtained as \( r_{LQR} = 3.6832 \). Using the LMI based approach of Theorem 4.3 and taking advantage of CVX [26] results in the feedback gain

\[
K = \begin{bmatrix} -4.7728 & -7.3727 & -10.6468 \end{bmatrix}
\]

and the corresponding maximum stability radius becomes \( r_{max} = 5.5193 \). An important exercise is to investigate the robust stability radius of LQR by choosing proper \( Q \) and \( R \) matrices in order to achieve the maximum stability radius.

### 6.2. Positive Stabilization with Maximum Stability Radius for Time-Delay Case

**Example 6.4.** Consider the following uncertain delay system (Case 1- Section 5.1.1)

\[
dx(t) = \sum_{i=0}^{2} \left( A_{i} + E_{i} \Delta_{i} F_{i} \right) x(t - \tau_{i}) + Bu(t)
\]  

(6.4)

where

\[
A_{0} = \begin{bmatrix} 1.5 & 1.75 \\ 4.75 & -0.75 \end{bmatrix}, \quad A_{1} = \begin{bmatrix} 0.5 & 0 \\ -1 & 0 \end{bmatrix}, \quad A_{2} = \begin{bmatrix} 0.45 & -0.5 \\ -0.25 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

and \( E_{i} = F_{i} = I \) for all \( i \).

Using the approach of Theorem 5.7 and by the help of CVX software [26], feedback gains are obtained as

\[
K_{0} = \begin{bmatrix} -4.75 & -1.75 \end{bmatrix}, \quad K_{1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad K_{2} = \begin{bmatrix} 0.25 & 0.5 \end{bmatrix}
\]
which leads to closed-loop Metzlerian delay system with \( A_{ci} = A + BK_i \); \( i = 0, 1, 2 \). The stability radius for the closed-loop system is easily computed as

\[
 r_{max} = \frac{1}{\| (A_{c0} + A_{c1} + A_{c2})^{-1} \|} = 1.05
\]

**Example 6.5.** Consider the following MIMO uncertain system (Case 2 - Section 5.1.2)

\[
\dot{x}(t) = \sum_{i=0}^{2} (A_i + E_i F_i) x(t - \tau_i) + B_i u(t - \tau_i)
\] (6.5)

with

\[
\begin{align*}
A_0 &= \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}, & A_1 &= \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
B_0 &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0.25 & 0 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0.5 & 0 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \\ 0.25 & 0.25 \end{bmatrix}
\end{align*}
\]

Assume the uncertainty structure as follow

\[
E_i = E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad F_i = F = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad i = 0, \ldots, 2
\]

Using the result of Case 2 with LMI implementation through CVX software [26], the state feedback gain is computed as

\[
K = \begin{bmatrix} -1.5606 & -0.5262 & 0.0006 \\ -0.4394 & -1.4737 & -2.9994 \end{bmatrix}
\]

and the closed-loop system matrix \( A_c \) becomes a Metzlerian stable with the eigenvalues \( \{-3.6445, -1.9988, -0.3899\} \) achieving the maximum stability radius of \( r_{max} = 3 \).
Note that, this example gave us same results as part 1 of example 6.2. This is due to fact that positive stabilization for time-delay case is actually a positive stabilization for delay-free case with $A$ and $B$ matrices properly defined as discussed in Section 5.2.
Chapter 7 – Conclusion

This section will be completed soon.
References


