ON THE REGULARITY OF SOLUTIONS TO THE BELTRAMI EQUATION IN THE PLANE

by

Bindu K Veetel
B. Sc. in Mathematics, Bangalore University, India
M. Sc. in Mathematics, Bangalore University, India

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Robert McOwen
Professor of Mathematics
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We obtain estimates on regularity of solutions to the inhomogeneous Beltrami equation in the plane namely $\bar{\partial} f - \mu \partial f - \nu \overline{\partial f} = h$ when $h$ and the coefficients $\mu$ and $\nu$ are Dini continuous; the coefficients satisfy an ellipticity condition $|\mu(z)| + |\nu(z)| \leq \kappa < 1$. In the case when $h$ and the coefficients $\mu$ and $\nu$ are Hölder continuous with exponent $\alpha, 0 < \alpha < 1$, it is well known (cf.[3]) that the solutions and their first order derivatives are Hölder continuous. In our case, we find that although the solutions are in $C^1$, their derivatives are less regular than the coefficients. Nevertheless, we are able to get a priori estimates. One application of this work is to stability analysis of the inverse problem of Calderon in two dimensions.
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1. **Introduction**

In this thesis, we study the regularity of solutions to the inhomogeneous Beltrami equation in the plane, namely

\[ \overline{\partial} f - \mu \partial f - \nu \overline{\partial} f = h, \]

where \( \overline{\partial} = \partial_z = \frac{1}{2}(\partial_x + i \partial_y), \quad \partial = \partial_z = \frac{1}{2}(\partial_x - i \partial_y). \)

We assume the ellipticity condition, \( |\mu(z)| + |\nu(z)| \leq \kappa < 1 \) for all \( z \in \Omega \subset \mathbb{C} \) where \( \Omega \) is a bounded domain in \( \mathbb{C} \). When the coefficients \( \mu \) and \( \nu \) and the function \( h \) are Hölder continuous with exponent \( \alpha \in (0, 1) \) i.e. in the Hölder space \( C^\alpha(\Omega) \), it is well known (cf.[3]) that the first order derivatives of a solution to the Beltrami equation (1) also lie in the space \( C^\alpha(\Omega) \). In this thesis, we consider the coefficients \( \mu \) and \( \nu \) and \( h \) to have modulus of continuity \( \omega \) satisfying the Dini condition \( \int_0^\epsilon \frac{\omega(r)}{r} dr < \infty \). The result that we obtain is that the first order derivatives of a solution of (1) will have modulus of continuity \( \sigma \) that could be weaker than \( \omega \), in particular, need not satisfy the Dini condition. Nevertheless, we are able to obtain interior estimates of the form

\[ \| \partial f \|_{C^\sigma(D)} + \| \overline{\partial} f \|_{C^\sigma(D)} \leq C(\| h \|_{C^\omega(\Omega)} + \| f \|_{C^0(\Omega)}), \]

where the constant \( C \) depends on \( \mu, \nu \) and the domains \( D, U \) where \( D \) is compactly contained in \( U \) and \( U \) is compactly contained in \( \Omega \). (In (2), we have used \( C^\omega \) to denote functions continuous with respect to the modulus of continuity \( \omega \), even though this is inconsistent with the notation \( C^\alpha \) for Hölder continuity; cf. Section 3). The techniques we use are mostly based on Fourier analysis, Littlewood-Paley theory and
Fourier multiplier operators on the torus are used to describe and work with functions that we encounter. Our main result is given as Theorem 5 in section 6.

One of the applications of (2) is to stability of the inverse problem of Calderon in two dimensions. An inverse problem assumes a direct problem that is well-posed: a solution exists, is unique and stable under perturbation of parameters. It uses information about the solutions to obtain a parameter in the problem.

To describe the Calderon problem, let $u$ be the unique solution to the Dirichlet problem

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in a bounded domain } U \subset \mathbb{R}^2,$$

(3)

$$u = f \quad \text{on } \partial U.$$

If we let $\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu}$ where $\nu$ is the exterior unit normal on $\partial U$, then the map $\Lambda_\gamma : H^{1/2}(\partial U) \to H^{-1/2}(\partial U)$ is called the the Dirichlet to Neumann map. If $\gamma$ represents the electrical conductivity of $U$, then $\Lambda_\gamma$ is the current induced at the boundary by applying a voltage $f$ at the boundary of $U$. Hence experimental data involving current and voltage measurements at the boundary determines $\Lambda_\gamma$. The inverse problem of Calderon is the determination of conductivity $\gamma$ in $U$ from the boundary measurements, i.e. from $\Lambda_\gamma$. In particular, solving the inverse problem requires showing that the Dirichlet to Neumann map $\Lambda_\gamma$ uniquely determines $\gamma$. Stability is a stronger statement than
uniqueness; it says that
\[
\|\gamma_1 - \gamma_2\|_{L^\infty} \leq V(\|\Lambda \gamma_1 - \Lambda \gamma_2\|_*),
\]
where \(\|\cdot\|_*\) denotes the operator norm \(H^{1/2}(\partial U) \to H^{-1/2}(\partial U)\) and \(V(\rho)\) is a “stability function” that satisfies \(V(\rho) \to 0\) as \(\rho \to 0\). Hence stability implies that small changes in the Dirichlet to Neumann map will only correspond to small changes in the conductivity. This inverse problem has important real world applications like in medical imaging to detect tumors in the human body, reconstruction of the interior of human body using exterior measurements.

Astala and Päiväranta [5] showed that \(\Lambda_\gamma\) uniquely determines \(\gamma \in L^\infty\) by reducing the problem to a Beltrami equation. In fact, they recover the complex solutions of (3) by letting
\[
(5) \quad u_\gamma = \Re(f_\nu) + i \Im(f_{-\nu})
\]
where \(f_\nu\) is the complex geometric optics solution of the \(\mathbb{R}\)-linear Beltrami equation,
\[
(6) \quad \overline{\partial} f = \nu \overline{\partial f},
\]
and \(\nu\) is defined in terms of conductivity \(\gamma\) as \(\nu = \frac{1-\gamma}{1+\gamma}\). Notice that the above equation is a special case of (1) where the coefficient \(\mu\) and the function \(h\) are zero. In [5], it is shown that for each \(k \in \mathbb{C}\), there exits a unique solution \(f_\nu(z,k)\) of (6) of the form
\[
f_\nu(z,k) = e^{ikz}M_\nu(z,k)
\]
where
\[ M_\nu(z, k) = 1 + O(z^{-1}) \quad \text{as } |z| \to \infty. \]

In order to control the behaviour of \( M_\nu \) as \( |z| \to \infty \), [5] shows that \( f_\nu(z, k) = e^{ik\phi(z, k)} \) where for each fixed \( k \), \( \phi \) satisfies a Beltrami equation.

The work of Barcelo, Faraco and Ruiz [4], stem from that of Astala and Päivärinta in [5]. They use regularity results for (6) to obtain stability in the case of Hölder continuous conductivity. In particular, they obtain that when the coefficients \( \mu, \nu \) and the function \( h \) are Hölder continuous with exponent \( \alpha \in (0, 1) \), any solution \( f \) in the Sobolev space \( W^{1, 2}(\Omega) \) satisfies the estimate
\[
\| \partial f \|_{C^\alpha(D)} + \| \overline{\partial} f \|_{C^\alpha(D)} \leq C \| f \|_{C^0(D)}
\]
where the constant \( C \) depends on \( \mu, \nu, \alpha \) and the domains \( D, U \) such that \( D \) is compactly contained in \( U \) and \( U \) is compactly contained in \( \Omega \). This estimate is then used to obtain regularity results for (6). The interior estimate (2) obtained in this thesis can be used to obtain corresponding estimates needed for stability of the complex solutions of the inverse problem of Calderon in two dimensions in the Dini case. This will be further discussed in the upcoming paper [14].

The basic structure of our work is as follows: In section 2, we define Dini continuous functions and Banach spaces associated with these functions. In section 3, we obtain the regularity of solutions to the
inhomogeneous Beltrami equation with constant coefficients, namely

\[(7) \quad \overline{\partial}f - \mu_0 \partial f - \nu_0 \overline{\partial}f = h \quad \text{where } h \quad \text{is Dini continuous.}\]

This is done first for the C-R equation, the special case of the Beltrami equation where the coefficients \( \mu \) and \( \nu \) are zero, and then translated to that of the Beltrami equation (7). But these estimates do not easily generalize to variable coefficients. Hence in section 4, we follow [15] by introducing a space of functions \( C^{(\lambda)} \) on a torus using Fourier multipliers and a Littlewood-Paley partition of unity; here the function \( \lambda \) is defined in terms of modulus of continuity \( \omega \). These functions need not be Dini continuous but are well behaved under the Beurling transform. We first obtain regularity estimates in terms of \( C^{(\lambda)} \) on solutions of the Beltrami equation with constant coefficients \( \mu_0 \) and \( \nu_0 \) but with \( h \in C^{(\lambda)} \). We then use these estimates to obtain similar estimates on solutions of the Beltrami equation (1) with variable coefficients in \( C^{(\lambda)} \). We make use of tools like the Beurling transform, Fourier transform and paraproduct. In section 5, we translate our results on the torus to regularity estimates for (1) with Dini continuous coefficients in a bounded domain \( \Omega \).

The following are some terms used in our work:

Modulus of continuity:
It is a continuous, increasing function on \([0, \infty)\), usually denoted by \( \omega \) such that \( \omega(0) = 0 \). The definition of modulus of continuity was first given by H. Lebesgue in 1910 for functions of one real variable. It is a
precise way to measure smoothness of a function.

Beltrami Equation:
Beltrami equations play an important part in complex function theory and theory of differential equations. Various theorems like the measurable Riemann mapping theorem can be proved using the Beltrami equation. The first global solution in $C$ of the Beltrami equation of the form

$$\bar{\partial} f = \mu \partial f$$

was given by Venkua [17] for compactly supported $\mu$. The $L^p$ properties of the operators associated with the Beltrami equation have been used to prove the measurable Riemann mapping theorem in [3].

Littlewood-Paley partition of unity:
This is a tool in analysis which allows us to decompose a function into pieces such that the frequency supports of these pieces are almost disjoint. On $\mathbb{R}^n$, it can be defined as a collection $\{\psi_k(\xi)\}$ of smooth functions with compact support such that for $k > 0$, each $\psi_k(\xi)$ has support in $2^{k-1} < |\xi| < 2^{k+1}$, $\psi_0(\xi)$ has support in $|\xi| < 2$ and $\sum_{k=0}^{\infty} \psi_k(\xi) = 1$.

Fourier Transform:
Let $S(\mathbb{R}^2)$ denote Schwartz space i.e. the space of smooth functions such that all derivatives decay rapidly at infinity. The Fourier transform $\mathcal{F} : S(\mathbb{R}^2) \rightarrow S(\mathbb{R}^2)$ is defined by

$$\langle \mathcal{F} f \rangle(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(z) e^{-iz.\xi} \, dz \quad \text{for } \xi \in \mathbb{R}^2$$
and the inverse Fourier transform \( \mathcal{F}^{-1} g \) is given by

\[
(\mathcal{F}^{-1}g)(z) = \hat{g}(z) = \int_{\mathbb{R}^2} g(\xi) e^{iz \cdot \xi} d\xi.
\]

If \( S'(\mathbb{R}^2) \) denote the dual of the Schwartz space, i.e. the space of tempered distributions, the Fourier transform can be extended to \( \mathcal{F} : S'(\mathbb{R}^2) \to S'(\mathbb{R}^2) \) by defining

\[
(\hat{u}(\phi)) = \hat{u}(\hat{\phi})
\]

for \( \phi \in S(\mathbb{R}^2) \).

If \( f, g \in L^1(\mathbb{R}^2) \) and one of them has compact support, then

\[
(fg)^\vee(z) = (f^\vee * g^\vee)(z) = \int_{\mathbb{R}^2} f^\vee(z - \tilde{z})g^\vee(\tilde{z}) d\tilde{z}.
\]

Fourier Multiplier Operator:

Using the Fourier transform, we can define Fourier multiplier operators \( M(D) \) for any bounded, measurable function \( m(\xi) \) as

\[
(M(D)f)^\wedge(\xi) = m(\xi) \hat{f}(\xi).
\]

The function \( m(\xi) \) is called its symbol or multiplier. Examples of Fourier multiplier operators include differential operators. Fourier multiplier operators are a special case of the pseudodifferential operators which are extensions of the concept of differential operators and that of singular integral operators and are used extensively in the field of partial differential equations and quantum field theory.

Paraproduct:

According to S. Jansen and J. Peetre [11], “The name paraproduct denotes an idea rather than a unique definition; several definitions exist
and can be used for the same purpose.” The term paraproduct means beyond product. It can be considered as a bilinear, non-commutative operator that is used to reconstruct a product such that it generally has better properties than the usual product of functions. The first version of a paraproduct appeared in A. P. Calderon’s work on commutators [7]. Several versions have appeared since then. J.-M. Bony introduced a paraproduct in his work, “Theory of paradifferential operators” [6], which came to be known as Bony’s paraproduct. It is defined using a Littlewood-Paley partition of unity and helps in reconstructing a product \( fg \) based on the frequency supports of \( f \) and \( g \). It separates out the parts where frequency supports of \( f \) and \( g \) are disjoint and hence enables analysis of each part of the product appropriately. The part where the supports are not disjoint is considered as the error term and hence plays the role of the best part of the product reconstruction.

Smoothing operator:
It is an operator that maps non-smooth functions or even distributions to smooth functions. Any Fourier multiplier operator \( \psi(D) \in L^1(\mathbb{R}^n) \) with compact support can be considered as a smoothing operator. The Fourier multiplier operator \( \psi_0(D) \) associated with the Littlewood-Paley partition of unity \( \{\psi_j(\xi)\} \) is an example of a smoothing operator used in this work.

Fredholm operators:
A bounded linear operator \( T : X \to Y \) between two Banach spaces \( X \) and \( Y \) is said to be Fredholm if it has finite dimensional kernel and
cokernel. The index of the Fredholm operator $T$ is given by $\text{ind}(T) = \text{nul}(T) - \text{def}(T)$ where $\text{nul}(T)$ is the dimension of kernel of $T$ and $\text{def}(T)$ is the dimension of cokernel of $T$. The composition of Fredholm operators is also Fredholm with index of the composition equal to the sum of index of each operator. If $K : X \to X$ is a compact operator, then the operator $T = I + K$ is Fredholm with $\text{ind}(T)=0$. A bounded linear operator $T : X \to Y$ between two Banach spaces $X$ and $Y$ is said to be semi-Fredholm if the range of $T$ is a closed subspace of $Y$ and atleast one of kernel $T$ or cokernel $T$ is of finite dimension. It is well known that if an operator $T$ satisfies an a priori estimate of the form $\|f\|_X \leq C(\|Tf\|_Y + \|f\|_Y)$ where $X$ is compactly contained in $Y$, then $T$ has a finite dimensional kernel and closed range, i.e. is a semi-Fredholm operator.

2. **Dini Continuous Functions**

Consider a function $\omega$ with the following properties: $\omega(0) = 0$, $\omega(r)$ is continuous and strictly increasing for $0 \leq r \leq 1$ and satisfies the Dini condition

$$
\int_0^\epsilon \frac{\omega(r)}{r} dr < \infty
$$

for some $\epsilon > 0$. For technical reasons, also assume

$$
0 < \omega(r) \leq 1
$$

and that

$$
r^{-\beta}\omega(r) \quad \text{is decreasing on } (0, 1] \text{ for some } \beta \in (0, 1).
$$
When convenient, we extend $\omega$ to $r > 1$ by letting $\omega(r) = \omega(1)$. This extension of $\omega$ will still satisfy (13) on $(0, \infty)$. As a consequence of (13), it is easy to see that $\omega(2r) \leq 2^{\beta} \omega(r)$. Hence there exists constants $C_1$ and $C_2$ such that $C_1 \omega(2r) \leq \omega(r) \leq C_2 \omega(2r)$.

For $0 \leq r \leq 1$, define a function $\sigma$ by

$$\sigma(r) := \int_0^r \frac{\omega(s)}{s} \, ds.$$  \hspace{1cm} (14)

Using (13), it is easy to see that

$$C_1 \sigma(2r) \leq \sigma(r) \leq C_2 \sigma(2r)$$  \hspace{1cm} (15)

for some constants $C_1, C_2$ and $\omega(r) \leq \beta \sigma(r)$ certainly implies

$$\omega(r) \leq \sigma(r).$$  \hspace{1cm} (16)

We shall use $\omega$ and $\sigma$ as modulii of continuity for functions. Let $U$ be a bounded domain in $\mathbb{R}^2$.

**Definition 1.** For any modulus of continuity $\omega$, let $C^\omega(U)$ denote those functions $f \in C(U)$ for which $|f(x) - f(y)| \leq C \omega(|x - y|)$ for all $x, y \in U$. $C^\omega(U)$ is a Banach space under the norm

$$\|f\|_{C^\omega(U)} := \sup_{x \in U} |f(x)| + \sup_{x, y \in U, x \neq y} \frac{|f(x) - f(y)|}{\omega(|x - y|)}.$$  \hspace{1cm} (17)

**Definition 2.** For any domain $\Omega$, the collection of all functions $f$ such that $f \in C^\omega(U)$ for all $U$ compactly contained in $\Omega$ is denoted by $C^\omega(\Omega)$.  


Definition 3. Let $C^{1,\omega}(\mathcal{U})$ denote those functions $f \in C^1(\mathcal{U})$ whose first order derivatives $\partial f / \partial z$ and $\partial f / \partial \overline{z}$ are in $C^\omega(\mathcal{U})$. $C^{1,\omega}(\mathcal{U})$ is a Banach space under the norm

$$
\| f \|_{C^{1,\omega}(\mathcal{U})} := \| \partial f \|_{C^\omega(\mathcal{U})} + \| \overline{\partial f} \|_{C^\omega(\mathcal{U})} + \| f \|_{C^0(\mathcal{U})}.
$$

Definition 4. For any domain $\Omega$, the collection of all functions $f$ such that $f \in C^{1,\omega}(\mathcal{U})$ for all $U$ compactly contained in $\Omega$ is denoted by $C^{1,\omega}(\Omega)$.

When $\omega$ satisfies the Dini condition (11), the functions in $C^\omega(\Omega)$ are called **Dini continuous**.

Throughout this paper, we shall use $\omega$ to denote the modulus of continuity with the properties (11), (12) and (13).

Using (14), $C^\sigma(\mathcal{U})$ and $C^{1,\sigma}(\mathcal{U})$ are Banach spaces defined as above using $\sigma$ as the modulus of continuity. In fact, using (16), we see that $C^\omega(\Omega)$ is contained in $C^\sigma(\Omega)$. When $\omega(r) = r^\alpha$ with $\alpha \in (0, 1)$, we get the special case of Holder continuous functions. In this case, $\sigma(r)$ is so equivalent to $r^\alpha$ and hence $C^\omega(\Omega)$ coincides with $C^\sigma(\Omega)$.

In this work, we also consider the collection of all functions $f$ whose first-order derivatives are square-integrable in every $U$ compactly contained in any domain $\Omega$, which is generally denoted by $W^{1,2}_{loc}(\Omega)$.
3. Constant coefficients in bounded domain

Throughout this work, we shall denote an open disk with center \( z_0 \) and radius \( r \) as \( \mathbb{D}_r(z_0) \) and an open disk with center as the origin and radius \( r \) as \( \mathbb{D}_r \). We shall consider \( \Omega \) to be a bounded domain in \( \mathbb{R}^2 \).

**Theorem 1.** Let \( h \in C^\omega(\Omega) \) and \( \mu_0 \) and \( \nu_0 \) be constants satisfying

\[ |\mu_0| + |\nu_0| \leq \kappa < 1. \]

Let \( f \in W^{1,2}_{loc}(\Omega) \) satisfy

\[ (19) \quad \overline{\partial} f - \mu_0 \partial f - \nu_0 \overline{\partial} f = h. \]

Then \( f \in C^{1,\sigma}(\Omega) \) and satisfies

\[ (20) \quad \|\partial f\|_{C^{\sigma}(\overline{D})} + \|\overline{\partial} f\|_{C^{\sigma}(\overline{D})} \leq C(\kappa, D, U) \left( \|h\|_{C^{\omega}(\overline{U})} + \|f\|_{C^{\omega}(\overline{U})} \right) \]

for any domain \( D, U \) such that \( \overline{D} \) is compactly contained in \( U \) and \( \overline{U} \) is compactly contained in \( \Omega \).

We first consider the special case of the Beltrami equation where the coefficients \( \mu \) and \( \nu \) are zero. This is the inhomogeneous Cauchy-Riemann equation of the form

\[ (21) \quad \overline{\partial} p = q. \]

If \( q \) is integrable on a domain \( \Omega \), then the domain potential of \( q \) is given by

\[ (22) \quad p(z) = \frac{1}{\pi} \int_{\Omega} \frac{q(\tau)}{z - \tau} \, dA \]

and is a distribution solution of the equation (21) in \( \Omega \) since \( \frac{1}{\pi z} \) is the fundamental solution for the C-R operator \( \overline{\partial} \).
An important result that we use in this section is that for any disk $\mathbb{D}_R(z_0)$,

$$
\int_{\mathbb{D}_R(z_0)} \frac{1}{(z_1 - \tau)^2} \, dA_\tau = 0 \quad \text{for } z_1 \in \mathbb{D}_R(z_0).
$$

Using polar coordinates, this can easily be seen for the case when $z_1 = z_0$. When $z_1 \neq z_0$, we can write

$$
\int_{\mathbb{D}_R(z_0)} \frac{1}{(z_1 - \tau)^2} \, dA_\tau = \lim_{\epsilon \to 0} \int_{\mathbb{D}_R(z_0) \setminus \mathbb{D}_\epsilon(z_1)} \frac{1}{(z_1 - \tau)^2} \, dA_\tau.
$$

Now Green’s theorem can be used to get (23).

We first prove results which are necessary to prove the above theorem.

**Lemma 1.** Consider a function $\phi \in C^\omega(\overline{\mathbb{D}}_1)$. Then

$$
\partial_z \int_{\mathbb{D}_1} \frac{\phi(\tau)}{(z_1 - \tau)^2} \, dA_\tau = -\int_{\mathbb{D}_1} \frac{\partial(\phi)}{(z_1 - \tau)^2} \, dA_\tau \quad \text{for } z \in \mathbb{D}_1.
$$

**Proof:** Let

$$
w(z) := \int_{\mathbb{D}_1} \frac{\phi(\tau)}{(z_1 - \tau)} \, dA_\tau \quad \text{and} \quad u(z) := -\int_{\mathbb{D}_1} \frac{\partial(\phi(z))}{(z_1 - \tau)^2} \, dA_\tau.
$$

Note that using (23), $u(z) = -\int_{\mathbb{D}_1} \frac{\phi(z)}{(z_1 - \tau)^2} \, dA_\tau$.

Consider a function $\eta \in C^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $0 \leq \eta' \leq 2$, and

$$
\eta(t) = \begin{cases} 
0, & \text{for } t \leq 1 \\
1, & \text{for } t \geq 2
\end{cases}
$$

Denote $\eta(\frac{|z_1 - \tau|}{\epsilon})$ as $\eta_\epsilon(z_1 - \tau)$. For $\epsilon > 0$, define

$$
w_\epsilon(z) := \int_{\mathbb{D}_1} \frac{\eta_\epsilon(z_1 - \tau) \phi(\tau)}{(z_1 - \tau)^2} \, dA_\tau.
$$
Note that \( w_\epsilon(z) \) is smooth as it is the convolution with the smooth function \( \frac{1}{z} \eta(\frac{|z|}{\epsilon}) \). Now \( w \) is the uniform limit of \( w_\epsilon \) since

\[
|w_\epsilon(z) - w(z)| = \left| \int_{|z-\tau|<2\epsilon} \frac{1 - \eta_\epsilon(z-\tau)}{(z-\tau)} \phi(\tau) dA_\tau \right|
\leq \int_{|z-\tau|<2\epsilon} \frac{|1 - \eta_\epsilon(z-\tau)|}{|z-\tau|} |\phi(\tau)| dA_\tau
\leq C'(\epsilon) \sup_{\tau \in \partial \mathbb{D}_1} |\phi(\tau)|
\]

where \( C'(\epsilon) \) tends to 0 as \( \epsilon \) tends to 0. We now prove that \( u \) is the uniform limit of \( \partial_z w_\epsilon \). Consider

\[
\partial_z w_\epsilon(z) = \int_{\mathbb{D}_1} \partial_z \left[ \frac{\eta_\epsilon(z-\tau)}{(z-\tau)} \right] (\phi(\tau) - \phi(z)) dA_\tau
+ \phi(z) \int_{\mathbb{D}_1} \partial_z \left[ \frac{\eta_\epsilon(z-\tau)}{(z-\tau)} \right] dA_\tau
\tag{25}
\]

The second integral can be written as

\[
\int_{\mathbb{D}_1} \partial_z \left[ \frac{\eta_\epsilon(z-\tau)}{(z-\tau)} \right] dA_\tau = -\int_{\mathbb{D}_1} \partial_\tau \left[ \frac{\eta_\epsilon(z-\tau)}{(z-\tau)} \right] dA_\tau
= -\frac{i}{2} \int_{\partial \mathbb{D}_1} \left[ \frac{\eta_\epsilon(z-\tau)}{(z-\tau)} \right] d\bar{\tau}
= -\frac{i}{2} \int_{\partial \mathbb{D}_1} \frac{1}{(z-\tau)} d\bar{\tau}
\]

since \( \epsilon \) can be chosen so that \( \eta_\epsilon(z-\tau) = 1 \) for \( \tau \) on \( \partial \mathbb{D}_1 \) for fixed \( z \).

Now using polar coordinates, it can be seen that

\[
\int_{\partial \mathbb{D}_1} \frac{1}{(z-\tau)} d\bar{\tau} = 0
\]

Hence

\[
\int_{\mathbb{D}_1} \partial_z \left[ \frac{\eta_\epsilon(z-\tau)}{(z-\tau)} \right] dA_\tau = 0.
\tag{26}
\]
Using (26) in (25), we get

\[ \partial_z w_\epsilon(z) = \int_{D_1} \partial_z \left[ \frac{\eta_\epsilon(z - \tau)}{(z - \tau)} \right] (\phi(\tau) - \phi(z)) dA_\tau \]

We now have

(27)

\[ |u(z) - \partial_z w_\epsilon(z)| = \left| \int_{|z - \tau| < 2\epsilon} \partial_z \left[ \frac{1 - \eta_\epsilon(z - \tau)}{z - \tau} \right] (\phi(\tau) - \phi(z)) dA_\tau \right| \]

\[ \leq \int_{|z - \tau| < 2\epsilon} \left| \partial_z \frac{1 - \eta_\epsilon(z - \tau)}{z - \tau} \right| |\phi(\tau) - \phi(z)| dA_\tau \]

But

\[ \left| \partial_z \frac{1 - \eta_\epsilon(z - \tau)}{z - \tau} \right| \leq \left( \frac{1}{\epsilon |z - \tau|} \right) \eta'(\frac{|z - \tau|}{\epsilon}) + \frac{1 - \eta_\epsilon(|z - \tau|)}{|z - \tau|^2} \]

\[ \leq \left( \frac{2}{\epsilon |z - \tau|} + \frac{1}{|z - \tau|^2} \right) \]

Substituting the above in (27), we get

\[ |u(z) - \partial_z w_\epsilon(z)| \]

\[ \leq \int_{|z - \tau| < 2\epsilon} \left( \frac{2}{\epsilon |z - \tau|} + \frac{1}{|z - \tau|^2} \right) |\phi(\tau) - \phi(z)| dA_\tau \]

\[ \leq \|\phi\|_{C^{\infty}(\overline{D}_1)} \int_{|z - \tau| < 2\epsilon} \left( \frac{2}{\epsilon |z - \tau|} + \frac{1}{|z - \tau|^2} \right) \omega(|\tau - z|) dA_\tau \]

\[ \leq \|\phi\|_{C^{\infty}(\overline{D}_1)} (4\omega(2\epsilon) + 2\pi \sigma(2\epsilon)) \]

which tends to 0 as \( \epsilon \) tends to 0. Hence we can conclude that \( \partial_z w(z) = u(z) \) which is our required result. \[\square\]

In the case of Holder continuous functions, it is well known (cf.[3]) that the Holder norm of derivative of the solution \( p \) to the inhomogeneous C-R equation (21) is bounded by the Holder norm of \( q \). We now obtain
similar bounds in the case of (21) for Dini continuous functions. This is where we get to see the role of the $C^\sigma$ norm.

**Lemma 2.** For $q \in C^\omega(\overline{D}_e)$, the domain potential $p(z) = \frac{1}{\pi} \int_{D_\epsilon} \frac{q(\tau)}{(z-\tau)^2} \, dA_{\tau}$ lies in $C^{1,\sigma}(\overline{D}_{e/2})$ and satisfies the estimate

\begin{equation}
||\partial p||_{C^\sigma(\overline{D}_{e/2})} + ||\partial p||_{C^\omega(\overline{D}_{e/2})} \leq K(\epsilon) || q ||_{C^\omega(\overline{D}_e)}.
\end{equation}

**Proof:** For $z \in D_{e/2}$, using (24) and (23), we can write

$$\partial p(z) = -\frac{1}{\pi} \int_{D_{\epsilon}} \frac{q(\tau)}{(z-\tau)^2} \, dA_{\tau} = -\frac{1}{\pi} \int_{D_{\epsilon}} \frac{q(\tau) - q(z)}{(z-\tau)^2} \, dA_{\tau}.$$  

Since $\partial p(z) = q(z)$, we have $||\partial p||_{C^\sigma(\overline{D}_{e/2})} \leq K || q ||_{C^\omega(\overline{D}_e)}$. Now let us estimate $||\partial p||_{C^\sigma(\overline{D}_{e/2})}$. For any $z_1, z_2 \in D_{e/2}$, we get

$$||\partial p(z_1) - \partial p(z_2)|| = \frac{1}{\pi} \left| \int_{D_{\epsilon}} \frac{q(\tau) - q(z_1)}{(z_1 - \tau)^2} \, dA_{\tau} - \int_{D_{\epsilon}} \frac{q(\tau) - q(z_2)}{(z_2 - \tau)^2} \, dA_{\tau} \right|.$$  

Let $|z_1 - z_2| = \delta < \epsilon$, $z_3 = \frac{z_1 + z_2}{2}$ so that $D_{\delta}(z_3)$ is contained in $D_{\epsilon}$. The above equation can now be estimated as

\begin{equation}
||\partial p(z_1) - \partial p(z_2)|| \leq \frac{1}{\pi} \left| \int_{D_{\delta}(z_3)} \frac{q(\tau) - q(z_1)}{(z_1 - \tau)^2} \, dA_{\tau} \right| + \frac{1}{\pi} \left| \int_{D_{\delta}(z_3)} \frac{q(\tau) - q(z_2)}{(z_2 - \tau)^2} \, dA_{\tau} \right|

+ \frac{1}{\pi} \left| \int_{D_\epsilon - D_\delta(z_3)} \frac{q(\tau) - q(z_1)}{(z_1 - \tau)^2} \, dA_{\tau} \right| + \frac{1}{\pi} \left| \int_{D_\epsilon - D_\delta(z_3)} \frac{q(\tau) - q(z_2)}{(z_2 - \tau)^2} \, dA_{\tau} \right|
\end{equation}

Now $|q(\tau) - q(z_1)| \leq || q ||_{C^\omega(D_{\epsilon})} \omega(|\tau - z_1|)$ gives

\begin{equation}
\left| \int_{D_{\delta}(z_3)} \frac{q(\tau) - q(z_1)}{(z_1 - \tau)^2} \, dA_{\tau} \right| \leq || q ||_{C^\omega(\overline{D}_e)} \int_{D_{\delta}(z_3)} \frac{\omega(|\tau - z_1|)}{|z_1 - \tau|^2} \, dA_{\tau}

\leq || q ||_{C^\omega(\overline{D}_e)} \int_{D_{\delta}(z_1)} \frac{\omega(|\tau - z_1|)}{|z_1 - \tau|^2} \, dA_{\tau}

= 2\pi || q ||_{C^\omega(\overline{D}_e)} \sigma(2\delta).
\end{equation}
Using (15), (30) can be written as

\[
\left| \int_{D_\delta(z_3)} \frac{q(\tau) - q(z_1)}{(z_1 - \tau)^2} \, dA_\tau \right| \leq C \|q\|_{C^\infty(\Omega_4)} \sigma(\delta). \tag{31}
\]

Similarly

\[
\left| \int_{D_\delta(z_3)} \frac{q(\tau) - q(z_2)}{(z_2 - \tau)^2} \, dA_\tau \right| \leq C \|q\|_{C^\infty(\Omega_4)} \sigma(\delta). \tag{32}
\]

Now we estimate the last term in (29). First observe that using (23), we get

\[
\left| \int_{D_\epsilon - D_\delta(z_3)} \left\{ \frac{1}{(z_1 - \tau)^2} - \frac{1}{(z_2 - \tau)^2} \right\} \left( q(\tau) - q(z_2) \right) \, dA_\tau \right| = \left| \int_{D_\epsilon - D_\delta(z_3)} \frac{1}{(z_1 - \tau)^2} \, dA_\tau - \int_{D_\epsilon - D_\delta(z_3)} \frac{1}{(z_2 - \tau)^2} \, dA_\tau \right| \left( q(\tau) - q(z_2) \right).
\]

Now by the mean value theorem, there exists \( \tilde{z}_\tau \) between \( z_1 \) and \( z_2 \) such that

\[
\frac{1}{(z_1 - \tau)^2} - \frac{1}{(z_2 - \tau)^2} = (z_1 - z_2). \frac{-2}{(\tilde{z}_\tau - \tau)^3}.
\]

Hence we get

\[
\left| \int_{D_\epsilon - D_\delta(z_3)} \frac{1}{(z_1 - \tau)^2} \, dA_\tau - \int_{D_\epsilon - D_\delta(z_3)} \frac{1}{(z_2 - \tau)^2} \, dA_\tau \right| \left( q(\tau) - q(z_2) \right) dA_\tau \leq \|q\|_{C^\infty(\Omega_4)} \left| z_1 - z_2 \right| \int_{D_\epsilon - D_\delta(z_3)} \left| \frac{2}{(\tilde{z}_\tau - \tau)^3} \right| \omega(|\tau - z_2|) \, dA_\tau.
\]

But

\[
|z_2 - \tau| \geq |z_2 - z_3| - |\tau - z_3| \geq |2\epsilon - 3\delta| / 2 \geq \delta / 2 = \frac{|z_1 - z_2|}{2}.
\]

Using property (13) of \( \omega(|\tau|) \), we get

\[
|z_2 - \tau|^{-\beta} \omega(|\tau - z_2|) \leq \left( \frac{|z_1 - z_2|}{2} \right)^{-\beta} \omega(|z_1 - z_2|).
\]
Using these we get

\[
\int_{D_\epsilon - B_\delta(z_3)} \frac{\omega(|\tau - z_2|)}{|\tilde{z}_r - \tau|^3} \, dA_r \leq \int_{D_\epsilon - B_\delta(z_3)} \frac{\omega(|z_1 - z_2|)}{|\tilde{z} - \tau|^\beta |z_2 - \tau|^{-\beta}} \, dA_r
\]

\[
\leq 2^3 \delta^{-\beta} \omega(\delta) \int_{D_\epsilon - B_\delta(z_3)} \frac{1}{|\tilde{z}_r - \tau|^3 |z_2 - \tau|^{-\beta}} \, dA_r.
\]

Using (33), it can be seen that \(|z_3 - \tau| \leq 2|\tilde{z}_r - \tau|\). Also

\[
|z_2 - \tau| \leq \frac{\delta}{2} + |z_3 - \tau| \leq \frac{3|z_3 - \tau|}{2}.
\]

Using these, the above integral can be written as

(35)

\[
\int_{D_\epsilon - B_\delta(z_3)} \frac{\omega(|\tau - z_2|)}{|\tilde{z}_r - \tau|^3} \, d\tau \leq 2^3 \delta^{-\beta} \omega(\delta) \int_{D_\epsilon - B_\delta(z_3)} \frac{3^3 2^3 - \beta}{|z_3 - \tau|^{3-\beta}} \, dA_r
\]

\[
\leq 2^3 3^3 \delta^{-\beta} \omega(\delta) \int_{|z_3 - \tau| \geq \delta} \frac{1}{|z_3 - \tau|^{3-\beta}} \, dA_r
\]

\[
= 2^4 3^3 \delta^{-\beta} \omega(\delta) \pi \frac{\delta^{-1+\beta}}{1 - \beta}
\]

\[
\leq C\delta^{-1} \omega(\delta).
\]

Now using (35) in (34) we get

(36)

\[
\left| \int_{D_\epsilon - B_\delta(z_3)} \left[ \frac{1}{(z_1 - \tau)^2} - \frac{1}{(z_2 - \tau)^2} \right] (q(z_2) - q(\tau)) \, dA_r \right| \leq C_1 \|q\|_{C^\omega(\overline{\Pi}_1)} \omega(\delta).
\]

Using (31), (32), (36) and (16) in (29), we get

\[
|\partial p(z_1) - \partial p(z_2)| \leq K \|q\|_{C^\omega(\overline{\Pi}_1)} \sigma(\delta)
\]

or

\[
\|\partial p\|_{C^\omega(\overline{\Pi}_{i/2})} \leq K(\epsilon) \|q\|_{C^\omega(\overline{\Pi}_1)}.
\]

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Hence we get our required result (28). □

To further understand the regularity properties of solutions to the Cauchy-Riemann equation and the Beltrami equation, it is convenient to make use of analytic properties of operators that are defined in the whole complex plane. For any bounded function \( \phi \) with compact support on \( \mathbb{C} \), the Cauchy transform is defined by

\[
(C\phi)(z) := \frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(\tau)}{(z-\tau)} dA_{\tau}
\]

and the Beurling transform is the singular integral operator defined by

\[
(S\phi)(z) := -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(\tau)}{(z-\tau)^2} dA_{\tau}.
\]

When \( \phi \) is integrable in a domain \( \Omega \subset \mathbb{C} \) with compact support in \( \Omega \) and if \( \phi \) is extended by zero outside \( \Omega \), then the domain potential of \( \phi \) coincides with the Cauchy transform of \( \phi \). The Cauchy transform \( p = Cq \) solves the C-R equation (21) when \( q \) is bounded and has compact support. Note that using (23) in Lemma 1, we can conclude that for any \( \phi \in C^\omega(U) \) with compact support in \( K \subset U \) and extended by zero outside \( K \),

\[
(\partial_z C\phi)(z) = S\phi(z) \quad \text{for } z \in \mathbb{C}.
\]

Also for any \( \phi \in C^{1,\omega}(U) \) with compact support in \( K \subset U \),

\[
\phi(z) = C(\bar{\partial}\phi(z)) \quad \text{for } z \in \mathbb{C}.
\]
Proposition 1. For any domain $\Omega$ and $q \in C^\omega(\Omega)$, every solution $p \in W^{1,2}_{\text{loc}}(\Omega)$ of the equation (21) lies in $C^{1,\sigma}(\Omega)$ and satisfies the estimate

$$\|\partial p\|_{C^\sigma(\mathbb{D}_\epsilon)} + \|\partial p\|_{C^\sigma(\mathbb{D}_{\epsilon/2})} \leq C(\|q\|_{C^\omega(\mathbb{D}_\epsilon)} + \|p\|_{C^\omega(\mathbb{D}_{\epsilon/2})})$$

where $C = C(\epsilon)$ and $\mathbb{D}_\epsilon, \mathbb{D}_{\epsilon/2}$ are concentric disks of radius $\epsilon$ and $\epsilon/2$ respectively that are compactly contained in $\Omega$.

Proof: We first prove that the solution $p$ lies in $C^{1,\sigma}(\Omega)$. Note that it is sufficient to prove that $p$ lies in $C^{1,\sigma}(\mathbb{D}_{\epsilon/2})$. Consider a smooth function $\phi$ with compact support in $\mathbb{D}_\epsilon$ such that $\phi \equiv 1$ in $\mathbb{D}_{\epsilon/2}$. Let $q_1 = \phi q$. Now since $q_1$ has compact support in $\mathbb{D}_\epsilon$ then let $p_1 = C q_1$, which is also the domain potential, so we know by Lemma 2 that $p_1 \in C^{1,\sigma}(\mathbb{D}_{\epsilon/2})$.

Hence for $z \in \mathbb{D}_{\epsilon/2}$, $\partial(p - p_1)(z) = (q - q_1)(z) = 0$. So $(p - p_1)$ is holomorphic in $\mathbb{D}_{\epsilon/2}$ and hence is smooth in $\mathbb{D}_{\epsilon/2}$. Thus $p \in C^{1,\sigma}(\mathbb{D}_{\epsilon/2})$.

Consider the smooth function $\phi$ introduced above. Define $\tilde{p} = \phi p$.

Then

$$\tilde{\partial p} = \partial p + \phi q.$$ 

Since $\partial p + \phi q \in C^\omega(\mathbb{D}_\epsilon)$ with compact support in $\mathbb{D}_\epsilon$, using (40), $\tilde{p} = C(\tilde{\partial p})$ solves (42). For $z \in \mathbb{D}_{\epsilon/2}$, $p(z)$ can be written as

$$p(z) = \tilde{p}(z) = C \tilde{\partial p}(\tau) = \frac{1}{\pi} \int_{\mathbb{D}_\epsilon} \frac{\tilde{\partial p}(\tau)}{(z - \tau)} dA_\tau = \frac{1}{\pi} \int_{\mathbb{D}_\epsilon} \frac{(\partial p + \phi q)(\tau)}{(z - \tau)} dA_\tau.$$ 

Using Lemma 2, $\tilde{p}$ satisfies the estimate

$$\|\partial p\|_{C^\sigma(\mathbb{D}_{\epsilon/2})} + \|\partial p\|_{C^\sigma(\mathbb{D}_{\epsilon/2})} \leq K \|\partial p + \phi q\|_{C^\omega(\mathbb{D}_{\epsilon/2})}.$$ 

But $\tilde{p} = p$ in $\mathbb{D}_{\epsilon/2}$. Hence

$$\|\partial p\|_{C^\sigma(\mathbb{D}_{\epsilon/2})} + \|\partial p\|_{C^\sigma(\mathbb{D}_{\epsilon/2})} \leq K(\|\partial p\|_{C^\omega(\mathbb{D}_{\epsilon})} + \|\phi q\|_{C^\omega(\mathbb{D}_{\epsilon})}).$$
For $z_1, z_2 \in \overline{\mathbb{D}}_\epsilon$,

$$
\frac{|(\bar{\partial}\phi p)(z_1) - (\bar{\partial}\phi p)(z_2)|}{\omega(|z_1 - z_2|)} = \frac{|\bar{\partial}\phi(z_1)(p(z_1) - p(z_2)) + p(z_2)(\bar{\partial}\phi(z_1) - \bar{\partial}\phi(z_2))|}{\omega(|z_1 - z_2|)} \leq |\bar{\partial}\phi(z_1)| \frac{|p(z_1) - p(z_2)|}{\omega(|z_1 - z_2|)} + |p(z_2)| \frac{|\bar{\partial}\phi(z_1) - \bar{\partial}\phi(z_2)|}{\omega(|z_1 - z_2|)}.
$$

We now get

$$
\|\bar{\partial}\phi p\|_{C_\omega(\overline{\mathbb{D}}_\epsilon)} \leq \|\bar{\partial}\phi\|_{C^0(\overline{\mathbb{D}}_\epsilon)} \|p\|_{C_\omega(\overline{\mathbb{D}}_\epsilon)} + \|p\|_{C^0(\overline{\mathbb{D}}_\epsilon)} \|\bar{\partial}\phi\|_{C_\omega(\overline{\mathbb{D}}_\epsilon)} \leq C_1\|p\|_{C_\omega(\overline{\mathbb{D}}_\epsilon)}
$$

(44)

where $C_1 = C_1(\epsilon)$. Similarly we can see that $\|\phi q\|_{C_\omega(\overline{\mathbb{D}}_\epsilon)} \leq C_2\|q\|_{C_\omega(\overline{\mathbb{D}}_\epsilon)}$ where $C_2 = C_2(\epsilon)$. Using this and (44) in (43), we get the desired result (41).

**Proof of Theorem 1:** As in [3], we first reduce the Beltrami equation (19) to an inhomogeneous Cauchy-Riemann equation and then try to obtain the bounds for the solution of (19) using the solution of the inhomogeneous Cauchy-Riemann equation. For any constants $a, b,$ define

$$
p(z) := f(\xi) + b\bar{f}(\xi), \quad \text{where } \xi = z + a\bar{z}
$$

(45)

which can be written as

$$
f(\xi) = \frac{p(z) - b\bar{p}(z)}{1 - |b|^2}, \quad \text{where } z = \frac{\xi - a\bar{\xi}}{1 - |a|^2}.
$$

(46)
Let $u(z) := h(\xi) = h(z + a\bar{z})$. Choose constants $a, b$ such that $f(\xi)$ satisfying (19) can be reduced to the form

$$\partial p(z) = u(z) + ab\overline{u(z)}.$$  

We choose the pair of values for $a$ and $b$ as

$$a = \frac{-2\mu_0}{1 + |\mu_0|^2 - |\nu_0|^2 + (1 + |\mu_0|^2 - |\nu_0|^2)^2 - 4|\mu_0|^2)^{1/2}},$$

$$b = \frac{-2\nu_0}{1 + |\nu_0|^2 - |\mu_0|^2 + (1 - |\nu_0|^2 + |\mu_0|^2 - 4|\mu_0|^2)^{1/2}}.$$  

This gives $|a|, |b| \leq \kappa < 1.$

Using Proposition 1, any solution $p \in W^{1,2}_{\text{loc}}(\Omega)$ of (47) lies in $C^{1,\sigma}(\overline{D}_{\epsilon/2})$ and satisfies

$$\|\partial p\|_{C^\sigma(\overline{D}_{\epsilon/2})} + \|\partial p\|_{C^\sigma(\overline{D}_{\epsilon/2})} \leq K(\epsilon)(\|u\|_{C^\infty(\overline{D}_{\epsilon/2})} + \|p\|_{C^\infty(\overline{D}_{\epsilon/2})}).$$

Using the definition of $u(z)$, the above relation can be written as

$$\|\partial p\|_{C^\sigma(\overline{D}_{\epsilon/2})} + \|\partial p\|_{C^\sigma(\overline{D}_{\epsilon/2})} \leq K(\epsilon)(\|h\|_{C^\infty(\overline{D}_{\epsilon/2})} + \|p\|_{C^\infty(\overline{D}_{\epsilon/2})}).$$

Using (85) and Proposition 1, we can see that any solution $f \in W^{1,2}_{\text{loc}}(\Omega)$ of (19) lies in $C^{1,\sigma}(\overline{D}_{\epsilon/2}).$

Now we estimate $\|\partial f\|_{C^\sigma(\overline{D}_{\epsilon/2})} + \|\partial f\|_{C^\sigma(\overline{D}_{\epsilon/2})}$ in terms of $\|\partial p\|_{C^\sigma(\overline{D}_{\epsilon/2})} + \|\partial p\|_{C^\sigma(\overline{D}_{\epsilon/2})}$. We first note that the map from $z$ to $\xi$ takes unit circles to ellipses which are contained in circles with double the radius. Since $\xi = z + a\bar{z}$, we get

$$|\xi_1 - \xi_2| \leq |z_1 - z_2|(1 + |a|) \leq 2|z_1 - z_2|.$$
This gives

\begin{equation}
\omega(|\xi_1 - \xi_2|) \leq \omega(2|z_1 - z_2|) \leq C\omega(|z_1 - z_2|).
\end{equation}

Again since \( z = \frac{\xi - a\bar{\xi}}{1 - |a|^2} \), we get \(|z_1 - z_2| \leq \frac{|\xi_1 - \xi_2|}{1 - \kappa} \). Hence

\begin{equation}
\sigma(|z_1 - z_2|) \leq \sigma\left(\frac{|\xi_1 - \xi_2|}{1 - \kappa}\right) \leq (1 - \kappa)^{-\beta} \sigma(|\xi_1 - \xi_2|)
\end{equation}

using the definition of sigma. Now let us obtain a relation between sigma norm of \( p \) and sigma norm of \( f \). Let \( \xi = \phi(z) \). Hence (85) can be written as

\[ p(z) = f(\phi(z)) + b\overline{f(\phi(z))}. \]

Now the \( C^\omega \) norm of \( p \) can be written as

\begin{equation}
\|p\|_{C^\omega(\overline{B}_{\epsilon/2})} \leq \|f \circ \phi\|_{C^\omega(\overline{B}_{\epsilon/2})} + |b|\|\overline{f \circ \phi}\|_{C^\omega(\overline{B}_{\epsilon/2})}
\end{equation}

\[ \leq C_1(1 + |b|)\|f\|_{C^\omega(\overline{B}_{\epsilon})} \]

where the last inequality is obtained using (51) and the fact that \( \phi(\overline{B}_{\epsilon/2}) \) is contained in \( \overline{D}_\epsilon \).

Now we get a relation between sigma norm of the derivatives of \( f \) and \( p \). Let the map from \( \xi \) to \( z \) be denoted by \( \psi \). Now using (46), we get

\[ f_\xi(\xi) = \frac{(1 + ab) p_z(z) - (a + b) p_z(\xi)}{(1 - |b|^2)(1 - |a|^2)} \]

\[ = \frac{(1 + ab) p_z\psi(\xi) - (a + b) p_z\psi(\xi)}{(1 - |b|^2)(1 - |a|^2)} \]

where we have used \( z = \psi(\xi) \) in the last equality. Using \( z = \frac{\xi - a\xi}{1 - |a|^2} \), we get \((1 - \kappa)|z| \leq |\xi|\). Choose \( \delta \) such that \( \overline{B}_\delta \) is contained in \( \phi(\overline{B}_{\epsilon/2}) \).
The last inequality is obtained using (50). Using (53), we now get

\[ \| f \|_{C^* (\mathbb{R})} \leq \frac{1 + ab \| p \cdot \psi \|_{C^* (\mathbb{R})}}{(1 - |b|^2)(1 - |a|^2)} + |a + b| \| p \cdot \psi \|_{C^* (\phi (\mathbb{R}))} \].

Using (52), the above inequality can be written as

\[ \| f \|_{C^* (\mathbb{R})} \leq \frac{1 + ab(1 - \kappa)^{-\beta}}{(1 - |b|^2)(1 - |a|^2)} \| p \cdot \psi \|_{C^* (\mathbb{R})} + |a + b| \| p \cdot \psi \|_{C^* (\phi (\mathbb{R}))} \].

Adding (54) and (55), we get

\[ \| f \|_{C^* (\mathbb{R})} \leq \frac{1 + ab(1 - \kappa)^{-\beta}}{(1 - |b|^2)(1 - |a|^2)} \| p \cdot \psi \|_{C^* (\mathbb{R})} + |a + b| \| p \cdot \psi \|_{C^* (\phi (\mathbb{R}))} \].

The last inequality is obtained using (50). Using (53), we now get

\[ \| f \|_{C^* (\mathbb{R})} \leq \frac{K_2(\epsilon)(\| h \|_{C^* (\mathbb{R})} + \| f \|_{C^* (\mathbb{R})})}{(1 - \kappa)^{\beta}(1 + |b|)(1 + |a|)} \].

For any \( D \) compactly contained in \( U \), we now need to get a similar inequality with norms in \( D \) and \( U \). Consider any point \( \xi \in D \). Choose
\( \epsilon \) such that \( \overline{D}_2(\xi) \) is contained in \( U \). Now for \( \xi_1, \xi_2 \in D \) such that 
\( \xi = (\xi_1 + \xi_2)/2 \) and \( |\xi_1 - \xi_2| < \delta \), we get

\[
\frac{|\partial f(\xi_1) - \partial f(\xi_2)|}{\sigma(|\xi_1 - \xi_2|)} \leq \sup_{\eta, \zeta \in \overline{D}_\delta(\xi)} \frac{|\partial f(\eta) - \partial f(\zeta)|}{\sigma(|\eta - \zeta|)}.
\]

Combining the above two inequalities, we get that for any \( \xi \in D \),

\[
\frac{|\partial f(\xi_1) - \partial f(\xi_2)|}{\sigma(|\xi_1 - \xi_2|)} + |\partial f(\xi)| \leq \|\partial f\|_{C^\ast(\overline{D}_2(\xi))}.
\]

Similarly,

\[
\frac{|\partial f(\xi_1) - \partial f(\xi_2)|}{\sigma(|\xi_1 - \xi_2|)} + |\partial f(\xi)| \leq \|\partial f\|_{C^\ast(\overline{D}_2(\xi))}.
\]

Adding the above two inequalities and using (56), we get

\[
\frac{|\partial f(\xi_1) - \partial f(\xi_2)|}{\sigma(|\xi_1 - \xi_2|)} + |\partial f(\xi)| + \frac{|\partial f(\xi_1) - \partial f(\xi_2)|}{\sigma(|\xi_1 - \xi_2|)} + |\partial f(\xi)|
\]

\[
\leq \|\overline{\partial} f\|_{C^\ast(\overline{D}_2(\xi))} + \|\partial f\|_{C^\ast(\overline{D}_2(\xi))}
\]

\[
\leq C(\kappa, \epsilon)(\|h\|_{C^\ast(\overline{D}_2(\xi))} + \|f\|_{C^\ast(\overline{D}_2(\xi))})
\]

\[
\leq C(\kappa, \epsilon)(\|h\|_{C^\ast(\overline{D}_2(\xi))} + \|f\|_{C^\ast(\overline{D}_2(\xi))})
\]

\[
\leq C(\kappa, \epsilon)(\|h\|_{C^\ast(\overline{D}_2(\xi))} + \|f\|_{C^\ast(\overline{D}_2(\xi))})
\]

Taking the supremum over \( \xi \) and dependant \( \xi_1, \xi_2 \), we get

\[
\|\partial f\|_{C^\ast(\overline{D}_2(\xi))} + \|\overline{\partial} f\|_{C^\ast(\overline{D}_2(\xi))} \leq C(\kappa, D, U)(\|h\|_{C^\ast(\overline{D}_2(\xi))} + \|f\|_{C^\ast(\overline{D}_2(\xi))}.
\]

\[ \square \]

In case of the Beltrami equation with variable coefficients, to obtain the estimates on domains \( D, U \) compactly contained in \( \Omega \), the standard method for generalising from constant to variable coefficients is the following: We first freeze the coefficients at a point in \( \Omega \) and consider functions with support in a small disk around this point: the disk is
chosen such that the difference between the variable coefficients and value of the coefficient at the center of the disk are sufficiently small.

We now look at the Beltrami equation satisfied by this function and then obtain estimates similar to (20). In obtaining this estimate, the standard absorption trick cannot be used since it requires having the same norms on the right hand side as the left hand side of the inequality. The norms on the right hand side and left hand side of (20) are not the same. We have the $C^{1,\sigma}$ norm of the solution bounded by the $C^\omega$ norm of the solution. Hence the above mentioned absorption cannot be done in this case.

To overcome this problem, we now look at functions that are less regular than the functions in the $C^\omega$ space but are more regular than the functions in the $C^\sigma$ space. The idea is that when the coefficients of the Beltrami equation are functions with this kind of regularity, we can obtain a result where the norm of the derivative of solution to the Beltrami equation is bounded by the same norm of the solution. This space of functions is defined using Fourier multipliers.

We first work with functions on a torus, obtain the required estimates to solutions of the Beltrami equation and then use this to get estimates on any bounded domain. We work with functions on a torus rather than the whole complex plane since this gives us boundedness of the Beurling transform which will be a key tool used in the next section.

4. The space $C^{(\lambda)}$ on the torus

In [15], the function space $C^{(\lambda)}(\mathbb{R}^n)$ with

\begin{equation}
\lambda(j) = \omega(2^{-j})
\end{equation}

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was defined for a general modulus of continuity $\omega$ using a Littlewood-Paley decomposition $\{\psi_j\}$ as the collection of all $f \in \mathbb{R}^n$ such that

$$\| (\psi_j(D)f) \|_{L^\infty(\mathbb{R}^n)} \leq C\lambda(j),$$

where $\psi_j(D)$ is the Fourier multiplier associated with the Littlewood-Paley partition of unity $\psi_j(\xi)$. $C^{(\lambda)}(\mathbb{R}^n)$ is a Banach space under the norm

$$\| f \|_{C^{(\lambda)}(\mathbb{R}^n)} := \sup_j \| (\psi_j(D)f) \|_{L^\infty(\mathbb{R}^n)} \lambda(j).$$

The classical results in Fourier analysis on $\mathbb{R}^n$ were used in [15] to analyse these functions and their properties. We can use these results to study functions on a torus by considering them as periodic functions on $\mathbb{R}^2$.

Let $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ denote the 2-dimensional torus. We identify $\mathbb{T}^2$ with $[-\pi, \pi) \times [-\pi, \pi) \subset \mathbb{R}^2$ and functions on $\mathbb{T}^2$ with functions on $\mathbb{R}^2$ that are $2\pi$-periodic in each of the coordinate directions.

For functions on a torus, the toroidal Fourier transform is defined from $C^\infty(\mathbb{T}^2)$ to $S(\mathbb{Z}^2)$ where $S(\mathbb{Z}^2)$ denotes the space of rapidly decaying functions on $\mathbb{Z}^2$. However, in our work we will be using the Fourier transform on $\mathbb{R}^2$. But since periodic functions on $\mathbb{R}^2$ are not $L^1$, while taking the Fourier transform, we consider them as tempered distributions, for which the Fourier transform is given by (10).

In order to define the $C^{(\lambda)}$ space, we first need a smooth Littlewood-Paley decomposition which is obtained as follows: Choose a compactly
supported function $\Psi_0$ such that

$$\Psi_0(\xi) = \begin{cases} 1 \text{ for } |\xi| \leq 1 \\ 0 \text{ for } |\xi| \geq 2 \end{cases}$$

for $\xi \in \mathbb{R}^2$. For $k \geq 1$, set

$$\Psi_k(\xi) = \Psi_0(2^{-k}\xi). \quad (58)$$

Let

$$\psi_0(\xi) = \Psi_0(\xi) \quad \text{and} \quad \psi_k(\xi) = \Psi_k(\xi) - \Psi_{k-1}(\xi) \quad \text{for } k \geq 1 \quad (59)$$

so that

$$\Psi_k(\xi) = \psi_0(\xi) + \ldots + \psi_k(\xi) \quad \text{and} \quad \sum_{k=0}^{\infty} \psi_k(\xi) = 1. \quad (60)$$

The collection $\{\psi_k(\xi)\}$ forms a Littlewood-Paley partition of unity. Note that for each $k$, $\Psi_k(\xi)$ is supported on a disk $|\xi| < 2^{k+1}$. For $k > 0$, each $\psi_k(\xi)$ is supported on the annulus $2^{k-1} < |\xi| < 2^{k+1}$ and $\psi_0(\xi)$ is supported on $|\xi| < 2$.

Since $\psi_j(\xi)$ has compact support, we consider $\psi_j(D) : S'(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ as a Fourier multiplier operator associated with the function $\psi_j(\xi)$. Here since we want to define $\psi_j(D)$ on periodic functions on $\mathbb{R}^2$, we again consider the functions as tempered distributions. It can also be written as a convolution given by

$$(\psi_j(D)f)(z) = \int_{\mathbb{R}^2} \psi_j^{\vee}(\tilde{z}) \hat{f}(z - \tilde{z}) d\tilde{z}.$$ 

Note that a Fourier multiplier operator preserves periodicity, as can be seen from the following argument: Let $f$ be a function on $\mathbb{R}^2$ with
period $2\pi$ and $P(D)$ be a Fourier multiplier operator with symbol $p(\xi)$. Then we have

\[
(P(D)f)(z) = \int_{\mathbb{R}^2} p^\vee(\tilde{z}) f(z - \tilde{z}) d\tilde{z} \\
= \int_{\mathbb{R}^2} p^\vee(\tilde{z}) f(z - \tilde{z} + 2\pi) d\tilde{z} \\
= \int_{\mathbb{R}^2} p^\vee(\tilde{z}) f((z + 2\pi) - \tilde{z}) d\tilde{z} \\
= (P(D)f)(z + 2\pi).
\]

Hence $\psi_j(D)$ can be considered as an operator on the torus.

We are now ready to define the function space $C(\lambda)(\mathbb{T}^2)$. We assume that $\omega$ is a Dini modulus of continuity, i.e. satisfies the Dini condition (11).

**Definition 5.** $C(\lambda)(\mathbb{T}^2)$ is the collection of all periodic functions $f \in \mathbb{R}^2$ such that

\[
\|\psi_j(D)f\|_{L^\infty(\mathbb{R}^2)} \leq C\lambda(j)
\]

and is a Banach space under the norm

\[
\|f\|_{C(\lambda)(\mathbb{T}^2)} := \sup_j \frac{\|\psi_j(D)f\|_{L^\infty(\mathbb{R}^2)}}{\lambda(j)}.
\]

The space $C(\lambda)(\mathbb{T}^2)$ is identified with $C_p(\lambda)(\mathbb{R}^2)$ where the subscript $P$ refers to periodic functions on $\mathbb{R}^2$.

**Definition 6.** We denote by $C^{1,\lambda}(\mathbb{T}^2)$, those functions $f \in C^1(\mathbb{T}^2)$ whose first order derivatives $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ are in $C(\lambda)(\mathbb{T}^2)$. $C^{1,\lambda}(\mathbb{T}^2)$ is a Banach space under the norm

\[
\|f\|_{C^{1,\lambda}(\mathbb{T}^2)} := \|\partial f\|_{C(\lambda)(\mathbb{T}^2)} + \|\bar{\partial} f\|_{C(\lambda)(\mathbb{T}^2)} + \|f\|_{C^0(\mathbb{T}^2)}.
\]
The relation between the spaces \( C(\lambda), C^{\omega} \) and \( C^{\sigma} \) that holds on \( \mathbb{R}^n \) (as shown in [15]) also holds in the case of a torus \( T^2 \) i.e.

\[
C^{\omega}(T^2) \subset C(\lambda)(T^2) \subset C^{\sigma}(T^2). \tag{64}
\]

This is true since the spaces \( C^{\omega}(T^2), C(\lambda)(T^2) \) and \( C^{\sigma}(T^2) \) are just the periodic functions on \( \mathbb{R}^2 \) which belong to the corresponding spaces on \( \mathbb{R}^2 \).

We observe here that \( \lambda(j) \) has the following properties:

**Lemma 3.** The positive decreasing sequence \( \lambda(j) \) given by \( \lambda(j) = \omega(2^{-j}) \) satisfies the following properties:

\[
\sum_j \lambda(j) < \infty, \tag{65}
\]

\[
\sum_{j \geq l} \lambda^2(j) \leq c\lambda(l), \tag{66}
\]

and \( \lambda(j) \) is slowly varying, i.e.

\[
\lambda(j) \leq C\lambda(j+1). \tag{67}
\]

**Proof:**

Proof of (65):

\[
\int_0^1 \frac{\omega(t)}{t} dt \geq \sum_{j=0}^{\infty} \frac{\omega(2^{-j})}{2^{-j}} (2^{-j} - 2^{-(j+1)}) = \frac{1}{2} \omega(2^{-j}) = \frac{1}{2} \sum_j \lambda(j).
\]

Using (11), we get \( \sum_{j=0}^{\infty} \lambda(j) < \infty \).

Proof of (66): Since \( \lambda(j) \) is a decreasing sequence, we can write

\[
\sum_{j \geq l} \lambda^2(j) \leq \lambda(l) \sum_{j \geq l} \lambda(j).
\]
Using (65), we get the final result.

Proof of (67): Using (13), we get

\[ \lambda(j) = \omega(2^{-j}) \leq 2^\beta \omega(2^{-(j+1)}) \leq C \lambda(j + 1). \]

\[ \square \]

A tool that we will use extensively in this section is the Beurling transform. It is defined for functions on a torus as a Fourier multiplier operator by

\[ (\widehat{Sf})(\xi) = m(\xi) \hat{f}(\xi) \text{ where } m(\xi) = \xi/\bar{\xi} \text{ for } \xi \in \mathbb{R}^2. \]

In working with Fourier multiplier operators, convolution of functions will be an operation that we will use frequently in this section. We shall make use of the following property of convolution of functions:

If \( f \in L^1(\mathbb{R}^2), \ g \in L^\infty(\mathbb{R}^2) \) and one of them has compact support, then

\[ \|f * g\|_{L^\infty(\mathbb{R}^2)} \leq \|f\|_{L^1(\mathbb{R}^2)} \|g\|_{L^\infty(\mathbb{R}^2)}. \]

We now prove the boundedness of the Beurling transform which plays a vital role in obtaining our estimates.

**Lemma 4.** If \( h \in C^{(\lambda)}(\mathbb{T}^2) \), then

\[ \|Sh\|_{C^{(\lambda)}(\mathbb{T}^2)} \leq C \|h\|_{C^{(\lambda)}(\mathbb{T}^2)}. \]

**Proof:** We first observe that the symbol of \( S \) has a singularity at \( \xi = 0 \). We handle this singularity by writing \( S \) in terms of a smoothing operator. Using the Littlewood-Paley partition of unity, we get

\[ S = \sum_{k=0}^{\infty} \psi_k(D)S = \psi_0(D)S + \sum_{k=1}^{\infty} \psi_k(D)S. \]
Let \( \tilde{S} := \sum_{k=1}^{\infty} \psi_k(D)S \). We now have

\[
\|Sh\|_{C^{(\lambda)}(\mathbb{T}^2)} \leq \|(\psi_0(D)S)h\|_{C^{(\lambda)}(\mathbb{T}^2)} + \|\tilde{S}h\|_{C^{(\lambda)}(\mathbb{T}^2)}
\]

(71)

Observe that as the symbol of \( \psi_0(D)S \) has compact support, \( \psi_0(D)S \) is a smoothing operator which takes a periodic function \( h \in C^{(\lambda)}(\mathbb{R}^2) \) to a smooth periodic function and so is bounded from \( C^{(\lambda)}(\mathbb{R}^2) \) to \( C^{(\lambda)}(\mathbb{T}^2) \), i.e.

\[
\|\psi_0(D)Sh\|_{C^{(\lambda)}(\mathbb{T}^2)} \leq C_2 \|h\|_{C^{(\lambda)}(\mathbb{T}^2)}.
\]

(72)

Let us now look at the boundedness of \( \tilde{S} \). To estimate \( \|\tilde{S}h\|_{C^{(\lambda)}(\mathbb{T}^2)} \), we first consider

\[
\|\psi_l(D)\tilde{S}h\|_{L^{\infty}(\mathbb{R}^2)} = \|\psi_l(D)\sum_{k=1}^{\infty} \psi_k(D)Sh\|_{L^{\infty}(\mathbb{R}^2)}
\]

\[
= \left\| \mathcal{F}^{-1}\left( \psi_l(\xi) \sum_{k=1}^{\infty} \frac{\psi_k(\xi)}{\xi} \mathcal{F}h \right) \right\|_{L^{\infty}(\mathbb{R}^2)}.
\]

Using the fact that \( \psi_l(\xi) \) is supported only in \( 2^{l-1} \leq |\xi| \leq 2^{l+1} \), we get

\[
\|\psi_l(D)\tilde{S}h\|_{L^{\infty}(\mathbb{R}^2)} = \left\| \mathcal{F}^{-1}\left( \sum_{k=l+1}^{k+1} \psi_k(\xi) \frac{\psi(\xi)}{\xi} \mathcal{F}h \right) \right\|_{L^{\infty}(\mathbb{R}^2)}
\]

\[
= \left\| \mathcal{F}^{-1}\left( \sum_{k=l+1}^{k+1} \psi_k(\xi) \frac{\psi(\xi)}{\xi} \mathcal{F}h \right) \right\|_{L^{\infty}(\mathbb{R}^2)}
\]

\[
= \left\| \mathcal{F}^{-1}\left( \sum_{k=l+1}^{k+1} \psi_k(\xi) \frac{\psi(\xi)}{\xi} \right) * \mathcal{F}^{-1}(\psi_l(\xi)\mathcal{F}h) \right\|_{L^{\infty}(\mathbb{R}^2)}.
\]

We now use (69) to get

\[
\|\psi_l(D)\tilde{S}h\|_{L^{\infty}(\mathbb{R}^2)} \leq \left\| \mathcal{F}^{-1}\left( \sum_{k=l+1}^{k+1} \psi_k(\xi) \frac{\psi(\xi)}{\xi} \right) \right\|_{L^{1}(\mathbb{R}^2)} \left\| \mathcal{F}^{-1}(\psi_l(\xi)\mathcal{F}h) \right\|_{L^{\infty}(\mathbb{R}^2)},
\]

(73)
Let us now estimate the term \( \| \mathcal{F}^{-1} \left( \sum_{k=l-1}^{k=l+1} \psi_k(\xi) \frac{\xi}{\xi} \right) \|_{L^1(\mathbb{R}^2)} \).

Using the definition of the inverse Fourier transform, (58) and (59), we get

\[
\mathcal{F}^{-1} \left( \sum_{k=l-1}^{k=l+1} \psi_k(\xi) \frac{\xi}{\xi} \right) = \int_{\mathbb{R}^2} (\Psi_0(2^{-l+1} \xi) - \Psi_0(2^{-l-2} \xi)) \frac{\xi}{\xi} e^{iz \cdot \xi} d\xi
\]

Let \( \tilde{\Psi}(\eta) := \Psi_0(2^{-1} \eta) - \Psi_0(2^{2} \eta) \). Observe that \( \tilde{\Psi}(\eta) \) is supported in \( 2^{-1} < |\eta| < 2^2 \). The above expression can now be written as

\[
\mathcal{F}^{-1} \left( \sum_{k=l-1}^{k=l+1} \psi_k(\xi) \frac{\xi}{\xi} \right) \leq \int_{\mathbb{R}^2} \tilde{\Psi}(\eta) \frac{\eta}{\eta} e^{iz \cdot \eta} d\eta 
\]

where \( m(\eta) = \frac{\eta}{\eta} \).

Now the \( L_1 \) norm of \( \mathcal{F}^{-1} \left( \sum_{k=l-1}^{k=l+1} \psi_k(\xi) \frac{\xi}{\xi} \right) \) can be written as

\[
\left\| \mathcal{F}^{-1} \left( \sum_{k=l-1}^{k=l+1} \psi_k(\xi) \frac{\xi}{\xi} \right) \right\|_{C(\mathbb{R}^2)} = \left\| 2^{l} \mathcal{F}^{-1}(m \tilde{\Psi})(2^l z) \right\|_{C(\mathbb{R}^2)}
\]

\[
= \int_{\mathbb{R}^2} 2^{2l} \left( \mathcal{F}^{-1}(m \tilde{\Psi})(2^l z) \right) d\xi
\]

\[
= 2^{2l} \int_{\mathbb{R}^2} \left( \mathcal{F}^{-1}(m \tilde{\Psi})(2^l z) \right) 2^{-2l} d\xi
\]

\[
= \left\| \mathcal{F}^{-1}(m \tilde{\Psi}) \right\|_{L^1(\mathbb{R}^2)} \leq C
\]
where $C$ is independent of $l$. This is true due to the fact that as $m\tilde{\Psi}$ is smooth and has compact support, it is in Schwartz class and hence in $L^1(\mathbb{R}^2)$.

Hence

$$\|\psi_l(D)\tilde{S}h\|_{L^\infty(\mathbb{R}^2)} \leq C_1\|\psi_l(D)h\|_{L^\infty(\mathbb{R}^2)}$$

which gives

(74) \[ \|\tilde{S}h\|_{C^0(\mathbb{T}^2)} \leq C_2\|h\|_{C^0(\mathbb{T}^2)}. \]

Using (72) and (74) in (71), we get

(75) \[ \|S\tilde{h}\|_{C^0(\mathbb{T}^2)} \leq C\|h\|_{C^0(\mathbb{T}^2)}. \]

Taylor had shown in [15] that $C^0(\mathbb{R}^2)$ is a Banach algebra. The following lemma uses a slightly different proof to show that $C^0(\mathbb{T}^2)$ is a Banach algebra.

**Lemma 5.** Assume $f,g \in C^0(\mathbb{T}^2)$. Then

(76) \[ \|fg\|_{C^0(\mathbb{T}^2)} \leq C\|f\|_{C^0(\mathbb{T}^2)}\|g\|_{C^0(\mathbb{T}^2)}. \]

**Proof:** Here we use Bony’s paraproduct decomposition. For the product $fg$, it is given by

$$fg = T_f g + T_g f + R_f g$$

where $T_f$ is Bony’s paraproduct defined by

$$T_f g = \sum_{k \geq 5} \Psi_{k-5}(D)f \cdot \psi_{k+1}(D)g$$
and $R_fg$, which is used to denote the remainder is given by

$$R_fg = \sum_j \phi_j(D)f \cdot \psi_j(D)g$$

where $\phi_j(D)f = \sum_{|j-k|\leq 3} \psi_k(D)f$

Hence

$$(77) \quad \|fg\|_{C(\lambda)(T^2)} \leq \|Tfg\|_{C(\lambda)(T^2)} + \|Tgf\|_{C(\lambda)(T^2)} + \|R_fg\|_{C(\lambda)(T^2)}.$$

Let us first estimate $\|R_fg\|_{C(\lambda)(T^2)}$. We have

$$\|(\psi_l(D)R_fg)\|_{L^\infty(\mathbb{R}^2)} = \|\psi_l(\sum_j \phi_j(D)f \cdot \psi_j(D)g)\|_{L^\infty(\mathbb{R}^2)}$$

$$= \|\psi_l(D) \sum_j (\sum_{|j-k|\leq 3} \psi_k(D)f) \cdot \psi_j(D)g\|_{L^\infty(\mathbb{R}^2)}.$$

Now let us consider the term in the right hand side of the above equation. The transform of the $j^{th}$ term of $R_fg$ is

$$\mathcal{F}(\sum_{|j-k|\leq 3} \psi_k(D)f \cdot \psi_j(D)g)(\xi)$$

$$= \mathcal{F}((\psi_{j-3} + \ldots + \psi_{j+3})(D)f.\psi_j(D)g)(\xi)$$

$$= [(\psi_{j-3} + \ldots + \psi_{j+3}) * \hat{f}) * (\psi_j * \hat{g})](\xi)$$

$$= |\psi_l(\xi) \sum_j [((\psi_{j-3} + \ldots + \psi_{j+3}) * \hat{f}) * (\psi_j * \hat{g})](\xi)]|$$

The term $\sum_j (\psi_{j-3} + \ldots + \psi_{j+3})(D)f$ is supported in $2^{l-4} \leq |\xi| \leq 2^{l+4}$ and the support of $\psi_j(D)g$ is in $2^{j-1} \leq |\xi| \leq 2^{j+1}$. Hence the support of transform of the $j^{th}$ term of $R_fg$ is in $|\xi| \leq C2^{j+4}$. Now $\psi_l(\xi)$ is supported in $2^{l-1} < |\xi| \leq 2^{l+1}$. Hence for the annulus to intersect the disc $|\xi| \leq 2^{l+4}$, we need $C2^{j+4} \geq 2^{l-1}$. We can now say that there
exists some integer \( N \) such that \( j \geq l - N \). Hence we can write

\[
\| \psi_l(D) R_f g \|_{L^\infty(\mathbb{R}^2)} = \| (\psi_l(D) \sum_{j \geq l-N} (\sum_{|j-k| \leq 3} \psi_k(D) f) \psi_j(D) g) \|_{L^\infty(\mathbb{R}^2)}
\]

\[
= \| \mathcal{F}^{-1}(\psi_l(\xi)) \ast \sum_{j \geq l-N} (\mathcal{F} \sum_{|j-k| \leq 3} \psi_k(D) f) \psi_j(D) g) \|_{L^\infty(\mathbb{R}^2)}.
\]

Now we can use (69) to get

\[
\| \psi_l(D) R_f g \|_{L^\infty(\mathbb{R}^2)}
\]

\[
\leq \| \psi_l \|_{L^1(\mathbb{R}^2)} \sum_{j \geq l-N} \| (\psi_{j-3} + \ldots + \psi_{j+3})(D) f) \|_{L^\infty(\mathbb{R}^2)} \| \psi_j(D) g \|_{L^\infty(\mathbb{R}^2)}
\]

\[
\leq C \| f \|_{C^{(\lambda)}(T^2)} \| g \|_{C^{(\lambda)}(T^2)} \sum_{j \geq l-N} (\lambda(j - 3) + \ldots + \lambda(j + 3)) \lambda(j)
\]

\[
\leq C \| f \|_{C^{(\lambda)}(T^2)} \| g \|_{C^{(\lambda)}(T^2)} \sum_{j \geq l-N} (k_1 \lambda(j) + k_2 \lambda(j) + k_2 \lambda(j) + 4 \lambda(j)) \lambda(j)
\]

\[
\leq C \| f \|_{C^{(\lambda)}(T^2)} \| g \|_{C^{(\lambda)}(T^2)} \sum_{j \geq l-N} C_1 \lambda^2(j).
\]

The last inequality is obtained using (67) and the property that \( \lambda(j) \) is a positive decreasing sequence. Now \( \sum_{j \geq l-N} (k_1 \lambda(j) + k_2 \lambda(j) + k_2 \lambda(j) + 4 \lambda(j)) \lambda(j) \leq C_1 \lambda^2(j) \). We now use (66) to get

\[
\| (\psi_l(D) R_f g \|_{L^\infty(\mathbb{R}^2)} \leq C_2 \| f \|_{C^{(\lambda)}(T^2)} \| g \|_{C^{(\lambda)}(T^2)} \lambda(l).
\]

Using the above inequality we get

\[
R_f g \|_{C^{(\lambda)}(T^2)} \leq K_1 \| f \|_{C^{(\lambda)}(T^2)} \| g \|_{C^{(\lambda)}(T^2)}.
\]

Now let us estimate \( T_f g \|_{C^{(\lambda)}(T^2)} \). We have

\[
\| (\psi_l(D) T_f g \|_{L^\infty(\mathbb{R}^2)} = \| \psi_l(D) \sum_{k \geq 5} \Psi_{k-5}(D) f \psi_{k+1}(D) g \|_{L^\infty(\mathbb{R}^2)}
\]

\[
= \| \psi_l(D) \sum_{k \geq 5} (\psi_0 + \ldots + \psi_{k-5})(D) f \psi_{k+1}(D) g \|_{L^\infty(\mathbb{R}^2)}
\]

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Since $\psi_l$ is supported in the annulus $2^{l-1} \leq |\xi| \leq 2^{l+1}$, we must have $k - 5 \geq l - 1$ or $k \geq l + 4$. Let $m = \min(k - 5, l + 1)$. Now we have

$$\|(\psi_l(D)T_f g)\|_{L^\infty(\mathbb{R}^2)}$$

$$= \|\psi_l(D) \sum_{k \geq l+4} \sum_{j = l-1}^m \psi_j(D) f \psi_k+1(D) g\|_{L^\infty(\mathbb{R}^2)}$$

$$\leq \|\psi_l\|_{L^1(T^2)} \sum_{k \geq l+4} \|\psi_j(D) f\|_{L^\infty(\mathbb{R}^2)} \|\psi_k+1(D) g\|_{L^\infty(\mathbb{R}^2)}$$

$$\leq C \|f\|_{C^{(\lambda)}(T^2)} \|g\|_{C^{(\lambda)}(T^2)} \sum_{k \geq l+4} (\sum_{j = l-1}^m \lambda(j)) \lambda(k + 1)$$

$$\leq C \|f\|_{C^{(\lambda)}(T^2)} \|g\|_{C^{(\lambda)}(T^2)} \sum_{k \geq l+4} \lambda(l - 1) + \lambda(l) + \lambda(l + 1) \lambda(k + 1)$$

$$\leq C \|f\|_{C^{(\lambda)}(T^2)} \|g\|_{C^{(\lambda)}(T^2)} \sum_{k \geq l+4} C_1 \lambda(l) \lambda(k + 1)$$

$$\leq C_2 \|f\|_{C^{(\lambda)}(T^2)} \|g\|_{C^{(\lambda)}(T^2)} \lambda(l)$$

The last inequality is obtained by using $\sum_{k \geq l+4} \lambda(k + 1) \leq C$ for some constant $C$. We now get

$$(79) \quad \|T_f g\|_{C^{(\lambda)}(T^2)} \leq K_2 \|f\|_{C^{(\lambda)}(T^2)} \|g\|_{C^{(\lambda)}(T^2)}.$$  

Similarly we can get

$$(80) \quad \|T_g f\|_{C^{(\lambda)}(T^2)} \leq K_3 \|f\|_{C^{(\lambda)}(T^2)} \|g\|_{C^{(\lambda)}(T^2)}.$$  

Using (78), (79) and (80) in (77), we get (76).

We now obtain a result on the $\bar{\partial}$ operator which will be useful in analysing the Beltrami equation.

**Proposition 2.** The operator $\bar{\partial} : C^{1,(\lambda)} \rightarrow C^{(\lambda)}$ is Fredholm of index zero with kernel and cokernel consisting of constants.
Proof: Define a parametrix \( P \) for \( \bar{\partial} \) by \( \bar{\partial}h = (\phi(\xi) \hat{h}(\xi))^{\vee} \) where \( \phi \) is a smooth function satisfying \( \phi(0) = 0 \) and \( \phi(\xi) = 1 \) for \( |\xi| \geq 1 \). Using a similar proof as in lemma 4 for the Beurling transform, it can be shown that for \( h \in C^{(\lambda)}(\mathbb{T}^2) \),

\[
\|P h\|_{C^{1,\lambda}(\mathbb{T}^2)} \leq C \|h\|_{C^{\lambda}(\mathbb{T}^2)}.
\]

Now

\[
\bar{\partial}P h = (\phi(\xi) \hat{h}(\xi))^{\vee} = ((1 - \Phi(\xi)) \hat{h}(\xi))^{\vee}
\]

where \( \Phi \) is a smooth function with compact support defined by \( 1 - \Phi(\xi) = \phi(\xi) \). Hence \( P \bar{\partial} = \bar{\partial}P = I - K \) where \( K = \Phi(D) \). But \( \Phi(D) : C^{(\lambda)}(\mathbb{T}^2) \to C^\infty(\mathbb{T}^2) \) is a smoothing operator. Since \( C^\infty(\mathbb{T}^2) \) is compactly contained in \( C^{(\lambda)}(\mathbb{T}^2) \), we get that \( K \) is a compact operator on \( C^{(\lambda)}(\mathbb{T}^2) \). Hence \( \bar{\partial} : C^{1,\lambda} \to C^{(\lambda)} \) is invertible modulo compact operators and is a Fredholm operator of index zero.

Let \( \bar{\partial}f = 0 \) on \( \mathbb{T}^2 \). Hence \( f \) is analytic and periodic. This implies that \( f \) is a constant. Hence kernel \( \bar{\partial} \) consists of constants.

Let \( \bar{\partial}f = h \). Using integration by parts, we see that

\[
\int_{\mathbb{T}^2} h = \int_{\mathbb{T}^2} \bar{\partial}f = 0.
\]

This shows that \( h \) is orthogonal to the constants. But the cokernel is one-dimensional (since the map has index zero and one-dimensional kernel), so the cokernel is exactly the constants. \( \square \)

We now have the necessary tools to obtain estimates for the constant coefficient Beltrami equation.
Theorem 2. Let $h \in C^{(\lambda)}(\mathbb{T}^2)$ and $\mu_0$ and $\nu_0$ be constants satisfying $|\mu_0| + |\nu_0| \leq \kappa < 1$. Let $L_0$ denote the operator given by $L_0 f = \overline{\partial} f - \mu_0 \partial f - \nu_0 \overline{\partial} f$. Then $L_0 : C^{1,(\lambda)}(\mathbb{T}^2) \to C^{(\lambda)}(\mathbb{T}^2)$ is Fredholm of index zero with kernel and cokernel consisting of constants. Moreover, if $f \in C^{1,(\lambda)}(\mathbb{T}^2)$ satisfies the equation

\begin{equation}
\overline{\partial} f - \mu_0 \partial f - \nu_0 \overline{\partial} f = h,
\end{equation}

then we have the estimate

\begin{equation}
\|\overline{\partial} f\|_{C^{(\lambda)}(\mathbb{T}^2)} + \|\partial f\|_{C^{(\lambda)}(\mathbb{T}^2)} \leq K \|L_0 f\|_{C^{(\lambda)}(\mathbb{T}^2)},
\end{equation}

where $K$ is a constant depending on $\mu_0$ and $\nu_0$.

Proof: We first observe that as in Section 4, for functions on $\mathbb{R}^2$, the equation $L_0 f = h$ can be reduced to an inhomogeneous Cauchy-Riemann equation of the form (47), i.e. we have

\begin{equation}
A^{-1} L_0 A p = \overline{\partial} p.
\end{equation}

Here $A$ is a transformation given by

\begin{equation}
A p(z) := \frac{p(z) - b \overline{p(z)}}{1 - |b|^2} = f(\xi) \quad \text{where} \quad z = \frac{\xi - a \overline{\xi}}{1 - |a|^2}
\end{equation}

and $A^{-1}$ is given by

\begin{equation}
A^{-1} f(\xi) := f(\xi) + b \overline{f(\xi)} = p(z), \quad \text{where} \quad \xi = z + a \overline{z}.
\end{equation}

Note that the constants $a$ and $b$ are chosen as in (48) and (49) such that $|a|, |b| \leq \kappa < 1$. We first prove that there exists constants $c_1$ and $c_2$ such that

\begin{equation}
c_1 \|A p\|_{C^{(\lambda)}(\mathbb{R}^2)} \leq \|p\|_{C^{(\lambda)}(\mathbb{R}^2)} \leq c_2 \|A p\|_{C^{(\lambda)}(\mathbb{R}^2)}.
\end{equation}
This can be proved as follows: Consider

\[
\| f \|_{C^\lambda(\mathbb{R}^2)} = \left\| \frac{p(z) - b \overline{p(z)}}{1 - |b|^2} \right\|_{C^\lambda(\mathbb{R}^2)} \\
\leq \frac{1 + |b|}{1 - |b|^2} \| p \|_{C^\lambda(\mathbb{R}^2)} \\
\leq \frac{1}{1 - |b|} \| p \|_{C^\lambda(\mathbb{R}^2)} \\
\leq \frac{1}{1 - \kappa} \| p \|_{C^\lambda(\mathbb{R}^2)}
\]

Hence we obtain

(87) \[ \| Ap \|_{C^\lambda(\mathbb{R}^2)} \leq \frac{1}{1 - \kappa} \| p \|_{C^\lambda(\mathbb{R}^2)} \]

or

\[ c_1 \| Ap \|_{C^\lambda(\mathbb{R}^2)} \leq \| p \|_{C^\lambda(\mathbb{R}^2)}. \]

Now consider

\[
\| p \|_{C^\lambda(\mathbb{R}^2)} = \| f(\xi) + b \overline{f(\xi)} \|_{C^\lambda(\mathbb{R}^2)} \\
\leq (1 + |b|) \| f \|_{C^\lambda(\mathbb{R}^2)} \\
\leq (1 + \kappa) \| f \|_{C^\lambda(\mathbb{R}^2)}.
\]

This gives the estimate

(88) \[ \| p \|_{C^\lambda(\mathbb{R}^2)} \leq (1 + \kappa) \| Ap \|_{C^\lambda(\mathbb{R}^2)} \leq c_2 \| Ap \|_{C^\lambda(\mathbb{R}^2)}, \]

Hence we have proved (86).

Similarly we can prove

(89) \[ c_3 \| A^{-1} f \|_{C^\lambda(\mathbb{R}^2)} \leq \| f \|_{C^\lambda(\mathbb{R}^2)} \leq c_4 \| A^{-1} f \|_{C^\lambda(\mathbb{R}^2)}.
\]
We now obtain the estimates (82) on $\mathbb{R}^2$. First observe that in Lemma 4, if we let $h = \overline{\partial}p$ and use $S\overline{\partial} = \partial$, we obtain
\[
\|\partial p\|_{C^\lambda(\mathbb{R}^2)} \leq C\|\overline{\partial}p\|_{C^\lambda(\mathbb{R}^2)}.
\]
Using (86), we can now obtain
\[
\|\partial p\|_{C^\lambda(\mathbb{R}^2)} + \|\overline{\partial}p\|_{C^\lambda(\mathbb{R}^2)} \leq C\|\overline{\partial}p\|_{C^\lambda(\mathbb{R}^2)}
= C\|A^{-1}L_0Ap\|_{C^\lambda(\mathbb{R}^2)}
\leq C_1\|L_0Ap\|_{C^\lambda(\mathbb{R}^2)}.
\]
Hence we have
\[
(90) \quad \|\partial p\|_{C^\lambda(\mathbb{R}^2)} + \|\overline{\partial}p\|_{C^\lambda(\mathbb{R}^2)} \leq C_1\|L_0f\|_{C^\lambda(\mathbb{R}^2)}.
\]
Now need to get a relation between the $C^\lambda$ norms of the first order derivatives of $f$ and $p$. Now using (46), we get
\[
f_\xi(\xi) = \frac{(1 + ab)p_\xi(z) - (a + b)p_z(z)}{(1 - |b|^2)(1 - |a|^2)}
\]
Now the $C^\lambda$ norm can be obtained as
\[
(91) \quad \|f_\xi\|_{C^\lambda(\mathbb{R}^2)} \leq \frac{1 + ab\|p_\xi\|_{C^\lambda(\mathbb{R}^2)} + |a + b|\|p_z\|_{C^\lambda(\mathbb{R}^2)}}{(1 - |b|^2)(1 - |a|^2)}
\]
Similarly we can get
\[
(92) \quad \|f_\xi\|_{C^\lambda(\mathbb{R}^2)} \leq \frac{|a + b|\|p_\xi\|_{C^\lambda(\mathbb{R}^2)} + |1 + ab|\|p_z\|_{C^\lambda(\mathbb{R}^2)}}{(1 - |b|^2)(1 - |a|^2)}.
\]
Adding (91) and (92), we obtain
\[
\|f_\xi\|_{C^\lambda(\mathbb{R}^2)} + \|f_\xi\|_{C^\lambda(\mathbb{R}^2)} \leq \frac{\|p_\xi\|_{C^\lambda(\mathbb{R}^2)} + \|p_z\|_{C^\lambda(\mathbb{R}^2)}}{(1 + |b|)(1 + |a|)}
\leq \frac{C_1\|L_0f\|_{C^\lambda(\mathbb{R}^2)}}{(1 + |b|)(1 + |a|)}.
\]
The last inequality is obtained using (90). Hence

\begin{equation}
\|f_\xi\|_{C^1(\lambda;\mathbb{R}^2)} + \|f_\xi\|_{C^1(\lambda;\mathbb{R}^2)} \leq K \|L_0 f\|_{C^1(\lambda;\mathbb{R}^2)}.
\end{equation}

In particular, for periodic functions on $\mathbb{R}^2$, we get the estimate

\begin{equation}
\|\bar{\partial}f\|_{C^1(\lambda;\mathbb{T}^2)} + \|\partial f\|_{C^1(\lambda;\mathbb{T}^2)} \leq K \|L_0 f\|_{C^1(\lambda;\mathbb{R}^2)}.
\end{equation}

which is our required estimate. Now adding the term $\|f\|_{C^0(\mathbb{T}^2)}$ on both sides of the above estimate, we obtain

\begin{equation}
\|f\|_{C^1(\lambda;\mathbb{T}^2)} \leq K (\|L_0 f\|_{C^1(\lambda;\mathbb{T}^2)} + \|f\|_{C^0(\mathbb{T}^2)}) \leq K (\|L_0 f\|_{C^1(\lambda;\mathbb{T}^2)} + \|f\|_{C^1(\lambda;\mathbb{T}^2)}).
\end{equation}

Now $C^1(\lambda;\mathbb{T}^2)$ is compactly contained in $C^1(\lambda;\mathbb{T}^2)$. Since $L_0$ satisfies the above estimate, we can conclude that dimension of kernel of $L_0$ is finite and the range is closed, i.e. $L_0$ is semi-Fredholm. But as we have seen, $A^{-1}L_0A = \bar{\partial}$. Consider the homotopy $G_t = (1-t)\bar{\partial} + tL_0$ for $0 \leq t \leq 1$. Since we know from Proposition (2) that $\bar{\partial}$ is an operator with index 0, using $G_t$, we can conclude that $L_0$ also has index 0. Hence we have proved that $L_0 : C^1(\lambda;\mathbb{T}^2) \to C^1(\lambda;\mathbb{T}^2)$ is Fredholm of index zero.

We use the estimate (82) to prove that the kernel consists of constants. Let $L_0 f = 0$. Then using (82), we obtain

\begin{equation}
\|\bar{\partial}f\|_{C^1(\lambda;\mathbb{T}^2)} + \|\partial f\|_{C^1(\lambda;\mathbb{T}^2)} \leq 0.
\end{equation}

This shows that $f$ has to be a constant and hence the kernel of $L_0$ consists of constants.
Now we prove that the cokernel of $L_0$ consists of constants. Consider $L_0 f = h$. Since $\mu_0$ and $\nu_0$ are constants, we can use integration by parts to obtain

$$\int_{T^2} h \, dz = \int_{T^2} (\overline{\partial f} - \mu_0 \partial f - \nu_0 \overline{\partial f}) \, dz = 0.$$  

This shows that $h$ is orthogonal to the constants. But we have already proved that the map $L_0$ has index zero and that the kernel is one dimensional. This indicates that the cokernel of $L_0$ should also be one-dimensional and hence it is exactly the constants.

We shall hereafter use $L$ to denote the Beltrami operator defined by

$$(95) \quad L f := \overline{\partial f} - \mu \partial f - \nu \overline{\partial f}.$$  

We need the following two lemmas in the proof of Theorem 3 below:

**Lemma 6.** Let $\mu, \nu \in C^\omega(T^2)$ satisfy $|\mu(\xi)| + |\nu(\xi)| \leq \kappa < 1$, for all $\xi \in T^2$. If $f \in W^{1,2}(T^2)$ satisfies $L f = 0$, then $f$ is a constant.

**Proof:** If $f \in W^{1,2}(T^2)$ satisfies $L f = 0$, then

$$\overline{\partial f} = \mu \partial f + \nu \overline{\partial f}.$$  

Taking an inner product with $\overline{\partial f}$ on both sides, we obtain

$$(96) \quad (\overline{\partial f}, \overline{\partial f}) = (\mu \partial f, \overline{\partial f}) + (\nu \overline{\partial f}, \overline{\partial f}).$$  

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But using the definition of inner product, we have

\[(\overline{\partial f}, \overline{\partial f}) = \int_{\mathbb{T}^2} \overline{\partial f}(z) \overline{\partial f}(z) dz \]

\[= \frac{1}{4} \int_{\mathbb{T}^2} (\partial_x + i\partial_y)f(x + iy) (\partial_x - i\partial_y)f(x + iy) dx dy \]

\[= \frac{1}{4} \int_{\mathbb{T}^2} |\partial_x f|^2 + |\partial_y f|^2 + i(\partial_y f \partial_x \overline{f} - \partial_x f \partial_y \overline{f}) dx dy \]

\[= \frac{1}{4} \|\nabla f\|_{L^2(\mathbb{T}^2)}^2 \]

where the term \(\int_{\mathbb{T}^2} (\partial_y f \partial_x \overline{f} - \partial_x f \partial_y \overline{f}) dx dy = 0\), using integration by parts. Hence we can write (96) as

\[\|\nabla f\|^2_{L^2(\mathbb{T}^2)} \leq (\|\mu\|_{L^\infty(\mathbb{T}^2)} + \|\nu\|_{L^\infty(\mathbb{T}^2)}) \|\nabla f\|_{L^2(\mathbb{T}^2)}^2 \]

\[\leq \kappa \|\nabla f\|_{L^2(\mathbb{T}^2)}^2.\]

The above inequality implies that \(\nabla f = 0\), i.e. \(f = 0\). \(\Box\)

**Lemma 7.** Let \(g \in C^\infty(\mathbb{T}^2)\) such that \(\|g - g_0\|_{C^\infty(\mathbb{D}_\epsilon)} < C(\epsilon)\) where \(g_0 = g\) at the center of \(\mathbb{D}_\epsilon\). Then we can find an extension \(\tilde{g}\) of \(g - g_0\) restricted to \(\mathbb{D}_\epsilon\) such that \(\tilde{g}\) has support only in \(\mathbb{D}_{2\epsilon}\) and

\[(97) \quad \|\tilde{g}\|_{C^\infty(\mathbb{T}^2)} \leq \|g - g_0\|_{C^\infty(\mathbb{D}_\epsilon)}.\]
Proof: We obtain the extension $\tilde{g}$ with the desired properties by reflecting the values of $g - g_0$ on $\mathbb{D}_2 \setminus \mathbb{D}_\epsilon$ i.e for each $z \in \mathbb{D}_2 \setminus \mathbb{D}_\epsilon (\xi_0)$, choose $\xi = 2\epsilon \frac{z}{|z|} - z$ on $\mathbb{D}_2 \setminus \mathbb{D}_\epsilon$ so that as $|z| \to \epsilon$, $\xi \to z$ and as $|z|$ approaches the origin, $\xi$ approaches the boundary of $\mathbb{D}_2 \epsilon$. We define $\tilde{g}$ by

$$
\tilde{g}(\xi) = \begin{cases} 
(g - g_0)(\xi) \text{ for } \xi \in \mathbb{D}_\epsilon \\
(g - g_0)(z) \text{ for } \xi \in \mathbb{D}_2 \setminus \mathbb{D}_\epsilon 
\end{cases}
$$

where $\xi$ is the reflection of $z$ in $\mathbb{D}_2 \setminus \mathbb{D}_\epsilon (\xi_0)$. We extend $\tilde{g}$ by zero outside $\mathbb{D}_2 \epsilon$. First we need to show that this extension is $C^\omega$ at the boundary of each of the disks $\mathbb{D}_\epsilon$ and $\mathbb{D}_2 \epsilon$. First we consider the boundary of $\mathbb{D}_\epsilon$. Let $z_0$ be a point on the boundary of $\mathbb{D}_\epsilon$. For $\delta < \epsilon$ and sufficiently small, consider a small disk $\mathbb{D}_\delta(z_0)$.

We just need to consider points on either side of the boundary. For $z$, $\xi_1 \in \mathbb{D}_\delta(z_0)$ such that $z \in \mathbb{D}_\epsilon$ and $\xi_1 \in \mathbb{D}_2 \setminus \mathbb{D}_\epsilon$, we obtain

$$
\|\tilde{g}\|_{C^\omega(\mathbb{D}_\delta(z_0))} = \sup_{\tilde{z} \in \mathbb{D}_\delta(z_0)} |\tilde{g}(\tilde{z})| + \sup_{z, \xi_1 \in \mathbb{D}_\delta(z_0)} \frac{|\tilde{g}(\xi_1) - \tilde{g}(z)|}{\omega(|\xi_1 - z|)}
$$

$$
= \sup_{\tilde{z} \in \mathbb{D}_\delta(z_0)} |\tilde{g}(\tilde{z})| + \sup_{z, \xi_1 \in \mathbb{D}_\delta(z_0)} \frac{|\tilde{g}(z_1) - \tilde{g}(z)|}{\omega(|\xi_1 - z|)}
$$

where $z_1$ is the reflection of $\xi_1$ and $\tilde{z}$ is the reflection of $\tilde{z}$ in $\mathbb{D}_\epsilon$. As $\delta$ tends to 0, $z$ and $\xi_1$ approach the boundary of $\mathbb{D}_\epsilon$ and $z_1$ approaches the boundary of $\mathbb{D}_\epsilon$. Now $|\xi_1 - z| = |2\epsilon \frac{z}{|z|} - z_1 - z| = |z_1 (2\epsilon \frac{1}{|z_1|} - 1) - z|$. But $|z_1| \leq \epsilon$ and $|z_1| \to \epsilon$ as $\delta \to 0$. Hence $2\epsilon \frac{1}{|z_1|} - 1 \geq 1$. We now obtain $|\xi_1 - z| \geq |z_1 - z|$ which gives $\omega(|\xi_1 - z|) \geq \omega(|z_1 - z|)$. Hence we conclude that

$$
\|\tilde{g}\|_{C^\omega(\mathbb{D}_\delta(z_0))} \leq \|g\|_{C^\omega(\mathbb{D}_\epsilon)}
$$
Next we consider the boundary of $\mathbb{D}_{2\epsilon}$. Let $\tilde{z}_0$ be a point on the boundary of $\mathbb{D}_\epsilon$. For $\delta < \epsilon$ and sufficiently small, consider a small disk $\mathbb{D}_\delta(\tilde{z}_0)$. Again we only need to consider points on either side of the boundary. For $\xi_1, \xi_2 \in \mathbb{D}_\delta(\tilde{z}_0)$ such that $\xi_1 \in \mathbb{D}_{2\epsilon}$ and $\xi_2$ lies outside $\mathbb{D}_{2\epsilon}$, we get

$$
\|\tilde{g}\|_{C^\omega(\mathbb{D}_\delta(\tilde{z}_0))} = \sup_{\xi \in \mathbb{D}_\delta(\tilde{z}_0)} |\tilde{g}(\tilde{\xi})| + \sup_{\xi_2, \xi_1 \in \mathbb{D}_\delta(\tilde{z}_0)} \left| \frac{\mu(\xi_1) - \tilde{\mu}(\xi_2)}{\omega(\xi_1 - \xi_2)} \right|
$$

where $\tilde{\xi}$ and $\tilde{z}_1$ are the reflections of $\tilde{\xi}$ and $\xi_1$ (which has already been chosen) respectively in $\mathbb{D}_\epsilon$. Now $\tilde{g}(\xi_2) = 0$ and hence we obtain

$$
\sup_{\xi_2, \xi_1 \in \mathbb{D}_\delta(\tilde{z}_0)} \left| \frac{\tilde{g}(\xi_1) - \tilde{g}(\xi_2)}{\omega(\xi_1 - \xi_2)} \right| = \sup_{\xi_2, \xi_1 \in \mathbb{D}_\delta(\tilde{z}_0)} \left| \frac{g(z_1) - g_0}{\omega(\xi_1 - \xi_2)} \right|
$$

$$
\leq \sup_{\xi_2, \xi_1 \in \mathbb{D}_\delta(\tilde{z}_0)} \frac{\|g\|_{C^\omega(\mathbb{D}_\delta(z_0))} \omega(|z_1 - \xi_0|)}{\omega(\xi_1 - \xi_2)}
$$

Now $|z_1 - \xi_0| \leq |\xi_1 - \xi_2|$ which gives $\omega(|z_1 - \xi_0|) \leq \omega(|\xi_1 - \xi_2|)$. Hence we get $\|\tilde{g}\|_{C^\omega(\mathbb{D}_\delta(\tilde{z}_0))} \leq \|g\|_{C^\omega(\mathbb{D}_\delta(z_0))}$ which is finite.

Now we consider the $C^\omega$ norm of $\tilde{g}$ in $\mathbb{D}_{2\epsilon}$. Using the definition, we have

$$
\|\tilde{g}\|_{C^\omega(\mathbb{D}_{2\epsilon})} = \sup_{\tilde{\xi} \in \mathbb{D}_{2\epsilon}} |\tilde{g}(\tilde{\xi})| + \sup_{z, \xi_1 \in \mathbb{D}_{2\epsilon}} \left| \frac{\tilde{g}(z) - \tilde{g}(\xi_1)}{\omega(|z - \xi_1|)} \right|
$$

where $\tilde{z}$ and $z_1$ are the reflections of $\tilde{\xi}$ and $\xi_1$ respectively in $\mathbb{D}_\epsilon$. Now $|z - \xi_1| \geq |z - z_1|$ which gives $\omega(|z - \xi_1|) \geq \omega(|z - z_1|)$. Hence the above equation can be written as

$$
\|\tilde{g}\|_{C^\omega(\mathbb{D}_{2\epsilon})} \leq \sup_{\tilde{z} \in \mathbb{D}_\epsilon} |\tilde{g}(\tilde{z})| + \sup_{z, z_1 \in \mathbb{D}_\epsilon} \left| \frac{g(z) - g(z_1)}{\omega(|z - z_1|)} \right| = \|g\|_{C^\omega(\mathbb{D}_\epsilon)}
$$
Theorem 3. Let \( h \in C^{(\lambda)}(\mathbb{T}^2) \) and \( \mu, \nu \in C^{\omega}(\mathbb{T}^2) \) satisfy \(|\mu(\xi)| + |\nu(\xi)| \leq \kappa < 1\), for all \( \xi \in \mathbb{T}^2 \). Then \( L : C^{1,(\lambda)}(\mathbb{T}^2) \to C^{(\lambda)}(\mathbb{T}^2) \) is Fredholm of index zero with kernel consisting of constants. If \( f \in C^{1,(\lambda)}(\mathbb{T}^2) \) is a solution to the equation

\[
\overline{\partial} f - \mu \partial f - \nu \overline{\partial f} = h
\]

and \( \|\mu\|_{C^{\omega}(\mathbb{T}^2)} + \|\nu\|_{C^{\omega}(\mathbb{T}^2)} < \Gamma_0 \), then there exists \( K = K(\Gamma_0) \) such that

\[
\|f\|_{C^{1,(\lambda)}(\mathbb{T}^2)} \leq K (\|h\|_{C^{(\lambda)}(\mathbb{T}^2)} + \|f\|_{C^{(\lambda)}(\mathbb{T}^2)})
\]

Proof: We start with the assumption that \( f \in C^{1,(\lambda)}(\mathbb{T}^2) \) and show that (99) holds. Let \( \xi_0 \in \mathbb{T}^2 \). Let \( \mu_0 = \mu(\xi_0) \) and \( \nu_0 = \nu(\xi_0) \). Choose \( \epsilon \) such that the \( C^\omega \) norms of \((\mu(z) - \mu_0)\) and \((\nu(z) - \nu_0)\) are small enough in a disk \( D_\epsilon(\xi_0) \). Without loss of regularity, we assume here that \( \xi_0 = 0 \).

Define \( F := \phi f \) where \( \phi \in C_0^\infty(D_\epsilon) \).

\[
\overline{\partial} F - \mu_0 \partial F - \nu_0 \overline{\partial F} = H_0 \quad \text{where} \quad H_0 := H + (\mu - \mu_0) F_\xi + (\nu - \nu_0) \overline{F_\xi}.
\]

Hence \( F \) satisfies the Beltrami equation in \( \mathbb{T}^2 \). Using (82), we can write

\[
\|\overline{\partial} F\|_{C^{(\lambda)}(\mathbb{T}^2)} + \|\partial F\|_{C^{(\lambda)}(\mathbb{T}^2)} \leq K \|L_0 F\|_{C^{(\lambda)}(\mathbb{T}^2)}
\]
where $L_0 F := H$ and $K$ is a constant. This can be written as

\[
\|\partial F\|_{C^{(1)}(T^2)} + \|\partial F\|_{C^{(1)}(T^2)} \\
\leq K\|(L - L_0) F\|_{C^{(1)}(T^2)} + \|LF\|_{C^{(1)}(T^2)}.
\]

(102)

Now

\[
\|(L - L_0) F\|_{C^{(1)}(T^2)} \leq \|(\mu - \mu_0) \partial F\|_{C^{(1)}(T^2)} + \|(\nu - \nu_0) \partial F\|_{C^{(1)}(T^2)}
\]

(103)

Now we need to be able to write the terms $\|(\mu - \mu_0) \partial F\|_{C^{(1)}(T^2)}$ and $\|(\nu - \nu_0) \partial F\|_{C^{(1)}(T^2)}$ in terms of $\|(\mu - \mu_0)\|_{C^\omega(D_\epsilon)}$ and $\|(\nu - \nu_0)\|_{C^\omega(D_\epsilon)}$ respectively. This will enable us to make use of the fact that these terms are small enough to get absorbed into on the left hand side of (102).

We first do this for $\|(\mu - \mu_0) \partial F\|_{C^{(1)}(T^2)}$. For this, we first observe that if we can find an extension $\tilde{\mu}$ to $T^2$ of $\mu - \mu_0$ restricted to $D_\epsilon$, such that $\|\tilde{\mu}\|_{C^{(1)}(T^2)} \leq \|(\mu - \mu_0)\|_{C^\omega(D_\epsilon)}$, then since $F$ has support only in $D_\epsilon$. We have

\[
\|(\mu - \mu_0) \partial F\|_{C^{(1)}(T^2)} = \|\tilde{\mu} \partial F\|_{C^{(1)}(T^2)} \leq \|\tilde{\mu}\|_{C^{(1)}(T^2)} \|\partial F\|_{C^{(1)}(T^2)}
\]

from which can obtain our desired estimate

(104) \[\|(\mu - \mu_0) \partial F\|_{C^{(1)}(T^2)} \leq \|\mu - \mu_0\|_{C^\omega(D_\epsilon)} \|\partial F\|_{C^{(1)}(T^2)}\]

For a similar extension $\tilde{\nu}$ of $\nu - \nu_0$, we can obtain

(105) \[\|(\nu - \nu_0) \partial F\|_{C^{(1)}(T^2)} \leq \|\nu - \nu_0\|_{C^\omega(D_\epsilon)} \|\partial F\|_{C^{(1)}(T^2)}\]

Lemma 8 enables us to obtain extensions $\tilde{\nu}$ and $\tilde{\mu}$ with the desired properties such that (104) and (105) hold good. Hence we can use (104) and (105) in (103) to get

\[
\|(L - L_0) F\|_{C^{(1)}(T^2)} \leq (\|\mu - \mu_0\|_{C^\omega(D_{2\epsilon})} + \|\nu - \nu_0\|_{C^\omega(D_{2\epsilon})}) \|\partial F\|_{C^{(1)}(T^2)}.
\]
Substituting the above in (102) gives
\[ \| \partial F \|_{C^2(T^2)} + \| \partial F \|_{C^2(T^2)} \leq K(\| \mu - \mu_0 \|_{C^2(D_{2\epsilon})} + \| \nu - \nu_0 \|_{C^2(D_{2\epsilon})}) \]

Now we chose \( \epsilon \) small enough so that \( K(\| \mu - \mu_0 \|_{C^2(D_{2\epsilon})} + \| \nu - \nu_0 \|_{C^2(D_{2\epsilon})}) < 1/2 \) and hence the corresponding term can be absorbed into \( (\| \partial F \|_{C^2(T^2)} + \| \partial F \|_{C^2(T^2)}) \). Hence we obtain
\[ (106) \quad \| \partial F \|_{C^2(T^2)} + \| \partial F \|_{C^2(T^2)} \leq K_1(\| \partial F \|_{C^2(T^2)}).
\]

Now consider an open cover of \( T^2 \) by \( \epsilon \) disks. Then \( T^2 \) has a finite subcover i.e.
\[ T^2 \subset \bigcup_{j=1}^N D_{\epsilon_j}(\xi_j) \]
where \( \xi_j \in T^2, j = 1, \ldots N \). Consider a partition of unity \((\phi_j), j = 1, \ldots N \) subordinate to this subcover. We can now write
\[ \| \partial f \|_{C^2(T^2)} = \left\| \sum_{j=1}^N \phi_j \partial f \right\|_{C^2(T^2)} \]
\[ \leq \sum_{j=1}^N \| \phi_j \partial f \|_{C^2(T^2)} \]
\[ \leq \sum_{j=1}^N (\| \partial(\phi_j f) \|_{C^2(T^2)} + \| \partial \phi_j f \|_{C^2(T^2)}) \]
\[ \leq \sum_{j=1}^N (\| \partial(\phi_j f) \|_{C^2(T^2)} + C_1 \| \partial \phi_j f \|_{C^2(T^2)} \| f \|_{C^2(T^2)}) \]
\[ \leq C_2 \sum_{j=1}^N (\| \partial(\phi_j f) \|_{C^2(T^2)} + \| f \|_{C^2(T^2)}) \]

Similarly we can get
\[ \| \partial f \|_{C^2(T^2)} \leq C_3 \sum_{j=1}^N (\| \partial(\phi_j f) \|_{C^2(T^2)} + \| f \|_{C^2(T^2)}) \]
Hence

\[ \| \overline{\partial} f \|_{C^0(T^2)} + \| \partial f \|_{C^0(T^2)} \]

\[ \leq C_5 \sum_{j=1}^{N} (\| \overline{\partial}(\phi_j f) \|_{C^0(T^2)} + \partial(\phi_j f) \|_{C^0(T^2)} + \| f \|_{C^0(T^2)}). \]

Now \( \phi_j f \) is supported only in \( \mathcal{D}_{\epsilon_j}(\xi_j) \). Hence (106) can be applied to \( \phi_j f \) to get the estimate

\[ \| \overline{\partial} f \|_{C^0(T^2)} + \| \partial f \|_{C^0(T^2)} \]

\[ \leq C_4 \sum_{j=1}^{N} (\| L(\phi_j f) \|_{C^0(T^2)} + \| f \|_{C^0(T^2)}). \]

We have \( L(\phi_j f) = \phi_j h + \overline{\partial}\phi_j f - \mu \partial\phi_j f - \nu \overline{\partial\phi_j f}. \) Hence \( \| L(\phi_j f) \|_{C^0(T^2)} \)
can be estimated as

\[ \| L(\phi_j f) \|_{C^0(T^2)} \]

\[ \leq \| \phi_j h \|_{C^0(T^2)} + \| \overline{\partial}\phi_j f \|_{C^0(T^2)} + \| \mu \partial\phi_j f \|_{C^0(T^2)} + \| \nu \overline{\partial\phi_j f} \|_{C^0(T^2)} \]

\[ \leq C_6 (\| h \|_{C^0(T^2)} + \| f \|_{C^0(T^2)} + \| \mu \|_{C^0(T^2)} + \| \nu \|_{C^0(T^2)} \| f \|_{C^0(T^2)}). \]

Now we have

\[ \| L(\phi_j f) \|_{C^0(T^2)} \leq C_6 (\| Lf \|_{C^0(T^2)} + \| f \|_{C^0(T^2)}) \]

where \( C_6 = C_6(\Gamma_0). \)

Substituting (108) in (107), we get

\[ \| \overline{\partial} f \|_{C^0(T^2)} + \| \partial f \|_{C^0(T^2)} \leq C_7 (\| h \|_{C^0(T^2)} + \| f \|_{C^0(T^2)}). \]

We can now obtain

\[ \| \overline{\partial} f \|_{C^0(T^2)} + \| \partial f \|_{C^0(T^2)} + \| f \|_{C^0(T^2)} \]

\[ \leq C_7 (\| h \|_{C^0(T^2)} + \| f \|_{C^0(T^2)} + \| f \|_{C^0(T^2)} \]

\[ \leq K (\| h \|_{C^0(T^2)} + \| f \|_{C^0(T^2)}). \]
where $K = K(\Gamma_0)$. Hence we obtain the final estimate (99).

We now use the regularity estimates (99) to prove that $L$ is Fredholm of index zero. Consider a homotopy $L_t$ given by $L_t = (1 - t)\bar{\partial} + tL = \bar{\partial} + t(-\mu \partial - \nu \bar{\partial})$ where $0 \leq t \leq 1$. The ellipticity condition is satisfied in the case of $L_t$ and hence it satisfies the estimate

$$(110) \quad \|f\|_{C^{1,\lambda}(\mathbb{T}^2)} \leq K (\|L_t f\|_{C^{1,\lambda}(\mathbb{T}^2)} + \|f\|_{C^{1,\lambda}(\mathbb{T}^2)}).$$

Now $C^{1,\lambda}$ is compactly contained in $C^\lambda$ and $L_t$ satisfies (110). Hence $L_t$ is semi-Fredholm i.e. dimension of kernel of $L$ is finite and the range is closed. Now $L_0 = \bar{\partial}$. From Proposition (2), $\bar{\partial}$ is an operator with index 0 and hence we can conclude that $L$ also has index 0.

We now need to prove that the kernel of $L$ consists of constants. But Lemma 6 shows that the kernel consists of constants.

**Corollary 1.** Let $h \in C^{(\lambda)}(\mathbb{T}^2)$ and $\mu, \nu \in C^\omega(\mathbb{T}^2)$ satisfying $|\mu(\xi)| + |\nu(\xi)| \leq \kappa < 1$, for all $\xi \in \mathbb{T}^2$. If $f \in W^{1,2}(\mathbb{T}^2)$ satisfies $Lf = h$, then $f \in C^{1,(\lambda)}(\mathbb{T}^2)$.

**Proof:** Using Theorem 3, let $f_1 \in C^{1,(\lambda)}(\mathbb{T}^2)$ satisfy $Lf = h$. Hence $L(f - f_1) = 0$. By Lemma (6), we infer that $f - f_1$ is a constant which in turn implies that $f \in C^{1,(\lambda)}(\mathbb{T}^2)$.

**Theorem 4.** Let $h \in C^\omega(\mathbb{T}^2)$ and $\mu, \nu \in C^\omega(\mathbb{T}^2)$ satisfy $|\mu(\xi)| + |\nu(\xi)| \leq \kappa < 1$ for all $\xi \in \mathbb{T}^2$. Let $f \in W^{1,2}(\mathbb{T}^2)$ be a solution to
the equation

\[ \partial f - \mu \partial f - \nu \overline{\partial f} = h. \]  

Then \( f \in C^{1,\sigma}(\mathbb{T}^2) \). If \( \|\mu\|_{C^{\infty}(\mathbb{T}^2)} + \|\nu\|_{C^{\infty}(\mathbb{T}^2)} < \Gamma_0 \), then there exists \( K = K(\Gamma_0) \) such that

\[ \|f\|_{C^{1,\sigma}(\mathbb{T}^2)} \leq K \left( \|Lf\|_{C^{\infty}(\mathbb{T}^2)} + \|f\|_{C^{0}(\mathbb{T}^2)} \right). \]  

**Proof:** Using Corollary 1, we get that if \( f \in W^{1,2}(\mathbb{T}^2) \) is a solution to (111), then \( f \in C^{1,(\lambda)}(\mathbb{T}^2) \). Using Theorem 3, we obtain that \( f \) satisfies the estimate

\[ \|f\|_{C^{1,(\lambda)}(\mathbb{T}^2)} \leq K \left( \|Lf\|_{C^{(\lambda)}(\mathbb{T}^2)} + \|f\|_{C^{(\lambda)}(\mathbb{T}^2)} \right). \]

Using (64), we obtain that \( f \in C^{1,\sigma}(\mathbb{T}^2) \) and satisfies

\[ \|f\|_{C^{1,\sigma}(\mathbb{T}^2)} \leq K \left( \|Lf\|_{C^{\infty}(\mathbb{T}^2)} + \|f\|_{C^{\infty}(\mathbb{T}^2)} \right). \]

We now need to replace the term \( \|f\|_{C^{\infty}(\mathbb{T}^2)} \) with \( \|f\|_{C^{0}(\mathbb{T}^2)} \) in the above inequality.

For \( 1 < q < \infty \), we have the elliptic estimate for Sobolev spaces

\[ \|f\|_{W^{1,q}(\mathbb{T}^2)} \leq K_1 \left( \|Lf\|_{L^q(\mathbb{T}^2)} + \|f\|_{L^q(\mathbb{T}^2)} \right) \]

for coefficients \( \mu \) and \( \nu \) that are just bounded and measurable. We also have

\[ W^{1,q}(\mathbb{T}^2) \subset C^{\alpha}(\mathbb{T}^2) \quad \text{for} \quad 1 - \frac{2}{q} > \alpha \]

and

\[ C^{\alpha}(\mathbb{T}^2) \subset C^{\omega}(\mathbb{T}^2) \subset C^{0}(\mathbb{T}^2) \subset L^q(\mathbb{T}^2). \]
Using (115) and (116), we obtain
\[ \|f\|_{C^\omega(T^2)} \leq K_2 \|f\|_{C^\alpha(T^2)} \leq K_3 \|f\|_{W^{1,\eta}(T^2)} \]

Using (114) and also (116) we now obtain
\[ \|f\|_{C^\omega(T^2)} \leq K_4 (\|L_f\|_{L^\eta(T^2)} + \|f\|_{L^\eta(T^2)}) \]
\[ \leq K_5 (\|L_f\|_{C^\alpha(T^2)} + \|f\|_{C^\alpha(T^2)}) \]
\[ \leq K_6 (\|L_f\|_{C^\omega(T^2)} + \|f\|_{C^0(T^2)}). \]

Substituting the above in (112), we now obtain our required estimate
\[ \|f\|_{C^{1,\sigma}(T^2)} \leq K_7 (\|L_f\|_{C^\omega(T^2)} + \|f\|_{C^0(T^2)}). \]

We are now ready to obtain similar estimates for the Beltrami equation in case of a bounded domain.

5. **Non constant coefficients in a bounded domain**

Let \( f \) be defined in a bounded domain \( \Omega \). Without loss of generality, we can assume that \( \overline{\Omega} \subset (-\pi, \pi) \times (\pi, \pi) \).

We now prove the main theorem of estimates on solutions of Beltrami equation in \( \Omega \).

**Theorem 5.** Let \( \mu, \nu, h \in C^\omega(\Omega) \) satisfying \(|\mu(\xi)| + |\nu(\xi)| \leq \kappa < 1 \) for all \( \xi \in \Omega \). Let \( f \in W^{1,2}_{\text{loc}}(\Omega) \) be a solution to the equation
\[ (117) \quad \overline{\partial}f - \mu \partial f - \nu \overline{\partial}f = h. \]

Then \( f \in C^{1,\sigma}(\Omega) \). In fact, for any domains \( D, U \) such that \( \overline{D} \subset U \) and \( \overline{U} \subset \Omega \), we have \( f \in C^{1,\sigma}(\overline{D}) \) and if \( \|\mu\|_{C^\omega(\overline{D})} + \|\nu\|_{C^\omega(\overline{D})} < \Gamma_0 \), then there exists \( K = K(\kappa, \Gamma_0, D, U) \) such that
\[ (118) \quad \|f\|_{C^{1,\sigma}(\overline{D})} \leq K (\|h\|_{C^\omega(\overline{D})} + \|f\|_{C^0(\overline{D})}). \]
**Proof:** We shall use estimates obtained on a torus to get the above result. Let \( \phi \in C_0^\infty(V) \) and \( \phi \equiv 1 \) on \( D \) where \( \overline{D} \subset V \) and \( \overline{V} \subset U \). We can consider \( \phi f \in W^{1,2}(\mathbb{T}^2) \subset L^q(\mathbb{T}^2) \) for \( 1 \leq q < \infty \). Now \( \phi f \) satisfies a Beltrami equation with \( L(\phi f) = \phi h + \overline{\partial} \phi f - \mu \partial \phi f - \nu \overline{\partial} \phi f \in L^q(\mathbb{T}^2) \). Now by using (114), we have \( \phi f \in W^{1,q}(\mathbb{T}^2) \). Using (115), we can take \( q \) large enough to get \( \phi f \in C^\alpha(\mathbb{T}^2) \) for some \( \alpha > 0 \) which is contained in \( C^\omega(\mathbb{T}^2) \). Now we can use Theorem 4 to infer that \( \phi f \in C^{1,\sigma}(\mathbb{T}^2) \). But \( \phi f = f \) in \( D \). Hence \( f \in C^{1,\sigma}(\overline{D}) \). We observe that since \( D \) is any domain compactly contained in \( \Omega \), we also obtain that \( f \in C^{1,\sigma}(\Omega) \).

We have

\[
\|f\|_{C^{1,\sigma}(\overline{D})} = \|\phi f\|_{C^{1,\sigma}(\overline{D})} \leq \|\phi f\|_{C^{1,\sigma}(\overline{V})} = \|\phi f\|_{C^{1,\sigma}(\mathbb{T}^2)}.
\]

Now \( \phi f \) satisfies a Beltrami equation defined on the whole of \( T^2 \) with \( L(\phi f) \in C^\omega(\mathbb{T}^2) \) and \( L \) is the Beltrami operator as given by (95). Hence using (112), we have

\[
\|\phi f\|_{C^{1,\sigma}(\mathbb{T}^2)} \leq K (\|L(\phi f)\|_{C^\omega(\mathbb{T}^2)} + \|\phi f\|_{C^0(\mathbb{T}^2)}).
\]

Using (119), we get the estimate

\[
\|f\|_{C^{1,\sigma}(\overline{D})} \leq K (\|L(\phi f)\|_{C^\omega(\overline{V})} + \|\phi f\|_{C^0(\overline{V})}).
\]

We now estimate \( \|L(\phi f)\|_{C^\omega(\overline{V})} \).

\[
\|L(\phi f)\|_{C^\omega(\overline{V})} \leq \|\phi h\|_{C^\omega(\overline{V})} + \|f \overline{\partial} \phi\|_{C^\omega(\overline{V})} + \|\mu \partial \phi f\|_{C^\omega(\overline{V})} + \|\nu \overline{\partial} \phi f\|_{C^\omega(\overline{V})}.
\]

Let us estimate each term above. We obtain

\[
\|\phi h\|_{C^\omega(\overline{V})} \leq \|\phi\|_{C^\omega(\overline{V})} \|h\|_{L^\infty(\overline{V})} + \|h\|_{C^\omega(\overline{V})} \|\phi\|_{L^\infty(\overline{V})}.
\]
Similarly we find

$$\parallel f \phi \parallel_{C^\omega(V)} \leq c_1 \parallel \phi \parallel_{C^\omega(V)} \parallel f \parallel_{L^\infty(V)} + \parallel f \parallel_{C^\omega(V)} \parallel \phi \parallel_{L^\infty(V)}.$$  

We can also obtain

$$\parallel \mu \phi \parallel_{C^\omega(V)} \leq c_2 (\parallel \phi \parallel_{C^\omega(V)} \parallel \mu \parallel_{L^\infty(V)} + \parallel \mu \parallel_{C^\omega(V)} \parallel \phi \parallel_{L^\infty(V)})$$

$$\leq c_3 \parallel \mu \parallel_{L^\infty(V)} (\parallel \phi \parallel_{C^\omega(V)} \parallel f \parallel_{L^\infty(V)} + \parallel f \parallel_{C^\omega(V)} \parallel \phi \parallel_{L^\infty(V)})$$

$$+ \parallel \mu \parallel_{C^\omega(V)} \parallel \phi \parallel_{L^\infty(V)} \parallel f \parallel_{L^\infty(V)}$$

$$\leq c_3 \parallel f \parallel_{C^\omega(V)} \parallel \mu \parallel_{L^\infty(V)} (\parallel \phi \parallel_{C^\omega(V)} + \parallel \phi \parallel_{L^\infty(V)})$$

$$+ \parallel \mu \parallel_{C^\omega(V)} \parallel \phi \parallel_{L^\infty(V)}.$$

This now simplifies to

$$\parallel \mu \phi \parallel_{C^\omega(V)} \leq c_4 (D,U) \parallel f \parallel_{C^\omega(V)} (\parallel \mu \parallel_{L^\infty(V)} + \parallel \mu \parallel_{C^\omega(V)}).$$

Similarly we can obtain

$$\parallel \nu \phi \parallel_{C^\omega(V)} \leq c_5 (D,U) \parallel f \parallel_{C^\omega(V)} (\parallel \nu \parallel_{L^\infty(V)} + \parallel \nu \parallel_{C^\omega(V)}).$$

Substituting (122),(123), (124) and (126) in (121) we obtain

$$\parallel L\phi \parallel_{C^\omega(V)} \leq C_1 (D,U) \parallel Lf \parallel_{C^\omega(V)} + C_2 (\kappa, \Gamma_0, D) \parallel f \parallel_{C^\omega(V)}.$$  

where $C_3 = C_3(U, D)$.

Substituting (127) in (120), we get the estimate

$$\parallel f \parallel_{C^{1, \sigma}(\bar{\Omega})} \leq C_4 (\parallel Lf \parallel_{C^\omega(V)} + \parallel f \parallel_{C^\omega(V)} + \parallel \phi \parallel_{C^0(V)})$$

$$\leq C_4 (\parallel Lf \parallel_{C^\omega(V)} + \parallel f \parallel_{C^\omega(V)} + \parallel f \parallel_{C^0(V)}).$$

where $C_4 = C_4(\kappa, \Gamma_0, U, D)$.  

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We now use the similar calculations as for (113) to replace the \( C^\omega \) norm with the \( C^0 \) norm on \( f \) and improve the estimate above. For \( 1 < q < \infty \), we use the elliptic estimate for Sobolev spaces

\[
\|f\|_{W^{1,q}(\mathcal{V})} \leq K_1(\|Lf\|_{L^q(\mathcal{V})} + \|f\|_{L^q(\mathcal{V})}.
\]

As in Theorem 5, we also use

\[
W^{1,q}(\mathcal{V}) \subset C^\alpha(\mathcal{V}) \quad \text{for } 1 - \frac{2}{q} > \alpha
\]

and

\[
C^\alpha(\mathcal{V}) \subset C^\omega(\mathcal{V}) \subset C^0(\mathcal{V}) \subset L^q(\mathcal{V}).
\]

Using (130) and (131), we obtain

\[
\|f\|_{C^\omega(\mathcal{V})} \leq K_2\|f\|_{C^\alpha(\mathcal{V})} \leq K_3\|f\|_{W^{1,q}(\mathcal{V})}.
\]

Using (129) and also (131) we now obtain

\[
\|f\|_{C^\omega(\mathcal{V})} \leq K_4(\|Lf\|_{L^q(\mathcal{V})} + \|f\|_{L^q(\mathcal{V})})
\]

\[
\leq K_5(\|Lf\|_{C^0(\mathcal{V})} + \|f\|_{C^0(\mathcal{V})})
\]

\[
\leq K_6(\|Lf\|_{C^\omega(\mathcal{V})} + \|f\|_{C^0(\mathcal{V})}).
\]

Substituting the above estimate in (128), we get the required estimate.

\[\square\]
REFERENCES