Three contributions to topology, algebraic geometry and representation theory: homological finiteness of abelian covers, algebraic elliptic cohomology theory, and monodromy theorems in the elliptic setting

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A dissertation submitted to

The Faculty of
the College of Science of
Northeastern University
in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

April 18, 2014

Dissertation directed by

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Acknowledgments

I would like to express my deepest gratitude to my advisor, Prof. Valerio Toledano Laredo, and unofficial advisors, Prof. Alex Suciu and Prof. Marc Levine. During the past 5 years, I have received tremendous support and encouragement from them. They showed me the road and helped to get me started on Mathematics. They were always available for my questions and were very positive, patient and gave generously of their time and knowledge. I wrote my first paper with Prof. Alex Suciu, and gave my first talk under the help of Prof. Alex Suciu. I am still benefit from his way of thinking and his attitude to mathematics. I am very grateful to Prof. Valerio Toledano Laredo for his enormous help, who treats everything seriously to make it perfect and always reminds me the seminars, workshops, talks, deadlines and etc. He has done me a great favor (even spent more time than me) in the process of finding me a postdoctoral position.

I would like to thank Prof. Pavel Etingof and Prof. Ivan Losev for being members of the committee. Their enthusiasm and passion for mathematics always inspires me. Their responsible, humor and pizza make the seminar enjoyable and memorable. I am particularly grateful to Prof. Pavel Etingof for encouragement and insights with admiration, who provides valuable advices to the work and generously shares his ideas, and always ready to help.

I would also like to thank my professors at Northeastern University, Prof. Venkatraman Lakshmibai, Prof. Alina Marian, Prof. Jonathan Weitsman, Prof. Ben Webster, Prof. Jerzy Weyman and Prof. Andrei Zelevinsky for offering excellent courses and seminars over the years, for their guidance and help.
In addition, I would like to give many thanks to my friends and colleagues, Gufang, Andrea, Sachin, Salvo, Shih-Wei, Yinbang, Andy, Kavita, Federico, Thomas, Jason, Andras, Nate, Jeremy, Barbara, Simone, Adina, Chen, Brian, Rueven, Floran, Undine, Nicholas, Ryan and Ryan, Gouri, Anupam, Jose, Rahul, Liang, He, Ruoran, Huijun, Sasha, Saif, Andrew, Peng, Yong, Thi, Toan, Lei, Annabelle, Tian, Jiuzu, Yi, John, Lee-Peng, and many others. Many years with them, full of happiness and joy. Especially, thanks Gufang and Sasha who are the potentially speakers whenever I couldn’t find enough speakers of the graduate student seminar, thanks Andrea and Hanai for sharing their wonderful templates, thanks Shih-Wei for liking every post on my facebook.

And finally, I would like to thank my family, my parents, my sister, my brother in-law, my lovely niece for their love and support. I am particularly grateful to my fiance, and collaborator Gufang, for cooking up good math and delicious food for us over the past 5 years.

Yaping Yang

Northeastern University

April 182014
Abstract of Dissertation

My doctoral work consists of three projects.

The first project is joint work with A. Suciu and G. Zhao described in more details in Chapter 2 and 3 of this dissertation. Chapter 2 is based on the paper [82], where we exploit the classical correspondence between finitely generated abelian groups and abelian complex algebraic reductive groups to study the intersection theory of translated subgroups in an abelian complex algebraic reductive group, with special emphasis on intersections of (torsion) translated subtori in an algebraic torus.

Chapter 3 is based on the paper [83], we present a method for deciding when a regular abelian cover of a finite CW-complex has finite Betti numbers. To start with, we describe a natural parameter space for all regular covers of a finite CW-complex $X$, with group of deck transformations a fixed abelian group $A$, which in the case of free abelian covers of rank $r$ coincides with the Grassmanian of $r$-planes in $H^1(X, \mathbb{Q})$. Inside this parameter space, there is a subset $\Omega^i_A(X)$ consisting of all the covers with finite Betti numbers up to degree $i$.

Building on work of Dwyer and Fried, we show how to compute these sets in terms of the jump loci for homology with coefficients in rank 1 local systems on $X$. For certain spaces, such as smooth, quasi-projective varieties, the generalized Dwyer–Fried invariants that we introduce here can be computed in terms of intersections of algebraic subtori in the character group. For many spaces of interest, the homological finiteness of abelian covers can be tested through the corresponding free abelian covers. Yet in general, abelian covers exhibit different homological finiteness properties than their free abelian counterparts.
The second project is joint work with M. Levine and G. Zhao described in more details in Chapter 4 of this dissertation, which is based on the paper [91]. We define the algebraic elliptic cohomology theory coming from Krichever’s elliptic genus as an oriented cohomology theory on smooth varieties over an arbitrary perfect field. We show that in the algebraic cobordism ring with rational coefficients, the ideal generated by differences of classical flops coincides with the kernel of Krichever’s elliptic genus. This generalizes a theorem of B. Totaro in the complex analytic setting.

The third project consists of two parts. Chapter 5 is joint work with N. Guay, where we give a double loop presentation of the deformed double current algebras, which are deformations of the central extension of the double current algebras \( \mathfrak{g}[u, v] \), for a simple Lie algebra \( \mathfrak{g} \). We prove some nice properties of the algebras using the double loop presentation. Especially, we construct a central element of the deformed double current algebra.

Chapter 6 is joint work with V. Toledano Laredo. In [10], Calaque-Enriquez-Etingof constructed the universal KZB equation, which is a flat connection on the configuration space of \( n \) points on an elliptic curve. They show that its monodromy yields an isomorphism between the completions of the group algebra of the elliptic braid group of type \( A_{n-1} \) and the holonomy algebra of coefficients of the KZB connection. We generalized this connection and the corresponding formality result to an arbitrary root system in [91]. We also gave two concrete incarnations of the connection: one valued in the rational Cherednik algebra of the corresponding Weyl group, the other in the double deformed current algebra \( D(\mathfrak{g}) \) of the corresponding Lie algebra \( \mathfrak{g} \). The latter is a deformation of the double current algebra \( \mathfrak{g}[u, v] \) recently defined by Guay in [34, 35], and gives rise to an elliptic version of the Casimir connection.
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Disclaimer

I hereby declare that the work in this thesis is that of the candidate alone, except where indicated in the text, and as described below.


Chapter 4 is based on a joint work with Dr. Marc N. Levine and Gufang Zhao, published as [56] *Algebraic Elliptic cohomology theory and flops I*, 33 pages, available at arXiv:1311.2159.

Chapter 5 is based on a joint work with Dr. Nicolas Guay, published as [36] *On deformed double current algebras*, preprint.

Chapter 6 is based on a joint work with Dr. Valerio Toledano Laredo, published as [91] *Universal KZB equations for arbitrary root systems and the elliptic Casimir connection*, 45 pages, preprint.
Chapter 1

Introduction

1.1 Brief Overview

My doctoral work consists of three parts described in more details in Chapter 2, 3, 4, 5, 6 respectively of this dissertation.

The first project is joint work with A. Suciu and G. Zhao. In [70, 81], S. Papadima and A. Suciu developed a method for determining when a regular free abelian cover of a CW-complex $X$ has finite Betti numbers. They showed that it can be tested through the characteristic variety of $X$, which are the jump loci for the cohomology with coefficients in rank 1 local systems on $X$. We generalized this method to the set of regular abelian covers of $X$ in [82, 83].

The second project is joint work with M. Levine and G. Zhao. Totaro in [84] showed that the kernel of the elliptic genus from the complex cobordism ring to the elliptic cohomology ring is generated by difference of classical flops. In [56], we generalized Totaro’s theorem to the algebraic setting. Our proof can be applied to a field with arbitrary characteristic.

The third project is joint work with V. Toledano Laredo. In [10], Calaque-Enriquez-Etingof constructed the universal KZB equation, which is a flat connection on the configuration space of $n$ points on an elliptic curve. They show that its monodromy yields an
isomorphism between the completions of the group algebra of the elliptic braid group of type $A_{n−1}$ and the holonomy algebra of coefficients of the KZB connection. We generalized this connection and the corresponding formality result to an arbitrary root system in [91]. We also gave two concrete incarnations of the connection: one valued in the rational Cherednik algebra of the corresponding Weyl group, the other in the double deformed current algebra $D(\mathfrak{g})$ of the corresponding Lie algebra $\mathfrak{g}$. The latter is a deformation of the double current algebra $\mathfrak{g}[u,v]$ recently defined by Guay in [34, 35], and gives rise to an elliptic version of the Casimir connection.

1.2 Characteristic varieties and Betti numbers of abelian covers

1.2.1 Motivations

Let $X$ be a connected finite CW-complex. Let $G = \pi_1(X, x_0)$ be the fundamental group. Consider the regular covers of $X$, with group of deck transformations $A$. In the case when $A \cong \mathbb{Z}^r$, the parameter set for regular, $A$-covers can be identified with $\text{Gr}_r(\mathbb{Q}^n)$, the Grassmannian of $r$-planes in $H^1(X, \mathbb{Q}) \cong \mathbb{Q}^n$.

In a foundational paper [18], Dwyer and Fried isolated the subsets $\Omega^i_r(X)$ of the Grassmannian $\text{Gr}_r(\mathbb{Q}^n)$, consisting of those covers for which the Betti numbers up to degree $i$ are finite.

The Dwyer–Fried sets $\Omega^i_r(X)$ have been studied in depth in [70, 81], using the characteristic varieties of $X$. These varieties, $V^i(X)$, are Zariski closed subsets of the character group $\hat{H} = \text{Hom}(H, \mathbb{C}^*)$, where $H = H_1(X, \mathbb{Z})$; they consist of those rank 1 local systems on $X$ for which the corresponding cohomology groups do not vanish, for some degree less or equal to $i$. 
Theorem 1.2.1 (Papadima–Suciu [70, 81]). Let $X$ be a connected, finite CW-complex, and let $H = H_1(X, \mathbb{Z})$. Then,

$$\Omega^i_r(X) = \{ P \in \text{Gr}_r(\mathbb{Q}^n) \mid \exp(P \otimes \mathbb{C}) \cap W^i(X) \text{ is finite} \},$$

where $\exp(P \otimes \mathbb{C})$ lies inside the identity component $\hat{H}^0$ of the character group $\text{Hom}(H, \mathbb{C}^\ast)$ and $W^i(X) = V^i(X) \cap \hat{H}^0$ is the Alexander variety of $X$.

The above theorem gives a method for deciding when a regular free abelian cover of $X$ has finite Betti numbers. In particular, when the Alexander variety $W^i(X)$ of $X$ is finite, all regular, free abelian covers of $X$ have finite Betti numbers.

A natural question is the following.

**Question 1.2.2.** For any quotient group $A$ of $\pi_1(X, x_0)$, is there a similar description of the set of regular $A$-covers of $X$ for which the Betti numbers are finite?

### 1.2.2 Results achieved

In the joint papers [82, 83] with Alex Suciu, and Gufang Zhao, we answered the Question 1.2.2 in the case when $A$ is an abelian group.

The regular covers of $X$ with group of deck transformations isomorphic to $A$ can be parameterized by the set

$$\Gamma(G, A) = \text{Epi}(G, A) / \text{Aut}(A),$$

where $\text{Epi}(G, A)$ is the set of all epimorphisms from $G$ to $A$ and $\text{Aut}(A)$ is the group of automorphisms of $A$, acting on $\text{Epi}(G, A)$ by composition. For an epimorphism $\nu: G \to A$, we write its class in $\Gamma(G, A)$ by $[\nu]$, and the corresponding cover by $X^\nu \to X$.

We first identify this set $\Gamma(G, A)$ with a (set-theoretical) twisted product over a rational Grassmannian, whose fiber is a concrete finite set. We define the generalized Dwyer–Fried invariants of $X$ to be the subsets $\Omega^i_A(X)$ of $\Gamma(G, A)$ consisting of those regular $A$-covers having finite Betti numbers up to degree $i$. In the case when $A$ is a finitely generated (not necessarily
torsion-free) abelian group, we establish a similar formula as Theorem 1.2.1, computing the invariants $\Omega^i_A(X)$, viewed now as subsets of $\Gamma(G, A)$, in terms of the characteristic varieties of $X$.

**Theorem 1.2.3** (Suciu–Yang–Zhao [83]). Let $X$ be a connected, finite CW-complex, and let $H = H_1(X, \mathbb{Z})$. Suppose $\nu: H \twoheadrightarrow A$ is an epimorphism to an abelian group $A$. Then

$$\Omega^i_A(X) = \{[\nu] \in \Gamma(H, A) \mid \text{im}(\hat{\nu}) \cap V^i(X) \text{ is finite}\},$$

where $\hat{\nu}: \hat{A} \to \hat{H}$ is the induced morphism between character groups.

As shown by Arapura [3], the characteristic varieties of a connected, smooth, quasi-projective variety $X$ consist of translated subtori of the character torus $\hat{H}$. Understanding the way translated subtori intersect gives valuable information on the Betti numbers of regular, abelian covers of $X$.

In [82], building on the approach taken by E. Hironaka in [38], we use Pontrjagin duality of algebraic subgroups of $\hat{H}$ and subgroups of $H$ to study intersections of translated subtori in a complex algebraic torus. This allows us to decide whether a finite collection of translated subgroups intersect non-trivially, and, if so, what the dimension of their intersection is.

Use the method developed in [82], we are able to compare generalized Dwyer–Fried invariants $\Omega^i_A(X)$ of $X$ with their classical counterparts $\Omega^i_r(X)$. Topologically, this comparison gives a criterion whether the finiteness of the Betti numbers of an $A$-cover can be tested through the corresponding $A/\text{Tors}(A)$-cover. Some nice classes of spaces to which our theory applies is that of toric complexes and quasi-projective varieties.

### 1.2.3 Organization of Chapter 3

Chapter 3 is organized as follows:

In §3.1 and §3.2, we describe the structure of the parameter set $\Gamma(H, A)$ for regular $A$-covers of a finite, connected CW-complex $X$ with $H_1(X, \mathbb{Z}) = H$, while in §3.3 we define the
generalized Dwyer–Fried invariants $\Omega^i_A(X)$, and study their basic properties.

In §3.4, we review the Pontryagin correspondence between subgroups of $H$ and algebraic subgroups of the character group $\hat{H}$, while in §3.5 we associate to each subvariety $W \subset \hat{H}$ a family of subgroups of $H$ generalizing the exponential tangent cone construction. In §3.6 and §3.7, we introduce and study several subsets of the parameter set $\Gamma(H, A)$, which may be viewed as analogues of the special Schubert varieties and the incidence varieties from classical algebraic geometry.

In §3.8 we revisit the Dwyer–Fried theory in the more general context of (not necessarily torsion-free) abelian covers, while in §3.9 we show how to determine the sets $\Omega^i_A(X)$ in terms of the jump loci for homology in rank 1 local systems on $X$. In §3.10, we compare the Dwyer–Fried invariants $\Omega^i_A(X)$ with their classical counterparts, $\Omega^i(X)$, while in §3.11 we discuss in more detail these invariants in the case when rank $A = 1$.

Finally, in §3.12 we study the situation when all irreducible components of the characteristic varieties of $X$ are (possibly translated) algebraic subgroups of the character group, while in §3.13 we consider the particular case when $X$ is a smooth, quasi-projective variety.

1.3 The algebraic elliptic cohomology theory

1.3.1 Motivations

In the category of differentiable manifolds, the cohomology theories have been studied by topologist for a long time, e.g., the singular cohomology groups, topological K-theory and etc.

The analog cohomology theories in algebraic geometry are also important tools to understand algebraic varieties. Let $k$ be a field and $\text{Sm}_k$ be the category of smooth, quasi-projective schemes over $k$. Let $R^*$ denote the category of commutative, graded rings with unit. One can define the oriented cohomology theory on $\text{Sm}_k$ to be an additive functor $A^* : \text{Sm}_k^{\text{op}} \to R^*$
with smooth pull-backs and proper push-forwards, which satisfy some formal Axioms. See [54] for details. The fundamental insight of Quillen is that it is not true in general that one has the formula

$$c_1(L \otimes M) = c_1(L) + c_1(M),$$

for line bundles $L$ and $M$ over the same base $X$.

In general, the power series

$$F_A(c_1(L), c_1(M)) := c_1(L \otimes M)$$

defines a formal group law on the ring $A^*(k)$. The Chow ring $X \mapsto \text{CH}^*(X)$ is a basic example of an oriented cohomology theory on $\text{Sm}_k$. The formal group law on $\text{CH}^*(k) = \mathbb{Z}$ is the additive formal group law $F_a(u, v) = u + v$. The Grothendieck $K^0$ functor $X \mapsto K^0(X)$ is another fundamental example of an oriented cohomology theory. The formal group law on $K^*(k) = \mathbb{Z}[[\beta, \beta^{-1}]]$ is the multiplicative formal group law $F_m(u, v) = u + v - \beta uv$.

Assume $k$ has characteristic zero. Levine and Morel construct the universal oriented cohomology theory, whose formal group law is the universal formal group law $F_\Omega$ on the Lazard ring $\text{Laz} := \mathbb{Z}[p_1, p_2, \cdots]$.

**Theorem 1.3.1** (Levine and Morel). Assume $k$ has characteristic zero. Then, there exists a universal oriented cohomology theory on $\text{Sm}_k$, denoted by

$$X \mapsto \Omega^*(X),$$

which called the algebraic cobordism. Thus, given an oriented cohomology theory $A^*$ on $\text{Sm}_k$, there is a unique morphism $\Omega^* \to A^*$ of oriented cohomology theories.

Let the power series $\lambda(x) \in A^*(\{\text{pt}\}) \otimes \mathbb{Q}[x]$ be the exponential of the formal group law $F$, that is,

$$F(u, v) = \lambda(\lambda^{-1}(u) + \lambda^{-1}(v)).$$
The map \( \phi : \Omega^*(\{pt\}) \to A^*(\{pt\}) \) called the Hirzebruch genus, is given by the Hirzebruch characteristic power series \( Q(x) = \frac{x}{\lambda(x)} \); more explicitly, the map \( \phi \) is defined to be

\[
\phi(X) := \int_X \left( \prod_{i=1}^n \frac{x_i}{\lambda(x_i)} \right),
\]

where \( x_1, \ldots, x_n \) are the Chern roots of \( X \).

In the case the \( A^* \) is the \( K \)-theory. The Hirzebruch genus specializes to the classical Todd genus with characteristic power series

\[
Q(x) = \frac{x}{\lambda(x)} = \frac{\beta x}{1 - e^{-\beta x}}.
\]

Let

\[
\phi_E : \Omega^* \to \mathbb{Q}(e^{2\pi i z})[e^{2\pi i \tau}, k]
\]

be the elliptic genus studied by Krichever in [49]. The Krichever elliptic genus has the rigidity property, see [49] and [43]. Use this rigidity property, Höhn and Totaro studied the kernel of the genus \( \phi_E \), see [43] and [84].

**Theorem 1.3.2** (Totaro). Let \( I \) be the ideal in the complex cobordism ring \( MU^* \otimes \mathbb{Q} \), which is additively generated by differences \( X_1 - X_2 \), where \( X_1 \) and \( X_2 \) are smooth projective varieties related by a classical flop. Then the complex elliptic genus, viewed as a ring homomorphism

\[
\phi_E : MU^* \otimes \mathbb{Q} \to \mathbb{Q}[x_1, x_2, x_3, x_4],
\]

is surjective with kernel equal to \( I \).

It is worth mentioning that in Totaro’s work, the proof depends on the category of weakly complex manifolds, and topological constructions which apparently does not lend themselves to a field of positive characteristic.

### 1.3.2 Results achieved

In the joint paper [56] with Levine and Zhao, we study the algebraic version of the elliptic cohomology theory over an arbitrary field \( k \). Based on a theorem of Landweber, any formal
group law gives rise to a \( \mathbb{P}^1 \)-spectrum, if certain flatness assumption called \textit{Landweber exactness} is satisfied. The elliptic formal group law corresponding to the Krichever’s elliptic genus is defined on the ring \( \mathbb{Z}[a_1, a_2, a_3, a_4] \), with explicit descriptions of the four elements \( a_i \). We show that this formal group law is Landweber exact if we enlarge the coefficient ring by taking \( \text{Ell}[1/2] = \mathbb{Z}[1/2][a_1, a_2, a_3, a_4][\Delta^{-1}] \), where \( \Delta \) is the discriminant. As a consequence, the elliptic cohomology theory can be defined as \( \text{Ell}^*(X) := \Omega^*(X) \otimes_{\text{Laz}} \text{Ell}[1/2] \).

We extended Totaro’s Theorem 1.3.2 to the algebraic setting, which also works when the base field \( k \) has positive characteristic. In the case when \( k \) has positive characteristic, though the algebraic cobordism theory \( \Omega^* \) is not constructed. There is another theory called \textit{the algebraic cobordism spectra} \( \text{MGL}^{*,*} \) in the motivic stable homotopy category of \( \mathbb{P}^1 \)-spectra of Morel and Voevodsky (see [63]), whose geometric part \( \bigoplus_n \text{MGL}^{2n,n}(X) \) coincides with the algebraic cobordism \( \Omega^*(X) \) when the base field has characteristic zero. The geometric part \( \bigoplus_n \text{MGL}^{2n,n}(\{\text{pt}\}) \) is denoted by \( \Omega^*(\{\text{pt}\}) \).

The following is the main theorem of the project we obtained so far.

\textbf{Theorem 1.3.3 (Levine–Yang–Zhao [56]).} \textit{The kernel of the algebraic elliptic genus}

\[ \phi_Q : \Omega^*(\{\text{pt}\}) \otimes \mathbb{Q} \to \text{Ell}^*(\{\text{pt}\}) \otimes \mathbb{Q} \]

\textit{is generated by the difference of classical flops, and its image is the free polynomial ring}

\[ \mathbb{Q}[a_1, a_2, a_3, a_4]. \]

The main tools we use to show the above theorem are

- the double point relations formula of \( \Omega^* \) found by Levine and Pandharipande, see [55].
- the Quillen’s push-forward formula, see [77].

For \( F \subset X \) be a subscheme of \( X \), the double point relation yields the following blow-up formula in \( \Omega^* \):

\[ X = Bl_F X + \mathbb{P}(N_F X \oplus \mathcal{O}) - \mathbb{P}(\mathcal{O}_{N_F X}(1) \oplus \mathcal{O}). \]
where $Bl_F X$ is blow-up of $X$ along $F$, and $N_F X$ is the normal bundle of $F$. This allows us to express the difference of flops $X_1 - X_2$ in terms of linear combinations of projective spaces. Then, we pushforward those projective spaces to $\text{Ell}^*\{\text{pt}\}$ using Quillen’s formula, which yields zero due to a classical identity of sigma-functions. Thus, $(X_1 - X_2) \subset \ker \phi_\mathbb{Q}$.

### 1.3.3 Future directions

In this section, we describe future directions of the ongoing project of the author joint with Levine and Zhao.

One may wonder whether the Theorem 1.3.3 holds over then coefficient ring $\mathbb{Z}$ without tensoring $\mathbb{Q}$. In the category of complex manifolds, as pointed out by Totaro that it is more natural to work with the $SU$-cobordism ring $\text{MSU}^*$ rather than the cobordism ring $\text{MU}^*$, for example because the image of $\text{MU}^*$ under the complex elliptic genus is not finitely generated, although after tensoring with $\mathbb{Q}$ it becomes the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, x_4]$.

As shown by Totaro in [84],

$$\text{MSU}^* \otimes \mathbb{Z}[1/2]/SU\text{-flops} \cong \mathbb{Z}[1/2][x_2, x_3, x_4]$$

In algebraic setting, for an integral questions such as this, the analogous of $SU$-cobordism is the algebraic cobordism spectra $\text{MSL}^{*,*}$, see [74] for the definition. To extend the Theorem 1.3.3 to the integral setting, we need to know the coefficient ring of $\text{MSL}^{*,*}(\{\text{pt}\})$, at least the geometric part $\text{MSL}(\{\text{pt}\}) := \oplus_n \text{MSL}^{2n,n}(\{\text{pt}\})$. Based on the result of complex $SU$-cobordism ring $\text{MSU}^*$, see [84], we make the following conjecture.

**Conjecture 1.3.4.** The coefficient ring of the MSL-cobordism ring

$$\text{MSL}(\{\text{pt}\}) \otimes \mathbb{Z}[1/2] \cong \mathbb{Z}[1/2][x_2, x_3, x_4, \ldots]$$
where the polynomial generators $x_i$ are determined as follows. For $X \in \text{MGL}({\{\text{pt}\}}) \otimes \mathbb{Z}[1/2]$ is a polynomial generator of $\text{MSL}({\{\text{pt}\}}) \otimes \mathbb{Z}[1/2]$ in degree $n$ if and only if

$$s^n(X) = \begin{cases} 
\pm p(\text{a power of } 2), & \text{if } n \text{ is a power of an odd prime } p; \\
\pm p(\text{a power of } 2), & \text{if } n + 1 \text{ is a power of an odd prime } p; \\
\pm (\text{a power of } 2), & \text{otherwise.}
\end{cases}$$

where $s^n(X)$ is the Chern number of the tangent bundle of $X$, that is,

$s^n(X) := \langle x_1^n + \cdots + x_n^n, [X] \rangle$, with $x_i$ being the Chern roots of the tangent bundle of $X$.

Since the $SU$-cobordism ring is obtained by Novikov using the Adams spectral sequences, to prove the Conjecture 1.3.4, it would be helpful to compare the classical Adams spectral sequence with the Motivic Adams spectral sequences via the Betti and étale realizations.

Conjecture 1.3.4 implies the following conjecture.

**Conjecture 1.3.5.** The kernel of the algebraic elliptic genus on $\text{MSL}^*(\{\text{pt}\}) \otimes \mathbb{Z}[1/2]$ is equal to the ideal of $\text{SL}$-flops. The quotient ring is a polynomial ring $\mathbb{Z}[1/2][a_2, a_3, a_4]$.

Nakajima quiver varieties form a large class of complex algebraic varieties arising from representation theory. The cotangent bundle of flag varieties, Hilbert scheme of points on surfaces are special cases of the Nakajima quiver varieties.

In [64], Nakajima used these varieties to provide a geometric construction of the irreducible integrable representations of the Kac-Moody Lie algebras $\mathfrak{g}$. In other words, there is an action of the Kac-Moody Lie algebras $\mathfrak{g}$ on the top degree of the Borel-Moore homology of the Nakajima quiver variety associated to the Dynkin quiver of $\mathfrak{g}$ and this action gives all the irreducible integrable representations of $\mathfrak{g}$.

In [94], the action of the Yangians on the equivariant Borel-Moore homology of quiver varieties is established by Varagnolo. In [65], Nakajima constructed the action of the quantum loop algebra on the equivariant K-theory of quiver varieties. Those actions give geometric
realizations of the algebras Yangians and quantum loop algebra respectively. As a corol-
lary, there is a set-theoretic map from the category of finite dimensional representations of
Yangians \(\text{Rep}_{fd}(Y_{\hbar}(\mathfrak{g}))\) to the category of finite dimensional representations of quantum loop
algebra \(\text{Rep}_{fd}(U_{\hbar}(L\mathfrak{g}))\).

Motivated by this work, Gautam and Toledano Laredo [30] constructed a functor from
\(\text{Rep}_{fd}(Y_{\hbar}(\mathfrak{g}))\) to \(\text{Rep}_{fd}(U_{\hbar}(L\mathfrak{g}))\), when \(\hbar\) is a formal parameter, and showed it’s an equivalence
of appropriate subcategories. In [29], Gautam and Toledano Laredo constructed a functor
from \(\text{Rep}_{fd}(Y_{\hbar}(\mathfrak{g}))\) to \(\text{Rep}_{fd}(U_{\hbar}(L\mathfrak{g}))\) for numerical parameter \(\hbar\), using monodromy of abelian
difference equations.

Guided by the same principal, one may ask the following question.

**Question 1.3.6.** Is there a quantum algebra action on the equivariant elliptic cohomology
of the Nakajima quiver varieties?

It is expected that the elliptic quantum groups act on the equivariant elliptic cohomol-
yogy of quiver varieties, as suggested to the author by Toledano Laredo. While the elliptic
quantum groups have not been well-studied yet. The twisted constructions of the equi-
variant elliptic cohomology theory (that is, the equivariant elliptic cohomology theory is the
equivariant Chow theory tensoring with the elliptic cohomology ring) constructed in [54]
is supposed to be useful to obtain such an action. The product of the equivariant elliptic
cohomology of the quiver varieties can be computed using the twisted constructions of the
equivariant elliptic cohomology theory.

1.3.4 Organization of Chapter 4

Chapter 4 is organized as follows: In §4.1 we recall some foundational material concerning
oriented cohomology and motivic oriented cohomology. We describe two methods of con-
struction oriented cohomology theories from a given formal group law: specialization and
twisting, and recall the construction of a motivic oriented cohomology theory through spe-
cialization to a Landweber exact formal group law. We use the twisting construction to construct the universal oriented cohomology theory on $\text{Sm}_k$ for $k$ a field of positive characteristic, after restriction to theories with coefficient ring a $\mathbb{Q}$-algebra; for $k$ of characteristic zero, one already has the universal theory among all oriented theories, namely, $\Omega^*$. In §4.2 we apply these results to give our construction of elliptic cohomology as an oriented cohomology theory on $\text{Sm}_k$ for $k$ an arbitrary perfect field, and we prove our main result on the existence of a motivic oriented theory representing elliptic cohomology (Theorem ??). In §4.3 we introduce the double-point cobordism and use this theory to give an explicit description of the difference of two flops in $\text{MGL}^*$-theory. We then apply this formula to show that the difference of two flops vanishes in elliptic cohomology. In §4.4 we use Höhn’s and Totaro’s algebraic computations to prove our main result (Theorem ??) on rational elliptic cohomology and its relation to $\text{MGL}^*_{\mathbb{Q}}$ (we recall Totaro’s computations in an appendix). We conclude in §4.5 with another application of algebraic cobordism, showing that birational smooth projective symplectic varieties that arise from different specializations of generically isomorphic families have the same class in algebraic cobordism, and in particular, have the same Chern numbers (see proposition 4.5.2 and corollary 4.5.5).

1.4 Universal KZB equations and the elliptic Casimir connection

1.4.1 Motivations

The KZ connection

Around 1990, T. Kohno and V. G. Drinfeld proved the Kohno-Drinfeld Theorem, see [48] [19]. Roughly speaking, the theorem states that quantum groups can be used to describe monodromy of certain first order Fuchsian PDEs known as Knizhnik- Zamolodchikov (KZ)
To be more precise, given a simple Lie algebra $\mathfrak{g}$, a representation $V$ of $\mathfrak{g}$ and a positive integer $n$, the KZ equations are the following system of PDEs:

$$
\frac{d\Phi}{dz_i} = \hbar \sum_{i \neq j} \Omega_{ij} \frac{\Phi}{z_i - z_j}, \quad (1.4.1)
$$

where

1. $\Phi$ is a function on the configuration space of $n$-ordered points on $\mathbb{C}$ (denoted by $C(\mathbb{C}, n)$) with values in $V^\otimes n$.

2. $\Omega_{ij} \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir operator.

3. $\hbar$ is a complex deformation parameter.

This system is integrable and invariant under the natural symmetric group $S_n$ action and hence defines a one-parameter family of monodromy representations of Artin’s braid group $B_n = \pi_1(C(\mathbb{C}, n)/S_n)$. The Kohno-Drinfeld theorem asserts that this representation is equivalent to the $R$-matrix representation of $B_n$ arising from the quantum group $U_\hbar(\mathfrak{g})$.

**The rational Casimir connection**

In subsequent years, J. Millson and V. Toledano Laredo [62, 87], and independently C. De Concini (unpublished) and G. Felder *et al* [26] constructed another flat connection $\nabla_C$, the *Casimir connection* of a simple Lie algebra $\mathfrak{g}$, which is described as follows.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}_{\text{reg}}$ be the complement of the root hyperplanes in $\mathfrak{h}$. For a finite-dimensional $\mathfrak{g}$ module $V$, the Casimir connection $\nabla_C$ is a connection on the trivial vector bundle $V$ of the following form:

$$
\nabla_C := d - \hbar \sum_{\alpha > 0} \frac{d\alpha}{\alpha} C_\alpha,
$$

where
1. the summation is over a set of chosen positive roots of \( \mathfrak{g} \).

2. \( C_\alpha \in \mathfrak{g} \otimes \mathfrak{g} \) is the Casimir operator of the \( \mathfrak{sl}_2 \) subalgebra of \( \mathfrak{g} \) corresponding to the root \( \alpha \).

This connection is flat and equivariant with respect to the Weyl group \( W \), thus, gives rise to a one-parameter family of monodromy representations of the generalized braid group \( B_\mathfrak{g} := \pi_1(\mathfrak{h}_{reg}/W) \).

In this setting a Kohno-Drinfeld type theorem was obtained by V. Toledano Laredo [85, 87, 88, 89, 90], stating that the monodromy of the Casimir connection is described by the quantum Weyl group operators of the quantum group \( U_h(\mathfrak{g}) \).

The trigonometric Casimir connection

V. Toledano Laredo constructed a trigonometric extension of \( \nabla_C \) in [86] recently, now known as the trigonometric Casimir connection. Let \( \mathfrak{h}_{reg} \) be the complement of the root hypertori of a maximal torus \( H \) of the simply connected Lie group \( G \) with Lie algebra \( \mathfrak{g} \).

This trigonometric Casimir connection \( \nabla_{\text{trig},C} \) is a connection on the trivial vector bundle \( \mathfrak{h}_{reg} \times V \), where the fiber \( V \) is a finite-dimensional representation of the Yangian \( Y_h(\mathfrak{g}) \), which is a deformation of \( U(\mathfrak{g}[s]) \). Let \( \{u^i\} \) and \( \{u_i\} \) be the dual basis of \( \mathfrak{h} \) and \( \mathfrak{h}^* \),

\[
\nabla_{\text{trig},C} = d - \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^{\alpha} - 1} \kappa_\alpha - A(u^i) du_i,
\]

is flat and \( W \)-equivariant, where

- \( A(u^i) du_i \) is a translation invariant 1-form on \( H \);

- \( A : \mathfrak{h} \to Y_h(\mathfrak{g}) \) is a linear map such that \( A(u) \equiv u \otimes s \mod \hbar \).

The connection \( \nabla_{\text{trig},C} \) is flat and \( W \)-equivariant and its monodromy yields a one parameter family of monodromy representations of the affine braid group

\[
B_\mathfrak{g}^{\text{Aff}} := \pi_1(\mathfrak{h}_{reg}/W).
\]
By analogy with the monodromy theorem of rational Casimir connection [87], Toledano Laredo formulated the following [86]

**Conjecture 1.4.1** (Toledano Laredo). The monodromy of the trigonometric Casimir connection is described by the quantum Weyl group operators of the quantum loop algebra \( U_h(Lg) \).

Partial results of the Conjecture 1.4.1 was proved by Gautam and Toledano Laredo in [29].

**The elliptic Casimir connection**

My joint project with V. Toledano Laredo [91] is aimed at addressing the following

**Problem 1.4.2.** Construct an elliptic analogue of the Casimir connection, and describe its monodromy in terms of quantum groups.

**The universal KZB connection**

In type \( A_n \), elliptic connections were constructed by Calaque-Enriquez-Etingof [10] in the following context. Let \( C(X, n) \) be the configuration space of \( n \)-distinct points on a Riemann surface \( X \). Bezrukavnikov showed that there is an isomorphism between the completion of the group algebra of \( \pi_1(C(X, n)) \) and that of the universal enveloping algebra of an explicit Lie algebra \( \mathfrak{t}_{g,n} \), where \( g \) is the genus of the Riemann surface \( X \). We call this isomorphism Bezrukavnikov’s isomorphism.

In the case when \( X = \mathcal{E} \) is an elliptic curve, Bezrukavnikov’s isomorphism is constructed explicitly by Calaque-Enriquez-Etingof using the monodromy of a flat connection on a suitable principal bundle over the configuration space \( C(\mathcal{E}, n) \) in [10]. The flat connection, known as the *universal KZB connection*, valued in the Lie algebra \( \mathfrak{t}_{g,n} \), is given by:

\[
\nabla_{\text{KZB}} = d - \sum_{i=1}^{n} \left( \sum_{\{j|j \neq i\}} k(z_i - z_j, \text{ad} x_{\tau}(t_{ij}))dz_i + \sum_{i=1}^{n} y_idz_i, \right) \tag{1.4.2}
\]
where:

1. $z_i$’s are coordinates of $\mathbb{C}^n$,

2. the elements $t_{ij}$, $x_i$, and $y_i$ are generators of $t_{1,n}$,

3. the function $k(z, x|\tau)$ is expressed as a ratio of theta functions. $k(z, x|\tau)$ has a simple pole at $z = 0$ with residue 1, but no pole at $x = 0$.

Calaque-Enriquez-Etingof showed that the monodromy of $\nabla_{KZB}$ induces an isomorphism from the completion of the group algebra of $\pi_1(C(\mathcal{E}, n))$ to the completion of the universal enveloping algebra of $t_{1,n}$.

A similar construction of Bezrukavnikov’s isomorphism for Riemann surfaces of higher genus case using the monodromy of flat connections was done by Enriquez in [22].

### 1.4.2 Results achieved

#### The universal KZB-connection

In joint work with Toledano Laredo, we generalized the universal KZB-connection and the corresponding formality result to the root system associated to an arbitrary simple Lie algebra $\mathfrak{g}$.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $P^\vee \subset \mathfrak{h}$ the corresponding coweight lattice, and $\mathcal{H}_r^{\text{reg}}$ the complement of the root hyperplanes in $\mathfrak{h}/(P^\vee + \tau P^\vee)$, where $\tau$ is a complex number with positive imaginary part. Note that $\mathcal{H}_r^{\text{reg}}$ is a complement of a divisor in the $r$th power of the elliptic curve $\mathcal{E}_r = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$, where $r = \dim \mathfrak{h}$. By definition, $\mathcal{H}_r^{\text{reg}}$ is the elliptic configuration space.

Let $A$ be an algebra endowed with the following data:

- a set of elements $\{t_\alpha\}_{\alpha \in \Phi} \subset A$ such that $t_{-\alpha} = t_\alpha$,

- a linear map $x : \mathfrak{h} \to A$, 

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• a linear map $y : \mathfrak{h} \to A$.

Consider the $A$-valued connection on $\mathcal{H}_{\tau}^{\text{reg}}$ given by

$$\nabla_{\tau} = d - \sum_{\alpha > 0} k(\alpha, \text{ad}(\frac{x_{\alpha^\vee}}{2})|\tau)(t_\alpha)d\alpha + \sum_{i=1}^{n} y(u^i)du_i, \quad (1.4.3)$$

where the summation is over a chosen system of positive roots, and $\alpha^\vee \in \mathfrak{h}$ are the coroots of $\mathfrak{g}$.

In [91], we give a criteria, in terms of relations of the algebra $A$, of the flatness and $W$-equivariance of the connection $\nabla_{\tau}$. This motives the definition of the holonomy Lie algebra $A$ by imposing those relations which are satisfied to make the connection $\nabla_{\tau}$ flat.

**Theorem 1.4.3** (Toledano Laredo–Yang [91]). The connection $\nabla_{\tau,n}$ in (6.2.3) is flat if and only if the following relations holds:

1. For any rank 2 root subsystem $\Psi \subset \Phi$, and $\alpha \in \Psi$,

$$[t_\alpha, \sum_{\beta \in \Psi^+} t_\beta] = 0;$$

2. $[x(u), x(v)] = 0, [y(u), y(v)] = 0$, for any $u, v \in \mathfrak{h}$;

3. $[y(u), x(v)] = \sum_{\gamma \in \Phi^+} \langle v, \gamma \rangle \langle u, \gamma \rangle t_\gamma$.

4. $[t_\alpha, x(u)] = 0, [t_\alpha, y(u)] = 0$, if $\langle \alpha, u \rangle = 0$.

The connection $\nabla_{\tau,n}$ in (6.2.3) is $W$-equivariant, if and only if:

1. $s_i(t_\alpha) = t_{s_i\alpha}$;

2. $s_i(x(u)) = x(s_iu)$;

3. $s_i(y(v)) = y(s_iv)$

We extended the formality result in this case.

**Theorem 1.4.4** (Toledano Laredo–Yang [91]). The induced map $\mu : \mathbb{C}[\pi_1(\mathcal{H}_{\text{reg}})] \to \hat{A}$ is an isomorphism of Hopf algebras.
Rational Cherednik algebras

We give in [91] a concrete incarnation $\nabla_{\text{RCA}}$ of the connection $\nabla_\tau$ with values in the rational Cherednik algebra of $W$. The connection $\nabla_{\text{RCA}}$ is flat and $W$-equivariant. Its monodromy yields a one parameter family of monodromy representations of the elliptic braid group $\pi_1(\mathcal{H}_{\text{reg}}/W)$, which factors through the double affine Hecke algebra of $W$ [14]. Thus, the monodromy induces a functor from the category of finite-dimensional representations of rational Cherednik algebra to the category of finite-dimensional representations of the double affine Hecke algebra.

The elliptic Casimir connection

In [91], we also give another concrete incarnation of the universal connection $\nabla_\tau$ related to the Lie algebra $\mathfrak{g}$. Thus, solving the first part of Problem 1.4.2. This elliptic Casimir connection takes values in the deformed double current algebra $D(\mathfrak{g})$, a deformation of the universal central extension of the double current algebra $\mathfrak{g}[u,v]$ recently introduced by Guay[34, 35]. The construction of a map from the algebra of coefficients of $\nabla_\tau$ to $D(\mathfrak{g})$ relies crucially on a double loop presentation of $D(\mathfrak{g})$. Such a presentation was obtained by Guay for $\mathfrak{sl}_n$ in [34] by relying on Schur-Weyl duality but was otherwise unknown. In work in progress with Guay [36], I obtained the following double loop presentation of the deformed double current algebra $D(\mathfrak{g})$.

**Theorem 1.4.5** (Guay–Yang [36]). The algebra $D(\mathfrak{g})$ is generated by elements $X, K(X), Q(X), P(X)$ such that

- $K(X), X \in \mathfrak{g}$ generate a subalgebra which is an image of $\mathfrak{g} \otimes \mathbb{C}[u]$ with $X \otimes u \mapsto K(X)$;
- $Q(X), X \in \mathfrak{g}$ generate a subalgebra which is an image of $\mathfrak{g} \otimes \mathbb{C}[v]$ with $X \otimes v \mapsto Q(X)$;
- $P(X)$ is linear in $X$, and for any $X, X' \in \mathfrak{g}$, $[P(X), X'] = P[X, X']$. 

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and the following relation holds for all root vectors \( X_{\beta_1}, X_{\beta_2} \in \mathfrak{g} \) with \( \beta_1 \neq -\beta_2 \):

\[
[K(X_{\beta_1}), Q(X_{\beta_2})] = P([X_{\beta_1}, X_{\beta_2}]) - \frac{(\beta_1, \beta_2)}{4} S(X_{\beta_1}, X_{\beta_2}) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}]),
\]

where \( S(a_1, a_2) = a_1 a_2 + a_2 a_1 \in U(\mathfrak{g}) \).

Based on the double loop presentation of the deformed double current algebra \( D_{\lambda}(\mathfrak{g}) \), we define the elliptic Casimir connection as follows:

**Theorem 1.4.6 ([91]).** The elliptic Casimir connection valued in the deformed double current algebra \( D_{\lambda}(\mathfrak{g}) \)

\[
\nabla_{\text{Ell}, C} = -\frac{\lambda}{2} \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(Q(\alpha^\vee)/2)|\tau)(C_\alpha) d\alpha + \sum_{i=1}^{n} K(u^i) du_i,
\]

is a flat and \( W \)-equivariant, where \( C_\alpha \in U(\mathfrak{g}) \) is the Casimir operator.

### 1.4.3 Future directions

#### Extension to the Moduli spaces

The universal KZB-connection \( \nabla_{\text{KZB}} \) (1.4.2) and \( \nabla_{\tau} \) (6.2.3) both depend on the modular parameter \( \tau \). As shown in [10], the KZB-connection \( \nabla_{\text{KZB}} \) (1.4.2) can be extended to the moduli space of elliptic curves with \( n \) marked points. In [91], we extended this result to the universal connection \( \nabla_{\tau} \) (6.2.3).

With regards to the two concrete incarnations described in § 1.4.2 and § 1.4.2, we also show in [91] that the connection \( \nabla_{\text{RCA}} \) with values in the rational Cherednik algebra can be extended to the moduli space of elliptic curves.

**Problem 1.4.7.** Extend the elliptic Casimir connection \( \nabla_{\text{Ell}, C} \) to the moduli space of elliptic curves with \( n \) marked points with values in the deformed double current algebra.

#### Deformed double current algebras

The deformed double current algebras is yet to be fully understood. One open problem is to show that it satisfies the PBW property.
The monodromy of the elliptic Casimir connection

By analogy with the Kohno-Drinfeld theorem [48, 19], and the Conjecture 1.4.1. We state the following

**Conjecture 1.4.8.** The monodromy of the elliptic Casimir connection $\nabla_{\text{Ell},C}$ is described by the quantum Weyl group operators of the quantum toroidal algebra $U_h(\mathfrak{g}^{\text{tor}})$.

One problem in addressing Conjecture 1.4.8 is to find a correct category of representations of $D(\mathfrak{g})$ on which to take monodromy. These should have in particular finite dimensional weight spaces under $\mathfrak{h}$ so that the monodromy is well-defined and be invariant under some action of $SL(2,\mathbb{Z})$ so that the connection has a chance to be extended in the modular direction.

In [31], Gautam and Toledano Laredo constructed an explicit equivalence of categories between finite dimensional representations of the Yangian $Y_h(\mathfrak{g})$ and finite dimensional representations of the quantum loop algebra $U_h(L\mathfrak{g})$, when $\mathfrak{g}$ is a finite dimensional simple Lie algebra. In the case when $\mathfrak{g}$ is a Kac-Moody Lie algebra, the functor in [31] establishes an equivalence between subcategories of the two categories $\mathcal{O}_{\text{int}}(Y_h(\hat{\mathfrak{g}}))$ of the affine Yangian and $\mathcal{O}_{\text{int}}(U_h(\mathfrak{g}^{\text{tor}}))$ of the quantum toroidal algebra.

**Problem 1.4.9.** Construct a functor from a suitable category of representations of the affine Yangian $Y_h(\hat{\mathfrak{g}})$ to a suitable category of the deformed double current algebra $D_h(\mathfrak{g})$.

This should lead us to the correct category of representations of the deformed double current algebra and the quantum toroidal algebra in Problem 1.4.8.

**The $\mathfrak{gl}_k$, $\mathfrak{gl}_n$ duality**

There is a $(\mathfrak{gl}_k, \mathfrak{gl}_n)$ duality between the rational KZ-connection $\nabla_{KZ}$ for $\mathfrak{g} = \mathfrak{gl}_k$ and the rational Casimir connection $\nabla_C$ for $\mathfrak{g} = \mathfrak{gl}_n$ found by Toledano Laredo [87]. Such a duality is a powerful tool to study the monodromy of the Casimir connection. Using this duality,
Toledano Laredo reduces the proof of the monodromy theorem of Casimir connections for $\mathfrak{sl}_n$ to the case of the Kohno-Drinfeld theorem for $\mathfrak{g} = \mathfrak{sl}_k$, see [87] for details. Such a $(\mathfrak{gl}_k, \mathfrak{gl}_n)$ duality also exists between the trigonometric KZ-connection $\nabla_{\text{trig,KZ}}$ for $\mathfrak{g} = \mathfrak{gl}_k$ and the trigonometric Casimir connection $\nabla_{\text{trig,C}}$ for $\mathfrak{g} = \mathfrak{gl}_n$ [29]. Guided by the same principle, we state the following

**Problem 1.4.10.** Establish a $(\mathfrak{gl}_k, \mathfrak{gl}_n)$ duality between the universal KZB connection for $\mathfrak{g} = \mathfrak{gl}_k$ and the elliptic Casimir connection for $\mathfrak{g} = \mathfrak{gl}_n$. 
Chapter 2

Intersections of translated algebraic subtori

2.1 Finitely generated abelian groups and abelian reductive groups

In this section, we describe an order-reversing isomorphism between the lattice of subgroups of a finitely generated abelian group $H$ and the lattice of algebraic subgroups of the corresponding abelian, reductive, complex algebraic group $T$.

2.1.1 Abelian reductive groups

We start by recalling a well-known equivalence between two categories: that of finitely generated abelian groups, $\text{AbFgGp}$, and that of abelian, reductive, complex algebraic groups, $\text{AbRed}$. For a somewhat similar approach, see also [38] and [61].

Let $\mathbb{C}^*$ be the multiplicative group of units in the field $\mathbb{C}$ of complex numbers. Given a group $G$, let $\widehat{G} = \text{Hom}(G, \mathbb{C}^*)$ be the group of complex-valued characters of $G$, with pointwise multiplication inherited from $\mathbb{C}^*$, and identity the character taking constant value $1 \in \mathbb{C}^*$. For a somewhat similar approach, see also [38] and [61].
for all $g \in G$. If the group $G$ is finitely generated, then $\hat{G}$ is an abelian, complex reductive algebraic group.

Note that $\hat{G} \cong \hat{H}$, where $H$ is the maximal abelian quotient of $G$. If $H$ is torsion-free, say, $H = \mathbb{Z}^r$, then $\hat{H}$ can be identified with the complex algebraic torus $(\mathbb{C}^*)^r$. If $A$ is a finite abelian group, then $\hat{A}$ is, in fact, isomorphic to $A$.

Given a homomorphism $\phi: G_1 \to G_2$, let $\hat{\phi}: \hat{G}_2 \to \hat{G}_1$ be the induced morphism between character groups, given by $\hat{\phi}(\rho) = \rho \circ \phi$. Since $\mathbb{C}^*$ is a divisible abelian group, the functor $H \mapsto \hat{H} = \text{Hom}(H, \mathbb{C}^*)$ is exact.

Now let $T$ be an abelian, complex algebraic reductive group. We can then associate to $T$ its weight group, $\hat{T} = \text{Hom}_{\text{alg}}(T, \mathbb{C}^*)$, where the hom set is taken in the category of algebraic groups. It turns out that $\hat{T}$ is a finitely generated abelian group, which can be described concretely, as follows.

According to the classification of abelian reductive groups over $\mathbb{C}$ (cf. [80]), the identity component of $T$ is an algebraic torus, i.e., it is of the form $(\mathbb{C}^*)^r$ for some integer $r \geq 0$. Furthermore, this identity component has to be a normal subgroup. Thus, the algebraic group $T$ is isomorphic to $(\mathbb{C}^*)^r \times A$, for some finite abelian group $A$.

The coordinate ring $\mathcal{O}[T]$ decomposes as

$$\mathcal{O}[(\mathbb{C}^*)^r \times A] \cong \mathcal{O}[(\mathbb{C}^*)^r] \otimes \mathcal{O}[A] \cong \mathbb{C}[\mathbb{Z}^r] \otimes \mathbb{C}[\hat{A}],$$

(2.1.1)

where $\mathbb{C}[G]$ denotes the group ring of a group $G$. Let $\mathcal{O}[T]^*$ be the group of units in the coordinate ring of $T$. By (2.1.1), this group is isomorphic to $\mathbb{C}^* \times \mathbb{Z}^r \times A$, where $\mathbb{C}^*$ corresponds to the non-zero constant functions. Taking the quotient by this $\mathbb{C}^*$ factor, we get isomorphisms

$$\hat{T} \cong \mathcal{O}[T]^*/\mathbb{C}^* \cong \mathbb{Z}^r \times A.$$  

(2.1.2)

Clearly, maxSpec $(\mathbb{C}[\hat{T}]) = \text{Hom}_{\text{alg}}(\mathbb{C}[\hat{T}], \mathbb{C}) = \text{Hom}_{\text{group}}(\hat{T}, \mathbb{C}^*) = T$.

Now let $f: T_1 \to T_2$ be a morphism in $\text{AbRed}$. Then the induced morphism on coordinate rings, $f^*: \mathcal{O}[T_2] \to \mathcal{O}[T_1]$, restricts to a group homomorphism, $f^*: \mathcal{O}[T_2]^* \to \mathcal{O}[T_1]^*$, which
takes constants to constants. Under the identification from (2.1.2), $f^\ast$ induces a homomorphism $\hat{f}: \hat{T}_2 \to \hat{T}_1$ between the corresponding weight groups.

The following proposition is now easy to check.

**Proposition 2.1.1.** The functors $H \sim \hat{H}$ and $T \sim \hat{T}$ establish a contravariant equivalence between the category of finitely generated abelian groups and the category of abelian reductive groups over $\mathbb{C}$.

Recall now that the functor $H \sim \hat{H}$ is exact. Hence, the functor $T \sim \hat{T}$ is also exact.

**Remark 2.1.2.** The above functors behave well with respect to (finite) direct products. For instance, let $\alpha: A \to C$ and $\beta: B \to C$ be two homomorphisms between finitely generated abelian groups, and consider the homomorphism $\delta: A \times B \to C$ defined by $\delta(a, b) = \alpha(a) + \beta(b)$. The morphism $\hat{\delta}: \hat{C} \to \hat{A} \times \hat{B} = \hat{A} \times \hat{B}$ is then given by $\hat{\delta}(f) = (\hat{\alpha}(f), \hat{\beta}(f))$.

### 2.1.2 The lattice of subgroups of a finitely generated abelian group

Recall that a poset $(L, \leq)$ is a **lattice** if every pair of elements has a least upper bound and a greatest lower bound. Define operations $\lor$ and $\land$ on $L$ (called join and meet, respectively) by $x \lor y = \sup\{x, y\}$ and $x \land y = \inf\{x, y\}$.

The lattice $L$ is called **modular** if, whenever $x < z$, then $x \lor (y \land z) = (x \lor y) \land z$, for all $y \in L$. The lattice $L$ is called **distributive** if $x \lor (y \land z) = (x \lor y) \land (x \lor z)$ and $x \land (y \lor z) = (x \land y) \lor (x \land z)$, for all $x, y, z \in L$. A lattice is modular if and only if it does not contain the pentagon as a sublattice, whereas a modular lattice is distributive if and only if it does not contain the diamond as a sublattice.

Finally, $L$ is said to be a **ranked lattice** if there is a function $r: L \to \mathbb{Z}$ such that $r$ is constant on all minimal elements, $r$ is monotone (if $\xi_1 \leq \xi_2$, then $r(\xi_1) \leq r(\xi_2)$), and $r$ preserves covering relations (if $\xi_1 \leq \xi_2$, but there is no $\xi$ such that $\xi_1 < \xi < \xi_2$, then $r(\xi_2) = r(\xi_1) + 1$).
Given a group $G$, the set of subgroups of $G$ forms a lattice, $\mathcal{L}(G)$, with order relation given by inclusion. The join of two subgroups, $\gamma_1$ and $\gamma_2$, is the subgroup generated by $\gamma_1$ and $\gamma_2$, and their meet is the intersection $\gamma_1 \cap \gamma_2$. There is unique minimal element—the trivial subgroup, and a unique maximal element—the group $G$ itself. The lattice $\mathcal{L}(G)$ is distributive if and only if $G$ is locally cyclic (i.e., every finitely generated subgroup is cyclic). Similarly, one may define the lattice of normal subgroups of $G$; this lattice is always modular. We refer to [79] for more on all this.

Now let $H$ be a finitely generated abelian group, and let $\mathcal{L}(H)$ be its lattice of subgroups. In this case, the join of two subgroup, $\xi_1$ and $\xi_2$, equals the sum $\xi_1 + \xi_2$. By the above, the lattice $\mathcal{L}(H)$ is always modular, but it is not a distributive lattice, unless $H$ is cyclic. Furthermore, $\mathcal{L}(H)$ is a ranked lattice, with rank function $\xi \mapsto \text{rank}(\xi) = \dim_{\mathbb{Q}}(\xi \otimes \mathbb{Q})$ enjoying the following property: $\text{rank}(\xi_1) + \text{rank}(\xi_2) = \text{rank}(\xi_1 \wedge \xi_2) + \text{rank}(\xi_1 \vee \xi_2)$.

### 2.1.3 The lattice of algebraic subgroups of a complex algebraic torus

Now let $T$ be a complex abelian reductive group, and let $(\mathcal{L}_{\text{alg}}(T), \leq)$ be the poset of algebraic subgroups of $T$, ordered by inclusion. It is readily seen that $\mathcal{L}_{\text{alg}}(T)$ is a ranked modular lattice. For any algebraic subgroups $P_1$ and $P_2$ of $T$, the join $P_1 \vee P_2 = P_1 \cdot P_2$ is the algebraic subgroup generated by the two subgroups $P_1$ and $P_2$, while the meet $P_1 \wedge P_2 = P_1 \cap P_2$ is the intersection of the two subgroups. Furthermore, the rank of a subgroup $P$ is its dimension.

The next theorem shows that the natural correspondence from Proposition 2.1.1 is lattice-preserving. Recall that, if $H$ is a finitely generated abelian group, the character group $\hat{H} = \text{Hom}_{\text{group}}(H, \mathbb{C}^*)$ is an abelian reductive group over $\mathbb{C}$, and conversely, if $T$ is an abelian reductive group, the weight group $\check{T} = \text{Hom}_{\text{alg}}(T, \mathbb{C}^*)$ is a finitely generated abelian group.

**Theorem 2.1.3.** Suppose $H \cong \check{T}$, or equivalently, $T \cong \hat{H}$. There is then an order-reversing isomorphism between the lattice of subgroups of $H$ and the lattice of algebraic subgroups of $T$.\text{\textit{...}}
Proof. For any subgroup $\xi \leq H$, let
\[ V(\xi) = \text{maxSpec}(\mathbb{C}[H/\xi]) \] (2.1.3)
be the set of closed points of $\text{Spec}(\mathbb{C}[H/\xi])$. Clearly, the variety $V(\xi)$ embeds into $\text{maxSpec}(\mathbb{C}[H]) \cong T$ as an algebraic subgroup. This subgroup can be naturally identified with the group of characters of $H/\xi$, that is, $V(\xi) \cong \hat{H}/\xi$, or equivalently,
\[ \hat{\xi} \cong T/V(\xi). \] (2.1.4)

For any algebraic subset $W \subset T$, define
\[ \epsilon(W) = \ker \left( \text{Hom}_{\text{alg}}(T, \mathbb{C}^*) \twoheadrightarrow \text{Hom}_{\text{alg}}(W, \mathbb{C}^*) \right). \] (2.1.5)

Write $T = (\mathbb{C}^*)^r \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_s}$, where $\mathbb{Z}_{k_i}$ embeds in $\mathbb{C}^*$ as the subgroup of $k_i$-th roots of unity. Using the standard coordinates of $(\mathbb{C}^*)^{r+s}$, we can identify $\epsilon(W)$ with the subgroup
\[ \{ \lambda \in H : t^\lambda - 1 \text{ vanishes on } W \}. \]

The proof of the theorem is completed by the next three lemmas.

**Lemma 2.1.4.** If $\xi$ is a subgroup of $H$, then $\epsilon(V(\xi)) = \xi$.

*Proof.* The inclusion $\epsilon(V(\xi)) \supseteq \xi$ is clear. Now suppose $\lambda \in H \setminus \xi$. Then we may define a character $\rho \in \hat{H}$ such that $\rho(\lambda) \neq 1$, but $\rho(\xi) = 1$. Evidently, $\lambda \notin \epsilon(V(\xi))$, and we are done.

Given two algebraic subgroups, $P$ and $Q$ of $T$, let $\text{Hom}_{\text{alg}}(P, Q)$ be the set of morphisms from $P$ to $Q$ which preserve both the algebraic and multiplicative structure.

**Lemma 2.1.5.** Any algebraic subgroup $P$ of $T$ is of the form $V(\xi)$, for some subgroup $\xi \subseteq H$.

*Proof.* Let $\iota : P \hookrightarrow T$ be the inclusion map. Applying the functor $\text{Hom}_{\text{alg}}(-, \mathbb{C}^*)$, we obtain an epimorphism $\iota^* : H \twoheadrightarrow \text{Hom}_{\text{alg}}(P, \mathbb{C}^*)$. Set $\xi = \ker(\iota^*)$; then $P = V(\xi)$.
The above two lemmas (which generalize Lemmas 3.1 and 3.2 from [38]), show that we have a natural correspondence between algebraic subgroups of $T$ and subgroups of $H$. This correspondence preserves the lattice structure on both sides. That is, if $\xi_1 \leq \xi_2$, then $V(\xi_1) \supseteq V(\xi_2)$, and similarly, if $P_1 \subseteq P_2$, then $\epsilon(P_1) \geq \epsilon(P_2)$.

**Lemma 2.1.6.** The natural correspondence between algebraic subgroups of $T$ and subgroups of $H$ is an order-reversing lattice isomorphism. In particular,

$$V(\xi_1 + \xi_2) = V(\xi_1) \cap V(\xi_2) \quad \text{and} \quad V(\xi_1 \cap \xi_2) = V(\xi_1) \cdot V(\xi_2).$$

**Proof.** The first claim follows from the two lemmas above. The two equalities are consequences of this. \qed

The above correspondence reverses ranks, i.e., $\text{rank}(\xi) = \text{codim} V(\xi)$ and $\text{corank}(\xi) = \dim V(\xi)$.

### 2.1.4 Counting algebraic subtori

As a quick application, we obtain a counting formula for the number of algebraic subgroups of an $r$-dimensional complex algebraic torus, having precisely $k$ connected components, and a fixed, $n$-dimensional subtorus as the identity component. It is convenient to restate such a problem in terms of a zeta function.

**Definition 2.1.7.** Let $T$ be an abelian reductive group, and let $T_0$ be a fixed connected algebraic subgroup. Define the zeta function of this pair as

$$\zeta(T, T_0, s) = \sum_{k=1}^{\infty} \frac{a_k(T, T_0)}{k^s},$$

where $a_k(T, T_0)$ is the number of algebraic subgroups $W \leq T$ with identity component equal to $T_0$ and such that $|W/T_0| = k$.

This definition is modeled on that of the zeta function of a finitely generated group $G$, which is given by $\zeta(G, s) = \sum_{k=1}^{\infty} a_k(G) k^{-s}$, where $a_k(G)$ is the number of index-$k$ subgroups of $G$, see for instance [78].
Corollary 2.1.8. Suppose $T \cong (\mathbb{C}^*)^r$ and $T_0 \cong (\mathbb{C}^*)^n$, for some $0 \leq n \leq r$. Then
\[
\zeta(T, T_0, s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-r+n+1),
\]
where $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ is the usual Riemann zeta function.

Proof. First assume $n = 0$, so that $T_0 = \{1\}$. By Theorem 2.3.3, $a_k(T, \{1\})$ is the number of algebraic subgroups $W \leq T$ of the form $W = V(\xi)$, where $\xi \leq \mathbb{Z}^r$ and $[\mathbb{Z}^r : \xi] = k$. Clearly, this number equals $a_k(\mathbb{Z}^r)$, and so $\zeta(T, \{1\}, s) = \zeta(\mathbb{Z}^r, s)$. By a classical result of Bushnell and Reiner (see [78]), we have that $\zeta(\mathbb{Z}^r, s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-r+1)$. This proves the claim for $n = 0$.

For $n > 0$, we simply take the quotient of the group $T$ by the fixed subtorus $T_0$, to get
\[
\zeta(T, T_0, s) = \zeta(T/T_0, \{1\}, s) = \zeta(\mathbb{Z}^{r-n}, s).
\]
This ends the proof.

2.2 Primitive lattices and connected subgroups

In this section, we show that the correspondence between $\mathcal{L}(H)$ and $\mathcal{L}_{\text{alg}}(T)$ restricts to a correspondence between the primitive subgroups of $H$ and the connected algebraic subgroups of $T$. We also explore the relationship between the connected components of $V(\xi)$ and the determinant group, $\xi/\xi$, of a subgroup $\xi \leq H$.

2.2.1 Primitive subgroups

As before, let $H$ be a finitely generated abelian group. We say that a subgroup $\xi \leq H$ is primitive if there is no other subgroup $\xi' \leq H$ with $\xi < \xi'$ and $[\xi' : \xi] < \infty$. In particular, a primitive subgroup must contain the torsion subgroup, $\text{Tors}(H) = \{\lambda \in H \mid \exists n \in \mathbb{N} \text{ such that } n\lambda = 0\}$.

The intersection of two primitive subgroups is again a primitive subgroup, but the sum of two primitive subgroups need not be primitive (see, for instance, the proof of Corollary 2.5.6). Thus, the set of primitive subgroups of $H$ is not necessarily a sublattice of $\mathcal{L}(H)$. 

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When $H$ is free abelian, then all subgroups $\xi \leq H$ are also free abelian. In this case, $\xi$ is primitive if and only if it has a basis that can be extended to a basis of $H$, or, equivalently, $H/\xi$ is torsion-free. It is customary to call such a subgroup a primitive lattice.

Returning to the general situation, let $H$ be a finitely generated abelian group. Given an arbitrary subgroup $\xi \leq H$, define its primitive closure, $\overline{\xi}$, to be the smallest primitive subgroup of $H$ containing $\xi$. Clearly,

$$\overline{\xi} = \{ \lambda \in H : \exists n \in \mathbb{N} \text{ such that } n\lambda \in \xi \}. \tag{2.2.1}$$

Note that $H/\overline{\xi}$ is torsion-free, and thus we have a split exact sequence,

$$0 \longrightarrow \overline{\xi} \longrightarrow \hat{\xi/\xi} \longrightarrow H/\xi \longrightarrow 0. \tag{2.2.2}$$

By definition, $\xi$ is a finite-index subgroup of $\overline{\xi}$; in particular, $\text{rank}(\xi) = \text{rank}(\overline{\xi})$. We call the quotient group, $\overline{\xi}/\xi$, the determinant group of $\xi$. We have an exact sequence,

$$0 \longrightarrow H/\overline{\xi} \longrightarrow H/\xi \longrightarrow \overline{\xi}/\xi \longrightarrow 0, \tag{2.2.3}$$

with $\overline{\xi}/\xi$ finite. The inclusion $\overline{\xi} \hookrightarrow H$ induces a monomorphism $\overline{\xi}/\xi \hookrightarrow H/\xi$, which yields a splitting for the above sequence, showing that $\overline{\xi}/\xi \cong \text{Tors}(H/\xi)$. Since the group $\overline{\xi}/\xi$ is finite, it is isomorphic to its character group, $\hat{\overline{\xi}/\xi}$, which in turn can be viewed as a (finite) subgroup of $\hat{H} = T$.

Using an approach similar to the one from [38, Lemma 3.3], we sharpen and generalize that result, as follows.

**Lemma 2.2.1.** For every subgroup $\xi \leq H$, we have an isomorphism of algebraic groups,

$$V(\xi) \cong \hat{\overline{\xi}/\xi} \cdot V(\overline{\xi}). \tag{2.2.4}$$

Moreover,

1. $V(\xi) = \bigcup_{\rho \in \hat{\overline{\xi}/\xi}} \rho V(\overline{\xi})$ is the decomposition of $V(\xi)$ into irreducible components, with $V(\overline{\xi})$ as the component of the identity.
2. \( V(\xi)/V(\xi) \cong \hat{\xi}/\xi \). In particular, if \( \text{rank } \xi = \dim H \), then \( V(\xi) \cong \hat{\xi}/\xi \).

**Proof.** As noted in §2.1.1, the functor \( \text{Hom}(-, \mathbb{C}^*) \) is exact. Applying this functor to sequence (3.4.4), we obtain an exact sequence in \( \text{AbRed} \),

\[
0 \rightarrow \hat{\xi}/\xi \rightarrow \hat{H}/\xi \rightarrow \hat{H}/\xi \rightarrow 0.
\]

Since sequence (3.4.4) is split, sequence (2.2.5) is also split, and thus we get decomposition (3.4.5).

Now recall that \( H/\xi \) is torsion-free; thus, \( V(\xi) = \text{maxSpec}(\mathbb{C}[H/\xi]) \) is a connected algebraic subgroup of \( T \). Claims (1) and (2) readily follow. \( \Box \)

**Example 2.2.2.** Let \( H = \mathbb{Z} \) and identify \( \hat{H} = \mathbb{C}^* \). If \( \xi = 2\mathbb{Z} \), then \( V(\xi) = \{\pm 1\} \subset \mathbb{C}^* \), whereas \( \overline{\xi} = H \) and \( V(\overline{\xi}) = \{1\} \subset \mathbb{C}^* \).

Clearly, the subgroup \( \xi \) is primitive if and only if \( \xi = \overline{\xi} \). Thus, \( \xi \) is primitive if and only if the variety \( V(\xi) \) is connected. Putting things together, we obtain the following corollary to Theorem 2.1.3 and Lemma 2.2.1.

**Corollary 2.2.3.** Let \( H \) be a finitely generated abelian group, and let \( T = \hat{H} \). The natural correspondence between \( \mathcal{L}(H) \) and \( \mathcal{L}_{\text{alg}}(T) \) restricts to a correspondence between the primitive subgroups of \( H \) and the connected algebraic subgroups of \( T \).

### 2.2.2 The dual lattice

Given an abelian group \( A \), let \( A^\vee = \text{Hom}(A, \mathbb{Z}) \) be the dual group. Clearly, if \( H \) is a finitely generated abelian group, then \( H^\vee \) is torsion-free, with \( \text{rank } H^\vee = \text{rank } H \).

Now suppose \( \xi \leq H \) is a subgroup. By passing to duals, the projection map \( \pi: H \to H/\xi \) yields a monomorphism \( \pi^\vee: (H/\xi)^\vee \hookrightarrow H^\vee \). Thus, \( (H/\xi)^\vee \) can be viewed in a natural way as a subgroup of \( H^\vee \). In fact, more is true.
Lemma 2.2.4. Let $H$ be a finitely generated abelian group, and let $\xi \leq H$ be a subgroup. Then $(H/\xi)^\vee$ is a primitive lattice in $H^\vee$.

Proof. Dualizing the short exact sequence $0 \to \xi \to H \xrightarrow{\pi} H/\xi \to 0$, we obtain a long exact sequence,

$$0 \longrightarrow (H/\xi)^\vee \xrightarrow{\pi^\vee} H^\vee \longrightarrow \xi^\vee \longrightarrow \text{Ext}(H/\xi, \mathbb{Z}) \longrightarrow 0 .$$

(2.2.6)

Upon identifying $\text{Ext}(H/\xi, \mathbb{Z}) = \text{Tors}(H/\xi) = \overline{\xi}/\xi$ and setting $K = \text{coker}(\pi^\vee)$, the above sequence splits into two short exact sequences, $0 \to (H/\xi)^\vee \to H^\vee \to K \to 0$ and

$$0 \longrightarrow K \longrightarrow \xi^\vee \longrightarrow \overline{\xi}/\xi \longrightarrow 0 .$$

(2.2.7)

Now, since $K$ is a subgroup of $\xi^\vee$, is must be torsion free. Thus, $(H/\xi)^\vee$ is a primitive lattice in $H^\vee$. \hfill \Box

Given two primitive subgroups $\xi_1, \xi_2 \leq H$, their sum, $\xi_1 + \xi_2$, may not be a primitive subgroup of $H$. Likewise, although both $(H/\xi_1)^\vee$ and $(H/\xi_2)^\vee$ are primitive subgroups of $H^\vee$, their sum may not be primitive. Nevertheless, the following lemma shows that the respective determinant groups are the same.

Proposition 2.2.5. Let $H$ be a finitely generated abelian group, and let $\xi_1$ and $\xi_2$ be primitive subgroups of $H$, with $\xi_1 \cap \xi_2$ finite. Set $\xi = (H/\xi_1)^\vee + (H/\xi_2)^\vee$, and let $\overline{\xi}$ be the primitive closure of $\xi$ in $H^\vee$. Then

$$\overline{\xi}/\xi \cong \overline{\xi_1 + \xi_2}/\xi_1 + \xi_2 .$$

Proof. Replacing $H$ by $H/\text{Tors}(H)$ and $\xi_i$ by $\xi_i/\text{Tors}(H)$ if necessary, we may assume that $H$ is free abelian, and $\xi_1$ and $\xi_2$ are primitive lattices with $\xi_1 \cap \xi_2 = \{0\}$. Furthermore, choosing a splitting of $H^\vee/\overline{\xi} \hookrightarrow H^\vee$ if necessary, we may assume that $\overline{\xi} \cong H^\vee$, or equivalently, $\overline{\xi_1 + \xi_2} = H$. Write $s = \text{rank} \xi_1$ and $t = \text{rank} \xi_2$. Then $s + t = n$, where $n = \text{rank} H$.

Choose a basis $\{e_1, \ldots, e_s\}$ for $\xi_1$. Since $\xi_1$ is a primitive lattice in $H$, we may extend this basis to a basis $\{e_1, \ldots, e_s, f_1, \ldots, f_t\}$ for $H$. Picking a suitable basis for $\xi_2$, we may assume
that the inclusion \( \iota: \xi_1 + \xi_2 \hookrightarrow H \) is given by a matrix of the form

\[
\begin{pmatrix}
I_s & C \\
0 & D
\end{pmatrix},
\]

(2.2.8)

where \( I_s \) is the \( s \times s \) identity matrix and \( D = \text{diag}(d_1, \ldots, d_t) \) is a diagonal matrix with positive diagonal entries \( d_1, \ldots, d_t \) such that \( 1 = d_1 = \cdots = d_{m-1} \) and \( 1 \neq d_t \mid d_{t-1} \mid \cdots \mid d_m \), for some \( 1 \leq m \leq t + 1 \). Then \( \frac{\xi_1 + \xi_2}{\xi_1 + \xi_2} = \mathbb{Z}/d_m\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_t\mathbb{Z} \).

Notice that the columns of the matrix \( \begin{pmatrix} C \\ D \end{pmatrix} \) form a basis for \( \xi_2 \). Since \( \xi_2 \) is a primitive lattice in \( H \), the last \( t - m \) columns of \( C \) must have a minor of size \( t - m \) equal to \( \pm 1 \). Without loss of generality, we may assume that the corresponding rows are also the last \( t - m \) ones.

The canonical projection \( \pi: H \twoheadrightarrow H/\xi_1 + H/\xi_2 \) is given by a matrix of the form

\[
\begin{pmatrix}
X & Y \\
0 & I_t
\end{pmatrix}.
\]

(2.2.9)

Using row and column operations, the matrix \( X \) can be brought to the diagonal form \( \text{diag}(x_1, \ldots, x_s) \), where \( x_i \) are positive integers with \( 1 = x_1 = \cdots = x_{a-1} \) and \( 1 \neq x_s \mid x_{s-1} \mid \cdots \mid x_a \). Moreover, the submatrix of \( Y \) involving the last \( s - a + 1 \) rows and columns is invertible. Taking the dual basis of \( H^\vee = \text{Hom}(H, \mathbb{Z}) \), the inclusion \( \xi = (H/\xi_1)^\vee + (H/\xi_2)^\vee \hookrightarrow H^\vee \) is given by the matrix \( \begin{pmatrix} X^T & 0 \\ Y^T & I_t \end{pmatrix} \). It follows that \( \overline{\xi}/\xi = \mathbb{Z}/x_a\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/x_s\mathbb{Z} \).

Evidently, the restriction of \( \pi \circ \iota \) to \( \xi_2 \) is the zero map; hence, \( XC = -YD \). For a fixed integer \( k \leq s \), set \( \delta_k := d_t d_{t-1} \cdots d_{k-s+t} \) and \( y_k := x_s x_{s-1} \cdots x_k \). Clearly, \( \delta_k \) is the gcd of all minors of size \( s - k + 1 \) of the submatrix in \( YD \) involving the last \( s - a + 1 \) rows and columns. Hence, \( \delta_k \) is also the gcd of all minors of size \( s - k + 1 \) of the corresponding submatrix of \( XC \). Thus, \( \delta_k = y_k \). Hence, \( d_t = x_s, \ldots, d_m = x_{s-t+m} \), and \( 1 = d_{m-1} = x_{s-t+m-1} \), which implies \( s - t + m = a \). This yields the desired conclusion. \( \Box \)
2.3 Categorical reformulation

In this section, we reformulate Theorem 2.1.3 using the language of categories. In order to simultaneously consider the category of all finitely generated abelian groups, and the lattice structure for all the subgroups of a fixed abelian group, we need the language of fibered categories from [33, §5.1], which we briefly recall here.

2.3.1 Fibered categories

Recall that a poset \((L, \leq)\) can be seen as a small category, with objects the same as the elements of \(L\), and with an arrow \(p \to p'\) in the category \(L\) if and only if \(p \leq p'\).

Let \(\mathcal{E}\) be a category. We denote by \(\text{Ob}\mathcal{E}\) the objects of \(\mathcal{E}\), and by \(\text{Mor}\mathcal{E}\) its morphisms. A category over \(\mathcal{E}\) is a category \(\mathcal{F}\), together with a functor \(\Phi: \mathcal{F} \to \mathcal{E}\). For \(T \in \text{Ob}\mathcal{E}\), we denote by \(\mathcal{F}(T)\) the subcategory of \(\mathcal{F}\) consisting of objects \(\xi\) with \(\Phi(\xi) = T\), and morphisms \(f\) with \(\Phi(f) = \text{id}_T\).

**Definition 2.3.1.** Let \(\Phi: \mathcal{F} \to \mathcal{E}\) be a category over \(\mathcal{E}\). Let \(f: v \to u\) be a morphism in \(\mathcal{F}\), and set \((F: V \to U) = (\Phi(f: v \to u))\). Then \(f\) is said to be a Cartesian morphism if for any \(h: v' \to u\) with \(\Phi(h: v' \to u) = F\), there exists a unique \(h': v' \to v\) in \(\text{Mor}\mathcal{F}(F)\) such that \(h = f \circ h'\).

The above definition is summarized in the following diagram:

\[
\begin{array}{ccc}
V & \xrightarrow{f} & U \\
\downarrow h' & & \downarrow h \\
V' & \xrightarrow{f} & U
\end{array}
\]

**Definition 2.3.2.** We say that \(\Phi: \mathcal{F} \to \mathcal{E}\) is a fibered category if for any morphism \(F: V \to U\) in \(\mathcal{E}\), and any object \(u \in \text{Ob}(\mathcal{F}(U))\), there exists a Cartesian morphism \(f: v \to u\), with \(\Phi(f) = F\). Moreover, the composition of Cartesian morphisms is required to be a Cartesian morphism.
We say \( \mathcal{F} \) is a **lattice over** \( \mathcal{E} \) (or less succinctly, a category fibered in lattices over \( \mathcal{E} \)) if \( \mathcal{F} \) is a fibered category, and every \( \mathcal{F}(T) \) is a lattice.

### 2.3.2 A lattice over \( \text{AbRed} \)

We now construct a category fibered in lattices over the category of abelian complex algebraic reductive groups, \( \text{AbRed} \). Let \( \text{SubAbRed} \) be the category with objects

\[
\text{Ob} \text{SubAbRed} = \{ i: W \hookrightarrow P \mid i \text{ is a closed immersion of algebraic subgroups} \}, \quad (2.3.2)
\]

and morphisms between \( i: W \rightarrow P \) and \( i': W' \rightarrow P' \) the set of pairs

\[
\{(f, g) \mid f: W \rightarrow W' \text{ and } g: P \rightarrow P' \text{ such that } i' \circ f = g \circ i \}. \quad (2.3.3)
\]

Projection to the target,

\[
\begin{array}{ccc}
W & \xrightarrow{i} & P \\
\downarrow{f} & & \downarrow{g} \\
W' & \xrightarrow{i'} & P'
\end{array}
\]

(2.3.4)

defines a functor from \( \text{SubAbRed} \) to \( \text{AbRed} \). One can easily check that this functor is a category fibered in lattices over \( \text{AbRed} \), with Cartesian morphisms obtained by taking preimages of subtori. More precisely, suppose \( F: W \rightarrow P \) is a morphism in \( \text{AbRed} \), and \( \theta \hookrightarrow P \) is an object in \( \text{Ob} \text{SubAbRed} \); the corresponding Cartesian morphism is then

\[
\begin{array}{ccc}
W & \xrightarrow{F^{-1}(\theta)} & \theta \\
\downarrow{i} & & \downarrow{i} \\
W & \xrightarrow{F} & P
\end{array}
\]

(2.3.5)

A similar construction works for \( \text{AbFgGp} \), the category of finitely generated abelian groups. That is, we can construct a category \( \text{SubAbGp} \) fibered in lattices over \( \text{AbFgGp} \) by taking injective morphisms of finitely generated abelian groups.
2.3.3 An equivalence of fibered categories

We now construct an explicit (contravariant) equivalence between these two fibered categories considered above. For any inclusion of subgroups \( \eta \hookrightarrow \xi \), let

\[
V(\eta \hookrightarrow \xi) := (\text{maxSpec}(\mathbb{C}[\xi/\eta]) \hookrightarrow \text{maxSpec}(\mathbb{C}[\xi])). \tag{2.3.6}
\]

This algebraic subgroup is naturally identified with the subgroup of characters of \( \xi/\eta \), i.e., the closed embedding of subtori \( \text{Hom}(\xi/\eta, \mathbb{C}^*) \rightarrow \text{Hom}(\xi, \mathbb{C}^*) \).

Finally, for any algebraic subgroups \( W \hookrightarrow P \), let

\[
\epsilon(W \hookrightarrow P) := \{ \lambda \in \text{Hom}(P, \mathbb{C}^*) : t^\lambda - 1 \text{ vanishes on } W \} \subseteq \text{Hom}(P, \mathbb{C}^*). \tag{2.3.7}
\]

For a fixed algebraic group \( T \) in \( \text{AbRed} \), with character group \( H = \hat{T} \), the subgroup \( \epsilon(W \hookrightarrow T) \) of \( H \) coincides with the subgroup \( \epsilon(W) \) defined in (2.1.5), and the algebraic subgroup \( V(\xi \hookrightarrow H) \) of \( T \) coincides with the algebraic subgroup \( V(\xi) \) defined in (2.1.3).

Tracing through the definitions, we obtain the following result, which reformulates Theorem 2.1.3 in this setting.

**Theorem 2.3.3.** The two fibered categories \( \text{SubAbRed} \) and \( \text{SubAbGp} \) are equivalent, with (contravariant) equivalences given by the functors \( V \) and \( \epsilon \) defined above.

2.4 Intersections of translated algebraic subgroups

In Sections 2.1 and 2.2, we only considered intersections of algebraic subgroups. In this section, we consider the more general situation where translated subgroups intersect.

2.4.1 Two morphisms

As usual, let \( T \) be a complex abelian reductive group, and let \( H = \hat{T} \) be the weight group corresponding to \( T = \hat{H} \).
Let \(\xi_1, \ldots, \xi_k\) be subgroups of \(H\). Set \(\xi = \xi_1 + \cdots + \xi_k\), and let \(\sigma: \xi_1 \times \cdots \times \xi_k \to \xi\) be the homomorphism given by \((\lambda_1, \ldots, \lambda_k) \mapsto \lambda_1 + \cdots + \lambda_k\). Consider the induced morphism on character groups,

\[
\hat{\sigma}: \hat{\xi} \longrightarrow \hat{\xi}_1 \times \cdots \times \hat{\xi}_k.
\] (2.4.1)

Using Remark 2.1.2, the next lemma is readily verified.

**Lemma 2.4.1.** Under the isomorphisms \(\hat{\xi}_i \cong T/V(\xi_i)\) and \(\hat{\xi} \cong T/V(\xi) = T/(\bigcap_i V(\xi_i))\), the morphism \(\hat{\sigma}\) gets identified with the morphism \(\delta: T/(\bigcap_i V(\xi_i)) \to T/V(\xi_1) \times \cdots \times T/V(\xi_k)\) induced by the diagonal map \(\Delta: T \to T^k\).

Next, let \(\gamma: \xi_1 \times \cdots \times \xi_k \to H \times \cdots \times H\) be the product of the inclusion maps \(\gamma_i: \xi_i \hookrightarrow H\), and consider the induced homomorphism on character groups,

\[
\hat{\gamma}: \hat{H} \times \cdots \times \hat{H} \longrightarrow \hat{\xi}_1 \times \cdots \times \hat{\xi}_k.
\] (2.4.2)

The next lemma is immediate.

**Lemma 2.4.2.** Under the isomorphisms \(\hat{\xi}_i \cong T/V(\xi_i)\), the morphism \(\hat{\gamma}\) gets identified with the projection map \(\pi: T^k \to T/V(\xi_1) \times \cdots \times T/V(\xi_k)\).

### 2.4.2 Translated algebraic subgroups

Given an algebraic subgroup \(P \subseteq T\), and an element \(\eta \in T\), denote by \(\eta P\) the translate of \(P\) by \(\eta\). In particular, if \(C\) is an algebraic subtorus of \(T\), then \(\eta C\) is a translated subtorus. If \(\eta\) is a torsion element of \(T\), we denote its order by \(\text{ord}(\eta)\). Finally, if \(A\) is a finite group, denote by \(c(A)\) the largest order of any element in \(A\).

We are now in a position to state and prove the main result of this section (Theorem ?? from the Introduction). As before, let \(\xi_1, \ldots, \xi_k\) be subgroups of \(H = \hat{T}\). Let \(\eta_1, \ldots, \eta_k\) be elements in \(T\), and consider the translated subgroups \(Q_1, \ldots, Q_k\) of \(T\) defined by

\[
Q_i = \eta_i V(\xi_i).
\] (2.4.3)
Clearly, each $Q_i$ is a subvariety of $T$, but, unless $\eta_i \in V(\xi_i)$, it is not an algebraic subgroup.

**Theorem 2.4.3.** Set $\xi = \xi_1 + \cdots + \xi_k$ and $\eta = (\eta_1, \ldots, \eta_k) \in T^k$. Then

$$Q_1 \cap \cdots \cap Q_k = \begin{cases} \emptyset & \text{if } \hat{\gamma}(\eta) \notin \text{im}(\hat{\sigma}), \\ \rho V(\xi) & \text{otherwise,} \end{cases} \tag{2.4.4}$$

where $\rho$ is any element in the intersection $Q = Q_1 \cap \cdots \cap Q_k$. Furthermore, if the intersection is non-empty, then

1. The variety $Q$ decomposes into irreducible components as $Q = \bigcup_{\tau \in \xi/\xi} \rho \tau V(\xi)$, and $\dim(Q) = \dim(T) - \text{rank}(\xi)$.

2. If $\eta$ has finite order, then $\rho$ can be chosen to have finite order, too. Moreover, $\text{ord}(\eta) | \text{ord}(\rho) | c(\xi/\xi)$.

**Proof.** We have:

$$Q_1 \cap \cdots \cap Q_k \neq \emptyset \iff \eta_1 a_1 = \cdots = \eta_k a_k, \text{ for some } a_i \in V(\xi_i)$$

$$\iff \hat{\gamma}_1(\eta_1) = \cdots = \hat{\gamma}_k(\eta_k) \quad \text{by Lemma 2.4.2}$$

$$\iff \hat{\gamma}(\eta) \in \text{im}(\hat{\sigma}) \quad \text{by Lemma 2.4.1.}$$

Now suppose $Q_1 \cap \cdots \cap Q_k \neq \emptyset$. For any $\rho \in Q_1 \cap \cdots \cap Q_k$, and any $1 \leq i \leq k$, there is a $\rho_i \in V(\xi_i)$ such that $\rho = \eta_i \rho_i$; thus, $\rho^{-1} \eta_i \in V(\xi_i)$. Therefore,

$$\rho^{-1}(Q_1 \cap \cdots \cap Q_k) = \rho^{-1}(\eta_1 V(\xi_1) \cap \cdots \cap \eta_k V(\xi_k))$$

$$= \rho^{-1}(\eta_1 V(\xi_1)) \cap \cdots \cap \rho^{-1}(\eta_k V(\xi_k))$$

$$= V(\xi_1) \cap \cdots \cap V(\xi_k)$$

$$= V(\xi_1 + \cdots + \xi_k).$$

Hence, $Q_1 \cap \cdots \cap Q_k = \rho V(\xi)$.
Finally, suppose $\eta$ has finite order. Let $\rho$ be an element in $T/V(\xi)$ such that $\hat{\sigma}(\rho) = \hat{\gamma}(\eta)$. Note that $\text{ord}(\rho) = \text{ord}(\eta)$. Using the exact sequence (2.2.5) and the third isomorphism theorem for groups, we get a short exact sequence,

$$
0 \longrightarrow \frac{\hat{\xi}}{\hat{\xi}} \longrightarrow T/V(\xi) \xrightarrow{q} T/V(\eta) \longrightarrow 0 .
$$

Applying the $\text{Hom}_{\text{group}}(-, \mathbb{C}^*)$ functor to the split exact (2.2.2), we get a split exact sequence,

$$
0 \longrightarrow V(\xi) \longrightarrow T \xrightarrow{s} T/V(\xi) \longrightarrow 0 .
$$

(2.4.6)

Now pick an element $\tilde{\rho} \in q^{-1}(\rho)$. We then have $q(\tilde{\rho}^{\text{ord}(\eta)}) = \tilde{\rho}^{\text{ord}(\eta)} = 1$, which implies that $\tilde{\rho}^{\text{ord}(\eta)} \in \frac{\hat{\xi}}{\hat{\xi}}$. Hence, $\tilde{\rho}$ has finite order in $T/V(\xi)$, and, moreover, $\text{ord}(\eta) \mid \text{ord}(\tilde{\rho}) \mid \text{ord}(\tilde{\rho}) \cdot c(\xi/\xi)$. Setting $\rho = s(\tilde{\rho})$ gives the desired translation factor.

When $k = 2$, the theorem takes a slightly simpler form.

**Corollary 2.4.4.** Let $\xi_1$ and $\xi_2$ be two subgroups of $H$, and let $\eta_1$ and $\eta_2$ be two elements in $T = \hat{H}$. Then

1. The variety $Q = \eta_1 V(\xi_1) \cap \eta_2 V(\xi_2)$ is non-empty if and only if $\eta_1 \eta_2^{-1}$ belongs to the subgroup $V(\xi_1) \cdot V(\xi_2)$.

2. If the above condition is satisfied, then $\dim Q = \text{rank } H - \text{rank}(\xi_1 + \xi_2)$.

In the special case when $H = \mathbb{Z}^r$ and $T = (\mathbb{C}^*)^r$, Theorem 2.4.3 allows us to recover Proposition 3.6 from [38].

**Corollary 2.4.5 (Hironaka [38]).** Let $\xi_1, \ldots, \xi_k$ be subgroups of $\mathbb{Z}^r$, let $\eta = (\eta_1, \ldots, \eta_k)$ be an element in $(\mathbb{C}^*)^r$, and set $Q_i = \eta_i V(\xi_i)$. Then

$$
Q_1 \cap \cdots \cap Q_k \neq \emptyset \iff \hat{\gamma}(\eta) \in \text{im}(\hat{\sigma}) .
$$

(2.4.7)

Moreover, for any connected component $Q$ of $Q_1 \cap \cdots \cap Q_k$, we have:
1. \( Q = \rho V(\xi) \), for some \( \rho \in (\mathbb{C}^*)^r \).

2. \( \dim(Q) = r - \text{rank}(\xi) \).

3. If \( \eta \) has finite order, then \( \text{ord}(\eta) | \text{ord}(\rho) | \text{ord}(\eta) \cdot c(\xi/\xi) \).

### 2.4.3 Some consequences

We now derive a number of corollaries to Theorem 2.4.3. Fix a complex abelian reductive group \( T \). To start with, we give a general description of the intersection of two arbitrary unions of translated subgroups.

**Corollary 2.4.6.** Let \( W = \bigcup_i \eta_i V(\xi_i) \) and \( W' = \bigcup_j \eta'_j V(\xi'_j) \) be two unions of translated subgroups of \( T \). Then

\[
W \cap W' = \bigcup_{i,j} \eta_i V(\xi_i) \cap \eta'_j V(\xi'_j),
\]

where \( \eta_i V(\xi_i) \cap \eta'_j V(\xi'_j) \) is either empty (which this occurs precisely when \( \hat{\gamma}_{i,j}(\eta_i,\eta_j) \) does not belong to \( \text{im}(\hat{\sigma}_{i,j}) \)), or equals \( \eta_{i,j} V(\xi_i + \xi'_j) \), for some \( \eta_{i,j} \in T \).

**Corollary 2.4.7.** With notation as above, \( W \cap W' \) is finite if and only if \( W \cap W' = \emptyset \) or \( \text{rank}(\xi_i + \xi'_j) = \text{dim}(T) \), for all \( i,j \).

**Corollary 2.4.8.** Let \( W \) and \( W' \) be two unions of (torsion-) translated subgroups of \( T \). Then \( W \cap W' \) is again a union of (torsion-) translated subgroups of \( T \).

The next two corollaries give a comparison between the intersections of various translates of two fixed algebraic subgroups of \( T \). Both of these results will be useful in another paper [83].

**Corollary 2.4.9.** Let \( T_1 \) and \( T_2 \) be two algebraic subgroups in \( T \). Suppose \( \alpha, \beta, \eta \) are elements in \( T \), such that \( \alpha T_1 \cap \eta T_2 \neq \emptyset \) and \( \beta T_1 \cap \eta T_2 \neq \emptyset \). Then

\[
\text{dim} (\alpha T_1 \cap \eta T_2) = \text{dim} (\beta T_1 \cap \eta T_2). \quad (2.4.9)
\]
Proof. Set $\xi = \epsilon(T_1) + \epsilon(T_2)$. From Theorem 2.4.3, we find that both $\alpha T_1 \cap \eta T_2$ and $\beta T_1 \cap \eta T_2$ have dimension equal to the corank of $\xi$. This ends the proof. \hfill \Box

**Corollary 2.4.10.** Let $T_1$ and $T_2$ be two algebraic subgroups in $T$. Suppose $\alpha_1$ and $\alpha_2$ are torsion elements in $T$, of coprime order. Then

$$ T_1 \cap \alpha_2 T_2 = \emptyset \implies \alpha_1 T_1 \cap \alpha_2 T_2 = \emptyset. \quad (2.4.10) $$

Proof. Set $H = \tilde{T}$, $\xi_i = \epsilon(T_i)$, and $\xi = \xi_1 + \xi_2$. By Theorem 2.4.3, the condition that $T_1 \cap \alpha_2 T_2 = \emptyset$ implies $\hat{\gamma}(1, \alpha_2) \notin \text{im}(\hat{\sigma})$, where $\sigma: \xi_1 \times \xi_2 \to \xi$ is the sum homomorphism, and $\gamma: \xi_1 \times \xi_2 \hookrightarrow H \times H$ is the inclusion map.

Suppose $\alpha_1 T_1 \cap \alpha_2 T_2 \neq \emptyset$. Then, from Theorem 2.4.3 again, we know that $\hat{\gamma}(\alpha_1, \alpha_2) \in \text{im}(\hat{\sigma})$; thus, $(\hat{\gamma}(\alpha_1, \alpha_2))^n \in \text{im}(\hat{\sigma})$, for any integer $n$. From our hypothesis, the orders of $\alpha_1$ and $\alpha_2$ are coprime; thus, there exist integers $p$ and $q$ such that $p \text{ord}(\alpha_1) + q \text{ord}(\alpha_2) = 1$. Hence,

$$ \hat{\gamma}(1, \alpha_2) = \hat{\gamma}(\alpha_1^{p \text{ord}(\alpha_1)}, \alpha_2^{1-q \text{ord}(\alpha_2)}) = (\hat{\gamma}(\alpha_1, \alpha_2))^{p \text{ord}(\alpha_1)} \in \text{im}(\hat{\sigma}), $$

a contradiction. \hfill \Box

### 2.4.4 Abelian covers

In [83], we use Corollaries 2.4.9 and 2.4.10 to study the homological finiteness properties of abelian covers. Let us briefly mention one of the results we obtain as a consequence.

Let $X$ be a connected CW-complex with finite 1-skeleton. Let $H = H_1(X, \mathbb{Z})$ the first homology group. Since the space $X$ has only finitely many 1-cells, $H$ is a finitely generated abelian group. The characteristic varieties of $X$ are certain Zariski closed subsets $V^i(X)$ inside the character torus $\tilde{H} = \text{Hom}(H, \mathbb{C}^*)$. The question we study in [83] is the following: Given a regular, free abelian cover $X' \to X$, with $\dim_{\mathbb{Q}} H_i(X', \mathbb{Q}) < \infty$ for all $i \leq k$, which regular, finite abelian covers of $X'$ have the same homological finiteness property?
Proposition 2.4.11 ([83]). Let $A$ be a finite abelian group, of order $e$. Suppose the characteristic variety $V(X)$ decomposes as $\bigcup_j \rho_j T_j$, with each $T_j$ an algebraic subgroup of $\hat{H}$, and each $\rho_j$ an element in $\hat{H}$ such that $\tilde{\rho}_j \in \hat{H}/T_j$ satisfies $\gcd(\text{ord}(\tilde{\rho}_j), e) = 1$. Then, given any regular, free abelian cover $X^\nu$ with finite Betti numbers up to some degree $k \geq 1$, the regular $A$-covers of $X^\nu$ have the same finiteness property.

2.5 Exponential interpretation

In this section, we explore the relationship between the correspondence $H \rightsquigarrow T = \hat{H}$ from §2.1 and the exponential map $\text{Lie}(T) \rightarrow T$.

2.5.1 The exponential map

Let $T$ be a complex abelian reductive group. Denote by $\text{Lie}(T)$ the Lie algebra of $T$. The exponential map $\exp: \text{Lie}(T) \rightarrow T$ is an analytic map, whose image is $T_0$, the identity component of $T$.

As usual, set $H = \tilde{T}$, and consider the lattice $\mathcal{H} = H^\vee \cong H/\text{Tors}(H)$. We then can identify $T_0 = \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*)$ and $\text{Lie}(T) = \text{Hom}(\mathcal{H}^\vee, \mathbb{C})$. Under these identifications, the corestriction to the image of the exponential map can be written as

$$\exp = \text{Hom}(-, e^{2\pi i z}): \text{Hom}(\mathcal{H}^\vee, \mathbb{C}) \rightarrow \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*), \quad (2.5.1)$$

where $\mathbb{C} \rightarrow \mathbb{C}^*$, $z \mapsto e^z$ is the usual complex exponential. Finally, upon identifying $\text{Hom}(\mathcal{H}^\vee, \mathbb{C})$ with $\mathcal{H} \otimes \mathbb{C}$, we see that $T_0 = \exp(\mathcal{H} \otimes \mathbb{C})$.

The correspondence $T \rightsquigarrow \mathcal{H} = (\hat{T})^\vee$ sends an algebraic subgroup $W$ inside $T$ to the sublattice $\chi = (\tilde{W})^\vee$ inside $\mathcal{H}$. Clearly, $\chi = \text{Lie}(W) \cap \mathcal{H}$ is a primitive lattice; furthermore, $\exp(\chi \otimes \mathbb{C}) = W_0$.

Now let $\chi_1$ and $\chi_2$ be two sublattices in $\mathcal{H}$. Since the exponential map is a group
homomorphism, we have the following equality:

$$\exp((\chi_1 + \chi_2) \otimes \mathbb{C}) = \exp(\chi_1 \otimes \mathbb{C}) \cdot \exp(\chi_2 \otimes \mathbb{C}).$$  \hfill (2.5.2)

On the other hand, the intersection of the two algebraic subgroups \(\exp(\chi_1 \otimes \mathbb{C})\) and \(\exp(\chi_2 \otimes \mathbb{C})\) need not be connected, so it cannot be expressed solely in terms of the exponential map. Nevertheless, we will give a precise formula for this intersection in Theorem 2.5.3 below.

### 2.5.2 Exponential map and Pontrjagin duality

First, we need to study the relationship between the exponential map and the correspondence from Proposition 2.1.1.

**Lemma 2.5.1.** Let \(T\) be a complex abelian reductive group, and let \(\mathcal{H} = (\hat{T})^\vee\). Let \(\chi \leq \mathcal{H}\) be a sublattice. We then have an equality of connected algebraic subgroups,

$$V((\mathcal{H}/\chi)^\vee) = \exp(\chi \otimes \mathbb{C}),$$  \hfill (2.5.3)

inside \(T_0 = \exp(\mathcal{H} \otimes \mathbb{C})\).

**Proof.** Let \(\pi: \mathcal{H} \rightarrow \mathcal{H}/\chi\) be the canonical projection, and let \(K = \text{coker}(\pi^\vee)\). As in (2.2.7), we have an exact sequence, \(0 \rightarrow K \rightarrow \chi^\vee \rightarrow \mathcal{H}/\chi \rightarrow 0\). Applying the functor \(\text{Hom}(-, \mathbb{C}^*)\) to this sequence, we obtain a new short exact sequence,

$$0 \longrightarrow \text{Hom}(\mathcal{H}/\chi, \mathbb{C}^*) \longrightarrow \text{Hom}(\chi^\vee, \mathbb{C}^*) \longrightarrow \text{Hom}(K, \mathbb{C}^*) \longrightarrow 0. \hfill (2.5.4)$$

From the way the functor \(V\) was defined in (2.1.3), we have that \(\text{Hom}(K, \mathbb{C}^*) = V((\mathcal{H}/\chi)^\vee)\). Composing with the map \(\exp: \mathbb{C} \rightarrow \mathbb{C}^*\), we obtain the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}(\chi^\vee, \mathbb{C}) & \xrightarrow{\exp} & \text{Hom}(\chi^\vee, \mathbb{C}^*) \\
\downarrow \cong & & \downarrow \\
\text{Hom}(K, \mathbb{C}) & \xrightarrow{\exp} & \text{Hom}(K, \mathbb{C}^*) = V((\mathcal{H}/\chi)^\vee) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathcal{H}^\vee, \mathbb{C}) & \xrightarrow{\exp} & \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*)
\end{array}
\]
Identify now \( \exp(\chi \otimes \mathbb{C}) \) with the image of \( \text{Hom}(\chi^\vee, \mathbb{C}) \) in \( \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*) \). Clearly, this image coincides with the image of \( V((\mathcal{H}/\chi)^\vee) \) in \( T_0 = \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*) \), and so we are done. \( \square \)

**Corollary 2.5.2.** Let \( H \) be a finitely generated abelian group, and let \( \xi \leq H \) be a subgroup. Consider the sublattice \( \chi = (H/\xi)^\vee \) inside \( \mathcal{H} = H^\vee \). Then

\[
V(\xi) = \exp(\chi \otimes \mathbb{C}). \tag{2.5.6}
\]

**Proof.** Note that \( \xi = (\mathcal{H}/\chi)^\vee \), as subgroups of \( H/\text{Tors}(H) = \mathcal{H}^\vee \). The desired equality follows at once from Lemma 2.5.1. \( \square \)

### 2.5.3 Exponentials and determinant groups

We are now in a position to state and prove the main result of this section (Theorem ?? from the Introduction).

**Theorem 2.5.3.** Let \( T \) be a complex abelian reductive group, and let \( \chi_1 \) and \( \chi_2 \) be two sublattices of \( \mathcal{H} = \mathcal{T}^\vee \).

1. Set \( \xi = (\mathcal{H}/\chi_1)^\vee + (\mathcal{H}/\chi_2)^\vee \leq \mathcal{H}^\vee \) and \( \chi = (\mathcal{H}^\vee/\xi)^\vee \leq \mathcal{H} \). Then

\[
\exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) = \frac{\xi}{\xi} \cdot \exp(\chi \otimes \mathbb{C}),
\]

as algebraic subgroups of \( T_0 \). Moreover, the identity component of both these groups is \( V(\xi) = \exp(\chi \otimes \mathbb{C}) \).

2. Now suppose \( \chi_1 \) and \( \chi_2 \) are primitive sublattices of \( \mathcal{H} \), with \( \chi_1 \cap \chi_2 = 0 \). We then have an isomorphism of finite abelian groups,

\[
\exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) \cong \frac{\chi_1 + \chi_2}{\chi_1 + \chi_2}.
\]
Proof. To prove part (1), note that
\[ \exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) = V((\mathcal{H}/\chi_1)') \cap V((\mathcal{H}/\chi_2)') \]
by Lemma 2.5.1
\[ = V((\mathcal{H}/\chi_1)' + (\mathcal{H}/\chi_2)') \]
by Lemma 2.1.6
\[ = \widehat{\xi}/\xi \cdot V(\xi) \]
by Lemma 2.2.1.

Finally, note that \((\mathcal{H}/\chi)' = \bar{\xi} \); thus, \(V(\bar{\xi}) = \exp(\chi \otimes \mathbb{C})\), again by Lemma 2.5.1.

To prove part (2), note that \(\xi = H'\), since we are assuming \(\chi_1 \cap \chi_2 = 0\). Hence, \(V(\xi) = \{1\}\). Since we are also assuming that the lattices \(\chi_1\) and \(\chi_2\) are primitive, Proposition 2.2.5 applies, giving that \(\bar{\xi}/\xi \cong \chi_1 + \chi_2/\chi_1 + \chi_2\). Using now part (1) finishes the proof.

The next corollary follows at once from Theorem 2.5.3.

**Corollary 2.5.4.** Let \(H\) be a finitely generated abelian group, and let \(\xi_1\) and \(\xi_2\) be two subgroups. Let \(\mathcal{H} = H'\) be the dual lattice, and \(\chi_i = (H/\xi_i)'\) the corresponding sublattices.

1. Set \(\xi = \bar{\xi_1} + \bar{\xi_2}\). Then:

\[ \exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) = \bar{\xi}/\xi \cdot V(\xi). \]

2. Now suppose \(\text{rank}(\xi_1 + \xi_2) = \text{rank}(H)\). Then

\[ \exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C}) \cong \chi_1 + \chi_2/\chi_1 + \chi_2 \cong \bar{\xi}/\xi. \]

**2.5.4 Some applications**

Let us now consider the case when \(T = (\mathbb{C}^*)^r\). In this case, \(\mathcal{H} = \mathbb{Z}^r\), and the exponential map (3.5.3) can be written in coordinates as \(\exp: \mathbb{C}^r \to (\mathbb{C}^*)^r, (z_1, \ldots, z_r) \mapsto (e^{2\pi i z_1}, \ldots, e^{2\pi i z_r})\).

Applying Theorem 2.5.3 to this situation, we recover Theorem 1.1 from [67].

**Corollary 2.5.5 (Nazir [67]).** Suppose \(\chi_1\) and \(\chi_2\) are primitive lattices in \(\mathbb{Z}^r\), and \(\chi_1 \cap \chi_2 = 0\).

Then \(\exp(\chi_1 \otimes \mathbb{C}) \cap \exp(\chi_2 \otimes \mathbb{C})\) is isomorphic to \(\chi_1 + \chi_2/\chi_1 + \chi_2\).
As another application, let us show that intersections of subtori can be at least as complicated as arbitrary finite abelian groups.

**Corollary 2.5.6.** For any integer $n \geq 0$, and any finite abelian group $A$, there exist subtori $T_1$ and $T_2$ in some complex algebraic torus $(\mathbb{C}^*)^r$ such that $T_1 \cap T_2 \cong (\mathbb{C}^*)^n \times A$.

**Proof.** Write $A = \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k}$. Let $\chi_1$ and $\chi_2$ be the lattices in $\mathbb{Z}^{2k}$ spanned by the columns of the matrices $(I_k^0)$ and $(I_k^D)$, where $D = \text{diag}(d_1, \ldots, d_k)$. Clearly, both $\chi_1$ and $\chi_2$ are primitive, and $\chi_1 \cap \chi_2 = 0$, yet the lattice $\chi = \chi_1 + \chi_2$ is not primitive if the group $A = \chi / \chi$ is non-trivial.

Now consider the subtori $P_i = \exp(\chi_i \otimes \mathbb{C})$ in $(\mathbb{C}^*)^{2k}$, for $i = 1, 2$. By Corollary 2.5.5, we have that $P_1 \cap P_2 \cong A$. Finally, consider the subtori $T_i = (\mathbb{C}^*)^n \times P_i$ in $(\mathbb{C}^*)^{n+2k}$. Clearly, $T_1 \cap T_2 \cong (\mathbb{C}^*)^n \times A$, and we are done. \(\square\)

In particular, if $A$ is a finite cyclic group, there exist 1-dimensional subtori, $T_1$ and $T_2$, in $(\mathbb{C}^*)^2$ such that $T_1 \cap T_2 \cong A$.

### 2.6 Intersections of torsion-translated subtori

In this section, we revisit Theorem 2.4.3 from the exponential point of view. In the case when the translation factors have finite order, the criterion from that theorem can be refined, to take into account certain arithmetic information about the translated tori in question.

#### 2.6.1 Virtual belonging

We start with a definition.

**Definition 2.6.1.** Let $\chi$ be a primitive lattice in $\mathbb{Z}^r$. Given a vector $\lambda \in \mathbb{Q}^r$, we say $\lambda$ virtually belongs to $\chi$ if $d \cdot \lambda \in \mathbb{Z}^r$, where $d = |\det [\chi \mid \chi_0]|$ and $\chi_0 = \mathbb{Q}\lambda \cap \mathbb{Z}^r$. 
Here, $[\chi \mid \chi_0]$ is the matrix obtained by concatenating basis vectors for the sublattices $\chi$ and $\chi_0$ of $Z^r$. Note that $\chi_0$ is a (cyclic) primitive lattice, generated by an element of the form $\lambda_0 = m\lambda \in Z^r$, for some $m \in \mathbb{N}$.

In the next lemma, we record several properties of the notion introduced above.

**Lemma 2.6.2.** With notation as above,

1. $\lambda$ virtually belongs to $\chi$ if and only if $d\lambda \in \chi_0$.

2. $d = 0$ if and only if $\lambda \in \chi \otimes \mathbb{Q}$, which in turn implies that $\lambda$ virtually belongs to $\chi$.

3. If $d > 0$, then $d$ equals the order of the determinant group $\frac{\chi + \chi_0}{\chi_0}$.

4. If $d = 1$, then $\lambda$ virtually belongs to $\chi$ if and only if $\lambda \in \mathbb{Z}^r$, which happens if and only if $\lambda \in \chi_0$.

Next, we give a procedure for deciding when a torsion element in a complex algebraic torus belongs to a subtorus.

**Lemma 2.6.3.** Let $P \subset (\mathbb{C}^*)^r$ be a subtorus, and let $\eta \in (\mathbb{C}^*)^r$ be an element of finite order. Write $P = \exp(\chi \otimes \mathbb{C})$, where $\chi$ is a primitive lattice in $\mathbb{Z}^r$. Then $\eta \in P$ if and only if $\eta = \exp(2\pi i \lambda)$, for some $\lambda \in Q^r$ which virtually belongs to $\chi$.

**Proof.** Set $n = \text{rank } \chi$, and write $\eta = \exp(2\pi i \lambda)$, for some $\lambda = (\lambda_1, \ldots, \lambda_r) \in Q^r$. Let $\lambda_0$ be a generator of $\chi_0 = Q\lambda \cap Z^r$, and write $\lambda = q\lambda_0$. Finally, set $d = |\det [\chi \mid \chi_0]|$.

First suppose $d = 0$. Then, by Lemma 2.6.2(2), $\lambda$ virtually belongs to $\chi$. Since $\lambda \in \chi \otimes \mathbb{Q}$, we also have $\eta \in P$, and the claim is established in this case.

Now suppose $d \neq 0$. As in the proof of Proposition 2.2.5, we can choose a basis for $Z^r$ so that the inclusion of $\chi + \chi_0$ into $\chi + \chi_0$ has matrix of the form (2.2.8), with $D = d$. In this basis, the lattice $\chi$ is the span of the first $n$ coordinates, and $\chi_0$ lies in the span of the first $n + 1$ coordinates. Also in these coordinates, $\lambda_{n+1} = q$, and so $\eta_{n+1} = e^{2\pi i q}$; furthermore, $\eta_{n+2} = \cdots = \eta_r = 1$. On the other hand, $P = \{z \in (\mathbb{C}^*)^r \mid z_{n+1} = \cdots = z_r = 1\}$. Therefore, $\eta \in P$ if and only if $dq$ is an integer, which is equivalent to $d\lambda \in \chi_0$. \qed
2.6.2 Torsion-translated tori

We are now ready to state and prove the main results of this section (Theorem ?? from the Introduction).

**Theorem 2.6.4.** Let $\xi_1$ and $\xi_2$ be two sublattices in $\mathbb{Z}^r$. Set $\varepsilon = \xi_1 \cap \xi_2$, and write $\widehat{\varepsilon}/\varepsilon = \{\exp(2\pi i \mu_k)\}_{k=1}^s$. Also let $\eta_1$ and $\eta_2$ be two torsion elements in $(\mathbb{C}^*)^r$, and write $\eta_j = \exp(2\pi i \lambda_j)$. The following are equivalent:

1. The variety $Q = \eta_1 V(\xi_1) \cap \eta_2 V(\xi_2)$ is non-empty.

2. One of the vectors $\lambda_1 - \lambda_2 - \mu_k$ virtually belongs to the lattice $(\mathbb{Z}^r/\varepsilon)^\vee$.

If either condition is satisfied, then $Q = \rho V(\xi_1 + \xi_2)$, for some $\rho \in Q$.

**Proof.** Follows from Corollary 2.4.4 and Lemma 2.6.3. \qed

This theorem provides an efficient algorithm for checking whether two torsion-translated tori intersect. We conclude with an example illustrating this algorithm.

**Example 2.6.5.** Fix the standard basis $e_1, e_2, e_3$ for $\mathbb{Z}^3$, and consider the primitive sublattices $\xi_1 = \text{span}(e_1, e_2)$ and $\xi_2 = \text{span}(e_1, e_3)$. Then $\varepsilon = \text{span}(e_1)$ is also primitive, and so $s = 1$ and $\mu_1 = 0$. For selected values of $\eta_1, \eta_2 \in (\mathbb{C}^*)^3$, let us decide whether the set $Q = \eta_1 V(\xi_1) \cap \eta_2 V(\xi_2)$ is empty or not, using the above theorem.

First take $\eta_1 = (1, 1, 1)$ and $\eta_2 = (1, e^{2\pi i/3}, 1)$, and pick $\lambda_1 = 0$ and $\lambda_2 = (1, 1/3, 0)$. One easily sees that $d = \det\begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = -3$, and $d(\lambda_1 - \lambda_2 - \mu_1) = (3, 1, 0) \in \mathbb{Z}^3$. Thus, $Q \neq \emptyset$.

Next, take the same $\eta_1$ and $\lambda_1$, but take $\eta_2 = (e^{3\pi i/2}, 1, 1)$ and $\lambda_2 = (3/4, 0, 1)$. In this case, $d = \det\begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} = -3$ and $d(\lambda_1 - \lambda_2 - \mu_1) = (0, 0, 3) \notin \mathbb{Z}^3$. Thus, $Q = \emptyset$. 

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Chapter 3

Homological finiteness of abelian covers

3.1 A parameter set for regular abelian covers

We start by setting up a parameter set for regular covers of a CW-complex, with special emphasis on the case when the deck-transformation group is abelian.

3.1.1 Regular covers

Let $X$ be a connected CW-complex with finite 1-skeleton. Without loss of generality, we may assume $X$ has a single 0-cell, which we will take as our basepoint, call it $x_0$. Let $G = \pi_1(X, x_0)$ be the fundamental group. Since the space $X$ has only finitely many 1-cells, the group $G$ is finitely generated.

Consider an epimorphism $\nu: G \rightarrow A$ from $G$ to a (necessarily finitely generated) group $A$. Such an epimorphism gives rise to a regular cover of $X$, which we denote by $X^{\nu}$. Note that $X^{\nu}$ is also a connected CW-complex, the projection map $p: X^{\nu} \rightarrow X$ is cellular, and the group $A$ is the group of deck transformations of $X^{\nu}$.

Conversely, every (connected) regular cover $p: (Y, y_0) \rightarrow (X, x_0)$ with group of deck
transformations $A$ defines a normal subgroup $p_\gamma(p_1(Y, y_0)) \triangleleft \pi_1(X, x_0)$, with quotient group $A$. Moreover, if $\nu: G \to A$ is the projection map onto the quotient, then the cover $Y$ is equivalent to $X^\nu$, that is, there is an $A$-equivariant homeomorphism $Y \cong X^\nu$.

Let $\text{Epi}(G, A)$ be the set of all epimorphisms from $G$ to $A$, and let $\text{Aut}(A)$ be the automorphism group of $A$. The following lemma is standard.

**Lemma 3.1.1.** Let $G = \pi_1(X, x_0)$, and let $A$ be a group. The set of equivalence classes of connected, regular $A$-covers of $X$ is in one-to-one correspondence with

$$\Gamma(G, A) := \text{Epi}(G, A)/\text{Aut}(A),$$

the set of equivalence classes $[\nu]$ of epimorphisms $\nu: G \to A$, modulo the right-action of $\text{Aut}(A)$.

When the group $A$ is finite, the parameter set $\Gamma(G, A)$ is of course also finite. Efficient counting methods for determining the size of this set were pioneered by P. Hall in the 1930s. New techniques (involving, among other things, characteristic varieties over finite fields) were introduced in [59]. In the particular case when $A$ is a finite abelian group, a closed formula for the cardinality of $\Gamma(G, A)$ was given in [59], see Theorem 3.1.3 below.

### 3.1.2 Functoriality properties

The above construction enjoys some (partial) functoriality properties in both arguments. First, suppose that $\varphi: G_1 \to G_2$ is an epimorphism between two groups. Composition with $\varphi$ gives a map $\text{Epi}(G_2, A) \to \text{Epi}(G_1, A)$, which in turn induces a well-defined map,

$$\Gamma(G_2, A) \xrightarrow{\varphi^*} \Gamma(G_1, A), \quad [\nu] \mapsto [\nu \circ \varphi]. \quad (3.1.1)$$

Under the correspondence from Lemma 3.1.1, this map can be interpreted as follows. Let $f: (X_1, x_1) \to (X_2, x_2)$ be a basepoint-preserving map between connected CW-complexes, and suppose $f$ induces an epimorphism $\varphi = f_*: G_1 \to G_2$ on fundamental groups. Then the
map $\varphi^*$ sends the equivalence class of the cover $p_\nu: X_\nu^2 \to X_2$ to that of the pull-back cover, $p_{\nu \circ \varphi} = f^*(p_\nu): X_1^{\nu \circ \varphi} \to X_1$.

Next, recall that a subgroup $K < A$ is characteristic if $\alpha(K) = K$, for all $\alpha \in \text{Aut}(A)$.

**Lemma 3.1.2.** Suppose we have an exact sequence of groups, $1 \to K \to A \xrightarrow{\pi} B \to 1$, with $K$ a characteristic subgroup of $A$. There is then a well-defined map between the parameter sets for regular $A$-covers and $B$-covers of $X$,

$$\Gamma(G, A) \xrightarrow{\tilde{\pi}} \Gamma(G, B),$$

which sends $[\nu]$ to $[\pi \circ \nu]$.

**Proof.** Suppose $\nu_1, \nu_2: G \to A$ are two epimorphisms so that $\alpha \circ \nu_1 = \nu_2$, for some $\alpha \in \text{Aut}(A)$. Since the subgroup $K = \ker(\pi)$ is characteristic, the automorphism $\alpha$ induces and automorphism $\bar{\alpha} \in \text{Aut}(B)$ such that $\bar{\alpha} \circ \pi = \pi \circ \alpha$. Hence, $\bar{\alpha} \circ (\pi \circ \nu_1) = \pi \circ \nu_2$, showing that $q$ is well-defined.

The correspondence of Lemmas 3.1.1 and 3.1.2 is summarized in the following diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\nu} & A \\
\downarrow{\pi \circ \nu} & \xrightarrow{\pi} & B \\
& \uparrow{\pi} & \\
X & \xrightarrow{p_\nu} & X \\
\downarrow{p_{\pi \circ \nu}} & \quad & \downarrow{p_{\pi \circ \nu}} \\
\end{array}
$$

Under this correspondence, the map $\tilde{\pi}$ sends the equivalence class of the cover $p_\nu$ to that of the cover $p_{\pi \circ \nu}$.

### 3.1.3 Regular abelian covers

Let $H = G^{ab}$ be the abelianization of our group $G$. Recall we are assuming $G$ is finitely generated; thus, $H$ is a finitely generated abelian group.

Now suppose $A$ is any other (finitely generated) abelian group. In this case, every homomorphism $G \to A$ factors through the abelianization map, ab: $G \to H$. Composition with
this map gives a bijection between $\text{Epi}(H, A)$ and $\text{Epi}(G, A)$, which induces a bijection

$$\Gamma(H, A) \xrightarrow{\cong} \Gamma(G, A), \quad [\nu] \mapsto [\nu \circ \text{ab}].$$

(3.1.4)

In view of Lemma 3.1.1, we obtain a bijection between the set of equivalence classes of connected, regular $A$-covers of a CW-complex $X$ and the set $\Gamma(H, A)$, where $H = H_1(X, \mathbb{Z})$ is the abelianization of $G = \pi_1(X, x_0)$. Note that this parameter set is empty, unless $A$ is a quotient of $H$.

Let $\text{Tors}(A)$ be the torsion subgroup, consisting of finite-order elements in $A$; clearly, this is a characteristic subgroup of $A$. Let $\overline{A} = A/\text{Tors}(A)$ be the quotient group, and let $\pi: A \to \overline{A}$ be the canonical projection.

Under the correspondence from (3.1.4), an epimorphism $\nu: H \twoheadrightarrow A$ determines a regular cover, $p_{\nu \text{ab}}: X^{\nu \text{ab}} \to X$, which, for economy of notation, we will write as $p_{\nu}: X^\nu \to X$. There is also a free abelian cover, $p_{\overline{\nu}}: X^{\overline{\nu}} \to X$, corresponding to the epimorphism $\overline{\nu} = \pi \circ \nu: H \to \overline{A}$.

The projection $\pi: A \to \overline{A}$ defines a map $\text{Epi}(H, A) \to \text{Epi}(H, \overline{A}), \nu \mapsto \overline{\nu}$, which in turn induces maps between the parameter spaces for $A$ and $\overline{A}$ covers,

$$\Gamma(H, A) \xrightarrow{\overline{\eta}} \Gamma(H, \overline{A}),$$

(3.1.5)

sending the cover $p_{\nu}$ to the cover $p_{\overline{\nu}}$. Notice that this map is compatible with the morphism $\tilde{\pi}$ from (3.1.2), induced by an epimorphism $\pi: A \twoheadrightarrow B$ with characteristic kernel.

### 3.1.4 Splitting the torsion-free part

For a topological group $G$, let $G \to EG \to BG$ be the universal principal $G$-bundle; the total space $EG$ a contractible CW-complex endowed with a free $G$-action, while the base space $BG$ the quotient space under this action. We will only consider here the situation when $G$ is discrete, in which case $BG = K(G, 1)$ and $EG = \widetilde{K}(G, 1)$. 

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As before, let $H$ be a finitely generated abelian group. Let $\text{Tors}(H)$ be its torsion subgroup, and let $\overline{H} = H/\text{Tors}(H)$. Fix a splitting $\overline{H} \to H$; then $H \cong \overline{H} \oplus \text{Tors}(H)$.

Now consider an epimorphism $\nu: H \to A$. After fixing a splitting $\overline{A} \hookrightarrow A$, we obtain a decomposition $A \cong \overline{A} \oplus \text{Tors}(A)$. We may view $\overline{A}$ as a subgroup of $H$ by choosing a splitting $\overline{A} \hookrightarrow H$ of the projection $H \to \overline{A}$. With these identifications, $\nu$ induces an epimorphism $\tilde{\nu}: H/\overline{A} \to A/\overline{A}$. This observation leads us to consider the set

$$\Gamma = \Gamma(H/\overline{A}, A/\overline{A}). \quad \text{(3.1.6)}$$

Clearly, the set $\Gamma$ is finite. Theorem 3.1 from [59] yields an explicit formula for the size of this set. Given a finite abelian group $K$, and a prime $p$, write the $p$-torsion part of $K$ as $K_p = \mathbb{Z}_{p^{\lambda_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\lambda_s}}$, for some positive integers $\lambda_1 \geq \cdots \geq \lambda_s$, where $s = 0$ if $p \nmid |K|$. Thus, $K_p$ determines a partition $\pi(K_p) = (\lambda_1, \ldots, \lambda_s)$. For each such partition $\lambda$, write $l(\lambda) = s$, $|\lambda| = \sum_{i=1}^s \lambda_i$, and $\langle \lambda \rangle = \sum_{i=1}^s (i - 1) \lambda_i$.

**Theorem 3.1.3** ([59]). Set $n = \text{rank } H$ and $r = \text{rank } A$. For each prime $p$ dividing the order of $A/\overline{A}$, let $\lambda = \pi((A/\overline{A})_p)$ and $\tau = \pi(((\text{Tors}(H/\overline{A}))_p)$ be the corresponding partitions. Then,

$$|\Gamma(H/\overline{A}, A/\overline{A})| = \prod_{p \mid |A/\overline{A}|} p^{l(\lambda)} \left( \frac{p^{n-r} - p^{l(\lambda) - r}}{p^{l(\lambda)}} \prod_{i=1}^{l(\lambda)} \frac{p^{\frac{n-r}{i} - r} - p^{-1}}{\varphi_m(1) - 1} \right),$$

where $m_k(\lambda) = \# \{ i \mid \lambda_i = k \}$, $\varphi_m(t) = \prod_{i=1}^m (1 - t^i)$, $\lambda^-$ is the partition with $\lambda^-_i = \lambda_i - 1$, $\theta_1(\lambda, \tau) = \sum_{j=1}^{l(\tau)} \min(\lambda_j, \tau_j)$, and $\theta(\lambda, \tau) = \sum_{i=1}^{l(\lambda)} \theta_i(\lambda, \tau)$.

For instance, $|\Gamma(\mathbb{Z}^n, \mathbb{Z}_p^r)| = \frac{p^{n-r} - p^{l(\lambda) - r}}{p^{l(\lambda) - 1}}$, whereas $|\Gamma(\mathbb{Z}^n, \mathbb{Z}_p^s)| = \prod_{i=0}^{s-1} \frac{p^{n-r} - p^i}{p^{l(\lambda) - 1}}$.

Each homomorphism $H/\overline{A} \to A/\overline{A}$ defines an action of $H/\overline{A}$ on $A/\overline{A}$. This action yields a fiber bundle,

$$A/\overline{A} \to E(H/A) \times_{H/\overline{A}} A/\overline{A} \to B(H/\overline{A}) \quad \text{(3.1.7)}$$

associated to the principal bundle $H/\overline{A} \to E(H/\overline{A}) \to B(H/\overline{A})$. The set $\Gamma = \Gamma(H/\overline{A}, A/\overline{A})$ parameterizes all such associated bundles.
3.1.5 A pullback diagram

We return now to diagram (3.1.5), which relates the parameter sets for regular $A$ and $\overline{A}$ covers of a connected CW-complex $X$. This diagram can be further analyzed by using pullbacks from the universal principal bundles over the classifying spaces for the discrete groups $A$ and $\overline{A}$.

Let $A \to EA \to BA$ and $\overline{A} \to E\overline{A} \to B\overline{A}$ be the respective classifying bundles, and let $X \to BA$ and $X \to B\overline{A}$ be classifying maps for the covers $X^\nu \to X$ and $X^\varrho \to X$, respectively. Upon identifying $\text{Tors}(A)$ with $A/A$, we obtain the following diagram:

\[
\begin{array}{ccc}
\text{Tors}(A) & \xrightarrow{\cong} & A/A \\
\downarrow & & \downarrow \\
A & \xrightarrow{\pi} & X \xrightarrow{\beta} E(H/A) \times_{H/\overline{A}} A/\overline{A} \\
\downarrow & & \downarrow \\
\overline{A} & \xrightarrow{\pi} & X^\nu \xrightarrow{\alpha} B(H/\overline{A}) \\
\downarrow & & \downarrow \\
EA & \xrightarrow{p_{\varrho}} & X^\varrho \xrightarrow{\beta} B(H/A) \\
\downarrow & & \downarrow \\
BA & \xrightarrow{p_{\nu}} & X \xrightarrow{\beta} BH \\
\end{array}
\]

(3.1.8)

Here, the map $X \to BH$ realizes the abelianization morphism, $\text{ab}: \pi_1(X, x_0) \to H$, while $\alpha$ denotes the composite $X^\varrho \to X \to BH \to B(H/\overline{A})$.

**Proposition 3.1.4.** The marked square in diagram (3.1.8) is a pullback square. That is, the cover $p_{\varrho}: X^\nu \to X^\varrho$ is the pullback along the map $\alpha: X^\varrho \to B(H/\overline{A})$ of the cover $\beta: E(H/A) \times_{H/\overline{A}} A/\overline{A} \to B(H/\overline{A})$ corresponding to the epimorphism $\tilde{\nu}: H/\overline{A} \to A/\overline{A}$.

**Proof.** Clearly, $(\alpha \circ p_{\varrho})(\pi_1(X^\nu)) = \text{im}(\ker \nu \hookrightarrow \ker \tilde{\nu})$, while $\beta_{\pi}(\pi_1(E(H/A) \times_{H/\overline{A}} A/\overline{A})) = \ker(\tilde{\nu}: H/\overline{A} \to A/\overline{A})$. After picking a splitting $\overline{A} \hookrightarrow H$, and identifying the group $H$ with $\ker \tilde{\nu} \oplus \overline{A}$, we see that

\[(\alpha \circ p_{\varrho})(\pi_1(X^\nu)) \subseteq \beta_{\pi}(\pi_1(E(H/A) \times_{H/\overline{A}} A/\overline{A})).\]
The existence of the dashed arrow in the diagram follows then from the lifting criterion for covers. It is readily seen that this arrow is equivariant with respect to the actions on source and target by \( \text{Tors}(A) \) and \( A/A \); thus, a morphism of covers. This completes the proof. \( \Box \)

Using this proposition, and the discussion from §3.1.4, we obtain the following corollary.

**Corollary 3.1.5.** With notation as above,

\[
\Gamma(H/A, A/A) \longrightarrow \Gamma(H, A) \overset{q_H}{\longrightarrow} \Gamma(H, A)
\]

is a set fibration; that is, all fibers of \( q_H \) are in bijection with the set \( \Gamma(H/A, A/A) \).

Let us identify topologically the fiber of \( q_H \). A regular, \( \overline{A} \)-cover of our space \( X \) corresponds to an epimorphism \( \bar{\nu} : H \rightarrow \overline{A} \). A regular, \( \text{Tors}(A) \)-cover of \( X^\varphi \) corresponds to an epimorphism \( \ker \bar{\nu} \rightarrow \text{Tors}(A) \). Given these data, and the chosen splitting \( \overline{A} \hookrightarrow A \), we can find an epimorphism \( \nu : H \rightarrow A \), such that the following diagram commutes:

\[
\begin{array}{ccc}
\ker \bar{\nu} & \longrightarrow & H \overset{\varphi}{\longrightarrow} \overline{A} \\
\downarrow & & \downarrow \nu \\
\text{Tors}(A) & \longrightarrow & A \overset{\pi}{\longrightarrow} \overline{A}
\end{array}
\]

(3.1.9)

Thus, any regular, \( \text{Tors}(A) \)-cover \( X^\nu \rightarrow X^\varphi \) defines a regular \( A \)-cover \( X^\nu \rightarrow X \), whose corresponding free abelian cover is \( X^\varphi \). Consequently, the fiber of \([\bar{\nu}]\) under the map \( q_H : \Gamma(H, A) \rightarrow \Gamma(H, \overline{A}) \) coincides with the set

\[
\{ [\nu] \in \Gamma(H, A) \mid X^\nu \text{ is a regular } \text{Tors}(A) \text{-cover of } X^\varphi \}.
\]

(3.1.10)

### 3.2 Reinterpreting the parameter set for \( A \)-covers

In this section, we give a geometric description of the parameter set for regular abelian covers of a space.
3.2.1 Splittings

As before, let $H$ and $A$ be finitely generated abelian groups, and assume there is an epimorphism $H \twoheadrightarrow A$.

**Lemma 3.2.1.** The action $\text{Aut}(A)$ on the set of all splittings $A/Tors(A) \hookrightarrow A$ induced by the natural action of $\text{Aut}(A)$ on $A$ is transitive.

**Proof.** Set $\overline{A} = A/Tors(A)$, and fix a splitting $s: \overline{A} \hookrightarrow A$. Using this splitting, we may decompose the group $A$ as $\overline{A} \oplus Tors(A)$, and view $s: \overline{A} \hookrightarrow \overline{A} \oplus Tors(A)$ as the map $a \mapsto (a, 0)$.

An arbitrary splitting $\sigma: \overline{A} \hookrightarrow \overline{A} \oplus Tors(A)$ is given by $a \mapsto (a, \sigma_2(a))$, for some homomorphism $\sigma_2: \overline{A} \rightarrow Tors(A)$. Consider the automorphism of $\alpha \in \text{Aut}(\overline{A} \oplus Tors(A))$ given by the matrix $\begin{pmatrix} \text{id} & 0 \\ \sigma_2 & \text{id} \end{pmatrix}$. Clearly, $\alpha \circ s = \sigma$, and we are done. \qed

Denote by $n$ the rank of $H$, and by $r$ the rank of $A$. Fixing splittings $\overline{H} \hookrightarrow H$ and $\overline{A} \hookrightarrow A$, we have $H = \overline{H} \oplus Tors(H)$, with $\overline{H} = \mathbb{Z}^n$, and $A = \overline{A} \oplus Tors(A)$, with $\overline{A} = \mathbb{Z}^r$.

Now identify the automorphism group $\text{Aut}(\overline{H})$ with the general linear group $\text{GL}_n(\mathbb{Z})$. Let $P$ be the parabolic subgroup of $\text{GL}_n(\mathbb{Z})$, consisting all matrices of the form $\begin{pmatrix} \ast_1 & 0 \\ \ast_2 & \ast_3 \end{pmatrix}$, where $\ast_1$ is of size $(n - r) \times (n - r)$. Then $\text{GL}_n(\mathbb{Z})/P$ is isomorphic to the Grassmannian $\text{Gr}_{n-r}(\mathbb{Z})$. It is readily checked that the left action of $P$ on $\mathbb{Z}^{n-r}$, given by multiplication of $\{\ast_1\} \cong \text{GL}_{n-r}(\mathbb{Z})$ on $\mathbb{Z}^{n-r}$, induces an action of $P$ on the set $\Gamma = \Gamma(H/A, A/\overline{A})$. Also note that, if $A$ is torsion-free, then the set $\Gamma$ is a singleton.

3.2.2 A fibered product

We are now ready to state and prove the main result of this section.

**Theorem 3.2.2.** There is a bijection

$$\Gamma(H, A) \leftrightarrow \text{GL}_n(\mathbb{Z}) \times_p \Gamma$$
between the parameter set $\Gamma(H, A) = \text{Epi}(H, A)/\text{Aut}(A)$ and the twisted product of $\text{GL}_n(\mathbb{Z})$ with the set $\Gamma = \Gamma(H/\overline{A}, A/\overline{A})$ over the parabolic subgroup $P$. Under this bijection, the map $q: \Gamma(H, A) \to \Gamma(H, \overline{A})$ induced by the projection $\pi: A \to \overline{A}$ corresponds to the canonical projection

$$\text{GL}_n(\mathbb{Z}) \times_P \Gamma \to \text{GL}_n(\mathbb{Z})/P = \text{Gr}_{n-r}(\mathbb{Z}^n).$$

**Proof.** Define a map $\theta: \text{GL}_n(\mathbb{Z}) \times \Gamma \to \Gamma(H, A)$ as follows. Given an element $(M, [\gamma])$ of $\text{GL}_n(\mathbb{Z}) \times \Gamma$, let $(\gamma_1, \gamma_2)$ be a representative of $[\gamma]$, with $\gamma_1: \mathbb{Z}^{n-r} \to \text{Tors}(A)$ and $\gamma_2: \text{Tors}(H) \to \text{Tors}(A)$. Let $\alpha_1, \ldots, \alpha_n$ be the column vectors of the matrix $M$, which forms a basis of $H \cong \mathbb{Z}^n$, we can write $H = \mathbb{Z}^{n-r} \oplus \mathbb{Z}^r \oplus \text{Tors}(H)$, where $\mathbb{Z}^{n-r}$ is the subspace of $H$ generated by the first $n - r$ column vectors of $M$. Now define $\theta(M, [\gamma]) = [\nu]$, where $\nu: \mathbb{Z}^{n-r} \oplus \mathbb{Z}^r \oplus \text{Tors}(H) \to \mathbb{Z}^r \oplus \text{Tors}(A)$ is the homomorphism given by the matrix $N = (0 \ id \ 0 \ 0 \ \gamma_1 \ 0 \ \gamma_2)$. It is straightforward to check that the map $\theta$ is well-defined, i.e., $\theta$ is independent of the splitting and representative we chose.

Now let’s check that the map $\theta$ factors through $\text{GL}_n(\mathbb{Z}) \times_P \Gamma$. Suppose we have two elements $(M, [\gamma_1, \gamma_2])$ and $(M', [\gamma'_1, \gamma'_2])$ of $\text{GL}_n(\mathbb{Z}) \times \Gamma$ which are equivalent, that is, there is a matrix $Q = (Q_1 \ 0 \ Q_2) \in P$ such that $M = M'Q$ and $(\gamma_1 Q_1^{-1}, \gamma_2) = (\gamma'_1, \gamma'_2)$. By definition, the map $\theta$ takes the pair $(M, [\gamma_1, \gamma_2])$ to the homomorphism $\nu$ given by the matrix $N$ above. Changing the basis of $H$ to the basis given by the column vectors of $M'$, the map $\nu$ is given by the matrix

$$NQ^{-1} = \begin{pmatrix} 0 & Q_3^{-1} & 0 \\ \gamma_1 Q_1^{-1} & \gamma_1 Q_4 & \gamma_2 \end{pmatrix} = \begin{pmatrix} Q_3^{-1} & 0 \\ \gamma_1 Q_4 & \text{id} \end{pmatrix} \begin{pmatrix} 0 & \text{id} & 0 \\ \gamma'_1 & 0 & \gamma'_2 \end{pmatrix}.$$}

Clearly, the matrix $\begin{pmatrix} Q_3^{-1} & 0 \\ \gamma_1 Q_4 & \text{id} \end{pmatrix}$ defines an automorphism of $A$. Thus, the map $\theta$ factors through a well-defined map, $\theta: \text{GL}_n(\mathbb{Z}) \times_P \Gamma \to \Gamma(H, A)$, which is readily seen to be a bijection. It is now a straightforward matter to verify the last assertion. \[\square\]
3.2.3 Further discussion

A particular case of the above theorem is worth singling out. Recall $\mathcal{H} = \mathbb{Z}^n$ and $\mathcal{A} = \mathbb{Z}^r$.

**Corollary 3.2.3.** Suppose $\text{Tors}(H) = \mathbb{Z}_p^s$ and $\text{Tors}(A) = \mathbb{Z}_p^t$, for some prime $p$. Then, the parameter set $\Gamma(H, A)$ is in bijective correspondence with $\text{GL}_n(\mathbb{Z}) \times_p \text{Gr}_t(\mathbb{Z}_p^{n-r+s})$.

**Proof.** In this case, the set $\Gamma = \Gamma(H/A, A/A)$ is in bijection with $\text{Epi}(\mathbb{Z}_p^{n-r} \oplus \mathbb{Z}_p^s, \mathbb{Z}_p^t)/\text{Aut}(\mathbb{Z}_p^t)$. This bijection is established using the diagram

\[
\begin{array}{ccc}
\mathbb{Z}_p^{n-r} \oplus \mathbb{Z}_p^s & \rightarrow & \mathbb{Z}_p^t \\
\downarrow & & \downarrow \\
\mathbb{Z}_p^{n-r} \oplus \mathbb{Z}_p^s & \rightarrow & \mathbb{Z}_p^t
\end{array}
\]  

(3.2.1)

Therefore, $\Gamma = \text{Gr}_t(\mathbb{Z}_p^{n-r+s})$, and we are done. 

**Remark 3.2.4.** Consider the projection map $q: \text{GL}_n(\mathbb{Z}) \times_p \Gamma \rightarrow \text{GL}_n(\mathbb{Z})/\text{P} = \text{Gr}_{n-r}(\mathbb{Z})$ from Theorem 3.2.2. It is readily seen that, for each subspace $Q \in \text{Gr}_{n-r}(n, \mathbb{Z})$, the cardinality of the fiber $q^{-1}(Q)$ is the same as the the cardinality of the set $\Gamma$. In particular, $q^{-1}(Q)$ is finite, for all $Q \in \text{Gr}_{n-r}(\mathbb{Z})$.

3.3 Dwyer–Fried sets and their generalizations

In this section, we define a sequence of subsets $\Omega^i_A(X)$ of the parameter set for regular $A$-covers of $X$. These sets, which generalize the Dwyer–Fried sets $\Omega^i_r(X)$, keep track of the homological finiteness properties of those covers.

3.3.1 Generalized Dwyer–Fried sets

Throughout this section, $X$ will be a connected CW-complex with finite 1-skeleton, and $G = \pi_1(X, x_0)$ will denote its fundamental group.
Definition 3.3.1. For each group $A$ and integer $i \geq 0$, the corresponding Dwyer–Fried set of $X$ is defined as

$$\Omega^i_A(X) = \{ [\nu] \in \Gamma(G, A) \mid b_j(X^\nu) < \infty, \text{ for all } 0 \leq j \leq i \}.$$ 

In other words, the sets $\Omega^i_A(X)$ parameterize those regular $A$-covers of $X$ having finite Betti numbers up to degree $i$. In the particular case when $A$ is a free abelian group of rank $r$, we recover the standard Dwyer–Fried sets, $\Omega^i_r(X) = \Omega^i_{\mathbb{Z}^r}(X)$, viewed as subsets of the Grassmannian $\text{Gr}_r(\mathbb{Q}^n)$, where $n = b_1(X)$.

By our assumption on the 1-skeleton on $X$, the group $G = \pi_1(X, x_0)$ is finitely generated. Thus, we may assume $A$ is also finitely generated, for otherwise $\text{Epi}(G, A) = \emptyset$, and so $\Omega^i_A(X) = \emptyset$, too.

The $\Omega$-sets are invariant under homotopy equivalence. More precisely, we have the following lemma, which generalizes the analogous lemma for free abelian covers, proved in [81].

Lemma 3.3.2. Let $f : X \to Y$ be a (cellular) homotopy equivalence. For any group $A$, the homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induces a bijection $f^*_\sharp : \Gamma(\pi_1(Y, y_0), A) \to \Gamma(\pi_1(X, x_0), A)$, sending each subset $\Omega^i_A(Y)$ bijectively onto $\Omega^i_A(X)$.

Proof. Since $f$ is a homotopy equivalence, the induced homomorphism on fundamental groups, $f_*$, is a bijection. Thus, the corresponding map between parameter sets, $f^*_\sharp$, is a bijection. To finish the proof, it remains to verify that $f^*_\sharp(\Omega^i_A(Y)) = \Omega^i_A(X)$.

Let $\nu : \pi_1(Y, y_0) \to A$ be an epimorphism. Composing with $f_*$, we get an epimorphism $\nu \circ f_* : \pi_1(X, x_0) \to A$. By the lifting criterion, $f$ lifts to a map $\bar{f}$ between the respective $A$-covers. This map fits into the following pullback diagram:

$$\begin{array}{ccc}
X^{\nu \circ f_*} & \xrightarrow{f} & Y^\nu \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y.
\end{array} \quad (3.3.1)$$
Clearly, \( \tilde{f} : X^{\nu \circ f_2} \to Y^\nu \) is also a homotopy equivalence. Thus, \( b_j(Y^\nu) < \infty \) if and only if \( b_j(X^{\nu \circ f_2}) < \infty \), which means that \( f_*^\nu(\Omega^i_A(Y)) = \Omega^i_A(X) \).

Based on this lemma, we may define the \( \Omega \)-sets of a (discrete, finitely generated) group \( G \) as \( \Omega^i_A(G) := \Omega^i_A(BG) \), where \( BG = K(G, 1) \) is a classifying space for \( G \).

### 3.3.2 Naturality properties

The Dwyer–Fried sets (or their complements) enjoy certain naturality properties in both variables, which we now describe.

**Proposition 3.3.3.** Let \( \varphi : G \to Q \) be an epimorphism of groups. Then, for each group \( A \), there is an inclusion \( \Omega^1_A(Q)^c \hookrightarrow \Omega^1_A(G)^c \).

**Proof.** Let \( \nu : Q \to A \) be an epimorphism. Composing with \( \varphi \), we get an epimorphism \( \varphi \circ \varphi : G \to A \). So there is an epimorphism \( \varphi : \ker(\nu \circ \varphi) \to \ker(\nu) \). Taking abelianizations, we get an epimorphism \( \varphi_{ab} : \ker(\nu \circ \varphi)_{ab} \to \ker(\nu)_{ab} \). Thus, if \( \ker(\nu)_{ab} \) has infinite rank, then \( \ker(\nu \circ \varphi)_{ab} \) also has infinite rank. The desired conclusion follows.

Before proceeding, let us recall a well-known result regarding the homology of finite covers, which can be proved via a standard transfer argument (see, for instance, [37]).

**Lemma 3.3.4.** Let \( p : Y \to X \) be a regular cover defined by a properly discontinuous action of a finite group \( A \) on \( Y \), and let \( \mathbb{k} \) be a coefficient field of characteristic 0, or a prime not dividing the order of \( A \). Then, the induced homomorphism in cohomology, \( p^* : H^*(X; \mathbb{k}) \to H^*(Y; \mathbb{k}) \), is injective, with image the subgroup \( H^*(Y; \mathbb{k})^A \) consisting of those classes \( \alpha \) for which \( \gamma^*(\alpha) = \alpha \), for all \( \gamma \in A \).

**Corollary 3.3.5.** Let \( p : Y \to X \) be a finite, regular cover. Then \( b_i(X) \leq b_i(Y) \), for all \( i \geq 0 \).
Now fix a CW-complex $X$ as above, with fundamental group $G = \pi_1(X, x_0)$. Suppose $1 \to K \to A \xrightarrow{\pi} B \to 1$ is a short exact sequence of groups, with $K$ a characteristic subgroup of $A$. As noted in Lemma 3.1.2, the homomorphism $\pi$ induces a map $\tilde{\pi}: \Gamma(G, A) \to \Gamma(G, B)$, $[\nu] \mapsto [\pi \circ \nu]$, between the parameter sets for regular $A$-covers and $B$-covers of $X$.

**Proposition 3.3.6.** Suppose $K = \ker(\pi: A \to B)$ is a finite, characteristic subgroup of $A$. Then the map $\tilde{\pi}: \Gamma(G, A) \to \Gamma(G, B)$ restricts to a map $\tilde{\pi}: \Omega_i^A(X) \to \Omega_i^B(X)$ between the respective Dwyer–Fried sets.

**Proof.** Let $\nu: G \to A$ be an epimorphism, and suppose $[\nu] \in \Omega_i^A(X)$, that is, $b_j(X^\nu) < \infty$, for all $j \leq i$. Then $X^\nu \to X^{\pi \circ \nu}$ is a regular $K$-cover. By Corollary 3.3.5, we have that $b_j(X^{\pi \circ \nu}) < \infty$, for all $j \leq i$; in other words, $[\pi \circ \nu] \in \Omega_i^B(X)$. \hfill $\square$

Proposition 3.3.6 may be summarized in the following commuting diagram:

$$
\begin{array}{ccc}
\Omega_i^A(X) & \hookrightarrow & \Gamma(G, A) \\
\downarrow{\tilde{\pi}|_{\Omega_i^A(X)}} & & \downarrow{\tilde{\pi}} \\
\Omega_i^B(X) & \hookrightarrow & \Gamma(G, B)
\end{array}
$$

This diagram is a pullback diagram precisely when

$$
\tilde{\pi}^{-1}(\Omega_i^B(X)) = \Omega_i^A(X).
$$

As we shall see later on, this condition is not always satisfied. For now, let us just single out a simple situation when (3.3.2) is tautologically a pullback diagram.

**Corollary 3.3.7.** With notation as above, if $\Omega_i^B(X) = \emptyset$, then $\Omega_i^A(X) = \emptyset$.

### 3.3.3 Abelian versus free abelian covers

Let us now consider in more detail the case when $A$ is an abelian group. As usual, we are only interested in the case when $A$ is a quotient of the (finitely generated) group $H = G_{ab}$, and thus we may assume $A$ is also finitely generated.
Consider the exact sequence $0 \to \text{Tors}(A) \to A \to \overline{A} \to 0$. Clearly, $\text{Tors}(A)$ is a finite, characteristic subgroup of $A$. Thus, Proposition 3.3.6 applies, giving a map

$$q: \Omega^i_A(X) \to \Omega^i_{\overline{A}}(X).$$  \hspace{1cm} (3.3.4)

In particular, if $\Omega^i_A(X) = \emptyset$, then $\Omega^i_{\overline{A}}(X) = \emptyset$.

**Example 3.3.8.** Let $\Sigma_g$ be a Riemann surface of genus $g \geq 2$. It is readily seen that $\Omega^i_r(\Sigma_g) = \emptyset$, for all $r \geq 1$ and $i \geq 1$, cf. [81]. Thus, if $A$ is any finitely generated abelian group with rank $A \geq 1$, then $\Omega^i_A(\Sigma_g) = \emptyset$, for all $i \geq 1$.

Suppose now we have a short exact sequence $1 \to K \to A \xrightarrow{\pi} B \to 1$, with $K$ characteristic. Let $\overline{\pi}: \overline{A} \to \overline{B}$ be the induced epimorphism between maximal torsion-free quotients. Since $K = \ker(\pi)$ is finite, $\overline{\pi}$ is an isomorphism. Using Proposition 3.3.6 again, and the identification from (3.1.4), we obtain the following commutative diagram:

$$
\begin{array}{cccc}
\Omega^i_B(X) & \xrightarrow{\pi} & \Gamma(H, B) & \\
\Omega^i_A(X) & \xrightarrow{q_A} & \Gamma(H, A) & \\
\Omega^i_{\overline{A}}(X) & \xrightarrow{q_{\overline{B}}} & \Gamma(H, \overline{B}) & \\
\end{array}
$$

**Proposition 3.3.9.** Assume the function $\overline{\pi}: \Gamma(H, A) \to \Gamma(H, B)$ is surjective. Then, if the front square in diagram (3.3.5) is a pullback square, so is the back square; that is,

$$q_A^{-1}(\Omega^i_A) = \Omega^i_A \implies q_B^{-1}(\Omega^i_{\overline{B}}) = \Omega^i_B.$$

**Proof.** Suppose the back square is not pullback square. Then there exist elements $[\bar{\nu}] \in \Omega^i_{\overline{B}}$ and $[\nu] \in \Gamma(H, B) \setminus \Omega^i_B$ such that $q_B([\nu]) = [\bar{\nu}]$. By assumption, the map $\overline{\pi}$ is surjective; thus, $\overline{\pi}^{-1}([\nu])$ is nonempty. Pick an element $[\sigma] \in \overline{\pi}^{-1}([\nu])$. Then $[\sigma] \in \Gamma(H, A) \setminus \Omega^i_A$, for otherwise $[\nu] = \overline{\pi}([\sigma]) \in \Omega^i_B$. On the other hand,

$$q_A([\sigma]) = q_B(\overline{\pi}([\sigma])) = q_B([\nu]) = [\bar{\nu}] \in \Omega^i_{\overline{B}} = \Omega^i_B.$$

Thus, the front square is not a pullback diagram, either. \hfill \Box
3.3.4 The comparison diagram

Now fix a splitting $\overline{A} \hookrightarrow A$, which gives rise to an isomorphism $A \cong \overline{A} \oplus \text{Tors}(A)$. Similarly, after fixing a splitting $\overline{H} \hookrightarrow H$, the abelianization $H = G_{\text{ab}}$ also decomposes as $H \cong \overline{H} \oplus \text{Tors}(H)$. Theorem 3.2.2 yields an identification

$$\Gamma(H, A) \cong \text{GL}_n(\mathbb{Z}) \times_P \Gamma,$$

where $n = \text{rank } H$, the group $P$ is a parabolic subgroup of $\text{GL}_n(\mathbb{Z})$ so that $\text{GL}_n(\mathbb{Z})/P = \text{Gr}_{n-r}(\mathbb{Z}^n)$, and $\Gamma = \Gamma(H/\overline{A}, A/\overline{A})$.

Putting things together, we obtain a commutative diagram, which we shall refer to as the comparison diagram,

$$
\Omega^i_A(X) \xrightarrow{\partial^i} \Gamma(H, A) \cong \text{GL}_n(\mathbb{Z}) \times_P \Gamma \xrightarrow{q} \Omega^i_{\overline{A}}(X) \cong \Gamma(H, \overline{A}) \cong \text{Gr}_{n-r}(\mathbb{Z}^n)
$$

The next result reinterprets the condition that this diagram is a pull-back in terms of Betti numbers of abelian covers.

**Proposition 3.3.10.** The following conditions are equivalent:

(i) Diagram (3.3.7) is a pull-back diagram.

(ii) $q^{-1}(\Omega^i_{\overline{A}}(X)) = \Omega^i_A(X)$.

(iii) If $X^\nu$ is a regular $\overline{A}$-cover with finite Betti numbers up to degree $i$, then any regular $\text{Tors}(A)$-cover of $X^\nu$ has the same finiteness property.

**Proof.** The equivalence (i) $\iff$ (ii) is immediate. To prove (ii) $\iff$ (iii), consider an epimorphism $\nu: G \to A$, and let $\bar{\nu} = \pi \circ \nu: G \to \overline{A}$. We know from (3.1.10) that $q^{-1}(\bar{\nu})$ coincides with the set of equivalence classes of regular $\text{Tors}(A)$-covers $X^\nu \to X^\nu$. The desired conclusion follows. \qed
In other words, (3.3.7) is a pull-back diagram if and only if the homological finiteness of an arbitrary abelian cover of \( X \) can be tested through the corresponding free abelian cover.

### 3.4 Pontryagin duality

Following the approach from [38, 82], we now discuss a functorial correspondence between finitely generated abelian groups and abelian, complex algebraic reductive groups.

#### 3.4.1 A functorial correspondence

Let \( \mathbb{C}^* \) be the multiplicative group of units in the field of complex numbers. Given a group \( G \), let \( \hat{G} = \text{Hom}(G, \mathbb{C}^*) \) be the group of complex-valued characters of \( G \), with pointwise multiplication inherited from \( \mathbb{C}^* \), and identity the character taking constant value 1 \( \in \mathbb{C}^* \) for all \( g \in G \). If the group \( G \) is finitely generated, then \( \hat{G} \) is an abelian, complex reductive algebraic group. Given a homomorphism \( \varphi: G_1 \to G_2 \), let \( \hat{\varphi}: \hat{G}_2 \to \hat{G}_1 \), \( \rho \mapsto \rho \circ \varphi \) be the induced morphism between character groups. Since the group \( \mathbb{C}^* \) is divisible, the functor \( G \mapsto \hat{G} = \text{Hom}(H, \mathbb{C}^*) \) is exact.

Now let \( H = G_{ab} \) be the maximal abelian quotient of \( G \). The abelianization map, \( \text{ab}: G \to H \), induces an isomorphism \( \text{ab}: \hat{H} \cong \hat{G} \). If \( H \) is torsion-free, then \( \hat{H} \) can be identified with the complex algebraic torus \( (\mathbb{C}^*)^n \), where \( n = \text{rank}(H) \). If \( H \) is a finite abelian group, then \( \hat{H} \) is, in fact, isomorphic to \( H \).

More generally, let \( \overline{H} \) be the maximal torsion-free quotient of \( H \). Fixing a splitting \( \overline{H} \to H \) yields a decomposition \( H \cong \overline{H} \oplus \text{Tors}(H) \), and thus an isomorphism \( \hat{H} \cong \hat{\overline{H}} \times \text{Tors}(H) \).

For simplicity, write \( T = \hat{H} \), and \( T_0 \) for the identity component of this abelian, reductive, complex algebraic group; clearly, \( T_0 = \hat{H} \) is an algebraic torus.

Conversely, we can associate to \( T \) its weight group, \( \check{T} = \text{Hom}_{\text{alg}}(T, \mathbb{C}^*) \), where the hom set is taken in the category of algebraic groups. The (discrete) group \( \check{T} \) is a finitely generated
abelian group. Let $\mathbb{C}[\hat{T}]$ be its group algebra. We then have natural identifications,

$$\text{maxSpec} (\mathbb{C}[\hat{T}]) = \text{Hom}_{\text{alg}}(\mathbb{C}[\hat{T}], \mathbb{C}) = \text{Hom}_{\text{group}}(\hat{T}, \mathbb{C}^*) = T.$$  \hfill (3.4.1)

The correspondence $H \hookrightarrow \hat{T}$ extends to a duality

\[
\begin{array}{ccc}
\text{Subgroups of } H & \leftrightarrow & \text{Algebraic subgroups of } \hat{T} \\
\text{V} & & \epsilon \\
\end{array}
\]

where $V$ sends a subgroup $\xi \leq H$ to $\text{Hom}(H/\xi, \mathbb{C}^*) \subseteq \hat{T}$, while $\epsilon$ sends an algebraic subgroup $C \subseteq \hat{T}$ to $\ker(\hat{T} \rightarrow \hat{C}) \leq H$.

Both sides of (3.4.2) are partially ordered sets, with naturally defined meets and joins. As showed in [82], the above correspondence is an order-reversing equivalence of lattices.

### 3.4.2 Primitive lattices and connected subgroups

Given a subgroup $\xi \leq H$, set

$$\bar{\xi} := \{ x \in H \mid mx \in \xi \text{ for some } m \in \mathbb{N} \}.$$  \hfill (3.4.3)

Clearly, $\bar{\xi}$ is again a subgroup of $H$, and $H/\bar{\xi}$ is torsion-free. By definition, $\xi$ is a finite-index subgroup of $\bar{\xi}$; in particular, $\text{rank}(\xi) = \text{rank}(\bar{\xi})$. The quotient group, $\bar{\xi}/\xi$, called the determinant group of $\xi$, fits into the exact sequence

$$0 \longrightarrow H/\bar{\xi} \longrightarrow H/\xi \longrightarrow \bar{\xi}/\xi \longrightarrow 0.$$  \hfill (3.4.4)

The inclusion $\bar{\xi} \hookrightarrow H$ induces a splitting $\bar{\xi}/\xi \hookrightarrow H/\xi$, showing that $\bar{\xi}/\xi \cong \text{Tors}(H/\xi)$. Since the (abelian) group $\bar{\xi}/\xi$ is finite, it is isomorphic to its character group, $\hat{\bar{\xi}}/\xi$, which in turn can be viewed as a (finite) subgroup of $\hat{H} = T$.

The subgroup $\xi$ is called primitive if $\bar{\xi} = \xi$. Under the correspondence $H \hookrightarrow \hat{T}$, primitive subgroups of $H$ correspond to connected algebraic subgroups of $T$. For an arbitrary subgroup $\xi \leq H$, we have an isomorphism of algebraic groups,

$$V(\xi) \cong \hat{\bar{\xi}}/\xi \cdot V(\bar{\xi}).$$  \hfill (3.4.5)
In particular, the irreducible components of \( V(\xi) \) are indexed by the determinant group, \( \xi/\xi \), while the identity component is \( V(\bar{\xi}) \).

### 3.4.3 Pulling back algebraic subgroups

Now let \( \nu: H \to A \) be an epimorphism, and let \( \bar{\nu}: \bar{H} \to \bar{A} \) be the induced epimorphism between maximal torsion-free quotients. Applying the \( \text{Hom}(\_, \mathbb{C}^*) \) functor to the left square of (3.4.6) yields the commuting right square in the display below:

\[
\begin{array}{ccc}
H \xrightarrow{\nu} A & \sim & \hat{H} \xleftarrow{\hat{\nu}} \hat{A} \\
\downarrow & & \downarrow \\
\bar{H} \xrightarrow{\bar{\nu}} \bar{A} & & \bar{H} \xleftarrow{\bar{\hat{\nu}}} \bar{A}
\end{array}
\] (3.4.6)

The morphism \( \hat{\nu}: \hat{A} \to \hat{H} \) sends the identity component \( \hat{A}_0 \) to the identity component \( \hat{H}_0 \), thereby defining a morphism \( \hat{\nu}_0: \hat{A}_0 \to \hat{H}_0 \). Fixing a splitting \( \mathcal{A} \to A \) yields an isomorphism \( \hat{\mathcal{A}} \cong \hat{\mathcal{A}} \times \text{Tors}(A) \). The following lemma is now clear.

**Lemma 3.4.1.** Let \( \nu: H \to A \) be an epimorphism. Upon identifying \( \hat{\mathcal{A}} = \hat{A}_0 \) and \( \hat{\mathcal{H}} = \hat{H}_0 \), we have:

(i) \( \hat{\nu} = \hat{\nu}_0 \).

(ii) \( \text{im}(\hat{\nu}) = V(\ker(\nu)) \).

Consequently, \( \text{im}(\hat{\nu}) = V(\ker(\nu))_0 \).

### 3.4.4 Intersections of translated subgroups

Before proceeding, we need to recall some results from [82], which build on work of Hironaka [38]. In what follows, \( T \) will be an abelian, reductive complex algebraic group.

**Proposition 3.4.2 ([82]).** Let \( \xi_1 \) and \( \xi_2 \) be two subgroups of \( H \), and let \( \eta_1 \) and \( \eta_2 \) be two elements in \( T = \hat{H} \). Then \( \eta_1 V(\xi_1) \cap \eta_2 V(\xi_2) \neq \emptyset \) if and only if \( \eta_1 \eta_2^{-1} \in V(\xi_1 \cap \xi_2) \), in which
case
\[ \dim \eta_1 V(\xi_1) \cap \eta_2 V(\xi_2) = \text{rank } H - \text{rank}(\xi_1 + \xi_2). \]

**Proposition 3.4.3 ([82]).** Let \( C \) and \( V \) be two algebraic subgroups of \( T \).

(i) Suppose \( \alpha_1, \alpha_2, \) and \( \eta \) are torsion elements in \( T \) such that \( \alpha_i C \cap \eta V \neq \emptyset \), for \( i = 1, 2 \).

Then
\[ \dim (\alpha_1 C \cap \eta V) = \dim (\alpha_2 C \cap \eta V). \]

(ii) Suppose \( \alpha \) and \( \eta \) are torsion elements in \( T \), of coprime order. Then
\[ C \cap \eta V = \emptyset \implies \alpha C \cap \eta V = \emptyset. \]

Here is a corollary, which will be useful later on.

**Corollary 3.4.4.** Let \( C \) and \( V \) be two algebraic subgroups of \( T \). Suppose \( \alpha \) and \( \rho \) are torsion elements in \( T \), such that \( \rho \notin CV, \alpha^{-1} \rho \in CV, \) and \( \dim(C \cap V) > 0 \). Then \( C \cap \rho V = \emptyset \) and \( \dim(\alpha C \cap \rho V) > 0 \).

**Proof.** By Proposition 3.4.2,
\[ \rho \notin C \cdot V \iff C \cap \rho V = \emptyset \text{ and } \alpha^{-1} \rho \in C \cdot V \iff \alpha C \cap \rho V \neq \emptyset. \]

By Proposition 3.4.3, \( \dim(\alpha C \cap \rho V) = \dim(C \cap V) > 0 \). The conclusion follows.

### 3.5 An algebraic analogue of the exponential tangent cone

We now associate to each subvariety \( W \subset T \) and integer \( d \geq 1 \) a finite collection, \( \Xi_d(W) \), of subgroups of the weight group \( H = \tilde{T} \), which allows us to generalize the exponential tangent cone construction from [16].
3.5.1 A collection of subgroups

Let $T$ be an abelian, reductive, complex algebraic group, and consider a Zariski closed subset $W \subset T$. The translated subtori contained in $W$ define an interesting collection of subgroups of the discrete group $H = \hat{T}$.

**Definition 3.5.1.** Given a subvariety $W \subset T$, and a positive integer $d$, let $\Xi_d(W)$ be the collection of all subgroups $\xi \leq H$ for which the following two conditions are satisfied:

(i) The determinant group $\bar{\xi}/\xi$ is cyclic of order dividing $d$.

(ii) There is a generator $\eta \in \bar{\xi}/\xi$ such that $\eta \cdot V(\bar{\xi})$ is a maximal, positive-dimensional, torsion-translated subtorus in $W$.

Clearly, if $d | m$, then $\Xi_d(W) \subseteq \Xi_m(W)$. Although this is not a priori clear from the definition, we shall see in Proposition 3.5.8 that $\Xi_d(W)$ is finite, for each $d \geq 1$.

To gain more insight into this concept, let us work out what the sets $\Xi_d(W)$ look like in the case when $W$ is a coset of an algebraic subgroup of $T$.

**Lemma 3.5.2.** Suppose $W = \eta V(\chi)$, where $\chi$ is a subgroup of $H$ and $\eta \in \hat{H}$ is a torsion element. Write $V(\chi) = \bigcup_{\rho \in \bar{\chi}/\chi} \rho V(\chi)$. Then

$$\Xi_d(W) = \left\{ \xi \leq H \mid \exists \rho \in \bar{\chi}/\chi \text{ such that } \text{ord}(\eta \rho) | d \text{ and } \bar{H}/\xi = \bigcup_{m \geq 1} (\eta \rho)^m V(\chi) \right\}.$$ 

**Corollary 3.5.3.** If $\chi$ is a primitive subgroup of $H$ and $\eta$ is an element of order $d$ in $\hat{H}$, then $\Xi_d(\eta V(\chi))$ consists of the single subgroup $\xi \leq \chi$ for which $\bar{\xi} = \chi$ and $\bar{\chi}/\xi = \langle \eta \rangle$.

Now note that $\Xi_d$ commutes with unions: if $W_1$ and $W_2$ are two subvarieties of $T$, then

$$\Xi_d(W_1 \cup W_2) = \Xi_d(W_1) \cup \Xi_d(W_2). \tag{3.5.1}$$

Lemma 3.5.2, then, provides an algorithm for computing the sets $\Xi_d(W)$, whenever $W$ is a (finite) union of torsion-translated algebraic subgroups of $T$. 

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Example 3.5.4. Let $H = \mathbb{Z}^2$, and consider the subvariety $W = \{(t, 1) \mid t \in \mathbb{C}^*\} \cup \{(-1, t) \mid t \in \mathbb{C}^*\}$ inside $T = (\mathbb{C}^*)^2$. Note that $W = V(\xi_1) \cup \eta V(\xi_2)$, where $\xi_1 = 0 \oplus \mathbb{Z}$, $\xi_2 = \mathbb{Z} \oplus 0$, and $\eta = (-1, 1)$. Hence, $\Xi_d(W) = \{\xi_1\}$ if $d$ is odd, and $\Xi_d(W) = \{\xi_1, 2\xi_2\}$ if $d$ is even.

3.5.2 The exponential map

Consider now the lattice

$$\mathcal{H} = H^\vee := \text{Hom}(H, \mathbb{Z}).$$ (3.5.2)

Evidently, $\mathcal{H} \cong H/\text{Tors}(H)$. Moreover, each subgroup $\xi \leq H$ gives rise to a sublattice $(H/\xi)^\vee \leq H^\vee$.

Let $\text{Lie}(T)$ be the Lie algebra of the complex algebraic group $T$. The exponential map $\exp : \text{Lie}(T) \to T$ is an analytic map, whose image is $T_0$. Let us identify $T_0 = \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*)$ and $\text{Lie}(T) = \text{Hom}(\mathcal{H}^\vee, \mathbb{C})$. Under these identifications, the corestriction to the image of the exponential map can be written as

$$\exp = \text{Hom}(-, e^{2\pi i z}) : \text{Hom}(\mathcal{H}^\vee, \mathbb{C}) \to \text{Hom}(\mathcal{H}^\vee, \mathbb{C}^*),$$ (3.5.3)

where $z \mapsto e^z$ is the usual exponential map from $\mathbb{C}$ to $\mathbb{C}^*$. Finally, upon identifying $\text{Hom}(\mathcal{H}^\vee, \mathbb{C})$ with $\mathcal{H} \otimes \mathbb{C}$, we see that $T_0 = \exp(\mathcal{H} \otimes \mathbb{C})$.

The correspondence $T \leadsto \mathcal{H} = (\hat{T})^\vee$ sends an algebraic subgroup $W$ inside $T$ to the sublattice $\chi = (\hat{W})^\vee$ inside $\mathcal{H}$. Clearly, $\chi = \text{Lie}(W) \cap \mathcal{H}$ is a primitive lattice; furthermore, $\exp(\chi \otimes \mathbb{C}) = W_0$. As shown in [82], we have

$$V((\mathcal{H}/\chi)^\vee) = \exp(\chi \otimes \mathbb{C}),$$ (3.5.4)

where both sides are connected algebraic subgroups inside $T_0 = \exp(\mathcal{H} \otimes \mathbb{C})$. 

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3.5.3 Exponential interpretation

The construction from §3.5.1 allows us to associate to each subvariety $W \subset T$ and each integer $d \geq 1$ a subset $\tau_d(W) \subseteq H^\vee$, given by

$$
\tau_d(W) = \bigcup_{\xi \in \Xi_d(W)} (H/\xi)^\vee. \quad (3.5.5)
$$

The next lemma reinterprets the set $\tau_1(W)$ in terms of the “exponential tangent cone” construction introduced in [16] and studied in detail in [81].

Lemma 3.5.5. For every subvariety $W \subset T$,

$$
\tau_1(W) = \{ x \in H^\vee \mid \exp(\lambda x) \in W, \text{ for all } \lambda \in \mathbb{C} \}. \quad (3.5.6)
$$

Proof. Denote by $\tau$ the right-hand side of (3.5.6). Given a non-zero homomorphism $x: H \to \mathbb{Z}$ such that $x \in \tau$, the subgroup $\ker(x) \leq H$ is primitive and $V(\ker(x)) = \exp(\mathbb{C}x) \subseteq W$. Hence, we can find a subgroup $\xi \leq H$ such that $\xi$ is primitive, $V(\ker(x)) \subseteq V(\xi)$, and $V(\xi) \subseteq W$ is a maximal subtorus. By (3.5.4), we have that $V(\xi) = \exp((H/\xi)^\vee \otimes \mathbb{C})$, which implies $x \in \tau_1(W)$.

Conversely, for any non-zero element $x \in \tau_1(W)$, there is a subgroup $\xi \leq H$ such that $\xi \in \Xi_1(W)$ and $x \in (H/\xi)^\vee$. Thus, $\mathbb{C}x \subseteq (H/\xi)^\vee \otimes \mathbb{C}$, and so $\exp(\mathbb{C}x) \subseteq \exp((H/\xi)^\vee \otimes \mathbb{C}) = V(\xi) \subseteq W$. Since $x \neq 0$, the map $x: H \to \mathbb{Z}$ is surjective; thus, $V(\ker(x)) = V((H/\chi)^\vee)$, where $\chi$ is the rank 1 sublattice of $H$ generated by $x$. Hence, $V(\ker(x)) = \exp(\chi \otimes \mathbb{C}) \subseteq W$, and so $x \in \tau$.

Using now the characterization of exponential tangent cones given in [16, 81], we obtain the following immediate corollary.

Corollary 3.5.6. $\tau_1(W)$ is a finite union of subgroups of $H^\vee$.

Thus, the set $\tau_1^Q(W) = \bigcup_{\xi \in \Xi_1(W)} (H/\xi)^\vee \otimes \mathbb{Q}$ is a finite union of linear subspaces in the vector space $\mathbb{Q}^n$, where $n = \text{rank } H$.  

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Example 3.5.7. Let $T = (\mathbb{C}^*)^n$, and suppose $W = Z(f)$, for some Laurent polynomial $f$ in $n$ variables. Write $f(t_1, \ldots, t_n) = \sum_{a \in S} c_a t_1^{a_1} \cdots t_n^{a_n}$, where $S$ is a finite subset of $\mathbb{Z}^n$, and $c_a \in \mathbb{C}^*$ for each $a = (a_1, \ldots, a_n) \in S$. We say a partition $p = (p_1 \cdots | p_q)$ of the support $S$ is admissible if $\sum_{a \in p_j} c_a = 0$, for each $1 \leq j \leq q$. To such a partition, we associate the subgroup

$$L(p) = \{ x \in \mathbb{Z}^n \mid (a - b) \cdot x = 0, \forall a, b \in p_j, \forall 1 \leq j \leq q \}. \quad (3.5.7)$$

Then $\tau_1(W)$ is the union of all subgroups $L(p)$, where $p$ runs through the set of admissible partitions of $S$. In particular, if $f(1) \neq 0$, then $\tau_1(W) = \emptyset$.

Proposition 3.5.8. For each $d \geq 1$, the set $\Xi_d(W)$ is finite.

Proof. Fix an integer $d \geq 1$. For any torsion point $\eta \in T$ whose order divides $d$, consider the set $\Xi_d(W, \eta)$ of subgroups $\xi \leq H$ for which $\widetilde{\xi}/\xi = \langle \eta \rangle$ and $\eta \cdot V(\xi)$ is a maximal, positive-dimensional, torsion-translated subtorus in $W$. Then

$$\Xi_d(W) = \bigcup_{\eta} \Xi_d(W, \eta), \quad (3.5.8)$$

where the union runs over the (finite) set of torsion points $\eta \in T$ whose order divides $d$.

For each such point $\eta$, we have a map $\Xi_d(W, \eta) \to \Xi_1(\eta^{-1}W)$, $\xi \mapsto \widetilde{\xi}$. Clearly, this map is an injection. Now, Corollary 3.5.6 insures that the set $\Xi_1(W)$ is finite. Thus, the set $\Xi_1(\eta^{-1}W)$ is also finite, and we are done. \qed

3.6 The incidence correspondence for subgroups of $H$

We now single out certain subsets $\sigma_A(\xi)$ and $U_A(\xi)$ of the parameter set $\Gamma(H, A)$, which may be viewed as analogues of the special Schubert varieties in Grassmann geometry.

3.6.1 The sets $\sigma_A(\xi)$

We start by recalling a classical geometric construction. Let $V$ be a variety in $\mathbb{Q}^n$ defined by homogeneous polynomials. Set $m = \dim V$, and assume $m > 0$. Consider the locus of
$r$-planes in $\mathbb{Q}^n$ intersecting $V$ non-trivially,

$$\sigma_r(V) = \{ P \in \text{Gr}_r(\mathbb{Q}^n) \mid \dim(P \cap V) > 0 \}. \quad (3.6.1)$$

This set is a Zariski closed subset of $\text{Gr}_r(\mathbb{Q}^n)$, called the *variety of incident $r$-planes* to $V$. For all $0 < r < n - m$, this an irreducible subvariety, of dimension $(r - 1)(n - r) + m - 1$.

Particularly simple is the case when $V$ is a linear subspace $L \subset \mathbb{Q}^n$. The corresponding incidence variety, $\sigma_r(L)$, is known as the *special Schubert variety* defined by $L$. Clearly, $\sigma_1(L) = \mathbb{P}(L)$, viewed as a projective subspace in $\mathbb{Q}P^{n-1} := \mathbb{P}(\mathbb{Q}^n)$.

Now let $H$ be a finitely generated abelian group, let $A$ be a factor group of $H$, and let $\Gamma(H, A) = \text{Epi}(H, A)/\text{Aut}(A)$.

**Definition 3.6.1.** Given a subgroup $\xi \leq H$, let $\sigma_A(\xi)$ be the set of all $[\nu] \in \Gamma(H, A)$ for which $\text{rank}(\ker(\nu) + \xi) < \text{rank} H$.

When $A$ is torsion-free, we recover the classical definition of special Schubert varieties. More precisely, set $n = \text{rank} H$ and $r = \text{rank} A$. We then have the following lemma.

**Lemma 3.6.2.** Under the natural isomorphism $\Gamma(H, \overline{A}) \cong \text{Gr}_r(\mathbb{Q}^n)$, the set $\sigma_{\overline{A}}(\xi)$ corresponds to the special Schubert variety $\sigma_r((H/\xi)^\vee \otimes \mathbb{Q})$.

**Proof.** Let $\text{Gr}_r(H^\vee \otimes \mathbb{Q})$ be the Grassmannian of $r$-dimensional subspaces in the vector space $H^\vee \otimes \mathbb{Q} \cong \mathbb{Q}^n$. Given an epimorphism $\nu: H \twoheadrightarrow \overline{A}$ and a subgroup $\xi \leq H$, we have

$$\text{rank}(\ker(\nu) + \xi) < \text{rank} H \iff \dim((H/\ker(\nu))^\vee \otimes \mathbb{Q} \cap (H/\xi)^\vee \otimes \mathbb{Q}) > 0.$$ 

Thus, the isomorphism

$$\Gamma(H, \overline{A}) \cong \text{Gr}_r(H^\vee \otimes \mathbb{Q}), \quad [\nu] \mapsto (H/\ker(\nu))^\vee \otimes \mathbb{Q}$$

establishes a one-to-one correspondence between $\sigma_{\overline{A}}(\xi)$ and $\sigma_r((H/\xi)^\vee \otimes \mathbb{Q})$. \hfill $\square$

For instance, if $A$ is infinite cyclic, then the parameter set $\Gamma(H, \mathbb{Z})$ may be identified with the projective space $\mathbb{P}(H^\vee)$, while the set $\sigma_{\mathbb{Z}}(\xi)$ coincides with the projective subspace $\mathbb{P}((H/\xi)^\vee)$.

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Example 3.6.3. Let $\xi \leq \mathbb{Z}^2$ be the sublattice spanned by the vector $(a, b) \in \mathbb{Z}^2$. Then $\sigma_Z(\xi) \subset \Gamma(\mathbb{Z}^2, \mathbb{Z})$ corresponds to the point $(-b, a) \in \mathbb{Q}\mathbb{P}^1$.

The sets $\sigma_A(\xi)$ can be reconstructed from the classical Schubert varieties $\sigma_{\overline{A}}(\xi)$ associated to the lattice $\overline{A} = A/\text{Tors}(A)$ by means of the set fibration described in Theorem 3.2.2. More precisely, we have the following proposition.

Proposition 3.6.4. Let $q: \Gamma(H, A) \to \Gamma(H, \overline{A})$ be the natural projection map. Then

(i) $q(\sigma_A(\xi)) = \sigma_{\overline{A}}(\xi)$.

(ii) $q^{-1}(\sigma_{\overline{A}}(\xi)) = \sigma_A(\xi)$.

Therefore, $\sigma_A(\xi)$ fibers over the Schubert variety $\sigma_{\overline{A}}(\xi)$, with each fiber isomorphic to the set $\Gamma(H/\overline{A}, A/\overline{A})$.

3.6.2 The sets $U_A(\xi)$

Although simple to describe, the sets $\sigma_A(\xi)$ do not behave too well with respect to the correspondence between subgroups of $H$ and algebraic subgroups of $T = \hat{H}$. This is mainly due to the fact that the $\sigma_A$-sets do not distinguish between a subgroup $\xi \leq H$ and its primitive closure, $\overline{\xi}$. To remedy this situation, we identify certain subsets $U_A(\xi) \subseteq \sigma_A(\xi)$ which turn out to be better suited for our purposes.

Definition 3.6.5. Given a subgroup $\xi \leq H$, let $U_A(\xi)$ be the set of all $[\nu] \in \sigma_A(\xi)$ for which $\ker(\nu) \cap \overline{\xi} \subseteq \xi$.

In particular, if $\overline{\xi} = \xi$, then $U_A(\xi) = \sigma_A(\xi)$. In general, though, $U_A(\xi) \subseteq \sigma_A(\xi)$. In order to reinterpret this definition in more geometric terms, we need a lemma.

Lemma 3.6.6. Let $\xi \leq H$ be a subgroup such that $\overline{\xi}/\xi$ is cyclic, and let $\chi \leq H$ be another a subgroup. Then the following conditions are equivalent.
(i) $\chi \cap \bar{\xi} \subseteq \xi$.

(ii) $V(\chi) \cap \eta V(\bar{\xi}) \neq \emptyset$, for some generator $\eta$ of $\widehat{\xi}/\xi$.

(iii) $V(\chi) \cap \eta V(\bar{\xi}) \neq \emptyset$, for any generator $\eta$ of $\widehat{\xi}/\xi$.

(iv) $\eta \in V(\ker(\nu) \cap \xi)$, for some generator $\eta$ of $\widehat{\xi}/\xi$.

(v) $\eta \in V(\ker(\nu) \cap \xi)$, for any generator $\eta$ of $\widehat{\xi}/\xi$.

Proof. Let $\eta$ be a generator of the finite cyclic group $\widehat{\xi}/\xi$. We then have

$$\epsilon(\langle \eta \rangle) \cap \bar{\xi} = \xi. \quad (3.6.2)$$

By Proposition 3.4.2, the intersection $V(\chi) \cap \eta V(\bar{\xi})$ is non-empty if and only if $\eta \in V(\chi \cap \bar{\xi})$, that is, $\langle \eta \rangle \subseteq V(\chi \cap \bar{\xi})$, which in turn is equivalent to

$$\epsilon(\langle \eta \rangle) \supseteq \chi \cap \bar{\xi}. \quad (3.6.3)$$

In view of equality (3.6.2), inclusion (3.6.3) is equivalent to $\chi \cap \bar{\xi} \subseteq \xi$. This shows $(5.1.5) \iff (5.2.1)$.

The other equivalences are proved similarly.

\[ \square \]

**Corollary 3.6.7.** Let $\xi \leq H$ be a subgroup, and assume $\bar{\xi}/\xi$ is cyclic. Let $\nu: H \to A$ be an epimorphism. Then

$$[\nu] \in U_A(\xi) \iff \dim (V(\ker(\nu)) \cap \eta V(\bar{\xi})) > 0$$

for any (or, equivalently, for some) generator $\eta \in \widehat{\xi}/\xi$.

Despite their geometric appeal, the sets $U_A(\xi)$ do not enjoy a naturality property analogous to the one from Proposition 3.6.4. In particular, the projection map $q: \Gamma(H, A) \to \Gamma(H, \bar{A})$ may not restrict to a map $U_A(\xi) \to U_{\bar{A}}(\xi)$. Here is a simple example.

**Example 3.6.8.** Let $\nu: H \to A$ be the canonical projection from $H = \mathbb{Z}^2$ to $A = \mathbb{Z} \oplus \mathbb{Z}_2$, and let $\xi = \ker(\nu)$. Then $[\nu] \in U_A(\xi)$, but $[\bar{\nu}] \notin U_{\bar{A}}(\xi)$. 

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3.7 The incidence correspondence for subvarieties of $\hat{H}$

In this section, we introduce and study certain subsets $\Upsilon_A(W) \subseteq \Gamma(H, A)$, which can be viewed as the toric analogues of the classical incidence varieties $\sigma_r(V) \subseteq \text{Gr}_r(\mathbb{Q}^n)$.

3.7.1 The sets $U_A(W)$

Let us start by recalling some constructions we discussed previously. In §3.5.1, we associated to each subvariety $W$ of the algebraic group $T = \hat{H}$ and each integer $d \geq 1$ a certain collection $\Xi_d(W)$ of subgroups of $H$. In §3.6.2, we associated to each subgroup $\xi \leq H$ and each abelian group $A$ a certain subset $U_A(\xi)$ of the parameter set $\Gamma(H, A) = \text{Epi}(H, A)/\text{Aut}(A)$. Putting together these two constructions, we associate now to $W$ a family of subsets of $\Gamma(H, A)$, as follows.

**Definition 3.7.1.** Given a subvariety $W \subset T$, an abelian group $A$, let

$$U_A(W) = \bigcup_{d \geq 1} U_{A,d}(W),$$

(3.7.1)

where

$$U_{A,d}(W) = \bigcup_{\xi \in \Xi_d(W)} U_A(\xi).$$

(3.7.2)

By Proposition 3.5.8, the union in (3.7.2) is a finite union.

**Lemma 3.7.2.** The set $U_{A,d}(W)$ consists of all $[\nu] \in \Gamma(H, A)$ for which there is a subgroup $\xi \leq H$ and an element $\eta \in \hat{H}$ of order dividing $d$ such that $\eta V(\xi)$ is a maximal, positive-dimensional translated subtorus in $W$, and $\dim (V(\ker(\nu)) \cap \eta V(\xi)) > 0$.

**Proof.** Let $\nu: H \twoheadrightarrow A$ be an epimorphism such that $[\nu] \in U_A(\xi)$, for some $\xi \in \Xi_d(W)$. According to Definition 3.5.1, this means that the group $\tilde{\xi}/\xi$ is cyclic of order dividing $d$, and there is a generator $\eta \in \tilde{\xi}/\xi$ such that $\eta V(\xi)$ is a maximal, positive-dimensional translated subtorus in $W$. In view of Corollary 3.6.7, the fact that $[\nu] \in U_A(\xi)$ insures that $V(\ker(\nu)) \cap \eta V(\xi)$ has positive dimension. □
The case \( d = 1 \) is worth singling out.

**Corollary 3.7.3.** Let \( W \subset T \) be a subvariety. Set \( n = \text{rank } H \) and \( r = \text{rank } A \). Then:

(i) \( U_{A,1}(W) = \bigcup_{\xi \in \Xi_1(W)} \sigma_A(\xi) \).

(ii) Under the isomorphism \( \Gamma(H,A) \cong \text{Gr}_r(\mathbb{Q}^n) \), the set \( U_{A,1}(W) \) corresponds to the incidence variety \( \sigma_r(\tau_{1}^{\mathbb{Q}}(W)) \).

In general, the set \( U_A(W) \) is larger than \( U_{A,1}(W) \). Here is a simple example; a more general situation will be studied in §3.11.3.

**Example 3.7.4.** Let \( H = \mathbb{Z}^2 \) and let \( W \subset (\mathbb{C}^*)^2 \) be the subvariety from Example 3.5.4. Pick \( A = \mathbb{Z} \oplus \mathbb{Z}_2 \), and identify \( \Gamma(H,A) \) with \( \mathbb{Q} \mathbb{P}^1 \). Then \( U_{A,d}(W) = \{(1,0)\} \) or \( \{(1,0),(0,1)\} \), according to whether \( d \) is odd or even.

### 3.7.2 The sets \( \Upsilon_A(W) \)

As before, let \( H \) be a finitely-generated abelian group, and let \( T = \widehat{H} \) be its Pontryagin dual. The next definition will prove to be key to the geometric interpretation of the (generalized) Dwyer–Fried invariants.

**Definition 3.7.5.** Given a subvariety \( W \subset T \), and an abelian group \( A \), define a subset \( \Upsilon_A(W) \) of the parameter set \( \Gamma(H,A) \) by setting

\[
\Upsilon_A(W) = \{ [\nu] \in \Gamma(H,A) \mid \text{dim}(V(\ker \nu) \cap W) > 0 \}.
\]  

Roughly speaking, the set \( \Upsilon_A(W) \subset \Gamma(H,A) \) associated to a variety \( W \subset T \) is the toric analogue of the incidence variety \( \sigma_r(V) \subset \text{Gr}_r(\mathbb{Q}^n) \) associated to a homogeneous variety \( V \subset \mathbb{Q}^n \).

It is readily seen that \( \Upsilon_A \) commutes with unions: if \( W_1 \) and \( W_2 \) are two subvarieties of \( T \), then

\[
\Upsilon_A(W_1 \cup W_2) = \Upsilon_A(W_1) \cup \Upsilon_A(W_2).
\]
Moreover, $\Upsilon_A(W)$ depends only on the positive-dimensional components of $W$. Indeed, if $Z$ is a finite algebraic set, then $\Upsilon_A(W \cup Z) = \Upsilon_A(W)$.

The next result gives a convenient lower bound for the $\Upsilon$-sets.

**Proposition 3.7.6.** Let $A$ be a quotient of $H$. Then

$$U_A(W) \subseteq \Upsilon_A(W). \tag{3.7.5}$$

**Proof.** Let $\nu: H \twoheadrightarrow A$ be an epimorphism such that $[\nu] \in U_{A,d}(W)$, for some $d \geq 1$. By Lemma 3.7.2, we have that $\dim(V(\ker \nu) \cap W) > 0$. Thus, $[\nu] \in \Upsilon_A(W)$. \qed

As we shall see in Example 3.9.8, inclusion (3.7.5) may well be strict.

### 3.7.3 Translated subgroups

If the variety $W$ is a torsion-translated algebraic subgroup of $T$, we can be more precise.

**Theorem 3.7.7.** Let $W = \eta V(\xi)$, where $\xi \leq H$ is a subgroup, and $\eta \in \hat{H}$ has finite order. Then $\Upsilon_A(W) = U_{A,c}(W)$, where $c = \text{ord}(\eta) \cdot c(\xi/\xi)$.

**Proof.** Inclusion $\supseteq$ follows from Proposition 3.7.6, so we only need to prove the opposite inclusion. Write

$$V(\xi) = \bigcup_{\rho \in \xi/\xi} \rho V(\xi).$$

Let $\nu: H \twoheadrightarrow A$ be an epimorphism such that $[\nu] \in \Upsilon_A(\eta V(\xi))$. Hence, there is a character $\rho: \xi/\xi \rightarrow \mathbb{C}^*$ such that $\dim(V(\ker(\nu)) \cap \eta \rho V(\xi)) > 0$. Consider the subgroup

$$\chi = \epsilon \left( \bigcup_m (\eta \rho)^m V(\xi) \right).$$

Lemma 3.5.2 implies that $\chi \in \Xi_d(\eta V(\xi))$, where $d = \text{ord}(\eta \rho)$. Using Corollary 3.6.7, we conclude that $[\nu] \in U_A(\chi)$.

Finally, set $c = \text{ord}(\eta) \cdot c(\xi/\xi)$. Then clearly $[\nu] \in U_{A,c}(W)$, and we are done. \qed
Here is an alternate description of the set $\Upsilon_A(W)$, in the case when $W$ is an algebraic subgroup of $T$, translated by an element $\eta \in T$, not necessarily of finite order.

**Theorem 3.7.8.** Let $\xi \leq H$ be a subgroup, and let $\eta \in \hat{H}$. Then

$$\Upsilon_A(\eta V(\xi)) = \sigma_A(\xi) \cap \{[\nu] \in \Gamma(H, A) \mid \eta \in V(\ker(\nu) \cap \xi)\}.$$  

In particular, $\Upsilon_A(V(\xi)) = \sigma_A(\xi)$.

**Proof.** By Proposition 3.4.2, we have

$$\{[\nu] \mid \dim(V(\ker(\nu)) \cap \eta V(\xi)) > 0\} = \{[\nu] \mid V(\ker(\nu)) \cap \eta V(\xi) \neq \emptyset\} \cap \{[\nu] \mid \text{rank}(\ker(\nu) + \xi) < \text{rank } H\}.$$  

Moreover, $V(\ker(\nu)) \cap \eta V(\xi) \neq \emptyset \iff \eta \in V(\ker(\nu) \cap \xi)$, and we are done. \qed

**Remark 3.7.9.** In the case when $A$ is a free abelian group of rank $r$ and $\xi$ is a primitive subgroup of $H = \mathbb{Z}^n$, the set $\Upsilon_A(\eta V(\xi))$ coincides with the set $\sigma_r(\xi, \eta)$ defined in [81].

When the translation factor $\eta$ from Theorem 3.7.8 has finite order, a bit more can be said.

**Corollary 3.7.10.** Let $W = \eta V(\xi)$ be a torsion-translated subgroup of $T$. Then

$$\Upsilon_A(W) = \sigma_A(\xi) \cap \{[\nu] \in \Gamma(H, A) \mid \epsilon(\langle \eta \rangle) \supseteq \ker(\nu) \cap \xi\}.$$  

### 3.7.4 Deleted subgroups

We now analyze in more detail the case when the variety $W$ is obtained from an algebraic subgroup of $T$ by deleting its identity component. First, we need to introduce one more bit of notation.

**Definition 3.7.11.** Given a subgroup $\xi \leq H$, and a quotient $A$ of $H$, consider the subset $\theta_A(\xi) \subseteq \Gamma(H, A)$ given by

$$\theta_A(\xi) = \bigcup_{\xi \leq \xi' \leq \xi \leq \xi' \text{ is cyclic}} \{[\nu] \in \Gamma(H, A) \mid \nu(x) \neq 0 \text{ for all } x \in \xi \setminus \xi'\}.$$  

(3.7.6)
Note that the indexing set for this union is a finite set, which is empty if $\xi$ is primitive. On the other hand, the condition that $\nu(x) \neq 0$ depends on the actual element $x$ in the (typically) infinite set $\overline{\xi} \setminus \xi'$, not just on the class of $x$ in the finite group $\overline{\xi}/\xi'$. Thus, even when $A = \mathbb{Z}^r$, the set $\theta_A(\xi)$ need not be open in the Grassmannian $\Gamma(H, A) = \text{Gr}_r(\mathbb{Q}^n)$, where $n = \text{rank}(H)$, although each of the sets $\{[\nu] \mid \nu(x) \neq 0\}$ is open.

**Proposition 3.7.12.** Suppose $W = V(\xi) \setminus V(\overline{\xi})$, for some non-primitive subgroup $\xi \leq H$. Then

$$\Upsilon_A(W) = \sigma_A(\xi) \cap \theta_A(\xi).$$  \hfill (3.7.7)

**Proof.** Write $W = \bigcup_{\eta \in \overline{\xi}/\xi \setminus \{1\}} \eta V(\overline{\xi})$. By Theorem 3.7.8 and Lemma 3.6.6, we have

$$\Upsilon_A(W) = \Upsilon_A \left( \bigcup_{\eta \in \overline{\xi}/\xi \setminus \{1\}} \eta V(\overline{\xi}) \right)$$

$$= \bigcup_{\eta \in \overline{\xi}/\xi \setminus \{1\}} \left( \sigma_A(\overline{\xi}) \cap \{[\nu] \in \Gamma(H, A) \mid \eta \in V(\ker(\nu) \cap \overline{\xi})\} \right)$$

$$= \sigma_A(\overline{\xi}) \cap \bigcup_{\eta \in \overline{\xi}/\xi \setminus \{1\}} \{[\nu] \in \Gamma(H, A) \mid \ker(\nu) \cap \overline{\xi} \neq \emptyset\}$$

$$= \sigma_A(\xi) \cap \bigcup_{\xi \leq \xi' \leq \overline{\xi} : \xi/\xi' \text{ is cyclic}} \{[\nu] \in \Gamma(H, A) \mid \ker(\nu) \cap \overline{\xi} \leq \xi'\}.$$

The desired conclusion follows at once. \(\square\)

### 3.7.5 Comparing the sets $\Upsilon_A(W)$ and $\Upsilon_{\overline{A}}(W)$

Fix a decomposition $A = \overline{A} \oplus \text{Tors}(A)$. Clearly, the projection map $q = q_A : \Gamma(H, A) \to \Gamma(H, \overline{A})$ sends $\Upsilon_A(W)^c$ to $\Upsilon_{\overline{A}}(W)^c$. On the other hand, as we shall see in Example 3.9.8, the map $q$ does not always send $\Upsilon_A(W)$ to $\Upsilon_{\overline{A}}(W)$. Nevertheless, in some special cases it does. Here is one such situation.

**Proposition 3.7.13.** Suppose $W = \rho T$, where $T \subset \hat{H}$ is an algebraic subgroup, and $\rho \in \hat{H}/T$ has finite order, coprime to the order of $\text{Tors}(A)$. Then

$$q(\Upsilon_A(W)) = \Upsilon_{\overline{A}}(W) \quad \text{and} \quad q^{-1}(\Upsilon_{\overline{A}}(W)) = \Upsilon_A(W).$$
Therefore, $\Upsilon_A(W)$ fibers over $\Upsilon_\pi(W)$, with each fiber isomorphic to $\Gamma(H/\overline{A}, A/\overline{A})$.

Here, $\bar{\rho}$ is the image of $\rho$ under the quotient map $\hat{H} \to \hat{H}/T$.

**Proof.** Let $\nu: H \to A$ be an epimorphism such that $[\bar{\nu}] = q([\nu])$ does not belong to $\Upsilon_\pi(W)$, that is, the subtorus $\text{im}(\hat{\nu}) = \text{im}(\hat{\nu})_1$ intersects $W$ in only finitely many points. We want to show that $\text{im}(\hat{\nu})_\alpha \cap W$ is also finite, for all $\alpha \in \text{Tors}(A)$.

First assume $\text{im}(\hat{\nu})_1 \cap W$ is non-empty. If $\text{im}(\hat{\nu})_\alpha \cap W = \emptyset$, we are done. Otherwise, using Proposition 3.4.3(i) with $C = \text{im}(\hat{\nu})_1$, $\alpha_1 = 1$, $\alpha_2 = \alpha$ and $\eta V = \rho T$, we infer that $\dim (\text{im}(\hat{\nu})_\alpha \cap \rho T) = \dim (\text{im}(\hat{\nu})_1 \cap \rho T)$, and the desired conclusion follows.

Now assume $\text{im}(\hat{\nu})_1 \cap W$ is empty. Using Proposition 3.4.3(ii) with $C = \text{im}(\hat{\nu})_1$ and $\eta V = \rho T$, our assumption that $\text{im}(\hat{\nu})_1 \cap \rho T = \emptyset$ implies that $\text{im}(\hat{\nu})_\alpha \cap \rho T = \emptyset$. Thus, the desired conclusion follows in this case, too, and we are done. $\square$

**Corollary 3.7.14.** For every subgroup $\xi \leq H$, the set $q(\Upsilon_A(V(\xi)))$ is contained in $\Upsilon_\pi(V(\xi))$.

In general, though, the projection map $q: \Gamma(H, A) \to \Gamma(H, \overline{A})$ does not restrict to a map $\Upsilon_A(W) \to \Upsilon_\pi(W)$. Proposition 3.7.16 below describes a situation when this happens. First, we need a lemma, whose proof is similar to the proof of Proposition 3.3.9.

**Lemma 3.7.15.** Let $\pi: A \to B$ be an epimorphism, and let $\bar{\pi}: \Gamma(H, A) \to \Gamma(H, B)$ be the induced homomorphism. Then

$$q_A(\Upsilon_A(W)) \subseteq \Upsilon_\pi(W) \implies q_B(\Upsilon_B(W)) \subseteq \Upsilon_\overline{\pi}(W).$$

**Proposition 3.7.16.** Let $H$ be a finitely generated, free abelian group, and let $A$ be a quotient of $H$ such that $\text{rank } A < \text{rank } H$. Let $W$ be a subvariety of $\hat{H}$ of the form $\rho T \cup Z$, where $Z$ is a finite set, $T$ is an algebraic subgroup, and $\rho$ is a torsion element whose order divides $c(A)$. Then $q(\Upsilon_A(W)) \not\subseteq \Upsilon_\pi(W)$.

**Proof.** We need to construct an epimorphism $\nu: H \to A$ such that $[\nu] \in \Upsilon_A(W)$, yet $[\bar{\nu}] \notin \Upsilon_\pi(W)$.
Step 1. First, we assume $\text{Tors}(A)$ is a cyclic group. In this case, we claim there exists a subtorus $C$ of $\hat{H}$, and a torsion element $\alpha \in \hat{H}$, such that $\text{ord } \alpha = \text{ord } \rho$ and $\text{dim}(C \cap \rho T) \leq 0$, yet $\text{dim}(\alpha C \cap \rho T) > 0$.

To prove the claim, set $\epsilon(\langle \rho \rangle) = L$ and $\xi = \epsilon(T)$. Since $\rho \notin T$, we have that $\xi \not\subseteq L$. Thus, there exist a sublattice $\chi$ of rank 1, such that $\chi \subseteq \xi$ and $\chi \not\subseteq L$. Set $T' = V(\chi)$. Then $T'$ is a codimension 1 subgroup with $T \subseteq T' \subset \hat{H}$, and $\rho \notin T'$. Thus, $T_0 \subseteq T'_0 \subset \hat{H}$.

Let $C$ be any dimension $r$ subtorus of $T'_0$ intersecting $T$ with positive dimension. Then $\rho \notin CT \subset T'$ and $\text{dim}(C \cap T) > 0$. Choose an element $\alpha \in \hat{H}$ such that $\alpha^{-1} \rho = 1 \in CT$. Clearly, $\text{ord}(\alpha) = \text{ord}(\rho) | c(A)$. Using Corollary 3.4.4, we conclude that $C \cap \rho T = \emptyset$ and $\text{dim}(\alpha C \cap \rho T) > 0$, thus finishing the proof of the claim.

Now, the algebraic subgroup $\bigcup_k \alpha^k C$ corresponds to an epimorphism $H \twoheadrightarrow \overline{A} \oplus \mathbb{Z}_d$, where $d = \text{ord}(\alpha)$. Since $H$ is torsion-free, $\text{Tors}(A)$ is cyclic, and $d$ divides $c(A)$, this epimorphism can be lifted to an epimorphism $\nu: H \twoheadrightarrow A$.

Step 2. For the general case, let $B$ be the cyclic subgroup of $\text{Tors}(A)$ for which $|B| = c(A)$ and $\text{ord}(\rho) | |B|$. Notice that $B$ is a direct summand of $\text{Tors}(A)$. We then have the following commuting diagram:

\[
\begin{array}{c}
\Gamma(H/\overline{A}, B) \xrightarrow{\pi_1} \Gamma(H/\overline{A}, \overline{A} \oplus B) \xrightarrow{q_{\overline{A} \oplus B}} \Gamma(H/\overline{A})
\end{array}
\]

\[
\begin{array}{c}
\Gamma(H/\overline{A}, \text{Tors}(A)) \xrightarrow{\pi_2} \Gamma(H, A) \xrightarrow{q_A} \Gamma(H, \overline{A})
\end{array}
\]

Since $H/\overline{A}$ is torsion-free, the group $\text{Aut}(H/\overline{A})$ acts transitively on $\Gamma(H/\overline{A}, B)$. Using the assumption that $\Gamma(H/\overline{A}, \text{Tors}(A)) \neq \emptyset$, we deduce that the map $\pi_1$ is surjective. Thus, the map $\pi_2$ is surjective. From Step 1, we know that $q(\Upsilon_{\overline{A} \oplus B}(W))$ is not contained in $\Upsilon_{\overline{A}}(W)$. Using Lemma 3.7.15, we conclude that $q(\Upsilon_A(W))$ is not contained in $\Upsilon_{\overline{A}}(W)$, either. \(\Box\)
3.8 Support varieties for homology modules

We now switch gears, and revisit the Dwyer–Fried theory in a slightly more general context. In particular, we show that the support varieties of the homology modules of two related chain complexes coincide.

3.8.1 Support varieties

Let $H$ be a finitely generated abelian group, and let $F$ be a finitely generated $\mathbb{C}$-algebra. Then the group ring $R = F[H]$ is a Noetherian ring. Let $\text{maxSpec}(R)$ be the set of maximal ideals in $R$, endowed with the Zariski topology.

Given a module $M$ over $F[H]$, denote by $\text{supp} M$ its support, consisting of those maximal ideals $m \in \text{maxSpec}(F[H])$ for which the localization $M_m$ is non-zero.

Now let $A$ be another finitely generated abelian group, and let $\nu: H \twoheadrightarrow A$ be an epimorphism. Denote by $S = F[A]$ the group ring of $A$. The extension of $\nu$ to group rings, $\nu: R \twoheadrightarrow S$, is a ring epimorphism. Let $\nu^*: \text{maxSpec}(S) \hookrightarrow \text{maxSpec}(R)$ be the induced morphism between the corresponding affine varieties.

In the case when $F = \mathbb{C}$, the group ring $R = \mathbb{C}[H]$ is the coordinate ring of the algebraic group $\hat{H} = \text{Hom}(G, \mathbb{C}^*)$, and $\text{maxSpec}(R) = \hat{H}$. Furthermore, if $M$ is an $R$-module, then

$$\text{supp}(M) = Z(\text{ann} M),$$

where $\text{ann} M \subset R$ is the annihilator of $M$, and $Z(\text{ann} M) \subset \hat{H}$ is its zero-locus.

**Lemma 3.8.1.** If $\nu: H \twoheadrightarrow A$ is an epimorphism, then

$$(\nu^*)^{-1}(\text{supp}(M)) \cong \text{im}(\hat{\nu}) \cap Z(\text{ann} M).$$

**Proof.** From the definitions, we see that the diagram

$$\begin{array}{ccc}
\text{maxSpec}(S) & \xhookrightarrow{\nu^*} & \text{maxSpec}(R) \\
\Downarrow & & \Downarrow \\
\hat{A} & \xrightarrow{\hat{\nu}} & \hat{H}
\end{array}$$

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commutes. The conclusion readily follows.

3.8.2 Homology modules

We are now ready to state and prove the main result of this section. An abbreviated proof was given by Dwyer and Fried in [18], in the special case when $A$ is free abelian. For the convenience of the reader, we give here a complete proof, modeled on the one from [18].

**Theorem 3.8.2.** Let $F$ be a finitely generated $C$–algebra, let $C_\bullet$ be a chain complex of finitely generated free modules over $F[H]$, and let $\nu : H \to A$ be an epimorphism. Viewing $F[A]$ as a module over $F[H]$ by extension of scalars via $\nu$, we have

$$\text{supp } H_*(C_\bullet \otimes_{F[H]} F[A]) = (\nu^*)^{-1}(\text{supp } H_*(C_\bullet)).$$

(3.8.4)

**Proof.** Set $n = \text{rank } H$ and $r = \text{rank } A$. There are three cases to consider.

**Case 1:** $H$ is torsion-free. We use induction on $n - r$ to reduce to the case $r = n - 1$, in which case $\text{Tors}(A) = \mathbb{Z}_q$, for some $q \geq 1$ (if $A$ is torsion-free, $q = 1$). We then have a short exact sequence of chain complexes,

$$0 \to C_\bullet \xrightarrow{\phi} C_\bullet \xrightarrow{\otimes_{F[H]} F[A]} 0,$$

which yields a long exact sequence of homology groups. Consider the map $\phi_* : M \to M$, where $M = H_*(C_\bullet)$, viewed as a module over $F[H]$. Localizing at a maximal ideal $m$, we obtain an endomorphism $\phi_m$ of the finitely-generated module $M_m$ over the Noetherian ring $F[H]_m$.

As a standard fact, if $\phi_m$ is surjective, then $\phi_m$ is injective. Using the exact sequence

$$0 \to \ker \phi_m \to M_m \to M_m \to \coker \phi_m \to 0,$$

which yields a long exact sequence of homology groups. Consider the map $\phi_* : M \to M$, where $M = H_*(C_\bullet)$, viewed as a module over $F[H]$. Localizing at a maximal ideal $m$, we obtain an endomorphism $\phi_m$ of the finitely-generated module $M_m$ over the Noetherian ring $F[H]_m$.

As a standard fact, if $\phi_m$ is surjective, then $\phi_m$ is injective. Using the exact sequence

$$0 \to \ker \phi_m \to M_m \to M_m \to \coker \phi_m \to 0,$$
we see that \( \text{coker } \phi_m = 0 \Rightarrow \ker \phi_m = 0 \). Therefore, \( \text{supp } \ker \phi_* \subseteq \text{supp } \text{coker } \phi_* \), and so

\[
\text{supp } H_\ast (C_\ast \otimes_{F[H]} F[A]) = \text{supp } \text{coker } \phi_* \cup \text{supp } \ker \phi_* \\
= \text{supp } \ker \phi_* \\
= \text{supp } M/(x^q - 1)M \\
= (\text{supp } M) \cap Z(x^q - 1) \\
= (\text{supp } M) \cap \text{im}(\hat{\nu}).
\]

When \( n-r > 1 \), one can change the basis of \( H \) and \( A \) so that, the epimorphism \( \nu: H \to A \) is the composite

\[
H' \oplus \mathbb{Z} \overset{\nu_1}{\longrightarrow} H' \oplus \mathbb{Z}_q \overset{\nu_2}{\longrightarrow} A' \oplus \mathbb{Z}_q,
\]

where \( \nu_1 \) is of the form \( \begin{pmatrix} \text{id} & 0 \\ 0 & \nu_1 \end{pmatrix} \), and \( \nu_2 \) is of the form \( \begin{pmatrix} \nu_1 \nu_2 & 0 \\ 0 & \text{id} \end{pmatrix} \). By the induction hypothesis, equality (3.8.4) holds for \( \nu_1 \) and \( \nu_2 \). Thus, the theorem holds for the map \( \nu = \nu_2 \circ \nu_1 \).

**Case 2: \( H \) is finite.** In this situation, \( \nu: H \to A \) is an epimorphism between two finite abelian groups. As above, \( \nu \) induces a ring epimorphism \( \nu: F[H] \to F[A] \). The corresponding map, \( i = \nu^*: \text{maxSpec } F[A] \hookrightarrow \text{maxSpec } F[H] \), is a closed immersion. Consider the commuting diagram

\[
\text{maxSpec } F[A] \hookrightarrow \text{maxSpec } F[H] \\
\downarrow \\
\text{maxSpec } \mathbb{C}[A] \hookrightarrow \text{maxSpec } \mathbb{C}[H],
\]

where the morphism \( j \) is induced by \( \nu: H \to A \). Clearly, \( j \) is an open immersion. By commutativity of (3.8.5), we have that \( \text{maxSpec } F[A] \) is an open subset of \( \text{maxSpec } F[H] \).

It suffices to show that

\[
\text{supp}(H_k(i^*\tilde{C}_\ast)) = i^{-1}(\text{supp } H_k(\tilde{C}_\ast)) \tag{3.8.6}
\]

for any \( k \in \mathbb{Z} \), where \( \tilde{C}_\ast \) is the sheaf of modules over \( \text{maxSpec } (F[H]) \) corresponding to the module \( C_\ast \) over \( F[H] \), and \( i^*\tilde{C}_\ast \) is the sheaf of modules over \( \text{maxSpec } (F[A]) \) obtained by pulling back the sheaf \( \tilde{C}_\ast \).
For any \( m \in \text{maxSpec}(F[A]) \), we have \((H_k(\tilde{C}_\bullet))_m = H_k((\tilde{C}_\bullet)_m)\), since localization is an exact functor, and also \((\tilde{C}_\bullet)_m = (i^*\tilde{C}_\bullet)_m\), since \( i \) is an open immersion. Thus,

\[
\text{supp } H_k(i^*\tilde{C}_\bullet) = \{ m \in \text{maxSpec}(F[A]) \mid (H_k(i^*\tilde{C}_\bullet))_m \neq 0 \}
\]

\[
= \{ m \in \text{maxSpec}(F[A]) \mid (i^*\tilde{C}_\bullet)_m \text{ is not exact at } k \}
\]

\[
= \{ m \in \text{maxSpec}(F[A]) \mid (H_k(\tilde{C}_\bullet))_m \neq 0 \}
\]

\[
= \text{supp } H_k(\tilde{C}_\bullet) \cap \text{maxSpec}(F[A])
\]

\[
= i^{-1}(\text{supp } H_k(\tilde{C}_\bullet)).
\]

**Case 3: \( H \) is arbitrary.** As in the proof of Theorem 3.2.2, we can choose splittings \( H = \overline{H} \oplus \text{Tors}(H) \) and \( A = \overline{A} \oplus \text{Tors}(A) \) such that \( \overline{H} = \overline{A} \oplus H' \), and the epimorphism \( \nu: H \to A \) is the composite

\[
\overline{A} \oplus H' \oplus \text{Tors}(H) \xrightarrow{\nu} \overline{A} \oplus \nu(H') \oplus \text{Tors}(H) \xrightarrow{\nu_2} \overline{A} \oplus \text{Tors}(A),
\]

where \( \nu(H') \) is finite, \( \nu_1 \) is of the form \( \begin{pmatrix} \text{id} & 0 & 0 \\ 0 & \nu|_{H'} & 0 \\ 0 & 0 & \text{id} \end{pmatrix} \), and \( \nu_2 \) is of the form \( \begin{pmatrix} \text{id} & 0 & 0 \\ 0 & i \nu|_{\text{Tors}(H)} & 0 \\ 0 & 0 & \text{id} \end{pmatrix} \), with \( i: \nu(H') \hookrightarrow \text{Tors}(A) \) the inclusion map.

Set \( F = \mathbb{C}[\overline{A} \oplus \text{Tors}(H)] \); by Case 1, equality (3.8.4) holds for \( \nu_1 \). Now set \( F = \mathbb{C}[\overline{A}] \); by Case 2, equality (3.8.4) holds for \( \nu_2 \). Thus, the theorem holds for the map \( \nu = \nu_2 \circ \nu_1 \). \( \square \)

### 3.8.3 Finite supports

We conclude this section with a result which is presumably folklore. For completeness, we include a proof.

**Proposition 3.8.3.** Let \( A \) be a finitely generated abelian group, and let \( M \) be a finitely generated module over the group ring \( S = \mathbb{C}[A] \). Then \( M \) as a \( \mathbb{C} \)-vector space is finite-dimensional if and only if \( \text{supp}(M) \) is finite.

**Proof.** Since \( M \) is a finitely generated \( S \)-module, we can argue by induction on the number of generators of \( M \). Using the short exact sequence \( 0 \to \langle m \rangle \to M \to M/\langle m \rangle \to 0 \), where
$m$ is a generator of $M$, and the fact that $\text{supp}(M) = \text{supp}(\langle m \rangle) \cup \text{supp}(M/\langle m \rangle)$, we see that it suffices to consider the case when $M$ is a cyclic module. In this case, $M = S/\text{ann}(M)$ and $\text{supp}(M) = \text{Z}(\text{ann}(M)) = \text{maxSpec}(S/\text{ann}(M))$. From the assumption that $\text{supp} M$ is finite, and using the Noether Normalization lemma, we infer that $S/\text{ann}(M)$ is an integral extension of $\mathbb{C}$. Thus, $\dim_{\mathbb{C}}(S/\text{ann}(M)) < \infty$.

Conversely, suppose $\text{supp}(M)$ is infinite. Then $\text{maxSpec}(S/\text{ann}(M))$ is infinite, which implies $\text{maxSpec}(S/\text{ann}(M))$ has positive dimension. Choose a prime ideal $p$ containing $\text{ann}(M)$, such that the Krull dimension of $S/p$ is positive. From the condition that $\dim_{\mathbb{C}} S/\text{ann}(M) < \infty$, we deduce that $\dim_{\mathbb{C}} S/p < \infty$. By the Noether Normalization lemma, $S/p$ is an integral extension of $\mathbb{C}[x_1, \ldots, x_n]$, with $n > 0$. Thus, $\dim_{\mathbb{C}} S/p = \infty$. This is a contradiction, and so we are done.

\section{Characteristic varieties and generalized Dwyer–Fried sets}

In this section, we finally tie together several strands, and show how to determine the sets $\Omega_{\Lambda}^i(X)$ in terms of the jump loci for homology in rank 1 local systems on $X$.

\subsection{The equivariant chain complex}

Let $X$ be a connected CW-complex. As usual, we will assume that $X$ has finite $k$-skeleton, for some $k \geq 1$. Without loss of generality, we may assume that $X$ has a single 0-cell $e^0$, which we will take as our basepoint $x_0$. Moreover, we may assume that all attaching maps $(S^i, \ast) \to (X^i, x_0)$ are basepoint-preserving. Let $G = \pi_1(X, x_0)$ be the fundamental group of $X$, and denote by $(C_i(X, \mathbb{C}), \partial_i)_{i \geq 0}$ the cellular chain complex of $X$, with coefficients in $\mathbb{C}$.

Let $p: X^{\text{ab}} \to X$ be the universal abelian cover. The cell structure on $X$ lifts in a natural fashion to a cell structure on $X^{\text{ab}}$. Fixing a lift $\tilde{x}_0 \in p^{-1}(x_0)$ identifies the group $H = G_{\text{ab}}$
with the group of deck transformations of $X^{ab}$, which permute the cells. Therefore, we may view the cellular chain complex $C_\bullet = C_\bullet(X^{ab}, \mathbb{C})$ as a chain complex of left-modules over the group algebra $R = \mathbb{C}[H]$. This chain complex has the form

$$\cdots \longrightarrow C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

(3.9.1)

The first two boundary maps can be written down explicitly. Let $e_1^1, \ldots, e_m^1$ be the 1-cells of $X$. Since we have a single 0-cell, each $e_i^1$ is a loop, representing an element $x_i \in G$. Let $\tilde{e}_0^0 = \tilde{x}_0$, and let $\tilde{e}_i^1$ be the lift of $e_i^1$ at $\tilde{x}_0$; then $\partial_1(\tilde{e}_i^1) = (x_i - 1)\tilde{e}_0^0$. Next, let $e^2$ be a 2-cell, and let $\tilde{e}_i^2$ be its lift at $\tilde{x}_0$; then

$$\partial_2(\tilde{e}_i^2) = \sum_{i=1}^m \phi(\partial r / \partial x_i) \cdot \tilde{e}_i^1,$$

(3.9.2)

where $r$ is the word in the free group $F_m = \langle x_1, \ldots, x_m \rangle$ determined by the attaching map of the 2-cell, $\partial r / \partial x_i \in \mathbb{C}[F_m]$ are the Fox derivatives of $r$, and $\phi: \mathbb{C}[F_m] \rightarrow \mathbb{C}[H]$ is the extension to group rings of the projection map $F_m \rightarrow G \rightarrow H$, see [27].

### 3.9.2 Characteristic varieties

Since $X$ has finite 1-skeleton, the group $H = H_1(X, \mathbb{Z})$ is finitely generated, and its dual, $\widehat{H} = \text{Hom}(H, \mathbb{C}^*)$, is a complex algebraic group. As is well-known, the character group $\widehat{H}$ parametrizes rank 1 local systems on $X$: given a character $\rho: H \rightarrow \mathbb{C}^*$, denote by $\mathbb{C}_\rho$ the 1-dimensional $\mathbb{C}$-vector space, viewed as a right $R$-module via $a \cdot g = \rho(g)a$, for $g \in H$ and $a \in \mathbb{C}$. The homology groups of $X$ with coefficients in $\mathbb{C}_\rho$ are then defined as

$$H_i(X, \mathbb{C}_\rho) := H_i(C_\bullet(X^{ab}, \mathbb{C}) \otimes_R \mathbb{C}_\rho).$$

(3.9.3)

**Definition 3.9.1.** The characteristic varieties of $X$ (over $\mathbb{C}$) are the sets

$$V^i(X) = \{ \rho \in \text{Hom}(H, \mathbb{C}^*) \mid \dim_\mathbb{C} H_j(X, \mathbb{C}_\rho) \neq 0 \text{ for some } 1 \leq j \leq i \}.$$

The identity component of the character group $T = \widehat{H}$ is a complex algebraic torus, which we will denote by $T_0$. Let $\overline{H} = H / \text{Tors}(H)$ be the maximal torsion-free quotient of
The projection map \( \pi : H \rightarrow \widetilde{H} \) induces an identification \( \hat{\pi} : \widetilde{H} \cong \hat{H}_0 \). Denote by \( W^i(X) \) the intersection of \( V^i(X) \) with \( T_0 = \hat{H}_0 \). If \( H \) is torsion-free, then \( W^i(X) = V^i(X) \); in general, though, the two varieties differ.

For each \( 1 \leq i \leq k \), the set \( V^i(X) \) is a Zariski closed subset of the complex algebraic group \( T \), and \( W^i(X) \) is a Zariski closed subset of the complex algebraic torus \( T_0 \). Up to isomorphism, these varieties depend only on the homotopy type of \( X \). Consequently, we may define the characteristic varieties of a group \( G \) admitting a classifying space \( K(G,1) \) with finite \( k \)-skeleton as \( V^i(G) = V^i(K(G,1)) \), for \( i \leq k \). It is readily seen that \( V^1(X) = V^1(\pi_1(X)) \). For more details on all this, we refer to [81].

The characteristic varieties of a space can be reinterpreted as the support varieties of its Alexander invariants, as follows.

**Theorem 3.9.2 ([70]).** For each \( 1 \leq i \leq k \), the characteristic variety \( V^i(X) \) coincides with the support of the \( \mathbb{C}[H] \)-module \( \bigoplus_{j=1}^i H_j(X^{ab}, \mathbb{C}) \), while \( W^i(X) \) coincides with the support of the \( \mathbb{C}[\widetilde{H}] \)-module \( \bigoplus_{j=1}^i H_j(X^{fab}, \mathbb{C}) \).

### 3.9.3 The first characteristic variety of a group

Let \( G \) be a finitely presented group. The chain complex (3.9.1) corresponding to a presentation \( G = \langle x_1, \ldots, x_q \mid r_1, \ldots, r_m \rangle \) has second boundary map, \( \tilde{\partial}_2 \), an \( m \) by \( q \) matrix, with rows given by (3.9.2). Making use of Theorem 3.9.2, we see that \( V^1(G) \) is defined by the vanishing of the codimension 1 minors of the Alexander matrix \( \tilde{\partial}_2 \), at least away from the trivial character 1. This interpretation allows us to construct groups with fairly complicated characteristic varieties.

**Lemma 3.9.3.** Let \( f = f(t_1, \ldots, t_n) \) be a Laurent polynomial. There is then a finitely presented group \( G \) with \( G^{ab} = \mathbb{Z}^n \) and \( V^1(G) = \{ z \in (\mathbb{C}^*)^n \mid f(z) = 0 \} \cup \{1\} \).

**Proof.** Let \( F_n = \langle x_1, \ldots, x_n \rangle \) be the free group of rank \( n \), with abelianization map \( ab : F_n \rightarrow \mathbb{Z}^n, x_k \mapsto t_k \). Recall the following result of R. Lyndon (as recorded in [27]): if \( v_1, \ldots, v_n \)
are elements in the ring $\mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, satisfying the equation $\sum_{k=1}^{n} (t_k - 1)v_k = 0$, then there exists an element $r \in F'_n$ such that $v_k = ab(\partial r / \partial x_k)$, for $1 \leq k \leq n$.

Making use of this result, we may find elements $r_{i,j} \in F'_n$, $1 \leq i < j \leq n$ such that

$$ab(\partial r_{i,j} / \partial x_k) = \begin{cases} 
  f \cdot (t_i - 1), & \text{if } k = i \\
  f \cdot (1 - t_j), & \text{if } k = j \\
  0, & \text{otherwise.}
\end{cases}$$

It is now readily checked that the group $G$ with generators $x_1, \ldots, x_n$ and relations $r_{ij}$ has the prescribed first characteristic variety. 

In certain situations, one may realize a Laurent polynomial as the defining equation for the characteristic variety by a more geometric construction.

**Example 3.9.4.** Let $L$ be an $n$-component link in $S^3$, with complement $X$. Choosing orientations on the link components yields a meridian basis for $H_1(X, \mathbb{Z}) = \mathbb{Z}^n$. Then

$$V^1(X) = \{ z \in (\mathbb{C}^*)^n \mid \Delta_L(z) = 0 \} \cup \{1\},$$

where $\Delta_L = \Delta_L(t_1, \ldots, t_n)$ is the (multi-variable) Alexander polynomial of the link.

### 3.9.4 A formula for the generalized Dwyer–Fried sets

Recall that, in Definition 3.7.5 we associated to each subvariety $W \subset \hat{H}$, and each abelian group $A$ a subset

$$\Upsilon_A(W) = \{ \nu \in \Gamma(H, A) \mid \dim(\text{im}(\nu) \cap W) > 0 \}. \quad (3.9.5)$$

The next theorem expresses the Dwyer–Fried sets $\Omega^i_A(X)$ in terms of the $\Upsilon$-sets associated to the $i$-th characteristic variety of $X$.

**Theorem 3.9.5.** Let $X$ be a connected CW-complex, with finite $k$-skeleton. Set $G = \pi_1(X, x_0)$ and $H = G_{ab}$. For any abelian group $A$, and for any $i \leq k$,

$$\Omega^i_A(X) = \Gamma(H, A) \setminus \Upsilon_A(V^i(X)).$$
Proof. Fix an epimorphism $\nu: H \to A$, and let $X^\nu \to X$ be the corresponding cover. Recall the cellular chain complex $C_\bullet = C_\bullet(X^{\text{ab}}, \mathbb{C})$ is a chain complex of left modules over the ring $R = \mathbb{C}[H]$. If we set $S = \mathbb{C}[A]$, the cellular chain complex $C_\bullet(X^\nu, \mathbb{C})$ can be written as $C_\bullet \otimes_R S$, where $S$ is viewed as a right $R$-module via extension of scalars by $\nu$.

Consider the $S$-module

$$M = \bigoplus_{j=1}^i H_j(X^\nu, \mathbb{C}) = \bigoplus_{j=1}^i H_j(C_\bullet \otimes_R S).$$

By definition, $[\nu]$ belongs to $\Omega^i_A(X)$ if and only if the Betti numbers $b_1(X^\nu), \ldots, b_i(X^\nu)$ are all finite, i.e., $\dim_\mathbb{C} M < \infty$. By Proposition 3.8.3, this condition is equivalent to $\text{supp } M$ being finite.

Now let $\nu^* : \text{maxSpec}(S) \hookrightarrow \text{maxSpec}(R)$ be the induced morphism between the corresponding affine schemes. We then have

$$\text{supp } M = (\nu^*)^{-1} \text{supp} \left( \bigoplus_{j=1}^i H_j(C_\bullet) \right)$$

by Theorem 3.8.2

$$\cong \text{im}(\hat{\nu}) \cap Z \left( \text{ann} \left( \bigoplus_{j=1}^i H_j(X^{\text{ab}}, \mathbb{C}) \right) \right)$$

by Lemma 3.8.1

$$= \text{im}(\hat{\nu}) \cap V^i(X)$$

by Theorem 3.9.2.

This ends the proof.

\[ \square \]

Remark 3.9.6. If $H$ is torsion-free, then $W^i(X) = V^i(X)$; thus, the set $\Omega^i_A(X)$ depends only the variety $W^i(X)$ and the abelian group $A$. On the other hand, if $\text{Tors}(H) \neq 0$, the variety $W^i(X)$ may be strictly included in $V^i(X)$, in which case the set $\Omega^i_A(X)$ may depend on information not carried by $W^i(X)$. We shall see examples of this phenomenon in §3.10.3.

### 3.9.5 An upper bound for the $\Omega$-sets

We now give a computable “upper bound” for the generalized Dwyer-Fried sets $\Omega^i_A(X)$, in terms of the sets introduced in §3.7.1.
Theorem 3.9.7. Let $X$ be a connected CW-complex with finite $k$-skeleton. Set $H = H_1(X,\mathbb{Z})$, and fix a degree $i \leq k$. Let $A$ be a quotient of $H$. Then

\[ \Omega^i_A(X) \subseteq \Gamma(H, A) \setminus U_A(V^i(X)). \] (3.9.6)

Proof. Follows at once from Proposition 3.7.6 and Theorem 3.9.5. \qed

In general, inclusion (3.9.6) is not an equality. Indeed, the fact that $V(\ker \nu) \cap V^i(X)$ is infinite cannot guarantee there is an algebraic subgroup in the intersection. Here is a concrete instance of this phenomenon.

Example 3.9.8. Using Lemma 3.9.3, we may find a 3-generator, 3-relator group $G$ with abelianization $H = \mathbb{Z}^3$ and characteristic variety

\[ V^1(G) = \{(t_1, t_2, t_3) \in (\mathbb{C}^*)^3 \mid (t_2 - 1) = (t_1 + 1)(t_3 - 1)\}. \]

This variety has a single irreducible component, which is a complex torus passing through the origin; nevertheless, this component does not embed as an algebraic subgroup in $(\mathbb{C}^*)^3$.

There are 4 maximal, positive-dimensional torsion-translated subtori contained in $V^1(G)$, namely $\eta_1 V(\xi_1), \ldots, \eta_4 V(\xi_4)$, where $\xi_1, \ldots, \xi_4$ are the subgroups of $\mathbb{Z}^3$ given by

\[ \xi_1 = \text{im} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \xi_2 = \text{im} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \xi_3 = \text{im} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \xi_4 = \text{im} \begin{pmatrix} 2 & -2 & 0 \\ -1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \]

and $\eta_1 = (-1,1,1), \ \eta_2 = \eta_3 = (1,1,1), \ \eta_4 = (-1,1,-1)$.

We claim that, for $A = \mathbb{Z}^2 \oplus \mathbb{Z}_2$, inclusion (3.9.6) from Theorem 3.9.7 is strict, i.e.,

\[ \Omega^1_A(G) \not\subseteq \Gamma(H, A) \setminus U_A(V^1(G)). \]

To prove this claim, consider the epimorphism $\nu: \mathbb{Z}^3 \to \mathbb{Z}^2 \oplus \mathbb{Z}_2$ given by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Note that $\ker(\nu) = \text{im} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, and so $\text{im}(\nu) = \{ t \in (\mathbb{C}^*)^3 \mid t_2 = \pm 1 \}$. The intersection of $\text{im}(\nu)$ with $V^1(G)$ consists of all points of the form $(t_1, \pm 1, t_3)$ with $(t_1 + 1)(t_3 - 1)$ equal to 0 or $-2$. Clearly, this is an infinite set; therefore, $[\nu] \notin \Omega^1_A(G)$. 

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On the other hand, \( \text{im}(\hat{\nu}) \cap \eta_1 V(\xi_1) = \text{im}(\hat{\nu}) \cap \eta_2 V(\xi_2) = \emptyset \), and \( \text{rank}(\ker(\nu) + \xi_3) = \text{rank}(\ker(\nu) + \xi_4) = \text{rank}(H) \). Hence, \( \text{im}(\hat{\nu}) \cap \eta_j V(\xi_j) \) is finite, for all \( j \). In view of Lemma 3.7.2, we conclude that \( [\nu] \notin U_A(V^1(G)) \).

### 3.10 Comparison with the classical Dwyer–Fried invariants

#### 3.10.1 The Dwyer–Fried invariants \( \Omega^i_r(X) \)

In the case of free abelian covers and the usual \( \Omega \)-sets, Theorem 3.9.5 allows us to recover the following result from [18], [70], [81].

**Corollary 3.10.1.** Set \( n = b_1(X) \). Then, for all \( r \geq 1 \),

\[
\Omega^i_r(X) = \{ [\nu] \in \text{Gr}_r(\mathbb{Z}^n) \mid \text{im}(\hat{\nu}) \cap W^i(X) \text{ is finite} \}.
\]

Using now the identification from Example 3.6.3, and taking into account Lemma 3.5.5, Theorem 3.9.7 yields the following corollary.

**Corollary 3.10.2 ([70, 81]).** Let \( X \) be a connected CW-complex with finite \( k \)-skeleton, and set \( n = b_1(X) \). Then \( \Omega^i_r(X) \subseteq \mathbb{P}^{n-1} \setminus \mathbb{P}(\tau_1(W^i(X))) \), for all \( i \leq k \) and all \( r \leq n \).

**Remark 3.10.3.** As noted in [81], if the variety \( W^i(X) \) is a union of algebraic subtori, then the Dwyer–Fried sets \( \Omega^i_r(X) \) are open subsets of \( \text{Gr}_r(\mathbb{Z}^n) \), for all \( r \geq 1 \). In general, though, examples from [18], [81] show that the sets \( \Omega^i_r(X) \) with \( r > 1 \) need not be open. On the other hand, as noted in [18, 70, 81], the sets \( \Omega^i_1(X) \) are always open subsets of \( \text{Gr}_1(\mathbb{Z}^n) = \mathbb{P}^{n-1} \).

We will come back to this phenomenon in §3.11, in a more general context.

#### 3.10.2 The comparison diagram, revisited

As may be expected, the generalized Dwyer–Fried invariants carry more information about the homotopy type of a space and the homological finiteness properties of its regular abelian
covers than the classical ones. To make this more precise, let \( A \) be a quotient of the group \( H = H_1(X, \mathbb{Z}) \). Fix a decomposition \( A = \widehat{A} \oplus \text{Tors}(A) \), and identify \( \Omega^i_r(X) = \Omega^i_{\widehat{A}}(X) \), where \( r = \text{rank}(A) \). As we saw in §3.3.4, for each \( i \leq k \) we have a comparison diagram

\[
\begin{array}{ccc}
\Omega^i_A(X) & \hookrightarrow & \Gamma(H, A) \\
\downarrow^{q|_{\Omega^i_A(X)}} & & \downarrow^q \\
\Omega^i_r(X) & \hookrightarrow & \Gamma(H, \widehat{A})
\end{array}
\]  

(3.10.1)

between the respective Dwyer–Fried invariants, viewed as subsets of the parameter sets for regular \( A \)-covers and \( \widehat{A} \)-covers, respectively.

We are interested in describing conditions under which the set \( \Omega^i_A(X) \) contains more information than \( \Omega^i_r(X) \). This typically happens when the comparison diagram (3.10.1) is not a pull-back diagram, i.e., there is a point \([\bar{\nu}] \in \Omega^i_r(X)\) for which the fiber \( q^{-1}([\bar{\nu}]) \) is not included in \( \Omega^i_A(X) \). In fact, the number of points in the fiber which lie in \( \Omega^i_A(X) \) may vary as we move about \( \Omega^i_r(X) \).

In view of Theorem 3.9.5, we have the following criterion.

**Proposition 3.10.4.** Diagram (3.10.1) fails to be a pull-back diagram if and only if \( q(\Upsilon_A(V^i(X))) \) is not included in \( \Upsilon_r(V^i(X)) \), i.e., there is an epimorphism \( \nu: H \twoheadrightarrow A \) such that

\[
\dim(\text{im} \hat{\nu} \cap V^i(X)) > 0, \quad \text{yet} \quad \dim(\text{im} \hat{\nu} \cap V^i(X)) = 0.
\]

From Proposition 3.3.10, we know diagram (3.10.1) is a pull-back diagram precisely when the homological finiteness of an arbitrary \( A \)-cover of \( X \) can be tested through the corresponding \( \widehat{A} \)-cover. In order to quantify the discrepancy between these two types of homological finiteness properties, let us define the “singular set”

\[
\Sigma^i_A(X) = \{ [ar{\nu}] \in \Omega^i_{\widehat{A}}(X) | \#(q^{-1}([\bar{\nu}]) \cap \Omega^i_A(X)) < \#(q^{-1}([\bar{\nu}])) \}.
\]  

(3.10.2)

We then have:

\[
\Sigma^i_A(X) = q(\Omega^i_A(X)^c) \cap \Omega^i_{\widehat{A}}(X).
\]  

(3.10.3)
3.10.3 Maximal abelian versus free abelian covers

We now investigate the relationship between the finiteness of the Betti numbers of the maximal abelian cover \( X^{ab} \) and the finiteness of the Betti numbers of the corresponding free abelian cover \( X^{fab} \) of our space \( X \).

As before, write \( H = H_1(X; \mathbb{Z}) \) and identify the character group \( \hat{H} \) with \((\mathbb{C}^*)^n \times \text{Tors}(H)\), where \( n = b_1(X) \).

**Proposition 3.10.5.** Suppose \( \text{Tors}(H) \neq 0 \). Furthermore, assume that \( W^i(X) \) is finite, whereas \( V^i(X) = W^i(X) \cup (\hat{H} \setminus \hat{H}_0) \). Then

(i) \( \Omega^i_\epsilon(X) = \text{Gr}_r(\mathbb{Z}^n) \), for all \( r \geq 1 \).

(ii) If \( \text{rank}(A) = \text{rank}(H) \) and \( \text{Tors}(A) \neq 0 \), then \( \Omega^1_A(X) = \emptyset \).

**Proof.** By Theorem 3.9.5, an element \([\nu] \in \text{Gr}_r(\mathbb{Z}^n)\) belongs to \( \Omega^i_\epsilon(X) \) if and only if \( \text{im}(\hat{\nu}) \cap W^i(X) \) is finite. By assumption, \( W^i(X) \) is finite; thus, the intersection \( \text{im}(\hat{\nu}) \cap W^i(X) \) is also finite. This proves (i).

Again by Theorem 3.9.5, an element \([\nu] \in \Gamma(H, A)\) belongs to \( \Omega^i_A(X) \) if and only if \( \text{im}(\hat{\nu}) \cap V^i(X) \) is finite. By assumption, \( V^i(X) = W^i(X) \cup (\hat{H} \setminus \hat{H}_0) \); moreover, \( \text{rank}(A) = \text{rank}(H) \) and \( \text{Tors}(A) \neq 0 \). Thus, the intersection \( \text{im}(\hat{\nu}) \cap V^i(X) \) contains at least one component of \( \hat{H} \setminus \hat{H}_0 \), which is infinite. This proves (ii).

A similar argument yields the following result.

**Proposition 3.10.6.** Suppose \( H_1(X, \mathbb{Z}) \) has non-trivial torsion, \( W^1(X) \) is finite, and \( V^1(X) \) is infinite. Then \( b_1(X^{fab}) < \infty \), yet \( b_1(X^{ab}) = \infty \).

**Proposition 3.10.7.** Suppose \( X = X_1 \vee X_2 \), where \( H_1(X_1, \mathbb{Z}) \) is free abelian and non-trivial, and \( H_1(X_2, \mathbb{Z}) \) is finite and non-trivial. If \( V^1(X_1) \) is finite, then \( b_1(X^{fab}) < \infty \), yet \( b_1(X^{ab}) = \infty \).
Proof. Let $H = H_1(X, \mathbb{Z})$; then $\overline{H} = H_1(X_1, \mathbb{Z}) \neq 0$ and $\text{Tors}(H) = H_1(X_2, \mathbb{Z}) \neq 0$. Thus, the character group $\hat{H}$ decomposes as $\hat{H}_0 \times \text{Tors}(H)$, with both factors non-trivial.

Now, $W^1(X) = V^1(X_1) \times \{1\}$ is finite, and thus $\Omega^1_{\overline{H}}(X)$ is a singleton. On the other hand, $V^1(X) = W^1(X) \cup \hat{H}_0 \times (\text{Tors}(H) \setminus \{1\})$ is infinite, and thus $\Omega^1_H(X) = \emptyset$. \hfill $\square$

Example 3.10.8. Consider the CW-complexes $X = S^1 \vee \mathbb{R}P^2$ and $Y = S^1 \times \mathbb{R}P^2$. Then $H_1(X, \mathbb{Z}) \cong H_1(Y, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}_2$. Clearly, the free abelian covers $X^{\text{fab}}$ and $Y^{\text{fab}}$ are rationally acyclic; thus both $\Omega^1_i(X)$ and $\Omega^1_i(Y)$ consist of a single point, for all $i \geq 0$. On the other hand, the free abelian cover $X^{\text{ab}}$ has the homotopy type of a countable wedge of $S^1$'s and $\mathbb{R}P^2$'s, whereas $Y^{\text{ab}} \simeq S^2$. Therefore, $\Omega^1_{\mathbb{Z} \oplus \mathbb{Z}_2}(X) = \emptyset$, while $\Omega^1_{\mathbb{Z} \oplus \mathbb{Z}_2}(Y) = \{\text{point}\}$.

This example shows that the generalized Dwyer–Fried invariants $\Omega^i_A(X)$ may contain more information than the classical ones.

3.11 The rank 1 case

In this section, we discuss in more detail the invariants $\Omega^i_A(X)$ in the case when $A$ has rank 1, and push the analysis even further in some particularly simple situations.

3.11.1 A simplified formula for $\Omega^i_A(X)$

Recall that, for a finitely generated abelian group $A$, the integer $c(A)$ denotes the largest order of an element in $A$. Recall also that, for every subvariety $W \subset \hat{H}$ and each index $d \geq 1$, we have a subset $U_{A,d}(W) \subset \Gamma(H, A)$, described geometrically in Lemma 3.7.2.

Theorem 3.11.1. Let $X$ be a connected CW-complex with finite $k$-skeleton. Set $H = H_1(X, \mathbb{Z})$, and fix a degree $i \leq k$. If rank($A$) = 1, then

$$\Omega^i_A(X) = \Gamma(H, A) \setminus U_{A, c(A)}(V^i(X)). \quad (3.11.1)$$
Proof. The inclusion $\subseteq$ follows from Theorem 3.9.7, so we only need to prove the opposite inclusion.

Let $\nu: H \to A$ be an epimorphism such that $[\nu] \notin \Omega^i_A(X)$. By Theorem 3.9.5, the variety $\text{im} \hat{\nu} \cap V^i(X)$ has positive dimension. Since $\text{rank}(A) = 1$, there exists a component of the 1-dimensional algebraic subgroup $\text{im} \hat{\nu}$ contained in $V^i(X)$; that is, there exists a character $\rho \in \text{Tors}(A)$ such that

$$\hat{\nu}(\rho) \cdot \text{im} \hat{\nu} \subseteq V^i(X).$$

Now, we may find a primitive subgroup $\chi \leq H$ and a torsion character $\eta \in \hat{H}$ such that $\eta V(\chi) = \hat{\nu}(\rho) V(\chi)$ is a maximal translated subtorus in $V^i(X)$ which contains $\hat{\nu}(\rho) \cdot \text{im} \hat{\nu}$. Set

$$\xi = \epsilon \left( \bigcup_{m \geq 1} \eta^m V(\chi) \right).$$

Clearly, the subgroup $\xi \leq H$ belongs to $\Xi_d(V^i(X))$, where $d := \text{ord}(\rho)$ divides $c(A)$. Since $\hat{\nu}(\rho) V(\chi) \supseteq \hat{\nu}(\rho) \cdot V(\ker(\hat{\nu}))$, we must also have $V(\chi) \supseteq V(\ker(\hat{\nu}))$, and so $[\nu] \in U_A(\xi)$. Therefore, $[\nu] \in U_{A,c(A)}(V^i(X))$, and we are done. \hfill $\square$

Corollary 3.11.2 ([70, 81]). Let $X$ be a connected CW-complex with finite $k$-skeleton, and set $n = b_1(X)$. Then $\Omega^i_1(X) = \mathbb{Q}P^{n-1} \setminus \mathbb{P}(\tau_1(W^i(X)))$, for all $i \leq k$.

In particular, $\Omega^i_1(X)$ is an open subset of the projective space $\mathbb{Q}P^{n-1}$.

3.11.2 The singular set

Recall from §3.10.2 that we measure the discrepancy between the generalized Dwyer–Fried invariant $\Omega^i_A(X)$ and its classical counterpart, $\Omega^i_A(X)$, by means of the “singular set,” $\Sigma^i_A(X) = q(\Omega^i_A(X)^c) \cap \Omega^i_A(X)$. When the group $A$ has rank 1, this set can be expressed more concretely, as follows.

Proposition 3.11.3. Suppose $H$ is torsion-free, and rank($A$) = 1. Then $\Sigma^i_A(X)$ consists of all $[\sigma] \in \Gamma(H, \overline{A})$ satisfying the following two conditions:
(i) \( \ker(\sigma) \supseteq \xi \), for some \( \xi \in \Xi_{c(A)}(V^i(X)) \), and

(ii) \( V(\ker(\sigma)) \not\subseteq V^i(X) \).

Proof. Without loss of generality, we may assume that \( \Gamma(H, A) \neq \emptyset \). From Theorem 3.11.1, we know that \( \Omega^i_A(X) = \Gamma(H, A) \setminus U \), where \( U = U_{A,c(A)}(V^i(X)) \). It follows that \( \Sigma^i_A(X) = q(U) \cap \Omega^i_A(X) \).

Now let \( S \) be the set of all \([\sigma] \in \Gamma(H, \overline{A})\) satisfying conditions (i) and (ii). It suffices to show that \( q(U) \cap \Omega^i_A(X) = S \).

The inclusion \( S \supseteq q(U) \cap \Omega^i_A(X) \) is straightforward. To establish the reverse inclusion, let \( \sigma: H \to \mathbb{Z} \) represent an element in \( S \). Condition (ii) implies that \([\sigma] \in \Omega^i_A(X) \). Recall that \( \pi: A \to \overline{A} \) is the natural projection. To prove that \([\sigma] \in q(U)\), it is enough to find an epimorphism \( \nu: H \to A \) such that \( \pi \circ \nu = \sigma \) and \( V(\ker(\nu)) \cap \eta V(\overline{\xi}) \neq \emptyset \), where \( \eta \) is a generator of \( \overline{\xi}/\xi \).

Set \( d := \text{ord}(\eta) \). Writing \( A = \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k} \), with \( d_1 | d_2 | \cdots | d_k \), we have that \( d \mid d_k \).

Denote by \( \iota \) the embedding of the cyclic group \( \langle \eta \rangle \) into \( \widehat{H} \). Under the correspondence from (3.4.2), there is a map \( \iota: H \to \mathbb{Z}_d \); clearly, this map factors as the composite

\[
H \xrightarrow{f_1} \mathbb{Z}^{n-1} \oplus \mathbb{Z} \xrightarrow{\text{id} \oplus \kappa_1} \mathbb{Z}^{n-1} \oplus \mathbb{Z}_{d_k}^{(0\ \kappa_2)} \to \mathbb{Z}_d,
\]

for some isomorphism \( f_1: H \to \mathbb{Z}^n \), where \( \kappa_1 \) and \( \kappa_2 \) are the canonical projections.

Recall we are assuming \( \Gamma(H, A) \neq \emptyset \); thus, there is an epimorphism \( \gamma: H \to A \). We then have a commuting diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\gamma} & A \\
\downarrow{f_2} & & \downarrow{f_2} \\
\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{(0\ \kappa_1)} & \xrightarrow{\text{id}} & \mathbb{Z}_{d_k}
\end{array}
\]

for some isomorphism \( f_2: H \to \mathbb{Z}^n \). The composite \( \nu = \gamma \circ f_2^{-1} \circ f_1: H \to A \), then, is the required epimorphism. \( \square \)

As we shall see in Examples 3.11.5 and 3.11.6, the singular set \( \Sigma^i_A(X) \) may be non-empty; in fact, as we shall see in Example 3.11.7, this set may even be infinite.
3.11.3 A particular case

Perhaps the simplest situation when such a phenomenon may occur is the one when \( H = \mathbb{Z}^2 \) and \( A = \mathbb{Z} \oplus \mathbb{Z}_2 \). In this case, the set

\[
\Gamma(H/\overline{A}, A/\overline{A}) = \text{Epi}(\mathbb{Z}, \mathbb{Z}_2)/\text{Aut}(\mathbb{Z}_2)
\]

is a singleton, and so the map \( q_H: \Gamma(H, A) \to \Gamma(H, \overline{A}) \) is a bijection. In other words, if \( H_1(X, \mathbb{Z}) = \mathbb{Z}^2 \), there is a one-to-one correspondence between regular \( \mathbb{Z} \oplus \mathbb{Z}_2 \)-covers of \( X \) and regular \( \mathbb{Z} \)-covers of \( X \), both parametrized by the projective line \( \mathbb{Q}\mathbb{P}^1 = \text{Gr}_1(\mathbb{Z}^2) \). The comparison diagram, then, takes the form

\[
\begin{array}{ccc}
\Omega_i^i_{\mathbb{Z} \oplus \mathbb{Z}_2}(X) & \xrightarrow{\cdot} & \mathbb{Q}\mathbb{P}^1 \\
\downarrow & & \downarrow \\
\Omega_i^1(X) & \xrightarrow{\cdot} & \mathbb{Q}\mathbb{P}^1
\end{array}
\]

(3.11.2)

Let \( V^i(X) \subset (\mathbb{C}^*)^2 \) be the \( i \)-th characteristic variety of \( X \). For each pair \((a, b) \in \mathbb{Z}^2\), consider the (translated) subtori \( T_{a,b}^\pm = \{(t_1, t_2) \in (\mathbb{C}^*)^2 | t_1^a t_2^b = \pm 1\} \).

**Proposition 3.11.4.** Suppose \( H = \mathbb{Z}^2 \) and \( A = \mathbb{Z} \oplus \mathbb{Z}_2 \). Then,

\[
\begin{align*}
\Omega^i_A(X) &= \{(a, b) \in \mathbb{Q}\mathbb{P}^1 | T_{-b,a}^+ \subset V^i(X) \text{ or } T_{-b,a}^- \subset V^i(X)\}^c \\
\Omega^i_1(X) &= \{(a, b) \in \mathbb{Q}\mathbb{P}^1 | T_{-b,a}^+ \subset V^i(X)\}^c.
\end{align*}
\]

**Proof.** Let \((a, b) \in H\) and let \( \xi \leq H \) be the subgroup generated by \((-b, a)\). Then \( \xi \in \Xi_1(V^i(X)) \) if and only if \( \xi = \overline{\xi} \) and \( V^i(X) \) contains a component of the form \( t_1^{-b} t_2^a = 1 \), whereas \( \xi \in \Xi_2(V^i(X)) \) if and only if \( \xi \) has index at most 2 in \( \overline{\xi} \) and \( V^i(X) \) contains a component of the form \( t_1^{-b} t_2^a = \pm 1 \). The conclusions follow from Theorem 3.11.1. \( \square \)

In particular, \( \Sigma_{\mathbb{Z} \oplus \mathbb{Z}_2}^i(X) = \Omega^i_1(X) \setminus \Omega^i_{\mathbb{Z} \oplus \mathbb{Z}_2}(X) \), and diagram (3.11.2) is not a pull-back diagram if and only if \( V^i(X) \) has a component of the form \( t_1^{a+b} t_2^a + 1 = 0 \).

A nice class of examples is provided by 2-components links. Let \( L = (L_1, L_2) \) be such a link, with complement \( X_L \). As we saw in Example 3.9.4, the characteristic variety \( V^1(X_L) \)
consists of the zero-locus in \((\mathbb{C}^*)^2\) of the Alexander polynomial \(\Delta_L(t_1, t_2)\), together with the identity.

**Example 3.11.5.** Let \(L\) be the 2-component link denoted 4_7 in Rolfsen’s tables, and let \(X_L\) be its complement. Then \(\Delta_L = t_1 + t_2\), and so \(V^1(X_L) = \{1\} \cup \{(t_1, t_2) \in (\mathbb{C}^*)^2 \mid t_1 t_2^{-1} = -1\}\). Hence, \(\Omega^1_\mathbb{A}(X_L) = \mathbb{P}^1\), but \(\Omega^1_{\mathbb{Z} \oplus \mathbb{Z}_2}(X_L) = \mathbb{P}^1 \setminus \{(1, 1)\}\).

### 3.11.4 Another particular case

The next simplest situation is the one when \(H = \mathbb{Z}^3\) and \(A = \mathbb{Z} \oplus \mathbb{Z}_2\). In this case, the set

\[
\Gamma(H/\mathbb{A}, A/\mathbb{A}) = \text{Epi}(\mathbb{Z}^2, \mathbb{Z}_2)/\text{Aut}(\mathbb{Z}_2) = (\mathbb{Z}_2 \oplus \mathbb{Z}_2)^*
\]

consists of 3 elements. In other words, if \(H_1(X, \mathbb{Z}) = \mathbb{Z}^3\), there is a three-to-one correspondence between the regular \(\mathbb{Z} \oplus \mathbb{Z}_2\)-covers of \(X\) and the regular \(\mathbb{Z}\)-covers of \(X\).

**Example 3.11.6.** Let \(G\) be the group from Example 3.9.8. With notation as before, we have that \(\Xi_1(V^1(G)) = \{\xi_2, \xi_3\}\) and \(\Xi_2(V^1(G)) = \{\xi_1, \xi_2, \xi_3, \xi_4\}\). Consequently, \(\tau_1(V^1(G)) = \{(1, 0, 0), (1, 2, 1)\}\), and so \(\Omega^1_\mathbb{A}(G) = \mathbb{P}^1 \setminus \{(1, 0, 0), (1, 2, 1)\}\).

Let \(A = \mathbb{Z} \oplus \mathbb{Z}_2\). Given an element \([\nu] \in \Gamma(H, A)\) such that \(\overline{\ker(\nu)} \supseteq \overline{\xi}_1\), we have that \([\bar{\nu}] = (0, 0, 1) \in \Omega^1_\mathbb{A}(G)\). Furthermore, \(q^{-1}([\bar{\nu}])\) consists of 3 representative classes: \(\nu_1 = (\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix})\), \(\nu_2 = (\begin{smallmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix})\), and \(\nu_3 = (\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix})\). By calculation \(\nu_1 \in U\), but \(\nu_2, \nu_3 \notin U\). Thus,

\[
\Omega^1_{\mathbb{A}}(G) = \Gamma(H, A) \setminus \{q^{-1}(1, 0, 0), q^{-1}(1, 2, 1), \nu_1\}.
\]

**Example 3.11.7.** Consider the group from [81, Example 9.7], with presentation

\[
G = \langle x_1, x_2, x_3 \mid [x_1^2, x_2], [x_1, x_3], x_1[x_2, x_3]x_1^{-1}[x_2, x_3]\rangle.
\]

The characteristic variety \(V^1(G) \subset (\mathbb{C}^*)^3\) consists of the origin, together with the translated torus \(\{(t_1, t_2, t_3) \in (\mathbb{C}^*)^3 \mid t_1 = -1\}\); hence, \(\Omega^1_\mathbb{A}(G) = \mathbb{P}^2\).

Let \(A = \mathbb{Z} \oplus \mathbb{Z}_2\). The singular set \(\Sigma = \Omega^1_{\mathbb{A}}(G)\) consists of those points \([0, b, c] \in \mathbb{P}^2\) with \(b\) and \(c\) coprime; thus, \(\Sigma \cong \mathbb{P}^1\) is infinite. Moreover, the restriction \(q: \Omega^1_{\mathbb{A}}(G) \setminus q^{-1}(\Sigma) \rightarrow\)
\( \Omega^1_1(G) \setminus \Sigma \) is three-to-one. On the other hand, if \([\nu] \in q^{-1}(\Sigma)\), then either \(\nu\) is of the form \(\begin{pmatrix} 0 & b & c \\ 1 & \epsilon_2 & \epsilon_3 \end{pmatrix}\), in which case \([\nu] \notin \Omega^1_1(G)\), or \(\nu\) is of the form \(\begin{pmatrix} 0 & b & c \\ 0 & \epsilon_2 & \epsilon_3 \end{pmatrix}\), in which case \([\nu] \in \Omega^1_1(G)\).

Thus, the restriction \(q: \Omega^1_1(G) \cap q^{-1}(\Sigma) \to \Sigma\) is one-to-one.

### 3.12 Translated tori in the characteristic varieties

Throughout this section, we assume all irreducible components of the characteristic varieties under consideration are (possibly translated) algebraic subgroups of the character group, a condition satisfied by large families of spaces.

#### 3.12.1 A refined formula for the \(\Omega\)-sets

In Theorem 3.9.5 we gave a general description of the Dwyer–Freed invariants \(\Omega^i_A(X)\) in terms of the characteristic variety \(W = V^i(X)\), while in Theorem 3.9.7 we gave a upper bound for those invariants, in terms of certain sets \(\Xi_d(W)\) and \(U_A(\xi)\), introduced in Definitions 3.5.1 and 3.6.1, respectively.

We now refine those results in the special case when all components of the characteristic variety are torsion-translated subgroups of the character group. The next theorem shows that inclusion (3.9.6) from Theorem 3.9.7 holds as equality in this case, with the union of all sets \(U_{A,d}(W)\) with \(d \geq 1\) replaced by a single constituent \(U_{A,c}(W)\), for some integer \(c\) depending only on \(W\) (and not on \(A\)).

**Theorem 3.12.1.** Let \(X\) be a connected CW-complex with finite \(k\)-skeleton. Set \(H = H_1(X, \mathbb{Z})\), and fix a degree \(i \leq k\). Suppose \(V^i(X)\) is a union of torsion-translated subgroups of \(\hat{H}\). There is then an integer \(c > 0\) such that, for every abelian group \(A\),

\[
\Omega^i_A(X) = \Gamma(H, A) \setminus U_{A,c}(V^i(X)).
\]  

(3.12.1)

**Proof.** By assumption, \(V^i(X) = \bigcup_{j=1}^n \eta_j V(\xi_j)\), for some subgroups \(\xi_j \leq H\) and torsion
elements $\eta_j \in \hat{H}$. In the special case when $s = 1$, the required equality is proved in Theorem 3.7.7, with $c = \text{ord}(\eta_1) \cdot c(\xi_1/\xi_1)$.

The general case follows from a similar argument, with $c$ replaced by the lowest common multiple of $\text{ord}(\eta_1) \cdot c(\xi_1/\xi_1), \ldots, \text{ord}(\eta_s) \cdot c(\xi_s/\xi_s)$.

3.12.2 Another formula for the $\Omega$-sets

We now present an alternate formula for computing the sets $\Omega^i_A(X)$ in the case when the $i$-th characteristic variety of $X$ is a union of torsion-translated subgroups of the character group. Although somewhat similar in spirit to Theorem 3.12.1, the next theorem uses different ingredients to express the answer.

**Theorem 3.12.2.** Let $X$ be a connected CW-complex with finite $k$-skeleton. Suppose there is a degree $i \leq k$ such that $V^i(X) = \bigcup_{j=1}^s \eta_j V(\xi_j)$, where $\xi_1, \ldots, \xi_s$ are subgroups of $H = H_1(X, \mathbb{Z})$, and $\eta_1, \ldots, \eta_s$ are torsion elements in $\hat{H}$. Then, for each abelian group $A$,

$$\Omega^i_A(X) = \bigcap_{j=1}^s \left( \sigma_A(\xi_j)^c \cup \{ [\nu] \in \Gamma(H, A) \mid \epsilon((\eta_j)) \not\in \ker(\nu) \cap \xi_j \} \right).$$

(3.12.2)

**Proof.** The result follows from Theorems 3.9.5 and 3.7.8, as well as formula (3.7.4).

The simplest situation in which the above theorem applies is that in which there are no translation factors in the subgroups comprising the characteristic variety.

**Corollary 3.12.3.** Suppose $V^i(X) = V(\xi_1) \cup \cdots \cup V(\xi_s)$ is a union of algebraic subgroups of $\hat{H}$. Then

$$\Omega^i_A(X) = \Gamma(H, A) \setminus \bigcup_{j=1}^s \sigma^{-1}(\sigma_A(\xi_j)).$$

(3.12.3)

In particular if $X^\nu$ is a free abelian cover with finite Betti numbers up to degree $i$, then any finite regular abelian cover of $X^\nu$ has the same finiteness property.

**Proof.** By Theorem 3.12.2,

$$\Omega^i_A(X) = \Gamma(H, A) \setminus \bigcup_{j=1}^s \sigma_A(\xi_j).$$

(3.12.4)
Indeed, for each \( j \) we have \( \eta_j = 1 \), and thus \( \{ \nu \mid \epsilon(\langle \eta_j \rangle) \not\in \ker(\nu) \cap \xi_j \} \) is the empty set. Applying now Proposition 3.6.4 ends the proof.

3.12.3 Toric complexes

We illustrate the above corollary with a class of spaces arising in toric topology. Let \( L \) be a simplicial complex with \( n \) vertices, and let \( T^n = S^1 \times \cdots \times S^1 \) be the \( n \)-torus, with the standard product cell decomposition. The toric complex associated to \( L \), denoted \( T_L \), is the union of all subcomplexes of the form

\[
T^\sigma = \{(x_1, \ldots, x_n) \in T^n \mid x_i = * \text{ if } i \notin \sigma \},
\]

where \( \sigma \) runs through the simplices of \( L \), and * is the (unique) 0-cell of \( S^1 \). Clearly, \( T_L \) is a connected CW-complex, with unique 0-cell corresponding to the empty simplex \( \emptyset \). The fundamental group of \( T_L \) is the right-angled Artin group

\[
G_L = \langle v \in V \mid vw = wv \text{ if } \{v, w\} \in E \rangle,
\]

where \( V \) and \( E \) denote the 0-cells and 1-cells of \( L \). Furthermore, a classifying space for \( \pi_1(T_L) \) is the toric complex \( T_{\Delta(L)} \), where \( \Delta(L) \) is the flag complex associated to \( L \).

Evidently, \( H_1(T_L, \mathbb{Z}) = \mathbb{Z}^n \); thus, we may identify the character group of \( \pi_1(T_L) \) with the algebraic torus \((\mathbb{C}^*)^V := (\mathbb{C}^*)^n\). For any subset \( W \subset V \), let \((\mathbb{C}^*)^W \subset (\mathbb{C}^*)^V\) be the corresponding subtorus; in particular, \((\mathbb{C}^*)^\emptyset = \{1\}\). Note that \((\mathbb{C}^*)^W = V(\xi_W)\), where \( \xi_W \) is the sublattice of \( \mathbb{Z}^n \) spanned by the basis vectors \( \{e_i \mid i \notin W\} \). From [69], we have the following description of the characteristic varieties of our toric complex:

\[
V^i(T_L) = \bigcup_W V(\xi_W),
\]

where the union is taken over all subsets \( W \subset V \) for which there is a simplex \( \sigma \in L_{V\setminus W} \) and an index \( j \leq i \) such that \( \tilde{H}_{j-1-|\sigma|}(\text{lk}_{L_W}(\sigma), \mathbb{C}) \neq 0 \). Here, \( L_W \) denotes the subcomplex induced by \( L \) on \( W \), and \( \text{lk}_K(\sigma) \) denotes the link of a simplex \( \sigma \in L \) in a subcomplex \( K \subset L \).
From the above, we see that the assumptions from Corollary 3.12.3 are true for the classifying space of a right-angled Artin group, and, in fact, for any toric complex. Hence, we obtain the following corollaries.

**Corollary 3.12.4.** Let $T_L$ be a toric complex. Then,

$$\Omega^i_A(T_L) = \Gamma(H, A) \setminus \bigcup_W q^{-1}(\sigma_{\bar{A}}(\xi_W))$$

where each $\sigma_{\bar{A}}(\xi_W) \subseteq \text{Gr}_{n-r}(\mathbb{Q}^n)$ is the special Schubert variety corresponding to the coordinate plane $\xi_W \otimes \mathbb{Q}$, and the union is taken over all subsets $W \subseteq V$ for which there is a simplex $\sigma \in L_{V \setminus W}$ and an index $j \leq i$ such that $\tilde{H}_{j-1-|\sigma|}(\text{lk}_{L_W}(\sigma), \mathbb{C}) \neq 0$.

**Corollary 3.12.5.** Let $T_L$ be a toric complex. If $T^\nu_L$ is a free abelian cover with finite Betti numbers up to some degree $i$, then all regular, finite abelian covers of $T^\nu_L$ also have finite Betti numbers up to degree $i$.

### 3.12.4 When the translation order is coprime to $|\text{Tors}(A)|$

We now return to the general situation, where the characteristic variety is a union of torsion-translated subgroups. In the next proposition, we identify a condition on the order of translation of these subgroups, insuring that diagram (3.3.7) is a pull-back diagram.

As usual, let $A$ be a quotient of $H = H_1(X, \mathbb{Z})$, and let $\overline{A} = A/\text{Tors}(A)$ be its maximal torsion-free quotient. Recall that the canonical projection, $q : \Gamma(H, A) \to \Gamma(H, \overline{A})$, restricts to a map $q|_{\Omega^i_A(X)} : \Omega^i_A(X) \to \Omega^i_{\overline{A}}(X)$, and that resulting commuting square is a pull-back diagram if and only if $\Omega^i_A(X)$ is the full pre-image of $q$.

Using Proposition 3.7.13 and Theorem 3.9.5, we obtain the following consequence.

**Proposition 3.12.6.** Suppose the characteristic variety $V^i(X)$ is of the form $\bigcup_j \rho_j T_j$, where each $T_j \subset \widehat{H}$ is an algebraic subgroup, and each $\rho_j \in \widehat{H}/T_j$ has finite order, coprime to the order of $\text{Tors}(A)$. Then $\Omega^i_A(X) = q^{-1}\left(\Omega^i_{\overline{A}}(X)\right)$. 

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Here is an application. As usual, let $X$ be a connected CW-complex with finite $k$-skeleton. Assume $H = H_1(X, \mathbb{Z})$ has no torsion, and identify the character torus $\hat{H}$ with $(\mathbb{C}^*)^n$, where $n = b_1(X)$.

**Corollary 3.12.7.** Suppose that, for some $i \leq k$, there is an $(n - 1)$-dimensional subspace $L \subseteq H^1(X; \mathbb{Q})$ such that $V^i(X) = (\bigcup \alpha \rho_\alpha T) \cup Z$, where $Z$ is a finite set, $T = \exp(L \otimes \mathbb{C})$ and $\rho_\alpha$ is a torsion element in $(\mathbb{C}^*)^n$ of order coprime to that of $\text{Tors}(A)$, for each $\alpha$. Let $r = \text{rank } A$. Then,

$$\Omega^i_A(X) = \begin{cases} 
\Gamma(H, A), & \text{if } r = 1; \\
q^{-1}(\text{Gr}_r(L)), & \text{if } 1 < r < n; \\
\emptyset, & \text{if } r \geq n.
\end{cases}$$

**Proof.** From [81, Proposition 9.6], we have that

$$\Omega^i_r(X) = \begin{cases} 
\mathbb{Q}P^{r-1}, & \text{if } r = 1; \\
\text{Gr}_r(L), & \text{if } 1 < r < n; \\
\emptyset, & \text{if } r \geq n.
\end{cases}$$

On the other hand, Proposition 3.12.6 shows that $\Omega^i_A(X) = q^{-1}(\Omega^i_r(X))$, and so the desired conclusion follows. 

\[
\square
\]

**3.12.5 When the translation order divides $|\text{Tors}(A)|$**

To conclude this section, we give some sufficient conditions on the groups $H$ and $A$, and on the order of translation of the subgroups comprising $V^i(X)$, insuring that diagram (3.3.7) is not a pull-back diagram.

Proposition 3.7.16 and Theorem 3.9.5 yield the following immediate application.

**Corollary 3.12.8.** Let $X$ be a connected CW-complex with finite $k$-skeleton. Assume that

(i) The group $H = H_1(X, \mathbb{Z})$ is torsion-free, and $A$ is a quotient of $H$. 

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(ii) There is a degree $i \leq k$ such that the positive-dimensional components of $V^i(X)$ form a torsion-translated subgroup $\rho T$ inside $\widehat{H}$.

(iii) The rank of $A$ is less than the rank of $H$, and $\operatorname{ord}(\rho)$ divides $c(A)$.

Then $\Omega_A^i(X) \subseteq q^{-1}\left(\Omega_T^i(X)\right)$.

Note that the hypothesis of Corollary 3.12.8 are satisfied in Examples 3.11.5 and 3.11.7, thus explaining why, in both cases, diagram (3.3.7) is not a pull-back diagram. Here is one more situation when that happens.

**Corollary 3.12.9.** Suppose $V^i(X) = \rho_1 T_1 \cup \cdots \cup \rho_s T_s$, with each $T_j$ an algebraic subgroup of $\widehat{H}$ and each $\rho_j$ a torsion element in $\widehat{H} \setminus T_j$. Furthermore, suppose that

(i) The identity component of $T_1$ is not contained in $T' = T_2 \cdots T_s$ (internal product in $\widehat{H}$).

(ii) The order of $\rho_1$ divides $c(A)$.

(iii) rank $A < \operatorname{rank} H - \dim T'$.

Then $\Omega_A^i(X) \subseteq q^{-1}\left(\Omega_T^i(X)\right)$.

**Proof.** Split $H$ as a direct sum, $H' \oplus H''$, so that $\widehat{H}' = T'$ and $\widehat{H}'' = \widehat{H}/T'$. Let $p: H \twoheadrightarrow H''$ be the canonical projection, and let $\hat{p}: \widehat{H}'' \twoheadrightarrow \widehat{H}$ be the induced morphism. By assumption (1), we have that $\hat{p}^{-1}(T_1)$ is a positive-dimensional algebraic subgroup of $\widehat{H}''$. Thus, the positive-dimensional components of $W = V^i(X) \cap \widehat{H}''$ form a torsion-translated subgroup of $\widehat{H}''$, namely, $\hat{p}^{-1}(\rho_1 T_1)$. Moreover, assumptions (2) and (iii) imply that rank $A < \operatorname{rank} H''$ and $\operatorname{ord}(\hat{p}^{-1}(\rho_1))$ divides $c(A)$.

Applying now Proposition 3.7.16 to the algebraic group $\widehat{H}''$ and to the subvariety $W$ yields an epimorphism $\mu: H'' \twoheadrightarrow A$ such that $\operatorname{im}(\hat{\mu}) \cap \hat{p}^{-1}(\rho_1 T_1)$ is finite, and $\operatorname{im}(\hat{\mu}) \cap \hat{p}^{-1}(\rho_1 T_1)$ is infinite. Setting $\nu = \mu \circ p$, we see that $\operatorname{im}(\hat{\nu}) = \operatorname{im}(\hat{\mu})$. Thus, $\operatorname{im}(\hat{\nu}) \cap V^i(X)$ is finite, while $\operatorname{im}(\hat{\nu}) \cap V^i(X)$ is infinite. \qed
3.13 Quasi-projective varieties

We conclude with a discussion of the generalized Dwyer–Fried sets of smooth, quasi-projective varieties.

3.13.1 Characteristic varieties

A space $X$ is said to be a (smooth) quasi-projective variety if there is a smooth, complex projective variety $\bar{X}$ and a normal-crossings divisor $D$ such that $X = \bar{X} \setminus D$. For instance, $X$ could be the complement of an algebraic hypersurface in $\mathbb{CP}^d$.

The structure of the characteristic varieties of such spaces was determined through the work of Beauville, Green and Lazarsfeld, Simpson, Campana, and Arapura in the 1990s, and further refined in recent years. We summarize these results, essentially in the form proved by Arapura.

**Theorem 3.13.1** ([3]). Let $X = \bar{X} \setminus D$, where $\bar{X}$ is a smooth, projective variety and $D$ is a normal-crossings divisor.

(i) If either $D = \emptyset$ or $b_1(\bar{X}) = 0$, then each characteristic variety $V^i(X)$ is a finite union of unitary translates of algebraic subtori of $T = H^1(X, \mathbb{C}^*)$.

(ii) In degree $i = 1$, the condition that $b_1(\bar{X}) = 0$ if $D \neq \emptyset$ may be lifted. Furthermore, each positive-dimensional component of $V^1(X)$ is of the form $\rho \cdot S$, where $S$ is an algebraic subtorus of $T$, and $\rho$ is a torsion character.

For instance, if $C$ is a connected, smooth complex curve of negative Euler characteristic, then $V^1(C)$ is the full character group $H^1(C, \mathbb{C}^*)$. For an arbitrary smooth, quasi-projective variety $X$, each positive-dimensional component of $V^1(X)$ arises by pullback along a suitable orbifold fibration (or, pencil). More precisely, if $\rho \cdot S$ is such a component, then $S = f^*(H^1(C, \mathbb{C}^*))$, for some curve $C$, and some holomorphic, surjective map $f: X \to C$ with connected generic fiber.
Using this interpretation, together with recent work of Dimca, Artal-Bartolo, Cogolludo, and Matei (as recounted in [81]), we can describe the variety $V^1(X)$, as follows.

**Theorem 3.13.2.** Let $X$ be a smooth, quasi-projective variety. Then

$$V^1(X) = \bigcup_{\xi \in \Lambda} V(\xi) \cup \bigcup_{\xi \in \Lambda'} (V(\xi) \setminus V(\tilde{\xi})) \cup Z,$$

where $Z$ is a finite subset of $T = H^1(X; \mathbb{C}^*)$, and $\Lambda$ and $\Lambda'$ are certain (finite) collections of subgroups of $H = H_1(X, \mathbb{Z})$.

### 3.13.2 Dwyer–Fried sets

Using now Theorem 3.13.2 (and keeping the notation therein), Proposition 3.7.12 yields the following structural result for the degree 1 Dwyer–Fried sets of a smooth, quasi-projective variety.

**Theorem 3.13.3.** Let $A$ be a quotient of $H$. Then

$$\Omega^1_A(X) = \Gamma(H, A) \setminus \left( \bigcup_{\xi \in \Lambda} q^{-1}(\sigma_A(\xi)) \cup \bigcup_{\xi \in \Lambda'} \left( q^{-1}(\sigma_A(\tilde{\xi})) \cap \theta_A(\xi) \right) \right).$$

In certain situations, more can be said. For instance, Proposition 3.12.6 yields the following corollary.

**Corollary 3.13.4.** Suppose the order of $\bar{\xi}/\xi$ is coprime to $c(A)$, for each $\xi \in \Lambda'$. Then

$$\Omega^1_A(X) = q^{-1}(\Omega^1_A(X)).$$

Similarly, Corollary 3.12.9 has the following consequence.

**Corollary 3.13.5.** Suppose $H$ is torsion-free, and there is a subgroup $\chi \in \Lambda'$ such that

(i) $V(\bar{\chi})$ is not contained in $T' := V(\bigcap_{\xi \in \Lambda \cup \Lambda' \setminus \{\chi\}} \xi)$.

(ii) There is a non-zero element in $\bar{\chi}/\chi$ whose order divides $c(A)$.

(iii) $\text{rank } A < \text{codim } T'$. 

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Then $\Omega^1_A(X) \subsetneq q^{-1}\left(\Omega^1_A(X)\right)$.

For the remainder of this section, we shall give some concrete examples of quasi-projective manifolds $X$ for which the computation of the sets $\Omega^1_A(X)$ can be carried out explicitly.

3.13.3 Brieskorn manifolds

Let $(a_1, \ldots, a_n)$ be an $n$-tuple of integers, with $a_j \geq 2$. Consider the variety $X$ in $\mathbb{C}^n$ defined by the equations $c_{j1}x_1^{a_1} + \cdots + c_{jn}x_n^{a_n} = 0$, for $1 \leq j \leq n - 2$. Assuming all maximal minors of the matrix $(c_{jk})$ are non-zero, $X$ is a quasi-homogeneous surface, with an isolated singularity at 0.

The space $X$ admits a good $\mathbb{C}^*$-action. Set $X^* = X \setminus 0$, and let $p: X^* \to C$ be the corresponding projection onto a smooth projective curve. Then $p^*: H^1(C, \mathbb{C}) \to H^1(X^*, \mathbb{C})$ is an isomorphism, the torsion subgroup of $H = H_1(X^*, \mathbb{Z})$ coincides with the kernel of $p_*: H_1(X^*, \mathbb{Z}) \to H_1(C, \mathbb{Z})$.

By definition, the Brieskorn manifold $M = \Sigma(a_1, \ldots, a_n)$ is the link of the quasi-homogeneous singularity $(X, 0)$. As such, $M$ is a closed, smooth, oriented 3-manifold homotopy equivalent to $X^*$. Put

$$l = \text{lcm}(a_1, \ldots, a_n), \quad l_j = \text{lcm}(a_1, \ldots, \hat{a}_j, \ldots, a_n), \quad a = a_1 \cdots a_n.$$

The $S^1$-equivalent homeomorphism type of $M$ is determined by the following Seifert invariants associated to the projection $p|_M: M \to C$:

- The exceptional orbit data, $(s_1(\alpha_1, \beta_1), \ldots, s_n(\alpha_n, \beta_n))$, with $\alpha_j = l/l_j$, $\beta_j l \equiv a_j \mod \alpha_j$ and $s_j = a/(a_j l_j)$, where $s_j = (\alpha_j \beta_j)$ means $(\alpha_j \beta_j)$ repeated $s_j$ times, unless $\alpha_j = 1$, in which case $s_j = (\alpha_j \beta_j)$ is to be removed from the list.

- The genus of the base curve, given by $g = \frac{1}{2} \left(2 + (n - 2)a/l - \sum_{j=1}^n s_j\right)$.

- The (rational) Euler number of the Seifert fibration, given by $e = -a/l^2$. 

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The group $H = H_1(M, \mathbb{Z})$ has rank $2g$, and torsion part of order $\alpha_1^{s_1} \cdots \alpha_n^{s_n} \cdot |e|$. Identify the character group $\hat{H}$ with a disjoint union of copies of $\hat{H}_0 = (\mathbb{C}^*)^{2g}$, indexed by $\text{Tors}(H)$, and set $\alpha = \alpha_1^{s_1} \cdots \alpha_n^{s_n} / \text{lcm}(\alpha_1, \ldots, \alpha_n)$.

**Proposition 3.13.6 ([17]).** The positive-dimensional components of $V^1(M)$ are as follows:

(i) $\alpha - 1$ translated copies of $\hat{H}_0$, if $g = 1$.

(ii) $\hat{H}_0$, together with $\alpha - 1$ translated copies of $\hat{H}_0$, if $g > 1$.

Denote the elements in $\text{Tors}(H)$ corresponding to the $\alpha - 1$ translated copies of $\hat{H}_0$ by $h_1, \ldots, h_{\alpha-1}$. We then have the following corollaries.

**Corollary 3.13.7.** Let $M = \Sigma(a_1, \ldots, a_n)$ be a Brieskorn manifold, and let $A$ be a quotient of $H = H_1(M, \mathbb{Z})$, with $r = \text{rank } A$.

(i) If $g > 1$, then $\Omega^1_A(M) = \Omega^1_{H} = \emptyset$.

(ii) If $g = 1$, then $\Omega^1_A(M) = \text{Gr}_r(\mathbb{Q}^{2g})$, while

$$\Omega^1_A(M) = \{[\nu] \in \Gamma(H,A) \mid \nu(h_i) = 0 \text{ for } i = 1, \ldots, \alpha - 1\}.$$ 

**Corollary 3.13.8.** Suppose $g = 1$ and $\alpha > 1$. Then $\Omega^1_H(M) = \{\text{pt}\}$, yet $\Omega^1_H(M) = \emptyset$; that is, $b_1(M^{\text{fab}}) < \infty$, yet $b_1(M^{\text{ab}}) = \infty$.

**Example 3.13.9.** Consider the Brieskorn manifold $M = \Sigma(2, 4, 8)$. Using the algorithm described by Milnor in [61], we see that the fundamental group of $M$ has presentation

$$G = \langle x_1, x_2, x_3 \mid x_1x_2^2 = x_3x_1, x_2x_3^2 = x_3^2x_2, x_3^2(x_3x_1x_2x_1^{-1}x_2^{-1})^2 = 1 \rangle,$$

while its abelianization is $H = \mathbb{Z}^2 \oplus \mathbb{Z}_4$. Identifying $\hat{H} = (\mathbb{C}^*)^2 \times \{\pm1, \pm i\}$, we find that $V^1(M) = \{1\} \cup (\mathbb{C}^*)^2 \times \{-1\}$. (The positive-dimensional component in $V^1(M)$ arises from an elliptic orbifold fibration $X \to \Sigma_1$ with two multiple fibers, each of multiplicity 2.) By Proposition 3.10.6, then, $b_1(M^{\text{fab}}) < \infty$, while $b_1(M^{\text{ab}}) = \infty$. 

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Now take $A = \mathbb{Z} \oplus \mathbb{Z}_4$. Applying Corollary 3.13.7 (with $g = 1$ and $\alpha = 2$), we conclude that $\Omega^1_A(M) = \mathbb{Q} \mathbb{P}^1$, while $\Omega^1_A(M)$ consists of two copies of $\Gamma(\mathbb{Z}^2, A)$, naturally embedded in $\Gamma(H, A)$.

### 3.13.4 The Catanese–Ciliberto–Mendes Lopes surface

We now give an example of a smooth, complex projective variety $M$ for which the generalized Dwyer–Fried sets exhibit the kind of subtle behavior predicted by Corollary 3.13.5.

Let $C_1$ be a (smooth, complex) curve of genus 2 with an elliptic involution $\sigma_1$ and $C_2$ a curve of genus 3 with a free involution $\sigma_2$. Then $\Sigma_1 = C_1/\sigma_1$ is a curve of genus 1, and $\Sigma_2 = C_2/\sigma_2$ is a curve of genus 2. The group $\mathbb{Z}_2$ acts freely on the product $C_1 \times C_2$ via the involution $\sigma_1 \times \sigma_2$; let $M$ be the quotient surface. This variety, whose construction goes back to Catanese, Ciliberto, and Mendes Lopes [11], is a minimal surface of general type with $p_g(M) = q(M) = 3$ and $K_M^2 = 8$.

The projection $C_1 \times C_2 \to C_1$ descends to an orbifold fibration $f_1: M \to \Sigma_1$ with two multiple fibers, each of multiplicity 2, while the projection $C_1 \times C_2 \to C_2$ descends to a holomorphic fibration $f_2: M \to \Sigma_2$. It is readily seen that $H = H_1(M, \mathbb{Z})$ is isomorphic to $\mathbb{Z}_6$; fix a basis $e_1, \ldots, e_6$ for this group. A computation detailed in [81] shows that the characteristic variety $V^1(M) \subset (\mathbb{C}^*)^6$ has two components, corresponding to the above two pencils; more precisely,

$$ V^1(M) = V(\xi_1) \cup (V(\xi_2) \setminus V(\bar{\xi_2})), \quad (3.13.1) $$

where $\xi_1 = \text{span}\{e_1, e_2\}$ and $\xi_2 = \text{span}\{2e_3, e_4, e_5, e_6\}$.

Now suppose $A$ is a quotient of $H = \mathbb{Z}_6$, and let $q: \Gamma(H, A) \to \Gamma(H, \overline{A})$ be the canonical projection. By Theorem 3.13.3,

$$ \Omega^1_A(M) = \Gamma(H, A) \setminus (q^{-1}(\sigma_A(\xi_1)) \cup (q^{-1}(\sigma_A(\xi_2)) \cap \theta_A(\xi_2))). \quad (3.13.2) $$

Let us describe explicitly this set in a concrete situation.
Example 3.13.10. Let $A = \mathbb{Z} \oplus \mathbb{Z}_2$, and identify $\overline{A} = \mathbb{Z}$ and $\Gamma(H, \mathbb{Z}) = \mathbb{Q}P^5$. The fiber of $q$ is the set $\Gamma = (\mathbb{Z}_2^5)^*$. Given an epimorphism $\nu: H \to \mathbb{Z} \oplus \mathbb{Z}_2$, let $\bar{\nu}: H \to \mathbb{Z}$ and $\nu': H \to \mathbb{Z}_2$ be the composites of $\nu$ with the projections on the respective factors. The terms on the right-side of (3.13.2) are as follows:

- $\sigma_\mathbb{Z}(\xi_1)$ is the projective subspace $\mathbb{Q}P^3 = \mathbb{P}((H/\xi_1)^\vee \otimes \mathbb{Q})$ spanned by $e_3, \ldots, e_6$.

- $\sigma_\mathbb{Z}(\xi_2)$ is the projective line $\mathbb{Q}P^1 = \mathbb{P}((H/\xi_2)^\vee \otimes \mathbb{Q})$ spanned by $e_1$ and $e_2$.

- $q^{-1}(\mathbb{Q}P^1) \cap \theta_A(\xi_2)$ consists of those $[\nu]$ satisfying $\nu(e_3) = \cdots = \nu(e_6) = 0$, and $\nu'(e_3) = 1, \nu'(e_4) = \nu'(e_5) = \nu'(e_6) = 0$.

This completes the description of the set $\Omega^1_{\mathbb{Z} \oplus \mathbb{Z}_2}(M)$. Clearly, $\Omega^1_M = \mathbb{Q}P^5 \setminus \mathbb{Q}P^3$. Furthermore, note that the restriction map $q^{-1}(\mathbb{Q}P^1) \cap \theta_A(\xi_2) \to \mathbb{Q}P^1$ is a 2-to-1 surjection. Thus, the restriction map $\Omega^1_{\mathbb{Z} \oplus \mathbb{Z}_2}(M) \to \Omega^1_M$ is not a set fibration: the fiber over $\Omega^1_M \setminus \mathbb{Q}P^1$ has cardinality 31, while the fiber over $\mathbb{Q}P^1$ has cardinality 29.
4.1 Oriented cohomology

4.1.1 Oriented cohomology theories and algebraic cobordism

In this subsection we collect preliminary notions and results we will use. The main goal is to fix the notations and conventions.

Recall that a formal group law over a commutative ring \( R \) with unit is an element \( F(u, v) \in R[[u, v]] \) satisfying the following conditions

1. \( F(u, 0) = u, F(0, v) = v; \)

2. \( F(F(u, v), w) = F(u, F(v, w)); \)

3. \( F(u, v) = F(v, u). \)

Lazard pointed out the existence of a universal formal group law \((\text{Laz}, F_{\text{Laz}})\). He also proved that the ring \text{Laz}, called Lazard ring, is a polynomial ring with integral coefficients on a
countable set of variables (see [57]). That is, for any formal group law \((R, F)\) over any ring \(R\), there exists a unique ring homomorphism \(r : \text{Laz} \rightarrow R\) such that \(F = r(F_{\text{Laz}})\).

Now let \(F(u, v) \in R[[u, v]]\) be an arbitrary formal group law over a commutative ring \(R\); we will always assume that \(R\) is graded and that, giving both \(u\) and \(v\) degree 1, \(F(u, v)\) is homogeneous of degree 0. As the Lazard ring is generated as a \(\mathbb{Z}\)-algebra by the coefficients of the universal formal group law, \(F_{\text{Laz}} \in \text{Laz}[u, v]\) this convention gives \(\text{Laz}\) a uniquely defined grading, concentrated in non-positive degrees, and the classifying homomorphism \(\phi_F : \text{Laz} \rightarrow R\) associated to \(F\) preserves the grading.

Let \(k\) be a field, and \(\text{Sm}_k\) be the category of smooth, quasi-projective schemes over \(k\). Let \(\text{Comm}\) denote the category of commutative, graded rings with unit. The following definition can be found in [54].

**Definition 4.1.1.** An oriented cohomology theory on \(\text{Sm}_k\) is given by the following data.

**D1** An assignment \(A^*\) sending any \(X \in \text{Sm}_k^{\text{op}}\) to an object in \(\text{Comm}\).

**D2** For any smooth morphism \(f : Y \rightarrow X\) in \(\text{Sm}_k\), a ring homomorphism

\[
f^* : A^*(X) \rightarrow A^*(Y).
\]

**D3** For any projective morphism \(f : Y \rightarrow X\) in \(\text{Sm}_k\) of relative codimension \(d\), a homomorphism of graded \(A^*(X)\)-modules

\[
f_* : A^*(Y) \rightarrow A^{*+d}(X).
\]

These data satisfy the axioms: functoriality for \(f^*\) with respect to arbitrary morphisms in \(\text{Sm}_k\), functoriality for \(f_*\) with respect to projective morphisms, base change for transverse morphisms, a projective bundle formula, and extended homotopy property. We should mention that, for \(L \rightarrow X\) a line bundle over some \(X \in \text{Sm}_k\), the first Chern class of \(L\) in the theory \(A\), denote \(c_1^A(L)\), is defined by the formula \(c_1^A(L) := s^*s_*(1_X)\), where \(s : X \rightarrow L\) is
the zero section and $1_X \in A^0(X)$ is the unit in $A^*(X)$. We often write simply $c_1(L)$ if the theory $A$ is understood.

For an oriented cohomology theory $A^*$, there is a unique power series

$$F(u, v) \in A^*(k)[[u, v]]$$

satisfying

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M)$$

for each pair line bundles $L$ and $M$ on a scheme $X \in \text{Sm}_k$; moreover, $F$ defines a formal group law over the ring $A^*(k)$ [54, Lemma 1.1.3]. As we will often consider cohomology theories with rational coefficients, we denote the ring $A^*(X) \otimes \mathbb{Q}$ simply by $A^*_{\mathbb{Q}}(X)$.

For any $X \in \text{Sm}_k$, Levine and Morel constructed a commutative graded ring $\Omega^*(X)$ as follows. Let $M(X)$ be the set of isomorphism classes of projective morphisms $f : Y \to X$ with $Y \in \text{Sm}_k$. $M(X)$ becomes a monoid under the disjoint union. Let $M_+^*(X)$ be its group completion, graded by the relative codimension of the map $f : Y \to X$. Then $\Omega^*(X)$ is constructed as a quotient of $M_+^*(X)$ [54, Lemma 2.5.11].

The following is a resumé of some of the main results from [54].

**Theorem 4.1.2** (Levine and Morel). Assume the base field $k$ has characteristic zero.

(i) The assignment

$$X \mapsto \Omega^*(X)$$

extends to define an oriented cohomology theory on $\text{Sm}_k$, called algebraic cobordism.

(ii) $\Omega^*$ is the universal oriented cohomology theory on $\text{Sm}_k$.

(iii) The canonical ring homomorphism $\text{Laz} \to \Omega^*(k)$ induced by the formal group law $F_{\Omega}$ of the algebraic cobordism $\Omega^*$ is an isomorphism.
When the base field $k$ has positive characteristic, the construction of $\Omega^*(X)$ described above is not known to give a oriented cohomology theory. However, the construction of $\Omega^*(X)$ in [54] leads to a notion of a “universal oriented Borel-Moore Laz-functor on $\text{Sm}_k$ of geometric type” (see [54, Definition 2.2.1]; here we index by codimension rather than by dimension), and $\Omega^*$ is the universal such theory [54, Theorem 2.4.13]. The structures enjoyed by such a theory $A^*$ are:

1. (Projective pushforward) For $f : Y \to X$ a projective morphism in $\text{Sm}_k$ of relative dimension $d$, there is a graded homomorphism $f_* : A^*(Y) \to A^{*-d}(X)$.

2. (Smooth pullback) For a smooth morphism $f : Y \to X$ in $\text{Sm}_k$, there is a graded homomorphism $f^* : A^*(X) \to A^*(Y)$.

3. (1st Chern class operators) For each line bundle $L \to X$, $X \in \text{Sm}_k$, there is a graded homomorphism $\tilde{c}_1(L) : A^*(X) \to A^{*+1}(X)$. For line bundles $L, M$ on $X$, the operators $\tilde{c}_1(L)$ and $\tilde{c}_1(M)$ commute.

4. (External products) For $X,Y \in \text{Sm}_k$, there is a graded homomorphism $A^*(X) \otimes_\mathbb{Z} A^*(Y) \to A^*(X \times_k Y)$, which is commutative and associative in the obvious sense. There is an element $1 \in A^0(k)$ which, together with the external product $A^*(k) \otimes_\mathbb{Z} A^*(k) \to A^*(k)$, makes $A^*(k)$ into a commutative graded ring with unit.

5. (Fundamental class) For $X \in \text{Sm}_k$ with structure morphism $p : X \to \text{Spec} \, k$, we denote $p^*(1)$ by $1_X$, and for $L \to X$ a line bundle, write $c_1(L)$ for $\tilde{c}_1(L)(1_X)$.

6. (Dimension axiom) For each collection of line bundles $L_1, \ldots, L_r$ on $X$ with $r > \dim_k X$, one has $\prod_{i=1}^r \tilde{c}_1(L_i)(1_X) = 0$.

7. (Formal group law) There is a homomorphism $\phi_A : \text{Laz} \to A^*(k)$ of graded rings, such that, letting $F_A(u,v) \in A^*(k)[u,v]$ be the image of the universal formal group law...
with respect to $\phi_A$, for each $X \in \text{Sm}_k$ and each pair of line bundles $L, M$ on $X$, we have
\[ c_1(L \otimes M) = F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_X). \]
These all satisfy a number of compatibilities, detailed in [54, §2.1, 2.2].

An oriented cohomology theory has a canonical structure of an oriented Borel-Moore Laz-functor on $\text{Sm}_k$ of geometric type [54, Proposition 5.2.1, Proposition 5.2.6], where the Chern class operator $\tilde{c}_1(L)$ is multiplication by the 1st Chern class $c_1(L)$. In particular, each oriented cohomology theory $A^*$ admits a canonical classifying map
\[ \Theta_A : \Omega^* \to A^*, \quad (4.1.1) \]
which is compatible with all push-forward maps $f_*$ for projective $f$, all pull-back maps $f^*$ for smooth $f$, first Chern classes and first Chern class operators, external products and the formal group law. Explicitly, for a generator $[f : Y \to X] \in M^*_+(X), \Theta_A([f : Y \to X]) = f^A_*(1^A_Y)$, where $f^A_*$ is the pushforward in the theory $A^*$ and $1_A \in A^0(Y)$ is the unit.

### 4.1.2 Motivic oriented cohomology theories

In positive characteristic, the universal oriented cohomology theory is not available; we use instead an approach via motivic homotopy theory.

Let $\text{SH}(k)$ be the motivic stable homotopy category of $\mathbb{P}^1$-spectra [63, 71, 96]. $\text{SH}(k)$ is a triangulated tensor category with tensor product $(E, F) \mapsto E \wedge F$ and unit the motivic sphere spectrum $S_k$. There is a functor
\[ \Sigma_{\mathbb{P}^1}^\infty : \text{Sm}_k \to \text{SH}(k) \]
called infinite $\mathbb{P}^1$-suspension; for example $S_k = \Sigma_{\mathbb{P}^1}^\infty \text{Spec } k$. We have as well the auto-equivalences $\Sigma_{S^1}$ and $\Sigma_{\mathbb{G}_m}$, and the translation in the triangulated structure is given by $\Sigma_{S^1}$. 115
An object \( E \) of \( \text{SH}(k) \) and integers \( n, m \) define a functor \( E^{n,m} : \text{Sm}^\text{op}_k \to \text{Ab} \) by
\[
E^{n,m}(X) := \text{Hom}_{\text{SH}(k)}(\Sigma_{P^1}^\infty X_+, \Sigma_{S^1}^{n-m} \Sigma_{G_m}^m E).
\]

An object \( E \) together with morphisms \( \mu : E \land E \to E \) (multiplication) and \( 1 : S_k \to E \) (unit) define a ring cohomology theory on \( \text{Sm}_k \) if they make the bi-graded group \( E^{*,*}(X) := \bigoplus_{n,m \in \mathbb{Z}} E^{n,m}(X) \) into a bi-graded ring which is graded commutative with respect to the first grading and commutative with respect to the second one.

Given a ring cohomology theory as above, an element \( \vartheta \in E^{2,1}(\mathbb{P}^\infty/0) \) is called an orientation if the restriction of \( \vartheta \) to \( \vartheta|_{\mathbb{P}^1/0} \in E^{2,1}(\mathbb{P}^1/0) \) (via the embedding \( \mathbb{P}^1 \to \mathbb{P}^\infty, (x_0 : x_1) \mapsto (x_0 : x_1 : 0 : 0 \ldots) \)) agrees with the image of \( 1 \in E^{0,0}(k) \) under the suspension isomorphism \( E^{2,1}(\mathbb{P}^1/0) \cong E^{0,0}(k) \). Here \( 0 := (1 : 0 : \ldots : 0) \in \mathbb{P}^n \), including \( n = \infty \).

In [75, Theorem 2.15] it is shown how an orientation \( \vartheta \) for a ring cohomology theory \( E \) on \( \text{Sm}_k \) gives rise to functorial pushforward maps \( f_* : E^{a,b}(Y) \to E^{a-2d,b-d}(X) \) for each projective morphism \( f : Y \to X \) in \( \text{Sm}_k \), where \( d \) is the relative dimension of \( f \) (\( d := \dim_k Y - \dim_k X \) in case \( X \) and \( Y \) are irreducible).

**Definition 4.1.3.** A motivic oriented cohomology theory on \( \text{Sm}_k \) is an object \( E \in \text{SH}(k) \) together with morphisms \( \mu : E \land E \to E, 1 : S_k \to E \) defining a ring cohomology theory, plus an orientation \( \vartheta \in E^{2,1}(\mathbb{P}^\infty/0) \).

**Remarks 4.1.4.**

1. The notion of a motivic oriented cohomology theory on \( \text{Sm}_k \) is referred to as an “oriented ring cohomology theory” in [72]. We find this too similar to the term “oriented cohomology theory”, hence our relabelling.

2. The algebraic cobordism \( \mathbb{P}^1 \)-spectrum \( MGL \in \text{SH}(k) \) is the universal motivic oriented cohomology theory. See [71, 96] for the construction of \( MGL \) and [72] for the proof of universality.

It follows from the results of [75] that, if \((E, \mu, 1, \vartheta)\) is a motivic oriented cohomology theory on \( \text{Sm}_k \), then the contravariant functor \( X \mapsto E^*(X) := E^{2*,*}(X) \) from \( \text{Sm}_k \) to \( \text{Comm} \),
together with the maps $f_*$ for projective morphisms $f : Y \to X$ in $\text{Sm}_k$, defines an oriented cohomology theory on $\text{Sm}_k$, which we denote by $E^\ast$. We call $E^\ast$ the oriented cohomology theory on $\text{Sm}_k$ represented by the motivic oriented cohomology theory $(E, \mu, 1, \vartheta)$ (or just $E$ for short). In particular, for each motivic oriented cohomology theory $E$ on $\text{Sm}_k$, there is a canonical homomorphism $\phi_E : \text{Laz} \to E^\ast(k)$ classifying the formal group law of the oriented cohomology theory $E^\ast$.

**Remark 4.1.5.** In general, an oriented cohomology theory on $\text{Sm}_k$ is not always represented by a motivic oriented theory (see [42]).

It follows from a theorem of Hopkins-Morel, recently established in detail by Hoyois [41], that for $k$ of characteristic zero, the ring homomorphism $\text{Laz} \to \text{MGL}(k)$ classifying the formal group law for $\text{MGL}^\ast$ is an isomorphism. When $k$ has characteristic $p > 0$, it is shown in [41] that after inverting $p$, the classifying map $\text{Laz}[1/p] \to \text{MGL}^\ast(k)[1/p]$ is an isomorphism. Conjecturally, for any field $k$, the classifying map is an isomorphism, but at present, this is not known.

### 4.1.3 Specialization of the formal group law

From the oriented cohomology theory $\text{MGL}^\ast$, and a given formal group law $F(u, v) \in R[u, v]$ such that the exponential characteristic $p$ of $k$ is invertible in $R$, we may construct an oriented cohomology theory $R^\ast$ on $\text{Sm}_k$ with $R^\ast(k) = R$ and formal group law $F$ as follows.

Take $X \in \text{Sm}_k$ and let $p_X : X \to \text{Spec} k$ be the structure morphism. The classifying map $\phi_{\text{MGL}} : \text{Laz} \to \text{MGL}^\ast(k)$ composed with $p_X^\ast$ defines the ring homomorphism $p_X^\ast \circ \phi_{\text{MGL}} : \text{Laz} \to \text{MGL}^\ast(X)$. Using the classifying homomorphism $\phi_F : \text{Laz} \to R$, we may form the tensor product ring $R^\ast(X) := \text{MGL}^\ast(X) \otimes_{\text{Laz}} R$. Since $p$ is invertible in $R$ and $\phi_{\text{MGL}}[1/p]$ is an isomorphism, it follows that the canonical map $R \to R^\ast(k)$ is an isomorphism. Extending the pull-back and push-forward maps for the theory $\text{MGL}^\ast$ define the pull-back and push-forward maps for the theory $R^\ast$. Since the functor $- \otimes_{\text{Laz}} R$ is additive and preserves isomorphisms,
it follows easily that the assignment $X \mapsto R^*(X)$ with pull-back and push-forward maps as described above defines an oriented cohomology theory on $\text{Sm}_k$. Similarly, it is easy to see that for $L \to X$ a line bundle, $c_1^R(L)$ is just the image of $c_1^\text{MGL}(L)$ in $R^*(X)$, and therefore the formal group law for $R^*$ is equal to $F(u,v) \in R[[u,v]]$. This gives us the following result:

**Lemma 4.1.6.** Let $k$ be a perfect field, $F \in R[u,v]$ a formal group law over a commutative (graded) ring $R$. Assume that the exponential characteristic of $k$ is invertible in $R$. Then there is an oriented cohomology theory $R^*$ on $\text{Sm}_k$ with $R^*(X) = \text{MGL}^*(X) \otimes_{\text{Laz}} R$. Moreover, $R^*(k) = R$ and $R^*$ has formal group law $F$. Finally, if the characteristic of $k$ is zero, then $R^*$ is the universal oriented cohomology theory on $\text{Sm}_k$ with formal group law $F \in R[u,v]$.

**Proof.** We have proved everything except for the last statement. This follows by noting that the isomorphism of oriented cohomology theories $\Theta : \Omega^* \to \text{MGL}^*$ shows that $R^*$ is isomorphic to the oriented cohomology theory $X \mapsto \Omega^*(X) \otimes_{\text{Laz}} R$. The fact that this latter theory is the universal oriented cohomology theory on $\text{Sm}_k$ with formal group law $F \in R[u,v]$ follows immediately from the fact that $\Omega^*$ is the universal oriented cohomology theory on $\text{Sm}_k$. \qed

There is a nice formula for push-forward in the cohomology theory $\text{MGL}^*$, in the case of the structure morphism of a projective space bundle.

**Theorem 4.1.7 (Quillen).** Let $X$ be a smooth quasi-projective variety, $V$ be some $n$-dimensional vector bundle on $X$, and $\pi : \mathbb{P}_X(V) \to X$ be the corresponding projective bundle. Let $f(t) \in \text{MGL}^*(X)[t]$. Then,

$$\pi_*(f(c_1(\mathcal{O}(1)))) = \sum_i \frac{f(-\Omega \lambda_i)}{\prod_{j \neq i}(\lambda_j - \Omega \lambda_i)},$$

(4.1.2)

where $\lambda_i$ are the Chern roots of $V$, and $x + \Omega y := F_{\Omega}(x,y)$, where $F_{\Omega}$ is the formal group law of the cobordism theory $\text{MGL}^*$.
A proof of this theorem, in the context of complex cobordism, can be found in [95], page 50. The proof goes through word for word in our setting, so we will not repeat it here. Clearly, for any formal group law \((F, R)\) such that the exponential characteristic is invertible in \(R\), the same formula as above is valid for the push-forward in the theory \(R^*\).

### 4.1.4 Landweber exactness

We now describe a sufficient condition for the theory \(X \mapsto R^*(X)\) to arise from a motivic oriented cohomology theory; this is the well-known condition of Landweber exactness.

For any prime \(l > 0\), we expand the \(l\)-series of the this formal group law \(x + F \cdots + F x\) (summing \(l\) copies of \(x\)), as \(\sum_{i \geq 1} a_i x^i\), and for all \(n \geq 0\) we write \(v_n := a_{l^n}\). In particular we have \(v_0 = a_1 = l\).

**Definition 4.1.8.** The formal group \((R, F)\) is said to be Landweber exact if for all primes \(l\) and for all integers \(n\), the multiplication map

\[
v_n : R/(v_0, \ldots, v_{n-1}) \to R/(v_0, \ldots, v_{n-1})
\]

is injective.

**Theorem 4.1.9.** Let \(k\) be a perfect field, \(F \in R[[u, v]]\) a formal group law. If the formal group law \((R, F)\) is Landweber exact and the exponential characteristic of \(k\) is invertible in \(R\), then the oriented cohomology theory \(X \mapsto R^*(X)\) on \(\text{Sm}_k\) is represented by a motivic oriented cohomology theory.

The classical precursor of this result is due to Landweber [50]; in our setting this result follows from [66, Theorem 7.3]. We denote the motivic cohomology theory associated to a Landweber exact formal group law by \(\text{MGL} \otimes_\text{Laz} R\) and the canonical morphism given by the universality of \(\text{MGL}\) by \(\Theta_{F,R} : \text{MGL} \to \text{MGL} \otimes_\text{Laz} R\).
4.1.5 Exponential and logarithm

Let \((R,F)\) be the formal group law. A logarithm of the formal group law \(F\) is a series \(g(u) = u + \text{higher order terms} \in R[u]\) satisfying the equation

\[ g(F(u,v)) = g(u) + g(v). \]

Novikov [68] showed that every formal group law with coefficients in a \(\mathbb{Q}\)-algebra has a logarithm. The functional inverse \(\lambda(u) \in R[[u]]\) of the logarithm \(g(u)\) is called the exponential of the formal group law. The expansion of \(\lambda(u)\) takes the form \(u + \text{higher order terms}\). With our grading conventions, if we give \(u\) degree one, then the power series \(g(u)\) and \(\lambda(u)\) are both homogeneous of degree one. Thus, if we write \(\lambda(u) = u + \sum_{i \geq 1} \tau_i u^{i+1}\), then \(\tau_i \in R\) has degree \(-i\).

In fact, the formal group law and the exponential power series uniquely determine each other, assuming that \(R \to R_\mathbb{Q}\) is injective. As ring homomorphisms \(\text{Laz} \to R\) are in bijection with formal group laws with coefficients in \(R\), it is noted by Hirzebruch that ring homomorphisms \(\text{Laz} \to R_\mathbb{Q}\) are in one to one correspondence with power series \(\lambda\) as above. For any formal group law \((R,F)\) with exponential \(\lambda(u) \in R \otimes \mathbb{Q}[u]\), the corresponding ring homomorphism \(\phi_F : \text{MU}^{2*} \to R_\mathbb{Q}^*(k)\) given by Quillen’s identification \(\text{MU}^{2*} \cong \text{Laz}\) is called the Hirzebruch genus. In terms of algebraic geometry, \(\phi_F\) sends the class of smooth projective irreducible variety \(X\) of dimensional \(n\) to \(\prod_{i=1}^n \frac{\xi_i}{\lambda(\xi_i)} \cdot [X]\), where \(\xi_1, \ldots, \xi_n\) are the Chern roots of the tangent bundle \(T_X\) (for \(\text{CH}^*(X)\)) and \(\langle -, [X] \rangle\) means evaluation on the fundamental class of \(X\). Explicitly, if we let \(\text{Td}_r(u) := \frac{u}{\phi(\xi_i)} \in R[[u]]\) and write \(\text{Td}_r(u) = 1 + \sum_{n \geq 0} t_d^n u^i\), with \(t_d^n \in R^{-i}\), then \(\phi_F([X]) = \deg_k(N_n(T_X)) \cdot t_d^n\), where \(N_n(T_X)\) is the Newton class, \(N_n(T_X) := \sum_j \xi_j^n\).

**Example 4.1.10.** Let \(R = \mathbb{Q}[b_1, b_2, \ldots]\), with \(b_n\) of degree \(-n\). Let \(\lambda_b(u) = u + \sum_{n \geq 1} b_n u^{n+1}\) and let \(\lambda_b^{-1}(u)\) be the functional inverse. We let \(F_b(u,v) \in R[u,v]\) be the formal group law \(\lambda_b(\lambda_b^{-1}(u) + \lambda_b^{-1}(v))\). One can show (see e.g. [1, theorem 7.8]) that the subring \(R_0\) of \(R\)
generated by the coefficients of $F_b$ is isomorphic to Laz via the homomorphism Laz $\to R_0$ classifying $F_b$; in particular, $(F_b, R_0)$ is the universal formal group law.

4.1.6 Twisting a cohomology theory

Let $k$ be an arbitrary perfect field and let $R^*$ be an oriented cohomology theory on Sm$_k$. Suppose that the coefficient ring $R^*(k)$ is a $\mathbb{Q}$-algebra. There is a twisting construction due to Quillen (see e.g. [54] for a detailed description), which enables one to construct the oriented cohomology theory $R^*$ from the theory given by the Chow ring $X \mapsto \text{CH}^*(X)$. In this section, we describe the twisting construction and its analog for motivic oriented cohomology theories.

Let $A^*$ be an oriented cohomology theory on Sm$_k$ and $\tau = (\tau_i) \in \prod_{i=0}^\infty A^{-i}(k)$, with $\tau_0 = 1$. Let $\lambda_{\tau}(u) = \sum_{i=0}^\infty \tau_i u^{i+1}$. The Todd genus $\text{Td}_{\tau}(t)$ is by definition the power series

$$\text{Td}_{\tau}(t) := \frac{t}{\lambda_{\tau}(t)}.$$

For a vector bundle $E$ on some $Y \in \text{Sm}_k$, the Todd class of a vector bundle $E$ on $Y$ is giving by

$$\text{Td}_{\tau}(E) = \prod_{i=1}^r \text{Td}_{\tau}(\xi_i)$$

where $\xi_1, \ldots, \xi_r$ are the Chern roots of $E$ in $A^*(Y)$. The assignment $E \mapsto \text{Td}_{\tau}(E)$ is multiplicative in exact sequences, hence descends to a well-defined homomorphism $\text{Td}_{\tau} : K_0(Y) \to (1 + A^{\geq 1}(Y))^\times$.

Define the twisted oriented cohomology theory $A^*_\tau$ on Sm$_k$ with $A^*_\tau(X) = A^*(X)$, for $X \in \text{Sm}_k$, and with pull-backs unchanged ($f^*_{\tau} = f^*$). For a projective morphism $f : Y \to X$, we set

$$f^*_{\tau} := f_* \circ \text{Td}_{\tau}(T_f),$$

where

$$T_f := [T_Y] - [f^*T_X] \in K_0(Y)$$
is the relative tangent bundle. It is not difficult to show that this does indeed define an oriented cohomology theory on $\text{Sm}_k$.

Let $\lambda_\tau(u) = \sum_{i=0}^\infty \tau_i u^{i+1}$. Then for a line bundle $L$, the first Chern class in the new cohomology theory $A^*_\tau$ is given by $c_1^\tau(L) := \lambda_\tau(c_1(L))$, from which one easily sees that the formal group law for $A^*_\tau$ is given by

$$
F^\tau_{A}(u,v) = \lambda_\tau(F_A(\lambda_\tau^{-1}(u), \lambda_\tau^{-1}(v))).
$$

In particular, if $F_A$ is the additive group law, $F_A(u,v) = u + v$, then $\lambda_\tau(u)$ is the exponential map for the twisted group law $(F^\tau_A, A^*(k))$.

The twisting construction is also available for motivic oriented cohomology theories. Indeed, let $(E, \mu, 1, \vartheta)$ be a motivic oriented cohomology theory, and $(\tau_i \in E^{-2i-1}(k))_i$ a sequence of elements, with $\tau_0 = 1$.

Form the orientation $\vartheta_\tau \in E^{2,1}(\mathbb{P}^\infty/0)$,

$$
\vartheta_\tau := \lambda_\tau(\vartheta) = \sum_{i \geq 0} \tau_i \vartheta^{i+1}.
$$

We note that the projective bundle formula implies that $\vartheta^m$ goes to zero in $E^{2,1}(\mathbb{P}^N/0)$ for $m > N$, from which it follows that $\lambda_\tau(\vartheta)$ is a well-defined element in $E^{2,1}(\mathbb{P}^\infty/0) = \lim_{\leftarrow} E^{2,1}(\mathbb{P}^N/0)$ and that $\vartheta_\tau$ is indeed an orientation. Let $\tau E^*$ be the oriented cohomology theory corresponding to $(E, \mu, 1, \vartheta_\tau)$.

It follows immediately from the definitions that, for $L \to X$ a line bundle, $X \in \text{Sm}_k$, the first Chern class $c_1^{\tau E^*}(L)$ in the theory $\tau E^*$ is the same as in the $\tau$-twist of $E^*$, namely $c_1^{\tau E^*}(L) = \lambda_\tau(c_1^{E^*}(L)) = c_1^\tau(L)$. From this it follows easily that the identity map on the graded groups $E^*_\tau(X) = E^*(X)$ defines an isomorphism of $\tau E^*$ with the twisted oriented cohomology theory $E^*_\tau$; we will henceforth drop the notation $\tau E^*$.

Let $R$ be a graded $\mathbb{Q}$-algebra. Define

$$
H R = \bigoplus_{n \in \mathbb{Z}} \Sigma_p^n H R^{-n}
$$

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where $HR^{-n}$ is the $\mathbb{P}^1$ spectrum representing motivic cohomology with coefficients in the $\mathbb{Q}$ vector space $R^{-n}$. We make $HR$ a motivic oriented cohomology theory by using the orientation induced by that of $HZ$.

Now let $(R, F)$ be a formal group law, where $R$ is given the grading following our conventions and is as above a $\mathbb{Q}$-algebra. Taking $(\tau_i \in R^{-i} = HR^{-2i-i}(k))_i$ to be the sequence such that $\lambda_r(u)$ is the exponential function for $(R, F)$, we form the twisted motivic oriented cohomology theory $HR_\tau$; by construction $HR_\tau$ has associated formal group law $(R, F)$. We note that $(R, F)$ is Landweber exact, since $R$ is a $\mathbb{Q}$-algebra.

We may also form the motivic oriented cohomology theory $MGL \otimes_{Laz} R$ associated to the Landweber exact formal group law $(R, F)$. As this is the universal motivic oriented cohomology theory with group law $(R, F)$, the classifying map $\Theta_{HR_\tau} : MGL \to HR_\tau$ factors through $\Theta_{F,R} : MGL \to MGL \otimes_{Laz} R$, giving the induced classifying map $\tilde{\Theta}_{HR_\tau} : MGL \otimes_{Laz} R \to HR_\tau$, unique up to a phantom map.

**Lemma 4.1.11.** The map $\tilde{\Theta}_{HR_\tau} : MGL \otimes_{Laz} R \to HR_\tau$ induces an isomorphism of bigraded cohomology theories $MGL^{*,*} \otimes_{Laz} R \to HR^{*,*}_\tau$, in particular, we have the isomorphism of associated oriented cohomology theories on $Sm_k$

$$\tilde{\Theta}_{HR_\tau} : R^* = MGL^* \otimes_{Laz} R \to HR^*_\tau$$

**Proof.** We apply the slice spectral sequence to $MGL \otimes_{Laz} R$ and $HR_\tau$; the map $\tilde{\Theta}_{HR_\tau}$ induces a map of spectral sequences. For a $\mathbb{P}^1$ spectrum $E$ denote the $n$th layer in the slice tower for $E$ as $s_n E$. For an abelian group $A$, it follows from [98] that

$$s_n HA = \begin{cases} HA & \text{for } n = 0 \\ 0 & \text{for } n \neq 0. \end{cases}$$

Also $s_n \sum_{\mathbb{P}^1} E = \sum_{\mathbb{P}^1} s_{n-m} E$. As $HR = \oplus_{n \in \mathbb{Z}} \sum_{\mathbb{P}^1} HR^{-n}$, it follows that the $n$th layer in the slice tower for $HR$ is given by $s_n HR = \sum_{\mathbb{P}^1} HR^{-n}$. By Spitzweck’s computation of the layers in the slice tower for a Landweber exact theory, we have the same $s_n MGL \otimes_{Laz} R = \sum_{\mathbb{P}^1} HR^{-n}$.
as well. The maps on the layers of the slice tower induced by \( \bar{\Theta}_{HR} \) are \( H\mathbb{Q} \)-module maps (by results of Pelaez [76]), and one knows by a result of Cisinski-Deglise [12, Theorem 16.1.4] that

\[
\text{Hom}_{H\mathbb{Q}-\text{Mod}}(\Sigma^n_{p_1}HR^{-n}, \Sigma^n_{p_1}HR^{-n}) \cong \text{Hom}_{\mathbb{Q}-\text{Vec}}(R^{-n}, R^{-n})
\]

In particular, the map \( s_n\bar{\Theta}_{HR} : \Sigma^n_{p_1}HR^{-n} \rightarrow \Sigma^n_{p_1}HR^{-n} \) is determined by the induced map after applying the functor \( H^{-2n,-n}(k,-) \), that is, on the coefficient rings of the theories \( \text{MGL}^* \otimes_{\text{Laz}} R \) and \( HR^* \). However, by construction, this is the map \( \bar{\Theta} : R \rightarrow R \) induced by the classifying map \( \text{Laz} \rightarrow R \) associated to the formal group \( F_r(u,v) = \lambda_r(g_r(u) + g_r(v)) \). As this latter formal group law is equal to \( F \) by construction, the map \( \bar{\Theta} : R \rightarrow R \) is the identity map.

Thus \( \Theta_{HR} \) induces an isomorphism of the (strongly convergent) slice spectral sequences, and hence an isomorphism of bi-graded cohomology theories on \( \text{Sm}_k \).

4.1.7 Universality

Since \( \text{MGL}^* \) is an oriented cohomology theory, we have the canonical comparison morphism

\[
\Theta_{\text{MGL}} : \Omega^*(X) \rightarrow \text{MGL}^*(X).
\]

Relying on the Hopkins-Morel-Hoyois isomorphism \( \text{Laz} \cong \text{MGL}^*(k) \), Levine has shown that for \( k \) a field of characteristic zero, \( \Theta_{\text{MGL}} : \Omega^* \rightarrow \text{MGL}^* \) is an isomorphism [53, theorem 3.1]. Thus, for a field of characteristic zero, \( \text{MGL}^* \) is the universal oriented cohomology theory on \( \text{Sm}_k \). At present, there is no proof of the existence of a universal oriented cohomology theory on \( \text{Sm}_k \) if \( k \) has positive characteristic. In this subsection, we use the universality of \( \text{MGL}^* \) as a motivic oriented cohomology theory plus some tricks with formal group laws to show that \( \text{MGL}_{\mathbb{Q}}^* \) is the universal oriented cohomology theory for theories in \( \mathbb{Q} \)-algebras on \( \text{Sm}_k \). We also show that the two constructions of oriented cohomology theories: construction by specialisation of the formal group law from \( \text{MGL}^* \) and construction by extending coefficients
for $\text{CH}^*$ and then twisting, are “equivalent”, assuming the coefficient ring is a $\mathbb{Q}$-algebra.

Lemma 4.1.12. Suppose $k$ has characteristic zero. Then $\text{CH}^*$ is the universal oriented cohomology theory on $\text{Sm}_k$ with formal group law $(u+v,\mathbb{Z})$. If $k$ has characteristic $p > 0$, then $\text{CH}^*_\mathbb{Q}$ is the universal oriented cohomology theory on $\text{Sm}_k$ with formal group law $(u + v, \mathbb{Q})$.

Of course, one would expect that over an arbitrary field, $\text{CH}^*$ is the universal oriented cohomology theory on $\text{Sm}_k$ with formal group law $(u + v, \mathbb{Z})$. This does not seem to be known.

Proof. The case of characteristic zero is proven in [54, Theorem 1.2.2]. In characteristic $p > 0$, let $A^*$ be an oriented cohomology theory with additive formal group law $F_A(u,v) = u + v$ and with $A^*(k)$ a $\mathbb{Q}$-algebra. Extend the coefficients in the theory $A^*$ by a Laurent polynomial ring, forming the theory $A^*[t,t^{-1}]$, with $t$ of degree -1. Then take the twist with respect to the modified exponential function

$$
\lambda_t(u) := t^{-1}(1 - e^{-tu})
$$

that is, $\tau_t := (-1)^i t^i / (i + 1)!$. A simple computation shows that theory $A^*[t,t^{-1}]_\tau$ has the multiplicative group law $F(u,v) = u + v - tuv$, and that the twisted first Chern class is given by $c_1^t(L) = t^{-1}(1 - e^{tc_1^t(L)})$.

One can define the modified Chern character

$$
\text{ch}_t^A : K_0[t, t^{-1}]_\mathbb{Q} \to A^*[t,t^{-1}]_\tau,
$$

which sends a vector bundle $E$ of rank $r$ to

$$
\text{ch}_t^A(E) := r - tc_1^t(E^\vee).
$$

For a line bundle $L$, we have

$$
\text{ch}_t^A(L) = 1 - tc_1^t(L^\vee) = e^{tc_1^t(L)}.
$$
Using the splitting principle, the fact that $A^*$ has the additive formal group law implies that $\text{ch}_t^A$ is a natural transformation of functors to graded $\mathbb{Q}[t, t^{-1}]$-algebras. Since $c^R_{K_0[t, t^{-1}]}(L) = t^{-1}(1 - L^{-1})$, we have

$$\text{ch}_t^A(c^K_1(L)) = c_1^t(L)$$

for all line bundles $L$. By Panin’s Riemann-Roch theorem [73, Corollary 1.1.10], this shows that $\text{ch}_t^A$ is a natural transformation of oriented cohomology theories.

We have the Adams operations $\psi_k$, $k = 1, 2, \ldots$, on $K_0(X)$, which we extend to Adams operations on $K_0(X)[t, t^{-1}]$ by $\mathbb{Q}[t, t^{-1}]$-linearity. Define the operation $\psi_k^A$ on $A^*(X)[t, t^{-1}]$ to be the $\mathbb{Q}[t, t^{-1}]$-linear map which is multiplication by $k^n$ on $A^n(X)$; it is easy to see that $\psi_k^A$ is a natural $\mathbb{Q}[t, t^{-1}]$-algebra homomorphism. As $A$ has the additive group law, $c_k^A(L^\otimes k) = kc_1^A(L)$ and thus

$$\text{ch}_t^A(\psi_k(L)) = \text{ch}_t^A(L^\otimes k) = e^{tc_1^A(L^\otimes k)} = e^{ktc_1^A(L)} = \psi_k^A(e^{tc_1^A(L)}) = \psi_k^A(\text{ch}_t^A(L))$$

for all line bundles $L$. By the splitting principle, this gives the identity

$$\text{ch}_t^A \circ \psi_k = \psi_k^A \circ \text{ch}_t^A.$$

If we take $A^* = \text{CH}_Q^*$, $\text{ch}_t^A$ is a modified version of the classical Chern character; thus by Grothendieck’s classical result, the natural transformation

$$\text{ch}_t^{\text{CH}_Q^*} : K_0[t, t^{-1}] \to \text{CH}_Q^*[t, t^{-1}]$$

is an isomorphism. This gives us the natural transformation of oriented cohomology theories

$$\text{ch}_t^A \circ (\text{ch}_t^{\text{CH}_Q^*})^{-1} : \text{CH}_Q^*[t, t^{-1}] \to A^*[t, t^{-1}]$$

Twisting back gives us the natural transformation of oriented cohomology theories

$$\vartheta_A^{\text{CH}_t} : \text{CH}_Q^*[t, t^{-1}] \to A^*[t, t^{-1}]$$

We give $\text{CH}_Q^*[t, t^{-1}]$ a bi-grading by putting $\text{CH}_Q^n \cdot t^m$ in bi-degree $(n, m)$, and do the same for $A^*[t, t^{-1}]$. Since both $\text{ch}_t^{\text{CH}_Q^*}$ and $\text{ch}_t^A$ commute with the Adams operations, we have

$$\vartheta_A^{\text{CH}_t} \circ \psi_k^{\text{CH}_Q} = \psi_k^A \circ \vartheta_A^{\text{CH}_t}.$$
and thus $\vartheta^\text{CH}_A$ respects the bi-grading. Passing to the respective quotients by the ideal $(t - 1)$ gives us the natural transformation of oriented cohomology theories

$$\vartheta^\text{CH}_A : CH^*_Q \to A^*$$

where we now use the original grading on $CH^*_Q$ and $A^*$.

The uniqueness of $\vartheta^\text{CH}_A$ follows from Grothendieck-Riemann-Roch. Indeed, as a natural transformation of oriented cohomology theories, $\vartheta^\text{CH}_A(c^\text{CH}_n(E)) = c^A_n(E)$ for all vector bundles $E$ on $X \in \text{Sm}_k$, and all $n$. But for irreducible $X \in \text{Sm}_k$, the Grothendieck-Riemann-Roch theorem implies that $CH^*(X)_Q$ is generated as a $Q$-vector space by the elements of the form $c^\text{CH}_n(E)$, $E$ a vector bundle on $X$, $n \geq 1$ an integer, together with the identity element $1 \in CH^0(X)$. Thus $\vartheta^\text{CH}_A$ is unique.

We now consider the generic twist $CH^*_Q[b]_b$, where $b = \{b_i\}_i$. We have as well the motivic oriented cohomology theory $H^Q \in SH(k)$, representing rational motivic cohomology, $H^*(X, Q(*))$. The orientation $v_H \in H^Q_{2,1}(\mathbb{P}^n/0)$ is given by the sequence of hyperplane classes $c^\text{CH}_1(O_{\mathbb{P}^n}(1)) \in H^Q_{2,1}(\mathbb{P}^n) = CH^1(\mathbb{P}^n)_Q$.

We may form the generic twist $H^Q[b]_b$ by taking the orientation $\lambda_b(v_H) := \sum_{n \geq 0} b_n v_H^{n+1}$.

**Proposition 4.1.13.**

1. Let $k$ be a field of characteristic zero. Then $\text{MGL}^*$ is the universal oriented cohomology theory on $\text{Sm}_k$.

2. Let $k$ be a perfect field of characteristic $p > 0$. Then $\text{MGL}^*_Q$ is the universal oriented cohomology theory in $Q$-algebras on $\text{Sm}_k$.

3. Let $(F, R)$ be a formal group law, with $R$ a $Q$-algebra and let $\lambda_\tau(u) = \sum_{n \geq 0} \tau_n u^{n+1}$ be the associated exponential function. Let $k$ be a perfect field. Then the classifying map $\text{MGL}^*_Q \otimes_{\text{Laz}} R \to (CH^* \otimes R)_\tau$ is an isomorphism.

4. Let $k$ be a perfect field, and consider the generic twist $CH^*_Q[b]_b$. For an arbitrary perfect field $k$, the classifying map $\text{MGL}^*_Q \to CH^*_Q[b]_b$ is an isomorphism.
Proof. (1) follows from the isomorphism of oriented cohomology theories on $\text{Sm}_k$, $\Omega^* \to \text{MGL}^*$ [53, Theorem 3.1], and the universal property of $\Omega^*$ [54, Theorem 7.1.3].

(4) follows from (3), noting that the classifying map $\text{Laz} \to \mathbb{Z}[b_1, b_2, \ldots]$ for the formal group law $F_b(u, v) := \lambda_b(\lambda_b^{-1}(u) + \lambda_b^{-1}(v))$ induces an isomorphism $\text{Laz}_Q \to \mathbb{Q}[b_1, b_2, \ldots]$ (see e.g. [1, Theorem 7.8]), hence $\text{MGL}_Q^* \to \text{MGL}_Q^* \otimes \text{Laz}_Q \mathbb{Q}$ is an isomorphism. The assertion (3) follows immediately from Lemma 4.1.11, as the isomorphism $\text{CH}^* \cong H^{2*,*}(-, \mathbb{Z})$ gives rise to a canonical isomorphism $HR^* \cong (\text{CH}^* \otimes R)_\tau$.

To prove (2), it suffices by (4) to show that $\text{CH}_Q^*[b]_b$ is the universal oriented cohomology theory in $\mathbb{Q}$-algebras on $\text{Sm}_k$. This follows by applying the twisting construction. Indeed, let $A^*$ be an oriented cohomology theory on $\text{Sm}_k$, such that $A^*(k)$ is a $\mathbb{Q}$-algebra. Let $(\tau_n^{-1} \in A^{-n})_n$ be the sequence such that $\lambda_{\tau^{-1}}(u)$ is the logarithm of the formal group law $(F_{A^*}, A^*(k))$. The twist $A_{\tau^{-1}}^*$ thus has the additive formal group law and hence by Lemma 4.1.12 we have the (unique) classifying map $\theta_{\tau^{-1}} : \text{CH}_Q^* \to A_{\tau^{-1}}^*$. As we have already mentioned above, the map $\text{Laz}_Q \to \mathbb{Q}[b_1, b_2, \ldots]$ classifying the formal group law $F_b(u, v)$ is an isomorphism, hence there is a unique ring homomorphism $\phi : \mathbb{Q}[b_1, b_2, \ldots] \to A^*(k)$ with $\phi(F_b(u, v)) = F_{A^*}(u, v)$. Extend $\theta_{\tau^{-1}}$ to $\Theta_{\tau^{-1}} : \text{CH}_Q^*[b] \to A_{\tau^{-1}}^*$ by using $\phi$, and then twist back by $b$ and $\phi(b) = \tau$ to get the map $\Theta_A : \text{CH}_Q^*[b]_b \to A^*$ of oriented cohomology theories on $\text{Sm}_k$. The uniqueness of $\Theta_A$ follows from the uniqueness of $\theta_{\tau^{-1}}$ and that of $\phi$. This completes the proof.

Given a formal group law $(R, F)$ with exponential $\lambda(t) \in R_Q[t]$. Write $\lambda(t) = t + \sum_{i \geq 1} \tau_it^{i+1}$ with $\tau_i \in R_Q^{-i}$; we set $\tau_0 = 1$. We have two methods of constructing an oriented cohomology theory with formal group law $(R_Q, F)$: the specialisation construction $\text{MGL}^* \otimes \text{Laz} R_Q$ or the twisting construction $(\text{CH}^* \otimes R_Q)_\tau$. As these two oriented cohomology theories are canonically isomorphic, we will denote both of them by $R_Q^*$:

$$\text{MGL}^* \otimes \text{Laz} R_Q =: R_Q^* := (\text{CH}^* \otimes R_Q)_\tau. \quad (4.1.3)$$
4.2 The algebraic elliptic cohomology theory

4.2.1 The elliptic formal group law

An algebraic elliptic cohomology theory is the cohomology theory corresponding to an elliptic formal group law. More precisely, let \( R \) be a ring and let \( p : E \to \text{Spec} R \) be a smooth projective morphism with section \( e : \text{Spec} R \to E \) and addition map \( E \times_R E \to E \) defining a commutative group scheme over \( R \) with geometrically connected fibers of dimension one. We assume in addition that we have chosen a local uniformizer \( t \) around the identity section. The expansion of the group law of \( E \) in terms of the coordinate \( t \) gives a formal group law \( F_E \) with coefficients in \( R \).

There is a well-studied elliptic formal group law and the corresponding cohomology theory. Namely, taking \( R = \mathbb{Z}[\mu_1,\mu_2,\mu_3,\mu_4,\mu_6] \) and taking \( E \) to be the Weierstrass curve

\[
y^2 + \mu_1 xy + \mu_3 y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6
\]

over the ring \( R \), and using \( t = y/x \) as the local uniformizer. This formal group law will be referred as the TMF elliptic formal group law. It has been studied by Franke, Hopkins, Landweber, Miller, Ravenel, Stong, etc. See [42] and [40] for survey of this theory. In particular, it has been shown that, the map of rings

\[
F_E : \text{Laz} \to R[\Delta^{-1}]
\]

given by this formal group law is Landweber exact, where \( \Delta \) is the discriminant.

The elliptic formal group law we are using is called the Krichever’s elliptic formal group law in literature. It is related to, but different from the TMF elliptic formal group law. We recall the genus corresponding to this formal group law, following the convention in [84]. Let

\[
\sigma(z,\tau) := z \prod_{w \in \mathbb{Z} + Z\tau, w \neq 0} (1 - z/w) e^{\frac{\pi i}{2} \left( \frac{(z/w)^2}{2} \right)}
\]
be the Weierstrass sigma function. Note that the sigma function defined above, as a function in \( z \), is an entire odd function, with zero set equal to the lattice \( \mathbb{Z} + \mathbb{Z} \tau \subseteq \mathbb{C} \). It has the following property

\[
\sigma(z, \tau) = -e^{\eta_1(z + \frac{1}{2})} \sigma(z + 1, \tau),
\]

and

\[
\sigma(z, \tau) = -e^{\eta_2(z + \frac{1}{2} \tau)} \sigma(z + \tau, \tau),
\]

for some functions \( \eta_j = \eta_j(\tau), \ j = 1, 2 \).

We define the algebraic elliptic genus as the ring homomorphism

\[
\phi_E : \text{Laz} \rightarrow \mathbb{Q}(\langle e^{2\pi i z} \rangle [e^{2\pi i \tau}, \frac{k}{2\pi i}])
\]

associated to the power series \( \lambda_E(t) := \frac{t}{Q(t)} \) under the Hirzebruch correspondence, where \(^1\)

\[
Q(t) := \frac{t}{2\pi i} e^{kt} e^{\zeta(z) \frac{i}{t}} \frac{\sigma(t \frac{1}{2\pi i} - z, \tau)}{\sigma(t \frac{1}{2\pi i}, \tau) \sigma(-z, \tau)},
\]

here \( \zeta(z) = \frac{d \log \sigma(z)}{dz} \) is the zeta function and the function

\[
\Phi(t, z) = e^{\zeta(z) \frac{i}{t}} \frac{\sigma(t \frac{1}{2\pi i} - z, \tau)}{\sigma(t \frac{1}{2\pi i}, \tau) \sigma(-z, \tau)}
\]

is the Baker-Akhiezer function. Explicitly, the elliptic genus of a \( n \)-dimensional smooth variety \( X \) is defined by \( \phi(X) := \langle \prod_{i=1}^{n} Q(\xi_i), [X] \rangle \), where the \( \xi_i \) are the Chern roots of the tangent bundle \( T_X \) of \( X \).

The coefficient ring \( \tilde{\text{Ell}} \) of the corresponding formal group law, i.e., the elliptic formal group law, is by definition the image of \( \phi_E \). Moreover, the elliptic formal group law can be written as

\[
x +_E y = \lambda(\lambda^{-1}(x) + \lambda^{-1}(y)),
\]

\(^1\)The series \( Q(t) \) we use here is slightly different from the one used in [43] and [84], where the series is

\[
Q(t) := \frac{t}{2\pi i} e^{kt} e^{\eta_1(z) \frac{i}{t}} \frac{\sigma(t \frac{1}{2\pi i} - z, \tau)}{\sigma(t \frac{1}{2\pi i}, \tau) \sigma(-z, \tau)},
\]

for the eta function \( \eta_1 \). Note that they both enjoy the rigidity property.
for the series \( \lambda(t) = \frac{t}{Q(t)} \) as above. The algebraic elliptic cohomology theory associated to this formal group law, \( \text{MGL}^* \otimes_{\text{Laz}} \tilde{\text{Ell}} \), is denoted by \( \tilde{\text{Ell}}^* \).

**Remark 4.2.1.** The coefficient ring \( \tilde{\text{Ell}}^*(k) \) depends only on the characteristic of \( k \). For \( k \) having characteristic zero, \( \tilde{\text{Ell}}^*(k) = \tilde{\text{Ell}} \) and for \( k \) having characteristic \( p > 0 \), \( \tilde{\text{Ell}}^*(k) = \tilde{\text{Ell}}[1/p] \).

When \( k = \mathbb{C} \), the elliptic genus has the rigidity property, proved by Krichever in [49] and Höhn in [43]. A consequence of the rigidity property is that given a fiber bundle \( F \to E \to B \) of closed connected weakly complex manifolds, with structure group a compact connected Lie group \( G \), and if \( F \) is a \( SU \)-manifold, then the elliptic genus is multiplicative with respect to this fibration.

The coefficient ring \( \tilde{\text{Ell}} \) has been studied in [9]. We summarize their results here. The subring \( \tilde{\text{Ell}} \) of \( \mathbb{Q}(e^{2\pi iz})(e^{2i\pi \tau}, \frac{k}{2\pi i}) \) is contained in a polynomial subring \( \mathbb{Z}[a_1, a_2, a_3, a_4] \) of \( \mathbb{Q}(e^{2\pi iz})(e^{2i\pi \tau}, \frac{k}{2\pi i}) \). The elements \( a_i \) can be described explicitly in terms of elements in \( \mathbb{Q}(e^{2\pi iz})(e^{2i\pi \tau}, \frac{k}{2\pi i}) \) as follows. Let \( g_i \) be the weight \( 2i \) Eisenstein series where \( i = 2, 3, \) and let \( X \) be the Weierstrass \( p \)-function and \( Y \) its derivative. The power series expansions of these functions are as follows.

\[
X(\tau, z) = \frac{1}{12} + \frac{y}{(1-y)^2} - 2 \prod_{m,n \geq 1} n q^{mn} + \prod_{m,n \geq 1} n q^{mn}(y^n + y^{-n});
\]
\[
Y(\tau, z) = \prod_{m \geq 0} q^m y (1 + q^m y) \left( \frac{1}{1-q^m y} \right)^3 - \prod_{m \geq 1} \frac{q^m / y (1 + q^m / y)}{(1 - q^m / y)^3};
\]
\[
g_2(\tau) = \frac{1}{12} \left[ 1 + 240 \prod_{m \geq 1} \frac{m^3 q^m}{1-q^m} \right],
\]
where \( y = e^{2\pi iz} \) and \( q = e^{2\pi i \tau} \). Then \( a_1 = -\frac{k}{2\pi i}, \ a_2 = X, \ a_3 = Y, \ a_4 = \frac{1}{2} g_2 \).

**Remark 4.2.2.** We shall see below that \( \tilde{\text{Ell}}_{\mathbb{Q}} = \mathbb{Q}[a_1, a_2, a_3, a_4] \). However, as noted by Totaro [84, §6], \( \tilde{\text{Ell}} \) itself is not even finitely generated over \( \mathbb{Z} \).

Let \( R = \mathbb{Z}[a_1, a_2, a_3, \frac{1}{2} a_4] \). We let \( E_R \to \text{Spec} \, R \) be the elliptic curve over \( R \) defined as the base change from the Weierstrass curve on \( A = \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6] \) via the map of rings
\[ \varphi : A \rightarrow R \]

\[
\begin{align*}
\mu_1 & \mapsto 2a_1 \\
\mu_2 & \mapsto 3a_2 - a_1^2 \\
\mu_3 & \mapsto -a_3 \\
\mu_4 & \mapsto -\frac{1}{2}a_4 + 3a_2^2 - a_1a_3 \\
\mu_6 & \mapsto 0.
\end{align*}
\]

The chosen parameter \( t \) on the Weierstrass curve gives by base-change a parameter \( t_R \) along the zero section of \( E_R \). This gives us a formal group law over \( R \), this just being the one induced from the TMF formal group law through change of coefficient ring via the map \( \varphi \). It is shown in [9, Lemma 44] that this formal group law is isomorphic to the Krichever’s elliptic formal group law (after extended the coefficient ring from \( \Ell \) to \( R \)). This isomorphism is explicitly given in [9].

Let \( \Ell = \mathbb{Z}[a_1, a_2, a_3, \frac{1}{2}a_4][\Delta^{-1}] \), where \( \Delta \) is the discriminant.

**Theorem 4.2.3 (Theorem ??).** The Krichever’s elliptic formal group law \( (\Ell[1/2], F_{Kr}) \) is Landweber exact. Therefore, the oriented cohomology theory \( \Ell[1/2]^* := \text{MGL}^* \otimes_{\text{Laz}} \Ell[1/2]^* \) on \( \text{Sm}_k \) is represented by a motivic oriented cohomology theory \( \text{Ell}[1/2] \) on \( \text{Sm}_k \).

This theorem can be proved using the same idea as the proof of the Landweber exactness of the TMF formal group law. Nevertheless, for sake of completeness and the reader’s convenience, we include a sketch of the proof here.

**Proof.** According to Theorem 4.1.9, it suffices to show that for all primes \( l > 2 \) and integers \( n \geq 0 \), the multiplication map

\[
v_n : \Ell[1/2]/(v_0, \ldots, v_{n-1}) \rightarrow \Ell[1/2]/(v_0, \ldots, v_{n-1})
\]

is injective.
Note that \( v_0 = l \) and \( \text{Ell}[1/2] \) is an integral domain, hence, multiplication by \( v_0 \) is always injective.

Consider the ring

\[
R_l := \mathbb{F}_l[a_1, \ldots, a_4] = R/(l)
\]

and the family of elliptic curves \( E_l := E_R \otimes_R R_l \) over \( R_l \). The injectivity of \( v_i \) for \( i > 0 \) is related to the height of the formal group law of these curves. Note that the only possible height of these curves are 1, 2, or infinity. We need first to remove the curves with infinite height formal group laws. This can be done by inverting the discriminant \( \Delta \). If we fix a geometric point \( x \in \text{Spec} \ R_l[\Delta^{-1}] \), the fiber is a supersingular elliptic curve if and only if the corresponding formal group law has height 1, i.e., \( v_1 \) vanishes when restricted to the residue field of \( x \). As \( R_l[\Delta^{-1}] \) is an integral domain, \( v_1 \) is injective if and only if it is not zero. This, in turn, is equivalent to the condition that on \( R_l[\Delta^{-1}] \) there is at least one geometric fiber which is not supersingular. Note that the residue field of \( x \) has characteristic \( l > 2 \). The family \( E_l \to \text{Spec} \ R_l \) contains the Legendre family, which is dominant over the moduli space when passing to the separable closure. Therefore, the family \( E_l \to \text{Spec} \ R_l \) is non-constant, hence contains a non-supersingular member.

Finally, we claim that \( v_2 \) in \( R_l[\Delta^{-1}]/(v_1) \) is a unit. This claim implies that multiplication by \( v_2 \) is injective and that \( R_l[\Delta^{-1}]/(v_1, v_2) = 0 \), which implies the required injectivity for \( n \geq 3 \). To verify the claim, assume otherwise, then \( v_2 \) is contained in a maximal ideal \( m \subset R_l[\Delta^{-1}]/(v_1) \). Therefore, the fiber of the family of curves over this closed point has associated formal group law with height greater than 2. This contradicts with the fact that the height of the formal group law of elliptic curves over a field of characteristic \( l > 0 \) can only be 1 or 2.

\[\square\]

**Remark 4.2.4.** The Landweber exactness condition for the prime \( l = 2 \) fails, since \( v_1 \) vanishes in \( \text{Ell}/(2) \). To see this, one can easily calculate the \( j \)-invariant of the family of elliptic curves \( E_{\text{Ell}} \otimes_{\text{Ell}} \text{Ell}/(2) \) over \( \text{Ell}/(2) \) to see that the \( j \)-invariant is a constant. Passing
to the separable closure, one finds that the curve $y^2 + y = x^3$, which is supersingular, is a member in the family, and hence every member in this family is supersingular.

**Remark 4.2.5.** Inverting $\Delta$ is an overkill, as the locus $\Delta = 0$ contains not only the curves with infinite height formal group law, but also certain curves whose formal group law has height 1. This can be fixed using the modern approach [42]. Taking the $j$-invariant for the family $E_R \otimes_R R[1/2]$ we obtain a morphism of $\text{Spec } \mathbb{Z}[1/2][a_1,a_2,a_3,a_4]$ to the coarse moduli space of elliptic curves and thereby obtain a stack with coarse moduli space $\text{Spec } \mathbb{Z}[1/2][a_1,a_2,a_3,a_4]$. Let $Kr$ be the open substack obtained by removing the locus consisting of curves with infinite height formal group laws. This stack $Kr$ is flat over the stack of formal group laws, by the same argument as in the proof of Theorem 4.2.3. Therefore, any flat morphism of stacks from a commutative ring $R$ to $Kr$ defines a motivic oriented Krichever elliptic cohomology theory.

### 4.3 Flops in the cobordism ring

#### 4.3.1 The double point formula

Following [55], we define the ring $\omega^*(X)$ of double-point cobordism to be the graded abelian group $M^*_+(X)$ of smooth projective schemes $Y \to X$ as in Subsection 4.1.1, modulo the relations generated by the following double point relation.

Let $Y \in \text{Sm}_k$ be of pure dimension. A morphism $\pi : Y \to \mathbb{P}^1$ is a double point degeneration over $0 \in \mathbb{P}^1$, if $\pi^{-1}(0) = A \cup B$ with $A$ and $B$ smooth, of codimension one, intersecting transversely along $A \cap B = D$.

We denote the normal bundles of $D$ in $A$ and $B$ by $N_DA$ and $N_DB$ respectively. Then, the two projective bundles

$$\mathbb{P}(\mathcal{O}_D \oplus N_DA) \to D \text{ and } \mathbb{P}(\mathcal{O}_D \oplus N_DB) \to D$$

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are isomorphic and will be denoted by $\mathbb{P}(\pi) \to D$. Let $g : Y \to X \times \mathbb{P}^1$ be a projective morphism for which $\pi = p_2 \circ g : Y \to \mathbb{P}^1$ is a double point degeneration over $0 \in \mathbb{P}^1$. The double point relation associated to $g$ is

$$[Y_\zeta \to X] - [A \to X] - [B \to X] + [\mathbb{P}(\pi) \to X],$$

where $Y_\zeta$ is the fiber over an arbitrary regular value $\zeta \in \mathbb{P}^1$.

It is shown in [55] that if $k$ has characteristic zero, then $\omega^*(X)$ is isomorphic to the cobordism ring $\Omega^*(X)$. We will use a weaker version of this theorem, which holds in a characteristic free fashion.

**Proposition 4.3.1** (Levine and Pandharipande). *For any field $k$ and scheme $X \in \text{Sm}_k$ of finite type over $k$, the natural projection $\Pi : M^*_+(X) \to \Omega^*(X)$ factors through $\omega^*(X)$.***

The proof in [55] uses only the existence of smooth pull-back, projective push-forward, the first Chern class of a line bundle, and external product, which means it does not depend on any assumption on $k$. Thus, the double point relation also holds when the field $k$ has positive characteristic.

We note that $\omega^*$ has the following structures:

1. pullback maps $f^* : \omega^*(X) \to \omega^*(Y)$ for each smooth morphism $f : Y \to X$ in $\text{Sm}_k$.

2. push-forward maps $f_* : \omega^*(Y) \to \omega^{*-d}(X)$ for each projective morphism $f : Y \to X$ of relative dimension $d$ in $\text{Sm}_k$.

3. associative, commutative external products, and an identity element $1 \in \omega^0(k)$.

The pullback and pushforward maps are functorial, and are compatible with the external products.

Composing the map $\omega^* \to \Omega^*$ with the natural transformation $\Theta_{\text{MGL}}$ gives us the transformation

$$\theta_{\text{MGL}} : \omega^* \to \text{MGL}^*,$$
natural with respect to smooth pullback, projective push-forward, external products and unit.

Let \( F \subseteq X \) be a smooth closed subscheme of some \( X \in \text{Sm}_k \). The double point relation yields the following blow-up formula in \( \omega^*(X) \), and hence in \( \Omega^*(X) \) and \( \text{MGL}^*(X) \):

\[
1_X = [Bl_F X \to X] + [\mathbb{P}(N_F X \oplus \mathcal{O}) \to X] - [\mathbb{P}(\mathcal{O}_{N_F X}(1) \oplus \mathcal{O}) \to X] \tag{4.3.1}
\]

where \( Bl_F X \) is blow-up of \( X \) along \( F \), and \( N_F X \) is the normal bundle of \( F \). This is proved by the usual method of deformation to the normal cone. In case \( X \) is projective over \( k \), pushing forward to \( \text{Spec} \, k \) gives the relation in \( \omega^*(k) \), \( \Omega^*(k) \) and \( \text{MGL}^*(k) \)

\[
[X] = [Bl_F X] + [\mathbb{P}(N_F X \oplus \mathcal{O})] - [\mathbb{P}(\mathcal{O}_{N_F X}(1) \oplus \mathcal{O})] \tag{4.3.2}
\]

Let \( Q^3 \subset \mathbb{P}^4 \) denote the 3-dimensional quadric with an ordinary double point \( v \), defined by the equation \( x_1 x_2 = x_3 x_4 \). We say two smooth projective \( n \)-folds \( X_1 \) and \( X_2 \) are related by a flop if we have the following diagram of projective birational morphisms:

\[
\begin{array}{ccc}
\tilde{X} & \xleftarrow{p_1} & X_1 \\
\downarrow & & \downarrow \scriptstyle{p_2} \\
X_2 & \xrightarrow{Y} & Y
\end{array}
\]

(4.3.3)

Here \( Y \) is a singular projective \( n \)-fold with singular locus \( Z \) such that along \( Z \), the pair \( (Y, Z) \) is étale locally isomorphic to an étale neighbourhood of \( (v, 0) \) in the pair \((Q^3 \times \mathbb{A}^{n-3}, v \times \mathbb{A}^{n-3})\). In particular, \( Z \) is smooth of dimension \( n - 3 \). We assume in addition that there exist rank 2 vector bundles \( A \) and \( B \) on \( Z \), such that the exceptional locus \( F_1 \) in \( X_1 \) is the \( \mathbb{P}^1 \)-bundle \( \mathbb{P}(A) \) over \( Z \), with normal bundle \( N_{F_1} X_1 = B \otimes \mathcal{O}(-1) \). Similarly, the the exceptional locus \( F_2 \) in \( X_2 \) is \( \mathbb{P}(B) \), with normal bundle \( N_{F_2} X_2 = A \otimes \mathcal{O}(-1) \). We say that \( X_1 \) and \( X_2 \) are related by a classical flop if in addition along \( Z \), \( (Y, Z) \) is Zariski locally isomorphic to \((Q^3 \times Z, v \times Z)\).

We assume now \( X_1 \) and \( X_2 \) are related by a flop. Let \( X = X_1 \) and \( F = F_1 \), the terms on
the right hand side of formula (4.3.2) become:

$$Bl_F X = \tilde{X},$$

$$\mathbb{P}(N_F X \oplus \mathcal{O}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}(A)}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \oplus \mathcal{O}),$$

$$\mathbb{P}^b_{N_F X}(1) \oplus \mathcal{O}) = \mathbb{P}^b_{\mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1))}(1) \oplus \mathcal{O}),$$

where $\mathbb{P}(\mathcal{O}_{\mathbb{P}(A)}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \oplus \mathcal{O})$ is a projective bundle over $\mathbb{P}(A)$, which in turn is a projective bundle over $Z$; and $\mathbb{P}^b_{\mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1))}(1) \oplus \mathcal{O})$ is a projective bundle over $\mathbb{P}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1))$, which is a projective bundle over $\mathbb{P}(A)$.

Thus, we get the following immediate lemma.

**Lemma 4.3.2.** In the cobordism ring $\text{MGL}^*(k)$, we have

$$X_1 - X_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}(A)}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \oplus \mathcal{O}) - \mathbb{P}^b_{\mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1))}(1) \oplus \mathcal{O})$$

$$- \mathbb{P}(\mathcal{O}_{\mathbb{P}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1))}(1) \oplus \mathcal{O}).$$

In particular, $X_1 - X_2$ in $\text{MGL}^*(k)$ comes from an element in $\text{MGL}^*(Z)$.

The second claim follows from the observation that each of the projective bundles are smooth varieties over $Z$. We will abuse notation by denoting a lifting of $X_1 - X_2$ to $\text{MGL}^*(Z)$ by $X_1 - X_2$ itself.

### 4.3.2 Flops in the cobordism ring

Since each term in the formula in Lemma 4.3.2 is a iterated projective bundle over $Z$, we will apply Quillen’s formula iteratively to each term, to calculate the fundamental class of the iterated projective bundles in $\text{MGL}^*(Z)$.

**Proposition 4.3.3.** Let $a_1, a_2$ be the Chern roots of the bundle $A$ over $Z$ and let $b_1, b_2$ be the Chern roots of the bundle $B$ over $Z$, all Chern roots to be taken in $\text{MGL}^*$. Then in
MGL*(Z), we have
\[
X_1 - X_2 = \frac{1}{(a_1 + \Omega b_1)(b_2 + \Omega a_1)(a_2 - \Omega a_1)} + \frac{1}{(a_2 + \Omega b_1)(b_2 + \Omega a_2)(a_1 - \Omega a_2)}
\]
\[
- \frac{1}{(b_1 + \Omega a_1)(a_2 + \Omega b_1)(b_2 - \Omega b_1)} - \frac{1}{(b_2 + \Omega a_1)(a_2 + \Omega b_2)(b_1 - \Omega b_2)}.
\]

The rest of this subsection is devoted to prove this proposition.

**The term** \(\mathbb{P}(\mathcal{O}_{\mathbb{P}(B\otimes\mathcal{O}(-1))}(1) \oplus \mathcal{O})\)

We first prove the following

**Lemma 4.3.4.** In MGL*(Z), we have \(\mathbb{P}(\mathcal{O}_{\mathbb{P}(B\otimes\mathcal{O}(-1))}(1) \oplus \mathcal{O}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}(A\otimes\mathcal{O}(-1))}(1) \oplus \mathcal{O}).\)

Let \(\pi_1 : \mathbb{P}(\mathcal{O}_{\mathbb{P}(B\otimes\mathcal{O}(-1))}(1) \oplus \mathcal{O}) \to \mathbb{P}(B \otimes \mathcal{O}(-1))\) be the natural projection. Let the first Chern class of the bundle \(\mathcal{O}_{\mathbb{P}(B\otimes\mathcal{O}(-1))}(1)\) be \(u_B\). Then the two Chern roots of \(\mathcal{O}_{\mathbb{P}(B\otimes\mathcal{O}(-1))}(1) \oplus \mathcal{O}\) are \(u_B\) and 0. Applying Quillen’s formula (4.1.2) with \(f_1(t) \equiv 1\) being the fundamental class, we have

\[
\pi_1^*([\mathbb{P}(\mathcal{O}_{\mathbb{P}(B\otimes\mathcal{O}(-1))}(1) \oplus \mathcal{O})]) = \frac{1}{0 - \Omega u_B} + \frac{1}{u_B - \Omega 0} = \frac{1}{-\Omega u_B} + \frac{1}{u_B}.
\]

Next, let \(\pi_2 : \mathbb{P}(A)(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1)) \to \mathbb{P}(A)\) be the projection. The two Chern roots of the bundle \(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1)\) are \(b_1 - \Omega v_A\) and \(b_2 - \Omega v_A\), where \(v_A := c_1(\mathcal{O}_{\mathbb{P}(A)}(1))\), and \(b_1, b_2\) are two Chern roots of bundle \(B\). Applying Quillen’s formula (4.1.2) with \(f_2(t) = \frac{1}{-\Omega t} + \frac{1}{t}\), we get

\[
\pi_2^*(f_2) = \frac{1}{-\Omega(b_1 - \Omega v_A)} + \frac{1}{-\Omega(b_2 - \Omega v_A)} + \frac{1}{-\Omega(b_2 - \Omega v_A)} - \frac{1}{b_1 - \Omega v_A} - \frac{1}{b_2 - \Omega v_A} - \frac{1}{b_2 - \Omega v_A} = \frac{1}{b_1 - \Omega v_A} + \frac{1}{b_2 - \Omega v_A} + \frac{1}{-\Omega b_2 + \Omega v_A}.
\]

Finally, let \(\pi_3 : \mathbb{P}(A) \to Z\) be the projection. Recall the two Chern roots of bundle \(A\) are denoted by \(a_1, a_2\). Let \(f_3(t) := \frac{1}{b_1 - \Omega b_1} + \frac{1}{b_2 - \Omega b_2} + \frac{1}{b_2 - \Omega b_1} + \frac{1}{b_1 - \Omega b_2}\), then Quillen’s formula (4.1.2)
yields
\[
\pi_3(f_3) = \frac{1}{b_1 + \Omega a_1} + \frac{1}{-\Omega b_1 - \Omega a_1} + \frac{1}{b_2 + \Omega a_1} + \frac{1}{-\Omega b_2 - \Omega a_1} + \frac{1}{b_1 + \Omega a_2} + \frac{1}{-\Omega b_1 - \Omega a_2} + \frac{1}{b_2 + \Omega a_2} + \frac{1}{-\Omega b_2 - \Omega a_2} + \frac{1}{(a_1 - \Omega a_2)(b_2 - \Omega b_1)} + \frac{1}{(a_1 - \Omega a_2)(b_1 - \Omega b_2)},
\]
which is \(\pi_*(\mathbb{P}(\mathcal{O}(\mathbb{P}(B \otimes \mathcal{O}(-1))(1) \oplus \mathcal{O}))).\)

Similarly, for the projective bundle \(\pi' : \mathbb{P}(\mathcal{O}(\mathbb{P}(A \otimes \mathcal{O}(-1))(1) \oplus \mathcal{O}) \to Z\), we have:
\[
\pi'_*([\mathbb{P}(\mathcal{O}(\mathbb{P}(A \otimes \mathcal{O}(-1))(1) \oplus \mathcal{O})]) = \frac{1}{a_1 + \Omega b_1} + \frac{1}{-\Omega a_1 - \Omega b_1} (b_2 - \Omega b_1)(a_2 - \Omega a_1) + \frac{1}{a_2 + \Omega b_1} + \frac{1}{-\Omega a_2 - \Omega b_1} (b_2 - \Omega b_1)(a_1 - \Omega a_2)
\]
\[
+ \frac{1}{a_1 + \Omega b_2} + \frac{1}{-\Omega a_1 - \Omega b_2} (b_1 - \Omega b_2)(a_2 - \Omega a_1) + \frac{1}{a_2 + \Omega b_2} + \frac{1}{-\Omega a_2 - \Omega b_2} (b_1 - \Omega b_2)(a_1 - \Omega a_2).
\]
Now comparing the two formulas of \(\pi_*(\mathbb{P}(\mathcal{O}(\mathbb{P}(B \otimes \mathcal{O}(-1))(1) \oplus \mathcal{O})))\) and \(\pi'_*([\mathbb{P}(\mathcal{O}(\mathbb{P}(A \otimes \mathcal{O}(-1))(1) \oplus \mathcal{O})))\), the lemma follows.

**The term** \(\mathbb{P}(A)(B \otimes \mathcal{O}(A))(-1) \oplus \mathcal{O}\)

Let \(\pi_1 : \mathbb{P}(A)(B \otimes \mathcal{O}(A))(-1) \oplus \mathcal{O} \to \mathbb{P}(A)\) be the natural projection. The three Chern roots of the bundle \(B \otimes \mathcal{O}(A)(-1) \oplus \mathcal{O}\) are \(b_1 - \Omega v_A, b_2 - \Omega v_A,\) and \(0,\) where \(v_A := c_1(\mathcal{O}(A)(1))\), and \(b_1, b_2\) are two Chern roots of the bundle \(B\). Applying Quillen's formula (4.1.2) with \(f_1(t) \equiv 1\), we get
\[
\pi_{1*}(\mathbb{P}(B \otimes \mathcal{O}(A)(-1) \oplus \mathcal{O})) = \frac{1}{(b_2 - \Omega b_1)(v_A - \Omega b_1)} + \frac{1}{(b_1 - \Omega b_2)(v_A - \Omega b_2)} + \frac{1}{(b_1 - \Omega v_A)(b_2 - \Omega v_A)}.
\]

Now let \(\pi_2 : \mathbb{P}(A) \to Z\) be the natural projection. Recall the two Chern roots of bundle \(A\) are denoted by \(a_1, a_2\). Let \(f_2(t) := \frac{1}{(b_2 - \Omega b_1)(t - \Omega b_1)} + \frac{1}{(b_1 - \Omega b_2)(t - \Omega b_2)} + \frac{1}{(b_1 - \Omega t)(b_2 - \Omega t)};\) then
Quillen’s formula (4.1.2) gives

\[
\pi_2^*(f_2) = \frac{1}{(b_2 - b_1)(a_2 - a_1)(-\Omega a_1 - b_2)} + \frac{1}{(b_1 - b_2)(a_2 - a_1)(-\Omega a_1 - b_2)}
\]

\[
+ \frac{1}{(a_1 + b_1)(b_2 + a_1)(a_2 - a_1)} + \frac{1}{(b_1 - b_2)(a_1 - a_2)(-\Omega a_2 - b_1)}
\]

\[
+ \frac{1}{(b_1 - a_2)(a_1 - a_2)(-\Omega a_2 - b_2)} + \frac{1}{(a_2 + b_1)(b_2 + a_2)(a_1 - a_2)}.
\]

Similarly, for the bundle \(\pi' : \mathbb{P}(B)\langle A \otimes \mathcal{O}_B(-1) \oplus \mathcal{O}\rangle \rightarrow \mathbb{Z}\), we have:

\[
\pi'_*([\mathbb{P}(B)\langle A \otimes \mathcal{O}_B(-1) \oplus \mathcal{O}\rangle])
\]

\[
= \frac{1}{(a_2 - a_1)(b_2 - b_1)(-\Omega b_1 - a_1)} + \frac{1}{(a_1 - a_2)(b_2 - b_1)(-\Omega b_1 - a_2)}
\]

\[
+ \frac{1}{(b_1 + a_1)(a_2 + b_1)(b_2 - b_1)} + \frac{1}{(a_2 - a_1)(b_1 - a_2)(-\Omega a_2 - b_1)}
\]

\[
+ \frac{1}{(a_1 - a_2)(b_1 - b_2)(-\Omega b_2 - a_2)} + \frac{1}{(b_2 + a_1)(a_2 + b_2)(b_1 - b_2)}.
\]

Therefore,

\[
\pi_*(\mathbb{P}(A)\langle B \otimes \mathcal{O}_A(-1) \oplus \mathcal{O}\rangle) - \pi'_*(\mathbb{P}(B)\langle A \otimes \mathcal{O}_B(-1) \oplus \mathcal{O}\rangle)
\]

\[
= \frac{1}{(a_1 + b_1)(b_2 + a_1)(a_2 - a_1)} + \frac{1}{(a_2 + b_1)(b_2 + a_2)(a_1 - a_2)}
\]

\[
- \frac{1}{(b_1 + a_1)(a_2 + b_1)(b_2 - b_1)} - \frac{1}{(b_2 + a_1)(a_2 + b_2)(b_1 - b_2)}.
\]

This finishes the proof of the proposition.

### 4.3.3 Flops in the elliptic cohomology ring

In this subsection, we prove the following Proposition.

**Proposition 4.3.5.** Suppose \(X_1\) and \(X_2\) are smooth projective varieties related by a flop.

Notations as above, in the ring \(\text{Ell}_Q(Z)\), we have

\[
\mathbb{P}(A)\langle B \otimes \mathcal{O}_A(-1) \oplus \mathcal{O}\rangle = \mathbb{P}(B)\langle A \otimes \mathcal{O}_B(-1) \oplus \mathcal{O}\rangle.
\]
In particular, we have $X_1 - X_2 = 0$ in $\text{Ell}_Q^*(Z)$, and hence $X_1 - X_2 = 0$ in $\tilde{\text{Ell}}^*(k)$.

**Remark 4.3.6.** Once we know that $X_1 - X_2 = 0$ in $\text{Ell}_Q^*(Z)$, it follows by pushing forward to $\text{Spec } k$ that $X_1 - X_2 = 0$ in $\text{Ell}_Q^*(k)$. But $X_1 - X_2$ is a well-defined element in $\tilde{\text{Ell}}^*(k)$ and $\tilde{\text{Ell}}^*(k) \to \text{Ell}_Q^*(k)$ is injective, since $\tilde{\text{Ell}}^*(k)$ is by construction $\mathbb{Z}$-torsion free, hence $X_1 - X_2 = 0$ in $\tilde{\text{Ell}}^*(k)$, as claimed above.

**Proof of the proposition.** Thanks to Proposition 4.3.3, we can reduce $X_1 - X_2$ to an explicit element in $\text{MGL}^*(Z)$. Applying the canonical map $\text{MGL}^*(Z) \to \text{Ell}^*(Z)$ to this element, we have, in the ring $\text{Ell}^*(Z)$,

$$X_1 - X_2 = \frac{1}{(a_1 + E b_1)(b_2 + E a_1)(a_2 - E a_2)} + \frac{1}{(a_2 + E b_1)(b_2 + E a_2)(a_1 - E a_2)}$$

$$- \frac{1}{(b_1 + E a_1)(a_2 + E b_1)(b_2 - E b_1)} - \frac{1}{(b_2 + E a_1)(a_2 + E b_2)(b_1 - E b_2)},$$

here the $a_i$'s and $b_i$'s are the Chern roots in the elliptic cohomology. We would like to show the above expression is 0 in $\text{Ell}_Q^*(Z)$.

Recall that $x + Ey = \lambda(\lambda^{-1}(x) + \lambda^{-1}(y))$ where the exponential $\lambda(t)$ is given by the power series $\frac{t}{Q(t)}$.

Let $\lambda^{-1}(a_i) = A_i$, and $\lambda^{-1}(b_i) = B_i$ for $i = 1, 2$. Then in $\text{Ell}_Q^*(Z)$, $X_1 - X_2$ becomes:

$$\frac{1}{\lambda}(A_1 + B_1)\frac{1}{\lambda}(B_2 + A_1)\frac{1}{\lambda}(A_2 - A_1) + \frac{1}{\lambda}(B_1 + A_2)\frac{1}{\lambda}(B_2 + A_2)\frac{1}{\lambda}(A_1 - A_2)$$

$$- \frac{1}{\lambda}(A_1 + B_1)\frac{1}{\lambda}(A_2 + B_1)\frac{1}{\lambda}(B_2 - B_1) - \frac{1}{\lambda}(A_1 + B_2)\frac{1}{\lambda}(B_2 + A_2)\frac{1}{\lambda}(B_1 - B_2).$$

Plugging-in

$$\frac{1}{\lambda}(t) = \frac{Q(t)}{t} = \frac{1}{2\pi i} e^{kt} e^{\zeta(z) \frac{t}{2\pi i}} \frac{\sigma(\frac{t}{2\pi i} - z, \tau)}{\sigma(\frac{t}{2\pi i}, \tau) \sigma(-z, \tau)},$$
and cancelling some obvious factors, $X_1 - X_2$ becomes

$$\frac{\sigma(A_1 + B_1 - z) \sigma(B_2 + A_1 - z) \sigma(A_2 - A_1 - z)}{\sigma(A_1 + B_1) \sigma(B_2 + A_1) \sigma(A_2 - A_1)} + \frac{\sigma(B_1 + A_2 - z) \sigma(B_2 + A_2 - z) \sigma(A_1 - A_2 - z)}{\sigma(B_1 + A_2) \sigma(B_2 + A_2) \sigma(A_1 - A_2)}$$

$$- \frac{\sigma(A_1 + B_1 - z) \sigma(A_2 + B_1 - z) \sigma(B_2 - B_1 - z)}{\sigma(A_1 + B_1) \sigma(A_2 + B_1) \sigma(B_2 - B_1)} - \frac{\sigma(A_1 + B_2 - z) \sigma(B_2 + A_2 - z) \sigma(B_1 - B_2 - z)}{\sigma(A_1 + B_2) \sigma(B_2 + A_2) \sigma(B_1 - B_2)}.$$

Now let

$$x_1 = -A_1, x_2 = B_2, x_3 = B_1, x_4 = -A_2.$$

and

$$y_1 = B_2 - z, y_2 = B_1 - z, y_3 = -A_2 + z, y_4 = -A_1 + z.$$

Using the fact that $\sigma(z)$ is an odd function, we get

$$\frac{\sigma(A_1 + B_1 - z) \sigma(B_2 + A_1 - z) \sigma(A_2 - A_1 - z) \sigma(-z)}{\sigma(A_1 + B_1) \sigma(B_2 + A_1) \sigma(A_2 - A_1)} = \frac{\sigma(x_1 - y_1) \sigma(x_1 - y_2) \sigma(x_1 - y_3) \sigma(x_1 - y_4)}{\sigma(x_1 - x_2) \sigma(x_1 - x_3) \sigma(x_1 - x_4)};$$

$$- \frac{\sigma(A_1 + B_2 - z) \sigma(B_2 + A_2 - z) \sigma(B_1 - B_2 - z) \sigma(-z)}{\sigma(A_1 + B_2) \sigma(B_2 + A_2) \sigma(B_1 - B_2)} = \frac{\sigma(x_2 - y_1) \sigma(x_2 - y_2) \sigma(x_2 - y_3) \sigma(x_2 - y_4)}{\sigma(x_2 - x_1) \sigma(x_2 - x_3) \sigma(x_2 - x_4)};$$

$$- \frac{\sigma(A_1 + B_1 - z) \sigma(A_2 + B_1 - z) \sigma(B_2 - B_1 - z) \sigma(-z)}{\sigma(A_1 + B_1) \sigma(A_2 + B_1) \sigma(B_2 - B_1)} = \frac{\sigma(x_3 - y_1) \sigma(x_3 - y_2) \sigma(x_3 - y_3) \sigma(x_3 - y_4)}{\sigma(x_3 - x_1) \sigma(x_3 - x_2) \sigma(x_3 - x_4)};$$

$$\frac{\sigma(B_1 + A_2 - z) \sigma(B_2 + A_2 - z) \sigma(A_1 - A_2 - z) \sigma(-z)}{\sigma(B_1 + A_2) \sigma(B_2 + A_2) \sigma(A_1 - A_2)} = \frac{\sigma(x_4 - y_1) \sigma(x_4 - y_2) \sigma(x_4 - y_3) \sigma(x_4 - y_4)}{\sigma(x_4 - x_1) \sigma(x_4 - x_2) \sigma(x_4 - x_3)}.$$

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The Proposition follows from the following classical identity of sigma-function with \( n = 4 \). (See § 20.53, Example 3 of [100].) Assuming \( \sum_{r=1}^{n} x_r = \sum_{r=1}^{n} y_r \), we have

\[
\sum_{r=1}^{n} \frac{\sigma(x_r - y_1)\sigma(x_r - y_2)\cdots \sigma(x_r - y_n)}{\sigma(x_r - x_1)\sigma(x_r - x_2)\cdots \sigma(x_r - x_n)} = 0
\]

with the \( * \) denoting that the vanishing factor \( \sigma(x_r - x_r) \) is to be omitted.

\[\square\]

4.4 The algebraic elliptic cohomology ring with rational coefficients

Let \( \mathcal{I}_{\text{ff}} \) be the ideal in \( \text{MGL}^*_Q(k) \) generated by differences \( X_1 - X_2 \), where \( X_1 \) and \( X_2 \) are related by a flop and \( \mathcal{I}_{\text{cl ff}} \subset \mathcal{I}_{\text{ff}} \) the ideal in \( \text{MGL}^*_Q(k) \) generated by differences \( X_1 - X_2 \), where \( X_1 \) and \( X_2 \) are related by a classical flop. Section 4.3.3 shows that the elliptic genus \( \phi : \text{MGL}^*(k) \to \text{Ell}^*(k) \) factors through the quotient \( \text{MGL}^*(k)/\mathcal{I}_{\text{ff}} \).

Proposition 4.4.1. The ideal \( \mathcal{I}_{\text{cl ff}} \) in \( \text{MGL}^*_Q(k) \) contains a system of polynomial generators \( x_n \) of \( \text{MGL}^*_Q(k) \) in degree \( n \leq -5 \).

This proposition follows from same the calculation as in [84]. Nevertheless, for the convenience of the readers and to be as explicit as possible, we include the calculation in the Appendix.

Following Höhn ([43]), we define the four elements \( W_i, i = 1, 2, 3, 4 \) in \( \text{MGL}^*(k) \) via their Chern numbers as follows,

\[
\begin{align*}
c_1[W_1] &= 2; \\
c_1^2[W_2] &= 0, c_2[W_2] = 24; \\
c_1^3[W_3] &= 0, c_1c_2[W_3] = 0, c_3[W_3] = 2; \\
\end{align*}
\]
In fact, solving the systems of linear equations, one can write down \( W_i \) explicitly as rational linear combinations of products of projective spaces,

\[
\begin{align*}
W_1 & := [\mathbb{P}^1]; \\
W_2 & := -16[\mathbb{P}^2] + 18[\mathbb{P}^1 \times \mathbb{P}^1]; \\
W_3 & := \frac{3}{2}[\mathbb{P}^3] - 4[\mathbb{P}^2 \times \mathbb{P}^1] + \frac{5}{2}([\mathbb{P}^1]^3); \\
W_4 & := -4[\mathbb{P}^4] + 12.5[\mathbb{P}^3 \times \mathbb{P}^1] + 6[\mathbb{P}^2 \times \mathbb{P}^2] - 26[\mathbb{P}^2 \times ([\mathbb{P}^1]^2)] + 11.5([\mathbb{P}^1]^4).
\end{align*}
\]

Write the Hirzebruch characteristic power series \( Q(t) = 1 + f_1 t + f_2 t^2 + \cdots \). Let \( A, B, C, \) and \( D \) be such that

\[
\begin{align*}
f_1 &= \frac{1}{2} A; \\
f_2 &= \frac{1}{2^4 \cdot 3} (6A^2 - B); \\
f_3 &= \frac{1}{2^5 \cdot 3} (2A^3 - AB + 16C); \\
f_4 &= \frac{1}{2^9 \cdot 3^2 \cdot 5} (60A^4 - 60A^2B + 1920AC + 7B^2 - 1152D).
\end{align*}
\]

Then, a calculation shows (see, e.g., § 2.2 of [43]), the lower degree parts of the elliptic genus \( \phi \) are as follows

\[
\begin{align*}
K_1 &= \frac{1}{2} Ac_1; \\
K_2 &= \frac{1}{2^4 \cdot 3} ((6A^2 - B)c_1^2 + 2Bc_2); \\
K_3 &= \frac{1}{2^5 \cdot 3} ((2A^3 - AB + 16C)c_1^3 + (2AB - 48C)c_2c_1 + 48Cc_3); \\
K_4 &= \frac{1}{2^9 \cdot 3^2 \cdot 5} ((60A^4 - 60A^2B + 1920AC + 7B^2 - 1152D)c_1^4 \\
&\quad+ (24B^2 - 2304D)c_2^2 + (120A^2B - 5760AC - 28B^2 + 4608D)c_1^2c_2 \\
&\quad+ (5760AC + 8B^2 - 4608D)c_3c_1 + (-8B^2 + 4608D)c_4),
\end{align*}
\]

where \( K_i \) is the homogeneous degree \( i \) part of the elliptic genus \( \phi \). Therefore, \( \phi(W_1) = A, \phi(W_2) = B, \phi(W_3) = C \) and \( \phi(W_4) = D \).

We then compare the elements \( A, B, C, D \) with the polynomial generators of \( \text{Ell}_Q^*(k) \). Let \( g_i \) be the weight \( 2i \) Eisenstein series where \( i = 2, 3, \) and let \( X \) be the Weierstrass \( p \)-function.
and $Y$ its derivative. The same calculations as in §2 of [43] show $B = 24X$, $C = Y$, and $D = 6X^2 - g_2/2$. Recall that $a_2 = X$, $a_3 = Y$, $a_4 = \frac{1}{2}g_2$. Note that the image of $W_1 = \mathbb{P}^1$, which is $\frac{k}{\pi i}$, involves non-trivially the variable $k$, but the images of $W_i$ for $i = 2, 3, 4$ lies in the subring $\mathbb{Q}((e^{2\pi i z}))[e^{2i\pi \tau}]$. In particular, $A$, $B$, $C$, and $D$ are algebraically independent.

Summarizing all the above, we have proved the following proposition.

**Proposition 4.4.2.** The classifying map $\phi_E : \text{MGL}^*(k) \to \mathbb{Q}((e^{2\pi iz}))[e^{2i\pi \tau}, \frac{k}{2\pi i}]$ descends to define an isomorphism of $\text{MGL}^*_Q(k)/\mathcal{I}_{\text{clf}}$ with a polynomial subalgebra $\mathbb{Q}[a_1, a_2, a_3, a_4]$ of $\mathbb{Q}((e^{2\pi i z}))[e^{2i\pi \tau}, \frac{k}{2\pi i}]$, with $a_i$ in degree $-i$.

This implies the following corollary.

**Corollary 4.4.3.** The natural ring homomorphism $\text{MGL}^*_Q(k)/\mathcal{I}_{\text{clf}} \to \text{Ell}^*_Q(k)$ is injective and $\mathcal{I}_{\text{clf}} = \mathcal{I}_{\text{fl}}$.

**Proof.** The first assertion follows immediately from proposition 4.4.2; the second from the first, noting that $\text{MGL}^*_Q(k)/\mathcal{I}_{\text{clf}} \to \text{Ell}^*_Q(k)$ factors through the quotient $\text{MGL}^*_Q(k)/\mathcal{I}_{\text{fl}}$ by proposition 4.3.5. \qed

## 4.5 Birational symplectic varieties

In this section we work over a base field $k$ with characteristic zero.

### 4.5.1 Specialization in algebraic cobordism theory

We will use the specialization morphism in algebraic cobordism theory. The existence of a specialization morphism in algebraic cobordism theory is folklore; for lack of a reference, we sketch a construction here.

**Proposition 4.5.1.** Let $C$ be a smooth curve, and $p : \mathcal{X} \to C$ a smooth projective morphism. Let $o \in C$ be a closed point with fiber $\mathcal{X}_o$ and $\eta$ be the generic point of $C$ whose fiber is denoted...
by $X$. Let $i : X_0 \to X$ and $j : X_\eta \to X$ be the natural embeddings. Then there is a natural morphism $\sigma : \Omega^*(X_\eta) \to \Omega^*(X_0)$ such that $\sigma \circ j^* = i^*$ where $j^*$ is the pull-back and $i^*$ is the Gysin morphism.

**Proof.** Let $R$ be the local ring of $C$ at $o \in C$. Although in this case neither $X_R$ nor $X_\eta$ are $k$-schemes of finite type, they are both projective limits of such, which allows us to define $\Omega^*(X_R)$ and $\Omega^*(X_\eta)$ as

$$\Omega^*(X_R) := \lim_{0 \in U \subset C} \Omega^*(p^{-1}(U)); \quad \Omega^*(X_\eta) := \lim_{0 \neq U \subset C} \Omega^*(p^{-1}(U)).$$

Here $U \subset C$ is an open subscheme. We may then replace $C$ with Spec $R$, $X$ with $X_R$.

For any integer $n$, there is a localization short exact sequence

$$\Omega^{n-1}(X_0) \xrightarrow{i^*} \Omega^n(X) \xrightarrow{j^*} \Omega^n(X_\eta) \to 0.$$

Let $i^* : \Omega^n(X) \to \Omega^n(X_0)$ be the pull-back. In order to show it factors through $j^*$, it suffices to check that $i^* \circ i_* = 0$. This is true since $i^* \circ i_* \cong c_1 \mathcal{O}_{X_0}|_{X_0} = 0$ (see [54, Lemma 3.1.8]).

Indeed, as the respective restriction maps $j^*$ are surjective, the fact that $\sigma$ is a ring homomorphism and the compatibility (1) follows from the fact that pullback maps are functorial
ring homomorphisms. For (2), we note that the diagram

\[
\begin{array}{ccc}
Y_o & \xrightarrow{i_Y} & Y \\
\downarrow f_o & & \downarrow f \\
X_o & \xrightarrow{i_X} & X
\end{array}
\]

is cartesian, and then the compatibility (2) follows from the base-change identity \(i_X^* \circ f_* = f_o^* \circ i_Y^*\) and the surjectivity mentioned above.

### 4.5.2 Cobordism ring of birational symplectic varieties

Consider two birational irreducible symplectic varieties \(X_1\) and \(X_2\) satisfying the following condition: There exist smooth projective algebraic varieties \(X_1\) and \(X_2\), flat over a smooth quasi-projective curve \(C\) with a closed point \(o \in C\), such that:

1. the fiber of \(X_i\) over \(o\) is \((X_i)_o = X_i\);
2. there is an isomorphism \(\Psi : (X_1)_{C \setminus \{o\}} \to (X_2)_{C \setminus \{o\}}\) over \(C\).

The counterpart of the following Proposition in Chow theory is proved in [32]. Let \(\Omega^*\) be the algebraic cobordism with \(\mathbb{Z}\) coefficient. When the base field \(k\) has characteristic zero, then \(\Omega^* = \text{MGL}^*\).

**Proposition 4.5.2.** Let \(X_1\) and \(X_2\) be two birational symplectic varieties satisfying conditions (i) and (ii). The deformation data induce an isomorphism

\[\Omega^*(X_1) \cong \Omega^*(X_2)\]

It was proved in [44] that the above conditions (i) and (ii) hold, when:

- either \(X_1\) and \(X_2\) are connected by a general Mukai flop,
- or \(X_1\) and \(X_2\) are isomorphic in codimension two, that is, there exist isomorphic open subsets \(U_1 \subset X_1\) and \(U_2 \subset X_2\) with \(\text{codim}_{X_i}(X_i \setminus U_i) \geq 3\), for \(i = 1, 2\).
The proof follows the same idea as in [32], nevertheless, for the convenience of the readers, we include the proof.

Let $\sigma_i : \Omega^*(X_{i\eta}) \to \Omega^*(X_{i\iota})$ be the specialization map, where $i = 1, 2$. Let $\Delta \subset (X_{1\eta} \times X_{2\eta})$ be the diagonal, or the graph of the isomorphism $\Psi$ as in condition (ii) restricted to the generic fiber. We define an element

$$Z := (\sigma_1 \times \sigma_2)([\Delta]) \in \Omega^*(X_1 \times X_2).$$

The projection $X_1 \times X_2 \to X_i$ for $i = 1, 2$ is denoted by $p_i$. The element $Z \in \Omega^*(X_1 \times X_2)$ defines a map

$$[Z] : \Omega^*(X_1) \to \Omega^*(X_2)$$

by $\alpha \mapsto p_2^*(p_1^*(\alpha) \cap Z)$. By symmetry, we also have a map $[Z^{op}] : \Omega^*(X_2) \to \Omega^*(X_1)$, given by $\beta \mapsto p_1^*(p_2^*(\beta) \cap Z^{op})$, where $Z^{op} \in \Omega^*(X_2 \times X_1)$ is the image of $Z$ by the symmetry morphism $X_1 \times X_2 \to X_2 \times X_1$.

We summarize the notations in the following diagram:

![Diagram](4.5.1)

Now we check that $[Z^{op}] \circ [Z] = 1$, i.e., for any $\alpha \in \Omega^*(X_1)$ we have

$$p_{1*}\left(p_2^*(p_2^*(p_1^*(\alpha) \cap Z) \cap Z^{op})\right) = \alpha.$$
The diagonal in $X_i \times X_i$ will be denoted by $\Delta_{X_i}$. Similarly we have $\Delta_{X_{i\eta}} \subseteq X_{i\eta} \times X_{i\eta}$ as the diagonal. We have:

$$p_1\left(p_2^*(p_2^*(p_1^*\alpha \cap Z)) \cap Z^{op}\right) = p_1\left((p_{23}^*(p_{12}^*(p_1^*\alpha \cap Z)) \cap Z^{op}\right)$$

$$= (p_1p_{23})_*\left((p_{13}p_1^*)^*\alpha \cap p_{12}^*Z \cap p_{23}^*Z^{op}\right)$$

$$= (p_{23})_*\left(p_1^*\alpha \cap p_{12}^*Z \cap p_{23}^*Z^{op}\right)$$

$$= (p_{23})_*\left(p_1^*\alpha \cap \Delta_{X_1}\right)$$

$$= \alpha.$$

The only less clear equality is the second last one. It follows from the following lemma.

**Lemma 4.5.3.** Notations as in the diagram, we have:

$$p_{13}^*(p_{12}^*Z \cap p_{23}^*Z^{op}) = \Delta_{X_1} \in \Omega(X_1 \times X_1).$$

**Proof.** Consider the same diagram as (4.5.1) with $X_i$ replaced by $X_{i\eta}$. We make the convention here in the proof that for any map $p$ in diagram (4.5.1), the corresponding map for generic fibers will be denoted by $\tilde{p}$. Note that

$$\tilde{p}_{13}^*(\tilde{p}_{12}^*\Delta \cap \tilde{p}_{23}^*\Delta^{op}) = \Delta_{X_{i\eta}} \in \Omega((X_1)_{\eta} \times (X_1)_{\eta}).$$

Applying the specialization map $\sigma$ on both sides, we get:

$$p_{13}^*(p_{12}^*Z \cap p_{23}^*Z^{op}) = \Delta_{X_1} \in \Omega(X_1 \times X_1).$$


Note that the following proposition has no direct counterpart in the Chow theory.

Let $\pi_i : \Omega^*(X_i) \to \Omega^*(k)$ be the structure map of $X_i \to k$, where $i = 1, 2$.

**Proposition 4.5.4.** Let $[Z] : \Omega^*(X_1) \to \Omega^*(X_2)$ be the map constructed as above. We have,

1. $[Z](1_{X_1}) = 1_{X_2}$,
2. \( \pi_1(1_X_1) = \pi_2(1_X_2) \).

\textit{Proof.} We have
\[
\bar{p}_{2*}(\bar{p}_1^*(1_{(X_1)\eta}) \cap \Delta) = 1_{(X_2)\eta} \in \Omega((X_2)_{\eta}).
\]
Applying the specialization map \( \sigma \) on both sides, we get:
\[
p_{2*}(p_1^*(1_X_1) \cap Z) = 1_{X_2} \in \Omega^*(X_2).
\]
This finishes the proof of (1).

Let \( \Psi : X_1 \eta \cong X_2 \eta \) be the isomorphism as in condition (ii), we have \( \Psi_*(1_{X_1}) = 1_{X_2} \), and \( \bar{\pi}_1(1_{X_1}) = \bar{\pi}_2(1_{X_2}) \). Applying the specialization map, (2) follows.

For a vector bundle \( E \) on some \( X \in \text{Sm}_k \), let \( c_{i_1, \ldots, i_r}(E) \) denote the product \( c_{i_1}(E) \cdots c_{i_r}(E) \) in \( \text{CH}^*(X) \). For \( X \) a smooth projective variety over \( k \), the Chern number \( c^I(X) \) associated to an index \( I = (i_1, \ldots, i_r) \) with \( \sum_j i_j = \text{dim}_k X \) and \( i_j > 0 \) is \( \deg_k (c^I(T_X)) \).

\textbf{Corollary 4.5.5.} Let \( X_1 \) and \( X_2 \) be two birational symplectic varieties satisfying conditions (i) and (ii). Then \( X_1 \) and \( X_2 \) have the same Chern numbers.

\textit{Proof.} For an integer \( d > 0 \), let \( P_d \) denote the number of partitions of \( d \). It is well-known that the function \( X \mapsto \prod_I c^I(X) \) on smooth projective irreducible \( k \)-schemes of dimension \( d \) over \( k \) descends to well-defined homomorphism \( c^{*,d} : \Omega^{-d}(k) \to \mathbb{Z}^{P_d} \) via the map sending \( X \) to its class \( [X] \in \Omega^{-d}(k) \). In fact, \( c^{*,d} \) is injective and has image a subgroup of \( \mathbb{Z}^{N_d} \) of finite index, but we will not use this. The result is now an immediate consequence of Proposition 4.5.4.

\textbf{4.6 Appendix: The ideal generated by differences of flops}

Let \( \mathcal{I}_{\text{diff}} \) be the ideal in \( \text{MGL}_{\mathbb{Q}}^*(k) \) generated by those \( [X_1] - [X_2] \) with \( X_1 \) and \( X_2 \) related by a classical flop.
Proposition 4.6.1. The ideal $I_{clfl}$ in $\text{MGL}^*_Q(k)$ contains a system of polynomial generators $x_n$ of $\text{MGL}^*_Q(k)$ in all degrees $n \leq -5$.

This proposition was originally proved in Section 5 of [84]. Their proof is based on explicit calculation that lends itself with slight adjustment to our setting. Note that this proof is independent of the fact that the Krichever’s elliptic formal group law is defined over $\mathbb{Z}[a_1, a_2, a_3, a_4]$.

The way to show this is to use the fact that an element $x$ of $\text{MGL}^*_{-n}(k)$ is a polynomial generator of the ring $\text{MGL}^*_Q(k)$ if and only if the Chern number $s^n$ is not zero on $x$. Here for a smooth irreducible projective variety $X$ over $k$, $s^n(X) = \langle \xi^n_1 + \cdots + \xi^n_n, [X] \rangle$, with $\xi_i$ being the Chern roots of the tangent bundle of $X$ (in the Chow ring $\text{CH}^*(X)$).

The following lemma can be found in [28], Page 47.

Lemma 4.6.2. For any space $X$ with a vector bundle $V$ of rank $r$, let $\pi : \mathbb{P}(V) \to X$ be the projective bundle and let $u = c_1(\mathcal{O}(1)) \in \text{CH}^1(\mathbb{P}(V))$. Then, for all $i \geq 0$,

$$\pi^*(u^i) = s_{i-(r-1)}(V),$$

where $s_k$ is the $k$-th Segre class.

Recall that we have shown that, in $\text{MGL}^*(k)$,

$$X_1 - X_2 = \mathbb{P}_{\mathbb{P}(A)}(B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \oplus \mathcal{O}) - \mathbb{P}_{\mathbb{P}(B)}(A \otimes \mathcal{O}_{\mathbb{P}(B)}(-1) \oplus \mathcal{O}).$$

For each smooth $Z$ and rank 2 vector bundles $A$ and $B$, there is a pair $X_1$ and $X_2$, related by a classical flop with exceptional fibers equal to $\mathbb{P}(A)$ and $\mathbb{P}(B)$ respectively. Indeed, consider the $\mathbb{P}^1 \times \mathbb{P}^1$ bundle $q : \mathbb{P}(A) \times Z \mathbb{P}(B) \to Z$, which we embed in the $\mathbb{P}^3$ bundle $\mathbb{P}(A \oplus B) \to Z$ via the line bundle $p_1^*\mathcal{O}_{\mathbb{P}(A)}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}(B)}(1)$. We then take $Y^0$ to be the affine $Z$ cone in $A \oplus B$ associated to $\mathbb{P}(A) \times Z \mathbb{P}(B) \subset \mathbb{P}(A \oplus B)$, and $Y$ the closure of $Y^0$ in $\mathbb{P}(A \oplus B \oplus \mathcal{O}_Z)$. $Y$ thus contains the $\mathbb{P}^2$ bundles $P_1 := \mathbb{P}(A \oplus \mathcal{O}_Z)$ and $P_2 := \mathbb{P}(B \oplus \mathcal{O}_Z)$; we take $X_i \to Y$ to be the blow-up of $Y$ along $P_i$, $i = 1, 2$ and $\tilde{X}$ the blow-up of $Y$ along $Z$. 

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We will, for each \( n \geq 5 \), find an \( n - 3 \)-fold \( Z \) and rank two vector bundles \( A \) and \( B \) over \( Z \), such that \( s^n(\mathbb{P}_A(B \otimes \mathcal{O}_A(-1) \oplus \mathcal{O})) \neq s^n(\mathbb{P}_A(B \otimes \mathcal{O}_A(1) \oplus \mathcal{O})) \).

We start with an arbitrary choice of \( Z \) and \( A, B \). Recall we denote the Chern roots (in \( \text{CH}^* \)) of \( A \) by \( a_1, a_2 \) and Chern roots of \( B \) by \( b_1, b_2 \). Set \( v_A := c_1(\mathcal{O}_A(1)), v_B = c_1(\mathcal{O}_B(1)), w_A := c_1(\mathcal{O}_A(1) \oplus \mathcal{O}_A(1)), \) and \( w_B = c_1(\mathcal{O}_B(1) \oplus \mathcal{O}_B(1)). \) Let the Chern roots of the tangent bundle of \( Z \) be \( z_1 \cdots, z_{n-3} \). Then, the Chern roots of the tangent bundle of \( \mathbb{P}_A(B \otimes \mathcal{O}_A)(-1) \oplus \mathcal{O} \) are

\[
b_1 - v_A + w_B, b_2 - v_A + w_B, a_1 + v_A, a_2 + v_A, z_1, \ldots, z_{n-3}.
\]

For the bundle \( \pi_1 : \mathbb{P}_A(B \otimes \mathcal{O}_A)(-1) \oplus \mathcal{O} \to \mathbb{P}_A(1) \), Lemma 4.6.2 yields

\[
\pi_1^*(w_B^n) = s_{i-2}(B \otimes \mathcal{O}_A)(-1) \oplus \mathcal{O}) = \sum_{i_1 + i_2 = i - 2} (-b_1 + v_A)^{i_1} (-b_2 + v_A)^{i_2}.
\]

Similarly, for \( \pi_2 : \mathbb{P}_A(1) \to Z \), we have

\[
\pi_2^*(v_A^i) = s_{i-1}(A) = \sum_{i_1 + i_2 = i - 1} (-a_1)^{i_1} (-a_2)^{i_2}.
\]

Thus

\[
n_1^n(\mathbb{P}_A(B \otimes \mathcal{O}_A)(-1) \oplus \mathcal{O}))
\]

\[
= \pi_1 \left[ (b_1 - v_A + w_B)^n + (b_2 - v_A + w_B)^n + a_1 + v_A)^n + (a_2 + v_A)^n + \sum_{i=1}^{n-3} z_i^n \right]
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} (b_1 - v_A)^{n-i} \pi_1^*(w_B^i) + \sum_{i=0}^{n} \binom{n}{i} (b_2 - v_A)^{n-i} \pi_1^*(w_B^i) + \pi_1^*(w_B^n)
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} (b_1 - v_A)^{n-i} \sum_{i_1 + i_2 = i - 2} (-b_1 + v_A)^{i_1} (-b_2 + v_A)^{i_2}
\]

\[
+ \sum_{i=2}^{n} \binom{n}{i} (b_2 - v_A)^{n-i} \sum_{i_1 + i_2 = i - 2} (-b_1 + v_A)^{i_1} (-b_2 + v_A)^{i_2}
\]

\[
+ \sum_{i_1 + i_2 = n-2} (-b_1 + v_A)^{i_1} (-b_2 + v_A)^{i_2}.
\]
In the case when $a_2 = b_1 = b_2 = 0$, we have:

$$\pi_1 s^n(\mathbb{P}(A)(B \otimes \mathcal{O}(A)(-1) \oplus \mathcal{O})) = 2 \sum_{i=2}^{n} \binom{n}{i} (-v_A)^{n-i} \sum_{i_1 + i_2 = i-2} (v_A)^{i-2} + \sum_{i_1 + i_2 = n-2} (v_A)^{n-2}$$

$$= 2 \sum_{i=2}^{n} \binom{n}{i} (-v_A)^{n-i}(i-1)(v_A)^{i-2} + (n-1)(v_A)^{n-2}$$

$$= (v_A)^{n-2}(2(-1)^n + n - 1).$$

According to Lemma 4.6.2, $\pi_2 s^n(v_A)^{n-2} = (-a_1)^{n-3}$, since $a_2 = 0$. Therefore,

$$\pi_2 s^n(\mathbb{P}(A)(B \otimes \mathcal{O}(A)(-1) \oplus \mathcal{O})) = (-a_1)^{n-3}(2(-1)^n + n - 1)$$

$$= (a_1)^{n-3}(-2 + (-1)^{n-3}(n - 1)).$$

We do the same for the projection $\pi_1': \mathbb{P}(A)\otimes \mathcal{O}(B)(-1) \oplus 1 \rightarrow \mathbb{P}(B)$, and $\pi_2': \mathbb{P}(B) \rightarrow Z$, in the case of $a_2 = b_1 = b_2 = 0$. By Lemma 4.6.2, $\pi_1'(w_A^i) = \sum_{i_1 + i_2 = i-2} (-a_1^{i_1}v_B^{i_2})$, and hence,

$$\pi_1' s^n(\mathbb{P}(B)(A \otimes \mathcal{O}(B)(-1) \oplus 1))$$

$$= \pi_1'((a_1 - v_B + w_A)^n + (-v_B + w_A)^n + w_A^n + 2(v_B)^n + \sum_{i=1}^{n-3} (z_i)^n)$$

$$= \sum_{i=0}^{n} \binom{n}{i} (a_1 - v_B)^{n-i}\pi_1(w_A^i) + \sum_{i=0}^{n} \binom{n}{i} (-v_B)^{n-i}\pi_1(w_A^i) + \pi_1(w_A^n)$$

$$= \sum_{i=1}^{n} \binom{n}{i} (a_1 - v_B)^{n-i} \sum_{i_1 + i_2 = i-2} (-a_1 + v_B)^{i_1}(v_B)^{i_2}$$

$$+ \sum_{i=2}^{n} \binom{n}{i} (-v_B)^{n-i} \sum_{i_1 + i_2 = i-2} (-a_1 + v_B)^{i_1}(v_B)^{i_2} + \sum_{i_1 + i_2 = n-2} (-a_1 + v_B)^{i_1}(v_B)^{i_2}. $$

For $\pi_2'$, Lemma 4.6.2 tells us that $\pi_2'(v_B) = 1$, and $\pi_2'(v_B^i) = 0$ for $i \neq 1$. Finally, we obtain:

$$\pi_2 s^n(\mathbb{P}(B)(A \otimes \mathcal{O}(B)(-1) \oplus 1)) = a_1^{n-3}(-(n-1)^2 + \binom{n}{2}) - (-1)^{n-3} + (n-1)(-1)^{n-3}$$

$$= a_1^{n-3}(-(n-1)^2 + \binom{n}{2} + (n-2)(-1)^{n-3})$$

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To show this can be non-vanishing for certain choice of $Z$ and $a_1 \in CH^1(Z)$, we simply take $Z = \mathbb{P}^{n-3}$ and $a_1 = c_1(\mathcal{O}_{\mathbb{P}^{n-3}}(1))$. We then have

$s^n[X_1 - X_2] = \langle a_1^{n-3}(-2 + (-1)^{n-3}(n - 1) + (n - 1)^2 - \binom{n}{2} - (n - 2)(-1)^{n-3}), [\mathbb{P}^{n-3}] \rangle$

$$= \frac{n^2 - 3n - 2 + 2(-1)^{n-1}}{2},$$

so for $n \geq 5$, $s^n[X_1 - X_2] \neq 0$, as desired.
Chapter 5

On deformed double current algebras
for simple Lie algebras

5.1 Deformed double current algebras

For a simple Lie algebra \( g \) over \( \mathbb{C} \) with rank \( \geq 3 \), let \( \Delta \) be the set of roots for \( g \), and \( \Delta^+ \) be a set of positive roots. Let \( C = (c_{ij})_{i,j=0}^{N} \) be the affine Cartan matrix of the affine Lie algebra \( \hat{g} \) and the numbers \( d_0, d_1, \ldots, d_N \) are such that \( (d_i c_{ij})_{i,j=0}^{N} \) is a symmetric matrix.

Proposition 5.1.1 ([35], Lemma 4.1, 4.2). The universal central extension \( \hat{g}(\mathbb{C}[u,v]) \) of \( g(\mathbb{C}[u,v]) := g \otimes \mathbb{C}[u,v] \) is isomorphic to the Lie algebra generated by elements \( X_{i,r}^+, H_{i,r} \) for \( 1 \leq i \leq N, r = 0, 1 \) and \( X_{0,0}, X_{0,1}^+ \) subject to the following relations:

\[
\begin{align*}
[H_{i_1,r_1}, H_{i_2,r_2}] &= 0, \quad [H_{i_1,0}, X_{i_3,r_3}^\pm] = \pm d_{i_1} c_{i_1,i_3} X_{i_3,r_3}^\pm, \\
&\quad \text{for } 1 \leq i_1, i_2 \leq N, 0 \leq i_1, i_2 \leq N, r_1, r_2, r_3 = 0 \text{ or } 1 \quad (5.1.1) \\
[H_{i_1,1}, X_{i_2,0}^\pm] &= [H_{i_1,0}, X_{i_2,1}^\pm], \quad [X_{i_1,1}^\pm, X_{i_2,0}^\pm] = [X_{i_1,0}^\pm, X_{i_2,1}^\pm], \quad 0 \leq i_1, i_2 \leq N \quad (5.1.2) \\
[X_{i_1,r_1}^+, X_{i_2,r_2}^-] &= \delta_{i_1i_2} H_{i_1,r_1+r_2}, \quad 0 \leq i_1 \leq N, 1 \leq i_2 \leq N, r_1 + r_2 = 0, 1 \quad (5.1.3) \\
[X_{i_1,0}^+, [X_{i_1,0}^+, \ldots, [X_{i_1,0}^+, X_{i_2,0}^\pm] \ldots]] &= 0. \quad (5.1.4)
\end{align*}
\]
In (5.1.2), (5.1.4), when \( i_1 = 0, i_2 = 0 \) or \( i_3 = 0 \), there is a relation only in the +–case.

Assume that \( \mathfrak{g} \) is not of Dynkin type \( A \). Then there is a unique \( k \in \{1, \ldots, N\} \) such that \( c_{0k} \neq 0 \). In other words, \( k \) is the label of the unique vertex in the Dynkin diagram of \( \widehat{\mathfrak{g}} \) to which the zero node is connected. Let \( \theta \) be the highest root of \( \mathfrak{g} \). Let

\[
\omega^\pm_i := \pm \frac{1}{4} \sum_{\alpha \in \Delta^+} S([X^\pm_i, X^\pm_\alpha], X^\pm_\alpha) - \frac{1}{4} S(X^\pm_i, H_i) \in \mathfrak{u}(\mathfrak{g})
\]  

(5.1.5)

and \( \nu_i := [\omega^+_i, X^-_i] \in \mathfrak{u}(\mathfrak{g}) \).

**Definition 5.1.1** ([35], Definition 5.1). The deformed double current algebra \( \mathfrak{D}(\mathfrak{g}) \) is the algebra generated by \( X^\pm_{i,r}, H_{i,r}, X^\pm_{0,r} \) for \( 1 \leq i \leq N, r = 0,1 \) subjected to the same relations as those in Proposition 5.1.1, except that the following relations involving \( X^\pm_{0,r} \) must be modified:

\[
[X^+_{k,1}, X^+_{0,0}] - [X^+_{k,0}, X^+_{0,1}] = -\frac{d_0}{2} S(X^+_{k,0}, X^-_\theta) + [\omega^+_k, X^-_\theta] + [X^+_0, \omega^+_0]
\]  

(5.1.6)

\[
[H_{k,1}, X^+_{0,0}] - [H_{k,0}, X^+_{0,1}] = -\frac{d_0}{2} S(H^+_k, X^-_\theta) + d_0 \omega^+_0 + [\nu_k, X^-_\theta]
\]  

(5.1.7)

\[
[X^+_{0,1}, X^-_{k,0}] = [X^-_{k,0}, \omega^+_0], \quad [X^+_0, X^+_0, X^-_0] = 2d_0X^+_{0,0}X^-_\theta
\]  

(5.1.8)

\[
[X^+_{0,0}, X^\pm_{i,0}] = [X^-_\theta, \omega^+_i], \quad [X^+_0, X^+_i, X^-_0] = -[\omega^+_0, X^+_i] \text{ for } i \neq k
\]  

(5.1.9)

The elements \( X^-_\theta \) and \( \omega^+_0 \) are defined in the following way. We write \( X^-_\theta \) as \( [X^-_{k,0}, X^-_{\theta,\alpha_k}] \) (it may be necessary to rescale \( X^\pm_{-\alpha_k} \) to achieve this) and

\[
X^-_\theta = [X^-_{k,0}, X^-_{\theta,\alpha_k}] \in \mathfrak{D}(\mathfrak{g}). \quad \text{(Here, } X^-_{\theta,\alpha_k} \in \mathfrak{g} \subset \mathfrak{D}(\mathfrak{g}).)
\]

Set \( \omega^+_0 = -[\omega^-_k, X^-_{\theta,\alpha_k}] \).

Let’s consider another algebra:

**Definition 5.1.2.** \(^1\) The algebra \( \mathfrak{D}(\mathfrak{g}) \) is generated by elements \( X, K(X), Q(X), P(X), X \in \mathfrak{g} \), such that

\[
\{K(X), X | X \in \mathfrak{g}\} \quad \text{generate a subalgebra which is an image of } \mathfrak{g} \otimes \mathbb{C}[u] \text{ with } X \otimes u \mapsto K(X);
\]

\(^1\)This definition is first found in [34] in the case of \( \mathfrak{sl}_n \), for \( n \geq 4 \).
\begin{itemize}
\item \{Q(X), X | X \in \mathfrak{g}\} generate a subalgebra which is an image of \(\mathfrak{g} \otimes \mathbb{C}[v]\) with \(X \otimes v \mapsto Q(X)\);
\item \(P(X)\) is linear in \(X\), and for any \(X, X' \in \mathfrak{g}\), \([P(X), X'] = P[X, X']\),
\end{itemize}

and the following relation holds for all root vectors \(X_{\beta_1}, X_{\beta_2} \in \mathfrak{g}\) with \(\beta_1 \neq -\beta_2\):
\[
[K(X_{\beta_1}), Q(X_{\beta_2})] = P([X_{\beta_1}, X_{\beta_2}]) - \frac{(\beta_1, \beta_2)}{4} S(X_{\beta_1}, X_{\beta_2}) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}]),
\]

\textbf{Theorem 5.1.3.} There exists an algebra isomorphism \(\varphi : D(\mathfrak{g}) \rightarrow D(\mathfrak{g})\) given by
\[
\varphi(X_{i,0}^\pm) = X_{i,0}^\pm, \quad \varphi(H_i,0) = H_i
\]
\[
\varphi(X_{i,1}^\pm) = Q(X_{i,1}^\pm), \quad \varphi(H_i,1) = Q(H_i), \text{ for } 1 \leq i \leq N
\]
\[
\varphi(X_{0,0}^\pm) = K(X_{0}^-), \quad \varphi(X_{0,1}^+ - \omega_0^-).
\]

In particular, the presentation in Definition 5.1.2 gives a double loop presentation of the deformed double current algebra.

\section{5.2 The map \(\varphi\) is a homomorphism of algebras}

This section is devoted to prove that the map \(\varphi\) defined in Theorem 5.1.3 is a homomorphism of algebras.

\textbf{Lemma 5.2.1.} Let \(\omega_i^\pm\) be the element in \(U(\mathfrak{g})\) defined in (5.1.5), we have
\[
\omega_i^\pm = \pm \frac{1}{4} \sum_{\alpha \in \Delta^+} S([X_i^\pm, X_\alpha^\mp], X_\alpha^\pm).
\]

\textbf{Proof.} The Casimir element of \(\mathfrak{g}\) is
\[
\Omega = \sum_{\alpha \in \Delta^+} S(X_\alpha^+, X_\alpha^-) + \sum_{i=1}^N \hat{h}_i \bar{\hat{h}}_i
\]
where \( \{\tilde{h}_1, \ldots, \tilde{h}_N\} \) is a basis of \( \mathfrak{h} \) and \( \{\tilde{h}^1, \ldots, \tilde{h}^N\} \) is the dual basis. \( \Omega \) is in the center of \( \U(g) \), so

\[
0 = \pm \frac{1}{4} [X^\pm_i, \Omega] = \omega^\pm_i \pm \frac{1}{4} \sum_{\alpha \in \Delta^+} S(X^\pm_i, [X^\pm_i, X^\pm_\alpha])
\]

because

\[
[X^\pm_i, \sum_{j=1}^N \tilde{h}^j \tilde{h}_j] = \sum_{j=1}^N ([X^\pm_i, \tilde{h}^j] \tilde{h}_j + \tilde{h}^j [X^\pm_i, \tilde{h}_j])
\]

\[
= \mp \sum_{j=1}^N ((H_i, \tilde{h}^j) X^\pm_i \tilde{h}_j + \tilde{h}^j (\tilde{h}_j, H_i) X^\pm_i) = \mp S(X^\pm_i, H_i).
\]

The conclusion follows.

We have to verify that \( \varphi \) is a homomorphism of algebras, that is, that it respects the defining relations of \( D(g) \).

Let’s start with (5.1.6). We have to check that

\[
[Q(X^+_k), K(X^-_\theta)] - [X^+_k, P(X^-_\theta)] = -\frac{d_0}{2} S(X^+_k, X^-_\theta) - \frac{1}{4} \sum_{\alpha \in \Delta^+} S([X^+_k, X^-_\alpha], [X^+_\alpha, X^-_\theta])
\]

\[
- \frac{1}{4} S([X^+_k, X^-_\theta], X^+_k).
\]

(Observe that \( [X^-_k, X^-_\theta] = 0 \) and that if \( [[X^+_k, X^-_\alpha], X^-_\theta] \) is a root vector (where \( \alpha \) is positive), then \( \alpha_k - \alpha - \theta \geq -\theta \), so \( \alpha \leq \alpha_k \); since \( \alpha_k \) is simple and both are positive, this forces \( \alpha \) to be equal to \( \alpha_k \).)

The previous equality is equivalent to

\[
[Q(X^+_k), K(X^-_\theta)] - [X^+_k, P(X^-_\theta)] = -\frac{d_0}{4} S(X^+_k, X^-_\theta) - \frac{1}{4} \sum_{\alpha \in \Delta^+} S([X^+_k, X^-_\alpha], [X^+_\alpha, X^-_\theta])
\]

and to

\[
[K(X^-_\theta), Q(X^+_k)] = P([X^-_\theta, X^+_k]) + \frac{d_0}{4} S(X^+_k, X^-_\theta) + \frac{1}{4} \sum_{\alpha \in \Delta^+} S([X^+_k, X^-_\alpha], [X^+_\alpha, X^-_\theta])
\]

This is the same as

\[
[K(X^-_\theta), Q(X^+_k)] = P([X^-_\theta, X^+_k]) - \frac{(-\theta, \alpha_k)}{4} S(X^+_k, X^-_\theta) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X^-_\theta, X^-_\alpha], [X^-_\alpha, X^-_k])
\]

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which is true in $\mathcal{D}(g)$. Thus, $\varphi$ preserves relation (5.1.6).

(5.1.7) follows from (5.1.6). For the same reason, the first relation in (5.1.8) holds because

$$[X^{-}_\theta, X^{-}_{\kappa,0}] = 0.$$ 

For the second relation in (5.1.8), we need to compute $[P(X^{-}_\theta) - \omega^+_\alpha, K(X^{-}_\theta)]$. Notice that if $[X^{-}_\alpha, X^{-}_{\theta - \alpha}] \neq 0$ for a positive root $\alpha$, then it is a root vector in $g$ for the root $-\alpha - \theta + \alpha_k$, so $-\alpha - \theta + \alpha_k \geq -\theta$ and thus $\alpha \leq \alpha_k$: this forces $\alpha = \alpha_k$ because $\alpha_k$ is a simple root. Similarly, if $[[X^{-}_k, X^{-}_\alpha], X^{-}_{\theta - \alpha}] \neq 0$ then it is a root vector in $g$ for the root $-\alpha_k + \alpha - \theta - \alpha + \alpha_k \geq -\theta$ hence $\alpha \geq \theta$: this implies that $\alpha = \theta$. These two observations serve to obtain the third equality below.

$$[\omega^+_\alpha, K(X^{-}_\theta)] = -[[\omega^{-}_\alpha, X^{-}_{\theta - \alpha}], K(X^{-}_\theta)] = -[[\omega^{-}_\alpha, K(X^{-}_\theta)], X^{-}_{\theta - \alpha}]$$

$$= -\frac{1}{4} \sum_{\alpha \in \Delta^+} [S([[X^{-}_k, X^{-}_\alpha], K(X^{-}_\theta)], X^{-}_\alpha), X^{-}_{\theta - \alpha}]$$

$$= -\frac{1}{4} S([[X^{-}_k, X^{-}_\alpha], K(X^{-}_\theta)], X^{-}_{\theta - \alpha}], X^{-}_\theta) - \frac{1}{4} S([[X^{-}_k, X^{-}_\alpha], K(X^{-}_\theta)], [X^{-}_k, X^{-}_{\theta - \alpha}])$$

$$= -\frac{1}{4} S([X^{-}_k, K(H_\theta)], X^{-}_{\theta - \alpha}], X^{-}_\theta) + \frac{1}{4} S([H_k, K(X^{-}_\theta)], X^{-}_\theta)$$

$$= -\frac{1}{4} d_0 S([K(X^{-}_\theta), X^{-}_{\theta}], X^{-}_\theta) - \frac{1}{4} d_0 S(K(X^{-}_\theta), X^{-}_\theta)$$

$$= -d_0 K(X^{-}_\theta) X^{-}_\theta$$

(5.2.2)

We show that $[P(X^{-}_\theta), K(X^{-}_\theta)] = d_0 K(X^{-}_\theta) X^{-}_\theta$ as follows. Start with

$$[K(X^{-}_{\alpha_k})] = P(X^{-}_\theta) - \frac{(\alpha_k, \theta - \alpha_k)}{4} S(X^{-}_{\alpha_k}, X^{-}_{\theta - \alpha_k}) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X^{-}_{\alpha_k}, X^{-}_{\theta}], [X^{-}_{\alpha}, X^{-}_{\theta - \alpha}])$$

(5.2.3)

Observe that if $[X^{-}_{\alpha_k}, X^{-}_\alpha] \neq 0$ and $[[X^{-}_{\alpha - \alpha}, X^{-}_{\theta - \alpha}], X^{-}_\theta] \neq 0$, then both are root vectors for the roots $-\alpha_k + \alpha$ and $-\alpha - \theta + \alpha_k - \theta$, so $-\alpha_k + \alpha \geq -\theta$ and $-\alpha - \theta + \alpha_k - \theta \geq -\theta$, hence $\alpha \geq -\theta + \alpha_k$, $-\theta + \alpha_k \geq \alpha$ and $\alpha = -\theta + \alpha_k$.

Similarly, if $[[X^{-}_{\alpha_k}, X^{-}_\alpha], X^{-}_\theta] \neq 0$ and $[X^{-}_{\alpha}, X^{-}_{\theta - \alpha}] \neq 0$, then both are root vectors for the roots $-\alpha_k + \alpha - \theta$ and $-\alpha - \theta + \alpha_k$; it follows that $-\alpha_k + \alpha - \theta \geq -\theta$ and $-\alpha - \theta + \alpha_k \geq -\theta$, so $\alpha \geq \alpha_k$ and $\alpha_k \geq \alpha$, thus $\alpha = \alpha_k$. 

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Using these two observations, we now apply \([\cdot, K(X_{\theta}^-)]\) to (5.2.3) to get
\[
[K(X_{\alpha_k}^-), [Q(X_{\theta-\alpha_k}^-), K(X_{\theta}^-)]]
\]
\[
= [P(X_{\theta}^-), K(X_{\theta}^-)] + \frac{1}{4} S([X_{\alpha_k}^-, X_{\theta-\alpha_k}], [[X_{\theta-\alpha_k}, X_{\theta-\alpha_k}], K(X_{\theta}^-)])
\]
\[
+ \frac{1}{4} S([X_{\alpha_k}^-, X_{\alpha_k}], K(X_{\theta}^-)), [X_{\theta-\alpha_k}, X_{\theta-\alpha_k}])
\]
\[
= [P(X_{\theta}^-), K(X_{\theta}^-)] + \frac{1}{4} S(X_{\theta-\alpha_k}, [H_{\theta-\alpha_k}, K(X_{\theta}^-)])
\]
\[
+ \frac{d_0}{4} S(K(X_{\theta}^-), X_{\theta}^-)
\]
\[
= [P(X_{\theta}^-), K(X_{\theta}^-)] - \frac{d_0}{4} S(X_{\theta-\alpha_k}, K(X_{\theta}^-)) + \frac{d_0}{4} S(K(X_{\theta}^-), X_{\theta}^-)
\]
\[
= [P(X_{\theta}^-), K(X_{\theta}^-)]
\]

We compute the left-hand side of the above equality (5.2.5).
\[
[Q(X_{\theta-\alpha_k}^-), K(X_{\theta}^-)] = \frac{(\theta - \alpha_k, \theta)}{4} S(X_{\theta-\alpha_k}^-, X_{\theta}^-) - \frac{1}{4} S([X_{\theta}^-, X_{\alpha_k}], [X_{\theta-\alpha_k}, X_{\theta-\alpha_k}])
\]
\[
= \frac{d_0}{4} S(X_{\theta-\alpha_k}^-, X_{\theta}^-) - \frac{1}{4} S([X_{\theta}^-, X_{\alpha_k}], X_{\theta}^-)
\]
so
\[
[K(X_{\alpha_k}^-), [Q(X_{\theta-\alpha_k}^-), K(X_{\theta}^-)] = \frac{d_0}{4} S(K([X_{\alpha_k}^-, X_{\theta-\alpha_k}^-], X_{\theta}^-)) + \frac{1}{4} S([X_{\theta}^-, K(H_{\theta}^-)], X_{\theta})
\]
\[
= \frac{d_0}{2} S(K(X_{\theta}^-), X_{\theta-\theta}) = d_0 K(X_{\theta}^-) X_{\theta-\theta}.
\]

Substituting (5.2.6) into (5.2.5) yields \([P(X_{\theta}^-), K(X_{\theta}^-)] = d_0 K(X_{\theta}^-) X_{\theta}^-\) and it follows from (5.2.2) that \([P(X_{\theta}^-) - \omega_k^+, K(X_{\theta}^-)] = 2d_0 K(X_{\theta}^-) X_{\theta}^-\) as desired.

Finally, we check the first relation in (5.1.9), so with \(i \neq 0, k\) (hence \((\theta, \alpha_i) = 0\).
\[
[K(X_{\theta}^-), Q(X_{i}^-)] = -[\omega_i^-, X_{\theta}] = \frac{1}{4} \sum_{\alpha \in \Delta^+} S([X_{\alpha}^- i], [X_{\alpha}^+, X_{\theta}^-])
\]
using (5.1.5) and
\[
[K(X_{\theta}^-), Q(X_{i}^+)] = -[\omega_i^+, X_{\theta}] = \frac{1}{4} \sum_{\alpha \in \Delta^+} S([X_{\alpha}^i], [X_{\alpha}^+, X_{\theta}^-])
\]
this time using (5.2.1). Both of these relations do hold in \(\mathfrak{D}(\mathfrak{g})\).

The second relation in (5.1.9) is also satisfied since \([P(X_{\theta}^-), X_{i,0}^\pm] = P([X_{\theta}^-, X_{i,0}^\pm]) = 0\) when \(i \neq k\).
5.3 The map $\varphi$ is an isomorphism of algebras.

In this section, we construct the inverse map $\psi : \mathcal{O}(g) \to \mathcal{D}(g)$ of $\varphi$.

Let us first describe the images $\psi(X)$, $\psi(K(X))$, $\psi(Q(X))$ and $\psi(P(X))$ of the generators of $\mathcal{D}(g)$ under the map $\psi$, where $X$ is any element in $g$.

**Lemma 5.3.1** ([ ]). The current Lie algebra $g[u]$ is isomorphic to the Lie algebra generated by elements $e_i, f_i, h_i$ for $1 \leq i \leq N$ and $e_0$ subject to the following relations: for $1 \leq i \leq N$,

- $e_i, f_i, h_i$ satisfy the usual relations for Serre’s Theorem for $g$;
- $[h_i, e_0] = d_i e_0 e_0$, $[e_0, f_i] = 0$;
- $ad(e_i)^{1-c_{0i}}(e_0) = 0$ and $ad(e_0)^{1-c_{0i}}(e_i) = 0$.

Note that the above relations of $g[u]$ are the defining relations of the Kac-Moody algebra $\hat{g}$, except those involving $f_0, h_0$.

In the defining relations of $\mathcal{D}(g)$, $\{K(X), X \mid X \in g\}$ generate a subalgebra $g[u]$. By Lemma 5.3.1, we have homomorphisms $\mathcal{U}(g[u]) \to \mathcal{D}(g)$, given by $X \mapsto X$, for $X \in g$, and $K(X_0^{-}) \mapsto X_{0,0}$. Likewise, we have homomorphisms $\mathcal{U}(g[u]) \to \mathcal{D}(g)$, given by $X \mapsto X$, for $X \in g$, and $Q(X_i^{+}) \mapsto X_{i,1}^{+}$, for $1 \leq i \leq N$. The above homomorphisms tell us how to define $\psi(K(X))$ and $\psi(Q(X))$ in $\mathcal{D}(g)$, for any $X \in g$.

**Lemma 5.3.2.** There is an $ad(g)$-module morphism $P : g \to \mathcal{D}(g)$, such that $X_{-\theta} \mapsto P(X_{-\theta}) := X_{0,1}^{+} + \omega_0^{+}$.

In particular, let $P(X)$ to be the image of $X \in g$ under the morphism $P$, then

1. $P(X)$ is linear in $X$;
2. $[X, P(X')] = P([X, X'])$, for any $X, X' \in g$.

**Proof.** Let $g = n^- \oplus h \oplus n$ be the triangular decomposition of $g$, and $b^- := n^- \oplus h$. Let $M(-\theta) := \mathcal{U}(g) \otimes_{\mathcal{U}(b^-)} \mathbb{C}_{-\theta}$ be the Verma module with lowest weight $-\theta$, where $\mathbb{C}_{-\theta}$

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2This proof is given by Valerio Toledano Laredo.
is the 1–dimensional $b^–$–module with trivial $n^–$–action. Thanks to the PBW Theorem, 
$\mathfrak{U}(\mathfrak{g}) \cong \mathfrak{U}(\mathfrak{n}) \otimes b^– \mathbb{C}_{–\theta}$, which allows us to write $M(–\theta) \cong \mathfrak{U}(\mathfrak{n}) \otimes \mathbb{C}_{–\theta}$. Denote the lowest weight vector of $M(–\theta)$ by $v^– := 1 \otimes 1$. Then, $M(–\theta)$ as a $\mathfrak{U}(\mathfrak{g})$ is generated by $v^–$, such that

$$n^– \cdot v^– = 0 \text{ and } h \cdot v^– = h(–\theta)v^–, \text{ for all } h \in \mathfrak{h}. \quad (5.3.1)$$

As in the following diagram $M(–\theta) \xrightarrow{P} D(\mathfrak{g})$, we first construct a morphism $\tilde{P} : M(–\theta) \to D(\mathfrak{g})$, given by $v^– \mapsto P(X_{–\theta}) := X_{0,1}^+ + \omega_0^+$, then show $\tilde{P}$ factors through the adjoint representation of $\mathfrak{g}$, which gives the desired map $P : \mathfrak{g} \to D(\mathfrak{g})$.

To show the morphism $\tilde{P}$ is well-defined, it suffices to verify that $P(X_{–\theta})$ satisfies (5.3.1).

Rewriting the defining relation (5.1.8) of $D(\mathfrak{g})$, we have: $[P(X_{–\theta}), X_{k,0}^-] = 0$, and similarly, relation (5.1.9) gives the following relation: for $i \neq k$, $[P(X_{–\theta}), X_{i,0}^+] = 0$. It remains to check

$$[H_{i,0}, P(X_{–\theta})] = (\alpha_i, –\theta) P(X_{–\theta}), \quad (5.3.2)$$

This follows from the relation in Proposition 5.1.1 that $[H_{i,0}, X_{0,1}^+] = d_i c_{i,0} X_{0,1}^+ = (\alpha_i, –\theta) X_{0,1}^+$ and

$$[H_{i,0}, \omega_0^+] = [H_{i,0}, [\omega_k^–, X_{\theta–\alpha_k}^-]]$$

$$= – ([H_{i,0}, \omega_k^–], X_{\theta–\alpha_k}^-) [\omega_k^–, [H_{i,0}, X_{\theta–\alpha_k}^-]]$$

$$= – (\alpha_i, –\alpha_k)[\omega_k^–, X_{\theta–\alpha_k}^-] – (\alpha_i, –\theta + \alpha_k)[\omega_k^–, X_{\theta–\alpha_k}^-]$$

$$= – (\alpha_i, –\theta)[\omega_k^–, X_{\theta–\alpha_k}^-]$$

$$= – (\alpha_i, \theta) \omega_0^+$$

Thus, we have a (non-zero) $\mathfrak{g}$–equivariant map from the Verma module $M(–\theta)$ with lowest weight $–\theta$ to $D(\mathfrak{g})$ mapping the lowest weight vector $v^–$ to $P(X_{–\theta})$.

We show $D(\mathfrak{g})$ is locally finite as an $\text{ad}(\mathfrak{g})$-module, that is, any vector is contained in a finite-dimensional $\mathfrak{g}$-module. Note that the algebra $D(\mathfrak{g})$ is bigraded, with the following grading:
\[
\begin{align*}
& \text{deg}(X_{i,0}^\pm) = (0, 0), \text{ for } i = 1, \ldots, n. \\
& \text{deg}(X_{i,1}^\pm) = (1, 0), \text{ for } i = 1, \ldots, n. \text{ and } \text{deg}(X_{0,0}^+) = (0, 1). \\
& \text{deg}(\lambda) = (1, 1) \text{ and } \text{deg}(X_{0,1}^+) = (1, 1).
\end{align*}
\]

Now each graded piece is invariant under the adjoint action of \( g \), since the degree of the elements in \( g \) is \((0, 0)\). The degree \((0, 0)\) piece of \( D(g) \) coincide with the enveloping algebra \( \mathcal{U}(g) \) and each graded piece as a \( \mathcal{U}(g) \)-module is finite generated. Since we know the enveloping algebra \( \mathcal{U}(g) \) as \( \text{ad}(g) \)-module is locally finite, thus, each graded piece of \( D(g) \) is also locally finite as an \( \text{ad}(g) \)-module.

Thus, this \( g \)-equivariant map \( \tilde{P} \) must factor through a finite dimensional quotient of the Verma module \( M(-\theta) \), and there is only one such quotient, the adjoint representation of \( g \).

The remainder of the proof consists in checking that, for any two roots \( \beta_1 \neq -\beta_2 \),

\[
\begin{align*}
[\psi(K(X_{\beta_1})), \psi(Q(X_{\beta_2}))] &= \psi(P([X_{\beta_1}, X_{\beta_2}])) \\
&= -\frac{\langle \beta_1, \beta_2 \rangle}{4} S(\psi(X_{\beta_1}), \psi(X_{\beta_2})) + \frac{1}{4} \sum_{\alpha \in \Delta} S([\psi(X_{\beta_1}), \psi(X_\alpha)], [\psi(X_\alpha), \psi(X_{\beta_2})]).
\end{align*}
\]

From the defining relations of \( D(g) \), this is known to be true in the cases when \( \beta_1 = -\theta \) and \( \beta_2 = \pm \alpha_1, \ldots, \pm \alpha_N \), where the \( \alpha_i \) are the simple roots of \( g \). In order to see that it’s true in general, we can use the operators \( s_i \) which have the property that \( s_i(X_\alpha) \) is a root vector for the root \( s_i(\alpha) \).

Suppose that (5.1.10) holds when \( \beta_1 = -\theta \) and \( \beta_2 = \pm \alpha_1, \ldots, \pm \alpha_N \). (These cases correspond under \( \varphi \) to defining relations of \( D(g) \) given in definition 5.1.1.) The goal is to show, using this assumption and \([K(X_1), X_2] = K([X_1, X_2]), [Q(X_1), X_2] = Q([X_1, X_2]), [P(X_1), X_2] = P([X_1, X_2])\), that (5.1.10) must hold in full generality for any two roots \( \beta_1, \beta_2 \) with \( \beta_1 \neq -\beta_2 \).
Let $m : \mathfrak{u}(g) \otimes_{\mathbb{C}} \mathfrak{u}(g) \to \mathfrak{u}(g)$ be the multiplication map. The following observation will be useful below:

$$
\sum_{\alpha \in \Delta} S([X_{\beta_1}, X_\alpha], [X_{-\alpha}, X_{\beta_2}]) = m \left( \sum_{\alpha \in \Delta} [[X_{\beta_1} \otimes 1, X_\alpha \otimes X_{-\alpha}], 1 \otimes X_{\beta_2}] \right)
$$

$$+ m \left( \sum_{\alpha \in \Delta} [[1 \otimes X_{\beta_1}, X_{-\alpha} \otimes X_\alpha], X_{\beta_2} \otimes 1] \right)
$$

$$= m \left( [[X_{\beta_1} \otimes 1, \Omega], 1 \otimes X_{\beta_2}] - \sum_{i=1}^{N} [[X_{\beta_1} \otimes 1, \tilde{h}_i \otimes \bar{h}_i], 1 \otimes X_{\beta_2}] \right)
$$

$$+ m \left( [1 \otimes X_{\beta_1}, \Omega], X_{\beta_2} \otimes 1] - \sum_{i=1}^{N} [1 \otimes X_{\beta_1}, \tilde{h}_i \otimes \bar{h}_i], X_{\beta_2} \otimes 1] \right)
$$

where we view $\Omega$ as an element of $\mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g}$. Consequently, for any $X \in \mathfrak{g}$, we have $[\Omega, X \otimes 1 + 1 \otimes X] = 0$ and

$$
\sum_{\alpha \in \Delta} [S([X_{\beta_1}, X_\alpha], [X_{-\alpha}, X_{\beta_2}]), X_\gamma] - \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_\gamma], [X_{-\alpha}, X_{\beta_2}])
$$

$$- \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_\alpha], [X_{-\alpha}, [X_{\beta_2}, X_\gamma]]) \quad (5.3.3)
$$

$$= - (\gamma, \beta_2)S([X_{\beta_1}, X_\gamma], X_{\beta_2}) - (\gamma, \beta_1)S(X_{\beta_1}, [X_{\beta_2}, X_\gamma]), \quad (5.3.4)
$$

$$\quad (5.3.5)
$$

Here we write $X = X_\gamma \in \mathfrak{g}_\gamma$, for $\gamma \in \Delta \cup \{0\}$. The above equality follows from the following computation:

$$
(6.16.1) = - \sum_{i=1}^{N} m \left( [[X_{\beta_1} \otimes 1, [\tilde{h}_i \otimes \bar{h}_i, X_\gamma \otimes 1 + 1 \otimes X_\gamma]], 1 \otimes X_{\beta_2}] \right)
$$

$$- \sum_{i=1}^{N} m \left( [1 \otimes X_{\beta_1}, [\tilde{h}_i \otimes \bar{h}_i, X_\gamma \otimes 1 + 1 \otimes X_\gamma]], X_{\beta_2} \otimes 1] \right)
$$

$$= - \sum_{i=1}^{N} m \left( [[X_{\beta_1} \otimes 1, (\tilde{h}_i, H_\gamma)X_\gamma \otimes \tilde{h}_i + (\tilde{h}_i, H_\gamma)\tilde{h}_i \otimes X_\gamma]], 1 \otimes X_{\beta_2}] \right)
$$

$$- \sum_{i=1}^{N} m \left( [1 \otimes X_{\beta_1}, (\tilde{h}_i, H_\gamma)X_\gamma \otimes \tilde{h}_i + (\tilde{h}_i, H_\gamma)\tilde{h}_i \otimes X_\gamma], X_{\beta_2} \otimes 1] \right)
$$

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= - m \left( \left[ [X_{\beta_1} \otimes 1, X_\gamma \otimes H_\gamma + H_\gamma \otimes X_\gamma], 1 \otimes X_{\beta_2} \right] \right) \\
= - m \left( \left[ [1 \otimes X_{\beta_1}, X_\gamma \otimes H_\gamma + H_\gamma \otimes X_\gamma], X_{\beta_2} \otimes 1 \right] \right) \\
= - m \left( [X_{\beta_1}, X_\gamma] \otimes [H_\gamma, X_{\beta_2}] + [X_{\beta_1}, H_\gamma] \otimes [X_\gamma, X_{\beta_2}] \right) \\
= - m \left( [X_\gamma, X_{\beta_2}] \otimes [X_{\beta_1}, H_\gamma] + [H_\gamma, X_{\beta_2}] \otimes [X_{\beta_1}, X_\gamma] \right) \\
= - m \left( (\gamma, \beta_2)[X_{\beta_1}, X_\gamma] \otimes X_{\beta_2} + (\gamma, \beta_1)X_{\beta_1} \otimes [X_{\beta_2}, X_\gamma] \right) \\
= - m \left( (\gamma, \beta_1)[X_{\beta_2}, X_\gamma] \otimes X_{\beta_1} + (\gamma, \beta_2)X_{\beta_2} \otimes [X_{\beta_1}, X_\gamma] \right) \\
= - (\gamma, \beta_2)S([X_{\beta_1}, X_\gamma], X_{\beta_2}) - (\gamma, \beta_1)S(X_{\beta_1}, [X_{\beta_2}, X_\gamma]) = (6.16.2).

We are supposing that (5.1.10) holds when \( \beta_1 = -\theta \) and \( \beta_2 = \pm \alpha_k \). For convenience, let’s write down these two relations here:

\[ [K(X_\theta^r), Q(X_k^r)] = P([X_\theta^r, X_k^r]) + \frac{(\theta, \alpha_k)}{4}S(X_\theta^r, X_k^r) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_\theta^r, X_\alpha], [X_\alpha, X_k^r]) \]  

(5.3.6)

and

\[ [K(X_\theta^r), Q(X_k^-)] = -\frac{(\theta, \alpha_k)}{4}S(X_\theta^r, X_k^-) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_\theta^r, X_\alpha], [X_\alpha, X_k^-]) \]

Let’s apply \([X_\theta^+], \cdot \) to this second relation to obtain, note that:

\[ ([X_\theta^+, X_k^-], X_{-\theta+\alpha_k}) = (X_\theta^+, [X_k^-, X_{-\theta+\alpha_k}]) = (X_\theta^+, X_k^-) = 1 \implies [X_\theta^+, X_k^-] = X_{-\alpha_k}. \]

\[ [K(H_\theta), Q(X_k^-)] + [K(X_\theta^r), Q(X_{\theta-\alpha_k})] = -\frac{(\theta, \alpha_k)}{4}S(H_\theta, X_k^-) - \frac{(\theta, \alpha_k)}{4}S(X_\theta^r, X_{\theta-\alpha_k}) \]

\[ + \frac{1}{4} \sum_{\alpha \in \Delta} S([H_\theta, X_\alpha], [X_\alpha, X_k^-]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_\theta^r, X_\alpha], [X_\alpha, X_{\theta-\alpha_k}]) \]

\[ + \frac{(\theta, \alpha_k)}{4}S(H_\theta, X_k^-) + \frac{d_0}{2}S(X_\theta^r, X_{\theta-\alpha_k}) \quad \text{by (6.16.2)} \]

\[ = \frac{1}{4} \sum_{\alpha \in \Delta} S([H_\theta, X_\alpha], [X_\alpha, X_k^-]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_\theta^r, X_\alpha], [X_\alpha, X_{\theta-\alpha_k}]) + \frac{d_0}{4}S(X_\theta^r, X_{\theta-\alpha_k}). \]  

(5.3.7)

Set \( e_i = \sqrt{\frac{2}{(\alpha_i, \alpha_i)}}X_i^+ \) and \( f_i = \sqrt{\frac{2}{(\alpha_i, \alpha_i)}}X_i^- \) for \( i \neq 0 \). Consider the operators in the adjoint representation of \( \mathfrak{g} \) given by \( s_i = \exp(\text{ad}(f_i)) \exp(-\text{ad}(e_i)) \exp(\text{ad}(f_i)) \). It is known
that \( X \in \mathfrak{g}_\alpha \implies s_i(X_\alpha) \in \mathfrak{g}_{s_i(\alpha)} \) (where \( \mathfrak{g}_\alpha \) is the root subspace of \( \mathfrak{g} \) for the root \( \alpha \)) and \( s_i(H_j) = H_j - \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} H_i \). We will use \( s_i \) to denote either a simple reflection in the Weyl group \( W \) of \( \mathfrak{g} \) or the corresponding operator in its adjoint representation.

Let \( w_0 \) be the longest element of the Weyl group. Then \( -w_0 \) is a permutation of the simple roots \( \alpha_1, \ldots, \alpha_k \) and it can be checked that \( -w_0(\alpha_k) = \alpha_k \) in all types (except type \( A \), which we are not considering). Express \( w_0 \) as a product of simple reflections \( w_0 = s_{i_1} \cdots s_{i_\ell} \) and denote also by \( w_0 \) the corresponding operator in the adjoint representation of \( \mathfrak{g} \) for this choice of decomposition. Applying \( w_0 \) to (5.3.6) shows that the following relation also holds (after rescaling):

\[
[K(X_\theta^+, Q(X_k^-)] = P([X_\theta^+, X_k^-]) + \frac{(\theta, \alpha_k)}{4} S(X_\theta^+, X_k^-) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_\theta^+, X_\alpha], [X^-_\alpha, X_k^-])
\]  

(5.3.8)

(For any \( w \in W \), one can choose a decomposition into simple transpositions \( w = s_{j_1} \cdots s_{j_k} \) and obtain a corresponding operator in the adjoint representation with the property that \( w(X_\alpha) \) is a scalar multiple of \( X_{w(\alpha)} \); moreover, since the Killing form is invariant for the adjoint action, \( (w(X_\alpha), w(X^-_\alpha)) = (X_\alpha, X^-_\alpha) = 1 \), so \( w(X_\alpha) = a X_{w(\alpha)}, w(X^-_\alpha) = a^{-1} X^-_{w(\alpha)} \) for some non-zero scalar \( a \) and \( S([X_\theta^+, X_{w(\alpha)}], [X^-_{w(\alpha)}, X_k^-]) = S([X_\theta^+, w(X_{w(\alpha)}), [w(X^-_{w(\alpha)}), X_k^-]). \)

A similar observation applies to \( w(X^-_\theta) \) and \( w(X_k^+) \).

Applying \([\cdot, X_\theta^-]\) to (5.3.8) gives

\[
[K(H_{\theta}), Q(X_k^-)] = P([H_{\theta}, X_k^-]) + \frac{(\theta, \alpha_k)}{4} S(H_{\theta}, X_k^-) + \frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\theta}, X_\alpha], [X^-_\alpha, X_k^-])
\]  

\[- \frac{d_0}{4} S(H_{\theta}, X_k^-)
\]  

\[= P([H_{\theta}, X_k^-]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\theta}, X_\alpha], [X^-_\alpha, X_k^-]) \text{ because } (-\theta, \alpha_k) = d_0 c_{0k} = -d_0 \quad (5.3.9)
\]
(6.16.2) was useful again here to obtain this relation. Indeed,
\[
\sum_{\alpha \in \Delta} [S([X^+, X_\alpha], [X^-, X^+_k], X^-_\theta)] - \sum_{\alpha \in \Delta} [S([X^+, X^-_\theta], X^-_\alpha)\] \[X^-_\alpha, X^-_k)]
\] - \sum_{\alpha \in \Delta} [S([X^+, X_\alpha], [X^-_\alpha, X^-_k, X^-_\theta)])
\] = - (\theta, \alpha_k)S([X^+, X^-_\theta], X^-_k) - (\theta, \theta)S(X^+, [X^-_\theta, X^-_\theta])
\] = - d_0S(H_\theta, X^-_k)
\]

Combining (5.3.9) with (5.3.7) yields
\[
[K(X^-_\theta), Q(X^-_{\theta - \alpha_k})] = -P([H_\theta, X^-_k]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X^-_\theta, X^-_\alpha, [X^-_\alpha, X^-_{\theta - \alpha_k}]) + \frac{d_0}{4} S(X^-_\theta, [X^+, X^-_{\theta - \alpha_k}])
\]
which is equivalent to
\[
[K(X^-_\theta), Q(X^-_{\theta - \alpha_k})]
\] = P([X^-_\theta, X^-_{\theta - \alpha_k})] - \frac{\theta, \alpha_k - \theta}{4} S(X^-_\theta, X^-_{\theta - \alpha_k}) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X^-_\theta, X^-_\alpha, [X^-_\alpha, X^-_{\theta - \alpha_k}]) \quad (5.3.10)
\]
because $[H_\theta, X^-_k] = - (\theta, \alpha_k)X^-_k = d_0c_{0k}X^-_k = -X^-_k$ and $[X^-_\theta, X^-_{\theta - \alpha_k}$ is a scalar multiple of $X^-_k$ with scalar given by $([X^-_\theta, X^-_{\theta - \alpha_k}, X^+_k])$, which in turn equals:
\[
([X^-_\theta, X^-_{\theta - \alpha_k}, X^+_k] = -([X^-_{\theta - \alpha_k}, X^-_\theta, X^+_k]) = - (X^-_{\theta - \alpha_k}, [X^-_\theta, X^+_k])
\] = (\theta, \alpha_k)(X^-_{\theta - \alpha_k}, X^-_{\theta - \alpha_k}) = (\theta, \alpha_k) = d_0,
\]
where the third equality follows from $[X^-_\theta, X^+_k] = - (\theta, \alpha_k)X^-_{\theta - \alpha_k}$, since
\[
([X^-_\theta, X^+_k, X^-_{\theta - \alpha_k}) = -([X^+_k, X^-_\theta], X^-_{\theta - \alpha_k}) = -(X^+_k, [X^-_\theta, X^-_{\theta - \alpha_k}])
\] = -(X^+_k, [X^-_\theta, [X^+_k, X^-_\theta]])
\] = (X^+_k, [X^+_k, [X^-_\theta, X^-_\theta]]) + (X^+_k, [X^-_\theta, X^-_\theta])
\] = -(\theta, \alpha_k)(X^+_k, X^-_k) = -(\theta, \alpha_k).
\]
Now let $\beta$ be any positive root different from $\theta - \alpha_k$. Then there exist simple roots $\beta_1, \ldots, \beta_\ell$ such that $[X^-_{\beta_1}, \ldots, [X^-_{\beta_1}, X^-_{\theta - \alpha_k}]]$ is a root vector in $\mathfrak{g}_\beta$; let’s denote it by $\tilde{X}_\beta$. (This is true
for $X_\theta$ instead of $X_{\theta-\alpha_k}$ because $\theta$ is the highest root of $g$, but the only simple root $\beta_i$ such that $[X_{\beta_i}^-, X_{\theta}^+] \neq 0$ is $\beta_i = \alpha_k$, so we might as well start with $X_{\theta-\alpha_k}$.) Applying $[X_{\beta_i}^-, \cdot]$ to (5.3.10) and using that $[X_{\beta_i}, X_\theta] = 0$, we obtain in the end (after perhaps rescaling since $X_\beta$ and $\tilde{X}_\beta$ may differ by a non-zero constant) that

$$[K(X_{\theta}^-), Q(X_\beta)] = P([X_{\theta}^- , X_\beta]) + \frac{(\theta, \beta)}{4} S(X_{\theta}^-, X_\beta) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\theta}^-, X_\alpha], [X_{-\alpha}, X_\beta]) \quad (5.3.11)$$

Let’s prove this by induction on $\ell$. We already know that this is true when $\ell = 0$ by (5.3.10), so let’s assume it’s true for $\ell - 1$. Set $\tilde{X}_\beta = [X_{\beta_{\ell-1}}, \ldots, [X_{\beta_1}, X_{\theta-\alpha_k}]]$, so $\tilde{X}_\beta = [X_{\beta_1}, \ldots, \tilde{X}_\beta]$ and

$$[K(X_{\theta}^-), Q(\tilde{X}_\beta)] = P([X_{\theta}^- , \tilde{X}_\beta]) + \frac{(\theta, \tilde{\beta})}{4} S(X_{\theta}^-, \tilde{X}_\beta) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\theta}^-, X_\alpha], [X_{-\alpha}, \tilde{X}_\beta])$$

We want to apply $[X_{\beta_i}^-, \cdot]$ to both sides and the only term which is unclear is the last one. The following computation helps to understand the difference $D$ between $\sum_{\alpha \in \Delta} S([X_{\beta_i}^-, X_\alpha], [X_{-\alpha}, \tilde{X}_\beta])$ and $\sum_{\alpha \in \Delta} S([X_{\theta}^-, X_\alpha], [X_{-\alpha}, \tilde{X}_\beta])$. Using (6.16.2), we obtain that

$$D = (-\beta_\ell, \tilde{\beta}) S([X_{\theta}^-, X_{\beta_i}], \tilde{X}_\beta) + (\beta_\ell, \theta) S(X_{\theta}^-, [\tilde{X}_\beta^-, X_{\beta_i}])$$

$$= - (\beta_\ell, \theta) S(X_{\theta}^-, \tilde{X}_\beta)$$

To complete the induction step, note that $\beta = \tilde{\beta} - \beta_\ell$.

We still have to prove (5.3.11) when $\beta$ is a negative root. Write $X_\beta = [X_{\beta_1}^-, \ldots, [X_{\beta_1}, X_{-\alpha_i}]]$ for some simple roots $\alpha_i, \beta_1, \ldots, \beta_\ell$. This can be done by induction as above, this time starting with relation (5.1.10) with $\beta_1 = -\theta, \beta_2 = -\alpha_i$ and applying successively $[X_{\beta_j}^-, \cdot], j = 1 \ldots, \ell$ to both sides. We have now proved that (5.3.11) holds for any root $\beta$ of $g$ different from $\theta$. Let $\beta_1$ be any long root for $g$. Then there exists $w \in W$ ($W$ being the Weyl group of $g$) such that $\beta_1 = w(-\theta)$.

Let $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ be a decomposition of $w$ into a product of simple reflections. Let’s denote also by $w$ the corresponding operator in the adjoint representation of $g$ for this choice.
of decomposition. Relation (5.1.10) when $\beta_1$ is a long root now follows by applying $w$ to (5.3.11) with $\beta = w^{-1}(\beta_2)$.

Therefore, (5.1.10) is true when $\beta_1$ is a long root. Using (5.3.8), it could also be proved that it is true when $\beta_2$ is a long root. If $g$ is simply laced, this means that it holds for any two roots $\beta_1, \beta_2$ with $\beta_1 \neq -\beta_2$.

It remains to deal with the case when both $\beta_1, \beta_2$ are short roots, with $\beta_1 \neq -\beta_2$.

Apply $[X^+, \cdot]$ to relation (5.3.11) in the case that $\beta$ is any positive root but $\beta \neq \theta$. Then,

$$[K(H_\theta), Q(X_\beta)] = P([H_\theta, X_\beta]) + \frac{(\theta, \beta)}{4} S([H_\theta, X_\alpha], [X_\alpha, X_\beta]) - \frac{(\theta, \beta)}{4} S(H_\theta, X_\beta)$$

$$= P([H_\theta, X_\beta]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([H_\theta, X_\alpha], [X_\alpha, X_\beta])$$

In the case that $\beta$ is a negative root with $\beta \neq -\theta$, we can apply $[\cdot, X_\beta]$ to relation (5.1.10) with $\beta_1 = \theta$, and $\beta_2 = \beta$, the same argument as above shows that for any root $\beta$, such that $\beta \neq \pm \theta$, then

$$[K(H_\theta), Q(X_\beta)] = P([H_\theta, X_\beta]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([H_\theta, X_\alpha], [X_\alpha, X_\beta]).$$

We are in a similar situation as before, use the action of Tits extension $\widetilde{W}$ of Weyl group on $g$, we obtain for any long root $\gamma$, for any root $\beta \neq \pm \gamma$,

$$[K(H_\gamma), Q(X_\beta)] = P([H_\gamma, X_\beta]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([H_\gamma, X_\alpha], [X_\alpha, X_\beta])$$

For any root $\eta \in \Delta$ of $g$, we will take $\eta$ to be a short root later, there exist some long root $\gamma \in \Delta$, such that $(\gamma, \eta) \neq 0$. For $\eta \neq -\beta$, apply $[\cdot, X_\eta]$ to relation (5.3.13) with such $\gamma$,


\[
[X_\beta, X_\eta] = aX_{\beta + \eta}, \text{ where } a \in \mathbb{C}. \text{ Then,}
\]

\[
[[K(H_\gamma), Q(X_\beta)], X_\eta] = (\gamma, \eta)P([X_\eta, X_\beta]) + aP([H_\gamma, X_{\beta+\eta}]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([H_\gamma, X_\alpha], [X_{-\alpha}, X_\beta]), X_\eta
\]

\[
= (\gamma, \eta)P([X_\eta, X_\beta]) + aP([H_\gamma, X_{\beta+\eta}]) + \frac{1}{4} (\gamma, \eta) \sum_{\alpha \in \Delta} S([X_\eta, X_\alpha], [X_{-\alpha}, X_\beta])
\]

\[
+ \frac{1}{4} a \sum_{\alpha \in \Delta} S([H_\gamma, X_\alpha], [X_{-\alpha}, X_{\beta+\eta}]) - \frac{1}{4} (\gamma, \eta)(\beta, \eta)S(X_\eta, X_\beta) \quad \text{by (6.16.2)}
\]

\[(5.3.14)\]

On the other hand, if \(\beta + \eta \neq \pm \gamma\), using (5.3.13), we get:

\[
[[K(H_\gamma), Q(X_\beta)], X_\eta] = (\gamma, \eta)[K(X_\eta), Q(X_\beta)] + a[K(H_\gamma), Q(X_{\beta+\eta})]
\]

\[
= (\gamma, \eta)[K(X_\eta), Q(X_\beta)] + aP([H_\gamma, X_{\beta+\eta}]) + \frac{1}{4} a \sum_{\alpha \in \Delta} S([H_\gamma, X_\alpha], [X_{-\alpha}, X_{\beta+\eta}]) \quad (5.3.15)
\]

Combine (5.3.14) with (5.3.15), if \(\beta + \eta \neq \pm \gamma\), (\(\beta\) and \(\eta\) could be short roots), we get

\[
[K(X_\eta), Q(X_\beta)] = P([X_\eta, X_\beta]) - \frac{(\beta, \eta)}{4} S(X_\eta, X_\beta) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_\eta, X_\alpha], [X_{-\alpha}, X_\beta]) \quad (5.3.16)
\]

Note that the root \(\gamma\) we are choosing is not unique, we can vary the root \(\gamma\) as long as \(\gamma\) being a long root and \((\eta, \gamma) \neq 0\), which implies (5.3.16) holds for all roots \(\eta\), and \(\beta\), with \(\eta \neq -\beta\).

Therefore, the relation (5.1.10) holds in \(D(g)\).

### 5.4 Some properties of the deformed double current algebras

**Corollary 5.4.1.** For any roots \(\beta_1, \beta_2\), the following relations hold in the deformed double current algebra \(D(g)\):

\[
[K(H_{\beta_1}), Q(X_{\beta_2})] = P([H_{\beta_1}, X_{\beta_2}]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\beta_1}, X_\alpha], [X_{-\alpha}, X_{\beta_2}]),
\]

\[(5.4.1)\]
and
\[ [K(X_{\beta_1}), Q(H_{\beta_2})] = P([X_{\beta_1}, H_{\beta_2}]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, H_{\beta_2}]). \]

**Proof.** In the case $\beta_1 \neq -\beta_2$, we can assume $[X_{\beta_2}, X_{\beta_1}] = aX_{\beta_2-\beta_1}$, for some $a \in \mathbb{C}$, when $\beta_2 - \beta_1$ is not a root, the number $a = 0$. Now applying $[\cdot, X_{\beta_1}]$ to relation (5.1.10), we then get

\[
[K(H_{\beta_1}), Q(X_{\beta_2})] + [K(X_{\beta_1}), aQ(X_{\beta_2-\beta_1})] = P([H_{\beta_1}, X_{\beta_2}]) + P([X_{\beta_1}, aX_{\beta_2-\beta_1}])
\]

\[- \frac{(\beta_1, \beta_2)}{4} S(H_{\beta_1}, X_{\beta_2}) - \frac{(\beta_1, \beta_2)}{4} S(X_{\beta_1}, aX_{\beta_2-\beta_1}) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2-\beta_1}])
\]

\[= P([H_{\beta_1}, X_{\beta_2}]) + P([X_{\beta_1}, aX_{\beta_2-\beta_1}]) - \frac{(\beta_1, \beta_2)}{4} S(H_{\beta_1}, X_{\beta_2}) - a \frac{(\beta_1, \beta_2)}{4} S(X_{\beta_1}, X_{\beta_2-\beta_1})
\]

\[
\frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2-\beta_1}]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, aX_{\beta_2-\beta_1}])
\]

\[- \frac{(\beta_1, \beta_2)}{4} S(H_{\beta_1}, X_{\beta_2}) + a \frac{(\beta_1, \beta_1)}{4} S(X_{\beta_1}, X_{\beta_2-\beta_1})
\]

\[= P([H_{\beta_1}, X_{\beta_2}]) + P([X_{\beta_1}, aX_{\beta_2-\beta_1}]) - a \frac{(\beta_1, \beta_2 - \beta_1)}{4} S(X_{\beta_1}, X_{\beta_2-\beta_1})
\]

\[
\frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2-\beta_1}]) + a \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2-\beta_1}])
\]

Relation (5.1.10) gives the following:

\[[K(X_{\beta_1}), Q(X_{\beta_2-\beta_1})]
\]

\[= P([X_{\beta_1}, X_{\beta_2-\beta_1}]) - \frac{(\beta_1, \beta_2 - \beta_1)}{4} S(X_{\beta_1}, X_{\beta_2-\beta_1}) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2-\beta_1}])
\]

Combining the above calculations, for $\beta_1 \neq -\beta_2$, we obtain:

\[[K(H_{\beta_1}), Q(X_{\beta_2})] = P([H_{\beta_1}, X_{\beta_2}]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}]),
\]

The first relation in the corollary follows from linearity of the factor $H_{\beta_1}$. Similar argument implies the second relation in the Corollary. □

**Corollary 5.4.2.** For any roots $\beta_1, \beta_2$, such that $(\beta_1, \beta_2) = 0$, the following relations hold
in the deformed double current algebra $D(\mathfrak{g})$:

$$[K(H_{\beta_1}), Q(H_{\beta_2})] = \frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\beta_1}, X_{\alpha}], [X_{-\alpha}, H_{\beta_2}])$$

$$= \frac{1}{2} \sum_{\alpha \in \Delta^+} (\beta_1, \alpha)(\beta_2, \alpha)S(X_{\alpha}, X_{-\alpha}),$$

where $S(X_{\alpha}, X_{-\alpha}) = X_{\alpha}X_{-\alpha} + X_{-\alpha}X_{\alpha}$ is called the truncated Casimir operator corresponding to root $\alpha$.

Proof. Applying $[\cdot, X_{-\beta_2}]$ to the relation (5.4.1), we have,

$$[K(H_{\beta_1}), Q(H_{\beta_2})] + (\beta_1, -\beta_2)K(X_{-\beta_2}), Q(X_{\beta_2})$$

$$= (\beta_1, -\beta_2)P([X_{-\beta_2}, X_{\beta_2}]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}], X_{-\beta_2})$$

$$= \frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\beta_1}, X_{\alpha}], [X_{-\alpha}, H_{\beta_2}])$$

by (6.16.2). The second equality follows from the fact that $[H_{\beta}, X_{\alpha}] = (\beta, \alpha)X_{\alpha}$. \hfill $\square$

**Proposition 5.4.1.** We have an isomorphism $D(\mathfrak{g})_{\lambda=0} \cong U(\hat{\mathfrak{g}}[u, v])$. In particular, (add some conditions of $\mathfrak{g}$) the universal central extension of $\mathfrak{g}[u, v]$ is generated by elements $X, K(X), Q(X), P(X), X \in \mathfrak{g}$, such that

- $\{K(X), X \mid X \in \mathfrak{g}\}$ generate a subalgebra which is an image of $\mathfrak{g} \otimes \mathbb{C} \mathbb{C}[u]$ with $X \otimes u \mapsto K(X)$;

- $\{Q(X), X \mid X \in \mathfrak{g}\}$ generate a subalgebra which is an image of $\mathfrak{g} \otimes \mathbb{C} \mathbb{C}[v]$ with $X \otimes v \mapsto Q(X)$;

- $P(X)$ is linear in $X$, and for any $X, X' \in \mathfrak{g}$, $[P(X), X'] = P[X, X']$,

and the following relation holds for all root vectors $X_{\beta_1}, X_{\beta_2} \in \mathfrak{g}$ with $\beta_1 \neq -\beta_2$:

$$[K(X_{\beta_1}), Q(X_{\beta_2})] = P([X_{\beta_1}, X_{\beta_2}]).$$
Proof. It is obvious that the algebra $D_{\lambda=0}(g)$ is isomorphic to the universal central extension $\Omega(\hat{g}[u,v])$ of $g[u,v]$ by the presentations of the two algebras. The assertion follows from Theorem 5.1.3. \qed

**Corollary 5.4.3** (See [34], Proposition 12.1). The following relation holds in $D(g)$:

$$[K(X_{\beta_1}), Q(X_{\beta_2})] + [Q(X_{\beta_1}), K(X_{\beta_2})] = 2P([X_{\beta_1}, X_{\beta_2}]),$$

if $\beta_1 \neq -\beta_2$.

Proof. The conclusion follows directly from relation (5.1.10), using the property that the $S(a_1, a_2) = S(a_2, a_1)$. \qed

**Proposition 5.4.2** ([91]). There is a well defined automorphism of $D(g)$ of order 4, which is defined by $z \mapsto z$, $K(z) \mapsto -Q(z)$, $Q(z) \mapsto K(z)$, and $P(z) \mapsto -P(z)$, for $z \in g$.

More generally, see [91], there is a right action of $SL_2(C)$ on $D(g)$. For an element $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL_2(C)$, the action of $A$ is defined by $z \mapsto z$, for any $z \in g$ and

$$K(z) \mapsto a_{11}K(z) + a_{12}Q(z), \quad Q(z) \mapsto a_{22}Q(z) + a_{21}K(z),$$

$$P(z) \mapsto (a_{11}a_{22} + a_{12}a_{21})P(z) + a_{11}a_{21}[K(y), K(w)] + a_{12}a_{22}[Q(y), Q(w)], \text{ where } z = [y, w].$$

In particular,

Proof. We first check that under the action of $A$ preserves the defining relations of $D(g)$. We only check it preserves relation (5.1.10), other relations are obvious. We set

$$C(X_{\beta_1}, X_{\beta_2}) := -\frac{1}{4}(\beta_1, \beta_2)S(X_{\beta_1}, X_{\beta_2}) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}]),$$

The property $S(a_1, a_2) = S(a_2, a_1)$ implies that $C(X_{\beta_1}, X_{\beta_2}) = C(X_{\beta_2}, X_{\beta_1})$. The relation (5.1.10) can be rewritten as $[K(x), Q(y)] = P([x, y]) + C(x, y)$.

---

3 The proof is copied from [91].
Now under the action of $A \in \text{SL}_2(\mathbb{C})$,

\[ [K(x), Q(y)] \mapsto [a_{11}K(x) + a_{12}Q(x), a_{22}Q(y) + a_{21}K(y)] \]

\[
= a_{11}a_{22}[K(x), Q(y)] + a_{12}a_{21}[Q(x), K(y)] + a_{11}a_{21}[K(x), K(y)] + a_{12}a_{22}[Q(x), Q(y)] \\
= a_{11}a_{22}(P([x, y]) + C(x, y)) - a_{12}a_{21}(P([y, x]) + C(y, x)) \\
+ a_{11}a_{21}[K(x), K(y)] + a_{12}a_{22}[Q(x), Q(y)] \\
= (a_{11}a_{22} + a_{12}a_{21})P([x, y]) + a_{11}a_{21}[K(x), K(y)] + a_{12}a_{22}[Q(x), Q(y)] + C(x, y).
\]

and by definition

\[
P([x, y]) + C(x, y) \mapsto \quad (a_{11}a_{22} + a_{12}a_{21})P([x, y]) \\
+ a_{11}a_{21}[K(x), K(y)] + a_{12}a_{22}[Q(x), Q(y)] + C(x, y).
\]

Thus, the action of $A$ preserves the relation (5.1.10).

We then check it defines a right action of $\text{SL}_2$. It’s obvious that $z \cdot \text{Id} = z$, for any $z \in \mathfrak{D}(\mathfrak{g})$, where $\text{Id}$ is the $2 \times 2$–identity matrix.

For any $A, B \in \text{SL}_2(\mathbb{C})$, and for any $z \in \mathfrak{U}(\mathfrak{g})$, it’s a direct calculation to show $(zA)B = z(AB)$. We show the less obvious case when $z = P([x, y])$ as follows.

Set $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, one can check that

\[ (P([x, y]).A).B = P([x, y]).(AB). \]

\[ \square \]

### 5.5 A central element of $\mathfrak{D}(\mathfrak{g})$

We introduce some elements in $\mathfrak{D}(\mathfrak{g})$. Set

\[ C(\beta_1, \beta_2) := [K(H_{\beta_1}), Q(H_{\beta_2})] - \frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\beta_1}, X_\alpha], [X_-\alpha, H_{\beta_2}]), \]
we denote $C(\beta) := C(\beta, \beta)$, and set

$$B(\beta) := [K(X_{\beta}), Q(X_{-\beta})] - P(H_{\beta}) - \frac{(\beta, \beta)}{4} S(X_{\beta}, X_{-\beta}) - \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta}, X_{\alpha}], [X_{-\alpha}, X_{-\beta}]) \in \mathfrak{D}(g).$$

**Proposition 5.5.1.** (i) For any simple Lie algebra $g$, the following relations holds in $\mathfrak{D}(g)$,

$$C(\beta_1, \beta_2) = (\beta_1, \beta_2) B(\beta_2).$$

In particular, when $\beta_1 = \beta_2 = \beta$, the above relation gives $\frac{C(\beta)}{(\beta, \beta)} = B(\beta)$.

(ii) For any two roots $\beta_1, \beta_2$ in $\Delta$, we have

$$\frac{C(\beta_1)}{(\beta_1, \beta_1)} = \frac{C(\beta_2)}{(\beta_2, \beta_2)}.$$

**Proof.** To show (i), applying $[\cdot, X_{\beta_2}^-]$ to relation (5.4.1), we then get:

$$[K(H_{\beta_1}), Q(H_{\beta_2})] + (\beta_1, -\beta_2)[K(X_{\beta_2}^-), Q(X_{\beta_2})] = P([H_{\beta_1}, H_{\beta_2}]) + (\beta_1, -\beta_2) P([X_{\beta_2}^-, X_{\beta_2}]) + [\frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}], X_{\beta_2}^-)]$$

$$= \frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\beta_1}, X_{\alpha}], [X_{-\alpha}, H_{\beta_2}]) + (\beta_1, \beta_2) P(H_{\beta_2})$$

$$+ \frac{1}{4} (\beta_1, -\beta_2) \sum_{\alpha \in \Delta} S([X_{\beta_2}^-, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}]) - (\beta_2, \beta_2)(\beta_1, \beta_2) S(X_{\beta_2}^-, X_{\beta_2}).$$

The claim (i) follows from rewriting the above equality.

To show (ii), using (i), it suffices to show $B(\beta_1) = B(\beta_2)$, for any two roots $\beta_1, \beta_2$. Assume $\beta_1 + \beta_2$ is a root, and $[X_{\beta_2}, X_{-\beta_1-\beta_2}] = aX_{-\beta_1}$, for some scalar $a \in \mathbb{C}$. Then $[X_{\beta_1}, X_{-\beta_1-\beta_2}] = -aX_{-\beta_1}$, since

$$([X_{\beta_1}, X_{-\beta_1-\beta_2}], X_{\beta_2}) = (X_{\beta_1}, [X_{-\beta_1-\beta_2}, X_{\beta_2}])$$

$$= - a (X_{\beta_1}, X_{-\beta_1}) = -a.$$
Applying $[,] , X - \beta_1 - \beta_2 ]$ to relation (5.1.10), we obtain,

\[
[K(X_{\beta_1}), Q(X_{-\beta_1})] - [K(X_{-\beta_2}), Q(X_{\beta_2})]
= P([X_{\beta_1}, X_{-\beta_1}]) - P([X_{-\beta_2}, X_{\beta_2}]) - \frac{(\beta_1, \beta_2)}{4} S(X_{\beta_1}, X_{-\beta_1}) - \frac{(\beta_1, \beta_2)}{4} S(X_{-\beta_2}, X_{\beta_2})
+ \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}], X_{-\beta_2})
+ \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{-\beta_2}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}], X_{-\beta_2}).
\]

Rewriting the above equality, we conclude that if $\beta_1 + \beta_2$ is a root, then $B(\beta_1) = B(-\beta_2)$.

Note by definition $C(\beta) = C(-\beta)$, the above implies

\[
\frac{C(\beta_1)}{(\beta_1, \beta_1)} = \frac{C(\beta_2)}{(\beta_2, \beta_2)}, \text{ if } \beta_1 + \beta_2 \text{ is a root. (\text{*})}
\]

It’s straightforward to check that when the root system $\Delta$ has rank 2, the condition (\text{*}) implies $\frac{C(\beta_1)}{(\beta_1, \beta_1)} = \frac{C(\beta_2)}{(\beta_2, \beta_2)}$, for any two roots $\beta_1, \beta_2 \in \Delta$. In general, we use induction on the rank of the root system. For any root system $\Delta$ of rank $n > 2$ of a simple Lie algebra. There exists a decomposition $\Delta = \Delta_1 \cup \Delta_2$, such that $\Delta_1 \cap \Delta_2 \neq \emptyset$, and $\Delta_i$ is a root system of some simple Lie algebra, for $i = 1, 2$. Therefore, the condition (\text{*}) implies the assertion (ii). \qed

\textbf{Theorem 5.5.1.} Some conditions on $g$, for any root $\beta$ of $g$, the element

\[
C(\beta) = [K(H_\beta), Q(H_\beta)] - \frac{1}{4} \sum_{\alpha \in \Delta} S([H_\beta, X_\alpha], [X_{-\alpha}, H_\beta])
\]

is a central element of the algebra $\mathfrak{D}(g)$.

\textbf{Proof.} It suffices to show $C(\beta)$ commutes with the generators of $\mathfrak{D}(g)$.

\textbf{Lemma 5.5.2.} For any $X \in g$, we have $[C(\beta), X] = 0$.

\textbf{Proof.} It suffices to take $X = X_\gamma \in g_\gamma \subset g$, for a root $\gamma$. We then have,
\[ \left[ [K(H_\beta), Q(H_\beta)], X_\gamma \right] = (\beta, \gamma)[K(X_\gamma), Q(H_\beta)] + \left[ [K(H_\beta), Q(H_\beta)], X_\gamma \right] \]
\[ = (\beta, \gamma)P([X_\gamma, H_\beta]) + \left( \frac{\beta, \gamma}{4} \right) \sum_{\alpha \in \Delta} S([X_\gamma, X_\alpha], [X_{-\alpha}, H_\beta]) + (\beta, \gamma)P([H_\beta, X_\gamma]) \]
\[ + \left( \frac{\beta, \gamma}{4} \right) \sum_{\alpha \in \Delta} S([H_\beta, X_\alpha], [X_{-\alpha}, X_\gamma]) \]
\[ = \frac{1}{4} \sum_{\alpha \in \Delta} S([H_\beta, X_\gamma], [X_{-\alpha}, H_\beta]) + \frac{1}{4} \sum_{\alpha \in \Delta} S([H_\beta, X_\alpha], [X_{-\alpha}, [H_\beta, X_\gamma]]) \]
\[ = \left[ \frac{1}{4} \sum_{\alpha \in \Delta} S([H_\beta, X_\alpha], [X_{-\alpha}, H_\beta]), X_\gamma \right] \text{ by Corollary 5.4.2} \]

Thus, the conclusion follows. \( \Box \)

**Lemma 5.5.3.** Assume there exist two roots \( \beta, \gamma \) in \( \Delta \), such that \( (\beta, \gamma) = 0 \), then \( [C(\beta), K(H_\gamma)] = 0 \).

The above condition holds for \( g = \mathfrak{sl}_n \), for \( n \geq 4 \), and any non-type \( A \) simple Lie algebra \( g \).

**Proof.** From the defining relations of \( D(g) \), the elements \( \{ K(X), X \mid X \in g \} \) generate the subalgebra \( g \otimes \mathbb{C}[u] \) of \( D(g) \), with \( X \otimes u = K(X) \). Therefore, we have \( [K(H_\beta), K(H_\gamma)] = 0 \) in \( D(g) \). As a result,

\[ [K(H_\gamma), [K(H_\beta), Q(H_\beta)]] = [K(H_\beta), [K(H_\gamma), Q(H_\beta)]] \]
\[ = [K(H_\beta), \frac{1}{4} \sum_{\alpha \in \Delta} (\gamma, \alpha)(\beta, \alpha)S(X_\alpha, X_{-\alpha})] \text{ by Corollary 5.4.2} \]
\[ = \frac{1}{4} \sum_{\alpha \in \Delta} (\gamma, \alpha)(\beta, \alpha)S([K(H_\beta), X_\alpha], X_{-\alpha}) + \frac{1}{4} \sum_{\alpha \in \Delta} (\gamma, \alpha)(\beta, \alpha)S(X_\alpha, [K(H_\beta), X_{-\alpha}]) \]
\[ = \frac{1}{4} \sum_{\alpha \in \Delta} (\gamma, \alpha)(\beta, \alpha)^2S(K(X_\alpha), X_{-\alpha}) - \frac{1}{4} \sum_{\alpha \in \Delta} (\gamma, \alpha)(\beta, \alpha)^2S(X_\alpha, K(X_{-\alpha})) \]
\[ = \frac{1}{2} \sum_{\alpha \in \Delta} (\gamma, \alpha)(\beta, \alpha)^2S(K(X_\alpha), X_{-\alpha}) \]
On the other hand,

\[
[K(H_\gamma), \frac{1}{4} \sum_{\alpha \in \Delta} S([H_\beta, X_\alpha], [X_{-\alpha}, H_\beta])] = [K(H_\gamma), \frac{1}{4} \sum_{\alpha \in \Delta} (\beta, \alpha)^2 S(X_\alpha, X_{-\alpha})]
\]

\[
= \frac{1}{4} \sum_{\alpha \in \Delta} (\beta, \alpha)^2 S([K(H_\gamma), X_\alpha], X_{-\alpha}) + \frac{1}{4} \sum_{\alpha \in \Delta} (\beta, \alpha)^2 S(X_\alpha, [K(H_\gamma), X_{-\alpha}])
\]

\[
= \frac{1}{4} \sum_{\alpha \in \Delta} (\gamma, \alpha)(\beta, \alpha)^2 S(K(X_\alpha), X_{-\alpha}) + \frac{1}{4} \sum_{\alpha \in \Delta} (\gamma, -\alpha)(\beta, \alpha)^2 S(X_\alpha, K(X_{-\alpha}))
\]

\[
= \frac{1}{2} \sum_{\alpha \in \Delta} (\gamma, \alpha)(\beta, \alpha)^2 S(K(X_\alpha), X_{-\alpha})
\]

Combining the above computations, the assertion follows. \[\square\]

As a corollary, under the assumption of Lemma 5.5.3, we have

\[\left[ C(\beta), K(X) \right] = 0, \text{ for any } X \in \mathfrak{g}.\]

Since by Lemma 5.5.2, \(C(\beta)\) commutes with \(\mathfrak{g}\), and the elements \(K(X)\) satisfies the relation \([K(X_1), X_2] = K[X_1, X_2]\), for any elements \(X_1, X_2 \in \mathfrak{g}\).

Similar argument shows

\[\left[ C(\beta), Q(X) \right] = 0, \text{ for any } X \in \mathfrak{g}.\]

Note that \(\mathfrak{D}(\mathfrak{g})\) can be generated by \(X, K(X), Q(X)\), for \(X \in \mathfrak{g}\), thus, \(C(\beta)\) is central in \(\mathfrak{D}(\mathfrak{g})\). \[\square\]

**Remark 5.5.4.** Consider a quotient algebra of \(\mathfrak{D}(\mathfrak{g})\), by modulo the central element \(C(\beta)\). Then, by Proposition 5.5.1, the presentation of the quotient algebra is the same as \(\mathfrak{D}(\mathfrak{g})\) with the following modified relations:

\[\left[ K(X_{\beta_1}), Q(X_{\beta_2}) \right] = P([X_{\beta_1}, X_{\beta_2}]) - \frac{(\beta_1, \beta_2)}{4} S(X_{\beta_1}, X_{\beta_2}) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_\alpha], [X_{-\alpha}, X_{\beta_2}]),\]

for any roots \(\beta_1, \beta_2\), without the assumption that \(\beta_1 \neq -\beta_2\).
For any $\beta_1, \beta_2$, in the quotient algebra, one has

$$[K(H_{\beta_1}), Q(H_{\beta_2})] = \frac{1}{2} \sum_{\alpha \in \Delta^+} (\beta_1, \alpha)(\beta_2, \alpha) \kappa_\alpha,$$

where $\kappa_\alpha := S(X_\alpha, X_{-\alpha})$ is the truncated Casimir operator.

### 5.6 Deformed double current algebras with two parameters

The following definition of deformed double current algebras in type $A$ with two parameters are introduced in [34] Definition 12.1. We first fix the notations. Let the Lie algebra $g = sl_n$, and $\{\epsilon_1, \ldots, \epsilon_n\}$ be the standard orthogonal basis of $\mathbb{C}^n$. The set of roots of $sl_n$ is denoted by $\Delta = \{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq n\}$, with a choice of positive roots $\Delta^+ = \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq n\}$. The longest positive root $\theta$ equals $\epsilon_1 - \epsilon_n$. The elementary matrices will be written as $E_{ij} \in sl_n$. So $X_i^+ = E_{i,i+1}$, $X_i^- = E_{i,i-1}$ and $H_i = E_{ii} - E_{i+1,i+1}$ for $2 \leq i \leq n - 1$. We set $E_\theta = E_{1n}.$

**Definition 5.6.1.** Let $\lambda, \beta \in \mathbb{C}$, and $n \geq 4$. We define $\mathfrak{D}_{\lambda, \beta}(sl_n)$ to be the algebra generated by elements

$$z, K(z), Q(z), P(z), z \in sl_n,$$

which satisfy the following relations: the elements $z_1, K(z_2), \forall z_1, z_2 \in sl_n$, satisfy the relations for $\mathfrak{U}(sl_n[u])$ so that we have a map $\mathfrak{U}(sl_n[u]) \rightarrow \mathfrak{D}_{\lambda, \beta}(sl_n)$ given by $z \otimes u \mapsto K(z)$. Similarly, $z_1, Q(z_2)$ satisfy the relations of $\mathfrak{U}(sl_n[v])$, so that we have a map $\mathfrak{U}(sl_n[v]) \rightarrow \mathfrak{D}_{\lambda, \beta}(sl_n)$ given by $z \otimes v \mapsto Q(z)$.

The elements $z_1, P(z_2)$ satisfies the relations of the Yangians of finite type $A_{n-1}$. For
\[ a \neq b, c \neq d, \text{ and } (a, b) \neq (d, c), \text{ the following relations hold:} \]

\[
[K(E_{ab}), Q(E_{cd})] = P([E_{ab}, E_{cd}]) + (\beta - \frac{\lambda}{2})(\delta_{bc}E_{ad} + \delta_{ad}E_{cb}) - \frac{\lambda}{4}(\epsilon_a - \epsilon_b, \epsilon_c - \epsilon_d)S(E_{ab}, E_{cd}) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{ab}, E_{ij}], [E_{ji}, E_{cd}]).
\]

When \( \beta = \frac{\lambda}{2} \), the relation (5.6.1) coincides with the relation (5.1.10) of the deformed double current algebra \( \mathfrak{D}_\lambda(\mathfrak{sl}_n) \) in Definition 5.1.2.

In the above definition, the elements \( z_1, P(z_2) \) satisfies the relations of the Yangians \( Y(\mathfrak{sl}_n) \). While, in Definition 5.1.2 for a general simple Lie algebra \( \mathfrak{g} \neq \mathfrak{sl}_n \), it's not known that the elements \( z_1, P(z_2) \) satisfies the relations of the Yangians \( Y(\mathfrak{g}) \).

We first list some relations of \( \mathfrak{D}_{\lambda, \beta}(\mathfrak{sl}_n) \), which are parallel to Corollary 5.4.1. For the convenience of the reader, we include the proof.

**Corollary 5.6.2.** For any \( a \neq b, \text{ and } c \neq d \), the following relations hold in the algebra \( \mathfrak{D}_{\lambda, \beta}(\mathfrak{sl}_n) \):

\[
[K(E_{ab}), Q(H_{cd})] = P([E_{ab}, H_{cd}]) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{ab}, E_{ij}], [E_{ji}, H_{cd}]) + (\beta - \frac{\lambda}{2})(\epsilon_a + \epsilon_b, \epsilon_c - \epsilon_d)E_{ab}.
\]

and

\[
[K(H_{ab}), Q(E_{cd})] = P([H_{ab}, E_{cd}]) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([H_{ab}, E_{ij}], [E_{ji}, E_{cd}]) + (\beta - \frac{\lambda}{2})(\epsilon_a - \epsilon_b, \epsilon_c + \epsilon_d)E_{cd}.
\]

In particular, \([K(E_{ab}), Q(H_{ab})] = -2P(E_{ab}) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{ab}, E_{ij}], [E_{ji}, H_{ab}]).\)
Proof. Assume that \((a, b) \neq (d, c)\), acting \([\cdot, E_{dc}]\) to relation (5.6.1), we then get

\[
[K(E_{ab}), Q(H_{cd})] + [K([E_{ab}, E_{dc}]), Q(E_{cd})]
\]
\[
= P([E_{ab}, H_{cd}]) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{ab}, E_{ij}], [E_{ji}, H_{cd}]) + P([E_{ab}, E_{dc}], E_{cd})
\]
\[
+ \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{ab}, E_{dc}], E_{ij}, [E_{ji}, E_{cd}])
\]
\[
- \frac{\lambda}{4} (\epsilon_a - \epsilon_b + \epsilon_d - \epsilon_c, \epsilon_c - \epsilon_d) S([E_{ab}, E_{dc}], E_{cd})
\]
\[
+ (\beta - \frac{\lambda}{2})(\delta_{bc} - \delta_{ad}) E_{ab},
\]

where the term involving \(\beta - \frac{\lambda}{2}\) is obtained by:

\[
[\delta_{bc}E_{ad} + \delta_{ad}E_{cb}, E_{dc}] = \delta_{bc}(E_{ac} - \delta_{ca}E_{dd}) + \delta_{ad}(\delta_{bd}E_{cc} - E_{db})
\]
\[
= \delta_{bc}E_{ac} - \delta_{ad}E_{db} = (\delta_{bc} - \delta_{ad})E_{ab}.
\]

On the other hand, under the assumption of \((a, b) \neq (d, c)\), the relation (5.6.1) gives

\[
[K([E_{ab}, E_{dc}]), Q(E_{cd})]
\]
\[
= P([E_{ab}, E_{dc}], E_{cd}) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{ab}, E_{dc}], E_{ij}, [E_{ji}, E_{cd}])
\]
\[
- \frac{\lambda}{4} (\epsilon_a - \epsilon_b + \epsilon_d - \epsilon_c, \epsilon_c - \epsilon_d) S([E_{ab}, E_{dc}], E_{cd}) + (\beta - \frac{\lambda}{2})(\delta_{bd} - \delta_{ac}) E_{ab},
\]

where the term involving \((\beta - \frac{\lambda}{2}\)) is obtained by the following:

Writing \([E_{ab}, E_{dc}] = \delta_{bd}E_{ac} - \delta_{ac}E_{db}\), note that under the assumption \((a, b) \neq (d, c)\), the two numbers \(\delta_{bd}, \delta_{ac}\) can not both be 1.

\[
\delta_{bd}(E_{ad} + \delta_{ad}E_{cc}) - \delta_{ac}(\delta_{bc}E_{dd} + E_{cb})
\]
\[
= \delta_{bd}E_{ad} - \delta_{ac}E_{cb} = (\delta_{bd} - \delta_{ac})E_{ab}
\]
gives the term \((\beta - \frac{\lambda}{2})(\delta_{bd} - \delta_{ac})E_{ab}\).
Combining the above calculations, under the assumption of \((a, b) \neq (d, c)\), we obtain

\[
[K(E_{ab}), Q(H_{cd})] = P([E_{ab}, H_{cd}]) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{ab}, E_{ij}], [E_{ji}, H_{cd}])
\]

\[
+ (\beta - \frac{\lambda}{2})(\delta_{bc} - \delta_{ad} - \delta_{bd} + \delta_{ac}) E_{ab}.
\]

The first relation follows from the linearity of the factor \(H_{cd}\).

The proof of the second relation is similar. Applying \([\cdot, E_{ba}]\) to \([K(E_{ab}), Q(E_{cd})]\), then we get a relation of \([K(H_{ab}), Q(E_{cd})] + [K(E_{ab}), Q([E_{cd}, E_{ba}])],\) where the coefficient of \((\beta - \frac{\lambda}{2})\) is giving by:

\[
[\delta_{bc}E_{ad} + \delta_{ad}E_{cb}, E_{ba}] = \delta_{bc}(\delta_{db}E_{aa} - E_{bd}) + \delta_{ad}(E_{ca} - \delta_{ac}E_{bb})
\]

\[
= - \delta_{bc}E_{bd} + \delta_{ad}E_{ca} = (\delta_{ad} - \delta_{bc}) E_{cd}.
\]

While in \([K(E_{ab}), Q([E_{cd}, E_{ba}])] = [K(E_{ab}), \delta_{db}Q(E_{ca}) - \delta_{ac}Q(E_{bd})],\) the coefficient of \((\beta - \frac{\lambda}{2})\) is giving by:

\[
\delta_{db}(\delta_{bc}E_{aa} + E_{cb}) - \delta_{ac}(E_{ad} + \delta_{ad}E_{bb})
\]

\[
= \delta_{db}E_{cb} - \delta_{ac}E_{ad} = (\delta_{db} - \delta_{ac}) E_{cd}.
\]

Thus, combining the above computations, the term of \((\beta - \frac{\lambda}{2})\) in \([K(H_{ab}), Q(E_{cd})]\) is

\[
(\beta - \frac{\lambda}{2})(\delta_{ad} - \delta_{bc} - \delta_{db} + \delta_{ac}) E_{cd} = (\beta - \frac{\lambda}{2})(\epsilon_a - \epsilon_b, \epsilon_c + \epsilon_d) E_{cd},
\]

which gives the second relation in the Corollary.

In the remainder of this section, we construct a central element of \(D_{\lambda, \beta}(\mathfrak{sl}_n)\). Set

\[
Z_{ab,cd} := [K(H_{ab}), Q(H_{cd})] - \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([H_{ab}, E_{ij}], [E_{ji}, H_{cd}]).
\]

and denote \(Z_{ab,ab}\) by \(Z_{ab}\) for short. Set

\[
W_{ab} := [K(E_{ab}), Q(E_{ba})] - P(H_{ab}) - \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{ab}, E_{ij}], [E_{ji}, E_{ba}]) - \frac{\lambda}{2} S(E_{ab}, E_{ba})
\]

The following Proposition is parallel to Proposition 5.5.1, we include the proof for the convenience of the reader.
Proposition 5.6.1. (i) For any $1 \leq a \neq b \leq n$, and $1 \leq c \neq b \leq n$, the following relations hold in $\mathfrak{d}(\mathfrak{sl}_n)$, $n \geq 4$.

\[
Z_{ab,cd} = -(\epsilon_c - \epsilon_d, \epsilon_b - \epsilon_a)W_{ab} + (\beta - \frac{\lambda}{2})(\epsilon_a + \epsilon_b, \epsilon_c - \epsilon_d)H_{ab}.
\]

In particular, we have $Z_{ab} = 2W_{ab}$, and when $a, b, c, d$ are distinct, the element $Z_{ab,cd} = 0$.

(ii) For $1 \leq a \neq b \leq n$, and $1 \leq c \neq d \leq n$, in $\mathfrak{d}(\mathfrak{sl}_n)$, $n \geq 4$, we have:

\[
W_{ab} - W_{cd} = (\beta - \frac{\lambda}{2})(H_{ac} + H_{bd}).
\]

(iii) For $1 \leq a \neq b \leq n$, and $1 \leq c \neq d \leq n$, in $\mathfrak{d}(\mathfrak{sl}_n)$, $n \geq 4$, we have:

\[
Z_{ab} - Z_{cd} = 2(\beta - \frac{\lambda}{2})(H_{ac} + H_{bd}).
\]

Proof. The relation in (i) is obtained by applying $[\cdot, E_{ba}]$ to the first relation in Corollary 5.6.2.

The relation in (iii) is obtained by (ii) and the fact $Z_{ab} = 2W_{ab}$ in (i).

It remains to show (ii). Let’s start with (5.6.1) and apply $[\cdot, E_{ca}]$ to $[K(E_{ab}), Q(E_{bc})]$ to obtain:

\[
[K(E_{ab}), Q(E_{ba})] - [K(E_{cb}), Q(E_{bc})]
= P(H_{ac}) + (\beta - \frac{\lambda}{2})H_{ac} + \frac{\lambda}{2}S(E_{ab}, E_{ba}) - \frac{\lambda}{2}S(E_{cb}, E_{bc})
- \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{cb}, E_{ij}], [E_{ji}, E_{bc}]) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{ab}, E_{ij}], [E_{ji}, E_{ba}])
\]

Starting with (5.6.1) and applying $[\cdot, E_{bd}]$ to $[K(E_{cb}), Q(E_{dc})]$, we obtain:

\[
[K(E_{cd}), Q(E_{dc})] - [K(E_{cb}), Q(E_{bc})]
= P([H_{bd}]) - (\beta - \frac{\lambda}{2})H_{bd} + \frac{\lambda}{2}S(E_{cd}, E_{de}) - \frac{\lambda}{2}S(E_{cb}, E_{bc})
+ \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{cd}, E_{ij}], [E_{ij}, E_{dc}]) - \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{cb}, E_{ij}], [E_{ji}, E_{bc}]).
\]
Therefore,

\[ [K(E_{ab}), Q(E_{ba})] - [K(E_{cd}), Q(E_{dc})] = P(H_{ab}) - P(H_{cd}) + (\beta - \frac{\lambda}{2})H_{ac} + \frac{\lambda}{2}S(E_{ab}, E_{ba}) \]

\[ + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{ab}, E_{ij}], [E_{ji}, E_{ba}]) + (\beta - \frac{\lambda}{2})H_{bd} \]

\[ - \frac{\lambda}{2}S(E_{cd}, E_{dc}) - \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{cd}, E_{ij}], [E_{ji}, E_{dc}]) \]

The conclusion in (ii) follows from rewriting the above equality.

Theorem 5.6.3. Set

\[ Z := \sum_{a=1}^{n} Z_{a,a+1} = \sum_{a=1}^{n} \left( [K(H_a), Q(H_a)] - \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([H_a, E_{ij}], [E_{ji}, H_a]) \right) \]

where \( (a, a + 1) = (n, 1) \), when \( a = n \). The element \( Z \) is central in \( \mathfrak{D}_{\lambda, \beta}(\mathfrak{sl}_n) \).

Proof. We first show the element \( Z \) commutes with elements \( E_{cd} \) in \( \mathfrak{sl}_n \). We have

\[ [Z_{ab}, E_{cd}] = 2(\beta - \frac{\lambda}{2})(\epsilon_a - \epsilon_b, \epsilon_c - \epsilon_d)(\epsilon_c + \epsilon_d, \epsilon_a - \epsilon_b)E_{cd} \]

Set \( \epsilon_{ab} := \epsilon_a - \epsilon_b \), and \( \overline{\epsilon_{ab}} := \epsilon_a + \epsilon_b \), we have

\[ [[K(H_{ab}), Q(H_{ab})], E_{cd}] \]

\[ = (\epsilon_{ab}, \epsilon_{cd})[K(E_{cd}), Q(H_{ab})] + (\epsilon_{ab}, \epsilon_{cd})[K(H_{ab}), Q(E_{cd})] \]

\[ = (\epsilon_{ab}, \epsilon_{cd}) \left( P([E_{cd}, H_{ab}]) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{cd}, E_{ij}], [E_{ji}, H_{ab}]) + (\beta - \frac{\lambda}{2})(\overline{\epsilon_{cd}}, \epsilon_{ab})E_{cd} \right) \]

\[ + (\epsilon_{ab}, \epsilon_{cd}) \left( P([H_{ab}, E_{cd}]) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([H_{ab}, E_{ij}], [E_{ji}, E_{cd}]) + (\beta - \frac{\lambda}{2})(\epsilon_{ab}, \overline{\epsilon_{cd}})E_{cd} \right) \]

\[ = \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} [S([H_{ab}, E_{ij}], [E_{ji}, H_{ab}]), E_{cd}] + 2(\beta - \frac{\lambda}{2})(\epsilon_{ab}, \epsilon_{cd})(\overline{\epsilon_{cd}}, \epsilon_{ab})E_{cd} \]

Thus,

\[ [Z, E_{i,i+1}] = \sum_{a=1}^{n} [Z_{a,a+1}, E_{i,i+1}] = 2(\beta - \frac{\lambda}{2}) \sum_{a=1}^{n} (\epsilon_a - \epsilon_{a+1}, \epsilon_i - \epsilon_{i+1})(\epsilon_i + \epsilon_{i+1}, \epsilon_a - \epsilon_{a+1})E_{i,i+1}. \]
It is not hard to show that, for any $1 \leq i < n$,

$$
\sum_{a=1}^{n} (\epsilon_a - \epsilon_{a+1}, \epsilon_i - \epsilon_{i+1})(\epsilon_i + \epsilon_{i+1}, \epsilon_a - \epsilon_{a+1}) \equiv 0,
$$

where $(a, a + 1) = (n, 1)$, if $a = n$.

Since the summand $(\epsilon_a - \epsilon_b, \epsilon_i - \epsilon_{i+1})(\epsilon_i + \epsilon_{i+1}, \epsilon_a - \epsilon_{b})$ is non-zero, only if either $a + 1 = i$, or $a = i + 1$. In the case $a + 1 = i$, the summand is 1, while the summand is $-1$, in the case $a = i + 1$.

Therefore, we have $[Z, E_{i,i+1}] = 0$, for any $1 \leq i < n$. Thus, the element $Z$ commutes with any elements in $\mathfrak{sl}_n$.

In the following, we show the element $Z$ commutes with $K(h)$, for some diagonal matrix $h$ of $\mathfrak{sl}_n$.

By the defining relations of $D_{\lambda, \beta}(\mathfrak{sl}_n)$, we have $[K(h), h'] = 0$, for any two diagonal matrices $h, h'$. By Proposition 5.6.1 (iii),

$$Z_{ab} - Z_{cd} = 2(\beta - \lambda / 2)(H_{ac} + H_{bd}).$$

Thus, $[Z_{ab}, K(h)] = [Z_{cd}, K(h)]$, for any diagonal matrix $h$.

To show $[Z, K(H_{cd})] = 0$, it suffices to show $[Z_{ab}, K(H_{cd})] = 0$, for distinct integers $1 \leq a \neq b \neq c \neq d \leq n$, with $(a, b) = (a, a + 1)$. On one hand,

$$[K(H_{cd}), K(H_{ab}), Q(H_{ab})]$$

$$= [K(H_{ab}), K(H_{cd}), Q(H_{ab})]$$

$$= [K(H_{ab}), \frac{1}{4} \sum_{1 \leq i,j \leq n} (\epsilon_c - \epsilon_d, \epsilon_i - \epsilon_j)(\epsilon_a - \epsilon_b, \epsilon_i - \epsilon_j)S(E_{ij}, E_{ji})]$$

by Proposition 5.6.1 (i)

$$= \frac{1}{4} \sum_{1 \leq i \neq j \leq n} (\epsilon_c - \epsilon_d, \epsilon_i - \epsilon_j)(\epsilon_a - \epsilon_b, \epsilon_i - \epsilon_j) (S([K(H_{ab}), E_{ij}], E_{ji}) + S(E_{ij}, [K(H_{ab}), E_{ji}]])$$

$$= \frac{1}{4} \sum_{1 \leq i \neq j \leq n} (\epsilon_c - \epsilon_d, \epsilon_i - \epsilon_j)(\epsilon_a - \epsilon_b, \epsilon_i - \epsilon_j)^2 (S(K(E_{ij}), E_{ji}) - S(E_{ij}, K(E_{ji})))$$

$$= \frac{1}{2} \sum_{1 \leq i \neq j \leq n} (\epsilon_c - \epsilon_d, \epsilon_i - \epsilon_j)(\epsilon_a - \epsilon_b, \epsilon_i - \epsilon_j)^2 S(K(E_{ij}), E_{ji})$$

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On the other hand, we have:

\[
[K(H_{cd}), \frac{1}{4} \sum_{1 \leq i \neq j \leq n} S([H_{ab}, E_{ij}], [E_{ji}, H_{ab}])] = [K(H_{cd}), \frac{1}{4} \sum_{1 \leq i \neq j \leq n} (\epsilon_a - \epsilon_b, \epsilon_i - \epsilon_j)^2 S(E_{ij}, E_{ji})]
\]

\[
= \frac{1}{4} \sum_{1 \leq i \neq j \leq n} (\epsilon_a - \epsilon_b, \epsilon_i - \epsilon_j)^2 (S([K(H_{cd}), E_{ij}], E_{ji}) + S(E_{ij}, [K(H_{cd}), E_{ji}]))
\]

\[
= \frac{1}{4} \sum_{1 \leq i \neq j \leq n} (\epsilon_a - \epsilon_b, \epsilon_i - \epsilon_j)^2 (\epsilon_c - \epsilon_d, \epsilon_i - \epsilon_j) (S(K(E_{ij}), E_{ji}) - S(E_{ij}, K(E_{ji})))
\]

\[
= \frac{1}{2} \sum_{1 \leq i \neq j \leq n} (\epsilon_c - \epsilon_d, \epsilon_i - \epsilon_j)(\epsilon_a - \epsilon_b, \epsilon_i - \epsilon_j)^2 S(K(E_{ij}), E_{ji})
\]

Combine the above computations, we conclude that \([K(H_{cd}), Z_{ab}] = 0\). Thus, \([K(H_{cd}), Z] = 0\).

As a corollary, we have

\([Z, K(X)] = 0\), for any \(X \in \mathfrak{sl}_n\).

Since \(Z\) commutes with \(\mathfrak{sl}_n\), and the elements \(K(X)\) satisfies the relation \([K(X_1), X_2] = K[X_1, X_2]\), for any elements \(X_1, X_2 \in \mathfrak{sl}_n\).

Similar argument shows

\([Z, Q(X)] = 0\), for any \(X \in \mathfrak{sl}_n\).

Note that \(\mathfrak{D}_{\lambda, \beta}(\mathfrak{sl}_n)\) can be generated by \(X, K(X), Q(X)\), for \(X \in \mathfrak{sl}_n\), thus, \(Z\) is central in \(\mathfrak{D}_{\lambda, \beta}(\mathfrak{sl}_n)\).
Chapter 6

Universal KZB Equations and the elliptic Casimir connection

6.1 The generalized holonomy Lie algebra $A_{\text{ell}}$

Let $\mathfrak{h}$ be a Euclidean vector space, $\Phi \subset \mathfrak{h}^*$ a reduced, crystallographic root system. Let $Q \subset \mathfrak{h}^*$ be the lattice generated by the roots $\alpha$, $\alpha \in \Phi$ and $P^\vee \subset \mathfrak{h}$ the dual lattice, called the coweight lattice. Let $Q^\vee \subset \mathfrak{h}$ be the lattice generated by the coroots $\alpha^\vee$, with the inner product $(\alpha^\vee, \alpha) = 2$, $\alpha \in \Phi$ and $P \subset \mathfrak{h}^*$ the dual weight lattice.

For a subset $\Psi \subset \Phi$ and subring $R \subset \mathbb{R}$, let $\langle \Psi \rangle_R \subset \mathfrak{h}^*$ be the $R$–span of $\Psi$.

**Definition 6.1.1.** A root subsystem of $\Phi$ is a subset $\Psi \subset \Phi$ such that $\langle \Psi \rangle_R \cap \Phi = \Psi$. $\Psi$ is complete if $\langle \Psi \rangle_R \cap \Phi = \Psi$. If $\Psi \subset \Phi$ is a root subsystem, we set $\Psi_+ = \Psi \cap \Phi_+$.

The algebra $A_{\text{ell}}^\Phi$ associated for a root system $\Phi$ is defined as follows,

**Definition 6.1.2.** Let $A_{\text{ell}}^\Phi$ be the Lie algebra generated by:

- a set of elements $\{t_\alpha\}_{\alpha \in \Phi}$, such that $t_\alpha = t_{-\alpha}$,
- two linear maps $x : \mathfrak{h} \to A$, $y : \mathfrak{h} \to A$, such that:

1. For any root subsystem $\Psi$ of $\Phi$, $[t_\alpha, \sum_{\beta \in \Psi} t_\beta] = 0$;
2. \([x(u), x(v)] = 0, [y(u), y(v)] = 0\), for any \(u, v \in \mathfrak{h}\);

3. \([y(u), x(v)] = \sum_{\gamma \in \Phi^+} \langle v, \gamma \rangle \langle u, \gamma \rangle t_\gamma\).

4. \([t_\alpha, x(u)] = 0, [t_\alpha, y(u)] = 0\), if \(\langle \alpha, u \rangle = 0\).

Note that algebra \(A^\Phi_{\text{ell}}\) is bigraded, with the grading \(\text{deg}(x(u)) = (1, 0)\), \(\text{deg}(y(v)) = (0, 1)\), and \(\text{deg}(t_\alpha) = (1, 1)\).

Remark 6.1.3. In topology, there is a explicit construction of the holonomy Lie algebra, see [?, Section 2]. By construction, the holonomy Lie algebra is a quadratic Lie algebra, that is, it has a presentation with generators in degree 1 and relations in degree 2 only. While, the Lie algebra \(A_{\text{ell}}\) we defined here is not a quadratic Lie algebra. We call \(A_{\text{ell}}\) a generalized holonomy Lie algebra in the rest of the paper.

Let \(\tilde{\alpha}\) be the highest root in \(\Psi\). Write \(\tilde{\alpha}^\vee = \sum_{i=1}^n g_i \alpha_i^\vee\), and let \(g(\Psi) := 1 + \sum_{i=1}^n g_i\).

Lemma 6.1.4. Let \(\Phi\) be of rank 2, and \(T \subset A\) the linear subspace spanned by the elements \(t_\alpha, \alpha \in \Phi_+\). Then the image of the map \(\mathfrak{h} \otimes \mathfrak{h} \to T\) given by \(u \otimes v \to [x(u), y(v)]\) is contained in the subspace of \(T\) given by

\[
\{ \sum_{\gamma \in \Phi_+} c_\gamma t_\gamma \mid \text{for any } \Psi_1 \neq \Psi_2 \subseteq \Phi \text{ rank 2 root systems}, \sum_{\gamma \in \Psi_1^+} c_\gamma / g(\Psi_1) = \sum_{\gamma \in \Psi_2^+} c_\gamma / g(\Psi_2) \}
\]

Proof. write \([x(u), y(v)] = \sum_{\gamma \in \Phi_+} c_\gamma(u, v)t_\gamma\), where \(c_\gamma(u, v) = \langle u, \gamma \rangle \langle v, \gamma \rangle\). Then, for any \(\Psi \subset \Phi\), we have \(\sum_{\gamma \in \Psi} c_\gamma = g(\Psi)(u|v)\). (see Lemma 6.1.3.4 for this equality.)

6.2 The universal connection for arbitrary root systems

6.2.1 Principal bundles

Let \(\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau\), and \(\mathcal{E}_\tau := \mathbb{C}/\Lambda_\tau\) be the elliptic curve with modular parameter \(\tau \notin \mathbb{R}\).
Let

\[ T := P^\vee \otimes \mathbb{Z} \mathcal{E}_\tau \]

be the simply-connected torus.

For any root \( \alpha \in Q \subset \mathfrak{h}^* \), we have a natural map \( \chi : P^\vee \otimes \mathbb{Z} \mathcal{E}_\tau \rightarrow \mathcal{E}_\tau \). Denote kernel of \( \chi \) by \( T_\alpha \), which is a divisor of \( T \). Denote \( T_{\text{reg}} = T \setminus \bigcup_{\alpha \in \Phi} T_\alpha \).

Consider the principal bundle \( \mathfrak{h} \times_{\Lambda_\tau \otimes P^\vee} \exp(\widehat{A}_{\text{ell}}^\Phi) \) over \( T \) with structure group \( \exp(\widehat{A}_{\text{ell}}^\Phi) \), where the action of \( \Lambda_\tau \otimes P^\vee \) on \( \mathfrak{h} = \mathbb{C} \otimes P^\vee \) is by translations, and the action on group \( \exp(\widehat{A}_{\text{ell}}^\Phi) \) is given by:

\[ \lambda_i^\vee(g) = g \text{ and } \tau \lambda_i^\vee(g) = e^{-2\pi i x(\lambda_i)} g. \]

Denote by \( P_{\tau,n} \) the restricting the bundle \( \mathfrak{h} \times_{\Lambda_\tau \otimes P^\vee} \exp(\widehat{A}_{\text{ell}}^\Phi) \) over \( T_{\text{reg}} \).

Equivalently, denote by \( \pi : \mathbb{C} \otimes P^\vee \rightarrow \mathcal{E}_\tau \otimes P^\vee \) the natural projection. Let \( C(\mathcal{E}, \tau) \) be the bundle on \( \mathcal{E}_\tau \otimes P^\vee \) with sections on \( U \subset \mathcal{E}_\tau \otimes P^\vee \) given by:

\[ \{ f : \pi^{-1}(U) \rightarrow \exp(\widehat{A}_{\text{ell}}^\Phi) \mid f(z + \lambda_i^\vee) = f(z), f(z + \tau \lambda_i^\vee) = e^{-2\pi i x(\lambda_i)} f(z) \}. \]

Then, the bundle \( P_{\tau,n} \) is the restricting of the bundle \( C(\mathcal{E}, \tau) \) under the inclusion

\[ (\mathcal{E}_\tau \otimes P^\vee)^{\text{reg}} \hookrightarrow \mathcal{E}_\tau \otimes P^\vee. \]

### 6.2.2 Some facts about theta functions

We recall some basic facts about theta functions in this section. Let \( q := e^{2\pi i \tau} \). Denote \( \vartheta_1(z, q) \) a classical Theta-function (See [100] Page 463-465.), that is, \( \vartheta_1(z, q) = 2 \sum_{n=0}^{\infty} q^{\frac{n^2}{8}} \sin(nz) \).

We have:

\[ \frac{\partial \vartheta_1}{\partial z}(0, q) = 2q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1 - q^n)^3. \]

Now consider the following theta function considered in [10]:

\[ \theta(z|\tau) := \frac{\vartheta_1(\pi z, q)}{\vartheta_1(0, q)}. \]
If \( \eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n) \), then \( \vartheta_1(z, \tau) = 2\eta(\tau)^3 \theta(z|\tau) \). Note that \( \vartheta_1(z, \tau) \) satisfies the partial differential equation \( \frac{1}{4\pi i} \frac{\partial^2 \vartheta_1(z, \tau)}{\partial z^2} = \frac{\partial \vartheta_1(z, \tau)}{\partial \tau} \).

Let \( \Lambda_\tau := \mathbb{Z} + \mathbb{Z} \tau \subset \mathbb{C} \) and \( \mathfrak{H} \) be the upper half plane, i.e. \( \mathfrak{H} := \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \).

The following properties of \( \theta(z|\tau) \) is straightforward from properties of \( \vartheta_1(z, q) \).

1. \( \theta(z|\tau) \) is a holomorphic function \( \mathbb{C} \times \mathfrak{H} \to \mathbb{C} \), such that \( \{ z \mid \theta(z|\tau) = 0 \} = \Lambda_\tau \).

2. \( \frac{\partial \theta}{\partial z}(0|\tau) = 1 \).

3. \( \theta(z + 1|\tau) = -\theta(z|\tau) = \theta(-z|\tau) \), and \( \theta(z + \tau|\tau) = -e^{-\pi i \tau} e^{-2\pi iz} \theta(z|\tau) = \theta(-z|\tau) \).

4. \( \theta(z|\tau + 1) = \theta(z|\tau) \), while \( \theta(-z/\tau - 1/\tau) = -(1/\tau) e^{(\pi i/\tau)z^2} \theta(z|\tau) \).

5. The following product formula holds:

\[
\theta(z|\tau) = u^{1/2} \prod_{s > 0} (1 - q^s u) \prod_{s \geq 0} (1 - q^s u^{-1}) \frac{1}{2\pi i} \prod_{s > 0} (1 - q^s)^{-2}, \tag{6.2.1}
\]

where \( u = e^{2\pi iz} \).

### 6.2.3 General form

Set

\[
k(z, x|\tau) := \frac{\theta(z + x|\tau)}{\theta(z|\tau) \theta(x|\tau)} - \frac{1}{x}. \tag{6.2.2}
\]

Consider the \( A_{\text{nil}} \)-valued connection on \( H_{\text{reg}} \) given by

\[
\nabla_{\text{KZB}} = d - \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(\frac{x_\alpha}{2})|\tau) (t_\alpha) d\alpha + \sum_{i=1}^n y(u^i) du_i, \tag{6.2.3}
\]

where \( \Phi^+ \subset \Phi \) is a chosen system of positive roots, \( \{ u_i \} \), and \( \{ u^i \} \) are dual basis of \( \mathfrak{h}^* \) and \( \mathfrak{h} \) respectively.

Note that the form is independent of the choice of \( \Phi^+ \), which follows from \( k(z, x|\tau) = -k(-z, -x|\tau) \) using the theta function \( \theta(z|\tau) \) being an odd function.
Rewriting the connection (6.2.3) as the form

\[ \nabla_{\text{KZB}} = d - \sum_{i=1}^{n} \left( \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) k(\alpha, \text{ad}(\frac{x_{\alpha^\vee}}{2})|\tau)(t_{\alpha}) - y(\alpha_i) \right) d\lambda_i^\vee \]

**Lemma 6.2.1.** For any \( i \), the function

\[ K_i := \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) k(\alpha, \text{ad}(\frac{x_{\alpha^\vee}}{2})|\tau)(t_{\alpha}) - y(\alpha_i) \]

satisfies the conditions

\[ K_i(z + \lambda_j^\vee) = K_i(z) \text{ and } K_i(z + \tau \lambda_j^\vee) = e^{-2\pi ix(\lambda_j^\vee)} K_i(z). \]

**Proof.** We use the fact that for any integer \( m \in \mathbb{Z} \), \( k(z + m, x|\tau) = k(z, x|\tau) \) and

\[ k(z + \tau m, x|\tau) = e^{-2\pi imx} k(z, x|\tau) + \frac{e^{-2\pi imx} - 1}{x}. \]

Then it’s obvious that \( K_i(\alpha + \lambda_j^\vee) = K_i(\alpha) \). We have

\[
\begin{align*}
K_i(\alpha + \tau \lambda_j^\vee) &= \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) k(\alpha + \tau(\alpha, \lambda_j^\vee), \text{ad}(\frac{x_{\alpha^\vee}}{2})|\tau)(t_{\alpha}) - y(\alpha_i) \\
&= \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) \left( \exp \left( -2\pi i(\alpha, \lambda_j^\vee) \frac{x(\alpha^\vee)}{2} \right) k(\alpha, \text{ad}(\frac{x_{\alpha^\vee}}{2})|\tau) + \frac{\exp \left( -2\pi i(\alpha, \lambda_j^\vee) x_{\alpha^\vee}/2 \right) - 1}{x_{\alpha^\vee}/2} \right)(t_{\alpha}) - y(\alpha_i) \\
&= \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) \left( \exp \left( -2\pi ix(\lambda_j^\vee) \right) k(\alpha, \text{ad}(\frac{x_{\alpha^\vee}}{2})|\tau) + \frac{-\exp \left( -2\pi ix(\lambda_j^\vee) \right) - 1}{x(\lambda_j^\vee)} \right)(t_{\alpha}) - y(\alpha_i)
\end{align*}
\]

by the relation \([- (\alpha, \lambda_j^\vee) \frac{x(\alpha^\vee)}{2} + x(\lambda_j^\vee), t_{\alpha}] = 0. \]

\[
\begin{align*}
&= \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) \exp \left( -2\pi ix(\lambda_j^\vee) \right) k(\alpha, \text{ad}(\frac{x_{\alpha^\vee}}{2})|\tau)(t_{\alpha}) + \frac{\exp \left( -2\pi ix(\lambda_j^\vee) \right) - 1}{x(\lambda_j^\vee)} \left( \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i)(\alpha, \lambda_j^\vee)t_{\alpha} \right) - y(\alpha_i) \\
&= \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) \exp \left( -2\pi ix(\lambda_j^\vee) \right) k(\alpha, \text{ad}(\frac{x_{\alpha^\vee}}{2})|\tau)(t_{\alpha}) - (\exp \left( -2\pi ix(\lambda_j^\vee) \right) - 1)(y(\alpha_i)) - y(\alpha_i)
\end{align*}
\]

by the relation \([y(\alpha_i), x(\lambda_j^\vee)] = \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i)(\alpha, \lambda_j^\vee)t_{\alpha}. \]

\[
\begin{align*}
&= \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) \exp \left( -2\pi ix(\lambda_j^\vee) \right) k(\alpha, \text{ad}(\frac{x_{\alpha^\vee}}{2})|\tau)(t_{\alpha}) - \exp \left( 2\pi ix(\lambda_j^\vee) \right)(y(\alpha_i)) \\
&= \exp(-2\pi ix(\lambda_j^\vee)) K_i(\alpha).
\end{align*}
\]
Theorem 6.2.2. The connection $\nabla_{KZB}$ (6.2.3) is flat if and only if the relations in $A_{EL}$ hold.

Proof. Set $A := \sum_{\alpha \in \Phi} k(\alpha, \text{ad}(\frac{x^{\alpha}}{2})|\tau)(t_\alpha) d\alpha - \sum_{i=1}^{n} y(u^i) du_i$. Since $dA = 0$, the connection (6.2.3) is flat if and only if the curvature $\Omega = A \land A = 0$.

Write $\Omega = \Omega_1 - \Omega_2 + \Omega_3$, where
\[
\Omega_1 = \frac{1}{2} \sum_{\alpha \neq \beta} \left[ k(\alpha, \text{ad}(\frac{x^{\alpha}}{2})|\tau)(t_\alpha), k(\beta, \text{ad}(\frac{x^{\beta}}{2})|\tau)(t_\beta) \right] d\alpha \land d\beta \quad (6.2.4)
\]
\[
\Omega_2 = \sum_{\alpha, i} \left[ k(\alpha, \text{ad}(\frac{x^{\alpha}}{2})|\tau)(t_\alpha), y(u^i) \right] d\alpha \land du_i \quad (6.2.5)
\]
\[
\Omega_3 = \frac{1}{2} \sum_{i \neq j} \left[ y(u^i), y(u^j) \right] du_i \land du_j \quad (6.2.6)
\]

The relation $[y(u^i), y(u^j)] = 0$ implies that $\Omega_3 = 0$. In order to show $\Omega = 0$, we now need to show $\Omega_1 = \Omega_2$. \qed

Corollary 6.2.3. In the case $g = \mathfrak{sl}_n$, connection (6.2.3) can be rewritten as $\nabla_{KZB} := d - \sum_{i=1}^{n} K_i(z|\tau) dz_i$, where
\[
K_i(z|\tau) = -y_i + \sum_{j \neq i} K_{ij}(z_{ij}|\tau) := -y_i + \sum_{j \neq i} k(z_{ij}, \text{ad} x_i|\tau)(t_{ij}).
\]

Which coincides with the connection constructed in [10].

Proof. In the case $g = \mathfrak{sl}_n$, write $\alpha = \epsilon_i - \epsilon_j \in \Phi$, where $\{\epsilon_1, \ldots, \epsilon_n\}$ is a standard orthogonal basis of $\mathbb{C}^n$. Notice that we have $K_{ij}(z_{ij}|\tau) = -K_{ji}(z_{ji}|\tau)$, since $k(z, x|\tau) = -k(-z, -x|\tau)$.

Using the relation $[x_i + x_j, t_{ij}] = 0$, we get
\[
(\text{ad} x_i)^k(t_{ij}) = (-\text{ad} x_j)^k(t_{ij}) = (\text{ad} \frac{x_i - x_j}{2})^k(t_{ij}),
\]
for any $k > 0$. The rest of the proof is simply by calculation. \qed
6.3 Flatness of the universal connection, part 1

For two vector $u \parallel v \in \mathfrak{h}^*$, set $u^\vee := \frac{2u}{(u,u)}$, now define a new element $\omega(u^\vee, v)$, which is spanned by $u$ and $v$, such that

$$(u^\vee + \omega(u^\vee, v)) \perp u^\vee, \omega(u^\vee, v) \perp v.$$  

More explicitly, we have $\omega(u^\vee, v) = \frac{-2(v,v)}{(u,u)(v,v)-(u,v)^2} u + \frac{2(u,v)}{(u,u)(v,v)-(u,v)^2} v$. The following identities, which follow from a direct calculation, will be used later.

**Lemma 6.3.1.**

1. $\omega(u^\vee, v) - \omega(v^\vee, u) = \omega(u^\vee, u + v)$;

2. $\omega((au + bv)^\vee, cu) = \frac{1}{b} \omega(v^\vee, u)$;

3. $\frac{u^\vee + \omega(u^\vee, v)}{(v^\vee, v)} = \frac{v^\vee + \omega(u^\vee, u + v)}{(u^\vee, u + v)}$;

4. $\frac{v^\vee + \omega(u^\vee, v)}{(v^\vee, v)} = \frac{\omega(v^\vee, u)}{-2}$.

**Proposition 6.3.2.** Modulo the relations $(tx)$, $(xx)$, the following identity holds for any $\alpha, \beta \in \Phi^+$:

$$[k(\alpha, \text{ad}(\frac{x_{\alpha^\vee}}{2})|\tau)(t_\alpha), k(\beta, \text{ad}(\frac{x_{\beta^\vee}}{2})|\tau)(t_\beta)] = k(\alpha, \text{ad}(\frac{x_{\omega(\alpha^\vee, \beta)}}{-2})|\tau)k(\beta, \text{ad}(\frac{x_{\omega(\beta^\vee, \alpha)}}{-2})|\tau)[t_\alpha, t_\beta].$$

**Proof.** If $\alpha = \beta$, both sides is obviously equal to 0.

If $\alpha \neq \beta$, since $k(z, x)$ is a formal power series in $x$, with coefficient in $\mathbb{C}[[z]]$, it suffices to show above identity replacing $k(z, x)$ by $x^n$ for any $n \in \mathbb{N}$. Now we have:

$$[\text{ad } x_{\alpha^\vee}]^n(t_\alpha), [\text{ad } x_{\beta^\vee}]^m(t_\beta)$$  

$$= [(- \text{ad } x_{\omega(\alpha^\vee, \beta)})^n(t_\alpha), (- \text{ad } x_{\omega(\beta^\vee, \alpha)})^m(t_\beta)]$$  

$$= (- \text{ad } x_{\omega(\alpha^\vee, \beta)})^n(- \text{ad } x_{\omega(\beta^\vee, \alpha)})^m[t_\alpha, t_\beta]$$

So the assertion follows.
Corollary 6.3.3. The curvature term $\Omega_1$ is equal to

$$\Omega_1 = \frac{1}{2} \sum_{\alpha \neq \beta} k(\alpha, \text{ad}(\frac{x_{\omega(\alpha,\beta)}}{2}|\tau)) k(\beta, \text{ad}(\frac{x_{\omega(\beta,\alpha)}}{2}|\tau))[t_\alpha, t_\beta] d\alpha \wedge d\beta \quad (6.3.1)$$

Proposition 6.3.4. Suppose $u \perp \alpha$, modulo the relations $(yt), (yx), (xx)$, the following identity holds for any $\alpha \in \Phi^+, u \in h^*$:

$$[y(u), k(\alpha, \text{ad}(\frac{x_{\alpha \gamma}}{2}|\tau))(t_\alpha)] = \sum_{\gamma \in \Phi^+} (\alpha^\vee, \gamma)(u, \gamma) \frac{k(\alpha, \text{ad}(\frac{x_{\alpha \gamma}}{2}|\tau)) - k(\alpha, \text{ad}(\frac{x_{\omega(\alpha \gamma)}}{2}|\tau))}{adx_{\alpha^\vee} + adx_{\omega(\alpha \gamma)}}[t_\gamma, t_\alpha].$$

Proof. Since $k(z, x)$ is a formal power series in $x$, with coefficient in $\mathbb{C}[[z]]$, it suffices to show above identity replacing $k(z, x)$ by $x^n$ for any $n \in \mathbb{N}$. Now we have:

$$[y(u), (\text{ad } x_{\alpha \gamma})^n(t_\alpha)]$$

$$= \sum_{s=0}^{n-1} (\text{ad } x_{\alpha \gamma})^s(\text{ad } y(u), x_{\alpha \gamma})(\text{ad } x_{\alpha \gamma})^{n-1-s}(t_\alpha)$$

$$= \sum_{\gamma \in \Phi^+} (\alpha^\vee, \gamma)(u, \gamma) \sum_{s=0}^{n-1} (\text{ad } x_{\alpha \gamma})^s(\text{ad } t_\gamma)(\text{ad } x_{\alpha \gamma})^{n-1-s}(t_\alpha)$$

$$= \sum_{\gamma \in \Phi^+} (\alpha^\vee, \gamma)(u, \gamma) \sum_{s=0}^{n-1} (\text{ad } x_{\alpha \gamma})^s(\text{ad } t_\gamma)(-\text{ad } x_{\omega(\alpha \gamma)})(\text{ad } x_{\alpha \gamma})^{n-1-s}(t_\alpha)$$

$$= \sum_{\gamma \in \Phi^+} (\alpha^\vee, \gamma)(u, \gamma) \sum_{s=0}^{n-1} (\text{ad } x_{\alpha \gamma})^s(-\text{ad } x_{\omega(\alpha \gamma)})(\text{ad } x_{\alpha \gamma})^{n-1-s}(t_\alpha)$$

$$= \sum_{\gamma \in \Phi^+} (\alpha^\vee, \gamma)(u, \gamma) f(\text{ad } x_{\alpha \gamma}, -\text{ad } x_{\omega(\alpha \gamma)})([t_\gamma, t_\alpha]),$$

where $f(u, v) = \frac{u^n - v^n}{u - v}$. So the assertion follows. \qed

Corollary 6.3.5. The curvature term $\Omega_2$ is equal to

$$\Omega_2 = \sum_{\alpha \neq \gamma \in \Phi^+} (\alpha^\vee, \gamma) \frac{k(\alpha, \text{ad}(\frac{x_{\omega(\alpha \gamma)}}{2}|\tau)) - k(\alpha, \text{ad}(\frac{x_{\omega(\alpha \gamma)}}{2}|\tau))}{adx_{\alpha^\vee} + adx_{\omega(\alpha \gamma)}}[t_\alpha, t_\gamma] d\alpha \wedge d\gamma.$$ 

Proof. By definition,

$$\Omega_2 = \sum_{\alpha} \sum_{i} \left[k(\alpha, \text{ad}(\frac{x_{\alpha \gamma}}{2}|\tau))(t_\alpha), y(u^i)\right] d\alpha \wedge du_i$$

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Choose the basis \( \{ u_i \} \subset \mathfrak{h}^* \) to be such that \( u_1 = \alpha \) and \( u_i \perp \alpha \) for \( i \geq 2 \). Then since \( d\alpha \land du_1 = 0 \), by Proposition 6.3.4, the above is equal to

\[
- \sum_{\alpha, i \geq 2} \sum_{\gamma \in \Phi^+} (u^i, \gamma)(\alpha^\vee, \gamma) \frac{k(\alpha, \text{ad}(\frac{x_\alpha}{2})|\tau) - k(\alpha, \text{ad}(\frac{x_\omega(\alpha^\vee, \gamma)}{2})|\tau)}{adx_{\alpha^\vee} + adx_{\omega(\alpha^\vee, \gamma)}}[t_\gamma, t_\alpha]d\alpha \land du_i
\]

which is equal to the claimed result since \( \gamma = (\gamma, u^i)u_i \). 

\[\square\]

### 6.4 Flatness of the universal connection, part 2

From last section, we have an expression of \( \Omega_1 \) and \( \Omega_2 \). In this section, we will show \( \Omega_1 - \Omega_2 = 0 \). Write

\[
\Omega_1 - \Omega_2
\]

\[
= \sum_{\alpha \neq \beta} \left( \frac{1}{2} k(\alpha, \text{ad}(\frac{x_\omega(\alpha^\vee, \beta)}{2}))[\beta, \text{ad}(\frac{x_\omega(\beta^\vee, \alpha)}{2})] - (\alpha^\vee, \beta) \frac{k(\alpha, \text{ad}(\frac{x_\omega(\alpha^\vee, \beta)}{2})) - k(\alpha, \text{ad}(\frac{x_\omega(\alpha^\vee, \beta)}{2}))}{adx_{\alpha^\vee} + adx_{\omega(\alpha^\vee, \beta)}} \right)
\]

\[\cdot \ [t_\alpha, t_\beta]d\alpha \land d\beta.\]

Let \( k(\alpha, \beta)[t_\alpha, t_\beta] \) be the subtraction of coefficient of \( d\alpha \land d\beta \) and coefficient of \( d\beta \land d\alpha \) in the formula of \( \Omega_1 - \Omega_2 \), that is:

\[
k(\alpha, \beta) = k(\alpha, \text{ad}(\frac{x_\omega(\alpha^\vee, \beta)}{2})]|\tau)k(\beta, \text{ad}(\frac{x_\omega(\beta^\vee, \alpha)}{2})]|\tau)
\]

\[
- (\alpha^\vee, \beta) \frac{k(\alpha, \text{ad}(\frac{x_\omega(\alpha^\vee, \beta)}{2})]|\tau) - k(\alpha, \text{ad}(\frac{x_\omega(\alpha^\vee, \beta)}{2})]|\tau)}{adx_{\alpha^\vee} + adx_{\omega(\alpha^\vee, \beta)}}
\]

\[
- (\beta^\vee, \alpha) \frac{k(\beta, \text{ad}(\frac{x_\omega(\beta^\vee, \alpha)}{2})]|\tau) - k(\beta, \text{ad}(\frac{x_\omega(\beta^\vee, \alpha)}{2})]|\tau)}{adx_{\beta^\vee} + adx_{\omega(\beta^\vee, \alpha)}}
\]

### Proposition 6.4.1

Notations as above, for any \( \alpha, \beta \in \Phi \), we have

\[
k(\alpha, \beta)d\alpha \land d\beta + k(\alpha + \beta)d(\alpha + \beta) \land d\alpha + k(\beta, \alpha + \beta)d\beta \land d(\alpha + \beta) = 0. \quad (6.4.1)
\]
Proof. We only need to show: 
\[ -k(\alpha, \beta) + k(\alpha, \alpha + \beta) + k(\beta, \alpha + \beta) = 0. \]
By definition,

\[
-k(\alpha, \beta) + k(\alpha, \alpha + \beta) + k(\beta, \alpha + \beta) \\
= -k(\alpha, \text{ad}\left(\frac{\omega(\alpha^\vee, \beta)}{-2}\right)\triangledown)k(\beta, \text{ad}\left(\frac{\omega(\beta^\vee, \alpha)}{-2}\right)\triangledown) + (\alpha^\vee, \beta)\frac{k(\alpha, \text{ad}\left(\frac{\omega(\alpha^\vee)}{-2}\right)\triangledown) - k(\alpha, \text{ad}\left(\frac{\omega(\alpha^\vee, \beta)}{-2}\right)\triangledown)}{adx_{\alpha^\vee} + adx_{\omega(\alpha^\vee, \beta)}} \\
+ (\beta^\vee, \alpha)\frac{k(\beta, \text{ad}\left(\frac{\omega(\beta^\vee)}{-2}\right)\triangledown) - k(\beta, \text{ad}\left(\frac{\omega(\beta^\vee, \alpha)}{-2}\right)\triangledown)}{adx_{\beta^\vee} + adx_{\omega(\beta^\vee, \alpha)}} \\
+k(\alpha, \text{ad}\left(\frac{\omega(\alpha^\vee, \alpha + \beta)}{-2}\right)\triangledown)k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee, \alpha)}{-2}\right)\triangledown) - (\alpha^\vee, \alpha + \beta)\frac{k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee)}{-2}\right)\triangledown) - k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee, \alpha)}{-2}\right)\triangledown)}{adx_{(\alpha + \beta)^\vee} + adx_{\omega((\alpha + \beta)^\vee, \alpha)}} \\
- ((\alpha + \beta)^\vee, \alpha)\frac{k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee)}{-2}\right)\triangledown) - k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee, \alpha)}{-2}\right)\triangledown)}{adx_{(\alpha + \beta)^\vee} + adx_{\omega((\alpha + \beta)^\vee, \alpha)}}. 
\]

Now use the identities in Lemma 6.3.1, We have:

\[
\frac{\alpha^\vee + \omega(\alpha^\vee, \beta)}{(\alpha^\vee, \beta)} = \frac{\alpha^\vee + \omega(\alpha^\vee, \alpha + \beta)}{(\alpha^\vee, \alpha + \beta)}, \frac{\beta^\vee + \omega(\beta^\vee, \alpha)}{(\beta^\vee, \alpha)} = \frac{\beta^\vee + \omega(\beta^\vee, \alpha + \beta)}{(\beta^\vee, \alpha + \beta)}, \frac{\omega(\alpha^\vee, \alpha + \beta)}{(\alpha^\vee, \alpha + \beta)} = -k(\alpha, \text{ad}\left(\frac{\omega(\alpha^\vee, \alpha + \beta)}{-2}\right)\triangledown)k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee, \alpha)}{-2}\right)\triangledown) \\
- ((\alpha + \beta)^\vee, \alpha)\frac{k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee)}{-2}\right)\triangledown) - k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee, \alpha)}{-2}\right)\triangledown)}{adx_{(\alpha + \beta)^\vee} + adx_{\omega((\alpha + \beta)^\vee, \alpha)}}. 
\]

Thus, using the fact that \( x : h \to A_{\text{nil}} \) is a linear function, we are able to make some cancelations:

\[
-k(\alpha, \beta) + k(\alpha, \alpha + \beta) + k(\beta, \alpha + \beta) \\
= -k(\alpha, \text{ad}\left(\frac{\omega(\alpha^\vee, \beta)}{-2}\right)\triangledown)k(\beta, \text{ad}\left(\frac{\omega(\beta^\vee, \alpha)}{-2}\right)\triangledown) + (\alpha^\vee, \beta)\frac{k(\alpha, \text{ad}\left(\frac{\omega(\alpha^\vee, \beta)}{-2}\right)\triangledown) - k(\alpha, \text{ad}\left(\frac{\omega(\alpha^\vee, \beta)}{-2}\right)\triangledown)}{adx_{\alpha^\vee} + adx_{\omega(\alpha^\vee, \beta)}} \\
+ (\beta^\vee, \alpha)\frac{k(\beta, \text{ad}\left(\frac{\omega(\beta^\vee, \alpha)}{-2}\right)\triangledown) - k(\beta, \text{ad}\left(\frac{\omega(\beta^\vee, \alpha)}{-2}\right)\triangledown)}{adx_{\beta^\vee} + adx_{\omega(\beta^\vee, \alpha)}} \\
+k(\alpha, \text{ad}\left(\frac{\omega(\alpha^\vee, \alpha + \beta)}{-2}\right)\triangledown)k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee, \alpha)}{-2}\right)\triangledown) - (\alpha^\vee, \alpha + \beta)\frac{k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee, \alpha)}{-2}\right)\triangledown) - k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee, \alpha)}{-2}\right)\triangledown)}{adx_{(\alpha + \beta)^\vee} + adx_{\omega((\alpha + \beta)^\vee, \alpha)}} \\
- ((\alpha + \beta)^\vee, \alpha)\frac{k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee, \alpha)}{-2}\right)\triangledown) - k(\alpha + \beta, \text{ad}\left(\frac{\omega((\alpha + \beta)^\vee, \alpha)}{-2}\right)\triangledown)}{adx_{(\alpha + \beta)^\vee} + adx_{\omega((\alpha + \beta)^\vee, \alpha)}}. 
\]
Now, set $u := \frac{\omega(\alpha, \beta)}{-2}$ and $v := \frac{\omega(\beta, \alpha + \beta)}{-2}$. Using the identities from Lemma 6.3.1, we get:

1. $u = \frac{\omega(\alpha, \beta)}{-2} = \frac{\beta + \omega(\beta, \alpha)}{\alpha, \alpha} - 2 = \beta + \omega(\beta, \alpha) - 2$.

2. $v = \frac{\omega(\beta, \alpha + \beta)}{-2} = \frac{\alpha + \omega(\alpha, \beta)}{\beta, \beta} - 2$.

Rewrite the above formula, we get:

$$
-k(\alpha, \beta) + k(\alpha, \alpha + \beta) + k(\beta, \alpha + \beta) = -k(\alpha, u|\tau)k(\beta, u + v|\tau) + k(\alpha, -v|\tau)k(\alpha + \beta, u + v|\tau) + k(\beta, v|\tau)k(\alpha + \beta, u|\tau) + \frac{k(\alpha, -v|\tau)}{u + v} + \frac{k(\beta, v|\tau)}{u} - \frac{k(\alpha + \beta, u|\tau)}{-v} = 0,
$$

which is 0 by the following theta function identity used in [10].

$$
k(z, -v)k(z', u + v) - k(z, u)k(z' - z, u + v) + k(z', u)k(z' - z, v) + \frac{k(z' - z, v)}{u} + \frac{k(z' - z, u + v)}{u + v} + \frac{k(z, u) - k(z, -v)}{u + v} = 0
$$

This ends the proof.

**Lemma 6.4.2.** Let $\Phi$ be a root system. For any two roots $\alpha, \beta \in \Phi$, then $\Psi := (\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi$ is a root subsystem of rank 2.

**Proof.** It’s straightforward to check all the axioms of root system hold for $\Psi$. 

For any $\alpha, \beta \in \Phi$, let $\Psi_{\alpha, \beta} := (\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi$. Whenever we have $\Psi_{\alpha', \beta'} \not\subseteq \Psi_{\alpha, \beta}$, delete $\Psi_{\alpha', \beta'}$; for $\Psi_{\alpha', \beta'} = \Psi_{\alpha, \beta}$, just keep one set $\Psi_{\alpha, \beta}$. Thus, we have a finite collection of rank 2 subsystems. List them as $\Psi_1, \ldots, \Psi_m$. We will show for any $\alpha, \beta \in \Phi$, there exist exactly one $\Psi_k$, such that $\alpha, \beta \in \Psi_k$.

It’s obvious that for any $\alpha, \beta \in \Phi$, there exist one $\Psi_k$, such that $\alpha, \beta \in \Psi_k$. 

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Suppose $\alpha, \beta \in \Psi_{k_1}$, and $\alpha, \beta \in \Psi_{k_2}$, and $\Psi_{\alpha, \beta} \neq \Psi_{k_i}, i = 1, 2$, then $\Psi_{k_1}, \Psi_{k_2}$ are both of type $B_2$, or $G_2$. If $\Psi_{k_1} \neq \Psi_{k_2}$, then there exists one root $\alpha$, such that $2\alpha \in \Phi$, which contradicts with the definition of root system. Thus, $\Psi_{k_1} = \Psi_{k_2}$.

Thus, we are able to write: $\Omega_1 - \Omega_2 = \sum_{i=1}^{n} \sum_{\alpha, \beta \in \Psi^+_i} (\Omega_1 - \Omega_2)^\Psi_i$, where

$$(\Omega_1 - \Omega_2)^\Psi_i = \sum_{\alpha \neq \beta \in \Psi^+} \frac{1}{2} k(\alpha, \text{ad}(\frac{x_\omega(\alpha \cdot \beta)}{-2})|\tau) k(\beta, \text{ad}(\frac{x_\omega(\beta \cdot \alpha)}{-2})|\tau)[t_\alpha, t_\beta] d\alpha \wedge d\beta.$$

$$- \sum_{\alpha \neq \beta \in \Psi^+} (\alpha \cdot \beta) \frac{k(\alpha, \text{ad}(\frac{x_\omega(\alpha \cdot \beta)}{2})|\tau) - k(\alpha, \text{ad}(\frac{x_\omega(\alpha \cdot \beta)}{-2})|\tau)}{\text{ad}x_\alpha + \text{ad}x_\omega(\alpha \cdot \beta)} [t_\alpha, t_\beta] d\alpha \wedge d\beta.$$

So, in order to show $\Omega_1 - \Omega_2 = 0$, we need to show $(\Omega_1 - \Omega_2)^\Psi = 0$, for $\Psi$ rank 2 root system.

### 6.4.1 Case-A2

Let $\epsilon_1, \epsilon_2, \epsilon_3$ be the standard orthogonal basis of $\mathbb{R}^3$. Let $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3$ be the simple roots for $A_2$. The root system $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$.

Now, by definition we have:

$$\Omega_1 - \Omega_2 = k(\alpha_1, \alpha_2)[t_{\alpha_1}, t_{\alpha_2}] d\alpha_1 \wedge d\alpha_2 + k(\alpha_1, z_{\alpha_1+\alpha_2})[t_{\alpha_1}, t_{\alpha_1+\alpha_2}] d\alpha_1 \wedge d(\alpha_1 + \alpha_2)$$

$$+ k(\alpha_2, z_{\alpha_1+\alpha_2})[t_{\alpha_2}, t_{\alpha_1+\alpha_2}] d\alpha_2 \wedge d(\alpha_1 + \alpha_2)$$

Using Proposition 6.4.1, we can write

$$k(\alpha_1, \alpha_1 + \alpha_2) d\alpha_1 \wedge d(\alpha_1 + \alpha_2) = k(\alpha_1, \alpha_2) d\alpha_1 \wedge d\alpha_2 + k(\alpha_2, \alpha_1 + \alpha_2) d\alpha_2 \wedge d(\alpha_1 + \alpha_2).$$

Plug the above expression into the definition of $\Omega_1 - \Omega_2$, we have

$$\Omega_1 - \Omega_2 = k(\alpha_1, \alpha_2)[t_{\alpha_1}, t_{\alpha_2} + t_{\alpha_1+\alpha_2}] d\alpha_1 \wedge d\alpha_2$$

$$+ k(\alpha_2, z_{\alpha_1+\alpha_2})[t_{\alpha_1} + t_{\alpha_2}, t_{\alpha_1+\alpha_2}] d\alpha_2 \wedge d(\alpha_1 + \alpha_2)$$

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Using (tt)-relations, we have \(\Omega_1 - \Omega_2 = 0\) in the case of \(A_2\). The above procedure will be represented as the graph:

\[
\begin{align*}
\text{(6.4.2)} & \quad (\alpha_1, \alpha_1 + \alpha_2) \\
& \quad \quad \downarrow \\
(\alpha_1, \alpha_2) & \quad \quad \longrightarrow \quad \quad (\alpha_2, \alpha_1 + \alpha_2)
\end{align*}
\]

### 6.4.2 Case-B2

Let \(\epsilon_1, \epsilon_2\) be the standard orthogonal basis of \(\mathbb{R}^2\). Let \(\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2\) be the simple roots for \(B_2\). The root system \(\Phi^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}\). Now, by definition we have:

\[
\begin{align*}
\Omega_1 - \Omega_2 &= k(\alpha_1, \alpha_2)[t_{\alpha_1}, t_{\alpha_2}]d\alpha_1 \wedge d\alpha_2 + k(\alpha_1, z_{\alpha_1+\alpha_2})[t_{\alpha_1}, t_{\alpha_1+\alpha_2}]d\alpha_1 \wedge d(\alpha_1 + \alpha_2) \\
&+ k(z_{\alpha_1+\alpha_2}, \alpha_2)[t_{\alpha_1+\alpha_2}, t_{\alpha_2}]d(\alpha_1 + \alpha_2) \wedge d\alpha_2 \\
&+ k(\alpha_1, \alpha_1 + 2\alpha_2)[t_{\alpha_1}, t_{\alpha_1+2\alpha_2}]d\alpha_1 \wedge d(\alpha_1 + 2\alpha_2) \\
&+ k(z_{\alpha_1+\alpha_2}, \alpha_1 + 2\alpha_2)[t_{\alpha_1+\alpha_2}, t_{\alpha_1+2\alpha_2}]d(\alpha_1 + \alpha_2) \wedge d(\alpha_1 + 2\alpha_2) \\
&+ k(\alpha_1 + 2\alpha_2, \alpha_2)[t_{\alpha_1+2\alpha_2}, t_{\alpha_2}]d(\alpha_1 + 2\alpha_2) \wedge d\alpha_2.
\end{align*}
\]

Using the graph

\[
\begin{align*}
\text{(6.4.3)} & \quad (\alpha_1, \alpha_2) \\
& \quad \quad \downarrow \\
(\alpha_1, \alpha_1 + \alpha_2) & \quad \quad \longrightarrow \quad \quad (\alpha_1 + \alpha_2, \alpha_2) \\
& \quad \quad \quad \downarrow \\
(\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2) & \quad \quad \longrightarrow \quad \quad (\alpha_1 + 2\alpha_2, \alpha_2)
\end{align*}
\]

We have:

\[
\begin{align*}
\Omega_1 - \Omega_2 &= k(\alpha_1, \alpha_2)[t_{\alpha_1}, t_{\alpha_2} + t_{\alpha_1+\alpha_2}]d\alpha_1 \wedge d\alpha_2 \\
&+ k(\alpha_2, \alpha_1 + \alpha_2)[t_{\alpha_1} + t_{\alpha_2} + t_{\alpha_1+2\alpha_2}, t_{\alpha_1+\alpha_2}]d\alpha_2 \wedge d(\alpha_1 + \alpha_2) \\
&+ k(\alpha_2, \alpha_1 + 2\alpha_2)[t_{\alpha_2} + t_{\alpha_1+\alpha_2}, t_{\alpha_1+2\alpha_2}]d\alpha_2 \wedge d(\alpha_1 + 2\alpha_2)
\end{align*}
\]
Note that each of the coefficient is of the form

$$[t_{\alpha}, \sum_{\beta \in \Psi^+} t_{\beta}],$$

where $\Psi = B_2$ root system. By (tt)-relation, we have $\Omega_1 - \Omega_2 = 0$ in the case of $B_2$.

**Remark 6.4.3.** Above is only one connected component of graph, the components are labeled by the root subsystem of $\Phi$. Here, there is another one corresponding to $A_1 \times A_1$ and it’s just the one vertex graph: $(\alpha_1, \alpha_1 + 2\alpha_2)$.

### 6.4.3 Case-G2

Let $\epsilon_1, \epsilon_2, \epsilon_3$ be the standard orthogonal basis of $\mathbb{R}^3$ The simple roots $\alpha_1 = 2\epsilon_2 - \epsilon_1 - \epsilon_3, \alpha_2 = \epsilon_1 - \epsilon_2$. The root system $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}$.

The connected component of the graph corresponding to $G_2$ is:

$$\begin{align*}
(\alpha_1, \alpha_2) \\
(\alpha_1, \alpha_1 + \alpha_2) &\rightarrow (\alpha_1 + \alpha_2, \alpha_2) \\
(\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2) &\rightarrow (\alpha_1 + 2\alpha_2, \alpha_2) \\
(\alpha_1 + \alpha_2, 2\alpha_1 + 3\alpha_2) &\rightarrow (2\alpha_1 + 3\alpha_2, \alpha_1 + 2\alpha_2) \\
(\alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2) &\rightarrow (\alpha_1 + 3\alpha_2, \alpha_2)
\end{align*}$$

(6.4.4)
Use the above graph, and Proposition 6.4.1, We have:

\[
\Omega_1 - \Omega_2 \\
= k(\alpha_1, \alpha_2)[t_{\alpha_1}, t_{\alpha_2} + t_{\alpha_1 + \alpha_2}]d\alpha_1 \wedge d\alpha_2 + k(\alpha_2, \alpha_1 + \alpha_2)[t_{\alpha_1} + t_{\alpha_2}, t_{\alpha_1 + \alpha_2}]d\alpha_1 \wedge d(\alpha_1 + \alpha_2) \\
+ k(\alpha_1 + 2\alpha_2, \alpha_2)([t_{\alpha_1 + 2\alpha_2}, t_{\alpha_2} + t_{\alpha_1 + \alpha_2} + t_{\alpha_1 + 3\alpha_2}] - [t_{\alpha_1 + \alpha_2}, t_{2\alpha_1 + 3\alpha_2}])d(\alpha_1 + 2\alpha_2) \wedge d\alpha_2 \\
+ k(2\alpha_1 + 3\alpha_2, \alpha_1 + 2\alpha_2)[t_{2\alpha_1 + 3\alpha_2}, t_{\alpha_1 + 2\alpha_2} + t_{\alpha_1 + \alpha_2}]d(2\alpha_1 + 3\alpha_2) \wedge d(\alpha_1 + 2\alpha_2) \\
+ k(\alpha_1 + 3\alpha_2, \alpha_2)[t_{\alpha_1 + 3\alpha_2}, t_{\alpha_2} + t_{\alpha_1 + 2\alpha_2}]d(\alpha_1 + 3\alpha_2) \wedge d\alpha_2
\]

Note that this term

\[
[t_{\alpha_1 + 2\alpha_2}, t_{\alpha_2} + t_{\alpha_1 + \alpha_2} + t_{\alpha_1 + 3\alpha_2}] - [t_{\alpha_1 + \alpha_2}, t_{2\alpha_1 + 3\alpha_2}]
\]

\[
= [t_{\alpha_1 + 2\alpha_2}, t_{\alpha_2} + t_{\alpha_1 + \alpha_2} + t_{\alpha_1 + 3\alpha_2} + t_{2\alpha_1 + 3\alpha_2}] - [t_{\alpha_1 + \alpha_2} + t_{\alpha_1 + 2\alpha_2}, t_{2\alpha_1 + 3\alpha_2}]
\]

and all the coefficients have the required form, that is, \([t_\alpha, \sum_{\beta \in \Psi} t_\beta]\), where \(\Psi = G_2\) root system. Apply the (tt)-relation, we have \(\Omega_1 - \Omega_2 = 0\) in the case of \(G_2\).

### 6.5 Equivariance under Weyl group action

Let \(W\) be the Weyl group. Assume that \(W\) acts on the algebra \(A_{\text{ell}}^\Phi\) by the following way:

For any simple reflection \(s_i \in W\) corresponding to a simple root \(\alpha_i\),

1. \(s_i(t_\alpha) = t_{s_i\alpha}\);

2. \(s_i(x(u)) = x(s_i u)\);

3. \(s_i(y(v)) = y(s_i v)\)

for any \(\alpha \in \Phi\), and \(u, v \in \h\).

**Proposition 6.5.1.** The connection \(\nabla\) is \(W\)-equivariant.
Proof. Note that $s_i$ permutes the set $\Phi^+ \setminus \{\alpha_i\}$, and $k(-\alpha, -\text{ad} \frac{\alpha}{2}) = k(\alpha, \text{ad} \frac{\alpha}{2})$. We have

$$s_i^*\nabla = d - \sum_{\alpha \in \Phi^+} k(s_i\alpha, \text{ad} \left(\frac{s_i\alpha}{2}\right))|\tau\rangle(s_i\alpha)d(s_i\alpha) + \sum_{j=1}^{n-1} (s_iy(u^j))d(s_iu_j)$$

$$= d - \sum_{\beta \in \Phi^+} k(\beta, \text{ad} \left(\frac{s_i\beta}{2}\right))|\tau\rangle(s_i\beta)d\beta + \sum_{j=1}^{n-1} (s_iy(u^j))d(s_iu_j)$$

$$= d - \sum_{\beta \in \Phi^+} k(\beta, \text{ad} \left(\frac{\beta}{2}\right))|\tau\rangle(t_\beta)d\beta + \sum_{j=1}^{n-1} (y((s_iu^j))d(s_iu_j)$$

$$= \nabla$$

Thus, the connection is $W$-equivariant. 

\[\Box\]

### 6.6 Degeneration of the elliptic connection

We show in this section that as $\tau \to +i\infty$, the connection $\nabla$ degenerates to a trigonometric connection of the form considered in [85].

#### 6.6.1 The trigonometric connection

In [85], Toledano Laredo introduced the trigonometric connection, which we recall it here. Let $H = \text{Hom}_\mathbb{Z}(P, \mathbb{C}^*)$ be the complex algebraic torus with Lie algebra $\mathfrak{h}$ and coordinate ring given by the group algebra $\mathbb{C}P$. We denote the function corresponding to $\lambda \in P$ by $e^\lambda \in \mathbb{C}[H]$, and set

$$H_{\text{reg}} = H \setminus \bigcup_{\alpha \in \Phi} \{e^\alpha = 1\} \quad (6.6.1)$$

Let $A_{\text{trig}}$ be an algebra endowed with the following data:

- a set of elements $\{t_\alpha\}_{\alpha \in \Phi} \subset A_{\text{trig}}$ such that $t_{-\alpha} = t_\alpha$
- a linear map $\tau : \mathfrak{h} \to A_{\text{trig}}$
Consider the $A_{\text{trig}}$–valued connection on $H_{\text{reg}}$ given by

$$\nabla = d - \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^\alpha - 1} t_\alpha - du_i \tau(u^i)$$  \hfill (6.6.2)

where $\Phi_+ \subset \Phi$ is a chosen system of positive roots, $\{u_i\}$ and $\{w^i\}$ are dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, the differentials $du_i$ are regarded as translation–invariant one–forms on $H$ and the summation over $i$ is implicit.

**Theorem 6.6.1** (Toledano Laredo [85]).

(1) The connection (6.6.2) is flat if, and only if the following relations hold:

\begin{itemize}
  \item[(tt)] For any rank 2 root subsystem $\Psi \subset \Phi$, and $\alpha \in \Psi$,
  $$[t_\alpha, \sum_{\beta \in \Psi_+} t_\beta] = 0.$$
  
  \item[(\tau\tau)] For any $u, v \in \mathfrak{h}$,
  $$[\tau(u), \tau(v)] = 0.$$
  
  \item[(t\tau)] For any $\alpha \in \Phi_+$, $w \in W$ such that $w^{-1} \alpha$ is a simple root and $u \in \mathfrak{h}$, such that $\alpha(u) = 0$,
  $$[t_\alpha, \tau_w(u)] = 0,$$
  where $\tau_w(u) = \tau(u) - \sum_{\beta \in \Phi_+ \cap \mathfrak{s} \Phi} \beta(u) t_\beta$.
\end{itemize}

(2) Modulo the relations (tt), the relations (t\tau) are equivalent to

\begin{itemize}
  \item[(tY)]
  $$[t_\alpha, Y(v)] = 0,$$
  for any $\alpha \in \Phi$ and $v \in \mathfrak{h}$ such that $\alpha(v) = 0$.
\end{itemize}

**Example 6.6.2.** In the case of $B_2$, the (tt) relations become:

$$[t_{\alpha_1}, t_{\alpha_1+2\alpha_2}] = 0, \quad [t_\gamma, \sum_{\beta \in \Psi_+} t_\beta] = 0,$$
while
\[ [t_{\alpha_2}, t_{\alpha_1 + \alpha_2}] \neq 0. \]

Assume now that the algebra \( A_{\text{trig}} \) is acted upon by the Weyl group \( W \) of \( \Phi \).

**Proposition 6.6.3.**

1. **The connection** \( \nabla_{\text{trig}} \) **is** \( W \) **equivariant if, and only if** For any \( \alpha \in \Phi \), simple reflection \( s_i \in W \) and \( x \in h \),
   \[
   s_i(t_{\alpha}) = t_{s_i(\alpha)}, \quad (6.6.3)
   \]
   \[
   s_i(\tau(x)) - \tau(s_i x) = (\alpha_i, x) t_{\alpha_i}. \quad (6.6.4)
   \]

2. **Modulo** (6.6.3), **the relation** (6.6.4) **is equivalent to** \( W \) **equivariance of** the linear map \( Y : h \to A_{\text{trig}} \).

**6.6.2**

We study the degeneration of the connection \( \nabla_{\text{KZB}} \) (6.2.3) as the modular parameter \( \tau \) changes as \( \text{Im } \tau \to +\infty \).

Let \( \wp(z) \) be the Weierstrass function with respect to the lattice \( \mathbb{Z} + \tau \mathbb{Z} \),
\[
\wp(z) = \frac{1}{z^2} + \sum_{m,n} \left( \frac{1}{(z - m\tau + n)^2} - \frac{1}{(m\tau + n)^2} \right).
\]
Differentiating \( \wp(z) \) term by term, we get \( \wp'(z) \).

Let \( q = e^{2\pi i \tau} \), the points \( (\wp(z), \wp'(z)) \) lie on the curve defined by the equation
\[
y^2 = 4x^3 - g_2x - g_3,
\]
where
\[
g_2 = \frac{(2\pi i)^4}{12} \left( 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} \right),
\]
and
\[
g_3 = \frac{(2\pi i)^6}{6^3} \left( -1 + 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} \right).
\]
The cubic polynomial $4x^3 - g_2x - g_3$ has a discriminant given by

$$
\Delta = g_2^3 - 27g_3^2.
$$

The map

$$
\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \to E \subset \mathbb{P}(C),
$$
given by $z \mapsto (1, \wp(z), \wp'(z))$ is an isomorphism of complex Lie groups, where $E$ is defined by the equation $y^2 = 4x^3 - g_2x - g_3$.

Let the modular parameter $\tau$ changes as $\text{Im} \tau \to +\infty$, then $q \to 0$. The elliptic curve $E$ degenerates to $\tilde{E}$ defined by the equation

$$
y^2 = 4x^3 - \frac{(2\pi i)^4}{12} x + \frac{(2\pi i)^6}{6^3} = -\frac{4}{27}(2\pi^2 - 3x)(3x + \pi^2)^2,
$$
with discriminant $\Delta = 0$. It has a singular point at $(x, y) = (\frac{2\pi^2}{3}, 0)$.

Removing the singular point $x_0$ of the degenerating elliptic curve $\tilde{E}_\tau$, topologically, the open subset $\tilde{E}_\tau \setminus x_0$ is homeomorphic to the complex torus $\mathbb{C}^*$. Thus, we have a continues map on topological spaces $\phi : \mathbb{C}^* \otimes P^\vee \to \tilde{E}_\tau \otimes P^\vee$.

Pulling-back the degenerating bundle $\tilde{P}_{\tau,n}$ under the map $\phi$, we get a trivial principal bundle on $\mathbb{C}^* \otimes P^\vee$ with group $\exp(\hat{A}_{\alpha})$. The section of the trivial bundle can be described as

$$
\{ f(z) : \pi^{-1}(U) \to \exp(\hat{A}_{\alpha}) \mid f(z + 1) = f(z) \},
$$
where $U \subset \mathbb{C}^* \otimes P^\vee$ and $\pi = \exp : \mathbb{C} \to \mathbb{C}^*$ be the natural exponential map.

### 6.6.3

We describe the degeneration of the connection (6.2.3) as $\text{Im} \tau \to +\infty$ in this subsection.

We denote the **Bernoulli numbers** by $B_n$, which are given by the power series expansion

$$
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.
$$
We have
\[ \pi z \cot(\pi z) = \sum_{n=0}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} B_{2n} z^{2n} = 1 + \sum_{n=1}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} B_{2n} z^{2n}. \]

Some special values of the Bernoulli numbers are given by
\[ B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad \text{and} \quad B_{2n+1} = 0, \quad \text{for} \quad n \geq 1. \]

As \( \text{Im} \, \tau \rightarrow +\infty \), using the product formula (6.2.1), one can show that the theta function \( \theta(z|\tau) \) tends to
\[ \theta(z|\tau) \rightarrow u^{\frac{1}{2}}(1 - u^{-1}) \frac{1}{2\pi i} = \frac{\sin(\pi z)}{\pi}. \]

Thus, by (6.6.5),
\[ k(\alpha, \text{ad}(\frac{x_{\alpha}^\vee}{2})|\tau) \rightarrow \pi \frac{\sin(\pi \alpha + \pi \text{ad}(\frac{x_{\alpha}^\vee}{2}))}{\sin(\pi \alpha) \sin(\pi \text{ad}(\frac{x_{\alpha}^\vee}{2}))} - \frac{2}{\text{ad} \, x_{\alpha}^\vee}
= \pi \cot(\pi \alpha) + \pi \cot(\pi \text{ad}(\frac{x_{\alpha}^\vee}{2})) - \frac{1}{\text{ad} \, x_{\alpha}^\vee}
= \pi \cot(\pi \alpha) + \sum_{n=1}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} B_{2n} \text{ad}(\frac{x_{\alpha}^\vee}{2})^{2n-1}. \]

As \( \text{Im} \, \tau \rightarrow +\infty \), the connection \( \nabla_{\text{KZB}} \) (6.2.3) degenerates to a flat connection \( \tilde{\nabla} \) on the regular points
\[ (\mathbb{C}^* \otimes P^\vee) \setminus \{ \cup_{\alpha \in \Phi \{ e^{2\pi i \alpha} = 1 \}} \}. \]

The degenerate connection \( \tilde{\nabla} \) is on the trivial principal bundle with fibers the group \( \exp(\hat{A}_{\text{trig}}) \).

Using \( \cot(\pi \alpha) = \frac{2i}{e^{2\pi i \alpha} - 1} + i \), we get:

**Proposition 6.6.4.** The degenerate connection \( \tilde{\nabla} \) takes the following form:
\[ \tilde{\nabla} = d - \sum_{\alpha \in \Phi^+} (\pi \cot(\pi \alpha) + \sum_{n=1}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} B_{2n} \text{ad}(\frac{x_{\alpha}^\vee}{2})^{2n-1}(t_\alpha)) d\alpha + \sum_{i=1}^{n} y(u^i) du_i
= d - \sum_{\alpha \in \Phi^+} \left( \frac{2\pi i t_\alpha}{e^{2\pi i \alpha} - 1} + \pi i t_\alpha + \sum_{n=1}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} B_{2n} \text{ad}(\frac{x_{\alpha}^\vee}{2})^{2n-1}(t_\alpha) \right) d\alpha + \sum_{i=1}^{n} y(u^i) du_i. \]

We modify the trigonometric connection (6.6.2) slightly by pulling-back to
\[ (\mathbb{C}^* \otimes P^\vee) \setminus \bigcup_{\alpha \in \Phi \{ e^{2\pi i \alpha} = 1 \}} \].
We then get an $A_{\text{trig}}$-valued flat trigonometric connection:

\[
\nabla = d - \sum_{\alpha \in \Phi_+} \frac{2\pi i d\alpha}{e^{2\pi i \alpha} - 1} t_\alpha - du \tau(u^i). \tag{6.6.6}
\]

Theorem 6.6.1 and Proposition 6.6.3 are the same with the modified connection.

By universality of the trigonometric holonomy algebra $A_{\text{trig}}$, Proposition 6.6.4 gives rise to a map $A_{\text{trig}} \to A_{\text{ell}}$ given by

\[
t_\alpha \mapsto t_\alpha
\]

\[
\tau(u) \mapsto -y(u) + \sum_{\alpha \in \Phi^+} (\alpha, u) \left( \pi i t_\alpha + \sum_{n=1}^{\infty} \frac{(2\pi)^{2n}}{B_{2n}} B_{2n} \text{ad} \left( \frac{x_{\alpha^\vee}}{2} \right)^{2n-1} t_\alpha \right).
\]

Note that this map does not preserve the usual gradings of $A_{\text{trig}}, A_{\text{ell}}$, but it preserves the following gradings of $A_{\text{trig}}, A_{\text{ell}}$. The grading on $A_{\text{trig}}$ is given by: $\deg(t_\alpha) = \deg(\tau(u)) = 1$. The grading on $A_{\text{ell}}$ is given by: $\deg(t_\alpha) = \deg(y(u)) = 1$ and $\deg(x(u)) = 0$.

### 6.7 Generalized Formality Isomorphism

The following definition can be found in [16, Definition 2.4]. See [5, ?] for more details.

**Definition 6.7.1.** A finitely generated group $G$ is 1-formal if its Malcev Lie algebra is isomorphic to the completion of its holonomy Lie algebra as filtered Lie algebras.

**Remark 6.7.2.** 1. By construction (see [16]), the holonomy Lie algebra is a quadratic Lie algebra, that is, it has a presentation with generators in degree 1 and relations in degree 2 only.

2. As showed in [5], the fundamental group $G$ of configuration spaces of $n$ points on a smooth compact Riemann surface is 1-formal except when the genus $g = 1$, and $n \geq 3$. See also [16, Example 10.1] for the failing of 1-formality when $g = 1$, and $n \geq 3$. 

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3. When $g = 1$, and $n \geq 3$, there is a weaker statement in [5, 10] that the Malcev Lie algebra of $G$ is isomorphic to the completion of the generalized holonomy Lie algebra, which is not a quadratic Lie algebra.

In this section, we follow the method in [10] and establish the weaker Formality Isomorphism.

6.7.1 The double affine braid group

We first recall some results from [13] about the topological interpretation of the double affine braid group.

Let $i^2 = -1$, and set

$$U = \{z \in \mathbb{C}^n \mid (\alpha, z) \notin \mathbb{Z} + i\mathbb{Z}, \text{ for any root } \alpha \in \Phi\}, \quad \bar{U} = \mathbb{C}^* \times U.$$  

Let $\bar{W}$ be the 2-extended Weyl group $\bar{W} := W \ltimes (P^\vee \oplus iP^\vee)$. For any element $\bar{w} = (w, a, b) \in \bar{W}$, the action of $\bar{w}$ on $\bar{z} = (z_\ast, z) \in \bar{U}$ is given by:

$$\bar{w}(\bar{z}) = ((-1)^{l'(w)}, w(z) + a + ib), \quad \text{where } l'(w) \text{ is the modified length of } w.$$ 

We fix a base point $\bar{z}^0 = (z_\ast^0, z^0)$ such that the real and imaginary parts of $\{z_\ast^0, z_j^0, 1 \leq j \leq n\}$ are positive and sufficiently small.

Note that the action of $\bar{W}$ on $\bar{U}$ is not free.

**Definition 6.7.3.** [13, Definition 2.3] The double affine braid group $\mathfrak{B}$ is formed by the paths $\gamma \subset \bar{U}$ joining $\bar{z}^0$ with points from $\{\bar{w}(\bar{z}^0), \bar{w} \in \bar{W}\}$ modulo the homotopy and the action of $\bar{W}$.

We introduce the elements $t_j := t_{\alpha_j}, x_j, y_j$ for $1 \leq j \leq n$ and $c \in \mathfrak{B}$ by the paths, for $1 \leq \psi \leq 1$:

$$t_j(\psi) = (\exp(l'(j)\pi\nu\psi)z_\ast^0, z^0 + (\exp(\pi\nu\psi) - 1)(z^0, \alpha_j)\alpha_j),$$
\[ x_j(\psi) = (z_0^0, z_0^0 + \psi b_j), \quad y_j(\psi) = (z_0^0, z_0^0 + i \psi b_j), \]
\[ c(\psi) = (\exp(-2\pi i \psi) z_0^0, z_0^0). \]

**Theorem 6.7.4.** [13, Theorem 2.4] The group \( B \) is generated by \( \{ t_i, 1 \leq i \leq n \} \), the two sets \( \{ x_i \}, \{ y_i \} \) of pairwise commutative elements and central \( c \) with the following relations:

1. \( t_i t_j t_i \cdots = t_j t_i t_j \cdots, \) \( m_{ij} \) factors on each side.

2. \( t_i x_i t_i = x_i x_j^{-1} c_i^{-1} \) and \( t_i^{-1} y_i t_i^{-1} = y_i y_j^{-1} c_i, \)

3. \( p_x x_j p_x^{-1} = x_j x_i^{-(b_i, \theta)} \) and \( p_x x_j p_x^{-1} = x_j^{-1}, \)

where \( c_i = c_{\theta(i)} \), \( p_x := y_r t_w^{-1} \) for \( w = \sigma_r^{-1}. \)

As is discussed in [13], the map \( T_i \mapsto t_i, \) \( X_b \mapsto x_b c^{(\rho', b)} \) and \( Y_b \mapsto y_b c^{-(\rho', b)}, \) \( \delta \mapsto c^d, \) where
\[ \rho' := \sum_{\alpha \in \Phi^+} \frac{\nu(\alpha) \alpha}{2}, \quad d = \frac{(\rho', \theta) + 1}{m} \in \mathbb{Z}, \quad b \in P^\vee, \]
identifies the double affine Hecke group (see [13] for the definition) with a subgroup of \( \mathfrak{B}. \) In particular, we have the relations:

\[ T_i X_b = X_b T_i \quad T_i Y_b = Y_b T_i, \quad \text{if } (b, \alpha_i) = 0. \]

We work on the spaces of the regular points of the adjoint torus

\[ X_{\text{reg}}^{\text{reg}} = (P^\vee \otimes \mathcal{E})^{\text{reg}} / W = U/W. \]

**Corollary 6.7.5.** The fundamental group of the space \( X_{\text{reg}}^{\text{reg}} \) is isomorphic to the double affine braid group \( \mathfrak{B} \) modulo the central element \( c. \)

**6.7.2**

We have the following short exact sequence

\[ 0 \to P \mathfrak{B} \to \mathfrak{B} \to W \to 0, \tag{6.7.1} \]

where \( P \mathfrak{B} := \pi_1(X_{\text{reg}}, x_0). \)

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Let $\gamma_\alpha \in P^B$ be the path around the hypertorus $H_\alpha$, then under the inclusion

$$P^B \hookrightarrow B,$$

we gave $\gamma_\alpha \mapsto (t_i)^2$, $X_i \mapsto X_i$, and $Y_i \mapsto Y_i$, for any $1 \neq i \leq n$. In particular, the following relations holds in $\pi_1(X_{\text{reg}}, x_o)$:

1. $\gamma_\alpha X(u) = X(u)\gamma_\alpha$, if $(\alpha, u) = 0$;

2. $\gamma_\alpha Y(u) = Y(u)\gamma_\alpha$, if $(\alpha, u) = 0$;

3. $X(u)X(v) = X(v)X(u)$, and $Y(u)Y(v) = Y(v)Y(u)$.

The flatness of the universal connection (6.2.3) gives rise to the monodromy map

$$\mu : P^B \to \exp \hat{A}_{\text{ell}},$$

where $\hat{A}_{\text{ell}}$ is the completion of $A_{\text{ell}}$ with respect to the grading $\deg(x(u)) = \deg(y(u)) = 1$ and $\deg(t_\alpha) = 2$.

Let $J \subseteq \mathbb{C}[P^B]$ be the augmentation ideal of group ring $\mathbb{C}[P^B]$, and let $\hat{\mathbb{C}}[P^B]$ be the completion with respect to the augmentation ideal $J$.

Let $U(A_{\text{ell}})$ be the universal enveloping algebra of the generalized holonomy Lie algebra $A_{\text{ell}}$. The monodromy map $\mu$ induces the following map on the completions:

$$\hat{\mu} : \hat{\mathbb{C}}[P^B] \to \hat{U}(A_{\text{ell}})$$

**Theorem 6.7.6.** The induced map $\hat{\mu} : \hat{\mathbb{C}}[P^B] \to \hat{U}(A)$ is an isomorphism of Hopf algebras.

Taking the primitive elements of the Hopf algebras $\hat{\mathbb{C}}[P^B]$ and $\hat{U}(A_{\text{ell}})$, we get an isomorphism of the Malcov Lie algebra of $P^B$ and the generalized holonomy Lie algebra $A_{\text{ell}}$ defined in Definition 6.1.2.

**Corollary 6.7.7.** The defining relations of the generalized holonomy Lie algebra $A_{\text{ell}}$ are necessary to make the connection (6.2.3) flat.
The rest of the section is devoted to prove the generalized formality isomorphism in Theorem 6.7.6.

Proof. Since both $\hat{C}[\pi_1]$ and $\hat{A}_{\text{ell}}$ is $\mathbb{N}$–filtered. It suffices to show the associated graded

$$\text{gr}(\mu) : \text{gr}(\hat{C}[P]) \to \text{gr}(\hat{U}(A_{\text{ell}})) = \hat{U}(A_{\text{ell}})$$

is an isomorphism.

We construct a map:

$$p : \hat{U}(A_{\text{ell}}) \to \text{gr}(\hat{C}[\pi_1]),$$

by sending $t_\alpha \mapsto T_\alpha$, $x(\lambda^\vee_i) \mapsto X_i$, and $y(\lambda^\vee_i) \mapsto Y_i$. We need to check the following:

1. $p$ is a homomorphism, that is, $p$ sends the relations in $A_{\text{ell}}$ to 0.

2. $p$ is surjective.

3. $\text{gr}(\mu) \circ p$ is an isomorphism.

Let $T := \text{Hom}_\mathbb{Z}(Q, \mathcal{E}) = P^\vee \otimes_\mathbb{Z} \mathcal{E}$ be the torus with adjoint type. Denote $\ker(\chi_\alpha)$ by $T_\alpha \subseteq T$.

Lemma 6.7.8. There exist a component $(T_\alpha \cap T_\beta)_\chi$, such that $(T_\alpha \cap T_\beta)_\chi$ contained in a subtorus $T_\gamma$ if and only if $\gamma \in \mathbb{Z}\alpha + \mathbb{Z}\beta$.

Proof. We can assume $\alpha$ is a simple root. Note $\mathbb{Z}\alpha + \mathbb{Z}\beta$ may not be a primitive sublattice in $Q$. Pick a vector $e_2$ such that $\alpha, e_2$ generate the primitive closure $\mathbb{Z}\alpha + \mathbb{Z}e_2$ of $\mathbb{Z}\alpha + \mathbb{Z}\beta$. Write $\beta = aa + be_2$. Use $\alpha, e_2$ generates a set of basis of $Q$, call it $\{e_1 := \alpha, e_2, e_3, \ldots, e_n\}$.

Choose a dual basis $\{f_1, f_2, f_3, \ldots, f_n\}$, which is a set of basis of $P^\vee$. Since the coweight lattice $P^\vee$ is dual to root lattice $Q$.
Identify $T = P^\vee \otimes_{\mathbb{Z}} \mathcal{E}$ with $(\mathcal{E})^n$ using the basis $\{f_1, f_2, f_3, \ldots, f_n\}$. Then, the maps $\chi_\alpha, \chi_\beta$ are given by:

$$\chi_\alpha : (\mathcal{E})^n \to \mathcal{E}, \Sigma_{i=1}^n z_i f_i \mapsto z_1.$$ 

and

$$\chi_\beta : (\mathcal{E})^n \to \mathcal{E}, \Sigma_{i=1}^n z_i f_i \mapsto az_1 + bz_2.$$ 

Thus, $T_\alpha \cong \{0\} \times (\mathcal{E})^{n-1} \subseteq (\mathcal{E})^n$, and $T_\alpha \cap T_\beta = \{(0, z_2, \ldots, z_n) \subseteq (\mathcal{E})^n \mid bz_2 = 0\}$. It’s clear that the number of connected components of $T_\alpha \cap T_\beta$ is $b^2$.

Take the component of $(T_\alpha \cap T_\beta)_\chi = \{(0, \frac{1}{b}, z_3, \ldots, z_n) \subseteq (\mathcal{E})^n\}$. Write $\gamma = \sum_{i=1}^n m_i e_i$, then,

$$(T_\alpha \cap T_\beta)_\chi \subseteq T_\gamma \iff \frac{1}{b} m_2 + \Sigma_{i=3}^n m_i z_i \in \mathbb{Z} + \tau \mathbb{Z}, \forall z_i \in \mathbb{C}, i = 3, \ldots, n.$$ 

$$\iff b \mid m_2, m_3 = \cdots = m_n = 0.$$ 

$$\iff \gamma \in \mathbb{Z} \alpha + \mathbb{Z} \beta.$$ 

Let $H := \text{Hom}_{\mathbb{Z}}(P, \mathcal{E}) = Q^\vee \otimes_{\mathbb{Z}} \mathcal{E}$ be the torus with simply connected type. Denote $\ker(\chi_\alpha)$ by $H_\alpha \subseteq H$. We can modify the above proof such that a similar lemma also holds for simply connected case.

**Lemma 6.7.9.** There exist a component $(H_\alpha \cap H_\beta)_\chi$, such that $(H_\alpha \cap T_\beta)_\chi$ contained in a subtorus $H_\gamma$ if and only if $\gamma \in \mathbb{Z} \alpha + \mathbb{Z} \beta$.

**Proof.** Use the same notation as in the proof of Lemma 6.7.8. We know the coroot lattice $Q^\vee$ is a sublattice of the coweight lattice $P^\vee$. Write the transition matrix of the basis $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ of $Q^\vee$ and the basis $\{f_1, \ldots, f_n\}$ of $P^\vee$ by $(a_{ij})_{i=1,\ldots,n,j=1,\ldots,n}$, which is an integral matrix.
We do row transformations to the matrix \((a_{ij})_{i=1,\ldots,n,j=1,\ldots,n}\). From linear algebra, we can get an integral upper triangular matrix. With abuse of notation, we still denote the resulted matrix by \((a_{ij})_{i=1,\ldots,n,j=1,\ldots,n}\).

The process is equivalent to say we can find a set of basis of \(Q^\vee\), called it, \(\{g_1, \ldots, g_n\}\), such that,

\[
g_1 = a_{11}f_1 + a_{12}f_2 + \cdots + a_{1n}f_n, g_2 = a_{22}f_2 + \cdots + a_{2n}f_n, \ldots, g_n = a_{nn}f_n.
\]

Identify \(H = Q^\vee \otimes \mathbb{Z} \mathcal{E}\) with \((\mathcal{E})^n\) using the basis \(\{g_1, \ldots, g_n\}\). Then, the maps \(\chi_\alpha, \chi_\beta\) are giving by:

\[
\chi_\alpha : (\mathcal{E})^n \to \mathcal{E}, \sum_{i=1}^n z_i g_i \mapsto a_{11}z_1.
\]

and

\[
\chi_\beta : (\mathcal{E})^n \to \mathcal{E}, \sum_{i=1}^n z_i g_i \mapsto (aa_{11} + ba_{12})z_1 + ba_{22}z_2.
\]

Take the component of \((H_\alpha \cap H_\beta)_\chi = \{(0, \frac{1}{ba_{22}}, z_3, \ldots, z_n) \subseteq (\mathcal{E})^n\}\). Then the rest of the proof is exactly the same as the proof of Lemma 6.7.8.

\[\square\]

**Proposition 6.7.10.** The \((tt)\)-relation \([t_\alpha, \sum_{\beta \in \Psi^+} t_\beta] = 0\) holds in \(\text{gr}(\hat{C}[P\mathcal{B}])\).

**Proof.** Fix two roots \(\alpha, \beta\), take the component \((H_\alpha \cap H_\beta)_\chi\) as in Lemma 6.7.9. Choose a point \(x_0 \in (H_\alpha \cap H_\beta)_\chi\), such that \(x_0 /\notin H_\gamma\), for any \(H_\gamma\) satisfying \((H_\alpha \cap H_\beta)_\chi \notin H_\gamma\).

Take an open disc neighborhood \(D\) of \(x_0\), such that \(D \cap H_\gamma = \emptyset\), for those \(H_\gamma\), satisfying \((H_\alpha \cap H_\beta)_\chi \notin H_\gamma\). By Lemma 6.7.9, we have

\[
D \cap \left( \bigcup_{\gamma \in \Phi^+} H_\gamma \right) = \bigcup_{\gamma \in Z\alpha + Z\beta} (D \cap H_\gamma).
\]

Now using Lemma 1.45 in [86], we get the \((tt)\) relation in \(\text{gr}(\hat{C}[P\mathcal{B}])\).

\[\square\]

**Proposition 6.7.11.** The relation \([y(u), x(v)] = \Sigma_{\gamma \in \Phi^+} \langle v, \gamma \rangle \langle u, \gamma \rangle t_\gamma\) holds in \(\text{gr}(\hat{C}[P\mathcal{B}])\).

**Proof.** Note that the loop \([y(\lambda_\gamma^\vee), x(\lambda_\gamma^\vee)]\) of \(H_{\text{reg}}\) is a loop wrapping around the hypertori \(H_\gamma\), \(\gamma \in \Phi^+\). How it wrap those hypertori maybe complicated. But in \(\text{gr}(\hat{C}[P\mathcal{B}])\), we have

\[
[y(\lambda_\gamma^\vee), x(\lambda_\gamma^\vee)] = \Sigma_{\gamma \in \Phi^+} C_{i,j,\gamma} t_\gamma.
\]

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Now let’s determine the coefficients $C_{i,j,\gamma}$: Consider the map $\chi_\alpha : Q^\vee \otimes_\mathbb{Z} \mathcal{E} \rightarrow \mathcal{E}$, which induces a map

$$\chi_\alpha : (P^\vee \otimes_\mathbb{Z} \mathcal{E}) \setminus \ker \chi_\alpha \rightarrow \mathcal{E} \setminus \{0\}.$$ 

Denote by $\varphi_\alpha$ the composition

$$\varphi_\alpha : T_{\text{reg}} := (P^\vee \otimes_\mathbb{Z} \mathcal{E}) \setminus \bigcup_{\gamma \in \Phi^+} \ker \chi_\gamma \hookrightarrow (P^\vee \otimes_\mathbb{Z} \mathcal{E}) \setminus \ker \chi_\alpha \rightarrow \mathcal{E} \setminus \{0\}$$

Write a loop $x(\lambda^\vee_i) = (t + \frac{1}{2} + \frac{1}{2} \tau)\lambda^\vee_i$, where $0 \leq t \leq 1$, and a loop $y(\lambda^\vee_j) = (s\tau + \frac{1}{2} + \frac{1}{2} \tau)\lambda^\vee_j$, where $0 \leq s \leq 1$.

Using the method in Corollary 5.1, in [6], it’s not hard to see in $\mathcal{E} \setminus \{0\}$, the following relation holds:

$$[\frac{1}{(\lambda^\vee_j, \alpha)} \varphi_\alpha(y(\lambda^\vee_j)), \frac{1}{(\lambda^\vee_i, \alpha)} \varphi_\alpha(x(\lambda^\vee_i))] = t_\alpha,$$

where $t_\alpha$ is the loop around the removed point $\{0\}$. The above relation can be rewritten as

$$[\varphi_\alpha(y(u)), \varphi_\alpha(x(v))] = (u, \alpha)(v, \alpha)t_\alpha.$$

So we have $C_{i,j,\gamma} = (\lambda^\vee_j, \gamma)(\lambda^\vee_i, \gamma)$.

The relations $[x(u), x(v)] = 0$, $[y(u), y(v)] = 0$, for any $u, v \in \mathfrak{h}$ and the relations $[t_\alpha, x(u)] = 0$, $[t_\alpha, y(u)] = 0$ are supposed to be easier.

The following proposition can be found in [8](Proposition A.1).

**Proposition 6.7.12** (M. Broué, G. Malle, R. Rouquier). Let $i$ be the injection of an irreducible divisor $D$ in a smooth connected complex variety $Y$ and base point $x_0 \in Y - D$. Then, the kernel of the morphism

$$\pi_1(i) : \pi_1(Y - D, x_0) \rightarrow \pi_1(Y, x_0)$$

is generated by all the generators of the monodromy around $D$. 

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Corollary 6.7.13. Suppose \( X = Y - \bigcup_{i=1}^{n} D_i \), where \( D_i \) are irreducible divisors of \( Y \). Then, the fundamental group \( \pi_1(X, x_0) \) is generated by all the generators of the monodromy around the divisors \( D_i \) and generators of \( \pi_1(Y, x_0) \).

Proof. By induction on \( n \): From the above proposition, the kernel of the morphism

\[
\pi_1((Y \setminus D_1) - (D_2 \setminus D_1), x_0) \to \pi_1(Y - D_1, x_0)
\]

is generated by all the generators of the monodromy around \( D_2 \). Thus, \( \pi_1(Y - D_1 - D_2, x_0) \) is generated by generators of the monodromy around \( D_2 \) and generators of \( \pi_1(Y - D_1, x_0) \).

Induction step implies that \( \pi_1(Y - D_1 - D_2, x_0) \) is generated by generators of the monodromy around \( D_1, D_2 \) and generators of \( \pi_1(Y, x_0) \). \(\square\)

Remark 6.7.14. The presentation of \( \hat{C}[\mathcal{PB}] \) can be described as follows: Suppose \( \mathcal{PB} \) is presented by generators \( g_1, \ldots, g_n \) and relations \( R_i(g_1, \ldots, g_n) \), \( i = 1, \ldots, p \). Then \( \hat{C}[\mathcal{PB}] \) is the quotient of the free Lie algebra generated by \( \gamma_1, \ldots, \gamma_n \) by the ideal generated by \( \log(R_i(e^{\gamma_1}, \ldots, e^{\gamma_n})) \), \( i = 1, \ldots, p \).

Thus, the generators of \( \hat{C}[\mathcal{PB}] \) can be chosen as \( T_\alpha, X_1, \ldots, X_n, Y_1, \ldots, Y_n \), where \( \alpha \in \Phi^+ \).

Let \( \deg(X_i) = \deg(Y_i) = 1 \) and \( \deg(T_\alpha) = 2 \). From the definition of \( p \), we know \( p \) is surjective, since we can pick the generators of \( \text{gr}(\hat{C}[\mathcal{PB}]) \) to be the projection of the generators of \( \hat{C}[\mathcal{PB}] \) \( T_\alpha, X_1, \ldots, X_n, Y_1, \ldots, Y_n \), where \( \alpha \in \Phi^+ \).

Lemma 6.7.15. Write the connection in the following form: \( \nabla = d - \sum_{i=1}^{n} k_i(\alpha)d\alpha_i \), where

\[
k_i(\alpha) = \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(x_{\alpha_i}^{\alpha})(\tau)(t_\alpha)) \frac{d\alpha}{d\alpha_i} - y(\lambda_j^\vee).
\]

Then

\[
k_i(\alpha + \alpha_j) = k_i(\alpha), k_i(\alpha + \tau \alpha_j) = e^{-2\pi i x(\lambda_j^\vee)} k_i(\alpha).
\]

Proof. Note we have \( k(z + \tau, x|\tau) = e^{-2\pi i x} k(z, x|\tau) + \frac{e^{-2\pi i x - 1}}{x} \).
We will also use the following fact: if \([x', t] = 0, [x', t] = 0\), then \(e^{-2\pi i \text{ad} x'} k(z, x|\tau)(t) = k(z, x|\tau)(t)\). Now

\[
k(\alpha + \tau \alpha_j, \text{ad}(\frac{x_{\alpha_j}}{2})|\tau)(t_\alpha) \frac{d\alpha}{d\alpha_i} = \left( e^{-2\pi i \text{ad}(\frac{x_{\alpha_j}}{2})} k(\alpha, \text{ad}(\frac{x_{\alpha_j}}{2})|\tau)(t_\alpha) + \frac{e^{-2\pi i \text{ad}(x_{\alpha_j})}}{\text{ad}(\frac{x_{\alpha_j}}{2})} - 1(t_\alpha) \right) \frac{d\alpha}{d\alpha_i} = \left( e^{-2\pi i \text{ad}(x_{\alpha_j})} k(\alpha, \text{ad}(\frac{x_{\alpha_j}}{2})|\tau)(t_\alpha) \right) \frac{d\alpha}{d\alpha_i},
\]

where the second equality follows from the fact \((- \frac{a_{\alpha_j}}{2} + \lambda_{\alpha_j}, \alpha) = - \frac{a_{\alpha_j}}{2} + (\lambda_{\alpha_j}, \alpha) = 0\). So

\[
k_i(\alpha + \tau \alpha_j) = \sum_{\alpha \in \Phi^+} k(\alpha + \tau \alpha_j, \text{ad}(\frac{x_{\alpha_j}}{2})|\tau)(t_\alpha) \frac{d\alpha}{d\alpha_i} - y(\lambda_{\alpha_j}^\vee)
\]

\[
= e^{-2\pi i \text{ad}(x_{\alpha_j})} \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(\frac{x_{\alpha_j}}{2})|\tau)(t_\alpha) \frac{d\alpha}{d\alpha_i} + \frac{e^{-2\pi i \text{ad}(x_{\alpha_j})}}{\text{ad}(x_{\alpha_j})} - 1 \left( \sum_{\alpha \in \Phi^+} t_\alpha \frac{d\alpha}{d\alpha_j} \frac{d\alpha}{d\alpha_i} \right) - y(\lambda_{\alpha_j}^\vee)
\]

\[
= e^{-2\pi i \text{ad}(x_{\alpha_j})} \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(\frac{x_{\alpha_j}}{2})|\tau)(t_\alpha) \frac{d\alpha}{d\alpha_i} + \frac{1 - e^{-2\pi i \text{ad}(x_{\alpha_j})}}{\text{ad}(x_{\alpha_j})} [x(\lambda_{\alpha_j}^\vee), y(\lambda_{\alpha_j}^\vee)] - y(\lambda_{\alpha_j}^\vee)
\]

\[
= e^{-2\pi i \text{ad}(x_{\alpha_j})} \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(\frac{x_{\alpha_j}}{2})|\tau)(t_\alpha) \frac{d\alpha}{d\alpha_i} - e^{-2\pi i \text{ad}(x_{\alpha_j})} y(\lambda_{\alpha_j}^\vee)
\]

\[
= e^{-2\pi i \text{ad}(x_{\alpha_j})} k_i(\alpha).
\]

Check: \(\text{gr}(\hat{\mu}) \circ p\) is isomorphism.

**Lemma 6.7.16.** The map \(\text{gr}(\hat{\mu}) : \text{gr}(\mathbb{C}[P_{2\mathbb{Z}}]) \rightarrow U(A_{ul})\) is given by: \(X_u \mapsto -y(u), Y_u \mapsto 2\pi i x(u) - \tau y(u),\) and \(T_\alpha \mapsto 2\pi i t_\alpha\).

**Proof.** Recall the connection is

\[
\nabla = d - \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(\frac{x_{\alpha_j}}{2})|\tau)(t_\alpha) d\alpha + \sum_{i=1}^n y(u^i) du_i,
\]

where \(k(\alpha, \text{ad}(\frac{x_{\alpha_j}}{2})|\tau)(t_\alpha) = \frac{1}{\alpha} t_\alpha + \text{terms of degree } \geq 3\).
Let $F_{\mu_0}(u)$ be the horizontal section. Then, $\log F_{\mu_0}(u) = -\sum_{i=1}^{n}(u_i - u_0^i)y(u^i) + \text{terms of degree } \geq 2$.

**Corollary 6.7.17.** For some open sets $U, V$, then following holds:

$$F_{u^i}^U(u + \delta_i) = F_{u^i}^U(u)\mu_{u_0}(x(u^i)),$$

and

$$F_{u^i}^V(u + \tau\delta_i) = e^{-2\pi ix(u^i)}F_{u^i}^V(u)\mu_{u_0}(y(u^i)).$$

**Proof.** This follows directly from Lemma 6.7.15.

If so, then we have $\log \mu_{u_0}(x(u^i)) = -y(u^i) + \text{terms of degree } \geq 2$ and $\log \mu_{u_0}(y(u^i)) = 2\pi ix(u^i) - \tau y(u^i) + \text{terms of degree } \geq 2$. So $\text{gr}(\hat{\mu})(x(u^i)) = -y(u^i)$ and $\text{gr}(\hat{\mu})(y(u^i)) = 2\pi ix(u^i) - \tau y(u^i)$.

Note that

$$\hat{\mu}(T_\alpha) = \int_{T_\alpha} \left(\frac{1}{t_\alpha} + \text{terms of degree } \geq 3\right) d\alpha = 2\pi it_\alpha + \text{terms of degree } \geq 3.$$ 

So $\text{gr}(\hat{\mu})(T_\alpha) = 2\pi it_\alpha$.

**Corollary 6.7.18.** The composition $\text{gr}(\hat{\mu}) \circ p : \widehat{U(A_{n\ell})} \to \widehat{U(A_{n\ell})}$ is given by: $x(u^i) \mapsto -y(u^i), y(u^i) \mapsto 2\pi ix(u^i) - \tau y(u^i)$, and $t_\alpha \mapsto 2\pi it_\alpha$. In particular, it’s an isomorphism.

### 6.8 Some set $S$

**Lemma 6.8.1.** Let $\Phi$ be a root system of rank 2. Then, for any $\alpha \in \Phi_+$, the sum

$$\sum_{\beta \in \Phi_+: (\alpha, \beta)_{\mathbb{Z}} = \Phi} [t_\alpha, t_\beta]$$

lies in the subspace spanned by the $(tt)$ relations.

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Proof. The statement is clearly true if $\Phi$ does not properly contain a root subsystem, for in that case the sum ranges over all $\beta \in \Phi_+ \setminus \{\alpha\}$. In general, write
\[ 0 = [t_\alpha, \sum_{\beta \neq \alpha} t_\beta] = \sum_{\Psi \subset \Phi} [t_\alpha, \sum_{\langle \alpha, \beta \rangle Z = \Psi} t_\beta] \]
where $\Psi$ ranges over all subsystems of $\Phi$ containing $\alpha$. Write above as
\[ [t_\alpha, \sum_{\langle \alpha, \beta \rangle Z = \Psi} \langle \alpha, \beta \rangle Z = \Psi] t_\beta = \sum_{\Psi \subset \Phi} [t_\alpha, \sum_{\langle \alpha, \beta \rangle Z = \Psi} t_\beta] \]
By induction on $\Psi$, the second summand is zero, and so the first one must be too. \hfill \Box

Let $\Phi$ be a rank 2 subsystem, for any root $\beta \in \Phi$, we define a set $S_\Psi^{(\beta)}$ by
\[ S_\Psi^{(\beta)} = \{ \alpha \in \Phi \mid \langle \alpha, \beta \rangle Z = \Psi \}, \]
where $\Psi$ is a rank 2 subsystem of $\Phi$.

Lemma 6.8.2. The set $\{ S_\Psi^{(\beta)} \}$ is one to one correspondence to the set
\[ \{ \Psi \subset \Phi \text{ of rank 2 subsystem} \mid \beta \in \Psi \} \]
Proof. Sending $\Psi$ in the second set to $S_\Psi = (\Psi \setminus \{\beta\}) \setminus \cup_{\Psi \subset \Psi'} \Psi'$ gives us the bijection. \hfill \Box

Lemma 6.8.3. The element $\omega(\alpha \lor, \beta)$ is independent up of $\alpha, (\text{modulo } \pm 1)$ for any $\alpha \in S_\Psi^{(\beta)}$.
Proof. For $u_i \in S_\Psi^{(\beta)}$, $i = 1, 2$. Since $\langle u_i, \beta \rangle Z = \Psi$, the transition matrix between the two integral basis $\{u_1, \beta\}$ and $\{u_2, \beta\}$ is of the form $\begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix}$, thus $x = \pm 1$. Write $u_1 = \pm u_2 + y \beta$. Then, $\omega(u_1 \lor, \beta) = \omega((\pm u_2 + y \beta) \lor, \beta) = \pm \omega(u_2 \lor, \beta)$. So the assertion follows. \hfill \Box

6.9 Derivation of the Lie algebra $A_{ell}^\Phi$

Let $\mathfrak{d}$ be the Lie algebra with generators $\Delta_0, d, X$, and $\delta_{2m}(m \geq 1)$, and relations
\[ [d, X] = 2X, \ [d, \Delta_0] = -2\Delta_0, \ [X, \Delta_0] = d, \]
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\[ [\delta_{2m}, X] = 0, \quad [d, \delta_{2m}] = 2m\delta_{2m}, \quad (\mathrm{ad} \Delta_0)^{2m+1}(\delta_{2m}) = 0. \]

Now, define a Lie algebra morphism \( \mathfrak{d} \to \text{Der}(A_{\Phi}^{\ell}) \), denoted by \( \xi \mapsto \tilde{\xi} \). Defined as follows,

\[
\begin{cases}
\tilde{\xi}(x(u)) = x(u); \\
\tilde{\xi}(y(u)) = y(u); \\
\tilde{\xi}(t_\alpha) = 0.
\end{cases}
\]

Proposition 6.9.1. The above map \( \mathfrak{d} \to \text{Der}(A_{\Phi}^{\ell}) \) is a Lie algebra morphism.

Proof. To check the map is well defined, we need to check that \( \tilde{\xi} \) is a derivation of \( A_{\Phi}^{\ell} \); for any \( \xi \) being generators of \( \mathfrak{d} \). The fact that \( \tilde{d}, \tilde{X} \) and \( \tilde{\Delta}_0 \) are derivations are clear. It’s also clear the \( \tilde{\delta}_{2m} \) preserves the relations \([x(u), x(v)] = 0 \) and \([t_\alpha, x(u)] = 0 \), if \((\alpha, u) = 0 \). It remains to show that \( \tilde{\delta}_{2m} \) preserves the following relations

1. \([y(u), x(v)] = \sum_{\gamma \in \Phi^+} (u, \gamma)(v, \gamma)t_\gamma; \]

2. \([t_\alpha, \sum_{\beta \in \Psi^+} t_\beta]; \]

3. \([t_\alpha, y(u)] = 0 \), if \((\alpha, u) = 0; \]

4. \([y(u), y(v)] = 0. \]

\[ \tilde{\delta}_{2m}\] preserve the commutating relation of \([y, x]\).

We check \([\tilde{\delta}_{2m}(y(u)), x(v)] + [y(u), \tilde{\delta}_{2m}(x(v))] = \sum_{\gamma \in \Phi^+} (u, \gamma)(v, \gamma)\tilde{\delta}_{2m}(t_\gamma) \) in this subsection.

By definition of \( \tilde{\delta}_{2m} \), we have

\[
[\tilde{\delta}_{2m}(y(u)), x(v)] + [y(u), \tilde{\delta}_{2m}(x(v))] = \frac{1}{2} \sum_{\gamma} \gamma(u) \sum_{p+q=2m-1} [(\mathrm{ad} \frac{x_\gamma}{2})^p(t_\gamma), (\mathrm{ad} - \frac{x_\gamma}{2})^q(t_\gamma)], x(v) \]

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Notice that we can write $x(v) = \frac{1}{2}(v, \gamma)x_{\gamma\nu} + x(v')$, where $(\gamma, v') = 0$, using the (tx)-relation, we have

$$[(\text{ad} \frac{x_{\gamma\nu}}{2})^p(t_\gamma), (\text{ad} - \frac{x_{\gamma\nu}}{2})^q(t_\gamma), x(v')] = 0$$

Thus,

$$[\tilde{\delta}_{2m}(y(u)), x(v)] + [y(u), \tilde{\delta}_{2m}(x(v))]$$

$$= \frac{1}{2} \sum_{\gamma} \gamma(u) \sum_{p+q=2m-1} \left[ (\text{ad} \frac{x_{\gamma\nu}}{2})^p(t_\gamma), (\text{ad} - \frac{x_{\gamma\nu}}{2})^q(t_\gamma), (v, \gamma)\frac{x_{\gamma\nu}}{2} \right]$$

$$= \frac{1}{2} \sum_{\gamma} (v, \gamma)\gamma(u) \sum_{p+q=2m-1} \left[ (\text{ad} \frac{x_{\gamma\nu}}{2})^p(t_\gamma), (\text{ad} - \frac{x_{\gamma\nu}}{2})^q(t_\gamma), (v, \gamma)\frac{x_{\gamma\nu}}{2} \right] - \frac{1}{2} \sum_{\gamma} (v, \gamma)\gamma(u) [t_\gamma, \frac{x_{\gamma\nu}}{2}]^m(t_\gamma)$$

$$= \sum_{\gamma} (v, \gamma)\gamma(u)[t_\gamma, \frac{x_{\gamma\nu}}{2}]^m(t_\gamma) = \sum_{\gamma \in \Phi}(v, \gamma)(u, \gamma)\tilde{\delta}_{2m}(t_\gamma).$$

6.9.2 Check $[\tilde{\delta}_{2m}(t_\alpha), \sum_{\beta \in \Psi^+} t_\beta] + [t_\alpha, \sum_{\beta \in \Psi^+} \tilde{\delta}_{2m}(t_\beta)] = 0$

Suppose we have a relation $[t_\alpha, \sum_{\beta \in \Psi^+} t_\beta] = 0$, where $\Psi$ is a rank 2 system. We first claim that:

$$\tilde{\delta}_{2m}(t_\beta) = \sum_{\gamma \in \Psi^+} [t_\beta, (\text{ad} \frac{x_{\gamma\nu}}{2})^m(t_\gamma)]$$

Proof of the claim: It suffices to check that: $\sum_{\gamma \in \Psi^+, \gamma \neq \beta} [t_\beta, (\text{ad} \frac{x_{\gamma\nu}}{2})^m(t_\gamma)] = 0$.

By definition,

$$\sum_{\gamma \in \Psi^+, \gamma \neq \beta} [t_\beta, (\text{ad} \frac{x_{\gamma\nu}}{2})^m(t_\gamma)] = \sum_{\gamma \in \Psi^+, \gamma \neq \beta} [t_\beta, (\text{ad} \frac{x_{\omega(\gamma, \beta)}}{2})^m(t_\gamma)]$$

$$= \sum_{\gamma \in \Psi^+, \gamma \neq \beta} [t_\beta, (\text{ad} \frac{x_{\omega(\gamma, \beta)}}{2})^m(t_\gamma)] = 0,$$

where $c_{\text{aut}}^{(\beta)} := \omega(\gamma, \beta)$, for $\gamma \in S^{(\beta)}_{\psi^+}$. According to Lemma 6.8.3, we know $\omega(\gamma, \beta)$ is a constant (up to a sign), so $c_{\text{aut}}^{(\beta)}$ is well-defined.

The last equality follows from (tx) relation $[t_\beta, x(c_{\text{aut}}^{(\beta)})] = 0$, and (tt) relation $[t_\beta, \sum_{\gamma \in S^{(\beta)}_{\psi^+}} t_\gamma] = 0$. So that finish the proof of claim.
Now, we have,

\[
[\tilde{\delta}_{2m}(t_\alpha), \sum_{\beta \in \Psi^+} t_\beta] + [t_\alpha, \sum_{\beta \in \Psi^+} \tilde{\delta}_{2m}(t_\beta)]
\]

\[= - [t_\alpha, \sum_{\beta \in \Psi^+} [t_\beta, \ad \frac{x_\alpha}{2}^\gamma_{2m}(t_\gamma)]] + [t_\alpha, \sum_{\beta \in \Psi^+} \sum_{\gamma \in \Psi^+} [t_\beta, (\ad \frac{x_\gamma}{2}^\gamma_{2m}(t_\gamma)]]
\]

\[=[t_\alpha, \sum_{\beta \in \Psi^+} [t_\beta, \sum_{\gamma \in \Psi^+} (\ad \frac{x_\gamma}{2}^\gamma_{2m}(t_\gamma)]] = [t_\alpha, \sum_{\beta \in \Psi^+} \sum_{\gamma \in \Psi^+} \sum_{\{\gamma \neq \alpha\}} (\ad \frac{x_\gamma}{2}^\gamma_{2m}(t_\gamma))]
\]

\[= \sum_{\Psi \subset \Psi} [t_\alpha, \sum_{\beta \in \Psi^+} \sum_{\{\gamma \neq \alpha\}} (\ad \frac{x_{\psi(\alpha)}}{2})^{2m} \sum_{\gamma \in \Psi^+} t_\gamma] = 0,
\]

The last equality follows from (tx) relation \([t_\alpha, x(e_{\psi(\alpha)}^{\alpha})] = 0\), and (tt) relations \([t_\alpha, \sum_{\gamma \in S_{\psi}^{(\alpha)}} t_\gamma] = 0\), and \([t_\alpha, \sum_{\beta \in \Psi^+} t_\beta] = 0\).

**6.9.3 Check that** \([y(u), \tilde{\delta}_{2m}(t_\alpha)] + [\tilde{\delta}_{2m}(y(u)), t_\alpha] = 0, \text{ if } (\alpha, u) = 0\)

Now, let’s calculate the second term \([\tilde{\delta}_{2m}(y(u)), t_\alpha]\) first. Write \(\Phi^+ = \{\alpha, \gamma_1, \cdots, \gamma_m\}\). The purpose for written like this is giving an order for the positive roots in \(\Phi^+ \setminus \{\alpha\}\).

**Lemma 6.9.2.** Now we assume we are in the rank 2 situation, then

\[
[\tilde{\delta}_{2m}(y(u)), t_\alpha] = \frac{1}{2} [t_\alpha, \sum_{\Psi \subset \Phi} \sum_{\{j \neq i, \gamma_j \in S_{\psi}^{(\alpha)}\}} \gamma_i(u) (\ad \frac{x_{\omega(\gamma_i^\gamma_{\gamma_j})}}{2})^{2m} - (\ad \frac{x_{\omega(\gamma_j^\gamma_{\gamma_i})}}{2})^{2m} + \frac{x_{\omega(\gamma_j^{\gamma_i})}}{2} + \frac{x_{\omega(\gamma_i^{\gamma_j})}}{2}] [t_{\gamma_i}, t_{\gamma_j}]
\]

**Proof.** For fixed \(p, q\), such that \(p + q = 2m - 1\), note we have

\[
[(\ad \frac{x_{\gamma^\gamma}}{2})^p(t_\gamma), (\ad - \frac{x_{\gamma^\gamma}}{2})^q(t_\gamma)], t_\alpha
\]

\[= [(\ad \frac{x_{\omega(\gamma^\gamma, \alpha)}}{2})^q[t_\alpha, t_\gamma], (\ad \frac{x_{\gamma^\gamma}}{2})^p(t_\gamma)] - [(\ad \frac{x_{\omega(\gamma^\gamma, \alpha)}}{2})^p[t_\alpha, t_\gamma], (\ad \frac{x_{\gamma^\gamma}}{2})^q(t_\gamma)].
\]

Now for \(\gamma_i \in S_{\psi}^{(\alpha)}\), plugging the (tt)-relation

\[
[t_\alpha, t_{\gamma_i}] = - [t_\alpha, \sum_{\{j \neq i, \gamma_j \in S_{\psi}^{(\alpha)}\}} t_{\gamma_j}]
\]

(6.9.1)
Proof.
For any $\omega(\gamma_j, \alpha) = \omega(\gamma_i, \alpha)$, we use $+; \text{if } \omega(\gamma_j, \alpha) = -\omega(\gamma_i, \alpha)$, we use $-$. 

The following corollary of Lemma 6.9.2 will be used later (to prove Claim 6.9.4).

Corollary 6.9.3. For any $p, q \in \mathbb{N}$, such that $p + q = 2m - 1$, for two different sets $S^{(a)}_\Psi$, $S^{(a)}_{\Psi'}$, we have:

$$\left[ \sum_{\gamma_i \in S^{(a)}_\Psi} \sum_{\gamma_j \in S^{(a)}_{\Psi'}} \left[ (\text{ad} \frac{x_{\gamma_i}}{2})^p(t_{\gamma_i}), (\text{ad} \frac{x_{\gamma_j}}{2})^q(t_{\gamma_j}) \right], t_\alpha \right] = 0$$

Proof. For any $\gamma_i \in S^{(a)}_\Psi$, in the proof of Lemma 6.9.2, instead of using the (tt)-relation
(6.9.1), we use
\[ [t_\alpha, t_\gamma] = -[t_\alpha, \sum_{\gamma' \neq \gamma} t_{\gamma'}] - C[t_\alpha, \sum_{\beta \in \Omega} t_{\beta}], \]
where \( C = \frac{lcm(n(S_\Phi^{(\alpha)}), n(S_\Phi^{(\alpha')}))}{n(S_\Phi^{(\alpha)})} \) and \( n(S_\Phi^{(\alpha)}) := \text{Index}(\Psi \subseteq \Phi_\mathbb{Z}) \), in particular, \( n(S_\Phi^{(\alpha)}) = 1 \). The benefit of choosing this constant \( C \) is that we are able to write
\[ \gamma_i(u)(\text{ad} \frac{x_\omega(\gamma_i^{\alpha}, \alpha)}{2})^q C[t_\alpha, t_\gamma] = \frac{lcm(n(S_\Phi^{(\alpha)}), n(S_\Phi^{(\alpha')}))}{n(S_\Phi^{(\alpha)})} \gamma_i(u)[t_\alpha, (\text{ad} \frac{x_\omega(\gamma_i, \alpha)}{2})^q t_\gamma] \]
Above equality follows from \( \gamma_i = \pm \frac{n(S_\Phi^{(\alpha)})}{n(S_\Phi^{(\alpha')})} \gamma_j + k\alpha \), for some \( k \in \mathbb{Z} \). So we get \( \omega(\gamma_i^{\alpha}, \alpha) = \frac{n(S_\Phi^{(\alpha)})}{n(S_\Phi^{(\alpha')})} \omega(\pm \gamma_i^{\alpha}, \alpha) \). Now do the same thing as in the proof of Lemma 6.9.2, we can get:
\[ \left[ \sum_{\gamma} \gamma_i(u)[(\text{ad} \frac{x_\omega(\gamma_i^{\alpha}, \alpha)}{2})^p(t_\gamma), (\text{ad} \frac{x_\omega(\gamma_i^{\alpha}, \alpha)}{2})^q(t_\gamma)], t_\alpha \right] = -\sum_{k} \sum_{\gamma_i \in \Omega \land \gamma_j \in \Omega} \gamma_i(u)[(\text{ad} \frac{x_\omega(\gamma_i^{\alpha}, \alpha)}{2})^p(t_\gamma), (\text{ad} \frac{x_\omega(\gamma_i^{\alpha}, \alpha)}{2})^q(t_\gamma)], t_\alpha \]
\[ = \left[ \sum_{\gamma_i \in \Omega \land \gamma_j \in \Omega} \frac{lcm(n(S_\Phi^{(\alpha)}), n(S_\Phi^{(\alpha')}))}{n(S_\Phi^{(\alpha)})} \gamma_i(u)[(\text{ad} \frac{x_\omega(\gamma_i^{\alpha}, \alpha)}{2})^p(t_\gamma), (\text{ad} \frac{x_\omega(\gamma_i^{\alpha}, \alpha)}{2})^q(t_\gamma)], t_\alpha \right], \]
Note that \( \frac{lcm(n(S_\Phi^{(\alpha)}), n(S_\Phi^{(\alpha')}))}{n(S_\Phi^{(\alpha)})} \gamma_i(u) \) is a constant. Now compare it with the result in Lemma 6.9.2, the conclusion follows.

Let’s calculate the first term \([y(u), \tilde{\delta}_{2m}(t_\alpha)]\), since \([y(u), t_\alpha] = 0\), we have
\[ [y(u), \tilde{\delta}_{2m}(t_\alpha)] = [t_\alpha, y(u), \text{ad} \frac{x_\omega(\gamma_i^{\alpha}, \alpha)}{2}(t_\alpha)] \]
\[ = [t_\alpha, \frac{1}{2} \sum_{\gamma \neq \alpha} (\alpha^{\gamma}, \gamma)(u, \gamma) \sum_{s=0}^{2m} \text{ad} \frac{x_\omega(\alpha^{\gamma}, \alpha)}{2}(t_\alpha)^{2m-1-s}[t_\gamma, t_\alpha]] \]
\[ = [t_\alpha, \sum_{\gamma \neq \alpha} (u, \gamma) \frac{\text{ad} \frac{x_\omega(\gamma_i^{\alpha}, \alpha)}{2}}{\text{ad} \frac{x_\omega(\gamma_i^{\alpha}, \alpha)}{2}}[t_\gamma, t_\alpha]] \]
Let \( \Phi^+ = \{\alpha, \gamma_1, \ldots, \gamma_m\} \). The following claim gives the formula for the first term.
Claim 6.9.4. We claim that in rank 2 case, we have the following equality:

\[ [y(u), \tilde{\delta}_{2m}(t_\alpha)] = -[t_\alpha, \sum_{\gamma \in \Phi} \sum_{i<j, \gamma_i, \gamma_j \in S^{(\alpha)}_\Psi} (u, \gamma_i) \left( \text{ad} \frac{x_{\omega(\alpha, \gamma_j)} 2} 2 \right)^{2m} - \left( \text{ad} \frac{x_{\omega(\alpha, \gamma_i)} 2} 2 \right)^{2m} [t_{\gamma_i}, t_{\gamma_j}] ] \]

Proof.

Lemma 6.9.5. The following (tt)-relations hold:

1. For \( \gamma_i \in S^{(\alpha)}_\Psi \), then \([t_{\gamma_i}, \alpha] = -\sum_{j \neq i} [t_{\gamma_i}, t_{\gamma_j}] \).

2. For \( \gamma_i \in S^{(\alpha)}_\Psi \neq S^{(\alpha)}_\Psi \), then \([t_{\gamma_i}, \alpha] = -\sum_{\{\gamma_j \in S^{(\alpha)}_\Psi \}} [t_{\gamma_i}, t_{\gamma_j}] - \frac{1}{n(S^{(\alpha)}_\Psi)^2} \sum_{\gamma_j \in S^{(\alpha)}_\Psi} [t_{\gamma_i}, t_{\gamma_j}] \).

Proof. The relation (1) follows from the relation \([t_{\gamma_i}, \alpha + \sum_{j \neq i} t_{\gamma_j}] = 0 \).

For \( \gamma_i \in S^{(\alpha)}_\Psi \), where \( \Psi \) is a root subsystem. In this case, \( S^{(\alpha)}_\Psi \cup \{\alpha\} \) is the root system \( \Psi \) provided that \( \Psi \neq \Phi \). The relation \([t_{\gamma_i}, \alpha + \sum_{\gamma_j \in S^{(\alpha)}_\Psi} t_{\gamma_j}] = 0 \) implies \([t_{\gamma_i}, \alpha] = -\sum_{\gamma_j \in S^{(\alpha)}_\Psi} [t_{\gamma_i}, t_{\gamma_j}] \).

Note that we also have \([t_{\gamma_i}, \alpha + \sum_{j \neq i} t_{\gamma_j}] = 0 \), which implies \(\sum_{\gamma_j \notin S^{(\alpha)}_\Psi} [t_{\gamma_i}, t_{\gamma_j}] = 0 \). Thus, the relation (2) follows.

If we denote the coefficient in (1) and (2) in Lemma 6.9.5 by \( \epsilon_{ij} \), that is: \([t_{\gamma_i}, t_{\alpha}] = -\sum_{j \neq i} \epsilon_{ij} [t_{\gamma_i}, t_{\gamma_j}] \). Then, the following equality holds: \( \frac{(u, \gamma_i)}{\omega(\gamma_i, \alpha)} \epsilon_{ij} = \frac{(u, \gamma_j)}{\omega(\gamma_j, \alpha)} \epsilon_{ji} \), for any \( i, j \), for \( (\alpha, u) = 0 \). If \( \gamma_i, \gamma_j \in S^{(\alpha)}_\Psi \), then \( \epsilon_{ij} = 1 \), the equality is obvious.

If \( \gamma_i \in S^{(\alpha)}_\Psi, \gamma_j \in S^{(\alpha)}_\Psi, \Phi \neq \Psi' \), then \( \epsilon_{ij} = \frac{1}{n(S^{(\alpha)}_\Psi)^2} \) and \( \epsilon_{ji} = \frac{1}{n(S^{(\alpha)}_\Psi)^2} \). The equality follows from the fact that \( \frac{(u, \gamma_i)}{\omega(\gamma_i, \alpha)} = n(S^{(\alpha)}_\Psi)^2 \frac{(u, \gamma_j)}{\omega(\gamma_j, \alpha)} \), for some \( \gamma_0 \in S^{(\alpha)}_\Psi \).

Plugging relation (1) and (2) into the equation above Claim 6.9.4, we get:

\[
[t_\alpha, \sum_{\gamma \neq \alpha} (u, \gamma) \left( \text{ad} \frac{x_{\omega(\alpha, \gamma)} 2} 2 \right)^{2m} - \left( \text{ad} \frac{x_{\omega(\alpha, \gamma)} 2} 2 \right)^{2m} [t_{\gamma}, t_\alpha]]
\]

\[
= -[t_\alpha, \sum_{i<j} \epsilon_{ij} \frac{(u, \gamma_i)}{\omega(\gamma_i, \alpha)} \left( \text{ad} \frac{x_{\omega(\alpha, \gamma)} 2} 2 \right)^{2m} - \left( \text{ad} \frac{x_{\omega(\alpha, \gamma)} 2} 2 \right)^{2m} [t_{\gamma}, t_{\gamma_j}]]
\]

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Suppose $a_{ij} \gamma_i + a_{ji} \gamma_j = s_{ij} \alpha$, for some integers $a_{ij}, a_{ji}, s_{ij}$ having no common divisor. By calculation, we have

$$\left[t_{\alpha}, \sum_{i<j} \epsilon_{ij} \frac{a_{ij}}{\text{ad} \frac{x_{\omega(\gamma_j)}}{2}} \left( \left( \text{ad} \frac{x_{\omega(\gamma_i)}}{2} \right)^{2m} - \left( \text{ad} \frac{x_{\omega(\gamma_j)}}{2} \right)^{2m} \right) \right] [t_{\gamma_i}, t_{\gamma_j}]$$

$$= \left[t_{\alpha}, \sum_{i<j} \epsilon_{ij} \frac{a_{ij}}{\text{ad} \frac{x_{\omega(\gamma_j)}}{2}} \left( \sum_{p+q=2m-1} \left( \text{ad} \frac{x_{\omega(\gamma_j)}}{2} \right)^p \left( \text{ad} \frac{x_{\omega(\gamma_j)}}{2} \right)^q \right) \right] [t_{\gamma_i}, t_{\gamma_j}]$$

Notice that $\gamma_i \in S_{\Psi}^{(a)}$, $\gamma_j \in S_{\Psi'}^{(a)}$ with $[t_{\gamma_i}, t_{\gamma_j}] \neq 0$, then the term $\frac{\epsilon_{ij} a_{ij}^{2m} (u, \gamma_i)}{(a_{ij})^{p+1} (\pm a_{ij})^q}$ is constant. By corollary 6.9.3, the above summation for $\gamma_i, \gamma_j$ coming from different sets $S_{\Psi}, S_{\Psi'}$ is 0. Thus, the conclusion of claim 6.9.4 follows.

Now, we compare the results in Lemma 6.9.2 and in Claim 6.9.4. Assume that $\gamma_i = \pm \gamma_j + \alpha$, then:

- $\omega(\gamma_i^\vee, \gamma_j) = \omega(\alpha^\vee, \gamma_j), \omega(\gamma_j^\vee, \gamma_i) = \mp \omega(\alpha^\vee, \gamma_i)$;
- $\omega(\alpha^\vee, \gamma_j) - \omega(\alpha^\vee, \gamma_i) = \omega(\gamma_j^\vee, \alpha)$.

Thus, under this assumption, we have $[y(u), \delta_{2m}(t_{\alpha})] + [\delta_{2m}(y(u)), t_{\alpha}] = 0$.

The above assumption doesn’t hold only in the case $G_2$, when $\alpha$ is the short root. In which case we have: $\gamma_i = \pm \gamma_j + 3\alpha$. Thus, in this case, the lemma can be rewritten as

$$\left[\frac{1}{2} \sum_{\gamma} \gamma(u) \sum_{p+q=2m-1} \left( \left( \text{ad} \frac{x_{\gamma^\vee}}{2} \right)^p (t_{\gamma}), \left( \text{ad} - \frac{x_{\gamma^\vee}}{2} \right)^q (t_{\gamma}) \right), t_{\alpha} \right]$$

$$= \frac{1}{2} \left[t_{\alpha}, \sum_{k \neq i \mid \gamma_i, \gamma_j \in S_{\Psi}^{(a)}} \gamma_i (u) \frac{\left( \text{ad} \frac{x_{\omega(3\alpha, \gamma_j)}}{2} \right)^{2m} - \left( \text{ad} \frac{x_{\omega(3\alpha, \gamma_j)}}{2} \right)^{2m}}{\text{ad} \frac{x_{\omega(\gamma_i^\vee, \gamma_j)}}{2}} \right] [t_{\gamma_i}, t_{\gamma_j}]$$

Notice that $\frac{\gamma_i (u)}{\omega(\gamma_i^\vee, \alpha)}$ is a constant, for $\gamma_i$ are three long roots in $G_2$. We can use the (tt)-relation for the three long roots to show that above summation is 0. Similarly, the summation in Claim 6.9.4 that involving the three long roots is 0.
By definition, we have

\[
[y(u), \tilde{\delta}_{2m}(y(v))] + [\tilde{\delta}_{2m}(y(u)), y(v)] = 0
\]

Notice that \([y_{\eta}, t_{\gamma}] = 0\), thus,

\[
[y_{\eta}, \tilde{\delta}_{2m}(y(u))] + [y(u), \tilde{\delta}_{2m}(y(v))] = 0
\]

where \(y_{\eta} = \gamma(v)y(u) - \gamma(u)y(v)\).

Therefore, we have to show

\[
\sum_{\gamma} \left[ \sum_{p+q=2m-1} \text{det}(\beta, \gamma)_{ij}(\beta, \gamma') \sum_{s} (ad \frac{x_{\gamma'}}{2})^{p}(ad \frac{x_{\omega(\gamma', \beta)}}{2})^{p-1-s}[t_{\beta}, t_{\gamma}], (ad - \frac{x_{\gamma'}}{2})^{q}(t_{\gamma}) \right] = 0
\]

(6.9.2)

We are in the situation of rank 2 case.

**Lemma 6.9.6.** For three fixed roots, \(\beta, \gamma, \beta_l \in \Phi\). Assume \(\beta \in S^{(\beta_l)}_{\Psi}\), for some root subsystem \(\Psi\) of \(\Phi\). For any \(\beta' \in S^{(\beta_l)}_{\Psi}\). If we have \(a\beta_l + b\beta + c\gamma = 0\), for some \(a, b, c \in \mathbb{Z}\), with \(gcd(a, b, c) = 1\), then, there exits an integral \(a' \in \mathbb{Z}\), such that the following equation holds

\[
a'\beta_l \pm b\beta' + c\gamma = 0.
\]
Remark 6.9.7. Based on the above lemma, the integer $c$ is independent of the elements in $S^{(β)}_Ψ$. So we can write $c(S^{(β)}_Ψ)$. For fixed root $γ ∈ Φ$, where $Φ^+ = \{γ, β_1, \cdots, β_m\}$. Now, similar as Lemma 6.9.5, we use the following (tt)-relation. For $β_t ∈ S^{(γ)}_Ψ$, then:

$$[t_{β_t}, γ] = - \sum_{s \neq t} [t_{β_t}, t_{β_s}]$$ (6.9.3)

For $β_t ∈ S^{(γ)}_Ψ \neq S^{(γ)}_Ψ$,

$$[t_{β_t}, γ] = - \sum_{β_s ∈ S^{(γ)}_Ψ} [t_{β_t}, t_{β_s}] - \frac{1}{n(S^{(γ)}_Ψ)^2} \left( \sum_{Ψ' ≤ Φ, Ψ' ≠ Ψ} c(S^{(β_t)}_Ψ) \sum_{β_s ∈ S^{(β_t)}_Ψ} [t_{β_t}, t_{β_s}] \right)$$ (6.9.4)

Proof. The above equation is obtained by using the fact that $S^{(β_t)}_Ψ$ is a root subsystem, provided that $Ψ' ≤ Φ, Ψ' ≠ Ψ$.

Claim 6.9.8. Assume that $a_{sl}β_s + b_{ls}β_t + c_{dl}γ = 0$, then,

$$\sum_{l} \sum_{p+q=2m-1} \det(β_t, γ)_{ij} \frac{(ad x^{γ}_{β_t})^p}{ad x^{γ}_{β_t}} \left[ t_{β_t}, t_{γ} \right] = \sum_{l<s} \sum_{r+n+q=2m-2} C_{l,s,γ} \left[ (ad x^{γ}_{β_t})^r, (ad x^{γ}_{β_t})^n \right]$$

where $C_{l,s,γ} = \begin{cases} \frac{\det(β_t, γ)_{ij}}{c_{sl}b_{ls}}, & \text{if } β_t, β_s ∈ S^{(γ)}_Ψ; \\ \frac{\det(β_t, γ)_{ij}}{n(S^{(γ)}_Ψ)^2 c_{sl}b_{ls}}, & \text{if } β_t ∈ S^{(γ)}_Ψ, β_s ∈ S^{(γ)}_Ψ. \end{cases}$

Proof. Plug the (tt)-relations (6.9.3) and (6.9.4) and the calculation is exactly the same as the calculations in the proof of Claim 6.9.4.

Lemma 6.9.9. $C_{l,s,γ} = C_{s,l,γ} = C_{γ,l,s}.$

Proof. It suffices to show that $C_{l,s,γ} = -C_{s,l,γ}$, and $C_{l,s,γ} = -C_{γ,s,l}$.

Proof of: $C_{l,s,γ} = -C_{s,l,γ}$: Let $β_t ∈ S^{(γ)}_Ψ$, and $β_s ∈ S^{(γ)}_Ψ$, then $β_t = n(S^{(γ)}_Ψ)β_0 + k_lγ$, and $β_s = ± n(S^{(γ)}_Ψ)β_0 + k_sγ$, for some $β_0$. The fact $\frac{b}{a} = \frac{± n(S^{(γ)}_Ψ)}{n(S^{(γ)}_Ψ)}$ implies the equality.
Proof of $C_{s,l} = -C_{l,s}$: First notice that $\det(\beta_l, \gamma)_{ij} = -\det(\gamma, \beta_l)_{ij}$.

Case 1: $\beta_s, \beta_l \in S^{(\gamma)}_{\Psi}$ and $\beta_s, \gamma \in S^{(\beta_l)}_{\Psi}$. The equality holds.

Case 2: $\beta_s, \beta_l \in S^{(\gamma)}_{\Psi}$, $\beta_s \in S^{(\beta_l)}_{\Psi}$ or $\beta_s, \gamma \in S^{(\beta_l)}_{\Psi}$, $\beta_s \in S^{(\gamma)}_{\Psi}$. This case can only happen in root system $G_2$, when $\langle \gamma, \beta_l \rangle_Z = \Phi$, thus, $n(S^{(\gamma)}_{\Psi}) = n(S^{(\beta_l)}_{\Psi}) = 1$. Therefore, the equality holds.

Case 3: $\beta_l \in S^{(\gamma)}_{\Psi}$, $\beta_s \in S^{(\gamma)}_{\Psi}$, $\gamma \in S^{(\beta_l)}_{\Psi}$. The equality holds in this case simply because $n(S^{(\gamma)}_{\Psi}) = n(S^{(\beta_l)}_{\Psi})$, since $\langle \beta_l, \gamma \rangle_Z = \Psi$.

Finally, take the summation over all positive roots $\gamma$, there will be two terms of the following:

$$C_{s,l} = [(\text{ad} \frac{x(a_s b_s)^\gamma}{2})^n(t_{\beta_s}), (\text{ad} \frac{x(c_s t_{\gamma})^\gamma}{2})^q(t_{\gamma}), (\text{ad} \frac{x(b_s t_{\beta_l})^\gamma}{2})^r(t_{\beta_l})]$$

and

$$C_{s,l} = [(\text{ad} \frac{x(c_s t_{\gamma})^\gamma}{2})^q(t_{\gamma}), (\text{ad} \frac{x(b_s t_{\beta_l})^\gamma}{2})^r(t_{\beta_l}), (\text{ad} \frac{x(a_s b_s)^\gamma}{2})^n(t_{\beta_s})]$$

Using Jacobi identity and Lemma 6.9.9, we know the summation of the three terms is 0. Thus, the equation (6.9.2) that we are trying to show is 0.

### 6.10 Bundle on the moduli space $\mathcal{M}_{1,n}$

Let $e, f, h$ be the standard basis of $\mathfrak{sl}_2$. Then, there is a Lie algebra morphism $\mathfrak{d} \rightarrow \mathfrak{sl}_2$ defined by $\delta_{2m} \mapsto 0$, $d \mapsto h$, $X \mapsto e$, $\Delta_0 \mapsto f$. We denote by $\mathfrak{d}_+ \subset \mathfrak{d}$ its kernel.

Since there is a section of the morphism $\mathfrak{d} \rightarrow \mathfrak{sl}_2$ (given by $e, f, g \mapsto X, \Delta_0, d$), we have a semidirect decomposition $\mathfrak{d} = \mathfrak{d}_+ \ltimes \mathfrak{sl}_2$. Then, we have

$$A_{\text{ali}} \ltimes \mathfrak{d} = (A_{\text{ali}} \ltimes \mathfrak{d}_+) \ltimes \mathfrak{sl}_2.$$  

The following lemma is similar to Lemma 3.2 in [10].

**Lemma 6.10.1.** $A_{\text{ali}} \ltimes \mathfrak{d}_+$ is positively graded.
Proof. Define $\mathbb{Z}^2$–grading of $\mathfrak{d}$ and $A_{\mathfrak{d}}$ by $\deg(\Delta_0) = (-1, 1)$, $\deg(d) = (0, 0)$, $\deg(X) = (1, -1)$, $\deg(\delta_{2m}) = (2m + 1, 1)$, $\deg(x(u)) = (1, 0)$, $\deg(y(u)) = (0, 1)$ and $\deg(t_\alpha) = (1, 1)$.

We can form the following semidirect products

$$G_n := \exp(A_{\mathfrak{d}} \rtimes \mathfrak{d}_+) \rtimes \text{SL}_2(\mathbb{C}).$$

The semidirect product $(P^\vee)^2 \rtimes \text{SL}_2(\mathbb{Z})$ acts on $\mathfrak{h} \times \mathfrak{g}$ by $(n, m) \star (z, \tau) := (z + n + \tau m, \tau)$ for $(n, m) \in (P^\vee)^2$ and $(a\ b\ c\ d) \star (z, \tau) := (z + \frac{\alpha + b}{c + d}, \tau)$, for $(a\ b\ c\ d) \in \text{SL}_2(\mathbb{Z})$.

Let $\chi_{\alpha,\tau} : P^\vee \times \mathbb{Z} \times \tau \to \mathbb{Z}$ be the natural morphism. We define $\tilde{H}_{\alpha,\tau}$ to be

$$\tilde{H}_{\alpha,\tau} = \{(z, \tau) \in \mathfrak{h} \times \mathfrak{g} \mid \alpha(z) \in \Lambda_{\tau}\}.$$

**Lemma 6.10.2.** The group $(P^\vee)^2 \rtimes \text{SL}_2(\mathbb{Z})$ preserves $\mathfrak{h} \times \mathfrak{g} \setminus \bigcup_{\alpha,\tau} \tilde{H}_{\alpha,\tau}$.

**Proof.** Take $(z, \tau) \in \mathfrak{h} \times \mathfrak{g} \setminus \bigcup_{\alpha} H_{\alpha}$, that is, for any $\alpha \in \Phi$, $\chi_{\alpha,\tau}(z) \notin P^\vee \times \Lambda_{\tau}$.

It’s obvious that $\chi_{\alpha,\tau}(z + n + \tau m) \notin P^\vee \times \Lambda_{\tau}$, for any $\alpha \in \Phi$.

Suppose $\chi_{\alpha,\tau}\frac{\alpha + b}{c + d} \in (P^\vee \times \Lambda_{\tau})^\vee$.

Case 1, if $c = 0$, then, $d = \pm 1$. we have $\chi_{\alpha,\tau}(z) \in \mathbb{Z} \times \mathbb{Z}(a\tau + b)$. It’s a contradiction.

Case 2, if $c \neq 0$, then, $\chi_{\alpha,\tau}(z) = (c\tau + d)\chi_{\alpha,\tau}\frac{\alpha + b}{c + d} \in (c\tau + d)\mathbb{Z}(a\tau + b) \subset \mathbb{Z} \times \mathbb{Z}\tau$. It’s a contradiction.

We can identify the quotient with the moduli space $\mathcal{M}_{1,n}$ of our space $X$.

Denote by $\pi : \mathfrak{h} \times \mathfrak{g} \setminus \bigcup_{\alpha} H_{\alpha} \to \mathcal{M}_{1,n}$ the projection map. We will define a principal $G_n$–bundle with flat connection $(P_n, \nabla_{P_n})$ over $\mathcal{M}_{1,n}$.

For $u \in \mathbb{C}^\times$, $u^d := \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right) \in \text{SL}_2(\mathbb{C}) \subset G_n$ and for $v \in \mathbb{C}$, $v^x := \left(\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}\right) \in \text{SL}_2(\mathbb{C}) \subset G_n$.

**Proposition 6.10.3.** There exists a unique principal $G_n$–bundle $P_n$ over $\mathcal{M}_{1,n}$, such that a section of $U \subset \mathcal{M}_{1,n}$ is a function $f : \pi^{-1}(U) \to G_n$, such that $f(z + \alpha_i^\vee|\tau) = f(z|\tau)$, $f(z + \tau\alpha_i^\vee|\tau) = e^{-2\pi i x_i \tau}/f(z|\tau)$, $f(z|\tau + 1) = f(z|\tau)$ and $f(\frac{z}{\tau} - \frac{1}{\tau}) = \tau^df(\frac{2\pi a_i}{\tau}(\sum_{\tau} z|\tau \alpha_i^\vee + X))f(z|\tau)$.
Proof. In [10], Calaque, Enriquez and Etingof showed there exists a principal bundle over \(\mathfrak{h} \times \mathfrak{h}/((\mathbb{Z}^n)^2 \times \text{SL}_2(\mathbb{Z}))\) with the structure group \(G_n\). Pulling back of this bundle along the map \(\mathcal{M}_{1,\,n} \hookrightarrow \mathfrak{h} \times \mathfrak{h}/((\mathbb{Z}^n)^2 \times \text{SL}_2(\mathbb{Z}))\) gives us the desired vector bundle \(P_n\).

6.11 Flat connection on the moduli space \(\mathcal{M}_{1,\,n}\)

Denote Diag\(_n\) := \(\{(z, \tau) \in \mathbb{C}^n \times \mathfrak{h} \mid \text{for some } i \neq j, \ z_i - z_j \in \Lambda_\tau\}\). Set \(g(z, x|\tau) := k(z, x|\tau) = \frac{\partial k(z, x|\tau)}{\partial x}\). We have \(g(z, x|\tau) \in \text{Hol}(\mathbb{C} \times \mathfrak{h}) - \text{Diag}_1[[x]]\).

For \(\psi(x) = \sum_{n \geq 1} b_{2n} x^{2n}\), we set \(\delta_\psi := \sum_{n \geq 1} b_{2n} \delta_{2n}\), \(\Delta_\psi := \Delta_0 + \sum_{n \geq 1} b_{2n} \delta_{2n}\). If we set \(\varphi(x) = g(0, 0|\tau) - g(0, x|\tau)\), then, \(\varphi(x) = \sum_{n \geq 1} a_{2n} E_{2n+2}(\tau) x^{2n}\). Denote \(\delta_\varphi := \sum_{n \geq 1} a_{2n} E_{2n+2}(\tau) \delta_{2n}\). We have \(\Delta_\varphi = \Delta_0 + \delta_\varphi\).

For \(u \in \mathfrak{h}\), define

\[
\Delta = \Delta(u, \tau) := -\frac{1}{2\pi i} \Delta_0 - \frac{1}{2\pi i} \sum_{n \geq 1} a_{2n} E_{2n+2}(\tau) \delta_{2n} + \frac{1}{2\pi i} \sum_{\beta \in \Phi^+} g(\beta, \text{ad} \frac{x_{\beta^\vee}}{2}|\tau)(t_\beta)
\]

\[
= -\frac{1}{2\pi i} \Delta_\varphi + \frac{1}{2\pi i} \sum_{\beta \in \Phi^+} g(\beta, \text{ad} \frac{x_{\beta^\vee}}{2}|\tau)(t_\beta),
\]

where \(a_{2n} = -(2n + 1)B_{2n+1}(2i\pi)^{2n+1}/(2n + 1)!\) and \(B_n\) are the Bernoulli numbers given by \(\frac{x}{e^x - 1} = \sum_{r \geq 0} (B_r/r!)x^r\). This is a a meromorphic function \(\mathbb{C}^n \times \mathfrak{h} \to (\mathfrak{a}_n \times \mathfrak{h}_+ \times \mathfrak{n}_+) \subset \text{Lie}(G_n)\), (where \(\mathfrak{n}_+ = \mathbb{C}\Delta_0 \subset \mathfrak{sl}_2\)) with only poles at \(\bigcup_{\alpha \in \Phi^+} H_\alpha\).

Theorem 6.11.1. The following \(A_{\text{ell}} \times \mathfrak{d}\)-valued connection on \(\mathcal{M}_{1,\,n}\) is flat.

\[
\nabla = d - \Delta d\tau - \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad} \frac{x_{\alpha^\vee}}{2}|\tau)(t_\alpha) d\alpha + \sum_{i=1}^n y(u^i) du_i.
\]

(6.11.1)
Note we have already known the flatness of $\nabla = d - \sum_{\alpha \in \Phi} k(\alpha, \text{ad}_{\frac{x_\alpha}{2}} \tau)(t_\alpha) d\alpha + \sum_{i=1}^n y(u^i) du_i$. Set

$$A := \Delta d\tau + \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}_{\frac{x_\alpha}{2}} \tau)(t_\alpha) d\alpha - \sum_{i=1}^n y(u^i) du_i.$$ 

The rest of this section is devoted to show the Theorem 6.11.1. By definition of the flatness of the connection $\nabla$, we need to show: $dA + A \wedge A = 0$.

### 6.11.1 Check that $dA = 0$

\[
dA = \frac{\partial A}{\partial \tau} d\tau + \sum_{i=1}^n \frac{\partial A}{\partial \alpha_i} d\alpha_i
\]

\[
= \sum_{\alpha \in \Phi^+} \frac{\partial}{\partial \tau} k(\alpha, \text{ad}_{\frac{x_\alpha}{2}} \tau)(t_\alpha) d\tau \wedge d\alpha + \sum_i \frac{\partial}{\partial \alpha_i} \Delta(\alpha, \tau) d\alpha_i \wedge d\tau
\]

\[
= \sum_{\alpha \in \Phi^+} \frac{\partial}{\partial \tau} k(\alpha, \text{ad}_{\frac{x_\alpha}{2}} \tau)(t_\alpha) - \frac{1}{2\pi i} \sum_i \sum_{\beta} \frac{\partial}{\partial \alpha_i} g(\beta, \text{ad}_{\frac{x_\alpha}{2}} \tau)(t_\alpha) d\tau \wedge d\alpha_i
\]

\[
= \sum_{\alpha \in \Phi^+} \left( \frac{\partial}{\partial \tau} k(\alpha, \text{ad}_{\frac{x_\alpha}{2}} \tau)(t_\alpha) - \frac{1}{2\pi i} \frac{\partial}{\partial \alpha} g(\alpha, \text{ad}_{\frac{x_\alpha}{2}} \tau)(t_\alpha) \right) d\tau \wedge d\alpha
\]

which is 0 by the following identity $(\partial_\tau k)(z, x | \tau) = \frac{1}{2\pi i} (\partial_{z \tau}) g(z, x | \tau)$, which can be found in [10], on Page 190.
6.11.2 Check that $A \land A = 0$

By definition, we have,

$$A \land A = \sum_{\alpha \in \Phi^+} [\Delta(\alpha, \tau), k(\alpha, \text{ad} \frac{x_{\alpha^\vee}}{2}|\tau)(t_\alpha)]d\tau \land d\alpha - \sum_i [\Delta(\alpha|\tau), y_{\chi^\vee}]d\tau \land d\alpha_i$$

$$= -\frac{1}{2\pi i} \left( \sum_{\alpha \in \Phi^+} [\Delta(\alpha, \tau), k(\alpha, \text{ad} \frac{x_{\alpha^\vee}}{2}|\tau)(t_\alpha)]d\tau \land d\alpha - \sum_i [\Delta(\alpha|\tau), y_{\chi^\vee}]d\tau \land d\alpha_i \right)$$

$$- \frac{1}{2\pi i} \left( \sum_{\alpha \in \Phi^+} [\Delta(\alpha, \tau), k(\alpha, \text{ad} \frac{x_{\alpha^\vee}}{2}|\tau)(t_\alpha)]d\tau \land d\alpha - \sum_i [\Delta(\alpha|\tau), y_{\chi^\vee}]d\tau \land d\alpha_i \right)$$

$$= -\frac{1}{2\pi i} \left( \sum_{\alpha \in \Phi^+} \sum_{\beta \in \Phi^+} g(\beta, \text{ad} \frac{x_{\beta^\vee}}{2}|\tau)(t_\beta), k(\alpha, \text{ad} \frac{x_{\alpha^\vee}}{2}|\tau)(t_\alpha)]d\tau \land d\alpha - \sum_i \sum_{\beta \in \Phi^+} g(\beta, \text{ad} \frac{x_{\beta^\vee}}{2}|\tau)(t_\beta), y_{\chi^\vee}]d\tau \land d\alpha_i \right)$$

Now let’s calculate each terms in the above expression of $A \land A$.

Lemma 6.11.2. Modulo the relations $(tx), (xx)$, the following identity holds for any $\alpha \neq \beta \in \Phi^+$:

$$[g(\beta, \text{ad} \frac{x_{\beta^\vee}}{2}|\tau)(t_\beta), k(\alpha, \text{ad} \frac{x_{\alpha^\vee}}{2}|\tau)(t_\alpha)] = g(\beta, (\text{ad} \frac{x_{\omega(\beta^\vee, \alpha)}|\tau}{-2})|\tau)k(\alpha, (\text{ad} \frac{x_{\omega(\alpha^\vee, \beta)}|\tau}{-2})|\tau)[t_\beta, t_\alpha]$$

Lemma 6.11.3. Modulo the relations $(yt), (yx), (tx), (xx)$, the following identity holds:

$$\sum_i [y(u^i), g(\beta, \text{ad} \frac{x_{\beta^\vee}}{2}|\tau)(t_\beta)]d\tau \land du_i - \sum_i [y(\beta^\vee), g(\beta, \text{ad} \frac{x_{\beta^\vee}}{2}|\tau)(t_\beta)]d\tau \land d\beta$$

$$= \sum_{\gamma \neq \beta \in \Phi^+} g(\beta, \text{ad} \frac{x_{\beta^\vee}}{2}|\tau) - g(\beta, (\text{ad} \frac{x_{\omega(\beta^\vee, \alpha)}|\tau}{-2})|\tau)\frac{t_\gamma, t_\beta}{\text{ad} \frac{x_{\omega(\gamma^\vee, \beta)}}{-2}}d\tau \land d\gamma$$

$$- \sum_{\gamma \neq \beta \in \Phi^+} g(\beta, \text{ad} \frac{x_{\beta^\vee}}{2}|\tau) - g(\beta, (\text{ad} \frac{x_{\omega(\beta^\vee, \alpha)}|\tau}{-2})|\tau)\frac{t_\gamma, t_\beta}{\text{ad} \frac{x_{\omega(\gamma^\vee, \beta)}}{-2}}d\tau \land d\beta$$

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Proof. If $\beta(u^i) = 0$, we have:

$$[y(u^i), g(\beta, \text{ad} \frac{x_{\gamma^i}}{2}|\tau)(t_\beta)] = \sum_{\gamma \in \Phi^+} (\beta^\wedge, \gamma) \gamma(u^i) \frac{g(\beta, \text{ad} \frac{x_{\gamma^i}}{2}|\tau) - g(\beta, (\text{ad} \frac{x_{\omega(\gamma, \gamma)}}{-2})|\tau)}{\text{ad} x_{\beta^\wedge} + \text{ad} x_{\omega(\beta^\wedge, \gamma)}} [t_\gamma, t_\beta]$$

For a fixed root $\beta$, choose $\{u_1, \ldots, u_n\}$, such that $u_1 = \beta$. Then, $\beta(u^i) = 0$, for $i \neq 1$. Then, we have:

$$\sum_{i=1}^n [y(u^i), g(\beta, \text{ad} \frac{x_{\gamma^i}}{2}|\tau)(t_\beta)] d\tau \land du_i$$

$$= \sum_{i=1}^n \sum_{\gamma \in \Phi^+} (\beta^\wedge, \gamma) \gamma(u^i) \frac{g(\beta, \text{ad} \frac{x_{\gamma^i}}{2}|\tau) - g(\beta, (\text{ad} \frac{x_{\omega(\gamma, \gamma)}}{-2})|\tau)}{\text{ad} x_{\beta^\wedge} + \text{ad} x_{\omega(\beta^\wedge, \gamma)}} [t_\gamma, t_\beta] d\tau \land du_i$$

$$- \sum_{\gamma \in \Phi^+} (\beta^\wedge, \gamma) \gamma(\beta^\wedge) \frac{g(\beta, \text{ad} \frac{x_{\gamma^i}}{2}|\tau) - g(\beta, (\text{ad} \frac{x_{\omega(\gamma, \gamma)}}{-2})|\tau)}{\text{ad} x_{\beta^\wedge} + \text{ad} x_{\omega(\beta^\wedge, \gamma)}} [t_\gamma, t_\beta] d\tau \land d\beta$$

$$= [y(\beta^\wedge), g(\beta, \text{ad} \frac{x_{\gamma^i}}{2}|\tau)(t_\beta)] d\tau \land d\beta + \sum_{\gamma \neq \beta \in \Phi^+} \frac{g(\beta, \text{ad} \frac{x_{\gamma^i}}{2}|\tau) - g(\beta, (\text{ad} \frac{x_{\omega(\gamma, \gamma)}}{-2})|\tau)}{\text{ad} \frac{x_{\omega(\gamma^i, \beta)}}{-2}} [t_\gamma, t_\beta] d\tau \land d\gamma$$

Thus, the conclusion follows. \qed

Lemma 6.11.4. Modulo the relation $(\delta y)$, we have:

$$\sum_{i} [\Delta_\phi, y(u^i)] du_i = \sum_{\alpha \in \Phi^+} \sum_{s} [f_s \text{ad} \frac{x_{\alpha^i}}{2}(t_\alpha), g_s (- \text{ad} \frac{x_{\alpha^i}}{2})(t_\alpha)] d\tau \land d\alpha,$$

where $\frac{1}{2} \psi(u^i - \psi(v)) = \sum_{s} f_s(u) g_s(v)$. 233
Proof. By definition, we have

\[ [\delta_{2m}, y(u^i)] = \frac{1}{2} \sum_{\alpha} \alpha(u^i) \sum_{p+q=2m-1} [\text{ad} \frac{x_{\alpha^\vee}}{2}(t_\alpha), (\text{ad} - \frac{x_{\alpha^\vee}}{2})^q(t_\alpha)], \]

Then:

\[ [\Delta_\varphi, y(u^i)] = \sum_{\alpha} \alpha(u^i) \sum_{s} [f_s \text{ad} \frac{x_{\alpha^\vee}}{2}(t_\alpha), g_s (\text{ad} - \frac{x_{\alpha^\vee}}{2})(t_\alpha)]. \]

Lemma 6.11.5. We have:

\[ [\delta_\varphi, k(\alpha, \text{ad} \frac{x_{\alpha^\vee}}{2}|\tau)(t_\alpha)] = \sum_{s} [l_s^\alpha (\text{ad} \frac{x_{\alpha^\vee}}{2})(t_\alpha), m_s^\alpha (\text{ad} \frac{x_{\alpha^\vee}}{2})(t_\alpha)], \]

where \( k(\alpha, u + v)\varphi(v) = \sum_s l_s^\alpha(u)m_s^\alpha(v). \)

Proof.

Lemma 6.11.6. Modulo the relation \([\Delta_0, x], [\Delta_0, y], [\Delta_0, t], \) and \((yx), \) we have:

\[ [\Delta_0, k(\alpha, \text{ad} \frac{x_{\alpha^\vee}}{2}|\tau)(t_\alpha)] = [\frac{y_{\alpha^\vee}}{2}, g(\alpha, \text{ad} \frac{x_{\alpha^\vee}}{2}|\tau)(t_\alpha)] - \sum_s [h_s(\text{ad} \frac{x_{\alpha^\vee}}{2})(t_\alpha), k_s(\text{ad} \frac{x_{\alpha^\vee}}{2})(t_\alpha)] \]

\[ - \sum_{\gamma \neq \alpha} (\alpha^\vee, \gamma)^2 k(\alpha, \text{ad} \frac{x_{\alpha^\vee}}{2} - k(\alpha, (\text{ad} \frac{x_{\omega(\alpha^\vee, \gamma)}}{2} - (\text{ad} \frac{x_{\alpha^\vee}}{2} + \text{ad} \frac{x_{\omega(\alpha^\vee, \gamma)}}{2})k_s(\alpha, (\text{ad} \frac{x_{\omega(\alpha^\vee, \gamma)}}{2}))(t_\gamma, t_\alpha) \]

where \( \frac{1}{2} (\frac{k(\alpha, u + v) - k(\alpha, u + v) - k(\alpha, u + v)}{v^2} - \frac{k(\alpha, u + v) - k(\alpha, u + v) - k(\alpha, u + v)}{u^2}) = -\sum_s h_s(u)k_s(v). \)

Proof. By definition \([\Delta_0, x_{\alpha^\vee}] = y_{\alpha^\vee}, [\Delta_0, y_\alpha] = 0 \) and \([\Delta_0, t_\alpha] = 0. \) Since \([\Delta_0, t_\alpha] = 0 \) and
\[ [x_\alpha^\vee, y_\alpha^\vee] = -\sum_{\gamma \in \Phi^+} (\alpha^\vee, \gamma)^2 t_\gamma, \text{ we have:} \]

\[ [\Delta_0, (\text{ad} x_\alpha^\vee)^n(t_\alpha)] \]

\[ = \sum_{s=0}^{n-1} (\text{ad} x_\alpha^\vee)^s(\text{ad}[\Delta_0, x_\alpha^\vee])(\text{ad} x_\alpha^\vee)^{n-1-s}(t_\alpha) = \sum_{s=0}^{n-1} (\text{ad} x_\alpha^\vee)^s(\text{ad} y_\alpha^\vee)(\text{ad} x_\alpha^\vee)^{n-1-s}(t_\alpha) \]

\[ = n \text{ ad } y_\alpha^\vee(\text{ad} x_\alpha^\vee)^{n-1}(t_\alpha) + \sum_{s=1}^{n-1} \sum_{i=0}^{s-1} (\text{ad} x_\alpha^\vee)^i \text{ ad}[x_\alpha^\vee, y_\alpha^\vee](\text{ad} x_\alpha^\vee)^{n-2-i}(t_\alpha) \]

\[ = n \text{ ad } y_\alpha^\vee(\text{ad} x_\alpha^\vee)^{n-1}(t_\alpha) - \sum_{s=1}^{n-1} \sum_{i=0}^{s-1} (\alpha^\vee, \gamma)^2(\text{ad} x_\alpha^\vee)^i \text{ ad } t_\gamma(\text{ad} x_\alpha^\vee)^{n-2-i}(t_\alpha) \]

\[ = n \text{ ad } y_\alpha^\vee(\text{ad} x_\alpha^\vee)^{n-1}(t_\alpha) - \sum_{s=1}^{n-1} \sum_{i=0}^{s-1} 4(\text{ad} x_\alpha^\vee)^i \text{ ad } t_\alpha(\text{ad} x_\alpha^\vee)^{n-2-i}(t_\alpha) \]

\[ - \sum_{s=1}^{n-1} \sum_{i=0}^{s-1} \sum_{\gamma \neq \alpha} (\alpha^\vee, \gamma)^2(\text{ad} x_\alpha^\vee)^i \text{ ad } t_\gamma(\text{ad} x_\alpha^\vee)^{n-2-i}(t_\alpha) \]

The third equality follows from:

\[ (\text{ad} x_\alpha^\vee)^s(\text{ad} y_\alpha^\vee)(\text{ad} x_\alpha^\vee)^{n-1-s}(t_\alpha) \]

\[ = (\text{ad} x_\alpha^\vee)^{s-1}(\text{ad}[x_\alpha^\vee, y_\alpha^\vee])(\text{ad} x_\alpha^\vee)^{n-1-s}(t_\alpha) + (\text{ad} x_\alpha^\vee)^{s-1}(\text{ad} y_\alpha^\vee)(\text{ad} x_\alpha^\vee)^{n-s}(t_\alpha) \]

The conclusion follows from the following identity which can be showed by induction:

\[ \sum_{s=1}^{n-1} \sum_{i=0}^{s-1} a^i b^{n-2-i} = \frac{a^n - b^n - n b^{n-1}(a - b)}{(a - b)^2}. \]

Now plug all the above lemmas into the formula of \( A \wedge A \), we get:

\[-2\pi i A \wedge A = \sum_{s=2}^n \sum_{\alpha \in \Phi^+} (F^\alpha_s(\text{ad} x_\alpha^\vee/2), C^\alpha_s(\text{ad} x_\alpha^\vee/2)) d\tau \wedge d\alpha + \sum_{\alpha \in \Phi^+} (\sum_{\gamma \neq \alpha} H(\alpha, \gamma)[t_\gamma, t_\alpha])d\tau \wedge d\alpha, \]

where \( \sum_s F_s(u)^\alpha C_s^\alpha(v) = -L(\alpha, u, v) \), and

\[ L(z, u, v) = \frac{1}{2} \varphi(u) - \varphi(v) + \frac{1}{2} k(z, u + v)(\varphi(u) - \varphi(v)) \]

\[ + \frac{1}{2} (g(z, u)k(z, u) - k(z, u)g(z, v)) \]

\[ - \frac{1}{2} \frac{k(z, u + v) - k(z, u) - vg(z, u)}{v^2} = \frac{k(z, u + v) - k(z, v) - vg(z, v)}{u^2}. \]

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As showed in [10] that, \( L(z, u, v) = 0 \).

We have

\[
H(\alpha, \gamma) = -k(\alpha, \operatorname{ad} \frac{\omega(\gamma, \alpha)}{2}) - k(\alpha, \operatorname{ad} \frac{\omega(\gamma, \alpha)}{-2}) + \frac{\gamma(\alpha) g(\alpha, \operatorname{ad} \frac{\omega(\gamma, \alpha)}{2})}{2} \operatorname{ad} \frac{\omega(\gamma, \alpha)}{-2}
+ g(\gamma, \operatorname{ad} \frac{\omega(\gamma, \alpha)}{2}) - g(\gamma, \operatorname{ad} \frac{\omega(\gamma, \alpha)}{-2}) - g(\gamma, \operatorname{ad} \frac{\omega(\gamma, \alpha)}{-2}) k(\alpha, \operatorname{ad} \frac{\omega(\gamma, \alpha)}{-2}).
\]

**Proposition 6.11.7.** The following identity holds:

\[
H(\alpha, \gamma) - H(\alpha, \alpha + \gamma) - H(\gamma + \alpha, \gamma) + H(\gamma + \alpha, \alpha) \equiv 0
\]

**Proof.** By calculation, we have:

\[
H(\alpha, \gamma) - H(\alpha, \alpha + \gamma) - H(\gamma + \alpha, \gamma) + H(\gamma + \alpha, \alpha)
= -k(\alpha, \operatorname{ad} \frac{\omega(\alpha + \gamma, \alpha)}{2}) - k(\alpha, \operatorname{ad} \frac{\omega(\alpha + \gamma, \alpha)}{-2})
+ \frac{\gamma(\alpha) g(\alpha, \operatorname{ad} \frac{\omega(\alpha + \gamma, \alpha)}{2})}{2} \operatorname{ad} \frac{\omega(\alpha + \gamma, \alpha)}{-2}
+ g(\gamma, \operatorname{ad} \frac{\omega(\alpha + \gamma, \alpha)}{2}) - g(\gamma, \operatorname{ad} \frac{\omega(\alpha + \gamma, \alpha)}{-2})
- g(\gamma, \operatorname{ad} \frac{\omega(\alpha + \gamma, \alpha)}{-2}) k(\alpha, \operatorname{ad} \frac{\omega(\alpha + \gamma, \alpha)}{-2}).
\]

Now use the fact (showed in [10]) that

\[
H(z, z', u, v) = \frac{k(z, u + v) - k(z, u) - vg(z, u)}{v^2} - \frac{k(z', u + v) - k(z', v) - u g(z', v)}{u^2}
+ \frac{g(z' - z, -u) - g(z' - z, v)}{u + v} - g(-z', -v) k(-z, -u) + g(-z, -u) k(-z', -v)
- g(z' - z, -v) k(z, u + v) + g(z' - z, -u) k(z', u + v) \equiv 0
\]

Note that \( g(z, x) = g(-z, -x) \). Thus, the conclusion holds.

Now similar as the argument in §6.4, it suffices to check the flatness for \( \Phi \) is a rank 2 root systems. We will check the flatness case by case.
6.11.3 Case $A_2$

Use the graph $(\alpha_1, \alpha_2) \rightarrow \begin{cases} (\alpha_1, \alpha_1 + \alpha_2) \\ (\alpha_1 + \alpha_2, \alpha_1) \\ (\alpha_1 + \alpha_2, \alpha_2) \end{cases}$, meaning write

$$H(\alpha_1, \alpha_2) = H(\alpha_1, \alpha_1 + \alpha_2) - H(\alpha_1 + \alpha_2, \alpha_1) - H(\alpha_1 + \alpha_2, \alpha_2)$$

according to Proposition 6.11.7.

Then, in the expression of $\sum_{\alpha \in \Phi^+} (\sum_{\gamma \neq \alpha} H(\alpha, \gamma)[t_\alpha, t_\gamma]) d\tau \wedge d\alpha$, the coefficient of $d\tau \wedge d\alpha_1$ becomes:

$$H(\alpha_1, \alpha_1 + \alpha_2)[t_{\alpha_1}, t_{\alpha_1 + \alpha_2} + t_{\alpha_2}] + H(\alpha_1 + \alpha_2, \alpha_1)[t_{\alpha_1 + \alpha_2} + t_{\alpha_2}, t_{\alpha_1}]$$
$$+ H(\alpha_1 + \alpha_2, \alpha_2)[t_{\alpha_1 + \alpha_2} + t_{\alpha_1}, t_{\alpha_2}] = 0.$$

Use the graph $(\alpha_2, \alpha_1) \rightarrow \begin{cases} (\alpha_2, \alpha_1 + \alpha_2) \\ (\alpha_1 + \alpha_2, \alpha_2) \\ (\alpha_1 + \alpha_2, \alpha_1) \end{cases}$, then the coefficient of $d\tau \wedge d\alpha_2$ becomes:

$$H(\alpha_2, \alpha_1 + \alpha_2)[t_{\alpha_2}, t_{\alpha_1 + \alpha_2} + t_{\alpha_1}] + H(\alpha_1 + \alpha_2, \alpha_1)[t_{\alpha_1 + \alpha_2} + t_{\alpha_2}, t_{\alpha_1}]$$
$$+ H(\alpha_1 + \alpha_2, \alpha_2)[t_{\alpha_1 + \alpha_2} + t_{\alpha_1}, t_{\alpha_2}] = 0.$$

Thus, the connection is flat in the case of $A_2$. 

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6.11.4 Case $B_2$

Use the graph $(\alpha_1, \alpha_2) \rightarrow \begin{cases} (\alpha_1, \alpha_1 + \alpha_2) \\ (\alpha_1 + \alpha_2, \alpha_1) \\ (\alpha_1 + \alpha_2, \alpha_2) \end{cases}$, and $(\alpha_1 + 2\alpha_2, \alpha_2) \rightarrow \begin{cases} (\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2) \\ (\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2) \\ (\alpha_1 + \alpha_2, \alpha_2) \end{cases}$

then the coefficient of $d\tau \wedge d\alpha_1$ is:

\[
H(\alpha_1, \alpha_1 + \alpha_2)[t_{\alpha_1}, t_{\alpha_1+\alpha_2} + t_{\alpha_2}] + H(\alpha_1 + \alpha_2, \alpha_1)[t_{\alpha_1+\alpha_2} + t_{\alpha_2}, t_{\alpha_1}] \\
+ H(\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2)[t_{\alpha_1+\alpha_2} + t_{\alpha_2}, t_{\alpha_1+2\alpha_2}] + H(\alpha_1 + \alpha_2, \alpha_2)[t_{\alpha_1+\alpha_2} + t_{\alpha_1} + t_{\alpha_1+2\alpha_2}, t_{\alpha_2}] \\
+ H(\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2)[t_{\alpha_1+\alpha_2}, t_{\alpha_1+\alpha_2} + t_{\alpha_2}] = 0.
\]

Use the graph $(\alpha_2, \alpha_1) \rightarrow \begin{cases} (\alpha_2, \alpha_1 + \alpha_2) \\ (\alpha_1 + \alpha_2, \alpha_2) \\ (\alpha_1 + \alpha_2, \alpha_1) \end{cases}$, $(\alpha_2, \alpha_1 + 2\alpha_2) \rightarrow \begin{cases} (\alpha_1 + 2\alpha_2, \alpha_2) \\ (\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2) \\ (\alpha_2, \alpha_1 + \alpha_2) \end{cases}$, then, the coefficient of $d\tau \wedge d\alpha_2$ is:

\[
H(\alpha_2, \alpha_1 + \alpha_2)[t_{\alpha_2}, t_{\alpha_1+\alpha_2} + t_{\alpha_1} + t_{\alpha_1+2\alpha_2}] + H(\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2)[t_{\alpha_1+2\alpha_2}, t_{\alpha_1+\alpha_2} + t_{\alpha_2}] \\
+ 2H(\alpha_1 + 2\alpha_2, \alpha_2)[t_{\alpha_1+2\alpha_2}, t_{\alpha_2} + t_{\alpha_1+\alpha_2}] + H(\alpha_1 + \alpha_2, \alpha_1)[t_{\alpha_1+\alpha_2} + t_{\alpha_2}, t_{\alpha_1}] \\
+ H(\alpha_1 + \alpha_2, \alpha_2)[t_{\alpha_1+\alpha_2} + t_{\alpha_1+\alpha_2} + t_{\alpha_1}, t_{\alpha_2}] = 0.
\]

6.11.5 Case $G_2$

Use the following graphs in order:

$(\alpha_1, \alpha_1 + 3\alpha_2) \rightarrow \begin{cases} (\alpha_1, 2\alpha_1 + 3\alpha_2) \\ (2\alpha_1 + 3\alpha_2, \alpha_1) \\ (2\alpha_1 + 3\alpha_2, \alpha_1 + 3\alpha_2) \end{cases}$, $(\alpha_1, \alpha_2) \rightarrow \begin{cases} (\alpha_1, \alpha_1 + \alpha_2) \\ (\alpha_1 + \alpha_2, \alpha_1) \\ (\alpha_1 + \alpha_2, \alpha_2) \end{cases}$
\[
(a_1 + 3a_2, a_1) \rightarrow \begin{cases} (a_1 + 3a_2, 2a_1 + 3a_2) \\ (2a_1 + 3a_2, a_1) \\ (2a_1 + 3a_2, a_1 + 3a_2) \end{cases},
(a_1 + 3a_2, a_2) \rightarrow \begin{cases} (a_1 + 2a_2, a_2) \\ (a_1 + 2a_2, a_1 + 3a_2) \end{cases},
(a_1 + 2a_1 + 3a_2) \rightarrow \begin{cases} (a_1 + a_2, a_1 + 2a_2) \\ (2a_1 + 3a_2, a_1 + 2a_2) \\ (2a_1 + 3a_2, a_1 + a_2) \end{cases},
(a_1 + a_2, a_1 + 2a_2) \rightarrow \begin{cases} (a_1 + a_2, a_1 + a_2) \end{cases},
(a_1 + a_2, a_1 + a_2) \rightarrow \begin{cases} (a_1 + 2a_2, a_1 + a_2) \end{cases}.
\]

Then the coefficient of \( d\tau \wedge da_1 \) is

\[
H(a_1, a_1 + a_2)[t_{a_1}, t_{a_1 + a_2} + t_{a_2}] + H(a_1, 2a_1 + 3a_2)[t_{a_1}, t_{2a_1 + 3a_2} + t_{a_1 + 3a_2}]
+ H(a_1 + a_2, a_1)[t_{a_1 + a_2} + t_{a_2}, t_{a_1}] + H(a_1 + a_2, a_1 + 2a_2)[t_{a_1 + a_2}, t_{a_1 + 2a_2} + t_{2a_1 + 3a_2} + t_{a_1} + t_{a_2}]
+ H(a_1 + 2a_2, a_2)[t_{a_1 + 2a_2} + t_{a_1 + a_2} + t_{a_1} + t_{a_1 + 3a_2} + t_{a_2}]
+ H(a_1 + 2a_2, a_1 + a_2)[t_{a_1 + 2a_2} + t_{2a_1 + 3a_2} + t_{a_1} + t_{a_1 + a_2} + t_{a_2}],
+ H(a_1 + 3a_2, a_1 + a_2)[t_{a_1 + 3a_2} + t_{2a_1 + 3a_2} + t_{a_1} + 3a_2]
+ H(a_1 + 3a_2, 2a_1 + 3a_2)[t_{a_1 + 3a_2} + t_{2a_1 + 3a_2} + t_{a_1}]
+ 2H(2a_1 + 3a_2, a_1)[t_{a_1 + 3a_2} + t_{2a_1 + 3a_2} + t_{a_1}],
+ H(2a_1 + 3a_2, a_1 + a_2)[t_{2a_1 + 3a_2} + t_{a_1 + a_2} + t_{a_1 + 2a_2}],
+ H(2a_1 + 3a_2, a_1 + 2a_2)[t_{2a_1 + 3a_2} + t_{a_1 + a_2} + t_{a_1 + 2a_2}],
+ 2H(2a_1 + 3a_2, a_1 + 3a_2)[t_{a_1} + t_{2a_1 + 3a_2} + t_{a_1 + 3a_2}]
= 0
\]

Use the following graphs in order:

\[
3(a_1 + 3a_2, a_1) \rightarrow \begin{cases} 3(a_1 + 3a_2, 2a_1 + 3a_2) \\ 3(a_1 + 3a_2, a_1) \end{cases},
(a_2, a_1 + a_2) \rightarrow \begin{cases} (a_1 + a_2, a_2) \end{cases}.
\]

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the coefficient of $d\tau \wedge d\alpha_2$ is:

$$H(\alpha_2, \alpha_1 + 2\alpha_2)[t_{\alpha_2}, t_{\alpha_1+2\alpha_2} - t_{\alpha_1+2\alpha_2} + t_{\alpha_1+3\alpha_2} - t_{\alpha_1+3\alpha_2}] + H(\alpha_1 + \alpha_2, \alpha_1)[t_{\alpha_1+\alpha_2} + t_{\alpha_1+2\alpha_2}]$$

$$+ H(\alpha_1 + \alpha_2, 2\alpha_1 + 3\alpha_2)[t_{\alpha_1+\alpha_2}, t_{2\alpha_1+3\alpha_2} - t_{2\alpha_1+3\alpha_2}]$$

$$+ 2H(\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2)[t_{\alpha_1+2\alpha_2}, t_{\alpha_1+\alpha_2} + t_{\alpha_1+3\alpha_2} + t_{\alpha_2} + t_{\alpha_1+3\alpha_2}]$$

$$+ 2H(\alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2)[t_{\alpha_1+2\alpha_2} + t_{\alpha_1+\alpha_2}, t_{\alpha_1+3\alpha_2}]$$

$$+ 3H(\alpha_1 + 3\alpha_2, \alpha_1 + 2\alpha_2)[t_{\alpha_1+3\alpha_2}, t_{\alpha_1+2\alpha_2} + t_{\alpha_1} + 3\alpha_2] + 3H(\alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2)[t_{\alpha_1+3\alpha_2}, t_{2\alpha_1+3\alpha_2} + t_{\alpha_1}]$$

$$+ 3H(2\alpha_1 + 3\alpha_2, \alpha_1)[t_{2\alpha_1+3\alpha_2} + t_{\alpha_1} + 3\alpha_2, t_{\alpha_1}] + 3H(2\alpha_1 + 3\alpha_2, \alpha_1 + 2\alpha_2)[t_{2\alpha_1+3\alpha_2}, t_{\alpha_1+2\alpha_2} + t_{\alpha_1+\alpha_2}]$$

$$+ 3H(2\alpha_1 + 3\alpha_2, \alpha_1 + 3\alpha_2)[t_{2\alpha_1+3\alpha_2} + t_{\alpha_1}, t_{\alpha_1+3\alpha_2}] = 0.$$
6.12 Simply connected type and adjoint type

Let $\beta \in \Phi$ be a root, and $s_\beta \in W$ be the corresponding element in the Weyl group. The action of $W$ on $Q^\lor_Q$ is given by $s_\beta : Q^\lor_Q \rightarrow Q^\lor_Q$, sending $y \mapsto y - y(\beta)\beta^\lor$.

The induced action of $W$ on $X = Q^\lor \otimes \mathcal{E}$ is

$$
\sum_i z_i \alpha_i^\lor \mapsto \sum_i z_i \alpha_i^\lor - \sum_i z_i \beta(\alpha_i^\lor)\beta^\lor,
$$

for $s_\beta \in W, z_i \in \mathbb{C}$. Thus,

$$
s_\beta\left(\sum_i z_i \alpha_i^\lor\right) = \sum_i z_i \alpha_i^\lor
$$

$$
\iff \sum_i z_i \beta(\alpha_i^\lor)\beta^\lor \in Q^\lor \otimes \Lambda_r
$$

$$
\iff \beta\left(\sum_i z_i \alpha_i^\lor\right) \in \Lambda_r
$$

$$
\iff \chi_\beta\left(\sum_i z_i \alpha_i^\lor\right) = 0
$$

From the above, we know, the hyperplanes $H_\beta$ consisting of $s_\beta$ fixed points.

**Proposition 6.12.1.** The Weyl group action on the regular points of the simply-connected torus

$$
T = \mathfrak{h}/(Q^\lor \oplus \tau Q^\lor) = Q^\lor \otimes \mathcal{E}
$$

is free.

**Proof.** To show the action is free, we need to prove for every regular point $z$ of $T = \mathfrak{h}/(Q^\lor \oplus \tau Q^\lor)$, the stabilizer of $z$ in $W$

$$
\{w \in W \mid wz = z\}
$$

consists of the identity.
We have \( \mathfrak{h} = E \oplus \tau E \), where \( E \) is the real vector space. Let \( \tilde{z} = x + y\tau \in \mathfrak{h} \) be a lifting of \( z \in T \), then \( w(z) = z \) implies that \( w(\tilde{z}) = \tilde{z} + t \), for some \( t = t_1 + \tau t_2 \in Q^\vee \oplus \tau Q^\vee \), which is equivalent to \( w(x) = x + t_1 \), and \( w(y) = y + t_2 \).

Since \( z \) is regular, thus, either \( x \) is regular or \( y \) is regular. We can assume \( x \) is regular, which by definition

\[
x \in E \setminus \left\{ \bigcup_{\alpha \in \Phi, k \in \mathbb{Z}} H_{\alpha,k} \right\}, \quad \text{where } H_{\alpha,k} = \{ \lambda \in E \mid (\alpha, \lambda) = k \}.
\]

Let \( W_a := W \rtimes Q^\vee \) be the affine Weyl group, then the element \( w_a = (w, -t_1) \) can be viewed as an element in \( W_a = W \rtimes Q^\vee \). The fact \( w(x) = x + t_1 \) implies that \( w_a(x) = x \).

Let \( \mathcal{A} \) be the set of alcoves in \( E \). We know that the action \( W_a \) on \( \mathcal{A} \) is simply transitive. See [43], chapter 4, Theorem 4.5, page 93. Let \( x \in A \), for some alcove \( A \), then,

\[
w_a(x) = x \Rightarrow w_a(A) \cap A \neq \emptyset
\]

\[
\Rightarrow w_a(A) = A
\]

\[
\Rightarrow w_a = id
\]

\[
\Rightarrow w = id
\]

Thus, the action of \( W \) is free on \( T^{\text{reg}} \).

The following example shows for adjoint torus, the action of \( W \) on \( X^{\text{reg}} \) is not free.

**Example 6.12.2.** Consider \( A_1 \) case, let \( \alpha \) be the unique positive root. For adjoint type, \( X = P^\vee \otimes \mathcal{E} \cong \mathcal{E} \), where \( P^\vee = \mathbb{Z} \frac{\alpha^\vee}{2} \) be the coweight lattice. In this case

\[
H_\alpha = \left\{ z \frac{\alpha^\vee}{2} \mid \alpha(z \frac{\alpha^\vee}{2}) \in \Lambda_r \right\}
\]

\[
= \left\{ z \frac{\alpha^\vee}{2} \mid z \in \Lambda_r \right\}
\]

Thus, \( X \setminus H_\alpha \cong \mathcal{E} \setminus \text{pt} \).

For \( s_\alpha \) fixed point \( z \frac{\alpha^\vee}{2} \), we have \( z\alpha^\vee \in P^\vee \otimes \Lambda_r \), which implies \( z \in \frac{1}{2} \Lambda_r \). Thus, \( X \setminus X^W \cong \mathcal{E} \setminus 4\text{pts} \). So if we consider adjoint type, the action of \( W \) on \( X^{\text{reg}} \) is not free.
6.13 The Homomorphism from $A_{\text{ell}}$ to rational Cherednik algebras

6.13.1 The rational Cherednik algebra

Let $W$ be a Weyl group, for any reflection $s \in W$, fix $\alpha_s \in \mathfrak{h}^*$, such that $s(\alpha_s) = -\alpha_s$. Write $S$ for the collection of linear functions

$$\{\pm \alpha_s \mid s \text{ is reflections in } W\}.$$

Let $t \in \mathbb{C}$, and $c : S \to \mathbb{C}, s \mapsto c_s$ be a $W$-invariant function.

**Definition 6.13.1.** The algebra $H_{\hbar, c}$ is the quotient of the algebra $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)[\hbar]$ (where $T$ denotes the tensor algebra) by the ideal generated by the relations

$$[x, x'] = 0; [y, y'] = 0; [y, x] = h \langle y, x \rangle - \sum_{s \in S} c_s \langle \alpha_s, y \rangle \langle \alpha_s^\vee, x \rangle s,$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$.

We recall some facts of the grading element of the rational Cherednik algebras. Let $y_i$ be the basis of $\mathfrak{h}$, and $x_i$ be the corresponding dual basis of $\mathfrak{h}^*$. Let

$$h := \sum_i x_i y_i + \frac{1}{2} \dim \mathfrak{h} - \sum_{s \in S} c_s s,$$

and

$$E := -\frac{1}{2} \sum_i x_i^2, \quad F := \frac{1}{2} \sum_i y_i^2.$$

**Proposition 6.13.2** ([25], Proposition 3.18, 3.19, Page 17-18.). We have

(i) For any $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, $[h, x] = x$, $[h, y] = -y$;

(ii) $h = \sum_i \frac{x_i y_i + y_i x_i}{2}$;

(iii) $h, E, F$ form a $\mathfrak{sl}_2$-triple.
6.13.2 The homomorphism from $A_{\text{ell}}$ to the rational Cherednik algebra

Let $\tilde{\alpha}$ be the highest root of $g$. Write $\tilde{\alpha}^\vee = \sum_{i=1}^n g_i \alpha_i^\vee$, and let $g := 1 + \sum_{i=1}^n g_i$. In case all roots have the same length, we have $g$ is the Coxeter number.

**Proposition 6.13.3.** For any $a, b \in \mathbb{C}$, we have a homomorphism of Lie algebras

$$\xi_{a,b} : A_{\text{ell}} \to H_{h,c},$$

defined as follows

$$x(v) \mapsto a \pi(v), \quad y(u) \mapsto bu$$

$$t_\gamma \mapsto ab \left( \frac{h}{g} - \frac{2c_s}{(\gamma|\gamma)} s_\gamma \right),$$

where $\pi : h \to h^*$ be the isomorphism induced by the non-degenerate bilinear form $(\cdot|\cdot)$ on $h$. and $s_\gamma$ is the reflection which sending $\gamma$ to $-\gamma$, and fixes the hyperplane $\ker(\gamma)$.

For simplicity and without lost of generality we can assume $a = b = 1$.

Note that in $A_{\text{ell}}$, the generators $x(v), y(u)$ are labeled by $u, v \in h$. We have the following relation of $A_{\text{ell}}$:

$$[y(u), x(v)] = \sum_{\gamma \in \Phi^+} \langle v, \gamma \rangle \langle u, \gamma \rangle t_\gamma.$$

We show that the map $\xi$ preserve the above relation of $A_{\text{ell}}$.

**Lemma 6.13.4.** The following equality holds:

$$(\cdot|\cdot) = \frac{1}{g} \sum_{\gamma \in \Phi^+} \langle \cdot, \gamma \rangle \langle \cdot, \gamma \rangle.$$

**Proof.** We have

$$[y(u), x(v)] \mapsto [u, \pi(v)]$$

$$= h \langle u, \pi(v) \rangle - \sum_{s \in S} c_s \langle \alpha_s, u \rangle \langle \alpha_s^\vee, \pi(v) \rangle s$$

$$= h \langle u|v \rangle - \sum_{\gamma \in \Phi^+} c_s \langle \gamma, u \rangle \langle \gamma^\vee, \pi(v) \rangle s_\gamma$$
\[
\sum_{\gamma \in \Phi^+} \langle v, \gamma \rangle \langle u, \gamma \rangle t_\gamma \mapsto \sum_{\gamma \in \Phi^+} \langle v, \gamma \rangle \langle u, \gamma \rangle (kh - \frac{2c_s}{(\gamma|\gamma)} s_\gamma)
\]

\[
= \sum_{\gamma \in \Phi^+} \langle v, \gamma \rangle \langle u, \gamma \rangle kh - \sum_{\gamma \in \Phi^+} c_s \langle \pi(v), \gamma^\vee \rangle \langle u, \gamma \rangle s_\gamma
\]

Now \((\cdot|\cdot) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}\), and \(\sum_{\gamma \in \Phi^+} \langle \cdot, \gamma \rangle \langle \cdot, \gamma \rangle : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}\) are two symmetric bilinear forms on \(\mathfrak{h}\), both are positive definite, and invariant under the Weyl group \(W\). Thus,

\[
(\cdot|\cdot) = k \sum_{\gamma \in \Phi^+} \langle \cdot, \gamma \rangle \langle \cdot, \gamma \rangle,
\]

for some constant \(k \in \mathbb{C}\).

It remains to show the constant \(k\) is equal to \(\frac{1}{g}\), which follows immediately from the following lemma. [See [57], Lemma 1.2.]

**Lemma 6.13.5.** \(\sum_{\gamma \in \Phi^+} \langle \check{\alpha}^\vee, \gamma \rangle \langle \check{\alpha}^\vee, \gamma \rangle = 2g\).

**Remark 6.13.6.** In [57], the summation is over the set of \(\gamma \in \Phi\), thus the result is

\[
\sum_{\gamma \in \Phi} \langle \check{\alpha}^\vee, \gamma \rangle \langle \check{\alpha}^\vee, \gamma \rangle = 4g.
\]

We take the summation over \(\gamma \in \Phi^+\), so the result is half of \(4g\).

Since \((\check{\alpha}^\vee|\check{\alpha}^\vee) = \frac{4}{(\check{\alpha}|\check{\alpha})}\). We can pick \(k := \frac{2}{g(\check{\alpha}|\check{\alpha})}\).

We normalize the inner product such that \((\check{\alpha}|\check{\alpha}) = 2\), then the constant \(k\) can be chosen to be \(k := \frac{1}{g}\). Note, in the special case of type \(A_n\), we have \(k = \frac{1}{n+1}\), which coincide with the constant in [10].

**6.13.3 The extension of the homomorphism**

**Proposition 6.13.7.** Let \(y_i\) be an orthogonal basis of \(\mathfrak{h}\), and let \(x_i\) be the corresponding dual basis of \(\mathfrak{h}^*\). Then, The homomorphism \(\xi_{a,b}\) can be extended to the homomorphism

\[
\xi_{a,b}^\sim : U(A_{el} \rtimes \mathfrak{d}) \rtimes W \to H_{h=1,c}
\]
by the following formulas

\[ w \mapsto w, \]
\[ d \mapsto h, \quad X \mapsto ab^{-1} E, \quad \Delta_0 \mapsto ba^{-1} F, \]
\[ \delta_{2m} \mapsto -\frac{1}{2}a^{2m-1}b^{-1} \sum_{\alpha \in \Phi^+} \frac{4mc_\alpha^2}{(\alpha, \alpha)}(x_{\alpha^\vee})^{2m}. \]

**Proof.** For simplicity, we assume that \( a = b = 1 \). We first show the above homomorphism preserve the relations of \( \varnothing \). From Proposition 6.13.2, we know the triple \( h, E, F \) form a \( \mathfrak{sl}_2 \)-triple. Thus, the homomorphism \( \tilde{\xi} \) preserve the relations of the triple \( d, X, \Delta_0 \).

It’s obvious that \( \tilde{\xi} \) preserve the relation \( [\delta_{2m}, X] = 0 \). The fact that \( [h, x] = x \) implies \( \tilde{\xi} \) preserves \( [d, \delta_{2m}] = 2m\delta_{2m} \).

Now we check \( \tilde{\xi} \) preserve the relation \( (\text{ad} \Delta_0)^{2m+1}(\delta_{2m}) = 0 \). We have the following relation:

\[ [F, x_\alpha] = 2hy_\alpha, \text{ for any } j. \]

Since

\[
[F, x_j] = \left[ \sum_i y_i^2, x_j \right] = \sum_i (y_i[y_i, x_j] + [y_i, x_j]y_i) = 2hy_j - \sum_i \sum_s c_s(y_i, \alpha_s)(x_j, \alpha_s^\vee)(y_i s + sy_i) = 2hy_j - \sum_i \sum_s c_s(\langle x_i, \alpha_s \rangle)(x_j, \alpha_s^\vee)(y_i s + sy_i) \quad \text{by } (y_i, \alpha_s) = (x_i, \alpha_s) = \sum_i \langle x_i, \alpha_s \rangle(y_i s + sy_i)
\]
\[
= 2hy_j - \sum_{\alpha \in \Phi^+} c_s(x_j, \alpha_s^\vee)(\sum_i \langle x_i, \alpha_s \rangle(y_i s + sy_i)) = 2hy_j - \sum_{\alpha \in \Phi^+} c_s(x_j, \alpha_s^\vee)(\alpha s + s \alpha) \quad \text{by } s_\alpha \alpha = -\alpha s_\alpha
\]

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Using above relation we obtain:

\[
(ad F)^{2m+1} x_{\alpha^\vee}^{2m+1} = (ad F)^{2m} [F, x_{\alpha^\vee}^{2m}] \\
= 2h (ad F)^{2m} (x_{\alpha^\vee}^{2m-1} y_{\alpha^\vee} + y_{\alpha^\vee} x_{\alpha^\vee}^{2m-1}) \\
= 2h ((ad F)^{2m} x_{\alpha^\vee}^{2m-1}) y_{\alpha^\vee} + y_{\alpha^\vee} ((ad F)^{2m} x_{\alpha^\vee}^{2m-1}) \\
= 0 \text{ by induction.}
\]

Now it remains to show that the derivation \( \mathfrak{d} \to \text{Der}(A_{31}) \) \( \zeta \mapsto \tilde{\zeta} \) corresponding to the operators \( ad(\tilde{\xi}_{a,b}(\zeta)) \), for \( \tilde{\xi}_{a,b}(\zeta) \in H_{h,c} \). Proposition 6.13.2 together with \([h, w] = 0\), for any \( w \in W \), implies that \( \tilde{\xi}_{a,b}(d) \) preserves the relations

\[
\begin{align*}
\tilde{d}(x(u)) &= x(u) ; \\
\tilde{d}(y(u)) &= -y(u) ; \\
\tilde{d}(t_{\alpha}) &= 0.
\end{align*}
\]

[\( h, w \)] = 0, for any \( w \in W \). Let \( u_i \) be another basis of \( \mathfrak{h}^* \), and \( v_i \) be the corresponding dual basis of \( \mathfrak{h} \). We have

\[
\sum_i x_i y_i = \sum_i \left( \sum_k \langle x_i, v_k \rangle u_k \right) y_i \\
= \sum_k u_k \left( \sum_i \langle x_i, v_k \rangle y_i \right) = \sum_k u_k v_k.
\]

Similarly, \( \sum_i y_i x_i = \sum_k v_k u_k \). For any \( w \in W \), \( w(x_i) \) are a basis of \( \mathfrak{h}^* \), and \( w(y_i) \) are the corresponding dual basis of \( \mathfrak{h} \). Thus,

\[
\mathfrak{h} = w(\mathfrak{h}).
\]
Check $[F, w] = 0$, for any $w \in W$.

$$
\sum_i [y_i^2, w] = \sum_i ([y_i, w]y_i + y_i[y_i, w]) = \sum_i (y_i - w(y_i))wy_i + y_i(y_i - w(y_i))w
$$

$$
= \sum_i \left((y_i - w(y_i))w(y_i) + y_i(y_i - w(y_i))\right)w
$$

$$
= \sum_i \left(y_iy_i - w(y_i)w(y_i)\right)w = 0
$$

$\{y_i\}$ are orthogonal basis.

Since $E$ and $F$ are symmetric. The relations involving $E$ can be proved using the same method.

We prove $\tilde{\xi}_{a,b}$ preserves relation

$$
\tilde{\delta}_{2m}(t_\alpha) = [t_\alpha, (\ad \frac{x(\alpha)}{2} )^{2m}(t_\alpha)]
$$

Check $[s_\alpha, (\ad \frac{x(\alpha)}{2} )^{2m}(s_\alpha)] = 0$.

$$
[s_\alpha, (\ad \frac{x(\alpha)}{2} )^{2m}(s_\alpha)] = [s_\alpha, x(\alpha)^{2m}s_\alpha]
$$

by $\frac{x(\alpha)}{2}, s_\alpha] = x(\alpha)s_\alpha$

$$
= s_\alpha x(\alpha)^{2m}s_\alpha - x(\alpha)^{2m}
$$

$$
= 0
$$

by $[s_\alpha x(\alpha)^2] = 0$

Check $[\sum_{\alpha \in \Phi^+}(x_\alpha)^{2m}, w] = 0$, for $w \in W$. Note that $w$ permute the root system $\Phi$ and preserve the killing form, thus we have

$$
\sum_{\alpha \in \Phi^+} (x_\alpha)^{2m} = \frac{1}{2} \sum_{\alpha \in \Phi} (x_\alpha)^{2m} = \frac{1}{2} \sum_{\alpha \in \Phi} (x_{\alpha^\vee})^{2m} = w(\sum_{\alpha \in \Phi^+} (x_{\alpha^\vee})^{2m})
$$

Finally, we show that $\tilde{\xi}_{a,b}$ preserves relation

$$
\tilde{\delta}_{2m}(y(u)) = \frac{1}{2} \sum_\alpha \alpha(u) \sum_{p+q=2m-1} [(\ad \frac{x(\alpha)}{2})^p(t_\alpha), (\ad - \frac{x(\alpha)}{2})^q(t_\alpha)]
$$

(6.13.1)
On one hand, we have

\[
\sum_{p+q=2m-1} [(\text{ad} \frac{x(\alpha^\vee)}{2})^p(s_\alpha), (\text{ad} - \frac{x(\alpha^\vee)}{2})^q(s_\alpha)]
\]

\[
= \sum_{p+q=2m-1} (-1)^q [x(\alpha^\vee)^p s_\alpha, x(\alpha^\vee)^q s_\alpha]
\]

by \[\frac{x(\alpha^\vee)}{2}, s_\alpha] = x(\alpha^\vee)s_\alpha
\]

\[
= \sum_{p+q=2m-1} (-1)^q \left( x(\alpha^\vee)^p s_\alpha x(\alpha^\vee)^q s_\alpha - x(\alpha^\vee))^q s_\alpha x(\alpha^\vee)^p s_\alpha \right)
\]

\[
= \sum_{p+q=2m-1} (-1)^q \left( (-1)^q - (-1)^p \right) x(\alpha^\vee)^{2m-1}
\]

by \(s_\alpha x(\alpha^\vee) = -x(\alpha^\vee)s_\alpha
\]

\[
= 4mx(\alpha^\vee)^{2m-1}.
\]

Thus, the right hand side of (6.13.1) is

\[
\frac{1}{2} \sum_\alpha x(\alpha^\vee)^{2m-1} \left( \frac{2c_\alpha}{(\alpha, \alpha)} \right)^2 = 4m \sum_\alpha \frac{c_\alpha^2}{(\alpha, \alpha)} x(\alpha^\vee)^{2m-1}
\]

On the other hand,

\[
[\sum_{\alpha \in \Phi^+} (x_\alpha^\vee)^{2m}, y(\alpha)] = \sum_{\alpha \in \Phi^+} \left( [x_\alpha^\vee, y(\alpha)]x_\alpha^\vee^{2m} + x_\alpha^\vee^{2m-1}[x_\alpha^\vee, y(\alpha)] \right)
\]

\[
= \sum_{\alpha \in \Phi^+} \left( \left( -\hbar\langle u, \alpha^\vee \rangle + \sum_{\gamma \in \Phi^+} c_\gamma \langle \gamma^\vee, u \rangle \langle \gamma, \alpha^\vee \rangle s_\gamma \right) x_\alpha^\vee^{2m-1}
\]

\[
+ x_\alpha^\vee^{2m-1} \left( -\hbar\langle u, \alpha^\vee \rangle + \sum_{\gamma \in \Phi^+} c_\gamma \langle \gamma^\vee, u \rangle \langle \gamma, \alpha^\vee \rangle s_\gamma \right) \right)
\]

\[
= \sum_{\alpha \in \Phi^+} -2\hbar x(\alpha^\vee)^{2m-1}.
\]

The last equality follows from the following calculation.

\[
\sum_{\alpha \in \Phi} \langle \gamma, \alpha^\vee \rangle \left( s_\gamma x_\alpha^\vee^{2m-1} + x_\alpha^\vee^{2m-1}s_\gamma \right)
\]

\[
= \sum_{\alpha \in \Phi} \left( \langle \gamma, \alpha^\vee \rangle (s_\gamma x_\alpha^\vee)^{2m-1}s_\gamma + \langle \gamma, s_\gamma(\alpha^\vee) \rangle (s_\gamma x_\alpha^\vee)^{2m-1}s_\gamma \right)
\]

\[
= \sum_{\alpha \in \Phi} \langle \gamma, 2\alpha^\vee - \langle \gamma, \alpha^\vee \rangle \gamma^\vee \rangle \left( s_\gamma x_\alpha^\vee \right)^{2m-1}s_\gamma
\]

\[
= 0.
\]
Thus, the left hand side of (6.13.1) is also
\[ 4m \sum_\alpha \frac{c_{\alpha}^2}{(\alpha, \alpha)} \alpha^\vee(u) x_{\alpha^\vee}^{2m-1} \]

\[ \square \]

6.14 The Homomorphism from \( A_{\text{ell}} \) to the deformed double current algebras

6.14.1 The deformed double current algebras

The deformed double current algebras are introduced by Guay in [34] [35], which is a deformation of the universal central extension of \( U(g[u, v]) \), for a semisimple Lie algebra \( g \). The double loop presentations of the deformed double current algebras are established in [34] for \( g = \mathfrak{sl}_n, n \geq 4 \) and in [36] for any simple Lie algebra \( g \).

We briefly recall the results here. Let \( g \) be a finite–dimensional, simple Lie algebra over \( \mathbb{C} \), in this section, we assume that \( g \neq \mathfrak{sl}_2, \mathfrak{sl}_3 \). Let \( (\cdot, \cdot) \) be the Killing form on \( g \) and let \( X_i^+, H_i, 1 \leq i \leq N \) be the Chevalley generators of \( g \) normalized so that \( (X_i^+, X_i^-) = 1 \) and \( [X_i^+, X_i^-] = H_i \). For each positive root \( \alpha \), we choose generators \( X_{\alpha}^\pm \) of \( g_{=\alpha} \) such that \( (X_{\alpha}^+, X_{\alpha}^-) = 1 \) and \( X_{\alpha}^\pm X_{\alpha}^\mp = X_i^\pm \). If \( \alpha > 0 \), set \( X_{\alpha} = X_{\alpha}^+ \); if \( \alpha < 0 \), set \( X_{\alpha} = X_{-\alpha}^- \).

Theorem 6.14.1. [34, 36] The deformed double current algebra \( \mathfrak{D}_\lambda(g) \) is generated by elements \( X, K(X), Q(X), P(X), X \in g \), such that

1. \( K(X), X \in g \) generate a subalgebra which is an image of \( g \otimes \mathbb{C}[u] \) with \( X \otimes u \mapsto K(X) \);

2. \( Q(X), X \in g \) generate a subalgebra which is an image of \( g \otimes \mathbb{C}[v] \) with \( X \otimes v \mapsto Q(X) \);

3. \( P(X) \) is linear in \( X \), and for any \( X, X' \in g \), \( [P(X), X'] = P[X, X'] \).
and the following relations hold for all root vectors $X_{\beta_1}, X_{\beta_2} \in \mathfrak{g}$ with $\beta_1 \neq -\beta_2$:

$$[K(X_{\beta_1}), Q(X_{\beta_2})] = P([X_{\beta_1}, X_{\beta_2}]) - \frac{\lambda}{4} \beta_1 \beta_2 S(X_{\beta_1}, X_{\beta_2}) + \frac{\lambda}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}]),$$

(6.14.1)

where $S(a_1, a_2) = a_1 a_2 + a_2 a_1 \in U(\mathfrak{g})$.

For any root $\beta$ of $\mathfrak{g}$, set

$$Z(\beta) := [K(H_{\beta}), Q(H_{\beta})] - \frac{1}{4} \sum_{\alpha \in \Delta} S([H_{\beta}, X_{\alpha}], [X_{-\alpha}, H_{\beta}]) \in \mathfrak{D}_\lambda(\mathfrak{g}).$$

**Theorem 6.14.2.** [36] In the deformed double current algebra $\mathfrak{D}_\lambda(\mathfrak{g})$, we have

1. $Z(\beta) = Z(\gamma)$, for any two roots $\beta, \gamma$ of $\mathfrak{g}$.

2. The element $Z(\beta)$ is central in $\mathfrak{D}_\lambda(\mathfrak{g})$.

Let $\mathfrak{QD}_\lambda(\mathfrak{g})$ be the quotient algebra of the deformed double current algebras $\mathfrak{D}_\lambda(\mathfrak{g})$ modulo the central element $Z(\beta)$. The presentation of the algebra $\mathfrak{QD}_\lambda(\mathfrak{g})$ has a nice form, which is given as follows.

**Proposition 6.14.3.** [36] The quotient algebra $\mathfrak{QD}_\lambda(\mathfrak{g})$ is generated by elements $X, K(X), Q(X), P(X)$, $X \in \mathfrak{g}$, such that relations (1), (2), (3) hold as in Theorem 6.14.1, and the relations (6.14.1) hold for all vectors $X_{\beta_1}, X_{\beta_2} \in \mathfrak{g}$, with $\beta_1, \beta_2 \in \Phi \cup \{0\}$.

In particular, for any $h, h' \in \mathfrak{g}$, we have:

$$[K(h), Q(h')] = \frac{\lambda}{4} \sum_{\alpha \in \Phi} S([h, X_{\alpha}], [X_{-\alpha}, h]) = \frac{\lambda}{2} \sum_{\alpha \in \Phi^+} (h, \alpha) (h', \alpha) \kappa_\alpha,$$

where $\kappa_\alpha = X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha$ is the truncated Casimir operator.
6.14.2 The homomorphism to the deformed double current algebras

Proposition 6.14.4. There is a right action of $SL_2(\mathbb{C})$ on $D$. For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL_2(\mathbb{C})$, the action is given by

$$z \mapsto z, K(z) \mapsto a_{11}K(z) + a_{12}Q(z), Q(z) \mapsto a_{22}Q(z) + a_{21}K(z),$$

$$P(z) \mapsto (a_{11}a_{22} + a_{12}a_{21})P(z) + a_{11}a_{21}[K(y), K(w)] + a_{12}a_{22}[Q(y), Q(w)],$$

where $z = [y, w]$, and for any $z \in g$. In particular, there is a well defined automorphism of $D$ by $K(z) \mapsto -Q(z); Q(z) \mapsto K(z); P(z) \mapsto -P(z);$ and $z \mapsto z$, for $z \in g$, which is of order 4.

Remark 6.14.5. We can make the right action of $SL_2(\mathbb{C})$ to a left action of $SL_2(\mathbb{C})$ by defining $A\hat{z} := zA^T$, where $A^T$ is the transpose of $A$. Thus, there is a left action of $SL_2(\mathbb{C})$ on $D$. For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL_2(\mathbb{C})$, the action is given by

$$z \mapsto z, K(z) \mapsto a_{11}K(z) + a_{21}Q(z), Q(z) \mapsto a_{22}Q(z) + a_{12}K(z),$$

$$P(z) \mapsto (a_{11}a_{22} + a_{21}a_{12})P(z) + a_{11}a_{12}[K(y), K(w)] + a_{21}a_{22}[Q(y), Q(w)],$$

where $z = [y, w]$, and for any $z \in g$.

Proof. We first check that under the action of $A$ preserves the defining relations of $D(g)$. We only check it preserves relation (6.14.1), other relations are obvious. We set

$$C(X_{\beta_1}, X_{\beta_2}) := -\frac{1}{4}(\beta_1, \beta_2)S(X_{\beta_1}, X_{\beta_2}) + \frac{1}{4} \sum_{\alpha \in \Delta} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}]),$$

The property $S(a_1, a_2) = S(a_2, a_1)$ implies that $C(X_{\beta_1}, X_{\beta_2}) = C(X_{\beta_2}, X_{\beta_1})$. The relation (6.14.1) can be rewritten as $[K(x), Q(y)] = P([x, y]) + C(x, y)$.

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Now under the action of $A \in \text{SL}_2(\mathbb{C})$,

$$[K(x), Q(y)]$$

$$\mapsto [a_{11}K(x) + a_{12}Q(x), a_{22}Q(y) + a_{21}K(y)]$$

$$= a_{11}a_{22}[K(x), Q(y)] + a_{12}a_{21}[Q(x), K(y)] + a_{11}a_{21}[K(x), K(y)] + a_{12}a_{22}[Q(x), Q(y)]$$

$$= a_{11}a_{22}(P([x, y]) + C(x, y)) - a_{12}a_{21}(P([y, x]) + C(y, x))$$

$$+ a_{11}a_{21}[K(x), K(y)] + a_{12}a_{22}[Q(x), Q(y)]$$

$$= (a_{11}a_{22} + a_{12}a_{21})P([x, y]) + a_{11}a_{21}[K(x), K(y)] + a_{12}a_{22}[Q(x), Q(y)] + C(x, y).$$

and by definition

$$P([x, y]) + C(x, y) \mapsto (a_{11}a_{22} + a_{12}a_{21})P([x, y])$$

$$+ a_{11}a_{21}[K(x), K(y)] + a_{12}a_{22}[Q(x), Q(y)] + C(x, y).$$

Thus, the action of $A$ preserves the relation (6.14.1).

We then check it defines a right action of $\text{SL}_2$. It’s obvious that $z \cdot \text{Id} = z$, for any $z \in \mathfrak{D}(\mathfrak{g})$, where Id is the $2 \times 2$-identity matrix.

For any $A, B \in \text{SL}_2(\mathbb{C})$, and for any $z \in \mathfrak{U}(\mathfrak{g})$, it’s a direct calculation to show $(zA)B = z(AB)$. We show the less obvious case when $z = P([x, y])$ as follows.

Set $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. It’s a direct computation to show that

$$(P([x, y]).A)B = P([x, y]).(AB).$$

$\square$

**Corollary 6.14.6.** The action of $\text{SL}_2(\mathbb{C})$ on $D$ induces a Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ action on $D$.

For $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$, the action is given by: $zX = 0$,

$$K(z)X = x_{11}K(z) + x_{21}Q(z), \quad Q(z)X = x_{22}Q(z) + x_{12}K(z),$$

$$P(z)X = x_{21}[K(y), K(w)] + x_{21}[Q(y), Q(w)],$$

where $z = [y, w]$.
In particular, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, for any $z \in \mathfrak{g}$, then: $hz = 0$, $hP(z) = 0$, $hK(z) = K(z)$, and $hQ(z) = -Q(z)$.

**Proposition 6.14.7.** There is a map from $A_{el} \to \mathfrak{OD}_\lambda(\mathfrak{g})$, giving by $x(u) \mapsto Q(u)$, $y(v) \mapsto K(v)$, and $t_\alpha \mapsto \frac{1}{2\lambda} \kappa_\alpha$, where $\kappa_\alpha := X_\alpha^+ X^-_\alpha + X^-_\alpha X^+_\alpha$ is the truncated Casimir operator.

**Proof.** It follows from the explicit relations of $\mathfrak{OD}_\lambda(\mathfrak{g})$ in Proposition 6.14.3. \qed

### 6.15 The extension of the elliptic Casimir connection

#### 6.15.1

Recall in Section 6.9 we introduced the derivation $\mathfrak{d}$ of generalized holonomy Lie algebra $A_{el}$. The generators of $\mathfrak{d}$ are denoted by $\Delta_0, d, X$, and $\delta_{2m}(m \geq 1)$.

In this section, we use the following action of $\mathfrak{sl}_2(\mathbb{C})$ on $D(\mathfrak{g})$. For any $z \in \mathfrak{g}$,

1. $hz = 0$, $hK(z) = -K(z)$ and $hQ(z) = Q(z)$.

2. $ez = 0$, $eK(z) = Q(z)$, $eQ(z) = 0$.

3. $fz = 0$, $fK(z) = 0$, $fQ(z) = K(z)$.

**Conjecture 6.15.1.** The map $A_{el} \to D(\mathfrak{g})$ can be extended to the following map

$$A_{el} \rtimes \mathfrak{d} \to D(\mathfrak{g}) \rtimes \mathfrak{sl}_2,$$

which is given by

$$d \mapsto h, \quad X \mapsto e, \quad \Delta_0 \mapsto f,$$

and

$$\delta_{2m} \mapsto \sum_{p=0}^{2m} (-1)^q \binom{2m}{p} \sum_i (h_i \otimes u^p)(h^i \otimes u^{2m-p}),$$

where $h_i, h^i$ are dual basis of $\mathfrak{h}$ and $h \otimes u := Q(u) \in \mathfrak{g}[u]$. 254
6.15.2

In this subsection, we show that for any $X \in \mathfrak{g}$, the necessary condition is

$$
\delta_{2m}(X) = \sum_{p=0}^{2m} (-1)^q \binom{2m}{p} \left[ \sum_i (h_i \otimes u^p)(h_i \otimes u^{2m-p}), X \right].
$$

In order to define the action of $\mathfrak{d}$ on $D(\mathfrak{g})$ which commutes with the map $A_{sl} \to D(\mathfrak{g})$, we must have

$$
\delta_{2m}S(X^+_{\alpha}, X^-_{\alpha}) = \frac{\lambda}{2} \left[ S(X^+_{\alpha}, X^-_{\alpha}), (\text{ad} \frac{Q(\alpha^\vee)}{2})^{2m}(S(X^+_{\alpha}, X^-_{\alpha})) \right].
$$

$\delta_{2m}$ is a derivation. Then

$$
\delta_{2m}S(X^+_{\alpha}, X^-_{\alpha}) = S(\delta_{2m}(X^+_{\alpha}), X^-_{\alpha}) + S(X^+_{\alpha}, \delta_{2m}(X^-_{\alpha})).
$$

On the other hand, we have

$$
[S(X^+_{\alpha}, X^-_{\alpha}), (\text{ad} \frac{Q(\alpha^\vee)}{2})^{2m}(S(X^+_{\alpha}, X^-_{\alpha}))]
= S \left( \left[ X^+_{\alpha}, (\text{ad} \frac{Q(\alpha^\vee)}{2})^{2m}(S(X^+_{\alpha}, X^-_{\alpha})) \right], X^-_{\alpha} \right) + S \left( X^+_{\alpha}, \left[ X^-_{\alpha}, (\text{ad} \frac{Q(\alpha^\vee)}{2})^{2m}(S(X^+_{\alpha}, X^-_{\alpha})) \right] \right)
$$

Thus,

$$
\delta_{2m}(X^+_{\alpha}) = \frac{\lambda}{2} \left[ X^+_{\alpha}, (\text{ad} \frac{Q(\alpha^\vee)}{2})^{2m}(S(X^+_{\alpha}, X^-_{\alpha})) \right]
= \frac{\lambda}{2} \sum_{p+q=2m} (-1)^q \binom{2m}{p} S(X^+_{\alpha} \otimes u^p, X^-_{\alpha} \otimes u^q)
= \frac{\lambda}{2} \sum_{p+q=2m} (-1)^q \binom{2m}{p} S(X^+_{\alpha} \otimes u^p, [X^+_{\alpha}, X^-_{\alpha}] \otimes u^q)
= \frac{\lambda}{2} \sum_{p+q=2m} (-1)^q \binom{2m}{p} S(X^+_{\alpha} \otimes u^p, H_{\alpha} \otimes u^q)
$$
Similarly,

\[
\delta_{2m}(X^-_\alpha) = \frac{\lambda}{2} \left[ X^-_\alpha, (\text{ad} \frac{Q(\alpha^\vee)}{2})^{2m}(S(X^+_\alpha, X^-_\alpha)) \right]
\]

\[
= \frac{\lambda}{2} \left[ X^-_\alpha, \sum_{p+q=2m} (-1)^q \binom{2m}{p} S(X^+_\alpha \otimes u^p, X^-_\alpha \otimes u^q) \right]
\]

\[
= \frac{\lambda}{2} \sum_{p+q=2m} (-1)^q \binom{2m}{p} S([-X^-_\alpha, X^+_\alpha] \otimes u^p, X^-_\alpha \otimes u^q)
\]

\[
= \frac{\lambda}{2} \sum_{p+q=2m} (-1)^q \binom{2m}{p} S(-H_\alpha \otimes u^p, X^-_\alpha \otimes u^q)
\]

Now

\[
\delta_{2m}(H_\alpha) = \delta_{2m}([X^+_\alpha, X^-_\alpha])
\]

\[
= [\delta_{2m}X^+_\alpha, X^-_\alpha] + [X^+_\alpha, \delta_{2m}X^-_\alpha]
\]

\[
= \frac{\lambda}{2} \left[ [X^+_\alpha, (\text{ad} \frac{Q(\alpha^\vee)}{2})^{2m}(S(X^+_\alpha, X^-_\alpha))] , X^-_\alpha \right] + \frac{\lambda}{2} \left[ X^+_\alpha, [X^-_\alpha, (\text{ad} \frac{Q(\alpha^\vee)}{2})^{2m}(S(X^+_\alpha, X^-_\alpha))] \right]
\]

\[
= \frac{\lambda}{2} \left[ H^+_\alpha, (\text{ad} \frac{Q(\alpha^\vee)}{2})^{2m}(S(X^+_\alpha, X^-_\alpha)) \right]
\]

\[
= \frac{\lambda}{2} (\text{ad} \frac{Q(\alpha^\vee)}{2})^{2m}([H_\alpha, S(X^+_\alpha, X^-_\alpha)])
\]

\[
= 0
\]

Thus, \( \delta_{2m}(\mathfrak{h}) = 0 \), for any \( h \in \mathfrak{h} \).

\textbf{6.15.3}

In this subsection, we show that for any \( X \in \mathfrak{g} \), we have:

\[
\delta_{2m}(Q(X)) = \sum_{p=0}^{2m} (-1)^q \binom{2m}{p} \left[ \sum_i (h_i \otimes u^p)(h^i \otimes u^{2m-p}), Q(X) \right].
\]

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It's obvious that $\delta_{2m}(Q(h)) = 0$, for $h \in \mathfrak{h}$. Thus, for $X_\alpha^+ \in \mathfrak{g}$, We can define

$$\delta_{2m}(Q(X_\alpha^+)) := \frac{Q(\alpha^\vee)}{2} \delta_{2m}(X_\alpha)$$

$$= \frac{\lambda}{2} \sum_{p+q=2m} (1-q^2) \left( \frac{2m}{p} \right) S(X_\alpha^+ \otimes u^p, H_\alpha \otimes u^q)$$

Similarly, one has:

$$\delta_{2m}(Q(X_\alpha^+)) = \frac{\lambda}{2} \sum_{p+q=2m} (1-q^2) \left( \frac{2m}{p} \right) S(-H_\alpha \otimes u^p, X_\alpha^- \otimes u^{q+1})$$

6.15.4

In this subsection, we show that for any $h \in \mathfrak{h}$, then

$$\delta_{2m}(K(h)) = \sum_{p=0}^{2m} (1-q^2) \left( \frac{2m}{p} \right) \left[ \sum_i (h_i \otimes u^p)(h^i \otimes u^{2m-p}), K(h) \right],$$

under the assumption of the following conjecture:

**Conjecture 6.15.2.** Let $h \otimes u := Q(h) \in \mathfrak{g}[u] \subset \mathfrak{O}_\lambda(\mathfrak{g})$, and $h \otimes v := K(h) \in \mathfrak{g}[v] \subset \mathfrak{O}_\lambda(\mathfrak{g})$, then the following relation holds in $\mathfrak{O}_\lambda(\mathfrak{g})$,

$$[h' \otimes v, h \otimes u^n] = \frac{1}{4} \sum_{\alpha} \sum_{p+q=n-1} S([h', X_\alpha] \otimes u^p, [X_{-\alpha}, h] \otimes u^q)$$

$$= \frac{1}{4} \sum_{\alpha} \sum_{p+q=n-1} (h, \alpha)(h', \alpha) \left( S(X_\alpha \otimes u^p, X_{-\alpha} \otimes u^q) \right)$$

In particular, we have

$$[h' \otimes v, h \otimes u^2] = \frac{1}{4} \sum_{\alpha} \left( S([h', X_\alpha] \otimes u, [X_{-\alpha}, h]) + S([h', X_\alpha], [X_{-\alpha}, h] \otimes u) \right)$$

$$= \frac{1}{4} \sum_{\alpha} (h, \alpha)(h', \alpha) \left( S(X_\alpha \otimes u, X_{-\alpha}) + S(X_\alpha, X_{-\alpha} \otimes u) \right)$$

$$= \frac{1}{2} \sum_{\alpha} (h, \alpha)(h', \alpha) \left( S(X_\alpha \otimes u, X_{-\alpha}) \right)$$

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Theorem 6.15.3. The Conjecture 6.15.2 implies the following:

\[ \delta_{2m}(K(h)) = \left[ \sum_i \sum_{p+q=2m} (-1)^q \left( \frac{2m}{p} \right) (h_i \otimes u^p)(h^i \otimes u^q), K(h) \right] \]

**Proof.** On one hand, using the conjecture 6.15.2, we have:

\[
[ h \otimes v, \sum_i \sum_{p+q=2m} (-1)^q \left( \frac{2m}{p} \right) (h_i \otimes u^p)(h^i \otimes u^q)]
\]

\[
= \sum_i \sum_{p+q=2m} (-1)^q \left( \frac{2m}{p} \right) (h_i \otimes u^p[h \otimes v, h^i \otimes u^q] + [h \otimes v, h_i \otimes u^p]h^i \otimes u^q)
\]

\[
\leq \frac{1}{4} \sum_{p+q=2m} (-1)^q \left( \frac{2m}{p} \right) \sum_{\alpha} (\alpha, h) \left( H_\alpha \otimes u^p \sum_{s+t=q-1} S(X_\alpha \otimes u^s, X_{\alpha} \otimes u^t) + S(X_\alpha \otimes u^s, X_{\alpha} \otimes u^t)H_\alpha \otimes u^q \right)
\]

\[
\leq \frac{1}{4} \sum_{\alpha} (\alpha, h) \sum_{p+q=2m} (-1)^q \left( \frac{2m}{p} \right) \left( H_\alpha \otimes u^p \sum_{s+t=q-1} S(X_\alpha \otimes u^s, X_{\alpha} \otimes u^t) + S(X_\alpha \otimes u^s, X_{\alpha} \otimes u^t)H_\alpha \otimes u^q \right)
\]

On the other hand,
We have for $h \in \mathfrak{h}$,

$$\delta_{2m}(K(h))$$

$$= \frac{\lambda^2}{8} \sum_{\beta} \beta(h) \sum_{p+q=2m-1} (-1)^q [(\text{ad} \frac{Q(\beta^\vee)}{2}) (\text{ad} \frac{Q(\beta^\vee)}{2})^p (S(X_\beta^+, X_\beta^-)), (\text{ad} \frac{Q(\beta^\vee)}{2})^q (S(X_\beta^+, X_\beta^-))]
$$

$$= \frac{\lambda^2}{8} \sum_{\beta} \beta(h) \sum_{p+q=2m-1} (-1)^q \sum_{s+t=p} (-1)^t \left( \binom{p}{s} \binom{q}{k} \right) S(X_\beta^+ \otimes u^s, X_\beta^- \otimes u^t),
$$

$$\sum_{k+j=q} (-1)^j \left( \binom{q}{k} \right) S(X_\beta^+ \otimes u^k, X_\beta^- \otimes u^j)]]
$$

$$= \frac{\lambda^2}{8} \sum_{\beta} \beta(h) \sum_{p+q=2m-1} (-1)^q \sum_{s+t=p} \sum_{k+j=q} (-1)^{t+j} \left( \binom{p}{s} \binom{q}{k} \right)
$$

$$\left[ S(X_\beta^+ \otimes u^s, X_\beta^- \otimes u^t), S(X_\beta^+ \otimes u^k, X_\beta^- \otimes u^j) \right]
$$

$$= \frac{\lambda^2}{8} \sum_{\beta} \beta(h) \sum_{p+q=2m-1} (-1)^q \sum_{s+t=p} \sum_{k+j=q} (-1)^{t+j} \left( \binom{p}{s} \binom{q}{k} \right)
$$

$$\left( S(S(X_\beta^+ \otimes u^k, H_\beta \otimes u^{s+j}), X_\beta^- \otimes u^l) - S(X_\beta^+ \otimes u^s, S(H_\beta \otimes u^{t+k}, X_\beta^- \otimes u^j)) \right)
$$

We compute the term as follows:

$$S(S(X_\beta^+ \otimes u^k, H_\beta \otimes u^{s+j}), X_\beta^- \otimes u^l) - S(X_\beta^+ \otimes u^s, S(H_\beta \otimes u^{t+k}, X_\beta^- \otimes u^j))
$$

$$= X_\beta^- \otimes u^l \left( 2(X_\beta^+ \otimes u^k)(H_\beta \otimes u^{s+j}) + (\beta, \beta) X_\beta^+ \otimes u^{k+s+j} \right)
$$

$$+ \left( 2(H_\beta \otimes u^{s+j})(X_\beta^+ \otimes u^k) - (\beta, \beta) X_\beta^+ \otimes u^{k+s+j} \right) X_\beta^- \otimes u^l
$$

$$- X_\beta^+ \otimes u^s \left( 2(X_\beta^- \otimes u^j) H_\beta \otimes u^{t+k} - (\beta, \beta) X_\beta^- \otimes u^{j+t+k} \right)
$$

$$- \left( 2(H_\beta \otimes u^{t+k})(X_\beta^- \otimes u^j) + (\beta, \beta) X_\beta^- \otimes u^{j+t+k} \right) X_\beta^+ \otimes u^s
$$

$$= 2(X_\beta^- \otimes u^l)(X_\beta^+ \otimes u^k) H_\beta \otimes u^{s+j} + 2H_\beta \otimes u^{s+j}(X_\beta^+ \otimes u^k)(X_\beta^- \otimes u^l)
$$

$$- 2(X_\beta^+ \otimes u^s)(X_\beta^- \otimes u^j) H_\beta \otimes u^{t+k} - 2H_\beta \otimes u^{t+k}(X_\beta^- \otimes u^j)(X_\beta^+ \otimes u^s)
$$

Now fix the root $\beta$, then note that

$$\sum_{p+q=2m-1} (-1)^q \sum_{s+t=p} \sum_{k+j=q} (-1)^{t+j} \left( \binom{p}{s} \binom{q}{k} \right) = \sum_{s+t+k+j=2m-1} (-1)^{k+j} \binom{s+t}{k} \binom{k+j}{k}
$$

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If we switch the pair \((s, j)\) with \((t, k)\), then the coefficient \((-1)^{k+t}\binom{s+t}{s}\binom{k+j}{k}\) is changed by a negative sign. Thus, we have two summands of the following form:

\[
2(X^-_\beta \otimes u^t)(X^+_\beta \otimes u^k)H_\beta \otimes u^{s+j} + 2H_\beta \otimes u^{s+j}(X^+_\beta \otimes u^k)(X^-_\beta \otimes u^t)
\]

\[
-2(X^+_\beta \otimes u^s)(X^-_\beta \otimes u^j)H_\beta \otimes u^{t+k} - 2H_\beta \otimes u^{t+k}(X^-_\beta \otimes u^j)(X^+_\beta \otimes u^s)
\]

and

\[
-2(X^-_\beta \otimes u^s)(X^+_\beta \otimes u^j)H_\beta \otimes u^{k+t} - 2H_\beta \otimes u^{k+t}(X^+_\beta \otimes u^j)(X^-_\beta \otimes u^s)
\]

\[
+2(X^+_\beta \otimes u^t)(X^-_\beta \otimes u^k)H_\beta \otimes u^{s+j} + 2H_\beta \otimes u^{s+j}(X^-_\beta \otimes u^k)(X^+_\beta \otimes u^t)
\]

Combine the above two summands with \((s, t, k, j)\) and \((t, s, j, k)\), note the coefficient is

\[
(-1)^{k+t}\binom{s+t}{s}\binom{k+j}{k},
\]

we get

\[
\left(2(X^-_\beta \otimes u^t)(X^+_\beta \otimes u^k) + 2(X^+_\beta \otimes u^t)(X^-_\beta \otimes u^k)\right)H_\beta \otimes u^{s+j}
\]

\[
+ H_\beta \otimes u^{s+j}\left(2(X^+_\beta \otimes u^k)(X^-_\beta \otimes u^t) + 2(X^-_\beta \otimes u^k)(X^+_\beta \otimes u^t)\right)
\]

\[
- \left(2(X^+_\beta \otimes u^s)(X^-_\beta \otimes u^j) + 2(X^-_\beta \otimes u^s)(X^+_\beta \otimes u^j)\right)H_\beta \otimes u^{t+k}
\]

\[
- H_\beta \otimes u^{t+k}\left(2(X^-_\beta \otimes u^j)(X^+_\beta \otimes u^s) + 2(X^+_\beta \otimes u^j)(X^-_\beta \otimes u^s)\right)
\]

\[
= \left(S(X^-_\beta \otimes u^t, X^+_\beta \otimes u^k) + S(X^+_\beta \otimes u^t, X^-_\beta \otimes u^k)\right)H_\beta \otimes u^{s+j}
\]

\[
+ H_\beta \otimes u^{s+j}\left(S(X^+_\beta \otimes u^k, X^-_\beta \otimes u^t) + S(X^-_\beta \otimes u^k, X^+_\beta \otimes u^t)\right)
\]

\[
- \left(S(X^+_\beta \otimes u^s, X^-_\beta \otimes u^j) + S(X^-_\beta \otimes u^s, X^+_\beta \otimes u^j)\right)H_\beta \otimes u^{t+k}
\]

\[
- H_\beta \otimes u^{t+k}\left(S(X^-_\beta \otimes u^j, X^+_\beta \otimes u^s) + S(X^+_\beta \otimes u^j, X^-_\beta \otimes u^s)\right)
\]

\[
= S\left(S(X^-_\beta \otimes u^t, X^+_\beta \otimes u^k), H_\beta \otimes u^{s+j}\right) + S\left(S(X^+_\beta \otimes u^t, X^-_\beta \otimes u^k), H_\beta \otimes u^{s+j}\right)
\]

\[
- S\left(S(X^+_\beta \otimes u^s, X^-_\beta \otimes u^j), H_\beta \otimes u^{t+k}\right) - S\left(S(X^+_\beta \otimes u^s, X^-_\beta \otimes u^j), H_\beta \otimes u^{t+k}\right)
\]
We have a identity of the binomial coefficients for any integers $j, k$, and $n$ satisfying $0 \leq j \leq k \leq n$, then

$$\sum_{m=0}^{n} \binom{m}{j} \binom{n-m}{k-j} = \binom{n+1}{k+1}$$

Now let fix the index $k$, and $t$, and fix the sum $s + j = \chi$, we change the index $p = 0, \ldots, 2m - 1$, then the index $s, j, q$ is changing according to $p$. The coefficient of the term

$$S\left( S(X^-_\beta \otimes u^t, X^+_\beta \otimes u^k), H_\beta \otimes u^{s+j} \right)$$

is

$$(-1)^{k+t} \sum_{p=0}^{2m-1} \binom{p}{t} \binom{2m-1-p}{2m-1-\chi-t} = (-1)^\chi \binom{2m}{\chi}.$$ 

Thus

$$\delta_{2m}(k(h)) = -\frac{\lambda^2}{4} \sum_{\chi=0}^{2m} (-1)^\chi \binom{2m}{\chi} \sum_{\beta, h} S(H_\beta \otimes u^\chi, \sum_{t+k=2m-1-\chi} S(X^-_\beta \otimes u^t, X^+_\beta \otimes u^k))$$

\[\square\]

### 6.16 Proof of Conjecture 6.15.2 in special cases

In this section, we show the Conjecture 6.15.2 when $(h, h') = 0$.

Let $m : U(\mathfrak{g}[u]) \otimes_{\mathbb{C}} U(\mathfrak{g}[u]) \to U(\mathfrak{g}[u])$ be the multiplication map. The following obser-
\[
\sum_{\alpha \in \Phi} S([X_{\beta_1}, X_\alpha] \otimes u^p, [X_{-\alpha}, X_{\beta_2}] \otimes u^q) \\
= m \left( \sum_{\alpha \in \Phi} [[X_{\beta_1} \otimes 1, X_\alpha(u^p) \otimes X_{-\alpha}(u^q)], 1 \otimes X_{\beta_2}] \right) \\
+ m \left( \sum_{\alpha \in \Phi} [[1 \otimes X_{\beta_1}, X_{-\alpha}(u^q) \otimes X_\alpha(u^p)], X_{\beta_2} \otimes 1] \right) \\
= m \left( [[X_{\beta_1} \otimes 1, \Omega_{p,q}], 1 \otimes X_{\beta_2}] - \sum_{i=1}^N [[X_{\beta_1} \otimes 1, \tilde{h}_i(u^p) \otimes \tilde{h}_i(u^q)], 1 \otimes X_{\beta_2}] \right) \\
+ m \left( [[1 \otimes X_{\beta_1}, \Omega_{q,p}], X_{\beta_2} \otimes 1] - \sum_{i=1}^N [[1 \otimes X_{\beta_1}, \tilde{h}_i(u^q) \otimes \tilde{h}_i(u^p)], X_{\beta_2} \otimes 1] \right)
\]

where we view
\[
\Omega_{p,q} := \sum_{\alpha \in \Phi} X_\alpha(u^p) \otimes X_{-\alpha}(u^q) + \sum_i \tilde{h}_i(u^p) \otimes \tilde{h}_i(u^q)
\]
as an element of \(U(g[u]) \otimes_C U(g[u])\). Consequently,

**Lemma 6.16.1.** For any \(X \in g\), we have \([\Omega_{p,q}, X \otimes 1 + 1 \otimes X] = 0\). But the above statement is not true for \(X(u^r)\). That is, we might have \([\Omega_{p,q}, X(u^r) \otimes 1 + 1 \otimes X(u^r)] \neq 0\).

Then we have:

\[
\sum_{\alpha \in \Phi} [S([X_{\beta_1}, X_\alpha(u^p)], [X_{-\alpha}(u^q), X_{\beta_2}]), X_\gamma] - \sum_{\alpha \in \Phi} S([[X_{\beta_1}, X_\gamma], X_\alpha(u^p)], [X_{-\alpha}(u^q), X_{\beta_2}]) \\
- \sum_{\alpha \in \Phi} [S([X_{\beta_1}, X_\alpha(u^p)], [X_{-\alpha}(u^q), [X_{\beta_2}, X_\gamma]])] \\
= - (\gamma, \beta_2) S([X_{\beta_1}, X_\gamma] \otimes u^p, X_{\beta_2} \otimes u^q) - (\gamma, \beta_1) S(X_{\beta_1} \otimes u^p, [X_{\beta_2}, X_\gamma] \otimes u^q),
\]

(6.16.1)

(6.16.2)

here we write \(X = X_\gamma \in g_\gamma\), for \(\gamma \in \Phi \cup \{0\}\). The above equality follows from the following computation:
\[ (6.16.1) = - \sum_{i=1}^{N} m \left( \left[ [X_{\beta_1} \otimes 1, \tilde{h}_i(u^p) \otimes \tilde{h}_i(u^q), X_\gamma \otimes 1 + 1 \otimes X_\gamma], 1 \otimes X_{\beta_2} \right] \right) \]

\[ - \sum_{i=1}^{N} m \left( \left[ [\tilde{h}_i(u^p) \otimes \tilde{h}_i(u^q), X_\gamma \otimes 1 + 1 \otimes X_\gamma], X_{\beta_2} \otimes 1 \right] \right) \]

\[ = - \sum_{i=1}^{N} m \left( \left[ [X_{\beta_1} \otimes 1, (\tilde{h}_i, H_\gamma) X_\gamma(u^p) \otimes \tilde{h}_i(u^q) + (\tilde{h}_i, H_\gamma) \tilde{h}_i(u^p) \otimes X_\gamma(u^q)], 1 \otimes X_{\beta_2} \right] \right) \]

\[ - \sum_{i=1}^{N} m \left( \left[ [1 \otimes X_{\beta_1}, (\tilde{h}_i, H_\gamma) X_\gamma(u^p) \otimes \tilde{h}_i(u^q) + (\tilde{h}_i, H_\gamma) \tilde{h}_i(u^p) \otimes X_\gamma(u^q)], X_{\beta_2} \otimes 1 \right] \right) \]

\[ = - m \left( \left[ [X_{\beta_1} \otimes 1, X_\gamma(u^p) \otimes H_\gamma(u^q) + H_\gamma(u^p) \otimes X_\gamma(u^q)], 1 \otimes X_{\beta_2} \right] \right) \]

\[ - m \left( \left[ [1 \otimes X_{\beta_1}, X_\gamma(u^p) \otimes H_\gamma(u^q) + H_\gamma(u^p) \otimes X_\gamma(u^q)], X_{\beta_2} \otimes 1 \right] \right) \]

\[ = - m \left( [(X_{\beta_1}, X_\gamma)(u^p) \otimes [H_\gamma, X_{\beta_2}](u^q) + [X_{\beta_1}, H_\gamma](u^p) \otimes [X_\gamma, X_{\beta_2}](u^q)] \right) \]

\[ - m \left( [(X_\gamma, X_{\beta_2})(u^p) \otimes [X_{\beta_1}, H_\gamma](u^q) + [H_\gamma, X_{\beta_2}](u^q) \otimes [X_{\beta_1}, X_\gamma](u^q)] \right) \]

\[ = - m \left( (\gamma, 2 \beta)[X_{\beta_1}, X_\gamma](u^p) \otimes X_{\beta_2}(u^q) + (\gamma, 1 \beta)X_{\beta_1}(u^p) \otimes [X_{\beta_2}, X_\gamma](u^q)] \right) \]

\[ - m \left( (\gamma, 1 \beta)[X_{\beta_2}, X_\gamma](u^p) \otimes X_{\beta_1}(u^q) + (\gamma, 2 \beta)X_{\beta_2}(u^q) \otimes [X_{\beta_1}, X_\gamma](u^q)] \right) \]

\[ = - \gamma, 2 \beta)S([X_{\beta_1}, X_\gamma] \otimes u^p, X_{\beta_2} \otimes u^q) - (\gamma, 1 \beta)S(X_{\beta_1} \otimes u^p, [X_{\beta_2}, X_\gamma] \otimes u^q) = (6.16.2). \]

\[
\sum_{p+q=n} [S([X_{\beta_1}, X_\alpha(u^p)], [X_{\alpha}(u^q), X_{\beta_2}]), h(u)] \\
= \sum_{p+q=n} S([X_{\beta_1}, h], X_\alpha(u^{p+1}), [X_{\alpha}(u^p), X_{\beta_2}]) + \sum_{p+q=n} S([X_{\beta_1}, X_\alpha(u^p), [X_{\alpha}, X_{\beta_2}, h]], u^{p+1}) \\
+ (h, \alpha)S([X_{\beta_1}, X_\alpha], [X_{\alpha}(u^{n+1})]) = (h, \alpha)S([X_{\beta_1}, X_\alpha], u^{n+1}, [X_{\alpha}, X_{\beta_2}])
\]

We would like to show in the case \((h, \gamma) = 0\), then:

\[ [h \otimes v, H_\gamma \otimes u^n] = \sum_{p+q=n-1} \sum_{\alpha \in \Phi} S([h, X_\alpha] \otimes u^p, [X_{\alpha}, H_\gamma] \otimes u^q) \]

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\[ [h \otimes v, H_\gamma \otimes u^n] = [h \otimes v, [X_\gamma, X_{-\gamma}] \otimes u^n] = [[h \otimes v, X_\gamma \otimes u], X_{-\gamma} \otimes u^{n-1}] + [X_\gamma \otimes u, [h \otimes v, X_{-\gamma} \otimes u^{n-1}]] = [[h \otimes v, X_\gamma \otimes u], X_{-\gamma} \otimes u^{n-1}] - \frac{1}{(\gamma, \gamma)} [X_\gamma \otimes u, [[h \otimes v, H_\gamma \otimes u^{n-1}], X_{-\gamma}]] = A + B. \]

where

\[ A = [[h \otimes v, X_\gamma \otimes u], X_{-\gamma} \otimes u^{n-1}] = \sum_\alpha [S([h, X_\alpha], [X_{-\alpha}, X_\gamma]), X_{-\gamma} \otimes u^{n-1}] \]

and

\[ B = -\frac{1}{(\gamma, \gamma)} [X_\gamma \otimes u, [[h \otimes v, H_\gamma \otimes u^{n-1}], X_{-\gamma}]] = -\frac{1}{(\gamma, \gamma)} [X_\gamma \otimes u, \left\{ \sum_{p+q=n-2} \sum_{\alpha \in \Phi} S([h, X_\alpha] \otimes u^p, [X_{-\alpha}, H_\gamma] \otimes u^q), X_{-\gamma} \right\}] = \sum_{p+q=n-2} \sum_{\alpha \in \Phi} [X_\gamma \otimes u, S([h, X_\alpha] \otimes u^p, [X_{-\alpha}, X_{-\gamma}] \otimes u^q)] = \sum_{\alpha \in \Phi} \sum_{p+q=n-2} S([h, [X_\gamma, X_\alpha]] \otimes u^{p+1}, [X_{-\alpha}, X_{-\gamma}] \otimes u^q) + \sum_{p+q=n-2} S([h, X_\alpha] \otimes u^p, [[X_\gamma, X_{-\alpha}], X_{-\gamma}] \otimes u^{q+1}) \]
For the last two terms above, we use the fact that $[X_\gamma, \Omega_{p,q}] = 0$, then we have

$$0 = [X_\gamma, \Omega_{p,q}]$$

$$= \sum_\alpha [X_\gamma, X_\alpha](u^p)X_{-\alpha}(u^q) + \sum_i [X_\gamma, h_i](u^p)h^i \otimes u^q$$

$$+ \sum_\alpha X_\alpha(u^p)[X_\gamma, X_{-\alpha}](u^q) + \sum h_i(u^p)[X_\gamma, h^i](u^q)$$

$$= \sum_\alpha [X_\gamma, X_\alpha](u^p)X_{-\alpha}(u^q) - \sum_i X_\gamma(u^p)H_\gamma(u^q)$$

$$+ \sum_\alpha X_\alpha(u^p)[X_\gamma, X_{-\alpha}](u^q) - \sum_i H_\gamma(u^p)X_\gamma(u^q)$$

Thus,

$$\sum_\alpha [X_\gamma, X_\alpha](u^p) \otimes X_{-\alpha}(u^q)$$

$$= -\sum_\alpha X_\alpha(u^p) \otimes [X_\gamma, X_{-\alpha}](u^q) + \sum X_\gamma(u^p) \otimes H_\gamma(u^q) + \sum_i H_\gamma(u^p) \otimes X_\gamma(u^q)$$

So, under the assumption of $(h, \gamma) = 0$, the last two term of $B$ can be simplified as follows:

$$\sum_{p+q=n-2} \sum_{\alpha \in \Phi} S([h, [X_\gamma, X_\alpha]] \otimes u^{p+1}, [X_{-\alpha}, X_{-\gamma}] \otimes u^q)$$

$$+ \sum_{p+q=n-2} \sum_{\alpha \in \Phi} S([h, X_\alpha] \otimes u^p, [[X_\gamma, X_{-\alpha}], X_{-\gamma}] \otimes u^{q+1})$$

$$= -\sum_{p+q=n-2} \sum_{\alpha \in \Phi} S([h, X_\alpha] \otimes u^{p+1}, [[X_\gamma, X_{-\alpha}], X_{-\gamma}] \otimes u^q)$$

$$+ \sum_{p+q=n-2} \sum_{\alpha \in \Phi} S([h, X_\alpha] \otimes u^p, [[X_\gamma, X_{-\alpha}], X_{-\gamma}] \otimes u^{q+1})$$

$$= -\sum_{\alpha \in \Phi} S([h, X_\alpha] \otimes u^{n-1}, [[X_\gamma, X_{-\alpha}], X_{-\gamma}]) + \sum_{\alpha \in \Phi} S([h, X_\alpha], [[X_\gamma, X_{-\alpha}], X_{-\gamma}] \otimes u^{n-1})$$

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Thus,

\[
\sum_a [S([h, X,\alpha], [X-\alpha, X,\gamma]), X-\gamma \otimes u^{n-1}] - \sum_{\alpha \in \Phi} S([h, X,\alpha] \otimes u^{n-1}, [[X,\gamma, X-\alpha], X-\gamma])
\]

\[
+ \sum_{\alpha \in \Phi} S([h, X,\alpha], [[X,\gamma, X-\alpha], X-\gamma] \otimes u^{n-1})
\]

\[
= \sum_a S([h, X,\alpha], [X-\alpha, [X,\gamma, X-\gamma]] \otimes u^{n-1}) + \sum_{\alpha \in \Phi} S([h, X,\alpha], [[X,\gamma, X-\alpha], X-\gamma] \otimes u^{n-1})
\]

\[
+ \sum_{\alpha \in \Phi} S([h, X,\alpha], [[X-\alpha, X-\gamma], X,\gamma] \otimes u^{n-1})
\]

\[
- \sum_{\alpha \in \Phi} S([h, X,\alpha] \otimes u^{n-1}, [[X,\gamma, X-\alpha], X-\gamma]) + \sum_{\alpha \in \Phi} S([h, [X,\alpha, X-\gamma] \otimes u^{n-1}, [X-\alpha, X,\gamma]])
\]

\[
= - \sum_{\alpha \in \Phi} S([h, X,\alpha] \otimes u^{n-1}, [[X,\gamma, X-\alpha], X-\gamma]) - \sum_{\alpha \in \Phi} S([h, [X,\alpha, X-\gamma] \otimes u^{n-1}, [X-\alpha, X-\gamma], X,\gamma])
\]

\[
= \sum_{\alpha \in \Phi} S([h, X,\alpha] \otimes u^{n-1}, [X-\alpha, H-\gamma]).
\]

Thus, when \((h, \gamma) = 0\), we have

\[
A + B = \sum_{\alpha \in \Phi} S([h, X,\alpha] \otimes u^{n-1}, [X-\alpha, H-\gamma]) + \sum_{p+q=n-2} \sum_{\alpha \in \Phi} S([h, X,\alpha] \otimes u^p, [X-\alpha, H-\gamma] \otimes u^q)
\]

\[
= \sum_{p+q=n-1} \sum_{\alpha \in \Phi} S([h, X,\alpha] \otimes u^p, [X-\alpha, H-\gamma] \otimes u^q).
\]
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