Derived category and cohomology of resolution of singularities: examples from representation theory

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Abstract of Dissertation

In this thesis, we study examples of noncommutative crepant resolutions of determinantal varieties, noncommutative symplectic varieties, and elliptic genera. All of these examples are motivated by the problem of comparing the derived categories, cohomology rings, and Chern numbers of two smooth varieties related by a flop.

In the first part of this thesis, we describe noncommutative desingularizations of determinantal varieties, determinantal varieties defined by minors of generic symmetric matrices, and pfaffian varieties defined by pfaffians of generic anti-symmetric matrices. For maximal minors of square matrices and symmetric matrices, this gives a non-commutative crepant resolution. Along the way, we describe a method to calculate the quiver with relations for any non-commutative desingularizations coming from exceptional collections over partial flag varieties.

In the second part of this thesis, we study $t$-structures coming from noncommutative symplectic resolutions. A localization theorem for the cyclotomic rational Cherednik algebra $H_c = H_c((\mathbb{Z}/l)^n \rtimes \mathfrak{S}_n)$ over a field of positive characteristic has been proved by Bezrukavnikov, Finkelberg, and Ginzburg. Localizations with different parameters give different $t$-structures on the derived category of coherent sheaves on the Hilbert scheme of points on a surface. In this short note, we concentrate on the comparison between different $t$-structures coming from different localizations. When $n = 2$, we show an explicit construction of tilting bundles that generates these $t$-structures. These $t$-structures are controlled by a real variation of stability conditions, a notion related to Bridgeland stability conditions.
We also show its relation to the topology of Hilbert schemes and irreducible representations of $H_c$.

In the last part of this thesis, we study Chern numbers of varieties related by flops. For this purpose, we define the algebraic elliptic cohomology theory coming from Krichever’s elliptic genus as an oriented cohomology theory on smooth varieties over an arbitrary perfect field. We show that in the algebraic cobordism ring with rational coefficients, the ideal generated by differences of classical flops coincides with the kernel of Krichever’s elliptic genus. This generalizes a theorem of B. Totaro in the complex analytic setting.
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Disclaimer

I hereby declare that the work in this thesis is that of the candidate alone, except where indicated in the text, and as described below.

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Chapter 1

Introduction

1.1 Motivation

In this thesis we address several aspects related to the following question. Let $Y$ be a variety with Gorenstein singularities, and let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be two crepant resolutions. We would like to understand how different or similar could $X_1$ and $X_2$ be. This question can be made precise by the following conjectures and problems. In this thesis, we will discuss examples coming from representation theory towards these conjectures and problems.

For a smooth quasi-projective variety over a separably closed field $k$, we denoted its derived category of coherent sheaves by $D^b(\text{Coh}(X))$.

**Conjecture 1.1.1** (Bondal and Orlov). There is an equivalence of derived categories

$$D^b(\text{Coh}(X_1)) \cong D^b(\text{Coh}(X_2)).$$

More generally, they conjectured that if $X_1$ and $X_2$ are smooth projective varieties which are $K$-equivalent, i.e., there is some smooth projective variety $\tilde{X}$ with $\pi_1 : \tilde{X} \to X_1$ and $\pi_2 : \tilde{X} \to X_2$ smooth resolutions such that $\pi_1^* K_{X_1} = \pi_2^* K_{X_2}$, then there is an equivalence of triangulated categories $D^b(\text{Coh}(X_1)) \cong D^b(\text{Coh}(X_2))$. 
Assuming $X_1$ or $X_2$ has maximal Kodaira dimension, i.e., when one of them (equivalently, both of them) has Kodaira dimension equal to its dimension, then it is proved by Kawamata that the Fourier-Mukai functor with kernel the structure sheaf of $\tilde{X}$ is an equivalence of derived categories. Assuming $X_1$ and $X_2$ are symplectic varieties related by Mukai flops, then it is shown by Namikawa that the Fourier-Mukai functor with kernel a suitable is an equivalence. It is shown by Bondal and Orlov that the same conclusion holds, assuming that $X_1$ and $X_2$ are related by a *standard flop*, that is, when there is a subvariety $\mathbb{P}^k \hookrightarrow X_1$ with normal bundle $\mathcal{O}(-1)^{\oplus k+1}$, and $\tilde{X}$ is the blow-up of $X_1$ along $\mathbb{P}^k$ and $X_2$ is the blow-down of $\tilde{X}$ along a different component of $\mathbb{P}^k$. In general, this conjecture is still open.

The Bonda-Orlov Conjecture has been proved when $\dim Y = 3$ by Tom Bridgeland. The proof has been reinterpreted by van den Bergh in [74]. We will review this in details in section 1.2.2 following the exposition of van den Bergh, together with closely related question. In [74], van den Bergh introduced the notion of *noncommutative crepant resolutions*. He proved that a three-dimensional Gorenstein singularity admits a noncommutative crepant resolution if and only if there is a commutative one. Moreover, in this case all crepant resolutions, commutative as well as noncommutative ones, are all derived equivalent to each other. Furthermore, he conjectured a more general statement.

**Conjecture 1.1.2** (van den Bergh). Let $R$ be a normal Gorenstein domain. All crepant resolutions of $\text{Spec } R$ (not necessarily of Krull dimension 3), commutative as well as non-commutative ones, are derived equivalent.

This conjecture has as a special case the McKay equivalence. It says, for $G \subseteq \text{SL}(V)$ with $V$ a vector space, for any crepant resolution $Y \to V/G$, there is a derived equivalence $D^b(\text{Coh}(Y)) \cong D^b_G(\text{Coh } V)$. This equivalence is still conjectural, except for some special cases, some which we will recall in details.

When $V$ is a symplectic vector space, and $Y \to V/G$ is a crepant (or symplectic) resolutions, it is proved by Bezrukavnikov and Kaledin that $D^b(\text{Coh}(Y)) \cong D^b_G(\text{Coh } V)$. This
equivalence comes from a tilting bundle on $Y$ that is constructed from a quantization of $Y$, or more precisely the lifting of $Y$ to a field of positive characteristic. In fact, quantizations come in families. It is natural to compare the different $t$-structures coming from different members of this family on the same derived category $D^b(\text{Coh}(X))$. This aspect will be treated in the body of this thesis.

From the point of view of cohomology theories, one can ask how different their cohomology could be. Any Weil cohomology of a smooth projective variety $X$ is determined by the Chow motive of $X$, as an object in the category of Chow motives over $k$. If one concentrate on Weil cohomology theories with rational coefficients, then the following is conjectured by Orlov.

**Conjecture 1.1.3** (Orlov). Assume $X_1$ and $X_2$ are two smooth projective varieties with $D^b(\text{Coh}(X_1)) \cong D^b(\text{Coh}(X_2))$, then the Chow motives with rational coefficients of $X_1$ and $X_2$ are isomorphic.

Although we will not address this direction in this thesis, it is an interesting subject closely related to questions discussed here.

Of similar nature, one can propose the following problem. We consider oriented cohomology theories, and the numerical invariants coming from to them.

**Problem 1.1.4.** Let $k$ be an algebraically closed field of characteristic zero. Let $\Omega^*(k)$ be the algebraic cobordism ring of the base field in the sense of Levine and Morel [54]. Let $I$ be the ideal in this ring generated by $[X_1] − [X_2]$ where $X_1$ and $X_2$ are two smooth quasi-projective varieties related by a classical flop. Give a description of the genus $\Omega^*(k) \to \Omega^*(k)/I$.

The solution to this problem is the first step towards another question asked by Totaro in [71].

**Question 1.1.5** (Totaro). Over a field of characteristic zero, what are the Chern numbers well-defined for a quasi-projective variety with Gorenstein singularities.
A complete answer to this question is given by Borisov and Libgober in [18], based on the work of Totaro in [71]. This question has a direct analogue when \( k \) is an arbitrary separably closed field, which will be spelled out more precisely in section 1.2.5. Although a complete answer is not known, partial results towards it has been achieved and will be discussed in the body of this thesis.

Birational symplectic varieties are more alike. Namikawa recently proved that any two symplectic resolutions are related to a sequence of Mukai flops. It is shown by Huybrechts that any two smooth symplectic varieties admit a common one parameter deformation. More precisely, if \( X_1 \) and \( X_2 \) are connected by a general Mukai flop, then, there exist smooth projective algebraic varieties \( X_1' \) and \( X_2' \), flat over a smooth quasi-projective curve \( C \) with a closed point \( o \in C \), such that:

(i) the fiber of \( X_i \) over 0 is \( (X_i)_o = X_i \);

(ii) there is an isomorphism \( \Psi : (X_1)_{C \setminus \{o\}} \to (X_2)_{C \setminus \{o\}} \) over \( C \).

It is shown in [37] that if conditions (i) and (ii) hold, then the varieties have isomorphic Chow rings. It is reasonable to make the following conjecture.

**Question 1.1.6.** Let \( X_1 \) and \( X_2 \) be smooth projective varieties such that there exist smooth projective algebraic varieties \( X_1' \) and \( X_2' \), flat over a smooth quasi-projective curve \( C \) with a closed point \( o \in C \) satisfying conditions (i) and (ii). Is there an isomorphism \( F : \Omega^*(X_1) \cong \Omega^*(X_2) \) of graded algebras satisfying the following properties?

Let \( p_1 : X_1 \to \text{pt} \) and \( p_2 : X_2 \to \text{pt} \) be their structure morphisms. We have \( p_1^*(1_{X_1}) = p_2^*(1_{X_2}) \in \Omega^*(k) \), where \( 1_{X_i} \) is the fundamental class of \( X_i \) in \( \Omega^*(X_i) \). In particular, all the Chern numbers of \( X_1 \) and \( X_2 \) are equal.

This question will be discussed in the body of this thesis in Section 4.6.
1.2 Preliminaries

1.2.1 Noncommutative crepant resolutions

Van den Bergh’s main idea in the proof of the Bondal-Orlov Conjecture in dimension three is to introduce the notion of noncommutative crepant resolutions, which will be recalled below, and use this notion as a media to prove the Bondal-Orlov Conjecture.

For a singular variety $X$, a non-commutative desingularization (Definition 2.1.2) is a coherent sheaf of associative algebras $\mathcal{A}$ over a proper scheme $Y$ over $X$ birationally equivalent to $X$, such that $\mathcal{A}$ is generically a sheaf of matrix algebras and has finite homological dimension. This notion arises from the study of the derived categories of coherent sheaves, Lie theory, maximal Cohen-Macaulay modules, as well as theoretical physics. In the case when $X$ has Gorenstein singularities, van den Bergh introduced a notion of non-commutative crepant resolution (Definition 2.1.4, c.f. also [74]), i.e., a non-commutative desingularization which is maximal Cohen-Macaulay as a coherent sheaf on $X$. This notion is a generalization of an existent notion of crepant resolution, and is introduced with the hope that any two crepant resolutions, commutative or not, are derived equivalent.

The idea of non-commutative desingularization can be traced back to the general theory of derived Morita equivalence due to Rickard et al.. Let $k$ be a field. An ordered set of objects $\Delta = \{\Delta_\alpha, \alpha \in I\}$ in a triangulated $k$-linear category $D$ is called exceptional if we have $\text{Ext}^n(\Delta_\alpha, \Delta_\beta) = 0$ for $\alpha < \beta$ and $\text{End}(\Delta_\alpha) = k$; it is said to be strongly exceptional if further $\text{Ext}^n(\Delta_\alpha, \Delta_\beta) = 0$ for $n \neq 0$. An exceptional set is said to be full if it generates $D$. If $X$ is a projective variety, $(E_0, \ldots, E_n)$ is a full strongly exceptional collection in the derived category of coherent sheaves on $X$. Then there is an equivalence of derived categories

$$R\text{Hom}(\oplus_i E_i, -) : D^b(\text{Coh } X) \to D^b(A\text{-mod}),$$

where $A\text{-mod}$ is the category of finitely generated left modules over $A := \text{End}(\oplus_i E_i)$ with the opposite multiplication. Note that by properness of $X$, the algebra $A$ is finite dimensional.
as vector space over $k$, and can be described as the path algebra of a quiver with relations.

Let $Z$ be a quasi-projective variety, e.g., the total space of a vector bundle over a projective variety $X$, then $Z$ rarely admits an exceptional collection. Nevertheless, if it does admit tilting objects, the category of coherent sheaves on $Z$ is derived equivalent to the endomorphism algebra of a tilting object, as has been explained in [45] and will be reviewed in Section 2.1. Here by a tilting object, we mean a perfect complex $T$ in $D^b(\text{Coh} \ Z)$ such that $\text{Ext}^i(T, T) = 0$ for all $i \neq 0$, and $T$ generates the entire derived category, in the sense that the smallest triangulated subcategory containing $T$ closed under direct summands is the entire derived category. In all cases we are interested in, the tilting object can always be chosen as a vector bundle, considered as a complex of coherent sheaves concentrated on degree 0. If this happens, the assumption that $T$ generates $D^b(\text{Coh} \ Z)$ can be replaced by that $\text{Ext}^\bullet(T, C) = 0$ for some complex $C$ implies $C$ is exact (see, e.g. [17]). But the fact that $Z$ is non-compact makes the endomorphism algebra slightly more complicated than the case when $Z$ proper.

If $X$ is a smooth projective Fano variety and $Z = \omega_X$ is the total space of its canonical bundle, the inverse images of some exceptional collections on $X$ to $\omega_X$ has been used by Tom Bridgeland et al. to study t-structures in the derived categories of coherent sheaves on $\omega_X$ (see, e.g. [16]).

There are a large number of examples of noncommutative crepant resolutions coming from tilting bundles on a commutative resolutions. Derived McKay equivalence is a typical example.

Let $V$ be a vector space, and $G \leq \text{GL}(V)$ a finite subgroup. Assume the characteristic of $k$ does not divide $|G|$, and $G$ contains no reflections. Denote the coordinate ring of $V$ by $S$, and let $R = S^G$ be the coordinate ring of $V/G$. It is shown by Auslander that $\text{End}_R(S)$ is isomorphic to the twisted group ring $S \# G$. As a consequence, $\text{End}_R(S)$ has finite global dimension and is maximal Cohen-Macaulay. In fact, the category of finitely generated
modules \text{Mod-} S\#G is canonically isomorphic to the category of equivariant coherent sheaves on \( V \), denoted by \( \text{Coh}_G(V) \). The indecomposable projective objects in it are globally free vector bundle on \( V \) carrying irreducible representations of \( G \). The simple objects are the skyscraper sheaves concentrated at the origin carrying irreducible representations of \( G \). The minimal free resolutions of the simple objects are given by the Koszul complexes.

When \( V \) has dimension 2 over \( k \) and \( G \) is a finite subgroup of \( \text{SL}(V) \), the quotient \( V/G \) is said to have Kleinian singularities. Such singularities are classified by the simply-laced Dynkin diagrams. It is observed by McKay that taking the minimal resolution \( \pi: \tilde{V}/G \rightarrow V/G \), the exceptional fiber of \( \pi \) consists of a tree of \( \mathbb{P}^1 \)'s, the number of irreducible components being one less than the number of irreducible representations of \( G \). Therefore, the Grothendieck group of coherent sheaves on \( \tilde{V}/G \) is isomorphic as a group to the equivariant K-theory of \( V \).

This isomorphism can be lifted to the level of derived categories, as proved by Kapranov and Vasserot. More precisely, they proved the existence of a derived equivalence \( D^b(\text{Coh}(\tilde{V}/G)) \cong D^b(\text{Coh}_G(V)) \).

1.2.2 Three dimensional terminal Gorenstein singularities

The first example of noncommutative crepant resolution of none-quotient singularities is constructed by van den Bergh. Assume \( X = \text{Spec} \, R \) for some complete local Gorenstein ring \( R \) with rational singularity, containing its residue field. Let \( f: Y \rightarrow X \) be a resolution of singularity such that \( e := f^{-1}(0) \) is a curve. Note that this generalizes the case of Kleinian singularities and includes 3-dimensional terminal Gorenstein singularities.

It is proved by van den Bergh that on \( Y \), there is a tilting bundle \( \mathcal{M} \), including \( \mathcal{O} \) as its direct summand, satisfying the following properties.

1. \( f_*\mathcal{M} \) is maximal Cohen-Macaulay as module over \( R \);

2. \( \text{End}_R(f_*\mathcal{M}) \cong \text{End}_Y(\mathcal{M}) \), and it is a noncommutative crepant resolution, i.e., maximal
Cohen-Macaulay as module over $R$ and has finite global dimension.

Consequently, there is a derived equivalence $D^b(\text{Coh}(Y)) \cong D^b(\text{Mod-End}_Y(\mathcal{M}))$.

The proof given by van den Bergh is illuminating. Let $C = \bigcup_{i=1}^n C_i$ be the decomposition of $C$ into its irreducible components. The assumption that $X$ has only rational singularities forces $C_i$’s to be a tree of $\mathbb{P}^1$’s. By the Grothendieck Existence Theorem, one has line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on $Y$, generated by global sections, satisfying the property that $\text{deg}(\mathcal{L}_i|_{C_j}) = \delta_{ij}$. Clearly $\mathcal{O}$ and $\bigoplus_{i=1}^n \mathcal{L}_i$ generate the category $D^b(\text{Coh}(Y))$. But there might be higher extension of $\mathcal{O}$ by $\mathcal{L}_i$. One define $\mathcal{M}_i$ to be the universal extension

$$0 \to \mathcal{O} \otimes H^1(Y, \mathcal{L}_i^{-1}) \to \mathcal{M}_i \to \mathcal{L}_i \to 0.$$ 

Then one can easily check that $\mathcal{M} := \bigoplus_{i=1}^n \mathcal{M}_i \oplus \mathcal{O}$ is a tilting bundle on $Y$.

Under the derived equivalence $D^b(\text{Coh}(Y)) \cong D^b(\text{Mod-End}_Y(\mathcal{M}))$, clearly the irreducible projective modules over $\text{End}_Y(\mathcal{M})$ corresponds to $\mathcal{O}$ and $\mathcal{M}_i$’s on $Y$. The complexes on $Y$ corresponding to the simple modules over $\text{End}_Y(\mathcal{M})$ can also be identified. They are $\mathcal{O}_C$ and $\mathcal{O}_C(-1)[1]$. They can also be viewed as simple perverse coherent sheaves on $Y$ for suitable perversity.

This construction gives, assuming the existence of commutative crepant resolution, a noncommutative crepant resolution. Assuming $\dim R \leq 3$, the reverse is also true. It is proved by van den Bergh that from a noncommutative crepant resolution, one can construct a commutative one as the moduli space of certain stable representations over this noncommutative ring. Furthermore, van den Bergh proved that, still in the case when $\dim R \leq 3$, all crepant resolutions of $\text{Spec } R$, commutative as well as noncommutative, are all derived equivalent to each other.

Note that this is property is a unique feature in dimension 3. In general, let $R = \mathbb{C}[x_0, \ldots, x_n]/(x_0^l + x_1^n + \cdots + x_n^n)$ with $l > n$ and $l = \equiv 1 \mod n$. For any $n$ there is a commutative crepant resolution of $\text{Spec } R$. But it is shown by Dao that there is no non-
commutative crepant resolution. Conversely, in Section 1.2.3, we will see examples admit noncommutative crepant resolutions but no commutative ones.

1.2.3 Noncommutative symplectic resolutions

Let $\Gamma$ be a finite group acting on a vector space $\mathfrak{h}$ generated by reflections. Let $V := \mathfrak{h} \oplus \mathfrak{h}^*$ be endowed with the natural symplectic form, with the diagonal $\Gamma$-action. The quotient $V/\Gamma$ has Gorenstein singularities, and the smooth part has a symplectic form. Let $W(\mathfrak{h})$ be the Weyl algebra, with the natural $\Gamma$-action. The algebra $W(\mathfrak{h})\#\Gamma$ is a noncommutative crepant resolution. When $|\Gamma|$ does not divide the characteristic of $k$, the algebra $W(\mathfrak{h})\#\Gamma$ is Morita equivalent to $W(\mathfrak{h})^\Gamma$.

The algebra $W(\mathfrak{h})\#\Gamma$ has a universal deformation, denoted by $H_c(\Gamma)$, called the rational Cherednik algebra with parameter $c$. The parameter space of this deformation is spanned by the conjugacy classes of reflections in $\Gamma$. There is also a spherical subalgebra $^s H_c(\Gamma)$.

The algebra $H_c(\Gamma)$ has PBW properties. Namely, for suitably defined filtration of $H_c(\Gamma)$, the associated graded algebra is isomorphic to $k[V]\#\Gamma$. When the characteristic of $k$ is positive and large enough, $^s H_c(\Gamma)$ has a big Frobenius center $k[V^{(1)}]^\Gamma$ where $V^{(1)}$ is the Frobenius twist of $V$.

For each value $c$, the algebra $H_c(\Gamma)$ has finite global dimension, hence is a noncommutative crepant resolution of $V/\Gamma$. But $^s H_c(\Gamma)$ could have infinite global dimension. In fact, $^s H_c(\Gamma)$ has finite global dimension if and only if it is Morita equivalent to $H_c(\Gamma)$. In this case, we say $c$ is a spherical value. Spherical value is generic.

Now we assume that $V/\Gamma$ admits a symplectic resolution $\pi : X \to V/\Gamma$, that is, a resolution of singularities endowed with a symplectic form which on the regular part coincide with the symplectic form on $V/\Gamma$. Clearly, the space $H^2(X; \mathbb{Q})$ is spanned by divisor classes coming from conjugacy classes of reflections in $\Gamma$. When the characteristic of $k$ is large enough, it is shown by Bezrukavnikov, Finkelberg, and Ginzburg in [12], that $X$ admits a
family of quantizations $\mathcal{O}_c$, parameterized by $c \in H^2(X; \mathbb{Q})$. The space of global sections $H^0(X, \mathcal{O}_c) \simeq \mathcal{H}_c(\Gamma)$. Let $\text{Fr} : X \to X^{(1)}$ be the Frobenius morphism. Then $\text{Fr}_* \mathcal{O}_c$ is a coherent sheaf of algebras on $X^{(1)}$, which splits on $X_0^\wedge$, the formal completion of $X$ at the fiber $\pi^{-1}(0)$. In other words, there is a tilting bundle $\mathcal{E}_c$ on $X$, such that $\mathcal{E}_{\text{nd}}(\mathcal{E}|_{X_0^\wedge}) \simeq \text{Fr}_* \mathcal{O}_c|_{X_0^\wedge}$. Therefore, if $\mathcal{H}_c(\Gamma)$ has finite global dimension, then there is a derived equivalence $D^b(\text{Coh}_0(X)) \simeq D^b(\text{Mod} \mathcal{H}_c(\Gamma))$.

It is shown by Ginzburg and Kaledin that when $\Gamma$ is the Coxeter group for a root system of type $D_n$, $E_n$, $F_4$, or $G_2$, the quotient $V/\Gamma$ has no symplectic or crepant resolutions. This provides an example where noncommutative crepant resolutions exist but the commutative one does not.

It is known (see e.g., [50]) that when $\Gamma = (\Gamma_1)^n \rtimes S_n$ with $\Gamma_1$ being a finite subgroup of $\text{SL}_2(k)$, the quotient $A^{2n}/\Gamma$ admits a symplectic resolution of Hilbert scheme type. This includes the Coxeter groups of type $A_n$, $B_n$, and $C_n$.

1.2.4 Noncommutative desingularization of determinantal varieties

Let $G$ be a reductive group and $P$ a parabolic subgroup. Let $p : Z \to G/P$ be an equivariant vector bundle. We use the inverse image of the exceptional collections on $G/P$ to $Z$, with special emphasis on the situation when the total space $Z$ is a commutative desingularization of an orbit closure in some representations of $G$.

When the vector bundle is the commutative desingularization of generic determinantal variety defined by maximal minors, a non-commutative desingularization of this nature has been studied by R. Buchweitz, G. Leuschke, and M. van den Bergh in [19].

The study of exceptional collections was initiated by Beilinson and Kapranov who dealt with the case of the projective spaces and partial flag varieties. Let $B_{u,v}$ be the set of partitions with no more than $u$ rows and $v$ columns. According to the result of Kapranov,
for certain ordering on $B_{r,n-r}$, the set
\[ \{ \mathbb{L}_\alpha Q \mid \alpha \in B_{r,n-r} \} \]
is an exceptional collection over Grass$_{n-r}(E)$, the Grassmannian of $(n-r)$-planes in an $n$-dimensional vector space $E$, where $\mathbb{L}_\lambda$ is the Schur functor applied to the partition $\lambda$, and $Q$ is the tautological rank $r$ quotient bundle over Grass$_{n-r}(E)$.

Let $H = \text{Hom}_k(E, F)$, where $E$ and $F$ are vector spaces over $k$ of dimension $m$ and $n$ respectively (with $n \leq m$). Let $\text{Spec} R \subseteq H$ be the subvariety consisting of \{ $\varphi \in \text{Hom}_k(E, F) \mid \text{rank} \varphi \leq n - 1$ \}. Let $0 \to \mathcal{R} \to F^* \otimes \mathcal{O} \to Q \to 0$ be the universal sequence over $\mathbb{P}(F^*)$. A desingularization of $\text{Spec} R$: $p' : Z \to \mathbb{P}(F^*)$, the total space of the vector bundle $E^* \otimes_k Q^*$.

It is proved by Buchweitz, Leuschke, and van den Bergh that in the case, the vector bundle $p'^* \oplus_{a=0}^n \wedge^a Q^*$ is a tilting bundle over $Z$. Its endomorphism algebra $\Lambda = \text{End}_{\mathcal{O}_Z}(p'^* \oplus_{a=0}^n \wedge^a Q^*)$ is maximal Cohen-Macaulay as an $R$-module, and has finite global dimension. In other words, $\Lambda$ is a noncommutative crepant resolution of $\text{Spec} R$. As a $k$-algebra, $\Lambda$ is isomorphic to the path algebra of the quiver
\[
\begin{array}{cccccc}
\bullet_1 & \xleftarrow{\alpha_1} & \bullet_2 & \xleftarrow{\alpha_2} & \cdots & \bullet_m \\
& \swarrow{\beta_m} & & \swarrow{\beta_m} & & \\
& \alpha_3 & & \alpha_3 & & \alpha_3 \\
& \swarrow{\beta_m} & & \swarrow{\beta_m} & & \\
& \alpha_4 & & \alpha_4 & & \alpha_4 \\
& \swarrow{\beta_m} & & \swarrow{\beta_m} & & \\
& \alpha_n & & \alpha_n & & \alpha_n \\
\end{array}
\]
with relations:
\[
\begin{align*}
\alpha_i \alpha_j + \alpha_j \alpha_i &= 0 = \alpha_i^2 \\
\beta_i \beta_j + \beta_j \beta_i &= 0 = \beta_i^2 \\
\alpha_k (\alpha_i \beta_j + \beta_j \alpha_i) &= (\alpha_i \beta_j + \beta_j \alpha_i) \alpha_k \\
gl(\beta_i \alpha_j + \alpha_j \beta_i) &= (\beta_i \alpha_j + \alpha_j \beta_i) \alpha_l
\end{align*}
\]
Moreover, the commutative desingularization can be constructed from the noncommutative one. More precisely, the variety $Z$ is the fine moduli space for the representations $W$ of this quiver of dimension vector $(1, m-1, (m-1)/2, \cdots, 1)$ that are generated by the component $W_m$ at the last vertex.
1.2.5 Elliptic genera and flops

The notion of complex oriented cohomology theory in the category of smooth manifolds was introduced by Quillen [69] in his study of the universal complex oriented cohomology theory, complex cobordism. The existence of a formal group law associated to a complex oriented theory allowed Quillen to define an isomorphism of the complex cobordism ring with the Lazard ring $\mathbb{L}_{\text{Laz}}$, the underlying ring of the universal formal group law $F_{\text{Laz}}(u, v) \in \mathbb{L}_{\text{Laz}}[u, v]$.

We use the notion of an oriented cohomology theory on the category of smooth quasi-projective schemes $\text{Sm}_k$ over a perfect field $k$ given by Levine and Morel in [54]. This is an algebraic analog of complex oriented cohomology, without however requiring an excision property or a Mayer-Vietoris sequence. When the underlying field $k$ has characteristic zero, the universal oriented cohomology theory, $X \mapsto \Omega^*(X)$, exists and is called algebraic cobordism. Just as for complex cobordism, the coefficient ring $\Omega^*(k)$ of algebraic cobordism is isomorphic to the Lazard ring. A construction of $\Omega^*(X)$ via explicit generators and relations is given in [54]; when the field $k$ has positive characteristic, it is not known if this construction yields an oriented cohomology theory.

To handle the situation in positive characteristic, we use motivic homotopy theory. Let $\text{SH}(k)$ be the motivic stable homotopy category of $\mathbb{P}^1$-spectra [58, 64, 76]. One has a refined notion of an oriented cohomology theory on $\text{Sm}_k$, which we refer to in this paper as a motivic oriented cohomology theory on $\text{Sm}_k$. This is given by an object $\mathcal{E} \in \text{SH}(k)$ together with a multiplication map $\mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$, a unit map $S_k \to \mathcal{E}$ and an orientation $\vartheta \in \mathcal{E}^{2,1}(\mathbb{P}^\infty/0)$ which gives the assignment $X \mapsto \mathcal{E}^{*,*}(X)$ the structure of an oriented ring cohomology theory in the sense of Panin [67]. Given an $(\mathcal{E}, \vartheta)$ and assuming that $k$ is perfect, the assignment $X \mapsto \mathcal{E}^*(X) := \mathcal{E}^{2*,*}(X)$ defines an oriented cohomology theory $\mathcal{E}^*$ on $\text{Sm}_k$, but not every oriented cohomology theory arises this way.

Let $\text{MGL} \in \text{SH}(k)$ be the algebraic cobordism $\mathbb{P}^1$-spectrum [64, 76]. MGL comes with a canonical orientation, $\vartheta_{\text{MGL}}$, and it has been shown by Panin-Pimenov-Röndigs [65] that
(MGL, \rho_{MGL}^*) defines the universal motivic oriented cohomology theory on Sm\_k. For a perfect field k, we thus have the oriented cohomology theory MGL\^* on Sm\_k, which by the main result of [53] is canonically isomorphic to \Omega^* in case k has characteristic zero.

We let p denote the exponential characteristic of the base field k, that is, p = 1 if k has characteristic zero, otherwise p is the characteristic of k.

It follows from a theorem of Hopkins-Morel, recently established in detail by Hoyois [42], that for k of characteristic zero, the ring homomorphism \mathbb{L}az \to MGL(k) classifying the formal group law for MGL\^* is an isomorphism. When k has characteristic p > 0, it is shown in [42] that MGL\^*(k)[1/p] \cong \mathbb{L}az[1/p]. Conjecturally, for any field k, the classifying map \mathbb{L}az \to MGL\^*(k) is an isomorphism, but at present, this is not known. In any case, for an arbitrary perfect field k, we have at our disposal the oriented cohomology theory MGL\^* on Sm\_k, which agrees with \Omega^* if k has characteristic zero, and for which the classifying homomorphism \phi_{MGL} : \mathbb{L}az \to MGL\^*(k) is an isomorphism after inverting the exponential characteristic of k.

Now let F(u, v) \in R[[u, v]] be an arbitrary formal group law over a commutative ring R; we assume that the exponential characteristic p is invertible in R. For each X \in Sm\_k, composing the classifying homomorphism \phi_{MGL} : \mathbb{L}az \to MGL(k) with the pull-back the structure morphism p\_X : X \to \text{Spec} k makes MGL\^*(X)[1/p] an algebra over \mathbb{L}az[1/p]; we have as well the classifying homomorphism \phi_F : \mathbb{L}az[1/p] \to R. The assignment

\[ X \mapsto MGL\^*(X) \otimes_{\mathbb{L}az[1/p]} R := R^*(X) \]

is easily seen to extend to an oriented cohomology theory on Sm\_k, which we denote by R^*. It is immediate that R^*(k) = R and that the formal group law associated to R^* is F(u, v) \in R[[u, v]].

We apply all this machinery to give algebraic versions of elliptic cohomology. Given a family of elliptic curves over some ring R, there is a formal group law over the ring R, coming from the additive structure of the elliptic curves and a choice of parameter along the
zero section over Spec $R$. The corresponding cohomology theory $X \mapsto R^*(X)$, an algebraic elliptic cohomology theory, is the main subject of this paper.

Such group laws are often constructed using a so-called elliptic genus rather than an explicit construction of the family of elliptic curves. The elliptic genus is simply a power series that transforms the additive group law $F(u, v) = u + v$ into the given elliptic group law by using it as a change of coordinates. There are various versions of elliptic genera. We concentrate on the elliptic genus studied by Krichever [49], whose Hirzebruch characteristic power series is given by

$$Q(t) := \frac{t}{2\pi i} e^{kt} \Phi(t, z) = \frac{t}{2\pi i} e^{kt} e^{\zeta(z)} \frac{\sigma(\frac{t}{2\pi i} - z, \tau)}{\sigma(\frac{t}{2\pi i}, \tau)\sigma(-z, \tau)},$$

where $\Phi(t, z)$ is the Baker-Akhiezer function, $\sigma$ is the Weierstrass sigma function and $\zeta(z) = \frac{d\log \sigma(z)}{dz}$. (See Section 4.2.1 for details.) We will call this the elliptic genus for the remainder of the paper.

When $k = \mathbb{C}$, the elliptic genus has a crucial property, called the rigidity property, proved by Krichever in [49] and Höhn in [40]. A consequence of the rigidity property is that given a fiber bundle $F \to E \to B$ of closed connected weakly complex manifolds, with structure group a compact connected Lie group $G$, and if $F$ is a $SU$-manifold, then the elliptic genus is multiplicative with respect to this fibration. The rigidity property states that the elliptic genus is multiplicative under the fibrations with fibers admitting an $SU$-structure. Moreover, the complex elliptic genus is the universal genus with this property. Using the rigidity property, Höhn showed that the elliptic cohomology ring (after $\mathbb{Q}$-localization) is isomorphic to a polynomial ring $\mathbb{Q}[a_1, a_2, a_3, a_4]$, with $a_i$ having degree $-i$ (using our conventions for the grading).

The elliptic genus also arises in the study of Chern numbers of singular varieties. Inspired by Höhn’s work, Totaro showed in [71] that the kernel of the elliptic genus coincides with the ideal in the complex cobordism ring $MU^*\mathbb{C}$ generated by the differences of classical flops. As a corollary, the characteristic numbers which can be defined for singular varieties
in a fashion which is compatible with small resolutions, are exactly the specializations of the elliptic genus. It is worth mentioning that in Totaro’s work, the proofs take place in the setting of weakly-complex manifolds and use topological constructions which do not lend themselves to the situation over a field of positive characteristic.

Based on Totaro’s work, Borisov and Libgober introduced the Orbifold elliptic genus and elliptic genus of singular varieties. elliptic genus $\phi(Y)$ for a projective variety with log-terminal singularities $Y$. They proved, for any crepant resolution $X \to Y$, we have $\phi(Y) = \phi(X)$. The way Borisov and Libgober defining the singular elliptic genus is by passing to a resolution with normal crossing divisor, and prove the independence of resolution using the weak factorization theorem of Abramovich, Karu, Matsuki, and Wlodarczyk [3]. Using the weak factorization theorem, Chin-Lung Wang proved in [79] a change of variable formula, and as a consequence, he proved the ideal in the complex cobordism ring $MU^*$ generated by differences of flops equals the ideal generated by $K$-equivalence. Moreover, he showed that the specializations of complex elliptic genus are the universal genera that could be defined through the change of variable formula on log-terminal singularities.

1.3 Summary of main results

1.3.1 Noncommutative desingularization of determinantal varieties

Let $G$ be a reductive group and $P$ a parabolic subgroup. Let $p : Z \to G/P$ be an equivariant vector bundle. We use the inverse image of the exceptional collections on $G/P$ to $Z$, with special emphasis on the situation when the total space $Z$ is a commutative desingularization of an orbit closure in some representations of $G$.

Recall that $B_{u,v}$ is the set of partitions with no more than $u$ rows and $v$ columns. For certain ordering on $B_{r,n-r}$, the set

$$\{\mathbb{L}_\alpha Q \mid \alpha \in B_{r,n-r}\}$$
is an exceptional collection over Grass\(_{n-r}(E)\), the Grassmannian of \((n-r)\)-planes in an \(n\)-dimensional vector space \(E\), where \(L_\lambda\) is the Schur functor applied to the partition \(\lambda\), and \(Q\) is the tautological rank \(r\) quotient bundle over Grass\(_{n-r}(E)\).

Let \(p : Z \to G/P\) be an equivariant vector bundle, and let \(\Delta(G/P) = \{\Delta_\alpha \mid \alpha \in I\}\) be a full exceptional collection on \(G/P\). On the total space \(Z\), the inverse image \(p^*(\bigoplus \alpha \Delta_\alpha)\) can fail to be a tilting object. Moreover, its endomorphism algebra \(\Lambda := \text{End}_{O_Z}(p^*(\bigoplus \alpha \Delta_\alpha))\) is usually infinite dimensional as a vector space.

It is easy to show (Proposition 2.1.7) that for any tilting bundle \(\mathcal{T}il\) on \(G/P\), (in particular for \(\bigoplus \alpha \Delta_\alpha\)) the inverse image \(p^*\mathcal{T}il\) is a tilting bundle on \(Z\) if the \(p^*\mathcal{E}\text{nd}_{O_{G/P}}(\mathcal{T}il)\) has no higher cohomology. Let \(R\) be a normal integral domain endowed with a resolution of singularity \(q : Z \to \text{Spec} R\) by \(Z\). If the vector bundle \(Z\) over \(G/P\) has rank higher than 1, then we show that \(\text{End}_{O_Z}(p^*\mathcal{T}il) \cong \text{End}_R(q_*p^*\mathcal{T}il)\) and \(\text{End}_R(q_*p^*\mathcal{T}il)\) is a non-commutative desingularization of \(\text{Spec} R\).

On a homogeneous space \(G/P\), an exceptional collection often consists of equivariant vector bundles. In such case the objects in \(D^b(Z)\) corresponding to the simples are obtained by pushing forward the dual exceptional collection (Definition 2.2.3) over \(G/P\).

Let \(\mathfrak{A}\) be a \(k\)-linear category. Recall that for an exceptional collection \(\Delta\), the dual collection \(\nabla = \{\nabla_\alpha \mid \alpha \in I\}\) is another subset of objects in \(D^b(\mathfrak{A})\), in bijection with \(\Delta\), such that \(\text{Ext}^*(\nabla_\beta, \Delta_\alpha) = 0\) for \(\beta > \alpha\), and there exists an isomorphism \(\nabla_\beta \cong \Delta_\beta \mod D_{<\beta}\), where \(D_{<\beta}\) is the full triangulated subcategory generated by \(\{\Delta_\alpha \mid \alpha < \beta\}\).

Let \(\Lambda = \text{End}_{O_Z}(p^*(\bigoplus \alpha \Delta_\alpha))\) and \(S_\beta = R\text{Hom}_{O_Z}(p^*(\bigoplus \alpha \Delta_\alpha), u_*\nabla_\beta)\), where \(u : G/P \to Z\) is the zero section.

We prove

**Theorem A** (Theorem 2.2.18). Let \(\Delta(G/P) = \{\Delta_\alpha \mid \alpha \in I\}\) be a full strongly exceptional collection consisting of equivariant sheaves over \(G/P\) with the dual collection \(\nabla\). Assume the resolution \(q : Z \to \text{Spec} R\) is \(G\)-equivariant with \(q^{-1}(0) = G/P\), and the only fixed closed
point of Spec $R$ is $0 \in \text{Spec } R$. Assume moreover that $p^*(\oplus \Delta_{\alpha})$ is a tilting bundle over $Z$, $\Lambda \cong \text{End}_R(q_*p^*(\oplus \Delta_{\alpha}))$, and $\text{End}_R(q_*p^*(\oplus \Delta_{\alpha}))$ is a non-commutative desingularization. Then,

1. $S_{\alpha}$'s are equivariant simple objects in $\Lambda$-$\text{mod}$;
2. a basis of the vector space $\text{Ext}^1_{\Lambda}(S_{\alpha}, S_{\beta})^*$ generates $\Lambda$ over $\oplus \alpha k_{\alpha}$;
3. with the generators for $\text{End}_R(q_*p^*(\oplus \Delta_{\alpha}))$ as above, $\text{Ext}^2_{\mathcal{O}_Z}(\nabla_i, \nabla_j)^*$ generates the relations.

In fact, the strongness assumption for $\Delta(G/P)$ is not essential. A more general statement can be find in Theorem 2.2.18.

If Spec $R$ is the closure of an orbit in a representation of $G$, and the commutative desingularization is an equivariant vector bundle over $G/P$, the above theorem and the discussions proceeding it give a general approach to calculate a non-commutative desingularization of orbit closures. The drawback of our approach, compared to [19], is that in order to do explicit calculation we need to use the Borel-Weil-Bott Theorem, which dose not have a counterpart in positive characteristic. Nevertheless, up to Subsection 2.3.4, the results are characteristic free, unless otherwise specified.

Exceptional collections over $G/P$ are difficult to find if the characteristic of the field is positive, and very few cases are known. However, in order to apply Theorem A to calculate the idempotents, generators, and relations of the non-commutative desingularization, we just need a set of objects satisfying much weaker condition than that of an exceptional collection. In the case $G/P$ is the Grassmannian, there is no known characteristic free exceptional collection. Nevertheless, in [21], Buchweitz, Leuschke, and van den Bergh constructed a characteristic free tilting bundle over the Grassmannian. Theorem A, along with Lemma 2.3.11, can be generalized slightly to calculating idempotents, generators, and relations of the non-commutative desingularization obtained from the tilting bundle constructed in [21], up to
Morita equivalence.

The non-commutative desingularization $\text{End}_R(q_*p^*\mathcal{T}il)$, expressed by idempotents, generators and relations as above, can also be expressed as the path algebra of a quiver with relations. Note that the vector space spanned by the set of arrows between any two vertexes is naturally a representation of $G$. We find it convenient to introduce the language of equivariant quivers, as all the Ext’s are naturally representations of $G$. For the precise definitions of equivariant quivers and their representations, see Section 2.4. Very roughly, an equivariant quiver is a triple $Q = (Q_0, Q_1, \alpha)$, where $(Q_0, Q_1)$ is a quiver and $\alpha$ is an assignment associating each arrow $q \in Q_1$ a finite dimensional irreducible representation of $G$. For any equivariant quiver, there is an underlying usual quiver, upon choosing a basis for each representation associated to each arrow. The path algebra of an equivariant quiver is endowed with a natural rational $G$-action, so that we can consider the equivariant representations of it. While imposing relations to an equivariant quiver, we require the relations to be subrepresentations. In fact, if the equivariant quiver gives the endomorphism ring of a tilting object over $G/P$, the derived category of equivariant sheaves over $G/P$ is equivalent to the derived category of equivariant representations of the path algebra with relations of the equivariant quiver.

Let $\text{Spec } R$ be a $GL_n$ orbit closure in the space of $n \times n$ symmetric matrices, we do get a non-commutative desingularization by pulling-back the Kapranov’s exceptional collection.\footnote{We use the upper script $s$ to remind us that we are in the symmetric matrix case. Similarly, later on we will use the upper script $a$ in the skew-symmetric matrix case.} More precisely, we assume $E$ is a vector spaces over $k$ of dimension $n$ and $H^s$ be the affine space $\text{Sym}_2(E^*)$. Upon choosing a set of basis for $E$, the coordinate ring of $H^s$ can be identified with $S^s = k[x_{ij}, i \leq j]$. Then $R^s$ is the quotient of $S^s$ by the ideal generated by the $(r + 1) \times (r + 1)$ minors of the generic matrix $(x_{ij})$.

Let $0 \to \mathcal{R} \to E \times \text{Grass} \to Q \to 0$ be the tautological sequence over Grass, where Grass is the Grassmannian of $(n - r)$-planes in $E$. Then the total space $Z^s$ of $\text{Sym}_2 Q^s$ is a
commutative desingularization of Spec $R^s$, as is proved in [80].

We consider the pull back of $\mathcal{T}il_K = \oplus_{\alpha \in B_{r,n-r}} \mathbb{L}_\alpha \mathcal{Q}^*$, the Kapranov’s tilting bundle over Grass$_{n-r}(E)$ by $p' : \mathcal{Z}^s \to \text{Grass}$.

**Theorem B** (Proposition 2.5.3, Proposition 2.5.5, and Proposition 2.5.6). Let $\mathcal{T}il_K$ be the Kapranov’s tilting bundle over Grass$_{n-r}(E)$, and $\mathcal{Z}^s$ is the total space of $\text{Sym}_2 \mathcal{Q}^*$ which desingularizes the rank $r$ determinantal variety Spec $R^s$ of symmetric matrices.

1. The bundle $p'^* \mathcal{T}il_K$ is a tilting bundle over $\mathcal{Z}^s$, i.e., $\text{Ext}^i_{\mathcal{Z}^s}(p'^* \mathcal{T}il_K, p'^* \mathcal{T}il_K) = 0$ for $i > 0$ and $\text{Ext}^i(p'^* \mathcal{T}il_K, C) = 0$ implies $C$ is an exact complex. In particular, $D^b(\text{Coh}(\mathcal{Z}^s)) \cong D^b(\text{End}_{\mathcal{Z}^s}(p'^* \mathcal{T}il_K)-\text{mod})$.

2. The $R^s$ module $q'_{\alpha}p'^* \mathbb{L}_\alpha \mathcal{Q}^*$ is maximal Cohen-Macaulay for any $\alpha \in B_{r,n-r}$.

3. The map $\text{End}_{\mathcal{Z}^s}(p'^* \mathcal{T}il_K) \to \text{End}_S(q'_{\alpha}p'^* \mathcal{T}il_K)$ is an isomorphism of $R^s$-algebra, and this algebra is a non-commutative desingularization of Spec $R^s$. In other words, it has finite global dimension. Moreover, if $r = n - 1$, it is maximal Cohen-Macaulay over $R^s$.

Now we describe the equivariant quiver with relations for the non-commutative desingularization. We will use $C^\beta_{\alpha, \mu}$ for the Littlewood-Richardson coefficient (see e.g., [?] for the definition). For a vector space $V$ and a non-negative integer $a$, we will denote $V^a$ simply by $aV$. For any two vertexes $\alpha$ and $\beta$, the space of paths from $\beta$ to $\alpha$ will be denoted by $\text{Hom}(\beta, \alpha)$.

When $k = \mathbb{C}$, the equivariant quiver with relations of the endomorphism ring $\text{End}_S(q'_{\alpha}p'^* \mathcal{T}il_K)$ is given as follows (see Proposition 2.5.16 and Proposition 2.5.17). The set of vertices is indexed by $B_{r,n-r}$.

- In the case $n - r = 1$, the set of arrows from $\beta$ to $\alpha$ is given by $E$ if $C^\alpha_{\beta, (1,0,\ldots,0)} \neq 0$ or $C^\beta_{\alpha, (1,0,\ldots,0)} \neq 0$. No arrows otherwise. The relations are generated by the following
representations in the space $\text{Hom}(\beta, \alpha)$

$$(C^\beta_{\beta, (1, 1, 0, \ldots, 0)} \mathbb{L}_2 E) \oplus (C^\alpha_{\alpha, (1, 1, 0, \ldots, 0)} \mathbb{L}_2 E) \oplus (\delta^\beta \wedge^2 E).$$

• In the case $n - r = 2$, arrows from $\beta$ to $\alpha$ is given by $E$ if $C^\alpha_{\beta, (1, 0, \ldots, 0)} \neq 0$ and $\mathbb{C}$ if $C^\beta_{\alpha, (1, 1, 0, \ldots, 0)} \neq 0$. No arrows otherwise. The relations are generated by the following representations in the group $\text{Hom}(\beta, \alpha)$

$$(C^\beta_{\alpha', (1, -1)} \wedge^2 E) \oplus (C^\beta_{\alpha', (-1, -2)} E) \oplus (C^\beta_{\alpha', (1, 1, 0, \ldots, 0)} \mathbb{L}_2 E) \oplus (C^\beta_{\alpha', (2, 0, \ldots, 0)} \wedge^2 E).$$

• In the case $n - r \geq 3$, The set of arrows has the same description as in the case $n - r = 2$. The relations are generated by the following representations in the group $\text{Hom}(\beta, \alpha)$

$$(C^\beta_{\alpha', (0, \ldots, 0, -1, 1, 0, \ldots, 0)} \mathbb{C}) \oplus (C^\beta_{\alpha', (1, 0, \ldots, 0, -1, -1, 1, 0, \ldots, 0)} E) \oplus (C^\beta_{\alpha', (2, 0, \ldots, 0)} \mathbb{L}_2 E) \oplus (C^\beta_{\alpha', (1, 1, 0, \ldots, 0)} \wedge^2 E).$$

Now let $\text{Spec } R^a$ be a $GL_n$-orbit closure in the space of $n \times n$ skew-symmetric matrices, we also get a non-commutative desingularization by pulling-back the Kapranov’s exceptional collection. Let $0 \to \mathcal{R} \to E \times \text{Grass} \to \mathcal{Q} \to 0$ be the tautological sequence over Grass, where Grass is the Grassmannian of $(n - r)$-planes in $E$. Then the total space $Z^a$ of $\wedge^2 \mathcal{Q}^*$ is a commutative desingularization of $\text{Spec } R^a$, as is proved in [80].

We consider the pull back of the Kapranov’s tilting bundle $\mathcal{T}il_K = \oplus_{\alpha \in B_{r, n-r}} \mathcal{L}_\alpha \mathcal{Q}^*$ over $\text{Grass}_{n-r}(E)$ by $p': Z^a \to \text{Grass}$.

**Theorem C** (Proposition 2.6.2, Proposition 2.6.3, and Proposition 2.6.4). Let $\mathcal{T}il_K$ be the Kapranov’s tilting bundle over $\text{Grass}_{n-r}(E)$, and $Z^a$ is the total space of $\wedge^2 \mathcal{Q}^*$ which desingularizes the rank $r$ Pfaffian variety $\text{Spec } R^a$ of skew-symmetric matrices.

1. The bundle $p'^* \mathcal{T}il_K$ is a tilting bundle over $Z^a$, i.e., $\text{Ext}^i_{Z^a}(p'^* \mathcal{T}il_K, p'^* \mathcal{T}il_K) = 0$ for $i > 0$ and $\text{Ext}^0_{Z^a}(p'^* \mathcal{T}il_K, C) = 0$ implies $C$ is an exact complex. In particular, $D^b(\text{Coh}(Z^a)) \cong D^b(\text{End}_{Z^a}(p'^* \mathcal{T}il_K)\text{-mod}).$
2. The $R^a$ module $q'_p p'^* \mathbb{L}_\alpha Q^*$ is maximal Cohen-Macaulay for any $\alpha \in B_{r,n-r}$.

3. The map $\text{End}_{Z^a}(p'^* \mathcal{T}il_K) \to \text{End}_{S}(q'_p p'^* \mathcal{T}il_K)$ is an isomorphism of $R^a$-algebra, and this algebra is a non-commutative desingularization of $\text{Spec } R^a$. In other words, it has finite global dimension.

When $k = \mathbb{C}$, the quiver with relations of the endomorphism ring is given as follows (see Proposition 2.6.8 and Proposition 2.6.9). The set of vertices is indexed by $B_{r,n-r}$.

- In the case $n - r = 1$, arrows from $\beta$ to $\alpha$ is given by $E$ if $C^\alpha_{\beta,(1,0,\ldots,0)} \neq 0$ and $\mathbb{C}$ if $C^\beta_{\alpha,(1,1,0,\ldots,0)} \neq 0$. No arrows otherwise. The relations are generated by the following representations in the group $\text{Hom}(\beta, \alpha)$
  \[ (C^\beta_{\alpha,t,(1,0,\ldots,0)} \wedge^3 E) \oplus (C^\beta_{\alpha,t,(2,0,\ldots,0)} \mathbb{L}_2 E). \]

- In the case $n - r = 2$, there is one more arrow from $\beta$ to $\alpha$ given by $\mathbb{C}$ for $C^\beta_{\alpha,t,(1,-1,0,\ldots,0)}$ besides the above ones. No arrows otherwise. The relations are generated by the following representations in the group $\text{Hom}(\beta, \alpha)$
  \[ (C^\beta_{\alpha,t,(0,\ldots,0,1,0)} \mathbb{C}) \oplus (C^\beta_{\alpha,t,(1,0,\ldots,0,2)} E) \oplus (C^\beta_{\alpha,t,(2,0,\ldots,0)} \mathbb{L}_2 E) \oplus (C^\beta_{\alpha,t,(1,1,0,\ldots,0)} \wedge^2 E). \]

- In the case $n - r \geq 3$, The set of arrows has the same description as in the case $n - r = 1$. The set of relations has the same description as in the case $n - r = 2$.

In the case of generic determinantal varieties, the quiver with relations is studied in [20]. We used the geometric technique in [80] to study the decomposition of the non-commutative desingularization as representations of the group.

1.3.2 Stability conditions for noncommutative symplectic resolutions

For a finite dimensional vector space $V$, equipped with a symplectic structure, and a finite subgroup of the symplectic group $\Gamma \subseteq \text{Sp}(V)$, the quotient $V/\Gamma$ is a Poisson variety and
the bracket is non-degenerate on the smooth part. Suppose that we have a resolution of singularity \( \pi : X \to V/\Gamma \), with a symplectic form on \( X \) which coincide with that of \( V/\Gamma \) on the smooth part. Such resolutions will be called *symplectic resolutions*.

Interesting examples includes minimal resolutions of Kleinian singularities. More precisely, for a finite subgroup \( \Gamma \subseteq \text{Sp}(\mathbb{A}^2) \), the quotient \( \mathbb{A}^2/\Gamma \) has a unique symplectic resolution, denoted by \( \tilde{\mathbb{A}}^2/\Gamma \). More generally, the symmetric product \( \text{Sym}^n(\mathbb{A}^2/\Gamma) \) has a symplectic resolution by the Hilbert scheme of points \( \text{Hilb}^n(\tilde{\mathbb{A}}^2/\Gamma) \).

Bezrukavnikov and Kaledin proved in [13], that for any symplectic resolution \( \pi : X \to V/\Gamma \), there exists a vector bundle \( V \) on \( X \), such that \( \text{End}_{\mathcal{O}_X}(V) \cong \mathcal{O}_V \# \Gamma \), and \( R\text{Hom}_{\mathcal{O}_X}(V, \cdot) \) induces an equivalence of triangulated categories \( D^b(\text{Coh}(X)) \cong D^b(\text{Mod-End}(V)) \). In the terminology of [23], \( \text{End}(V) \) is a noncommutative resolution of singularity, which is clearly a noncommutative crepant resolution. Consequently, all symplectic resolutions of \( V/\Gamma \) are derived equivalent to each other. When \( V = \mathbb{A}^2 \), the theorem of Bezrukavnikov and Kaledin specializes to the classical derived McKay correspondence. When \( V = \mathbb{A}^{2n} \) and \( \Gamma = \mathfrak{S}_n \) acting via the reflection representation, and \( X = \text{Hilb}^n(\mathbb{A}^2) \), the theorem of Bezrukavnikov and Kaledin is related to the Procesi bundles studied by Haiman in [38].

It is worth mentioning that the construction of the noncommutative resolutions given in [13] comes from quantization of symplectic varieties over fields of positive characteristic. The sheaf of algebras \( \mathcal{E}_{\text{nd}\mathcal{O}_X}(V) \) is Morita equivalent to \( \mathcal{A}_0(\Gamma) \), a suitable quantization of \( X \). There is a family of sheaves on non-commutative \( \mathcal{O}_X \)-algebras \( \mathcal{A}_c(\Gamma) \) parameterized by \( c \in H^2(X, \mathbb{Q}) \), such that \( \mathcal{A}_0(\Gamma) \cong \mathcal{E}_{\text{nd}\mathcal{O}_X}(V) \).

An example generalizing the \( V = \mathbb{A}^{2n} \) and \( \Gamma = \mathfrak{S}_n \) case is the following. We work over a separably closed field \( k \) of characteristic \( p >> 0 \). Let \( \Gamma_n := (\mathbb{Z}_r)^n \rtimes \mathfrak{S}_n \) acting on \( \mathfrak{h} = \mathbb{A}^n \) in the natural way. Let \( V = \mathfrak{h} \oplus \mathfrak{h}^* \cong \mathbb{A}^{2n} \) be endowed with the diagonal action of \( \Gamma_n \). The action preserves the natural symplectic form on \( V \). A symplectic resolution of \( \mathbb{A}^{2n}/\Gamma_n \) can be given as \( \text{Hilb}^n(\tilde{\mathbb{A}}^2/\mathbb{Z}_r) \) where \( \tilde{\mathbb{A}}^2/\mathbb{Z}_r \) is the minimal resolution of \( \mathbb{A}^2/\mathbb{Z}_r \). Let \( \mathcal{W}(\mathfrak{h}) \) be the
Weyl algebra. The algebra $\mathcal{W}(\mathfrak{h})^\Gamma_n$ is a noncommutative desingularization of $\mathbb{A}^{2n}/\Gamma_n$. So is the algebra $\mathcal{W}(\mathfrak{h})^\Gamma_n$, which is Morita equivalent to $\mathcal{W}(\mathfrak{h})\#\Gamma_n$.

The (spherical) Cherednik algebra $^*H_c$ is a deformation of $\mathcal{W}(\mathfrak{h})^\Gamma_n$. The parameter space of the deformation is spanned by the conjugacy classes of reflections in $\Gamma_n$. The algebra $^*H_c$ has a big Frobenius center $k[A^{2n(1)}\Gamma_n]$. For any central character $\chi$, the irreducible objects in the category $\text{Mod-}_\chi H_c$ are naturally labeled by $\text{Irrep}(\Gamma_n)$. The algebra $^sH_c := eH_ce \subset H_c$ is called the spherical Cherednik algebra, where $e := \sum_{\gamma \in \Gamma} \gamma$. If $^*H_c$ has finite global dimension, then the value $c$ is called spherical value. Otherwise we say $c$ is aspherical. The aspherical values form an affine hyperplane arrangement in the space of parameters.

Let $\text{Hilb}^{(1)}$ be the Frobenius twist of $\text{Hilb} := \text{Hilb}^n(\mathbb{A}^2/\mathbb{Z}_n)$.

Let $\text{Coh}_0 \text{Hilb}^{(1)}$ be the category of coherent sheaves on $\text{Hilb}^{(1)}$ set-theoretically supported on the zero-fiber of the Hilbert-Chow morphism. It is shown by Bezrukavnikov-Finkelberg-Ginzburg that there is a tilting bundle $\mathcal{E}_c$ on $\text{Hilb}^{(1)}$, such that $\text{End}(\mathcal{E}_c)_0 \cong (^*H_c)_0$. In particular, for spherical values $c$, $^*H_c$ has finite global dimension, therefore there is a derived equivalence

$$D^b(\text{Coh}_0 \text{Hilb}^{(1)}) \cong D^b(\text{Mod}_0 ^*H_c).$$

Let $\Gamma_1 \subseteq \text{Sp}(\mathbb{A}^2)$ be a finite subgroup, and let $\Gamma_n = (\Gamma_1)^n \rtimes S_n$ acting on $V = \mathbb{A}^{2n}$ in the obvious way via symplectic reflections. Let $X_{\Gamma_n}$ be $\text{Hilb}^n(\mathbb{A}^2/\Gamma_1)$, a symplectic resolution of $\mathbb{A}^n/\Gamma_n$. Assume the base field $k$ has characteristic $p > 0$. It is shown by Bezrukavnikov, Finkelberg, and Ginzburg in [12] and [11] that the algebra $H^0(X_{\Gamma_n}^{(1)}, \mathcal{A}_c(\Gamma_n))$ is isomorphic to the spherical rational Cherednik algebra (see [30]). Denote the rational Cherednik algebra over $k$ with parameter $c$ by $\mathcal{H}_c(\Gamma_n)$, and denoted the spherical rational Cherednik algebra by $^*\mathcal{H}_c(\Gamma_n)$. There is an affine hyperplane arrangement in the parameter space $H^2(X_{\Gamma_n}, \mathbb{Q})$, such that when the parameter $c \in H^2(X_{\Gamma_n}, \mathbb{Q})$ does not lie on the hyperplanes, $\mathcal{H}_c(\Gamma_n)$ is Morita equivalent to $^*\mathcal{H}_c(\Gamma_n)$. The special values of the parameter $c$ are called aspherical values, and the generic values, spherical ones. Further more, for large enough $p$, and for
spherical values $c$, $\mathcal{H}_c$, we have a derived equivalence $D^b(\text{Coh}_0 X^{(1)}_{\Gamma_n}) \cong D^b(\text{Mod}_{-0} \mathcal{H}_c(\Gamma_n))$ where $\text{Coh}_0 X^{(1)}_{\Gamma_n}$ means the category of coherent sheaves on $X_{\Gamma_n}$ set-theoretically supported on the fiber of zero.

Assume the characteristic of the base field $k$ is $p \gg 0$. Then for regular value of the parameter $c \in H^2(X_{\Gamma_n}; \mathbb{Q})$ in the sense that the spherical rational Cherednik algebra $\mathcal{H}_c(\Gamma)$ has finite homological dimension, the derived equivalence given by [10] endows $D^b(\text{Coh}(X_{\Gamma_n}))$ with a $t$-structure. If the parameter $c \in H^2(X_{\Gamma_n}; \mathbb{Q})$ varies inside an alcove, the $t$-structure is constant, and the dimension of the irreducible object $\dim_k L_c(\tau; p)$ is a polynomial in $c$.

We reparameterize the value $c$ by setting $x = cp$ and define

$$Z_x(x) = \lim_{p \to \infty} p^{-n} \dim_k L_c(\tau; p).$$

In this paper, we concentrate on the following problem in special cases.

**Conjecture 1.3.1.** We consider the collection of polynomials $\{Z_\tau(x) \mid \tau \in \text{Irrep}(\Gamma_n)\}$ as a polynomial map $H^2(X_{\Gamma_n}; \mathbb{R}) \to K_0(X_{\Gamma_n})^* \otimes \mathbb{R}$. Let $\phi$ be the assignment associating to each alcove the $t$-structure on $D^b(\text{Coh}(X_{\Gamma_n}))$ coming from $\text{Mod-} \mathcal{H}_c(\Gamma_n)$ for $c$ lying in this alcove. Find out whether the pair $(\phi, Z)$ is a real variation of stability conditions defined in [2].

More concretely, for any alcove $A$, let $\mathfrak{A} := \text{heart of } \phi(A)$. We have,

1. for any $x \in A$, $Z_L(x) > 0$ for any simple object $L \in \mathfrak{A}$;
2. for any $A'$, sharing a codimension 1 wall $H$ with $A$.

Let $\mathfrak{A} \supseteq \mathfrak{A}_i := \langle L \in \mathfrak{A} \mid Z_L(x) \text{ vanishes of order } \geq i \text{ on } H \rangle$. Then,

- the $T(A')$ is compatible with the filtration on $T(A)$;
- on $\text{gr}_i(\mathfrak{A}) = \mathfrak{A}_i/\mathfrak{A}_{i+1}$, $\phi(A')$ differs by $[i]$ from $\phi(A)$.

Conjecture 1.3.1 is related to the following conjecture by Bezrukavnikov and Okounkov. In order to state it, we introduce some notations. Let $\mathcal{H}_c(\Gamma_n)_{\mathbb{C}}$ be the spherical rational
Cherednik algebra over $\mathbb{C}$. Let $R$ be a subring of $\mathbb{C}$ which is finitely generated over $\mathbb{Z}$ such that $^s\mathcal{H}_c(\Gamma_n)_\mathbb{C}$ has an $R$-form $^s\mathcal{H}_c(\Gamma_n)_R$. Let $^s\mathcal{H}_c(\Gamma_n)_k$ be the base change of $^s\mathcal{H}_c(\Gamma_n)_R$ to a field $k$ of characteristic $p$. For an irreducible representation $\tau$ of $\Gamma_n$, let $L_c(\tau; \mathbb{C})$ be the corresponding irreducible object in the category $\mathcal{O}$ of $^s\mathcal{H}_c(\Gamma_n)_\mathbb{C}$.

**Conjecture 1.3.2.** For any aspherical value $c$, let $\text{Mod-}^s\mathcal{H}_c(\Gamma_n)_k^{\leq d}$ be the Serre subcategory generated by irreducible objects $L_c(\lambda; p)$ such that the corresponding polynomial $Z_\tau$ satisfies: $\deg(Z_\tau) \leq d$. Suppose the codimension of support in the sense of [30] of $L_c(\tau; \mathbb{C})$ is $d$. We take $L_c(\tau; R)$ be its $R$-form. Let $\overline{L_c(\tau; R)}_k$ be the central reduction of $L_c(\tau; R) \otimes_R k$. Then $\overline{L_c(\tau; R)}_k$ is a nonzero object in $\text{Mod-}^s\mathcal{H}_c(\Gamma_n)_k^{\leq d}/\text{Mod-}^s\mathcal{H}_c(\Gamma_n)_k^{\leq d+1}$.

It is shown in Proposition 3.4.3 that the solution to the Chern character problem determines the dimension polynomials $\dim L_c(\tau, p)$ of the irreducible objects.

It is well-known (see [50]) that a symplectic resolution of $\mathbb{A}^n/\Gamma_n$ can be constructed as a Nakajima quiver variety associated to the extended Dynkin quiver. For a suitable choice of the stability condition, the Nakajima variety is isomorphic to $\text{Hilb}^n(\widehat{\mathbb{A}^2/\Gamma_1})$, where $\widehat{\mathbb{A}^2/\Gamma_1}$ is the minimal resolution of the Kleinian singularity $\mathbb{A}^2/\Gamma_1$. As an intermediate step of studying the stability conditions, in the example when $\Gamma_1 = \mathbb{Z}/l\mathbb{Z}$ and $n = 2$, using this quiver description, the Chern character map has been written down explicitly in Proposition 3.4.4. In general the calculation of the Chern character map is difficult. But it is easier, at least in some cases, to calculate the dimension polynomials.

For an integral parameter $m$, let $Q_m$ be the $m$-quasi-invariants in $k[\hbar]$. As $\Gamma_n^{-s}H_m$ bimodule, $Q_m = \oplus_{\tau \in \text{Irrep}(\Gamma_n)} \tau^* \otimes M_m(\tau)$. Let $\overline{Q}_m$ be the quasi-invariants on the Frobenius neighborhood of 0. A resolution of $\overline{Q}_m$: $\cdots \rightarrow Q_m \otimes \wedge^2 \hbar^{(1)} \rightarrow Q_m \otimes \hbar^{(1)} \rightarrow Q_m$.

**Theorem D.** Fix a character $i$ of $\mathbb{Z}_r$. Let $\tau(i)$ be the 1-dimensional representation of $\Gamma_n = (\mathbb{Z}_r)^n \rtimes \mathfrak{S}_n$ on which $\mathbb{Z}_r$ acts by the character $i$ and $\mathfrak{S}_n$ acts by the sign representation.
The Poincaré series of $L_m(\tau(i))$ is
\[
\frac{t^{ni} \prod_{k=0}^{n-1} (1 - t^{rk+m_0+n+p+1+rm_{i+1}})}{\prod_{k=1}^{n} (1 - t^{kr})}.
\]

Using the induction and restriction functors, this theorem gives an algorithm to calculate the dimension polynomials of the irreducible objects as long as the parameter $m$ is in the fundamental alcove (the alcove containing 0). But away from the fundamental alcove, the combinatorics becomes complicated and we can only deal with the case when $n = 2$ in the current paper.

**Theorem E.** Conjecture 1.3.1 and Conjecture 1.3.2 are both true for $n = 2$ and $\Gamma_1 = \mathbb{Z}/l\mathbb{Z}$.

Proof in more general set-ups will be treated in subsequent papers. In Chapter 3 of this thesis, we concentrate on a more explicit description in this special case of the central charge polynomials, the hyperplane arrangements, and the tilting bundles generating the hearts of all the $t$-structures involved, and the derived equivalences for any two adjacent alcoves.

An explicit description of the derived equivalences for any two adjacent alcoves in this case can be found in Section 3.6. Where are only two types: $\mathbb{P}^2$-semi-reflection, and tilting with respect to suitable torsion theory. The question how the tilting generators change under $\mathbb{P}^n$-semi-reflection is studied in § 3.1, which is interesting on its own rights.

There are two prototypical examples of $\mathbb{P}^n$-semi-reflections.

**Example 1.3.3.** Let $\text{Perv}(\mathbb{P}^n)$ be the category of perverse constructible sheaves with respect to the usual stratification of $\mathbb{P}^n$. Similarly we have $\text{Perv}((\mathbb{P}^n)^\vee)$. Let $R : D^b(\text{Perv}(\mathbb{P}^n)) \to D^b(\text{Perv}((\mathbb{P}^n)^\vee))$ be the Radon transform with kernel the incidence locus. Then $R(\text{Perv}(\mathbb{P}^n))$ is the semi-reflection of $\text{Perv}((\mathbb{P}^n)^\vee)$ with respect to the $\mathbb{P}^n$-object $C_{\mathbb{P}^n}[n]$.

**Example 1.3.4.** Let $D^b(\text{Coh}_0 T^*\mathbb{P}^n)$ be the derived category of coherent sheaves on $T^*\mathbb{P}^n$ set-theoretically supported on the zero-section, and let $\mathfrak{A}$ be the heart of the $t$-structure whose projective generator are the Beilinson’s tilting bundles $\oplus_{i=0}^{n} O(-i)$. Similarly let $\mathfrak{A}'$
be the the heart of the \( t \)-structure in \( D^b(\text{Coh}_0 T^*(\mathbb{P}^n)^\vee) \) whose projective generator are the Beilinson’s tilting bundles on \((\mathbb{P}^n)^\vee\). Let \( FM : D^b(\text{Coh}_0 T^*\mathbb{P}^n) \to D^b(\text{Coh} T^*(\mathbb{P}^n)^\vee) \) be the Fourier-Mukai transform with kernel constructed by Namikawa in [62]. Then \( FM(\mathfrak{A}) \) is the semi-reflection of \( \mathfrak{A}' \) with respect to the \( \mathbb{P}^n \)-object \( \mathcal{O}_{\mathbb{P}^n}(-n) \). (See also [72].)

The following results, which is a scene from Section 3.1, tells us the projective generators about this \( t \)-structure obtained from \( \mathbb{P}^n \)-semi-reflection.

A more general set-up for the \( \mathbb{P}^n \)-semi-reflection is the following. Let \( X \) be a smooth variety which is projective over \( \text{Spec} \, A \). Also we assume the map \( \pi : X \to \text{Spec} \, A \) is \( \mathbb{G}_m \)-equivariant, such that \( X \) is deformation retracts to \( X = \pi^{-1}(\text{Spec} \, A/m) \), the fiber over \( A/m \) under this \( \mathbb{G}_m \)-action. Let \( \{ P_\alpha \mid \nabla \} \) be a collection of \( \mathbb{G}_m \)-equivariant tilting bundles on \( X \), and denote \( \text{End}(\bigoplus_{\alpha \in \nabla} P_\alpha) \) by \( E \). Let \( \mathfrak{A} \) be the category of finitely generated \( E \)-modules set-theoretically supported on \( A/m \).

The following fact about \( \mathbb{P}^n \)-semi-reflection is proved in Corollary 3.1.20. (The result also holds if \( \mathcal{A} \) is a finite length abelian category with enough projectives, e.g., the category of perverse constructible sheaves.) Assume \( S_\theta \) is a simple object has vanishing \( \text{Ext}^1(S_\theta, S_\theta) \). We endow \( \mathfrak{A} \) with the filtration that \( 0 = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 = \mathfrak{A} \) where \( \mathfrak{A}_1 = \langle S_\theta \rangle \). Assume for the perversity function \( p \) with \( p(1) = 0 \) and \( p(2) = n \) we have a perverse equivalence \( (t, t', p) \) such that the projective covers of the simple objects in the heart of \( t' \) have representatives lying in \( E\text{-mod} \). Then for any \( p' \) with \( p'(1) = 0 \) and \( p'(1) \leq n \) the perverse equivalence \( (t, t'', p') \) exists, and the projective covers of the simple objects in the heart of \( t' \) have representatives lying in \( E\text{-mod} \). Moreover, the projective generators of these \( t \)-structures are given by the truncated mutations defined in Section 3.1.

### 1.3.3 Algebraic elliptic cohomology theory and flops

In Chapter 4 of this thesis, we study the algebraic version of the elliptic cohomology over an arbitrary perfect field, and consider as well as the question of the existence of a cor-
responding motivic oriented cohomology theory representing elliptic cohomology. According to a theorem of Landweber (see Theorem 4.1.9), extended to the motivic setting by Naumann-Spitzweck-Østvær [59], a formal group law defines a motivic oriented cohomology theory if a certain flatness assumption called Landweber exactness is satisfied. The underlying ring of the elliptic formal group law corresponding to the Krichever’s elliptic genus is a certain subring of \( \mathbb{Z}[a_1, a_2, a_3, a_4] \); there is an explicit descriptions of the four generators \( a_i \) (see [6]) as elements in a formal power series ring \( \mathbb{Q}((e^{2\pi iz}))[[e^{2\pi \tau}, \frac{k}{2\pi i}]] \). In order to make this formal group law Landweber exact, we need to enlarge the coefficient ring. Let

\[
\text{Ell}[1/2] = \mathbb{Z}[1/2][a_1, a_2, a_3, a_4][\Delta^{-1}],
\]

where

\[
\Delta = 36(-4a_1a_3 - a_4 + 6a_2^2a_2^2 - 8(-4a_1a_3 - a_4 + 6a_2^2)^2 - 27a_3^4 + \\
+ 108(-4a_1a_3 - a_4 + 6a_2^2)a_2a_3^2 - 432a_2^3a_3^2
\]

is the discriminant.

**Theorem F** (Theorem 4.2.3). Let \( k \) be a perfect field. The oriented cohomology theory on \( \text{Sm}_k \) sending \( X \mapsto \text{MGL}^*(X) \otimes_{\text{Laz}} \text{Ell}[1/2] \) is represented by a motivic oriented cohomology theory on \( \text{Sm}_k \).

We will make precise in Remark 4.2.4 that inverting \( 2 \) is necessary to make the elliptic formal group law Landweber exact, hence it is not clear if the integral Krichever’s elliptic cohomology theory, although existing as an oriented cohomology theory on \( \text{Sm}_k \) (after inverting \( p \) in characteristic \( p > 0 \)), is represented by a motivic oriented cohomology theory on \( \text{Sm}_k \).

We say two smooth projective \( n \)-folds \( X_1 \) and \( X_2 \) are related by a flop if we have the
following diagram of projective birational morphisms:

\[
\begin{array}{c}
\tilde{X} \\
X_1 \downarrow p_1 \downarrow \downarrow p_2 \downarrow \downarrow X_2 \\
Y
\end{array}
\]

(1.1)

Here \( Y \) is a singular projective \( n \)-fold with singular locus \( Z \), such that \( Z \) is smooth of dimension \( n - k - 1 \). We assume in addition that there exist rank \( k \) vector bundles \( A \) and \( B \) on \( Z \), such that the exceptional locus \( F_1 \) in \( X_1 \) is the \( \mathbb{P}^{k-1} \)-bundle \( \mathbb{P}(A) \) over \( Z \), with normal bundle \( N_{F_1}X_1 = B \otimes \mathcal{O}(-1) \). Similarly, the exceptional locus \( F_2 \) in \( X_2 \) is \( \mathbb{P}(B) \), with normal bundle \( N_{F_2}X_2 = A \otimes \mathcal{O}(-1) \). Let \( Q^3 \subset \mathbb{P}^4 \) denote the 3-dimensional quadric with an ordinary double point \( v \), defined by the equation \( x_1x_2 = x_3x_4 \). We say that \( X_1 \) and \( X_2 \) are related by a \emph{classical} flop if in addition \( k = 2 \), and along \( Z \), \((Y, Z)\) is Zariski locally isomorphic to \((Q^3 \times Z, v \times Z)\).

Now we can state our main theorem in this paper as follows. Let \( \text{MGL}^*_{\mathbb{Q}}(k) := \text{MGL}^*(k) \otimes \mathbb{Z} \mathbb{Q} \) be the algebraic cobordism ring with \( \mathbb{Q} \) coefficients, and \( \text{Ell}^*_{\mathbb{Q}}(k) \) be the elliptic cohomology ring with \( \mathbb{Q} \) coefficients.

**Theorem G** (Proposition 4.3.5, Proposition 4.4.2, and Corollary 4.4.3). The kernel of the algebraic elliptic genus \( \phi_{\mathbb{Q}} : \text{MGL}^*_{\mathbb{Q}}(k) \to \text{Ell}^*_{\mathbb{Q}}(k) \) is generated by the differences of classical flops, and coincides with the ideal generated by the differences of flops. The image of \( \phi_{\mathbb{Q}} \) is the polynomial ring \( \mathbb{Q}[a_1, a_2, a_3, a_4] \).

The basic idea underlying our approach is to use the \emph{double-point cobordism} of Levine-Pandharipande [56] to give a simple explicit description of the difference of two flops in the algebraic cobordism ring \( \text{MGL}^*(k) \) (see Proposition 4.3.3 for a more precise statement), replacing Totaro’s topological constructions. The vanishing of the difference of two flops in elliptic cohomology is then a consequence of a classical identity satisfied by the \( \sigma \) function (see Proposition 4.3.5).
As another application of the algebraic cobordism theory, in Proposition 4.6.2, assuming the characteristic of the base field is zero, we will confirm Question 1.1.6 about birational symplectic varieties.
Chapter 2

Noncommutative desingularization of orbit closures for some representations of $GL_n$

2.1 Construction of non-commutative resolution

In this section we recall some basic notions about noncommutative desingularization. Then we introduce a construction that in many interesting situations provides noncommutative desingularizations. A criterion is given to test whether this construction gives a noncommutative desingularization or not.

2.1.1 General notions about tilting bundles

For a scheme $X$, a vector bundle (or more generally a perfect complex) $\mathcal{T}il$ is called a tilting bundle (resp. tilting complex) over $X$ if it satisfies the following:

1. $\mathcal{T}il^\perp = 0$ in $D^b(X)$, i.e., for any complex $M$, $\text{Hom}_{D^b(X)}(\mathcal{T}il, M[i]) = 0$ for all $i$ implies $M = 0$, where $[1]$ is the shifting functor;
2. Ext^i(\mathcal{F}il, \mathcal{F}il) = 0 for i > 0.

**Theorem 2.1.1** (7.6 in [45]). For X a projective scheme over a Noetherian affine scheme of finite type, and \mathcal{F}il \in D(Qcoh(X)) a tilting object. We have the following

1. \( R \text{Hom}_{O_X}(\mathcal{F}il, -) \) induces an equivalence

\[ D(Qcoh(X)) \cong D(\text{End}_{O_X}(\mathcal{F}il)-\text{Mod}). \]

2. This equivalence restricts to an equivalence

\[ D^b(Coh(X)) \cong D^b(\text{End}_{O_X}(\mathcal{F}il)-\text{mod}). \]

3. If X is smooth then \( \text{End}_{O_X}(\mathcal{F}il) \) has finite global dimension.

**Definition 2.1.2** (§ 5 in [23]). For an algebraic variety X, a noncommutative (birational) desingularization of X is a pair \((p, \mathcal{A})\) consisting of a proper birational morphism \( p : Y \to X \) and an algebra \( \mathcal{A} = \text{End}_{O_Y}(\mathcal{F}) \) on Y, the sheaf of local endomorphisms of a reflexive coherent \( O_Y \)-module \( \mathcal{F} \), such that the abelian category of sheaves of right modules \( \mathcal{A} \) has finite homological dimension.

**Remark 2.1.3.** In the original definition of [23], the sheaf \( \mathcal{F} \) is only required to be torsion free.

**Definition 2.1.4.** For a Gorenstein ring \( R \), we call an endomorphism ring \( A = \text{End}_R(M) \) for a reflexive \( R \)-module \( M \) a non-commutative crepant resolution if \( A \) has finite global dimension and is maximal Cohen-Macaulay.

### 2.1.2 Preimage of tilting bundles and noncommutative desingularization

Let \( G \) be a reductive group, \( P < G \) a parabolic subgroup. Suppose we have a tilting bundle \( \mathcal{F}il_{G/P} \) over the partial flag variety \( G/P \), and suppose the total space \( Z \) of a vector
subbundle of \( \mathbb{A}^n \times G/P \) over \( G/P \) desingularizes an affine subvariety \( \text{Spec} \, R \subseteq \mathbb{A}^n \), i.e., the projection \( \mathbb{A}^n \times G/P \to \mathbb{A}^n \) restricted to \( Z \) is birational onto \( \text{Spec} \, R \), so that the restriction is a desingularization. We seek the conditions for the inverse image of \( \mathcal{T}nil_{G/P} \) to be a tilting bundle over \( Z \), and to give a non-commutative crepant resolution of \( \text{Spec} \, R \).

\[
\begin{array}{ccc}
Z & \xrightarrow{j} & G/P \\
\downarrow & & \downarrow p' \\
\text{Spec} \, R & \xrightarrow{i} & \mathbb{A}^n
\end{array}
\]

\[\text{(2.1)}\]

**Lemma 2.1.5.** Assume \( p : X \to Y \) is the projection of a vector bundle over an algebraic scheme \( Y \), \( \mathcal{T} \) is a quasi-coherent sheaf over \( Y \) such that \( \mathcal{T}^\perp = 0 \) in \( D^b(Y) \), then,

\[ (p^* \mathcal{T})^\perp = 0 \]

in \( D^b(X) \).

**Proof.** The projection \( p \) is an affine morphism. Denote the corresponding sheaf of algebras of the affine morphism \( p \) by \( \mathcal{A} \). The push-forward functor \( p_* \) induces an equivalence between \( \text{Qcoh}(X) \) and the subcategory \( \mathcal{A}\text{-mod} \) of \( \text{Qcoh}(Y) \). In particular, \( p_* \) is exact.

The sheaf \( \mathcal{A} = \text{Sym}(X^*) \) is locally free. Therefore, \( p^* \) is also exact.

We have a pair of adjoint exact functors \( (p^*, p_*) \) between \( \text{Qcoh}(X) \) and \( \text{Qcoh}(Y) \). By a standard result in homological algebra (III.6 in [36]), they induce an adjoint pair between \( D^b(X) \) and \( D^b(Y) \), which we still denote by \( (p^*, p_*) \). They commute with the shifting functor [1].

For a complex \( C \) over \( X \), assume \( \text{Hom}_{D^b(X)}(p^* \mathcal{T}, C[i]) = 0 \) for all \( i \). Using the adjoint property, we get \( \text{Hom}_{D^b(Y)}(\mathcal{T}, p_* C[i]) = \text{Hom}_{D^b(X)}(p^* \mathcal{T}, C[i]) = 0 \). By assumption, \( p_* C \cong 0 \) in \( D^b(Y) \). This implies \( p_* C \cong 0 \) in \( D^b(\mathcal{A}) \), hence, \( C \cong 0 \) in \( D^b(X) \).

**Lemma 2.1.6.** Assume \( p : X \to Y \) is the projection of a vector bundle over an algebraic scheme \( Y \), \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) are two vector bundles over \( Y \) such that \( H^i(X, p^* \mathcal{H}om(\mathcal{T}_1, \mathcal{T}_2)) = 0 \) for all \( i > 0 \), then \( \text{Ext}^i(p^* \mathcal{T}_1, p^* \mathcal{T}_2) = 0 \) for all \( i > 0 \).
Proof. Denote the corresponding sheaf of algebras of the affine morphism $p$ by $\mathcal{A}$. We have the local-global spectral sequence $E$ with

$$E_2^{ij} = H^i(X, \mathcal{E}xt_X^j(p^*\mathcal{I}_1, p^*\mathcal{I}_2))$$

which converges to $\text{Ext}^{i+j}(p^*\mathcal{I}_1, p^*\mathcal{I}_2)$.

Since $p^*\mathcal{I}_1$ and $p^*\mathcal{I}_2$ are both locally free, $\mathcal{E}xt_X^j(p^*\mathcal{I}_1, p^*\mathcal{I}_2) = 0$ for all positive $j$. We have

$$\text{Ext}^i(p^*\mathcal{I}_1, p^*\mathcal{I}_2) = H^i(X, \mathcal{H}om_{\mathcal{O}_X}(p^*\mathcal{I}_1, p^*\mathcal{I}_2)).$$

Since $\mathcal{A}$ is locally free, we can identify

$$p^*\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_1, \mathcal{I}_2) \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_1, \mathcal{I}_2 \otimes \mathcal{O}_Y, \mathcal{A}) \cong \mathcal{H}om_{\mathcal{O}_X}(p^*\mathcal{I}_1, p^*\mathcal{I}_2).$$

Use the assumption that $H^i(X, p^*\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_1, \mathcal{I}_2)) = 0$, then, $\text{Ext}^i(p^*\mathcal{I}_1, p^*\mathcal{I}_2) = 0$ for $i > 0$. 

We get the following method to construct crepant noncommutative desingularizations.

**Proposition 2.1.7.** Notations as in diagram (2.1). Let $\mathcal{I}l$ be a tilting bundle over $G/P$.

1. If $H^i(Z, p^*\mathcal{E}nd_{\mathcal{O}_{G/P}}(\mathcal{I}l)) = 0$ for all positive $i$, then $p^*\mathcal{I}l$ is a tilting bundle over $Z$.

2. If moreover, $\mathcal{E}nd_{\mathcal{O}_Z}(p^*\mathcal{I}l)$ is maximal Cohen-Macaulay, and

$$\mathcal{E}nd_{\mathcal{O}_Z}(p^*\mathcal{I}l) \cong \mathcal{E}nd_R(q^p*p^*\mathcal{I}l),$$

then $\mathcal{E}nd_R(q^p*p^*\mathcal{I}l)$ gives a noncommutative crepant desingularization of Spec $R$.

3. Assume $H^i(Z, p^*\mathcal{E}nd_{\mathcal{O}_{G/P}}(\mathcal{I}l)) = 0$ for all $i > 0$. If the exceptional locus of $q^p : Z \to \text{Spec } R$ has codimension at least two in both $Z$ and $\text{Spec } R$, and $R$ is an integral domain, then $\mathcal{E}nd_{\mathcal{O}_Z}(p^*\mathcal{I}l) \cong \mathcal{E}nd_R(q^p*p^*\mathcal{I}l)$ and it gives a non-commutative desingularization of Spec $R$. 

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Proof. The first part of this proposition follows directly from Lemma 2.1.5, Lemma 2.1.6 and the definition of tilting bundles.

Note that if $p^* \mathcal{T}il$ is a tilting bundle, then $\text{End}_{O_Z}(p^* \mathcal{T}il)$ has finite global dimension by Theorem 2.1.1, since $Z$ is smooth. This shows the second part.

Now we prove the third part. First note that by Lemma 4.2.1 in [73], if the exceptional locus of $q'$ has codimension at least 2 in both $Z$ and Spec $R$, then $q'_*\text{ sends any reflexive sheaves to reflexive sheaves. Note also that for a finitely generated module } M \text{ over a commutative noetherian domain, } M \text{ being reflexive is equivalent to being torsion free and } M = \bigcap M_p, \text{ where the intersection is taken over all codimension 1 primes (7.4.2 of [22]). Both } Z \text{ and Spec } R \text{ are integral schemes, hence for any reflexive sheaf } N \text{ over one or the other, } \mathcal{H}om(M, N) \text{ is also reflexive for any } M. \text{ This is because } \mathcal{H}om(M, N) \text{ is torsion-free whenever } N \text{ is, and both intersection and localization commute with } \mathcal{H}om(M, -), \text{ given integrality of the scheme. Thus, we know } p^* \mathcal{T}il \text{ is reflexive, so are } \text{End}_{O_Z}(p^* \mathcal{T}il), q'_* \text{End}_{O_Z}(p^* \mathcal{T}il), q'_*p^* \mathcal{T}il, \text{ and } \text{End}_R(q'_*p^* \mathcal{T}il). \text{ The map } \text{End}_Z(p^* \mathcal{T}il) \rightarrow \text{End}_S(q'_*p^* \mathcal{T}il) \text{ is an isomorphism of rings in the complement of a codimension at least 2 subvariety. Both } \text{End}_Z(p^* \mathcal{T}il) \text{ and } \text{End}_S(q'_*p^* \mathcal{T}il) \text{ are reflexive, hence}
\[
\text{End}_{O_Z}(p^* \mathcal{T}il) = \bigcap \text{End}_{O_Z}(p^* \mathcal{T}il)_p \cong \bigcap \text{End}_R(q'_*p^* \mathcal{T}il)_p = \text{End}_R(q'_*p^* \mathcal{T}il),
\]
where the intersection is taken over all codimension 1 primes. \hfill \Box

In some situations we will deal with, the exceptional locus has codimension 1 in $Z$, in which case unfortunately Proposition 2.1.7 does not apply. Nevertheless, we have the following Lemma.

**Lemma 2.1.8.** Let $X = \text{Spec}(R)$ be an affine normal Gorenstein scheme and $f : Z \rightarrow X$ a crepant resolution with exceptional locus having codimension at least 2 in $X$. Let $\mathcal{T}il$ be a tilting bundle over $Z$. If $\mathcal{T}il(Z)$ is reflexive and $\text{End}_{O_Z}(\mathcal{T}il)$ is maximal Cohen-Macaulay, then
\[
\text{End}_{O_Z}(\mathcal{T}il) \cong \text{End}_R(\mathcal{T}il(Z))
\]
and it is a non-commutative crepant resolution of $X$.

Proof. The only thing we need to show is that taking global section and taking endomorphism ring commute in this case. First recall that (7.4.2 of [22]) for a finitely generated module $M$ over a commutative noetherian domain, $M$ being reflexive is equivalent to being torsion free and $M = \bigcap M_p$ where the intersection is taken over all codimension 1 primes.

There is a natural morphism of rings $\text{End}_{O_Z}(\mathcal{T}il) \to \text{End}_R(\mathcal{T}il(Z))$. The target is reflexive since it is a maximal Cohen-Macaulay module over a Gorenstein domain. This morphism is an isomorphism outside the exceptional locus of $f$ which has codimension at least 2 in $X$. If the source is also reflexive, then we have $\text{End}_{O_Z}(\mathcal{T}il) = \bigcap \text{End}_{O_Z}(\mathcal{T}il)_p \cong \bigcap \text{End}_R(\mathcal{T}il(Z))_p = \text{End}_R(\mathcal{T}il(Z))$, where the intersection is taken over all codimension 1 primes.

We have the following Lemma to test whether a non-commutative desingularization obtained as in Proposition 2.1.7 is crepant or not. Although it is known, (see e.g., [20],) we include the proof for completeness.

**Lemma 2.1.9.** Assume $Z$ is the total space of the vector bundle $V$ on $G/P$, and $\text{End}_{O_Z}(p^* \mathcal{T}il) \cong \text{End}_R(q'_*p'^* \mathcal{T}il)$ with $H^i(Z, \mathcal{E}nd_{O_Z}(p^* \mathcal{T}il)) = 0$. Then, $\text{End}_R(q'_*p'^* \mathcal{T}il)$ is maximal Cohen-Macaulay iff $H^i(G/P, \mathcal{T}il^* \otimes \mathcal{T}il \otimes \text{Sym}(V^*) \otimes \omega_Z) = 0$ for any $i > 0$, where $\omega$ is the dualizing complex.

Proof. By definition, $\text{End}_{O_Z}(p^* \mathcal{T}il)$ is maximal Cohen-Macaulay iff $\text{Ext}_R^i(\text{End}_{O_Z}(p^* \mathcal{T}il), \omega_R) = 0$ for all $i > 0$. As $q$ is a proper and $Rq_* \mathcal{E}nd_{O_Z}(p^* \mathcal{T}il) \cong \text{End}_{O_Z}(p^* \mathcal{T}il)$, using the Grothendieck duality for proper morphisms (1.2.22 of [80])

$$\text{Ext}_R^i(\text{End}_{O_Z}(p^* \mathcal{T}il), \omega_R) \cong \text{Ext}_{O_Z}^i(\mathcal{E}nd_{O_Z}(p^* \mathcal{T}il), \omega_Z).$$
We know \( \text{Ext}^i_R(\text{End}_{\mathcal{O}_Z}(p^*\mathcal{F}il), \omega_R) = 0 \) is equivalent to the following

\[
\text{Ext}^i_R(\text{End}_{\mathcal{O}_Z}(p^*\mathcal{F}il), \omega_R) \cong \text{Ext}^i_{\mathcal{O}_Z}(\mathcal{E}nd \mathcal{O}_Z(p^*\mathcal{F}il), \omega_Z) \\
\cong H^i(Z, \mathcal{E}nd\mathcal{O}_Z(p^*\mathcal{F}il) \otimes \omega_Z) \\
\cong H^i(G/P, \mathcal{F}il^* \otimes \mathcal{F}il \otimes \text{Sym}(V^*) \otimes \omega_Z) \\
\cong 0.
\]

\[\square\]

2.2 Quiver and relations of the noncommutative desingularization

In this section we describe a method to calculate the quiver and relations of the noncommutative desingularization constructed in Section 2.1. This method relies on the notion of quasi-hereditary structure, which will be recalled briefly.

2.2.1 Exceptional collections and dual collections

**Definition 2.2.1 ([45]).** Let \( \mathcal{C} \) be a triangulated category. A subset \( \Omega \) is called spanning if for each object \( a \in \mathcal{C} \), any of the following conditions implies \( a \cong 0 \):

1. \( \text{Ext}^i(a, b) = 0 \) for all \( b \in \Omega \) and all \( i \in \mathbb{Z} \);

2. \( \text{Ext}^i(b, a) = 0 \) for all \( b \in \Omega \) and all \( i \in \mathbb{Z} \).

Let \( \mathfrak{A} \) be a \( k \)-linear abelian category.

**Definition 2.2.2.** An ordered set of objects \( \Delta = \{ \Delta_\alpha, \alpha \in I \} \) in \( D^b(\mathfrak{A}) \) is called exceptional if we have \( \text{Ext}^\bullet(\Delta_\alpha, \Delta_\beta) = 0 \) for \( \alpha < \beta \) and \( \text{End}(\Delta_\alpha) = k \). An exceptional set is called strongly exceptional if in addition \( \text{Ext}^n(\Delta_\alpha, \Delta_\beta) = 0 \) for \( n \neq 0 \). It is said to be full if it is spanning for \( D^b(\mathfrak{A}) \).
In particular, in the definition above, we take $\mathfrak{A}$ to be the category of coherent sheaves over some scheme. It is not hard to see that if we have a finite full strongly exceptional set $\Delta = \{\Delta_\alpha, \alpha \in I\}$ consisting of vector bundles, then $T = \oplus_\alpha \Delta_\alpha$ is a tilting bundle.

**Definition 2.2.3.** For an exceptional collection $\Delta$, let $\nabla = \{\nabla_\alpha, \alpha \in I\}$ be another subset of objects in $D^b(\mathfrak{A})$, in bijection with $\Delta$. We say that $\nabla$ is the dual collection to $\Delta$ if $\text{Ext}^\bullet(\nabla_\beta, \Delta_\alpha) = 0$ for $\beta > \alpha$, and there exists an isomorphism $\nabla_\beta \cong \Delta_\beta \mod D_{<\beta}$, where $D_{<\beta}$ is the full triangulated subcategory generated by $\{\Delta_\alpha \mid \alpha < \beta\}$.

There are some well-known facts (see e.g., [8]):

1. $\text{Ext}^\bullet(\nabla_\alpha, \Delta_\alpha) = k$, where $k$ lies in homological degree 0;

2. $\text{Ext}^\bullet(\nabla_\alpha, \Delta_\beta) = 0$ for $\alpha \neq \beta$;

3. the dual collection is unique if it exists at all.

Note that even if $\Delta$ is a strongly exceptional collection, its dual collection $\nabla$ might not be strong.

### 2.2.2 Quasi-hereditary structure

Even if an exceptional collection is not strongly exceptional, under milder assumptions, we still have a tilting object. For an example of the general discussion in this section, we refer to Section 2.3.

Let $\mathfrak{A}$ be an abelian artinian $k$-linear category with a fixed complete set of pairwise distinct simple objects $\{S_\lambda \mid \lambda \in I\}$. Assume $\Lambda$ is finite for simplicity. Let $P_\lambda \rightarrow S_\lambda$ be the projective cover and $S_\lambda \rightarrow Q_\lambda$ be the injective envelop for each $\lambda \in I$. Endow $I$ with a partial order. We define the standard objects $\Delta_\lambda$ to be the largest quotient of $P_\lambda$ whose simple factors $S_\mu$ have $\mu < \lambda$. The costandard objects $\nabla_\lambda$ are defined to be the largest submodule of $Q_\lambda$ with all simple factors $S_\mu$ having $\mu \leq \lambda$. 
Note that when $P_\lambda$ coincide with $\Delta_\lambda$, the set $\Delta = \{\Delta_\lambda \mid \lambda \in I\}$ is a full strongly exceptional collection in $D^b(\mathfrak{A})$. In this case, $S_\lambda$ coincide with $\nabla_\lambda$.

**Definition 2.2.4.** The category $\mathfrak{A}$ with $\{\nabla_\lambda \mid \lambda \in I\}$ and $\{\Delta_\lambda \mid \lambda \in I\}$ is said to be *quasi-hereditary* if all the indecomposable projective objects $P_\lambda$ have filtrations with all the associated subquotients coincide with one of the standard objects.

The following proposition is proved in [29]. The version stated here is weaker than the original one proved in [29], but enough for our purpose. Let $\mathfrak{A}$ be a $k$-linear abelian category, and $\Delta = \{\Delta_\lambda \mid \lambda \in I\}$ a collection of objects, the additive full subcategory of $\mathfrak{A}$ consisting of objects admitting filtration with subquotients coincide with $\Delta_\lambda$'s will be denoted by $\mathcal{F}(\Delta)$.

**Proposition 2.2.5.** Let $\mathfrak{A}$ be a $k$-linear abelian category, and $\Delta = \{\Delta_\lambda \mid \lambda \in I\}$ is a finite exceptional collection in $\mathfrak{A}$ with the properties that $\dim_k \text{Hom}(\Delta_\alpha, \Delta_\beta) < \infty$ and $\dim_k \text{Ext}^1(\Delta_\alpha, \Delta_\beta) < \infty$ for any $\alpha, \beta \in I$. Then

1. there is a collection of objects $\Phi = \{\Phi_\lambda \mid \lambda \in I\}$ in $\mathcal{F}(\Delta)$, in bijection with $\{\Delta_\lambda\}$ such that the object $\Phi := \oplus_{\lambda \in I} \Phi_\lambda$ is a projective generator of $\mathcal{F}(\Delta)$, and $\text{End}(\Phi)\text{-mod}$ has a quasi-hereditary structure with the standard objects given by $\text{Hom}(\Phi, \Delta_\lambda)$.

2. If $\{\Delta_\lambda\}$ is a full exceptional collection, and $\text{Ext}^i(\Phi, \Delta_\lambda) = 0$ for all $i > 0$ and $\lambda \in I$, then $\Phi$ is a tilting object in $D^b(\mathfrak{A})$. In particular, we have an equivalence of triangulated categories $D^b(\mathfrak{A}) \cong D^b(\text{End}(\Phi)\text{-mod})$.

3. If $\{\nabla_\alpha\}$ is the dual collection, then the costandard objects in $\text{End}(\Phi)\text{-mod}$ are given by $R\text{Hom}(\Phi, \nabla_\lambda)$.

Under the assumptions of Proposition 2.2.5 (2) and (3), let $\Sigma_\lambda$ be the object in $\text{End}(\Phi)\text{-mod}$ that corresponds to the simple top of $R\text{Hom}(\Phi, \Delta_\lambda)$ under the equivalence $D^b(\mathfrak{A}) \cong D^b(\text{End}(\Phi)\text{-mod})$. Note that we automatically get natural maps $\Phi_\lambda \to \Delta_\lambda$, and $\Sigma_\lambda \to \nabla_\lambda$.

In particular, $\text{Ext}^\bullet(\Phi_\alpha, \Sigma_\alpha) = k$, where $k$ lies in homological degree 0.
2.2.3 A provisional method

Let $G$ be a reductive group, $P < G$ a parabolic subgroup. Suppose $p' : Z \to G/P$ is a vector bundle. Let $u : G/P \to Z$ be the zero section. Let $\Delta(G/P) = \{ \Delta_\alpha \mid \alpha \in I \}$ be a full exceptional collection over $G/P$ and $\nabla(G/P) = \{ \nabla_\alpha \mid \alpha \in I \}$ be the dual collection. Let $\Phi(G/P)$ and $\Sigma(G/P)$ be as in Proposition 2.2.5. We assume $\text{Ext}^i(\Phi, \Delta_\lambda) = 0$ for all $i > 0$ and $\lambda \in I$.

Let us start with a provisional method to get the shape of quivers for the non-commutative desingularizations.

Lemma 2.2.6. Let $\mathcal{T}il = \bigoplus_\alpha \Phi_\alpha$ over $G/P$. Suppose $p'^* \mathcal{T}il$ is a tilting bundle over $Z$. Then, $\text{Ext}^t_{\mathcal{O}_Z}(p'^* \Phi_\alpha, u_* \Sigma_\beta)$ vanishes unless $t = 0$ and $\alpha = \beta$ in which case it is 1-dimensional.

In particular, $R \text{Hom}_{\mathcal{O}_Z}(p'^* \mathcal{T}il, u_* \Sigma_\beta) = \text{Hom}_{\mathcal{O}_Z}(p'^* \Phi_\beta, u_* \Sigma_\beta) = k$.

Proof. Note that $\text{Ext}^t_{\mathcal{O}_Z}(p'^* \Phi_\alpha, u_* \Sigma_\beta) = \text{Ext}^t_{\mathcal{O}_{G/P}}(\Phi_\alpha, \Sigma_\beta)$. The conclusion then follows from the facts (1) and (2) about dual exceptional collections listed above.

From this lemma, we easily get the following Corollary.

Corollary 2.2.7. Suppose further more that $\{ R \text{Hom}(p'^* \mathcal{T}il, p'^* \Phi_\alpha), \alpha \in I \}$ is a complete set of (distinct) indecomposable projective objects in $\text{End}_{\mathcal{O}_Z}(p'^* \mathcal{T}il)_{-\text{mod}}$.

Then, for each $\alpha \in I$, $u_* \Sigma_\alpha$ is the simple object corresponds to the indecomposable projective $R \text{Hom}(p'^* \mathcal{T}il, p'^* \Phi_\alpha)$.

Remark 2.2.8. Even if $\Phi(G/P)$ is a full strongly exceptional collection over $G/P$, the collection of inverse images $\{ p'^* \Phi_\alpha \mid \alpha \in I \}$ is rarely an exceptional collection over $Z$. This is because there could be non-trivial maps $p'^* \Phi_\alpha \to p'^* \Phi_\beta$ for $\alpha < \beta$.

Throughout this paper, by radical, we mean the Jacobson radical.

Proposition 2.2.9. Let $\Lambda := \text{End}_{\mathcal{O}_Z}(p'^* \mathcal{T}il)$. Assume $\text{End}_{\mathcal{O}_Z}(p'^* \mathcal{T}il) \cong \text{End}_R(q_* p'^* \mathcal{T}il)$ and $\text{End}_R(q_* p'^* \mathcal{T}il)$ is a non-commutative desingularization of $\text{Spec } R$. Let $S_\alpha = R \text{Hom}(p'^* \mathcal{T}il, u_* \Sigma_\alpha)$.
Then $S_\alpha$'s are distinct simple objects in $\Lambda$-mod.

Assume further that $\Lambda/\text{rad}(\Lambda)$ is finite dimensional over $k$, and $\Lambda$ is Krull-Schmidt, semi-perfect (meaning projective covers exist in $\Lambda$-mod). Let $P_\alpha$ be the projective cover of $S_\alpha$. Then, $P_\alpha \cong R\text{Hom}(p^*Til, p^*\Phi_\alpha)$, and $\Lambda = \text{End}_\Lambda(\oplus_\alpha P_\alpha)$ is a basic algebra.

See [5] for the definitions of basic algebras.

Note that if $A/\text{rad}(A)$ is finite dimensional, then its radical is zero and hence its module category is semi-simple. We record here some facts that will be used in the proof of the proposition below.

**Proposition 2.2.10.** Notations as above.

1. For any $A$-module $M$, $\text{rad } M = \text{rad } A \cdot M$;

2. The assignment $M \mapsto M/\text{rad } M$ defines a functor from $A$-mod to $A/\text{rad } A$-mod;

3. For any idempotent element $e \in A$, the natural map $\text{Hom}_A(Ae, M) \to Me$ is an isomorphism of $eAe$-modules;

4. For any $A/\text{rad } A$-modules $M$ and $N$, let $P_N$ be the projective cover of $N$ in $A$-mod, we have $\text{Hom}_A(N, M) \cong \text{Hom}_A(P_N, M)$;

5. For any $A/\text{rad } A$-module $M$, any simple $A/\text{rad } A$-module $S$ and the projective cover $P$ of $S$ in $A$-mod, we have $\text{Hom}_A(M, S) \cong \text{Hom}_A(P, M)$.

In the case of finite dimensional algebra, this proposition can be found in [5]. But the same proof applied in this set-up.

**Proof of Proposition 2.2.9.** Recall that $R\text{Hom}(p^*Til, -)$ induces an equivalence between derived categories. Note that for any $\alpha$, $R\text{Hom}(p^*Til, p^*\Phi_\alpha)$ is concentrated in degree 0, and isomorphic to $\Lambda e_\alpha$, where $e_\alpha \in \Lambda$ is the identity element of $\text{End}_{\mathcal{O}_Z}(p^*\Phi_\alpha)$. Up to isomorphism, there are no more indecomposable projective $\Lambda$-modules because $\Lambda \cong$
⊕ \text{Hom}(p'\ast \mathcal{T}il, p'\ast \Phi_\alpha) and they are direct summands of \Lambda. Note also that \( S_\alpha = R \text{Hom}(p'\ast \mathcal{T}il, u_\ast \Sigma_\alpha) \) are all concentrated in degree 0 as well, and each isomorphic to \( \mathbb{C} \). Hence all of \( S_\alpha \)'s are simple objects in \( \Lambda\text{-mod} \). They are pairwise non-isomorphic, because \( u_\ast \Sigma_\alpha \)'s are distinct, as can be seen from Lemma 2.2.6.

Note that \( P_\alpha \)'s form a complete set of (distinct) indecomposable projectives. They are distinct as their tops are distinct. Clearly they are indecomposable as their tops are. These are all the indecomposable projectives (up to isomorphism) since

\[
\# I = \text{rank } K_0(G/P) = \text{rank } K_0(Z) = \text{rank } K_0(\Lambda\text{-mod}) = \# \{ P_\alpha | \alpha \in I \},
\]

where the 2nd equality comes from the Thom isomorphism (5.4 of [26]) and the 3rd one from the derived equivalence (i.e., any abelian category has the same \( K \)-group as its derived category).

It remains to show that \( \Lambda e_\alpha \cong P_\alpha \). As a projective module, \( \Lambda e_\alpha \) can be decomposed as direct sum of indecomposable projectives. And Lemma 2.2.6 yields \( \Lambda e_\alpha \cong P_\alpha \). \( \Box \)

\textbf{Remark 2.2.11.} Under the assumptions of the second part of Proposition 2.2.9, in the quiver with relation for \( \Lambda = \text{End}_{\mathcal{O}_Z}(p'\ast \mathcal{T}il) \), the arrows are given by a basis of \( \text{Ext}^1_{\mathcal{O}_Z}(\Sigma_i, \Sigma_j) \ast \). (See e.g., [4].) More precisely, \( e_\alpha(\text{rad } \Lambda / \text{rad}^2 \Lambda) e_\beta \cong \text{Ext}^1_\Lambda(S_\alpha, S_\beta) \ast \cong \text{Ext}^1_{\mathcal{O}_Z}(\Sigma_i, \Sigma_j) \ast \). And the relations are given by \( \text{Ext}^2_{\mathcal{O}_Z}(\Sigma_i, \Sigma_j) \ast \).

Unfortunately, the conditions in Proposition 2.2.9 and Remark 2.2.11 are rarely satisfied. We will see later on that there could be infinitely many simple modules over \( \text{End}_{\mathcal{O}_Z}(p'\ast \mathcal{T}il) \). However, the conclusion of Remark 2.2.11 is sometimes true in the situations we consider, as we assume that \( Z \) is the total space of an equivariant vector bundle \( V \) over \( G/P \), and both \( \Phi(G/P) \) and \( \Sigma(G/P) \) consist of equivariant sheaves over \( G/P \). Note that in this case, \( \Lambda = \text{End}_{\mathcal{O}_Z}(p'\ast \mathcal{T}il) \) is a representation of \( G \).
2.2.4 Calculation of quiver and relations

Definition 2.2.12. A pair \( (\Phi(G/P), \Sigma(G/P)) \) of collections of objects in \( D^b(Coh(G/P)) \) is called an equivariant dual pair if \( \Phi(G/P) = \{ \Phi_\alpha \mid \alpha \in I \} \) is a collection of equivariant sheaves over \( G/P \) and \( \Sigma(G/P) \) is another collection in \( D^b(G/P) \), also equivariant, such that \( \text{Ext}^*(\Phi_\alpha, \Sigma_\beta) = \delta_\alpha^\beta k \) where \( k \) lies in degree zero.

For an exceptional collection \( \Delta(G/P) \) consisting of equivariant sheaves and its dual collection \( \nabla(G/P) \) which is also equivariant, the pairs \( (\Delta(G/P), \nabla(G/P)) \) and \( (\Phi(G/P), \Sigma(G/P)) \) are both equivariant dual pair.

Proposition 2.2.13. Let \( \Phi(G/P) \) and \( \Sigma(G/P) \) be an equivariant dual pair such that \( \mathcal{T}il = \bigoplus_\alpha \Phi_\alpha \) is a tilting bundle. Assume the resolution \( q : Z \to \text{Spec} R \) is \( G \)-equivariant with \( q^{-1}(0) = G/P \), and the only \( G \)-fixed closed point of \( \text{Spec} R \) is \( 0 \in \text{Spec} R \). Assume moreover that \( p^* \mathcal{T}il \) is a tilting bundle over \( Z \) such that \( \text{End}_{\mathcal{O}_Z}(p^* \mathcal{T}il) \cong \text{End}_R(q_*p^* \mathcal{T}il) \).

Then, the only \( G \)-equivariant simple \( \text{End}_{\mathcal{O}_Z}(p^*(\bigoplus_\alpha \Phi_\alpha)) \) modules are given by
\[
S_\beta := R \text{Hom}_{\mathcal{O}_Z}(p^*(\bigoplus_\alpha \Phi_\alpha), u_*\Sigma_\beta) \cong \text{Hom}_{\mathcal{O}_Z}(p^*\Phi_\beta, u_*\Sigma_\beta)
\]
and they are pairwise distinct as \( \text{End}_{\mathcal{O}_Z}(p^*(\bigoplus_\alpha \Phi_\alpha)) \)-modules.

Proof. First we show that the simple equivariant \( \text{End}_{\mathcal{O}_Z}(p^*(\bigoplus_\alpha \Phi_\alpha)) \) modules have to be scheme theoretically supported on \( i : \{0\} \to \text{Spec} R \). Let \( M \) be a simple equivariant \( \text{End}_{\mathcal{O}_Z}(p^*(\bigoplus_\alpha \Phi_\alpha)) \) module whose support contains \( \{0\} \). Then \( M \) is an \( R \)-module. The surjective morphism \( M \to i_*i^*M \) has non-trivial target, because \( \{0\} \) is a subset of the support of \( M \). Note that his map is an \( G \)-equivariant morphism of \( \text{End}_{\mathcal{O}_Z}(p^*(\bigoplus_\alpha \Phi_\alpha)) \)-modules. It has trivial kernel since \( M \) is simple. Thus, \( M \cong i_*i^*M \), i.e., \( M \) has (scheme theoretical) support on \( \{0\} \).

Now we show that \( M \), with support on \( \{0\} \), is a simple module over
\[
i^* \text{End}_{\mathcal{O}_Z}(p^*(\bigoplus_\alpha \Phi_\alpha)) \cong i^*q_* \mathcal{E}nd_{\mathcal{O}_Z}(p^*(\bigoplus_\alpha \Phi_\alpha)).
\]
Using the cartesian diagram

\[
\begin{array}{ccc}
Z & \xleftarrow{u} & G/P \\
\downarrow{q} & \downarrow{q'} & \\
\text{Spec } R & \xleftarrow{i} & \{0\}
\end{array}
\]

and Remark III.9.3.1 in [39], we obtain a map

\[
i^* q_* \mathcal{E} \text{nd}_{O_Z}(p'^* (\oplus \alpha \Phi \alpha)) \to q'_* u^* \mathcal{E} \text{nd}_{O_Z}(p'^* (\oplus \alpha \Phi \alpha)).
\]

Note that \(i^* q_* \mathcal{E} \text{nd}_{O_Z}(p'^* (\oplus \alpha \Phi \alpha))\), as the inverse image through a proper morphism, is a finite dimensional algebra. There is a natural grading on it giving by the weights of \(G_m\)-action, where \(G_m\) acts on \(Z\) by scaling. The 0-th degree piece of \(i^* q_* \mathcal{E} \text{nd}_{O_Z}(p'^* (\oplus \alpha \Phi \alpha))\) is \(\text{End}_{G/P}(\oplus \alpha \Phi \alpha)\). Every element in the first degree piece can be easily checked to be in the Jacobson radical of \(i^* q_* \mathcal{E} \text{nd}_{O_Z}(p'^* (\oplus \alpha \Phi \alpha))\) using finite dimensionality. Therefore, we get an isomorphism

\[
i^* \text{End}_{O_Z}(p'^* (\oplus \alpha \Phi \alpha)) / \text{rad} \cong \text{End}_{G/P}(\oplus \alpha \Phi \alpha) / \text{rad}.
\]

So far we know \(M\) is a simple module over \(\text{End}_{G/P}(\oplus \alpha \Phi \alpha)\). As \(\Phi\) is a full exceptional collection over \(G/P\), its endomorphism ring is a finite dimensional basic algebra, with all the simple modules of the form \(\text{Hom}_{G/P}(\oplus \alpha \Phi \alpha, \Sigma_\beta) \cong \text{Hom}_Z(p'^* (\oplus \alpha \Phi \alpha), u_* \Sigma_\beta)\). Thus, \(M\) has to be isomorphic to one of them. (Note that the Jacobson radical acts trivially on the simple objects. Thus, two simple modules are isomorphic over \(\text{End}_{G/P}(\oplus \alpha \Phi \alpha)\) iff they are isomorphic over \(\text{End}_{O_Z}(p'^* (\oplus \alpha \Phi \alpha))\).)

For the claim that \(S_\alpha \neq S_\beta\) unless \(\alpha = \beta\), note that \(R \text{Hom}_{O_Z}(p'^* (\oplus \alpha \Phi \alpha), -)\) induces an equivalence of derived categories. To show \(S_\alpha \neq S_\beta\), it suffices to show \(u_* \Sigma_\alpha \neq u_* \Sigma_\beta\), and this is clear by Lemma 2.2.6.

\[\square\]

**Remark 2.2.14.** As can be seen from the proof, the finite dimensional algebra \(\text{End}_{O_{G/P}}(\oplus \alpha \Phi \alpha)\) is a subring of \(\text{End}_{O_Z}(p'^* (\oplus \alpha \Phi \alpha))\). The modules \(S_\alpha\) are also simple modules over \(\text{End}_{O_{G/P}}(\oplus \alpha \Phi \alpha)\). Actually, it is easier to see that \(S_\alpha\)'s are distinct as modules over \(\text{End}_{O_{G/P}}(\oplus \alpha \Phi \alpha)\).
Remark 2.2.15. In Proposition 2.2.13 we only characterized all the simple \( \text{End}_{\mathcal{O}_Z}(p^*\mathcal{T}il) \)-modules which happen to admit equivariance structure. There could be more simple objects in the abelian category of \( G \)-equivariant \( \text{End}_{\mathcal{O}_Z}(p^*\mathcal{T}il) \)-modules.

As \( R\text{Hom}(p^*\mathcal{T}il, p^*\Phi_\alpha) \) could be different than the projective covers of \( \mathbb{L}_\alpha \), from now on we make the convention that by \( P_\alpha \) we mean \( R\text{Hom}(p^*\mathcal{T}il, p^*\Phi_\alpha) \).

Definition 2.2.16. For an algebra with a rational \( G \)-action, and any \( G \)-equivariant module \( M \), we define \( \text{rad}_G M \) to be the intersection of all the \( G \)-equivariant maximal submodules of \( M \).

Caution: Note that \( \text{rad}_G M \) is not the intersection of all the maximal subobjects of \( M \) in the category of \( G \)-equivariant modules.

Lemma 2.2.17. Assume \( \Lambda \) is a \( k \)-algebra with a rational \( G \)-action such that \( \Lambda / \text{rad}_G \Lambda \) is semi-simple. For any equivariant module \( M \) over \( \Lambda = \text{End}_{\mathcal{O}_Z}(p^*\mathcal{T}il) \), we have \( \text{rad}_G M = \text{rad}_G \Lambda \cdot M \).

Proof. To show the inclusion \( \text{rad}_G M \supseteq \text{rad}_G \Lambda \cdot M \), it suffices to show that \( \text{rad}_G \Lambda \) is in the kernel of the map \( m \cdot : \Lambda \to M/N \) (multiplication by \( m \)) for any maximal submodule \( N \) and any element \( m \in M \). This is true since \( M/N \) is simple.

Because of the semi-simplicity of \( \Lambda / \text{rad}_G \Lambda \), the \( \Lambda / \text{rad}_G \Lambda \)-module \( M / \text{rad}_G M \) decomposes into direct sum of equivariant simple objects. And it is clear that \( \text{rad}_G M \) is the maximal submodule with this property. Hence we get the reverse inclusion. \( \square \)

Theorem 2.2.18. Let \( \Phi(G/P) \) and \( \Sigma(G/P) \) be an equivariant dual pair such that \( \mathcal{T}il = \bigoplus_\alpha \Phi_\alpha \) is a tilting bundle. Assume the resolution \( Z \to \text{Spec} R \) is \( G \)-equivariant with \( q^{-1}(0) = G/P \), and the only fixed closed point of \( \text{Spec} R \) is \( \{0\} \subset \text{Spec} R \). Let \( \Lambda := \text{End}_{\mathcal{O}_Z}(p^*\mathcal{T}il) \).

Assume moreover that \( p'^*(\bigoplus_\alpha \Phi_\alpha) \) is a tilting bundle over \( Z \) such that \( \Lambda \cong \text{End}_R(q_*p'^*\mathcal{T}il) \). Then,
1. we have $e_\alpha(\text{rad}_G\Lambda/\text{rad}^2_G\Lambda)e_\beta \cong \text{Ext}^1_\Lambda(S_\alpha, S_\beta)^*$. In particular, a lifting of a basis of this vector space to $\text{rad}_G\Lambda$ generates $\Lambda$ over $\oplus_\alpha k_\alpha$.

2. With generators of $\text{End}_{OZ}(p^*\mathcal{T}il)$ chosen as above, $\text{Ext}^2_{OZ}(\Sigma_i, \Sigma_j)^* \text{ generates the relations.}$

Proof of (1). We know that $S_\alpha$’s are all the simple $\Lambda$-modules which are $G$-equivariant. As $\text{Hom}_\Lambda(P_\beta, S_\alpha) = 0$ unless $\alpha = \beta$, in which case $\text{Hom}_\Lambda(P_\beta, S_\alpha) = k$ and $P_\alpha/\text{rad}_G P_\alpha \cong S_\alpha$.

Note that this also implies $\Lambda/\text{rad}_G\Lambda \cong \oplus k_\alpha$ which is semi-simple.

We take the short exact sequence

$$0 \rightarrow \text{rad}_G P_\alpha \rightarrow P_\alpha \rightarrow S_\alpha \rightarrow 0,$$

with $P_\alpha = R\text{Hom}(p^*\mathcal{T}il, p^*\Phi_\alpha)$. Hence we get an exact sequence

$$0 \rightarrow \text{Hom}(S_\alpha, S_\beta) \rightarrow \text{Hom}(P_\alpha, S_\beta) \rightarrow \text{Hom}(\text{rad}_G P_\alpha, S_\beta) \rightarrow \text{Ext}^1(S_\alpha, S_\beta) \rightarrow 0.$$

Hence

$$\text{Ext}^1_\Lambda(S_\alpha, S_\beta) \cong \text{Hom}_\Lambda(\text{rad}_G P_\alpha, S_\beta)$$

$$\cong \text{Hom}_\Lambda(\text{rad}_G P_\alpha/\text{rad}^2_G P_\alpha, S_\beta)$$

$$\cong \text{Hom}_\Lambda(\Lambda e_\beta, \text{rad}_G P_\alpha/\text{rad}^2_G P_\alpha)^*$$

$$\cong \text{Hom}_\Lambda(\Lambda e_\beta, (\text{rad}_G\Lambda/\text{rad}^2_G\Lambda)e_\alpha)^*$$

$$\cong e_\alpha(\text{rad}_G\Lambda/\text{rad}^2_G\Lambda)e_\beta^*.$$

The claim that $\text{rad}_G\Lambda$ generates $\Lambda$ over $\oplus_\alpha k_\alpha$ is clear from Lemma 2.2.17.

The second part of this theorem follows directly from the next Lemma, which gives an equivariant projective resolution of $S_\alpha$ and is interesting in its own right.

**Lemma 2.2.19.** There is a projective resolution of $S_\alpha$ of the form

$$0 \leftarrow S_\alpha \leftarrow P_\alpha \leftarrow \oplus_\beta \text{Ext}^1(S_\alpha, S_\beta)^* \otimes P_\beta \leftarrow \oplus_\beta \text{Ext}^2(S_\alpha, S_\beta)^* \otimes P_\beta \leftarrow \cdots.$$
Proof. We start with a surjective map $S_\alpha \leftarrow P_\alpha$ whose kernel is $\text{rad}_G P_\alpha$ which has a rational $G$-action, and therefore each vector lies in some finite dimensional subrepresentation. This implies the existence of a collection of finite dimensional representations $\{V^1_\gamma\}_\gamma$ of $G$ equipped with an equivariant surjective map $\text{rad}_G P_\alpha \leftarrow \bigoplus_\gamma V^1_\gamma \otimes P_\gamma$ which fits into an exact sequence

$$0 \leftarrow S_\alpha \leftarrow P_\alpha \leftarrow \bigoplus_\gamma V^1_\gamma \otimes P_\gamma.$$

We take the kernel and proceed to get an equivariant projective resolution of $S_\alpha$ with $i$-th term of the form $\bigoplus_\gamma V^i_\gamma \otimes P_\gamma$.

We apply $\text{Hom}_\Lambda(-, S_\beta)$ to get

$$0 \to \delta^3_k \to \delta^3_k \to V^1_{\beta^*} \to V^2_{\beta^*} \to \cdots,$$

which is a chain complex of $G$-representations. Hence, its $i$-th homology can be identified with a subquotient of the $i$-th term. In particular, (replacing $V^1_\beta$ with $(\ker d_2)^*$ if necessary,) we obtain

$$0 \leftarrow \text{coker} \leftarrow \bigoplus_\gamma V^1_\gamma \otimes P_\gamma \leftarrow \bigoplus_\gamma V^2_\gamma \otimes P_\gamma.$$

We claim that the composition $d_2 \circ \pi_1 : \bigoplus_\gamma V^2_\gamma \otimes P_\gamma \to \text{coker}$ is surjective. If so, we can replace $V^1_\gamma \otimes P_\gamma$ by $\text{Ext}^1(S_\alpha, S_\gamma) \otimes P_\gamma$, and $\bigoplus_\gamma V^2_\gamma \otimes P_\gamma$ by $d_2^{-1}(\text{Ext}^1(S_\alpha, S_\gamma) \otimes P_\gamma)$, and proceed iteratively to get the desired resolution.

To prove the surjectivity of $d_2 \circ \pi_1$, we can assume $\text{coker} = V^1_{\gamma'} \otimes P_\gamma \neq 0$, hence $V^1_{\gamma'} \neq 0$ for some $\gamma$. We show that $V^1_{\gamma'} \otimes P_\gamma$ is in the image of $d_2 \circ \pi_1$. Let $W_\gamma := V^1_{\gamma'} \otimes P_\gamma / V^1_{\gamma'} \otimes P_\gamma \cap \text{im}(d_2 \circ \pi_1)$. If this is not zero, $M/\text{rad}_G M$ would contribute to $\text{Ext}^1(S_\alpha, S_\beta)$ which makes it larger than it actually is. Hence, we are done.

Remark 2.2.20. Note that the complexes $u_* \Sigma_\alpha$’s are not in the heart of the usual $t$-structure in $D^b(\text{Coh}(Z))$, but $R\text{Hom}(p^* \mathcal{F}il, u_* \Sigma_\alpha)$’s are for the usual $t$-structure of $D^b(\Lambda\text{-mod})$. This means the functor $R\text{Hom}(p^* \mathcal{F}il, -)$ does not restrict to an equivalence $\text{Coh}(Z) \to \Lambda\text{-mod}$.
Remark 2.2.21. This derived equivalence gives the triangulated category $D^b(\Lambda\text{-mod})$ a $t$-structure, by lifting the tautological $t$-structure of $D^b(\Lambda\text{-mod})$ (see e.g. IV.4 in [36] for the definitions of $t$-structures of a triangulated categories and the tautological $t$-structures of derived categories).

Note that twisting the tilting bundle with any line bundle will give the same endomorphism ring. Consequently, any two different exceptional collections in the same $H^1(G/P, \mathcal{O}^*)$-orbit in the set of exceptional collections give the same non-commutative desingularization. But the induced $t$-structures on $D^b(Coh(Z))$ are different.

2.3 Grassmannians

In this section we collect some preliminary results about coherent sheaves on the Grassmannians. It can also be viewed as a collection of examples to the notions of exceptional collection, tilting bundle, and quasi-hereditary structure we recalled without illustrating examples in previous sections.

2.3.1 Representation theory of $GL_n$

We recall here some classical results about representation theory of reductive group $G$ over $k$, with emphasis on the case when $G = GL_n$. For the proofs of the results, we refer to [46].

Let $G$ be a split reductive group and $T$ a split maximal torus whose weight lattice will be denoted by $\hat{T}$. We fix a Borel subgroup $B$ containing $T$. The dominant chamber in $\hat{T}$ determined by $B$ will be denoted by $\hat{T}_+$. The category of finite dimensional representations of $G$ will be denoted by $\text{Rep}(G)$.

For a subgroup $H$ of $G$ such that $G/H$ is a scheme. The category of $G$-equivariant coherent sheaves on $G/H$ will be denoted by $Coh_G(G/H)$. We can define a functor $\mathfrak{L}_{G/H}: \text{Rep}(H) \to Coh_G(G/H), V \mapsto G \times_H V$. It is an equivalence of categories, with quasi-inverse
given by $E \mapsto \mathcal{E}_{[H]}$, i.e., taking the fiber at the point $[H] \in G/H$. In particular, both functors are exact functors.

For a $G$-scheme $X$, i.e., a scheme $X$ with an algebraic $G$-action, we have an adjoint pair $H^0(X, -): \text{Coh}_G(X) \to \text{Rep}(G)$ and $- \otimes \mathcal{O}_X: \text{Rep}(G) \to \text{Coh}_G(X)$. In general, $H^0(X, -)$ is only left exact and $- \otimes \mathcal{O}_X$ is only right exact. In the case when $X = G/H$, the composition $V \mapsto (V \otimes \mathcal{O}_X)_{[H]}$ is naturally equivalent to $\text{Res}^G_H: \text{Rep}(G) \to \text{Rep}(H)$, therefore, its adjoint $\text{Ind}^G_H: \text{Rep}(H) \to \text{Rep}(G)$ is the composition $H^0(G/H, -, \mathcal{O}_{G/H})$. These functors are summarized in the following diagram.

As the functor $\mathcal{L}_{G/H}$ is exact, we have the Borel-Weil Theorem.

**Theorem 2.3.1.** We have a natural isomorphism of functors $R^i \text{Ind}^G_H \cong H^i(G/H, \mathcal{L}_{G/H})$.

Now we take $H = B$. Note that $\text{Rep}(B) \cong \text{Rep}(T)$, i.e., irreducible representations of $B$ are indexed by weights $\alpha \in \hat{T}$. The following vanishing theorem is originally due to Kempf.

**Theorem 2.3.2.** If $\alpha \in \hat{T}_+$, then

$$R^i \text{Ind}^G_B \alpha = H^i(G/B, \mathcal{L}_{G/H}\alpha) = 0$$

for $i > 0$.

Now consider an $n$-dimensional vector space $V$. Let $G = \text{GL}(V^*)$, for $\lambda \in \hat{T}_+$, the induced representation $\text{Ind}^G_B(\lambda)$ is called the Schur module corresponds to the dominant weight $\lambda$, denoted by $\mathbb{L}_\lambda V$. Let $w_0$ be the longest element in the Weyl group, the representation $\text{Ind}^G_B(-w_0(\lambda))$ is called the Weyl module, denoted by $\mathbb{K}_\lambda V$. 

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If $k = \mathbb{C}$, these two modules coincide, and are both simple representations. But in general, $L_\lambda V$ has a simple socle which coincides with the simple top of $K_\lambda V$. Taking $K$ as the standard objects and $L$ as the costandard objects gives $\text{Rep}(G)$ a quasi-hereditary structure.

The following theorem tells us the tensor product of any two standard objects has a filtration by standard objects, and the multiplicity of each factor can be calculated.

**Theorem 2.3.3** (See [80], 2.3.2, 2.3.4). Let $V$ and $W$ be two vector spaces over $k$, and $\alpha$, $\beta$ be any two dominant integral weights.

1. There is a natural filtration on $\text{Sym}_t(V \otimes W)$ whose associated graded object is a direct sum with summands tensor products $L_\delta V \otimes L_\alpha W$ of Schur functors.

2. There is a natural filtration on $L_\alpha V \otimes L_\beta V$ whose associated graded object is a direct sum of Schur functors $L_\delta V$. The multiplicities can be computed using the usual Littlewood-Richardson rule.

We identify $GL_n$ with the space of all the invertible $n \times n$ matrices, and $T$ as the diagonal invertible matrices. Then $\hat{T}$ can be identified with the set of all $n$-tuples of integers $\mathbb{Z}^n$, and $\hat{T}_+$ consists of $n$-tuples of non-increasing integers. We go further to identify $n$-tuples of non-increasing non-negative integers with partitions of lengths no more than $n$. For a vector space of dimension $n$ and a partition $\lambda$ of length no more than $n$, upon choosing a basis for $E^*$, we can identity $GL(E^*)$ with $GL_n$ using this basis, then $L_\lambda E$ and $K_\lambda E$ can be define as above. This procedure behaves well with respect to change of basis, hence can be thought of as functors. For explicit descriptions of the Schur functor $L_\lambda$ and Weyl functor $K_\lambda$ associated to a partition $\lambda$, see e.g., [?] or [80]. In particular, they can be applied to vector bundles $h : E \to X$.

Now we consider the case when $P$ is a maximal parabolic subgroup, i.e., when $G/P$ is Grass = Grass($n-r,n$), the Grassmannian of $n-r$-planes in the $n$-dimensional vector space.
\( E^* \) (or equivalently the Grassmannian of \( r \)-planes in \( E \)). Let \( 0 \to R \to O \otimes E^* \to Q \to 0 \) be the tautological sequence on Grass. In this case, the Levi subgroup \( L = GL_{n-r} \times GL_r \).

For the defining representation \( k^{n-r} \) of \( GL_{n-r} \), the sheaf \( L_{Grass}(k^{n-r}) \) is \( R^* \), and similarly for the defining representation \( k^r \) of \( GL_r \), the sheaf \( L_{Grass}(k^r) \) is \( Q^* \).

A weight \( \alpha \) is called \((n-r)\)-dominant if it is dominant as a weight of \( GL_{n-r} \times GL_r \), i.e., \( \alpha_1 \geq \cdots \geq \alpha_{n-r} \) and \( \alpha_{n-r+1} \geq \cdots \geq \alpha_n \). For an \((n-r)\)-dominant \( \alpha \) weight we can consider two weights \( \beta = (\alpha_1, \cdots, \alpha_{n-r}) \) and \( \gamma = (\alpha_{n-r+1}, \cdots, \alpha_n) \). We define the vector bundle \( V(\alpha) = L_\beta R^* \otimes L_\gamma Q^* \).

The following is an easy corollary of the Kempf vanishing Theorem.

**Corollary 2.3.4.** Consider the integral dominant weight \( \alpha \) and the corresponding vector bundle \( V(\alpha) \) on \( Grass(n-r, E^*) \) defined as above. Then \( H^0(Grass, V(\alpha)) = L_{\alpha} E \otimes (\wedge^n E)^{\otimes \alpha_n} \) and \( H^i(Grass, V(\alpha)) = 0 \) for \( i > 0 \).

The symmetric group \( \Sigma_n \) acts on the set of weights. Let \( \alpha = (\alpha_1, \cdots, \alpha_n) \). The permutation \( \sigma_i = (i, i+1) \) acts on the set of weights by: \( \sigma_i \alpha = (\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1} - 1, \alpha_i + 1, \alpha_{i+2}, \cdots, \alpha_n) \).

Let \( \alpha \in \mathbb{Z}^n \) be a dominant integral weight. We denote \( \bar{\alpha} = (\alpha_1 - \alpha_n, \cdots, \alpha_{n-1} - \alpha_n, 0) \). By definition the weight \( \bar{\alpha} \) is a partition.

If \( k = \mathbb{C} \), we know more about cohomologies of bundles corresponding to non-dominant weights. Note that part of it is the Borel-Weil Theorem which holds in arbitrary characteristic.

**Theorem 2.3.5 (Bott).** Let \( k = \mathbb{C} \). We consider a weight \( \alpha \) satisfying \( \alpha_i \geq \alpha_{i+1} \) for \( i \neq n-r \) and the corresponding vector bundle \( V(\alpha) \) over Grass defined above. Then one of the two mutually exclusive possibilities occurs:

1. There exists an element \( \sigma \in \Sigma_n, \sigma \neq 1 \), such that \( \sigma(\alpha) = \alpha \). Then the higher direct images \( H^i(Grass, V(\alpha)) \) are zero for \( i \geq 0 \).
2. There exists a unique element $\sigma \in \Sigma_n$ such that $\sigma(\alpha) := (\beta)$ is a partition (i.e. is non-increasing). In this case all higher direct images $H^i(\text{Grass}, \mathcal{V}(\alpha))$ are zero for $i \neq l(\sigma)$, and

$$H^{l(\sigma)}(\text{Grass}, \mathcal{V}(\alpha)) = \mathbb{L}_\beta E.$$

### 2.3.2 A geometric technique

Now we review a geometric technique used throughout this paper. The reference for this subsection is [80].

For a projective variety $V$ of dimension $m$, $X = \mathbb{A}_k^N$ an affine space, the space $X \times V$ can be viewed as the total space of the trivial vector bundle $\mathcal{E}$ of dimension $N$ over $V$. Let us consider the subvariety $Z$ in $X \times V$ which is the total space of a subbundle $\mathcal{S}$ in $\mathcal{E}$. We denote by $q$ the projection $q : X \times V \to X$ and by $q'$ the restriction of $q$ to $Z$. Let $Y = q(Z)$. We get the basic diagram

$$
\begin{array}{c}
Z \\ q' \downarrow \\
Y \downarrow \\
\rightarrow \\
\rightarrow \\
X \\
\end{array}
$$

The projection from $X \times V$ onto $V$ is denoted by $p$, and the quotient bundle $\mathcal{E}/\mathcal{S}$ by $\mathcal{T}$. Thus we have the exact sequence of vector bundles on $V$,

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{T} \to 0.$$

The coordinate ring of $X$ will be denoted by $A$. It is a polynomial ring in $N$ variables over $k$. We will identify the sheaves on $X$ with $A$-modules. The direct image $p_*(\mathcal{O}_Z)$ can be identified with the sheaf of algebras $\text{Sym}(\eta)$, where $\eta = S^*$. For a vector bundle $\mathcal{V}$ over $V$, the $\mathcal{O}_{X \times V}$-module $\mathcal{O}_Z \otimes p^* \mathcal{V}$ will be denoted by $M(\mathcal{V})$.

**Theorem 2.3.6** (5.1.2 in [80]). For a vector bundle $\mathcal{V}$ on $V$, we define free graded $A$-modules

$$F(\mathcal{V})_i = \bigoplus_{j \geq 0} H^j(V, \bigwedge^{i+j} \xi \otimes \mathcal{V}) \otimes_k A(-i-j)$$

where $\xi = \mathcal{T}^*$ and $(i)$ means shifting by $i$. 
1. There exist minimal differentials
\[ d_i(V) := F(V)_i \to F(V)_{i-1} \]
of degree 0 such that \( F(V)_\bullet \) is a complex of free graded \( A \)-modules with
\[ H_{-i}(F(V)_\bullet) = R^i q_* M(V). \]
In particular, the complex \( F(V)_\bullet \) is exact in positive degrees.

2. The sheaf \( R^i q_* M(V) \) is equal to \( H^i(Z, M(V)) \) and it can be also identified with the graded \( A \)-module \( H^i(V, \text{Sym}(\eta) \otimes V) \).

3. If \( \phi : M(V) \to M(V')(n) \) is a morphism of graded sheaves, then there exists a morphism of complexes
\[ f_\bullet(\phi) : F(V)_\bullet \to F(V')_\bullet(n) \]
Its induced map \( H_{-i}(f_\bullet(\phi)) \) can be identified with the induced map
\[ H^i(Z, M(V)) \to H^i(Z, M(V'))(n). \]

This theorem will be mentioned as the basic theorem of geometric method in this paper.

Now we come to a criterion for maximal Cohen-Macaulayness in the context of geometric technique. The proof, which can be found in 5.1.5 of [80], is based on the basic theorem of geometric technique and Lemma 2.1.9.

Proposition 2.3.7. Let \( \mathcal{V} \) be a bundle over \( V \), and let \( \tilde{V} := \omega_V \otimes \wedge^{\text{top}} \xi \otimes \mathcal{V}^* \). Assume \( \dim Z = \dim Y \) and \( R^i q'_*(\mathcal{O}_z \otimes p^* \mathcal{V}) = 0 \) for all \( i > 0 \). Then, \( R^0 q'_*(\mathcal{O}_z \otimes p^* \tilde{V}) \) is a maximal Cohen-Macaulay module supported on \( Y \) iff \( R^i q'_*(\mathcal{O}_z \otimes p^* \tilde{V}) = 0 \) for all \( i > 0 \).

This proposition will be used in Section 2.8 to prove that some non-commutative desingularizations we study are crepant.
2.3.3 Exceptional collections on Grassmannians

The main reference for this section is [21].

Kapranov [49] constructed an exceptional collection over Grass = Grass\(n-r(E^*)\), the Grassmannian of \((n-r)\)-planes in the vector space \(E^*\) over \(\mathbb{C}\).

Let

\[
0 \to \mathcal{R} \to E^* \times \text{Grass} \to \mathcal{Q} \to 0
\]

be the tautological exact sequence over Grass. Recall that we will write \(B_{u,v}\) to mean the set of partitions with no more than \(v\) columns and no more than \(u\) rows. For a partition \(\lambda\), recall that \(L_{\lambda}\) is the Schur functor corresponding to it. For a partition \(\alpha\) we write \(\alpha'\) for its transpose, and for any weight \(\alpha\), we call \(\sum_i \alpha_i\) its area which is denoted by \(|\alpha|\).

**Theorem 2.3.8** (Kapranov, see also [21]). *For a suitable choice of ordering,

\[
\{L_{\alpha}^{\mathcal{R}^*} \mid \alpha \in B_{n-r,r}\}
\]

is an exceptional collection in \(D^b(\text{Coh(Grass)})\), whose dual exceptional collection is given by

\[
\{L_{\alpha'}^{\mathcal{Q}[[|\alpha|]]} \mid \alpha \in B(n-r,r)\}.
\]

If \(k = \mathbb{C}\), or \(r = 1\) or \(n - 1\), this exceptional collection is strong, hence

\[
\mathcal{Til}_K = \bigoplus_{\alpha \in B_{n-r,r}} L_{\alpha}^{\mathcal{R}^*}
\]

is a tilting bundle.

Observe that \(- \otimes \wedge^{\text{top}} \mathcal{Q}\) and \(- \otimes \wedge^{\text{top}} \mathcal{R}\) define a \(\mathbb{Z}^2\)-action on the triangulated category \(D^b(\text{Coh(Grass}_{n-r}(E^*))\)). This action sends one exceptional collection to another, and preserves duality.

Applying the \(\mathbb{Z}^2\)-action we can see that \(\Delta(\text{Grass}) = \{L_{\lambda}^{\mathcal{Q}^*} \mid \lambda \in B_{r,n-r}\}\) over Grass is also a full exceptional collection. As can be checked by definition, the dual collection is given
by $\nabla(\text{Grass}) = \mathbb{L}_{(n-r)-r}Q^* \otimes \mathbb{L}_{(\alpha' \gamma)} R[(n-r)r - |\alpha|]$, where for a subpartition $\alpha$ of $r^{n-r}$ we write $\alpha^c$ for its complement in the rectangle $(r^{n-r})$.

If $k$ has positive characteristic or $1 < r < n - 1$, Proposition 2.2.5(1) gives a collection $\Phi(\text{Grass}) = \{\Phi_{\alpha} \mid \alpha \in B_{r,n-r}\}$. For $\alpha \in B_{r,n-r}$, denote the simple top of $\mathbb{K}_\alpha k^{n-r}$ in $\text{Rep}(\text{GL}_{n-r})$ by $L_\alpha$ and its projective cover by $M_\alpha$, similarly denote the simple top of $\mathbb{K}_\lambda k^r$ by $L'_\alpha$. It is shown in [21] that $\Phi_{\alpha}$ given in Proposition 2.2.5(1) is equal to $L_{\text{Grass}}(M_\alpha)$, and $\Sigma_{\alpha} = \mathcal{L}_{\text{Grass}}(L'_\alpha)[|\alpha|]$. Also the hypothesis in Proposition 2.2.5(2) is satisfied, hence, $\mathcal{T}il = \oplus_{\alpha \in B_{r,n-r}} \Phi_{\alpha}$ is a tilting bundle.

Further more, there is a characteristic free collection of vector bundles on the Grassman-nian, whose all direct summands are $\Phi_{\alpha}$. For a partition $\alpha = (\alpha_1 \cdots \alpha_r)$ and any vector space $V$,

$$\wedge^\alpha V = \wedge^{\alpha_1} \otimes \cdots \otimes \wedge^{\alpha_r} V.$$ 

**Theorem 2.3.9 ([21]).** The vector bundle $\mathcal{T}il_0 = \oplus_{\alpha \in B_{r,n-r}} \wedge^{\alpha'} Q$ is a classical tilting bundle on Grass. In $\text{End}_{\text{Grass}}(\mathcal{T}il_0)-\text{mod}$, the pair of collections

$$\Delta_{\alpha} = R\text{Hom}_{\text{Grass}}(\mathcal{T}il_0, \mathbb{L}_\alpha Q)$$

$$\nabla_{\alpha} = R\text{Hom}_{\text{Grass}}(\mathcal{T}il_0, \mathbb{L}_{\alpha'} R^*[|\alpha|])$$

gives a quasi-hereditary structure. The simple objects are $R\text{Hom}_{\text{Grass}}(\mathcal{T}il_0, \mathcal{L}_{\text{Grass}}(L'_\alpha)[|\alpha|])$ and their projective covers are given by $R\text{Hom}_{\text{Grass}}(\mathcal{T}il_0, \mathcal{L}_{\text{Grass}}(M_\alpha))$.

In particular, $\text{End}_{\text{Grass}}(\mathcal{T}il_0)$ is Morita equivalent to the basic algebra $\text{End}_{\text{Grass}}(\oplus \mathcal{L}_{\text{Grass}}(M_\alpha))$.

The vector bundle $\mathcal{T}il_0$ in Theorem 2.3.9 will be referred to as the BLV’s tilting bundle, and $\mathcal{T}il_K$ the Kapranov’s tilting bundle.

The following proposition proved in [21] will be used later.

**Proposition 2.3.10.** Let $\alpha \in B_{r,n-r}$ and let $\delta$ be any partition. Then for all $i > 0$ one has

$$H^i(\text{Grass}, (\wedge^{\alpha'} Q)^* \otimes_{\mathcal{O}_{\text{Grass}}} \mathbb{L}_\delta Q) = 0.$$
Now, let us work in the set-up as in diagram 2.1 by taking $G/P$ to be the Grassmannian. We describe the Ext’s between the equivariant simples $S_\beta = R\text{Hom}_{\mathcal{O}_Z}(p^*\mathcal{T}il, u_*\Sigma_\beta)$ as in Proposition 2.2.13. We will use that to get the shape of quivers with relations of the non-commutative desingularization. The proof of the following lemma is along the same lines as the corresponding one in [19].

**Lemma 2.3.11.** Let $Z$ be the total space of the vector bundle $\mathbb{L}_\delta Q^*$ for some partition $\delta$ such that the conditions in Proposition 2.2.13 are satisfied for the tilting bundle $\mathcal{T}il_0$. Let $S_\alpha$’s be the simples as in Proposition 2.2.13. Then, the Ext’s among them are given by

$$\text{Ext}^t(S_\alpha, S_\beta) \cong \bigoplus_s H^{t-s-|\beta|+|\alpha|}(\text{Grass}, (\wedge^s \mathbb{L}_\delta Q)^* \otimes \mathcal{L}_{\text{Grass}}(L_\beta') \otimes \mathcal{L}_{\text{Grass}}(L_\alpha')^*).$$

**Proof.** We have

$$\text{Ext}^t(S_\alpha, S_\beta) = \text{Ext}^t(\mathcal{O}_Z(u_*\Sigma_\alpha, u_*\Sigma_\beta)) = \text{Ext}^t(\mathcal{O}_Z(u_*\mathcal{L}_{\text{Grass}}(L_\beta'), u_*\mathcal{L}_{\text{Grass}}(L_\alpha'))) = \bigoplus_s H^{t-s-|\beta|+|\alpha|}(\text{Grass}, (\wedge^s \mathbb{L}_\delta Q)^* \otimes \mathbb{L}_{(\beta')}\mathcal{R} \otimes \mathbb{L}_{(\alpha')}\mathcal{R}^*).$$

The only thing need to explain in the above is the third equality. To get that we take the Koszul resolution of $u_*V := u_*\mathbb{L}_{(n-r)}Q^* \otimes \mathbb{L}_{(r)}\mathcal{R}$ as follows

$$\cdots \to \wedge^2 \mathbb{L}_\delta Q \otimes V \otimes \mathcal{O}_Z \to \mathbb{L}_\delta Q \otimes V \otimes \mathcal{O}_Z \to V \otimes \mathcal{O}_Z,$$

apply $\mathcal{H}\text{om}_{\mathcal{O}_Z}(-, u_*U)$, and use adjunction formula to get

$$0 \to \mathcal{H}\text{om}_{\text{Grass}}(V, U) \to \mathcal{H}\text{om}_{\text{Grass}}(\mathbb{L}_\delta Q \otimes V, U) \to \mathcal{H}\text{om}_{\text{Grass}}(\wedge^2 \mathbb{L}_\delta Q \otimes V, U) \to \cdots.$$ 

Note that the $t$-th hypercohomology of this complex is exactly $\text{Ext}^t(S_\alpha, S_\beta)$. The third equality above is equivalent to the degeneracy of the hypercohomology spectral sequence

$$E_1^{r,s} = H^r(\mathcal{H}\text{om}_{\text{Grass}}(\wedge^s \mathbb{L}_\delta Q \otimes V, U)) \Rightarrow \text{Ext}^{r+s}(S_\alpha, S_\beta)$$

at $E_1$-page. We claim that the hypercohomology spectral sequence does degenerate at $E_1$-page. In fact, after plugging in $V = \mathcal{L}_{\text{Grass}}(L_\alpha')$ and $U = \mathcal{L}_{\text{Grass}}(L_\beta')$, all the differentials...
are equivariant under $GL_n$ and hence equivariant under the $\mathbb{G}_m$-action. Here the $\mathbb{G}_m \subset GL_n$ consists of the scalar matrices. Note that the weights of this $\mathbb{G}_m$-action in different columns in the spectral sequence are different. Therefore, there is no non-trivial differentials other than the vertical ones, all of which are in $E_0$.

**Corollary 2.3.12.** Assume $k = \mathbb{C}$, $Z$ be the total space of the vector bundle $\mathbb{L}_\delta \mathbb{Q}^*$ for some partition $\delta$, and $\nabla(\text{Grass}) = \{ \mathbb{L}_\lambda \mathbb{Q}^* \mid \lambda \in B_{r,n-r} \}$ satisfies the conditions in Proposition 2.2.13. Let $S_\alpha$'s be the simples as in Proposition 2.2.13. Then, the Ext's among them are given by

$$\text{Ext}^t(S_\alpha, S_\beta) \cong \bigoplus_{s} \bigoplus_{\lambda \in \wedge^s \mathbb{L}_\delta} H^{t-s-|\beta|+|\alpha|}(\text{Grass}, \mathbb{L}_\lambda \mathbb{Q}^* \otimes \mathbb{L}_\beta \mathbb{R}^* \otimes \mathbb{L}_\alpha \mathbb{R}),$$

where $\wedge^s \mathbb{L}_\delta$ stands for the decomposition of $\wedge^s \mathbb{L}_\delta \mathbb{C}^{n-r}$ into irreducible representations counting multiplicity.

**Proof.** This is because if $k = \mathbb{C}$, we have

$$\bigoplus_{s} H^{t-s-|\beta|+|\alpha|}(\text{Grass}, \wedge^s \mathbb{L}_\delta \mathbb{Q}^* \otimes \mathbb{L}_{(\beta^c)} \mathbb{R} \otimes \mathbb{L}_{(\alpha^c)} \mathbb{R}^*)$$

$$= \bigoplus_{s} \bigoplus_{\lambda \in \wedge^s \mathbb{L}_\delta} H^{t-s-|\beta|+|\alpha|}(\text{Grass}, \mathbb{L}_\lambda \mathbb{Q}^* \otimes \mathbb{L}_{(\beta^c)} \mathbb{R} \otimes \mathbb{L}_{(\alpha^c)} \mathbb{R}^*)$$

$$= \bigoplus_{s} \bigoplus_{\lambda \in \wedge^s \mathbb{L}_\delta} H^{t-s-|\beta|+|\alpha|}(\text{Grass}, \mathbb{L}_\lambda \mathbb{Q}^* \otimes \mathbb{L}_{\beta} \mathbb{R}^* \otimes \mathbb{L}_{\alpha} \mathbb{R}).$$

\[\square\]

### 2.3.4 Combinatorics from the Kapranov’s exceptional collection

In the rest of this section, we do some combinatorics with the Kapranov’s exceptional collection, which makes it easier to calculate the Ext’s. *From now on till the end this section, we assume $k = \mathbb{C}$.*

**Corollary 2.3.13.** Let $\mathbb{L}_\gamma \mathbb{R}^* \subset \mathbb{L}_{\beta} \mathbb{R}^* \otimes \mathbb{L}_\alpha \mathbb{R}$ and $k = t - s - |\gamma|$.

1. For a fixed $t$, $H^k(\text{Grass}, \mathbb{L}_\lambda \mathbb{Q}^* \otimes \mathbb{L}_\gamma \mathbb{R}^*) = 0$ for any $s > t$. 57
2. For a fixed \( t \), \( H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = 0 \) for any \(|\delta|s \geq k\).

3. For a fixed \( s \) and \( t \), \( H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = 0 \) for any \( \gamma = (\gamma_1, \cdots, \gamma_{n-r}) \) with the positive area of \( \gamma \) greater than \( t - s \).

4. For any \( \gamma = (\gamma_1, \cdots, \gamma_{n-r}) \), \( H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = 0 \) unless the negative part of \( \gamma' \) is contained in \( \lambda \).

Proof. According to Bott’s Theorem, there can be no more than one \( k \) such that \( H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) \neq 0 \), and this \( k \) is the number of adjacent transpositions for \( (\gamma_1, \cdots, \gamma_{n-r}, \lambda_1, \cdots, \lambda_r) \) to make it dominant. Assume \( H^k \neq 0 \), we know \( \gamma_{n-r} \geq -r \), and therefore the total negative area in \( (\gamma_1, \cdots, \gamma_{n-r}) \) is no larger than \( k \). So, \(-s + t = |\gamma| + k \geq 0\) which proves (1).

Again because \( \gamma_{n-r} \geq -r \), the total area of \( (\gamma_1, \cdots, \gamma_{n-r}, \lambda_1, \cdots, \lambda_r) \) is positive. Therefore, \(|\delta|s \geq k\).

We know that the total negative area in \( (\gamma_1, \cdots, \gamma_{n-r}) \) is no larger than \( k \). Let the positive area of \( \gamma \) be \( l \). Then we have \(-(|\gamma| - l) \leq k \). Hence \( l \leq t - s \).

The last part is clear. \( \square \)

The following examples, which are direct consequences of Corollary 2.3.13, will be used in Section 2.5 and Section 2.6. To state them, we introduce the following notations. For any Young diagram \( \alpha \), by \( \tau_l \alpha \) we mean \( \alpha \) delating the first \( l \) columns, and by \( r^l \) we mean \( \alpha \) deleting everything after the first \( l \) columns. For any partition \( \alpha = (\alpha_1, \cdots, \alpha_l) \), by \( (-\alpha) \) we mean a partition with \( (-\alpha)_i = -\alpha_{l-i} \).

**Example 2.3.14.** Notations as above. Let \( t = 1 \), then, the only non-zero \( H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) \) happens only in the following cases:

- \( s = 1 \): \( -\gamma' = \lambda \) in which case \( H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = \mathbb{C} \). Or length(\( \gamma \)) = \( n-r \), and \((r^{n-r}\lambda) = (-\gamma)' \). In this case, the corresponding \( H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = \mathbb{L}_{r_{n-r}} E \).

- \( s = 0 \): \( \gamma = (1, 0, \cdots, 0) \). In this case, the corresponding \( H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = E \).
Proof. According to Corollary 2.3.13, we only need to consider the $s = 1$ and $s = 0$ case. If $s = 1$, then part 3 and 4 Corollary 2.3.13 yield that $-\gamma' \subset \lambda$. We also have $k = -|\gamma|$. Note that $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) \neq 0$ implies that the weight becomes dominant after $k$ exchanges. Therefore, $(\lambda')_i = (-\gamma)_i$ for all $i \leq \text{length}(\gamma)$, and the corresponding $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = \mathbb{L}_{(\lambda_1 - (\gamma)_{n-r}, \lambda_2 - (\gamma)_{n-r-1}, \ldots, \lambda_{n-r} - (\gamma)_{1})}E$. The statement for $s = 0$ is clear.

Similarly, one has the following example.

Example 2.3.15. Notations as above. Let $t = 2$, then, the only non-zero $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*)$ happens only in the following cases.

- $s = 2$: $-\gamma' = \lambda$ in which case $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = \mathbb{C}$. Or $\text{length}(\gamma) = n - r$, and $(\tau^{n-r}\lambda) = (-\gamma)'$. In this case, the corresponding $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = \mathbb{L}_{\tau_{n-r}\lambda}E$.

- $s = 1$, the positive part of $\gamma$ is $(1, 0, \cdots, 0)$: The negative part of $\gamma$ is $(-\lambda)$ in which case $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = \mathbb{C}$. Or the length of the negative part of $(\gamma)$ is $n - r - 1$, $(\tau^{n-r-1}\lambda) = ((-\gamma')_1, \cdots, (-\gamma')_{n-r-1})$, and $\tau_{n-r-1}\lambda_1 \leq 1$, in which the corresponding $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = \mathbb{L}_{(1, \tau_{n-r-1}\lambda)}E$.

- $s = 1$, $\gamma$ has no positive part: $\text{length}(\gamma) = n - r$, and $((\tau^{n-r}\lambda)')_i = (-\gamma)_i$ for all $i \leq n - r - 1$, $((\tau^{n-r}\lambda)')_{n-r} = (-\gamma)_{n-r} + 1$, and $\lambda_{-\gamma} + 1 = (-\gamma')_{-\gamma} + 2$, in which case the corresponding $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = \mathbb{L}_{(\tau_{n-r}\lambda_1, \cdots, \tau_{n-r}\lambda_{-\gamma}, \tau_{n-r}\lambda_{-\gamma}, \lambda_{-\gamma}, \lambda_{-\gamma}, 1, 0, \cdots, 0)}E$. Or $\gamma = (0, -1, \cdots, -1)$, $\lambda_2 = 0$ and $\lambda_1 \geq n - r + 1$, in which case the corresponding $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = \mathbb{L}_{(\lambda_1, n+r, 1, 0, \cdots, 0)}E$.

- $s = 0$: $\gamma = (1, 1, 0, \cdots, 0)$ in which case, the corresponding $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = \wedge^2 E$. Or $\gamma = (2, 0, \cdots, 0)$ in which case, the corresponding $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) = \mathbb{L}_2 E$.

The following lemma and remark give an algorithm to calculate the higher Ext’s. We will fix $\lambda$ with $|\lambda| = s|\delta|$, and let $t = k - s - |\gamma|$. 

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Lemma 2.3.16. 1. For a fixed partition $\lambda$ with $l(\lambda) \leq r$, there is a unique dominant $GL_{n-r}$-weight $\gamma$ with $\gamma_{n-r} \geq -r$ such that $|\gamma|$ is minimal and $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma \mathcal{R}^*) \neq 0$ for some $k$.

2. Let the minimal $\gamma$ in (1) be $\gamma_{\min \lambda}$ and the corresponding $t$ be $t_{\min \lambda}$. Every other $\gamma$ with $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma \mathcal{R}^*) \neq 0$ for some $k$ has $\gamma_k \geq (\gamma_{\min \lambda})_k$ and the corresponding $t$ strictly greater than $t_{\min \lambda}$.

The proof of this lemma shows how to find this minimal $\gamma$, the corresponding $k$ and $t$.

Proof. We look at $\lambda + (r, r-1, \cdots, 1)$. Let $i_0 = r$, for $j > 0$ let $i_j = \lambda_r + r - j - \lambda_{r-j}$, i.e., $\lambda_r + j + 1 = \lambda_r + 1 + (r - i_j)$. Suppose the largest $j$ with positive $i_j$ is $p$ with the corresponding $i_p = q$. We construct the minimal $\gamma$ as follows.

Start with $k = r - 1$ and $l = n - r$. If $k = i_j$ for some $j = 1, \cdots, p$, we do nothing for $l$ and decrease $k$ by 1. Otherwise we have $k \neq i_j$ for any $j = 1, \cdots, p$, in which case we set $\gamma_l = \lambda_r + (1 + r - k) - (n - l + 1)$ and then decrease both $k$ and $l$ by 1. Repeat this process. Stop if we reach $l = 0$ or $k = 0$.

If the above process stopped with $k = 0$ and $l \neq 0$, we reset $k = p - 1$ and maintain the same $l$. If $\lambda_k + (1 + r - k) = \gamma_{l-1} + (n - l + 1) + 1$, we keep the same $l$ and decrease $k$ by 1. Otherwise we have $\lambda_k + (1 + r - k) > \gamma_{l-1} + (n - l + 1) + 1$, in which case we set $\gamma_l = \lambda_k + (1 + r - k) - (n - l + 1) + 1$ and then decrease both $k$ and $l$ by 1. Repeat this process. Stop if we reach $l = 0$ or $k = 0$.

Again if the above process stopped with $k = 0$ and $l \neq 0$, we set $\gamma_l = \gamma_{l+1}$ and decrease $l$ by 1. Repeat this process. Stop if we reach $l = 0$.

The number $k$ is the length of the permutation making $(\gamma, \lambda)$ dominant.

The second part is clear from the construction. \qed

The following remark gives a recursive method to find out all the other $\gamma$'s with some non-vanishing cohomology.
Remark 2.3.17. Let the minimal $\gamma$ in (1) be $\gamma_{\text{min} \lambda}$ and the corresponding $t$ be $t_{\text{min} \lambda}$. Every other $\gamma$ with $H^k(\text{Grass}, \mathbb{L}_\lambda Q^* \otimes \mathbb{L}_\gamma R^*) \neq 0$ for some $k$ can be obtained by a sequence of the following operations $\gamma \mapsto \hat{\gamma}$.

The operation depends on the parameter $s = 1, \ldots, n - r$. Set $\hat{\gamma}_t = (\gamma_{\text{min} \lambda})_t$ for all $t > s$.

Start with $l = s$. We find the largest $j$ with $\max\{\hat{\gamma}_{l+1} + (n-l) + 1, (\gamma_{\text{min} \lambda})_l + (n-l+1) + 1\} < \lambda_j + (r - j + 1)$ and set $\hat{\gamma}_l = \max\{\lambda_{j+1} + (r - j) - (n-l+1) + 1, \hat{\gamma}_{l+1} + (n-l) + 1, (\gamma_{\text{min} \lambda})_l + (n-l+1) + 1\}$. Then we decrease $l$ by 1, and repeat this process. Stop if we reach $l = 0$.

Note that in the calculation of Ext's we only care about those $\gamma$'s with $\gamma_1 \leq r$. We always get all the possible $\gamma$'s with $\gamma_1 \leq r$ after finitely many operations above.

2.4 Equivariant quivers

Here we introduce the basic notions on equivariant quivers and representations of them. They will provide convenient language for the description of non-commutative desingularizations, especially if we would like to consider the equivariant derived categories. Also, as we will see in Subsection 2.4.2, the derived category of coherent sheaves over homogeneous spaces are easier to describe in this way.

2.4.1 The notion of equivariant quivers

Let $G$ be a reductive group, which will be $GL_n$ later on.

Definition 2.4.1. An equivariant quiver is a triple $Q = (Q_0, Q_1, \alpha)$ where $(Q_0, Q_1)$ is a quiver and $\alpha$ is an assignment associating each arrow $q \in Q_1$ a finite dimensional irreducible representation of $G$.

Definition 2.4.2. Let $Q$ be an equivariant quiver. The path algebra $kQ$ of $Q$ is the $k$-algebra
whose underlying vector space is

\[ \bigoplus_{(q_1, \ldots, q_l) \text{ is a path}} \alpha((q_1, \ldots, q_l)), \]

where \( \alpha((q_1, \ldots, q_l)) := \bigotimes_{i=1}^{l} \alpha(q_i) \). We define the product \( kQ \otimes kQ \rightarrow kQ \) to be

\[ \alpha((q_1, \ldots, q_l)) \otimes \alpha((p_1, \ldots, p_h)) \rightarrow \alpha((r_1, \ldots, r_s)) \]

is given by

\[
\begin{cases}
\text{id}, & \text{if the path } (r_1 \ldots, r_s) = (q_1, \ldots, q_l, p_1, \ldots, p_h) \\
0, & \text{otherwise}
\end{cases}
\]

Obviously, \( kQ \) has an action by \( G \) and the multiplication is \( G \)-equivariant.

We will say \( Q \) is finite if both \( Q_0 \) and \( Q_1 \) are finite sets. We will concentrate on connected finite quivers.

Let \( Q \) be a finite quiver, and \( I \) be a two sided ideal of \( kQ \). We say \((Q, I)\) is an equivariant quiver with relations if \( I \) is generated by sub-representations of \( \alpha(q_1, \ldots, q_r) \) for paths \((q_1, \ldots, q_r)\). We usually specify a set of such sub-representations as generators of the ideal and call this set relations. We say \( I \) is admissible if it is generated by sub-representations of \( \alpha(q_1, \ldots, q_r) \) for paths of length 2 or longer and \( I \) contains some power of the arrow ideal.

In this case, the pair \((Q, I)\) will be called a bound equivariant quiver, and \( kQ/I \) the bound path algebra.

We would like to define two notions of representations of a bound equivariant quiver, depending on whether or not we will acknowledge \( G \)-action.

The first notion will be called simply a representation which is just a representation of the quiver obtained by replacing each arrow \( q \in Q_1 \) by as many arrows as the dimension of \( \alpha(q) \).

**Definition 2.4.3.** A representation of \((Q_0, Q_1, \alpha)\) is an assignment associating to each vertex \( a \in Q_0 \) a vector space \( V_a \), and to each arrow \( q : a \rightarrow b \) a \( k \)-linear morphism \( V_a \otimes \alpha(q) \rightarrow V_b \) (or equivalently \( \dim(\alpha(q)) \) linear maps \( V_a \rightarrow V_b \) according to a fixed basis of \( \alpha(q) \)).
We can as well define the notion of equivariant representations of an equivariant quiver.

**Definition 2.4.4.** Let \(Q = (Q_0, Q_1)\) be an equivariant quiver. An *equivariant representation* of \(Q\) is an assignment associating to each vertex \(a \in Q_0\) a representation \(V_a\) of \(G\), and to each arrow \(q : a \to b\) a \(G\)-morphism \(V_a \otimes \alpha(q) \to V_b\).

Let \((Q, I)\) be an equivariant quiver with relations. For a bound representation (resp. bound equivariant representation) we require that the generators of \(I\), which we can chose to be sub-representations by definition, act trivially, i.e., all the morphisms above involving subspaces of \(I\) are trivial maps.

If \(A\) is a commutative \(k\)-algebra with a rational \(G\)-action, we can also talk about bound representations (resp. bound equivariant representation) of \((Q, I)\) over \(A\). By this we mean associating to each vertex a projective \(A\)-module and all the maps have to be \(A\)-linear. For equivariant representations we require that the projective modules have a rational \(G\)-action compatible with the \(A\)-module structure and that all maps corresponding to arrows are \(G\)-equivariant.

**Proposition 2.4.5.** The category of bound representations (resp. bound equivariant representations) of \((Q, I)\) (over \(k\)) is equivalent to the category of modules (resp. \(G\)-equivariant modules) over the ring \(kQ/I\).

By an equivariant module over \(kQ/I\), we mean an \(G\)-equivariant \(kQ/I\) action \(kQ/I \otimes M \to M\).

### 2.4.2 Beilinson and Kapranov quivers

According to a result of Beilinson, we have a full exceptional collection

\[
\nabla(\mathbb{P}^{n-1}) := \{ \Omega^{i-1}(i) \mid i \in [0, n - 1]\}
\]
in the derived category of quasi-coherent sheaves over \( \mathbb{P}^{n-1} \). Thus, the endomorphism ring of \( \oplus_{i=0}^{n-1} \Omega^{i-1}(i) \) is derived equivalent to \( \mathbb{P}^{n-1} \), where \( \Omega^k \) is the \( k \)-th exterior power of the sheaf of Kähler differentials.

Let \( E \) be a vector space of dimension \( n \) and we take \( G = GL_n \) acting naturally on \( E \). The Beilinson equivariant quiver, which will be denoted by \( QB(n) \), is defined as follows:

\[
\begin{array}{cccc}
\bullet_0 & \overset{\alpha_0(E)}{\longrightarrow} & \bullet_1 & \overset{\alpha_1(E)}{\longrightarrow} & \cdots & \overset{\alpha_{n-1}(E)}{\longrightarrow} & \bullet_{n-1}
\end{array}
\]

with relations:

\[ \alpha_i \alpha_{i+1}(\wedge^2 E). \]

**Remark 2.4.6.** Let’s pick up a basis for \( E \), say, \( e_1, \cdots, e_n \). Then, the above quiver, with equivariant structure forgotten, has \( n \) arrows going from the \( i \)-th vertex to the \( i+1 \)-th, denoted by \( \alpha_i^1, \cdots, \alpha_i^n \), corresponding to the basis elements of \( E \). The relations \( \alpha_i \alpha_{i+1}(\wedge^2 E) \) can be written as \( \alpha_i^j \alpha_{i+1}^k - \alpha_{i+1}^k \alpha_i^j \) for all \( j, k \).

The bound path algebra of this quiver is isomorphic to \( \text{End}_{\mathcal{O}_{\mathbb{P}^{n-1}}} (\oplus_{i=0}^{n-1} \Omega^{i-1}(i)) \), as can be checked by \( \text{Hom}_{\mathcal{O}(E)} (\Omega^{a-1}(a), \Omega^{b-1}(b)) \cong \wedge^{a-b}(E^*) \) if \( a \leq b \) and 0 otherwise.

**Remark 2.4.7.** It is not hard to see and is proved in [19] that the category of representations of the Beilinson quiver is isomorphic to the category of graded modules over the exterior algebra. Also the same is true for equivariant representations.

Now let \( k = \mathbb{C} \). We would like to do the same thing for the Grassmannian \( \text{Grass}_r(n) \) with \( n-r > 1 \) and call the corresponding equivariant quiver the Kapranov quiver \( QK(r, n) \). The exceptional collection we take will be

\[ \nabla(\text{Grass}_r(n)) := \{ L_\alpha \mathcal{R}^*, \alpha \in B_{r,n-r} \}. \]

Let \( G = GL_n(\mathbb{C}) \) and \( E = \mathbb{C}^n \) with the natural \( G \) action.
The Kapranov quiver has the set of vertices corresponding to the set of subpartitions of \(((n - r)^r)\), and two vertices \(\lambda_1\) and \(\lambda_2\) are linked by an arrow \(\lambda_1 \to \lambda_2\) iff \(\lambda_2/\lambda_1\) is a single box, and in this case this arrow will be associated to the \(G\)-representation \(E\).

For any three partitions \(\alpha, \beta, \mu\), their Littlewood-Richardson coefficient will be denoted by \(C^\mu_{\alpha, \beta}\), i.e., the number of Littlewood-Richardson tableaux of shape \(\mu/\alpha\) and of weight \(\beta\) is \(C^\mu_{\alpha, \beta}\). With this notation, the relations in Kapranov quiver are generated by the sub-representations

\[
C^{\beta_1}_{\alpha_1, (2,0,\ldots,0)} \boxdot 2E^* \oplus C^{\beta_2}_{\alpha_2, (1,1,0,\ldots,0)} \wedge 2E
\]

of the arrows in \(\text{Hom}(\beta, \alpha)\).

The bound path algebra of \(QK(r, n)\) is isomorphic to \(\text{End}_{\text{Grass}}(\oplus_{\alpha \in B_{r,n-r}} \mathbb{L}_\alpha \mathcal{R}^*)\).

**Example 2.4.8.** We compute the Kapranov equivariant quiver of the Grassmannian Grass\(_2(4)\).

![Diagram](image)

with relations:

- \(\text{Hom}(\begin{array}{c|c}
\cdot & \\
\hline
\cdot & \end{array}, \emptyset)\): \(\alpha_1 \alpha_3 (\wedge^2 E)\);

- \(\text{Hom}(\begin{array}{c|c|c}
\cdot & & \\
\hline
\cdot & & \\
\cdot & & \\
\hline
\cdot & & \\
\end{array}, \emptyset)\): \(\alpha_1 \alpha_2 (\boxdot 2E)\);

- \(\text{Hom}(\begin{array}{c|c|c|c|c}
\cdot & & & & \\
\hline
\cdot & & & & \\
\cdot & & & & \\
\hline
\cdot & & & & \\
\end{array}, \begin{array}{|c|c|c|c|c}
\cdot & & & & \\
\hline
\cdot & & & & \\
\cdot & & & & \\
\hline
\cdot & & & & \\
\end{array})\): \(\alpha_5 \alpha_6 (\wedge^2 E)\);
Proposition 2.4.9. The functor \( \Phi := R\text{Hom}_{\text{Grass}(n)}(\bigoplus_{\alpha \in B_{r,n}} \mathbb{L}_{\alpha} \mathcal{R}^{}, -, -) \) induces an equivalence of triangulated categories

\[
D^b_G(\text{Coh}(X)) \cong D^b_G(\text{CQK}(r, n)/I\text{-mod}),
\]

with quasi-inverse given by \( \Psi := - \otimes L (\bigoplus_{\alpha \in B_{r,n}} \mathbb{L}_{\alpha} \mathcal{R}^{}). \)

Proof. First note that both functors are well-defined on this level. The Hom space between any two equivariant sheaves is naturally a representation of \( G \) and the \( G \)-action is compatible with multiplication by elements in \( \text{End}_{\text{Grass}}(\bigoplus_{\alpha \in B_{r,n}} \mathbb{L}_{\alpha} \mathcal{R}^{}) \cong \text{CQK}(r, n)/I \), and similar for \( \Psi \). Also both functors commute with the forgetful functors forgetting the \( G \)-action. Without the \( G \)-equivariance, this result has been discussed. The compatibility of \( G \)-actions can be checked directly.

\[\square\]

2.5 Determinantal varieties of symmetric matrices

In this section we study a non-commutative desingularization of determinantal varieties in the space of symmetric matrices.

2.5.1 Review of the commutative desingularization

Now we describe a desingularization of determinantal varieties in the space of symmetric matrices.
Let $E$ be a vector space over $k$ of dimension $n$ and $H^s$ be the subspace of $\text{Hom}_k(E,E^*)$ consisting of symmetric morphisms. This space can be identified with $\text{Sym}_2(E^*)$. Upon choosing a set of basis, $H^s$ can be identified with the set of symmetric $(n \times n)$-matrices $(x_{ij})$ with $x_{ij} = x_{ji}$, the coordinate ring of which can be identified with $S^s = k[x_{ij}, i \leq j]$. Similar to the case of determinantal varieties, we get a universal morphism $\varphi : E \to E^*$ over $H^s$. For $r \leq n$, let $\text{Spec } R^s \subseteq \text{Spec } S^s$ be the locus where $\varphi$ has rank $\leq r$. In other words, $R^s$ is the quotient of $S^s$ by the ideal generated by the $(r + 1) \times (r + 1)$ minors of $(x_{ij})$.

Let 

$$0 \to \mathcal{R} \to E \times \text{Grass} \to \mathcal{Q} \to 0$$

be the tautological sequence over Grass, where Grass is the Grassmannian of $n - r$ planes in $E$.

With a little abuse of notation, we will add an upper script $s$ for the varieties and keep the same notations for the maps. More precisely, let $\mathcal{Y}^s$ be the product $\text{Grass} \times H^s$, $p$ and $q$ be the projection to Grass and $H^s$ respectively. Inside of $\mathcal{Y}^s$, there is an incidence variety, denoted by $\mathcal{Z}^s$, defined as $\mathcal{Z}^s = \{(g, h) \in \text{Grass} \times H^s : \text{im } h \in g\}$. The inclusion $\mathcal{Z}^s \hookrightarrow \mathcal{Y}^s$ is denoted by $j$, and $\text{Spec } R^s \to H^s$ by $i$. The induced map $\mathcal{Z}^s \to \text{Spec } R^s$ from $q : \mathcal{Y}^s \to H^s$ is denoted by $q'$ and $\mathcal{Z}^s \to \text{Grass}$ from $p : \mathcal{Y}^s \to \text{Grass}$ by $p'$.

These notations are summarized by the following diagram.

![Diagram](https://example.com/diagram.png)

**Proposition 2.5.1 (6.3.2 in [80])**. The variety $\mathcal{Z}^s$ is a desingularization of $\text{Spec } R^s$.

The variety $\mathcal{Z}^s$ can be described as the total space of the vector bundle $\text{Sym}_2(\mathcal{Q}^*)$ over Grass. Equivalently, $p' : \mathcal{Z}^s \to \text{Grass}$ is an affine morphism with $p_* \mathcal{O}_{\mathcal{Z}^s}$ equal to the sheaf of algebra $\text{Sym}(\text{Sym}_2 \mathcal{Q})$. 67
2.5.2 A tilting bundle over the desingularization

We consider the inverse image of \( T_{\text{il}_0} = \oplus_{\alpha \in B_{r,n-r}} \wedge^{\alpha'} Q \), the BLV's tilting bundle over \( \text{Grass}_{n-r}(E) \), by \( p' : Z^s \to \text{Grass} \). We can show the following.

**Lemma 2.5.2.** For all \( i > 0 \), and any \( \alpha, \beta \in B_{n-r,r} \), we have

\[
H^i(Z^s, p'^* \mathcal{H}om_{\text{Grass}}(\wedge^\alpha Q, \wedge^\beta Q)) = 0.
\]

**Proof.** We have \( p'^* \mathcal{H}om_{\text{Grass}}(\wedge^\alpha Q, \wedge^\beta Q) \cong \mathcal{H}om_{\text{Grass}}(\wedge^\alpha Q, \wedge^\beta Q \otimes_{\text{Grass}} \text{Sym}(\text{Sym}_2 Q)) \). Use Theorem 2.3.3, we know that the sheaf \( \wedge^\beta Q \otimes_{\text{Grass}} \text{Sym}(\text{Sym}_2 Q) \) has a filtration with subquotients of the form \( L_{\gamma} Q \). By Proposition 2.3.10, all the higher cohomology of

\[
\mathcal{H}om_{\text{Grass}}(L_{\alpha} Q^*, L_{\beta} Q^* \otimes_{\text{Grass}} \text{Sym}(\text{Sym}_2 Q))
\]

vanishes. \( \square \)

From the lemma above, using Theorem 2.1.7, we get the following.

**Proposition 2.5.3.** Let the BLV’s tilting bundle over \( \text{Grass}_{n-r}(E) \) be denoted by \( \mathcal{T}_{\text{il}_0} \). Let the rank \( r \) determinantal variety \( \text{Spec } R^s \) of symmetric matrices and its desingularization \( Z^s \) be defined as above. The bundle \( p'^* \mathcal{T}_{\text{il}_0} \) is a tilting bundle over \( Z^s \).

By Theorem 7.6 in [45], since \( Z^s \) is smooth, \( \text{End}_{Z^s}(p'^* \mathcal{T}_{\text{il}_K}) \) has finite global dimension.

**Lemma 2.5.4.** Notations as above, we have the following.

1. If \( r = n - 1 \), \( \text{End}_{Z^s}(p'^* \mathcal{T}_{\text{il}_0}) \) is maximal Cohen-Macaulay over \( \text{Spec } R^s \).

2. If \( r < n - 1 \), \( \text{End}_{Z^s}(p'^* \mathcal{T}_{\text{il}_0}) \) is never maximal Cohen-Macaulay over \( \text{Spec } R^s \).

**Proof.** According to Lemma 2.1.9, it suffices to compute

\[
H^i(\text{Grass}, \wedge^\alpha Q^* \otimes \wedge^\beta Q \otimes \omega_{Z^s} \otimes \text{Sym}(\text{Sym}_2 Q))
\]

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for \(i > 1\) and \(\alpha, \beta \in B_{r,n-r}\). The sheaf \(\omega_{Z_s}\) has been computed in 6.7 of [80], which says

\[
\omega_{Z_s} \cong (\wedge^n E^*)^\otimes \cdots \otimes (\wedge^r Q^*)^\otimes \cdots ^{r-1},
\]

hence,

\[
\wedge^\alpha Q^* \otimes \wedge^\beta Q \otimes \omega_{Z_s} \otimes \text{Sym} \left( \text{Sym}^2 Q \right) \cong \wedge^\alpha Q^* \otimes \wedge^\beta Q \otimes (\wedge^r Q^*)^\otimes \cdots \otimes \text{Sym} \left( \text{Sym}^2 Q \right) (\wedge^n E^*)^\otimes \cdots ^{r-1}.
\]

Using Theorem 2.3.3, as well as Proposition 2.3.10, we see that when \(r = n - 1\) we have

\[
H^i(\text{Grass}, \wedge^\alpha Q^* \otimes \wedge^\beta Q \otimes \omega_{Z_s} \otimes \text{Sym} \left( \text{Sym}^2 Q \right)) = 0
\]

for all \(i > 0\).

Now let \(r < n - 1\). We want to show the non-vanishing of

\[
H^i(\text{Grass}, \wedge^\alpha Q^* \otimes \wedge^\beta Q \otimes \omega_{Z_s} \otimes \text{Sym} \left( \text{Sym}^2 Q \right))
\]

for some positive \(i\). By upper-semi-continuity, it suffices to show this non-vanishing in characteristic zero case. For \(k = \mathbb{C}\), taking \(\beta = 0\) and \(\alpha\) with \(\alpha_1' = 2\), the Bott Theorem tells us

\[
\wedge^\alpha Q^* \otimes (\wedge^r Q^*)^\otimes \cdots \otimes \text{Sym} \left( \text{Sym}^2 Q \right)
\]

does have non-vanishing higher cohomology. Therefore, in this case \(\text{End}_{Z_s}(p^* T_{l_0})\) is never maximal Cohen-Macaulay.

Using the same computation as in Lemma 2.1.9 and Lemma 2.5.4, we get

**Proposition 2.5.5.** The \(R^*\)-module \(q'_* p^* T_{l_0}\) is maximal Cohen-Macaulay. In particular, it is reflexive.

According to Proposition 2.1.7 and Lemma 2.1.8, Lemma 2.5.4 and Lemma 2.5.5 imply the following Proposition.

**Proposition 2.5.6.** Notations as above, we have the following.
1. The natural map
\[ \text{End}_{Z^s}(p^*\mathcal{T}il_0) \rightarrow \text{End}_{S}(q'_*p^*\mathcal{T}il_0) \]
is an isomorphism of rings.

2. If \( r = n-1 \), \( \text{End}_{Z^s}(p^*\mathcal{T}il_0) \) is a non-commutative crepant desingularization of \( \text{Spec } R^s \).

If \( r < n-1 \), \( \text{End}_{Z^s}(p^*\mathcal{T}il_0) \) is a non-commutative desingularization of \( \text{Spec } R^s \) but never maximal Cohen-Macaulay.

Before we discuss about the properties and the quiver with relations for this non-commutative desingularization, let us remark about the equivariant projectives and simples over this non-commutative desingularization.

Remark 2.5.7. When \( k = \mathbb{C} \), the tilting bundle over \( Z^s \) can be taken as the inverse image of an exceptional collection \( \Delta(\text{Grass}) = \{ L_{\lambda}Q^* \mid \lambda \in B_{r,n-r} \} \) over Grass. The dual collection is given by \( \nabla(\text{Grass}) = L_{(n-r)r}Q^* \otimes L_{\alpha^*R}|(n-r)r-|\alpha|] \).

Remark 2.5.8. When \( k = \mathbb{C} \), although \( p'^*(\mathbb{L}_{\alpha}Q^*) \) tend to have the same global sections for different \( \alpha \)'s, they differ as graded objects. This difference is essential. If there were \( \alpha \neq \beta \) with \( q'_*p'^*(\mathbb{L}_{\alpha}Q^*) \cong q'_*p'^*(\mathbb{L}_{\beta}Q^*) \), we would have
\[ R\text{Hom}_{Z^s}(p'^*\mathcal{T}il_K, p'^*(\mathbb{L}_{\alpha}Q^*)) \cong R\text{Hom}_{Z^s}(p'^*\mathcal{T}il_K, p'^*(\mathbb{L}_{\beta}Q^*)) \],
i.e., they would correspond to the same object in \( D^b(\text{End}_{Z^s}(p'^*\mathcal{T}il_K)) \). But as we have seen in Lemma 2.2.6, there is an object with all Ext’s to \( p'^*(\mathbb{L}_{\alpha}Q^*) \) vanishes but having non-trivial Ext to \( p'^*(\mathbb{L}_{\beta}Q^*) \). This is a contradiction.

More explicitly, we describe here the equivariant simple modules, considered as modules over \( \text{End}_{\mathcal{O}_Z^s}(p^*(\bigoplus_{\alpha \in B_{r,n-r}} \nabla_{\alpha})) \) and as representations.

Remark 2.5.9. As can be seen, in the set-up of this section, the conditions in Proposition 2.2.13 and Remark 2.2.14 are satisfied. Thus, \( S_{\beta} = R\text{Hom}_{\mathcal{O}_Z^s}(p^*(\bigoplus_{\alpha \in B_{r,n-r}} \Phi_{\alpha}), u^*_\Sigma_{\beta}) \).
are all the equivariant simple objects over \( \text{End}_{O_Z}(p^*(\oplus_{\alpha \in B_{r,n-r}} \Phi_{\alpha})) \). As has been remarked in 2.2.14, the finite dimensional algebra \( \text{End}_{O_{\text{Grass}}}(\oplus_{\alpha \in B_{r,n-r}} \Phi_{\alpha}) \) is a subring of \( \text{End}_{O_Z}(p^*(\oplus_{\alpha \in B_{r,n-r}} \Phi_{\alpha})) \). The modules \( S_\alpha \) are also simple modules over \( \mathbb{C}QK(n,n-r)/I \).

In fact, when \( k = \mathbb{C} \), as \( \text{End}_{O_{\text{Grass}}}(\oplus_{\alpha \in B_{r,n-r}} \Phi_{\alpha}) \cong \mathbb{C}QK(n,n-r)/I \) is a finite dimensional basic algebra, \( S_\alpha \) is the simple representation of the underlying quiver \( QK(n,n-r) \) corresponding to the vertex \( \alpha \). In this case, it is evident that \( S_\alpha \)'s are distinct as modules over \( \text{End}_{O_Z}(p^*(\oplus_{\alpha} \nabla_{\alpha})) \).

Let us look at the structure of \( S_\beta \)'s as \( GL_n \)-representations. By definition,

\[
S_\beta = R \text{Hom}_{O_Z}(p^*(\oplus_{\alpha} \nabla_{\alpha}), u_* \Delta_{\beta})
\]

\[
\cong \text{Hom}_{O_Z}(p^*\nabla_{\beta}, u_* \Delta_{\beta})
\]

\[
\cong \text{Hom}_{\text{Grass}}(\nabla_{\beta}, \Delta_{\beta})
\]

\[
\cong H^{(n-r)r-|\alpha|}(\text{Grass}, L_{\beta} \mathbb{Q} \otimes L_{(n-r)} \mathbb{Q}^* \otimes L_{\alpha,\epsilon} \mathcal{R})
\]

\[
\cong \mathbb{C}.
\]

All of them are 1-dimensional trivial representations.

Similar to Proposition 2.4.9, we have the following equivalence of equivariant derived categories.

**Proposition 2.5.10.** The functor \( R \text{Hom}_{O_Z}(p^* T_{il_{0}}, -) \) induces an equivalence between \( D^b_G(\text{Coh}(Z^*)) \) and \( D^b_G(\text{End}_{O_Z}(p^* T_{il_{0}})\text{-mod}) \).

The following Proposition is a direct consequence of Lemma 2.3.11.

**Proposition 2.5.11.** Let \( k = \mathbb{C} \) and let \( S_\alpha \) be as in Proposition 2.2.13. Then, the Ext's among them are given by

\[
\text{Ext}^t(S_\alpha, S_\beta) \cong \oplus_s \oplus_{\lambda \in Q_{-1}(2s)} H^{t-s-|\beta|+|\alpha|}(\text{Grass}, L_{\lambda} \mathbb{Q}^* \otimes L_{\beta} \mathcal{R}^* \otimes L_{\alpha,\epsilon} \mathcal{R}),
\]

where \( Q_{-1}(2s) = \{ \lambda \vdash 2s | \lambda = (a_1, \ldots, a_r | a_1 - 1, \ldots, a_r - 1) \} \) in the hook notation.
More explicitly, we have the following formula.

**Proposition 2.5.12.** Under the assumptions of Proposition 2.5.11. We have

\[
\Ext^1(S_\alpha, S_\beta) \cong \begin{cases} 
(C^\alpha_{\beta,(1,0,\ldots,0)} E^*) \oplus (C^\beta_{\alpha,(1,0,\ldots,0)} E^*), & \text{if } n - r = 1 \\
(E^*) \oplus (C^\alpha_{\beta,(1,0,\ldots,0)} C), & \text{if } n - r \geq 2.
\end{cases}
\]

\[
\Ext^2(S_\alpha, S_\beta) \cong \begin{cases} 
(C^\alpha_{\beta,(1,1,0,\ldots,0)} \Sym_2 E^*) \oplus (C^\beta_{\alpha,(1,1,0,\ldots,0)} \Sym_2 E^*) \oplus (\delta^\beta_\alpha \wedge^2 E^*), & \text{if } n - r = 1; \\
(C^\beta_{\alpha,(1,1,0,\ldots,0)} \wedge^2 E^*) \oplus (C^\beta_{\alpha,(1,1,0,\ldots,0)} \Sym_2 E^*) \oplus (C^\beta_{\alpha,(2,0,\ldots,0)} \wedge^2 E^*), & \text{if } n - r = 2; \\
(C^\beta_{\alpha,(0,\ldots,0,-1,1,0,\ldots,0)} C) \oplus (C^\beta_{\alpha,(1,0,\ldots,0,-1,1,0,\ldots,0)} E^*) \oplus (C^\beta_{\alpha,(2,0,\ldots,0)} \Sym_2 E^*) \oplus (C^\beta_{\alpha,(1,1,0,\ldots,0)} \wedge^2 E^*), & \text{if } n - r \geq 2.
\end{cases}
\]

**Proof.** Note that the only element in \( Q_{-1}(2) \) is \((2,0,\ldots,0)\) and the only element in \( Q_{-1}(4) \) is \((3,1,0,\ldots,0)\). One can easily calculate \( \Ext^1 \) and \( \Ext^2 \) with the aid of Lemma 2.3.16.

For \( \lambda = (3,1,0,\ldots,0) \), the \( \gamma_{\min(3,1,0,\ldots,0)} \) is given by Lemma 2.3.16 is \((-2)\) if \( n - r = 1 \) with the corresponding \( t_{\min(3,1,0,\ldots,0)} = 2; \gamma_{\min(3,1,0,\ldots,0)} = (-1,-2)\) if \( n - r = 2 \) with the corresponding \( t_{\min(3,1,0,\ldots,0)} = 2; \gamma_{\min(3,1,0,\ldots,0)} = (-1,-1,-2)\) if \( n - r \geq 3 \) with the corresponding \( t_{\min(3,1,0,\ldots,0)} = 2 \). Note that in any case above there is no operation described in Remark 2.3.17 satisfying the constrains given by Corollary 2.3.13.

For \( \lambda = (2,0,\ldots,0) \), the \( \gamma_{\min(2,0,\ldots,0)} \) is given by Lemma 2.3.16 is \((-1)\) if \( n - r = 1 \) with the corresponding \( t_{\min(2,0,\ldots,0)} = 1; \gamma_{\min(2,0,\ldots,0)} = (0,\ldots,0,-1,-1)\) if \( n - r \geq 2 \) with the corresponding \( t_{\min(2,0,\ldots,0)} = 1 \).

Note that in the case \( n - r = 1 \) there is one operation described in Remark 2.3.17 satisfying the constrains given by Corollary 2.3.13, which gives \( \gamma = (0) \) with \( t = 2 \). In the case \( n - r = 2 \) there is one operation described in Remark 2.3.17 satisfying the constrains given by Corollary 2.3.13 but can be applied successively, which gives \( \gamma = (1,-1) \) and \( \gamma = (2,-1) \)
with the corresponding $t = 2$ and $3$ respectively. In the case $n - r > 2$ there are a lot of operations. But if we only care about those with corresponding $t = 2$ and satisfying the constrains given by Corollary 2.3.13, there is only one which gives $\gamma = (1, 0, \cdots, 0, -1, -1)$.

Note that only $Q_{-1}(2s)$ with $s = 0, 1, 2$ contributes to $\text{Ext}^1$ and $\text{Ext}^2$. This finishes the proof.

2.5.3 Maximal minors

In the rest of this section, we illustrate Proposition 2.5.11 and Proposition 2.5.12 with explicit examples and numerical consequences.

We assume $r = n - 1$, $S^s = k[x_{i,j}]_{1 \leq i \leq j \leq n}$ and $R^s = S^s/(\det(x_{ij}))$. In this case, $R^s$ is always Gorenstein, since it is a hypersurface.

The inverse image of the tilting bundle from the Grassmannian is still a tilting bundle by the same argument as before. More explicitly, it is

$$\bigoplus_{i=0}^{n-1} \wedge^i Q^s \otimes \text{Sym}(\text{Sym}_2 Q).$$

Proposition 2.5.13. The endomorphism ring $\text{End}_S(q'_{q^*} \mathcal{T}K)$ is isomorphic to the path algebra of the quiver:

```
0 \alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_n
```

```
\beta_1 \leftarrow \beta_2 \leftarrow \cdots \leftarrow \beta_n
```

with relations:

$$\alpha_i \alpha_j + \alpha_j \alpha_i,$$

$$\beta_i \beta_j + \beta_j \beta_i,$$

$$(\alpha_i \beta_j + \beta_j \alpha_i) - (\alpha_j \beta_i + \beta_i \alpha_j),$$

where for any term not making sense at some vertex, it is to be understood as dropped.
In the language of equivariant quivers, the above quiver can be written as

\[
\begin{array}{ccc}
\bullet_0 & \xleftarrow{\beta_0(E)} & \bullet_1 & \xleftarrow{\beta_1(E)} & \cdots & \xleftarrow{\beta_{n-2}(E)} & \bullet_{n-1} \\
\alpha_0(E) & \xrightarrow{} & \alpha_1(E) & \xrightarrow{} & \alpha_{n-2}(E) & \xrightarrow{} & \cdots
\end{array}
\]

and the relations:

\[
\alpha_i \alpha_{i+1}(\wedge^2 E);
\beta_{i+1} \beta_i(\wedge^2 E);
(\beta_i \alpha_i + \alpha_{i+1} \beta_{i+1})(\Sym_2 E).
\]

**Proof.** See the proof of Proposition 2.5.16 and the proof of Proposition 2.5.17.

A description of the Ext’s between the simples in the module category over the path algebra of quiver with relations is given by Proposition 2.5.11.

**Example 2.5.14.** Let \( k = \mathbb{C} \). For any integer \( a \in [0, n-1] \), the minimal projective resolution of the simple object \( S_a \) is:

\[
\begin{array}{cccc}
P_a & \xleftarrow{E \otimes P_{a-1}} & \oplus & \L_3 E \otimes P_{a-3} \\
& \oplus & \L_2 E \otimes P_{a-2} & \oplus \L_4 E \otimes P_{a-4} \\
& \oplus & \L_2 E \otimes P_{a+2} & \oplus \L_3 E \otimes P_{a+3} \\
& \oplus & \L_22 E \otimes P_{a-2} & \oplus \L_32 E \otimes P_{a+3} \\
& \oplus & \L_221 E \otimes P_{a-1} & \oplus \L_42 E \otimes P_{a+4} \\
& \oplus & \L_2211 E \otimes P_a & \oplus \L_322 E \otimes P_{a+1} \\
& \oplus & \L_2111 E \otimes P_a & \oplus \L_43 E \otimes P_{a+1} \\
& \oplus & \wedge^2 E \otimes P_a & \oplus \L_2211 E \otimes P_a
\end{array}
\]

If \( a \) is close to the boundary (i.e., 0 and n-1), some terms in the above resolution does not exist. In those cases, the terms in question should be understood as dropped.

In this case, the commutative desingularization and the noncommutative one are further related in the sense that \( \mathcal{Z}^s \) is the fine moduli space of certain representations of the noncommutative one.
Example 2.5.15. Take \( n = 2 \) and \( r = 1 \). Then \( R^s \) is the nil-cone of \( \mathfrak{sl}_2 \), and the commutative resolution \( Z^s \) is the Springer resolution \( T^*\mathbb{P}^1 \). The quiver with relations:

\[
\begin{array}{c}
\bullet_0 & \overset{\alpha_0}{\longrightarrow} & \bullet_1 \\
\alpha_1 & \overset{\beta_0}{\longrightarrow} & \beta_1
\end{array}
\]

relations: \( \alpha_0 \beta_1 = \alpha_1 \beta_0; \beta_0 \alpha_1 = \beta_1 \alpha_0. \)

Representations \( W \) of dimension \((1, 1)\) generated by \( W_1 \) are parameterized by \( T^*\mathbb{P}^1 \). The irreducible such representations form the regular locus in \( Z^s \). These simple modules (there are infinitely many of them) do not admit \( GL_2 \)-equivariant structures.

2.5.4 Higher codimension cases

Now we start using Proposition 2.5.12 to describe the quiver with relations for this non-commutative desingularization.

Proposition 2.5.16. Let \( k = \mathbb{C} \). Assume \( n - r \geq 2 \). In the quiver with relations for the algebra \( \Lambda = \text{End}_{O_{Z^s}(p^*\mathfrak{T}il_K)} \) as in Proposition 2.2.13, the vertex set is indexed by \( B_{r,n-r} \), and the arrow from \( \beta \) to \( \alpha \) is given by \( E \) if \( C_{\beta,(1,0,\ldots,0)}^\alpha \neq 0 \) and given by \( C \) if \( C_{\beta,(1,1,0,\ldots,0)}^\alpha \neq 0 \). No arrows otherwise.

Proposition 2.5.17. Let \( k = \mathbb{C} \). In the quiver with relations for the algebra \( \Lambda = \text{End}_{O_{Z^s}(p^*\mathfrak{T}il_K)} \) as in Proposition 2.2.13, the relations are generated by the following sub-representations of the arrows in \( \text{Hom}(\beta, \alpha) \):

\[
(C_{\alpha^t,(1,1,0,\ldots,0)}^\beta \wedge^2 E) \oplus (C_{\alpha^t,(1,0,\ldots,0)}^\beta \wedge^2 E) \oplus (C_{\alpha^t,(1,1,0,\ldots,0)}^\beta \text{Sym}_2 E) \oplus (C_{\alpha^t,(2,0,\ldots,0)}^\beta \wedge^2 E)
\]

in the case \( n - r = 2 \);

\[
(C_{\alpha^t,(0,\ldots,0,-1,1,0,\ldots,0)}^\beta \wedge^2 E) \oplus (C_{\alpha^t,(1,0,\ldots,0,-1,1,0,\ldots,0)}^\beta \wedge^2 E) \oplus (C_{\alpha^t,(2,0,\ldots,0)}^\beta \text{Sym}_2 E) \oplus (C_{\alpha^t,(1,1,0,\ldots,0)}^\beta \wedge^2 E)
\]
in the case \( n - r \geq 3 \).

### 2.5.5 Examples

We study some combinatorial properties of the noncommutative desingularization. *In this subsection we assume \( k = \mathbb{C} \).*

We look at two examples with \( r = n - 1 \), \( S^* = \mathbb{C}[x_{i,j}]_{1 \leq i \leq j \leq n} \) and \( R^* = S^*/(\det) \).

The inverse image of the tilting bundle is

\[
\bigoplus_{i=0}^{n-1} \wedge^i Q^* \otimes \text{Sym}(\text{Sym}_2 Q).
\]

The presentation of \( H^0(Z^*, \wedge^i Q^* \otimes \text{Sym}(\text{Sym}_2 Q)) \) as a \( S \)-module is given by

\[
0 \leftarrow H^0(Z^*, \wedge^i Q^* \otimes \text{Sym}(\text{Sym}_2 Q)) \leftarrow L_{(1^1, 0^{n-1})} E^* \otimes S \leftarrow L_{(2^1, 1^{n-1})} E^* \otimes S.
\]

**Example 2.5.18.** Now we take \( n = 3 \) and \( r = 2 \) for a concrete example. Here \( S^* = \mathbb{C}[x_{i,j}]_{1 \leq i \leq j \leq 3} \) and \( R^* = S^*/(\det) \).

In this example, \( R \) is a normal Gorenstein domain since it is a hypersurface. Consequently, if \( \text{End}_Z(p^* \mathcal{I}l_K) \) is maximal Cohen-Macaulay and \( q_* p^* \mathcal{I}l_K \) is reflexive, then \( \text{End}_Z(p^* \mathcal{I}l_K) \) is a non-commutative crepant desingularization due to Proposition 2.5.6.

The inverse image of the tilting bundle consists of three direct summands: \( \text{Sym}(\text{Sym}_2 Q) \), \( Q^* \otimes \text{Sym}(\text{Sym}_2 Q) \), and \( \wedge^2 Q^* \otimes \text{Sym}(\text{Sym}_2 Q) \). Let us denote their global sections as \( R \)-modules, (and consequently as \( S^* \)-modules), by

\[
M_i := H^0(Z^*, \wedge^i Q^* \otimes \text{Sym}(\text{Sym}_2 Q))
\]

with \( i = 0, 1, 2 \). The presentations of their global sections as \( S^* \)-modules are given by

\[
0 \leftarrow H^0(Z^*, \wedge^i Q^* \otimes \text{Sym}(\text{Sym}_2 Q)) \leftarrow L_{(1^1, 0^{i-1})} E^* \otimes S^* \leftarrow L_{(2^1, 1^{i-1})} E^* \otimes S^*.
\]

The \( \text{Hom} \)'s between them are

\[
\text{Hom}_R(M_i, M_j) = H^0(Z^*, \wedge^i Q^* \otimes \wedge^j Q \otimes \text{Sym}(\text{Sym}_2 Q)).
\]

More explicitly,
• $\text{Hom}_R(M_0, M_i) = M_i$,

• $\text{Hom}_R(M_i, M_0) = M_i$,

• $\text{Hom}_R(M_2, M_i) = M_{2-i}$,

• $\text{Hom}_R(M_i, M_2) = M_{2-i}$.

Through some computations, we get

$$\text{Hom}_R(M_1, M_1) = M_0 \oplus N$$

and the presentation of $N$ is

$$0 \leftarrow N \leftarrow \mathbb{L}(200)E^* \oplus \mathbb{L}(110)E^* \otimes S^s \leftarrow \mathbb{L}(220)E^* \oplus \mathbb{L}(211)E^* \otimes S^s.$$

The endomorphism ring is isomorphic to the path algebra of the following quiver with relations.

The quiver is

```
\begin{tikzpicture}
    
    \node (0) at (0,0) {$\bullet_0$};
    \node (1) at (1,0) {$\bullet_1$};
    \node (2) at (2,0) {$\bullet_2$};

    \draw [->] (0) -- (1); 
    \draw [->] (1) -- (2); 
    \draw [->] (2) -- (0); 

\end{tikzpicture}
```

with relations:

$$\gamma_i \alpha_j + \gamma_j \alpha_i,$$

$$\beta_j \delta_i + \beta_i \delta_j,$$

$$\beta_j \alpha_i - \beta_i \alpha_j,$$

$$\gamma_j \delta_i - \gamma_i \delta_j,$$

$$(\alpha_i \beta_j + \delta_j \gamma_i) - (\alpha_j \beta_i + \delta_i \gamma_j).$$

The Hom between any two direct summands is graded, with grading given by the weight of $\mathbb{G}_m$-action on $E$. The grading defined this way is different but (strictly) finer than the one used in the proof of Proposition 2.2.13.

The Hilbert series of those modules can be computed from the presentations:
Putting the Hilbert series of the Hom's into a matrix, we get

\[
\frac{1}{(1-t^2)^5} \times \begin{pmatrix}
1 + t^2 + t^4 & 3t + 3t^3 & 3t^2 \\
3t + 3t^3 & 1 + 10t^2 + t^4 & 3t + 3t^3 \\
3t^2 & 3t + 3t^3 & 1 + t^2 + t^4
\end{pmatrix}
\]

The coefficients in front of each monomials in the entries of the inverse matrix gives the multiplicity of the projectives in the resolution of the simples. The matrix above has an inverse with polynomial entries, which reflects the fact that the derived category over the endomorphism ring has finite global dimension.

The inverse matrix is

\[
\begin{pmatrix}
-t^6 - 3t^4 + 3t^2 + 1 & 3t^5 - 3t & -6t^4 + 6t^2 \\
3t^5 - 3t & -t^6 - 3t^4 + 3t^2 + 1 & 3t^5 - 3t \\
-6t^4 + 6t^2 & 3t^5 - 3t & -t^6 - 3t^4 + 3t^2 + 1
\end{pmatrix}
\]

It is easy to guess the resolution of the simples from this matrix.

\[S_0: \quad P_0 \leftarrow E \otimes P_1 \leftarrow \bigotimes_{211} E \otimes P_0 \leftarrow \bigotimes_{212} E \otimes P_1 \leftarrow \bigotimes_{222} E \otimes P_0 \leftarrow 0; \]

\[S_1: \quad P_1 \leftarrow \bigotimes_{1} E \otimes P_0 \leftarrow \bigotimes_{211} E \otimes P_1 \leftarrow \bigotimes_{212} E \otimes P_0 \leftarrow \bigotimes_{222} E \otimes P_1 \leftarrow 0.\]

One can easily verify that this resolution coincide with the one given by Lemma 2.2.19 and Proposition 2.5.11.
Example 2.5.19. Let us look at one more example, with $n = 4$ and $r = 3$.

As before, the direct summands of the tilting bundle over the desingularization are $\wedge^i Q^* \otimes \text{Sym}_2(\mathbb{L}_2 Q)$ where $i = 0, \ldots, 3$.

The presentations of their global sections as $S^*$-modules are given by

$$0 \leftarrow M_i := H^0(Z, \wedge^i Q^* \otimes \text{Sym}(\mathbb{L}_2 Q)) \leftarrow \mathbb{L}_{(1,0^a-i)} E^* \otimes S^* \leftarrow \mathbb{L}_{(2,1^a-i)} E^* \otimes S^*.$$

The endomorphism ring is isomorphic to the path algebra of the quiver:

$\begin{array}{cccc}
\bullet_0 & \overset{\alpha_1}{\rightarrow} & \overset{\alpha_2}{\rightarrow} & \overset{\alpha_3}{\rightarrow} \\
\overset{\beta_1}{\leftarrow} & \overset{\beta_2}{\leftarrow} & \overset{\beta_3}{\leftarrow} & \overset{\beta_4}{\leftarrow}
\end{array}$

with relations:

$$\alpha_i \alpha_j + \alpha_j \alpha_i,$$

$$\beta_i \beta_j + \beta_j \beta_i,$$

$$(\alpha_i \beta_j + \beta_j \alpha_i) - (\alpha_j \beta_i + \beta_i \alpha_j).$$

To get the Hilbert polynomials of the Hom’s among them, we only need to compute
the presentations of two modules, i.e., Hom$(M_1, M_1)$ and Hom$(M_1, M_2)$. They are given as follows.

- Hom$(M_1, M_1) = M_0 \oplus C_1$, where

$$0 \leftarrow C_1 \leftarrow \mathbb{L}_{11} E \otimes S^* \oplus \mathbb{L}_2 E \otimes S^* \leftarrow \mathbb{L}_{2211} E \otimes S^* \oplus \mathbb{L}_{222} E \otimes S^*$$

is exact;

- Hom$(M_1, M_2) = M_1 \oplus C_2$, where

$$0 \leftarrow C_2 \leftarrow \mathbb{L}_{111} E \otimes S^* \oplus \mathbb{L}_{21} E \otimes S^* \leftarrow \mathbb{L}_{221} E \otimes S^* \oplus \mathbb{L}_{2111} E \otimes S^*$$

is exact.
The matrix of Hilbert polynomials between their Hom’s is

\[
\frac{1}{(1-t^2)^{10}} \begin{pmatrix}
  t^{18} & 4 - 4t^6 & 6 - 6t^4 & 4 - 4t^2 \\
  4 - 4t^6 & 17 - 16t^4 - t^8 & 28 - 24t^2 - 4t^6 & 6 - 6t^4 \\
  6 - 6t^4 & 28 - 24t^2 - 4t^6 & 17 - 16t^4 - t^8 & 4 - 4t^6 \\
  4 - 4t^2 & 6 - 6t^4 & 4 - 4t^6 & 1 - t^8
\end{pmatrix}.
\]

Its inverse matrix is

\[
\begin{pmatrix}
  15t^8 - 6t^{10} - t^{12} - 15t^4 + 6t^2 + 1 & 20t^5 - 20t^7 + 4t^{11} - 4t \\
  20t^5 - 20t^7 + 4t^{11} - 4t & 15t^8 - 6t^{10} - t^{12} - 15t^4 + 6t^2 + 1 \\
  -20t^5 + 20t^8 - 10t^{10} + 10t^2 & 20t^5 - 20t^7 + 4t^{11} - 4t \\
  -20t^5 + 60t^5 - 60t^7 + 20t^9 & -20t^5 + 20t^8 - 10t^{10} + 10t^2
\end{pmatrix}.
\]

One can guess the resolution of the simples from this matrix.

\[S_0 : \xrightarrow{P_0} E \otimes P_1 \xrightarrow{\wedge^2 E \otimes P_0} \xrightarrow{+} \xrightarrow{\otimes +} \xrightarrow{L_2 E \otimes P_2} \xrightarrow{+} \xrightarrow{\otimes +} \xrightarrow{L_3 E \otimes P_3} \xrightarrow{+} \xrightarrow{L_3 E \otimes P_3} \xrightarrow{+} \xrightarrow{L_2 E \otimes P_2} \xrightarrow{+} \xrightarrow{L_3 E \otimes P_3} \xrightarrow{+} \xrightarrow{L_2 E \otimes P_2} \xrightarrow{+} \xrightarrow{L_3 E \otimes P_3} \xrightarrow{+}\]

\[S_1 : \xrightarrow{P_1} E \otimes P_0 \xrightarrow{\wedge^2 E \otimes P_1} \xrightarrow{+} \xrightarrow{\otimes +} \xrightarrow{L_2 E \otimes P_3} \xrightarrow{+} \xrightarrow{L_2 E \otimes P_3} \xrightarrow{+} \xrightarrow{L_2 E \otimes P_3} \xrightarrow{+} \xrightarrow{L_2 E \otimes P_3} \xrightarrow{+} \xrightarrow{L_2 E \otimes P_3} \xrightarrow{+} \xrightarrow{L_2 E \otimes P_3} \xrightarrow{+} \xrightarrow{L_2 E \otimes P_3} \xrightarrow{+} \xrightarrow{L_2 E \otimes P_3} \xrightarrow{+}\]

Again, it is easy to verify that this resolution coincide with the one given by Lemma 2.2.19 and Proposition 2.5.11.

**Example 2.5.20.** Now we look at an example of higher codimension symmetric minors case.

Let’s take \( n = \dim E = 4 \) and \( r = 2 \).
The first two steps are relatively easy. By direct computation, we get the following quiver (recall that vertices are indexed by Young diagrams):

with relations:

- \( \text{Hom}(\begin{array}{c}
\end{array}, \emptyset): \alpha_1 \alpha_3 (\Lambda^2 E); \)
- \( \text{Hom}(\begin{array}{c}
\end{array}, \emptyset): \alpha_1 \alpha_2 (\Sym_2 E); \)
- \( \text{Hom}(\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}): \alpha_5 \alpha_6 (\Lambda^2 E); \)
- \( \text{Hom}(\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}): \alpha_4 \alpha_6 (\Sym_2 E). \)
- \( \text{Hom}(\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}): \alpha_2 \alpha_4 - \alpha_3 \alpha_5 (\Sym_2 E \oplus \Lambda^2 E); \)
- \( \text{Hom}(\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}): \alpha_3 \beta_1 \alpha_1 - \alpha_3 \alpha_5 \beta_2 (\Lambda^2 E); \)
- \( \text{Hom}(\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}): \alpha_6 \beta_3 \alpha_5 - \beta_2 \alpha_3 \alpha_5 (\Lambda^2 E); \)
Using Olver’s description of Pieri inclusions, (see, e.g., 1.2 of [70],) it is easy to check the above listed subrepresentations acts trivially on $p^* \oplus_{\alpha \in B_{r,n-r}} \mathbb{L}_\alpha \mathcal{Q}^*$. Then Proposition 2.5.17 yields that these are all the relations.

### 2.6 Pfaffian varieties of anti-symmetric matrices

What we will do in this section is parallel to the previous section.

#### 2.6.1 Review of the commutative desingularization

Let $E$ be a vector spaces over $k$ of dimension $n$ and $H^a$ be the subspace of $\text{Hom}_k(E^*, E)$ consisting of skew-symmetric morphisms. This space can be identified with $\wedge^2(E)$. Upon choosing a set of basis, $H^s$ can be identified with the set of skew-symmetric $(n \times n)$-matrices $(x_{ij})$ with $x_{ij} = -x_{ji}$, whose coordinate ring can be identified with $S^a = k[x_{ij}]_{i<j}$.

We get a universal morphism $\varphi : \mathcal{E}^* \rightarrow \mathcal{E}$ over $H^a$. For even $r = 2u$ satisfying $0 \leq r \leq n$, we desingularize the locus $\text{Spec } R^a$, where $\varphi$ has rank $\leq r$. In other words, $R^a$ is the quotient of $S^a$ by the ideal generated by the $(r+1) \times (r+1)$ minors of $(x_{ij})$.

Let

$$0 \rightarrow \mathcal{R} \rightarrow E \times \text{Grass} \rightarrow \mathcal{Q} \rightarrow 0$$

be the tautological sequence over Grass, where Grass is the Grassmannian of $(n-r)$-planes in $E$.

We will add an upper script $a$ for the varieties and keep the same notations for the maps. The incidence variety $\mathcal{Z}^a$ of $\mathcal{Y}^a$ desingularizing $\text{Spec } R^a$, is defined by $\mathcal{Z}^a = \{(g, h) \in$
Grass \times H^a : \text{im} \ h \in g \}$. We use the following diagram to illustrate all the notations.

\[
\begin{array}{ccc}
Z^a & \xrightarrow{j} & \mathcal{Y}^a = \text{Grass} \times H^a \\
\downarrow q' & & \downarrow p \\
\text{Spec } R^a & \xrightarrow{i} & H^a = \wedge^2 (E^*)
\end{array}
\]

**Proposition 2.6.1** (6.4.2 in [80]). *The variety $Z^a$ is a desingularization of Spec $R^a$.***

The variety $Z^a$ can be described as the total space of the vector bundle $\wedge^2 (Q^*)$ over Grass. Equivalently, $p' : Z^a \to \text{Grass}$ is an affine morphism with $p_* \mathcal{O}_{Z^a}$ equal to the sheaf of algebra $\text{Sym}(\wedge^2 Q)$.

### 2.6.2 A tilting bundle over the desingularization

As before, we pull back the BLV’s tilting bundle $\mathcal{I}l_0 = \bigoplus_{\lambda \in B_{r,n-r}} \wedge^\lambda Q$ over $\text{Grass}_{n-r}(E)$ by the projection $p' : Z^a \to \text{Grass}$. Similar to the symmetric case, we can show that for all $i > 0$, and all $\alpha, \beta \in B_{r,n-r}$,

$$H^i(Z^a, p'^* \mathcal{H}om_{\mathcal{O}_{\text{Grass}}} (\wedge^\alpha Q, \wedge^\beta Q)) = 0.$$  

From the claim above, using Lemma2.1.6 and Lemma2.1.5, we get the following proposition.

**Proposition 2.6.2.** *The BLV’s tilting bundle over Grass_{n-r}(F) is denoted by $\mathcal{I}l_0$. The rank r determinantal variety Spec $R^a$ of anti-symmetric matrices and its desingularization $Z^a$ are as above. The bundle $p'^* \mathcal{I}l_0$ is a tilting bundle over $Z^a$.***

The following proposition is proved along the same line.

**Proposition 2.6.3.** *Over Spec $R^a$, the coherent sheaf $q'_* p'^* \mathcal{I}l_0$ is maximal Cohen-Macaulay. But the coherent sheaf $\text{End}_{Z^a}(p'^* \mathcal{I}l_0)$ is never maximal Cohen-Macaulay.*
Proof. According to Lemma 2.1.9, it suffices to compute

\[ H^i(\text{Grass}, \wedge^\alpha Q^* \otimes \wedge^\beta Q \otimes \omega_{Z_a} \otimes \text{Sym}(\wedge^2 Q)) \]

for \( i > 0 \) and \( \alpha, \beta \in B_{n-r,r} \). The sheaf \( \omega_{Z_a} \) has been computed in 6.7 of [80], which says

\[ \omega_{Z_a} \cong (\wedge^n E^*)^{\otimes-n+1+r} \otimes (\wedge^r Q^*)^{\otimes-n-r+1}. \]

Hence,

\[ \wedge^\alpha Q^* \otimes \wedge^\beta Q \otimes \omega_{Z_a} \otimes \text{Sym}(\wedge^2 Q) \cong \wedge^\alpha Q^* \otimes \wedge^\beta Q \otimes (\wedge^n E^*)^{\otimes-n+1+r} \otimes (\wedge^r Q^*)^{\otimes-n-r+1} \otimes \text{Sym}(\wedge^2 Q). \]

If \( \alpha = 0 \), there is no higher cohomology, according to Proposition 2.3.10. This shows \( q'_* p^! \mathcal{T}il_0 \) is maximal Cohen-Macaulay.

In order to show \( \text{End}_{Z_a}(p^* \mathcal{T}il_0) \) is never maximal, we only need to find out the non-vanishing of \( H^i(\text{Grass}, \wedge^\alpha Q^* \otimes \wedge^\beta Q \otimes (\wedge^r Q^*)^{\otimes-n-r+1} \otimes \text{Sym}(\wedge^2 Q)) \) for some positive \( i \). By upper-semi-continuity, it suffices to show this non-vanishing in characteristic zero. For \( k = \mathbb{C} \), taking \( \beta = 0 \) and \( \alpha \neq 0 \), the Bott Theorem tells us

\[ \wedge^\alpha Q^* \otimes (\wedge^r Q^*)^{\otimes-n-r+1} \otimes \text{Sym}(\wedge^2 Q) \]

does have nonvanishing higher cohomology. We are done. \( \square \)

By Theorem 7.6 in [45], since \( Z^a \) is smooth, \( \text{End}_{Z_a}(p^* \mathcal{T}il_0) \) has finite global dimension. Using Proposition 2.1.7, we get the following.

**Proposition 2.6.4.** The map

\[ \text{End}_{Z_a}(p^* \mathcal{T}il_0) \to \text{End}_S(q'_* p^! \mathcal{T}il_0) \]

is an isomorphism of rings. In particular, \( \text{End}_{Z_a}(p^* \mathcal{T}il_0) \) is a non-commutative desingularization but never maximal Cohen-Macaulay.

Again, we have the following equivalence of equivariant derived categories.

**Proposition 2.6.5.** The functor \( R\text{Hom}_{\mathcal{O}_{Z_a}}(p^* \mathcal{T}il_0, -) \) induces an equivalence

\[ D^b_G(\text{Coh}(Z^a)) \cong D^b_G(\text{End}_{\mathcal{O}_{Z_a}}(p^* \mathcal{T}il_0)-\text{mod}). \]
2.6.3 Quiver and relations for the noncommutative desingularization

Now, we describe the Ext’s between the equivariant simples. Then, we use it to get the quiver with relations for the non-commutative desingularization.

**Proposition 2.6.6.** Let $k = \mathbb{C}$. Let $S_\alpha$ be as in Proposition 2.2.13. Then, the Ext’s among them are given by

$$\text{Ext}^i(S_\alpha, S_\beta) \cong \oplus_{s} \oplus_{\lambda \in Q_1(2s)} H^{t-s-|\beta|+|\alpha|}(\text{Grass}, \mathbb{L}_\lambda \mathcal{Q}^* \otimes \mathbb{L}_\beta \mathcal{R}^* \otimes \mathbb{L}_\alpha \mathcal{R}),$$

where $Q_1(2s) = \{ \lambda \vdash 2s \mid \lambda = (a_1, \cdots, a_r | a_1 + 1, \cdots, a_r + 1) \}$ in the hook notation.

More explicitly, we have the following proposition.

**Proposition 2.6.7.** Under the assumptions of Proposition 2.6.6. We have

$$\begin{align*}
\text{Ext}^1(S_\alpha, S_\beta) &\cong (C_{\beta,(1,0,\ldots,0)}^\alpha E^*) \oplus (C_{\alpha,(1,0,\ldots,0)}^\beta \mathbb{C}), \\
\text{Ext}^2(S_\alpha, S_\beta) &\cong \\
\begin{cases}
(C_{\alpha^t,(1,0,\ldots,0)}^\beta \wedge^3 E^*) \oplus (C_{\alpha^t,(2,0,\ldots,0)}^\beta \mathbb{L}_2 E^*), & \text{if } n-r = 1; \\
(C_{\alpha^t,(0,\ldots,0,1,0)}^\beta \mathbb{C}) \oplus (C_{\alpha^t,(1,0,\ldots,0,0)}^\beta E^*) \oplus (C_{\alpha^t,(2,0,\ldots,0)}^\beta \mathbb{L}_2 E^*) \oplus (C_{\alpha^t,(1,1,0,\ldots,0)}^\beta \wedge^2 E^*), & \text{if } n-r \geq 2.
\end{cases}
\end{align*}$$

**Proof.** Note that the only element in $Q_1(2)$ is $(1,1,0,\cdots,0)$ and the only element in $Q_1(4)$ is $(2,1,1,0,\cdots,0)$. One can easily calculate $\text{Ext}^1$ and $\text{Ext}^2$ with the aid of Lemma 2.3.16.

For $\lambda = (1,1,0,\cdots,0)$, the $\gamma_{\min(1,1,0,\ldots,0)}$ is given by Lemma 2.3.16 is $(0,\cdots,0,-2)$ with the corresponding $t_{\min(1,1,0,\ldots,0)} = 1$. There are a lot of operations described in Remark 2.3.17, but if we only care about those with corresponding $t = 2$ and satisfying the constrains given by Corollary 2.3.13, there is only one which gives $\gamma = (1,0,\cdots,0,-2)$. 

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For \( \lambda = (2, 1, 1, 0, \ldots, 0) \), the \( \gamma_{\min}(2,1,1,0,\ldots,0) \) is given by Lemma 2.3.16 is \((-3)\) if \( n-r = 1 \) with the corresponding \( t_{\min}(2,1,1,0,\ldots,0) = 2 \); \( \gamma_{\min}(2,1,1,0,\ldots,0) = (0,\ldots,0, -1, -3) \) if \( n-r \geq 2 \) with the corresponding \( t_{\min}(2,1,1,0,\ldots,0) = 2 \).

Note that only \( Q_1(2s) \) with \( s = 0, 1, 2 \) contributes to \( \text{Ext}^1 \) and \( \text{Ext}^2 \). This finishes the proof.

Now we start using Proposition 2.6.7 to describe the quiver with relations for this non-commutative desingularization.

**Proposition 2.6.8.** Let \( k = \mathbb{C} \). In the quiver with relations for the algebra \( \Lambda = \text{End}_{\mathcal{O}_{2^n}}(p^* \mathcal{F}il_K) \) as in Proposition 2.2.13, the set of vertices correspond to \( B_{r,n-r} \), and the arrow from \( \beta \) to \( \alpha \) is given by \( E \) if \( C_{\beta,(1,0,\ldots,0)}^\alpha \neq 0 \) and given by \( \mathbb{C} \) if \( C_{\alpha,(1,1,0,\ldots,0)}^\beta \neq 0 \). No arrows otherwise.

**Proposition 2.6.9.** Let \( k = \mathbb{C} \). In the quiver with relations for the algebra \( \Lambda = \text{End}_{\mathcal{O}_{2^n}}(p^* \mathcal{F}il_K) \) as in Proposition 2.2.13, the relations are generated by the following sub-representations of the arrows in \( \text{Hom}(\beta,\alpha) \):

- \( (C_{\alpha',(1,0,\ldots,0)}^{\beta'} \wedge^3 E) \oplus (C_{\alpha',(2,0,\ldots,0)}^{\beta'} \mathbb{L}_2 E) \)

  in the case \( n-r = 1 \);

- \( (C_{\alpha',(0,\ldots,0,-1,\ldots,-3)}^{\beta'} \mathbb{C}) \oplus (C_{\alpha',(1,0,\ldots,0,-2)}^{\beta'} E) \oplus (C_{\alpha',(2,0,\ldots,0)}^{\beta'} \mathbb{L}_2 E) \oplus (C_{\alpha',(1,1,0,\ldots,0)}^{\beta'} \wedge^2 E) \)

  in the case \( n-r \geq 2 \).

### 2.7 Determinantal varieties

In this section we study a non-commutative desingularization of determinantal varieties in the space of matrices.
2.7.1 Review of the commutative desingularization

Now we start describing a desingularization of determinantal varieties. We will follow the notations in the paper by Buchweitz, et al [19].

Let \( G \) and \( F \) be vector spaces of dimension \( m, n \) respectively with \( m \geq n \), and \( H = \text{Hom}_k(G,F) \). Upon choosing a set of basis \( f_1, \cdots, f_m \) and \( g_1, \cdots, g_n \), \( H \) is identified with the set of \((m \times n)\)-matrices \((x_{ij})\). The coordinate ring of \( H \) can be identified with \( S = \mathbb{C}[x_{ij}] \).

By base extension from \( \text{Spec} \, k \) to \( H \), we get two vector bundles \( F \) and \( G \) over \( H \), and a universal morphism \( \varphi : G \to F \). For \( r \leq n \), we desingularize the locus \( \text{Spec} \, R \), where \( \varphi \) has rank \( \leq r \). In other words, \( R \) is the quotient of \( S \) by the ideal generated by the \((r+1) \times (r+1)\) minors of \((x_{ij})\).

We take the Grassmannian of \( n-r \) planes in \( F^* \), denoted by \( \text{Grass} \). Let \( \mathcal{Y} \) be the product \( \text{Grass} \times H \), \( p \) and \( q \) be the projection to \( \text{Grass} \) and \( H \) respectively. Inside of \( \mathcal{Y} \), there is an incidence variety, denoted by \( Z \), defined by

\[
Z = \{(g,h) \in \text{Grass} \times H : \text{im} \, g \circ h = 0\}.
\]

The inclusion \( Z \hookrightarrow \mathcal{Y} \) is denoted by \( j \), and \( \text{Spec} \, R \to H \) by \( i \). The induced map \( Z \to \text{Spec} \, R \) from \( q : \mathcal{Y} \to H \) is denoted by \( q' \), and \( Z \to \text{Grass} \) from \( p : \mathcal{Y} \to \text{Grass} \) by \( p' \).

These notations are summarized by the following diagram.

\[
\begin{tikzcd}
Z \arrow{d}{q'} \arrow{r}{j} & \mathcal{Y} \arrow{r}{p} \arrow{d}{q} & \text{Grass} = \text{Grass}_{n-r}(F^*) \\
\text{Spec} \, R \arrow{r}{i} & H = \text{Hom}(G,F)
\end{tikzcd}
\]

**Proposition 2.7.1** (6.1.1 in [80]). The variety \( Z \) is a desingularization of \( \text{Spec} \, R \).

The variety \( Z \) can be described as the total space of a vector bundle over \( \text{Grass} \). Let

\[
0 \to R \to F^* \times \text{Grass} \to Q \to 0
\]
be the tautological sequence over Grass. We apply the functor $G^* \otimes -$ to the dualized tautological sequence $0 \to Q^* \to F \times \text{Grass} \to R^* \to 0$ to get

$$0 \to S \to E \to T \to 0.$$  

Following [80], let $\eta$ be the sheaf of sections of $S^* = G \otimes Q$. The desingularization $\mathcal{Z}$ is the total space of $S$. Equivalently, $p' : \mathcal{Z} \to \text{Grass}$ is an affine morphism with $p_* \mathcal{O}_\mathcal{Z}$ equal to the sheaf of algebra $\text{Sym}(\eta)$.

### 2.7.2 A tilting bundle over the desingularization

We will show, in this subsection, that $p^* \mathcal{T}il_0$ is a tilting bundle over the desingularization $\mathcal{Z}$.

Now we prove the following

**Proposition 2.7.2.** The BLV’s tilting bundle over $\text{Grass}_{n-r}(F^*)$ is denoted by

$$\mathcal{T}il_0 = \bigoplus_{\alpha \in \mathcal{B}_{r,n-r}} \wedge^\alpha Q^*.$$  

The rank $r$ determinantal variety $\text{Spec } R$ and its desingularization $\mathcal{Z}$ are as above. The bundle $p^* \mathcal{T}il_0$ is a tilting bundle over $\mathcal{Z}$.

The module $\text{End}_\mathcal{Z}(p^* \mathcal{T}il_0)$ is reflexive and maximal Cohen-Macaulay.

**Proof.** This has been proved in [20].

**Proposition 2.7.3.** The map

$$\text{End}_\mathcal{Z}(p^* \mathcal{T}il_0) \to \text{End}_S(q'_* p'^* \mathcal{T}il_0)$$

is an isomorphism of rings and it is a non-commutative desingularization of $\text{Spec } R$.

**Proof.** By Proposition 2.1.7, it suffices to show the exceptional locus of $q'$ has codimension at least 2 in both $\mathcal{Z}$ and $\text{Spec } R$. The codimension of $\text{Spec } R$ in $\text{Spec } S$ is $\binom{n-r+1}{2}$, and hence
the codimension of the singular locus of Spec $R$ has codimension at least 2 unless $n = r$ in which case Spec $R$ is smooth. The codimension of the exceptional locus in $Z$ is the rank of the bundle $Z \to \text{Grass}$, which is at least 2 unless $m = \dim G = 1$ and $r = 1$ in which case Spec $R$ is still smooth. 

By theorem 7.6 in [45], since $Z$ is smooth, $\text{End}_Z(p^*Til_0)$ has finite global dimension.

### 2.7.3 Minimal presentation for the non-commutative desingularization

We study the minimal presentation of the noncommutative desingularization $\text{End}_S(q'_*p^*Til_K) = \text{End}_Z(p^*Til_K)$ as module over $S = \mathbb{C}[x_{i,j}]$. In this subsection we assume $k = \mathbb{C}$.

Using a procedure similar to the one in Section 2.5, one can get the quiver with relations for the non-commutative desingularization. As the quiver with relations in this case is known to experts and yet to be published, instead, we study the endomorphism ring using the basic theorem of geometric technique.

Let $\eta = Q \otimes G$. The modules $H^0(\text{Grass}, \text{Sym}(\eta) \otimes L_\alpha Q^*)$ will be denoted by $M_\alpha$.

In the basic theorem of geometric method 2.3.6, we take the vector bundle $V$ to be $L_\alpha V = L_\alpha Q^*$, $V = \text{Grass}$, and $X$ to be $H$. By the same argument as in proposition 2.7.2, we get the vanishing of higher derived images for $M(L_\alpha V)$, i.e.,

$$R^iq_*M(L_\alpha V) = R^iq_*(O_Z \otimes p^*L_\alpha V) = H^i(\text{Grass}, \text{Sym}(\eta) \otimes L_\alpha V) = 0$$

through the expansion of $\text{Sym}(\eta) \otimes L_\alpha V$ using the Cauchy-Littlewood and the Littlewood-Richardson formulas, and Bott’s theorem.

According to theorem 2.3.6, there is a presentation of $M_\alpha$ given by $F_{\alpha,1} \to F_{\alpha,0}$ with $F_{\alpha,i}$
defined to be

\[ F_{\alpha,i} = \bigoplus_{j \geq 0} H^j(\text{Grass}, \wedge^{i+j}(\xi) \otimes \mathbb{L}_\alpha Q^*) \otimes S \]

\[ = \bigoplus_{j \geq 0} H^j(\text{Grass}, \oplus_{|\mu| = i+j} \mathbb{L}_\mu G \otimes \mathbb{L}_\mu R \otimes \mathbb{L}_\alpha Q^*) \otimes S \]

\[ = \bigoplus_{j \geq 0} \oplus_{|\mu| = i+j} H^j(\text{Grass}, \mathbb{L}_\mu R \otimes \mathbb{L}_\alpha Q^*) \otimes \mathbb{L}_\mu G \otimes S. \]

To compute the \( F_{\alpha,i} \)'s, all we need to do is to compute \( H^j(\text{Grass}, \mathbb{L}_\mu R \otimes \mathbb{L}_\alpha Q^*) \).

**Theorem 2.7.4.** Notations as above,

1. \( F_{\alpha,1} = \mathbb{L}_{(\alpha,1^{r+1-t},0^{m-r-1})} G \otimes \mathbb{L}_{(1^{r+1-t})} F^* \otimes S; \)

2. \( F_{\alpha,0} = \mathbb{L}_\alpha G \otimes S. \)

**Proof.** Assume \( l(\alpha) = t \), i.e., \( \alpha_t \neq 0 \) and \( \alpha_{t+1} = 0 \), \( |\mu| = j + 1 \). To show (1), it suffices to prove that \( H^j(\mathbb{L}_\mu E \otimes \mathbb{L}_\mu R \otimes \mathbb{L}_\alpha Q^*) \) is isomorphic to

\[
\begin{cases}
\mathbb{L}_{(\alpha,1^{r+1-t},0^{m-r-1})} G \otimes \mathbb{L}_{(1^{r+1-t})} F^*, & \text{if } j = |\alpha| + r - t - 1 \text{ and } \mu = (\alpha' + (r - t, 0^{n-r-1})); \\
0, & \text{otherwise}.
\end{cases}
\]

We have, using the language of 2.3.2,

\[
H^j(\mathbb{L}_\mu' G \otimes \mathbb{L}_\mu R \otimes \mathbb{L}_\alpha Q^*) = H^j(\mathbb{L}_\mu R \otimes \mathbb{L}_\alpha Q^*) \otimes \mathbb{L}_\mu' G
\]

\[ = H^j(\mathcal{V}(0/\mu, \alpha)) \otimes \mathbb{L}_\mu' G. \]

Since \( \alpha_1 \leq n - r \), one easily sees that \( l(\mu) = \alpha_1 \). In fact, if \( l(\mu) < \alpha_1 \), we would get \((0, \cdots, 0, 1, \cdots)\) performing the symmetric group action described in 2.3.2; and we would get negative entries followed by 0s performing the symmetric group action if \( l(\mu) > \alpha_1 \).

Note also that we must have \( \mu \supset \alpha \). Otherwise we would get 0's followed by positive entries performing the symmetric group action.
By performing adjacent transposition actions $\alpha_1$ times, we can delete the first row of $\alpha$, delete the first column of $\mu$, and move everything remains in $\mu$ down by one row. Inductively, after performing $|\alpha|$ transposition actions times, we will get $(0, \cdots, 0, \alpha'/\mu', 0^{r-t})$.

To get non-trivial $H^{|\mu|-1}$, we have to perform $|\mu| - |\alpha| - 1$ transposition actions to $(\alpha'/\mu', 0^{r-t})$ to make it dominant. The only possibility is $\alpha'/\mu' = (0, \cdots, 0, -r + t + 1)$, and the corresponding $\mu = (\alpha' + (r - t, 0^{n-r-1}))$ and $H^{|\mu|-1} = \mathbb{L}_{(\alpha, 1^{r+1-t}, 0^{n-r-1})} E \otimes \mathbb{L}_{(1^{r+1-t})} F^*$.

By the same argument, one proves (2). 

\section*{2.8 Other examples}

Here we look at some non-commutative desingularizations beyond representations with finitely many orbits. The mechanics we developed in Section 2.1 work in these examples. But as one have seen in Proposition 2.5.11, computing quiver with relations involves the problem of inner plethysm and cannot be done in general. For simplicity, we assume $k = \mathbb{C}$ in this section, although part of it works in a characteristic free fashion.

\subsection*{2.8.1 Rank varieties of anti-symmetric tensors}

Let $E$ be a vector space over $k$ of dimension $n$ and $H_a^{(d)}$ be the affine space $\wedge^d E^*$ consisting of anti-symmetric tensors of power $d$. Upon choosing a set of basis for $E$, it’s coordinate ring is identified with $A_a^{(d)} \text{Sym}(\wedge^d E)$. Let $X_a^{(d)} \subset H_a^{(d)}$ be the rank variety consisting of tensors of rank $\leq n - 1$. We will find a non-commutative desingularization for $X_a^{(d)}$.

Let us review a commutative desingularization. Let

$$0 \rightarrow \mathcal{R} \rightarrow E \times \text{Grass} \rightarrow Q \rightarrow 0$$

be the tautological sequence over Grass where Grass = Grass$(1, E)$ is the Grassmannian of lines in $E$. Let $\eta_a^{(d)}$ be the vector bundle $\wedge^d Q$. As can be found in Section 7.3 of [80], the total space $Z_a^{(d)}$ of the vector bundle $\eta_a^{(d)*}$ is a commutative desingularization, i.e.,
\[ Z_a^{(d)} = \text{Spec}_{\text{Grass}}(\text{Sym} \wedge^d Q) \]. Alternatively, it can also be defined as the incidence variety
\[ Z_a^{(d)} = \{ (S, \phi) \in \text{Grass} \times H_a^{(d)} \mid \phi \in \wedge^d S \subset \wedge^d E^* \} \).

As before, let \( \mathcal{Y}_a^{(d)} \) be the product \( \text{Grass} \times H_a^{(d)} \), \( p \) and \( q \) be the projections to \( \text{Grass} \) and \( H_a^{(d)} \) respectively. Inside of \( \mathcal{Y}_a^{(d)} \), there is an incidence variety, denoted by \( Z_a^{(d)} \). The inclusion \( Z_a^{(d)} \hookrightarrow \mathcal{Y}_a^{(d)} \) is denoted by \( j \), and \( X_a^{(d)} \rightarrow H_a^{(d)} \) by \( i \). The induced map \( Z_a^{(d)} \rightarrow X_a^{(d)} \) from \( q : \mathcal{Y}_a^{(d)} \rightarrow H_a^{(d)} \) is denoted by \( q' \) and \( Z_a^{(d)} \rightarrow \text{Grass} \) from \( p : \mathcal{Y}_a^{(d)} \rightarrow \text{Grass} \) by \( p' \).

These notations are summarized by the following diagram.

\[
\begin{array}{ccc}
Z_a^{(d)} & \xrightarrow{j} & \mathcal{Y}_a^{(d)} = \text{Grass} \times H_a^{(d)} \xrightarrow{p} \text{Grass} = \text{Grass}_1(E) \\
\downarrow & & \downarrow q \\
X_a^{(d)} & \xrightarrow{i} & H_a^{(d)} = \wedge^d(E^*)
\end{array}
\]

We consider the inverse image of \( \mathcal{T}il_K = \oplus_{i=0}^{n-1} \wedge^i Q^* \), the Kapranov’s tilting bundle over \( \text{Grass}(1, E) \), by \( p' : Z_a^{(d)} \rightarrow \text{Grass} \). We can show

**Lemma 2.8.1.** For all \( k > 0 \), and \( i, j = 0, \ldots, n-1 \), \( H^k(Z_a^{(d)} p'^* \mathcal{H}\text{om}_{\text{Grass}}(\wedge^i Q^*, \wedge^j Q^*)) = 0 \).

**Proof.** Use the Cauchy-Littlewood and the Littlewod-Richardson formulas, we can reduce the sheaf

\[ p'^* \mathcal{H}\text{om}_{\text{Grass}}(\wedge^i Q^*, \wedge^j Q^*) = \mathcal{H}\text{om}_{\text{Grass}}(\wedge^i Q^*, \wedge^j Q^* \otimes_{\text{Grass}} \text{Sym}(\wedge^d Q)) \]

into the form \( \oplus (L_\gamma Q^*)^{\otimes C_\gamma} \) for some coefficients \( C_\gamma \). By the Bott’s theorem 2.3.5, all the higher cohomology of

\[ \mathcal{H}\text{om}_{\text{Grass}}(\wedge^i Q^*, \wedge^j Q^* \otimes_{\text{Grass}} \text{Sym}(\wedge^d Q)) \]

vanishes. \qed

From the Lemma above, using Theorem 2.1.7, we get the following
Proposition 2.8.2. The Kapranov’s tilting bundle over $\text{Grass}(1, E)$ is denoted by $\mathcal{T}il_K$. The rank $n - 1$ subvariety $X^{(d)}_a$ of $d$-th anti-symmetric tensors and its desingularization $Z^{(d)}_a$ are as above. The bundle $p^* \mathcal{T}il_K$ is a tilting bundle over $Z^{(d)}_a$.

Since $Z^{(d)}_a$ is smooth, $\text{End}_{Z^{(d)}_a}(p^* \mathcal{T}il_K)$ has finite global dimension.

Lemma 2.8.3. Notations as above, $\text{End}_{Z^{(d)}_a}(p^* \mathcal{T}il_K)$ is maximal Cohen-Macaulay iff

$$(n - 2) - n - 1 \geq 0.$$ 

Remark 2.8.4. As can be easily checked, $(n - 2) - n - 1 \geq 0$ if and only if $n \geq 6$ and $3 \leq d \leq n - 3$.

Proof of Lemma 2.8.3. By Theorem 2.3.7, it suffices to compute

$$H^k(\text{Grass}, \wedge^i Q \otimes \wedge^j Q^* \otimes \omega_{\text{Grass}} \otimes \wedge^{\text{top}} \xi^* \otimes \text{Sym}(\wedge^d Q))$$

for $k > 0$ and $i, j = 0, \ldots, n - 1$, here $\xi = R \otimes \wedge^{d-1} Q$ according to 7.3.1 of [80]. The sheaf $\omega_{\text{Grass}} = \wedge^{n-1} Q^* \otimes R^{n-1}$. Using Cauchy-Littlewood and Littlewood-Richardson formulas,

$$\wedge^i Q \otimes \wedge^j Q^* \otimes \omega_{\text{Grass}} \otimes \wedge^{\text{top}} \xi^* \otimes \text{Sym}(\wedge^d Q)$$

$$\cong \wedge^{j-i} Q^* \otimes \text{Sym} \wedge^d Q \otimes \wedge^{n-1} Q^{(n-2)-(n-1)-1} \otimes \wedge^n E^{(n-1)-(n-1)}.$$ 

The conclusion now follows from the Bott’s Theorem. \qed

We know that $O_{\text{Grass}}$ is a direct summand of $\mathcal{T}il_K$. In the case $(n-2) - n - 1 \geq 0$, the module $q'_* p'^* \mathcal{T}il_K$ is maximal Cohen-Macaulay. In particular, it is reflexive. According to Proposition 2.1.7 and Lemma 2.1.8, using Lemma 2.8.3, we get the following Proposition.

Proposition 2.8.5. Notations as above, in the case $(n-2) - n - 1 \geq 0$, the map

$$\text{End}_{Z^{(d)}_a}(p^* \mathcal{T}il_K) \to \text{End}_{A^{(d)}_a}(q'_* p'^* \mathcal{T}il_K)$$

is an isomorphism of rings and $\text{End}_{Z^{(d)}_a}(p^* \mathcal{T}il_K)$ is a non-commutative crepant desingularization of $X^{(d)}_a$. 
Remark 2.8.6. In this case, the tilting bundle over $Z^{(d)}_d$ is the inverse image of an exceptional collection $\{\wedge^i Q^* \mid i = 0, \ldots, n - 1\}$ over Grass. As can be checked by definition, The dual collection is given by $\wedge^{n-1} Q^* \otimes \mathbb{L}_{n-i} R[n - 1 - i]$.

Let us take $n=6$ and $d=3$, for an example. One can prove as in Proposition 2.5.11, that for any $\alpha, \beta = 0, \ldots, 6$, $\text{Ext}^1(S_\alpha, S_\beta) = H^{1-s-\beta+\alpha}(\text{Grass}, \wedge^s \wedge^3 Q^* \otimes \mathbb{L}_{(6-\beta)} R \otimes \mathbb{L}_{(6-\alpha)} R^*)$. As it is known that $\wedge^2 \wedge^3 \mathbb{C}^6 \cong \mathbb{L}_{2,2,1,1,0,0} \mathbb{C}^6 \oplus \mathbb{L}_{1,1,1,1,1} \mathbb{C}^6$. This means, in this example $\text{Ext}^1$ and $\text{Ext}^2$ can be calculated very easily which gives the quiver with relations for the endomorphism algebra.

\begin{align*}
\alpha_i \alpha_{i+1} (\wedge^2 E); \\
\beta \alpha \beta \alpha + \alpha \beta \alpha \beta (\wedge^2 E).
\end{align*}

2.8.2 Cone over a rational normal curve

Let $E$ be a vector space over $k$ of dimension $n$ and $H^{(d)}_s$ be the affine space $\mathbb{L}_s E^*$ consisting of symmetric tensors of power $d$. Upon choosing a set of basis for $E$, it’s coordinate ring is identified with $A^{(d)}_s = \text{Sym}(\mathbb{L}_s E)$. Let $X^{(d)}_s \subset H^{(d)}_s$ be the rank variety consisting of symmetric tensors of rank $\leq n - 1$. We will find a non-commutative desingularization for $X^{(d)}_s$ and describe it in certain cases.

Let us review a commutative desingularization. Let

$$0 \rightarrow R \rightarrow E \times \text{Grass} \rightarrow Q \rightarrow 0$$

be the tautological sequence over Grass where Grass = Grass(1, $E$) is the Grassmannian of lines in $E$. Let $\eta^{(d)}_s$ be the vector bundle $\mathbb{L}_s Q$. As can be found in Section 7.3 of [80],
the total space $Z_s^{(d)}$ of the vector bundle $(\eta_s^{(d)})^*$ is a commutative desingularization, i.e.,

$$Z_s^{(d)} = \text{Spec}_{\text{Grass}}(\text{Sym} \mathbb{L}_d \mathcal{Q}).$$

Alternatively, it can also be defined as the incidence variety

$$Z_s^{(d)} = \{(S, \phi) \in \text{Grass} \times H_s^{(d)} \mid \phi \in \mathbb{L}_d S \subset \mathbb{L}_d E^*\}.

As before, let $Y_s^{(d)}$ be the product $\text{Grass} \times H_s^{(d)}$, $p$ and $q$ be the projections to Grass and $H_s^{(d)}$ respectively. Inside of $Y_s^{(d)}$, there is an incidence variety, denoted by $Z_s^{(d)}$. The inclusion $Z_s^{(d)} \hookrightarrow Y_s^{(d)}$ is denoted by $j$, and $X_s^{(d)} \rightarrow H_s^{(d)}$ by $i$. The induced map $Z_s^{(d)} \rightarrow X_s^{(d)}$ from $q : Y_s^{(d)} \rightarrow H_s^{(d)}$ is denoted by $q'$ and $Z_s^{(d)} \rightarrow \text{Grass}$ from $p : Y_s^{(d)} \rightarrow \text{Grass}$ by $p'$.

These notations are summarized by the following diagram.

$$
\begin{array}{ccc}
Z_s^{(d)} & \xrightarrow{j} & Y_s^{(d)} = \text{Grass} \times H_s^{(d)} & \xrightarrow{p} & \text{Grass} = \text{Grass}_1(E) \\
\downarrow{q'} & & \downarrow{q} & & \\
X_s^{(d)} & \xrightarrow{i} & H_s^{(d)} = \mathbb{L}_d(E^*) & & \\
\end{array}
$$

As before, we have:

**Lemma 2.8.7.** For all $k > 0$, and $i, j = 0, \ldots, n - 1$, $H^k(Z_s^{(d)} \xrightarrow{p^*} \text{Hom}_{\text{Grass}}(\wedge^i \mathcal{Q}^*, \wedge^j \mathcal{Q}^*)) = 0$.

From the Lemma above, using Theorem 2.1.7, we get the following

**Proposition 2.8.8.** The Kapranov’s tilting bundle over $\text{Grass}(1, E)$ is denoted by $\mathcal{T}il_K$.

The rank $n - 1$ subvariety $X_s^{(d)}$ of $d$-th symmetric tensors and its desingularization $Z_s^{(d)}$ as above. The bundle $p^* \mathcal{T}il_K$ is a tilting bundle over $Z_s^{(d)}$.

Since $Z_s^{(d)}$ is smooth, $\text{End}_{Z_s^{(d)}}(p^* \mathcal{T}il_K)$ has finite global dimension.

**Lemma 2.8.9.** Notations as above, $\text{End}_{Z_s^{(d)}}(p^* \mathcal{T}il_K)$ is maximal Cohen-Macaulay.

**Proof.** By Theorem 2.3.7, it suffices to compute

$$H^k(\text{Grass}, \wedge^i \mathcal{Q} \otimes \wedge^j \mathcal{Q}^* \otimes \omega_{\text{Grass}} \otimes \wedge^{\text{top}} \xi^* \otimes \text{Sym}(\mathbb{L}_d \mathcal{Q}))$$

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for \( k > 0 \) and \( i, j = 0, \ldots, n - 1 \), here \( \xi = R \otimes L_{d-1}E \) according to 7.2.1 of [80]. The sheaf 
\[
\omega_{\text{Grass}} = \wedge_{n-1}^{n-1}Q^* \otimes R^{n-1}.
\]
Using Cauchy-Littlewood and Littlewood-Richardson formulas,
\[
\wedge^i Q \otimes \wedge^j Q^* \otimes \omega_{\text{Grass}} \otimes \wedge^\text{top} \xi^* \otimes \text{Sym}(L_d Q) \cong \wedge^{i-j} Q^* \otimes \text{Sym} L_d Q \otimes \wedge_{n-1}^{n-1} Q \left( \frac{n+d-1}{n-1} \right) - (n-1)^{-1} \otimes \mathbb{L}_{\mu} E
\]
for some \( \mu \). Now it follows easily from the Bott’s Theorem that \( \text{End}_{\mathbb{Z}^d}(p^* \mathcal{T}il_K) \) is maximal Cohen-Macaulay iff 
\[
\left( \frac{n+d-1}{n-1} \right) - n - 1 \geq 0.
\]
But as can be easily seen, \( \left( \frac{n+d-1}{n-1} \right) - n - 1 \) is always positive.

In the case that \( n = 2 \), one can see that \( X_s^{(d)} \) is the cone over a rational normal curve. And in this case, \( \text{End}_{\mathbb{Z}^d}(p^* \mathcal{T}il_K) \) is always maximal Cohen-Macaulay grant that \( d \geq 2 \).

With similar calculation as in Proposition 2.5.11, we get the following:

**Proposition 2.8.10.** For any two simple objects \( S_i, S_j, i, j = 0, 1 \), we have
\[
\text{Ext}^t(S_i, S_j) = \bigoplus_{s=0}^1 H^{t-s-j+i}(\mathbb{P}^1, (Q^*)^{ds} \otimes (R^*)^{j-i}).
\]

Plug-in \( t = 1, 2 \) and \( i, j = 0, 1 \), we get the quiver with relations for the non-commutative desingularization.

\[
\begin{array}{c}
\beta(L_{d-1}E) \\
\bullet_0 \\
\alpha(E) \\
\bullet_1
\end{array}
\]

and the relations:
\[
\alpha \beta(L_{(d-1,1)}E);
\]
\[
\beta \alpha(L_{(d-1,1)}E).
\]
Chapter 3

Real variations of stability conditions for noncommutative symplectic resolutions

3.1 Truncated mutations

3.1.1 Tilting with respect to a simple object

Suppose $\mathcal{A} \subset D$ is the heart of a bounded $t$-structure and is a finite length abelian category. A torsion pair in $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ which satisfy $\text{Hom}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and every object $E \in \mathcal{A}$ fits into a short exact sequence

$$0 \to T \to E \to F \to 0.$$

The following Lemma is due to Happel, Reiten, and Smalø. (See also [24], Proposition 5.4.)

**Lemma 3.1.1.** Suppose $\mathcal{A} \subset D$ is the heart of a bounded $t$-structure on a triangulated category $D$. Given an object $E \in D$, suppose $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$. Then the full subcategory $R_\tau \mathcal{A} = \{ E \in D \mid H^i(E) = 0 \text{ for } i \notin \{-1, 0\}, H^{-1} \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T} \}$ is the heart of a bounded $t$-structure.
The new $t$-structure $R_T \mathcal{A}$ is called the tilting with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$.

The following Lemma gives a criterion for simple objects in the heart of the new $t$-structure $R_T \mathcal{A}$.

**Lemma 3.1.2** (See [81], Lemma 2.4). Let $\mathcal{T}$ be a torsion theory in the heart $\mathcal{A}$ of a $t$-structure. Then any simple object in $R_T \mathcal{A}$ lies either in $\mathcal{T}$ or in $\mathcal{F}[1]$ and

1. $T \in \mathcal{T}$ is simple in $R_T \mathcal{A}$ iff there are no exact triangles

   $$T' \to T \to T''[1] \text{ or } T' \to T \to F'[1] \to T'[1]$$

   with $T', T'' \in \mathcal{T}$ and $F' \in \mathcal{F}$ and all non-zero;

2. $F[1] \in \mathcal{F}[1]$ is simple in $R_T \mathcal{A}$ iff there are no exact triangles

   $$F' \to F \to F'' \to F'[1] \text{ or } T'[1] \to F' \to F \to T'$$

   with $F', F'' \in \mathcal{F}$ and $T' \in \mathcal{T}$ and all non-zero.

Given a simple object $S \in \mathcal{A}$ define $\langle S \rangle \subset \mathcal{A}$ to be the full subcategory consisting of objects $E \in \mathcal{A}$ all of whose simple factors are isomorphic to $S$. One can easily check $\mathcal{F} = \langle S \rangle$, and $\mathcal{T} = \{ E \mid \text{Hom}(E, S) = 0 \}$ is a torsion pair.

The (right) tilted subcategory of $\mathcal{A}$ with respect to $S$ is defined to be

$$R_S \mathcal{A} = \{ E \in D \mid H^i(E) = 0 \text{ for } i \neq -1, 0, H^{-1}(E) \in \langle S \rangle \text{ and } \text{Hom}(H^0(E), S) = 0 \}.$$

Similarly there is a notion of left tilting $L_S \mathcal{A}$.

In the heart $R_S \mathcal{A}$ of the new $t$-structure, $S[1]$ is a simple object. We can consider the tilting of $R_S \mathcal{A}$ with respect to $S[1]$. But for this to work we need $R_S \mathcal{A}$ to be of finite length. Now we give a sufficient condition to guarantee this property.

Fix a simple object $S_\theta$ in an abelian category $\mathcal{A}$, for another simple object $S_\alpha$ we use $S'_\alpha$ to denote the universal extension of $S_\theta$ by $S_\alpha$, which is the middle term in the tautological short exact sequence

$$0 \to S_\theta \otimes \text{Ext}^1(S_\alpha, S_\theta)^* \to S'_\alpha \to S_\alpha \to 0. \quad (3.1)$$
Note that $S'_{\alpha}$ has cohomology concentrated in degree zero.

**Lemma 3.1.3.** Let $\mathfrak{A}$ be an abelian category of finite length with simple objects $\{S_{\alpha} | \alpha \in \nabla\}$ for a finite indexing set $\nabla$. Suppose that $\text{Ext}^1(S_{\theta}, S_{\theta}) = 0$, then $R_{S_{\theta}} \mathfrak{A}$ is still a finite length category whose simple objects are $\{S_{\theta}[1]\} \cup \{S'_{\alpha} | \alpha \neq \theta\}$.

**Proof.** First note that $S_{\theta}[1]$ and $S'_{\alpha}$ are simple objects in $R_{S_{\theta}} \mathfrak{A}$. For $S_{\theta}[1]$, this is clear by Lemma 3.1.2. For $S'_{\alpha}$, applying $\text{Hom}(\cdot, S_{\theta}[1])$ to the short exact sequence (3.1), we get

$$\cdots \to \text{Hom}(S'_{\alpha}, S_{\theta}) \to \text{Hom}(S_{\theta}, S_{\theta}) \otimes \text{Ext}^1(S_{\alpha}, S_{\theta}) \to \text{Ext}^1(S_{\alpha}, S_{\theta}) \to \text{Ext}^1(S'_{\alpha}, S_{\theta}) \to 0.$$ 

This shows $\text{Hom}(S'_{\alpha}, S_{\theta}[1]) = 0$. The composition factors of $S'_{\alpha}$ are $S_{\alpha}$ and some copies of $S_{\theta}$, therefore, there is no exact triangle $T' \to S'_{\alpha} \to T'' \to T'[1]$ with $T'$ and $T'' \in \mathcal{T}$. So, Lemma 3.1.2 yields the simplicity of $S'_{\alpha}$.

We only need to show that any object $E$ in $R_{S_{\theta}} \mathfrak{A}$ has a finite filtration with sub-quotients isomorphic to $S_{\theta}[1]$ and $S'_{\alpha}$. We use induction on total number of copies of $S_{\alpha}$ with $\alpha \neq \theta$ in the composition factors of $H^0(E)$. If the length is zero, this means the cohomology of $E$ is concentrated in degree -1, and therefore, is a direct sum of $S_{\theta}[1]$. Otherwise, via taking cokernel of maps from $S_{\theta}[1]$ in the abelian category $R_{S_{\theta}} \mathfrak{A}$, we can assume the cohomology of $E$ is concentrated in degree zero. There is some $\alpha \neq \theta$ such that $\text{Hom}(H^0(E), S_{\alpha}) \neq 0$ which implies $\text{Hom}(E, S'_{\alpha}) \neq 0$. As $S'_{\alpha}$ is simple in $R_{S_{\theta}} \mathfrak{A}$, this map must be surjective. Let the kernel be $K$. Taking cohomology long exact sequence with respect to the original $t$-structure of the exact triangle

$$K \to E \to S'_{\alpha} \to K[1],$$

we know that in the composition factors of $H^0(K)$ the total number of copies of $S_{\alpha}$ with $\alpha \neq \theta$ has been reduced by 1. \qed

We will denote $S_{\theta}[1]$ by $S'_{\theta}$.

---

1 The author is grateful to Sasha Kuznetzov for pointing out a better set-up to carry out iterated tiltings studied in his work in preparation.
Now we assume $\text{Ext}^1(S_\theta, S_\theta)$ vanish. For a fixed $\theta \in \nabla$, let $S_\alpha^0 = S_\alpha$. Recursively we define $S_\alpha^i$ to be the universal extension of $S_\alpha^{i-1}$ by $S_\alpha^{i-1}$ for $\alpha \neq \theta$, and $S_\theta^i = S_\theta^{i-1}[1]$.

Since $\text{Ext}^i(S_\theta^i, S_\theta^i) = \text{Ext}^i(S_\theta, S_\theta)$, we have the following proposition.

**Proposition 3.1.4.** Let $\mathfrak{A}$ be an abelian category of finite length with simple objects $\{S_\alpha \mid \alpha \in \nabla\}$ for a finite set $\nabla$. Suppose that $\text{Ext}^1(S_\theta, S_\theta) = 0$, then $R S_\theta[i-1] R S_\theta[i-2] \cdots R S_\theta(\mathfrak{A})$ is still a finite length category whose simple objects are $\{S_\alpha^i \mid \alpha\}$.

Let $\mathfrak{A} \subset D$ be the heart of some $t$-structure of $D$, following Bridgeland, we denote the region in the stability space corresponding to $\mathfrak{A}$ by $U(\mathfrak{A})$. Suppose $(Z, \mathfrak{A})$ is a stability condition in the boundary of the region $U(\mathfrak{A})$. Then there is some $i$ such that $Z(S_i)$ lies on the real axis. Assume that $\text{im}Z(S_j) > 0$ for every $j \neq i$. Since each object $S_i$ is stable for all stability conditions in $U(\mathfrak{A})$, each $S_i$ is at least semistable in $(Z, \mathfrak{A})$, and hence $Z(S_i)$ is nonzero.

**Lemma 3.1.5** ([24], Lemma 5.2). Suppose the heat $\mathfrak{A} \subset D$ of a bounded $t$-structure has finite length and $n$ simple objects, then $U(\mathfrak{A})$ is isomorphic to $\mathbb{H}^n$ where $\mathbb{H}$ is the upper half plane in $\mathbb{C}$ with positive real axis.

For a stability condition $(\mathfrak{A}, Z)$ on a wall of codimension 1, then $Z(S)$ takes positive real values on that wall for some simple object $S$. If $R S \mathfrak{A}$ has the same finiteness property, then $U(\mathfrak{A})$ and $R S \mathfrak{A}$ glues together along this wall.

**Corollary 3.1.6.** If $S$ is a simple object in $\mathfrak{A}$ without self-extension, then $\text{Stab}(\mathfrak{A})$ has a locally closed subspace obtained by gluing $\mathbb{H}^n$’s together along the copy of $\mathbb{H}$ corresponding to the simple object $S$.

### 3.1.2 Perverse equivalences

Our main reference for this subsection is [27].
For a Serre subcategory $\mathcal{I}$ of an exact category $\mathcal{A}$, the thick subcategory in $D^b(\mathcal{A})$ generated by $\mathcal{I}$ will be denoted by $\langle \mathcal{I} \rangle$.

Let $\mathcal{A}$ and $\mathcal{A}'$ be two exact categories endowed with filtrations $0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \cdots \subseteq \mathcal{A}_r = \mathcal{A}$ and $0 = \mathcal{A}'_0 \subseteq \mathcal{A}'_1 \cdots \subseteq \mathcal{A}'_r = \mathcal{A}'$ by Serre subcategories. Let $p : \{0, \ldots, r\} \to \mathbb{Z}$ be any function. The notion of perverse equivalence with respect to this filtration and perversity function $p$ is defined in [27].

**Definition 3.1.7.** An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{A}')$ is perverse relative to the filtrations $(\mathcal{A}_*, \mathcal{A}'_*)$ and function $p$, if $F$ restricts to equivalences $\langle \mathcal{A}_i \rangle \cong \langle \mathcal{A}'_i \rangle$, and there is an equivalence $\mathcal{A}_i = \mathcal{A}_i / \mathcal{A}_{i-1} \to \mathcal{A}'_i = \mathcal{A}'_i / \mathcal{A}'_{i-1}$ compatible with the following equivalence induced by $F$:

$$F[p(i)] : \langle \mathcal{A}_i \rangle / \langle \mathcal{A}_{i-1} \rangle \cong \langle \mathcal{A}'_i \rangle / \langle \mathcal{A}'_{i-1} \rangle.$$ 

In the case when $\mathcal{A}'$ is not endowed with filtration, we make the following convention. We define the filtration on $\mathcal{A}'$ by $\mathcal{A}'_i = \mathcal{A}' \cap F(\mathcal{A}_i)$, and we talk about perverse equivalence only in the case that each $\mathcal{A}'_i$ defined this way is a Serre subcategory.

There is also a notion of perverse data when talking about two $t$-structures $t$ and $t'$ on the same triangulated category with a filtration $\mathfrak{T}_*$ with respect to a perversity function $p$ defined in [27]. We say the quadruple $(t, t', \mathfrak{T}_*, p)$ is a perverse data if both $t$ and $t'$ are compatible with the filtration $\mathfrak{T}_*$, and for each $i$ we have $t|_{\mathfrak{T}_i / \mathfrak{T}_{i-1}} = t'|_{\mathfrak{T}_i / \mathfrak{T}_{i-1}}[p(i)]$.

The followings are some basic properties of perverse equivalences.

**Proposition 3.1.8 (See [27]).** Notations as above, we have the following.

1. If $F$ is a perverse equivalence relative to $(\mathcal{A}_*, \mathcal{A}'_*, p)$, then $F^{-1}$ is perverse relative to $(\mathcal{A}'_*, \mathcal{A}_*, -p)$.

2. In this case, let $\mathcal{A}''$ be another exact category endowed with filtration $\mathcal{A}''_*$ by Serre subcategories, and let $p' : \{0, \ldots, r\} \to \mathbb{Z}$ be another map. Assume $F' : D^b(\mathcal{A}') \to D^b(\mathcal{A}'')$ is a perverse equivalence relative to $(\mathcal{A}'_*, \mathcal{A}''_*, p)$. Then $F' \circ F$ is a perverse equivalence relative to $(\mathcal{A}_*, \mathcal{A}''_*, p + p')$. 

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3. If we have two perverse data \((t, t'; \mathcal{X}_s, p_1)\) and \((t, t''; \mathcal{X}_s, p_2)\) with \(p_1 = p_2\), then \(t' = t''\).

### 3.1.3 Truncated mutations

Let \(E\) be an associative algebra over a base field \(k\). Let \(\{P_\alpha \mid \alpha \in \nabla\}\) be the set of (isomorphism classes of) indecomposable projective objects in the category \(E\text{-}mod\). We assume \(\nabla\) to be a finite set. Then it is well-know that \(E\) is Morita equivalent to \(\text{End}(\oplus P_\alpha)\).

Let \(\mathfrak{A} \hookrightarrow E\text{-}mod\) be a fully-faithful exact embedding of a finite length abelian subcategory with finite dimensional Hom’s, which preserves Ext’s. Assume the (isomorphism classes of) simple objects \(\{S_\alpha \mid \alpha \in \nabla\}\) in \(\mathfrak{A}\) are indexed by the same set \(\nabla\), such that each \(S_\alpha\) is simple in \(E\text{-}mod\) and its projective cover is \(P_\alpha\). We make one additional assumption: For each pair \((\theta, \alpha)\) in \(\nabla\), let \(S_{\alpha, \theta}\) be the universal extension, fitting into the short exact sequence

\[
0 \rightarrow \text{Ext}^1(S_\alpha, S_\theta)^* \otimes S_\theta \rightarrow S_{\alpha, \theta} \rightarrow S_\alpha \rightarrow 0.
\]

We assume the map \(\text{Hom}(P_\theta, P_\alpha) \rightarrow \text{Ext}^1(S_\alpha, S_\theta)^*\), induced by the composition morphism \(\text{Hom}(P_\alpha, S_{\alpha, \theta}) \otimes \text{Hom}(P_\theta, P_\alpha) \rightarrow \text{Hom}(P_\theta, S_{\alpha, \theta})\), is surjective.

In the case when \(E\) is finite dimensional over \(k\), the only example of such subcategory \(\mathfrak{A}\) is \(E\text{-}mod\) itself. A non-trivial example of such subcategory will be given in Subsection 3.1.5.

Fix an \(\theta \in \nabla\). For each \(\alpha \neq \theta\), we fix a section of the surjection \(\text{Hom}(P_\theta, P_\alpha) \rightarrow \text{Ext}^1(S_\alpha, S_{\theta, \text{theta}})^*\), and denote the image of the section by \(\text{Hom}(P_\theta, P_\alpha)_{t_\alpha}\). We define \(P'_\alpha\) to be \(P_\alpha\) if \(\alpha \neq \theta\), and \(P'_\theta\) to be the mapping cone in \(D^b(E\text{-}mod)\) of the natural map \(P_\theta \rightarrow \oplus_{\alpha \neq \theta} P_\alpha \otimes \text{Hom}(P_\theta, P_\alpha)_{t_\alpha}^*\).

**Definition 3.1.9.** If the natural map \(P_\theta \rightarrow \oplus_{\alpha \neq \theta} P_\alpha \otimes \text{Hom}(P_\theta, P_\alpha)_{t_\alpha}^*\) is injective, the set \(\{P'_\alpha \mid \alpha \in \nabla\}\) consists of objects in \(E\text{-}mod\). If moreover, \(P' := \oplus P'_\alpha\) has no higher self-extension, we say the set \(\{P'_\alpha \mid \alpha \in \nabla\}\) is the **truncated mutation** of \(\{P_\alpha \mid \alpha \in \nabla\}\) with respect to \(P_\theta\), if the natural map \(P_\theta \rightarrow \oplus_{\alpha \neq \theta} P_\alpha \otimes \text{Hom}(P_\theta, P_\alpha)_{t_\alpha}^*\) is injective.

Whether truncated mutations exist or not, the object \(P' := \oplus P'_\alpha\), considered as an
object in $D^b(E\text{-mod})$, always generates the triangulated category $D^b(E\text{-mod})$, in the sense that $P'^\perp = 0$ in $D^b(E\text{-mod})$. This can be easily verified from the fact that $\bigoplus \alpha P_\alpha$ generates $D^b(E\text{-mod})$. Therefore, we have the following Lemma.

**Lemma 3.1.10.** If the truncated mutation exists, then we get an equivalence of derived categories $D^b(E\text{-mod}) \cong D^b(\text{End}(P')\text{-mod})$. Also in this case, the projective objects in the $t$-structure coming from $\text{End}(P')\text{-mod}$ are objects in $E\text{-mod}$.

On the other hand, fixing a simple object $S_\theta$ in the abelian category $\mathfrak{A}$ such that $\text{Ext}^1(S_\theta, S_\theta) = 0$, we also have the tilting of $\mathfrak{A}$ with respect to $S_\theta$. Recall that the set of simple objects in $R_{S_\theta} \mathfrak{A}$ are given by Lemma 3.1.3, and they are denoted by $\{S'_\alpha \mid \alpha \in \nabla\}$. The $t$-structures obtained from truncated mutations and tiltings are related by the following Lemma.

**Lemma 3.1.11.** Assume the truncated mutation of $\{P_\alpha \mid \alpha \in \nabla\}$ with respect to $P_\theta$ exists, and $\text{Ext}^1(S_\theta, S_\theta) = 0$. Then the $t$-structure obtained from $\text{End}(P')\text{-mod}$ coincide with $R_{S_\theta} \mathfrak{A}$.

*Proof.* Two nested $t$-structures have to coincide. Therefore, it’s enough to show that $\text{Ext}^i(P'_\lambda, S'_\alpha) = 0$ for all $\lambda$, $\alpha$ and all $i > 0$. Clearly, for all $\alpha$ and all $i > 0$, we have $\text{Ext}^i(P_\lambda, S'_\alpha) = 0$ for all $\lambda \neq \theta$, and $\text{Ext}^i(P'_\theta, S_\theta[1]) = 0$. The only less clear point is the vanishing of $\text{Ext}^1(P'_\theta, S'_\alpha)$ for $\alpha \neq \theta$. For this we take the short exact sequence

$$0 \to P_\theta \to \bigoplus_{\alpha \neq \theta} P_\alpha \otimes \text{Hom}(P_\theta, P_\alpha)_{t_\alpha} \to P'_\theta \to 0,$$

and look at the long exact sequence associated to it. Note that $\text{Hom}(P_\lambda, S_\alpha) = \delta_{\lambda, \alpha} k$ and $\text{Ext}^1(P_\lambda, S'_\alpha) = 0$ for all $\lambda$, we get

$$\cdots \to \text{Hom}(P_\alpha, S'_\alpha) \otimes \text{Hom}(P_\theta, P_\lambda)_{t_\lambda} \to \text{Hom}(P_\theta, S'_\alpha) \to \text{Ext}^1(P'_\theta, S'_\alpha) \to 0 \to \cdots.$$

By the assumption that $\text{Hom}(P_\theta, P_\alpha)$ maps surjectively to $\text{Ext}^1(S_\alpha, S_\theta)^*$, the map

$$\text{Hom}(P_\alpha, S'_\alpha) \otimes \text{Hom}(P_\theta, P_\lambda)_{t_\lambda} \to \text{Hom}(P_\theta, S'_\alpha)$$

is also surjective, which conclude the vanishing of $\text{Ext}^1(P'_\theta, S'_\alpha)$. □
Remark 3.1.12. In fact, if we have more than one simple objects $S_1, \cdots, S_k$, we can define tilting with respect to all of them in a similar way. If $\text{Ext}^1(S_i, S_j) = 0$ for $i, j = 1, \cdots, k$, then the tilted subcategory of the derived category is also a finite length abelian category, by the same argument. And if the truncated mutations exist, they also give the projective objects in the tilted subcategory.

3.1.4 Iterated tilting and iterated truncated mutation

Fix a $\theta \in \nabla$ such that $\text{Ext}^1(S_{\theta}, S_{\theta}) = 0$. Recall that Proposition 3.1.4 says

$$R_{S_\theta[i-1]}R_{S_\theta[i-2]} \cdots R_{S_\theta}(\mathfrak{A})$$

is still a finite length category. There is a construction of its simple objects, and they are denoted by $\{S^i_\alpha \mid \alpha\}$. Similarly, let $P^0_\alpha = P_\alpha$. Recursively we define $P^i_\theta$ to be the mapping one of the natural map $P^{i-1}_{\theta} \to \oplus_{\alpha \neq \theta} P^{i-1}_\alpha \otimes \text{Hom}(P^{i-1}_{\theta}, P^{i-1}_\alpha)^*_{\alpha}$. For $\alpha \neq \theta$, we define $P^i_\alpha$ to be $P_\alpha$.

Lemma 3.1.13. Notations as above, we have $\text{Ext}^k(P^i_\alpha, S^i_\beta) = \delta^i_\beta^k$ for $k = 0$ and vanishes for $k \neq 0$.

Proof. We prove this by induction on $i$. For $i = 0$, this is clear.

Assume the statement for $i - 1$, now we show the corresponding statement for $k$. By chasing the Ext long exact sequence, we easily get $\text{Ext}^k(P^{i-1}_\lambda, S^i_\alpha) = 0$ for all $k \neq 0$ and $\lambda \neq \theta$, and $\text{Ext}^k(P^i_\theta, S^{i-1}_\theta[1]) = 0$ for $k \neq 0$.

We only need to show $\text{Ext}^k(P^i_\theta, S^i_\alpha) = 0$ for $\alpha \neq \theta$. Looking at the Ext long exact sequence, this is equivalent to the surjectivity of $\text{Hom}(P^{i-1}_\alpha, S^i_\alpha) \otimes \text{Hom}(P^{i-1}_\theta, P^{i-1}_\lambda)_{\lambda} \to \text{Hom}(P^{i-1}_\theta, S^i_\alpha)$. Note also that $\text{Hom}(P^{i-1}_\alpha, S^i_\alpha) \cong \text{Hom}(P^{i-1}_\alpha, S^{i-1}_\alpha)$, and $\text{Hom}(P^{i-1}_\theta, S^i_\alpha) \cong \text{Hom}(P^{i-1}_\theta, S^{i-1}_\theta) \otimes \text{Ext}^1(S^{i-1}_\alpha, S^{i-1}_\theta)^*$ for $\alpha \neq \theta$. This boils down to the surjectivity of $\text{Hom}(P^{i-1}_\theta, P^{i-1}_\alpha) \to \text{Ext}^1(S^{i-1}_\alpha, S^{i-1}_\theta)^*$.

$\square$
Corollary 3.1.14. Under the assumption of Lemma 3.1.13, if $E$ is a finite dimensional algebra, then truncated mutation with respect to $P_\theta$ exist as long as the natural map $P_\theta \rightarrow \bigoplus_{\alpha \neq \theta} P_\alpha \otimes \text{Hom}(P_\theta, P_\alpha)_{t_\alpha}$ is injective.

When we take the iterated mapping cone $P_i^i$, we assume that each time the truncated mutation exists. We know that $\text{End}(\bigoplus P_i \alpha)-\text{mod}$ is derived equivalent to $E-\text{mod}$, and $\{P_i^i \alpha | \alpha\}$ is a set of projective generators in $\text{End}(\bigoplus P_i^i \alpha)-\text{mod}$. In particular, all projective object in it has a representative in $E-\text{mod}$, and the indecomposable projective objects are projective covers of the simple object in $R_{S_\theta[i-1]} R_{S_\theta[i-2]} \cdots R_{S_\theta}(\mathcal{A})$. Conversely, we have the following Lemma.

Lemma 3.1.15. Suppose $P_\theta$ is the projective cover of $S_\theta$ in $E-\text{mod}$ and the truncated mutation exists up to $i-1$ iterations. Assume $R_{S_\theta[i-1]} R_{S_\theta[i-2]} \cdots R_{S_\theta}(\mathcal{A})$ is of finite length with simple objects $\{S_i^\alpha | \alpha \in \nabla\}$, and the projective covers of them have representatives in $E-\text{mod}$. Then the truncated mutation $\{P_i^i \alpha | \alpha \in \nabla\}$ exists.

Proof. We take the projective cover of $S_i^i$, denoted by $Q_i^i$, which can be chosen to be in $E-\text{mod}$. We know that $\text{Ext}^j(Q_i^i, S_\theta)$ vanish for $j \neq i$ and is one dimensional when $j = i$.

We take the minimal projective resolution of $Q_i^i$ in $\text{End}(\bigoplus P_i^i \alpha)-\text{mod}$. It has length 2 as the projective dimension of $Q_i^i$ is 1. The degree 1 term of the resolution has $P_i^{i-1}$ as a summand and the degree 0 term does not have summand $P_i^{i-1}$. This already implies the injectivity of $P_i^{i-1} \rightarrow \bigoplus_{\alpha \neq \theta} \text{Hom}(P_i^{i-1}, P_\alpha)_{t_\alpha} \otimes P_\alpha$. \hfill $\square$

Example 3.1.16. An example of Proposition 3.1.18 is the case when $\mathcal{A}$ is the category of perverse sheaves on $\mathbb{P}^n$ with the standard stratification. Let $S_n$ be the simple object $\mathbb{C}_{\mathbb{P}^n}[n]$ which is an $\mathbb{P}^n$ object in this category. The semi-reflection of $D^b(\mathcal{A})$ with respect to $S_n$ can be obtained by taking the image of $\text{Perv}(\mathbb{P}^{n*})$ under the Radon transform. In particular, the semi-reflection is derived equivalent to $\mathcal{A}$ and equivalence comes from a tilting generator in $\mathcal{A}$. In fact, according to Proposition 3.1.8, if one do tilting with respect to $S_n$ for $n$ times, one will get the same $t$-structure.
We will illustrate Proposition 3.1.18 by explicitly calculation of the tilting generator for the intermediate \( t \)-structures, i.e., those obtained from tilting with respect to \( S_n \) for \( i \) times for \( i < n \).

For simplicity, we take \( n = 2 \). The general case is similar. The category \( \mathfrak{A} \) is Morita equivalent to the module category of the quiver

\[
\begin{array}{ccc}
\bullet_{pt} & \xrightarrow{\alpha} & \bullet_{A_1} \\
& \xleftarrow{\beta} & \\
& \xleftarrow{\gamma} & \\
\bullet_{A_2} & \xrightarrow{\delta} & \bullet_{A_1}
\end{array}
\]

with relations \( \alpha \beta = 0, \delta \gamma = 0, \delta \alpha = 0, \) and \( \beta \gamma = 0 \). The projective objects in \( \mathfrak{A} \) are

\[
P_{pt} = \mathbb{C}_{pt}^2 \left[ \begin{array}{c} \mathbb{C}_{A_1} \end{array} \right];
\]

\[
P_{A_1} = \mathbb{C}_{pt} \left[ \begin{array}{c} \mathbb{C}_{A_1}^2 \mathbb{C}_{A_2} \end{array} \right];
\]

\[
P_{A_2} = \mathbb{C}_{A_1} \mathbb{C}_{A_2}.
\]

We consider the tilting with respect to \( S_2 \): The tilting generators are: \( P_{pt}, P_{A_1}, \) and

\[
coker(P_{A_2} \to P_{A_1}) \cong P'_{A_2} = \mathbb{C}_{pt} \left[ \begin{array}{c} \mathbb{C}_{A_1} \end{array} \right].
\]

Then we consider the tilting with respect to \( S_2[1] \): The tilting generators are: \( P_{pt}, P_{A_1}, \) and

\[
coker(P'_{A_2} \to P_{pt}) \cong P''_{A_2} = \mathbb{C}_{pt}.
\]

The hearts of all these \( t \)-structures are derived equivalent to \( \mathfrak{A} \).

If we do tilting with respect to \( S_2 \), the tilting generators of the new heart will be \( P_{A_1}, P_{A_2}, \) and the cokernel of the map \( P_{A_2} \to P_{A_1} \) which is \( P'_{A_2} = \mathbb{C}_{A_1} \left[ \begin{array}{c} \mathbb{C}_{A_2} \end{array} \right]. \) If we do tilting another time with respect to \( S_2[1] \), the tilting generators of the new heart will be \( P_{A_1}, P_{A_2}, \) and the cokernel of the map \( P'_{A_2} \to P_{pt} \) which is \( P''_{A_2} = \mathbb{C}_{pt}. \) The hearts of all these \( t \)-structures are derived equivalent to \( \mathfrak{A} \).
3.1.5 Truncated mutations from geometric origin

Now let $X$ be a smooth variety which is projective over $\text{Spec} \, A$. Also we assume the map $\pi : X \to \text{Spec} \, A$ is $\mathbb{G}_m$-equivariant, such that $X$ is deformation retracts to $X = \pi^{-1}(\text{Spec} \, A/m)$, the fiber over $A/m$ under this $\mathbb{G}_m$-action. Let $\{P_{\alpha} \mid \nabla\}$ be a collection of $\mathbb{G}_m$-equivariant vector bundles on $X$, which classically generates $\text{Qcoh}(X)$ and $\text{Ext}^i(\oplus P_{\alpha}, \oplus P_{\alpha}) = 0$ for all $i > 0$. Let $E = \text{End}(\oplus_{\alpha \in \nabla} P_{\alpha})$. Then [15] gives an equivalence of derived categories $D(\text{Qcoh}(X)) \cong D(E\text{-Mod})$, and it restricts to equivalences $D^b(E\text{-mod}) \cong D^b(\text{Coh}(X))$, and $D^b_{A/m}(E\text{-mod}) \cong D^b_X(\text{Coh}(X))$. Now we take $A$ to be the category of $E$-modules set theoretically supported at $A/m$.

Fix a $\theta \in \nabla$. Assume $S_{\theta}$ is a simple object in $\mathfrak{A}$ with $\text{Ext}^1(S_{\theta}, S_{\theta}) = 0$. We take $P^0_{\alpha}$ to be $P_{\alpha}$ for all $\alpha \in \nabla$. Recursively, we define $P^i_{\theta}$ to be the mapping cone of the natural map $P^{i-1}_{\theta} \to \oplus_{\alpha \neq \theta} \text{Hom}(P^{i-1}_{\theta}, P_{\alpha})^* \otimes P_{\alpha}$, and $P^i_{\alpha} = P^{i-1}_{\alpha}$ for $\alpha \neq \theta$. Define $P^i = \oplus_{\alpha \in \nabla} P^i_{\alpha}$.

**Lemma 3.1.17.** Assume $S_{\theta}$ is a simple object in $\mathfrak{A}$ with $\text{Ext}^1(S_{\theta}, S_{\theta}) = 0$. The map $\text{Hom}(P^{i-1}_{\theta}, P^{i-1}_{\alpha}) \to \text{Ext}^1(S_{\alpha}, S_{\beta})^*$ induced by the composition morphism $\text{Hom}(P_{\theta}, S_{\alpha, \theta}) \otimes \text{Hom}(P_{\theta}, P_{\alpha}) \to \text{Hom}(P_{\theta}, S_{\alpha, \theta})$ is surjective.

**Proof.** For a complex $N$, let $\text{red} \, N$ be the complex fit in the exact triangle $\text{red} \, N \to N \to \oplus \alpha \text{Hom}(N, S^{i-1}_{\alpha}) \otimes S^{i-1}_{\alpha}$. Then we have $\text{Ext}^i(\text{red} \, N, S^{i-1}_{\alpha}) = 0$ for all $i < 0$ if this property holds for $N$.

We have, from the exact triangle $\text{red} \, P^{i-1}_{\alpha} \to P^{i-1}_{\alpha} \to S^{i-1}_{\alpha}$, that $\text{Hom}(P^{i-1}_{\theta}, P^{i-1}_{\alpha}) \cong \text{Hom}(P^{i-1}_{\theta}, \text{red} \, P^{i-1}_{\alpha})$.

Also from the exact triangle $\text{red} \, P^{i-1}_{\alpha} \to \text{red} \, P^{i-1}_{\alpha} \to \oplus_{\alpha} \text{Hom}(\text{red} \, P^{i-1}_{\alpha}, S^{i-1}_{\alpha}) \otimes S^{i-1}_{\alpha}$, we have $\text{Hom}(\text{red} \, P^{i-1}_{\alpha}, S^{i-1}_{\theta}) \cong \text{Hom}(P^{i-1}_{\theta}, \oplus_{\alpha} \text{Hom}(\text{red} \, P^{i-1}_{\alpha}, S^{i-1}_{\alpha}) \otimes S^{i-1}_{\alpha})^*$. We only need to show

$$\text{Ext}^1(P^{i-1}_{\theta}, \text{red} \, P^{i-1}_{\alpha}) = 0.$$
For this purpose, note that the complex \( Q \cong \text{red red } P^{i-1}_{\alpha} \) can be chosen \( \mathbb{G}_m \)-equivariantly. Let \( Q_k \) be \( Q/m^k Q \). Then \( Q \) can be obtained by taking the \( \mathbb{G}_m \) finite part of \( \varprojlim Q_k \). Since \( P^{i-1}_{\alpha} \) is equivariant under \( \mathbb{G}_m \), we have the canonical isomorphism of complexes \( R \text{Hom}(P^{i-1}_{\theta}, Q) \cong R \text{Hom}(\varprojlim P^{i-1}_{\theta}, Q) \). Hence, \( H^1(R \text{Hom}(P^{i-1}_{\theta}, Q)) = 0 \) implies \( H^1(R \text{Hom}(P^{i-1}_{\theta}, Q)) \cong \varprojlim H^1(P^{i-1}_{\theta}, Q) = 0 \). Note that \( Q_k \) lies in \( D^b(\mathfrak{A}) \), and has the property that \( \text{Ext}^i(Q_k, S^{i-1}_{\alpha}) = 0 \) for all \( i < 0 \), all \( \alpha \), and large enough \( k \). This means \( Q_k \) can be chosen as a complex concentrated in non-positive degrees with respect to the \( t \)-structure \( R S_{\theta}^{i-2} R S_{\theta}^{i-3} \cdots R S_{\theta}(\mathfrak{A}) \). Therefore, we have \( \text{Ext}^1(P^{i-1}_{\theta}, Q_k) = 0 \).

Take \( P'_\theta \) to be the mapping cone of the natural map \( P_\theta \rightarrow \bigoplus_{\alpha \neq \theta} P_\alpha \otimes \text{Hom}(P_\theta, P_\alpha)^*_t \), and \( P'_\alpha = P_\alpha \) for \( \alpha \neq \theta \). Then, there is an equivalence between \( D(E\text{-mod}) \) and \( D(R \text{Hom}(P', P')) \), where \( R \text{Hom}(P', P') \) is understood as a DG-algebra. The DG-algebra \( R \text{Hom}(P', P') \) has homology concentrated in non-negative degrees not exceeding 1, and is concentrated in degree zero if and only if \( P_\theta \rightarrow \bigoplus_{\alpha \neq \theta} P_\alpha \otimes \text{Hom}(P_\theta, P_\alpha)^*_t \) is injective, namely, the truncated mutation exits. In general, we also have an equivalence \( D(E\text{-mod}) \cong D(R \text{Hom}(P^i, P^i)) \). Inductively, the DG-algebra \( R \text{Hom}(P^i, P^i) \) has homologies concentrated in non-negative degrees, and is concentrated in degree zero if and only if \( P^{j-1}_{\theta} \rightarrow \bigoplus_{\alpha \neq \theta} P^j_{\alpha} \otimes \text{Hom}(P^{j-1}_{\theta}, P^{j-1}_{\alpha})^*_t \) is injective for all \( j \leq i \).

**Proposition 3.1.18.** Assume \( S_\theta \) is a simple object in \( \mathfrak{A} \) with \( \text{Ext}^1(S_\theta, S_\theta) = 0 \). Assume the \( n \)-th iterated tilting with respect to \( S_\theta \) has a set of indecomposable projectives \( \{ Q_\alpha \} \) consists of objects concentrated in degree zero. Then the iterated truncated mutations up to \( n \) times exist.

**Proof.** As we have \( \text{Hom}(P^i_{\alpha}, S^j_{\beta}) = \delta_{\alpha}^{\beta} k \) according to Lemma 3.1.13. This means \( P^i_{\alpha} \cong Q_\alpha \) for all \( \alpha \), and hence \( P^i_{\alpha} \) is concentrated in degree zero.

**Remark 3.1.19.** If \( \mathfrak{A} \) is a finite length abelian category with enough projective objects, then the conclusion in Proposition 3.1.18 still holds.
Corollary 3.1.20. Assume $S_\theta$ is a simple object in $\mathfrak{A}$ with $\text{Ext}^1(S_\theta, S_\theta) = 0$. We endow $\mathfrak{A}$ with the filtration that $0 = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 = \mathfrak{A}$ where $\mathfrak{A}_1 = \langle S_\theta \rangle$. Assume for the perversity function $p$ with $p(1) = 0$ and $p(2) = n$ we have a perverse equivalence $(t, t', p)$ such that the projective covers of the simple objects in the heart of $t'$ have representatives lying in $E\text{-mod}$. Then for any $p'$ with $p'(1) = 0$ and $p'(1) \leq n$ the perverse equivalence $(t, t'', p')$ exists, and the projective covers of the simple objects in the heart of $t'$ have representatives lying in $E\text{-mod}$.

Example 3.1.21. Let $\pi : T^*\mathbb{P}^n \to \mathbb{P}^n$ and let $D = D^b(\text{Coh}_0 T^*\mathbb{P}^n)$. Let $\mathfrak{A} = \text{heart of the } t\text{-structure in } D\text{ induced by the tilting bundle } \pi^*(\bigoplus_{i=0}^n \mathcal{O}(i))\text{ on } T^*\mathbb{P}^n$. The simple objects in $\mathfrak{A}$ are $\{\wedge^i \mathcal{Q}|_{\mathbb{P}^n}[i] \mid i = 0, \ldots, n\}$. In $D^b(\text{Coh}_0(T^*\mathbb{P}^n))$ there is a $t\text{-structure } \mathfrak{A}'\text{ induced by the tilting object } \pi^*(\bigoplus_{i=0}^n \mathcal{O}_t(i))\text{ on } T^*\mathbb{P}^n$. The transform of $\mathfrak{A}'$ under the Fourier-Mukai transform of Namikawa in [62] is the semi-reflection of $\mathfrak{A}$ with respect to $S := \wedge^n \mathcal{Q}|_{\mathbb{P}^n}[n]$. Clearly $S$ is a $\mathbb{P}^n$-object.

$$
\begin{array}{ccc}
T^*\mathbb{P}^n & \to & T^*(\mathbb{P}^n) \\
\downarrow & & \downarrow \\
\mathcal{N} & \to & \mathcal{N}
\end{array}
$$

According to Proposition 3.1.18, this $t\text{-structure can alternatively be described as iterative tilting with respect to } S\text{ } n\text{-times. We study the projective generators in the hearts of all these intermediate } t\text{-structures using truncated mutation.}

For simplicity, we take $n = 2$. The algebra $E := \text{End}_{T^*\mathbb{P}^2}(p_i^*(\bigoplus_{i=0}^2 \mathcal{O}(i)))$ can be described by the following quiver

$$
\begin{array}{ccc}
\bullet_0 & \xrightarrow{\alpha(C^3)} & \bullet_1 \\
\wedge(C^3) & \xrightarrow{\delta(C^3)} & \gamma(C^3) \\
\bullet_2 & \xrightarrow{\beta(C^3)} & \beta(C^3)
\end{array}
$$

with relations

\[
\delta \alpha(\wedge^2 C^3); \quad \beta \gamma(\wedge^2 C^3); \\
\beta \alpha(C); \quad \delta \gamma(C); \quad \gamma \delta + \alpha \beta(C).
\]
The projective objects, $P_0$, $P_1$, and $P_2$ are spanned by paths starting at the vertices 0, 1, and 2 respectively.

Consider the $t$-structure obtained by tilting of $\mathcal{A}$ with respect to $S$. The indecomposable projective objects are $P_0$, $P_1$, and $\tilde{P}_2$, where $\tilde{P}_2$ is the mapping cone of the morphism $P_2 \to P_1 \otimes \mathbb{C}^3$. It can be visualized as pre-composing paths from 1 with the arrow $\delta$. As the relation indicates, the morphism $P_2 \to P_1 \otimes \mathbb{C}^3$ is injective. In terms of the quiver picture, this fact is equivalent to that pre-composing with $\delta$ does not kill any path from 1. Therefore, $\tilde{P}_2$ is the cokernel of $P_2 \to P_1 \otimes \mathbb{C}^3$, which in particular is in $E$-mod.

Consider the $t$-structure obtained by tilting of $R_S A$ with respect to $S[1]$. The indecomposable projective objects are $P_0$, $P_1$, and $\tilde{\tilde{P}}_2$, where $\tilde{\tilde{P}}_2$ is the mapping cone of the morphism $\tilde{P}_2 \to P_0 \otimes \mathbb{C}^3$. In terms of quivers, this map can be visualized as pre-composing paths from 1 with the arrow $\alpha$. Again it is easy to see that this map is injective, hence $\tilde{\tilde{P}}_2$ is the cokernel of $\tilde{P}_2 \to P_0 \otimes \mathbb{C}^3$. To summarize, this $t$-structure is the semi-reflection of $\mathcal{A}$ with respect to the $\mathbb{P}^2$-object $S$. The indecomposable projective objects in the semi-reflection are $P_0$, $P_1$, and $\tilde{\tilde{P}}_2$, all of which are in $E$-mod.

An example of truncated mutations from geometric origin as in the set-up of this subsection will be given in Section 3.6.

### 3.1.6 Koszulity of truncated mutations

**Lemma 3.1.22.** In the set up of Section 3.1.5, assume there is a choice of $\{S_{\alpha} \mid \alpha \in \nabla\}$ such that each one is graded, and $\Ext^1(S_{\alpha}, S_{\beta})$ has homogeneous degree one for any $\alpha$ and $\beta \in \nabla$. Then there is such a choice for $\{S'_{\alpha} \mid \alpha \in \nabla\}$ with the same properties.

**Proof.** We define the grading on $\{S'_{\alpha} \mid \alpha \in \nabla\}$ as follows. For $S'_{\theta} \cong S_{\theta}[1]$, we define the its degree to be the degree of $S_{\theta} - 1$. For $\alpha \neq \theta$, we define the degree of $S'_{\alpha}$, which is the universal extension of $S_{\alpha}$ by $S_{\theta}$, by keeping the degree of $S_{\alpha}$ and $S_{\theta}$ as they are, and declare $\Ext^1(S_{\alpha}, S_{\theta})^*$ to be in degree zero.
Then we get right away that \( \text{Ext}^1(S_\theta[1], S'_\alpha) \cong \text{Hom}(S_\theta, S_\theta) \otimes \text{Ext}^1(S_\alpha, S_\theta)^* \) has degree 1, since \( \text{Hom}(S_\theta, S_\theta) \) has degree 1 and \( \text{Ext}^1(S_\alpha, S_\theta)^* \) has degree zero.

As for \( \text{Ext}^1(S'_\alpha, S_\theta[1]) \cong \text{Ext}^2(S'_\alpha, S_\theta) \), where \( \alpha \neq \theta \), look at the following part of a long exact sequence

\[
\cdots \rightarrow \text{Ext}^2(S_\alpha, S_\theta) \rightarrow \text{Ext}^2(S'_\alpha, S_\theta) \rightarrow \text{Ext}^2(S_\theta, S_\theta) \otimes \text{Ext}^1(S_\alpha, S_\theta) \rightarrow \cdots ,
\]
we need to show the terms at two sides both have degree 1. For \( \text{Ext}^2(S_\alpha, S_\theta) \) this is clear by assumption and the fact that the degree of \( S_\theta \) has been reduced by 1. For the same reason, \( \text{Ext}^2(S_\theta, S_\theta) \) also has degree 1. The degree of \( \text{Ext}^1(S_\alpha, S_\theta) \) has been declared to be zero. So, the term \( \text{Ext}^2(S_\theta, S_\theta) \otimes \text{Ext}^1(S_\alpha, S_\theta) \) also has degree one.

In the case \( \alpha, \beta \neq \theta \), we first show that \( \text{Ext}^1(S'_\beta, S_\alpha) \) has degree 1. This can be done by observing the two sides of the following part of a long exact sequence

\[
\cdots \rightarrow \text{Ext}^1(S_\beta, S_\alpha) \rightarrow \text{Ext}^1(S'_\beta, S_\alpha) \rightarrow \text{Ext}^1(S_\theta, S_\alpha) \otimes \text{Ext}^1(S_\beta, S_\theta) \rightarrow \cdots .
\]
Then we look at the following part of a different long exact sequence

\[
\cdots \rightarrow \text{Ext}^1(S'_\beta, S_\theta) \otimes \text{Ext}^1(S_\alpha, S_\theta)^* \rightarrow \text{Ext}^1(S'_\beta, S'_\alpha) \rightarrow \text{Ext}^1(S'_\beta, S_\alpha) \rightarrow \cdots .
\]
Note that \( \text{Ext}^1(S'_\alpha, S_\theta) = 0 \). We conclude that \( \text{Ext}^1(S'_\beta, S'_\alpha) \) also has degree 1. \( \square \)

**Lemma 3.1.23.** Assume there is a choice for \( \{ P_\alpha \mid \alpha \in \nabla \} \) such that each \( P_\alpha \) is graded and \( \text{End}(P) \) has only non-negative degree pieces, with \( \text{Hom}(P_\theta, P_\alpha)_{t_\alpha} \) lies in homogeneous degree one. Assume further that the truncated mutation exists, then there is also such a choice for \( \{ P'_\alpha \mid \alpha \in \nabla \} \) such that \( \text{End}(P') \) has only non-negative degree pieces, and \( \text{Hom}(P'_\theta, P'_\alpha)_{t_\alpha} \) can be chosen to be in homogeneous degree one.

**Proof.** We define the grading on \( \{ P'_\alpha \mid \alpha \in \nabla \} \) as follows. For \( \alpha \neq \theta \), we have \( P'_\alpha \cong P_\alpha \), and we keep the grading of it as it is. For \( P'_\theta \) which is the cokernel of the map \( P_\theta \rightarrow \bigoplus_{\alpha \neq \theta} P_\alpha \oplus \text{Hom}(P_\theta, P_\alpha)^*_{t_\alpha} \), we use the grading of \( P_\theta \) and \( P_\alpha \) and declare \( \text{Hom}(P_\theta, P_\alpha)^*_{t_\alpha} \) to be in degree -1.

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Clearly, $\text{Hom}(P'_\alpha, P'_\beta)$ for $\alpha, \beta \neq \theta$ has not been influenced. Also it is clear that $\text{Hom}(P'_\theta, P'_\alpha)$ has non-negative grading. Note also that $\text{Hom}(P'_\theta, P'_\alpha) \otimes \text{Hom}(P'_\alpha, P'_\beta)$ embeds into

$$\oplus_{\alpha \neq \theta} \text{Hom}(P'_\alpha, P'_\theta) \otimes \text{Hom}(P'_\theta, P'_\alpha).$$

We only need to show the non-negativity of the grading of $\text{Hom}(P'_\alpha, P'_\theta)$ for $\alpha \neq \theta$. For this, we need to show the degree -1 part of $\text{Hom}(P'_\alpha, P'_\alpha) \otimes \text{Hom}(P'_\theta, P'_\alpha)^*$ maps injectively into $\text{Hom}(P'_\alpha, P'_\theta)$. This is clear from the construction of $\text{Hom}(P'_\theta, P'_\alpha)^*$.

\begin{flushright}
$\square$
\end{flushright}

\section{The $t$-structures from quantization in positive characteristic}

\subsection{Localization of rational Cherednik algebras}

We work over a separably closed field $k$ of characteristic $p$ which is large enough. Let $\Gamma_1 \subseteq SL(2)$ be a finite subgroup. Let $\Gamma_n := (\Gamma_1)^n \rtimes S_n$ acting on $\mathfrak{h} := \mathbb{A}^n$ in the natural way, i.e., the $i$-th copy of $\Gamma_1$ acts on the $i$-th $\mathbb{A}^1$ summand, and $S_n$ permutes the coordinates.

There is a natural symplectic form on $\mathbb{A}^{2n} \cong \mathfrak{h} \oplus \mathfrak{h}^*$. It is preserved by the diagonal action of $\Gamma_n$. A symplectic resolution of $\mathbb{A}^{2n}/\Gamma_n$ can be given as $\text{Hilb}^n(\tilde{\mathbb{A}}^2/\Gamma_1)$, where $\tilde{\mathbb{A}}^2/\Gamma_1$ is the minimal resolution of $\mathbb{A}^2/\Gamma_1$. Later on we will use the short hand notation $\text{Hilb}^{n\Gamma_1}$ or simply $\text{Hilb}^n$ when $\Gamma_1$ is clear from the context.

One of example is when $\Gamma = (\mathbb{Z}/l)^n \rtimes S_n$. It is well-known (see [50]) that a symplectic resolution of $\mathbb{A}^{2n}/\Gamma_n$ can be constructed as a Nakajima quiver variety of extended Dynkin quiver with suitable dimension vectors and stability conditions. Recall the Nakajima variety of a quiver $Q$ with dimension vectors $v$ and $w$, stability condition $\theta$ is the Hamiltonian reduction $T^*(\text{Rep}(Q, v) \oplus \text{Hom}(k^v, k^w))/(\theta GL(v)$. For suitable choice of stability condition, the Nakajima variety isomorphic to $\text{Hilb}^n(\tilde{\mathbb{A}}^2/\Gamma_1)$. In perticular, we know $H^2(\text{Hilb}^n(\tilde{\mathbb{A}}^2/\Gamma_1))$ is isomorphic to the character group of $GL(v)$, which is a free abelian group with a basis
indexed by the vertices of this quiver. The Weil divisors on Hilb\(^n\) corresponding to these basis elements are again in natural one to one correspondence with the conjugacy classes of symplectic reflections in the group \(\Gamma_n\) (see, e.g., [13, §4] for a description of this correspondence).

Write Hilb\(^n(1)\) for the Frobenius twist Hilb\(^n(\widehat{\mathbb{A}^2/\mathbb{Z}_d})^{(1)}\). Quantizations of Hilb\(^n(1)\) are related to rational Cherednik algebras. The precise relationship is given by [12] and [11] which we will briefly recollect here.

Let Ref be the set of reflections in \(\Gamma_n\). Decompose Ref = \(\bigsqcup_{i=0}^r\text{Ref}_i\) into conjugacy classes. Pick integers \(c = (c_0, c_1, \cdots, c_r)\), the rational Cherednik algebra

\[ H_c = H_c(\mathfrak{h}, \Gamma_n) := k[\mathfrak{h}]\langle \mathfrak{h}^* \rangle \#\Gamma_n / I \]

where \(I\) is generated by \([u, v] = \langle u, v \rangle - 2\sum_{i=1}^r c_i \sum_{\gamma \in \text{Ref}_i} \langle u, v \rangle_{\gamma} \cdot \gamma\) for \(u \in \mathfrak{h}\) and \(v \in \mathfrak{h}^*\), where \(\langle -, - \rangle_{\gamma}\) is the paring between im(\(\gamma - 1\)) and its dual.

The algebra \(H_c\) as a natural filtration, and the associated graded is \(k[\mathbb{A}^{2n}] \#\Gamma_n\). Let \(\mathbb{A}^{2n(1)}\) be the Frobenius twist of \(\mathbb{A}^{2n}\), then the algebra \(H_c\) has a big Frobenius center \(k[\mathbb{A}^{2n(1)}] \#\Gamma_n\).

For any central character \(\chi\) (i.e., an element in the maximal spectrum of \(k[\mathbb{A}^{2n(1)}] \#\Gamma_n\)) we can consider the category of finitely generated modules over \(H_c\), on which the Frobenius center acts by the central character \(\chi\). This category will be denoted by \(\text{Mod}-\chi H_c\). The irreducible objects in the category \(\text{Mod}-\chi H_c\) are naturally labeled by \(\text{Irrep}(\Gamma_n)\).

Let \(e := \sum_{\gamma \in \Gamma_n} \gamma\). For generic values of \(c\), \(H_c\) is Morita equivalent to \(^{s}H_c := eH_c e\). Such values of \(c\) are called spherical values. The special values are called aspherical values. Let \(\text{Mod}_{0}^{s}H_c\) be the category of \(^{s}H_c\)-modules with central character 0. The irreducible object in \(\text{Mod}_{0}^{s}H_c\) labeled by \(\tau \in \text{Irrep}(\Gamma_n)\) will be denoted by \(L_{c}(\tau)\).

Taking any \(\chi \in H^2(\text{Hilb}^n, \mathbb{Q})\), there is a quantization \(\mathcal{A}_\chi\) of Hilb\(^n(1)\) coming from the quantum Hamiltonian reduction of the sheaf of \(\chi\)-twisted differential operators on \(\text{Rep}(Q, v) \oplus \text{Hom}(v, w)\).

**Theorem 3.2.1** (Bezrukavnikov-Finkelberg-Ginzburg). *For each \(c\), there is a sheaf of alge-
bras $\mathcal{A}_c$ on $\text{Hilb} := \text{Hilb}^n(\mathbb{A}^2/\Gamma_1)$, which is an Azumaya algebra on $\text{Hilb}^{(1)}$.

1. $\mathcal{A}_x$ splits on the formal neighborhood of the fibers of the Hilbert-Chow morphism;

2. $H^i(\text{Hilb}^{(1)}, \mathcal{A}_c) = 0$ for $i > 0$;

3. for large enough $p$, one has an isomorphism

$$\phi_c : \Gamma(\text{Hilb}^{(1)}, \mathcal{A}_c) \cong {}^sH_c;$$

4. for spherical values $c$, $^sH_c$ has finite global dimension, in which case one has a derived equivalence $D^b(\text{Coh}_0\text{Hilb}^{(1)}) \cong D^b(\text{Mod}_{-0}{}^sH_c)$.

In the terminology of [23], the algebras $^sH_c$’s are called noncommutative resolutions of singularities when $c$ is spherical. They are derived equivalent to $\text{Coh}(\text{Hilb}^{(1)})$. As the splitting vector bundle on the formal neighborhood can be chosen to be $\mathbb{G}_m$-equivariant, therefore, a standard argument shows it extends as a vector bundle $\mathcal{E}_c$ on the whole $\text{Hilb}^{(1)}$ and gives a global derived equivalence between $\text{Coh}_0(\text{Hilb}^{(1)})$ and $\text{Mod}_c\text{End}(\mathcal{E}_c)$.

In particular, take $c = 0$ (which is always spherical) we get a derived equivalence

$$D^b(\text{Coh}\text{Hilb}_{\Gamma_1}^n) \cong D^b(\text{Cohr}_{\alpha}(\mathbb{A}^{2n})).$$

This equivalence is called the symplectic McKay correspondence. The splitting bundle $\mathcal{E}_0$ is called a Procesi bundle in the terminology of [55]. The non-uniqueness of choice of $\mathcal{E}_0$ has been studied in [55].

For each spherical value $c$, the derived equivalence

$$D^b(\text{Coh}_0\text{Hilb}^{(1)}) \cong D^b(\text{Mod}_{-0}{}^sH_c)$$

endows $D^b(\text{Coh}_0\text{Hilb}^{(1)})$ with a $t$-structure.

**Question 3.2.2.** For two different spherical values $c$ and $c'$, what is the relation between the $t$-structures on $D^b(\text{Coh}_0\text{Hilb}^{(1)})$?
The aspherical values form a union of affine hyperplanes. The open facets will be called alcoves, and codimension-1 facets will be called walls.

If \(c\) and \(c'\) are in the same alcove, then the translation functor induces a Morita equivalence; the \(t\)-structures are the same. In particular, the aspherical values are exactly the locus where the central charge applied to some simple object vanishes.

Let \(L_c(\tau)\) be the irreducible object in \(\text{Mod}_{0}^*H_c\) labeled by \(\tau \in \text{Irrep}(\Gamma_n)\). Under the BFG-derived equivalence, for any irreducible object \(L_c(\tau) \in \text{Mod}_{0}^*H_c\), let the corresponding complex in \(D^b(\text{Coh}_0\text{Hilb}^{(1)})\) be denoted by \(\mathcal{L}_c(\tau)\); for the projective cover of \(L_c(\tau)\) in \(\text{Mod-End}(\mathcal{E}_c)\), let the corresponding vector bundle on \(\text{Hilb}\) be denoted by \(\mathcal{V}\). We have

\[
\text{Ext}^i(\mathcal{V}_\alpha, \mathcal{L}_\beta) = \begin{cases} 
\delta_{\alpha, \beta}, & i = 0; \\
0, & i > 0.
\end{cases}
\]

As a corollary of the derived localization theorem, the translation functors, and the Hirzebruch-Riemann-Roch, we get the following Lemma.

**Lemma 3.2.3.** When \(c + \nu\) is in the same alcove as \(c\),

\[
\dim L_{c+\nu}(\tau) := \chi(\mathcal{L}_c(\tau) \otimes \mathcal{E}_0 \otimes \mathcal{O}(\nu))
\]

is a polynomial in \(\nu\). Here for any \(\nu \in H^2(\text{Hilb}^{n(1)})\), the corresponding line bundle is denoted by \(\mathcal{O}(\nu)\).

There polynomials will be refered to as the dimension polynomials.

We reparameterize the value \(c\) by setting \(x = cp\) and define

\[
Z_\tau(x) = \lim_{p \to \infty} p^{-n} \dim_k L_c(\tau; p).
\]

(3.2)

We consider the collection of polynomials \(\{Z_\tau(x) \mid \tau \in \text{Irrep}(\Gamma_n)\}\) as a polynomial map

\[
H^2(\text{Hilb}; \mathbb{Q}) \to \text{Hom}_\mathbb{Z}(K_0(\text{Hilb}), \mathbb{Q}).
\]

This polynomial map is called the central charge.

We will make precise of the slogan that the central charge controls the difference of the \(t\)-structures associated to different alcoves.
3.2.2 Real variation of stability conditions

Bridgeland introduced a notion of stability conditions, (see [24],) which parameterizes all bounded $t$-structures of the same triangulated category and goes along well with deformations. Recall that for an abelian category $\mathfrak{A}$, a stability function on it is a group homomorphism $Z : K(\mathfrak{A}) \to \mathbb{C}$ such that

$$0 \neq E \in \mathfrak{A} \Rightarrow Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi(E))$$

where the real number $\phi(E) \in (0, 1]$ is called the phase of $E$. A nonzero subobject is said to be semi-stable with respect to $Z$ if every subobject has smaller or equal phase. The stability function $Z$ is said to have the Harder-Narasimhan property if every nonzero object $E \in \mathfrak{A}$ has a finite filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ whose factors $F_j = E_j/E_{j-1}$ are semistable objects of $\mathfrak{A}$ with $\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n)$.

Defining a Bridgeland stability condition on a triangulated category $D$ is equivalent to giving a bounded $t$-structure on $D$ together with a stability function on its heart with the Harder-Narasimhan property. Bridgeland showed that the set $\text{Stab}(D)$ of all stability conditions on a triangulated category $D$ has a complex manifold structure, such that the function $\text{Stab}(D) \to K(D)^\ast$ sending any stability condition to its stability function is an local isomorphism to a subspace $V \subseteq K(D)^\ast$ on each connected component of $\text{Stab}(D)$.

There is a notion of real variation of stabilities defined in [2]. We briefly recall the definition here.

Let $D$ be a $k$-linear triangulated category with finite rank $K$-group and finite dimensional Hom’s, and $V$ a real vector space. Fix a discrete collection $\Sigma$ of affine hyperplanes in $V$. Let $V^0$ denote their complement. Let $\Sigma_{lin}$ be the set of their translations through zero, a collection of linear hyperplanes. Fix a component $V^+$ of $V \setminus \cup \Sigma_{lin}$. The choice of $V^+$ determines for each $H \in \Sigma$ the choice of the positive half-space $(V \setminus H)^+ \subset V \setminus H$. Let $Alc$ denote the set of all alcoves, i.e., connected components of $V^0$. 

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Definition 3.2.4. A real variation of stability conditions on $D$ parametrized by $V^0$ and directed to $V^+$ is the data $(Z, \tau)$, where $Z$ (the central charge) is a polynomial map $Z : V \to (K^0(D) \otimes \mathbb{R})^*$, and $\tau$ is a map from $Alc$ to the set of bounded $t$-structures on $D$ with finite length hearts, subject to the following conditions.

1. For $0 \neq M \in \tau(A)$ and $x \in A$, $\langle Z(x), [M] \rangle > 0$.

2. Suppose $A, A' \in Alc$ share a codimension one face and $A'$ is above $A$. Let $A_n \subseteq \tau(A)$ be the full subcategory $\{M \in A_n \mid \langle Z(x), [M] \rangle \text{ has zero of order at least } n\}$. Then we require:

   - The $t$-structure $\tau(A')$ is compatible with the filtration.
   - The $t$-structure on $\text{gr}_n(D) = D_n/D_{n+1}$ induced by $\tau(A)$ differs from that of $\tau(A')$ by $[n]$.

Now we can state our main theorem of this paper.

Theorem 3.2.5. Let $Z : H^2(\text{Hilb}_{Z_2}; \mathbb{Q}) \to K^2(\text{Hilb}_{Z_2})^\vee$ be defined as in (3.2). Let $\tau$ be the assignment associating each alcove in $H^2(\text{Hilb}_{Z_2}; \mathbb{Q})$ to the $t$-structure on $D^b(\text{Coh}_0(\text{Hilb}_{Z_2}))$ given by $\text{Mod}_{0}^*\text{H}_c$ for some $c$ in this alcove. Then, the pair $(Z, \tau)$ is a real variation of stability conditions on $D^b(\text{Coh}_0(\text{Hilb}_{Z_2}))$.

3.2.3 Comparison with category $\mathcal{O}_c$ in characteristic zero

Let $\Gamma$ be an arbitrary reflection group acting on $\mathfrak{h}$. Let $V = \mathfrak{h} \oplus \mathfrak{h}^*$ with the natural symplectic form and the diagonal $\Gamma$-action.

Let $Z \subseteq R \subset \mathbb{C}$ finitely generated over $\mathbb{Z}$, such that $H_c(\Gamma_2)_R$ exists. Let $\mathcal{O}_c$ be the category $\mathcal{O}$ of $H_c(\Gamma)_C$. For any $\tau \in \text{Irrep}(\Gamma)$ let the corresponding irreducible object in $\mathcal{O}_c$ be denoted by $L_c(\tau; \mathbb{C})$, and its $R$-form be $L_c(\tau; R)$. Let $\overline{L_c(\tau; R)_k}$ be the central reduction of $L_c(\tau; R) \otimes_R k$. 

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Lemma 3.2.6. For any parameter $c$ and any $\tau \in \text{Irrep}(\Gamma)$, we have $L_c(\tau; C)^\Gamma = 0$ if and only if $\overline{L_c(\tau; R)_k}^\Gamma = 0$.

Proof. If $L_c(\tau; C)^\Gamma = 0$ then we can choose $L_c(\tau; R)$ so that $L_c(\tau; R)^\Gamma = 0$. Therefore, clearly we have $\overline{L_c(\tau; R)_k}^\Gamma = 0$.

Conversely, for any weight space $L_c(\tau; C)[\alpha]$, for $p >> 0$ we have an isomorphism $L_c(\tau; C)[\alpha] \to \overline{L_c(\tau; R)_k}[\alpha]$. If $L_c(\tau; C)^\Gamma \neq 0$, then there is some weight $\alpha$ such that $L_c(\tau; C)[\alpha]^\Gamma \neq 0$, and therefore $\overline{L_c(\tau; R)_k}[\alpha]^\Gamma \neq 0$. \qed

Recall that in the terminology of [9], such representations is said to be asperical. The asperical locus in $H^2(\text{Hilb}; \mathbb{Q})$ (defined to be the locus where $^sH_c(\Gamma)_C$ has infinite global dimension) consists of values $c$ such that $H_c(\Gamma)$ has an asperical module.

For any aspherical value $c$, define a filtration on $O_c$ by Serre subcategories

$$O_c \leq d := \langle L_c(\tau; \mathbb{C}) \mid \text{codim supp } L_c(\tau; \mathbb{C}) \leq d \rangle.$$ 

Also we have a filtration on $\text{Mod-}^sH_c(\Gamma_2)_k$ by Serre subcategories

$$\text{Mod-}^sH_c(\Gamma_2)_k \leq d := \langle L_c(\tau; p) \mid \text{deg}(Z_\tau) \leq d \rangle.$$ 

These two filtrations are compatible in the following sense.

Let $A$ and $A'$ be two alcoves sharing a wall $H$. Assume $c$ is in alcove $A$, and $c'$ in $A'$. Assume moreover that $c_0$ is on $H$ but not any other walls.

Conjecture 3.2.7. Let $\Gamma_1 = \mathbb{Z}_l$. Suppose the codimension of support of $L_c(\tau; C)$ is $d$. Then $\overline{L_c(\tau; R)_k}$ is a nonzero object in

$$\text{Mod-}^sH_c(\Gamma_2)_k \leq d / \text{Mod-}^sH_c(\Gamma_2)_k \leq d+1.$$ 

The following Lemma, which is the only place where we use the condition $n = 2$ and $\Gamma_1 = \mathbb{Z}/l\mathbb{Z}$, is checked by explicit description of the central charge polynomials in Section 3.5.

\[\text{The author is grateful to Roman Bezrukavnikov for access to his unpublished work where the author learned this argument.}\]
Lemma 3.2.8. Let $\Gamma_1 = \mathbb{Z}/l\mathbb{Z}$. Let $H$ be a codimension-1 wall on which there is some $\theta \in \text{Irrep}(\Gamma_2)$ with $Z_{L_c(\theta; k)}(\nu)$ vanishes of degree 2. Then $L_c(\theta; \mathbb{C})$ is a finite dimensional representation of $^sH_c(\Gamma_2)$, and $\theta$ is the only irreducible representation of $\Gamma_2$ such that $Z_{L_c(\theta; k)}(\nu)$ vanishes on $H$.

If $L_c(\theta; \mathbb{C})$ is a finite dimensional irreducible representation of $^sH_c(\Gamma_2)$ with $T_{c \rightarrow c_0}(L_c(\theta; \mathbb{C})) = 0$, then for $k$ with large enough characteristic, $L_c(\theta; R)_k$ supported on $0 \in \mathbb{A}^{4(1)}/\Gamma_2$. Therefore we have $L_c(\theta; R)_k \cong \overline{L_c(\theta; R)_k}$, and $\overline{L_c(\theta; R)_k}$ is an irreducible representation of the same dimension as $\dim_{\mathbb{C}} L_c(\theta; \mathbb{C})$. By definition of the central charge polynomial, $Z_\theta(\nu)$ vanishes of degree 2 on $H$. In particular, taking into account of Lemma 3.2.8, we have the following lemma.

Lemma 3.2.9. The irreducible representation $L_c(\theta, \mathbb{C})$ of $^sH_c(\Gamma_2)$ is a finite dimensional with $T_{c \rightarrow c_0}(L_c(\theta; \mathbb{C})) = 0$ for any $c_0 \in H$ if and only if $Z_\theta(\nu)$ vanishes of degree 2 on $H$.

Now we are ready to prove Theorem 3.2.5 and Conjecture 3.2.7 in the case when $n = 2$. Hopefully part of the proof will generalize to a more general set-up.

There are two cases.

Case 1: In category $\mathcal{O}_c$ there is a unique finite dimensional irreducible object $L_c(\theta; \mathbb{C})$ for some $\theta \in \text{Irrep}(\Gamma_2)$ such that $T_{c \rightarrow c_0}(L_c(\theta; \mathbb{C})) = 0$. In this case, by Lemma 3.2.6, the only $\tau \in \text{Irrep}(\Gamma_2)$ such that $Z_\tau(\nu)$ that vanishes on $H$ is $\tau = \theta$. This in turn forces $Z_\theta(\nu)$ to have vanishing order 2 on the wall $H$. This proves Conjecture 3.2.7 in this case.

In characteristic zero, $T_{c \rightarrow c'}(L_c(\theta; \mathbb{C}))$ in concentrated in degree 2 as complex of $^sH_c(\Gamma_2)$-modules. Therefore, over a field $k$ with characteristic $p \gg 0$, the complex $T_{c \rightarrow c'}(\overline{L_c(\theta; R)_k})$ has non-trivial cohomology in degree 2, and all cohomologies in degree more than 2. Moreover, on the quotient $\text{Mod-}_0 ^sH_c(\Gamma_2)_k/\langle \overline{L_c(\theta; R)_k} \rangle$ the functor $T_{c \rightarrow c'}$ induces a Morita equiv-
alence, which is fits into a commutative diagram

\[
\begin{array}{ccc}
\text{Mod-}_0 \, ^s \text{H}_c(\Gamma_2)_k/\langle L_c(\theta; R) \rangle_k & \xrightarrow{T_{c \to c'}} & \text{Mod-}_0 \, ^s \text{H}_{c'}(\Gamma_2)_k/\langle L_{c'}(\theta; R) \rangle_k \\
\text{Mod-}_0 \, ^s \text{H}_{c_0}(\Gamma_2)_k & \xrightarrow{T_{c \to c_0}} & \xrightarrow{T_{c' \to c_0}} \\
\end{array}
\]

So, Theorem 3.2.5 is true in this case.

Case 2: There are more than one irreducible representation \( \tau \in \text{Irrep}(\Gamma_2) \) with \( T_{c \to c_0}(L_c(\tau; \mathbb{C})) = 0 \). In this case, none of such \( \tau \) can have \( L_c(\tau; \mathbb{C}) \) being finite dimensional. This means, for any such \( \tau \), the codimension of support of \( L_c(\tau; \mathbb{C}) \) has to be 1. This in turn, by Lemma 3.2.9, implies that \( Z(\nu) \) vanishes on \( H \) with order 1. Then Lemma 3.2.6 yields \( L_{c_0}(\tau; R) \) is aspherical. Therefore, \( Z_{L_c(\tau; R)_{k}}(\nu) \) vanishes on degree 1 on \( H \). This proves Conjecture 3.2.7 in this case.

In order to finish the proof, it only remains to show that for such \( \tau \), \( T_{c \to c'}(L_c(\tau; R)_{k}) \) as a complex in \( \text{Mod-}_0 \, ^s \text{H}_{c} \) is concentrated in degree 1. Note that \( T_{c \to c'}(L_c(\tau; \mathbb{C}) \) has homological degree no more than 1, therefore so is \( T_{c \to c'}(L_c(\tau; R)_{k}) \). Then similar to the previous case, the commutativity of \( T_{c \to c_0} = T_{c' \to c_0} \circ T_{c \to c'} \) implies that in homological degree zero \( T_{c \to c'}(L_c(\tau; R)_{k}) \) vanishes. Also, \( T_{c \to c'} \) induces a Morita equivalence when passing to \( \text{Mod-}_0 \, ^s \text{H}(\Gamma_2)_k/\text{Mod-}_0 \, ^s \text{H}(\Gamma_2)_k^{\leq 1} \). This finishes the proof.

### 3.3 Dimensions of irreducible objects

We already know that \( K_0(\text{Mod-}^s \text{H}_c(\Gamma_n)) \cong K_0(X) \cong K_0(\Gamma_n) \), and the irreducible objects in \( \text{Mod-}^s \text{H}_c \) are labeled by the irreducible representations of \( \Gamma \). Recall that for an irreducible representation \( \tau \) of \( \Gamma \), the corresponding irreducible object in \( \text{Mod-}^s \text{H}_c(\Gamma_n) \) will be denoted by \( L_c(\tau; p) \). As has been seen in Lemma 3.2.3, \( \dim_k(L_c(\tau; p)) \) is a polynomial in \( c \), as long as \( c \) varies in an alcove in the affine hyperplane arrangement.

**Problem 3.3.1.** Assume \( p \) is large enough, compute the characters of the irreducible representation \( L_c(\tau; p) \).
A weaker version of this Problem is: compute the Poincaré polynomial of the irreducible object $L_c(\tau; p)$ for regular parameter $c$.

Note that the Poincaré polynomial specializes to the dimension polynomial $\dim_k(L_c(\tau; p))$. When the parameter $c$ lies in the alcove containing 0, the category $\text{Mod-} \mathcal{H}_c(\Gamma_n)$ is a highest weight category. The irreducible modules $L_c(\tau; p)$ are quotients of the Verma modules, which are the $\tau$-isotypical components in the space of $c$-quasi-invariant polynomials. The Poincaré polynomials of the Verma modules have been calculated by Berest, Chalykh, Felder, and Veselov in [7] and [33].

The ring $K := \oplus_{n\geq 0}K_0(\Gamma_n)$, endowed with the parabolic induction and restriction functors of finite group representations, is a Hopf algebra. The results on parabolic induction and restriction functors in [9] show that in order to know the Poincaré polynomials of any $L_c(\tau; p)$, it suffices to calculate the Poincaré polynomials of the irreducible $\Gamma_n$-representations which are algebraic generators.

For $\Gamma_1 = \mathbb{Z}/l\mathbb{Z}$ and $c$ lying in the alcove containing 0, for a particular set of irreducible modules which generates $K$ multiplicatively, we construct the resolutions of those irreducible modules by Verma modules in this section. As a consequence, for such $\tau$, the Poincaré polynomials of $L_c(\tau; p)$ will be obtained.

### 3.3.1 Trivial representation of $\mathfrak{S}_n$

We work over a field of characteristic $p > 0$. Let $\mathfrak{h}$ be the reflection representation of $\mathfrak{S}_n$. Let $m$ be an integer and let $Q_m(\mathfrak{h}) := Q_m$ be the $m$-quasi-invariants on $\mathfrak{h}^*$. Let $\tilde{Q}_m$ be the quasi-invariants on the Frobenius neighborhood of the origin. The space $\tilde{Q}_m$ carries actions of $\mathfrak{S}_n$ and $eH_me$ which satisfies the Schur-Weyl duality. In other words, the multiplicity space on $\tilde{Q}_m$ corresponds to irreducible representations of $\mathfrak{S}_n$ gives the irreducible representations of the spherical rational Cherednik algebra.

We want to calculate the Poincaré series of the isotypical component on $\tilde{Q}_m$ corresponds
to the trivial representation. A resolution of \( \widetilde{Q}_m \) is given by the Koszul complex

\[
\cdots \to Q_m \otimes \Lambda^2 h^{(1)} \to Q_m \otimes h^{(1)} \to Q_m.
\]

Note that here \( h^{(1)} \) has degree \( p \). We decompose \( Q_m \) according to the \( S_n \)-eH\(_m\)e bimodule,

\[
Q_m = \bigoplus_{\tau \in \hat{S}_n} \tau^* \otimes M_m(\tau)e.
\]

Then [10] and [33] give the Poincaré series of \( M_m(\tau)e \).

\[
P_t(M_m(\tau)e) = t^{\xi_m(\tau)} K_{\tau}(t) \prod_{i=2}^{n-1} (1 - t^i),
\]

where \( K_{\tau}(t) \) is the Poincaré series of the \( \tau \)-component in the spaces of harmonic polynomials, and \( \xi_m(\tau) \) is the integer by which the element \( \sum_s \) a reflection in \( e_n(1 - s) \) acts on \( \tau \).

It is a classical formula that

\[
K_{\tau}(t) = \left( \prod_{k=1}^{n} (1 - t^k) \right) \left( \prod_{(i,j) \in \tau} \frac{l(i,j)}{1 - t^{h(i,j)}} \right),
\]

where \( l(i, j) \) is the leg length of the box \( (i, j) \) in the partition \( \tau \), and \( h(i, j) \) is the hook length.

So the Poincaré series of the irreducible representation of \( eH_m e \) corresponding to the trivial representation of \( S_n \) is

\[
\sum_{s=1}^{n} (-1)^{s-1} P_t(Q_m \otimes \Lambda^{s-1} h^{(1)})^{S_n}.
\]

This is the same as

\[
= \sum_{s=1}^{n} (-1)^{s-1} t^{s-1}(mn+p+\binom{s}{2}) \frac{1}{\prod_{i=1}^{s-1} (1 - t^i) \prod_{i=1}^{n-s} (1 - t^i)} \frac{1 - t}{1 - t^n}
\]

\[
= \frac{1 - t}{1 - t^n} \sum_{s=0}^{n-1} (-1)^s t^{s(mn+p+\binom{s+1}{2})} \prod_{i=1}^{s} (1 - t^i) \prod_{i=1}^{n-s} (1 - t^i).
\]

Now let us briefly recall an identity proved in [48]. Define \( [n] := \frac{n-1}{n-1} \), \( [n!] := [n][n-1] \cdots [1] \), and \( (x+a)_t^n := (x+a)(x+ta)(x+t^2a) \cdots (x+t^{n-1}a) \). Then, we have the following identity

\[
(x+a)_t^n = \sum_{j=0}^{n} \frac{[n]!}{[j]![n-j]!} t^{\binom{j}{2}} a^j x^{n-j}.
\]
Using notations in [48],

\[
1 - t \frac{\sum_{s=0}^{n-1} (-1)^st^{s(mn+p)+\binom{s+1}{2}}}{1 - t^n} = 1 - t \prod_{i=1}^{n-1} (1 - t^i) \prod_{i=1}^{n-1} \frac{[n-1]!}{[n-1-s]!} t^{\binom{s+1}{2}} (-t^{mn+p})^s.
\]

According to the identity from [48] recalled above, it is equal to

\[
1 - t \prod_{i=1}^{n-1} (1 - t^i) (1 + (-t^{mn+p})) t^{n-1} = 1 - t \prod_{i=1}^{n-1} \frac{1 - t^{mn+p+i}}{1 - t^i}.
\]

To summarize, we have the following Lemma.

**Lemma 3.3.2.** The Poincaré series of the irreducible representation of \( eH_m e \) corresponding to the trivial representation of \( S_n \) is

\[
\frac{1 - t}{1 - t^n} \prod_{i=1}^{n-1} \frac{1 - t^{mn+p+i}}{1 - t^i}.
\]

### 3.3.2 Characters of the wreath product \( \Gamma_n = (\mathbb{Z}/l)^n \rtimes S_n \)

Recall that the irreducible representations of \( \Gamma_n \) are in one-to-one correspondence with \( l \)-partitions of \( n \), i.e., \( \lambda = (\lambda^1, \ldots, \lambda^l) \) where \( \lambda^i \)'s are partitions such that \( \sum_{i=1}^{l} |\lambda^i| = n \).

More explicitly, for an \( l \)-partition \( \lambda = (\lambda^1, \ldots, \lambda^l) \), let \( l_r \) be the number of rows in \( \lambda^r \), and let \( I_\lambda(r) = \sum_{i=1}^{r-1} |\lambda^i| + 1, \sum_{i=1}^{r-1} |\lambda^i| + 2, \ldots, \sum_{i=1}^{r} |\lambda^i| \). Let \( S_\lambda = S_{I_\lambda(1)} \times \cdots \times S_{I_\lambda(l)} \).

Then, the \( l \)-partition \( \lambda \) corresponds to the irreducible representation of \( \Gamma_n \) constructed as

\[
\text{Ind}^\Gamma_n\langle \mathbb{Z}/l \rangle (1, 2, \ldots, n) \times S_\lambda (\phi^1 \cdot \lambda^1 \otimes \cdots \otimes \phi^l \cdot \lambda^l),
\]

where \( \phi^r \) is the character \( \det^r \) of \( (\mathbb{Z}/l \mathbb{Z})^{I_\lambda(r)} \). (We follow the convention in [35].)

In this section let \( \mathfrak{h} \cong \mathbb{C}^n \) be the reflection representation of \( \Gamma_n \). Let \( m \) be an integer valued function on the set of reflections in \( \Gamma_n \), constant on conjugacy classes, and let \( Q_m(\mathfrak{h}) := Q_m \) be the \( m \)-quasi-invariants on \( \mathfrak{h} \). Let \( \widetilde{Q}_m \) be the quasi-invariants on the Frobenius neighborhood of the origin. Again, the multiplicity space \( \widetilde{Q}_m(\tau) \) on \( \widetilde{Q}_m \) corresponds to irreducible representations \( \tau \) of \( \text{Gamma}_n \) gives the irreducible representations of the spherical rational Cherednik algebra. Section 8.2 of [7] gives the Poincaré series of \( Q_m(\tau') e \) as

\[
P_t(Q_m(\tau') e) = t^{\xi_m(\tau')} \cdot P_t((k[\mathfrak{h}] \otimes (\tau')^*) \Gamma_n),
\]

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where $\tau := k\pi_k(\tau)$ is the $k\pi$-twist defined in [63].

Let $\tau(i)$ be the $l$-partition $\lambda$ whose $\lambda^i = (n)$ and $\lambda^j$ is empty for all $j \neq i$. We want to calculate the Poincaré series of the isotypical component on $\widetilde{Q}_m$ corresponds to the $\tau(i)$. A resolution of $\widetilde{Q}_m$ is given by the Koszul complex

$$\cdots \to Q_m \otimes \wedge^2 \mathfrak{h}^{(1)} \to Q_m \otimes \mathfrak{h}^{(1)} \to Q_m.$$

The Poincaré series of the irreducible representation of $eH_m\mathfrak{e}$ corresponding to the representation $\tau(i)$ of $\text{Gamma}_n$ is

$$\sum_{s=0}^{n} (-1)^s P_i((Q_m \otimes \wedge^s \mathfrak{h}^{(1)} \otimes \tau(i)^s)^{\Gamma_n}).$$

The representation $\wedge^s \mathfrak{h}$ corresponds to the $l$-partition $((1)^s, \emptyset, \cdots, \emptyset, (n-s))$. Hence, $\wedge^s \mathfrak{h} \otimes \tau(i)^s$ corresponds to the $l$-partition $\lambda$ whose $i$-th component is $(n-s)$ and $i+1$-th component is $(1)^s$. By the adjoint of $\text{Ind}$ and $\text{Res}$, we have the following sequence of isomorphisms of graded modules $\text{Hom}_{\Gamma_n}(\lambda, k[\mathfrak{h}] ) \cong \text{Hom}_{(\mathbb{Z}/l\mathbb{Z})^n \times (\mathbb{Z}/l\mathbb{Z})^{n-s}}(\phi^i \cdot (n-s) \otimes \phi^{i+1} \cdot ((1)^s), \text{Res} k[\mathfrak{h}] ) \cong \text{Hom}_{\mathfrak{m}^{n-s}}((n-s), \otimes_{q=1}^{n-s} k[x_q^i]_{x_q^i}) \otimes \text{Hom}_{\mathfrak{m}^s}(((1)^s), \otimes_{q=1}^{s} k[x_q^i]_{x_q^i+1})$. Now, each individual Poincaré polynomial can be calculated using the hook-length formula. Note that on this $\lambda$, the value $\xi_m$ is $s(nm_0 + lm_{i+1})$.

$$\sum_{s=0}^{n} (-1)^s P_i((Q_m \otimes \wedge^s \mathfrak{h}^{(1)} \otimes \tau(i)^s)^{\Gamma_n})$$

$$= \sum_{s=0}^{n} (-1)^st^{\xi_m(\lambda)}spt^{s(i+1)} \prod_{k=1}^{s} \frac{t^{l(k-1)}}{1 - t^k} t^{(n-s)i} \prod_{k=1}^{n-s} \frac{1}{1 - t^k}$$

$$= \frac{t^{ni} \prod_{k=1}^{n} (1 - t^{kl})}{\prod_{k=1}^{n-l}(1 - t^{k-l})} \sum_{s=0}^{n} (-1)^s t^{m_{n+p+1}l_{n-i}} \prod_{k=1}^{n} (1 - t^{kl}) \prod_{k=1}^{n-s} (1 - t^{kl})$$

$$= \frac{t^{ni} \prod_{k=0}^{n-l}(1 - t^{k+lm_{n+p+1}})}{\prod_{k=1}^{n-l}(1 - t^{kl})}.$$
\[
\frac{t^{\mu_i} \prod_{k=0}^{n-1} (1 - t^{lk+mn+p+1+lm_i+1})}{\prod_{k=1}^{n} (1 - t^{kl})}.
\]

3.4 The Chern character map of the resolution

The central charge map \( Z : H^2(\text{Hilb}; \mathbb{Q}) \to \text{Hom}_{\mathbb{Z}}(K_0(\text{Hilb}), \mathbb{Q}) \), which is defined by modifying the dimension polynomials of the irreducible modules over \(^*H_c\), is related to the Chern character map \( \text{ch} : K_0(\text{Hilb})_{\mathbb{Q}} \to H^*(\text{Hilb}; \mathbb{Q}) \), as will be explained in more details in this section. Thanks to the work of Ginzburg and Kaledin, the multiplicative structure of \( H^*(\text{Hilb}, \mathbb{Q}) \) is easily described. The abelian group structure of \( K_0(\text{Hilb}) \) is given by the symplectic McKay correspondence. It is a long-stand question to calculate the Chern character map in terms of the natural bases of the two sides.

3.4.1 The central charge and the Chern character map

Let \( k \) be a seperably closed field of characteristic \( p > 0 \). Let \( \Gamma \) be an arbitrary symplectic reflection group acting on \( \mathbb{A}^{2n}_k \) by symplectic reflections. Let \( X \) be a symplectic resolution of \( \mathbb{A}^{2n}/\Gamma \). According to [13], there is a derived equivalence

\[
D^b(\text{Coh } X) \cong D^b(\text{Mod-} W_n^\Gamma)
\]

where \( W_n \) is the ring of differential operators on \( \mathbb{A}^n \). This derived equivalence is given by a vector bundle \( \mathcal{E}_0 \) on \( X \). It is shown in [13] that any of such a vector bundle \( \mathcal{E}_0 \) lifts to characteristic zero. Therefore, for simplicity in what follows in this section and the next one, we work over a field of characteristic zero. (This is not essential. One can replace \( H^*(X; \mathbb{Q}) \) by \( H^*(X; \mathbb{Q}_r) \) for \( r \neq p \), and all the statements in this section are still true.) The vector bundle \( \mathcal{E}_0 \) in [13] is not unique. The non-uniqueness has been studied by Losev in [55], together with a preferred choice. Under the derived correspondence \( D^b(\text{Coh } X) \cong D^b(\text{Mod-} W_n^\Gamma) \), there is a set of vector bundles \( \{ \mathcal{Y}_\alpha \} \) on the symplectic resolution \( X \) corresponding to the
indecomposable projective modules over $W^\Gamma_n$, which are in turn labeled by the irreducible representations of $\Gamma$. The classes of $\{\mathcal{V}_\alpha\}$ in the Grothendieck group form a basis of $K_0(X)$. 

On the other hand, the cohomology ring $H^*(X, \mathbb{Q})$ has an explicit description. Let $\mathbb{Q}[\Gamma]$ be the group ring. It is filtered by the codimension of the fixed point loci. This filtration induces a filtration on the center $\mathbb{Z}[\mathbb{Q}[\Gamma]]$, whose associated graded ring will be denoted by $\text{gr} \mathbb{Z}[\mathbb{Q}[\Gamma]]$.

**Theorem 3.4.1** ([30] and [34]). The algebra $H^*(X, \mathbb{Q})$ is isomorphic to the algebra $\text{gr} \mathbb{Z}[\mathbb{Q}[\Gamma_n]]$.

The following problem is raised by Etingof, Ginzburg, and Kaledin, and is referred to as the Chern character problem.

**Problem 3.4.2** ([30] and [34]). Express explicitly the map

$$K_0(\Gamma_n) \to \text{gr} \mathbb{Z}[\mathbb{Q}[\Gamma_n]]$$

induced by the Chern character

$$\text{ch} : K_0(X) \to H^*(X; \mathbb{Q}).$$

The character group of $\Gamma$ will be denoted by $\hat{\Gamma}$. We define polynomials on $H^2(X; \mathbb{Q})$

$$L_{\alpha}(n_{\tau})_{\tau \in \hat{\Gamma}} := \chi(\mathcal{L}_{\alpha} \otimes (\otimes_{\tau \in \hat{\Gamma}} \mathcal{V}_{\tau}^{n_{\tau}}))$$

in the variables $n_{\tau}$.

In characteristic $p$, the dimension of the irreducible representations can be calculated by modifying these polynomials. For an explicit illustration of this we refer to § 3.5.

**Proposition 3.4.3.** For an arbitrary basis $\{b\}$ of $H^*(X; \mathbb{Q})$ with $b = \sum h_{\alpha} \text{ch}(\mathcal{V}_{\alpha})$, we have $L_{\alpha}(b) = h_{\alpha}^b$.

**Proof.** To calculate the polynomial

$$L_{\alpha}(n_{\tau})_{\tau \in \hat{\Gamma}} := \chi(\mathcal{L}_{\alpha} \otimes (\otimes_{\tau \in \hat{\Gamma}} \mathcal{V}_{\tau}^{n_{\tau}}))$$

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in the variables $n_\tau$ from the Chern character map. In positive characteristic, the dimension of the irreducible representations can be calculated by modifying these polynomials, as will be done in later sections.

The group $\Gamma$ is generated by the classes of symplectic reflections, therefore, there is a basis of $H^*(\text{Hilb}^2, \mathbb{Q})$ given by $\{\text{ch}(\otimes_{\tau \in \hat{\Gamma}} \mathcal{V}\alpha_{\tau}) \mid (a_{\tau})_{\tau} \in I\}$ for some set $I \subset \mathbb{Z}^{[l]}$. Also $\{\otimes_{\tau \in \hat{\Gamma}} \mathcal{V}\alpha_{\tau} \mid (a_{\tau})_{\tau} \in I\}$ form a basis of $K_{\mathbb{Q}}(\text{Hilb}^2)$. For simplicity, we write $\mathcal{O}(a)$ for the line bundle $\otimes_{\tau \in \hat{\Gamma}} \mathcal{V}\alpha_{\tau}$. Let

$$[\mathcal{V}_a] = \sum_{a \in I} m^a_{\alpha}[\mathcal{O}(a)].$$

Therefor, $[\mathcal{V}_a^*] = \sum_{a \in I} m^a_{\alpha}[\mathcal{O}(a)^*]$.

The polynomials $I_{\mathcal{V}_a}(n_\tau)_{\tau \in \hat{W}}$, considered as functions on $H^*(\text{Hilb}^2, \mathbb{Q})$, are $\mathbb{Q}$-linear functions, hence,

$$\sum_{a \in I} m^a_{\beta} I_{\mathcal{V}_a}(a) = \delta_{\alpha,\beta}.$$

In other words, the value of the linear function $I_{\mathcal{V}_a}(n_\tau)_{\tau \in \hat{W}}$ at the basis element $\text{ch}(\mathcal{O}(a))$ is given by $m^{-1,2}_{\alpha}$, where $m^{-1,2}_{\alpha}$ is the $(a,\alpha)$-entry of the inverse matrix of $(m^a_{\beta})$. Therefore, for an arbitrary basis $\{b\}$ of $H^*(X; \mathbb{Q})$ with $b = \sum h^b_{\alpha} \text{ch}(\mathcal{V}_a)$, we have $I_{\mathcal{V}_a}(b) = h^b_{\alpha}$.

In this section and the next one, we study the Chern character map in some special cases.

### 3.4.2 The topology of the punctual Hilbert scheme

From this section on we concentrate on the case when $\Gamma_1 = \mathbb{Z}/l$ and $n = 2$. In this section we describe the cohomology ring and the $K$-group of the symplectic resolution, and give a formula of the Chern character map.

We present $\Gamma_2 = (\mathbb{Z}/l\mathbb{Z})^2 \rtimes \mathfrak{S}_2$ as $\{\xi, \eta, \sigma \mid \xi, \eta, \sigma^2, \sigma \eta \sigma = \xi\}$. Now we look at the conjugacy classes of elements in $\Gamma_2$. There are $l$ conjugacy classes in $\Gamma_2$ consists of symplectic reflections. They are represented by $\xi^i$ with $i = 1, \ldots, l - 1$, and $\sigma$. There are $\binom{l}{2} + 2(l - 1)$ conjugacy classes whose fixed point loci consist of the origin only. They are represented by
\( \sigma \eta^i \) with \( i = 1, \ldots, l - 1 \), \( \xi^i \eta^i \) with \( i = 1, \ldots, l - 1 \), and \( \xi^i \eta^j \) with \( i \neq j \).

If we write \([g]\) for \( \sum_{h \sim g} h \in \mathbb{Q}[\Gamma_2] \), the natural basis of \( \text{gr } Z \mathbb{Q}[\Gamma_2] \) is given by \( \{[g] \mid g \in W\} \). They satisfies \( [\xi^i] \cdot [\xi^i] = [\xi^i \eta^j], [\xi^i]^2 = 2[\xi^i \eta^i] \), and \( [\sigma] \cdot [\xi^i] = 2[\sigma \xi^i] \).

Let \( \widetilde{A^2/\mathbb{Z}_l} \to A^2/\mathbb{Z}_l \) be the minimal resolution of Kleinian singularity. Then a symplectic resolution of \( A^4/W \) is given by \( \text{Hilb}^2 = \text{Hilb}^2(\widetilde{A^2/\mathbb{Z}_l}) \). It fits in the basic diagram

\[
\begin{array}{ccc}
\text{Bl}_\Delta(\widetilde{A^2/\mathbb{Z}_l} \times \widetilde{A^2/\mathbb{Z}_l}) & \xrightarrow{q} & \widetilde{A^2/\mathbb{Z}_l} \times \widetilde{A^2/\mathbb{Z}_l} \\
\downarrow & & \downarrow \\
\text{Hilb}^2(\widetilde{A^2/\mathbb{Z}_l} \times \widetilde{A^2/\mathbb{Z}_l}) & \to & \widetilde{A^2/\mathbb{Z}_l} \times \widetilde{A^2/\mathbb{Z}_l}/\mathbb{G}_2.
\end{array}
\] (3.3)

Let \( C \subset \widetilde{A^2/\mathbb{Z}_l} \) be the exceptional divisor. Recall that \( C \) is a chain of \( \mathbb{P}^1 \)'s, each having self-intersection number \(-2\). We number them as \( C_1, \ldots, C_{l-1} \) such that \([C_i][C_{i+1}] = 1\) and \([C_i][C_j] = 0\) if \( \mid i - j \mid > 1 \).

We now describe the cohomology ring of \( \text{Hilb}^2 \). Since \( \text{Hilb}^2 \) deformation retracts to the punctual Hilbert scheme \( X = \text{Hilb}^2_C(\widetilde{A^2/\mathbb{Z}_l}) \), we concentrate on the latter. The scheme \( X \) has \( \binom{l}{2} + 2(l - 1) \) irreducible components, coming from the strict transform of \( C_i \times C_j \subset \widetilde{A^2/\mathbb{Z}_l} \times \widetilde{A^2/\mathbb{Z}_l} \) into \( \text{Hilb}^2 \) under the maps \( p \) and \( q \). We now describe these components.

For each irreducible component \( C_i \subset C \), there are two irreducible components coming out of the strict transform of \( C_i \times C_i \). One component is isomorphic to \( \mathbb{P}^2 \) which we will denote by \( \mathbb{P}^2_i \). The other component is isomorphic to the rational ruled surface \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-4)) \cong \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(-2)) \) which will be denoted by \( S_i \). We identify \( \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \) with \( T(\mathbb{T}^\ast \mathbb{P}^1)|_{\mathbb{P}^1} \) where \( \mathbb{P}^1 \subset \mathbb{T}^\ast \mathbb{P}^1 \) is the zero section. The fiber of \( S_i \to \mathbb{P}^1 \) over \( x \in \mathbb{P}^1 \) is \( \mathbb{P}(T_{x}T^\ast \mathbb{P}^1) \). These two components \( \mathbb{P}^2_i \) and \( S_i \) are glued together along a common divisor \( \mathbb{P}^1 \). This \( \mathbb{P}^1 \) sits inside \( \mathbb{P}^2_i \) as a degree 2 irreducible hypersurface. In \( S_i \) this divisor \( \mathbb{P}^1 \) is embedded as a section of this rational ruled surface which corresponds to the subbundle \( \mathcal{O}(2) \subseteq \mathcal{O}(2) \oplus \mathcal{O}(-2) \). (Over each point \( x \) in the zero section of \( \mathbb{T}^\ast \mathbb{P}^1 \), this \( \mathbb{P}^1 \) corresponds to the direction of \( T_x \mathbb{P}^1 \) in \( T_x \mathbb{T}^\ast \mathbb{P}^1 \).) In fact, the normal bundle \( N_{\mathbb{C},\mathbb{A}^2/\mathbb{Z}_l} \cong \mathcal{O}(C_i)|_{C_i} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \cong T^\ast \mathbb{P}^1 \), and also \( N_{\Delta}(\mathbb{A}^2/\mathbb{Z}_l)^2|_{C_i} \cong N_{\Delta}(N_{\mathbb{C},\mathbb{A}^2/\mathbb{Z}_l})^2 \). Therefore, the strict transform of \( C_i \times C_i \subset \widetilde{A^2/\mathbb{Z}_l} \times \widetilde{A^2/\mathbb{Z}_l} \) into \( \text{Hilb}^2 \) is
isomorphic to the strict transform of $\mathbb{P}^1 \times \mathbb{P}^1 \subset T^*\mathbb{P}^1 \times T^*\mathbb{P}^1$ into $\text{Bl}_\Delta(T^*\mathbb{P}^1 \times T^*\mathbb{P}^1)/\mathbb{Z}_2$, which is obviously $\mathbb{P}^2 \sqcup \mathbb{P}^1 S$ as described above. The common $\mathbb{P}^1$ in $S_i$ and $\mathbb{P}^2$ is the strict transform of the diagonal.

For $C_i \neq C_j$, there is an irreducible component of $X$ coming from the strict transform of $C_i \times C_j$, which will be called $P_{ij}$. We have $P_{ij} \cong \mathbb{P}^1_i \times \mathbb{P}^1_j$ if $[C_i][C_j] = 0$ in $\mathbb{A}^2/\mathbb{Z}_l$, and $P_{ij} \cong \text{Bl}_* \mathbb{P}^1_i \times \mathbb{P}^1_j$ if $[C_i][C_j] = [*]$ in $\mathbb{A}^2/\mathbb{Z}_l$ where $*$ is a point.

Let us write down a basis of the cohomology rings of each of the irreducible components. We take the canonical basis of $H^2(\mathbb{P}^2_i)$ as $q_i$, and $q_i^2 = p_i$. We denote the Poincaré dual of the zero section of $S_i \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1_i} \oplus \mathcal{O}_{\mathbb{P}^1_i}(-4))$ in $H^2(S_i)$ by $c_i$, the Poincaré dual of the fiber by $f_i$, and the fundamental class in $H^4(S_i)$ by $s_i$. (Here we follow the convention in Hartshorne and therefore $c_0^2 = \text{degree}$.) If $i = j - 1$, we denote the Poincaré dual of the exceptional divisor in $H^2(\text{Bl}_* \mathbb{P}^1_i \times \mathbb{P}^1_j)$ by $e_i$. No matter whether $i$ and $j$ are adjacent or not, we denote the Poincaré dual of $[* \times \mathbb{P}^1_j]$ by $l_{j,i}$, and the Poincaré dual of $[\mathbb{P}^1_i \times *]$ by $l_{i,j}$. The fundamental class will be denoted by $p_{i,j}$.

Besides the basis of $H^*(\text{Hilb}^2, \mathbb{Q})$ coming from the natural basis of $\text{gr}^{\text{top}} \mathbb{Z}\mathbb{Q}[W]$ described at the beginning of this section, there is another basis of $H^*(\text{Hilb}^2, \mathbb{Q})$ coming from the topology of $X$ which will be described here. The basis of $H^2(\text{Hilb}^2, \mathbb{Q})$ comes from the divisor classes, which in turn corresponds to conjugacy classes of symplectic reflections in $W$. The basis of $H^4(\text{Hilb}^2, \mathbb{Q})$ comes from irreducible component of $X$.

Note that in our case (and many other cases), the resolution $\text{Hilb}^2$ can be constructed as a Nakajima quiver variety (see, e.g., [50]). The basis of $H^4(\text{Hilb}^2, \mathbb{Q})$ (resp. $H^{\text{mid}}$) coming from irreducible components coincide with the basis given by Nakajima in [61]. It is a natural question to ask what the matrix of transform is between this basis and the one coming from conjugacy classes in $\text{gr}^{\text{top}} \mathbb{Z}\mathbb{Q}[\Gamma_2]$. In the case concerned in this paper, we will solve this problem by working out the multiplicative structure of $H^*(\text{Hilb}^2, \mathbb{Q})$ under the topological basis.
Now we can describe the basis of $H^2(X)$ coming from symplectic reflections more explicitly. The divisor class coming from the symplectic reflection $\sigma$ is $d_0 = \sum q_j + \sum c_j + 2 \sum f_j + \sum_{j=1}^{l-2} e_j$. The divisor coming from the symplectic reflection $\xi^i$ for $i = 1, \cdots, l - 1$ is $d_i = q_i + 2f_i + \sum_{j \neq i} l_{i,j}$. The non-trivial multiplications of them are given by

$$d_0^2 = \sum p_j - \sum p_{j,j+1};$$

$$d_0 \cdot d_i = p_i + 2s_i;$$

$$d_i^2 = p_i;$$

$$d_i \cdot d_j = p_{ij}.$$

Now we study the $K$-theory of $\text{Hilb}^2$.

Fix a primitive $l$-th root of unity $\omega$, the irreducible representations of $W$ can be written as:

- the trivial representation, whose corresponding vector bundle on $\text{Hilb}^2$ under the McKay correspondence is denoted by $\mathcal{V}_0$;

- the 1 dimensional representation $k_i$ acted trivially by $S_2$ and via $\omega^i$ by $\mathbb{Z}_l$, whose corresponding vector bundle is $\mathcal{V}_i$;

- the sign representation of $S_2$ tensor with $k_i$, whose corresponding vector bundle is $\mathcal{V}_{\sigma,i}$;

- the irreducible 2 dimensional representation acted via $\begin{pmatrix} \omega^i & 0 \\ 0 & \omega^j \end{pmatrix}$ by $\mathbb{Z}_l$, whose corresponding vector bundle is $\mathcal{V}_{i,j}$.

The main result of this section is the following proposition.
Proposition 3.4.4. We have

\[
\begin{align*}
\text{ch}(\mathcal{V}_0) & = 1; \\
\text{ch}(\mathcal{V}_i) & = 1 + d_i + p_i/2; \\
\text{ch}(\mathcal{V}_\sigma) & = 1 + d_0 + \sum_{j} p_j/2 - \sum_{j} p_{j,j+1}/2; \\
\text{ch}(\mathcal{V}_{\sigma,i}) & = 1 + d_i + d_0 + \sum_{j} p_j/2 - \sum_{j} p_{j,j+1}/2 + 3p_i/2 + 2s_i; \\
\text{ch}(\mathcal{V}_{0,i}) & = 2 + d_0 + d_i + \sum_{j} p_j/2 - \sum_{j} p_{j,j+1}/2 + p_i/2 + s_i; \\
\text{ch}(\mathcal{V}_{i,j}) & = 2 + d_j + d_i + d_0 + \sum_{k} p_k/2 - \sum_{k} p_{k,k+1}/2 + p_{i,j} + p_i/2 + p_j/2 + s_i + s_j.
\end{align*}
\]

The rest of this section is devoted to the proof of this proposition.

Lemma 3.4.5. In \( K(\text{Hilb}^2) \), we have \( \mathcal{V}_{i,j} = \mathcal{V}_i \otimes (\mathcal{O} \oplus \mathcal{V}_\sigma \oplus \mathcal{V}_{0,j} - \mathcal{V}_{0,i}) \).

Proof. We have, on the one hand, in \( K^S_2(\mathbb{A}/\mathbb{Z}) \),

\[
Rq_\star p^* \mathcal{V}_{i,j} = \mathcal{O}_i \boxtimes \mathcal{O}_j \boxplus \mathcal{O}_j \boxtimes \mathcal{O}_i
\]

\[
= \mathcal{O}_i \boxtimes \mathcal{O}_i \otimes (\mathcal{O} \oplus \mathcal{O}_\sigma \oplus (\mathcal{O} \boxtimes \mathcal{O}_j \oplus \mathcal{O}_j \boxtimes \mathcal{O}) - (\mathcal{O} \boxtimes \mathcal{O}_i \oplus \mathcal{O}_i \boxtimes \mathcal{O})),
\]

where \( \mathcal{O}_\sigma \) is endowed with the sign representation of \( S_2 \). On the other hand,

\[
\mathcal{O}_i \boxtimes \mathcal{O}_j \oplus \mathcal{O}_j \boxtimes \mathcal{O}_i
\]

\[
= \mathcal{O}_i \boxtimes \mathcal{O}_i \otimes (\mathcal{O} \oplus \mathcal{O}_\sigma \oplus (\mathcal{O} \boxtimes \mathcal{O}_j \oplus \mathcal{O}_j \boxtimes \mathcal{O}) - (\mathcal{O} \boxtimes \mathcal{O}_i \oplus \mathcal{O}_i \boxtimes \mathcal{O}))
\]

\[
= \mathcal{O}_i \boxtimes \mathcal{O}_i \otimes (Rq_\star p^* (\mathcal{O} \oplus \mathcal{V}_\sigma \oplus \mathcal{V}_{0,j} - \mathcal{V}_{0,i}))
\]

\[
= Rq_\star (q^* (\mathcal{O}_i \boxtimes \mathcal{O}_i) \otimes p^* (\mathcal{O} \oplus \mathcal{V}_\sigma \oplus \mathcal{V}_{0,j} - \mathcal{V}_{0,i}))
\]

\[
= Rq_\star (p^* (\mathcal{V}_i \otimes (\mathcal{O} \oplus \mathcal{V}_\sigma \oplus \mathcal{V}_{0,j} - \mathcal{V}_{0,i}))).
\]

Example 3.4.6. We take a concrete example, i.e., the case when \( \Gamma_2 = B_2 \). We present \( B_2 \) as \( \langle s_1, s_2, \sigma \mid s_1^2 = s_2^2 = 1, \sigma s_1 \sigma = s_2 \rangle \).

There are 5 irreducible representations of \( B_2 \): the trivial representation \( V_0 \), \( V_1 = k \) with \( \sigma = -1 \), \( V_2 = k \) with \( s_1 = s_2 = -1 \), \( V_3 = k \) with \( \sigma = s_1 = s_2 = -1 \), and \( V_4 = h \).
Let $V_i$ be the vector bundle on the symplectic resolution corresponding to the projective object $V_i \times \mathbb{A}^4$ under the McKay correspondence. Their corresponding simple object will be denoted by $L_i$.

The symplectic resolution of $\mathbb{A}^4/B_2$ is given by the Hilbert scheme $\text{Hilb}^2 = \text{Hilb}^2(\mathbb{C}^2/\mathbb{Z}_2)$ where $\mathbb{C}^2/\mathbb{Z}_2 \rightarrow \mathbb{C}^2/\mathbb{Z}_2$ is the minimal resolution of Kleinian singularity. More concretely, $\mathbb{C}^2/\mathbb{Z}_2 \cong T^*\mathbb{P}^1$. The Hilbert scheme $\text{Hilb}^2(\mathbb{C}^2/\mathbb{Z}_2) \cong \text{Bl}_\Delta(T^*\mathbb{P}^1 \times T^*\mathbb{P}^1)/\mathbb{Z}_2$. The punctual Hilbert scheme $X = \text{Hilb}^2\mathbb{P}^1 \cong \mathbb{P}^2 \sqcup S$ where $S \cong \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(-2)) \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-4))$ with the base $\mathbb{P}^1$ being the zero section of $T^*\mathbb{P}^1$ and fiber of $x \in \mathbb{P}^1$ being $\mathbb{P}(T_xT^*\mathbb{P}^1)$. These two surfaces are glued together over a common divisor $\mathbb{P}^1$ which are the proper transforms of the diagonal. We write $\pi : \mathbb{P}^2 \sqcup S \rightarrow \text{Hilb}^2$ as the natural map.

The cohomology ring $H^*(X)$ of the central fiber, which is canonically isomorphic to the cohomology ring of the entire resolution, has a basis as follows. We take the basis of $H^2(S)$ as $c_0$ and $f$, where $c_0$ is the Poincaré dual of the zero section, and $f$ is the Poincaré dual of the fiber. We denote the canonical basis of $H^2(\mathbb{P}^2)$ by $q$. Then the basis for $H^2(X)$ can be chosen as $d_1 = q + 2f$ and $d_2 = q + c_0 + 2f$. (The common divisor $\mathbb{P}^1$ is $2q$ in $H^2(\mathbb{P}^2)$ and $c_0 + 4f$ in $H^2(S)$.) We denote the fundamental class of $S$ by $s$ and the fundamental class of $\mathbb{P}^2$ by $p$.

There are 2 conjugacy classes of symplectic reflections in $B_2$, which give two divisors in $\text{Hilb}^2$, i.e., $D_1$ corresponding to $s_i$ and $D_2$ corresponding to $\sigma$. We can restrict these two divisors to the central fiber and get $\pi_s^*[D_1] = [2F]$, $\pi_s^*[D_2] = [C_0 + 2F]$, $\pi_{\mathbb{P}^2}^*[D_1] = [Q]$, and $\pi_{\mathbb{P}^2}^*[D_2] = [Q]$.

We will identify the sheaves $\pi^*\mathcal{V}_i$ as better-known sheaves over $\mathbb{P}^2$ and $S$. To identify the line bundles, we use the stratification of $\text{Hilb}^2$ to reduce to the 2-dimensional situations, as described in Section 4 of [13]. The stratification of $\text{Hilb}^2(T^*\mathbb{P}^1)$ is as the following. One open stratum, two divisors corresponding to the two classes of symplectic reflections, and one codimension 2 stratum which is the punctual Hilbert scheme $X$. We take the 2-dimmensional
complimentary $W_1$ to the fixed subspace of $s_1$, and the 2-dimensional complimentary $W_2$ to the fixed subspace of $\sigma$. The restriction of the line bundles $\mathcal{V}_i$ for $i = 1, 2, 3$ to $W_1$ and $W_2$, we get that $\mathcal{V}_1 = \mathcal{O}(D_2)$, $\mathcal{V}_2 = \mathcal{O}(D_1)$, and $\mathcal{V}_3 = \mathcal{O}(D_1 + D_2)$ where $D_1$ is the exceptional divisor coming from the class $s_i$ and $D_2$ from $\sigma$. To identify the rank 2 vector bundle, we use the quiver picture, as will be done in the next subsection.

### 3.4.3 Chern characters via quivers

To calculate the Chern characters, we want to identify the vector bundle $\mathcal{V}_\alpha$ as better-known vector bundles. For this, we look at the quiver variety that gives the central fiber of the resolution. According to [50], the resolution is the Nakajima variety associated to the quiver affine Dynkin $\tilde{A}_{l-1}$,

![Quiver Diagram]

with dimension vectors $v_j = 2$, $w_0 = 1$, and $w_j = 0$ for $j \neq 0$ and stability $(-1, \cdots, -1)$ (hence being stable means no invariant subrepresentations containing the image of $J: W_0 \to V_0$). The McKay correspondence is fixed if we ask the sub-bundle of the tautological bundle generated by the image of $J$ to be the trivial representation.

There is a $\mathbb{G}_m$-action on the quiver variety. On the quiver representation level, this action is given by sending $(X,Y,I,J)$ to $(tX, t^{-1}Y, I, J)$. Clearly the tautological bundles on the quiver variety are equivariant under this group action. And there are only finitely many fixed points. Therefore, we can calculate the second Chern classes of the tautological bundles using Atiyah-Bott-Berline-Vergne localization.

First let us look at the case when $n = 2$:

![Quiver Diagram]
with dimension vector \( \dim V = (2, 2) \) and \( \dim W = (1, 0) \). The condition \( \mu_C(X, Y, I, J) = 0 \)
means \( X_1 Y_1 = Y_2 X_2 \) and \( Y_1 X_1 = X_2 Y_2 \).

The central fiber is the moduli space of the nilpotent representations of this quiver satisfying these conditions. The rank-2 vector bundle is the summand of the tautological bundle corresponding to \( V_2 \). There are two possibilities to get such a representation.

Case 1: \( \ker X_1 = \ker Y_2 \) both 1-dim, and do not contain \( \text{im} J \) in \( V_0 \). And \( X_1(J) \) and \( Y_2(J) \) are linearly independent in \( V_1 \). (Here and in what follows we don’t distinguish between \( J \) and \( J(1) \).) We can take the basis for \( V \) as follows. Take \( J = (0, 1) \) and any non-zero vector in \( \ker X_1 = \ker Y_2 \) to be \( (1, 0) \). Take \( X_1(J) \) to be \( (1, 0) \) and \( Y_2(J) \) to be \( (0, 1) \). Thus, under this basis, \( X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), \( Y_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), \( Y_1 = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \), and \( X_2 = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \). The homogeneous coordinates \([a, b, c]\) do not depend on the choice of basis for \( \ker Y_2 \). Note that here we do not allow \( a^2 = bc \). Clearly the restriction of \( \mathcal{V}_4 \) to this part is trivial.

Case 2: \( X_1(J) \) and \( Y_2(J) \) are linearly independent in \( V_2 \), and \( Y_1|_{\text{im} X_1} = 0 \) \( X_2|_{\text{im} Y_2} = 0 \) \( Y_2|_{\text{im} X_2} = 0 \) and \( X_1|_{\text{im} Y_1} = 0 \). We take the basis for \( V_1 \) as before, \( J = (1, 0) \) \( \in V_0 \), and \( (0, 1) \) \( \in V_0 \) arbitrary. Under this basis, \( X_1 = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \), \( Y_2 = \begin{pmatrix} 0 & 0 \\ 1 & w \end{pmatrix} \), \( Y_1 = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \), and \( X_2 = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \). These coordinates satisfies the relations \( a + bx = 0 \) and \( c + dw = 0 \). The homogeneous coordinates \([b, d]\) does not depend on the choice of basis hence form a \( \mathbb{P}^1 \). All such representations form the total space of the bundle \( \mathcal{O}(1) \) over \( \mathbb{P}^1 \). Clearly the restriction of \( \mathcal{V}_4 \) to this part is also trivial.

There is a copy of \( \mathbb{P}^1 \) sitting as the boundaries of both quasi-projective varieties above. It correspond to the common divisor \( \mathbb{P}^1 \) in \( \mathbb{P}^2 \) and \( S \). This type of representations have \( X_1 \) proportional to \( Y_2 \) and rank \( X_2 \), rank \( Y_1 \) \( \leq 1 \). Hence \( X_2 \) is proportional to \( Y_1 \). The restriction of \( \mathcal{V}_4 \) to this part is \( \mathcal{O}(1) \oplus \mathcal{O}(3) \).

The weights of the tautological bundle and the tangent bundle at the fixed points on the irreducible component \( \mathbb{P}^2 \) are summarized in the following table.
The weights of the tautological bundle and the tangent bundle at the fixed points on the irreducible component $S = \mathbb{P}(T\mathbb{P}^1 \oplus T^*\mathbb{P}^1)$ are summarized in the following table.

<table>
<thead>
<tr>
<th>Fixed point</th>
<th>$[1, 0, 0]$</th>
<th>$[0, 1, 0]$</th>
<th>$[0, 0, 1]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{G_m}(T\mathbb{P}^2)$</td>
<td>$8u^2$</td>
<td>$8u^2$</td>
<td>$-4u^2$</td>
</tr>
<tr>
<td>$c_2^{G_m}(\nu)$</td>
<td>$3u^2$</td>
<td>$3u^2$</td>
<td>$-u^2$</td>
</tr>
</tbody>
</table>

Now we can calculate that $c_2(\nu_4) = s + p$ and $c_1(\nu_4) = 2q + 4f + c_0$ ($c_1(\nu_4)$ can also be obtained by using the stratification and passing to the dimension 2 case).

Summarizing all the discussions above in this section we get.

**Lemma 3.4.7.** The chern characters of $\nu_i$'s are as follows:

\[
\begin{align*}
\text{ch}(\nu_0) &= 1; \\
\text{ch}(\nu_1) &= 1 + c_0 + 2f + q + p/2; \\
\text{ch}(\nu_2) &= 1 + 2f + q + p/2; \\
\text{ch}(\nu_3) &= 1 + c_0 + 4f + 2q + 2p + 2s; \\
\text{ch}(\nu_4) &= 2 + 2q + 4f + c_0 + p + s. 
\end{align*}
\]

**Remark 3.4.8.** In the calculation we use the stability condition $(-1, \ldots, -1)$. The Hilbert scheme of point corresponds to Nakajima quiver variety but with a different stability condition. It is not hard to see, by analysing the weights to the tangent spaces, that the fixed points on the Hilbert scheme match up with the above table.

The natural embedding of $\text{Hilb}^0(T^*\mathbb{P}^1)$ into $\text{Hilb}^2(\widehat{\mathbb{A}^2}/\mathbb{Z}_l)$ as $S_i \coprod \mathbb{P}^2_i$ can be constructed on the quiver level. For any representation $\xi$ of $\widehat{\mathbb{A}_1}$ that lie in $\text{Hilb}^0(T^*\mathbb{P}^1)$, we associate a presentation $\eta$ of $\widehat{\mathbb{A}_{l-1}}$ as follows. Take $X_1, \cdots, X_{i-1}$ and $X_{i+2}, \cdots, X_l$ all to be the identity. Identify $V_0, \cdots, V_{i-1}, V_{i+1}, \cdots, V_{l-1}$ using those $X$ maps which we take to be the identity.
Let the $X_i$ map of $\eta$ to be the $X_1$ map of $\xi$, using the identification of $V_0$ and $V_{i-1}$ as above. Similarly, let the $Y_i$ map of $\eta$ to be the $Y_1$ map of $\xi$; the $X_{i+1}$ map of $\eta$ to be the $X_2$ map of $\xi$; the $Y_{i+1}$ map of $\eta$ to be the $Y_2$ map of $\xi$. Then, the ADHM equation uniquely determines $Y_1, \ldots, Y_{i-1}$ and $Y_{i+2}, \ldots, Y_l$. Note that this embedding is not equivariant under the $\mathbb{G}_m$-action.

Now we look at the $n = 3$ case.

We express the open part of the $P_{1,2}$ component, when $X_1$ and $X_3$ has rank 1, in terms of quiver varieties. In this case, the representations have $\ker X_1 = \text{im} Y_1 = \ker Y_3 = \text{im} X_3$, $\ker X_2 = \text{im} Y_2$, and $\text{im} X_2 = \ker Y_2$. We choose a basis for $V_0$ to be $J(1)$ and an arbitrary non-zero vector in $\text{im} Y_1$. We take the basis for $V_1$ to be $X_1 J(1)$ and $Y_2 Y_3 J(1)$, and the basis for $V_2$ to be $Y_3 J(1)$ and $X_2 X_1 J(1)$. Let $Y_1 = \left( \begin{array}{cc} 0 & 0 \\ a & b \end{array} \right)$, $X_3 = \left( \begin{array}{cc} 0 & 0 \\ c & a \end{array} \right)$. Under this choice of basis, the representations are determined by the homogenous coordinates $[a, b, c]$. An open part of this $\mathbb{P}^2$ is the open set of the component $P_{1,2}$ we are looking for. There is one fixed point in this open set which corresponds to $([1 : 0], [0 : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1$. The other fixed points all lie in the intersection of $P_{1,2}$ with $\mathbb{P}^2_1$ or $\mathbb{P}^2_2$, and are not in this open set.

<table>
<thead>
<tr>
<th>Fixed point</th>
<th>$[0, 1, 0]$ on $\mathbb{P}^2_1$</th>
<th>$[0, 0, 1]$ on $\mathbb{P}^2_1$</th>
<th>$[1, 0, 0]$ on $\mathbb{P}^2_2$</th>
<th>$[0, 0, 1]$ on $\mathbb{P}^2_2$</th>
<th>$([1, 0], [0, 1])$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{G_m}(TP_{1,2})$</td>
<td>$-8u^2$</td>
<td>$4u^2$</td>
<td>$-8u^2$</td>
<td>$4u^2$</td>
<td>$-4u^2$</td>
</tr>
<tr>
<td>$c_2^{G_m}(\mathcal{V})$</td>
<td>$-2u^2$</td>
<td>$-2u^2$</td>
<td>$-2u^2$</td>
<td>$-2u^2$</td>
<td>$-2u^2$</td>
</tr>
</tbody>
</table>

For $l \geq 3$, the natural embedding of $\text{Hilb}^3_C(\mathbb{A}^2/\mathbb{Z}_3)$ into $\text{Hilb}^2_C(\mathbb{A}^2/\mathbb{Z}_4)$ as

$$S_i \coprod S_{i+1} \coprod \mathbb{P}^2_i \coprod \mathbb{P}^2_{i+1} \coprod \mathbb{P}_{i,i+1}$$

gives the second Chern classes of the bundle $\mathcal{V}_{0,k}$ on any of the components $P_{i,i+1}$.

**Lemma 3.4.9.** The second Chern class of the rank 2 vector bundle $\mathcal{V}_{0,i}$ is:

$$c_2(\mathcal{V}_{0,i}) = s_i + p_i.$$
Proof of Proposition 3.4.4. By Lemma 3.4.5 we have

\begin{align*}
\text{ch}(\mathcal{V}_{i,j}) &= \text{ch}(\mathcal{V}_i) \cdot (\text{ch}(\mathcal{V}_0) + \text{ch}(\mathcal{V}_\sigma) + \text{ch}(\mathcal{V}_{0,i}) - \text{ch}(\mathcal{V}_{0,j})) \\
&= (1 + d_i + p_i/2) \cdot (2 + d_0 + d_j - d_i + p_j/2 - p_i/2 + s_j - s_i \\
&\quad + 1/2(\sum p_i - \sum p_{j,i+1})) \\
&= 2 + d_j + d_i + d_0 + \sum p_k/2 - \sum p_{k,k+1}/2 + p_{i,j} + p_i/2 + p_j/2 + s_i + s_j.
\end{align*}

\[\square\]

3.5 The central charge

As has been mentioned before, the central charge can be obtained from the knowledge of the Chern character map. In this section, we illustrate this in the special case \(n = 2\) and \(\Gamma_1 = \mathbb{Z}/l\).

3.5.1 The \(B_2\)-case

By calculation it is easy to know that

\[\text{ch}(\mathcal{V}_4) = 1/2 \text{ch}(\mathcal{V}_1) + 3/2 \text{ch}(\mathcal{V}_2) + 1/2 \text{ch}(\mathcal{V}_1 \otimes \mathcal{V}_2) - 1/2 \text{ch}(\mathcal{V}_2^2).\]

For any coherent sheaf \(\mathcal{F}\) on \(X\), we define the linear functional on \(H^*(\text{Hilb}^2) \ l_{\mathcal{F}}(\text{ch}(\mathcal{M})) = \chi(\mathcal{F} \otimes \pi^* \mathcal{M})\) for \(\mathcal{M} \in \text{Coh} (\text{Hilb}^2)\). We now calculate the polynomials

\[l_{\mathcal{F}}(a, b) := l_{\mathcal{F}}(\text{ch}(\mathcal{O}(aD_1 + bD_2))).\]

This polynomial can be written as

\[l_{\mathcal{F}}(a, b) = l_{\mathcal{F}}(1) + al_{\mathcal{F}}(d_1) + bl_{\mathcal{F}}(d_2) + (a^2 + b^2 + 2ab)l_{\mathcal{F}}(p/2) + 4abf_{\mathcal{F}}(s/2).\]

We denote the coefficients by

\[C^0_{\mathcal{F}} = l_{\mathcal{F}}(1),\ C^1_{\mathcal{F}} = l_{\mathcal{F}}(d_1),\ C^2_{\mathcal{F}} = l_{\mathcal{F}}(d_2),\ C^3_{\mathcal{F}} = l_{\mathcal{F}}(s),\ \text{and}\ C^4_{\mathcal{F}} = l_{\mathcal{F}}(p).\]
We know that $\chi(\mathcal{H}\text{om}(\mathcal{V}_i, \mathcal{L}_j)) = \delta_{i,j}$ for $i, j = 0, \cdots, 4$. This tells us the values of the polynomials $\mathcal{L}_j$ at the points $(0, 0), (-1, 0), (0, -1), (-1, -1)$ and $(-2, 0)$. Now we plug-in and solve these systems of linear equations to get

$$
\begin{align*}
C_0^0 &= 1 & C_0^1 &= 3/2 & C_0^2 &= 3/2 & C_0^3 &= 1/2 & C_0^4 &= 0 \\
C_1^0 &= 0 & C_1^1 &= 1/2 & C_1^2 &= -1/2 & C_1^3 &= 1/2 & C_1^4 &= -1/2 \\
C_2^0 &= 0 & C_2^1 &= -1/2 & C_2^2 &= 1/2 & C_2^3 &= 1/2 & C_2^4 &= -1/2 \\
C_3^0 &= 0 & C_3^1 &= 1/2 & C_3^2 &= 1/2 & C_3^3 &= 1/2 & C_3^4 &= 0 \\
C_4^0 &= 0 & C_4^1 &= -1 & C_4^2 &= -1 & C_4^3 &= -1 & C_4^4 &= 1/2
\end{align*}
$$

Plug-in these coefficients and we get

$$
\begin{align*}
\mathcal{L}_0 &= 1/2(a + b + 1)(a + b + 2); \\
\mathcal{L}_1 &= 1/2(a - b + 1)(a - b); \\
\mathcal{L}_2 &= 1/2(a - b - 1)(a - b); \\
\mathcal{L}_3 &= 1/2(a + b + 1)(a + b); \\
\mathcal{L}_4 &= -a - b - a^2 - b^2.
\end{align*}
$$

Now we define the dimension polynomials

$$
P_{\mathcal{L}_j}(a, b) = \chi(\mathcal{E}_0 \otimes \mathcal{L}_j \otimes \mathcal{O}(a, b)),
$$

where $\mathcal{E}_0$ is the self-dual splitting bundle on Hilb$^2$ in [12]. (Their reparametrizations $P_{\mathcal{L}_j}(\alpha, \beta)$ with $\alpha = ap$ and $\beta = bp$ gives the dimension of the irreducible objects.) They can be computed as

$$
\chi(\mathcal{E}_0 \otimes \mathcal{L}_j \otimes \mathcal{O}(a, b)) = \sum_i [V_i : (k[x]/(x^p))^\otimes 2] \chi(\mathcal{V}_i^* \otimes \mathcal{L}_j \otimes \mathcal{O}(a, b)),
$$

where $V_i$'s are the irreducible representations of $B_2$. We plug-in the equality in $K_Q$ that

$$
\mathcal{V}_4 = 1/2\mathcal{V}_1 + 3/2\mathcal{V}_2 + 1/2\mathcal{V}_3 - 1/2\mathcal{O}(2, 0),
$$

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and get \( P_{\mathcal{L}}(a, b) = \frac{p^2 - 1}{8} l_{a, b} + \frac{p^2 - 2p - 3}{4} l_{a, b - 1} + \frac{p^2 - p + 3}{4} l_{a - 1, b} + \frac{p^2 + 4p + 3}{4} l_{a - 1, b - 1} - \frac{p^2 - 1}{8} l_{a - 2, b} \).

Then we define the central charge polynomials as \( Z_j(a, b) = \lim_{p \to \infty} 1/p^2 P_{\mathcal{L}}(a, b) \). An easy calculation shows

\[
\begin{align*}
Z_0 &= 1/8(2a + 2b + 1)^2; \\
Z_1 &= 1/8(-2a + 2b - 1)^2; \\
Z_2 &= 1/8(-2a + 2b + 1)^2; \\
Z_3 &= 1/8(2a + 2b - 1)^2; \\
Z_4 &= -1/4(4a^2 + 4b^2 - 1).
\end{align*}
\]

One can see that \( Z_4 \) is an irreducible polynomial and its real zero locus is a circle, and it takes positive values in the region bounded by the zero locus of \( Z_0, \ldots, Z_3 \).

### 3.5.2 The \( \Gamma_1 = \mathbb{Z}/l \)-case

We solve for the geometric basis of \( H^*(\text{Hilb}_{\mathbb{Z}/l}^2, \mathbb{Q}) \) in terms of \( \text{ch}(\mathcal{V}_a) \).

\[
\begin{align*}
\sigma_i &= \text{ch}(\mathcal{V}_{0,i}) - \text{ch}(\mathcal{V}_\sigma) - \text{ch}(\mathcal{V}_i); \\
p_i &= \text{ch}(\mathcal{V}_0) + \text{ch}(\mathcal{V}_{\sigma,i}) + \text{ch}(\mathcal{V}_\sigma) + \text{ch}(\mathcal{V}_i) - 2 \text{ch}(\mathcal{V}_{0,i}); \\
p_{i,j} &= \text{ch}(\mathcal{V}_{\sigma}) + \text{ch}(\mathcal{V}_0) + \text{ch}(\mathcal{V}_{i,j}) - \text{ch}(\mathcal{V}_{0,i}) - \text{ch}(\mathcal{V}_{0,j}); \\
d_i &= \text{ch}(\mathcal{V}_i)/2 + \text{ch}(\mathcal{V}_{0,i}) - \text{ch}(\mathcal{V}_{\sigma,i})/2 - 3/2 \text{ch}(\mathcal{V}_0) - \text{ch}(\mathcal{V}_\sigma)/2; \\
d_0 &= -3/2 \text{ch}(\mathcal{V}_0) + 1/2 \text{ch}(\mathcal{V}_\sigma) + 1/2 \text{ch}(\mathcal{V}_{0,1}) + 1/2 \text{ch}(\mathcal{V}_{0,l-1}) \\
&\quad - \sum_{i=1}^{l-2} \text{ch}(\mathcal{V}_i)/2 - \sum_{j=1}^{l-2} \text{ch}(\mathcal{V}_{\sigma,i})/2 + 1/2 \sum_{j=1}^{l-2} \text{ch}(\mathcal{V}_{j,j+1}); \\
1 &= \text{ch}(\mathcal{V}_0).
\end{align*}
\]

We have

\[
\text{ch}(\mathcal{O}(n_0 D_0 + \sum n_i D_i))
\]

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\[ n_0d_0 + \sum_i n_id_i + n_0^2/2(\sum_i p_i - \sum p_{i,i+1}) + \sum_i n_i^2/2p_i + \sum_{i\geq j} n_in_jp_{i,j} + \sum_i n_0n_i(p_i + 2s_i). \]

Hence, the polynomials \( L_n \) can be calculated as below.

- \( L_0 = 1/2(n_0 + \sum n_i - 1)(n_0 + \sum n_i - 2); \)
- \( L_\sigma = 1/2(n_0 - \sum n_i + 1)(n_0 - \sum n_i); \)
- \( L_i = 1/2(n_i - n_0)(n_i - n_0 + 1); \)
- \( L_{\sigma,i} = 1/2(n_0 + n_i)(n_0 + n_i - 1); \)
- \( L_{0,i} = 1/2n_0(1 - n_0) + n_i(1 - \sum n_j) \text{ if } i = 1 \text{ or } l - 1; \)
- \( L_{0,i} = n_i(1 - \sum n_j) \text{ if } i \neq 1 \text{ or } l - 1; \)
- \( L_{j,i} = 1/2n_0(1 - n_0) + n_in_j \text{ if } i - j = \pm 1; \)
- \( L_{j,i} = n_in_j \text{ otherwise}. \)

Also, there is a basis given by Chern character of line bundles, which can be chosen as \( ch\mathcal{V}_i, ch\mathcal{V}_{\sigma,i}, ch\mathcal{V}_i^2, \) and \( ch(\mathcal{V}_i) \cdot ch(\mathcal{V}_j). \) As can be easily checked,

\[ ch(\mathcal{V}_i^2) = 1 + 2d_i + 2p_i, \]
\[ ch(\mathcal{V}_i) \cdot ch(\mathcal{V}_j) = 1 + d_i + d_j + 1/2(p_i + p_j) + p_{i,j}. \]

In the group \( K_G, \) we have the change of basis

\[ \mathcal{V}_{0,i} = 3/2\mathcal{V}_i + 1/2\mathcal{V}_\sigma + 1/2\mathcal{V}_{\sigma,i} - 1/2\mathcal{V}_i^2, \]
\[ \mathcal{V}_{i,j} = \mathcal{V}_i \cdot \mathcal{V}_j + 1/2\mathcal{V}_{\sigma,i} - 1/2\mathcal{V}_i^2 + 1/2\mathcal{V}_i + 1/2\mathcal{V}_{\sigma,j} - 1/2\mathcal{V}_j^2 + 1/2\mathcal{V}_j. \]

We will decompose \((k[x]/xp)^2\) into isotypical components. But in the decomposition, we only care about the behavior for \( p \) large enough. Therefore, we will keep only highest order terms with respect to \( p \) in the multiplicities of the irreducible representations. the multiplicity of irreducibles is \([(k[x]/xp)^2 : V_{i,j}] = (p/l)^2\) for \( i \neq j, \) and \([(k[x]/xp)^2 : V_i] = [(k[x]/xp)^2 : V_{\sigma,i}] = 1/2(p/l)^2.\)
The dimensional polynomials are, forgetting terms involving $p$’s power less than or equal to 2,

\[
P_{Z_{\alpha}}(n_0, n_1, \ldots, n_{l-1}) = \left(\frac{p}{l}\right)^2(1/2l_{\alpha}(n_0, n_1, \ldots, n_{l-1}) + l/2l_{\alpha}(n_0 - 1, n_1, \ldots, n_{l-1})
\]
\[
+ \sum_{i=1}^{l-1} (l + 2)/2l_{\alpha}(n_0, n_1, \ldots, n_i - 1, \ldots, n_{l-1})
\]
\[
+ \sum_{i=1}^{l-1} l/2l_{\alpha}(n_0 - 1, n_1, \ldots, n_i - 1 \ldots, n_{l-1})
\]
\[
+ \sum_{i=1}^{l-1} -(l - 1)/2l_{\alpha}(n_0, n_1, \ldots, n_i - 2, \ldots, n_{l-1})
\]
\[
+ \sum_{i>j} l_{\alpha}(n_0, n_1, \ldots, n_j - 1, \ldots, n_i - 1, \ldots, n_{l-1})).
\]

So the central charge polynomials are

\[
Z_0 = 1/2((n_0 + \sum n_i) - (3 - 1/l))^2;
\]
\[
Z_{\sigma} = 1/2((n_0 - \sum n_i) + (1 - 1/l))^2;
\]
\[
Z_i = 1/2((n_i - n_0) + (1 - 1/l))^2;
\]
\[
Z_{\sigma,i} = 1/2((n_0 + n_i) - (1 + 1/l))^2
\]
\[
Z_{0,i} = -l/2(n_0 - 1)^2 - (n_i - 1/l)(\sum n_j - (2 - 1/l)) \text{ if } i = 1 \text{ or } l - 1;
\]
\[
Z_{0,i} = -(n_i - 1/l)(\sum n_j - (2 - 1/l)) \text{ if } i \neq 1 \text{ or } l - 1;
\]
\[
Z_{j,i} = -l/2(n_0 - 1)^2 + (n_i - 1/l)(n_j - 1/l) \text{ if } i - j = \pm 1;
\]
\[
Z_{j,i} = (n_i - 1/l)(n_j - 1/l) \text{ otherwise.}
\]

### 3.6 The $t$-structures associated to alcoves

In this section we study the $t$-structures associated to the alcoves in the affine hyperplane arrangement defined in Section 3.2.

In the alcove containing the origin, we associate the $t$-structure coming from the derived
equivalence

\[ D^b(\text{Coh Hilb}^n) \cong D^b(\text{Coh}_{\Gamma_n} \mathbb{A}^{2n}). \]

We need to find out the \( t \)-structures associated to other alcoves. We start with the case when \( \Gamma_n \cong B_2 \).

### 3.6.1 The \( B_2 \)-case

We need to calculate the Ext’s among the simple objects in \( \text{Coh}_{B_2}^-\mathbb{C}^4 \). The result is summarized as follows:

\[
\text{Ext}^\bullet(\mathcal{L}_i, \mathcal{L}_j) =
\]

<table>
<thead>
<tr>
<th>deg</th>
<th>( \mathcal{L}_i = \mathcal{L}_j ) both rank 1</th>
<th>( \mathcal{L}_i \neq \mathcal{L}_j ) both rank 1</th>
<th>( \mathcal{L}_i \neq \mathcal{L}_j ) one having rank 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>deg = 0</td>
<td>( \mathbb{C} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>deg = 1</td>
<td>0</td>
<td>0</td>
<td>( \mathbb{C}^2 )</td>
</tr>
<tr>
<td>deg = 2</td>
<td>( \mathbb{C} )</td>
<td>( \mathbb{C} ) or ( \mathbb{C}^3 )</td>
<td>0</td>
</tr>
<tr>
<td>deg = 3</td>
<td>0</td>
<td>0</td>
<td>( \mathbb{C}^2 )</td>
</tr>
<tr>
<td>deg = 4</td>
<td>( \mathbb{C} )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In the case that both \( \mathcal{L}_i \) and \( \mathcal{L}_j \) have rank 1, \( \text{Ext}^2(\mathcal{L}_i, \mathcal{L}_j) \) is \( \mathbb{C}^3 \) only in the following cases: One of \( i, j \) is 0 and the other is 3, or one of them is 1 and the other is 2. The \( \text{Ext}^\bullet(\mathcal{L}_4, \mathcal{L}_4) \) is \( \mathbb{C} \) in degree 0 and 4, \( \mathbb{C}^6 \) in degree 2, and zero otherwise.

The bilinear pairing \( \langle [A], [B] \rangle := \sum_i (-1)^i \text{Ext}^i(A, B) \) can be expressed under the basis \( \{ \mathcal{L}_j \} \) as

\[
\begin{pmatrix}
3 & 1 & 1 & 3 & -4 \\
1 & 3 & 3 & 1 & -4 \\
1 & 3 & 3 & 1 & -4 \\
3 & 1 & 1 & 3 & -4 \\
-4 & -4 & -4 & -4 & 8
\end{pmatrix}
\]

Let us look at what happens when cross the wall defined by \( Z_0 = 0 \). The new abelian category has the same Grothendieck group. The simple objects have classes \([\mathcal{L}_0], [\mathcal{L}_1] - [\mathcal{L}_0], \ldots\).
\([L_2 - L_0], [L_3 - 3L_0], \text{and } [L_4 + 2L_0] \). One can calculate the dual basis to find the classes of their projective covers in the Grothendieck group. They are \([V_0] + [V_1] + 3[\mathcal{V}_3] - 2[\mathcal{V}_4], [\mathcal{V}_1], [\mathcal{V}_2], [\mathcal{V}_3], \text{and } [\mathcal{V}_4] \) respectively. One can also see, through the bilinear pairing under the new basis, that the new abelian category is not Morita equivalent to the original one, as the bilinear form

\[
\begin{pmatrix}
3 & -2 & -2 & -6 & 2 \\
-2 & 4 & 4 & 4 & -4 \\
-2 & 4 & 4 & 4 & -4 \\
-6 & 4 & 4 & 12 & -4 \\
2 & -4 & -4 & -4 & 4
\end{pmatrix}
\]

does not differ from the original one by a permutation.

The central charge polynomials corresponding to the simple objects in this abelian heart can be obtained from the original ones. More explicitly, they are, respectively,

\[
Z_0 = \frac{1}{8}(2a + 2b + 1)^2; \\
Z_1 - Z_0 = -(2a + 1)b; \\
Z_2 - Z_0 = -a(2b + 1); \\
Z_3 - 3Z_0 = \frac{1}{8}((2a + 2b - 1)^2 - 3(2a + 2b + 1)); \\
Z_4 + 2Z_0 = \frac{1}{2}(2a + 1)(2b + 1).
\]

One can see that \(Z_3 - 3Z_0\) takes positive values in the region bounded by the other 4 polynomials.

In fact, we can do iterated (right) tilting with respect to the simple object \(L_0\). The intermediate t-structure \(R_{L_0} \mathfrak{A}\) has simple objects \(L_0[1], L_j\) for \(j = 1, 2, 3\), and \(L_4^1\) fitting into the short exact sequence

\[
0 \to \text{Ext}^1(L_4, L_0)^* \otimes L_0 \to L_4^1 \to L_4 \to 0.
\]

The classes of their potential projective covers are \(-[\mathcal{V}_6] + 2[\mathcal{V}_4], [\mathcal{V}_1], [\mathcal{V}_2], [\mathcal{V}_3], \text{and } [\mathcal{V}_4] \) respectively. We can try to do the truncated mutation with respect to \(\mathcal{V}_0\). We get \(\mathcal{V}_j^1 = \mathcal{V}_j\)
for $j \neq 0$, and $\mathcal{V}^0_j$ being the cokernel of $\mathcal{V}_0 \to \text{Hom}(\mathcal{V}_0, \mathcal{V}_4)^* \otimes \mathcal{V}_4$. The injectivity of the map $\mathcal{V}_0 \to \text{Hom}(\mathcal{V}_0, \mathcal{V}_4)^* \otimes \mathcal{V}_4$ can be checked generically on $\mathbb{C}^4$. Proposition 3.1.18 tells us that the abelian heart $R_{Z_0} \mathcal{A}$ is derived equivalent to the original category.

If we do tilting again, we get $R_{Z_0[1]} R_{Z_0} \mathcal{A}$ whose simple objects are $\mathcal{L}_0[2]$, $\mathcal{L}_2$ fitting into the short exact sequence

$$0 \to \text{Ext}^2(\mathcal{L}_j, \mathcal{L}_0) \otimes \mathcal{L}_0[1] \to \mathcal{L}_j \to 0$$

for $j = 1, 2, 3$, and $\mathcal{L}_2 = \mathcal{L}_1^1$. This is exactly the new abelian category we obtained cross the wall $Z_0 = 0$.

Similarly, one can start from the initial region and go across the other walls. As an example, let us look at the $t$-structure associated to the region across the wall defined by $Z_1 = 0$. The simple objects have classes in the Grothendieck group $[\mathcal{L}_0] - [\mathcal{L}_1]$, $[\mathcal{L}_1]$, $[\mathcal{L}_2] - 3[\mathcal{L}_1]$, $[\mathcal{L}_3] - [\mathcal{L}_1]$, and $[\mathcal{L}_4] + 2[\mathcal{L}_1]$. Their corresponding projective covers have classes $[\mathcal{V}_0]$, $[\mathcal{V}_1] + [\mathcal{V}_0] + 3[\mathcal{V}_2] + [\mathcal{V}_3] - 2[\mathcal{V}_4]$, $[\mathcal{V}_2]$, $[\mathcal{V}_3]$, and $[\mathcal{V}_4]$.

We can do iterated tilting to find out the complexes in the original abelian category represents these simple objects. The simple object $\mathcal{L}_1^2$ corresponding to $\mathcal{L}_1$ is $\mathcal{L}_1[2]$. For $i = 0, 2, 3$, the simple object $\mathcal{L}_i^2$ fits into short exact sequences

$$0 \to \text{Ext}^2(\mathcal{L}_i, \mathcal{L}_1)^* \otimes \mathcal{L}_1[1] \to \mathcal{L}_i^2 \to \mathcal{L}_i \to 0.$$

And $\mathcal{L}_4^2$ fits into the short exact sequence

$$0 \to \text{Ext}^1(\mathcal{L}_4, \mathcal{L}_1)^* \otimes \mathcal{L}_1 \to \mathcal{L}_4^2 \to \mathcal{L}_4 \to 0.$$


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If we do iterated tilting, we can find that the simple object \( L_3^2 \) corresponding to \( L_3 \) is \( L_3[2] \). For \( i = 0, 1, 2 \), the simple object \( L_i^2 \) fits into short exact sequences

\[
0 \to \Ext^2(L_i, L_3)^* \otimes L_3[1] \to L_i^2 \to L_i \to 0.
\]

And \( L_4^2 \) fits into the short exact sequence

\[
0 \to \Ext^1(L_4, L_3)^* \otimes L_3 \to L_4^2 \to L_4 \to 0.
\]

Now it is turn to look at the \( t \)-structure associated to the region across the wall defined by \( Z_2 = 0 \). The simple objects have classes in the Grothendieck group \([L_0] - [L_2], [L_1] - 3[L_2], [L_2], [L_3] - [L_2], \) and \([L_4] + 2[L_2]\). Their corresponding projective covers have classes \([\mathcal{V}_0], [\mathcal{V}_1], [\mathcal{V}_2] + [\mathcal{V}_0] + 3[\mathcal{V}_2] + [\mathcal{V}_3] - 2[\mathcal{V}_4], [\mathcal{V}_3], \) and \([\mathcal{V}_4]\).

Similarly we can find that the simple object \( L_2^2 \) corresponding to \( L_2 \) is \( L_2[2] \). For \( i = 0, 1, 3 \), the simple object \( L_i^2 \) fits into short exact sequences

\[
0 \to \Ext^2(L_i, L_2)^* \otimes L_2[1] \to L_i^2 \to L_i \to 0.
\]

And \( L_4^2 \) fits into the short exact sequence

\[
0 \to \Ext^1(L_4, L_2)^* \otimes L_2 \to L_4^2 \to L_4 \to 0.
\]

Note that the region bounded by the walls \( a = \pm 1/2 \) and \( b = \pm 1/2 \) is a fundamental domain of the \( H^2(\text{Hilb}^2(T^*\mathbb{P}^1), \mathbb{Z}) \cong \mathbb{Z}^2 \) action on \( H^2(\text{Hilb}^2(T^*\mathbb{P}^1), \mathbb{R}) \). Denote this domain by \( D_0 \). Summarizing the discussion in this subsection, we get the following proposition.

**Proposition 3.6.1.** There is a real variation of stability conditions on \( D_0 \) (hence any translation of it by \( H^2(\text{Hilb}^2(T^*\mathbb{P}^1), \mathbb{Z}) \)) whose \( t \)-structure at the origin is \( \text{Coh}_{B_2}(\mathbb{C}^4) \) with central charge polynomials given by \( Z_i \) defined in Subsection 3.5.1.

To summarize the description of the hyperplane arrangement and the \( t \)-structures associated to alcoves in this case, the following picture is part of the hyperplane arrangement.
The $t$-structure associated to the alcove labeled by $t_0$ is the so called orbifold (or BKR) $t$-structure, whose tilting generator can be chosen to be the Haiman’s Procesi bundle. The four $t$-structure associated to alcoves adjacent to $t_0$ are obtained from $t_0$ by $\mathbb{P}^2$-semi-reflections. The tilting generators for these $t$-structures are obtained from the truncated mutations described in Section 3.1.

The rest of the alcoves can be obtained from them by a shifting. For example, the alcove $t_2$ and $t'_2$ differ by a translation, hence the corresponding $t$-structures differs by twisting by a line bundle.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- cycle;
\draw (0,0) -- (1,1) -- (-1,1) -- cycle;
\draw (2,0) -- (1,-1) -- (-1,-1) -- cycle;
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (-1,0) -- (-1,1) -- (0,1) -- cycle;
\draw (2,0) -- (1,-1) -- (1,1) -- (2,1) -- cycle;
\draw (2,0) -- (1,-1) -- (-1,-1) -- (-1,1) -- cycle;
\draw (0,0) -- (-1,0) -- (-1,1) -- (0,1) -- cycle;
\draw (2,0) -- (1,-1) -- (-1,-1) -- (-1,1) -- cycle;
\node at (1.5,1.5) {$b$};
\node at (-1.5,1.5) {$a$};
\node at (1.5,-1.5) {$t_3$};
\node at (-1.5,-1.5) {$t'_2$};
\node at (0.5,0.5) {$t_2$};
\node at (-0.5,0.5) {$t_1$};
\node at (-0.5,-0.5) {$t_0$};
\node at (0.5,-0.5) {$t'_2$};
\end{tikzpicture}
\end{center}

**Lemma 3.6.2.** The functor $F = - \otimes \mathcal{V}_2 : D^b(\text{Hilb}^2(T^*\mathbb{P}^1)) \to D^b(\text{Hilb}^2(T^*\mathbb{P}^1))$ with the source endowed with the $t$-structure coming from $\text{Coh}_{B_2}(\mathbb{C}^4)$ is a perverse equivalence for suitable filtration and perversity function.

**Proof.** Define a filtration on $\mathfrak{A} = \text{Coh}_{B_2}(\mathbb{C}^4)$ as follows: $\mathfrak{A}_1$ is the Serre subcategory generated by the simple objects $\mathcal{L}_0$ and $\mathcal{L}_1$; $\mathfrak{A}_2$ is generated by $\mathcal{L}_{0,1}$ in addition to $\mathfrak{A}_1$; and $\mathfrak{A}_3 = \mathfrak{A}$. Define the perversity function to be $p(1) = 2$, $p(2) = 1$, $p(3) = 0$.

Using the notations as in Diagram 3.3, the bundle $\mathcal{V}^2$ comes from $(T^*\mathbb{P}^1)^2$. Using projection formula, the complexes $F(\mathcal{V}_i)$ can be calculated in $\text{Coh}_{\mathfrak{A}_2}((T^*\mathbb{P}^1)^2)$. Let $\mathcal{O}_r$ be the trivial line bundle on $(T^*\mathbb{P}^1)^2$ endowed with the reflection representation. We have

$$F(\mathcal{V}_0) = \mathcal{L} \boxtimes \mathcal{L} \cong \mathcal{V}_1;$$
\[
F(\mathcal{V}_1) = (\mathcal{L}^2 \boxtimes (\mathcal{L}^2)) \cong (\mathcal{O} \to 2\mathcal{L}) \boxtimes (\mathcal{O} \to 2\mathcal{L}) \cong (\mathcal{O} \to 2\mathcal{V}_{0,1} \to 4\mathcal{V}_1);
\]
\[
F(\mathcal{V}_2) = \mathcal{O}_r \otimes (\mathcal{L} \boxtimes (\mathcal{L}^2)) \cong \mathcal{V}_3;
\]
\[
F(\mathcal{V}_3) = (\mathcal{L} \boxtimes (\mathcal{L}^2)) \otimes (\mathcal{O}_r \otimes (\mathcal{L} \boxtimes (\mathcal{L}^2))) = (\mathcal{V}_2 \to 2\mathcal{V}_{0,1} \to 4\mathcal{V}_3);
\]
\[
F(\mathcal{V}_4) = (\mathcal{L}^2 \boxtimes (\mathcal{L}^2)) \oplus (\mathcal{L} \boxtimes (\mathcal{L}^2)) \cong (\mathcal{V}_4 \to 4\mathcal{V}_1).
\]

### 3.6.2 The $\Gamma_1 = \mathbb{Z}_l$-case

In the initial alcove which is bounded by the walls
\[
1/2((n_0 + \sum n_i) - (3 - 1/l))^2 = 0
\]
\[
1/2((n_0 - \sum n_i) + (1 - 1/l))^2 = 0
\]
\[
1/2((n_i - n_0) + (1 - 1/l))^2 = 0 \text{ for all } i = 1, \cdots, l - 1
\]
\[
1/2((n_0 + n_i) - (1 + 1/l))^2 = 0 \text{ for all } i = 1, \cdots, l - 1,
\]
we associate to it the $t$-structure coming from the derived equivalence
\[
D^b(\text{Hilb}_{\mathbb{Z}_l}^2) \cong D^b(\text{Coh}_2(\mathbb{A}^{2n})).
\]

The simple objects are labeled by the irreducible representations.

If we go across the wall defined by $1/2((n_0 + \sum n_i) - (3 - 1/l))^2 = 0$, the $t$-structure is the one coming from $R_{\mathbb{Z}_l[1]}R_{\mathbb{Z}_l}\text{Coh}_2(\mathbb{A}^{2n})$. By Section 3.1.3, we know the heart of this new $t$-structure is also a finite length category. The simple objects are $\mathcal{L}_\alpha''$ as defined in Section 3.1.3. The classes of the simple objects in the Grothendieck group are $[\mathcal{L}_0], [\mathcal{L}_0] - 3[\mathcal{L}_0], [\mathcal{L}_1], [\mathcal{L}_{0,1}] + [\mathcal{L}_0], [\mathcal{L}_{0,l-1}] + [\mathcal{L}_0], [\mathcal{L}_{0,i}]$ for $i \neq 2, l - 2$, and $[\mathcal{L}_{i,j}]$. The new alcove is bounded by the walls
\[
\sum n_j - 2 + 1/l = 0
\]
\[
1/2((n_0 + \sum n_i) - (3 - 1/l))^2 = 0
\]
\[
1/2((n_i - n_0) + (1 - 1/l))^2 = 0 \text{ for all } i = 1, \cdots, l - 1
\]
where on the first wall, both $Z_{\mathcal{L}'_{0,1}}$ and $Z_{\mathcal{L}'_{0,l-1}}$ vanish, and both of order 1.

If we go across the wall $\sum n_j - 2 + 1/l = 0$, the new $t$-structure is obtained by doing tilting with respect to $\mathcal{L}_{0,1}$ and $\mathcal{L}_{0,l-1}$. The classes of the simple objects in the Grothendieck group are $[\mathcal{L}_0], [\mathcal{L}_\sigma] - 3[\mathcal{L}_0], [\mathcal{L}_i]$ for $i = 2, \cdots, l - 2, [\mathcal{L}_1] + [\mathcal{L}_{0,1}] + [\mathcal{L}_0], [\mathcal{L}_{l-1}] + [\mathcal{L}_{0,l-1}] + [\mathcal{L}_0], [\mathcal{L}_{\sigma,i}], -[\mathcal{L}_{0,1}] - [\mathcal{L}_0], -[\mathcal{L}_{0,l-1}] - [\mathcal{L}_0], [\mathcal{L}_{0,i}]$ for $i \neq 2, l - 2$, and $[\mathcal{L}_{i,j}]$. The new alcove is bounded by the walls

$$\frac{1}{2}((n_0 - 1) + \sum_{j=2}^{l-1} (n_j - 1/l) - 1)^2 = 0$$
$$\frac{1}{2}((n_0 - 1) + \sum_{j=1}^{l-2} (n_j - 1/l) - 1)^2 = 0$$
$$\frac{1}{2}((n_0 - \sum_{i=1}^{l} n_i) + (1 - 1/l))^2 = 0$$
$$\sum_{j=1}^{l-1} n_j - 2 + 1/l = 0$$
$$\frac{1}{2}((n_i - n_0) + (1 - 1/l))^2 = 0 \text{ for all } i = 1, \cdots, l - 1$$

Then similar to the $B_2$-case, symmetry gives the $t$-structures associated to the other alcoves.
Chapter 4

Algebraic elliptic cohomology theory
and flops

4.1 Oriented cohomology

4.1.1 Oriented cohomology theories and algebraic cobordism

In this subsection we collect preliminary notions and results we will use. The main goal is to fix the notations and conventions.

Recall that a formal group law over a commutative ring $R$ with unit is an element $F(u,v) \in R[u,v]$ satisfying the following conditions

1. $F(u,0) = u, F(0,v) = v$;

2. $F(F(u,v),w) = F(u,F(v,w))$;

3. $F(u,v) = F(v,u)$.

Lazard pointed out the existence of a universal formal group law $(\mathbb{Laz}, F_{\mathbb{Laz}})$. He also proved that the ring $\mathbb{Laz}$, called Lazard ring, is a polynomial ring with integral coefficients on a
countable set of variables (see [52]). That is, for any formal group law \((R, F)\) over any ring \(R\), there exists a unique ring homomorphism \(r : \text{Laz} \to R\) such that \(F = r(F_{\text{Laz}})\).

Now let \(F(u, v) \in R[[u, v]]\) be an arbitrary formal group law over a commutative ring \(R\); we will always assume that \(R\) is graded and that, giving both \(u\) and \(v\) degree 1, \(F(u, v)\) is homogeneous of degree 0. As the Lazard ring is generated as a \(\mathbb{Z}\)-algebra by the coefficients of the universal formal group law, \(F_{\text{Laz}} \in \mathbb{Laz}[u, v]\) this convention gives \(\text{Laz}\) a uniquely defined grading, concentrated in non-positive degrees, and the classifying homomorphism \(\phi_F : \text{Laz} \to R\) associated to \(F\) preserves the grading.

Let \(k\) be a field, and \(\text{Sm}_k\) be the category of smooth, quasi-projective schemes over \(k\). Let \(\text{Comm}\) denote the category of commutative, graded rings with unit. The following definition can be found in [54].

**Definition 4.1.1.** An oriented cohomology theory on \(\text{Sm}_k\) is given by the following data.

**D1** An assignment \(A^*\) sending any \(X \in \text{Sm}_k^{op}\) to an object in \(\text{Comm}\).

**D2** For any smooth morphism \(f : Y \to X\) in \(\text{Sm}_k\), a ring homomorphism

\[
f^* : A^*(X) \to A^*(Y).
\]

**D3** For any projective morphism \(f : Y \to X\) in \(\text{Sm}_k\) of relative codimension \(d\), a homomorphism of graded \(A^*(X)\)-modules

\[
f_* : A^*(Y) \to A^{*+d}(X).
\]

These data satisfy the axioms: functoriality for \(f^*\) with respect to arbitrary morphisms in \(\text{Sm}_k\), functoriality for \(f_*\) with respect to projective morphisms, base change for transverse morphisms, a projective bundle formula, and extended homotopy property. We should mention that, for \(L \to X\) a line bundle over some \(X \in \text{Sm}_k\), the first Chern class of \(L\) in the theory \(A\), denote \(c_1^A(L)\), is defined by the formula \(c_1^A(L) := s^*s_*(1_X)\), where \(s : X \to L\) is
the zero section and \(1_X \in A^0(X)\) is the unit in \(A^*(X)\). We often write simply \(c_1(L)\) if the theory \(A\) is understood.

For an oriented cohomology theory \(A^*\), there is a unique power series

\[
F(u, v) \in A^*(k)[[u, v]]
\]

satisfying

\[
F_A(c_1(L), c_1(M)) = c_1(L \otimes M)
\]

for each pair line bundles \(L\) and \(M\) on a scheme \(X \in \text{Sm}_k\); moreover, \(F\) defines a formal group law over the ring \(A^*(k)\) [54, Lemma 1.1.3]. As we will often consider cohomology theories with rational coefficients, we denote the ring \(A^*(X) \otimes \mathbb{Q}\) simply by \(A^*_Q(X)\).

For any \(X \in \text{Sm}_k\), Levine and Morel constructed a commutative graded ring \(\Omega^*(X)\) as follows. Let \(M(X)\) be the set of isomorphism classes of projective morphisms \(f : Y \to X\) with \(Y \in \text{Sm}_k\). \(M(X)\) becomes a monoid under the disjoint union. Let \(M^+_*(X)\) be its group completion, graded by the relative codimension of the map \(f : Y \to X\). Then \(\Omega^*(X)\) is constructed as a quotient of \(M^+_*(X)\) [54, Lemma 2.5.11].

The following is a resumé of some of the main results from [54].

**Theorem 4.1.2** (Levine and Morel). *Assume the base field \(k\) has characteristic zero.*

1. *The assignment*

   \[X \mapsto \Omega^*(X)\]

   *extends to define an oriented cohomology theory on \(\text{Sm}_k\), called algebraic cobordism.*

2. \(\Omega^*\) *is the universal oriented cohomology theory on \(\text{Sm}_k\).*

3. *The canonical ring homomorphism \(\mathbb{L}_{\text{az}} \to \Omega^*(k)\) induced by the formal group law \(F_{\Omega}\) of the algebraic cobordism \(\Omega^*\) is an isomorphism.*
When the base field $k$ has positive characteristic, the construction of $\Omega^*(X)$ described above is not known to give a oriented cohomology theory. However, the construction of $\Omega^*(X)$ in [54] leads to a notion of a “universal oriented Borel-Moore Laz-functor on $\text{Sm}_k$ of geometric type” (see [54, Definition 2.2.1]; here we index by codimension rather than by dimension), and $\Omega^*$ is the universal such theory [54, Theorem 2.4.13]. The structures enjoyed by such a theory $A^*$ are:

1. (Projective pushforward) For $f : Y \to X$ a projective morphism in $\text{Sm}_k$ of relative dimension $d$, there is a graded homomorphism $f_* : A^*(Y) \to A^{*-d}(X)$.

2. (Smooth pullback) For a smooth morphism $f : Y \to X$ in $\text{Sm}_k$, there is a graded homomorphism $f^* : A^*(X) \to A^*(Y)$.

3. (1st Chern class operators) For each line bundle $L \to X$, $X \in \text{Sm}_k$, there is a graded homomorphism $\tilde{c}_1(L) : A^*(X) \to A^{*+1}(X)$. For line bundles $L, M$ on $X$, the operators $\tilde{c}_1(L)$ and $\tilde{c}_1(M)$ commute.

4. (External products) For $X,Y \in \text{Sm}_k$, there is a graded homomorphism $A^*(X) \otimes_Z A^*(Y) \to A^*(X \times_k Y)$, which is commutative and associative in the obvious sense. There is an element $1 \in A^0(k)$ which, together with the external product $A^*(k) \otimes_Z A^*(k) \to A^*(k)$, makes $A^*(k)$ into a commutative graded ring with unit.

5. (Fundamental class) For $X \in \text{Sm}_k$ with structure morphism $p : X \to \text{Spec } k$, we denote $p^*(1)$ by $1_X$, and for $L \to X$ a line bundle, write $c_1(L)$ for $\tilde{c}_1(L)(1_X)$.

6. (Dimension axiom) For each collection of line bundles $L_1, \ldots, L_r$ on $X$ with $r > \text{dim}_k X$, one has $\prod_{i=1}^r \tilde{c}_1(L_i)(1_X) = 0$.

7. (Formal group law) There is a homomorphism $\phi_A : \mathbb{L}_{\text{az}} \to A^*(k)$ of graded rings, such that, letting $F_A(u,v) \in A^*(k)[[u,v]]$ be the image of the universal formal group law
with respect to $\phi_A$, for each $X \in \text{Sm}_k$ and each pair of line bundles $L, M$ on $X$, we have

$$c_1(L \otimes M) = F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_X).$$

These all satisfy a number of compatibilities, detailed in [54, §2.1, 2.2].

An oriented cohomology theory has a canonical structure of an oriented Borel-Moore Laz-functor on $\text{Sm}_k$ of geometric type [54, Proposition 5.2.1, Proposition 5.2.6], where the Chern class operator $\tilde{c}_1(L)$ is multiplication by the 1st Chern class $c_1(L)$. In particular, each oriented cohomology theory $A^*$ admits a canonical classifying map

$$\Theta_A : \Omega^* \to A^*, \quad (4.1)$$

which is compatible with all push-forward maps $f_*$ for projective $f$, all pull-back maps $f^*$ for smooth $f$, first Chern classes and first Chern class operators, external products and the formal group law. Explicitly, for a generator $[f : Y \to X] \in M_+^*(X)$, $\Theta_A([f : Y \to X]) = f_*A(1_A^1_Y)$, where $f_*A^1_Y$ is the pushforward in the theory $A^*$ and $1_A \in A^0(Y)$ is the unit.

### 4.1.2 Motivic oriented cohomology theories

In positive characteristic, the universal oriented cohomology theory is not available; we use instead an approach via motivic homotopy theory.

Let $\text{SH}(k)$ be the motivic stable homotopy category of $\mathbb{P}^1$-spectra [58, 64, 76]. $\text{SH}(k)$ is a triangulated tensor category with tensor product $(E, F) \mapsto E \wedge F$ and unit the motivic sphere spectrum $S_k$. There is a functor

$$\Sigma^\infty_{\mathbb{P}^1} : \text{Sm}_k \to \text{SH}(k)$$

called \textit{infinite} $\mathbb{P}^1$-\textit{suspension}; for example $S_k = \Sigma^\infty_{\mathbb{P}^1} \text{Spec} k$. We have as well the auto-equivalences $\Sigma_{S^1}$ and $\Sigma_{G_m}$, and the translation in the triangulated structure is given by $\Sigma_{S^1}$. 

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An object $E$ of $\text{SH}(k)$ and integers $n, m$ define a functor $\mathcal{E}^{n,m} : \text{Sm}_k^{\text{op}} \to \text{Ab}$ by

$$\mathcal{E}^{n,m}(X) := \text{Hom}_{\text{SH}(k)}(\Sigma_{P_1}^\infty X_+, \Sigma_{S^1}^n \Sigma_{G_m}^m E).$$

An object $E$ together with morphisms $\mu : E \otimes E \to E$ (multiplication) and $1 : S_k \to E$ (unit) define a ring cohomology theory on $\text{Sm}_k$ if they make the bi-graded group $\mathcal{E}^{*,*}(X) := \oplus_{n,m \in \mathbb{Z}} \mathcal{E}^{n,m}(X)$ into a bi-graded ring which is graded commutative with respect to the first grading and commutative with respect to the second one.

Given a ring cohomology theory as above, an element $\vartheta \in \mathcal{E}^{2,1}(P^\infty/0)$ is called an orientation if the restriction of $\vartheta$ to $\vartheta|_{P^1/0} \in \mathcal{E}^{2,1}(P^1/0)$ (via the embedding $P^1 \to P^\infty$, $(x_0 : x_1) \mapsto (x_0 : x_1 : 0 : 0 \ldots)$) agrees with the image of $1 \in \mathcal{E}^{0,0}(k)$ under the suspension isomorphism $\mathcal{E}^{2,1}(P^1/0) \cong \mathcal{E}^{0,0}(k)$. Here $0 := (1 : 0 : \ldots : 0) \in \mathbb{P}^n$, including $n = \infty$.

In [67, Theorem 2.15] it is shown how an orientation $\vartheta$ for a ring cohomology theory $\mathcal{E}$ on $\text{Sm}_k$ gives rise to functorial pushforward maps $f_* : \mathcal{E}^{a,b}(Y) \to \mathcal{E}^{a-2d,b-d}(X)$ for each projective morphism $f : Y \to X$ in $\text{Sm}_k$, where $d$ is the relative dimension of $f$ ($d := \dim_k Y - \dim_k X$ in case $X$ and $Y$ are irreducible).

**Definition 4.1.3.** A motivic oriented cohomology theory on $\text{Sm}_k$ is an object $\mathcal{E} \in \text{SH}(k)$ together with morphisms $\mu : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$, $1 : S_k \to \mathcal{E}$ defining a ring cohomology theory, plus an orientation $\vartheta \in \mathcal{E}^{2,1}(P^\infty/0)$.

**Remark 4.1.4.** 1. The notion of a motivic oriented cohomology theory on $\text{Sm}_k$ is referred to as an “oriented ring cohomology theory” in [65]. We find this too similar to the term “oriented cohomology theory”, hence our relabelling.

2. The algebraic cobordism $P^1$-spectrum $\text{MGL} \in \text{SH}(k)$ is the universal motivic oriented cohomology theory. See [64, 76] for the construction of $\text{MGL}$ and [65] for the proof of universality.

It follows from the results of [67] that, if $(\mathcal{E}, \mu, 1, \vartheta)$ is a motivic oriented cohomology theory on $\text{Sm}_k$, then the contravariant functor $X \mapsto \mathcal{E}^*(X) := \mathcal{E}^{2*,*}(X)$ from $\text{Sm}_k$ to $\text{Comm},$
together with the maps $f_*$ for projective morphisms $f : Y \to X$ in $\text{Sm}_k$, defines an oriented cohomology theory on $\text{Sm}_k$, which we denote by $\mathcal{E}^*$. We call $\mathcal{E}^*$ the oriented cohomology theory on $\text{Sm}_k$ represented by the motivic oriented cohomology theory $(\mathcal{E}, \mu, 1, \vartheta)$ (or just $\mathcal{E}$ for short). In particular, for each motivic oriented cohomology theory $\mathcal{E}$ on $\text{Sm}_k$, there is a canonical homomorphism $\phi_\mathcal{E} : \mathbb{Laz} \to \mathcal{E}^*(k)$ classifying the formal group law of the oriented cohomology theory $\mathcal{E}^*$.

**Remark 4.1.5.** In general, an oriented cohomology theory on $\text{Sm}_k$ is not always represented by a motivic oriented theory (see [43]).

It follows from a theorem of Hopkins-Morel, recently established in detail by Hoyois [42], that for $k$ of characteristic zero, the ring homomorphism $\mathbb{Laz} \to \text{MGL}(k)$ classifying the formal group law for $\text{MGL}^*$ is an isomorphism. When $k$ has characteristic $p > 0$, it is shown in [42] that after inverting $p$, the classifying map $\mathbb{Laz}[1/p] \to \text{MGL}^*(k)[1/p]$ is an isomorphism. Conjecturally, for any field $k$, the classifying map is an isomorphism, but at present, this is not known.

### 4.1.3 Specialization of the formal group law

From the oriented cohomology theory $\text{MGL}^*$, and a given formal group law $F(u, v) \in R[u, v]$ such that the exponential characteristic $p$ of $k$ is invertible in $R$, we may construct an oriented cohomology theory $R^*$ on $\text{Sm}_k$ with $R^*(k) = R$ and formal group law $F$ as follows.

Take $X \in \text{Sm}_k$ and let $p_X : X \to \text{Spec} k$ be the structure morphism. The classifying map $\phi_{\text{MGL}} : \mathbb{Laz} \to \text{MGL}^*(k)$ composed with $p_X^*$ defines the ring homomorphism $p_X^* \circ \phi_{\text{MGL}} : \mathbb{Laz} \to \text{MGL}^*(X)$. Using the classifying homomorphism $\phi_F : \mathbb{Laz} \to R$, we may form the tensor product ring $R^*(X) := \text{MGL}^*(X) \otimes_{\mathbb{Laz}} R$. Since $p$ is invertible in $R$ and $\phi_{\text{MGL}}[1/p]$ is an isomorphism, it follows that the canonical map $R \to R^*(k)$ is an isomorphism. Extending the pull-back and push-forward maps for the theory $\text{MGL}^*$ define the pull-back and push-forward maps for the theory $R^*$. Since the functor $\otimes_{\mathbb{Laz}} R$ is additive and preserves isomorphisms,
it follows easily that the assignment $X \mapsto R^*(X)$ with pull-back and push-forward maps as described above defines an oriented cohomology theory on $\text{Sm}_k$. Similarly, it is easy to see that for $L \to X$ a line bundle, $c_1^R(L)$ is just the image of $c_1^\text{MGL}(L)$ in $R^*(X)$, and therefore the formal group law for $R^*$ is equal to $F(u, v) \in R[[u, v]]$. This gives us the following result:

**Lemma 4.1.6.** Let $k$ be a perfect field, $F \in R[[u, v]]$ a formal group law over a commutative (graded) ring $R$. Assume that the exponential characteristic of $k$ is invertible in $R$. Then there is an oriented cohomology theory $R^*$ on $\text{Sm}_k$ with $R^*(X) = \text{MGL}^*(X) \otimes_{\text{Laz}} R$. Moreover, $R^*(k) = R$ and $R^*$ has formal group law $F$. Finally, if the characteristic of $k$ is zero, then $R^*$ is the universal oriented cohomology theory on $\text{Sm}_k$ with formal group law $F \in R[[u, v]]$.

**Proof.** We have proved everything except for the last statement. This follows by noting that the isomorphism of oriented cohomology theories $\Theta : \Omega^* \to \text{MGL}^*$ shows that $R^*$ is isomorphic to the oriented cohomology theory $X \mapsto \Omega^*(X) \otimes_{\text{Laz}} R$. The fact that this latter theory is the universal oriented cohomology theory on $\text{Sm}_k$ with formal group law $F \in R[[u, v]]$ follows immediately from the fact that $\Omega^*$ is the universal oriented cohomology theory on $\text{Sm}_k$. ⊓⊔

There is a nice formula for push-forward in the cohomology theory $\text{MGL}^*$, in the case of the structure morphism of a projective space bundle.

**Theorem 4.1.7 (Quillen).** Let $X$ be a smooth quasi-projective variety, $V$ be some $n$-dimensional vector bundle on $X$, and $\pi : \mathbb{P}_X(V) \to X$ be the corresponding projective bundle. Let $f(t) \in \text{MGL}^*(X)[t]$. Then,

$$\pi_*(f(c_1(\mathcal{O}(1)))) = \sum_i \frac{f(-\Omega \lambda_i)}{\prod_{j \neq i}(\lambda_j - \Omega \lambda_i)},$$

(4.2)

where $\lambda_i$ are the Chern roots of $V$, and $x + \Omega y := F_\Omega(x, y)$, where $F_\Omega$ is the formal group law of the cobordism theory $\text{MGL}^*$. 

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A proof of this theorem, in the context of complex cobordism, can be found in [75], page 50. The proof goes through word for word in our setting, so we will not repeat it here. Clearly, for any formal group law \((F, R)\) such that the exponential characteristic is invertible in \(R\), the same formula as above is valid for the push-forward in the theory \(R^\ast\).

### 4.1.4 Landweber exactness

We now describe a sufficient condition for the theory \(X \mapsto R^\ast(X)\) to arise from a motivic oriented cohomology theory; this is the well-known condition of Landweber exactness.

For any prime \(l > 0\), we expand the \(l\)-series of the this formal group law \(x +_F \cdots +_F x\) (summing \(l\) copies of \(x\)), as \(\sum_{i \geq 1} a_i x^i\), and for all \(n \geq 0\) we write \(v_n := a_{ln}\). In particular we have \(v_0 = a_1 = l\).

**Definition 4.1.8.** The formal group \((R, F)\) is said to be Landweber exact if for all primes \(l\) and for all integers \(n\), the multiplication map

\[
v_n : R/(v_0, \cdots, v_{n-1}) \rightarrow R/(v_0, \cdots, v_{n-1})
\]

is injective.

**Theorem 4.1.9.** Let \(k\) be a perfect field, \(F \in R[[u, v]]\) a formal group law. If the formal group law \((R, F)\) is Landweber exact and the exponential characteristic of \(k\) is invertible in \(R\), then the oriented cohomology theory \(X \mapsto R^\ast(X)\) on \(\text{Sm}_k\) is represented by a motivic oriented cohomology theory.

The classical precursor of this result is due to Landweber [51]; in our setting this result follows from [59, Theorem 7.3]. We denote the motivic cohomology theory associated to a Landweber exact formal group law by \(\text{MGL} \otimes_{\text{Laz}} R\) and the canonical morphism given by the universality of \(\text{MGL}\) by \(\Theta_{F, R} : \text{MGL} \rightarrow \text{MGL} \otimes_{\text{Laz}} R\).
4.1.5 Exponential and logarithm

Let \((R, F)\) be the formal group law. A *logarithm* of the formal group law \(F\) is a series \(g(u) = u + \text{higher order terms} \in R[[u]]\) satisfying the equation

\[ g(F(u, v)) = g(u) + g(v). \]

Novikov [60] showed that every formal group law with coefficients in a \(\mathbb{Q}\)-algebra has a logarithm. The functional inverse \(\lambda(u) \in R[[u]]\) of the logarithm \(g(u)\) is called the *exponential* of the formal group law. The expansion of \(\lambda(u)\) takes the form \(u + \text{higher order terms}\). With our grading conventions, if we give \(u\) degree one, then the power series \(g(u)\) and \(\lambda(u)\) are both homogeneous of degree one. Thus, if we write \(\lambda(u) = u + \sum_{i \geq 1} \tau_i u^{i+1}\), then \(\tau_i \in R\) has degree \(-i\).

In fact, the formal group law and the exponential power series uniquely determine each other, assuming that \(R \to R_\mathbb{Q}\) is injective. As ring homomorphisms \(\mathbb{L}az \to R\) are in bijection with formal group laws with coefficients in \(R\), it is noted by Hirzebruch that ring homomorphisms \(\mathbb{L}az \to R_\mathbb{Q}\) are in one to one correspondence with power series \(\lambda\) as above. For any formal group law \((R, F)\) with exponential \(\lambda(u) \in R \otimes \mathbb{Q}[u]\), the corresponding ring homomorphism \(\phi_F : MU^{2*} \to R^*_\mathbb{Q}(k)\) given by Quillen’s identification \(MU^{2*} \cong \mathbb{L}az\) is called the *Hirzebruch genus*. In terms of algebraic geometry, \(\phi_F\) sends the class of smooth projective irreducible variety \(X\) of dimensional \(n\) to \((\prod_{i=1}^n \xi_i^{N_n(T_X)}; [X])\), where \(\xi_1, \ldots, \xi_n\) are the Chern roots of the tangent bundle \(T_X\) (for \(\text{CH}^*(X)\)) and \(\langle -, [X] \rangle\) means evaluation on the fundamental class of \(X\). Explicitly, if we let \(\text{Td}_r(u) := \frac{u}{\text{X}_{[u]}^r} \in R[[u]]\) and write \(\text{Td}_r(u) = 1 + \sum_{n \geq 0} t d_i^n \cdot u^i\), with \(t d_i^n \in R^{-i}\), then \(\phi_F([X]) = \deg_k(N_n(T_X)) \cdot t d_i^n\), where \(N_n(T_X)\) is the Newton class, \(N_n(T_X) := \sum_j \xi_j^n\).

**Example 4.1.10.** Let \(R = \mathbb{Q}[b_1, b_2, \ldots]\), with \(b_n\) of degree \(-n\). Let \(\lambda_b(u) = u + \sum_{n \geq 1} b_n u^{n+1}\) and let \(\lambda_b^{-1}(u)\) be the functional inverse. We let \(F_b(u, v) \in R[[u, v]]\) be the formal group law \(\lambda_b(\lambda_b^{-1}(u) + \lambda_b^{-1}(v))\). One can show (see e.g. [1, theorem 7.8]) that the subring \(R_0\) of \(R\)
generated by the coefficients of \( F_b \) is isomorphic to \( \mathbb{L} \) via the homomorphism \( \mathbb{L} \to R_0 \) classifying \( F_b \); in particular, \((F_b, R_0)\) is the universal formal group law.

### 4.1.6 Twisting a cohomology theory

Let \( k \) be an arbitrary perfect field and let \( R^* \) be an oriented cohomology theory on \( \text{Sm}_k \). Suppose that the coefficient ring \( R^*(k) \) is a \( \mathbb{Q} \)-algebra. There is a twisting construction due to Quillen (see e.g. [54] for a detailed description), which enables one to construct the oriented cohomology theory \( R^* \) from the theory given by the Chow ring \( X \mapsto \text{CH}^*(X) \).

In this section, we describe the twisting construction and its analog for motivic oriented cohomology theories.

Let \( A^* \) be an oriented cohomology theory on \( \text{Sm}_k \) and \( \tau = (\tau_i) \in \prod_{i=0}^{\infty} A^{-i}(k) \), with \( \tau_0 = 1 \). Let \( \lambda_{\tau}(u) = \sum_{i=0}^{\infty} \tau_i u^{i+1} \). The Todd genus \( \text{Td}_\tau(t) \) is by definition the power series

\[
\text{Td}_\tau(t) := \frac{t}{\lambda_{\tau}(t)}.
\]

For a vector bundle \( E \) on some \( Y \in \text{Sm}_k \), the Todd class of a vector bundle \( E \) on \( Y \) is given by

\[
\text{Td}_\tau(E) = \prod_{i=1}^{r} \text{Td}_\tau(\xi_i)
\]

where \( \xi_1, \ldots, \xi_r \) are the Chern roots of \( E \) in \( A^*(Y) \). The assignment \( E \mapsto \text{Td}_\tau(E) \) is multiplicative in exact sequences, hence descends to a well-defined homomorphism \( \text{Td}_\tau : K_0(Y) \to (1 + A^{\geq 1}(Y))^\times \).

Define the twisted oriented cohomology theory \( A^*_\tau \) on \( \text{Sm}_k \) with \( A^*_\tau(X) = A^*(X) \), for \( X \in \text{Sm}_k \), and with pull-backs unchanged \((f^*_\tau = f^*)\). For a projective morphism \( f : Y \to X \), we set

\[
f^*_\tau := f^* \circ \text{Td}_\tau(T_f),
\]

where

\[
T_f := [T_Y] - [f^*T_X] \in K_0(Y)
\]
is the relative tangent bundle. It is not difficult to show that this does indeed define an oriented cohomology theory on $\text{Sm}_k$.

Let $\lambda_\tau(u) = \sum_{i=0}^{\infty} \tau_i u^{i+1}$. Then for a line bundle $L$, the first Chern class in the new cohomology theory $A^*_\tau$ is given by $c_1^\tau(L) := \lambda_\tau(c_1(L))$, from which one easily sees that the formal group law for $A^*_\tau$ is given by

$$F^\tau_A(u, v) = \lambda_\tau(F_A(\lambda_\tau^{-1}(u), \lambda_\tau^{-1}(v))).$$

In particular, if $F_A$ is the additive group law, $F_A(u, v) = u + v$, then $\lambda_\tau(u)$ is the exponential map for the twisted group law $(F^\tau_A, A^*(k))$.

The twisting construction is also available for motivic oriented cohomology theories. Indeed, let $(\mathcal{E}, \mu, 1, \vartheta)$ be a motivic oriented cohomology theory, and $(\tau_i \in \mathcal{E}^{-2i,-i}(k))_i$ a sequence of elements, with $\tau_0 = 1$.

Form the orientation $\vartheta_\tau \in \mathcal{E}^{2,1}(\mathbb{P}^\infty/0)$,

$$\vartheta_\tau := \lambda_\tau(\vartheta) = \sum_{i \geq 0} \tau_i \vartheta^{i+1}.$$ 

We note that the projective bundle formula implies that $\vartheta^m$ goes to zero in $\mathcal{E}^{2,1}(\mathbb{P}^N/0)$ for $m > N$, from which it follows that $\lambda_\tau(\vartheta)$ is a well-defined element in $\mathcal{E}^{2,1}(\mathbb{P}^\infty/0) = \lim_{\leftarrow} \mathcal{E}^{2,1}(\mathbb{P}^N/0)$ and that $\vartheta_\tau$ is indeed an orientation. Let $\tau \mathcal{E}^*$ be the oriented cohomology theory corresponding to $(\mathcal{E}, \mu, 1, \vartheta)$.

It follows immediately from the definitions that, for $L \to X$ a line bundle, $X \in \text{Sm}_k$, the first Chern class $c_1^{\tau \mathcal{E}^*}(L)$ in the theory $\tau \mathcal{E}^*$ is the same as in the $\tau$-twist of $\mathcal{E}^*$, namely $c_1^{\tau \mathcal{E}^*}(L) = \lambda_\tau(c_1^{\mathcal{E}^*}(L)) = c_1^\tau(L)$. From this it follows easily that the identity map on the graded groups $\mathcal{E}^*_\tau(X) = \mathcal{E}^*(X)$ defines an isomorphism of $\tau \mathcal{E}^*$ with the twisted oriented cohomology theory $\mathcal{E}^*_\tau$; we will henceforth drop the notation $\tau \mathcal{E}^*$.

Let $R$ be a graded $\mathbb{Q}$-algebra. Define

$$HR = \bigoplus_{n \in \mathbb{Z}} \Sigma^n_{\mathbb{P}^1} HR^{-n}$$

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where $HR^{-n}$ is the $\mathbb{P}^1$ spectrum representing motivic cohomology with coefficients in the $\mathbb{Q}$ vector space $R^{-n}$. We make $HR$ a motivic oriented cohomology theory by using the orientation induced by that of $HZ$.

Now let $(R, F)$ be a formal group law, where $R$ is given the grading following our conventions and is as above a $\mathbb{Q}$-algebra. Taking $(\tau_i \in R^{-i} = HR^{-2i-n}(k))_i$ to be the sequence such that $\lambda_r(u)$ is the exponential function for $(R, F)$, we form the twisted motivic oriented cohomology theory $HR_\tau$; by construction $HR_\tau$ has associated formal group law $(R, F)$. We note that $(R, F)$ is Landweber exact, since $R$ is a $\mathbb{Q}$-algebra.

We may also form the motivic oriented cohomology theory $\text{MGL} \otimes_{\text{Laz}} R$ associated to the Landweber exact formal group law $(R, F)$. As this is the universal motivic oriented cohomology theory with group law $(R, F)$, the classifying map $\Theta_{HR_\tau} : \text{MGL} \to HR_\tau$ factors through $\Theta_{F,R} : \text{MGL} \to \text{MGL} \otimes_{\text{Laz}} R$, giving the induced classifying map $\bar{\Theta}_{HR_\tau} : \text{MGL} \otimes_{\text{Laz}} R \to HR_\tau$, unique up to a phantom map.

**Lemma 4.1.11.** The map $\bar{\Theta}_{HR_\tau} : \text{MGL} \otimes_{\text{Laz}} R \to HR_\tau$ induces an isomorphism of bigraded cohomology theories $\text{MGL}^{*,*} \otimes_{\text{Laz}} R \to HR^{*,*}_{\tau}$, in particular, we have the isomorphism of associated oriented cohomology theories on $\text{Sm}_k$

$$\bar{\Theta}_{HR_\tau} : R^* = \text{MGL}^* \otimes_{\text{Laz}} R \to HR^*_{\tau}$$

**Proof.** We apply the slice spectral sequence to $\text{MGL} \otimes_{\text{Laz}} R$ and $HR_\tau$; the map $\bar{\Theta}_{HR_\tau}$ induces a map of spectral sequences. For a $\mathbb{P}^1$ spectrum $\mathcal{E}$ denote the $n$th layer in the slice tower for $\mathcal{E}$ as $s_n \mathcal{E}$. For an abelian group $A$, it follows from [78] that

$$s_n HA = \begin{cases} 
HA & \text{for } n = 0 \\
0 & \text{for } n \neq 0.
\end{cases}$$

Also $s_n \Sigma_{\mathbb{P}^1} \mathcal{E} = \Sigma_{\mathbb{P}^1} s_{n-m} \mathcal{E}$. As $HR = \oplus_{n \in \mathbb{Z}} \Sigma_{\mathbb{P}^1} HR^{-n}$, it follows that the $n$th layer in the slice tower for $HR$ is given by $s_n HR = \Sigma_{\mathbb{P}^1} HR^{-n}$. By Spitzweck’s computation of the layers in the slice tower for a Landweber exact theory, we have the same $s_n \text{MGL} \otimes_{\text{Laz}} R = \Sigma_{\mathbb{P}^1} HR^{-n}$.
as well. The maps on the layers of the slice tower induced by $\tilde{\Theta}_{\mathbb{H}R}$ are $\mathbb{H}\mathbb{Q}$-module maps (by results of Pelaez [68]), and one knows by a result of Cisinski-Deglise [25, Theorem 16.1.4] that

$$\text{Hom}_{\mathbb{H}\mathbb{Q}-\text{Mod}}(\Sigma^n_{p^1}HR^{-n}, \Sigma^n_{p^1}HR^{-n}) \cong \text{Hom}_{\mathbb{Q}-\text{Vec}}(R^{-n}, R^{-n})$$

In particular, the map $s_n\tilde{\Theta}_{\mathbb{H}R}: \Sigma^n_{p^1}HR^{-n} \to \Sigma^n_{p^1}HR^{-n}$ is determined by the induced map after applying the functor $H^{-2n,-n}(k,-)$, that is, on the coefficient rings of the theories $\text{MGL}^\ast \otimes_{\text{Laz}} R$ and $HR^\ast$. However, by construction, this is the map $\tilde{\Theta}: R \to R$ induced by the classifying map $\text{Laz} \to R$ associated to the formal group $F_\tau(u,v) = \lambda(\tau(u) + \tau(v))$. As this latter formal group law is equal to $F$ by construction, the map $\tilde{\Theta}: R \to R$ is the identity map.

Thus $\Theta_{\mathbb{H}R}$ induces an isomorphism of the (strongly convergent) slice spectral sequences, and hence an isomorphism of bi-graded cohomology theories on $\text{Sm}_k$.

\[\Box\]

### 4.1.7 Universality

Since $\text{MGL}^\ast$ is an oriented cohomology theory, we have the canonical comparison morphism (4.1)

$$\Theta_{\text{MGL}} : \Omega^\ast(X) \to \text{MGL}^\ast(X).$$

Relying on the Hopkins-Morel-Hoyois isomorphism $\text{Laz} \cong \text{MGL}^\ast(k)$, Levine has shown that for $k$ a field of characteristic zero, $\Theta_{\text{MGL}} : \Omega^\ast \to \text{MGL}^\ast$ is an isomorphism [53, theorem 3.1]. Thus, for a field of characteristic zero, $\text{MGL}^\ast$ is the universal oriented cohomology theory on $\text{Sm}_k$. At present, there is no proof of the existence of a universal oriented cohomology theory on $\text{Sm}_k$ if $k$ has positive characteristic. In this subsection, we use the universality of $\text{MGL}$ as a motivic oriented cohomology theory plus some tricks with formal group laws to show that $\text{MGL}^\ast_\mathbb{Q}$ is the universal oriented cohomology theory for theories in $\mathbb{Q}$-algebras on $\text{Sm}_k$.

We also show that the two constructions of oriented cohomology theories: construction by specialisation of the formal group law from $\text{MGL}^\ast$ and construction by extending coefficients
for $\text{CH}^*$ and then twisting, are “equivalent”, assuming the coefficient ring is a $\mathbb{Q}$-algebra.

**Lemma 4.1.12.** Suppose $k$ has characteristic zero. Then $\text{CH}^*$ is the universal oriented cohomology theory on $\text{Sm}_k$ with formal group law $(u+v, \mathbb{Z})$. If $k$ has characteristic $p > 0$, then $\text{CH}_Q^*$ is the universal oriented cohomology theory on $\text{Sm}_k$ with formal group law $(u + v, \mathbb{Q})$.

Of course, one would expect that over an arbitrary field, $\text{CH}^*$ is the universal oriented cohomology theory on $\text{Sm}_k$ with formal group law $(u + v, \mathbb{Z})$. This does not seem to be known.

**Proof.** The case of characteristic zero is proven in [54, Theorem 1.2.2]. In characteristic $p > 0$, let $A^*$ be an oriented cohomology theory with additive formal group law $F_A(u, v) = u + v$ and with $A^*(k)$ a $\mathbb{Q}$-algebra. Extend the coefficients in the theory $A^*$ by a Laurent polynomial ring, forming the theory $A^*[t, t^{-1}]$, with $t$ of degree $-1$. Then take the twist with respect to the modified exponential function

$$
\lambda_t(u) := t^{-1}(1 - e^{-t u})
$$

that is, $\tau_i := (-1)^i t^i / (i + 1)!$. A simple computation shows that theory $A^*[t, t^{-1}]_\tau$ has the multiplicative group law $F(u, v) = u + v - tuv$, and that the twisted first Chern class is given by $c_1^\tau(L) = t^{-1}(1 - e^{tc_1^\tau(L)})$.

One can define the modified Chern character

$$
\text{ch}_t^A : K_0[t, t^{-1}]_\mathbb{Q} \to A^*[t, t^{-1}]_\tau,
$$

which sends a vector bundle $E$ of rank $r$ to

$$
\text{ch}_t^A(E) := r - tc_1^\tau(E^\vee).
$$

For a line bundle $L$, we have

$$
\text{ch}_t^A(L) = 1 - tc_1^\tau(L^\vee) = e^{tc_1^\tau(L)}.
$$
Using the splitting principle, the fact that $A^*$ has the additive formal group law implies that $\text{ch}_A^t$ is a natural transformation of functors to graded $\mathbb{Q}[t, t^{-1}]$-algebras. Since $c^K_{t}^{K_0[t, t^{-1}]}(L) = t^{-1}(1 - L^{-1})$, we have

$$\text{ch}_A^t(c^K_t(L)) = c^t_1(L)$$

for all line bundles $L$. By Panin’s Riemann-Roch theorem [66, Corollary 1.1.10], this shows that $\text{ch}_A^t$ is a natural transformation of oriented cohomology theories.

We have the Adams operations $\psi_k$, $k = 1, 2, \ldots$, on $K_0(X)$, which we extend to Adams operations on $K_0(X)[t, t^{-1}]_\mathbb{Q}$ by $\mathbb{Q}[t, t^{-1}]$-linearity. Define the operation $\psi^A_k$ on $A^*(X)[t, t^{-1}]$ to be the $\mathbb{Q}[t, t^{-1}]$-linear map which is multiplication by $k^n$ on $A^n(X)$; it is easy to see that $\psi^A_k$ is a natural $\mathbb{Q}[t, t^{-1}]$-algebra homomorphism. As $A$ has the additive group law, $c^A_1(L \otimes k) = kc^A_1(L)$ and thus

$$\text{ch}_A^t(\psi_k(L)) = \text{ch}_A^t(L \otimes k) = e^{tc^A_1(L \otimes k)} = e^{ktc^A_1(L)} = \psi^A_k(e^{tc^A_1(L)}) = \psi^A_k(\text{ch}_A^t(L))$$

for all line bundles $L$. By the splitting principle, this gives the identity

$$\text{ch}_A^t \circ \psi_k = \psi^A_k \circ \text{ch}_A^t.$$ 

If we take $A^* = CH^*_\mathbb{Q}$, $\text{ch}_A^t$ is a modified version of the classical Chern character; thus by Grothendieck’s classical result, the natural transformation

$$\text{ch}_t^{CH_\mathbb{Q}} : K_0[t, t^{-1}]_\mathbb{Q} \to CH^*_\mathbb{Q}[t, t^{-1}]_r$$

is an isomorphism. This gives us the natural transformation of oriented cohomology theories

$$\text{ch}_t^A \circ (\text{ch}_t^{CH^*_\mathbb{Q}})^{-1} : CH^*_\mathbb{Q}[t, t^{-1}]_r \to A^*[t, t^{-1}]_r$$

Twisting back gives us the natural transformation of oriented cohomology theories

$$\vartheta_A^{CH_t} : CH^*_\mathbb{Q}[t, t^{-1}] \to A^*[t, t^{-1}].$$

We give $CH^*_\mathbb{Q}[t, t^{-1}]$ a bi-grading by putting $CH^m_n \cdot t^n$ in bi-degree $(n, m)$, and do the same for $A^*[t, t^{-1}]$. Since both $\text{ch}_t^{CH^*_\mathbb{Q}}$ and $\text{ch}_A^t$ commute with the Adams operations, we have

$$\vartheta_A^{CH_t} \circ \psi_k^{CH} = \psi^A_k \circ \vartheta_A^{CH}.$$
and thus \( \vartheta^\text{CH}_A \) respects the bi-grading. Passing to the respective quotients by the ideal \((t - 1)\) gives us the natural transformation of oriented cohomology theories

\[
\vartheta^\text{CH}_A : \text{CH}^*_Q \to A^*
\]

where we now use the original grading on \( \text{CH}^*_Q \) and \( A^* \).

The uniqueness of \( \vartheta^\text{CH}_A \) follows from Grothendieck-Riemann-Roch. Indeed, as a natural transformation of oriented cohomology theories, \( \vartheta^\text{CH}_A (c_n^\text{CH}(E)) = c_n^A(E) \) for all vector bundles \( E \) on \( X \in \text{Sm}_k \), and all \( n \). But for irreducible \( X \in \text{Sm}_k \), the Grothendieck-Riemann-Roch theorem implies that \( \text{CH}^*(X)_Q \) is generated as a \( Q \)-vector space by the elements of the form \( c_n^\text{CH}(E) \), \( E \) a vector bundle on \( X \), \( n \geq 1 \) an integer, together with the identity element \( 1 \in \text{CH}^0(X) \). Thus \( \vartheta^\text{CH}_A \) is unique.

We now consider the generic twist \( \text{CH}^*_Q[b]_b \), where \( b = \{b_i\}_i \). We have as well the motivic oriented cohomology theory \( H^Q \in \text{SH}(k) \), representing rational motivic cohomology, \( H^*(X, Q(*)) \). The orientation \( v_H \in H^Q(\mathbb{P}^1(\mathbb{P}^\infty/0) \) is given by the sequence of hyperplane classes \( c_1^\text{CH}(\mathcal{O}_{\mathbb{P}^n}(1)) \in H^Q(\mathbb{P}^1(\mathbb{P}^n)_Q \).

We may form the generic twist \( H^Q[b]_b \) by taking the orientation \( \lambda_b(v_H) := \sum_{n \geq 0} b_nv_H^{n+1} \).

**Proposition 4.1.13.** 1. Let \( k \) be a field of characteristic zero. Then \( \text{MGL}^*_Q \) is the universal oriented cohomology theory on \( \text{Sm}_k \).

2. Let \( k \) be a perfect field of characteristic \( p > 0 \). Then \( \text{MGL}^*_Q \) is the universal oriented cohomology theory in \( Q \)-algebras on \( \text{Sm}_k \).

3. Let \((F, R)\) be a formal group law, with \( R \) a \( Q \)-algebra and let \( \lambda_\tau(u) = \sum_{n \geq 0} \tau_n u^{n+1} \) be the associated exponential function. Let \( k \) be a perfect field. Then the classifying map \( \text{MGL}^*_Q \otimes_{\text{Laz}} R \to (\text{CH}^* \otimes R)_\tau \) is an isomorphism.

4. Let \( k \) be a perfect field, and consider the generic twist \( \text{CH}^*_Q[b]_b \). For an arbitrary perfect field \( k \), the classifying map \( \text{MGL}^*_Q \to \text{CH}^*_Q[b]_b \) is an isomorphism.
Proof. (1) follows from the isomorphism of oriented cohomology theories on $\text{Sm}_k$, $\Omega^* \to \text{MGL}^*$ [53, Theorem 3.1], and the universal property of $\Omega^*$ [54, Theorem 7.1.3].

(4) follows from (3), noting that the classifying map $\text{Laz} \to \mathbb{Z}[b_1, b_2, \ldots]$ for the formal group law $F_b(u, v) := \lambda_b(\lambda_b^{-1}(u) + \lambda_b^{-1}(v))$ induces an isomorphism $\text{Laz}_\mathbb{Q} \to \mathbb{Q}[b_1, b_2, \ldots]$ (see e.g. [1, Theorem 7.8]), hence $\text{MGL}^*_\mathbb{Q} \to \text{MGL}^*_\mathbb{Q} \otimes_{\text{Laz}_\mathbb{Q}} \mathbb{Q}[b]$ is an isomorphism. The assertion (3) follows immediately from Lemma 4.1.11, as the isomorphism $\text{CH}^* \cong H^{2*,-*}(-, \mathbb{Z})$ gives rise to a canonical isomorphism $HR^*_\tau \cong (\text{CH}^* \otimes R)_\tau$.

To prove (2), it suffices by (4) to show that $\text{CH}^*_\mathbb{Q}[b]_b$ is the universal oriented cohomology theory in $\mathbb{Q}$-algebras on $\text{Sm}_k$. This follows by applying the twisting construction. Indeed, let $A^*$ be an oriented cohomology theory on $\text{Sm}_k$, such that $A^*(k)$ is a $\mathbb{Q}$-algebra. Let $(\tau_n^{-1} \in A^{-n})_n$ be the sequence such that $\lambda_{n-1}(u)$ is the logarithm of the formal group law $(F_{A^*}, A^*(k))$. The twist $A^*_\tau$ thus has the additive formal group law and hence by Lemma 4.1.12 we have the (unique) classifying map $\theta_{\tau^{-1}} : \text{CH}^*_\mathbb{Q} \to A^*_\tau$. As we have already mentioned above, the map $\text{Laz}_\mathbb{Q} \to \mathbb{Q}[b_1, b_2, \ldots]$ classifying the formal group law $F_b(u, v)$ is an isomorphism, hence there is a unique ring homomorphism $\phi : \mathbb{Q}[b_1, b_2, \ldots] \to A^*(k)$ with $\phi(F_b(u, v)) = F_{A^*}(u, v)$. Extend $\theta_{\tau^{-1}}$ to $\Theta_{\tau^{-1}} : \text{CH}^*_\mathbb{Q}[b] \to A^*_\tau$ by using $\phi$, and then twist back by $b$ and $\phi(b) = \tau$ to get the map $\Theta_A : \text{CH}^*_\mathbb{Q}[b]_b \to A^*$ of oriented cohomology theories on $\text{Sm}_k$. The uniqueness of $\Theta_A$ follows from the uniqueness of $\theta_{\tau^{-1}}$ and that of $\phi$. This completes the proof. \hfill $\square$

Given a formal group law $(R, F)$ with exponential $\lambda(t) \in R_\mathbb{Q}[t]$. Write $\lambda(t) = t + \sum_{i \geq 1} \tau_it^{i+1}$ with $\tau_i \in R_\mathbb{Q}^{-i}$; we set $\tau_0 = 1$. We have two methods of constructing an oriented cohomology theory with formal group law $(R_\mathbb{Q}, F)$: the specialisation construction $\text{MGL}^* \otimes_{\text{Laz}} R_\mathbb{Q}$ or the twisting construction $(\text{CH}^* \otimes R_\mathbb{Q})_\tau$. As these two oriented cohomology theories are canonically isomorphic, we will denote both of them by $R^*_\mathbb{Q}$:

\[
\text{MGL}^* \otimes_{\text{Laz}} R_\mathbb{Q} =: R^*_\mathbb{Q} := (\text{CH}^* \otimes R_\mathbb{Q})_\tau.
\] (4.3)
4.2 The algebraic elliptic cohomology theory

4.2.1 The elliptic formal group law

An algebraic elliptic cohomology theory is the cohomology theory corresponding to an elliptic formal group law. More precisely, let $R$ be a ring and let $p : E \to \text{Spec } R$ be a smooth projective morphism with section $e : \text{Spec } R \to E$ and addition map $E \times_R E \to E$ defining a commutative group scheme over $R$ with geometrically connected fibers of dimension one. We assume in addition that we have chosen a local uniformizer $t$ around the identity section. The expansion of the group law of $E$ in terms of the coordinate $t$ gives a formal group law $F_E$ with coefficients in $R$.

There is a well-studied elliptic formal group law and the corresponding cohomology theory. Namely, taking $R = \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$ and taking $E$ to be the Weierstrass curve

$$y^2 + \mu_1 xy + \mu_3 y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6$$

over the ring $R$, and using $t = y/x$ as the local uniformizer. This formal group law will be referred as the TMF elliptic formal group law. It has been studied by Franke, Hopkins, Landweber, Miller, Ravenel, Stong, etc. See [43] and [41] for survey of this theory. In particular, it has been shown that, the map of rings

$$F_E : \mathbb{Laz} \to \mathbb{R}[\Delta^{-1}]$$

given by this formal group law is Landweber exact, where $\Delta$ is the discriminant.

The elliptic formal group law we are using is called the Krichever’s elliptic formal group law in literature. It is related to, but different from the TMF elliptic formal group law. We recall the genus corresponding to this formal group law, following the convention in [71].

Let

$$\sigma(z, \tau) := z \prod_{w \in \mathbb{Z} + \mathbb{Z} \tau, w \neq 0} (1 - z/w)e^{\frac{z}{w^2} + \frac{(z/w)^2}{2}}$$
be the Weierstrass sigma function. Note that the sigma function defined above, as a function in $z$, is an entire odd function, with zero set equal to the lattice $\mathbb{Z} + \mathbb{Z}\tau \subseteq \mathbb{C}$. It has the following property
\[ \sigma(z, \tau) = -e^{\eta_1(z + \frac{i}{2})}\sigma(z + 1, \tau), \]
and
\[ \sigma(z, \tau) = -e^{\eta_2(z + \frac{\tau}{2})}\sigma(z + \tau, \tau), \]
for some functions $\eta_j = \eta_j(\tau)$, $j = 1, 2$.

We define the algebraic elliptic genus as the ring homomorphism
\[ \phi_E : \mathbb{L}az \to \mathbb{Q}(\langle e^{2\pi i z} \rangle_{\mathbb{C}}, \frac{k}{2\pi i}) \]
associated to the power series $\lambda_E(t) := \frac{t}{Q(t)}$ under the Hirzebruch correspondence, where
\[ Q(t) := \frac{t}{2\pi i} e^{kt} e^{\zeta(z)} \frac{\sigma\left(\frac{t}{2\pi i} - z, \tau\right)}{\sigma\left(\frac{t}{2\pi i}, \tau\right)\sigma(-z, \tau)}, \]
here $\zeta(z) = \frac{d\log \sigma(z)}{dz}$ is the zeta function and the function
\[ \Phi(t, z) = e^{\zeta(z)} \frac{\sigma\left(\frac{t}{2\pi i} - z, \tau\right)}{\sigma\left(\frac{t}{2\pi i}, \tau\right)\sigma(-z, \tau)} \]
is the *Baker-Akhiezer function*. Explicitly, the elliptic genus of a $n$-dimensional smooth variety $X$ is defined by $\phi(X) := \langle \prod_{i=1}^n Q(\xi_i), [X] \rangle$, where the $\xi_i$ are the Chern roots of the tangent bundle $T_X$ of $X$.

The coefficient ring $\tilde{\mathbb{Ell}}$ of the corresponding formal group law, i.e., the elliptic formal group law, is by definition the image of $\phi_E$. Moreover, the elliptic formal group law can be written as
\[ x + E y = \lambda(\lambda^{-1}(x) + \lambda^{-1}(y)), \]

\[ Q(t) := \frac{t}{2\pi i} e^{kt} e^{\eta_1(z)} \frac{\sigma\left(\frac{t}{2\pi i} - z, \tau\right)}{\sigma\left(\frac{t}{2\pi i}, \tau\right)\sigma(-z, \tau)}, \]
for the eta function $\eta_1$. Note that they both enjoy the rigidity property.
for the series $\lambda(t) = \frac{t}{Q(t)}$ as above. The algebraic elliptic cohomology theory associated to this formal group law, $\text{MGL}^* \otimes_{\text{Laz}} \tilde{\text{Ell}}$, is denoted by $\tilde{\text{Ell}}^*$.

**Remark 4.2.1.** The coefficient ring $\tilde{\text{Ell}}^*(k)$ depends only on the characteristic of $k$. For $k$ having characteristic zero, $\tilde{\text{Ell}}^*(k) = \tilde{\text{Ell}}$ and for $k$ having characteristic $p > 0$, $\tilde{\text{Ell}}^*(k) = \tilde{\text{Ell}}[1/p]$.

When $k = \mathbb{C}$, the elliptic genus has the rigidity property, proved by Krichever in [49] and Höhn in [40]. A consequence of the rigidity property is that given a fiber bundle $F \to E \to B$ of closed connected weakly complex manifolds, with structure group a compact connected Lie group $G$, and if $F$ is a $SU$-manifold, then the elliptic genus is multiplicative with respect to this fibration.

The coefficient ring $\tilde{\text{Ell}}$ has been studied in [6]. We summarize their results here. The subring $\tilde{\text{Ell}}$ of $\mathbb{Q}(e^{2\pi i z})[e^{2\pi i \tau}, \frac{k}{2\pi i}]$ is contained in a polynomial subring $\mathbb{Z}[a_1, a_2, a_3, a_4]$ of $\mathbb{Q}(e^{2\pi i z})[e^{2\pi i \tau}, \frac{k}{2\pi i}]$. The elements $a_i$ can be described explicitly in terms of elements in $\mathbb{Q}(e^{2\pi i z})[e^{2\pi i \tau}, \frac{k}{2\pi i}]$ as follows. Let $g_i$ be the weight $2i$ Eisenstein series where $i = 2, 3$, and let $X$ be the Weierstrass $p$-function and $Y$ its derivative. The power series expansions of these functions are as follows.

$$X(\tau, z) = \frac{1}{12} + \frac{y}{(1 - y)^2} - 2 \prod_{m,n \geq 1} nq^{mn} + \prod_{m,n \geq 1} nq^{mn}(y^n + y^{-n});$$

$$Y(\tau, z) = \prod_{m \geq 0} \frac{q^m y (1 - q^m y)}{(1 - q^{2m} y)^3} - \prod_{m \geq 1} \frac{q^m y (1 - q^m y)}{(1 - q^m y)^3};$$

$$g_2(\tau) = \frac{1}{12} \left[ 1 + 240 \prod_{m \geq 1} \frac{m^3 q^m}{1 - q^m} \right],$$

where $y = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$. Then $a_1 = -\frac{k}{2\pi i}$, $a_2 = X$, $a_3 = Y$, $a_4 = \frac{1}{2} g_2$.

**Remark 4.2.2.** We shall see below that $\text{Ell}_Q = \mathbb{Q}[a_1, a_2, a_3, a_4]$. However, as noted by Totaro [71, §6], $\text{Ell}$ itself is not even finitely generated over $\mathbb{Z}$.

Let $R = \mathbb{Z}[a_1, a_2, a_3, \frac{1}{2} a_4]$. We let $E_R \to \text{Spec } R$ be the elliptic curve over $R$ defined as the base change from the Weierstrass curve on $A = \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$ via the map of rings
\( \varphi : A \to R \)

\[
\begin{align*}
\mu_1 & \mapsto 2a_1 \\
\mu_2 & \mapsto 3a_2 - a_1^2 \\
\mu_3 & \mapsto -a_3 \\
\mu_4 & \mapsto -\frac{1}{2}a_4 + 3a_2^2 - a_1a_3 \\
\mu_6 & \mapsto 0.
\end{align*}
\]

The chosen parameter \( t \) on the Weierstrass curve gives by base-change a parameter \( t_R \) along the zero section of \( E_R \). This gives us a formal group law over \( R \), this just being the one induced from the TMF formal group law through change of coefficient ring via the map \( \varphi \). It is shown in [6, Lemma 44] that this formal group law is isomorphic to the Krichever’s elliptic formal group law (after extended the coefficient ring from \( \tilde{\text{Ell}} \) to \( R \)). This isomorphism is explicitly given in [6].

Let \( \text{Ell} = \mathbb{Z}[a_1, a_2, a_3, \frac{1}{2}a_4][\Delta^{-1}] \), where \( \Delta \) is the discriminant.

**Theorem 4.2.3** (Theorem F). The Krichever’s elliptic formal group law \((\text{Ell}[1/2], F_{Kr})\) is Landweber exact. Therefore, the oriented cohomology theory \( \text{Ell}[1/2]^* := \text{MGL}^* \otimes_{\text{Lat}} \text{Ell}[1/2]^* \) on \( \text{Sm}_k \) is represented by a motivic oriented cohomology theory \( \mathcal{E}_{\text{Ell}}[1/2] \) on \( \text{Sm}_k \).

This theorem can be proved using the same idea as the proof of the Landweber exactness of the TMF formal group law. Nevertheless, for sake of completeness and the reader’s convenience, we include a sketch of the proof here.

**Proof.** According to Theorem 4.1.9, it suffices to show that for all primes \( l > 2 \) and integers \( n \geq 0 \), the multiplication map

\[
v_n : \text{Ell}[1/2]/(v_0, \ldots, v_{n-1}) \to \text{Ell}[1/2]/(v_0, \ldots, v_{n-1})
\]

is injective.
Note that $v_0 = l$ and $\text{Ell}[1/2]$ is an integral domain, hence, multiplication by $v_0$ is always injective.

Consider the ring

$$R_l := \mathbb{F}_l[a_1, \ldots, a_4] = R/(l)$$

and the family of elliptic curves $E_i := E_R \otimes_R R_l$ over $R_l$. The injectivity of $v_i$ for $i > 0$ is related to the height of the formal group law of these curves. Note that the only possible height of these curves are 1, 2, or infinity. We need first to remove the curves with infinite height formal group laws. This can be done by inverting the discriminant $\Delta$. If we fix a geometric point $x \in \text{Spec } R_l[\Delta^{-1}]$, the fiber is a supersingular elliptic curve if and only if the corresponding formal group law has height 1, i.e., $v_1$ vanishes when restricted to the residue field of $x$. As $R_l[\Delta^{-1}]$ is an integral domain, $v_1$ is injective if and only if it is not zero. This, in turn, is equivalent to the condition that on $R_l[\Delta^{-1}]$ there is at least one geometric fiber which is not supersingular. Note that the residue field of $x$ has characteristic $l > 2$. The family $E_l \to \text{Spec } R_l$ contains the Legendre family, which is dominant over the moduli space when passing to the separable closure. Therefore, the family $E_l \to \text{Spec } R_l$ is non-constant, hence contains a non-supersingular member.

Finally, we claim that $v_2$ in $R_l[\Delta^{-1}]/(v_1)$ is a unit. This claim implies that multiplication by $v_2$ is injective and that $R_l[\Delta^{-1}]/(v_1, v_2) = 0$, which implies the required injectivity for $n \geq 3$. To verify the claim, assume otherwise, then $v_2$ is contained in a maximal ideal $m \subset R_l[\Delta^{-1}]/(v_1)$. Therefore, the fiber of the family of curves over this closed point has associated formal group law with height greater than 2. This contradicts with the fact that the height of the formal group law of elliptic curves over a field of characteristic $l > 0$ can only be 1 or 2.

\[\square\]

**Remark 4.2.4.** The Landweber exactness condition for the prime $l = 2$ fails, since $v_1$ vanishes in $\text{Ell}/(2)$. To see this, one can easily calculate the $j$-invariant of the family of elliptic curves $E_{\text{Ell}} \otimes_{\text{Ell}} \text{Ell}/(2)$ over $\text{Ell}/(2)$ to see that the $j$-invariant is a constant. Passing
to the separable closure, one finds that the curve $y^2 + y = x^3$, which is supersingular, is a member in the family, and hence every member in this family is supersingular.

**Remark 4.2.5.** Inverting $\Delta$ is an overkill, as the locus $\Delta = 0$ contains not only the curves with infinite height formal group law, but also certain curves whose formal group law has height 1. This can be fixed using the modern approach [43]. Taking the $j$-invariant for the family $E_R \otimes_R R[1/2]$ we obtain a morphism of $\text{Spec} \mathbb{Z}[1/2][a_1, a_2, a_3, a_4]$ to the coarse moduli space of elliptic curves and thereby obtain a stack with coarse moduli space $\text{Spec} \mathbb{Z}[1/2][a_1, a_2, a_3, a_4]$. Let $K_r$ be the open substack obtained by removing the locus consisting of curves with infinite height formal group laws. This stack $K_r$ is flat over the stack of formal group laws, by the same argument as in the proof of Theorem 4.2.3. Therefore, any flat morphism of stacks from a commutative ring $R$ to $K_r$ defines a motivic oriented Krichever elliptic cohomology theory.

### 4.3 Flops in the cobordism ring

#### 4.3.1 The double point formula

Following [56], we define the ring $\omega^*(X)$ of double-point cobordism to be the graded abelian group $M^*_+(X)$ of smooth projective schemes $Y \to X$ as in Subsection 4.1.1, modulo the relations generated by the following double point relation.

Let $Y \in \text{Sm}_k$ be of pure dimension. A morphism $\pi : Y \to \mathbb{P}^1$ is a double point degeneration over $0 \in \mathbb{P}^1$, if $\pi^{-1}(0) = A \cup B$ with $A$ and $B$ smooth, of codimension one, intersecting transversely along $A \cap B = D$.

We denote the normal bundles of $D$ in $A$ and $B$ by $N_DA$ and $N_DB$ respectively. Then, the two projective bundles

$$\mathbb{P}(O_D \oplus N_DA) \to D \text{ and } \mathbb{P}(O_D \oplus N_DB) \to D$$
are isomorphic and will be denoted by $\mathbb{P}(\pi) \to D$. Let $g : Y \to X \times \mathbb{P}^1$ be a projective morphism for which $\pi = p_2 \circ g : Y \to \mathbb{P}^1$ is a double point degeneration over $0 \in \mathbb{P}^1$. The double point relation associated to $g$ is

$$[Y_\zeta \to X] - [A \to X] - [B \to X] + [\mathbb{P}(\pi) \to X],$$

where $Y_\zeta$ is the fiber over an arbitrary regular value $\zeta \in \mathbb{P}^1$.

It is shown in [56] that if $k$ has characteristic zero, then $\omega^*(X)$ is isomorphic to the cobordism ring $\Omega^*(X)$. We will use a weaker version of this theorem, which holds in a characteristic free fashion.

**Proposition 4.3.1** (Levine and Pandharipande). *For any field $k$ and scheme $X \in \text{Sm}_k$ of finite type over $k$, the natural projection $\Pi : M^+_\ast(X) \to \Omega^\ast(X)$ factors through $\omega^*(X)$.*

The proof in [56] uses only the existence of smooth pull-back, projective push-forward, the first Chern class of a line bundle, and external product, which means it does not depend on any assumption on $k$. Thus, the double point relation also holds when the field $k$ has positive characteristic.

We note that $\omega^*$ has the following structures:

1. pullback maps $f^* : \omega^*(X) \to \omega^*(Y)$ for each smooth morphism $f : Y \to X$ in $\text{Sm}_k$.

2. push-forward maps $f_* : \omega^*(Y) \to \omega^{*-d}(X)$ for each projective morphism $f : Y \to X$ of relative dimension $d$ in $\text{Sm}_k$.

3. associative, commutative external products, and an identity element $1 \in \omega^0(k)$.

The pullback and pushforward maps are functorial, and are compatible with the external products.

Composing the map $\omega^* \to \Omega^*$ with the natural transformation $\Theta_{\text{MGL}}$ gives us the transformation

$$\theta_{\text{MGL}} : \omega^* \to \text{MGL}^*,$$

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natural with respect to smooth pullback, projective push-forward, external products and unit.

Let \( F \subseteq X \) be a smooth closed subscheme of some \( X \in \text{Sm}_k \). The double point relation yields the following blow-up formula in \( \omega^*(X) \), and hence in \( \Omega^*(X) \) and \( \text{MGL}^*(X) \):

\[
1_X = [\text{Bl}_F X \to X] + [\mathbb{P}(N_F X \oplus \mathcal{O}) \to X] - [\mathbb{P}(\mathcal{O}_{N_F X}(1) \oplus \mathcal{O}) \to X] 
\]  

(4.4)

where \( \text{Bl}_F X \) is blow-up of \( X \) along \( F \), and \( N_F X \) is the normal bundle of \( F \). This is proved by the usual method of deformation to the normal cone. In case \( X \) is projective over \( k \), pushing forward to \( \text{Spec} \ k \) gives the relation in \( \omega^*(k) \), \( \Omega^*(k) \) and \( \text{MGL}^*(k) \)

\[
[X] = [\text{Bl}_F X] + [\mathbb{P}(N_F X \oplus \mathcal{O})] - [\mathbb{P}(\mathcal{O}_{N_F X}(1) \oplus \mathcal{O})] 
\]  

(4.5)

We say two smooth projective \( n \)-folds \( X_1 \) and \( X_2 \) are related by a flop if we have the following diagram of projective birational morphisms:

\[
\begin{array}{ccc}
X & \xrightarrow{p_1} & X_1 \\
\downarrow & & \downarrow \\
Y & \xleftarrow{p_2} & X_2
\end{array}
\]  

Here \( Y \) is a singular projective \( n \)-fold with singular locus \( Z \), such that \( Z \) is smooth of dimension \( n - k - 1 \). We assume in addition that there exist rank \( k \) vector bundles \( A \) and \( B \) on \( Z \), such that the exceptional locus \( F_1 \) in \( X_1 \) is the \( \mathbb{P}^{k-1} \)-bundle \( \mathbb{P}(A) \) over \( Z \), with normal bundle \( N_{F_1} X_1 = B \otimes \mathcal{O}(-1) \). Similarly, the the exceptional locus \( F_2 \) in \( X_2 \) is \( \mathbb{P}(B) \), with normal bundle \( N_{F_2} X_2 = A \otimes \mathcal{O}(-1) \). Let \( Q^3 \subset \mathbb{P}^4 \) denote the 3-dimensional quadric with an ordinary double point \( v \), defined by the equation \( x_1 x_2 = x_3 x_4 \). We say that \( X_1 \) and \( X_2 \) are related by a \textit{classical} flop if in addition \( k = 2 \), and along \( Z \), \( (Y,Z) \) is Zariski locally isomorphic to \( (Q^3 \times Z, v \times Z) \).

We assume now \( X_1 \) and \( X_2 \) are related by a flop. Let \( X = X_1 \) and \( F = F_1 \), the terms on
the right hand side of formula (4.5) become:

\[ Bl_F X = \tilde{X}, \]
\[ \mathbb{P}(N_F X \oplus \mathcal{O}) = \mathbb{P}_{\mathbb{P}(A)}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \oplus \mathcal{O}), \]
\[ \mathbb{P}(\mathcal{O}_{N_F X}(1) \oplus \mathcal{O}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O}), \]

where \( \mathbb{P}_{\mathbb{P}(A)}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \oplus \mathcal{O}) \) is a projective bundle over \( \mathbb{P}(A) \), which in turn is a projective bundle over \( Z \); and \( \mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O}) \) is a projective bundle over \( \mathbb{P}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1)) \), which is a projective bundle over \( \mathbb{P}(A) \).

Thus, we get the following immediate lemma.

**Lemma 4.3.2.** In the cobordism ring \( \text{MGL}^*(k) \), we have

\[ X_1 - X_2 = \mathbb{P}_{\mathbb{P}(A)}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \oplus \mathcal{O}) - \mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O}) \]
\[ - \mathbb{P}_{\mathbb{P}(B)}(A \otimes \mathcal{O}_{\mathbb{P}(B)}(-1) \oplus \mathcal{O}) + \mathbb{P}(\mathcal{O}_{\mathbb{P}(A \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O}). \]

In particular, \( X_1 - X_2 \) in \( \text{MGL}^*(k) \) comes from an element in \( \text{MGL}^*(Z) \).

The second claim follows from the observation that each of the projective bundles are smooth varieties over \( Z \). We will abuse notation by denoting a lifting of \( X_1 - X_2 \) to \( \text{MGL}^*(Z) \) by \( X_1 - X_2 \) itself.

### 4.3.2 Flops in the cobordism ring

Since each term in the formula in Lemma 4.3.2 is a iterated projective bundle over \( Z \), we will apply Quillen’s formula iteratively to each term, to calculate the fundamental class of the iterated projective bundles in \( \text{MGL}^*(Z) \).

**Proposition 4.3.3.** Let \( a_1, \ldots, a_k \) be the Chern roots of the bundle \( A \) over \( Z \) and let \( b_1, \ldots, b_k \) be the Chern roots of the bundle \( B \) over \( Z \), all Chern roots to be taken in \( \text{MGL}^* \). Then in \( \text{MGL}^*(Z) \), we have

\[ X_1 - X_2 = \sum_{m=1}^{k} \frac{1}{\prod_{i=1}^{k} (b_i + \Omega a_m) \prod_{l \neq m} (a_i - \Omega a_m)} - \frac{1}{\prod_{i=1}^{k} (a_i + \Omega b_m) \prod_{l \neq m} (b_i - \Omega b_m)}. \]
The rest of this subsection is devoted to prove this proposition.

The term $\mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O})$

We first prove the following

**Lemma 4.3.4.** In $\text{MGL}^*(Z)$, we have $\mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{A} \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O})$.

Let $\pi_1: \mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O}) \to \mathbb{P}(B \otimes \mathcal{O}(-1))$ be the natural projection. Let the first Chern class of the bundle $\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}(-1))}(1)$ be $u_B$. Then the two Chern roots of $\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O}$ are $u_B$ and 0. Applying Quillen’s formula (4.2) with $f_1(t) \equiv 1$ being the fundamental class, we have

$$
\pi_{1*}(\mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O})) = \frac{1}{0 - u_B} + \frac{1}{u_B - 0} = \frac{1}{-u_B} + \frac{1}{u_B}.
$$

(4.7)

Next, let $\pi_2: \mathbb{P}(\mathbb{A})(B \otimes \mathcal{O}_{\mathbb{P}(\mathbb{A})}(-1)) \to \mathbb{A}$ be the projection. The Chern roots of the bundle $B \otimes \mathcal{O}_{\mathbb{P}(\mathbb{A})}(-1)$ are $b_i - \Omega v_A$ for $i = 1, \ldots, k$, where $v_A := c_1(\mathcal{O}_{\mathbb{P}(\mathbb{A})}(1))$, and $b_i$ for $i = 1, \ldots, k$ are the Chern roots of bundle $B$. Applying Quillen’s formula (4.2) with $f_2(t) = \frac{1}{-t} + \frac{1}{t}$, we get

$$
\pi_{2*}(f_2) = \sum_{i=1}^{k} \prod_{j \neq i} \left( \frac{1}{b_i - \Omega v_A} \right) + \prod_{j \neq i} \left( \frac{1}{b_i - \Omega v_A} \right)
$$

$$
= \sum_{i=1}^{k} \frac{1}{b_i - \Omega v_A} + \frac{1}{b_i + \Omega v_A}.
$$

Finally, let $\pi_3: \mathbb{A} \to Z$ be the projection. Recall the Chern roots of bundle $A$ are denoted by $a_i$ for $i = 1, \ldots, k$. Let $f_3(t) := \sum_{i=1}^{k} \prod_{j \neq i} \left( \frac{1}{b_j - \Omega b_i} \right)$, then Quillen’s formula (4.2) yields

$$
\pi_3(f_3) = \sum_{i=1}^{k} \sum_{m=1}^{k} \frac{1}{b_i + \Omega a_m} + \frac{1}{b_i - \Omega a_m}.
$$

which is $\pi_*(\mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O}))$.

Similarly, for the projective bundle $\pi': \mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{A} \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O}) \to Z$, we have:

$$
\pi_*'([\mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{A} \otimes \mathcal{O}(-1))}(1) \oplus \mathcal{O})]) = \sum_{i=1}^{k} \sum_{m=1}^{k} \frac{1}{a_i + \Omega b_m} + \frac{1}{a_i - \Omega b_m}.
$$

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Now comparing the two formulas of $\pi_*(\mathbb{P}(\mathcal{O}_\mathbb{P}(B \otimes \mathcal{O})(-1) \oplus \mathcal{O}))$ and $\pi'_*(\mathbb{P}(\mathcal{O}_\mathbb{P}(A \otimes \mathcal{O})(-1) \oplus \mathcal{O}))$, the lemma follows.

**The term** $\mathbb{P}(\mathcal{O}_\mathbb{P}(A)(-1) \oplus \mathcal{O})$

Let $\pi_1 : \mathbb{P}(\mathcal{O}(1)) \to \mathbb{P}(A)$ be the natural projection. The Chern roots of the bundle $B \otimes \mathcal{O}(A)(-1) \oplus \mathcal{O}$ are $b_i - \Omega v_A$ for $i = 1, \ldots, k$, and 0, where $v_A := c_1(\mathcal{O}(A)(1))$, and $b_1, b_2$ are two Chern roots of the bundle $B$. Applying Quillen’s formula (4.2) with $f_1(t) \equiv 1$, we get

$$
\pi_1^*(\mathbb{P}(B \otimes \mathcal{O}(A)(-1) \oplus \mathcal{O})) = \sum_{i=1}^{k} \frac{1}{(v_A - \Omega b_i) \prod_{j \neq i}(b_j - \Omega b_i)} + \frac{1}{\prod_{i=1}^{k} (b_i - \Omega v_A)}.
$$

Now let $\pi_2 : \mathbb{P}(A) \to Z$ be the natural projection. Recall the Chern roots of bundle $A$ are denoted by $a_i$, $i = 1, \ldots, k$. Let $f_2(t) := \sum_{i=1}^{k} \frac{1}{(v_A - \Omega b_i) \prod_{j \neq i}(b_j - \Omega b_i)} + \frac{1}{\prod_{i=1}^{k} (b_i - \Omega v_A)}$, then Quillen’s formula (4.2) gives

$$
\pi_2^*(f_2) = \sum_{m=1}^{k} \frac{\sum_{i=1}^{k} \frac{1}{(-\Omega a_m - \Omega b_i) \prod_{j \neq i}(b_j - \Omega b_i)}}{\prod_{l \neq m}(a_l - \Omega a_m)} + \frac{1}{\prod_{i=1}^{k} (a_i + \Omega b_m)}.
$$

Similarly, for the bundle $\pi' : \mathbb{P}(\mathcal{O}(1)) \to Z$, we have:

$$
\pi'_*(\mathbb{P}(\mathcal{O}(A)(-1) \oplus \mathcal{O})) = \sum_{m=1}^{k} \frac{\sum_{i=1}^{k} \frac{1}{(-\Omega b_m - \Omega a_i) \prod_{j \neq i}(a_j - \Omega a_i)}}{\prod_{l \neq m}(b_l - \Omega b_m)} + \frac{1}{\prod_{i=1}^{k} (a_i + \Omega b_m)}.
$$

Therefore,

$$
\pi_*(\mathbb{P}(\mathcal{O}(A)(-1) \oplus \mathcal{O})) - \pi'_*(\mathbb{P}(\mathcal{O}(A)(-1) \oplus \mathcal{O})) = \sum_{m=1}^{k} \frac{\sum_{i=1}^{k} \frac{1}{(a_i - \Omega a_m)}}{\prod_{l \neq m}(b_l - \Omega b_m)} - \frac{1}{\prod_{i=1}^{k} (a_i + \Omega b_m) \prod_{l \neq m}(b_l - \Omega b_m)}.
$$

This finishes the proof of the proposition.

### 4.3.3 Flops in the elliptic cohomology ring

In this subsection, we prove the following Proposition.
Proposition 4.3.5. Suppose $X_1$ and $X_2$ are smooth projective varieties related by a flop. Notations as above, in the ring $\text{Ell}_Q^*(Z)$, we have

$$\mathbb{P}_{F(A)}(B \otimes O_{F(A)}(-1) \oplus O) = \mathbb{P}_{F(B)}(A \otimes O_{F(B)}(-1) \oplus O).$$

In particular, we have $X_1 - X_2 = 0$ in $\text{Ell}_Q^*(Z)$, and hence $X_1 - X_2 = 0$ in $\tilde{\text{Ell}}^*(k)$.

Remark 4.3.6. Once we know that $X_1 - X_2 = 0$ in $\text{Ell}_Q^*(Z)$, it follows by pushing forward to $\text{Spec} \, k$ that $X_1 - X_2 = 0$ in $\text{Ell}_Q^*(k)$. But $X_1 - X_2$ is a well-defined element in $\tilde{\text{Ell}}^*(k)$ and $\tilde{\text{Ell}}^*(k) \to \text{Ell}_Q^*(k)$ is injective, since $\tilde{\text{Ell}}^*(k)$ is by construction $\mathbb{Z}$-torsion free, hence $X_1 - X_2 = 0$ in $\tilde{\text{Ell}}^*(k)$, as claimed above.

Proof of the proposition. Thanks to Proposition 4.3.3, we can reduce $X_1 - X_2$ to an explicit element in $\text{MGL}^*(Z)$. Applying the canonical map $\text{MGL}^*(Z) \to \text{Ell}^*(Z)$ to this element, we have, in the ring $\text{Ell}^*(Z)$,

$$X_1 - X_2 = \sum_{m=1}^{k} \left( \prod_{l \neq m}^{k} \frac{1}{\lambda}(a_i - \Omega a_m) \prod_{l \neq m}^{k} \frac{1}{\lambda}(m - \Omega) \right),$$

here the $a_i$'s and $b_i$'s are the Chern roots in the elliptic cohomology. We would like to show the above expression is $0$ in $\text{Ell}_Q^*(Z)$.

Recall that $x + E y = \lambda^{-1}(x) + \lambda^{-1}(y)$ where the exponential $\lambda(t)$ is given by the power series $t \sum_{\ell \in \mathbb{Z}}$.

Let $\lambda^{-1}(a_i) = A_i$, and $\lambda^{-1}(b_i) = B_i$ for $i = 1, \ldots, k$. Then in $\text{Ell}_Q^*(Z)$, $X_1 - X_2$ becomes:

$$\sum_{m=1}^{k} \left( \prod_{i=1}^{k} \frac{1}{\lambda}(B_i + \Omega A_m) \prod_{l \neq m}^{k} \frac{1}{\lambda}(A_i - \Omega A_m) - \prod_{i=1}^{k} \frac{1}{\lambda}(A_i + \Omega B_m) \prod_{l \neq m}^{k} \frac{1}{\lambda}(B_i - \Omega B_m) \right).$$

Plugging-in

$$\frac{1}{\lambda}(t) = \frac{Q(t)}{t} = \frac{1}{2\pi i} e^{kt} e^{(z(t))} \frac{\sigma^{(z(t))/\pi t}}{\sigma^{(z(t))/\pi t}},$$

and cancelling some obvious factors, $X_1 - X_2$ becomes

$$\sum_{m=1}^{k} \left( \prod_{i=1}^{k} \frac{\sigma(B_i + \Omega A_m - z)}{\sigma(B_i + \Omega A_m)} \prod_{l \neq m}^{k} \frac{\sigma(A_i - \Omega A_m - z)}{\sigma(A_i - \Omega A_m)} - \prod_{i=1}^{k} \frac{\sigma(A_i + \Omega B_m - z)}{\sigma(A_i + \Omega B_m)} \prod_{l \neq m}^{k} \frac{\sigma(B_i - \Omega B_m - z)}{\sigma(B_i - \Omega B_m)} \right).$$
Now let

\[ x_1 = A_i, \quad x_{k+j} = -B_j \]

for \( i, j = 1, \ldots, k \); and

\[ y_i = A_i - z, \quad y_{k+j} = -B_j + z \]

for \( i, j = 1, \ldots, k \). Using the fact that \( \sigma(z) \) is an odd function, we get

\[
\sigma(-z) \prod_{i=1}^{k} \frac{\sigma(B_i + \Omega A_m - z)}{\sigma(B_i + \Omega A_m)} \prod_{l \neq m} \frac{\sigma(A_i - \Omega A_m - z)}{\sigma(A_i - \Omega A_m)} = \frac{\prod_{i=1}^{k} \sigma(y_i - x_m) \prod_{j=1}^{k} \sigma(y_{n+j} - x_m)}{\prod_{i \neq m} \sigma(x_i - x_m) \prod_{j=1}^{k} \sigma(x_{n+j} - x_m)}
\]

for any \( m = 1, \ldots, k \);

\[
- \sigma(-z) \prod_{i=1}^{k} \frac{\sigma(B_i + \Omega A_m - z)}{\sigma(B_i + \Omega A_m)} \prod_{l \neq m} \frac{\sigma(A_i - \Omega A_m - z)}{\sigma(A_i - \Omega A_m)} = \frac{\prod_{i=1}^{k} \sigma(y_i - x_{k+m}) \prod_{j=1}^{k} \sigma(y_{n+j} - x_{k+m})}{\prod_{i \neq m} \sigma(x_i - x_{k+m}) \prod_{j=1}^{k} \sigma(x_{n+j} - x_{k+m})}
\]

for any \( m = 1, \ldots, k \).

The Proposition follows from the following classical identity of sigma-function. (See § 20.53, Example 3 of [82].) Assuming \( \sum_{r=1}^{k} x_r = \sum_{r=1}^{k} y_r \), we have

\[
\sum_{r=1}^{k} \frac{\sigma(x_r - y_1)\sigma(x_r - y_2)\cdots\sigma(x_r - y_k)}{\sigma(x_r - x_1)\sigma(x_r - x_2)\cdots*\cdots\sigma(x_r - x_k)} = 0
\]

with the * denoting that the vanishing factor \( \sigma(x_r - x_r) \) is to be omitted. \( \square \)

### 4.4 The algebraic elliptic cohomology ring with rational coefficients

Let \( \mathcal{I}_{\text{fl}} \) be the ideal in \( \text{MGL}_Q^*(k) \) generated by differences \( X_1 - X_2 \), where \( X_1 \) and \( X_2 \) are related by a flop and \( \mathcal{I}_{\text{clfl}} \subset \mathcal{I}_{\text{fl}} \) the ideal in \( \text{MGL}_Q^*(k) \) generated by differences \( X_1 - X_2 \), where \( X_1 \) and \( X_2 \) are related by a classical flop. Section 4.3.3 shows that the elliptic genus \( \phi: \text{MGL}^*(k) \to \text{Ell}^*(k) \) factors through the quotient \( \text{MGL}^*(k)/\mathcal{I}_{\text{fl}} \).
**Proposition 4.4.1.** The ideal $I_{cfl}$ in $\text{MGL}_Q^*(k)$ contains a system of polynomial generators $x_n$ of $\text{MGL}_Q^*(k)$ in degree $n \leq -5$.

This proposition follows from same the calculation as in [71]. Nevertheless, for the convenience of the readers and to be as explicit as possible, we include the calculation in the Appendix.

Following Höhn ([40]), we define the four elements $W_i$, $i = 1, 2, 3, 4$ in $\text{MGL}^*(k)$ via their Chern numbers as follows,

\[
\begin{align*}
    c_1[W_1] &= 2; \\
    c_2^2[W_2] &= 0, c_2[W_2] = 24; \\
    c_1^3[W_3] &= 0, c_1c_2[W_3] = 0, c_3[W_3] = 2; \\
\end{align*}
\]

In fact, solving the systems of linear equations, one can write down $W_i$ explicitly as rational linear combinations of products of projective spaces,

\[
\begin{align*}
    W_1 &= [\mathbb{P}^1]; \\
    W_2 &= -16[\mathbb{P}^2] + 18[\mathbb{P}^1 \times \mathbb{P}^1]; \\
    W_3 &= \frac{3}{2}[\mathbb{P}^3] - 4[\mathbb{P}^2 \times \mathbb{P}^1] + \frac{5}{2}([\mathbb{P}^1]^3); \\
    W_4 &= -4[\mathbb{P}^4] + 12.5[\mathbb{P}^3 \times \mathbb{P}^1] + 6[\mathbb{P}^2 \times \mathbb{P}^2] - 26[\mathbb{P}^2 \times (\mathbb{P}^1)^2] + 11.5(\mathbb{P}^1)^4.
\end{align*}
\]

Write the Hirzebruch characteristic power series $Q(t) = 1 + f_1 t + f_2 t^2 + \cdots$. Let $A$, $B$, $C$, and $D$ be such that

\[
\begin{align*}
    f_1 &= \frac{1}{2} A; \\
    f_2 &= \frac{1}{24 \cdot 3} (6A^2 - B); \\
    f_3 &= \frac{1}{25 \cdot 3} (2A^3 - AB + 16C); \\
    f_4 &= \frac{1}{29 \cdot 32 \cdot 5} (60A^4 - 60A^2B + 1920AC + 7B^2 - 1152D).
\end{align*}
\]
Then, a calculation shows (see, e.g., § 2.2 of [40]), the lower degree parts of the elliptic genus \( \phi \) are as follows

\[
K_1 = \frac{1}{2} Ac_1;
K_2 = \frac{1}{2^4 \cdot 3} ((6A^2 - B)c_1^2 + 2Bc_2);
K_3 = \frac{1}{2^5 \cdot 3} ((2A^3 - AB + 16C)c_1^3 + (2AB - 48C)c_2c_1 + 48Cc_3);
K_4 = \frac{1}{2^9 \cdot 3^2 \cdot 5} ((60A^4 - 60A^2B + 1920AC + 7B^2 - 1152D)c_1^4
+ (24B^2 - 2304D)c_2^3 + (120A^2B - 5760AC - 28B^2 + 4608D)c_1^2c_2
+ (5760AC + 8B^2 - 4608D)c_3c_1 + (-8B^2 + 4608D)c_4),
\]

where \( K_i \) is the homogeneous degree \( i \) part of the elliptic genus \( \phi \). Therefore, \( \phi(W_1) = A, \phi(W_2) = B, \phi(W_3) = C \) and \( \phi(W_4) = D \).

We then compare the elements \( A, B, C, D \) with the polynomial generators of Ell\( Q^* (k) \). Let \( g_i \) be the weight \( 2i \) Eisenstein series where \( i = 2, 3 \), and let \( X \) be the Weierstrass \( p \)-function and \( Y \) its derivative. The same calculations as in §2 of [40] show \( B = 24X, C = Y, \) and \( D = 6X^2 - g_2/2 \). Recall that \( a_2 = X, a_3 = Y, a_4 = \frac{1}{2} g_2 \). Note that the image of \( W_1 = \mathbb{P}^1 \), which is \( \frac{k}{\pi i} \), involves non-trivially the variable \( k \), but the images of \( W_i \) for \( i = 2, 3, 4 \) lies in the subring \( \mathbb{Q}(e^{2\pi i})[e^{2i\pi \tau}] \). In particular, \( A, B, C, \) and \( D \) are algebraically independent.

Summarizing all the above, we have proved the following proposition.

**Proposition 4.4.2.** The classifying map \( \phi_E : \mathrm{MGL}^* (k) \rightarrow \mathbb{Q}(e^{2\pi i})[e^{2i\pi \tau}, \frac{k}{2\pi i}] \) descends to define an isomorphism of \( \mathrm{MGL}^*_Q(k)/I_{\mathrm{clfl}} \) with a polynomial subalgebra \( \mathbb{Q}[a_1, a_2, a_3, a_4] \) of \( \mathbb{Q}(e^{2\pi i})[e^{2i\pi \tau}, \frac{k}{2\pi i}] \), with \( a_i \) in degree \( -i \).

This implies the following corollary.

**Corollary 4.4.3.** The natural ring homomorphism \( \mathrm{MGL}^*_Q(k)/I_{\mathrm{clfl}} \rightarrow \mathrm{Ell}^*_Q(k) \) is injective and \( I_{\mathrm{clfl}} = I_{\mathrm{fl}} \).
Proof. The first assertion follows immediately from proposition 4.4.2; the second from the first, noting that $\text{MGL}_Q^*(k)/\mathcal{I}_{\text{clfl}} \to \text{Ell}_Q^*(k)$ factors through the quotient $\text{MGL}_Q^*(k)/\mathcal{I}_{\text{fl}}$ by proposition 4.3.5.

\section{4.5 The ideal generated by differences of flops}

Let $\mathcal{I}_{\text{clfl}}$ be the ideal in $\text{MGL}_Q^*(k)$ generated by those $[X_1] - [X_2]$ with $X_1$ and $X_2$ related by a classical flop.

\textbf{Proposition 4.5.1.} The ideal $\mathcal{I}_{\text{clfl}}$ in $\text{MGL}_Q^*(k)$ contains a system of polynomial generators $x_n$ of $\text{MGL}_Q^*(k)$ in all degrees $n \leq -5$.

This proposition was originally proved in Section 5 of [71]. Their proof is based on explicit calculation that lends itself with slight adjustment to our setting. Note that this proof is independent of the fact that the Krichever’s elliptic formal group law is defined over $\mathbb{Z}[a_1, a_2, a_3, a_4]$.

The way to show this is to use the fact that an element $x$ of $\text{MGL}_Q^{-n}(k)$ is a polynomial generator of the ring $\text{MGL}_Q^*(k)$ if and only if the Chern number $s^n$ is not zero on $x$. Here for a smooth irreducible projective variety $X$ over $k$, $s^n(X) = \langle \xi_1^n + \cdots + \xi_n^n, [X] \rangle$, with $\xi_i$ being the Chern roots of the tangent bundle of $X$ (in the Chow ring $\text{CH}^*(X)$).

The following lemma can be found in [32], Page 47.

\textbf{Lemma 4.5.2.} For any space $X$ with a vector bundle $V$ of rank $r$, let $\pi : \mathbb{P}(V) \to X$ be the projective bundle and let $u = c_1(\mathcal{O}(1)) \in \text{CH}^1(\mathbb{P}(V))$. Then, for all $i \geq 0$,

$$\pi^*(u^i) = s_{i-(r-1)}(V),$$

where $s_k$ is the $k$-th Segre class.

Recall that we have shown that, in $\text{MGL}^*(k)$,

$$X_1 - X_2 = \mathbb{P}_{\mathbb{P}(A)}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \oplus \mathcal{O}) - \mathbb{P}_{\mathbb{P}(B)}(A \otimes \mathcal{O}_{\mathbb{P}(B)}(-1) \oplus \mathcal{O}).$$
For each smooth $Z$ and rank 2 vector bundles $A$ and $B$, there is a pair $X_1$ and $X_2$, related by a classical flop with exceptional fibers equal to $\mathbb{P}(A)$ and $\mathbb{P}(B)$ respectively. Indeed, consider the $\mathbb{P}^1 \times \mathbb{P}^1$ bundle $q : \mathbb{P}(A) \times_Z \mathbb{P}(B) \to Z$, which we embed in the $\mathbb{P}^3$ bundle $\mathbb{P}(A \oplus B) \to Z$ via the line bundle $p_1^*O_A(1) \otimes p_2^*O_B(1)$. We then take $Y^0$ to be the affine $Z$ cone in $A \oplus B$ associated to $\mathbb{P}(A) \times_Z \mathbb{P}(B) \subset \mathbb{P}(A \oplus B)$, and $Y$ the closure of $Y^0$ in $\mathbb{P}(A \oplus B \oplus \mathcal{O}_Z)$. $Y$ thus contains the $\mathbb{P}^2$ bundles $P_1 := \mathbb{P}(A \oplus \mathcal{O}_Z)$ and $P_2 := \mathbb{P}(B \oplus \mathcal{O}_Z)$; we take $X_i \to Y$ to be the blow-up of $Y$ along $P_i$, $i = 1, 2$ and $\tilde{X}$ the blow-up of $Y$ along $Z$.

We will, for each $n \geq 5$, find an $n - 3$-fold $Z$ and rank two vector bundles $A$ and $B$ over $Z$, such that $s^n(\mathbb{P}(A))(B \otimes \mathcal{O}_A)(-1) \oplus \mathcal{O}) \neq s^n(\mathbb{P}(B))(A \otimes \mathcal{O}_B)(-1) \oplus \mathcal{O})$.

We start with an arbitrary choice of $Z$ and $A$, $B$. Recall we denote the Chern roots (in CH*) of $A$ by $a_1, a_2$ and Chern roots of $B$ by $b_1, b_2$. Set $v_A := c_1(\mathcal{O}_A(1))$, $v_B = c_1(\mathcal{O}_B(1))$, $w_A := c_1(\mathcal{O}_A \otimes \mathcal{O}_B(-1))(1))$, and $w_B = c_1(\mathcal{O}_A \otimes \mathcal{O}_B(-1))(1))$. Let the Chern roots of the tangent bundle of $Z$ be $z_1 \cdots, z_{n-3}$. Then, the Chern roots of the tangent bundle of $\mathbb{P}(A)(B \otimes \mathcal{O}_A(-1) \oplus \mathcal{O})$ are

$$b_1 - v_A + w_B, b_2 - v_A + w_B, w_B, a_1 + v_A, a_2 + v_A, z_1, \ldots, z_{n-3}.$$ 

For the bundle $\pi_1 : \mathbb{P}(A)(B \otimes \mathcal{O}_A)(-1) \oplus \mathcal{O}) \to \mathbb{P}(A)$, Lemma 4.5.2 yields

$$\pi_1^*(w_B^i) = s_{i-2}(B \otimes \mathcal{O}_A(-1) \oplus \mathcal{O}) = \sum_{i_1 + i_2 = i-2} (-b_1 + v_A)^{i_1}(-b_2 + v_A)^{i_2}.$$ 

Similarly, for $\pi_2 : \mathbb{P}(A) \to Z$, we have

$$\pi_2^*(v_B^i) = s_{i-1}(A) = \sum_{i_1 + i_2 = i-1} (-a_1)^{i_1}(-a_2)^{i_2}.$$ 

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Thus

\[ \pi_{1*} s^n \left( \mathbb{P}_{\mathbb{P}(A)} (B \otimes \mathcal{O}_{\mathbb{P}(A)} (-1) \oplus \mathcal{O}) \right) \]

\[ = \pi_{1*} \left[ (b_1 - v_A + w_B)^n + (b_2 - v_A + w_B)^n + w_B^n + (a_1 + v_A)^n + (a_2 + v_A)^n + \sum_{i=1}^{n-3} z_i^n \right] \]

\[ = \sum_{i=0}^{n} \binom{n}{i} (b_1 - v_A)^{n-i} \pi_{1*} (w_B^i) + \sum_{i=0}^{n} \binom{n}{i} (b_2 - v_A)^{n-i} \pi_{1*} (w_B^i) + \pi_{1*} (w_B^n) \]

\[ = \sum_{i=2}^{n} \binom{n}{i} (b_1 - v_A)^{n-i} \sum_{i_1+i_2=i-2} (-b_1 + v_A)^{i_1} (-b_2 + v_A)^{i_2} \]

\[ + \sum_{i=2}^{n} \binom{n}{i} (b_2 - v_A)^{n-i} \sum_{i_1+i_2=i-2} (-b_1 + v_A)^{i_1} (-b_2 + v_A)^{i_2} \]

\[ + \sum_{i_1+i_2=n-2} (-b_1 + v_A)^{i_1} (-b_2 + v_A)^{i_2}. \]

In the case when \( a_2 = b_1 = b_2 = 0 \), we have:

\[ \pi_{1*} s^n \left( \mathbb{P}_{\mathbb{P}(A)} (B \otimes \mathcal{O}_{\mathbb{P}(A)} (-1) \oplus \mathcal{O}) \right) = \frac{1}{2} \sum_{i=2}^{n} \binom{n}{i} (-v_A)^{n-i} \sum_{i_1+i_2=i-2} (v_A)^{i_2} + \sum_{i_1+i_2=n-2} (v_A)^{n-2} \]

\[ = 2 \sum_{i=2}^{n} \binom{n}{i} (-v_A)^{n-i} (i-1)(v_A)^{i-2} + (n-1)(v_A)^{n-2} \]

\[ = (v_A)^{n-2} (2(-1)^n + n - 1). \]

According to Lemma 4.5.2, \( \pi_{2*} (v_A)^{n-2} = (-a_1)^{n-3} \), since \( a_2 = 0 \). Therefore,

\[ \pi_{2*} \pi_{1*} s^n \left( \mathbb{P}_{\mathbb{P}(A)} (B \otimes \mathcal{O}_{\mathbb{P}(A)} (-1) \oplus \mathcal{O}) \right) = (-a_1)^{n-3} (2(-1)^n + n - 1) \]

\[ = (a_1)^{n-3} (-2 + (-1)^{n-3}(n - 1)). \]

We do the same for the projection \( \pi'_1 : \mathbb{P}_{\mathbb{P}(B)} (A \otimes \mathcal{O}_{\mathbb{P}(B)} (-1) \oplus 1) \rightarrow \mathbb{P}(B) \), and \( \pi'_2 : \mathbb{P}(B) \rightarrow Z \), in the case of \( a_2 = b_1 = b_2 = 0 \). By Lemma 4.5.2, \( \pi_{1*} (w_A) = \sum_{i_1+i_2=i-2} (-a_1 + v_B)^{i_1} (v_B)^{i_2}, \)
and hence,

\[
\pi_1' s^n(\mathbb{P}_\mathbb{P}^1(A \otimes \mathcal{O}_\mathbb{P}(-1) \oplus 1))
= \pi_1' ((a_1 - v_B + w_A)^n + (-v_B + w_A)^n + w_A^n + 2(v_B)^n + \sum_{i=1}^{n-3} (z_i)^n)
= \sum_{i=0}^{n} \binom{n}{i} (a_1 - v_B)^{n-i} \pi_1^*(w_A^i) + \sum_{i=0}^{n} \binom{n}{i} (-v_B)^{n-i} \pi_1^*(w_A^i) + \pi_1^*(w_A^n)
= \sum_{i=2}^{n} \binom{n}{i} (a_1 - v_B)^{n-i} \sum_{i_1 + i_2 = i - 2} (-a_1 + v_B)^{i_1} (v_B)^{i_2}
+ \sum_{i=2}^{n} \binom{n}{i} (-v_B)^{n-i} \sum_{i_1 + i_2 = n - 2} (-a_1 + v_B)^{i_1} (v_B)^{i_2}.
\]

For \(\pi_{2*}',\) Lemma 4.5.2 tells us that \(\pi_{2*}'(v_B) = 1,\) and \(\pi_{2*}'(v_B^i) = 0\) for \(i \neq 1.\) Finally, we obtain:

\[
\pi_{2*}' \pi_1' s^n(\mathbb{P}_\mathbb{P}^1(A \otimes \mathcal{O}_\mathbb{P}(-1) \oplus 1)) = a_1^{n-3}(-(n-1)^2 + \binom{n}{2} - (-1)^{n-3} + (n-1)(-1)^{n-3})
= a_1^{n-3}(-(n-1)^2 + \binom{n}{2} + (n-2)(-1)^{n-3})
\]

To show this can be non-vanishing for certain choice of \(Z\) and \(a_1 \in \text{CH}^1(Z),\) we simply take \(Z = \mathbb{P}^{n-3}\) and \(a_1 = c_1(\mathcal{O}_{\mathbb{P}^{n-3}}(1)).\) We then have

\[
s^n[X_1 - X_2] = \langle a_1^{n-3}(-2 + (-1)^{n-3}(n-1) + (n-1)^2 - \binom{n}{2} - (n-2)(-1)^{n-3}), [\mathbb{P}^{n-3}] \rangle
= \frac{n^2 - 3n - 2 + 2(-1)^{n-1}}{2},
\]

so for \(n \geq 5, s^n[X_1 - X_2] \neq 0,\) as desired.

### 4.6 Birational symplectic varieties

In this section we work over a base field \(k\) with characteristic zero.
4.6.1 Specialization in algebraic cobordism theory

We will use the specialization morphism in algebraic cobordism theory. The existence of a specialization morphism in algebraic cobordism theory is folklore; for lack of a reference, we sketch a construction here.

**Proposition 4.6.1.** Let $C$ be a smooth curve, and $p : X \to C$ a smooth projective morphism. Let $o \in C$ be a closed point with fiber $X_o$ and $\eta$ be the generic point of $C$ whose fiber is denoted by $X_\eta$. Let $i : X_o \to X$ and $j : X_\eta \to X$ be the natural embeddings. Then there is a natural morphism $\sigma : \Omega^*(X_\eta) \to \Omega^*(X_o)$ such that $\sigma \circ j^* = i^*$ where $j^*$ is the pull-back and $i^*$ is the Gysin morphism.

**Proof.** Let $R$ be the local ring of $C$ at $o \in C$. Although in this case neither $X_R$ nor $X_\eta$ are $k$-schemes of finite type, they are both projective limits of such, which allows us to define $\Omega^*(X_R)$ and $\Omega^*(X_\eta)$ as

$$\Omega^*(X_R) := \lim_{\emptyset \neq U \subset C} \Omega^*(p^{-1}(U)); \quad \Omega^*(X_\eta) := \lim_{\emptyset \neq U \subset C} \Omega^*(p^{-1}(U)).$$

Here $U \subset C$ is an open subscheme. We may then replace $C$ with $	ext{Spec } R$, $X$ with $X_R$.

For any integer $n$, there is a localization short exact sequence

$$\Omega^{n-1}(X_o) \xrightarrow{i_*} \Omega^n(X) \xrightarrow{j^*} \Omega^n(X_\eta) \to 0.$$

Let $i^* : \Omega^n(X) \to \Omega^n(X_o)$ be the pull-back. In order to show it factors through $j^*$, it suffices to check that $i^* \circ i_* = 0$. This is true since $i^* \circ i_* \cong c_1 \mathcal{O}_{X_o}|_{X_o} = 0$ (see [54, Lemma 3.1.8]).

We note that the specialization map $\sigma : \Omega^*(X_\eta) \to \Omega^*(X_o)$ is a ring homomorphism, and is natural with respect to pullback and push-forward in the following sense: Let $q : Y \to C$ be a smooth projective morphism, with $C$ as above, let $f : Y \to X$ be a morphism over $C$ and let $f_o : Y_o \to X_o$, $f_\eta : Y_\eta \to X_\eta$ be the respective restrictions of $f$. Let $\sigma_X : \Omega^*(X_\eta) \to \Omega^*(X_o)$, $\sigma_Y : \Omega^*(Y_\eta) \to \Omega^*(Y_o)$ be the respective specialization maps. Then
1. $f_\sigma^* \circ \sigma \chi = \sigma \gamma \circ f_\eta^*$.

2. Suppose $f$ is projective. Then $f_{\sigma} \circ \sigma \gamma = \sigma \chi \circ f_{\eta}$.

Indeed, as the respective restriction maps $j^*$ are surjective, the fact that $\sigma$ is a ring homomorphism and the compatibility (1) follows from the fact that pullback maps are functorial ring homomorphisms. For (2), we note that the diagram

$$
\begin{array}{ccc}
Y_o & \xrightarrow{i_y} & Y \\
\downarrow{f_o} & \downarrow{f} & \downarrow{f} \\
X_o & \xrightarrow{i_X} & X
\end{array}
$$

is cartesian, and then the compatibility (2) follows from the base-change identity $i_X^* \circ f_* = f_{\sigma} \circ i_Y^*$ and the surjectivity mentioned above.

### 4.6.2 Cobordism ring of birational symplectic varieties

Consider two smooth projective varieties $X_1$ and $X_2$ satisfying the following condition: There exist smooth projective algebraic varieties $X_1$ and $X_2$, flat over a smooth quasi-projective curve $C$ with a closed point $o \in C$, such that:

(i) the fiber of $X_i \times C$ over $o$ is $(X_i)_o = X_i$;

(ii) there is an isomorphism $\Psi : (X_1)_C \setminus \{o\} \to (X_2)_C \setminus \{o\}$ over $C$.

The counterpart of the following Proposition in Chow theory is proved in [37]. Let $\Omega^*$ be the algebraic cobordism with $\mathbb{Z}$ coefficient. When the base field $k$ has characteristic zero, then $\Omega^* = \text{MGL}^*$.

**Proposition 4.6.2.** Let $X_1$ and $X_2$ be two smooth projective varieties satisfying conditions (i) and (ii). The deformation data induce an isomorphism

$$\Omega^*(X_1) \cong \Omega^*(X_2).$$
It was proved in [44] that the above conditions (i) and (ii) hold, when $X_1$ and $X_2$ are irreducible symplectic varieties such that:

- either $X_1$ and $X_2$ are connected by a general Mukai flop,
- or $X_1$ and $X_2$ are isomorphic in codimension two, that is, there exist isomorphic open subsets $U_1 \subset X_1$, and $U_2 \subset X_2$ with $\text{codim}_{X_i}(X_i \setminus U_i) \geq 3$, for $i = 1, 2$.

The proof follows the same idea as in [37], nevertheless, for the convenience of the readers, we include the proof.

Let $\sigma_i : \Omega^*(X_{i\eta}(\mathcal{U}_{\mathcal{X}_{i\eta}})) \to \Omega^*(\mathcal{X}_{i\eta})$ be the specialization map, where $i = 1, 2$. Let $\Delta \subset (\mathcal{X}_{1\eta} \times \mathcal{X}_{2\eta})$ be the diagonal, or the graph of the isomorphism $\Psi$ as in condition (ii) restricted to the generic fiber. We define an element

$$Z := (\sigma_1 \times \sigma_2)([\Delta]) \in \Omega^*(X_1 \times X_2).$$

The projection $X_1 \times X_2 \to X_i$ for $i = 1, 2$ is denoted by $p_i$. The element $Z \in \Omega^*(X_1 \times X_2)$ defines a map

$$[Z] : \Omega^*(X_1) \to \Omega^*(X_2)$$

by $\alpha \mapsto p_2^*(p_1^*(\alpha) \cap Z)$. By symmetry, we also have a map $[Z^{op}] : \Omega^*(X_2) \to \Omega^*(X_1)$, given by $\beta \mapsto p_1^*(p_2^*(\beta) \cap Z^{op})$, where $Z^{op} \in \Omega^*(X_2 \times X_1)$ is the image of $Z$ by the symmetry morphism $X_1 \times X_2 \to X_2 \times X_1$.

We summarize the notations in the following diagram:

\begin{equation}
\begin{array}{c}
\begin{array}{ccc}
X_1 \times X_2 \times X_1 & \xrightarrow{p_{12}} & X_1 \times X_2 \\
\downarrow & & \downarrow p_{23} \\
X_1 \times X_2 & \xrightarrow{p_2} & X_1 \\
\downarrow & & \downarrow p_1 \\
X_1 & & X_1
\end{array}
\end{array}
\end{equation}

Now we check that $[Z^{op}] \circ [Z] = 1$, i.e., for any $\alpha \in \Omega^*(X_1)$ we have

$$p_{1+} \left(p_2^*(p_1^*(\alpha) \cap Z) \cap Z^{op}\right) = \alpha.$$
The diagonal in \( X_i \times X_i \) will be denoted by \( \Delta_{X_i} \). Similarly we have \( \Delta_{X_{i\eta}} \subseteq X_{i\eta} \times X_{i\eta} \) as the diagonal. We have:

\[
p_1^* \left( \left( p_2^* \left( p_2^* (p_1^* \alpha \cap Z) \right) \right) \cap Z^{\text{op}} \right) = p_1^* \left( \left( \left( p_{23}^* \left( p_{12}^* (p_1^* \alpha \cap Z) \right) \right) \right) \cap Z^{\text{op}} \right)
\]

\[
= (p_1p_{23})^* \left( \left( (p_1p_{13})^* \alpha \cap p_{12}^* Z \cap p_{23}^* Z^{\text{op}} \right) \right)
\]

\[
= (pr_2)^* \left( pr_1^* \alpha \cap p_1^* Z \cap p_{23}^* Z^{\text{op}} \right)
\]

\[
= (pr_2)^* (pr_1^* \alpha \cap \Delta_{X_1})
\]

\[
= \alpha.
\]

The only less clear equality is the second last one. It follows from the following lemma.

**Lemma 4.6.3.** Notations as in the diagram, we have:

\[
p_{13}^* (p_{12}^* Z \cap p_{23}^* Z^{\text{op}}) = \Delta_{X_1} \in \Omega(X_1 \times X_1).
\]

**Proof.** Consider the same diagram as (4.8) with \( X_i \) replaced by \( X_{i\eta} \). We make the convention here in the proof that for any map \( p \) in diagram (4.8), the corresponding map for generic fibers will be denoted by \( \tilde{p} \). Note that

\[
\tilde{p}_{13}^* (\tilde{p}_{12}^* \Delta \cap \tilde{p}_{23}^* \Delta^{\text{op}}) = \Delta_{X_{i\eta}} \in \Omega((X_i)_{\eta} \times (X_i)_{\eta}).
\]

Applying the specialization map \( \sigma \) on both sides, we get:

\[
p_{13}^* (p_{12}^* Z \cap p_{23}^* Z^{\text{op}}) = \Delta_{X_1} \in \Omega(X_1 \times X_1).
\]

Note that the following proposition has no direct counterpart in the Chow theory.

Let \( \pi_i : \Omega^*(X_i) \rightarrow \Omega^*(k) \) be the structure map of \( X_i \rightarrow k \), where \( i = 1, 2 \).

**Proposition 4.6.4.** Let \( [Z] : \Omega^*(X_1) \rightarrow \Omega^*(X_2) \) be the map constructed as above. We have,

1. \( [Z](1_{X_1}) = 1_{X_2} \),

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2. $\pi_1(1 \mathcal{X}_1) = \pi_2(1 \mathcal{X}_2)$.

**Proof.** We have

$$\tilde{p}_2*(\tilde{p}_1^*(1(1(\mathcal{X}_1)_\eta) \cap \Delta)) = 1_{(\mathcal{X}_2)_\eta} \in \Omega((\mathcal{X}_2)_\eta).$$

Applying the specialization map $\sigma$ on both sides, we get:

$$p_2*(p_1^*(1 \mathcal{X}_1) \cap Z) = 1_{\mathcal{X}_2} \in \Omega^*(\mathcal{X}_2).$$

This finishes the proof of (1).

Let $\Psi : \mathcal{X}_1 \cong \mathcal{X}_2$ be the isomorphism as in condition (ii), we have $\Psi_* (1_{\mathcal{X}_1}) = 1_{\mathcal{X}_2}$, and $\tilde{\pi}_1*(1_{\mathcal{X}_1}) = \tilde{\pi}_2*(1_{\mathcal{X}_2})$. Applying the specialization map, (2) follows.

For a vector bundle $E$ on some $X \in \text{Sm}_k$, let $c_{i_1, \ldots, i_r}(E)$ denote the product $c_{i_1}(E) \cdots c_{i_r}(E)$ in $\text{CH}^*(X)$. For $X$ a smooth projective variety over $k$, the Chern number $c^I(X)$ associated to an index $I = (i_1, \ldots, i_r)$ with $\sum_j i_j = \text{dim}_k X$ and $i_j > 0$ is $\text{deg}_k(c^I(T_X))$.

**Corollary 4.6.5.** Let $X_1$ and $X_2$ be two birational symplectic varieties satisfying conditions (i) and (ii). Then $X_1$ and $X_2$ have the same Chern numbers.

**Proof.** For an integer $d > 0$, let $P_d$ denote the number of partitions of $d$. It is well-known that the function $X \mapsto \prod_I c^I(X)$ on smooth projective irreducible $k$-schemes of dimension $d$ over $k$ descends to well-defined homomorphism $c^{*,d} : \Omega^{-d}(k) \to \mathbb{Z}^{P_d}$ via the map sending $X$ to its class $[X] \in \Omega^{-d}(k)$. In fact, $c^{*,d}$ is injective and has image a subgroup of $\mathbb{Z}^{N_d}$ of finite index, but we will not use this. The result is now an immediate consequence of Proposition 4.6.4. 

\[\square\]


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