Control-Parameter-Space Classification
for Delay-Independent-Stability of
Linear Time-Invariant Time-Delay Systems;
Theory and Experiments

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“It’s not that I’m so smart, it’s just that I stay with problems longer.”

Albert Einstein
Abstract

College of Engineering
Department of Mechanical and Industrial Engineering

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Control-Parameter-Space Classification for Delay-Independent-Stability of Linear Time-Invariant Time-Delay Systems; Theory and Experiments

by Payam Mahmoodi Nia
Reading the instantaneous reaction (states) of dynamic systems subjected to internal/external inputs is almost impossible in practice. To overcome the problem of this loss of information, in theory, it is preferred to include the information of the dynamic states of the system within its mathematical model. In literature, these system models are known as, “Delayed-Differential Equations” (DDEs), and “Functional-Differential Equations” (FDEs), and in some other sources, they are called “Differential-Difference Equations” (DDEs).

Delays frequently arise in many dynamical systems which can be a source of instability or unexpected/undesired response or performance. A careful control design is therefore critical in order to assure that a controlled system performs properly and remains stable despite the delays. The goal of this dissertation is to investigate methods to design controllers that can fulfill the need of stability for linear time-invariant (LTI) systems with multiple uncertain/unknown delays.

Even for linear time-invariant (LTI) systems, control design in the presence of delays can be challenging, especially within a non-conservative framework. Frequency domain technique (FDT) is one promising direction for obtaining non-conservative results as demonstrated for analysis and synthesis problems of LTI systems. However, in general, finding all the stabilizing controller gains for the system can be a major challenge since the corresponding eigenvalue problems are infinite dimensional and nonlinear. When the delays are uncertain/unknown, design of delay-independent stable (DIS) controllers is desirable with the expectation that the systems functionality, e.g., output feedback control, can still be maintained. Based on FDT, several approaches are proposed in the literature for testing delay-independent stability of LTI systems with constant delays. However some problems related to control design in order to render such systems stable independent of the amount of delays have not been thoroughly investigated especially for the multiple-delay cases.

Within this dissertation, a purely algebraic approach is developed for designing controllers for regulation purposes of the general class of linear time-invariant (LTI) systems with uncertain delays. The proposed approach applies on a given system with controller matrices, some enterers of which are parameters, and identifies the regions in the parametric controller gain space where the system at hand is delay-independent stable (DIS) for all finite constant delays, thereby guaranteeing stable operation under uncertain delays. The approach builds on the recent results developed by the research team, and is based on properties of the discriminant operations on multi-variable polynomials, and is computationally efficient as it bypasses the need to sweep the frequency as was done
in the literature for the same problem. While achieving these, the proposed approach complies with necessary and sufficient conditions of stability, and permit considering structured controllers.

The results are new and address the major issue of extending the DIS control design to increased number of discrete delays. This is achieved mainly by procedures based on algebraic tools which allow designing controllers that can stabilize such systems regardless of how large/small the delays are. That is, with these controllers, the system at hand can be rendered delay-independent stable (DIS). The essence of the control design, likewise in the single delay case, is based on the Rekasius transformation, however multiple delay case requires several non-trivial steps including algebraic tools, polynomial theory, elimination techniques, and Sturm sequences. The advantages of the design procedure are that it simplifies the control design to managing the roots of some single-variable polynomials while also preserving the controller structure and complying with the necessary and sufficient conditions of stability.

The developed approaches are demonstrated on several application problems, including simulation:

An LTI observer-based controller design is proposed to stabilize a plant with an output delay. Different than the existing work, we use the observer gains to influence the plant stability. This becomes possible simply by removing the delay terms from the observer part. Given the plant controller gains, our approach can find the parametric regions with respect to the observer controller gains so that gains selected from these regions make the combined plant-observer system asymptotically stable independent of the amount of the delay in the plant.

Application of control design for active vibration control (AVC) with the delays in sensors is also investigated. Especially, in remote sensing where delays are large, and in high-speed applications with even small delays, instability can be inevitable. It is shown that using DIS controllers one can secure the stability while using the advantage of delayed controllers in vibration control. In other words, we demonstrate via simulations that vibration suppression, within certain excitation frequency bands, can be improved or be as effective as those in AVC applications without delays.

Another application topic for DIS controllers investigated and tested with experiments is on a Networked Control System (NCS). NCS have applications in telesurgery, space exploration robots, outer-space robot arms, military/defense research, forestry, mining,
and satellites, where normal on-board manual control cannot be used, or it is too hazardous and/or expensive to use onboard operations. An application of DIS control design is demonstrated in a non-conservative control design approach for delay-independent stability of a LTI NCS with uncertain/unknown time delays, and tested. The control design approach can be utilized on both centralized and decentralized controlled systems while delays can exist in any part of the structure of the controller, and/or model. Finally, we demonstrate the consistency of analytical results by showing very good agreement between simulations and real-time experiments of a master-slave plant with set-point regulation control. The control design approach proposed here not only provides stability regardless of the amount of the constant delays, but also provides the optimal tracking performance measured by analyzing integral absolute error and rightmost root distributions of the closed-loop system.

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<th>Description</th>
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<tr>
<td>AVC</td>
<td>Active Vibration Control</td>
</tr>
<tr>
<td>DDE</td>
<td>Delayed-Differential Equation / Differential-Difference Equations</td>
</tr>
<tr>
<td>DDS</td>
<td>Delay-Dependent Stability</td>
</tr>
<tr>
<td>DIS</td>
<td>Delay-Independent Stability</td>
</tr>
<tr>
<td>FDE</td>
<td>Functional-Differential Equations</td>
</tr>
<tr>
<td>FDT</td>
<td>Frequency Domain Technique</td>
</tr>
<tr>
<td>IAE</td>
<td>Integral Absolute Error</td>
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<tr>
<td>LTI</td>
<td>Linear Time-Invariant</td>
</tr>
<tr>
<td>MIMO</td>
<td>Multi Input Multi Output</td>
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<tr>
<td>MPC</td>
<td>Model Predictive Control</td>
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<tr>
<td>NCS</td>
<td>Networked Control System</td>
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<tr>
<td>SC</td>
<td>Supply Chain</td>
</tr>
<tr>
<td>SISO</td>
<td>Single Input Single Output</td>
</tr>
<tr>
<td>TDS</td>
<td>Time-Delay System</td>
</tr>
<tr>
<td>deg</td>
<td>degree of $X$ in a polynomial $P(X)$</td>
</tr>
<tr>
<td>gcd</td>
<td>greatest common divisor</td>
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# Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\mathbb{C}$</td>
<td>Complex plane</td>
</tr>
<tr>
<td>$\mathbb{C}_+$</td>
<td>Right half open complex plane</td>
</tr>
<tr>
<td>$\mathbb{C}_-$</td>
<td>Left half open complex plane</td>
</tr>
<tr>
<td>$\mathbb{C}_0$</td>
<td>Imaginary axis of the complex plane</td>
</tr>
<tr>
<td>$\mathbb{N}$</td>
<td>Natural</td>
</tr>
<tr>
<td>$\mathbb{R}_{0+}$</td>
<td>Set of Non-negative real numbers</td>
</tr>
<tr>
<td>$\mathbb{R}_+$</td>
<td>Set of positive real numbers</td>
</tr>
<tr>
<td>$\mathbb{R}_\infty$</td>
<td>$\mathbb{R} \cup \pm \infty$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Single delay parameter</td>
</tr>
<tr>
<td>${\tau_\ell}_{\ell=1}^L = (\tau_1, \tau_2, \ldots, \tau_L)$</td>
<td>Delay vector space</td>
</tr>
<tr>
<td>$T$</td>
<td>Single pseudo-delay parameter</td>
</tr>
<tr>
<td>${T_\ell}_{\ell=1}^L = (T_1, T_2, \ldots, T_L)$</td>
<td>Pseudo-delay vector space</td>
</tr>
<tr>
<td>$\Re(\cdot)$</td>
<td>Real part of (·)</td>
</tr>
<tr>
<td>$\Im(\cdot)$</td>
<td>Imaginary part of (·)</td>
</tr>
<tr>
<td>$c_\ell$</td>
<td>Commensuracy order of delay $\tau_\ell$</td>
</tr>
<tr>
<td>$s$</td>
<td>Laplace variable</td>
</tr>
<tr>
<td>$CE$</td>
<td>Characteristic equation</td>
</tr>
<tr>
<td>$\omega$</td>
<td>crossing frequency</td>
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To my parents
Chapter 1

On LTI Time-Delay Systems

1.1 Introduction and Literature Review

In many dynamical systems involving networks, remote communication, decision-making, sensing and actuation, time delays naturally arise [106]. Due to delays, information can be available and actuation can be executed only after a certain period of time. While this may not be critical in open-loop systems in which decisions and actions are not based on the output of the system, in closed-loop systems, this is extremely critical as delays can cause poor performance and instability [45, 75].

Even for linear time-invariant systems with delays, many problems related to control design and stability analysis are open today [67, 106]. As it is well-known, this is primarily due to arising infinite-dimensional non-linear eigenvalue problems that are extremely cumbersome to solve and to efficiently compute. Both time-domain and frequency domain analysis are proposed in the literature to systematically address some of the major challenges [45, 61, 66, 75, 99, 106, 109]. Existing work relevant to this dissertation include stability analysis of systems with delays, which can be addressed without any conservatism using frequency domain techniques (FDT) [44, 66, 104, 109, 114], and control design which can be performed via eigenvalue-based optimization schemes [9, 64, 125].

Time-delay systems (TDS) have been broadly studied [49, 124]. Stability of TDS is one of the fundamental problems that triggered a 50 year research effort in control systems community with an increasing intensity in the last decade [7, 13, 31, 66, 78, 89, 114]. It is a fundamental problem since delays inevitably exist in many control problems around us. Since they affect system stability, whether their magnitudes be very small or quite large
the presence and effects of delays cannot be ignored. In order to perform more accurate stability analysis, methods to cope with delays are needed. In this dissertation, it is intended to develop new procedures for delay-independent stability design of TDS, and this section is devoted to explain how and where these delays exist in real-life systems.

This Chapter is provided to introduce the time delays in engineering applications as discussed in Section 1.2. In Section 1.3 we formulate these applications under a general model. We will utilize throughout this dissertation and Section 1.4 provides literature search details in stability criteria using frequency domain to achieve DIS. Finally, this chapter will conclude with Section 1.5 which provides author’s observations regarding this field of research according to the provided literature search and references therein, specifically on DIS analysis and control synthesis for LTI-TDS.

1.2 Effect of Time Delays in Engineering Applications

In this section we will provide some examples of engineering systems that delays introduce non-negligible effects on their functionality and performance. That is, for an accurate qualitative and quantitative control design, delay and its effects must be considered within the mathematical modeling of these dynamic systems.

1.2.1 Traffic Modeling

Traffic flow problem represents human-in-the-loop dynamics [107]. Since the human is a part of the dynamics, humans add delays to a car following system due to sensing, decision making, and performing the appropriate actions in driving. Average delay range is between 0.6 seconds and 2 seconds and its value depends on the drivers’ cognitive and physiological states [101, 104]. Two main classes of traffic modeling that exist in the literature are macroscopic (continuum) models mainly expressed by partial differential equations and microscopic (car-following) models some based on delay differential equations [82].

Car following models formulate a vehicle’s velocity and/or acceleration as a function of vehicle distance $h_n$ to other vehicles, their velocity differences $\dot{h}_n$, and vehicle’s own velocity $v_n$,

$$\dot{v}_n(t) = g(h_n(t - \tau_1), \dot{h}_n(t - \tau_2), v_n(t - \tau_3)),$$

(1.1)
where \(\tau_1, \tau_2,\) and \(\tau_3\) may represent driver reaction times to different stimuli. Linearizing the given traffic model in (1.1), one can study its equilibrium stability for achieving fixed headway \(h_n\) and design controllers to achieve this control goal.

1.2.2 Networked Control Systems

A wide range of real-world control problems suffer from the influence of time delays on dynamic system behavior. One such problem arises in Networked Control Systems (NCS)\([51, 73, 94, 128]\), which have applications in telesurgery \([63]\), teleoperation \([52, 59, 97]\), traffic control, and unmanned vehicle coordination systems \([12, 93]\). Due to the physical distance between the components of NCS, these systems may suffer from communication delays, which could be uncertain, and this could cause instability and/or poor performance \([8, 45, 52, 61, 75, 97, 106]\). Control design should therefore be carefully performed to assure that the system functions stably despite the delays \([29, 32, 48, 99, 127]\).

A case specific NCS is considered in this dissertation which mimics a master and slave type system. In general, we can state that this type of applications with coupled dynamics can be useful in many scenarios where the operator is separated from a robot by a large distance, but it can also refer to a change in scale, where for example, a surgeon may use micromanipulator technology to conduct surgery at microscopic level \([59]\).

When the master and slave are physically distant, then the communication between the two needs to be established, e.g., by a wireless communication protocol. Regardless of how the two sites communicate, however, communication delays will be inevitable \([8, 52, 97]\).

An example NCS model with multiple delays can be formulated as below,

\[
\begin{align*}
\text{Master plant} & \quad \dot{x}_{ms}(t) = A_{ms}x_{ms}(t) + B_{ms}u_{ms}(t) + R, \\
y_{ms}(t) &= \sum_{\ell=1}^{L}(C_{ms})_{\ell}x_{ms}(t - \tau_{\ell}), \\
u_{ms}(t) &= K_{ms}y_{ms}(t), \\
\text{Slave plant} & \quad \dot{x}_{sl}(t) = A_{sl}x_{sl}(t) + B_{sl}u_{sl}(t), \\
y_{sl}(t) &= \sum_{\ell=1}^{L}(C_{sl})_{\ell}x_{sl}(t - \tau_{\ell}), \\
u_{sl}(t) &= K_{sl}(y_{ms}(t) - y_{sl}(t)),
\end{align*}
\]

where \((\cdot)_{ms}\) and \((\cdot)_{sl}\) indices represent master and slave respectively. \(x(t) \in \mathbb{R}^n\) is the state vector, \(y(t) \in \mathbb{R}^q\) is the output, \(u(t) \in \mathbb{R}^m\) is the control input, \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\), and \((C)_\ell \in \mathbb{R}^{q \times n}\) are given according to the realized state space model of...
master and slave plants, \( R \in \mathbb{R}^n \) is the set point reference adjusted by a human operator, and \( K \in \mathbb{R}^{m \times q} \) is the controller matrix

One can illustrate the master and slave experimental system studied in this thesis, as shown in Figure 1.1, where communication and sensory delays exist. In particular, here we address a DC motor velocity regulation problem in a consensus setting, in which a transfer function representation of a virtual DC motor (master) and a hardware DC motor (slave) are designed to match their speeds to a set-point reference. In the communication line between master and slave, an unknown amount of delay exists, causing instability concerns. Moreover, a fault detection feedback line exists from slave to master, which is affected by another unknown communication delay. This fault detection line is used to monitor the speed of the master motor to detect anomalies in the speed behavior of the slave motor, e.g., a breakdown of the slave DC motor. When the delays are uncertain, it would be desirable that the NCS responds stably regardless of the amount of delays, along with system’s ability to satisfy certain performance conditions, e.g., reference tracking. In Chapter 6, we will design DIS controllers for this experimental system, and show the effectiveness of these controllers by means of experiments.

### 1.2.3 Supply Chain Management

Inventory dynamics in supply chains (SC) has a complex behavior since inventory level variations are the end results of combined decision-making, manufacturing, product shipment and information sharing components, which must be dynamically adapted
against unpredictable demands. A cost effective supply chain management naturally requires a fine understanding of these components that directly affect the underlying mechanisms of inventory behavior. One of the most critical parameters in supply chain management is the delay [90, 103, 115]. Delay is inevitable in SC due to physical constraints related to lead times (in manufacturing), transportation and delivery times (shipments), decision-making durations (human behavior) and information availability (communication delays, data collection delays) [27]. Without getting into details, in the cited work, an inventory-level dynamics with multiple delays was modeled, and shown to be governed by the following delay-differential equation,

$$\lambda \frac{d^2 e(t)}{dt^2} + \frac{de(t)}{dt} = -\alpha^{WIP}(c(t - \tau_1) - e(t - \tau_1 - \tau_2)) - c(t)e(t - \tau_1 - \tau_2 - \tau_3), \quad (1.3)$$

where $e(t)$ is the error dynamics related to how closely inventory levels are around a predefined safety inventory level, $c(t)$ is the inventory regulation function, $\alpha^{WIP}$ is inventory work-in-progress control constant.

### 1.2.4 Active Vibration Control (AVC) Systems

AVC systems have been broadly studied [1, 16, 17, 40, 42, 60, 62, 79, 92, 98, 135–137], including the presence of delays in the control design. Generally speaking, an AVC system consists of masses, springs and dampers subjected to harmonic excitation $f(t)$ as in Figure 1.2. The system shown in Figure 1.2 can be expressed as,

![Figure 1.2: Active suspension system with two controllers and cyclic input disturbance.](image-url)
\[ \dot{X}(t) = AX(t) + BK_1 X(t - \tau_1) + BK_2 X(t - \tau_2) + F(t), \] (1.4)

where,

\[ X = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \frac{v_1 + v_2}{m_1} & 0 & 0 \\ \frac{v_1 + v_2}{m_1} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \frac{v_2}{m_2} & \frac{c_2}{m_2} & \frac{v_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \]

\[ K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad F(t) = \begin{bmatrix} 0 \\ \sin(\mu t) \\ 0 \\ 0 \end{bmatrix}, \]

in which system’s output has sensor delays that will affect the control actions in system’s feedback loop. Sensor delays are however ubiquitous in many control applications including AVC [106]. The presence of delays can be extremely critical for such applications, especially for those with remote sensing where delays can be large, and in high-speed AVC applications even if delays are small. Since, with sensor delays, information cannot be available instantaneously [106], control actions designed with conventional techniques based on such delayed information may lead to poor performance and instability, as pointed out in [45, 75, 108]. The effects of delays and delay-independent stable control design for AVC systems will be covered in details in Chapter 5.

1.3 Problem Formulation

In this section, we define the linear time invariant (LTI) time-delay system with multiple delays \( \tau_\ell \) as the general model considered in this dissertation. This model, was also studied in some other similar forms in the past, [7, 13–15, 18, 26, 27, 45, 46, 50, 56, 102, 114, 119, 129] reads in state space as,

\[ \frac{d}{dt} x(t) = Ax(t) + \sum_{\ell=1}^L B_\ell x(t - \tau_\ell), \] (1.5)

where \( x(t) \) is the state vector, \( A \) and \( B_\ell \) are system matrices, respectively, all with appropriate dimensions. The matrices \( A \) and \( B_\ell \) have in general numerical entries. \( \tau_\ell \) are the nonnegative constant delays and they are basically shifting the operator in time and \( L \) is the maximum number of distinct delays. Figure 1.3 shows a simulated model of linear second-order system for two cases, one with a single output delay \( \tau \) and the other
with no delay. The characteristic equation of the introduced system in (1.5) is given by,

\[
f(s; e^{-s\tau_{\ell}}) = \sum_{k=0}^{K} P_k(s)e^{-s\sum_{\ell=1}^{L} v_{k\ell} \tau_{\ell}} = 0, \tag{1.6}
\]

where \(P_k\) are polynomials in terms of \(s\) with real coefficients, \(K \in \mathbb{Z}_+\) and \(v_{k\ell} \in \mathbb{N}\).

Since there exists no delay in the highest-order derivative of the states in (1.5), the characteristic equation (1.6) represents a \textit{retarded} class LTI system with multiple delays \([45, 114]\), and thus \(\xi_0\ell = 0\). Here the commensurability order of delay \(\tau_{\ell}\) is denoted with \(c_{\ell}\), which is given by \(c_{\ell} = \max_{0 \leq q \leq Q}(\xi_{q\ell})\).

It is proven in [26] that the roots, \(s\), of the characteristic equation introduced in (1.5) exhibit continuity property with respect to delay values \(\tau_{\ell}\). Utilizing these results authors in [27, 30, 100] proved that stability may only change when the roots cross the imaginary axis since the imaginary axis is the boundary separating the stable-unstable regions on the complex plane, see Figure 1.4. Consequently, stability may only change when the real part of the rightmost root of the system’s characteristic equation vanishes. That is, one should analyze the characteristic function on the imaginary axis, by setting \(s = j\omega\) for \(\omega \in \mathbb{R}\). The frequency parameter \(\omega\) indicates the pathways of the eigenvalues across the imaginary axis. Figure 1.4, indicates the crossing possibility for a second order retarded system, for which the characteristic equation is represented by,
\[ f(s, \tau) = s^2 + 6s + 5 + (7s + 100)e^{-\tau s} = 0. \]

As shown in Figure 1.4, on the left, for the delay-free system, stability is guaranteed. On the other hand, by sweeping the delay value from zero, the characteristic roots will tend to move on the complex plane and as illustrated on the right side, the roots will cross the imaginary axis for delay value \( \bar{\tau} = 0.12 \). This delay value is called delay margin for system stability in the literature. It is noted that the non-conservatism of FDT is appealing and has been an inspiration of many studies [66, 106], primarily for a mainstream of research problems, indicating delay-independent stability (DIS) [13, 104, 113]. In DIS analysis, the objective is to test whether or not a given system is stable regardless of the amount of delays [75]. In the single-delay case, DIS analysis can be performed based on necessary and sufficient conditions [13, 45, 66]. As regard to DIS analysis of single delay LTI systems, several techniques are available. In [13], spectral radius of some system matrices can be calculated to conclude on DIS property. In [129], frequency sweeping is used to verify whether the system can ever have eigenvalues on the imaginary axis, an indicative of loss of stability. Frequency sweeping is also applied to systems with multiple delays as demonstrated in [46, 104].

**Property 1.1** (DIS Property [65]). Given \( \{ k_i \}_{i=1}^{\bar{\tau}} \), system in (1.5) is delay-independent stable if and only if (1.6) has no roots on \( \mathbb{C}_{0+} \) for all delay values. That is,

\[ f(s, \{ \tau_i \}_{i=1}^{\bar{\tau}}, \{ k_i \}_{i=1}^{\bar{\tau}}) \neq 0, \quad (1.7) \]
Due to the presence of transcendental terms, the characteristic equation (1.6) possesses infinitely many roots, some or all of which may determine stability. Because of this infinite dimensionality, this stability analysis without introducing conservatism can be difficult, especially when $L > 1$ [27, 28, 30]. What is known about the stability of (1.5) is that the continuity property of the roots of (1.6) on the complex plane holds [26], which indicates that stability analysis of (1.5) requires detecting the critical values \(\{\tau^*_\ell\}_{\ell=1}^L\) for which at least one root of (1.6) lies on the imaginary axis, \(s = j\omega\), of the complex plane, where \(\omega \in \mathbb{R}_{0+}\) without loss of generality [7]. For a given delay, if such a critical \(s = j\omega\) root exists, then the system is known to be delay-dependent stable (DDS). Complimentary to this scenario is the case when the roots of (1.6) never lie on the imaginary axis for all nonzero delay values and also when the delay-free system (\(\{\tau_\ell\}_{\ell=1}^L = 0\)) is asymptotically stable [76]. While the analysis of latter is trivial, it is the former that is challenging to address. Once these two conditions can be verified to hold, then the system in (1.5) is categorized as delay-independent stable (DIS) [66].

### 1.4 Previous Studies

Many papers are published along the lines of delay-independent stability (DIS) with sufficient conditions of DIS [13, 15], and necessary and sufficient conditions of DIS [14]. When $L = 1$, some graphical inspection of Property 1.1 can be established, e.g., by frequency sweeping $\omega$, for $s = j\omega$, in characteristic equation, but when $L > 1$, graphical analysis can be challenging. There are other techniques to test DIS of TDS. DIS conditions are studied in [45] for systems introduced in (1.5). In another study, one of the most complicated MTDS is studied for robustness via frequency sweeping [14, 27]. Studies in [18, 47, 50, 56, 113, 119, 131] are on single-delay cases ($L = 1$), and the work in [129, 132] are feasible for two-delay cases with a case specific characteristic equations, but not in a general form of (1.5). In all the cited work above, extensions to $L > 2$ case is restrictive due to two main reasons; one is that the DIS test for $L > 1$ is an NP-hard problem since each delay needs to be treated as an independent parameter [120]. Second is that the number of available equations to be solved for DIS analysis is less than the number of unknowns in the respective analysis [27].
Recently, new results are reported in [27, 30] on delay-independent stability analysis with multiple delays for LTI-TDS with necessary and sufficient conditions of stability, addressing some of the existing challenges.

Taking FDT approach, authors in [27, 30, 100] noticed that stability analysis of the characteristic equation (1.6) is still arduous and challenging since the delay-independent analysis requires sweeping all the delays in the range $[0, \infty)$ and check the stability. They noticed that delay-independent stability analysis is convenient in algebraic domain. Instead of checking whether there exist stability switching for some delay $\tau_\ell$ values, one can easily check this in an alternative parameter domain, known as pseudo-delay domain, which is in algebraic domain after a proper substitution [27, 88, 100]. In delay-dependent stability analysis, for each frequency $\omega$, transformed equation has real and imaginary parts. Common zeros of these real and imaginary parts can be calculated using the Resultant concept [6, 11, 54, 68]. Next, since computing the crossing frequency sweeping range is crucial, in the cited work, this was achieved by iterated Discriminant concept as an optimization type problem in which one seeks for the extremum points of the crossing frequency instead of sweeping $\omega$. Existence of valid extremum crossing frequency reveals ranges of frequencies within which $\omega$ exists and that the characteristic equation possesses a $s = j\omega$ root that crosses the imaginary axis for a specific amount of delays. That is, they proved successfully that delay-independent stability is possible if no $\omega$ extrema exist [27, 30].

This dissertation is based on FDT for achieving necessary and sufficient conditions for DIS of LTI-TDS. Linear Matrix Inequality (LMI) based approaches [4, 45] and systems with time-varying delays [19, 39, 83] are kept outside the scope of this dissertation.

### 1.5 Observations and Motivations

According to the literature surveyed above, to the best of the author’s knowledge, there still exists openings and challenging problems to solve in the domain of DIS analysis and control design theory for LTI systems with multiple delays. Major observations for a successful dissertation are as follows:

(i) Control parameters must be incorporated into delay-independent stability synthesis. There still remain unresolved and important problem in the literature in control synthesis which is associated with rendering a system DIS, where structured control design is desired and one seeks the set of controller gains with which infinitely many eigenvalues
of the closed-loop system are stable, no matter how large or small the delays are. In [27], a theoretical framework for analysis is encouraging, and in [29] there exists a preliminary work in control synthesis utilizing the conservative Descartes rule of signs, which can reveal the DIS controller gains but covering sufficient conditions. Within this dissertation we seek non-conservative approaches, and also wish to find the analytical boundaries in controller gains space in which DIS is achieved.

(ii) Characterizing the DIS regions analytically and without approximation in the controller gain space is an open problem, and numerically sweeping \( \omega \), all the delays in their range and the controller parameters may not be computationally efficient and may not allow one to reveal the precise boundaries of these regions. The proposed framework here does not need to sweep \( \omega \), and in some cases even the controller gains (except for rendering the boundaries of the DIS regions), making it computationally efficient. That is, under certain conditions, it becomes possible to reveal the controller gain regions by finding this analytical boundaries, in the form of piecewise implicit polynomials as a function of controller gains. We discuss these conditions and find these boundaries with the synergy of FDT, algebraic geometry, elimination techniques, and discriminant operation of polynomials, all of which simply reduce the DIS control space categorization problem to the study of the roots of some single-variable polynomials.

(iii) DIS analysis for systems with multiple delays is still challenging under computational efficiency concerns. As is seen in the literature, there still exists no work within which one can analyze the DIS with \( L > 3 \). Controller space categorization is even more challenging in this sense since more control parameters will increase the number of unknowns in the problem at hand corresponding to increased infinite degree of the parameter space. A factorization technique is revealed in this dissertation, which can help circumvent such computational problems by allowing one to study DIS in factors of polynomials.

(iv) Optimization of DIS controllers is another part of this dissertation. The control design approach proposed here not only provides stability regardless of the amount of multiple constant delays, but also takes the optimal set-point tracking performance into account, which are measured by integral absolute error inspired from [138] and closed-loop delayed system rightmost root distributions numerically calculated via existing tools [9].

(v) Finally, on case studies, the vibration suppression capabilities of the DIS controllers is investigated, which we find them to be even more effective for some frequency bands compared with vibration suppression results based on the assumption that delays are
zero. Moreover, in light of previous studies in this field, we find out that experimental studies to investigate the effectiveness of delay-independent stable controllers even on LTI systems was not reported, where these controllers are derived based on a non-conservative stability analysis framework and in the existence of multiple delays. This dissertation is also aimed to address this opening. A non-conservative control design approach is taken to render a LTI NCS model with two unknown time delays to become DIS, while satisfactorily and optimally track set-point input references, all of which are then validated via experiments.
Chapter 2

Polynomial Theory

2.1 Detection of Real Root in Single Variable Polynomials

2.1.1 Review of Descartes Rule of Signs

Polynomial real root isolation is the task of computing disjoint intervals, each containing a single root, for all the real roots of a given univariate polynomial with real coefficients. In [123], Vincent showed that polynomial real root isolation can be performed using a test based on the Descartes Rule of Signs. The test in essence evaluates a condition that implies that a given interval contains a single root, and another condition that implies that the interval does not contain any roots. If neither condition is satisfied, the interval is bisected and each subinterval is tested recursively.

**Definition 2.1** ([3, 21, 58]). Let $a = (a_0, a_1, \ldots, a_n)$ be a finite sequence of real numbers. The number of sign variations $\text{var}(a)$ in $a$ is the number of pairs $(i, k)$ with $0 \leq i < k \leq n$ and $a_i \cdot a_k < 0$, with $a_{i+1} = \ldots = a_{k-1} = 0$, and for nonzero $a_i$ and $a_k$. It is noted that $a_k$ is skipped to $a_{k+1}$ when $a_k = 0$, to disregard the effects of zeros in the coefficient sequence. Let $A$ be the polynomial $a_0 + a_1 x + \ldots + a_n x^n$. The number of sign changes $\text{var}_D(A)$ in the coefficient sequence of $A$ is given by $\text{var}(a)$.

**Theorem 2.2** ([3]). (Descartes Rule of Signs [5, 41]): For any non-zero real polynomial $A$, the number of sign variations, $\text{var}_D(A)$, in the coefficient sequence of the polynomial is less than or equal to the number of positive real zeros of $A$, counting multiplicities, by a non-negative even integer.
Remark 2.3. If $\text{var}_D(A) = 0$, then the polynomial $A$ is guaranteed to have no positive real zeros, and if $\text{var}(A)$ is odd, then $A$ is guaranteed to have at least one positive real zero. When $\text{var}_D(A)$ is even, Descartes rule of signs is inconclusive on the number of positive real zeros of $A$. In such a case, $A$ has either no positive real zeros, or has even number of positive real zeros.

**Numerical Examples**

(a) $A_1 = x^4 + 2x^3 + 4x + 5 \rightarrow \text{var}_D(A_1) = 0$. $A_1$ has no positive real root.

(b) $A_2 = x^4 + 2x^3 - 4x - 5 \rightarrow \text{var}_D(A_2) = 1$. $A_2$ has exactly one positive real root.

(c) $A_3 = x^4 + 2x^3 - 4x + 5 \rightarrow \text{var}_D(A_3) = 2$. $A_3$ has either 2 or no positive real roots.

(d) $A_4 = x^4 - 2x^3 + 4x - 5 \rightarrow \text{var}_D(A_4) = 3$. $A_4$ has either 3 or 1 positive real root(s).

Notice that Descartes rule of signs is inconclusive to declare the exact number of positive real roots (see $A_3$ and $A_4$) except when $\text{var}_D(A_1) = 0$ or $\text{var}_D(A_2) = 1$. If $\text{var}_D(A)$ is odd, Descartes rule of signs can guarantee that $A$ must have at least one positive real root (see $A_4$).

### 2.1.2 Review of Sturm Sequences

Sturm sequences grant the necessary and sufficient conditions on the number of positive real roots of a polynomial [38, 117, 129].

**Definition 2.4.** $\text{rem}(p(x), q(x))$ denotes the remainder of the division of the two polynomials $p(x)$ and $q(x)$.

**Definition 2.5** ([38, 117]). Let $P(x)$ be a polynomial with real coefficients. The Sturm sequences associated with $P(x)$ is the sequence of polynomials

$$P_0 = P(x), P_1(x), P_2(x), \ldots, P_\lambda(x)$$
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determined by the following equations

\[ P_0(x) = P(x), \]
\[ P_1(x) = \frac{dP_0(x)}{dx}, \]
\[ P_2(x) = -\text{rem}(P_0(x), P_1(x)), \]
\[ \vdots \]
\[ P_i(x) = -\text{rem}(P_{i-2}(x), P_{i-1}(x)), \]
\[ \vdots \]
\[ P_\lambda(x) = -\text{rem}(P_{\lambda-2}(x), P_{\lambda-1}(x)), \]

where the leading Sturm function \( P_0(x) \) is the polynomial under investigation for its real roots, and \( P_{\lambda+1}(x) = 0 \), where \( \lambda \leq \deg(P(x)) \). In general, each \( P_i(x) \) named as Sturm function in the Sturm sequences is nonzero. In that case, excluding \( P_0(x) \) and \( P_1(x) \), the pairs \( P_{i-2}(x) \) and \( P_{i-1}(x) \) are relatively prime since the sequence from \( i = 2 \ldots \lambda \) is derived using division. Moreover, one can prove that \( P_\lambda(x) \) is a non-zero constant if and only if \( P_0(x) \) has no repeated roots. When \( P_0(x) \) has repeated roots, however, some modifications are needed. Due to the repeated roots, \( P_0(x) \) and \( P_1(x) \) will not be prime with respect to each other, that is, \( \gcd(P_0(x), P_1(x)) \) becomes a function of \( x \). In this case, one of the Sturm functions vanishes, \( P_i(x) = 0 \), with \( 2 \leq i \leq \lambda \), and \( \gcd(P_0(x), P_1(x)) \) is known to be equal to \( P_{i-1}(x) \). When a Sturm function vanishes, the modification is done as follows. \( P_0(x) \) and \( P_1(x) \) should be divided by \( P_{i-1}(x) = \gcd(P_0(x), P_1(x)) \), in order to remove the repeated factors of \( P_0(x) \) out of the analysis. Once the Sturm sequences analysis is completed, the count of the repeated roots can be separately studied from the repeated factor \( P_{i-1}(x) \).

Definition 2.6. Let \( \alpha \in \mathbb{R} \) and \( P(\alpha) \neq 0 \). The sign variations \( var_s(P(\alpha)) \) of the Sturm sequences associated with \( P(x) \) is the number of sign variations in the sequence \( P(\alpha) = P_0(\alpha), P_1(\alpha), \ldots, P_\lambda(\alpha) \).

Theorem 2.7. (Sturm sequences Theorem [38, 117]): Let the polynomial \( P(x) \) have no repeated roots. In the interval \([a, b], a, b \in \mathbb{R}\), the number of distinct real roots of \( P(x) \) is given by \( var_s(P(a)) - var_s(P(b)) \).

We illustrate the capabilities of Sturm sequences in a simple example. Let us count the real roots of \( P(x) = x^4 + 2x^3 - 4x + 5 \) in the interval \([0, +\infty)\). We first find the Sturm
functions $P_i(x)$ using MAPLE software package,

\begin{align*}
P_0(x) &= x^4 + 2x^3 - 4x + 5, \\
P_1(x) &= x^3 + 1.5x^2 - 1, \\
P_2(x) &= x^2 + 4x - 7.3333, \\
P_3(x) &= -x + 1.1154, \\
P_4(x) &= 1. \tag{2.1}
\end{align*}

We can calculate $\text{var}_S(P(0))$ and $\text{var}_S(P(\infty))$ by taking the limits $x \to 0$ and $x \to \infty$, respectively in the Sturm sequences. It is easy to see that $\text{var}_S(P(0)) = \text{var}_S(P(\infty)) = 2$. According to Sturm sequences theorem, this means that the polynomial has no positive real roots in the interval $x \in [0, \infty)$ since $\text{var}_S(P(0)) - \text{var}_S(P(\infty)) = 0$.

Notice that Descartes rule of signs is inconclusive here to declare the exact number of positive real roots since $\text{var}_D(P(x))$ is greater than one. On the other hand, Sturm sequences grant the necessary and sufficient conditions in identifying the number of positive distinct real roots of a polynomial [85].

### 2.2 Elimination Theory

In this section, we shall review the resultant computations in elimination theory and it’s properties. Taken from [68], Mishra described resultant in terms below:

Resultant is an important and classical idea in constructive algebra, whose development owes considerably to such luminaries as Bezout, Cayley, Euler, Hurwitz, and Sylvester, among others. In recent time, resultant has continued to receive much attention both as the starting point for the elimination theory as well as for the computational efficiency of various constructive algebraic algorithms these ideas lead to; fundamental developments in these directions are due to Hermann, Kronecker, Macaulay, and Noether. Some of the close relatives, e.g., discriminant and sub-resultant, also enjoy widespread applications. Other applications and generalizations of these ideas occur in Sturm sequences and Descartes Rule of Signs that will be further investigated in following chapters as a tool to introduce the framework of control design for delay-independent stability.

Burnside and Panton define a resultant as follows [11]:
Being given a system of \( n \) equations, homogeneous between \( n-1 \) variables, if we combine these equations in such a manner as to eliminate the variables, and obtain an equation \( R = 0 \) containing only the coefficients of the equations, the quantity \( R \) is, when expressed in a rational and integral form, called the Resultant or Eliminant.

That is, resultant operation is an algebraic condition developed in terms of the coefficients of a given polynomials set, which is satisfied if the given system of equations has a common solution. There are two ways to view the development of resultant: algebraic and geometric. For more information on this introduction, interested reader is recommended to follow [6, 23, 54, 68, 116].

In the sequel, we will introduce the reader with some handy definitions and properties that will be useful in the following chapters’ theorems and propositions.

**Definition 2.8 ([6, 54]).** Let \( P \) and \( Q \) be two non-zero polynomials of degree \( p \) and \( q \). Let,

\[
P = \alpha_p X^p + \alpha_{p-1} X^{p-1} + \cdots + \alpha_0 = \alpha_p \prod_{i=1}^{p} (X - a_i),
\]

\[
Q = \beta_q X^q + \beta_{q-1} X^{q-1} + \cdots + \beta_0 = \beta_q \prod_{j=1}^{q} (X - b_j),
\]

(2.2)

where \( a_i \) are the roots of \( P \) (counting multiplicities) and \( b_j \) are the roots of \( Q \) (counting with multiplicities). The Sylvester matrix associated to \( P \) and \( Q \) is defined as,

\[
\text{Sylv}(P, Q) = \begin{bmatrix}
\alpha_p & \alpha_{p-1} & \cdots & \alpha_0 & 0 & 0 & 0 \\
0 & \alpha_p & \alpha_{p-1} & \cdots & \alpha_1 \alpha_0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\beta_q & \beta_{q-1} & \cdots & \beta_0 & 0 & 0 & 0 \\
0 & \beta_q & \beta_{q-1} & \cdots & \beta_1 \beta_0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
& & & & & \beta_1 & \beta_0
\end{bmatrix}.
\]

(2.3)

The matrix in (2.4), has \( p+q \) columns and \( p+q \) rows, where the first \( q \) rows contain the coefficients \( \alpha_p, \alpha_{p-1}, \cdots, \alpha_0 \) of \( P \) shifted by 0, 1, \cdots, \( q-1 \) and padded with zeros, and the last \( p \) rows contain the coefficients \( \beta_q, \beta_{q-1}, \cdots, \beta_0 \) of \( P \) shifted by 0, 1, \cdots, \( p-1 \) and padded with zeros. In other words, the entry of the Sylvester matrix at \( (m,n) \) equals \( \alpha_{p+m-n} \) if \( 1 \leq m \leq q \) and \( \beta_{m-n} \) if \( q+1 \leq m \leq p+q \), with \( \alpha_m = 0 \) if \( m > p \) or \( m < 0 \) and \( \beta_n = 0 \) if \( m > q \) or \( m < 0 \).

The Resultant of \( P \) and \( Q \) is the the determinant of \( \text{Sylv}(P, Q) \) as,

\[
P,q \text{Res}_X(P, Q) = \det(\text{Sylv}(P, Q)) = \alpha_p^{q} \beta_q^{p} \prod_{i=1}^{p} \prod_{j=1}^{q} (a_i - b_j)
\]

(2.4)

where, \( X \) is the target variable to be eliminated from polynomials \( P \) and \( Q \).
It must be noted that, script suppression is performed occasionally for simplicity of equations.

**Theorem 2.9** ([20, 43, 68]). The resultant in (2.4) vanishes under at least one of the following conditions:

(I) There exists a common zero for $P$ and $Q$,

(II) leading coefficients of both $P$ and $Q$ vanish, $\alpha_p = \beta_q = 0$,

(III) all the coefficients in $P$ vanish, $\alpha_p = \ldots = \alpha_0 = 0$,

(IV) all the coefficients in $Q$ vanish, $\beta_q = \ldots = \beta_0 = 0$.

**Property 2.1** ([43, 68]). Let $P$ and $Q$ be two polynomials. Then $P$ and $Q$ have a common non-trivial factor if and only if $\text{Res}(P, Q) = 0$. Equivalently, $P$ and $Q$ are coprime if and only if $\text{Res}(P, Q) \neq 0$.

**Property 2.2** ([54]). If $P$, $Q$ and $R$ are polynomials with degrees $\deg(P) \leq p$, $\deg(Q) \leq q$, and $\deg(R) \leq r$ then,

$$
\begin{align*}
\text{Res}(P \times Q, R) &= \text{Res}(P, R) \times \text{Res}(Q, R), \\
\text{Res}(P, Q \times R) &= \text{Res}(P, Q) \times \text{Res}(P, R).
\end{align*}
$$

(2.5)

One can utilize this property to simplify large resultant operations by factoring them into two sub-resultant operations.

**Property 2.3** ([54]). Let $P$ and $Q$ be polynomials with degrees $\deg(P) \leq p$ and $\deg(Q) \leq q$. If $p \geq q$ and $R$ is any polynomial with $\deg(R) \leq p - q$, then

$$
\text{Res}(P + RQ, Q) = \text{Res}(P, Q).
$$

(2.6)

Similarly, If $p \leq q$ and $R$ is any polynomial with $\deg(R) \leq q - p$, then

$$
\text{Res}(P, Q + RP) = \text{Res}(P, Q).
$$

(2.7)

One can utilize this property to simplify special resultant operations.

**Definition 2.10** ([43, 68]). Let $P \in \mathbb{R}[X]$, where $\mathbb{R}$ is a real closed field, be a monic polynomial of degree $p$,

$$
P = X^p + \alpha_{p-1}X^{p-1} + \cdots + \alpha_0,
$$

(2.8)
and let $x_1, \ldots, x_p$ be the roots of $P$ in $\mathbb{C}$ (repeated according to their multiplicities). The Discriminant of $P$, $\text{Disc}_X(P)$ is defined by,

$$\text{Disc}_X(P) = \prod_{p \geq i > j \geq 1} (x_i - x_j)^2. \quad (2.9)$$

**Property 2.4** ([6, 68]). $\text{Disc}_X(P) = 0$ if and only if $\deg(\gcd(P(X), \frac{dP(X)}{dX})) > 0$. In the other words, $\text{Disc}_X(P)$ vanishes if and only if $P$ has a multiple root in $\mathbb{C}$.

**Property 2.5** ([68]). When all the roots of $P(X)$ are in $\mathbb{R}$ and distinct, $\text{Disc}_X(P) > 0$.

### 2.3 Case Study: Multivariate Polynomials Optimization

Here, we implement resultant properties in an optimization problem.

* Given two polynomials,

\[
\begin{align*}
P_1(x, y, z) &= xyz + 2xy + y + x + 3, \\
P_2(x, y, z) &= xyz + z^2 + 3yz + 1.
\end{align*}
\]

find all the extrema in $z \in \mathbb{R}$ satisfying $P_1(x, y, z) = 0$ and $P_2(x, y, z) = 0$ in $(x, y) \in \mathbb{R}^2$.

**Solution:** We start with the Resultant elimination. When

$$P_1(x, y, z) = 0 \quad \text{and} \quad P_2(x, y, z) = 0$$

have a common root $(\tilde{x}, \tilde{y}, \tilde{z})$ then the Resultant of $P_1(x, y, z)$ and $P_2(x, y, z)$ eliminating $x$ will vanish at $(\tilde{y}, \tilde{z})$, but the converse does not always hold. Therefore, for $P_1(x, y, z) = 0$ and $P_2(x, y, z) = 0$ to have common roots, it is necessary that the following Resultant holds,

$$\Phi_1(y, z) := R_x(P_1, P_2) = \begin{vmatrix} yz + 2y + 1 \\ yz \\ 3yz + z^2 + 1 \end{vmatrix} = z^3y + (1 + 3y^2 + 2y)z^2 + (y + 5y^2)z + 2y + 1 = 0.$$

Notice that all the common solutions of $P_1(x, y, z) = 0$ and $P_2(x, y, z) = 0$ are also the solutions of $\Phi_1(y, z) = 0$, hence, when one explores if $z$ possesses at least one extremum in $(x, y)$ domain, this can be done alternatively by studying $\Phi_1(y, z)$. If $z$ exhibits an
extremum, it is necessary that \( \frac{\partial z}{\partial y} = 0 \), which can be found from,

\[
\frac{\partial \Phi_1(y, z)}{\partial y} + \frac{\partial \Phi_1(y, z)}{\partial z} \frac{\partial z}{\partial y} = 0.
\]

whether or not \( \frac{\partial \Phi_1(y, z)}{\partial z} \) vanishes, one should now search for the common \( y \in \mathbb{R} \) roots of \( \Phi_1(y, z) = 0 \) and \( \frac{\partial \Phi_1(y, z)}{\partial y} = 0 \), which is necessary for an extremum to exist. If such common \( y \in \mathbb{R} \) solutions exist, they may correspond to extrema of \( z \) satisfying the original polynomials \( P_1(x, y, z) = 0 \) and \( P_2(x, y, z) = 0 \) as well as \( \Phi_1(y, z) = 0 \) and \( \frac{\partial \Phi_1(y, z)}{\partial y} = 0 \). To find those common roots, one needs to implement another Resultant elimination; this time between \( \Phi_1(y, z) = 0 \) and \( \frac{\partial \Phi_1(y, z)}{\partial y} = 0 \), by eliminating the variable \( y \).

It yields

\[
\Phi_2(z) := Ry\left( \Phi_1(y, z), \frac{\partial \Phi_1(y, z)}{\partial y} \right) = \begin{vmatrix}
3z^2+5z^3+2z^2+z+2 & z^2+1 \\
6z^2+10z & 0 \\
6z^2+10z & z^3+2z^2+z+2 \\
3z^8-17z^7-2z^6+66z^5+69z^4+63z^3+68z^2-20z & 0
\end{vmatrix} = 0.
\]

We now have a polynomial only in terms of the variable \( z \). Zeros of \( \Phi_2(z) \) must be investigated to explore the admissible solutions, if any. For this, two steps are performed:

(i) ignore the non-real solutions of \( \Phi_2(z) = 0 \), and

(ii) find all the real solutions that satisfy all the polynomials in the Resultant operations including solutions at infinity due to vanishing of the leading coefficients of those polynomials.

Here the solution set of real zeros of \( \Phi_2(z) \) is given as,

\[
\tilde{z} = \{ 0, \ -1.6667, \ 2.1823, \ 0.2301, \ -1.6854, \ -4.7269 \}.
\]

By back substitution of each element in \( \tilde{z} \) into \( \Phi_1(y, z) = 0 \) and \( \frac{\partial \Phi_1(y, z)}{\partial y} = 0 \), we obtain all the real \( \tilde{y} \) common solutions, and use \((\tilde{z}, \tilde{y})\) pairs to seek if \( P_1(x, y, z) = 0 \) and \( P_2(x, y, z) = 0 \) have common \( \tilde{x} \) solutions. Notice that this can be effectively performed since at each step, one deals with a single variable polynomial. If \((x, y, z)\) triplets satisfying all the polynomials \( P_1(x, y, z) = 0 \), \( P_2(x, y, z) = 0 \), \( \Phi_1(y, z) = 0 \) and \( \frac{\partial \Phi_1(y, z)}{\partial y} = 0 \) can be found, these triplets correspond to some critical points, including the extrema points, among the common roots of \( P_1(x, y, z) = 0 \) and \( P_2(x, y, z) = 0 \). We present all such common solutions \((\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3 \) in Table 2.1.

The optimization approach explained above establishes the foundation of the delay-independent stability analysis [27], and will be the starting point of this dissertation.
Table 2.1: The critical solutions satisfying $P_1(x,y,z) = 0$, $P_2(x,y,z) = 0$, $\Phi_1(y,z) = 0$, and $\frac{\partial \Phi_1(y,z)}{\partial y} = 0$ polynomials used to construct successive resultants in the case study.

| $\tilde{z}$ | $\Phi_1(y,z)|_{\tilde{z}} = 0$ | $\frac{\partial \Phi_1(y,z)}{\partial y}|_{\tilde{z}} = 0$ | $\tilde{\gamma}$ | $P_1(x,y,\tilde{z})|_{(\tilde{z},\tilde{\gamma})} = 0$ | $P_2(x,y,\tilde{z})|_{(\tilde{z},\tilde{\gamma})} = 0$ | $\tilde{x}$ |
|---|---|---|---|---|---|---|
| 0 | $(0)y^2 + 2y + 1 = 0$ | $(0)y + 2 = 0$ | $\infty$ | $x\tilde{z} + 2x + 1 = 0$ | $x\tilde{z} + 3\tilde{z} = 0$ | $-0.5$ |
| -1.6667 | $(0)y^2 + 1.2592y + 3.7777 = 0$ | $(0)y + 1.2592 = 0$ | $\infty$ | $x\tilde{z} + 2x + 1 = 0$ | $x\tilde{z} + 3\tilde{z} = 0$ | 2 |
| 2.1823 | 25.1991$y^2 + 24.1006y + 5.7625 = 0$ | $50.3981y + 24.1006 = 0$ | -0.4782 | $-x + 2.5218 = 0$ | $-1.0436x + 2.6317 = 0$ | 2.5218 |
| 0.23006 | 1.3091$y^2 + 2.3481y + 1.0529 = 0$ | $2.6182y + 2.3481 = 0$ | -0.8968 | $-x + 2.1032 = 0$ | $-0.2063x + 0.4339 = 0$ | 2.1031 |
| -1.6854 | 0.0949$y^2 + 1.2081y + 3.8407 = 0$ | $0.1899y + 1.2081 = 0$ | -6.3584 | $-x - 3.3584 = 0$ | $10.7168x + 35.9911 = 0$ | -3.3584 |
| -4.7269 | 43.3968$y^2 - 63.6565y + 23.3438 = 0$ | $66.7306y - 63.6565 = 0$ | 0.7334 | $-x + 3.7334 = 0$ | $-3.4968x + 12.9432 = 0$ | 3.7334 |
Chapter 3

DIS Synthesis/Design for LTI-TDS with Single Delay

This chapter presents algebraic methods to design controllers for a class of linear time-invariant systems with single output delay. The proposed methods identify the regions in the controller gain space within which the system at hand is delay-independent stable (DIS). While achieving these, the methods satisfy necessary and sufficient conditions of stability. Numerical examples illustrate the approaches and their effectiveness.

3.1 Introduction

There still remains unresolved and important problems in the literature in control synthesis which is associated with rendering a system DIS, where structured control design is desired and one seeks the set of controller gains with which infinitely many eigenvalues of the closed-loop system are stable, no matter how large or small the delays are. In this problem, considering the particular structure of the controller is critical since in many practical problems, controllers have specific structures. As evidenced in the literature [66], FDT naturally allows incorporating the controller structure, and this is what we follow here as well. Addressing this problem is challenging and open, and so far no work could express the boundaries of the feasible gain regions using implicit/explicit algebraic functions of the controller gains. Within Chapter 3, we develop an algebraic approach based on FDT, algebraic geometry, and elimination techniques in order to reveal some polynomials, roots of which indicate DIS property of a LTI system with a single output
DIS Synthesis/Design for LTI-TDS with Single Delay

delay. Next, these polynomials are connected with the results of Descartes rule of signs and Sturm sequences in order to systematically derive explicit representations of the controller-gain regions where the system remains DIS. While achieving control design for DIS, we also incorporate the specific structure of the controller matrices into the algebraic approach developed here, thanks to the FDT framework. Finally, it is noted that utilization of FDT also leads to necessary and sufficient stability conditions. For this reason, conservative time-domain approaches are left outside the scope of this work.

The chapter is organized as follows. In Section 3.2, problem formulation and some preliminaries are given. Section 3.3 presents methods on controller gains space categorization for DIS, and case studies are presented in Section 3.4.

3.2 Problem Formulation and Stability Analysis

Here we present LTI system with a single-delay. The control problem of the following general class multi-input multi-output LTI plant with output delay \( \tau \geq 0 \) is shown as,

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t), \\
u(t) &= Ky(t - \tau),
\end{align*}
\]  

(3.1)

where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times q} \) are the constant system and control matrices, respectively, \( C \in \mathbb{R}^{q \times n} \) is the output matrix, \( x(t) \in \mathbb{R}^n \) is the system state vector, \( y(t) \in \mathbb{R}^q \) is the output vector, \( u(t) \in \mathbb{R}^q \) is the control input to the plant, the entries of \( r(t) \in \mathbb{R}^q \) are taken as step inputs to be tracked by the entries of \( y(t) \) and \( K = \{k_i\}_{i=1}^z \in \mathbb{R}^i \) is the plant control law, where \( \{k_i\}_{i=1}^z \in \mathbb{R} \) are the controller gains of the plant. In (3.1), only \( y(t - \tau) \) is available as measurements and for use in control design. The objective here is to reveal \( \{k_i\}_{i=1}^z \) control gains that make (3.1) delay-independent stable, without imposing any conservatism in the stability analysis. One advantage of this method is for cases in practice that the designer’s hands are not open in manipulating the control structure of a given system and we want to assure the DIS of a system with only changing the entries of the system matrices.

Let us note that the number of distinct delays is not a limitation of the developed theories, but we decided to included them in a separate chapter to be able to separate specific properties of single delay systems here. Systems with multiple delays require addendum theories and propositions to manifest DIS property.
The main idea here is to start with the stability and use it to design the controller gains. Stability of (3.1) is studied over its characteristic equation given by

\[ f(s, \tau, \{k_i\}_{i=1}^z) = |sI - A - BKCe^{-\tau s}| = 0, \quad (3.2) \]

where \( \{k_i\}_{i=1}^z \) and delay \( \tau \) are unknown parameters. In (3.2), \( I \) is the \( n \times n \) identity matrix, \( s \) is the Laplace variable, and \( \tau \) appears in the exponent in Laplace domain. The characteristic equation (3.2) can also be written as

\[ f(s, \tau, \{k_i\}_{i=1}^z) = \sum_{r=0}^{c} P_r(s, \{k_i\}_{i=1}^z)e^{-r\tau s} = 0, \quad (3.3) \]

where \( P_r \) are polynomials in terms of \( s \) and the entries of \( K \), and \( c \) represents the order of commensurability of \( \tau \), which can be calculated as \( c = \text{rank}(BK) \) [80]. Furthermore, since the highest derivative of the system states in (3.1) is not influenced by \( \tau \), the largest power of \( s \) in (3.3) does not multiply any exponential terms. To account for this condition, \( P_0 \) has the largest power of \( s \) in (3.3).

Due to the fact that the highest-order derivative of the state is not influenced by delays, the infinitely many roots of the characteristic equation (3.3) exhibit continuity on the complex plain with respect to the delay parameter \( \tau \) [26]. The continuity property can thus be used in the stability analysis. It requires the detection of critical values \( \tau^* \) for which at least one characteristic root of (3.3) lies on the imaginary axis, \( s = j\omega \), where \( \omega \geq 0 \) without loss of generality [7]. For finding the stability transitions of the system with respect to \( \tau \), the movement of \( s = j\omega \) across the imaginary axis should be checked as \( \tau \) infinitesimally increases. Roots crossing the imaginary axis from left to right (or from right to left) favor instability (or stability). This information can be used to account for the unstable roots of the system, leading to the stability properties of the system with respect to \( \tau \) [66].

Solving \( s = j\omega \) roots from the transcendental characteristic equation (3.3) is however cumbersome [7], yet several enabling manipulations can be done [110]. For instance, we can convert the infinite dimensional characteristic equation (3.3) to a finite dimensional characteristic equation that has continuous coefficients. This conversion does not lose the infinite dimensional nature of the problem, and can be done via the exact Rekasius transformation [88],

\[ e^{-\tau s} \equiv \frac{1 - Ts}{1 + Ts}, \quad T \in \mathbb{R}, \quad s = j\omega, \quad \omega \geq 0, \quad (3.4) \]
which has a back-transformation constraint equation found by developing the phase conditions in (3.4), see [80]. It maps \((T, \omega)\) domain to \(\tau\) domain as

\[
\tau = \frac{2}{\omega} \left[ \arctan(\omega T) \pm L\pi \right],
\]

where \(L = 0, 1, 2, \ldots\), and \(0 \leq \arctan(\cdot) < \pi\). With the Rekasius substitution of (3.4) into (3.3), and with a manipulation to remove the fractions, we obtain the transformed characteristic equation, which is valid only on the imaginary axis \((s = j\omega)\) and given by

\[
g(j\omega, T, \{k_i\}_{i=1}^c) = \left( f \left| e^{-j\omega\tau - \frac{1-T\omega^2}{1+T\omega}} \right| (1 + Tj\omega)^c \right) = 0,
\]

where \(g\) is an algebraic polynomial.

Remark 3.1. For delay-independent stability of (3.1), it is necessary that the uncontrolled system and the delay-free controlled system are Hurwitz stable [13, 66]. This automatically guarantees that \(\omega = 0\) cannot be a feasible solution of the characteristic equation for both \(\tau = 0\) and any finite \(\tau\). In this case, \(\omega = 0\) solutions can be ignored, which technically corresponds to weak DIS analysis. If strong DIS analysis is pursued, one should also guarantee that \(\omega = 0\) cannot be a solution for \(\tau \to \infty\).

### 3.3 Controller Space Categorization Methods

In this section, algebraic approaches are developed to locate the controller gains \(\{k_i\}_{i=1}^c\), with which the system in (3.1) is delay-independent stable. For this objective, we first connect the transformed equation (3.6) to the results of Descartes. Recall that the imaginary spectrum of (3.1) is converted identically to \(s = j\omega\) roots of \(g\) in (3.6). This is the starting point, where \(g\) is as follows

\[
g(\omega, T, \{k_i\}_{i=1}^c) = g_R(\omega, T, \{k_i\}_{i=1}^c) + jg_\Im(\omega, T, \{k_i\}_{i=1}^c) = 0,
\]

with \(g_R\) and \(g_\Im\) respectively being the real and imaginary parts of \(g\), which are given by

\[
g_R(\omega, T, \{k_i\}_{i=1}^c) = \sum_{r=0}^c \alpha_r(\omega, \{k_i\}_{i=1}^c) T^r,
\]

\[
g_\Im(\omega, T, \{k_i\}_{i=1}^c) = \sum_{r=0}^c \beta_r(\omega, \{k_i\}_{i=1}^c) T^r.
\]
For DIS analysis, one should guarantee that (3.8) and (3.9) do not have any common 
\( (\omega, T) \in \mathbb{R}_+ \times \mathbb{R} \) roots. This is obviously not a trivial task, inspecting equations (3.8)-(3.9) and (3.4)-(3.5).

Let us first investigate whether or not one can reveal that (3.8)-(3.9) have common \( (\omega, T) \) solutions. In order to solve \( (\omega, T) \) pairs from (3.7), one should guarantee that 
\[
\mathcal{R} \mathcal{I} = 0, \\
\mathcal{I} \mathcal{I} = 0,
\]
concurrently. At this step, we can eliminate \( T \) from these two equations to simplify the analysis. The elimination can be done by using the resultant theory [43].

A 2c-order Sylvester matrix, \( S \), is constructed by using the coefficients of \( T \) in (3.8) and (3.9). In other words, the unknowns in \( S \) are \( \omega \) and \( \{k_i\}_{i=1}^{z} \). For the common \( \omega \) solutions of (3.8) and (3.9), Sylvester matrix \( S \) is singular (but not vice versa). That is, the resultant \( R_T(\omega, \{k_i\}_{i=1}^{z}) \), which is the determinant of \( S \), should be studied for its zeros,

\[
\det(S) = R_T(\omega, \{k_i\}_{i=1}^{z}) = \begin{vmatrix}
\alpha_c & \alpha_{c-1} & \ldots & \alpha_0 & 0 & 0 \\
0 & \alpha_c & \alpha_{c-1} & \ldots & \alpha_1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \beta_c & \beta_{c-1} & \ldots & \beta_1 & \beta_0 \\
0 & 0 & 0 & \beta_c & \beta_{c-1} & \ldots & \beta_1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{vmatrix}.
\] (3.10)

**Theorem 3.2.** [30, 104] The system in (3.1) possesses at least one pole on the imaginary axis for delay \( \tau \) if, for some \( \omega = \omega^* \in \mathbb{R} \) and \( T \in \mathbb{R} \), the following conditions are satisfied:

A) The resultant, \( R_T(\omega) \), vanishes, for known controller gains.

B) The transformed characteristic equation (3.7) is satisfied. i.e., \( g_\mathcal{R} = 0, g_3 = 0 \).

See [30, 104] for the treatment of singular points, which does not change the essence of Theorem 3.2.

By inspection of the product formula of the resultant, \( R_T \) is a polynomial in terms of the coefficients of \( T \) in \( g_\mathcal{R} \) and \( g_3 \). With respect to \( \omega \), \( g_\mathcal{R} \) is an even polynomial, and \( g_3 \) is an even polynomial multiplied by a factor of \( \omega \). Using the resultant formula,

\[
R_T(\omega, \{k_i\}_{i=1}^{z}) = (\alpha_c)^c \prod_{u=1}^{c} g_3(\delta_u),
\] (3.11)

where \( \delta_u \) are the zeros of \( g_\mathcal{R} \), one can see that the resultant yields an even polynomial with respect to \( \omega \), excluding \( \omega = 0 \) roots, which can be separately studied as per Remark 3.1 [30, 104]. Hence, \( R_T \) is an even polynomial in \( \omega \) with real coefficients. Therefore, after eliminating \( T \) and omitting \( \omega = 0 \) roots, the resultant becomes,

\[
R_T(\omega, \{k_i\}_{i=1}^{z}) = D(\omega^2) = \sum_{h} \gamma_h(\{k_i\}_{i=1}^{z})\omega^{2h}.
\] (3.12)
We can express $R_T$ by defining a new variable $y$ with $y = \omega^2$, where the number of real zeros of $R_T$ is equal to the number of positive zeros of the new polynomial $\Phi(y, \{k_i\}_{i=1}^z)$. By inspection of the existence of the positive real roots of $\Phi(y, \{k_i\}_{i=1}^z) = 0$ for a given $\{k_i\}_{i=1}^z$, one can have ample information about stability. If no roots of $\Phi = 0$ are positive real, this implies that no admissible $y = \omega^2 > 0$ real roots exist, hence a stability switch is impossible. If, by construction, the DIS conditions in Remark 3.1 also hold, we can then conclude that the system remains stable independent of the amount of delay $\tau$.

Presence of at least one positive real zero of $\Phi(y, \{k_i\}_{i=1}^z)$, however, requires checking whether $T \in \mathbb{R}$ solution exists satisfying (3.7). If such a $T$ does not exist, then we can still declare delay-independent stability. If such a $T$ exists, then delay-independent stability can not be rendered. In this case, the system can at best be stable within finite number of intervals along the delay axis. The boundaries of these intervals can be detected using (3.5)[80]. The problems related to finding the stability of delayed systems within a certain range are also known as Delay-Dependent Stability (DDS) and are out of the scope of this dissertation.

Descartes rule of signs and Sturm sequences are two methods that can assess the number of positive real roots of a polynomial with real coefficients, without solving for these roots. The following theorems establish the connections between these methods and the resultant to construct the necessary and sufficient conditions for delay-independent stability of system (3.1).

**Property 3.1 (DIS Property [65]).** Given $\{k_i\}_{i=1}^z$, system in (3.1) is delay-independent stable if and only if (3.3) has no roots on $\mathbb{C}_{0+}$ for all delay values. That is,

$$ f(s, \{\tau^L\}_{\ell=1}^L, \{k_i\}_{i=1}^z) \neq 0, \quad (3.13) $$

in $\{\tau^L\}_{\ell=1}^L \in \mathbb{R}_{0+}$ for $\forall s \in \mathbb{C}_{0+}$.

**Theorem 3.3.** There exists at least one control law $K$ with $\{k_i\}_{i=1}^z \neq 0$ that makes the system (3.1) delay-independent stable if and only if $A$ is Hurwitz, and $(A, BKC)$ is a controllable pair.

**Proof.** Let $K = 0$. In this case, we have $\dot{x}(t) = Ax(t) + BNr(t)$. Under this condition, system (3.1) is Hurwitz stable, if and only if $A$ is Hurwitz stable. The critical observation is that (3.1) is Hurwitz stable for all $\tau \geq 0$. This is possible only if $A$ is Hurwitz stable since delay effects vanish for $\{k_i\}_{i=1}^z = 0$. In other words, the point $\{k_i\}_{i=1}^z = 0$ lies in the DIS region in the parameter space of the controller gains. It is then straightforward
to show from the continuity of the characteristic roots of the system [26] and from controllability of the delay-free system [76] that there exist controller gains \( \{k_i \}_{i=1}^{z} \) such that the system (3.1) remains DIS. On the contrary, if \( A \) is not Hurwitz stable, DIS is impossible as per Remark 3.1.

**Theorem 3.4.** LTI-TDS in (3.1) is DIS in the delay parameter space \( \tau \) with controller gain \( K \), if

i) \( |sI - A| \) is Hurwitz stable,

ii) \( |sI - A - BKC| \) is Hurwitz stable,

iii) \( \text{var}_D(\Phi) = 0 \).

**Proof.** For the system (3.1) to be delay-independent stable, it is necessary that the uncontrolled system and the delay-free controlled system (\( \tau = 0 \)) are Hurwitz stable as per Remark 3.1. These conditions correspond to (i)-(ii) of the theorem. Furthermore, a system can never change its stability/instability with respect to delay \( \tau \) if it does not possess any characteristic roots on the imaginary axis for any \( \tau \). That is, \( s = j\omega \), \( \omega \in \mathbb{R} \), should not be a root of the corresponding characteristic equation \( f \). Based on the derivations above, \( s = j\omega \) root of \( f \) does not exist if \( y = \omega^2 > 0 \) roots of \( \Phi = 0 \) do not exist, which is guaranteed by (iii) of Theorem 3.4 as per Descartes rule of signs. \( \Box \)

Note that Theorem 3.4 states the weak DIS conditions, which can be extended to strong DIS by analyzing \( \omega \to 0 \) with \( \tau \to \infty \) in \( f \), see e.g. [111]. This extension is also discussed in Theorem 3.5 and thus suppressed here. Furthermore, application of Descartes rule of signs leads to sufficient conditions of weak delay-independent stability regions in the \( \{k_i \}_{i=1}^{z} \) gain space since this rule does not detect all \( \{k_i \}_{i=1}^{z} \) with which positive real roots of \( \Phi(y) = 0 \) do not exist. Consequently, this rule can reveal only a portion of the DIS regions in the controller gain space. On the other hand, implementation of the rule is practical; it requires only checking the signs of the coefficients \( \gamma_b(\{k_i \}_{i=1}^{z}) \) of \( \Phi \) in the gain space thereby allowing an algebraic way of implicitly expressing DIS regions in terms of only the controller gains \( \{k_i \}_{i=1}^{z} \).

### 3.3.1 Controller Space Categorization Using Sturm Sequences

Using Sturm sequences instead, we can find the DIS regions with less conservatism. Application of Sturm Sequence requires care, since \( R_T = 0 \) is only a necessary condition for (3.8) and (3.9) to have common roots.
Theorem 3.5. LTI-TDS in (3.1) is DIS in the delay parameter space $\tau$ with given controller gain $K$, if and only if

i) $|sI - A|$ is Hurwitz stable,

ii) $|sI - A - BK|$ is Hurwitz stable,

iii)-(a) $\text{var}_s(\Phi(0)) - \text{var}_s(\Phi(\infty)) = 0$,

iii)-(b) $\text{var}_s(\Phi(0)) - \text{var}_s(\Phi(\infty)) \neq 0$ for some $\omega = \omega^* \in \mathbb{R}_+$ where $(\Re(\omega^*))^2 + (g_3(\omega^*))^2 \neq 0$, $\forall T \in \mathbb{R} \cup \{\infty\}$,

iii)-(c) $\lim_{\omega \to 0 \ T \to \infty} \left[ (\Re) + (g_3) \right] \neq 0$, with $(\omega, T) \in \mathbb{R} \cup \{\infty\}$

Proof. Condition (i)-(ii) are obvious from Theorem 3.4. Furthermore a system can never change its stability/instability behavior with respect to delay $\tau$ if it does not possess any characteristic roots on the imaginary axis. That is, $s = j\omega$ should not be a root of (3.7).

Based on the derivations above, this is equivalent to guaranteeing three conditions; (a) polynomial $\Phi(y) = 0$ does not have $y = \omega^2 \in \mathbb{R}_+$ roots, which is guaranteed by (iii)-a. (b) Moreover, $\Phi(y) = 0$ may have roots $\omega = \omega^* \in \mathbb{R}_+$ that do not correspond to $T \in \mathbb{R} \cup \{\infty\}$ roots of (3.8)-(3.9). In such cases, condition (iii)-b holds. (c) Finally, for strong DIS, $\omega \to 0$ with corresponding $\tau \to \infty$ should be avoided. In $T$ domain, this corresponds to $\omega \to 0$ and $T \to \infty$ while the multiplication $(\omega, T)$ is a real number or goes to infinity [111]. Condition (iii)-c prevents this possibility. Therefore, conditions (i)-(iii) guarantee, with necessary and sufficient conditions, that the system (3.1) never switches its stability to instability for any finite or infinite delay value.

Notice that Theorem 3.5 captures the strong DIS conditions, with necessary and sufficient conditions, by covering all imaginary axis crossings for all $\tau$ including $\tau \to \infty$.

For weak DIS analysis with sufficient conditions, one can disregard (iii)-b and (iii)-c. This would still lead to less conservative conditions compared with Theorem 3.4, since Theorem 3.5 utilizes Sturm sequences.

Remark 3.6. Given $A$, $B$, $C$, and $K$, delay-independent stability test can be completed by checking only the conditions (ii)-(iii) of Theorems 3.4-3.5, since these conditions guarantee that condition (i) must hold [104]. It is noted however that in the design process of $K$, one needs to consider condition (i), hence the authors decide to include condition (i) in Theorems 3.4-3.5.

Theorem 3.3 can be extended to cases where the controller is structured, that is, some $\{k_i\}_{i=1}^z$ are non-zero. Assume that for some $\{k_i\}_{i=1}^z$, $\{k_i\}_{i=1}^z \neq 0$, i.e., we have $K = K_0 \neq 0$. In this case, one replaces $A$ being Hurwitz condition in Theorem 3.3 with $\dot{x}(t) = \ldots$. 
\( \mathbf{A}x(t) + \mathbf{B}K_0 \mathbf{C}x(t - \tau) \) being delay-independent stable, and check using Theorem 3.5 if this system with \( K_0 \) is delay-independent stable. If it is, then the system remains DIS for controller gains \( \{k_i\}_{i=1}^{\infty} + \{\epsilon\}_{i=1}^{\infty}, \|\epsilon\|_{i=1} \ll 0 \).

We now illustrate our weak DIS controller design step-by-step, see Figure 3.1. Our method is broken down into four major steps, namely, algebraic mapping, checking the DIS necessary conditions, elimination procedure, and weak/strong DIS conditions using Sturm sequences. At the algebraic mapping step, after the extraction of the characteristic equation (3.3), and using exact Rekasius substitution (3.4), we transform the transcendental equation into an algebraic one as in (3.6). At the second step, DIS necessary conditions are checked as per Remark 3.1. At step 3, utilizing the resultant technique, we are able to search for the existence of common roots of real and imaginary parts of (3.7). Using admissible controller gains that satisfy the DIS necessary conditions in Remark 3.1, \( \hat{k}_i \), we proceed to the final step. Substitution of the admissible controller gains into \( R_T \) allows us to seek for the roots of \( \Phi = 0 \) in terms of \( y = \omega^2 \). At the last step, we can categorize the admissible controller gains. Sturm sequences is used to seek if \( \Phi(y) \) has any positive real zeros. These roots may correspond to common roots of the real and imaginary parts of (3.7). If no such roots exist, DIS is guaranteed for the particular controller gains. If such roots exist, then one must check whether or not the condition (iii)-b in Theorem 3.5 holds. It must be noted that the flowchart in Figure 3.1 is for weak DIS analysis, with necessary and sufficient conditions. For strong DIS analysis, one must consider condition (iii)-c in Theorem 3.5 as well.

### 3.3.2 Analytical Boundaries Responsible for Stability Switching in Control-Gain Space

In this section, we show under certain conditions that DIS boundaries in controller gain space \( \{k_i\}_{i=1}^{\infty} \) can be identified. We start with the following proposition.

**Proposition 3.7.** Let (3.11) be rewritten as,

\[
\Phi(y, \{k_i\}_{i=1}^{\infty}) = \sum_{\ell=0}^{L} \gamma_{\ell}(\{k_i\}_{i=1}^{\infty})y^{\ell} = 0. \tag{3.14}
\]

where \( \gamma_{\ell} \) are constants in \( \mathbb{R} \). If, on the complex \( y \) plane and with respect to the parameter \( \{k_i\}_{i=1}^{\infty} \), there exists any crossing of the root loci of \( \Phi(y, \{k_i\}_{i=1}^{\infty}) = 0 \) with the positive real axis, this crossing occurs either (i) with at least one double positive real root of
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Figure 3.1: DIS controller design chart (necessary and sufficient conditions) from Theorem 3.5.
\( \Phi(y, \{k_i\}_{i=1}^z) = 0 \), or (ii) with at least one zero root of \( \Phi(y, \{k_i\}_{i=1}^z) = 0 \) at the origin of the y-plane.

The proof follows from the behavior of the roots of polynomials with real coefficients.

According to Property 3.1, Definition 2.10, and in light of Proposition 3.7, we are now able to state that if double roots of \( \Phi(y, \{k_i\}_{i=1}^z) \) exist for some \( \{k_i\}_{i=1}^z \in \mathbb{R} \), then the discriminant of \( \Phi(y, \{k_i\}_{i=1}^z) = 0 \) with eliminating \( y \) should vanish,

\[
\varphi(\{k_i\}_{i=1}^z) = \text{Discrim}_y(\Phi(y, \{k_i\}_{i=1}^z)) = 0. \tag{3.15}
\]

This formulation offers a way to investigate case (i) of Proposition 3.7. In other words, for a given \( \{k_i\}_{i=1}^z \), the roots of \( \varphi(\{k_i\}_{i=1}^z) = 0 \) can be double negative/positive real and/or double complex conjugate.

As per Proposition 3.7, the cases that yield a non-DIS system are when there exist either positive real roots \( y^* > 0 \) of (3.15) (for case (i) of Proposition 3.7), or \( \gamma_0(\{k_i\}_{i=1}^z) = 0 \) (for case (ii) of Proposition 3.7), and if, in each case, the corresponding \( \omega = \sqrt{y^*} > 0 \) and \( \{T^*_\ell\}_{\ell=1}^L \) satisfy the characteristic equation in (3.3). Otherwise, for a given \( \{k_i\}_{i=1}^z \) we may have a DIS property inside the regions bordered by the hypersurfaces defined by \( \varphi(\{k_i\}_{i=1}^z) = 0 \). Hence, one has to inspect all these enclosed regions arising due to the boundaries \( \varphi(\{k_i\}_{i=1}^z) = 0 \) and \( \gamma_0(\{k_i\}_{i=1}^z) = 0 \), and identify the segments of those boundaries encapsulating the DIS regions. Let us denote these refined segments with \( \Lambda(\{k_i\}_{i=1}^z) \), which is obviously a subset of \( \Gamma(\{k_i\}_{i=1}^z) = \{\{k_i\}_{i=1}^z \in \mathbb{R} | \varphi(\{k_i\}_{i=1}^z) = 0 \cup \gamma_0(\{k_i\}_{i=1}^z) = 0\} \), which is the union of all the hypersurfaces along which the system can be potentially switching from DIS to non-DIS property in the controller gain space. To identify all the DIS regions, one has to inspect all the arising enclosed regions in \( k_i, \) denoted by \( S_\chi, \chi = 1, \ldots, \kappa, \) and encapsulated by the boundaries of \( \Gamma(\{k_i\}_{i=1}^z) \).

**Theorem 3.8.** System in (3.1) is delay-independent stable (DIS) in the region \( S_\chi, \chi = 1, 2, \ldots, \theta, \theta < \kappa, \) if for a test point \( K =: \{k_i\}_{i=1}^z \in S_\chi, \) the following two conditions simultaneously hold:

i) \( |sI - A - BKC| \) is Hurwitz stable,

ii) Single point inspection in \( S_\chi \) when \( \{y > 0 | \Phi(y, \{k_i\}_{i=1}^z) = 0\} = \emptyset, \)

**Proof.** Condition (i) is obvious from Theorem 3.5. Furthermore, system (3.1) can never change its stability/instability behavior with respect to delay \( \tau_\ell \) if (3.3) does not possess
any roots on the imaginary axis. That is, $s = j\omega$ should not be a root of (3.7). Based on the features of the polynomial in (3.14), this can be guaranteed if the polynomial $\Phi(y, k_i) = 0$ does not have $y = \omega^2 \in \mathbb{R}_+$ roots for a single test point in each identified closed region $S_\chi$, $\chi = 1, 2, \ldots, \theta$. By continuity, this property is guaranteed to hold for all the points in the region enclosed by some parts of $\Gamma(\{k_i\}_{i=1}^z)$. The condition (ii) of theorem thus reveals the fundamental DIS regions $\{S_\chi\}_\chi^{\theta}$ in the controller gain space.

**Theorem 3.9.** System in (3.1) is delay-independent stable (DIS) in $\{k_i\}_{i=1}^z \in \mathbb{R}^z \backslash \{S_\chi\}_\chi^{\theta}$, if and only if the conditions below are satisfied:

i) $|sI - A - BKC|$ is Hurwitz stable,

ii) Point-wise sweep of controller gains in $\{k_i\}_{i=1}^z \in \mathbb{R}^z \backslash \{S_\chi\}_\chi^{\theta}$ whenever $\{y > 0 | \Phi(y, \{k_i\}_{i=1}^z) = 0\} \neq \emptyset$ for some $y^* = (\omega^*)^2 \in \mathbb{R}_+$, and when at least one condition in (B)-(C) of Theorem 3.2 does not hold.

**Proof.** The condition provided in this theorem guarantees the only case for which $\Phi(y, k_i) = 0$ have roots $y = (\omega^*)^2 \in \mathbb{R}_+$, but these roots do not correspond to a root in $T_\ell \in \mathbb{R} \cup \{-\infty\}$ that vanishes (3.8) and (3.9) simultaneously. This inspection however requires point-wise sweep in all the remaining regions in $\{k_i\}_{i=1}^z \in \mathbb{R}^z \backslash \{S_\chi\}_\chi^{\theta}$. Hence, along with the conditions (i)-(ii) of Theorem 3.8, all the DIS regions of system (3.1) can be identified.

**Remark 3.10.** Checking the DIS properties in the regions enclosed by $\varphi(\{k_i\}_{i=1}^z) = 0$ and $\gamma_0(\{k_i\}_{i=1}^z) = 0$ requires a simple single-point test in each arising enclosing region, as per Theorem 3.8. With this, some of the DIS regions can be identified and thus their boundaries $\Lambda(\{k_i\}_{i=1}^z)$ can be revealed. However, when a positive real root $y$ of $\Phi(y, k_i) = 0$ exists in a region $S_\chi$, then single-point test is not sufficient in this region since it is not possible to claim that all the $T_\ell$ corresponding to all $k_i$ in this region can remain in either real domain or complex domain. Hence, the point-wise scanning in such regions as covered by Theorem 3.9 is still necessary.

### 3.4 Case studies

We present examples to demonstrate the application of the developed tools to design delay-independent stable controllers in some case studies.
3.4.1 DIS Control For Set-point Tracking/Regulation

Consider the a system with representation in state space as,

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t), \\
u(t) &= Ky(t - \tau) + \tilde{N}r,
\end{align*}
\]

(3.16)

where \(r \in \mathbb{R}\) is the reference and \(\tilde{N} \in \mathbb{R}\) shape the reference to minimize the steady state error to a negligible value and it can be calculated by finding the DC gain of the closed loop system. Knowing that,

\[
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & -8 & -2 \\ 0 & 0 & -k_1 & -2 \\ 0 & 0 & -9 & -k_2 \end{bmatrix}, \quad BKC = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k_1 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 \end{bmatrix},
\]

(3.17)

one can find the characteristic equation as below,

\[
f(s, \tau, k_1, k_2) = (s^4 + (6 + k_1)s^3 + (k_2 + 6k_1 + 18)s^2 + (2k_2 + 22 + 9k_1)s + k_2 + 4k_1 + 9) \\
+ (8k_2 + (2k_2 + 8k_1)s + 14k_1)e^{-\tau s} = 0,
\]

It must be noted that the stability analysis does not have anything to do with the reference \(r\) and the shaper \(\tilde{N}\) as it is the non-homogeneous part of the differential equation and hence can be neglected for the DIS stability analysis. After the Rekasius substitution (3.4) into (3.18), and some algebraic manipulations, the real and imaginary parts of the transformed characteristic equation at \(s = j\omega\) are found as

\[
g_R(\omega, T, k_1, k_2) = \left( \sum_{\ell=2}^{4} a_{1\ell}(k_1, k_2)\omega^{\ell} \right) T + \left( \sum_{\ell=0}^{4} a_{0\ell}(k_1, k_2)\omega^{\ell} \right),
\]

(3.18)

\[
g_3(\omega, T, k_1, k_2) = \left( \sum_{\ell=1}^{5} b_{1\ell}(k_1, k_2)\omega^{\ell} \right) T + \left( \sum_{\ell=1}^{3} b_{0\ell}(k_1, k_2)\omega^{\ell} \right),
\]

(3.19)

where \(a_{0\ell}, a_{1\ell}, b_{0\ell}\) and \(b_{1\ell}\) are listed in the Appendix. The Sylvester matrix formed by using (3.18) and (3.19), and by eliminating \(T\) is found as

\[
S = \begin{bmatrix} \sum_{\ell=2}^{4} a_{1\ell}(k_1, k_2)\omega^{\ell} & \sum_{\ell=0}^{4} a_{0\ell}(k_1, k_2)\omega^{\ell} \\ \sum_{\ell=1}^{5} b_{1\ell}(k_1, k_2)\omega^{\ell} & \sum_{\ell=1}^{3} b_{0\ell}(k_1, k_2)\omega^{\ell} \end{bmatrix}.
\]

(3.20)
The entries of $S$ in (3.20) can be found from

\[
\begin{align*}
    a_{00} &= 9k_2 + 18k_1 + 9 & a_{10} &= 0 \\
    a_{02} &= -k_2 - 6k_1 - 18 & a_{12} &= -k_1 - 22 \\
    a_{04} &= 1 & a_{14} &= k_1 + 6 \\
    b_{01} &= 4k_2 + 17k_1 + 22 & b_{11} &= -7k_2 - 10k_1 + 9 \\
    b_{03} &= -k_1 - 6 & b_{13} &= -k_2 - 6k_1 - 18 \\
    b_{15} &= 1 
\end{align*}
\]

The rest of $a_{0\ell}, a_{1\ell}, b_{1\ell}$ and $b_{1\ell}$ are zero.

from which we calculate the resultant by computing the determinant of $S$. It is given by

\[
\Phi(y, k_1, k_2) = 4\sum_{i=0}^{4} \gamma_i(k_1, k_2)y^i = 0, 
\]

where $y = \omega^2$, and $\omega = 0$ solutions are dropped as per Remark 3.1. The expression $\Phi = 0$ in (3.21) is given by

\[
\Phi(y, k_1, k_2) = y^4 + (-2k_2 + k_1^2)y^3 + (14k_2 + 78 + 18k_1^2 + 72k_1 + 8k_1k_2 + k_2^2)y^2 \\
+ (-16k_1k_2 - 31k_2^2 + 144k_1 + 34k_2 - 2k_2^2 + 160)y \\
+ 81 + 72k_1 + 18k_2 - 216k_1k_2 - 180k_1^2 - 63k_2^2. 
\] (3.22)

For DIS regions to exist in $k_1 - k_2$ plane, it is necessary that $\mathbf{A}$ and $\mathbf{A} + \mathbf{BKC}$ are Hurwitz stable, see Theorem 3.4 and Theorem 3.5. That is, according to the represented characteristic equation (3.18) for the system and considering (3.3), $p_0(s)$ and $p_0(s) + p_1(s)$ should be stable polynomials. Using the well-known Routh-Hurwitz stability criterion [76], we can find the region in $k_1 - k_2$ where the conditions (i) and (ii) in Theorems 3.4-3.5 are satisfied, see Figure 3.2. In the following, we obtain the controller gains with which the system can be made DIS. We use both Descartes rule of signs and the Sturm sequence theorem on equation (3.22) to achieve this.

### 3.4.1.1 Extraction of DIS regions; Sufficient Conditions Using Descartes Rule of Signs

According to Theorem 3.4, all $\gamma_i(k_1, k_2)$ in (3.21) must have the same sign so that $\Phi = 0$ is guaranteed not to have any positive real $y$ roots. To apply the rule, one can first draw
the boundaries $\gamma_i(k_1, k_2) = 0$ in the controller gain plane. Next, by testing one point in each arising region, we can determine the parametric regions where all $\gamma_i(k_1, k_2)$ have the same sign. The identification leads to two types of regions in Figure 3.3, namely, the delay-independent stable region identifiable by Descartes rule of signs (black), and regions where the Descartes rule of signs is inconclusive (hatched). Some interesting observations are as follows. The DIS region (black) can neighbor a region that does not lead to Hurwitz stability of the delay-free system (white background). The boundaries that separate such regions present significant lack of robustness. This observation also
concludes that the delay-free system can have stable eigenvalues with significantly large negative real parts, but the system can still be made DIS by selecting $k_1$ and $k_2$ in the black region.

### 3.4.1.2 Extraction of DIS regions; Using Sturm Sequence Theorem

The DIS regions can be extracted with less conservatism following Theorem 3.5. To implement the theorem, one needs to check the controller gain space corresponding to the delay-free stability region (which is the gray region in Figure 3.2). Each controller-gain pair that complies with Theorem 3.5 makes the system DIS, see Figure 3.4. The results show the DIS regions in black revealed by using Theorem 3.5 based on Sturm sequences and using only the conditions (i), (ii), and (iii)-a of the theorem. Extensions with the conditions (iii)-b and (iii)-c can simply be done via numerical implementation. Notice that the hatched regions in Figure 3.3 could not be determined using the Descartes rule of signs. With Sturm sequences, we are able to conclude that those regions also qualify to be a part of the DIS region. Finally, the controller gains selected from the remaining gray regions in Figure 3.4 lead to delay-dependent stability of the system. In other words, for a given $(k_1, k_2)$ there exists some positive delay $\bar{\tau}$ such that the system remains stable for $0 \leq \tau < \bar{\tau}$. 

![Figure 3.4: Delay-independent stable points in $(k_1, k_2)$ identified by Sturm sequences.](image-url)
### 3.4.1.3 Simulation Results

We next provide simulations of the system with the designed controller where the single input is a step function of 10 units. As controller gains, we select $k_1 = 1$ and $k_2 = -1$ from the delay-independent stability region in Figure 3.4. In simulations, we choose three different delay values $\tau = 0.5$, $\tau = 2$, and $\tau = 10$ and present the results in Figure 3.5. As expected, the output of the system is stable in all the three cases, however, it is obviously impossible to verify delay-independent stability only by way of simulations.

![Figure 3.5: Step response of the system model for controlled system. Here $k_1 = 1$ and $k_2 = -1$.](image)

### 3.4.1.4 Rightmost Root Calculations

We next revisit the analytical results in Figure 3.4 and study the spectrum of the system. Notice that detecting the spectrum of LTI time-delay systems is not a trivial task, and this is a research topic alone [10, 134]. We use the TRACE DDE toolbox developed in [10] to compute the rightmost roots of the system at hand. In Figure 3.6, we present the real part of the rightmost roots as a function of different controller gains. In this figure, the gain plane, which is in the range of $[-5, 5]$, is provided with color coding where color represents the real part $\sigma$ of the rightmost root, and $\tau = 10$. For easier comparison, we also superimpose the DIS boundary (solid curves) found in Figure 3.4 onto Figure 3.6. We can see in Figure 3.6 that the controller gains can be selected to make the system...
DIS while still maintaining the stable rightmost roots sufficiently far from the imaginary axis.

**Figure 3.6:** Real part of the rightmost roots in $k_1 - k_2$ controller gain plane for $\tau = 10$. The black solid curves are the boundaries of the DIS region found in Figure 3.4. Color code identifies the real part of the rightmost roots of the system found using TRACE DDE [10].

### 3.4.2 DIS for Delayed Systems Via Delay-Free Observers

In many cases of practice, not all state variables are available for measurements. Therefore, a state observer is usually used in order to estimate the unavailable state variables [77]. Observer design for LTI time-delay systems (TDS) has attracted considerable interest over the years, and several design methods have been proposed [13, 25, 35, 36, 81, 91], including coordinate-change approach [53], reducing transformation technique [84], polynomial approaches [112, 130], frequency domain approaches [87, 121], and the factorization approach [133]. There also exist studies that propose to make observer states *unaffected* from delays [24, 33]. Another useful technique is to incorporate the delay of the plant into the observer [134]. Adding the delay in the observer dynamics allows an elegant decoupling of the stability properties of the plant and observer. Consequently, one can independently design the plant and observer controllers [95, 134]. This decoupling idea, which can be implemented in frequency domain, is practical as it simplifies the control design.
In this case study, however, we do the contrary; we remove the decoupling property. This in turns allows influencing the LTI plant stability by using the LTI observer controller gains. In other words, when the decoupling property is removed by not allowing the delay term in the observer, the observer becomes a stability facilitator for the plant. This approach not only leads to the design of a stable observer, but it also uses the observer gains advantageously to make the plant dynamics delay-independent stable. More specifically, given the plant controller gains, our approach can find the parametric regions with respect to the observer controller gains so that gains selected from these regions make the combined plant-observer system asymptotically stable independent of the amount of delay in the plant. That is, the approach reveals the gain space that achieves the stability of the combined system, no matter how large the delay is. Moreover, the approach is able to express the boundaries of these regions algebraically as a function of observer controller gains. The delay-independent stability (DIS) analysis becomes possible with the results of Descartes on algebra [5, 41], which allow designing the observer controller gains with sufficient stability conditions.

### 3.4.2.1 Delay-free observer design

The focus here is on the control of the following LTI plant with a single output delay $\tau \geq 0$,

\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &=Cx(t-\tau), \\
u(t) &= K\hat{x}(t),
\end{align*}

(3.23)

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are the constant system and control matrices, respectively, $C \in \mathbb{R}^{q \times n}$ is the output matrix, $x(t) \in \mathbb{R}^n$ is the system state vector, $u(t)$ is the control input to the plant, $y(t) \in \mathbb{R}^q$ is the output vector, and $K = \{k_{iv}\}$ is the plant control law, where $k_{iv} \in \mathbb{R}$ are the controller gains of the plant. In (3.23), the system states $x(t)$ are unavailable, hence the prediction of the states $\hat{x}(t) \in \mathbb{R}^n$ is used to control the plant. The prediction $\hat{x}(t)$ can be obtained from the delay-free observer,

\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + \hat{K}(\hat{y}(t) - y(t)), \\
\hat{y}(t) &= C\hat{x}(t),
\end{align*}

(3.24)
where \( \tilde{y}(t) \in \mathbb{R}^q \) is the observer output vector, and \( \mathbf{\tilde{K}} = \{ \tilde{k}_{iv} \} \) is the observer control law, where \( \tilde{k}_{iv} \in \mathbb{R} \) are the controller gains of the observer. Different from the literature, the observer output \( \tilde{y}(t) \) in (3.24) is not affected by \( \tau \). The dynamics of the combined system then becomes

\[
\dot{\tilde{x}}(t) = A^* \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + B^* \begin{bmatrix} x(t) \\ \tilde{x}(t-\tau) \end{bmatrix},
\]

with

\[
A^* = \begin{bmatrix} A & BK \\ 0 & A+BK+KC \end{bmatrix} \quad \text{and} \quad B^* = \begin{bmatrix} 0 & 0 \\ -KC & 0 \end{bmatrix},
\]

where \( 0 \) is an \( n \times n \) matrix with all its entries zero. Since we propose a delay-free observer, the observer dynamics cannot be decoupled from that of the plant. Therefore, the combined system in (3.25) should be considered as a whole. It must be noted that the selected LTI plant and its observer do not have disturbances and uncertainties. These conditions can be relaxed in future work.

**Theorem 3.11.** LTI-TDS in (3.25) is DIS in the delay parameter space \( \tau \) with observer gain \( \mathbf{\tilde{K}} \), if

i) \( |sI - A^*| \) is Hurwitz stable,

ii) \( |sI - A^* - B^*| \) is Hurwitz stable,

iii) \( \text{var}(\Phi) = 0 \).

**Theorem 3.12.** There exists at least one observer control law \( \mathbf{\tilde{K}} \) that makes the combined system in (3.25) delay-independent stable if \( A^* \) is Hurwitz.

Interested reader is directed to [70] for the proof of Theorems above.

Consider the plant dynamics

\[
A = \begin{bmatrix} -27 & 3.6 & 6 \\ 9.6 & -12.5 & 0 \\ 0 & 9 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.26 & 0.9 & 0.8 \\ 0 & -0.18 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.
\]

Let us set the plant and observer controllers as,

\[
K = \begin{bmatrix} 0 & -10 & 0 \\ -10 & -50 & -20 \end{bmatrix}, \quad \mathbf{\tilde{K}} = \begin{bmatrix} \tilde{k}_1 \\ \tilde{k}_2 \end{bmatrix}.
\]

\( A^* \) and \( B^* \) matrices are suppressed here as they can be constructed using (3.27) and (3.28) in (3.26). In this problem, we have \( n = 3 \) and \( c = \text{rank}(B^*) = 1 \). The objective is to find the regions in the parameter space of \( \tilde{k}_1 - \tilde{k}_2 \) so that a pair \((\tilde{k}_1, \tilde{k}_2)\) selected from these regions guarantees delay-independent stability of (3.25).
Following the procedure explained in the previous section, we start with the characteristic equation of the coupled plant-observer system given by

\[ f(s, \tau, \tilde{k}_1, \tilde{k}_2) = P_0 + P_1 e^{-\tau s} = 0, \]  

(3.29)

where \( P_0 \) and \( P_1 \) are polynomials in \( s \) with coefficients in terms of \( \tilde{k}_1 \) and \( \tilde{k}_2 \). After the Rekasius substitution in (3.29), and substituting \( s = j\omega \), the real and imaginary parts of the transformed characteristic equation are found as,

\[ g_R(\omega, T, \tilde{k}_1, \tilde{k}_2) = \left( \sum_{i=2}^{6} a_{1i}(\tilde{k}_1, \tilde{k}_2)\omega^i \right) T + \left( \sum_{i=0}^{6} a_{0i}(\tilde{k}_1, \tilde{k}_2)\omega^i \right), \]  

(3.30)

\[ g_I(\omega, T, \tilde{k}_1, \tilde{k}_2) = \left( \sum_{i=1}^{7} b_{1i}(\tilde{k}_1, \tilde{k}_2)\omega^i \right) T + \left( \sum_{i=1}^{5} b_{0i}(\tilde{k}_1, \tilde{k}_2)\omega^i \right), \]  

(3.31)

where \( a_{0i}, a_{1i}, b_{1i} \) and \( b_{0i} \) are suppressed for conciseness. The Sylvester matrix formed by using (3.30) and (3.31), and by eliminating \( T \) is a \( 2 \times 2 \) matrix

\[ S = \begin{bmatrix} \sum_{i=2}^{6} a_{1i}(\tilde{k}_1, \tilde{k}_2)\omega^i & \sum_{i=0}^{6} a_{0i}(\tilde{k}_1, \tilde{k}_2)\omega^i \\ \sum_{i=1}^{7} b_{1i}(\tilde{k}_1, \tilde{k}_2)\omega^i & \sum_{i=1}^{5} b_{0i}(\tilde{k}_1, \tilde{k}_2)\omega^i \end{bmatrix}, \]  

(3.32)

from which we calculate the resultant by computing the determinant of \( S \). It is given by

\[ \Phi(y, \tilde{k}_1, \tilde{k}_2) = \sum_{i=0}^{6} \gamma_i(\tilde{k}_1, \tilde{k}_2)y^i = 0, \]  

(3.33)

For DIS regions to exist in \( \tilde{k}_1 - \tilde{k}_2 \) plane, it is necessary that \( A^\star \) is Hurwitz stable, see Theorem 3.11. Using the well-known Routh-Hurwitz stability criterion [77], we find that \( A^\star \) remains stable in the light-gray-shaded region in Figure 3.7.

Next, we find the DIS regions of the plant-observer system in \( \tilde{k}_1 - \tilde{k}_2 \) plane using Theorem 3.11 on (3.33).

3.4.2.2 Extraction of DIS regions

According to Descartes rules of signs, all \( \gamma_i(\tilde{k}_1, \tilde{k}_2) \) in (3.33) must have the same sign so that \( \Phi = 0 \) is guaranteed not to have any positive real \( y \) roots. To apply the rule, one can first draw the boundaries \( \gamma_i(\tilde{k}_1, \tilde{k}_2) = 0 \) in the observer gain plane. Next, by
testing one point in each arising region, we can determine the parametric regions where all $\gamma_i(\hat{k}_1, \hat{k}_2)$ have the same sign. The identification leads to three types of regions in Figure 3.8, namely, the delay-independent stable region identifiable by Descartes rule of signs (dark gray), the region where DIS is impossible since $A^*$ is not Hurwitz (white region), and light gray regions where Descartes rule of signs is either inconclusive (since $\text{var}(\Phi)$ is even) or concludes that DIS property is impossible (since $\text{var}(\Phi)$ is odd).
Some interesting observations on Figure 3.8 are as follows. The DIS region (dark gray) can neighbor a region that does not lead to Hurwitz stability of the delay-free system (white region). The boundaries that separate such regions present significant lack of robustness. This observation also concludes that the delay-free controlled system can have stable eigenvalues that are significantly close to the imaginary axis, but the system can still be made DIS by selecting $\tilde{k}_1$ and $\tilde{k}_2$ from the dark gray region. This is again a consequence of proposing a delay-free observer, which strengthens the stability of the system against the delay $\tau$.

3.4.2.3 Simulations

We implement the designed observer controller gains in time domain simulations in order to inspect the output response $y(t)$. For this, we choose $\tilde{k}_1 = -20$ and $\tilde{k}_2 = -1$, which is a point in the DIS region (dark gray) shown in Figure 3.8. We next simulate the combined system using MATLAB/Simulink. With appropriate settings of the numerical integration method and with different initial conditions in the plant and the observer, the simulation results are obtained, see Figure 3.9 and Figure 3.10 for $\tau = 0.5$ and $\tau = 3$ sec cases, respectively.

**Figure 3.9:** Output response $y(t)$ of the coupled system model. Here $\tilde{k}_1 = -20$, $\tilde{k}_2 = -1$, and $\tau = 0.5$. 
Figure 3.10: Output response $y(t)$ of the coupled system model. Here $\tilde{k}_1 = -20$, $\tilde{k}_2 = -1$, and $\tau = 3$.

3.4.2.4 Rightmost Root Computation

We revisit our analytical results in Figure 3.8. We select points from different regions of this figure to show the spectrum of the system. Notice that detecting the spectrum of LTI time-delay systems is not a trivial task, and it is a research topic alone [10, 134].

Here, we use the TRACE DDE toolbox developed in [10] to compute the rightmost roots of the system at hand. The objective is to test the behavior of the rightmost roots of the system for two different choices of $(\tilde{k}_1, \tilde{k}_2)$ pairs. First pair $p_1 = (\tilde{k}_1, \tilde{k}_2) = (0, -11)$ is selected inside the DIS region (dark gray in Figure 3.8) and sufficiently close to the white regions (unstable system for $\tau = 0$). The second pair $p_2 = (\tilde{k}_1, \tilde{k}_2) = (0, -3.7)$ is selected in the neighborhood of the DIS region but in the light gray region, where we know that DIS does not hold but the controlled system is stable for $\tau = 0$.

The rightmost roots of the system with $\tau = 20$ and the observer controller gains at $p_1$ are shown in Figure 3.11. It is clear that the system is stable for $\tau = 20$ since the real part of the rightmost root is negative, as expected from our DIS analysis. The information in Figure 3.11 is obviously inconclusive to conclude DIS property. To further test DIS, we next compute the real part of the rightmost root with respect to $\tau$ for the given gains at $p_1$ and for $\tau \in [0, 1000]$. The results are again consistent with our DIS analysis and thus suppressed.
Figure 3.11: Twenty rightmost characteristic roots of the combined system for $\tilde{k}_1 = 0$, $\tilde{k}_2 = -11$, and $\tau = 20$.

Figure 3.12: Real part of the rightmost characteristic root of the combined system for delay $0 \leq \tau \leq 10$. The observer controller gains are selected at $p_2 = (\tilde{k}_1, \tilde{k}_2) = (0, -3.7)$ from the light gray region in Figure 3.8.
We study the rightmost root behavior for the gains at $p_2$, Figure 3.12. We see that for delay values in the range of $[0, 10]$, the real part of the rightmost characteristic root will change sign past a critical delay value. That is, the system will become unstable. This result confirms the analytical findings that the point $p_2$ does not guarantee delay-independent stability of the system.

It is important to note that it would be hard to fully confirm the DIS property of a TDS using rightmost root computation. On the other hand, these computational tools make it possible to reliably test stability in a finite range of $\tau$.

### 3.4.3 Effects of System Parameters on DIS Boundaries: A 2\textsuperscript{nd} order open-loop system Case with a PD-controller

Consider a second order system with PD-controller with an unknown output delay $\tau$, as,

\[
A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K = [k_P, k_D],
\]

(3.34)

where $k_P \in \mathbb{R}$ and $k_D \in \mathbb{R}$ are respectively the proportional and derivative controller gains, $\omega_n \in \mathbb{R}_{0+}$ is the natural frequency, and $\xi \in \mathbb{R}_{0+}$ is the damping ratio of the open-loop plant. The characteristic equation of the system becomes,

\[
f = s^2 + 2\xi\omega_n s + \omega_n^2 + (k_D s + k_P)e^{-\tau s},
\]

(3.35)

which we would like to render DIS by appropriate design of $(k_P, k_D)$ gains. This effort was followed in [29] but based on sufficient conditions and without developing the closed-form expressions of the DIS boundaries.

It must be noted that $\xi$ and $\omega_n$ are constant physical parameters are thus dropped from the arguments in the sequel. We dedicate this subsection to studying the effects of $\xi$ and $\omega_n$ on the closed-form DIS boundaries formed in the $(k_P, k_D)$ plane.

After applying the Rekasius substitution on (3.35), the transformed characteristic equation is found as,

\[
g(j\omega, T, k_D, k_P) = ((j\omega)^3 + (\omega_n^2 - k_P)j\omega + (-k_D + 2\xi) \\
\omega_n)(j\omega)^2 T + (j\omega)^2 + (k_D + 2\xi\omega_n)j\omega + \omega_n^2 + k_P.
\]

(3.36)
Separating the imaginary and real parts of (3.36) and excluding \(\omega = 0\) roots as per Remark 3.1, yields

\[
g_R = (k_D - 2\xi \omega_n)\omega^2 T + \omega_n^2 + k_P - \omega^2, \quad (3.37)
\]
\[
g_\Im = (\omega_n^2 - k_P - \omega^2)T + (k_D + 2\xi \omega_n)\omega. \quad (3.38)
\]

Resultant elimination of parameter \(T\) from (3.37)-(3.38), and considering the proposed change of variable \(y = \omega^2\) on the resultant \(R_T\) leads to the polynomial \(\Phi\),

\[
\Phi(y, k_D, k_P) = -y^2 + (-4\omega_n^2\xi^2 + 2\omega_n^2 + k_D^2)y + k_P^2 - \omega_n^4 = 0. \quad (3.39)
\]

where \(\gamma_0 := k_P^2 - \omega_n^4\).

One has to note that, since in general \(\Phi(y, k_D, k_P) = 0\) does not identically vanish, we expect to find points of intersection between the line equations \(g_R\) and \(g_\Im\) defined with respect to \(T\). We now move on finding the analytical boundaries in \((k_P, k_D)\) plane such that, inside the regions bordered by these boundaries, condition (ii) of Theorem 3.8 holds. According to Proposition 3.7, the potential DIS boundaries are obtained by,

\[
\varphi(k_D, k_P) = \text{Discrim}_y(\Phi(y, k_D, k_P)) = 4k_P^2 + 4\omega_n^2(1 - 2\xi^2)k_D^2 + k_D^4
+ 16\omega_n^4\xi^2(\xi^2 - 1) = 0. \quad (3.40)
\]

In simple terms, the analytical boundaries in PD-controller gain space for which there exists the potential for a change from delay-independent stability (DIS) to either delay-dependent stability or instability are represented by the curves \(\varphi(k_D, k_P) = 0\) and \(\gamma_0(k_D, k_P) = 0\), which we denote as

\[
\Gamma(k_D, k_P) = \{(k_D, k_P) \in \mathbb{R}^2 | \gamma_0 = 0 \cup \varphi = 0\}. \quad (3.41)
\]

We now identify the delay-independent stability property of system in (3.35) with the theorem below.

**Theorem 3.13.** The system in (3.35) is delay-independent stable (DIS) in the delay parameter space, if and only if, for a given \(k_D, k_P\) pair in any closed region formed by the boundaries \(\Gamma(k_D, k_P)\) in (3.41), the following conditions are simultaneously satisfied,

i) \(k_D > -2\xi \omega_n, \quad k_P > -\omega_n^2\),

ii) \(\{y > 0|\Phi(y) = 0\} = 0\).
**Proof.** System in (3.34) has a single delay with first order of commensurability, hence, the pseudo-delay parameter $T$ appears in first order in (3.37)-(3.38). Therefore, if there exists $\omega^* = \sqrt{y^*}$ for which condition (ii) of Theorem 3.8 is violated, then $\omega^*$ is guaranteed to correspond to a $T^* \in \mathbb{R}$. In other words, existence of $y = \omega^2 \in \mathbb{R}_+$ roots of $\Phi(y) = 0$ guarantees an admissible $T^* \in \mathbb{R}$ since all the possible conditions for existence of an admissible $T^* \in \mathbb{R}$ are captured within the resultant calculations. Hence, condition (ii) completes the identification of DIS regions by guaranteeing that no positive real root of $\Phi(y) = 0$ exists and there is no need to implement Theorem 3.9.

**Example:** Consider a case with $\omega_n = 1$ and $\xi = 0.95$. According to Theorem 3.13, the DIS region is the dark gray shaded region in Figure 3.13.

![Figure 3.13](image)

**Figure 3.13:** PD-controller gain plane with $\omega_n = 1$ and $\xi = 0.95$. Light gray: delay-dependent stable region. Dark gray: delay-independent stable region. Note: The straight solid boundary lines categorize the region that preserve delay-dependent stability (conditions (i)-(ii) on Theorem 3.13) and $\{k_P = \pm \omega_n^2 \cap \varphi(k_D, k_P) = 0\}$ are the boundaries which encapsulate the potential DIS regions.

**Influence of damping ratio and natural frequency on DIS boundaries**

Here we discuss the influence of damping ratio $\xi$ and natural frequency $\omega_n$ on $\Lambda(k_D, k_P)$:

A) $\omega_n = 0$ and $\xi \neq 0$: In this case the delay-free system has a pole at the origin, which violates the DIS necessary conditions in Remark 3.1. Furthermore, $\Phi(y) = y^2 - k_D^2 y - k_P^2$ possesses one positive real root according to Descartes rule of signs [6], for which there exists an admissible $T^* \in \mathbb{R}$, see proof of Theorem 3.13. Therefore, for $\omega_n = 0$, there
are no possible PD-controller gains with which the LTI-TDS described in (3.34) can be rendered DIS.

B) $\omega_n \neq 0$ and $\xi = 0$: In this case, the delay-free system poles are on the imaginary axis, which is a violation of DIS necessary conditions in Remark 3.1. Moreover, $k_P < -\omega_n^2$ does not comply with the DIS necessary condition in Remark 3.1, and if $k_P > \omega_n^2$, then $\Phi(y) = y^2 - (k_D^2 + 2\omega_n^2)y - (k_P^2 - \omega_n^2)$ has one positive real root according to Descartes rule of signs [6]. On the other hand, given $-\omega_n^2 < k_P < \omega_n^2$, the quadratic function $\Phi(y)$ has a negative minimum, and thus it is straightforward to prove that it has positive real zeros when $-\omega_n^2 < k_P < \omega_n^2$ and thus a feasible $T^* \in \mathbb{R}$ exists as per Theorem 3.13. With these arguments, we can conclude that a stability switching definitely occurs under this setting, and therefore rendering DIS is impossible.

Remark 3.14. Condition (i) of Theorem 3.13 and $\gamma_0(k_D, k_P) = k_P^2 - \omega_n^4$ extracted from (3.39) can also be used to check, first, the necessary conditions with which PD-controller gains exist to achieve DIS. To summarize, these conditions can be combined as: $k_D > -2\xi \omega_n$ and $-\omega_n^2 < k_P < \omega_n^2$. In one sense, increasing $\omega_n$ and $\xi$ expands the regions within which DIS regions may exist, that is, one may find larger DIS regions in the controller-gain space after incorporating the boundaries $\varphi(k_D, k_P) = 0$ into the analysis.

C) Other scenarios:

• $\omega_n = 1$ and $0 < \xi < 1$: For a fixed $\omega_n$ and various $\xi$ values, we draw the boundaries $\varphi(k_D, k_P) = 0$ in Figure 3.14. These boundaries are expanding along $k_D$ axis. There is a critical point $\xi = 0.5^{0.5} \approx 0.707$ for which the curve characteristics change since the coefficient of $k_D^2$ vanishes in $\varphi(k_D, k_P) = 0$. Also, when $\xi = 1$, a singularity arises as the constant term in $\varphi(k_D, k_P) = 0$ vanishes, see Figure 3.15. The influence of this singular point can be observed by comparing Figures 3.14-3.15. Notice also that, since $\varphi(k_D, k_P) = 0$ is an even polynomial in $(k_D, k_P)$, the curves $\varphi(k_D, k_P) = 0$ are symmetric with respect to both controller-gain axes.

• $\omega_n = 1$ and $\xi \geq 1$: Increasing $\xi$ may widen the DIS region in $k_D$ domain as per Remark 3.14. In this case, we see that $\varphi(k_D, k_P) = 0$ forms into two separate closed curves for $\xi > 1$. Figure 3.15 illustrates how the boundaries $\varphi(k_D, k_P) = 0$ change for different $\xi \geq 1$ values and for the given $\omega_n = 1$. We next take the case with $\omega_n = 1$ and $\xi = 1.2$ and study the DIS regions using Theorem 3.13. In this example, we find two separate ellipse-shaped curves represented by $\varphi(k_D, k_P) = 0$. Combining these curves with the constraint $\gamma_0(k_D, k_P) = 0$, as well
Figure 3.14: $\phi(k_D, k_P) = 0$ curve in the controller gain plane for various values of $\xi$ when $0 < \xi < 1$ and $\omega_n = 1$.

Figure 3.15: $\phi(k_D, k_P) = 0$ curve in the controller gain plane for varied $\xi > 1$ and $\omega_n = 1$. 
as the condition (i) of Theorem 3.13, and performing a point-wise check in each arising region, yields the dark gray region as the DIS region, see Figure 3.16.

**Figure 3.16:** PD-controller gain categorization for $\omega_n = 1$ and $\xi = 1.2$. Light gray: delay-dependent stable region. Dark gray: delay-independent stable region.
3.5 Conclusion

An algebraic technique to perform control design of single-delay LTI systems is proposed within this chapter, where the control gain space is decomposed into regions for which the system is either delay-independent stable or delay-dependent stable for a range of delay $0 \leq \tau \leq \bar{\tau}$. Since the purpose of this research is on DIS control gains space categorization, we did not categorize the spotted DDS region according to delay margin $\bar{\tau}$ and left it for future studies. The novelty of the presented approach on DIS control design is the utilization of algebraic geometry, with which the boundaries decomposing the gain space can be expressed in explicit form in terms of gain parameters. Utilization of optimization techniques within the spotted DIS region in control gain space, will lead to applications of these optimal controllers on real-time experiments. This chapter can be considered as a foundation for expanding the proposed approach in DIS regions detection in control gain space of LTI systems with multiple delays.
Chapter 4

DIS Synthesis/Design for
LTI-TDS with Multiple Delays

This chapter is an extension to the previous chapter which comprised single delay systems. Here, we present algebraic methods to design controllers and categorize controller gain space according to delay-independent stability property for general class of linear time-invariant systems with multiple delays. The methods are satisfying necessary and sufficient conditions of stability while achieving the control goals. Furthermore, we expand methods and theorems utilizing which one can improve the computational efficiency on stability analysis and controller gains space categorization steps for DIS analysis. Numerical examples within this chapter and two following chapters will illustrate the effectiveness of the presented approaches.

4.1 Introduction

In this section we study the stability analysis of linear time invariant (LTI) multiple time delay system with respect to delays $\tau_\ell$. The system is expressed in state space form as,

$$\frac{d}{dt} x(t) = Ax(t) + \sum_{\ell=1}^{L} B_\ell x(t - \tau_\ell),$$

where $x(t)$ is the state vector, $A$ and $B_\ell$ are system matrices, respectively, all with appropriate dimensions, and $\tau$ is the non-negative constant delay. The matrices $A$ and
\( B_\ell \) have in general numerical entries. \( \tau_\ell \) are the non-negative pure delays and they are basically shift operator in time and \( L \) is the maximum number of distinct delays \( \tau_\ell \).

The characteristic function of the introduced system in (4.1) is given by,

\[
f(s; e^{-s\tau_\ell}) = \sum_{k=0}^{K} P_k(s)e^{-s\sum_{\ell=1}^{L} \nu_{k\ell} \tau_\ell} = 0,
\]

where \( P_k \) are polynomials in terms of \( s \) with real coefficients, \( K \in \mathbb{Z}_+ \) and \( \nu_{k\ell} \in \mathbb{N} \). Since there exists no delay in the highest-order derivative of the states in (4.1), the characteristic equation (4.2) represents a retarded class LTI system with multiple delays \([45, 114]\), and thus \( \xi_{0\ell} = 0 \). Here the commensuracy order of delay \( \tau_\ell \) is denoted with \( c_\ell \), which is given by \( c_\ell = \max_{0 \leq q \leq Q} (\xi_{q\ell}) \).

It is noted that the non-conservatism of FDT is appealing and has been an inspiration of many studies \([66, 106]\), primarily for a mainstream of research problems, known as delay-independent stability (DIS) \([13, 104, 113]\). In DIS analysis, the objective is to test whether or not a given system is stable regardless of the amount of delays \([75]\). In the single-delay case, DIS analysis can be performed based on necessary and sufficient conditions \([13, 45, 66]\). As regard to DIS analysis of single delay LTI systems, several techniques are also proposed. In \([13]\), spectral radius of some system matrices can be calculated to conclude on DIS property. In \([129]\), frequency sweeping is used to verify whether the system can ever have eigenvalues on the imaginary axis, an indicative of loss of stability. Frequency sweeping is also applied to systems with multiple delays as demonstrated in \([46, 104]\).

Due to the presence of transcendental terms, the characteristic equation (4.2) possesses infinitely many roots, some or all of which may determine stability. Because of this infinite dimensionality, this stability analysis without introducing conservatism can be difficult, especially when \( L > 1 \). What is known about the stability of (4.1) is that the continuity property of the roots of (4.2) on the complex plane holds \([26]\), which indicates that stability analysis of (4.1) requires detecting the critical values \( \{\tau^*_\ell\}_{\ell=1}^{L} \) for which at least one root of (4.2) lies on the imaginary axis, \( s = j\omega \), of the complex plane, where \( \omega \in \mathbb{R}_{0+} \) without loss of generality \([7]\) to achieve delay-dependent stability (DDS). Complimentary to this scenario is the case when the roots of (4.2) “never lie” on the imaginary axis for all nonzero delay values (non-trivial part) and also when the delay-free system is asymptotically stable (trivial part) \([76]\). This is when the system in (4.1) can be categorized as delay-independent stable (DIS) \([66]\).
4.2 DIS Existence and Detection

DIS analysis for systems with many delays starts with finding the necessary and sufficient conditions guaranteeing that \( s = j\omega \) roots of (4.2) can never exist. This is complimentary to DDS conditions, where one calculates \( s = j\omega \) solutions of (4.2) (DDS analysis). An enabling approach to study such solutions is by converting the infinite-dimensional characteristic equation (4.2) to a finite dimensional characteristic equation that has continuous coefficients as was done in [88, 100, 109]. This conversion does not lose the infinite-dimensional nature of the problem, and can be done via the exact Rekasius transformation [88],

\[
e^{-\tau s} = \frac{1 - T_\ell s}{1 + T_\ell s}, \quad (4.3)
\]

where \( s = j\omega, T_\ell \in \mathbb{R}_\infty, \forall \omega \in \mathbb{R}_0^+ \). With the substitution of (4.3) into (4.2), and with a manipulation to remove the fractions, we obtain the transformed characteristic equation expressed on the imaginary axis \( s = j\omega \) as

\[
g(j\omega, \{T_\ell\}_{\ell=1}^L) = \left(CE \left| \begin{array}{c}
1 - j\omega T_\ell \\
1 + j\omega T_\ell
\end{array} \right|_{\ell = 1, \ldots, L} \right) \prod_{\ell=1}^L (1 + j\omega T_\ell)^{c_\ell}, \quad (4.4)
\]

where \( g \) is a polynomial with complex coefficients.

It must be noted that different from the Padé approximation [76], Rekasius transformation is an exact substitution for the imaginary roots of (4.2). All the \( s = j\omega \) roots of (4.2) and of (4.4) are identical. That is, \( s = j\omega \) roots are preserved under the transformation (4.3) [88, 109]. Guaranteeing DIS of (4.2) thus requires that no \( s = j\omega \) solution of (4.4) exists for all \( \{T_\ell\}_{\ell=1}^L \in \mathbb{R}_\infty \). Verification of this is still difficult, as it is known to be NP hard [120] and noticing that each \( T_\ell \) is in the range of \(( -\infty, +\infty )\). However, the algebraic form of (4.4) is easier to manage than studying DIS over (4.2), as we demonstrated in [30].

Remark 4.1. For delay-independent stability of LTI systems with multiple delays, it is necessary that the delay-free system (with \( \{\tau_\ell\}_{\ell=1}^L = 0 \)) and the uncontrolled system (with \( u(t) = 0 \)) are Hurwitz stable [13, 66]. This requires that \( A \) and \( A + \sum_{\ell=1}^L B_\ell \) are Hurwitz stable matrices. With these conditions guaranteed, \( \omega = 0 \) cannot be a feasible solution of the characteristic equation for \( \{\tau_\ell\}_{\ell=1}^L = 0 \) and for any finite \( \tau_\ell \) [37], and hence \( \omega = 0 \) solutions are ignored in the rest of the text. This technically corresponds to “weak” DIS analysis [14], and it ignores the case when \( \tau_\ell \to \infty \). Since, in many
practical control applications, delays remain finite even if they may be large, unknown, or uncertain, we argue that neglecting $\tau_\ell \to \infty$ is an acceptable assumption.

**Remark 4.2.** Equation (4.4) and its various forms were studied extensively for DDS and DIS analysis in [30, 88, 100, 104, 109] and the references therein, however for many constant delays there exists no non-conservative framework that are verified with case studies. Within this chapter we expand the DIS analysis framework for several discrete and constant delay parameters using the Resultant and Discriminant properties revised in previous chapter. Thanks to these properties, we will be able to test DIS of a system with 5 delays in a case study.

We now develop an algebraic approach to find out if the system in (4.1) is DIS. Recall from the previous section that DIS analysis of (4.1) can be converted to analyzing $s = j\omega$ roots of (4.4). This is where we start. We rewrite $g$ as follows,

$$g(j\omega, \{T_\ell\}_{\ell=1}^L) = g_R(\omega, \{T_\ell\}_{\ell=1}^L) + jg_\Im(\omega, \{T_\ell\}_{\ell=1}^L) = 0,$$

where $g_R$ and $g_\Im$ are respectively the real and imaginary parts of $g$. In order to solve $\omega, \{T_\ell\}_{\ell=1}^L$ pairs from (4.5), one should guarantee that $g_R = 0$ and $g_\Im = 0$, concurrently. At this step, in order to reduce the number of unknowns, we can eliminate $T_L$ from these coupled equations using the Resultant Theory [43], which leads to a $2c_L$-order Sylvester matrix denoted by $S$, where $c_L$ is the commensurate degree of $\tau_L$. When $g_R = 0$ and $g_\Im = 0$ have common solutions, then $S$ is singular (but the converse is not always true). That is, the resultant $R_{T_L}(g_R, g_\Im) = \det(S)$, which is a function of $\omega, \{T_\ell\}_{\ell=1}^{L-1}$ should be zero as a necessary condition for (4.5) to be satisfied.

DIS analysis for the multiple time-delay system at hand can actually be constructed as an optimization problem using the principles of resultant theory and starting with $R_{T_L}$, see [30].

We start with converting the DIS analysis to an optimization problem in $T_\ell$ domain. Since it is necessary that $R_{T_L} = 0$ for $g_R = 0$ and $g_\Im = 0$ to have common zeros for $\omega$ to exhibit an extremum, it is necessary that $\frac{\partial \omega}{\partial T_{L-1}} = 0$, which can be found from

$$\frac{\partial R_{T_L}(g_R, g_\Im)}{\partial T_{L-1}} + \frac{\partial R_{T_L}(g_R, g_\Im)}{\partial \omega} \frac{\partial \omega}{\partial T_{L-1}} = 0,$$

(4.6)

Under regular point assumption, we have $\frac{\partial R_{T_L}(g_R, g_\Im)}{\partial \omega} \neq 0$ [22].
**Definition 4.3.** Let $V_L = RT_L(g_\Re, g_\Im)$ and $W_L = \frac{\partial V_L}{\partial T_{L-1}}$, and for $d$ in descending order from $L - 1$ to $1$, $d = L - 1 \ldots 1$, calculate the following equations sequentially, $V_d = RT_d(V_{d+1}, W_{d+1})$, and $W_d = \frac{\partial V_d}{\partial T_{d-1}}$, $d \neq 1$.

**Theorem 4.4.** [30, 104] The system in (4.1) possesses at least one pole on the imaginary axis for some delays $\{\tau^*_\ell\}_{\ell=1}^L$ if, for some $\omega = \omega^* \in \mathbb{R}$ and $\{T^*_\ell\}_{\ell=1}^L \in \mathbb{R}$, the following conditions are satisfied:

A) The repeated resultant, $D(\omega) := RT_1 = V_1 = 0$, which is only a function of $\omega$, vanishes.

B) All the polynomials, $V_d = 0$, $W_d = 0$, $d = 2 \ldots L$, used to construct each one of the resultants in Definition 4.3 are satisfied.

C) The transformed characteristic equation (4.4) is satisfied: $g_\Re = 0$, $g_\Im = 0$.

See [30, 104] for the treatment of singular points, which does not change the essence of Theorem 4.4.

**Proof.** To study the extremum of $\omega$, one can look for the common solutions of $V_L = 0$ and $W_L = 0$ for the regular points of $V_L$ as discussed above. This allows implementing another resultant, this time between $V_L$ and $W_L$, by eliminating $T_{L-1}$. In this resultant, which is $V_{L-1}$, we now have $\{T^*_\ell\}_{\ell=1}^{L-2}$ and $\omega$, and we can use the same logic with other partial derivatives to vanish ($W_d = 0$) as necessary conditions for the extremum of $\omega$ to exist, and apply consecutive resultant operations by eliminating one variable $T^*_\ell$ in each resultant operation. Finally, after eliminating all the $T^*_\ell$ parameters and excluding $\omega = 0$ roots (see Remark 4.1), we obtain a polynomial $D(\omega)$ as defined in condition (A) of the theorem. For a given set of controller gains, if $\omega^* \in \mathbb{R}$ roots of $D(\omega) = 0$ exist and if these roots by back substitution satisfy all the arising single-variable polynomials in (B)-(C) of the theorem for some $T^*_\ell \in \mathbb{R}$, then $s = j\omega^*$ is an admissible solution and the root $\omega^*$ of $D(\omega)$ is one of the extremum points of $\omega$ while also satisfying (4.4) for some $T^*_\ell$. This guarantees that there will be at least one $\omega$ in the set of all possible $\omega$ solutions, such that $s = j\omega$ is a root of (4.4) for some $T^*_\ell \in \mathbb{R}$. This implies, as per properties of (4.3), that (4.2) has a root at $s = j\omega$ for some $\tau^*_\ell \in \mathbb{R}_+$. □

From above, we obtain the polynomial $D(\omega)$ first. For DIS, either one of the following conditions should hold [30],

- $D(\omega)$ has no $\omega \in \mathbb{R}$ roots.
- $D(\omega)$ has some $\omega \in \mathbb{R}$ roots but the corresponding $T^*_\ell$ values satisfying (4.4) are not all in $\mathbb{R}$. 
4.2.1 Detection of DIS Property

We know that imaginary roots of (4.4) are in complex conjugate form, that is, both \( s = j\omega \) and \( s = -j\omega \) satisfy (4.4). This implies that \( V_1 := D(\omega_{i=1}^2) \) is an even polynomial in \( \omega \). Thus, a variable change as \( y = \omega^2 \) becomes convenient for two reasons; one being for the sake of computational efficiency and the other for converting the problem into the investigation of positive real roots of the polynomial found after the change of variable. This is exactly when Sturm sequences become instrumental. Excluding \( \omega = 0 \) zeros of \( V_1 \) as per Remark 4.1, it is easy to see that \( V_1 \) is an even polynomial with respect to \( \omega \) [30]. Therefore, we can write

\[ \Phi(y) = V_1(\omega^2), \tag{4.7} \]

where the number of real zeros of \( V_1 \) is twice the number of positive real zeros of \( \Phi(y) \). Hence, instead of inspecting the real roots of \( V_1 = 0 \), we can inspect the existence of positive real roots of \( \Phi(y) = 0 \). This can be done as follows:

- If no roots of \( \Phi(y) = 0 \) are positive, this implies that no admissible \( y = \omega^2 > 0 \) roots exist, hence a stability switch is impossible. If, by construction, the conditions in Remark 4.1 also hold, then the system is stable independent of the amount of each delay \( \tau_\ell \).

- Presence of at least one positive real zero of \( \Phi(y) \), however, requires checking whether a solution in \( \{T_\ell\}_{\ell=1}^L \subset \mathbb{R} \) exists satisfying (4.5). If such a set of \( \{T_\ell\}_{\ell=1}^L \) does not exist, and the conditions in Remark 4.1 are satisfied, then the system is delay-independent stable.

Remark 4.5. It is noted that calculation of \( \Phi(y) \) in (4.7) requires multiple resultant operations according to the number of distinct delays in the system at hand, which is computationally expensive. That is, one cannot find solutions for systems when \( L > 3 \) and only in special cases there exists cases with \( L = 3 \) [30, 88, 100, 104, 109] and the references therein. In the sequel, we expand the DIS analysis framework for several delays using the Resultant and Discriminant properties revised in previous chapter. Thanks to these properties, we will be able to perform DIS analysis for a system with 5 delays as a case study.
4.2.2 Polynomial Factorization in Consecutive Resultant Operations

In this section we walk through the parameter elimination steps and show how we can extract factors of polynomials using properties in Chapter 1 to simplify the computations of the consecutive resultant operations. It is noted that these simplifications play important role in reducing the expensive computations of the resultant operations and make it possible for us to look at DIS analysis of problems with several delays. Before we start with polynomial factorization technique with resultant and discriminant properties, notice the property below.

**Property 4.1.** Let $P$ and $Q$ be two non-zero polynomials of degree $p = 2q$ and $q$ in domain of $X$. Then let us define,

$$P = \alpha X^{2q} + \alpha_{q-1}X^{2(q-1)} + \cdots + \alpha_0 = \alpha \prod_{i=1}^{2q} (X - a_i),$$

$$Q = \alpha X^{q} + \alpha_{q-1}X^{q-1} + \cdots + \alpha_0 = \alpha \prod_{j=1}^{q} (X - b_j),$$

where $a_i$ are the roots of $P$ (counting multiplicities) and $b_j$ are the roots of $Q$ (counting multiplicities). It holds that $a_i = b_j^2$ or $a_i$ in $P$ corresponds to two roots in $Q$ equal to $\pm \sqrt{b_j}$ where $i = 1, 2, \cdots, p = 2q$ and $j = 1, 2, \cdots, q$. That is, $\text{Disc}_X(P) = (\text{Disc}_X(Q))^2$.

**Proof.** We start with,

$$\text{Disc}_X P = \alpha^{(2q-2)} \prod_{l<k} (a_l - a_k)^2. \quad (4.9)$$

Considering (4.9), one can write,

$$\text{Disc}_X Q = \alpha^{(2q-2)} \prod_{n<m} (b_n - b_m)^2 =$$

$$\alpha^{(2q-2)} \prod_{l<k} ((\sqrt{a_k} + \sqrt{a_l})^2(\sqrt{a_l} + \sqrt{a_k})^2(\sqrt{a_l} + \sqrt{a_k})^2(\sqrt{a_i} - \sqrt{a_k})^2. \quad (4.10)$$

Hence, some simplifications and manipulations of (4.10) lead to

$$\text{Disc}_X(Q) = \alpha^{(2q-2)} \prod_{l<k} (a_k + a_l + 2\sqrt{a_k a_l})^2(a_k + a_l - 2\sqrt{a_k a_l})^2,$$

$$\Rightarrow \text{Disc}_X Q = \alpha^{(2q-2)} \prod_{l<k} ((a_k + a_l)^2 - 4a_k a_l)^2 = \alpha^{(2q-2)} \prod_{l<k} (a_k^2 + a_l^2 - 2a_k a_l)^2,$$

$$\Rightarrow \text{Disc}_X Q = \alpha^{(2q-2)} \prod_{l<k} (a_k - a_l)^4.$$
\[
\Rightarrow \text{Disc}_X Q = \alpha^{-(6q-2)} \prod_{l<k} (\text{Disc}_X(P))^2.
\]

Let us start right before the very first resultant operation and investigate the special characters of the developed sylvester matrix at hand for the general LTI-TDS system with characteristic equation given in (4.2). The first resultant operation is performed on \(g_R\) and \(g_3\) in (4.5) that leads to \(V_L = R_{T_L}(g_R, g_3)\). Next step is to eliminate \(T_{L-1}\) according to Theorem 4.4. One can define \(V_L\) and \(W_L\) in terms of \(T_{L-1}\) as,

\[
V_L = R_{T_L}(g_R, g_3) = \alpha_N T_{L-1}^N + \alpha_{N-1} T_{L-1}^{N-1} + \cdots + \alpha_0,
\]

\[
W_L = \frac{\partial V_L}{\partial T_{L-1}} = N \alpha_N T_{L-1}^N + (N - 1) \alpha_{N-1} T_{L-1}^{N-2} + \cdots + \alpha_1,
\]

That is, the sylvester matrix formed to compute \(V_{L-1}\) is,

\[
R_{T_{L-1}}(\omega, T_{L-2}, \cdots, T_1) = \begin{bmatrix}
\alpha_N & \alpha_{N-1} & \cdots & \alpha_0 & 0 & 0 & 0 \\
0 & \alpha_N & \alpha_{N-1} & \cdots & \alpha_1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
N \alpha_N & (N-1) \alpha_{N-1} & \cdots & \alpha_1 & 0 & 0 & 0 \\
0 & N \alpha_N & (N-1) \alpha_{N-1} & \cdots & \alpha_1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & \alpha_1
\end{bmatrix}
\]

If we set,

\[
\Xi_1 = \begin{bmatrix}
\alpha_{N-1} & \cdots & \alpha_0 & 0 & 0 & 0 \\
\alpha_N & \alpha_{N-1} & \cdots & \alpha_1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}, \quad \Xi_2 = \begin{bmatrix}
N \alpha_N & (N-1) \alpha_{N-1} & \cdots & \alpha_2 & 0 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

It can be concluded that the determinant of the sylvester matrix in (4.12) can be written as,

\[
R_{T_{L-1}}(\omega, T_{L-2}, \cdots, T_1) = \alpha_N (\text{det}(\Xi_2) + (-1)^{N+1} N \text{det}(\Xi_1)),
\]

It is noted that \(\alpha_N\) factor in Equation 4.14 is a function of \(T_{L-3}, \cdots, T_1\), and \(\omega\) and is a significant large factor in many case studies.

Remark 4.6. The use of Property 4.1 is preferred since \(V_1 := D(\omega)\) is an even polynomial in \(\omega\) according to Subsection 4.2.1. Next, it is proved that \(R_{T_{L-1}}\) is could be factored out. That is, one can use the Properties in Chapter 2, Subsection 2.2 presented within previous chapter to further simplify the consecutive resultant operations by factoring out the computed polynomials using theorem below.
\textbf{Theorem 4.7.} Set \( V_{L-1} = \Lambda(\omega, T_{L-2}, \cdots, T_1) \times \Theta(\omega, T_{L-2}, \cdots, T_1) \), where \( T_{L-2} \) is of the order of less or equal than \( m \) in \( \Lambda \), \( T_{L-2} \) is of the order of less or equal than \( n \) in \( \Theta \), where \( m \geq n \) and \( W_{L-1} = \frac{\partial V_{L-1}}{\partial T_{L-2}} \), then \( V_{L-2} =^{m,m+n+1} \text{Res}(V_{L-1}, W_{L-1}) \) according to Definition 4.3 and the operation can be simplified factored out in the simplified form below,

\[
V_{L-2}(\omega, T_{L-3}, \cdots, T_1) = \quad m,m+n-1 \text{Res} \left( \Lambda, \frac{\partial \Lambda}{\partial T_{L-2}} \Theta \right) \times \quad n,m+n-1 \text{Res} \left( \Theta, \frac{\partial \Theta}{\partial T_{L-2}} \Lambda \right). \tag{4.15}
\]

\textbf{Proof.} The proof will use Properties 2.2 and 2.3 of Chapter 2.

In (4.14), it is shown that,
\[
V_{L-1} = R_{T_{L-1}} = \Lambda(\omega, T_{L-2}, \cdots, T_1) \times \Theta(\omega, T_{L-2}, \cdots, T_1)
\]
and that,
\[
W_{L-1} = \frac{\partial V_{L-1}}{\partial T_{L-2}} = \frac{\partial \Lambda}{\partial T_{L-2}} \times \Theta + \frac{\partial \Theta}{\partial T_{L-2}} \times \Lambda.
\]

Using Property 2.2, we write
\[
V_{L-2}(\omega, T_{L-3}, \cdots, T_1) =^{m,m+n+1} \text{Res}(V_{L-1}, W_{L-1}) =
\]

\[
m,m+n-1 \text{Res} \left( \Lambda, \frac{\partial \Lambda}{\partial T_{L-2}} \times \Theta + \frac{\partial \Theta}{\partial T_{L-2}} \times \Lambda \right) \times \quad n,m+n-1 \text{Res} \left( \Theta, \frac{\partial \Lambda}{\partial T_{L-2}} \times \Theta + \frac{\partial \Theta}{\partial T_{L-2}} \times \Lambda \right),
\]

Note that,
\[
deg(\Lambda) \leq m \text{ and } deg(\Theta) \leq n
\Rightarrow deg \left( \frac{\partial \Lambda}{\partial T_{L-2}} \times \Theta \right) \leq (n + m - 1) \text{ and } deg \left( \frac{\partial \Theta}{\partial T_{L-2}} \right) \leq (m - 1)
\Rightarrow deg \left( \frac{\partial \Lambda}{\partial T_{L-2}} \times \Theta \right) - deg(\Lambda) = deg \left( \frac{\partial \Lambda}{\partial T_{L-2}} \right)
\]
and the same case for \( deg \left( \frac{\partial \Theta}{\partial T_{L-2}} \times \Lambda \right) - deg(\Theta) = deg \left( \frac{\partial \Theta}{\partial T_{L-2}} \right) \).

That is, using Property 2.3 it can be concluded that
\[
V_{L-2}(\omega, T_{L-3}, \cdots, T_1) =^{m,m+n+1} \text{Res} \left( \Lambda, \frac{\partial \Lambda}{\partial T_{L-2}} \Theta \right) \times \quad n,m+n-1 \text{Res} \left( \Theta, \frac{\partial \Theta}{\partial T_{L-2}} \Lambda \right). \quad \square
\]

Theorem 4.7 will lead to simplified resultant operations within the rest of the elimination problem at hand.

\textbf{Remark 4.8.} Note the two important facts here following from the resultant operations;

\begin{itemize}
  \item The factorization process may lead to factors of polynomials that have powers higher than unity. Since the higher powers will not influence the detection of DIS system and will only lead to resultant singularities, one should neglect the powers of each factor at each elimination step and for the rest of the elimination process.
  \item Using property 2.2 of Chapter 2, we can take the resultant of each factor separately
\end{itemize}
again to simplify the elimination steps.
That is, using the two notes within this remark, one can improve the computational efficiency and eliminate the calculation redundancies and singularities.

### 4.2.3 Existence of DIS Regions in Control Parameter Space

**Theorem 4.9** (Existence of DIS Gain Regions [72]). For a given set of delays \( \{\tau_\ell\}_{\ell=1}^L \), there exists at least one region in controller gain parameter space \( \{k_i\}_{i=1}^z \), where the system in (4.1) is DIS, if and only if \( A + \sum_{\ell=1}^L B_\ell e^{-\tau_\ell s} \) is DIS for some \( \{k_i\}_{i=1}^z = \{\tilde{k}_i\}_{i=1}^z \).

**Proof.** It follows from the continuity of the characteristic roots of the system [26] that if there exist controller gains \( \{\tilde{k}_i\}_{i=1}^z \) that render a system DIS, then that system remains DIS for \( \{k_i\}_{i=1}^z = \{k^*_i\}_{i=1}^z \), so long as \( |\{\tilde{k}_i\}_{i=1}^z - \{k^*_i\}_{i=1}^z| < \varepsilon_i, \varepsilon_i \to 0^+ \). \( \square \)

Note the special case when \( \{\tilde{k}_i\}_{i=1}^z = 0 \), i.e., at the origin of the controller gain space, system (4.1) is delay-independent stable, since all \( B_\ell = 0 \). Hence, in view of Remark 4.1, we know that at least one DIS region exists, which is around the point \( \{k_i\}_{i=1}^z = 0 \). To reveal this region as well as other possibly existing DIS regions, we start as follows:

We take \( g \) in (4.5) where \( g_R \) and \( g_3 \) are respectively given by,

\[
g_R(\omega, \{T_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) = \sum_{i=0}^{c_L} \alpha_i(\omega, \{T_\ell\}_{\ell=1}^{L-1}, \{k_i\}_{i=1}^z) T_L^i, \tag{4.16}
g_3(\omega, \{T_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) = \sum_{i=0}^{c_L} \beta_i(\omega, \{T_\ell\}_{\ell=1}^{L-1}, \{k_i\}_{i=1}^z) T_L^i, \tag{4.17}
\]

where \( c_L \) is the commensurate degree of \( \tau_L \), see Section II.

Recall that DIS requires that \( s = j\omega \) cannot be a solution of (4.2) in the entire \( \{\tau_\ell\}_{\ell=1}^L \in \mathbb{R}_+^L \) domain, which equivalently requires that \( s = j\omega \) does not satisfy (4.4) in \( \{T_\ell\}_{\ell=1}^L \in \mathbb{R}_\infty^L \) domain. To verify this condition in (4.4), one can seek solutions \( \omega \in \mathbb{R} \) by studying (4.16)-(4.17) in \( T_L \in \mathbb{R}_\infty \) domain. On the other hand, this is computationally/numerically cumbersome, if not impossible, when \( L > 1 \) as was discussed in [102]. Therefore, instead of computing all the \( \omega \) solutions, we wonder only if the max/min of \( \omega \) exists among all the common solutions of (4.16)-(4.17). If max/min of \( \omega \) does not exist in \( \{T_\ell\}_{\ell=1}^L \), it is guaranteed that \( s = j\omega \) is not a solution of (4.4) in \( \{T_\ell\}_{\ell=1}^L \). This
observation is critical in what follows, and it suggests to convert the DIS analysis to an optimization problem in $T_\ell$ domain.

### 4.3 DIS Control Design

We now discuss the DIS control design in light of the above discussions. In order to prevent disrupting the flow of the presentation, the main proof for the multiple delay case is provided in the next subsection and in the Appendix.

DIS control design for multiple time-delay system (4.1) in connection with (4.4) can actually be constructed using the principles of the elimination technique [6, 126], as demonstrated as a numerical example in the previous section. Given $\{k_i\}_{i=1}^z$, in order to find $(\omega, \{T_i\}_{i=1}^L)$ solutions satisfying (4.4), one should guarantee that $g_R = 0$ and $g_3 = 0$, concurrently. At this step, we can eliminate $T_L$ from these two equations to reduce the number of unknowns. The elimination can be done using the resultant theory [43]. For this, a $2c_L$-order Sylvester matrix is constructed by eliminating $T_L$ in (4.16) and (4.17). For the common solutions of (4.16) and (4.17), Sylvester matrix is singular (but not vice versa). That is, it is necessary that the resultant $R_{T_L}$, which is the determinant of the Sylvester matrix is zero.

Next, one implements the conditions $\frac{\partial \omega}{\partial T_\ell} = 0$, $\ell = 1, ..., L - 1$, as a necessary condition for $\omega$ to exhibit an extrema in $\omega \in \mathbb{R}$, and following the approach presented in the previous subsection, one calculates the partial derivatives of the resultants, to sequentially eliminate the $T_\ell$ parameters. This approach can then be expressed as in Theorem 4.4, which covers the cases with more than two delays ($L > 2$) and when the resultants have singularity points [30, 104].

What we have from Theorem 4.4 is the polynomial $D(\omega) = 0$, which is analogous to $\Phi_2(z) = 0$ in (2.10) as presented in Chapter 1. We next incorporate controller gains $\{k_i\}_{i=1}^z$ into $D(\omega)$ polynomial and develop an algebraic approach to select $\{k_i\}_{i=1}^z$ such that the system is DIS. For this, we know that when $s = j\omega$ is a root of (4.4), so is $s = -j\omega$. That is, the extremum points in $\omega \in \mathbb{R}$ is symmetric with respect to $\omega = 0$ point, and thus $D(\omega)$ is an even polynomial in terms of $\omega$. Therefore, a variable change as $y = \omega^2$ becomes convenient as explained next. With the change of variable, we obtain $D(\omega) := \Phi(y, \{k_i\}_{i=1}^z) = 0$, and the analysis of the real $\omega$ roots of $D(\omega)$ reduces to checking the positive real roots $y$ of $\Phi(y, \{k_i\}_{i=1}^z) = 0$. This is exactly when the Sturm sequences discussed in the previous section becomes instrumental.
Theorem 4.10. System in (4.1) is delay-independent stable (DIS) in the delay parameter space \( \{ \tau_\ell \}_{\ell=1}^L \) with given \( \{ k_i \}_{i=1}^z \) gains in \( B_\ell \), if

i) \( |sI - A - \sum_{\ell=1}^L B_\ell| \) is Hurwitz stable,

ii)-(a) \( \lim_{y \to 0} \text{var}_S(\Phi(y, \{ k_i \}_{i=1}^z)) - \lim_{y \to \infty} \text{var}_S(\Phi(y, \{ k_i \}_{i=1}^z)) = 0 \),

ii)-(b) \( \lim_{y \to 0} \text{var}_S(\Phi(y, \{ k_i \}_{i=1}^z)) - \lim_{y \to \infty} \text{var}_S(\Phi(y, \{ k_i \}_{i=1}^z)) \neq 0 \) for some \( y = \tilde{y} = (\omega^*)^2 \in \mathbb{R}_+ \) with \( \Phi(\tilde{y}, \{ k_i \}_{i=1}^z) = 0 \), but an admissible solution satisfying \( V_d = 0, W_d = 0, d = 2, ..., L \), and (4.16)-(4.17) in \( \{ T_\ell \}_{\ell=1}^L \in \mathbb{R}_\infty \cup \{ \mp \infty \} \) does not exist.

Proof. As per Remark 4.1, for the system in (4.1) to be DIS, condition (i) should hold. Furthermore, a system can never change its stability/instability behavior with respect to delay \( \{ \tau_\ell \}_{\ell=1}^L \) if it does not possess any characteristic roots on the imaginary axis. That is, \( s = j \omega, \omega \in \mathbb{R} \), should not be a root of the corresponding characteristic equation in (4.4). Based on the derivations above and in connection with Theorem 2.7 and Theorem 4.4, this is equivalent to not having \( y = \omega^2 \in \mathbb{R}_+ \) roots of \( \Phi(y, \{ k_i \}_{i=1}^z) = 0 \) (case (ii)-(a)), and even if \( \omega^2 \in \mathbb{R}_+ \) roots of \( \Phi(y, \{ k_i \}_{i=1}^z) = 0 \) exist, all the corresponding \( \{ T_\ell \}_{\ell=1}^L \) solutions satisfying all the polynomials used in successive resultants in Theorem 4.4 are not in the real domain, and (4.16)-(4.17) have no common solutions in \( T_\ell \in \mathbb{R} \) domain (case (ii)-(b)).

In Theorem 4.10, Sturm sequences is used to identify the number of positive real solutions \( y = \omega^2 \). This way, we use algebraic methods and Theorems 4.10-4.4 in order to effectively study the possibility of ever having \( s = j \omega \) roots of (4.2) for some \( \tau_\ell \). This is how an algebraic approach allows designing DIS controllers for the system with multiple delays. Finally, notice that, when conditions (i)-(ii) of Theorem 4.10 hold, then \( A \) is guaranteed to be Hurwitz stable [104], hence checking Hurwitz stability of \( A \) is not necessary.

We recall that Sturm sequences is used above to guarantee the necessary and sufficient conditions for \( \Phi(y, \{ k_i \}_{i=1}^z) = 0 \) not to have positive real roots, but this alone does not guarantee DIS with necessary and sufficient conditions. This is because there may exist controller gains for which \( \Phi(y, \{ k_i \}_{i=1}^z) \) possesses positive real zeros, but which may correspond to inadmissible \( T_\ell \), in case, \( T_\ell \notin \mathbb{R} \) holds. In such cases, the system can still be DIS, as covered by condition (ii)-(b) of Theorem 4.10.

Theorem 4.11. For a monic polynomial \( A \), to possess no positive real roots, it is necessary that the nonzero coefficient of the lowest power of the polynomial is positive.

Proof. Let \( A = \sum_{k=0}^n a_k x^k \), with \( a_n = 1 \) for \( A \) to be monic, and consider the three scenarios:
i) When \( a_0 < 0 \), then \( \text{var}_D(A) \) is odd regardless of \( n \). According to Descartes rule of signs as stated in Theorem 2.2, polynomial \( A \) possesses at least one positive real root.

ii) When \( a_0 > 0 \), then \( \text{var}_D(A) \) is either even or zero regardless of \( n \). Therefore, according to Descartes rule of signs, polynomial \( A \) has either no positive real roots or has even number of positive real roots.

iii) When \( a_0 = 0 \) and possibly some other coefficients are zero, one must check the previous conditions (i)-(ii) on the nonzero coefficient of the smallest power of \( x \).

\begin{remark}
 In order to efficiently identify the DIS regions in \( \{k_i\}_{i=1}^z \), it is suggested to implement (i) of Theorem 4.10 on the delay-free system followed by Theorem 2.2 on \( \Phi(y, \{k_i\}_{i=1}^z) = 0 \), prior to implementing (ii)-(a) of Theorem 4.10. This way, one can first find the candidate controller gain regions using Theorem 4.11, which can then be further processed in condition (ii)-(a) of Theorem 4.10. It must be noted that for checking the rest of the admissible DIS controller gain regions, one must run a point-wise test of case (ii)-(b) in Theorem 4.10, without considering the conditions imposed by Theorem 4.11.
\end{remark}

The DIS controller design approach explained above can be summarize in Fig 4.1. The approach is broken down into four major steps:

1. Algebraic mapping step: Based on the number of delays in the characteristic equation (4.2), utilize Rekasius substitution in (4.3) for each term with delay to transform the transcendental equation (4.2) into an algebraic one as in (4.4).

2. DIS necessary conditions: Find the admissible controller gains \( \{k_i^*\}_{i=1}^z \) that satisfy the DIS necessary conditions stated in Remark 4.1.

3. Elimination procedure step: Calculate the iterative resultants introduced above based on the number of delays, in order to eliminate all \( \{T_i\}_{i=1}^L \) and to find \( D(\omega, \{k_i\}_{i=1}^z) = 0 \) polynomial.

4. Controller gain design: Use each admissible \( \{k_i^*\}_{i=1}^z \) set found in Step 2 as candidate point, and follow the two steps below in light of (ii)-(a) and (ii)-(b) of Theorem 4.10.

4.1. Use the condition stated in Remark 4.12 to narrow down the admissible controller gain region found in Step 2, and then implement the Sturm sequences, which can be easily computed with MAPLE software, and seek if Sturm sequences of \( \Phi(y, \{k_i^*\}_{i=1}^z) \) have any sign changes. If there are no sign changes, the examined controller gains \( \{k_i\}_{i=1}^z = \{k_i^*\}_{i=1}^z \) will render the AVC system in (4.1) delay-independent stable as per condition (ii)-a in Theorem 4.10. If there exists at least one sign change then proceed to the next step.
Figure 4.1: Flowchart of controller gain design leading to DIS of the multiple time-delay system (4.1). See Theorem 4.4 for definitions.
4.2. Use candidate points \( \{ k_i^* \}_{i=1}^z \) to check whether or not the condition (ii)-b in Theorem 4.10 holds. Controller gains, \( \{ k_i^* \}_{i=1}^z \), that satisfy condition (ii)-b will render the system in (4.1) delay-independent stable.

### 4.3.1 Extraction of DIS regions; Sufficient Conditions Using Metzlerian Systems’ Properties

Among the special types of systems, the class of positive systems play an important role for continuous-time and discrete-time systems [34, 55]. The positive continuous-time systems also known as Metzlerian systems can also be defined even in the presence of delays.

**Definition 4.13.** The system (4.1) without the feedback control \( K \) is internally positive or Metzlerian delay system if for every initial condition and all input \( u(t) \in \mathbb{R}^m_+ \), \( t \geq 0 \), we have \( x(t) \in \mathbb{R}^n_+ \) and \( y(t) \in \mathbb{R}^r_+ \) for \( t \geq 0 \).

**Theorem 4.14.** The system (4.1) without the feedback gain \( K \) is internally positive if and only if \( A \) is a Metzler matrix and the matrices \( \bar{B}, C_\ell \) for all \( \ell = 1, \ldots, L \) are matrices with nonnegative entries, \( \bar{B} \in \mathbb{R}^{n \times m}_+ \), and \( C_\ell \in \mathbb{R}^{r \times n}_+ \).

Incorporating the feedback control \( K \) into (4.1), we obtain

\[
\dot{x}(t) = Ax(t) + \sum_{\ell=1}^L B_\ell x(t - \tau_\ell). \tag{4.18}
\]

Based on Definition 4.13 and Theorem 4.14, one can state that (4.18) is a Metzlerian delay system if and only if \( A \) is a Metzler matrix, and the matrices \( B_\ell \) have nonnegative entries, \( B_\ell \in \mathbb{R}^{n \times n}_+ \), \( \ell = 1, \ldots, L \).

**Lemma 4.15.** The delay-free Metzlerian system in (4.18) with \( \{ \tau_\ell \}_{\ell=1}^L = 0 \) is asymptotically stable if and only if one of the following equivalent conditions are satisfied:

1) Eigenvalues of \( \bar{A} = A + \sum_{\ell=1}^L B_\ell \) have negative real parts.
2) All coefficients of the characteristic equation \( \det(\lambda I - \bar{A}) = \lambda^p + \bar{a}_{p-1}\lambda^{p-1} + \ldots + \bar{a}_1\lambda + \bar{a}_0 \) are positive, i.e. \( \bar{a}_i > 0 \) for \( i = 1, \ldots, p - 1 \).
3) All principal minors of the matrix \( -\bar{A} \) are positive.
4) The matrix \( \bar{A} \) is nonsingular and \( -(\bar{A})^{-1} > 0 \).
5) There exists a positive definite diagonal matrix \( P \) such that \( \bar{A}^T P + P \bar{A} \) is negative definite.
6) There exists a positive vector \( v \in \mathbb{R}^n_+ \) such that \( \bar{A}v < 0 \).
Now, assuming that (4.18) is a Metzlerian delay system, we can state the following result:

**Theorem 4.16.** The Metzlerian delay system (4.18) is DIS if and only if the Metzlerian system without delay $\dot{x}(t) = \bar{A}x(t)$ is asymptotically stable.

The proof of this result can be established by writing the solution expression of (4.18) and show that if (4.18) is asymptotically stable then there exists a strictly positive vector $v \in \mathbb{R}_+^n$ satisfying $\bar{A}v < 0$, which is precisely the part (6) of Lemma 4.15 associated with the Metzlerian system without delay. See [86, 96] for the details of the proof. With the aid of Theorem 4.16, the Property 3.1 in Chapter 3 can further be simplified. Given $\{k_i\}_{i=1}^z$, the Metzlerian delay system (4.18) is delay-independent stable if and only if,

$$f(s, \{\tau_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) \big|_{\{\tau_\ell\}_{\ell=1}^L = 0} = |\lambda I - \bar{A}|,$$

has no roots on $\mathbb{C}_{0+}$. Consequently, the existence of DIS gain regions of Theorem 4.9 can also be simplified as follows.

**Corollary 4.17.** For a given set of delays $\{\tau_\ell\}_{\ell=1}^L$, there exists at least one region in controller gain parameter space $\{k_i\}_{i=1}^z$ where the Metzlerian delay system (4.18) is DIS if and only if $\bar{A} = A + \sum_{\ell=1}^L B_\ell$ is asymptotically stable for $\{k_i\}_{i=1}^z = \{\bar{k}_i\}_{i=1}^z$.

In a similar fashion Theorem 4.10 simplifies as follows.

**Corollary 4.18.** The Metzlerian delay system (4.18) is DIS in the delay parameter space $\{\tau_\ell\}_{\ell=1}^L$ with given gains $\{k_i\}_{i=1}^z = \{\bar{k}_i\}_{i=1}^z$ in $B_\ell$ if and only if $\bar{A} = A + \sum_{\ell=1}^L B_\ell$ is asymptotically stable.

### 4.3.2 Analytical Boundaries Responsible for Stability Switch in Control-Gain Space

In this section, we show under certain conditions that DIS boundaries in controller gain space $\{k_i\}_{i=1}^z$ can be identified. We start with $\Phi(y, \{k_i\}_{i=1}^z)$ and Proposition 3.7 previously stated in Chapter 3.

According to Property 3.1, Definition 2.10, and in light of Proposition 3.7, we are now able to state that if double roots of $\Phi(y, \{k_i\}_{i=1}^z)$ exist for some $\{k_i\}_{i=1}^z \in \mathbb{R}$, then the discriminant of $\Phi(y, \{k_i\}_{i=1}^z) = 0$ with eliminating $y$ should vanish, and one is able to calculate $\varphi(\{k_i\}_{i=1}^z) = 0$ as in (3.15). That is, utilizing all the consecutive resultant operation steps back in Section 4.2 and properties for simplifying the elimination operation.
stated in Subsection 4.2.2, one can calculate $\Phi(y, \{k_i\}_{i=1}^z)$ and follow theorems below to categorize the controller gains space for DIS property.

As per Proposition 3.7, the cases that yield a non-DIS system are when there exist either positive real roots $y^* > 0$ of (3.15) (for case (i) of Proposition 3.7), or $\gamma_0(\{k_i\}_{i=1}^z) = 0$ (for case (ii) of Proposition 3.7), and if, in each case, the corresponding $\omega = \sqrt{y^*} > 0$ and $\{T^*_\ell\}_{\ell=1}^L$ satisfy the characteristic equation in (4.4). Otherwise, for a given $\{k_i\}_{i=1}^z$ we may have a DIS property inside the regions bordered by the hypersurfaces defined by $\phi(\{k_i\}_{i=1}^z) = 0$. Hence, one has to inspect all these enclosed regions arising due to the boundaries $\phi(\{k_i\}_{i=1}^z) = 0$ and $\gamma_0(\{k_i\}_{i=1}^z) = 0$, and identify the segments of those boundaries encapsulating the DIS regions. Let us denote these refined segments with $\Lambda(\{k_i\}_{i=1}^z)$, which is obviously a subset of $\Gamma(\{k_i\}_{i=1}^z) = \{\{k_i\}_{i=1}^z \in \mathbb{R} | \phi(\{k_i\}_{i=1}^z) = 0 \cup \gamma_0(\{k_i\}_{i=1}^z) = 0\}$, which is the union of all the hypersurfaces along which the system can be potentially switching from DIS to non-DIS property in the controller gain space.

To identify all the DIS regions, one has to inspect all the arising enclosed regions in $k_i$, denoted by $S_{\chi}$, $\chi = 1, \ldots, \kappa$, and encapsulated by the boundaries of $\Gamma(\{k_i\}_{i=1}^z)$. Theorem 4.19. System in (4.1) is delay-independent stable (DIS) in the region $S_{\chi}$, $\chi = 1, 2, \ldots, \theta$, $\theta < \kappa$, if for a test point $\{k_i\}_{i=1}^z \in S_{\chi}$, the following two conditions simultaneously hold:

i) $|sI - A - B_\ell|$ is Hurwitz stable,

ii) Single point inspection in $\{S_{\chi}\}_{\chi=1}^\theta$ when $\{y > 0 | \Phi(y, \{k_i\}_{i=1}^z) = 0\} = \emptyset$,

Proof. Condition (i) is obvious from Theorem 4.10. Furthermore, system (4.1) can never change its stability/instability behavior with respect to delay $\tau_\ell$ if (4.2) does not possess any roots on the imaginary axis. That is, $s = j\omega$ should not be a root of (4.4). Based on the features of the polynomial in (3.14), this can be guaranteed if the polynomial $\Phi(y, k_i) = 0$ does not have $y = \omega^2 \in \mathbb{R}_+$ roots for a single test point in each identified closed region $S_{\chi}$, $\chi = 1, 2, \ldots, \theta$. By continuity, this property is guaranteed to hold for all the points in the region enclosed by some parts of $\Gamma(\{k_i\}_{i=1}^z)$. The condition (ii) of theorem thus reveals the fundamental DIS regions $\{S_{\chi}\}_{\chi=1}^\theta$ in the controller gain space. □

Theorem 4.20. System in (4.1) is DIS in $\{k_i\}_{i=1}^z \in \mathbb{R}^z \backslash \{S_{\chi}\}_{\chi=1}^\theta$, if and only if the conditions below are satisfied:

i) $|sI - A - B_\ell|$ is Hurwitz stable,

ii) Point-wise sweep of controller gains in $\{k_i\}_{i=1}^z \in \mathbb{R}^z \backslash \{S_{\chi}\}_{\chi=1}^\theta$ whenever $\{y >
0|Φ(y, {k_i}_{i=1}^z) = 0} \neq \emptyset \text{ for some } y^* = (\omega^*)^2 \in \mathbb{R}_+, \text{ and when at least one condition in (B)-(C) of Theorem 4.4 does not hold.}

**Proof.** The condition provided in this theorem guarantees the only case for which Φ(y, k_i) = 0 have roots y = (ω^*)^2 \in \mathbb{R}_+, but these roots do not correspond to a root in T_ℓ \in \mathbb{R} \bigcup \{-\infty\} that vanishes (4.16) and (4.17) simultaneously. This inspection however requires point-wise sweep in all the remaining regions in \{k_i\}_{i=1}^z \in \mathbb{R}^\chi \backslash \{S_\chi\}_{\chi=1}^\theta. \text{ Hence, along with the conditions (i)-(ii) of Theorem 4.19, all the DIS regions of system (4.1) can be identified. □}

**Remark 4.21.** Checking the DIS properties in the regions enclosed by \varphi({k_i}_{i=1}^z) = 0 and \gamma_0({k_i}_{i=1}^z) = 0 requires a simple single-point test in each arising enclosing region, as per Theorem 4.19. With this, some of the DIS regions can be identified and thus their boundaries \Lambda({k_i}_{i=1}^z) can be revealed. However, when a positive real root y of Φ(y, k_i) = 0 exists in a region S_\chi, then single-point test is not sufficient in this region since it is not possible to claim that all the T_ℓ corresponding to all k_i in this region can remain in either real domain or complex domain. Hence, the point-wise scanning in such regions as covered by Theorem 4.20 is still necessary.

Next two chapters provide applications of the developed theories in designing delay-independent stable LTI-TDS with multiple delays.

### 4.3.3 Review: Delay-independent Stable Control Space Categorization

**Steps**

Given the characteristic equation of an LTI-TDS with multiple delays, DIS controller gain design can be performed in 10 steps:

**Step 1.** Set all the delays equal to zero and find the characteristic equation representing the delay-free system. Stability analysis of the delay-free system is trivial. One can use the Ruth-Hurwitz method to find the regions in control-gain space within which the delay-free system is stable.

The non-trivial segment of the DIS analysis is starting from Step 2.

**Step 2.** Use the Rekasius transformation [88] to replace the exponential functions in characteristic equation with,

\[
e^{-\tau s}|_{s=j\omega} \equiv \frac{1 - T_\ell s}{1 + T_\ell s}|_{s=j\omega}, \quad (4.20)
\]
Notice that in the Padé approximation both sides of the identity are in terms of $\tau$. For example, first order Padé approximation at $s = j\omega$ is given by, $e^{-\tau s} \approx \frac{1 - \frac{\tau}{2s}}{1 + \frac{\tau}{2s}}$, which is different from (4.20). [100].

**Step 3.** Since the focus is to study stability vs instability, the objective is to investigate whether or not the system can ever possess eigenvalues on the imaginary axis $s = j\omega$, which is indication of loss of stability [72]. With the substitution of (4.20) into characteristic equation, and with a manipulation to remove the fractions, one obtains the transformed characteristic equation on the imaginary axis $s = j\omega$ as

$$g(j\omega, \{T_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) = \left( f \left( e^{-j\omega T_\ell} = \frac{1-j\omega T_\ell}{1+j\omega T_\ell} \right) \prod_{\ell=1}^L (1 + j\omega T_\ell)^{c_\ell} \right) = 0,$$  \hspace{1cm} (4.21)

where $g$ is a polynomial with complex coefficients. We suppress the details regarding various utilizations of (4.21), referring the readers to the citations within this chapter as well as [57, 69, 104, 113].

**Step 4.** Rekasius transformation is an *exact substitution* for the imaginary roots of the characteristic equation, and all the $s = j\omega$ roots of the characteristic equation and of (4.21) are identical [109]. This is in one sense a way of converting the stability problem to a robust control problem in (4.21) with respect to parameter $T_\ell$, owing to property that $s = j\omega$ roots are preserved under the transformation (4.20). Consequently, delay-independent stability can be studied using (4.21) instead of the given characteristic equation. For the characteristic equation not to have eigenvalues at $s = j\omega$ for all $\{\tau_\ell\}_{\ell=1}^L \in \mathbb{R}_+^L$ is thus analogous to requiring that no $s = j\omega$ solutions of (4.21) exist for all $\{T_\ell\}_{\ell=1}^L \in \mathbb{R}^L$ [30, 119]. To the author’s best knowledge, this information was utilized for the first time in [119] for single delay problems, $L = 1$. Extension to $L > 1$ is however not trivial [72].

**Step 5.** Rewrite $g$ as follows,

$$g(j\omega, \{T_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) = g_R(\omega, \{T_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) + jg_\Im(\omega, \{T_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) = 0,$$  \hspace{1cm} (4.22)

where $g_R$ and $g_\Im$ are respectively the real and imaginary parts of $g$. For (4.22) to hold, $g_R$ and $g_\Im$ must have at least one common solution.
Step 6. Eliminate the unknown $T_L$ from $g_R = 0$ and $g_\Im = 0$ using the Resultant Theory [43]. This can be done by building a Sylvester matrix, which is singular whenever $g_R$ and $g_\Im$ have a common solution (but the converse is not always true). That is, the resultant $R_{TL}(\omega; \{T_\ell\}_{\ell=1}^{L-1}, \{k_i\}_{i=1}^z)$, which is the determinant of the Sylvester matrix should be zero as a necessary condition for $g_R$ and $g_\Im$ to have at least one common solution,

$$R_{TL}(g_R, g_\Im) := \det(S) = 0.$$  \hspace{1cm} (4.23)

Step 7. DIS analysis for the multiple time-delay system at hand can actually be constructed using the principles of resultant theory and starting with (4.23). Recall from Step 4 that DIS requires that $s = j\omega$ should not be a solution of the characteristic equation, or in $T_\ell$ domain, $s = j\omega$ should not be a solution of (4.21), or equivalently (4.23), see [30]. Instead of numerically searching for whether or not feasible $\omega$ solutions of (4.23) exist, one can instead seek whether or not the max/min of $\omega$ ever exists in $\omega \in \mathbb{R}_+$. If max/min of $\omega$ does not exist for all $T_\ell \in \mathbb{R}$, it is then guaranteed that $s = j\omega$ cannot be a solution of (4.21) and (4.23) for all $T_\ell$, which would ultimately imply that $s = j\omega$ cannot be a root of the original characteristic equation in the characteristic equation for all $\tau_\ell$, as per Step 4.

Step 8. If, by construction, the delay-free system (with $\{\tau_\ell\}_{\ell=1}^L = 0$) is stable, and if from Step 7, one reveals that $s = j\omega$ is not a feasible solution of the characteristic equation, then the system at hand is guaranteed to be controlled independent of the amount of uncertain delays [70–72], since loss of stability is not possible by the existence of an eigenvalue of the system of the form $s = j\omega$.

Step 9. Construct the proposed optimization scheme in Step 7 starting with (4.23), which requires multiple elimination procedures on all the parameters $T_\ell$, leads to the polynomial $D(\omega) = 0$ as explained in Section 4.2.2 and from [30, 72]. Incorporate the controller gains $\{k_i\}_{i=1}^z$ into this polynomial, which yields $V_1 := D(\omega, \{k_i\}_{i=1}^z) = 0$.

Step 10. The roots of the polynomial $D(\omega) = 0$ carry critical information about system’s spectrum at $s = j\omega$. If none of these roots are real, and the condition in Step 8 holds, then the system is declared DIS. If a real root of $D(\omega) = 0$ exists, then one checks whether or not at least one of the corresponding $T_\ell$ values is not real, where $T_\ell$ is calculated sequentially from $V_2, \ldots, V_L$, see Section 4.2.2. If at least one $T_\ell \notin \mathbb{R}$, then the calculated $\omega \in \mathbb{R}$ is not feasible, and hence with the condition in Step 7 still holding, the system can be declared as DIS.
In the next section, we study categorization of control gains space for DIS. A third order system with three unknown output delays is investigated utilizing the method explained above within this chapter.

### 4.4 Case Study

Consider an LTI TDS as,

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= \sum_{\ell=1}^{3} C_{\ell}x(t - \tau_{\ell}), \\
u(t) &= K\hat{x}(t),
\end{align*}
\]

where the matrices are defined the forms below,

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad C_{1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_{2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_{3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad K = [k_{1} k_{2} k_{1} k_{2}].
\]

The characteristic equation of the system becomes,

\[
f(s, k_{1}, k_{2}, \tau_{1}, \tau_{2}, \tau_{3}) = |sI - A - (BK_{1}e^{-\tau_{1}s} + BK_{2}e^{-\tau_{2}s} + BK_{3}e^{-\tau_{3}s})|,
\]

which we would like to render DIS by appropriate design of \((k_{1}, k_{2})\) gains. In the following, we will use the developed theories to find the the closed-form expressions of the DIS boundaries in \((k_{1}, k_{2})\) gain space. First step as in Theorem 4.10 is to find the region in control gain space for which the delay-free system is stable, see Step 1 in previous section. The delay-free system is represented as in (4.27) and shown in Figure 4.2.

\[
f(s, k_{1}, k_{2}) = s^{3} + (6 - k_{1}k_{2} - k_{1} + k_{2})s^{2} + (11 + 5k_{2} - 5k_{1}k_{2} - 5k_{1})s - 6(k_{1}k_{2} + k_{1} - k_{2} - 1)
\]

Next, as in Step 2, we utilize the rekasius substitution. After simplification and finding the transformed characteristic equation, we separate the imaginary and real parts of (4.26) and excluding \(\omega = 0\) roots as per Remark 3.1 as in Step 5. Following Step 6-9, we find the DIS region within the delay-free shaded region, Figure 4.3.
Figure 4.2: Hurwitz stable region (light gray) in \((k_1, k_2)\) space when the delays are zero, \(\tau_1 = \tau_2 = \tau_3 = 0\).

Figure 4.3: DIS region of the combined system (dark gray) in \((\tilde{k}_1, \tilde{k}_2)\) found by using Theorem 3.11. The curves are determined by \(\gamma_i = 0\).
4.5 Conclusion

Algebraic approaches to perform DIS stability analysis and controller gain space categorization of LTI systems with multiple delays are proposed within this chapter, where the control gain space is decomposed into regions for which the system is delay-independent stable. The presented approach on DIS control design is the utilization of algebraic geometry according to founded theorems in previous chapter and expanding their applications on systems with multiple unknown delays. As before for single delay systems, the boundaries decomposing the gain space can be expressed in explicit form in terms of gain parameters. In the case of multiple delay systems, elimination process is getting more complex and without the extra treats, explained within this chapter, on resultant singularities and factorization properties in consecutive elimination operations, it is usually impossible to analyze complex systems with more than two delays. Utilization of optimization techniques within the spotted DIS region in control gain space, will lead to applications of these optimal controllers on real-time experiments. we will investigate some case studies along with real-time experiments/simulations in Chapters 5-6.
Chapter 5

DIS Controllers & Active Vibration Control (AVC)

One of the critical parameters that can deteriorate the effectiveness of active vibration control (AVC) is the delay in sensors or control commands. Especially, in remote sensing where delays are large, and in high-speed applications with even small delays, instability can be inevitable. This chapter presents algebraic approaches to design controllers in order to achieve stability regardless of the amount of delays for AVC applications modeled by linear time-invariant systems with “multiple” constant delays. The approach is based on developed tools in DIS controller design methods in Chapter 4.

5.1 Literature Review

In many environments where dynamical systems operate, vibrations are inevitable. Such dynamics are found in space applications, structural engineering, automotive design, helicopter rotor dynamics [16, 17, 60, 98, 135, 137]. Active vibration control (AVC) has matured as one of the ways to eliminate the detrimental effects of vibrations to structures [40, 42, 62]. For this, a feedback control system with sensing and actuation capabilities is designed to appropriately produce counteractions against vibrations. Since the controller can be designed to be adaptive to temporal changes in vibratory environment, AVC is quite effective and thus broadly utilized [1, 16, 17, 60, 79, 92, 98, 135–137]. Sensor delays are however ubiquitous in many control applications including AVC [106]. The presence of delays can be extremely critical for such applications, especially for those with remote
sensing where delays can be large, and in high-speed AVC applications even if delays are small. Since, with sensor delays, information cannot be available instantaneously [106], control actions designed with conventional techniques based on such delayed information may lead to poor performance and instability, as pointed out in [45, 75, 108].

Even for linear time-invariant systems, control design in the presence of “multiple” constant delays can be extremely challenging [67, 106]. As it is well known, this is primarily due to the need to deal with infinite-dimensional eigenvalue problems, which are cumbersome to analyze/solve efficiently. Both time-domain and frequency-domain analysis are proposed in the literature to systematically address some of the major challenges [45, 61, 66, 67, 75, 99, 106, 109, 114]. Existing work relevant to this chapter include stability analysis of systems with delays that can be addressed without any conservatism using frequency domain techniques (FDT) [46, 66, 104, 109, 114], and control design that can be performed via eigenvalue-based optimization schemes [9, 64, 125, 134].

Most of the DIS and DDS studies related to both analysis and control synthesis focus on single delay problems, see [13, 32, 45, 64, 66, 70]. DIS/DDS analysis for systems with multiple delays can be found, for instance, in [46, 104, 109, 114, 129]. Furthermore, in [99] a PID controller is proposed to stabilize a first-order single-input single-output (SISO) plant with a single delay. For a given delay in the plant dynamics, authors of the cited study utilize the extended Hermite-Biehler theorem in order to reveal the stabilizing parametric regions in the controller gain space. Many other PID design strategies are also proposed in the literature for stabilizing SISO plants, some of which are based on FDT, see [32] and the references therein. Other approaches include the work in [129], where FDT is used to perform delay-independent stability analysis and parametric design for a special class of characteristic equations with a maximum of two delays. Finally, the recent results in [30] address the DIS analysis part of the problem with multiple delays, with some initial attempts in DIS control design, but using a conservative framework based on Descartes Rule of Signs [29].

Another important aspect in control design is to have access to sensor measurements. For cost effectiveness reasons and due to design constraints, however, not all the measurements in an AVC system can be available. Hence, in many cases when designing the controller gains of AVC systems, it is mandatory that the structure of the controller is taken into account. To summarize, one of the problems in the literature that we wish to address in this chapter is on the non-conservative synthesis of structured controllers in order to achieve DIS of the closed-loop AVC systems with multiple constant delays.
One could appreciate the benefits of making such systems delay-independent stable, especially in scenarios where constant delays are uncertain and/or unknown. To the best of our knowledge, however, an approach to reveal the feasible controller gain regions for DIS of such systems based on a non-conservative framework has not been yet reported. Without limiting the number of delays, we develop an algebraic approach to achieve DIS control design, departing from FDT [66, 114], algebraic geometry [2, 20], and elimination techniques [20]. The approach is inspired from our recent work in [30, 70], where one reveals some polynomials, roots of which are parametrically dependent on controller gains and these roots are directly associated with the DIS analysis of the LTI system with multiple delays. In particular, here these polynomials are connected with the results of Sturm sequences [38, 117] in order to systematically identify the controller-gain regions where the vibration control system remains DIS. These results are then compared with our previous conservative results based on the Descartes rule of signs [29, 70]. While designing such controllers, we also incorporate their specific structure into the algebraic approach developed here. On case studies, we then investigate the vibration suppression capabilities of the DIS controllers, which we find to be even more effective for some frequency bands compared with vibration suppression results based on the assumption that delays are zero. Finally, it is noted that the utilization of FDT incorporates necessary and sufficient stability conditions into the design procedure, preventing any conservatism in the DIS control design.

5.2 DIS AVC Design

We start with the general form of a linear time-invariant vibration dynamics represented in state-space as

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + F(t), \\
u(t) &= \sum_{\ell=1}^{L} K_\ell x(t - \tau_\ell),
\end{align*}
\]

(5.1)

where \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\) are given constant matrices, \(x(t) \in \mathbb{R}^n\) is the state vector, \(u(t) \in \mathbb{R}^{m \times 1}\) is the control input, \(F(t)\) has entries in the form of continuous harmonic functions, \(\tau_\ell\) are the unknown constant delays that arise when measuring system states, \(L \in \mathbb{N}\) is the maximum number of delays in the system, and \(K_\ell\) are the controllers with constant entries \(k_i \in \mathbb{R}\), denoted as a set \(\{k_i\}_{i=1}^\ell\). Since all the states may not be measurable, we assume that \(K_\ell\) has a particular structure with zero and
nonzero entries, and we define $B_\ell = B K_\ell$ whose entries are parameterized by unknown controller gains, $k_i$. Here, the entries of $F(t)$ are assumed to be continuous, independent of system states, and are upper and lower bounded in time, hence $F(t)$ does not affect the stability of (5.1). In other words, stability of (5.1) is determined by its homogeneous part, which is associated with the following characteristic equation expressed in Laplace domain as

$$CE(s, \{\tau_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) = |s I - A - \sum_{\ell=1}^L B_\ell e^{-\tau_\ell s}| =$$

$$\sum_{q=0}^Q P_q(s, \{k_i\}_{i=1}^z) e^{-\sum_{\ell=1}^L \xi_{q\ell} \tau_\ell s} = 0, \quad (5.2)$$

where $P_q$ are polynomials in terms of $s$ and the controller gains $\{k_i\}_{i=1}^z$, $Q \in \mathbb{Z}_+$, and $\xi_{q\ell} \in \mathbb{N}$. Since there exists no delay in the highest-order derivative of the states in (5.1), the characteristic equation (5.2) represents a retarded class LTI system with multiple delays [45, 114], and thus $\xi_{0\ell} = 0$. Here the order of commensurability of delay $\tau_\ell$ is denoted with $c_\ell$, which is given by $c_\ell = \max_{0 \leq q \leq Q}(\xi_{q\ell})$. 

A two-degree-of-freedom active shock absorber system shown in Figure 5.1 is considered as a case study, with

$$m_1 \ddot{x}_1 = -v_1 x_1 - c_1 \dot{x}_1 - v_2 (x_1 - x_2) - c_2 (\dot{x}_1 - \dot{x}_2) + \sin(\mu t) + u_2 - u_1,$$

$$m_2 \ddot{x}_2 = -v_2 (x_2 - x_1) - c_2 (\dot{x}_2 - \dot{x}_1) - u_2,$$  

where $m_1$ and $m_2$ are the masses, $v_1$ and $v_2$ are spring constants, $c_1$ and $c_2$ are damping constants of the vibration system at hand, and the numerical values of these constants are taken as $m_1 = 3 \text{ kg}$, $m_2 = 1 \text{ kg}$, $v_1 = 100 \text{ N/m}$, $v_2 = 100 \text{ N/m}$, $c_1 = 50 \text{ N.s/m}$, and $c_2 = 40 \text{ N.s/m}$. 

In (5.3), $x_1$ and $x_2$ are respectively the vertical displacements of the masses $m_1$ and $m_2$ from their respective equilibrium points, and $u_1$ and $u_2$ are the control actions produced by controllers to be designed to attenuate the vibrations of mass $m_1$ against the sinusoidal force disturbance $f(t) = \sin(\mu t)$. In this example, we assume that the controllers receive some or all of the measurements $x_1$, $x_2$, $\dot{x}_1$ and $\dot{x}_2$ but with delays, making the design of $u_1$ and $u_2$ nontrivial.
5.2.1 Case 1: Active Vibration Suppression with Position Feedback

The state-space representation of (5.3) is given by,

\[ \dot{X}(t) = AX(t) + Bu(t) + F(t), \]  
(5.4a)
\[ y(t) = C_1X(t - \tau_1) + C_2X(t - \tau_2), \]  
(5.4b)
\[ u(t) = Ky(t), \]  
(5.4c)

where we assume that only \( x_1 \) and \( x_2 \) are measured but with delays \( \tau_1 \) and \( \tau_2 \), respectively. Under this condition we have,

\[
X = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{v_1v_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{v_2^2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{v_2^2}{m_2} & \frac{c_2}{m_2} & -\frac{v_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix},
\]

\[
K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad F(t) = \begin{bmatrix} 0 & \sin(\mu t) \\ 0 & 0 \end{bmatrix}.
\]
The system in (5.4) can be expressed compactly as

\[
\dot{X}(t) = AX(t) + BK_1X(t-\tau_1) + BK_2X(t-\tau_2) + F(t),
\]

from which the characteristic equation is found from the homogeneous part as,

\[
CE(s, \tau_1, \tau_2, k_1, k_2) = \left| sI - A - BK_1e^{-\tau_1s} - BK_2e^{-\tau_2s} \right| = \\
s^4 + 70s^3 + \left( k_1e^{-\tau_1s} + \frac{2500}{3} + k_2e^{-\tau_2s} \right)s^2 + \left( 3000 - 10k_2e^{-\tau_2s} + 40k_1e^{-\tau_1s} \right)s \\
+ 10000 - \frac{100}{3}k_2e^{-\tau_2s} + 100k_1e^{-\tau_1s} + k_1k_2e^{-(\tau_1+\tau_2)s} = 0.
\]

Recall from Remark 4.1 that for the system to be delay-independent stable, it is required that \(A\) is Hurwitz stable, which is satisfied in this case, since the eigenvalues of matrix \(A\) are all on the left half complex plane, \(s_1 = -56.0739, s_2 = -8.6668, s_3 = -2.8668,\) and \(s_4 = -2.3926.\) Next, we analyze the Hurwitz stability of the delay-free controlled system as per condition (i) of Theorem 4.10. This part of the analysis is trivial. Using the well-known Routh-Hurwitz stability criterion, we can find the hatched regions in \(k_1 - k_2\) where (5.6) is stable for \(\tau_1 = \tau_2 = 0,\) see Figure 5.2. In the shaded region in Figure 5.2,

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig.png}
\caption{Case 1. Hurwitz stable region for the delay-free system in the controller gain plane.}
\end{figure}

we next implement our proposed approach, starting with the Rekasius substitution (4.3) into (5.6), separation of the real and imaginary parts of the characteristic equation as in (4.16)-(4.17), and application of two consecutive resultant computations to eliminate \(T_1\) and \(T_2,\) see also Figure 4.1. This yields the function \(D(\omega, k_1, k_2) = \sum_{q=0}^{12} \gamma_q \omega^{2q}\) defined
in Theorem 4.4. According to Theorem 4.10, the next step is to find the $k_1 - k_2$ regions in the shaded region in Figure 5.2 where the constant term of $D(\omega, k_1, k_2)$ is positive, see Theorem 4.11, i.e., $\gamma_0 > 0$, which is presented in Figure 5.3.

![Figure 5.3: Case 1. The positivity of $\gamma_0$ coefficient in controller gain plane is satisfied in the shaded regions.](image)

Now we are at Step 4 of our DIS controller design for the case study at hand. Step 4.1 requires finding the regions where the conditions in both Figure 5.2 and Figure 5.3 hold, see Figure 5.4. These common regions are where one should explore DIS controller gains, by implementing the Sturm sequences on $D(\omega, k_1, k_2)$ for each $(k_1, k_2)$ pair, in the shaded region in Figure 5.4, as per case (ii)-(a) of Theorem 4.10. Finally, Step 4.2 of Summary subsection is taken into account. The admissible controller gains extracted from the shaded region in Figure 5.2, excluding the points already found to be DIS at Step 4.1, are to be tested by implementing the case (ii)-(b) of Theorem 4.10. The design process results in categorizing the whole controller gain space at hand where all the conditions in Theorem 4.10 are satisfied (black color in Figure 5.5). In other words, $(k_1, k_2)$ gains selected from this region render the controlled vibration system in (5.4) DIS.

The control design goal in this study is complete here, and future work can focus on choosing the optimal gain combinations from the DIS regions in Figure 5.5. Yet, we wish to demonstrate in the sequel that, with proper gain selection, DIS controller gains can effectively attenuate vibrations for a certain range of frequencies.
Figure 5.4: Case 1. Intersection of the delay-free stable region and the regions where $\gamma_0 > 0$. The shaded region is used at Step 4.1, given by Summary Subsection III.D.

Figure 5.5: Case 1. DIS region in black found by the application of Theorem 4.10.
5.2.2 Frequency Response Analysis

The outputs of the vibration control system in (5.4) are expressed in Laplace domain as

\[ X(s) = (sI - A - BKC_1 e^{-\tau_1 s} - BKC_2 e^{-\tau_2 s})^{-1} F(s), \]  
(5.7)

where \( k_1 \) and \( k_2 \) are to be selected such that (5.7) is DIS. Having Figure 5.5 available, this is now an easy task. Choosing a \((k_1, k_2) = (15, 15)\) pair from the black-colored region in Figure 5.5 guarantees DIS. Next, we focus on the behavior of \( x_1(t) \). Letting the delays be \( \tau_1 = 2 \) and \( \tau_2 = 1 \), the transfer function between the input force and the output displacement of mass \( m_1 \) is expressed as,

\[
\frac{x_1(s)}{F(s)} = \frac{3s^3 + 210s^2 + (2800 + 3k_2 e^{-s})s + 5000 - 30k_2 e^{-s}}{3s^4 + 210s^3 + (3k_1 e^{-2s} + 2500 + 3k_2 e^{-s})s^2 + (9000 - 30k_2 e^{-s} + 120k_1 e^{-2s})s + 10000 - 100k_2 e^{-s} + 300k_1 e^{-2s} + 3k_1 k_2 e^{-3s}}.
\]  
(5.8)

We next study (5.8) for the frequency response of the mass \( m_1 \) under harmonic excitation \( f(t) = \sin(\mu t) \). The frequency response of output \( x_1(t) \) is presented in Figure 5.6.

According to Figure 5.6, it is clear that vibration attenuation capacity of the closed-loop system is improved when delays do not exist, compared with the uncontrolled system. But with inevitable sensor delays, the system becomes infinite dimensional, with infinitely many vibration modes, as seen from the many peaks in the frequency response curve in Figure 5.6. We observe in this figure that one can still use the same controller
gains, while making the vibration system delay-independent stable and improving the frequency response of the controlled system within some frequency bands. For instance in a simulation case, we examine the amplitude of oscillations for both \( m_1 \) and \( m_2 \) for a sinusoidal input as, \( F(t) = [0, 5 \sin(2.6t), 0, 0]^T \). We use the same controller gains selected from the DIS region in Figure 5.5 as \((k_1, k_2) = (15, 15)\), and let the position-measurement delays be \( \tau_1 = 2 \) and \( \tau_2 = 1 \). The oscillation amplitudes for \( x_1 \) and \( x_3 \) in steady state show respectively a 25% and 10% reduction with the delayed controller compared with the delay-free controlled system, see Figure 5.7.

\[ \begin{align*}
\end{align*} \]

**Figure 5.7:** Case 1. Oscillations in the position of \( m_1 \) and \( m_2 \) with controller gains selected from the DIS region in Figure 5.5 as \((k_1, k_2) = (15, 15)\), and with the position-measurement delays as \( \tau_1 = 2 \) and \( \tau_2 = 1 \). Excitation applied on \( m_1 \) and it is \( 5 \sin(2.6t) \).

### 5.2.3 Case 2: Active Vibration Suppression with Velocity Feedback

In this case, we assume that only velocity feedback with two output delays is available for the controller design for the system in Figure 5.1. Consider the system with the specified characteristics as in the previous case study, but with \( C_1 \) and \( C_2 \) modified for velocity feedback as

\[ \begin{align*}
C_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{align*} \]  

\[ (5.9) \]

In this case, the delay-free system is found to be Hurwitz stable for the gains in the gray region in Figure 5.8. We then find that the coefficient \( \gamma_0 \) of the lowest power of
Figure 5.8: Case 2. Hurwitz stable region for the delay-free system in the controller gain plane.

Figure 5.9: Case 2. DIS region in black found by the application of Theorem 4.10.
\( \omega \) in \( D(\omega, k_1, k_2) \) is always positive, \( \gamma_0 > 0 \). Hence, to find the DIS region, we need to apply Theorem 4.10 only in the gray region in Figure 5.8, since \( \gamma_0 > 0 \) is guaranteed regardless of \( k_1 \) and \( k_2 \). This study leads to Figure 5.9, in which the black region shows the controller gains that can make the vibration control system DIS.

Studying the frequency response of the mass \( m_1 \) under harmonic excitation \( f(t) = \sin(\mu t) \), with delays picked as \( \tau_1 = 2 \) and \( \tau_2 = 1 \), and with controllers \( (k_1, k_2) = (-1, 35) \) applied on delayed velocity feedback leads to Figure 5.10, where we again reach similar conclusions as those found with only position feedback.

![Figure 5.10: Case 2. Frequency response of the system output \( x_1(t) \) with controller gains selected in the DIS region in Figure 5.9 for the controller gains selected as \( (k_1, k_2) = (-1, 35) \). Here the controller uses only the delayed velocity feedback, where \( \tau_1 = 2 \) and \( \tau_2 = 1 \). Excitation frequency is \( \mu \).](image)

**Remark 5.1.** The main contribution here is the way we establish connections between delay systems, algebraic polynomials, and the implementation of Sturm sequences as a necessary and sufficient condition to detect the positive real roots of \( \Phi(y, \{k_i\}_{i=1}^z) \), which are directly linked to DIS conditions of the original control problem with “multiple” delays. We then aim to show that vibration suppression characteristics with such DIS controllers can still exhibit satisfactory results (see Section IV). One can also replace condition (ii) of Theorem 4.10 with the Descartes rule of signs (instead of using Sturm sequences), see [70], but this would lead to sufficient conditions when identifying the DIS controller gain regions. That is, utilizing Sturm sequences theorem relaxes this limitation, especially in cases when Descartes rule of signs is inconclusive in detecting the number of real positive roots of \( \Phi(y, \{k_i\}_{i=1}^z) = 0 \). Nevertheless, there is also a trade-off here. Descartes rule of signs requires inspecting the coefficients of \( \Phi(y, \{k_i\}_{i=1}^z) = 0 \).
in terms of \( \{ k_i \}_{i=1}^z \), which can be in some cases easier to study than implementing Sturm sequences point-wise for each controller gain in \( \{ k_i \}_{i=1}^z \).

### 5.3 Conclusion

We perform control design for active vibration control systems with multiple uncertain sensor delays, in order to render delay-independent stability (DIS) according to theorems provided in Chapter 4. A DIS controller is designed for a shock absorber dynamics whose controller is subjected to uncertain multiple sensor delays. Using either position or velocity sensing only, we reveal that DIS can be rendered, while still achieving satisfactory vibration suppression of the closed-loop system within certain frequency ranges.
This work presents control design, simulations, and experimental results for a LTI system that mimics Networked Control Systems (NCS) with “multiple” uncertain communication/sensory delays. The design is based on algebraic tools and allows finding controllers that can optimally stabilize such systems for set-point tracking regardless of how large/small the delays are (Delay-Independent Stability). Design scheme, which complies with necessary and sufficient conditions of stability, is also applicable for systems with particular control structures, and also allows a way to detect slave failure in master behavior via a fault-detection type control architecture. Simulation and real-time experiment results are reported here for set-point tracking of this system using the DIS controllers optimized for improved tracking performance. Results of the experiments confirm the simulations, verifying the validity of the approach for practice.

6.1 Problem Overview

A wide range of real-world control problems suffer from the time delays influence on dynamic system behavior. One such problem arises in Networked Control Systems (NCS). NCSs are systems in which a control loop is closed via a communication channel [51, 94, 128]. These systems have applications in telesurgery [63], teleoperation [52, 59, 97], traffic control, and unmanned vehicle coordination systems [12, 93], etcetera.
Due to the physical distance between the components of NCS or the imperfect communication channels, these systems may suffer from communication delays, which could be uncertain, and this could cause instability and/or poor performance [8, 45, 52, 61, 75, 97, 106]. Control design should therefore be carefully performed to assure that the system functions stably despite the delays [29, 32, 48, 99, 127]. Even in the case of linear time-invariant (LTI) systems, control design can be challenging, especially within a non-conservative framework. This is mainly due to the fact that the eigenvalue problems associated with the ability to control are infinite dimensional, and hence difficult to handle. Frequency domain technique (FDT) could be promising in this endeavor as it provides a non-conservative framework for the analysis and synthesis of LTI systems with delays [9, 46, 64, 66, 104, 109, 114, 125].

In the presence of uncertain delays, one would appreciate that the closed-loop system operates in a delay-independent stable (DIS) setting [13, 30, 70, 75, 104, 113, 118], along with system’s capability to satisfy certain performance conditions [29, 72]. Based on FDT, the author recently developed several approaches to design controllers for LTI systems with delays, in order to render such systems DIS [71, 73, 74]. However, while many theoretical results exist in DIS analysis; to the best of our knowledge an experimental study to evaluate the effectiveness of DIS controllers has not been studied.

In light of previous studies in this field, we find out that an experimental study to investigate the effectiveness of delay-independent stable controllers even on LTI systems was not reported, where these controllers are derived based on a non-conservative stability analysis framework and in the existence of multiple delays in the NCS. This chapter is aimed to address this opening particularly on a velocity regulation problem, while also incorporating a fault detection feedback line from the slave to the master, which is also affected by a communication delay. Specifically, here a non-conservative control design approach is taken to render this LTI NCS with multiple uncertain time delays to become DIS, to satisfactorily track set-point velocity input references while detecting, from the master’s behavior, whether or not the slave exhibits an anomaly/fault, all of which are then validated via experiments.

The theoretical approach utilized here is on algebraic tools and a non-conservative FDT framework, as we reported in [30, 72, 73], by which we can reveal control gains, possibly arising in a structured control matrix, that preserve the stability of the overall system “independent of uncertain multiple delays”. This DIS control design approach can be utilized on both centralized and decentralized control systems while delays can exist in any part of the structure of the NCS model. The control design approach proposed
here not only provides stability regardless of the amount of multiple constant delays, but also provides the optimal tracking performance measured by integral absolute error and closed-loop delayed system rightmost root distributions numerically calculated via existing tools [9].

In summary, this chapter is motivated by the above observations and our desire to investigate the effectiveness of delay-independent controllers on LTI systems [72, 73]. Here, we design DIS controllers for an experimental LTI NCS based on a non-conservative framework while considering the presence of multiple sensory delays. We design these controllers so that the NCS at hand satisfactorily tracks set-point inputs, which are then validated on experiments. The NCS studied here is a master-slave system, where the master is a simulated transfer function and the slave is a brush-type DC motor with encoder hardware set-up controlled by a data acquisition system. The input and controlled output variables are voltage to DC motor and the angular velocity of the motor shaft, respectively for both the master and slave. When the master and slave are physically distant, then the communication between the two needs to be established, e.g., by a wireless communication protocol. Regardless of how the two sites communicate, however, communication delays will be inevitable [8, 52, 97].

### 6.2 LTI Networked Control Systems

The focus of this section is on the control design for output tracking purposes of a NCS while keeping the system DIS. We divide this section into several subsections, covering the preliminary theorems and the materials on algebraic delay-independent stable control design for a LTI system with multiple uncertain delays.

A sample NCS with multiple uncertain delays can be formulated as below,

\[
\begin{align*}
\text{Master plant} & \quad \dot{x}_{ms}(t) &= A_{ms}x_{ms}(t) + B_{ms}u_{ms}(t) + R, \\
& \quad y_{ms}(t) &= \sum_{\ell=1}^{L}(C_{ms})_{\ell}\cdot x_{ms}(t - \tau_{\ell}), \\
& \quad u_{ms}(t) &= K_{ms}y_{ms}(t).
\end{align*}
\]

\[
\begin{align*}
\text{Slave plant} & \quad \dot{x}_{sl}(t) &= A_{sl}x_{sl}(t) + B_{sl}u_{sl}(t), \\
& \quad y_{sl}(t) &= \sum_{\ell=1}^{L}(C_{sl})_{\ell}\cdot x_{sl}(t - \tau_{\ell}), \\
& \quad u_{sl}(t) &= K_{sl}(y_{ms}(t) - y_{sl}(t)),
\end{align*}
\]

where \((.)_{ms}\) and \((.)_{sl}\) indices represent master and slave, respectively. \(x(t) \in \mathbb{R}^{n}\) is the state vector, \(y(t) \in \mathbb{R}^{r}\) is the output, \(u(t) \in \mathbb{R}^{m}\) is the control input, \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\), and \((C)_{\ell} \in \mathbb{R}^{r \times n}\) are given according to the realized state-space model of
master and slave plants, \( \mathbf{R} \in \mathbb{R}^n \) is the set-point reference which would be adjusted, for instance, by a human operator, and \( \mathbf{K} \in \mathbb{R}^{m \times r} \) is the controller matrix, entries of which are constant gains \( k_i \in \mathbb{R} \), denoted as a set \( \{k_i\}_{i=1}^z \). Since all the states may not be measurable, we assume that \( \mathbf{K} \) has a particular structure with zero and nonzero entries, and in what follows all the delays in (6.1) form the set \( \{\tau_\ell\}_{\ell=1}^L \) where \( L \in \mathbb{N} \) is the maximum number of delays in the system. Here, the set-point reference \( \mathbf{R} \) is assumed to have entries that are continuous, independent of system states, and upper and lower bounded in time, and thus do not affect the stability of (6.1). Hence stability of (6.1) can be studied over its characteristic equation, which can be written in the following general form:

\[
f(s, e^{-\tau \epsilon s}, \{k_i\}_{i=1}^z) = P(s, \{k_i\}_{i=1}^z) + Q(s, e^{-\tau \epsilon s}, \{k_i\}_{i=1}^z) = 0,
\]

(6.2)

where the controller gains \( k_i \in \mathbb{R} \) are to be designed to achieve delay-independent stability in (6.1). Here, \( \{\tau_\ell\}_{\ell=1}^L \) are the unknown constant but finite delays, which arise in general when measuring states, and when sending/recieving feedback and control signals between the master and slave. Moreover, since the highest derivative of the states in (6.1) is not affected by delays, it is known that this type of systems are of “retarded” type [45, 114], and hence \( P(s, k_i) \), which is not affected by any delay terms, has the highest order of \( s \).

Since the stability of the general plant in (6.1) needs to be guaranteed before improving its tracking performance, its stability is discussed first.

**Property 6.1.** [65] Given \( \{k_i\}_{i=1}^z \), \( \text{LTI time-delay system represented by (6.1) is delay-independent stable if and only if (6.2) has no roots on the closed right-half plane for all delay values. That is,} \)

\[
f(s, e^{-\tau \epsilon s}, \{k_i\}_{i=1}^z) \neq 0, \quad \forall s \in \mathbb{C}_0^+, \forall \{\tau_\ell\}_{\ell=1}^L \in \mathbb{R}_0^+.
\]

(6.3)

Finding all \( \{k_i\}_{i=1}^z \) controller gains satisfying the DIS condition in Property 6.1 can be difficult, if not impossible. This is because (6.2) has infinitely many roots, and verifying (6.3) is computationally quite involved.

**Theorem 6.1** (Existence of DIS control gain region [72]). For a given set of delays \( \{\tau_\ell\}_{\ell=1}^L \), there exists at least one region in the controller gain parameter space \( \{k_i\}_{i=1}^z \in \mathbb{R}^z \), where the system in (6.1) is DIS, if and only if \( P(s, \{k_i\}_{i=1}^z) + Q(s, e^{-\tau \epsilon s}, \{k_i\}_{i=1}^z) \) is DIS for some \( \{k_i\}_{i=1}^z = \{\tilde{k}_i\}_{i=1}^z \).
Theorem 6.1 utilizes the well-known root continuity argument [26], and states that if we can design the system to be DIS at a point in \( \tilde{k}_i \in \mathbb{R}^z \) space, then we know that in a region around \( \tilde{k}_i \), the system remains DIS. To make a LTI system with multiple delays DIS, for instance, by taking \( \tilde{k}_i = 0 \), i.e., the origin of the parameter space, we recently developed an algebraic approach [72] that can be used to perform the design of \( k_i \) in (6.2), while bypassing the need to check the computationally expensive DIS condition in Property 6.1. In the next subsection, we summarize this approach.

### 6.2.1 Review on DIS Controller Design

DIS controller gain design can be performed as follows:

**Step 1.** Use the Rekasius transformation [88] to replace the exponential functions in (6.2) with:

\[
\begin{align*}
    e^{-\tau_s} &\bigg|_{s=j\omega} = \frac{1 - T_\ell s}{1 + T_\ell s} \bigg|_{s=j\omega}, \quad T_\ell \in \mathbb{R}, \quad \omega \geq 0, \\
    (6.4)
\end{align*}
\]

where \( T_\ell \) is known as pseudo-parameter, and (6.4) is an exact substitution at \( s = j\omega \), which is completely different from Padé approximation [100].

**Step 2.** With the substitution of (6.4) into (6.2), and with a manipulation to remove the fractions, one obtains the transformed characteristic equation on the imaginary axis \( s = j\omega \) as

\[
g(j\omega, \{T_\ell\}_{\ell=1}^L, \{k_i\}_i^z) = \left( \int_{\ell=1}^L e^{-j\omega\tau_\ell} = \frac{1 - j\omega T_\ell}{1 + j\omega T_\ell} \right) \prod_{\ell=1}^L (1 + j\omega T_\ell)^{c_\ell} = 0,
\]

(6.5)

where \( g \) is a polynomial with complex coefficients. Here, the analysis on the imaginary axis \( s = j\omega \) is for the purpose of checking whether or not the system can ever have eigenvalues on the imaginary axis, which is indication of loss of stability. This is, as expected, directly related to DIS conditions, as we studied in [72]. We suppress the details regarding various utilizations of (6.5), referring the readers to the citations above as well as to [57, 69, 104, 113].

**Step 3.** It must be noted that different from the Padé approximation [76], Rekasius transformation is an **exact substitution** for the imaginary roots of (6.2), and all the \( s = j\omega \) roots of (6.2) and of (6.5) are identical [109]. This is in one sense a way of converting the stability problem of (6.2) to a robust control problem in (6.5), which becomes possible...
since \( s = j\omega \) roots are preserved under the transformation (6.4). Consequently, delay-independent stability can be studied using (6.5) instead of (6.2). For (6.2) not to have eigenvalues at \( s = j\omega \) for all \( \{\tau_\ell\}_{\ell=1}^L \in \mathbb{R}_+^L \), see [30].

**Step 4.** Rewrite \( g \) as follows,

\[
g(j\omega, \{T_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) = g_R(\omega, \{T_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) + jg_\Im(\omega, \{T_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) = 0, \tag{6.6}
\]

where \( g_R \) and \( g_\Im \) are respectively the real and imaginary parts of \( g \). For (6.6) to hold, \( g_R \) and \( g_\Im \) must have at least one common solution.

**Step 5.** Eliminate the unknown \( T_L \) from \( g_R = 0 \) and \( g_\Im = 0 \) using the Resultant Theory [43]. This can be done by building a Sylvester matrix, which is singular whenever \( g_R \) and \( g_\Im \) have a common solution (but the converse is not always true). That is, the resultant \( R_{T_L}(\omega, \{T_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z) \), which is the determinant of the Sylvester matrix should be zero as a necessary condition for \( g_R \) and \( g_\Im \) to have at least one common solution,

\[
R_{T_L}(g_R, g_\Im) := \det(S) = 0. \tag{6.7}
\]

**Step 6.** DIS analysis for the multiple time-delay system at hand can actually be constructed using the principles of resultant theory and starting with (6.7). Recall from Step 3 that DIS requires that \( s = j\omega \) should not be a solution of (6.2), or in \( T_\ell \) domain, \( s = j\omega \) should not be a solution of (6.5), or equivalently (6.7), see [30]. Instead of numerically searching for whether or not feasible \( \omega \) solutions of (6.7) exist, one can instead seek whether or not the max/min of \( \omega \) exists in \( \omega \in \mathbb{R}_+ \). If max/min of \( \omega \) does not exist for all \( T_\ell \in \mathbb{R} \), it is then guaranteed that \( s = j\omega \) cannot be a solution of (6.5) and (6.7) for all \( T_\ell \), which would ultimately imply that \( s = j\omega \) cannot be a root of the original characteristic equation in (6.2) for all \( \tau_\ell \), as per Step 3.

**Step 7.** If, by construction, the delay-free system (with \( \{\tau_\ell\}_{\ell=1}^L = 0 \)) is stable, and if from step 6, one reveals that \( s = j\omega \) is not a feasible solution of (6.2), then the system at hand is guaranteed to be controlled independent of the amount of uncertain delays [70–72], since loss of stability is not possible by the existence of a \( s = j\omega \) solution.

**Step 8.** Construct the proposed optimization scheme in Step 6 starting with (6.7), which leads to the polynomial \( D(\omega) = 0 \) as summarized in the Appendix from [30, 72]. Incorporate the controller gains \( \{k_i\}_{i=1}^z \) into this polynomial to obtain

\[
V_1 := D(\omega, \{k_i\}_{i=1}^z) = 0.
\]
Step 9. The roots of the polynomial $D(\omega) = 0$ carry critical information about system’s spectrum at $s = j\omega$. If none of these roots are real, and the condition in Step 7 holds, then the system is declared DIS. If a real root of $D(\omega) = 0$ exists, however if at least one of the corresponding $T_\ell$ values is not real, as calculated sequentially from $V_2, \ldots, V_\ell$ as explained in Chapter 4, and while the condition in Step 7 still holds, then the system is DIS.

6.2.2 Delay-Independent Stability Requirements

For DIS, the delay-free system must be stable (see Step 7 in Subsection 6.2.1) which indicates that $\omega = 0$ is not a root of (6.2) for all finite $\tau_\ell \geq 0$ [14]. Hence, $\omega = 0$ zeros of $V_1$ can be ignored. Moreover, since the imaginary roots of (6.5) are in complex conjugate form, that is, both $s = j\omega$ and $s = -j\omega$ satisfy (6.5), we know that $D(\omega, \{k_i\}_{i=1}^z)$ is an even polynomial in $\omega$, after ignoring the $\omega = 0$ zeros, i.e., $D = D(\omega^2, \{k_i\}_{i=1}^z)$. Then a variable change as $y = \omega^2$ becomes convenient:

$$
\Phi(y, \{k_i\}_{i=1}^z) = D(\omega^2, \{k_i\}_{i=1}^z),
$$

where the number of real zeros of $D$ is twice the number of positive real zeros of $\Phi(y, \{k_i\}_{i=1}^z)$. Hence, instead of inspecting the real roots of $D = 0$ in Step 9 in Subsection 6.2.1, we can inspect the existence of positive real roots of $\Phi(y, \{k_i\}_{i=1}^z) = 0$ for a given set of controller gains $\{k_i\}_{i=1}^z$ [30, 72]. This modifies Step 9 as follows:

• If no roots of $\Phi(y, \{k_i\}_{i=1}^z) = 0$ are positive, this implies that no admissible $y = \omega^2 > 0$ roots exist, hence a stability switch is impossible. If the condition in Step 7 also holds, then the system is stable independent of the amount of delays $\tau_\ell$.

• Presence of at least one positive real zero of $\Phi(y, \{k_i\}_{i=1}^z)$, however, requires checking whether a solution in $\{T_\ell\}_{\ell=1}^L \in \mathbb{R}^L$ exists satisfying (6.6). If such a set of $\{T_\ell\}_{\ell=1}^L$ does not exist, and the condition in Step 7 is satisfied, then the system is delay-independent stable.

6.3 Networked Control System Design

In the sequel, we implement delay-independent stable controller design approach summarized in Section 2 on an experimental NCS, Figure 6.1. In this system, master and
slave plants are represented with identical transfer functions although this is not a limitation to the approach. Master plant, which is a virtual system represented by a Simulink transfer function, can receive operator command where the operator command here is a set-point reference for velocity control. Slave plant is a DC motor with shaft-mounted encoder. The communication ports link the DC motor with the computer using a National Instruments data acquisition board supported by QUARC embedded system software running with MATLAB/Simulink toolbox. Moreover, delays exist in master and slave communications, including delays in speed information sharing between the motors. To mimic a realistic scenario, these delays are artificially introduced within Simulink.

![Diagram of experimental NCS setup](image)

**Figure 6.1:** An illustration of the experimental NCS setup, where the slave is a DC motor hardware and the master is a simulated model in MATLAB/Simulink.

The control goals on this experimental system are:

(a) Design the closed loop to be stable regardless of the amount of communication and sensory delay values.

(b) Find the optimal controller gains such that set-point angular velocity tracking performance is improved.

According to these goals, this section is divided into three subsections:

- Control design for delay-independent stability of the NCS,
- Extraction of optimal DIS controller gains for set-point angular velocity tracking,
- Simulation results and real-time experiments.
6.3.1 Control Design for Delay-Independent Stability of NCS

6.3.1.1 Open Loop System Definition

To be able to analytically design DIS controller before real-time experiments, we need to create a mathematical model of the physical system. Here, we study the step response characteristics of the DC motor at hand, where the input is in Volts and the output is in radians per second. The DC motor transfer function can be reliably fit to a well-known first-order dynamic model [76],

\[
G(s) = \frac{\Omega(s)}{V(s)} = \frac{c_1}{c_2 s + 1},
\]

where \(c_1\) is the DC gain, and \(c_2\) is the time constant, given here as \(c_1 = 8.4\) and \(c_2 = 0.028\) [122].

6.3.1.2 Proposed Control Architecture for LTI NCS

The NCS here consists of a scalar set-point reference (operator), \(R = R \in \mathbb{R}\), master plant, \(M_1\), communication channels, slave plant, \(M_2\), and controllers \(C_1, C_2, C_3, C_4, I_1, I_2\) some of which are functions of control gains \(\{k_i\}_{i=1}^5\), see Figure 6.2 for the proposed control architecture. This particular structure is proposed to meet the following specific design goals: (a) create a closed-loop system for which DIS controllers exist, (b) achieve zero error in set-point tracking for the best possible performance.

Here, the controllers \(C_1\) and \(C_2\) are used to drive \(M_1\) towards the reference as well as to the state of the slave \(M_2\), while \(C_3\) and \(C_4\) drive the slave to track the state of the master plant. Moreover, \(I_1\) and \(I_2\) are used to shape the inputs, which help us create the necessary control structure meeting the specific goals (a)-(b) above. Finally, the state information from the master to the slave and back is delayed by \(\tau_1\), while a delay \(\tau_2 \neq \tau_1\) exists in the local controller around the slave. Although the proposed architecture does not address all possible problems associated with applications of Systems in network, it offers a benchmark problem that can be used to study various scenarios, and expand on, for future research.

The coupled master-slave system also shown in Figure 6.2 can be modeled as in Equation (6.1) considering the communication delays, \(\tau_1\), and feedback delay, \(\tau_2\), which are
assumed to be constant but uncertain. The described scenario is common in NCS dynamics as covered in the Introduction. Nevertheless, guaranteeing DIS and satisfactory tracking performance in the presence of two non-identical delays $\tau_1$ and $\tau_2$ is not trivial.

Remark 6.2. Notice here that fault detection is possible by the weak feedback from slave to the master plant. That is, the master and the slave are coupled and each sends a feedback signal to the other, as shown in Figure 6.2.

We start with extracting the characteristic equation of the system in Figure 6.2. We have

$$I_1R(t) + x_2(t - \tau_1) - C_2x_1(t) = \frac{x_1(t)}{C_1M_1}, \quad (6.10a)$$

$$I_2x_1(t - \tau_1) - C_4x_2(t - \tau_2) = \frac{x_2(t)}{C_3M_2}. \quad (6.10b)$$

Eliminating $x_2$ in (6.10) yields in Laplace domain

$$X_1 \left( e^{-2\tau_1 s} - (C_2 + \frac{1}{C_1M_1}) \left( \frac{C_4}{I_2} e^{-\tau_2 s} + \frac{1}{C_3M_2I_2} \right) \right) = -I_1R(s)e^{-(\tau_1 + \tau_2)s} \left( \frac{C_4}{I_2} + \frac{1}{C_3M_2I_2} \right). \quad (6.11)$$

Similarly, one can instead eliminate $x_1$ in (6.10), and find the Laplace transform of $x_2$ as

$$X_2 \left( e^{-2\tau_1 s} - (C_2 + \frac{1}{C_1M_1}) \left( \frac{C_4}{I_2} e^{-\tau_2 s} + \frac{1}{C_3M_2I_2} \right) \right) = -I_1R(s)e^{-\tau_1 s}. \quad (6.12)$$
The single-input multi-output system transfer function under investigation is then given by,

\[
G(s, \tau_1, \tau_2, \{k_i\}_{i=1}^z) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -I_1 e^{-(\tau_1 + \tau_2)s} \left( C_2 + \frac{1}{C_1} \right) + \frac{1}{C_3 M_2} \\ e^{-\tau_2 s - C_2 C_3 M_2 I_2} \end{bmatrix} R(s). \tag{6.13}
\]

Stability of this system needs to be guaranteed before improving its tracking performance, in particular for rendering it DIS. For this, we obtain the corresponding characteristic equation, which in Laplace domain, reads

\[
f(s, \tau_1, \tau_2, \{k_i\}_{i=1}^z) = M_1 C_1 C_2 (s, \{k_i\}_{i=1}^z) + 1 - M_1 M_2 C_1 C_3 I_2 (s, \{k_i\}_{i=1}^z) e^{-2\tau_1 s} + \frac{1}{C_1 C_2 C_3 C_4} e^{-\tau_2 s} = 0. \tag{6.14}
\]

We next establish conditions for zero steady-state set-point tracking. Once this is complete, we will use (6.14) along with Steps 1-9 in Subsection 6.2.1 in order to design \( k_i \) such that the NCS can be made DIS.

### 6.3.1.3 Input Shaper Design for Set-point Reference Tracking

Tracking the set-point reference provided by the operator input is one of the goals here. One way to force the stable system to track the reference is to dictate zero steady-state error with an appropriate input shaper. Assuming the system associated with the characteristic equation (6.14) is stable, the procedure starts with the calculation of the transfer function for each output with respect to the desired reference:

i) We require the output of \( M_2 \) to follow the output of \( M_1 \). That is, \( \lim_{t \to \infty} x_2(t) = \lim_{t \to \infty} x_1(t) \) is required in order to guarantee zero tracking error in the slave plant. Simplifying equation (6.10b) in Laplace domain and utilizing the final value theorem \([76]\) yields,

\[
\lim_{t \to \infty} x_2(t) = \lim_{s \to 0} s \left( I_2 e^{-\tau_1 s} \right) \frac{X_1}{1 + \frac{1}{C_3 M_2} + C_4 e^{-\tau_2 s}} = 1,
\]

Hence,

\[
I_2(k_1, k_2) = \left. \left( \frac{1}{C_3 M_2} + C_4 \right) \right|_{s=0}.
\]
ii) We also require the output of master $M_1$ to follow the set-point reference $R$, which is constant in this case study. That is, $\lim_{t \to \infty} \frac{x_1(t)}{R} \to 1$ must hold to guarantee zero tracking error. Simplifying equation (6.11) in Laplace domain yields,

$$\lim_{t \to \infty} x_1 = \lim_{s \to 0} \frac{s(-I_1 e^{-\tau_1 s}(\frac{C_4}{T_2} + \frac{1}{C_3 M_2 T_2})) R(s)}{(e^{-2\tau_1 s} - (C_2 + \frac{1}{C_1 M_1})(\frac{C_4}{T_2} e^{-\tau_2 s} + \frac{1}{C_3 M_2 T_2}))} = 1,$$

which calculates $I_1$ as:

$$I_1(k_1, k_2) = \left| \frac{1 - (C_2 + \frac{1}{C_1 M_1})(\frac{C_4}{T_2} + \frac{1}{C_3 M_2 T_2})}{-(\frac{C_4}{T_2} + \frac{1}{C_3 M_2 T_2})} \right|_{s=0}.$$  (6.16)

Based on the above constraints (6.15)-(6.16), one can further identify the characteristic equation in (6.14) and next apply the DIS control design approach according to described method in Chapter 4.

### 6.3.1.4 Control Design for Delay-Independent Stability

According to the general model of the controlled master-slave dynamics as introduced in Figure 6.2, identical first-order transfer functions are selected for $M_1$ and $M_2$ as master and slave plants. The structure of feed-forward and feedback controllers are considered as predefined dynamics given by,

$$M_1 = M_2 = \frac{8.4}{0.028s + 1},$$

$$C_1 = \frac{s + k_2}{50s + 12}, \quad C_2 = 1, \quad C_3 = \frac{s + k_2}{25s + k_1}, \quad C_4 = 1.$$  (6.17)

For the sake of plotting and concise illustration of the results, we use two unknown controller gains $k_1$ and $k_2$, however this is not a limitation of the presented approach. Controller gains $k_1$ and $k_2$ are to be designed to achieve stability independent of the amount of delays $\tau_1$ and $\tau_2$ while the coupled system attains the desired control goals:

1. Angular velocity of the master plant must follow the set-point reference.
2. Detect any faulty operation of slave automatically by observing anomalies in master behavior.
3. Angular velocity of the slave motor must follow the angular velocity of the master motor.
Based on (6.17), input shaper gains $I_1$ and $I_2$ can be calculated according to Subsection 6.3.1.3, as

$$I_2(k_1, k_2) = \left( \frac{1}{C_3 M_2} + C_4 \right) \bigg|_{s=0} = \frac{5k_1}{42k_2} + 1,$$

$$I_1(k_1, k_2) = \left( 1 - \left( C_2 + \frac{1}{C_3 M_1} \right) \left( \frac{C_4}{I_2} + \frac{1}{C_3 M_2 I_2} \right) \right) \bigg|_{s=0} = \frac{10}{7k_2}. \tag{6.18}$$

Next, DIS control design approach and Theorem 4.10 are implemented. We start with the calculation of the characteristic equation (6.14), which is found as

$$f(s, \tau_1, \tau_2, k_1, k_2) = s^4 + \left( \frac{13592}{175} + \frac{k_1}{25} \right) s^3 + \left( \frac{24111}{16} + \frac{553k_1}{178} + 6k_2 \right) s^2$$

$$+ \left( \frac{15000}{49} + \frac{14768k_1}{245} + \frac{6k_1 k_2}{25} + \frac{1500k_2}{7} \right) s + \frac{600k_1}{49} + \frac{60k_1 k_2}{7}$$

$$- \left( \frac{60k_1}{7k_2} + 72 \right)s^2 + \left( \frac{120k_1}{7} + 144k_2 \right) s + \frac{72k_2^2}{7} + \frac{60k_1 k_2}{7} e^{-2\tau_1 s}$$

$$+ \left( 12s^3 + \frac{15607}{31} + 12k_2 \right) s^2 + \left( \frac{720}{7} + \frac{17839k_2}{31} \right) s + \frac{720k_2}{7} + \frac{72k_2^2}{7} e^{-\tau_2 s}. \tag{6.19}$$

Citing [72], we know that for the controlled system to be delay-independent stable, it is required that the open-loop system must be stable, i.e., the system in (6.1) must be stable in the case when feedback becomes ineffective, which occurs in the limits $\tau_1 \to \infty$ and $\tau_2 \to \infty$. Next, we analyze the Hurwitz stability of the delay-free controlled system as per condition (i) of Theorem 4.10, setting the delay values to zero. The characteristic equation (6.19) becomes

$$f(s, k_1, k_2) = s^4 + \left( \frac{15692}{175} + \frac{k_1}{25} \right) s^3 + \left( \frac{34891}{18} - \frac{60k_1}{7k_2} + \frac{553k_1}{178} + \frac{18k_2}{2} \right) s^2$$

$$+ \left( \frac{20040}{49} + \frac{10568k_1}{245} + \frac{6k_1 k_2}{25} + \frac{12269k_2}{19} \right) s + \frac{600k_1}{49} + \frac{700k_2}{7} = 0, \tag{6.20}$$

from which stability conditions can be trivially obtained [76]. Conditions for open-loop stability and delay-free closed loop system stability are then reflected in Figure 6.3, and DIS control design is performed in $(k_1, k_2)$ parameter space where these two stability conditions hold. For this, we implement the DIS control design approach in Section 6.2.1, starting with the Rekasius substitution in Step 1, separation of the real and imaginary parts of the characteristic equation in (6.6) (Step 4), and application of two consecutive resultants to eliminate $T_1$ and $T_2$ (Step 5). After these operations, we obtain the polynomial $D(\omega, k_1, k_2)$ (Step 8) and by change of variable $\omega^2 = y$, we obtain $\Phi(y, k_1, k_2) = 0$ (Subsection 6.2.2), which must be studied for its positive real roots (Step 9). This
polynomial is found in the following form

\[ \Phi(y, k_1, k_2) = \sum_{i=0}^{12} \gamma_i(k_1, k_2)y^i = 0, \]  

(6.21)

where \( \gamma_i \) are polynomials in terms of \( k_1 \) and \( k_2 \).

We next apply Theorem 4.10 on (6.21) point-by-point on \( k_1 - k_2 \) plane, and identify the points for which condition (i), and (ii)-(a) or (ii)-(b) of Theorem 4.10 hold, and thus where delay-independent stability of the NCS in (6.10) is satisfied, see Figure 6.3. Note that two different boundaries are depicted in this figure. Blue curves are representing the coefficients of Laplace variable, \( s \), in \( f(s, k_1, k_2) \) in (6.20), and black dashed curves are the plots of \( \gamma_0(k_1, k_2) \) coefficients of \( \Phi(y, k_1, k_2) \) utilized in the application of Descartes rule of signs as explained in Theorem 4.11.

![Figure 6.3: Controller gain space: Black stars are representing delay-independent stable controller gains and blue dots are controller gains for stability of delay-free controlled system.](image)

At this step, the controller gain space is categorized into three different regions in Figure 6.3,

1) delay-independent stability (marked with black stars),
2) delay-dependent stable (marked with blue dots, viewed as a grey on black and white printout),
3) instability (blank, white background).

Using the controller gain sets marked by the blue dots in Figure 6.3, which correspond
to delay-dependent stability, may not be safe since, in the system at hand, it is assumed that the amount of delays $\tau_1$ and $\tau_2$ are uncertain and unknown. Therefore, using delay-independent stable controller gains shown with black stars in Figure 6.3 is the safest choice in this case, in order to guarantee stability for any delay values in the control loop.

6.3.1.5 Fault Detection

Here, we develop the analysis for a fault-detection system that reflects faulty operation of the slave to the behavior of the master. For this purpose, we utilize equation (6.10a), in Laplace domain, to further investigate the master-slave dynamics, as a function of given reference input $R$,

$$X_2 e^{-\tau_1 s} = \left(\frac{1}{C_1 M_1} + C_2\right) X_2 - I_1 R(s). \quad (6.22)$$

The above equation clearly shows that any anomaly in slave behavior $X_2$ affects $X_1$, thus can be picked up from the master behavior $X_1$. In the case when one is interested to study the effect of fault at slave to the master’s response in steady state, Equation (6.22) becomes as follows, after incorporating the transfer functions introduced in (6.17) and (6.22) and applying the final value theorem,

$$x_2(\infty) = x_1(\infty)(1 + \frac{10}{7k_2}) - \frac{10}{7k_2} R, \quad (6.23)$$

from which we obtain the master’s behavior as $x_1(\infty) = \frac{x_2(\infty) + \frac{10R}{7k_2}}{1 + \frac{10}{7k_2}}$. For this relationship, one can show that in steady state, an error in master at an amount of $\frac{7k_2 R}{10 + 7k_2}$ will arise in the case when the slave makes a complete stop, $x_2 \rightarrow 0$, e.g., due to power supply failure. Moreover, since the set-point reference $R$ and gain $k_2$ are known in an experimental setting, one can also use (6.23) to estimate slave’s state in steady state by reading the master’s state, and calculating how much anomaly the slave exhibits.

In the next subsection, we find optimized controller gains, within the DIS region in Figure 6.3, according to master-slave tracking performance, which we will then implement in simulations and real-time experiments to study various scenarios, including fault-detection.
6.3.2 Extraction of Optimal DIS Control-Gains for Set-Point Tracking

Now that DIS control gains are identified in Figure 6.3, we wish to find the DIS gains that yield improved tracking performance. This is a necessary step since not all the DIS controller gains may lead to desirably good tracking; e.g., some gains may cause significant overshoot and too long settling times. Finding the optimal DIS controller gains is challenging since an analytical solution of the closed loop system is impossible to obtain due to the infinite dimensional nature of the system in the presence of delays. We therefore will resort to numerical and computational tools in this optimization effort.

6.3.2.1 Rightmost Root Calculation

At this step, we calculate the rightmost root, $G$, of the characteristic equation inside of the DIS region. For this, we use TRACE-DDE tool [9] to compute $G$ by sweeping the controller gains in the identified DIS region, and for some predefined delay values $\tau_1$ and $\tau_2$. Plotting the real part of the rightmost root, $\Re(G)$, for the selected DIS gains generates a 3D graph, letting us identify the DIS gain combinations for which $|\Re(G)|$ is maximized, with $\Re(G) < 0$, ultimately corresponding to reduced settling time of the closed-loop system as approximated by the rightmost roots. In the DIS region, we find that $\varsigma \leq \Re(G) < 0$ holds.

6.3.2.2 Integral Absolute Error in Set-Point Tracking

Here, DIS controller gain region is used to simulate the closed loop system response with respect to a unit step reference, $R$. This leads to the simulated sum of tracking error in master and slave plants for each DIS controller gain. We can then calculate the Integral Absolute Error (IAE) in tracking dynamics for each simulation using,

$$ J_{IAE} = \int_0^T (|e_1(t)| + |e_2(t)|) \ dt, $$

$$ e_1(t) = R - x_1(t - \tau_1), $$

$$ e_2(t) = x_1(t - \tau_1) - x_2(t - \tau_1), $$

(6.24)

where $T$ is the simulation time, $e_1(t)$ and $e_2(t)$ are respectively the reference tracking errors of master and slave plants, and delays $\tau_1$ and $\tau_2$ are given. Note that, according to the NCS model introduced in Figure 6.2, the operator provides the set-point reference $R$ for the master plant, and the master plant is the reference of the slave plant, hence...
the formulation of $e_1(t)$ and $e_2(t)$ in (6.24). Plotting $\mathcal{J}_{IAE}$ as defined in (6.24) for the DIS controller gains at hand generates a 3D graph letting us identify the optimal DIS gains corresponding to minimum tracking error as measured by IAE metric, which is found to be between an upper and a lower bound, $0 \leq \varphi_1 \leq \mathcal{J}_{IAE} \leq \varphi_2$.

It must be noted that using either the rightmost roots or the IAE metric alone is not enough for optimization purposes since the rightmost root information is related to the settling time of the system but not necessarily to the system’s transient response. On the other hand, IAE alone will provide information more about the transient characteristics and the deviations in tracking the set-points [105].

### 6.3.2.3 Evaluating Tracking Performance

To define a single metric for evaluating tracking performance of the NCS, we first calibrate/normalize the values of the rightmost stable roots and integral absolute errors according to their respective ranges determined by $\varsigma$, $\varphi_1$, and $\varphi_2$. The new metric, $\mu$, is a weighted sum of both metrics, given by,

$$
\mu = \frac{1}{2} \left\{ \frac{\mathcal{R}(\Theta)}{\varsigma} + \frac{(\mathcal{J}_{IAE})^{-1}}{\frac{1}{\varphi_1} - \frac{1}{\varphi_2}} \right\},
$$

(6.25)

where inverse of $\mathcal{J}_{IAE}$ is used to make it compatible with $\mathcal{R}(\Theta)$ in order to represent improvements in tracking. We note that each DIS controller gain pair selected from Figure 6.3 has different $\varsigma$, $\varphi_1$, and $\varphi_2$ values, and in general yield different $\mu$ values, which again generate a 3D surface above the DIS region. The optimal DIS controller gain pair with the maximum $\mu$ value is the optimal one, which will be used in simulations and experiments.

**Remark 6.3.** It must be noted that we need to select a value for each delay parameter when studying the rightmost roots, performing simulations, and real-time experiments. However, verification of the analytical DIS design is impossible only by means of experiments and simulations, since one would need to scan infinitely many combinations of delays $\tau_1$ and $\tau_2$ ranging from zero to infinity. For this reason, in the sequel we will assume that satisfactory results are obtained if/when the simulations and experiments show strong agreement.

To better illustrate the effectiveness and application of DIS controllers, we create three different scenarios:
- Large communication delay $\tau_1$ compared with sensory delay $\tau_2$.
- Large sensory delay $\tau_2$ compared with communication delay $\tau_1$.
- Both communication and sensory delay values are large.

For each category defined above, we sweep the controller gains within the detected DIS region in controller gain space (Figure 6.3) to locate the DIS gains providing the optimal tracking performance as measured by (6.25). For this purpose, we focus inside the box $k_1 = [0, 15]$ and $k_2 = [-1, 1]$ in Figure 6.3, and detail the results of the above scenarios:

**Large $\tau_1$, small $\tau_2$: ($\tau_1 = 1$, $\tau_2 = 0.01$ seconds)**

Large communication delay and small feedback delay may occur when the slave controller is installed locally on the slave plant, but on the other hand, communication delay between master and slave may be large due to the physical distance between them. Furthermore, this analysis can reveal how much the NCS is sensitive to the small delay $\tau_2$, and how well the DIS controller gains perform.

Utilizing the TRACE-DDE tool and for the selected amount of constant delays $\tau_1 = 1$ and $\tau_2 = 0.01$ seconds, we compute the rightmost root range as $-0.24 \leq \Re(\Phi) < 0$, and after extracting the simulation results from the model shown in Figure 6.4, the IAE range is found in $2.77 \leq J_{IAE} \leq 52.21$, within the identified range $k_1 = [0, 15]$ and $k_2 = [-1, 1]$ and with a grid size of 0.5 for $k_1$ and 0.1 for $k_2$. Next, we calculate
the optimization cost function, $\mu$, according to the identified bounds $\varsigma$, $\varphi_1$, and $\varphi_2$ of $\Re(\mathcal{G})$ and $\mathcal{J}_{IAE}$ at each grid point. It is noted that the second order filter shown in simulation model in Figure 6.4, is used only for illustration purposes and to remove the gear connection noise and other negligible disturbances. Finally, we illustrate the optimization cost function, $\mu$, as a surface overlayed on the selected DIS controller gain space, shown from top in Figure 6.5, left. Using the colormap editor, we can identify the optimal set of gains maximizing $\mu$, and to be used in real-time experiments.

Large $\tau_2$, small $\tau_1$: ($\tau_1 = 0.01$, $\tau_2 = 1$ seconds)

Large feedback delay and small communication delay may occur when the master-slave plants are physically close to each other while the slave is receiving information from a long distance location. The other purpose for considering this scenario is to illustrate how much the NCS is sensitive to small communication delay, $\tau_1$, while the feedback delay, $\tau_2$, is large. In this case, the real part of the rightmost root ranges in $-0.25 \leq \Re(\mathcal{G}) < 0$, and the IAE ranges in $1.30 \leq \mathcal{J}_{IAE} \leq 48.50$, within the identified box in controller gain plane $k_1 = [0, 15]$ and $k_2 = [-1, 1]$, and with a grid size of 0.5 for $k_1$ and 0.1 for $k_2$. The cost function $\mu$ based on the same scale used is depicted in Figure 6.5, center. Using the colormap editor, we can identify the optimal sets of gains with the maximum $\mu$ value, and use these gains in real-time experiments.

Both communication and sensory delay values are relatively large: ($\tau_1 = 1$ and $\tau_2 = 4$ seconds)

Here, we investigate the effects of large communication and sensory delays in the NCS. In this case, the real part of the rightmost root range is found in $-0.242 \leq \Re(\mathcal{G}) < 0$, and the IAE ranges in $3.58 \leq \mathcal{J}_{IAE} \leq 54.39$, within the same gain range used in the previous scenarios. The corresponding cost function $\mu$ is provided in Figure 6.5, right.
Remark 6.4. We note that in Figure 6.5, $\mu$ is normalized separately in each sub-figure from 0 to 100 where color code changes linearly from dark blue ($\mu = 0$, the poorest performance) to dark red ($\mu = 100$, the best performance). Since the normalization is separately applied, the three plots in the figure and hence the $\mu$ values in each one of these plots are not comparable.

6.3.3 Experiments and Simulations

Real-time experiment of the NCS introduced in (6.10) and for the specific case study with master and slave plants and structured control matrices in (6.17) are shown in MATLAB/Simulink environment in Figure 6.6. In both simulations and real-time experiments, the reference $R$ is set to 15 rad/sec for 30 sec, which then switches to 50 rad/sec for 60 sec. Note that the saturation block is not active for the whole real-time process since the limits are set to $[-8, 8]$ Volts while the maximum voltage needed for tracking the desired angular velocity is calculated as 5.95 Volts according to the given DC gain in Subsection 6.3.1.

![Figure 6.6: Real-time experiment model using QUARC in MATLAB/Simulink. Master and Slave parts of the experiment are distinguished. The encoder is reading the angle of DC motor shaft with 4098 count/turn. The saturation block is set to maximum allowable input voltage for safety reasons.](image)

Here, we use the controller gains designed in the scenarios in Subsection 6.3.2.3. It is noted that for the real-time experiment purposes, we select the optimal DIS controller gains as $(k_1, k_2) = (7, 0.35)$, according to Figure 6.5 and based on all the three scenarios considered. The second-order filter shown in the model in Figure 6.6 with parameters
\( \zeta_f = 0.9 \) as the damping ratio and \( \omega_{cf} = 31.4159 \text{ rad/sec} \) as the cut-off frequency does not enter the closed-loop system, and is used for plotting purposes\(^1\).

In Figure 6.7, we compare the angular velocity of master and slave plants obtained via simulations and real-time experiments, where DIS optimal controller gains \((k_1, k_2) = \)

\(^1\)Note that the IAE metric is calculated based on the unfiltered data.
(7, 0.35) are implemented for constant delays $\tau_1 = 1$ and $\tau_2 = 0.01$ seconds. In this case, for the selected DIS controller gains, the real part of the rightmost root is at $\Re(\Theta) = -0.2156$, corresponding to about $4/0.21 \approx 19.04$ sec settling time based on 2% settling time criterion [76]. Simulation results are shown with dashed lines and experimental results are shown by solid lines. Simulation and real-time data are within 0.3% cumulative error for the master and 2% cumulative error for the slave DC motor. We believe that this error stems from unmodeled dynamics (e.g., dry friction in the gearbox attached to the motor shaft) and weak nonlinearities in the slave motor dynamics.

The tracking error in the simulations is provided separately in Figure 6.8, where the error between the desired reference and the simulated angular velocity of master and slave systems are depicted. Since we see in Figure 6.7 that simulations and experiments are in good agreement, another figure for the experimental results similar to Figure 6.8 is suppressed.

Figure 6.9 compares the simulation and real time angular velocity measurements of master and slave plants, where $\tau_1 = 0.01$ and $\tau_2 = 1$ seconds, and the same DIS optimal controller gains $(k_1, k_2) = (7, 0.35)$ are used here. In this case, for the selected DIS controller gains, the real part of the rightmost root is at $\Re(\Theta) = -0.2328$, corresponding to a settling time of about $4/0.23 \approx 17.04$ sec. Simulation results are shown with dashed

\[\text{Figure 6.9: Simulation and real-time experimental results. Angular velocity of master and slave motors.}\]

\[\text{Figure 6.9}\] The cumulative errors for master and slave are calculated by the summation of the absolute error between simulation and real-time values of angular velocity of each output value, for master and slave plants respectively, divided by the total number of output data.

\[\text{\footnote{The cumulative errors for master and slave are calculated by the summation of the absolute error between simulation and real-time values of angular velocity of each output value, for master and slave plants respectively, divided by the total number of output data.}}\]
lines and experimental results are shown by solid lines. Simulation and real-time data are within 0.3% cumulative error for the master and 2% cumulative error for the slave DC motor.

Figure 6.10 compares the simulation and real-time angular velocity measurements of master and slave plants, where $\tau_1 = 1$ and $\tau_2 = 4$ seconds, and the same DIS optimal controller gains $(k_1, k_2) = (7, 0.35)$ are used as before. In this case, for the selected DIS controller gains, the real part of the rightmost root is at $\Re(\zeta) = -0.2319$, which yields about $4/0.2319 \approx 17.25$ sec settling time. Simulation and experimental results are shown with respectively dashed and solid lines. Simulation and real-time data are within 0.3% cumulative error for the master and 2% cumulative error for the slave DC motor.

Before we close this section, we present experimental results on the fault detection aspect of the control loop. In this experiment, we have $R = 50$ rad/sec, optimal controller gains $(k_1, k_2) = (7, 0.35)$, and $\tau_1 = 3$ and $\tau_2 = 2$ seconds. Figure 6.11 shows the results of the experiment, which is separated into two halves. At the first half of the experiment, the system is functioning properly, while through the second half, the slave motor is at fault and its speed goes to zero, e.g., due to power loss. To mimic the conditions for the power loss, we use breaks to stop the slave motor. Validating equation (6.23), one is able to calculate master’s state as $40.16$ rad/sec by setting slave’s speed to zero.
6.4 Conclusion

A controller is designed for a LTI Networked Control System (NCS) with multiple delays in order to achieve optimal set-point velocity tracking and stability of the system independent of delays. The control design is then validated via several simulations and experiments, including a fault-detection mechanism by which we can automatically detect faulty slave from master’s behavior.
Bibliography


