Cube-like Regular Incidence Complexes

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Abstract of Dissertation

Incidence complexes are generalizations of abstract polytopes. Danzer's power complex construction, \( n^K \), generates incidence complexes that have a cube-like structure. In this dissertation we resolve the question of the structure of the automorphism group of a power complex and investigate its flag-transitive subgroups. We show that the twisting construction for \( 2^K \) when \( K \) is an abstract regular polytope generalizes to arbitrary \( n \) and a regular incidence complex \( K \). In addition we give sufficient conditions under which the more general twisting construction \( L^K \) of abstract regular polytopes generalizes to regular incidence complexes. We also study coverings of regular power complexes and describe conditions for when a covering of (exponent) incidence complexes induces a covering of power complexes.
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Chapter 1

Background

Combinatorial structures built from cubes or cube-like elements have attracted a lot of attention in geometry, topology, and combinatorics. In this thesis we study a particularly interesting class of cube-like structures known as power complexes. These power complexes were first discovered by Danzer in the early 1980’s (see [15, 32, 48]). Power complexes that are also abstract polytopes have repeatedly appeared somewhat unexpectedly in various contexts, although often under a different name; for example, see Coxeter [8], Effenberger-Kühnel [19], Kühnel [29] and McMullen-Schulte [32, Chapter 8]. Power complexes are also used to construct polytopes with specific properties; for example see Pisanski, Schulte and Weiss [42] and Monson and Schulte [36]. However, most power complexes are not abstract polytopes, and have not been very well researched.

An incidence complex has some of the key combinatorial properties of the face lattice of a convex polytope (see [10, 24]); in general, however, an incidence complex need not be a lattice, need not be finite, need not be an abstract polytope, and need not admit any familiar geometric realization. The notion of an incidence complex is originally due to Danzer [15, 16], inspired by Grünbaum [23]. Incidence complexes can also be viewed as incidence geometries or diagram geometries (see [4]) with a linear diagram, although here we study them from
the somewhat different discrete geometric and combinatorial perspective of polytopes and ranked partially ordered sets. Incidence complexes also arise as generalizations of complex polytopes([11, 50]). The structures arising from power complexes have also been used to describe fault-tolerant communication networks (see [3, 14, 20, 21]).

The groups that preserve the structure of convex or complex regular polytopes are known to be reflection groups. These groups are also of interest as the generators of group codes (see [28, 35, 37, 41]) and in Quantum Computing (see Planat-Kibler [43]).

Chapter One provides the background for incidence complexes and polytopes. In Chapter One we also present examples of rank two incidence complexes and describe classes of incidence complexes. In Chapter Two we define power complexes and describe the construction technique. We resolve the open question of the structure of the automorphism group of a power complex. Chapter Three shows that the twisting construction of power complexes for abstract polytopes carries over to incidence complexes and that the construction yields an alternate way to generate power complexes. Chapter Four describes the circumstances which guarantee the existence of covering maps between power complexes and of sequences of covering maps for sequences of power complexes.

1.1 Incidence Complexes and Polytopes

Following Danzer-Schulte [16] (also [46]), an incidence complex $\mathcal{K}$ of rank $k$, or briefly a $k$-complex, is defined by the properties (I1),..., (I4) below. The elements of $\mathcal{K}$ are called faces of $\mathcal{K}$.

(I1) $\mathcal{K}$ is a partially ordered set with a unique least face and a unique greatest face.

(I2) Every totally ordered subset of $\mathcal{K}$ is contained in a (maximal) totally ordered subset with exactly $k + 2$ elements.
These two conditions make $\mathcal{K}$ into a ranked partially ordered set, with a strictly monotone rank function with range $\{-1, 0, \ldots, k\}$. A face of rank $i$ is called an $i$-face; often $F_i$ will indicate an $i$-face. The least face and greatest face are the improper faces of $\mathcal{K}$ and have ranks $-1$ and $k$, respectively; all other faces of $\mathcal{K}$ are proper faces of $\mathcal{K}$. A face of rank 0, 1 or $n-1$ is also called a vertex, an edge or a facet, respectively. A flag of $\mathcal{K}$ is a maximal totally ordered subset of $\mathcal{K}$. We let $\mathcal{F}(\mathcal{K})$ denote the set of flags of $\mathcal{K}$.

(I3) $\mathcal{K}$ is strongly flag-connected, meaning that if $\Phi$ and $\Psi$ are two flags of $\mathcal{K}$, then there is a finite sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_{m-1}, \Phi_m = \Psi$, all containing $\Phi \cap \Psi$, such that successive flags are adjacent (differ in just one face).

Call two flags $i$-adjacent, for $i = 0, \ldots, k-1$, if they are adjacent and differ exactly in their $i$-faces. With this notion of adjacency, $\mathcal{F}(\mathcal{K})$ becomes the flag graph for $\mathcal{K}$ and acquires a natural edge-labeling where edges labeled $i$ represent pairs of $i$-adjacent flags.

Our last defining condition is a homogeneity requirement for the numbers of $i$-adjacent flags for each $i$.

(I4) There exist cardinal numbers $c_0, \ldots, c_{k-1} \geq 2$, for our purposes taken to be finite, such that, whenever $F$ is an $(i-1)$-face and $G$ a $(i+1)$-face with $F < G$, the number of $i$-faces $H$ with $F < H < G$ equals $c_i$.

Clearly, (I4) can be rephrased by requiring the existence of $c_0, \ldots, c_{k-1} \geq 2$ such that each flag of $\mathcal{K}$ has exactly $c_i - 1$ adjacent faces for $i = 1, \ldots, k-1$. An incidence complex $\mathcal{K}$ of rank $k$ with parameters $c_0, \ldots, c_{k-1}$ is said to be in the class $[c_0, \ldots, c_{k-1}]$.

If $F$ is an $i$-face and $G$ a $j$-face with $F < G$, we call

$$G/F := \{H \in \mathcal{K} \mid F \leq H \leq G\}$$

a section of $\mathcal{K}$. It follows that $G/F$ is an incidence complex in its own right, of rank $j-i-1$.
and with homogeneity parameters $c_{i+1}, \ldots, c_{j-1}$. It is useful to identify a $j$-face $G$ of $K$ with the $j$-complex $G/F_{-1}$. Likewise, if $F$ is an $i$-face, the $(k-i-1)$-complex $F_k/F$ is the co-face of $F$ in $K$; if $F$ is a vertex (and $i = 0$), this is also called the vertex-figure at $F$.

Given two complexes $K$ and $L$ a map\(^1\) $\varphi : K \to L$ is called a homomorphism if $F \varphi < G \varphi$ in $L$ whenever $F < G$ in $K$. An isomorphism $\varphi$ between $K$ and $L$ is a bijection such that $\varphi$ and $\varphi^{-1}$ are homomorphisms. An automorphism of $K$ is an isomorphism from $K$ to itself. The automorphism group $\Gamma(K)$ of an incidence complex $K$ consists of the order-preserving bijections of $K$.

An abstract $k$-polytope, or simply $k$-polytope, is an incidence complex of rank $k$ such that $c_i = 2$ for $i = 0, \ldots, k-1$ (see McMullen-Schulte [32]). Thus a polytope is a complex in which every flag has precisely one $i$-adjacent flag for each $i$. For polytopes, the last condition (I4) is also known as the diamond condition.

Abstract polytopes often admit a Schl"afli symbol. For $i = 1, 2, \ldots, k-1$, for any $(i-2)$-face $F$, and $(i+1)$-face $G$ of $K$ incident with $F$, let $p_i(F,G)$ denote the number of $i$-faces of a $k$-polytope $K$ (with $k \geq 2$) in the section $G/F$. Then $G/F$ is isomorphic to the 2-polytope with Schl"afli symbol $\{p_i(F,G)\}$. In our applications these numbers will only depend on $i$, not on $F$ and $G$, and then simply denoted by $p_i$. In this case, the polytope is said be equivelar of type $\{p_1, p_2, \ldots, p_{k-1}\}$. The polytope of rank 1 is assigned the Schl"afli symbol $\{\}$.\(^2\)

While (I3) expresses connectedness properties of $K$ in terms of flags, the following condition (I3') is based on the faces of $K$. For posets with properties (I1) and (I2), the conditions (I3) and (I3') are equivalent.

(I3') $K$ is strongly connected, meaning that every section $G/F$ of $K$ (including $K$ itself) is connected in the following sense: either $G/F$ has rank 1, or $G/F$ has rank $\geq 2$ and for any two proper faces $H$ and $H'$ of $G/F$ there exists a finite sequence of proper faces $H = H_0, H_1, \ldots, H_{l-1}, H_l = H'$ of $G/F$ such that $H_{i-1}$ and $H_i$ are incident for $i = 1, \ldots, l$.

\(^1\)Throughout this thesis maps are written on the right.

\(^2\)
The dual of an incidence complex $\mathcal{K}$ of rank $k$ is the incidence complex $\mathcal{K}^*$ of rank $k$ obtained from $\mathcal{K}$ by reversing the partial order. Clearly, $(\mathcal{K}^*)^* = \mathcal{K}$.

### 1.1.1 Automorphism Group

We say that an incidence complex $\mathcal{K}$ is regular if its automorphism group $\Gamma(\mathcal{K})$ is transitive on the flags of $\mathcal{K}$. Note that a regular complex need not have a simply flag-transitive automorphism group (in fact, $\Gamma(\mathcal{K})$ may not even have a simply flag-transitive subgroup), so in general $\Gamma(\mathcal{K})$ has nontrivial flag-stabilizer subgroups. However, the group of a regular polytope is always simply flag-transitive.

It was shown in [46] (for a proof for polytopes see also [32, Chapter 2]) that the group $\Gamma := \Gamma(\mathcal{K})$ of a regular $k$-complex $\mathcal{K}$ has a well-behaved system of generating subgroups. Let $\Phi := \{F_{-1}, F_0, \ldots, F_k\}$ be a fixed, or base flag, of $\mathcal{K}$, where $F_i$ designates the $i$-face in $\Phi$ for each $i$. For each $\Omega \subseteq \Phi$ let $\Gamma_\Omega$ denote the stabilizer of $\Omega$ in $\Gamma$. Then $\Gamma_\Phi$ is the stabilizer of the base flag $\Phi$, and $\Gamma_\emptyset = \Gamma$. Moreover, for $i = -1, 0, \ldots, k$ set

$$R_i := \Gamma_{\Phi \setminus \{F_i\}} = \langle \varphi \in \Gamma \mid F_j \varphi = F_j \text{ for all } j \neq i \rangle.$$

Then each $R_i$ contains $\Gamma_\Phi$, and coincides with $\Gamma_\Phi$ when $i = -1$ or $k$; in particular,

$$c_i := |R_i : \Gamma_\Phi| \quad (i = 0, \ldots, k - 1). \quad \text{(1.1.1)}$$

Moreover, these subgroups have the following commutation property:

$$R_i \cdot R_j = R_j \cdot R_i \quad (-1 \leq i < j - 1 \leq k - 1). \quad \text{(1.1.2)}$$

Note here that $R_i$ and $R_j$ commute as subgroups, but not generally at the level of elements.

The groups $R_{-1}, R_0, \ldots, R_k$ form a distinguished system of generating subgroups of $\Gamma$. 
that is, 
\[ \Gamma = \langle R_{-1}, R_0, \ldots, R_k \rangle. \] (1.1.3)

Here the subgroups \( R_{-1} \) and \( R_k \) are redundant when \( k > 0 \). More generally, if \( \Omega \) is a proper subset of \( \Phi \), then
\[ \Gamma_\Omega = \langle R_i \mid -1 \leq i \leq k, F_i \not\in \Omega \rangle. \]

Moreover, if \( F \) is an \( i \)-face and \( G \) a \( j \)-face with \( F \subseteq G \), and if \( \Omega \) is a chain in \( \mathcal{P} \) containing, besides \( F \) and \( G \), only faces of ranks from \( \{-1, 0, \ldots, i-1\} \cup \{j+1, \ldots, n\} \), then the stabilizer \( \Gamma_\Omega(\mathcal{P}) \) of \( \Omega \) in \( \mathcal{P} \) acts flag-transitively (but not necessarily faithfully) on the section \( G/F \).

For each nonempty subset \( I \) of \( \{-1, 0, \ldots, k\} \) define \( \Gamma_I := \langle R_i \mid i \in I \rangle \); and for \( I = \emptyset \) define \( \Gamma_I := R_{-1} = \Gamma_{\Phi} \). (As a warning, the notation \( \Gamma_{\emptyset} \) can have two meanings, either as \( \Gamma_\emptyset \) with \( \Omega = \emptyset \) or \( \Gamma_I \) with \( I = \emptyset \); the context should make it clear which of the two is being used.) Thus
\[ \Gamma_I = \Gamma_{\{F_j \mid j \not\in I\}} \quad (I \subseteq \{-1, 0, \ldots, k\}); \]
or equivalently,
\[ \Gamma_\Omega = \Gamma_{\{i \mid F_i \not\in \Omega\}} \quad (\Omega \subseteq \Phi). \]

The automorphism group \( \Gamma \) of \( \mathcal{K} \) and its distinguished generating system satisfy the following important intersection property:
\[ \Gamma_I \cap \Gamma_J = \Gamma_{I \cap J} \quad (I, J \subseteq \{-1, 0, \ldots, k\}). \] (1.1.4)

1.1.2 Combinatorial Structure

The combinatorial structure of \( \mathcal{K} \) can be completely described in terms of the distinguished generating system of \( \Gamma(\mathcal{K}) \). In fact, bearing in mind that \( \Gamma \) acts transitively on the
faces of each rank, the partial order is given by

$$F_i \varphi \leq F_j \psi \iff \psi^{-1} \varphi \in \Gamma_{\{i+1,\ldots,k\}} \cap \Gamma_{\{-1,0,\ldots,j-1\}} \quad (-1 \leq i \leq j \leq k; \ \varphi, \psi \in \Gamma),$$

or equivalently,

$$F_i \varphi \leq F_j \psi \iff \Gamma_{\{-1,0,\ldots,k\} \setminus \{i\}} \cap \Gamma_{\{-1,0,\ldots,k\} \setminus \{j\}} \psi \neq \emptyset \quad (-1 \leq i \leq j \leq k; \ \varphi, \psi \in \Gamma). \ (1.1.5)$$

Conversely, as stated more explicitly below, if $\Gamma$ is any group with a system of subgroups $R_{-1}, R_0, \ldots, R_k$ such that (1.1.2), (1.1.3) and (1.1.4) hold, and $R_{-1} = R_k$, then $\Gamma$ is a flag-transitive subgroup of the full automorphism group of a regular incidence complex $K$ of rank $k$ (see again [46], or [32, Chapter 2] for polytopes). The $i$-faces of $K$ are the right cosets of $\Gamma_{\{-1,0,\ldots,k\} \setminus \{i\}}$ for each $i$, and the partial order is given by (1.1.5); and two faces of $K$ are incident if and only if the corresponding cosets intersect in $\Gamma$. The homogeneity parameters $c_0, \ldots, c_{k-1}$ are determined by (1.1.1).

That is we have the following ([46, Theorem 3])

**Theorem 1.1.1.** Let $\Gamma$ be a group with subgroups $R_{-1}, R_0, \ldots, R_k$. For $I \subseteq \{-1,0,1,\ldots,k\}$ define the subgroup $\Gamma_I := \langle R_i \mid i \in I \cup \{-1\} \rangle$.

1. $R_{-1} = R_k$

2. $\Gamma = \langle R_{-1}, R_0, \ldots, R_k \rangle$

3. $\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J}$ whenever $I, J \subseteq \{-1,0,\ldots,k\}$

4. $R_i R_j = R_j R_i$ whenever $-1 \leq i < j - 1 \leq k - 1$

Then it is possible to construct a regular $k$-complex $K$ whose full automorphism group contains $\Gamma$ as a flag-transitive subgroup.
1.1.2.1 Abstract Polytopes

For abstract regular polytopes, these structure results lie at the heart of much research activity in this area (see [32]). In this case the flag stabilizer $\Gamma_\Phi$ is the trivial group, and each nontrivial subgroup $R_i$ (with $i \neq -1, k$) has order 2 and is generated by an involutory automorphism $\rho_i$ that maps $\Phi$ to its unique $i$-adjacent flag. The group of an abstract regular polytope is then a string C-group, meaning that the distinguished involutory generators $\rho_0, \ldots, \rho_{k-1}$ satisfy both the commutativity relations typical of a Coxeter group with string diagram, and the intersection property (1.1.4).

The rotation subgroup, $\Gamma^+(\mathcal{K})$, of the automorphism group $\Gamma(\mathcal{K})$ is the subgroup of $\Gamma(\mathcal{K})$ consisting of the even length words on the generators $\rho_0, \ldots, \rho_{k-1}$. If we define $\sigma_i = \rho_{i-1} \rho_i$ for $i = 1, \ldots, k - 1$, it is easy to see that $\Gamma^+(\mathcal{K}) = \langle \sigma_1, \ldots, \sigma_{k-1} \rangle$. If $\mathcal{K}$ is regular then the index of $\Gamma^+(\mathcal{K})$ in $\Gamma(\mathcal{K})$ is at most 2.

A $k$-polytope $\mathcal{K}$ is chiral (see [49]) if its automorphism group induces two orbits on the flags such that adjacent flags belong to different orbits. A chiral polytope $\mathcal{K}$ has an automorphism group, $\Gamma^+(\mathcal{K})$ generated by “abstract rotations $\sigma_0, \ldots, \sigma_{k-1}$. Chiral polytopes have recently attracted a lot of attention (see, for example [6], [13], [39], and [40]).

For on-line atlases of chiral and regular polytopes see [26], [27], and [25].

1.1.2.2 Additional Properties

Let $0 \leq d < k$. The $d$-skeleton $\text{ske}_d(\mathcal{K})$ of an incidence complex $\mathcal{K}$ of rank $k$ is the incidence complex of rank $d + 1$ obtained as follows. The faces of $\text{ske}_d(\mathcal{K})$ of rank at most $d$ are the faces of $\mathcal{K}$ of rank at most $d$, and the face of rank $d + 1$ is the $k$-face of $\mathcal{K}$; the partial order of $\text{ske}_d(\mathcal{K})$ is inherited from $\mathcal{K}$. Note that every rank $k$ incidence complex is (trivially) isomorphic to its own $(k - 1)$-skeleton.
The automorphism group of an incidence complex is equal to the automorphism group of its dual, \( \Gamma(\mathcal{K}^*) = \Gamma(\mathcal{K}) \).

1.2 Simple Incidence Complex Examples

1.2.1 Simple Examples in Ranks 1 and 2

The 1-dimensional incidence complexes are simply called \( v\)-edges, \( v \geq 2 \), with \( v \) vertices. This complex, denoted by \( \gamma_v \), is in the class \([v]\). This is a generalization of Shephard’s [50] complex \( m\)-line. When \( v = 2 \) this is the polytope \( \{\} \), also referred to as a 1-simplex.

The combinatorial automorphism group of the \( v\)-edge is the symmetric group on \( v \) letters \( S_v \). Considered as a complex 1-polytope in the sense of [50], its symmetry group is the cyclic group \( C_v \).

Example 1.2.1. Generalized Square

The \( v\)-square \( \gamma_v^v \) is the complex 2-cube with \( v^2 \) vertices and \( 2v \) edges. It is an example of a complex polygon in the unitary plane (see Section 1.2.1.2). This \( v\)-square is an incidence complex of rank 2. For example, Figure 1.1 shows the incidence complex \( \gamma_4^4 \). Its edges are the eight \( v\)-edges of the polygon, four colored blue and four colored red. Each vertex is in 2 edges and each edge consists of 4 vertices. If we label the vertices \((i, j)\) with \( i, j = 1, \ldots, v \) then the \( v\)-edges consist of the \( v \) vertices of the form \((i, V) = \{(i, j)| j = 1, \ldots, v\}\) and \((V, j) = \{(i, j)| i = 1, \ldots, v\} \), with \( V := \{1, 2, 3, 4\} \).

The automorphisms of \( \gamma_v^v \) either preserve or interchange the two sets of edges. Therefore, the automorphism group is isomorphic to \( S_v \wr S_2 \cong (S_v \times S_v) \rtimes S_2 \). This group has order \( 2(v!)^2 \). As we shall see later, the \( v\)-square is an instance of a power complex of rank 2 (isomorphic to \( v^{\{\}} \)).
1.2.1.1 Regular Graphs

The 1-skeleton of any incidence complex is a connected regular graph. The following proposition shows that the converse is also true.

Proposition 1.2.2. A connected $c$-regular graph $G$ augmented by unique minimal and maximal elements is a rank 2 incidence complex, denoted by $\tilde{G}$, in the class $[2,c]$.

Proof. We need to show that $\tilde{G}$ satisfies the conditions. Clearly, (I1) and (I2) are satisfied. Every vertex has $c$ edges incident, and every edge has two vertices, so (I4) is satisfied. We need to show that $\tilde{G}$ is strongly flag-connected. It is convenient to use the equivalent form (I3').

Let $u, v$ be two vertices of $G$. Since $G$ is a connected graph there is an edge-path with edges $e_1, \ldots, e_l$ connecting $u$ and $v$. For $i = 1, \ldots, l - 1$, let $u_i$ denote the common vertex of $e_i$ and $e_{i+1}$. Then it is clear that $u, e_1, u_1, \ldots, e_{l-1}, u_{l-1}, e_l, u_l, v$ is a sequence in which any two elements are incident in $\tilde{G}$. Hence $\tilde{G}$ is strongly connected.

The dual of the complex $v$-square is the 2-dimensional complex cross polytope $\beta_2^v$. This is isomorphic to the complete bipartite graph with $2v$ vertices $K_{v,v}$. Augmented as in Proposition 1.2.2. Figure 1.2 shows the case $v = 3$, the Thomsen Graph, $K_{3,3}$. The graph is
bipartite with vertex sets \( U = \{1, 3, 5\} \) and \( V = \{2, 4, 6\} \). The automorphism group acts on \( \{1, 2, 3, 4, 5, 6\} \), imprimitively, with blocks \( U \) and \( V \) each invariant under a subgroup \( S_3 \) of \( S_6 \), and an involution interchanging the blocks. Therefore, the automorphism group is \( (S_3 \times S_3) \rtimes S_2 \), as expected from the duality. The two copies of \( S_3 \) are \( \langle (1 \ 3), (1 \ 5) \rangle \) and \( \langle (2 \ 4), ((2 \ 6)) \rangle \) and the involution is given by conjugation in \( S_6 \) with \( (1 \ 4)(2 \ 5)(3 \ 6) \). The size of the automorphism group is 72. From the graph, by adding in elements \( F_{-1} \) and \( F_2 \) we get

![Figure 1.2: Thomsen Graph \( K_{3,3} = \beta_2^3 \)](image)

the regular 2-complex shown in Figure 1.3. It is again denoted by \( \beta_2^3 \). The Thomsen Graph is in the class \([2, 3]\) with a total of 18 flags. We can find the distinguished generator subgroups

![Figure 1.3: Thomsen Complex](image)
$R_0, R_1$ of $\Gamma(\beta_2^3)$ as follows. Let $\Phi = \{F_{-1}, 1, 12, F_2\}$ be the base flag. The elements that stabilize the flag are exactly those elements that keep 1 and 2 fixed. So $\Gamma_\Phi = \langle (3 \ 5), (4 \ 6) \rangle$, and

$$R_{-1} = \Gamma_\Phi = \langle (3 \ 5), (4 \ 6) \rangle \simeq S_2 \times S_2,$$

$$R_0 = \Gamma_{\Phi \setminus \{1\}} = \langle (3 \ 5), (1 \ 2)(3 \ 4)(5 \ 6) \rangle \simeq D_4,$$

$$R_1 = \Gamma_{\Phi \setminus \{12\}} = \langle (3 \ 5), (4 \ 6), (2 \ 4) \rangle \simeq D_6 \simeq S_3 \times S_2.$$  

We then have the following for the sizes of the generating subgroups

$$|R_{-1}| = 4$$

$$|R_0| = 8$$

$$|R_1| = 12$$

This gives us the parameters $c_0 = |R_0 : \Gamma_\Phi| = 2$ and $c_1 = |R_1 : \Gamma_\Phi| = 3$ as expected.

1.2.1.2 Regular Complex Polygons in the Unitary Plane

A complex polygon, or complex 2-polytope $P$ is a finite family of affine subspaces (points or lines) called (proper) faces (vertices or edges resp.), of the unitary plane $\mathbb{C}^2$, ordered by inclusion, such that the following conditions are satisfied (see [50], [11]). Every vertex is incident with at least two edges and every edge is incident with at least two vertices. Any two vertices are connected by a chain of successively incident edges and vertices. Note that if $P$ is augmented by $\emptyset$ and $\mathbb{C}^2$ as improper faces, and ordered by inclusion, then $P$ becomes a regular incidence complex of rank 2. However, the combinatorial automorphism group of this incidence complex is generally larger than the unitary symmetry group of the underlying complex polygon. We call $P$ regular (as a complex polygon) if its unitary symmetry group, $G(P)$, acts transitively on the flags (incident pairs of vertices and edges). As with regular
polytopes, the transitivity is sharp and the order of the symmetry group is equal to the number of flags of the complex polytope.

The symmetry group $G(P)$ of a regular complex polygon $P$ is generated by two elements, $R$ and $S$, where $R$ permutes the vertices of an edge, and $S$ permutes the edges through a vertex on that edge. If $P$ has $m$ vertices per edge, with $l$ edges meeting at a vertex, then $G(P)$ has a presentation of the form

$$R^m = S^l = 1, \quad RSRSR\ldots = SRSRS\ldots,$$

with $p$ terms $R$ or $S$ on either side of the last defining equation, and is denoted by $m[p]l$; moreover $P$ itself is denoted by $m\{p\}l$. See [11, Chapter 12] and [32, p. 293] for the possible choices of $m$, $l$, and $p$. The ordinary real regular $p$-gon occurs as the case $2\{p\}2$.

The interpretation of the symbol $m\{p\}l$ is: there are $m$ vertices per edge (that is the edges are $m$-edges), each vertex is in $l$ edges, and the shortest closed vertex-edge path with no two edges in the same $m$-edge is $p$.

**Example 1.2.3. Complex Square**

For example, the complex square $m\{4\}2$ has $m$ vertices per edge, with each vertex lying in 2 edges; the case $m = 3$ is shown in Figure 1.4. In the notation of Section 1.2.1, $m\{4\}2 = \gamma_{2}^{m}$. 

![Figure 1.4: Complex square with three vertices per edge](image)
The coordinates of its \(m^2\) vertices are \((\omega^j, \omega^k)\), where \(\omega = e^{2\pi/m}\) and \(1 \leq j, k \leq m\); in Figure 1.4, the vertices are denoted by \((jk)\). The \(m\)-edges are the sets of vertices \(\{(\omega^j, \omega^k) | 1 \leq j \leq m\}\) with fixed \(k = 1, \ldots, m\) and \(\{(\omega^j, \omega^k) | 1 \leq k \leq m\}\) with fixed \(j = 1, \ldots, m\). In Figure 1.4 the edges are the sets \(\{(jk) | 1 \leq j \leq m\}\) for fixed \(k = 1, \ldots, m\) and \(\{(jk) | 1 \leq k \leq m\}\) for a fixed \(j = 1, \ldots, m\).

The symmetry group has order \(2m^2\) and is isomorphic to \((C_m \times C_m) \rtimes C_2 = C_m \wr C_2\), a wreath product. Recall that the full automorphism group is isomorphic to \((S_m \times S_m) \rtimes S_2 = S_m \wr S_2\), which is a much larger group than the symmetry group. The corresponding 4-dimensional convex polytope in real 4-space is the direct product of two regular convex \(m\)-gons in orthogonal planes.

The regular complex polygons (and their analogues of higher rank) were classified in [52]. In Table 1.1 we list data for some examples we require later in the text.

<table>
<thead>
<tr>
<th>Group</th>
<th>Order</th>
<th>Vertices</th>
<th>Edges</th>
<th>Class</th>
<th>Symmetry Group</th>
<th>Automorphism Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>2[q]2</td>
<td>2q</td>
<td>q</td>
<td>q</td>
<td>[2, 2]</td>
<td>(D_q)</td>
<td>(D_q)</td>
</tr>
<tr>
<td>p[4]2</td>
<td>(p^2)</td>
<td>(p^2)</td>
<td>(2p)</td>
<td>([p, 2])</td>
<td>(C_p \rtimes C_2)</td>
<td>(S_p \rtimes S_2)</td>
</tr>
<tr>
<td>2[4]p</td>
<td>(2p^2)</td>
<td>(2p^2)</td>
<td>(p^2)</td>
<td>([2, p])</td>
<td>(C_p \rtimes C_2)</td>
<td>(S_p \rtimes S_2)</td>
</tr>
<tr>
<td>3[3]3</td>
<td>24</td>
<td>8</td>
<td>8</td>
<td>[3, 3]</td>
<td>(SL(2, 3))</td>
<td>(GL(2, 3))</td>
</tr>
<tr>
<td>3[4]3</td>
<td>72</td>
<td>24</td>
<td>24</td>
<td>[3, 3]</td>
<td>(SL(2, 3) \times C_3)</td>
<td>((SL(2, 3) \times C_3) \rtimes C_2)</td>
</tr>
</tbody>
</table>

Example 1.2.4. The polygon \(3\{3\}3\)

The complex polygon \(3\{3\}3\) (Figure 1.5) has the same vertices and edges, in a sense, as the regular 4-cross polytope, or 16-cell, when viewed in real 4-space. However, in the case of \(3\{3\}3\) the 24 edges of the 16 cell exist as eight 3-edges (see [9] and [11]).

The vertices of \(3\{3\}3\) are the non-zero elements of the finite field \(\mathbb{F}_{3^2}\) with nine elements. Construct the field \(\mathbb{F}_{3^2}\) from \(\mathbb{F}_3\) using the equation \(\alpha^2 + \alpha + 2 = 0\). View \(\mathbb{F}_{3^2}\) as \(\mathbb{F}_3^2\) and associate with the vertex label \(j\) the element \(\alpha^j = x_0\alpha + x_1 = (x_0, x_1) \in \mathbb{F}_3^2\). The vertices of the eight
3-edges are the sets of three vertices \( \{ \alpha^j, \alpha^j, \alpha^j \} \) whose coordinates in \( \mathbb{F}_3^2 \) sum to zero. For example for \( \alpha, \alpha^6 = \alpha + 2, \alpha^7 = \alpha + 1 \), we have \( \alpha + (\alpha + 2) + (\alpha + 1) = 0 \) so \( \{1, 6, 7\} \) is an edge. Once one edge has been found, the others can be found by multiplying each of its vertices by \( \alpha \), this has the effect of increasing by one each component. So \( \{2, 7, 8\} \) is also an edge. The edges are \( \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 8\}, \{6, 7, 1\}, \{7, 8, 2\}, \) and \( \{8, 1, 3\} \).

The geometric symmetry group is isomorphic to \( SL(2, 3) \), with the following action. The vertices of \( 3\{3\}3 \) are labeled as in Figure 1.5, with \( ij \) meaning \( (i, j) \in \mathbb{F}_3^2 \). For \( A \in SL(2, 3) \), the action of \( A \) on \( 3\{3\}3 \) is given by \( (x_0, x_1)A = (y_0, y_1) \) for each vertex and by association for the 3-edges.

Let \( \{1, \{1, 2, 4\}\} \), where \( \{1, 2, 4\} \) is the 3-edge with vertices 1, 2, and 4, be the base flag. Then taking \( R = (1 \ 2 \ 4)(5 \ 6 \ 8) \) and \( S = (2 \ 7 \ 8)(3 \ 4 \ 6) \) we have \( 3\{3\}3 = \langle R, S \rangle \) and \( R^3 = S^3 \) and \( RSR = SRS \) as desired. The complex reflection \( R \) cyclically permutes the vertices in the 3-edge \( \{1, 2, 4\} \) and \( S \) cyclically permutes the 3-edges containing the vertex 1, namely \( \{1, 2, 4\}, \{1, 6, 7\} \) and \( \{1, 3, 8\} \).
From [11, Section 11.2], we know that the combinatorial automorphism group of $3\{3\}3$ is a subgroup of $[3,3,4]^+$, the rotation subgroup of $[3,3,4]$. A search using GAP [22] of the subgroups of $[3,3,4]^+$ that include $3\{3\}3$ reveals two subgroups, one of order 48 and one of order 96. Only the subgroup, generated by $R, S$ and $Z = (1 2 3 4 5 6 7 8)$, of order 48 is flag-transitive. Hence this subgroup must be the automorphism group. With base flag \{1, \{1,2,3\}\}, it is generated by

\[
R_0 := \langle (2 4)(3 7)(6 8), (1 2 4)(5 6 8) \rangle, \quad (1.2.1)
\]
\[
R_1 := \langle (2 4)(3 7)(6 8), (2 7 8)(3 4 6) \rangle \quad (1.2.2)
\]

and is isomorphic to $GL(2,3)$. Note that $R_0$ and $R_1$ are both congruent to $S_3$ and $R_{-1} := \langle (2 4)(3 7)(6 8) \rangle$.

**Example 1.2.5.** The polygon $3\{4\}3$.

As with $3\{3\}3$, the complex polygon $3\{4\}3$ has vertices and edges, in a sense, in common with a regular rank 4 polytope. In this case it is $\{3,4,3\}$. However not all edges of $\{3,4,3\}$ occur as edges of $3\{4\}3$; there are twenty edges deleted from $\{3,4,3\}$ to get the edges that connect vertices in 3-edges of $3\{4\}3$.

This is a complex polygon with twenty-four vertices and twenty-four 3-edges (see Figure 1.6). On the figure, the 3-edges are the twelve rotations of $\{1,2,13\}$ and $\{1,15,22\}$. The geometric symmetry group of order 72 is generated by

\[
R = \langle (1 2 3)(14 18 22)(3 4 15)(16 20 24)(5 6 17)(7 8 9)(9 10 21)(11 12 23) \rangle \quad (1.2.3)
\]
\[
S = \langle (13 22 12)(2 15 24)(14 17 16 6 19)(18 8 21)(20 10 23) \rangle \quad (1.2.4)
\]

and is isomorphic to $C_3 \times SL(2,3)$.

From [11, Section 11.2], we know that the combinatorial automorphism group is a sub-
Figure 1.6: Complex polygon 3{4}3

group of [3, 4, 3]⁺, the rotation subgroup of [3, 4, 3] of size 576. Using GAP [22], of the subgroups of [3, 4, 3]⁺ that include the symmetry group, only the subgroup with 144 elements is flag-transitive on 3{4}3. Hence this must be the automorphism group of 3{4}3. In particular, for the base flag \{1, \{1, 2, 13\}\} the full automorphism group, of size 144, is generated by

\[
R_0 = \langle (1\ 2\ 3)(3\ 4\ 15)(5\ 6\ 17)(7\ 8\ 19)(9\ 10\ 21)(11\ 12\ 23)(14\ 18\ 22)(16\ 20\ 24), \\
(2\ 13)(3\ 11)(4\ 23)(5\ 9)(6\ 21)(8\ 19)(10\ 17)(12\ 15)(14\ 20)(16\ 18)(22\ 24) \rangle \\
R_1 = \langle (2\ 12)(3\ 11)(4\ 10)(5\ 9)(6\ 8)(13\ 24)(14\ 23)(15\ 22)(16\ 21)(17\ 20)(18\ 19), \\
(2\ 15\ 24)(4\ 17\ 14)(6\ 19\ 16)(8\ 21\ 18)(10\ 23\ 20)(12\ 13\ 22) \rangle.
\]

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1.2.2 Finite Geometries

A finite geometry is any “incidence geometry” that has a finite number of points. Not all finite geometries are incidence complexes. For example, in the the three-dimensional dihedral bow tie, two simplices joined at a common vertex \( F_0 \), the section \( \langle F_3/F_0 \rangle \), is not connected and therefore not an incidence complex.

Finite geometries constructed from vector spaces over finite fields are examples of incidence complexes; typical examples are affine and projective spaces. However, not all interesting geometries arise by this construction. There are projective planes whose construction is axiomatic. For the remainder of this section we will concentrate on finite geometries that arise from vector spaces over finite fields. Let \( \mathbb{F}_q \) be the finite field of order \( q = p^e \) for some prime \( p \) and positive integer \( e \).

It can be shown, for example [45, Chapter 9], that the automorphism group of \( \text{PG}(n,q) \), projective \( n \)-space over \( \mathbb{F}_q \), is given by \( \text{PGL}(n+1,q) = \text{PGL}(n+1,q) \rtimes \text{Gal}(\mathbb{F}_q/\mathbb{F}_p) \).

When \( q \) is prime and \( n = 2 \), this simplifies to \( \text{PGL}(3,q) = \text{PGL}(3,q) \). When \( q = 2 \) we have \( \text{PGL}(3,2) = \text{PSL}(3,2) \cong \text{PSL}(2,7) \), the simple group of order 168, and for \( q = 3 \), \( \text{PGL}(3,3) = \text{PSL}(3,3) \).

1.2.2.1 Affine and Projective Lines

The affine line \( \text{AG}(1,q) \) of order \( q \) is the set of points \( x \in \mathbb{F}_q \) and can be viewed as a \( q \)-edge. The geometric automorphism group consists of the maps of the form \( ax + b \), for \( a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \) and is isomorphic to \( \mathbb{F}_q \rtimes \mathbb{F}_q^\times \).

The projective line \( \text{PG}(1,q) \) of order \( q \) is the affine line of order \( q \) with a “point at infinity” added. It can be viewed as the set of \( q + 1 \) non-zero points \( (x,y) \) in \( \mathbb{F}_q^2 \) such that \( (x,y) \) and

Note that \( R_{-1} = \langle 2 13(3 11)(4 23)(5 9)(6 21)(8 19)(10 17)(12 15)(14 20)(16 18)(22 24) \rangle \).
\((lx, ly)\) with \(l \in \mathbb{F}_q^*\) are equivalent. That is the set of non-zero points \((x, y)\) modulo scalar multiplication. We can take this to be the set of points \((x, 1)\), \(x \in \mathbb{F}_q\), and \((1, 0)\), the “point at infinity”. The geometric automorphism group is \(PGL(2, q)\), the projective linear group.

1.2.2.2 Affine and Projective Planes

The affine plane, \(AG(2, q)\) of order \(q\) consists of the sets of vertices and lines (1-dimensional affine subspaces) in \(\mathbb{F}_q^2\). Every line has a representation of the form \(y = mx + b\) with \(m, b \in \mathbb{F}_q\), or \(x = a\) with \(a \in \mathbb{F}_q\); in the former case \(m\) is called the slope of the line, and in the later case we say that the slope is infinite. Two lines are parallel if they have no points in common. It is easy to see that this occurs only if the lines have the same slope, possibly infinite.

![Figure 1.7: The affine plane of order 3](image)

For example the affine plane of order 3 (Figure 1.7) has nine vertices and twelve lines. The lines \(\{(0, 0), (0, 1), (0, 2)\}\), \(\{(1, 0), (1, 1), (1, 2)\}\), \(\{(2, 0), (2, 1), (2, 2)\}\) are parallel with slope zero. The lines \(\{(0, 0), (1, 0), (2, 0)\}\), \(\{(0, 1), (1, 1), (2, 1)\}\), and \(\{(0, 2), (1, 2), (2, 2)\}\) are parallel with infinite slope. The line \(\{(0, 1), (1, 2), (2, 0)\}\) has slope 1. In all there are three parallel lines for each slope, a total of twelve lines. In the general case the affine plane of order \(q\) has \(q(q + 1)\) lines.
The projective plane, PG(2, q) of order q is the affine plane of order q with an “additional line at infinity”. The projective plane consists of the non-zero points \((x, y, z)\) in \(\mathbb{F}_q^3\) such that \((x, y, z)\) and \((lx, ly, lz), l \in \mathbb{F}_q^*\), are equivalent. We can then take as the points the elements in

\[
\{(x, y, 1) | x, y \in \mathbb{F}_q\} \cup \{(1, y, 0) | y \in \mathbb{F}_q\} \cup \{(0, 1, 0)\}.
\]

The points \((1, y, 0), y \in \mathbb{F}_q,\) and \((0, 1, 0)\) are the points on the “line at infinity”. As with the affine plane, the lines are the lines with slope \(m\) in the copy of the affine plane \(z = 1\) augmented by an additional point \((1, m, 0)\) at infinity, and the line at infinity \(\{(1, y, 0) | y \in \mathbb{F}_q\} \cup \{(1, 0, 0)\}\). The former lines in PG(2, q) are referred to as lines with slope \(m\).

### 1.2.2.3 Steiner Systems

The affine and projective planes can be described as Steiner systems, a subset of incidence geometries called block designs. Given a finite set \(X\) with \(v\) elements called points, and integers \(t, k, \lambda \geq 1\), a \(t\)-design is a collection, \(\mathcal{B}\), of distinct \(k\)-element subsets of \(X\) (called blocks), such that any \(t\)-element subset of \(X\) is contained in exactly \(\lambda\) blocks. This is also called (see for example [1], [5],[7]) a \(t\)−\((v,k,\lambda)\) design. We assume \(v > k\) so that no block contains all of the elements of \(X\). For a point \(p \in X\) we define the set of blocks incident to \(p\) as \((p) := \{B \in \mathcal{B} | p \in B\}\). Similarly, for a block \(B \in \mathcal{B}\) the set of points in \(X\) incident to \(B\) is \((B) := \{p \in X | p \in B\}\).

There are certain conditions that the parameters \(n, k, v, t,\) and \(\lambda\) must satisfy. The number of blocks that are incident with \(i\) distinct points, where \(0 \leq i \leq t\), is

\[
\lambda_i = \lambda \binom{v-i}{t-i} / \binom{k-i}{t-i}
\]
and is independent of the choice of points. The total number of blocks, \( b \), is given by

\[
b = \lambda \binom{v}{t} / \binom{k}{t}.
\]

Each point belongs to \( r(= \lambda_1) \) blocks, where

\[
bk = vr
\]

\[
\lambda(v - 1) = r(k - 1) \text{ if } t \geq 2.
\]

A symmetric design is a design where the number of points equals the number of blocks, and therefore also the number of points incident to a block equals the number of blocks incident at a point.

From a \( t - (v, k, \lambda) \) design \( D = (X, B) \) we can obtain a derived design or a residual design at either a point or block as follows.

The contraction \( D_p \) of \( D \) at a point \( p \) is the design obtained by deleting \( p \) but retaining only the blocks incident to \( p \), with the point \( p \) deleted from those blocks. Namely, we define \( X_p := X \setminus \{p\} \), \( B_p := \{B \setminus \{p\} \mid B \in (p)\} \) and \( D_p := (X_p, B_p) \). We say that a design \( D = (X, B) \) is an extension of a design \( D' = (X', B') \) if, there is a point \( p \in X \), such that the contraction \( D_p \) is isomorphic to \( D' \). The contraction of an \( t - (v, k, \lambda) \) design is a \((t-1) - (v-1, k-1, \lambda)\) design.

Similarly the contraction \( D_B \) of \( D \) at a block \( B \) is the design obtained by deleting \( B \), as a block and retaining the points incident with \( B \) and the blocks that intersect \( B \). Namely, we define \( X_B := (B) \), \( B_B := \{B' \in B \mid B' \cap B \neq \emptyset\} \) and \( D_B := (X_B, B_B) \).

The residual \( D^B \) of \( D \) at a block \( B \) is the design obtained by removing \( B \) and the points incident with \( B \). Namely, we define \( X^B := X \setminus (B) \), \( B^B := \{B' \setminus (B) \mid B' \in B, B' \neq B\} \) and \( D^B := (X^B, B^B) \). Since we have removed from \( B' \in B^B \) all points in the block \( B \), for \( D^B \) to
be a design, \(|B' \cap B|\) must be constant for all blocks \(B'\) distinct from \(B\). This occurs when \(D\) is a symmetric design, and necessarily \(t = 2\). By [1, Theorem 4.4.2] the residual at a block of a symmetric \(2-(v, k, \lambda)\) design is a \(2-(v-k, k-\lambda, \lambda)\) design.

Similarly the residual \(D^p\) of \(D\) at a point \(p\) is the design obtained by removing \(p\) and the blocks incident with \(p\). Namely, we define \(B^p := B \setminus (p)\) and \(D^p := (X_p, B^p)\). As an example, the incidence complex \(3\{3\}3\) (Example 1.2.4) is the residual of the affine plane of order 3, \(AG(2, 3)\), at the point \((0, 0)\).

If \(\lambda = 1\) and \(t \geq 2\) then the design is an \(S(t, k, v)\) Steiner system. The affine planes \(AG(2, q)\) and projective planes \(PG(2, q)\) provide important examples of Steiner systems.

The affine plane \(AG(2, q)\) of order \(q\) has \(q^2\) points and \(q^2 + q\) lines. Each line contains \(q\) points and each point is on \(q + 1\) lines. Every pair of points is coincident with exactly one line. If lines are called blocks, then \(AG(2, q)\) naturally gives a Steiner system \(S(2, q, q^2)\). Conversely, every Steiner system \(S(2, q, q^2)\) is isomorphic to an \(AG(2, q)\).

The projective plane \(PG(2, q)\) of order \(q\) has \(q^2 + q + 1\) points and lines, each line incident with \(q + 1\) points, with any two points incident to exactly one line. Again, if lines are called blocks, then \(PG(2, q)\) gives a symmetric \(S(2, q + 1, q^2 + q + 1)\) Steiner system. By taking the residual of the Steiner system for \(PG(2, q)\) at any block, we obtain an \(S(2, q, q^2)\) Steiner system, the affine plane \(AG(2, q)\) of order \(q\). Note that the parameter \(q\) does not generally determine a Steiner system \(S(2, q + 1, q^2 + q + 1)\) uniquely.

The affine plane \(AG(2, 2)\) is identical with the complete graph \(K_4\) and the (unique) Steiner system \(S(2, 2, 4)\) and can also be obtained by removing any block and its incident points from the unique Steiner system \(S(2, 3, 7)\). The affine plane \(AG(2, 3)\) (Figure 1.7) has nine points and twelve lines and arises from the (unique) Steiner system \(S(2, 3, 9)\).

There are projective planes that are not projective geometries, that is not isomorphic to \(PG(2, q)\) for some \(q\). These are \(2-(q + 1, q^2 + q + 1, 1)\) designs that satisfy an axiomatic
definition of projective plane. Projective planes do not exist for all orders. It is known that there are no projective planes of order six or ten. Furthermore, there are, up to isomorphism and duality three distinct projective planes of order \( q = 9 \), and so there are multiple \( S(2, 10, 91) \) Steiner systems.

**Example 1.2.6.** The Fano plane \( \text{PG}(2, 2) \)

The \( S(2, 3, 7) \) Steiner system is unique and is known as the *Fano plane* [1] (Figure 1.8). This is the projective plane of order 2. There are seven points (non-zero elements of \( \mathbb{F}_2^3 \)), the blocks are the seven lines consisting of three points that sum to \((0, 0, 0)\). Each pair of points belongs to a unique line. The Fano plane is the unique projective plane of order 2 (lines have three points), \( \text{PG}(2, 2) \).

![Figure 1.8: Fano plane](image)

Identifying the vertex \( abc \) with the integer \( 4a + 2b + c \), the vertices can be identified with the integers \( 1, \ldots, 7 \). The generating subgroups of the automorphism group of the Fano
plane, relative to the base flag \{1, \{1, 2, 3\}\}, are

\[
R_0 = \langle (45)(67), (46)(57), (23)(67), (24)(35) \rangle \\
R_1 = \langle (45)(67), (46)(57), (23)(67), (12)(56) \rangle
\]

The flag stabilizer, \(R_{-1} = \langle (45)(67), (46)(57), (23)(67) \rangle\), is isomorphic to \(D_4\). The generating subgroups \(R_0\) and \(R_1\) are each isomorphic to \(S_4\). The full automorphism group is \(PGL(3, 2) \cong PSL(3, 2) \cong PSL(2, 7)\) of order 168.

**Proposition 1.2.7.** An \(S = S(t, k, n)\) Steiner system is a rank 2 incidence complex, with vertices the points of \(S\) and edges the blocks of \(S\).

**Proof.** We only need to show that \(S = (X, B)\) is strongly flag connected. Given two flags \(\Phi = \{p_1, B_1\}\) and \(\Psi = \{p_2, B_2\}\). If \(p_1 \neq p_2\) and \(B_1 \neq B_2\) there is a block \(B\) that contains \(p_1\) and \(p_2\). Then \(\Phi, \{p_1, B\}, \{p_2, B\}, \Psi\) is a sequence of successively adjacent flags of \(S\). If either \(p_1 = p_2\) or \(B_1 = B_2\) then \(\Phi\) and \(\Psi\) are adjacent. \(\square\)

**Example 1.2.8.** The Projective Plane of order 3, \(PG(2, 3)\).

The projective plane of order 3, \(PG(2, 3) \cong S(2, 4, 13)\), has thirteen points and thirteen lines. Each line has four points, and each point is incident with four lines. The underlying affine plane is the \(S(2, 3, 9)\) Steiner system. Since the \(S(2, 4, 13)\) Steiner system is unique, there is a unique projective plane of order 3.

With the points labeled as in Figure 1.9, the generating subgroups of the automorphism
The subgroups $R_0$ and $R_1$ are isomorphic to the general affine group, $AGL(2, 3)$, of all invertible affine transformations.

**Example 1.2.9.** The Projective Plane of order 4, PG(2, 4), $S(2, 5, 21)$.

We define the field of 4 elements, $\mathbb{F}_4$, over $\mathbb{F}_2$ as $\mathbb{F}_4 \cong \mathbb{F}_2[x]/(x^2 + x + 1)$. Let $\omega$ be a root of $x^2 + x + 1$, then we have $\overline{\omega} := \omega^2 = \omega + 1$. We note that $\omega \overline{\omega} = 1$, that is $\omega^3 = \overline{\omega}^3 = 1$. Thus $\mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}$.

The Mathieu Group $M_{24}$ is a 5-transitive permutation group on 24 elements that is the automorphism group of the extended binary Golay code $C_{24}$ of length 24. For a codeword $C$ in $C_{24}$, a $C$-set is the support of the codeword. The $C$-sets with 8 elements are called octads.

The octads in $C_{24}$ form a unique $S(5, 8, 24)$ Steiner system also with automorphism group $M_{24}$. By contracting three points of the Steiner system in succession we can derive the unique $S(2, 5, 21)$ Steiner system; this is the finite projective plane of order 4, PG(2, 4). The stabilizer subgroup of $M_{24}$ of the three points is isomorphic to $PSL(3, 4)$, which is also
referred to as the Mathieu group $M_{21}$.

The unique $S(2,5,21)$ Steiner system consists of 21 points and 21 lines, each line coincident with 5 points, each point on 5 lines. The points are the non-zero elements of $\mathbb{F}_4^3$ modulo scalar multiplication so that the point $(x, y, z)$ and $(ax, ay, az)$ for $a \in \mathbb{F}_4$ are identical. We then have the following points.

1. $(x, y, 1)$ for $x, y \in \mathbb{F}_4$, the points of the affine plane of order 4, $AG(2, 4)$,

2. $(1, y, 0)$ for $y \in \mathbb{F}_4$, giving four points on the line at infinity

3. $(0, 1, 0)$, giving the fifth point on the line at infinity.

The 21 lines consist of five sets of four lines and the line at infinity. Each of the five sets of four lines is determined by a slope either in $\mathbb{F}_4$ or $\infty$. The four lines with slope $\infty$ have point sets $\{(x, y, 1) | y \in \mathbb{F}_4\} \cup \{(0, 1, 0)\}$ with $x \in \mathbb{F}_4$. The four lines with slope $m$, $m \in \mathbb{F}_4$
have four points of the form \((x, y, 1)\) with \(y = mx + b, \ b \in \mathbb{F}_4\), as well as a point at infinity, namely \((1, m, 0)\). Figure 1.10 shows PG(2, 4) with the points at infinity at the left.

![Figure 1.10: Projective plane of order 4](image)

The automorphism group of PG(2, 4) is \(P\Gamma L(3, 4)\). Its subgroup \(PSL(3, 4)\) is flag-transitive and \(P\Gamma L(3, 4) \cong PSL(3, 4) \rtimes S_3\) is a maximal subgroup of \(M_{24}\).

The action of \(P\Gamma L(3, 4)\) on PG(2, 4) includes mappings that involve scalar multiplication and the outer automorphism \(\omega \mapsto \overline{\omega}\) of \(Gal(\mathbb{F}_4/\mathbb{F}_2) \cong C_2\). Scalar multiplication extends \(PSL(3, 4)\) to \(PGL(3, 4)\) by \(C_3\). The outer automorphism simply extends \(PGL(3, 4)\) to \(P\Gamma L(3, 4) \cong PGL(3, 4) \rtimes C_2\).

Our construction of PG(2, 4) from the Golay code \(C_{24}\) started with \(S(5, 8, 24)\) and contracted to \(S(2, 5, 21)\). We note that we can reverse the construction and extend the Steiner system \(S(2, 5, 21)\) to the Steiner systems \(S(3, 6, 22), S(4, 7, 23)\) and finally \(S(5, 8, 24)\) (see [5] and [31]). Thus from \(PSL(3, 4)\) one can build up \(M_{24}\). (Note that from the automorphism group of PG(2, 3) one can similarly construct the Mathieu group \(M_{12}\) as the automorphism
group of the extended ternary Golay code.)

1.2.3 Incidence Complexes of Higher Ranks

1.2.3.1 Regular Polytopes in Unitary Complex Space

The generalized Schl"afli symbol $r_0\{q_1\}r_1\{q_2\}r_2\cdots r_{k-2}\{q_{k-1}\}r_{k-1}$ is associated with the group

$$\langle \rho_0, \rho_1, \ldots, \rho_{k-1} | \rho_i^{r_i} = 1, (\rho_i \rho_{i+1})^{q_{i+1}/2} = (\rho_{i+1} \rho_i)^{q_{i+1}/2}, \rho_i \rho_j = \rho_j \rho_i \text{ if } |i - j| \geq 2 \rangle.$$

Here the notation $(\rho_i \rho_{i+1})^{q_{i+1}/2} = (\rho_{i+1} \rho_i)^{q_{i+1}/2}$ means that $\rho_i \rho_{i+1} \rho_i \cdots = \rho_{i+1} \rho_i \rho_i \cdots$ with $q_{i+1}$ total terms $\rho_i$ or $\rho_{i+1}$ on each side ([11, Chapter 12]). This group can be represented by the diagram shown in Figure 1.11

Figure 1.11: Diagram for the symmetry group of a complex regular polytope

and is denoted $r_0[q_1]r_1[q_2]r_2\cdots r_{k-2}[q_{k-1}]r_{k-1}$.

Shephard [51] described all of the regular complex polytopes in unitary complex space. We present the following two examples of complex polyhedra for reference in Chapter Three.

**Example 1.2.10.** The Hessian Polyhedron, $3\{3\}3\{3\}3$

This is a three dimensional complex polyhedron with facets and vertex-figures isomorphic to $3\{3\}3$. There are 27 vertices and facets, and 72 edges. The symmetry group has order 648. For details see [11, Section 12.3].
Example 1.2.11. $2\{4\}3\{3\}3$

This is a complex polyhedron with facets isomorphic to the Thomsen Graph ($2\{4\}3$) and vertex-figures isomorphic to $3\{3\}3$. There are 54 vertices, 216 edges and 72 facets. Its dual, $3\{3\}3\{4\}2$, has facets isomorphic to $3\{3\}3$ and a vertex-figure isomorphic to the complex 3-square $3\{4\}2$. The dual has 72 vertices, 216 edges and 54 facets. For details see [11, Section 12.4]. Note that the symmetry group has order 1296 and that $2[4]3[3]3 = C_2 \rtimes 3[3]3[3]3$.

1.2.3.2 Complex Cubes of Higher Dimensions

The generalized complex cube of dimension $k$ with $v$ vertices per edge, denoted by $\gamma_v^k$, has $v^k$ vertices, and is in the class $[v, 2, \ldots, 2]$. The automorphism group is $S_v \wr S_k$, but its unitary symmetry group is $C_v \wr S_k$.

Again, for $V = \{1, \ldots, v\}$, if we label the $v^k$ vertices $(v_1, \ldots, v_k)$ with $v_i \in V$, the edges are of the form the

$$(V, v_2, \ldots, v_k), (v_1, V, v_3, \ldots, v_k), \ldots, (v_1, \ldots, v_{k-1}, V),$$

with

$$(v_1, \ldots, v_i-1, V, v_{i+1}, \ldots, v_k) := \{(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k) | v \in V\}$$

for each $i$. In general, the $j$-faces are of the form

$$(v_1, \ldots, v_{i_1-1}, V, v_{i_1+1}, \ldots, v_{i_2-1}, V, v_{i_2+1}, \ldots v_{i_j-1}, V, v_{i_j+1}, \ldots, v_k),$$

for all $i_1, \ldots, i_j$ with $1 \leq i_1 \leq \cdots \leq i_j \leq k$, where again the positions label the places where the coordinate is allowed to vary freely.

From this we see that $\gamma_v^k$ has $kv^{k-1}$ edges and in general $\binom{k}{j}v^{k-j}$ faces of dimension $j$. 29
The dual of $\gamma^\circ_k$ is the generalized cross polytope $\beta^\circ_k$.

1.2.3.3 Affine and Projective Space

As a point set, an affine space with dimension $m$ of order $q$, $AG(m, q)$, is an $m$-dimensional vector space over the finite field $\mathbb{F}_q$. The subspaces in $AG(m, q)$ are the affine subspaces of this vector space. The automorphism group is the general affine group $AGL(m, q)$.

When $m = q = 3$ we have $AG(3, 3)$ which consists of 27 points. Through each point there are 13 lines and 13 planes.

As a point set, the projective space of dimension $m$ and order $q$, $PG(m, q)$, is the quotient of $\mathbb{F}_q^{m+1} \setminus \{0\}$ by the equivalence relation $(a_0, \ldots, a_m) \sim (b_0, \ldots, b_m)$ if and only if there exists an $\alpha \in \mathbb{F}_q$ such that $a_i = \alpha b_i$ for $i = 0, \ldots, m$. In this way the points, lines, ..., hyperplanes of $PG(m, q)$ are the linear subspaces of dimension 1, 2, ..., $m$ of the vector space $\mathbb{F}_q^{m+1}$.

1.3 Quotients

Let $\mathcal{K}$ and $\mathcal{L}$ be $n$-incidence complexes. Recall that a homomorphism $\gamma : \mathcal{K} \rightarrow \mathcal{L}$ is a map that preserves incidences in one direction, that is $F \gamma \leq G \gamma$ in $\mathcal{L}$ whenever $F \leq G$ in $\mathcal{K}$. A homomorphism $\gamma : \mathcal{K} \rightarrow \mathcal{L}$ is adjacency preserving if $\gamma$ maps adjacent flags of $\mathcal{K}$ onto adjacent flags of $\mathcal{L}$. A homomorphism $\gamma$ is rank preserving if faces of $\mathcal{K}$ are mapped to faces of $\mathcal{L}$ of the same rank. A homomorphism $\gamma$ is a rap-map if $\gamma$ is rank preserving and adjacency preserving. A surjective rap-map $\varphi$ is called a covering (map).

We say that $\varphi$ is weakly adjacency preserving if $\gamma$ maps a pair of adjacent flags of $\mathcal{K}$ onto a pair of flags of $\mathcal{L}$ that are either adjacent or identical. A weakly adjacency and rank preserving homomorphism is a weak rap-map. A surjective weak rap-map $\varphi$ is called a weak covering map.
Figure 1.12 illustrates an example of a covering $\gamma: \{6\} \to \{3\}$ between the hexagon $\mathcal{K} = \{6\}$ and the triangle $\mathcal{L} = \{3\}$ given by $i, i + 3 \mapsto i$, for $i = 1, 2, 3$. The edges (vertex-pairs) are mapped by $(i, i + 1), (i + 3, i + 4) \mapsto (i, i + 1)$. Thus $\gamma$ wraps the hexagon twice around the triangle.

![Figure 1.12: Rap-map from \{6\} onto \{3\}](image)

Clearly $\gamma$ is surjective and rank preserving. To see that $\gamma$ is also adjacency preserving, note that $\gamma$ acts “locally” (in the neighborhood of a flag) like an isomorphism. Thus $\gamma$ is a covering.

Returning to the general discussion, let $\mathcal{K}$ be an $n$-incidence complex and $\Sigma$ be a subgroup of $\Gamma(\mathcal{K})$. The set of orbits of $\Sigma$ in $\mathcal{K}$ is denoted by $\mathcal{K}/\Sigma$, and the orbit of a face $F$ of $\mathcal{K}$ by $F \cdot \Sigma$. Introduce a partial ordering on $\mathcal{K}/\Sigma$ as follows: if $\hat{F}, \hat{G} \in \mathcal{K}/\Sigma$, then $\hat{F} \leq \hat{G}$ if and only if $\hat{F} = F \cdot \Sigma$ and $\hat{G} = G \cdot \Sigma$ for some faces $F$ and $G$ of $\mathcal{K}$ with $F \leq G$. The set $\mathcal{K}/\Sigma$ together with this partial order is the *quotient of $\mathcal{K}$ with respect to $\Sigma$*. The mapping $\pi: \mathcal{K} \to \mathcal{K}/\Sigma$ given by $F\pi = F \cdot \Sigma$ is called the *canonical projection*. A *quotient complex* of $\mathcal{K}$ is a quotient of $\mathcal{K}$ which is also a complex. It is shown in ([32], Section 2D), that for an abstract polytope $\mathcal{K}$, $\mathcal{K}/\Sigma$ is a flag-connected poset of rank $n$ which has the properties (I1) and (I2). This result carries over to incidence complexes (essentially with the same proof as in [32]).

Continuing the example of Figure 1.12, the image of the rap-map $\gamma$ is also describable as
the quotient of $\mathcal{K} = \{6\}$ derived from its central symmetry. In this case $\Sigma = \langle (1\ 4)(2\ 5)(3\ 6) \rangle$. The flag $f = \{F_{-1}, 1, (1, 6), \mathcal{K}\}$, becomes $f\Sigma = \{F_{-1}, 1, (1, 3), \mathcal{K}/\Sigma\}$. The orbit of $(1, 6)$ under $\Sigma$ is $(1, 6)\Sigma = \{(1, 6), (3, 4)\}$.

If instead, we take $\Sigma = \langle (1\ 6)(2\ 5)(3\ 4) \rangle$, we get a quotient $\mathcal{K}/\Sigma$ with three vertices given by the orbits
\[ \{1, 6\}, \{2, 5\}, \{3, 4\}, \]
and four edges
\[ \{\{(1, 6)\}, \{(1, 2), (5, 6)\}, \{(2, 3), (4, 5)\}, \{(3, 4)\}\} \]
Thus $\mathcal{K}/\langle (1\ 6)(2\ 5)(3\ 4) \rangle$ is not an incidence complex since property (I4) is not met. For example, the section $\langle\{(1, 6)\}/F_{-1}\rangle$ of rank 1 has only one face $H$, namely $\{1, 6\} = 1 \cdot \Sigma$, such that $F_{-1} < H < \{(1, 6)\}$. 
Chapter 2

Power Complexes

2.1 Basics

In this section we will review the construction of the power complexes $n^K$, a family of incidence complexes with $n$ vertices on each edge, and with each vertex-figure isomorphic to $K$ (see [47] [48] and [32, Section 8d] for $n = 2$). These power complexes were first discovered by Danzer in the early 1980’s; however, the construction announced in [15] was never published by Danzer, and first appeared in print in [48].

The power complexes with $n = 2$ and $K$ an abstract polytope, are abstract polytopes and have attracted a lot of attention (see [32, Chapter 8]). In a sense these power complexes are generalized cubes; and in the cases where $K$ has simplex facets, they can be viewed as cubical complexes.

For the case of arbitrary $n$ and $K$ a simplex, the resulting structures are complex hyper-cubes (see [50], [51] and [11]). They have also gained interest for redundant communications networks (see [3], [14] and [20]).
2.1.1 Construction of Power Complexes

A complex $\mathcal{K}$ is *vertex-describable* if its faces are uniquely determined by their vertex sets. Note that a complex is vertex-describable if and only if its underlying face poset can be represented by a family of subsets of the vertex-set ordered by inclusion. If a complex $\mathcal{K}$ is a lattice then it is vertex-describable. For example, the torus map $\mathcal{K} = \{4,4\}_{(s,0)}$ is vertex-describable if and only if $s \geq 3$. The faces of a vertex-describable complex are also vertex-describable.

Let $n \geq 2$ and define $N := \{1, \ldots, n\}$. Suppose $\mathcal{K}$ is a finite vertex-describable $d$-incidence complex with $v$ vertices and vertex set $V := \{1, \ldots, v\}$. Then $\mathcal{P} := n^\mathcal{K}$ is a finite $(d + 1)$-complex with vertex set

$$N^v = \bigotimes_{i=1}^v N,$$

the cartesian product of $v$ copies of $N$. The $n^v$ vertices of $n^\mathcal{K}$ are written as row vectors $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_v)$. Since $\mathcal{K}$ is vertex-describable we can view the faces $F$ of $\mathcal{K}$ as subsets of $V$. For the $j$-faces of $n^\mathcal{K}$, we take for any $(j - 1)$-face $F$ of $\mathcal{K}$ and any vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_v)$ in $N^v$, the subset $F(\varepsilon)$ of $N^v$ given by

$$F(\varepsilon) := \{(\eta_1, \ldots, \eta_v) \in N^v | \eta_i = \varepsilon_i \text{ if } i \notin F\}.$$  

That is, the $j$-face $F(\varepsilon)$ of $n^\mathcal{K}$ consists of the vectors in $N^v$ that coincide with $\varepsilon$ in precisely the components determined by the vertices of $\mathcal{K}$ not lying in the $(j - 1)$-face $F$ of $\mathcal{K}$. We can rewrite this as

$$F(\varepsilon)_j = \begin{cases} 
\varepsilon_j & j \notin F \\
N & \text{otherwise.}
\end{cases}$$

Or abusing notation,

$$F(\varepsilon) := \left(\bigotimes_{i \in F} N\right) \times \left(\bigotimes_{i \notin F} \{\varepsilon_j\}\right).$$
For the remainder of this section, unless otherwise stated all incidence complexes $K$ are vertex-describable. We start with the following basic results (see [47]).

**Lemma 2.1.1.** For any faces $F, F'$ of $K$ and $\varepsilon, \varepsilon' \in N^v$, we have $F(\varepsilon) \subseteq F'(\varepsilon')$ if and only if $F \leq F'$ and $\varepsilon_j = \varepsilon'_j$ for all $j \notin F'$. Therefore in designating the higher rank face we may always take $\varepsilon' = \varepsilon$.

*Proof.* Clearly $F(\varepsilon) \subseteq F'(\varepsilon')$ if and only if the vertex-set of $F$ lies in the vertex-set of $F'$ (that is $F \leq F'$ in $K$) and $\varepsilon$ and $\varepsilon'$ agree on the $j^{th}$ components with $j \notin F'$. This is what we have claimed.

**Lemma 2.1.2.** For any faces $F, F'$ of $K$ and $\varepsilon, \varepsilon' \in N^v$, we have $F(\varepsilon) = F'(\varepsilon')$ if and only if $F = F'$ and $\varepsilon_j = \varepsilon'_j$ for all $j \notin F(= F')$.

*Proof.* This follows from Lemma 2.1.1 since we have both $F(\varepsilon) \subseteq F'(\varepsilon')$ and $F'(\varepsilon') \subseteq F(\varepsilon)$. This is true if and only if $F \leq F'$ and $F' \leq F$, that is $F = F'$, and $\varepsilon_j = \varepsilon'_j$ for $j \notin F(= F')$.

**Proposition 2.1.3.** Let $K$ be a $d$-complex, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_v)$ a vertex of $n^K$. Then the vertex-figure at $\varepsilon$ in $n^K$ is isomorphic to $K$.

*Proof.* The vertex-figure at the vertex $\varepsilon$, identified with $F_{-1}(\varepsilon)$ is the section $F_d(\varepsilon)/\varepsilon$. By Lemma 2.1.1 we have $G(\varepsilon) \subseteq F(\eta)$ if and only if $G \leq F$ in $K$ and $\varepsilon_i = \eta_i$ for all $i \notin F$.

Define the mapping $\varphi : K \to F_d(\varepsilon)/\varepsilon$ by $G \mapsto G(\varepsilon)$. We show that $\varphi$ is an isomorphism of complexes.

For any face $G(\eta)$ of $F_d(\varepsilon)/\varepsilon$, since $\eta_i = \varepsilon_i$ for all $i$ not in $G$, we have $G\varphi = G(\varepsilon) = G(\eta)$, so $\varphi$ is surjective. By Lemma 2.1.2, $G(\varepsilon) = G'(\varepsilon)$ if and only if $G = G'$, so $\varphi$ is also injective.

More generally, by Lemma 2.1.1, $G(\varepsilon) \subseteq G'(\varepsilon)$ if and only if $G \leq G'$ in $K$, so $\varphi$ also preserves incidence. Thus $\varphi$ is an isomorphism.
Proposition 2.1.4. If $F$ is a $(j - 1)$-face of $K$ and $F = F/F_{-1}$ is the $(j - 1)$-complex determined by $F$, then the sections of $P = n^K$ corresponding to the $j$-faces of $P$ of the form $F(\varepsilon)$ with $\varepsilon \in N^v$ are isomorphic to the power complex $n^F$ of rank $j$.

Proof. Let $K$ be an incidence complex with $v$ vertices. Let $F$ be a $(j - 1)$-face of $K$ with $v_F$ vertices, and let $\varepsilon$ be a vector in $N^v$. By Lemma 2.1.1, for any face $G(\eta)$ of $P$ with $G(\eta) \subseteq F(\varepsilon)$, we have $G \leq F$ in $K$ and $\eta_i = \varepsilon_i$ for all $i \not\in F$; in other words, the vectors $\varepsilon$ and $\eta$ agree on each component representing a vertex $i$ of $K$ that lies outside of $F$. Recall that $\emptyset$ is the rank $-1$ face of $P$.

Let $\tau: F(\varepsilon)/\emptyset \to n^F$ be given by $G(\eta) \mapsto G(\eta')$, where $G(\eta) \subseteq F(\varepsilon)$ and the $v_F$-vector $\eta'$ is defined by

$$
\eta'_i = \begin{cases} 
N & i \in G \\
\eta_i & i \in F \setminus G.
\end{cases}
$$

Then it is straightforward to show that $\tau$ is an isomorphism. \hfill \square

Proposition 2.1.5. If $K$ is a vertex-describable incidence complex then $P = n^K$ is vertex-describable.

Proof. If $F(\varepsilon), G(\varepsilon')$ are two faces of $P$ with the same vertex set, then $F = G$ in $K$ and $\varepsilon, \varepsilon'$ are vertices of both $F(\varepsilon)$ and $G(\varepsilon')$; in particular, $\varepsilon_i = \varepsilon'_i$ for all $i \notin F = G$. It follows that $F(\varepsilon) = G(\varepsilon')$ in $P$. \hfill \square

Proposition 2.1.6. Let $K$ be a finite, vertex-describable, incidence complex of rank $d$ with $v$ vertices. Then the power complex $P = n^K$ is an incidence complex of rank $d + 1$.

Proof. Property (I1) for all incidence complexes clearly holds. By Lemma 2.1.1 all maximal chains of $P$ are of the form

$$
\emptyset < F_{-1}(\varepsilon) < F_0(\varepsilon) < \cdots < F_d(\varepsilon),
$$

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when $\varepsilon \in N^v$ and $F_{-1} < F_0 < \cdots < F_d$ is a flag of $\mathcal{K}$. Hence all flags of $\mathcal{P}$ have length $d + 2$. Thus (I2) holds.

We prove (I3) inductively, building on Propositions 2.1.3 and 2.1.4. In particular, as the faces of $n^\mathcal{K}$ are power complexes of lower rank and the vertex-figures are isomorphic to $\mathcal{K}$, it suffices to verify that $n^\mathcal{K}$ is connected.

Suppose $G(\varepsilon)$ and $G(\varepsilon')$ are two proper faces of $N^\mathcal{K}$. Clearly, since $\varepsilon$ and $\varepsilon'$ are vertices of $G(\varepsilon)$ and $G(\varepsilon')$ respectively, it suffices to show that $\varepsilon$ and $\varepsilon'$ can be joined by an edge path in the 1-skeleton of $n^\mathcal{K}$.

To this end, consider an edge path of $\mathcal{K}$ that passes through every vertex of $\mathcal{K}$ at least once, and let $F_1, \ldots, F_m$ be its vertices, occurring in this order. Now construct a sequence

$$\varepsilon = \varepsilon^0, \varepsilon^1, \ldots, \varepsilon^{m-1}, \varepsilon^m = \varepsilon'$$

as follows. In each step, at most one component will be changed. More precisely, for $i \geq 1$, the vertex $\varepsilon^i$ is obtained from $\varepsilon^{i-1}$ by either replacing its component corresponding to $F_i$ by the corresponding component of $\varepsilon'$ if they do not already coincide, or leaving it unchanged if they do coincide. Then $\varepsilon^{i-1}, \varepsilon^i$ are vertices of $F_i(\varepsilon^{i-1}) = F_i(\varepsilon^i)$ for $i = 1, \ldots, m$ and at the final stage $\varepsilon^m = \varepsilon'$. Thus $F_1(\varepsilon^1), F_2(\varepsilon^2), \ldots, F_m(\varepsilon^m)$ is the desired edge-path joining $\varepsilon$ and $\varepsilon'$ in $n^\mathcal{K}$.

Finally, for (I4) note that every edge (rank 1 face) has exactly $n$ vertices, and that the vertex-figures, $\mathcal{K}$, of $n^\mathcal{K}$ already satisfy the property (I4). It follows that $n^\mathcal{K}$ itself satisfies (I4). \qed
2.1.2 Properties of Power Complexes

If $K = \alpha_{v-1}$ (see [10]), the $(v - 1)$-simplex (with $v$ vertices), then $n^K$ is combinatorially equivalent to the complex $v$-cube

$$\gamma_v^n = n\{4\}2\{3\}2 \cdots 2\{3\}2$$

in unitary complex $v$-space $\mathbb{C}^v$. That is, $n^{\alpha_{v-1}} = \gamma_v^n$. The unitary complex symmetry group of $\gamma_v^n$ is isomorphic to $C_n \wr S_v$ (see Corollary 3.2.2, also [11] and [50]). However, the combinatorial automorphism group of $\gamma_v^n$ is much larger when $n > 2$ and we will show in Section 2.2 that it is in fact $S_n \wr S_v$. The case when $n = 2$ always gives the ordinary real $v$-cube $\gamma_v := \gamma_v^2 = \{4, 3^{v-2}\}$ (see [10]).

The combinatorics of the complex square $\gamma_2^n = n\{4\}2$ in $\mathbb{C}^2$ is illustrated in Figure 2.1 for $n = 3$. There are 9 vertices (denoted $ij$ with $i, j = 1, 2, 3$), each contained in 2 complex edges (drawn as 3-cycles), as well as 6 complex edges, each containing 3 vertices.

![Figure 2.1: Combinatorics of the complex square $\gamma_3^3$](image)

Now let $K$ be an arbitrary incidence complex of rank $d$ and let $0 \leq j \leq d - 1$. Recall that the $j$-skeleton, $\text{skel}_j(K)$, of $K$ is the incidence complex of rank $j + 1$ whose faces of rank less than or equal to $j$ are those of $K$, with the partial order inherited from $K$; as greatest

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face, of rank $j + 1$, we may simply take the greatest face of $\mathcal{K}$.

The following lemma says that taking skeletons and taking power complexes are commuting operations.

**Lemma 2.1.7.** Let $\mathcal{K}$ be a finite vertex-describable $d$-complex, let $0 \leq j \leq d - 1$ and let $n \geq 2$. Then

$$\text{skel}_{j+1}(n^\mathcal{K}) = n^{\text{skel}_j(\mathcal{K})}.$$  

**Proof.** The proof is straightforward. First note that a skeleton of a vertex-describable complex is again vertex-describable, with the same vertex set as the underlying complex. The proper faces of $\text{skel}_{j+1}(n^\mathcal{K})$ are the faces $F(\varepsilon)$ of $n^\mathcal{K}$, where $F$ has rank at most $j$ and $\varepsilon$ lies in $N^v$. On the other hand, the proper faces of $n^{\text{skel}_j(\mathcal{K})}$ are of the form $F(\varepsilon)$ where $F$ is a face of $\text{skel}_j(\mathcal{K})$ of rank at most $j$ and $\varepsilon$ lies in $N^v$. But the faces of $\mathcal{K}$ of rank at most $j$ are precisely the faces of $\text{skel}_j(\mathcal{K})$ of rank at most $j$. The lemma follows. 

Suppose $n \geq 2$ and $\mathcal{K} = \{\}_v$, the $v$-edge, that is the (unique) complex of rank 1 with $v$ vertices. Now identifying $\mathcal{K}$ with $\text{skel}_0(\alpha_v-1)$ we have

$$n^{\{\}_v} = n^{\text{skel}_0(\alpha_v-1)} = \text{skel}_1(n^{\alpha_v-1}) = \text{skel}_1(\gamma_v^n).$$ \hspace{1cm} (2.1.5)

Thus the 2-complex $n^{\{\}_v}$ is isomorphic to the 1-skeleton of the unitary complex $v$-cube $\gamma_v^n$ described above.

We have as a corollary to Lemma 2.1.7 the following more general result.

**Proposition 2.1.8.** Let $\{\}_v$ be the (unique) $v$-edge. For any complex $\mathcal{K}$ with $v$ vertices, $\text{skel}_1(n^\mathcal{K})$ is isomorphic to $n^{\{\}_v}$. That is, the edge-graph of the power complex derived from any incidence complex with $v$ vertices is isomorphic to the 2-incidence complex $n^{\{\}_v}$.

**Proof.** Let $\mathcal{K}$ be any incidence complex with $v$ vertices and $n \geq 2$. The edge-graph of a complex is its 1-skeleton. By the Lemma, $\text{skel}_1(n^\mathcal{K}) = n^{\text{skel}_0(\mathcal{K})}$. However, $\text{skel}_0(\mathcal{K}) = \{\}_v$. 

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For \( n = 2 \), since \( 2^{(p)} = \{4, p\mid 4^{\lfloor p/2 \rfloor}\} \) [32], the edge-graph of the power complex of any incidence complex with \( p \) vertices is isomorphic to \( \text{skel}_1(\{4, p\mid 4^{\lfloor p/2 \rfloor}\}) \).

### 2.2 The Automorphism Group

Let \( K \) be an arbitrary vertex-describable \( d \)-complex, and \( n \) an integer with \( n \geq 2 \). We are interested in determining the full automorphism group \( \Gamma(n^K) \) of \( n^K \).

For a vertex \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_v) \in N^v \) we let \( I := \{i \in V| \varepsilon_i \neq 1\} \) denote its support; if \( |I| = k \), we say that \( \varepsilon \) has weight \( k \).

**Lemma 2.2.1.** Let \( \varepsilon \) be a vertex of \( n^K \) and let \( \varphi \) be an automorphism of \( n^K \) fixing \( \varepsilon \). Then \( \varphi \) is induced by an automorphism \( \hat{\varphi} \) of \( K \). In particular, \( G(\varepsilon)\varphi = (G\hat{\varphi})(\varepsilon) \) for all faces \( G \) of \( K \).

**Proof.** Since \( \varphi \) fixes \( \varepsilon \), every face \( G(\varepsilon) \) must be mapped under \( \varphi \) to a face of the form \( G'(\varepsilon) \) (with the same \( \varepsilon \)), where \( G' \) is a face of \( K \). Moreover, if \( G(\varepsilon) \subseteq H(\varepsilon) \) for faces \( G \) and \( H \) of \( K \) (and thus \( G \leq H \)), then also \( G'(\varepsilon) = G(\varepsilon)\varphi \subseteq H(\varepsilon)\varphi = H'(\varepsilon) \), since \( \varphi \) is an automorphism of \( n^K \), and hence \( G' \leq H' \) in \( K \). Conversely, if \( G' \leq H' \) in \( K \), then \( G'(\varepsilon) \subseteq H'(\varepsilon) \) and hence \( G(\varepsilon) \subseteq H(\varepsilon) \) since \( \varphi \) is an automorphism, and therefore \( G \leq H \) in \( K \). Thus in \( K \) we have \( G \leq H \) if and only if \( G' \leq H' \). It follows that the mapping \( G\hat{\varphi} := G' \), for a face \( G \) of \( K \), is an automorphism of \( K \). \( \square \)

### 2.2.1 Main Theorem

**Theorem 2.2.2.** Let \( n \geq 2 \) and \( K \) a vertex-describable complex with automorphism group \( \Gamma(K) \). Then the power complex \( n^K \) has automorphism group isomorphic to \( S_n \wr \Gamma(K) \). Based on this result we also use the alternate notation for \( n^K \) of \( n \wr K \).
Proof. First we will show that $S_n \wr \Gamma(K)$ acts as a group of automorphisms on $n^K$.

Clearly $\Gamma(n^K)$ contains a subgroup isomorphic to $S_n^v$. For each $i$ in $V$, a copy of $S_n$ acts on the $i$th component of the vectors of $N^v$ leaving all other components unchanged. This action on the vertex-set $N^v$ induces an action as a group of automorphisms on $n^K$.

Additionally, $\Gamma(K)$ is naturally embedded in $\Gamma(n^K)$ as a subgroup of the vertex stabilizer of $\varepsilon = (1, \ldots, 1)$ in $\Gamma(n^K)$. An automorphism $\phi$ of $K$ determines an automorphism $\hat{\phi}$ of $n^K$ as follows. For a vertex $\eta = (\eta_1, \ldots, \eta_v)$ of $n^K$, define

\[ \eta \hat{\phi} := (\eta_{1\phi}, \ldots, \eta_{v\phi}) =: \eta_{\phi}, \quad (2.2.1) \]

and, more generally, for the faces $F(\eta)$ of $n^K$, define

\[ F(\eta) \hat{\phi} := (F(\phi))(\eta_{\phi}). \quad (2.2.2) \]

Then, for $\eta = (\eta_1, \ldots, \eta_v) = (1, \ldots, 1)$, we have $\eta_i = \eta_{i\phi}$ so $\hat{\phi}$ fixes $(1, \ldots, 1)$ as desired.

Since $\phi$ preserves rank, $\hat{\phi}$ preserves rank. We need to show that $\hat{\phi}$ is bijective and preserves incidence.

Since $\phi$ is an automorphism of $K$, for any $G \in K$ and $\eta \in N^v$ there is a $G'$ in $K$ and $\eta' \in N^v$ so that $G'\phi = G$ and $\eta = \eta'_{\phi}$. Therefore, $\hat{\phi}$ is surjective.

To prove that $\hat{\phi}$ is injective and incidence preserving let $G(\eta) \hat{\phi} \subseteq G'(\eta') \hat{\phi}$ in $n^K$, that is, $(G(\phi))(\eta_{\phi}) = (G'(\phi))(\eta'_{\phi})$. Then, by Lemma 2.1.1, $G\phi \subseteq G'\phi$ in $K$ and $\eta_{j\phi} = \eta'_{j\phi}$ for all $j$ with $j\phi$ not contained in $G'\phi$. Since $\phi$ is an automorphism of $K$ it follows that $G \subseteq G'$ in $K$ and $\eta_i = \eta'_i$ for all $i$ not contained in $G'$. Thus $G(\eta) \subseteq G'(\eta')$. Clearly, each of the steps can be reversed, so $G(\eta) \subseteq G'(\eta')$ also implies $G(\eta) \hat{\phi} \subseteq G'(\eta') \hat{\phi}$. Thus $\hat{\phi}$ preserves incidence. Moreover, applying these considerations with $G(\eta) \hat{\phi} = G'(\eta') \hat{\phi}$ establishes injectivity.

The two subgroups $S_n^v$ and $\Gamma(K)$ combine to give the wreath product $S_n \wr \Gamma(K) = S_n^v \times$
Γ(K). For σ ∈ Sn and ϕ ∈ Γ(K) the typical element θ = σϕ of Sn × Γ(K) acts on F(ε) according to

\[ F(ε)θ := F(εσ)ϕ = (Fϕ)((εσ)_ϕ). \] (2.2.3)

This is the wreath product action defined by ϕ and σ, so θ ∈ Sn ⋊ Γ(K). Since both Sn and Γ(K) are subgroups of Γ(nK), the wreath product Sn ⋊ Γ(K) is also a subgroup of Γ(nK).

We will now show that Γ(nK) is also a subgroup of Sn ⋊ Γ(K). To this end, let Φ := \{F_{-1}, F_0, \ldots, F_d\} be the base flag of K, and let Φ(1) := \{1, F_0(1), \ldots, F_{d-1}(1)\} be the corresponding base flag of nK with the vertex 1 := (1, \ldots, 1). (Here we suppressed the improper faces, ∅ and F_d(1) = Nv, of nK.) For i ∈ V and j ∈ N, let 1^j_i := (1, \ldots, 1, j, 1, \ldots, 1) be the vector obtained from 1 by replacing the i^{th} component by j. In designating edges of nK, we find it convenient to use two separate notations for vertices of K, namely 1, \ldots, v and G_1, \ldots, G_v, with the understanding that G_i is identified with i for i = 1, \ldots, v. The first notation views a vertex of K as an element of V, and the second emphasizes that a vertex is a face of K (of rank 0). Notice that G_i(1) is an edge of nK with vertices 1^j_i for j ∈ N.

Now suppose an element ρ ∈ Γ(nK) fixes the base flag Φ(1) of nK, so in particular 1^j_1 = 1. We will show that there are elements τ ∈ Γ(K) and σ ∈ Sn such that ρτσ is the identity automorphism of nK. We begin by constructing τ.

First note that, since ρ fixes 1, it induces an automorphism on the vertex-figure F_d(1)/1 of nK at 1, which is isomorphic to K by Proposition 2.1.3. If τ is the automorphism of K corresponding to the restriction of ρ^{-1} to the vertex-figure F_d(1)/1, then ρ' = ρτ is an automorphism of nK that acts trivially on the entire vertex-figure F_d(1)/1 at 1. Thus F(1)ρ' = F(1) for all faces F of K (see Figure 2.2). Furthermore, since each edge G_i(1) is fixed under ρ', the vertices 1^j_i of G_i(1) are permuted among each other by ρ' (see Figure 2.3).

Next we construct σ by viewing ρ' as a permutation on N^v. Recall that Sn is a subgroup
Figure 2.2: Faces containing \( 1 \), fixed under \( \rho' \)

Figure 2.3: Vertices of edge \( G_i(1) \)

of \( \Gamma(n^K) \). Define \( S_n^{(i)} := \langle 1 \rangle \times \cdots \times \langle 1 \rangle \times S_n \times \langle 1 \rangle \cdots \langle 1 \rangle \), where the \( S_n \) occurs in position \( i \). Then \( S_n^v = S_n^{(1)} \times \cdots \times S_n^{(v)} \).

For each \( i \in V \) there exists an automorphism \( \sigma_i \) of \( n^K \) in \( S_n^{(i)} \) such that \( 1^j_i \rho' \sigma_i = 1^j_i \) for all \( j \), and \( 1^j_i \rho' \sigma_i = 1^j_k \rho' \) for all \( k \neq i \) and all \( j \). Setting \( \sigma := \sigma_1 \sigma_2 \cdots \sigma_v \) and taking
\( \rho'' := \rho' \sigma = \rho \tau \sigma \), we observe that \( \rho'' \) fixes the vertices and edges in Figure 2.4, that is, all vertices \( \mathbf{1}_i^j \) with \( i \in V \) and \( j \in N \), and all edges \( G_i(1) \) with \( i \in V \).

![Figure 2.4: Vertices and edges fixed by \( \rho'' \)](image)

Recall that the weight of a vertex \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_v) \) is the number of components \( \varepsilon_i \) distinct from 1. Similarly define the weight of an edge \( G_j(\varepsilon) \) to be the number of \( \varepsilon_l, l \neq j \), that are distinct from 1. We have just established the fact that \( \rho'' \) fixes all vertices of weight one and edges of weight zero, so in particular \( \rho'' \) fixes all vertices and edges of weight zero.

Assume for induction that \( \rho'' \) fixes all vertices and edges of weight \( k - 1 \geq 0 \). We will show that then all vertices and edges of weight \( k \) are fixed by \( \rho'' \).

To begin with the proof for vertices note that the case \( k = 1 \) is already known to be true. Now let \( k \geq 2 \). Every vertex of weight \( k \) lies in exactly \( k \) edges of weight \( k - 1 \). Namely an arbitrary vertex \( \varepsilon \) of weight \( k \) with support \( I \) is incident to all edges of the form \( G_i(\varepsilon) \) with \( i \in I \). Let \( \varepsilon \) be such a vertex and let \( \varepsilon \rho'' = \eta \); we need to show that \( \eta = \varepsilon \). Since \( \rho'' \) fixes all edges of weight \( k - 1 \), we have \( \eta = \varepsilon \rho'' \in G_i(\varepsilon) \rho'' = G_i(\varepsilon) \) and hence \( \eta_j = \varepsilon_j \) for \( j \neq i \), for each \( i \in I \). Hence we have \( \varepsilon = \eta \) since \( k \geq 2 \), and \( \rho'' \) fixes \( \varepsilon \) as desired. This proves the statement for each vertex.

To establish the statement for edges, let \( \varepsilon \) be a vertex with support \( I \) of weight \( k \). An edge \( G_j(\varepsilon) \) has weight \( k - 1 \) or \( k \) according as \( j \in I \) or \( j \notin I \). By inductive hypothesis, for \( j \in I \), the edge \( G_j(\varepsilon) \) is fixed by \( \rho'' \). Let \( j \notin I \) be arbitrary and let \( G_j(\varepsilon) \rho'' = G_j'(\varepsilon) \) for
some $j'$ where necessarily $j' \notin I$; note then that $\varepsilon$ is fixed under $\rho''$. We need to show $j' = j$. Let $i_k$ denote the largest index in $I$, so in particular $j \neq i_k$. Define the two vertices $\mu$ and $\nu$ by

$$
\mu_i = \begin{cases} 
1 & \text{if } i = i_k, \\
2 & \text{if } i = j, \\
\varepsilon_i & \text{otherwise},
\end{cases} \quad (2.2.4)
$$

and

$$
\nu_i = \begin{cases} 
2 & \text{if } i = j, \\
\varepsilon_i & \text{otherwise}.
\end{cases} \quad (2.2.5)
$$

Then the vertices $\varepsilon, \nu, \mu$ are in a sequence of sequentially adjacent vertex-edge chains (see Figure 2.5):

$$
\{\varepsilon, G_j(\varepsilon)\} \leftrightarrow \{\nu, G_j(\varepsilon)\} \leftrightarrow \{\nu, G_{i_k}(\nu)\} \leftrightarrow \{\mu, G_{i_k}(\nu) = G_{i_k}(\mu)\}. \quad (2.2.6)
$$

Applying $\rho''$ to (2.2.6) we obtain the permuted sequence:

$$
\{\varepsilon \rho'', G_j(\varepsilon) \rho''\} \leftrightarrow \{\nu \rho'', G_j(\varepsilon) \rho''\} \leftrightarrow \{\nu \rho'', G_{i_k}(\nu) \rho''\} \leftrightarrow \{\mu \rho'', G_{i_k}(\nu) \rho''\}. \quad (2.2.7)
$$

Since $\varepsilon$ and $\mu$ have weight $k$ they are fixed by $\rho''$. This simplifies (2.2.7) to the sequence

\begin{center}
\begin{tikzpicture}
\node (A) at (0,0) {$\varepsilon$};
\node (B) at (0,1) {$G_j(\varepsilon)$};
\node (C) at (2,0) {$\nu$};
\node (D) at (2,1) {$G_{i_k}(\nu)$};
\node (E) at (4,0) {$\mu$};
\draw (A) -- (B); \draw (B) -- (C); \draw (C) -- (D); \draw (D) -- (E);
\end{tikzpicture}
\end{center}

Figure 2.5: Chain sequence

Applying $\rho''$ to (2.2.6) we obtain the permuted sequence:

$$
\{\varepsilon \rho'', G_j(\varepsilon) \rho''\} \leftrightarrow \{\nu \rho'', G_j(\varepsilon) \rho''\} \leftrightarrow \{\nu \rho'', G_{i_k}(\nu) \rho''\} \leftrightarrow \{\mu \rho'', G_{i_k}(\nu) \rho''\}. \quad (2.2.7)
$$

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\node (C) at (2,0) {$\nu$};
\node (D) at (2,1) {$G_{i_k}(\nu)$};
\node (E) at (4,0) {$\mu$};
\draw (A) -- (B); \draw (B) -- (C); \draw (C) -- (D); \draw (D) -- (E);
\end{tikzpicture}
\end{center}

Figure 2.5: Chain sequence

Applying $\rho''$ to (2.2.6) we obtain the permuted sequence:

$$
\{\varepsilon \rho'', G_j(\varepsilon) \rho''\} \leftrightarrow \{\nu \rho'', G_j(\varepsilon) \rho''\} \leftrightarrow \{\nu \rho'', G_{i_k}(\nu) \rho''\} \leftrightarrow \{\mu \rho'', G_{i_k}(\nu) \rho''\}. \quad (2.2.7)
$$

Since $\varepsilon$ and $\mu$ have weight $k$ they are fixed by $\rho''$. This simplifies (2.2.7) to the sequence
shown in Figure 2.6.

Substituting also $G_j(\varepsilon)\rho'' = G_{j'}(\varepsilon)$ then gives the sequence

\[
\{\varepsilon, G_{j'}(\varepsilon)\} \leftrightarrow \{\nu\rho'', G_{j'}(\varepsilon)\} \leftrightarrow \{\nu\rho'', G_{i_k}(\nu)\rho''\} \leftrightarrow \{\mu, G_{i_k}(\nu)\rho''\}
\]  

(2.2.8) shown in Figure 2.7. However, $\mu \in G_{i_k}(\nu)\rho''$ means that $G_{i_k}(\nu)\rho'' = G_l(\mu)$ for some $l \in V$; and $\nu\rho'' \in G_{i_k}(\nu)\rho'' = G_l(\mu)$ means $(\nu\rho'')_i = \mu_i$ for $i \neq l$. On the other hand, the permuted chain sequence (2.2.8) gives us $\varepsilon, \nu\rho'' \in G_{j'}(\varepsilon)$ and hence $(\nu\rho'')_i = \varepsilon_i$ for $i \neq j'$. Now since $(\nu\rho'')_i = \mu_i$ for $i \neq l$ and $(\nu\rho'')_i = \varepsilon_i$ for $i \neq j'$, we must have $\mu_i = \varepsilon_i$ for $i \neq j', l$. By the definition of $\mu$ this forces $\{l, j'\} = \{i_k, j\}$. Since $j, j' \notin I$ but $i_k \in I$ this then gives $j = j'$ (and $l = i_k$). But this is what we want to show. Thus all the edges of weight $k$ are fixed by $\rho''$ as well.

Therefore, by induction, $\rho''$ fixes all vertices and edges of $n^k$.  
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Now, since \( n^K \) is vertex-describable all faces of \( n^K \) are uniquely determined by their vertex-sets. It follows that \( \rho'' \) must fix every face of \( n^K \). Therefore, \( \rho'' \) is the identity on \( n^K \).

Finally, since \( \rho'' = \rho \tau \sigma = 1 \), we have \( \rho = \sigma^{-1} \tau^{-1} \in S_n \wr \Gamma(K) \). Therefore, we have \( \Gamma(n^K) \) is a subgroup of \( S_n \wr \Gamma(K) \) as desired. Thus \( \Gamma(n^K) = S_n \wr \Gamma(K) \).

\[ \Box \]

**Corollary 2.2.3.** If \( K \) is regular, then so is \( n^K \), and the action of \( S_n \wr \Gamma(K) \) is flag-transitive on \( n^K \).

*Proof.* First note that the subgroup \( S_n^\nu \) acts vertex-transitively on \( n^K \). Hence, if \( K \) is regular, then \( \Gamma(K) \) acts flag-transitively on \( K \) and so does \( S_n \wr \Gamma(K) \) on \( n^K \). Therefore, \( n^K \) is regular. \[ \Box \]

### 2.2.2 Generators of the Automorphism Group

We want to be able to explicitly describe a system of generating subgroups for the automorphism group of a power complex. We follow [47].

Let \( \Gamma(K) = \langle R_{-1}, R_0, \ldots, R_d \rangle \), where \( R_{-1}, R_0, \ldots, R_d \) are the generating subgroups for \( \Gamma(K) \) with respect to a flag

\[ \Phi = \{ F_{-1}, F_0, F_1, \ldots, F_d \} \]

of \( K \). Recall that \( R_{-1} = R_d \) is the stabilizer of \( \Phi \) in \( \Gamma(K) \). Without loss of generality we may take \( F_0 = 1 \) in the numbering of the vertices of \( K \).

#### 2.2.2.1 Flag Stabilizers

It is convenient to take \( \varepsilon = (n, \ldots, n) \) as the base vertex for \( n^K \). Then

\[ \Phi(\varepsilon) = \{ \emptyset, \varepsilon = F_{-1}(\varepsilon), F_0(\varepsilon), \ldots, F_d(\varepsilon) \} \]
is a flag of $n^K$. We also have $F_d(\varepsilon) = N^v$ and

$$F_0(\varepsilon) = N \times \{n\} \times \cdots \times \{n\}.$$  

Clearly, the subgroup $S_{n-1} \wr R_{-1}$ of $\Gamma(n^K)$ stabilizes $\Phi(\varepsilon)$, since $R_{-1}$ stabilizes $\Phi$ and the subgroup $S_{n-1}^v$ (with each $S_{n-1}$ acting on $1, \ldots, n - 1$) stabilizes $\varepsilon$.

Let $\theta = \psi \varphi$ fix $\Phi(\varepsilon)$ where $\varphi \in \Gamma(K)$ and $\psi = \theta_1 \times \cdots \times \theta_v \in S_n \times \cdots \times S_n$. Then for $i = -1, 0, \ldots, d$,

$$F_i(\varepsilon) = F_i(\varepsilon) \theta = F_i(\varphi(\psi \varepsilon)). \quad (2.2.9)$$

Hence, since $K$ is vertex-describable, $F_i \varphi = F_i$ for all $i = -1, 0, \ldots, d$ and so $\varphi$ lies in $R_{-1}$, the stabilizer of $\Phi$. Moreover, $\varepsilon \psi = \varepsilon \theta \varphi^{-1} = \varepsilon \varphi^{-1} = \varepsilon$, so $\psi$ lies in the subgroup $S_{n-1}^v$ of $S_n^v$ (with each $S_{n-1}$ acting on $1, \ldots, n - 1$). Thus the stabilizer of $\Phi(\varepsilon)$ in $\Gamma(n^K)$ is given by $S_{n-1} \wr R_{-1}$.

2.2.2.2 Face Stabilizers

If $\theta = \psi \varphi$ fixes $\Phi(\varepsilon) \setminus \{\varepsilon\}$ then $\varphi$ fixes $\Phi$ and thus $\varphi \in R_{-1}$, and $F_0(\varepsilon \psi) = F_0(\varepsilon)$ shows that $\varepsilon \psi$ and $\varepsilon$ coincide on all but possibly the first component. Hence $\psi \in S_n \times S_{n-1} \times \cdots \times S_{n-1}$, and

$$\theta \in \langle S_n \times S_{n-1} \times \cdots \times S_{n-1}, R_{-1} \rangle = (S_n \times S_{n-1}^v) \times R_{-1}. \quad (2.2.10)$$

Conversely, the elements of this group fix $\Phi(\varepsilon) \setminus \{\varepsilon\}$.

If $\theta$ fixes $\Phi(\varepsilon) \setminus \{F_j(\varepsilon)\}$ for some $j \neq -1, d$, then

$$F_i(\varepsilon) = F_i(\varepsilon) \theta = F_i(\varphi(\psi \varepsilon)) \text{ for } i \neq j. \quad (2.2.11)$$

For $i = -1$ this shows that $\varepsilon \theta = \varepsilon$, that is, $\varepsilon \psi = \varepsilon$ and hence $\psi \in S_{n-1} \times \cdots \times S_{n-1}$. Also
$F_i \varphi = F_i$ for $i \neq j$; so $\varphi \in R_j$, the stabilizer of $\Phi \setminus \{F_j\}$, and

$$\theta \in S_{n-1} \wr R_j.$$ \hfill (2.2.12)

Therefore, we have the following as the generating subgroups for $S_n \wr \Gamma(K) = \Gamma(n^K)$.

**Theorem 2.2.4.** If $K$ is a regular $d$-incidence complex with automorphism group $\Gamma(K) = \langle R_{-1}, R_0, \ldots, R_d \rangle$, then the automorphism group of $n^K$ is given by

$$\Gamma(n^K) = \langle \widehat{R}_{-1}, \widehat{R}_0, \ldots, \widehat{R}_d, \widehat{R}_{d+1} \rangle$$

where,

$$\widehat{R}_{-1} = \widehat{R}_{d+1} = S_{n-1} \wr R_{-1},$$
$$\widehat{R}_0 = \langle S_n \times S_{n-1} \times \cdots \times S_{n-1}, R_{-1} \rangle = (S_n \times S_{n-1}^v) \wr R_{-1},$$
$$\widehat{R}_j = S_{n-1} \wr R_{j-1} \text{ for } j = 1, \ldots, d.$$

### 2.2.3 Flag-Transitive Subgroups

We define a characterization of the flag-transitive subgroups of the automorphism group of $n^K$.

**Proposition 2.2.5.** Let $K$ be a regular incidence complex with $v$ vertices. If $U$ is a subgroup of $S_n$ acting transitively on $\{1, \ldots, n\}$ and $\Lambda$ a flag-transitive subgroup of $\Gamma(K)$, then the group $U \wr \Lambda = U^v \rtimes \Lambda$ is a flag-transitive subgroup of $\Gamma(n^K)$.

**Proof.** First we note that $U \wr \Lambda = U^v \rtimes \Lambda$, where $U^v$ is the direct product of $v$ copies of $U$ acting in the obvious way on the vertex set $N^v$ of $n^K$. In particular, $U^v$ acts transitively on $N^v$ and hence is a vertex transitive subgroup of $\Gamma(n^K)$. On the other hand, $\Lambda$ acts flag-transitively on
the vertex-figure at the base vertex. Hence $U \wr \Lambda = \langle U^v, \Lambda \rangle$ acts flag-transitively on $n^K$. 

\section{Power Complex Examples}

The simplest example arises when $K$ is an $m$-edge, that is, as the power complex $n \wr m$. This is a rank 2 structure with $n^m$ vertices with each vertex on $m$ $n$-edges. When $n = 2$ and $m = 3$ we have the 1-skeleton of the cube. When $m = 2$ we have the generalized square $\gamma_2^n$. The automorphism group is $S_n \wr S_m$.

\subsection{Rank Three}

Let $K$ be a rank 2 incidence complex in class $[c_0, c_1]$ with $v$ vertices. The edges of $K$ are $c_0$-edges with $c_1$ edges meeting at a vertex. We then have the following lemma.

\textbf{Lemma 2.3.1.} The power complex $n^K$ has $v \cdot n^{v-1} n$-edges, and has facets that are isomorphic to $n \wr c_0$. The number of facets is dependent on the number of edges $e$ of $K$ and is given by $n^{v-c_0} e$.

\textbf{Example 2.3.2.} $n^{3(3)}^3$ and $n^{(3,3,4)}$

Since $\text{ske}_{0}(3\{3\}3) = \text{ske}_{0}(\{3, 3, 4\})$ (see Example 1.2.4), the power complexes $n^{3(3)}^3$ and $n^{(3,3,4)}$ will have isomorphic 1-skeletons. For $n = 2$, by [32, Corollary 8C6] we know that $2^{(3,3,4)}$ is the cubical regular 5-toroid $\{4, 3, 3, 4\}_{(4,0^3)}$. This polytope has 256 vertices and 256 facets (of type $\{4, 3, 3\}$).

The incidence complex $2^{3(3)}^3$ has its 256 vertices and $8 \cdot 2^7$ edges (2-edges) coincident with the vertices and edges of $\text{ske}_{1}(\{4, 3, 3, 4\}_{(4,0^3)})$. Each facet is of type $2 \wr 3$, the 1-skeleton of the ordinary 3-cube, and there are 256 such facets.

\textsuperscript{1}Here $q^k$ is interpreted as $q q \cdots q$, a total of $k$ copies of $q$. 

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Figure 2.8 shows the vertex-figure at \( \varepsilon = (11111111) \) and the facet 124(\( \varepsilon \)). Around each vertex there are eight edges (2-edges) and eight facets, each isomorphic to the 1-skeleton of the 3-cube. The 1-skeleton associated with the facet 124(\( \varepsilon \)) is shown in the figure.

When \( n \geq 3 \) there are \( n^8 \) vertices, \( 8 \cdot n^7 \) n-edges and \( 8 \cdot n^5 \) facets of type \( n \wr 3 \). Around each vertex there are still eight edges and facets.

The automorphism group of \( n^{3\{3\}} \) is \( S_n \wr GL(2, 3) \cong S_n^8 \ltimes GL(2, 3) \) of order \( 48(n!)^8 \). The automorphism group of \( n^{\{3,3,4\}} \) is \( S_n \wr [3, 3, 4] \) of order \( 384(n!)^8 \).

**Example 2.3.3.** \( n^{3\{4\}} \)

The power complex \( 2^{3\{4\}} \) has 224 vertices, 24\( \cdot 2^{23} \) edges and 24\( \cdot 2^{21} \) facets. The 0-skeletons of \( 3\{4\}3 \) and \( \{3,4,3\} \) are isomorphic (see Example 1.2.5), so the 1-skeletons of \( 2^{3\{4\}} \) and the regular 5-toroid \( 2^{\{3,4,3\}} \) are isomorphic.

Figure 2.9 shows three facets of type \( 2 \wr 3 \), each the 1-skeleton of an ordinary cube, that share the edge 1(\( \varepsilon \)). In this figure only the vertices are labeled namely by the index of the position in which they are different from \( \varepsilon \). There are 24 edges joining \( \varepsilon \) to an adjacent
vertex (in the vertex-figure), and each edge is surrounded by three facets.

In general, each vertex lies in twenty-four $n$-edges. There are also twenty four facets of type $n \wr 3$. The automorphism group is $S_n \wr ((C_3 \times SL(2, 3)) \rtimes C_2) \cong S^{24}_n \rtimes ((C_3 \times SL(2, 3)) \rtimes C_2)$

Example 2.3.4. $n^{PG(2,2)}$

In the Fano plane, each pair of vertices is colinear with exactly one additional vertex. Therefore in the power complex each pair of edges forms a facet with exactly one additional edge. Around each vertex there are then seven facets.

In general, $n^{PG(2,2)}$ will have $n^7$ vertices. The $7 \cdot n^6$ edges are $n$-edges. The $7 \cdot n^4$ facets, seven around each vertex, are each isomorphic to $n \wr 3$.

In the power complex $2^{PG(2,2)}$ there are 128 vertices, 448 edges and 112 facets. Each facet is isomorphic to $2 \wr 3$, the 1-skeleton of the ordinary cube. Three facets meet at each edge. The vertex-figure (see Figure 2.10) is the Fano plane, so about each vertex $\varepsilon$ are seven edges $1(\varepsilon), \ldots, 7(\varepsilon)$.
The automorphism group is $S_n \wr PGL(3, 2) \cong S_n^7 \rtimes PGL(3, 2)$.

Example 2.3.5. $n^{PG(2, 3)}$

The power complex $n^{PG(2, 3)}$ has $n^{13}$ vertices. Around each vertex are thirteen $n$-edges, with vertex-figure $PG(2, 3)$. Since the lines of $PG(2, 3)$ have four vertices, the facets of $n^{PG(2, 3)}$ have type $n \wr 4$.

A facet of $2^{PG(2, 3)}$ is shown in Figure 2.11. It is isomorphic to the 1-skeleton of the 4-cube; in fact

$$2 \wr 4 = 2^{(4)} = 2^{\text{skel}_1(\{3,3\})} = \text{skel}_1(2^{\{3,3\}}) = \text{skel}_1(\{4,3,3\}).$$

Example 2.3.6. $n^{PG(2, 4)}$

The power complex $n^{PG(2, 4)}$ consists of $n^{21}$ vertices, $21 \cdot n^{20}$ edges and $21 \cdot n^{16}$ facets. Each edge is an $n$-edge and each facet is of type $n \wr 5$, with $n^5$ vertices. When $n = 2$ the facets are
isomorphic to 1-skeletons of 5-cubes, for reasons similar to those described in the previous example.

**Example 2.3.7.** $nK_{m,m}$

When viewed as a complex (see Proposition 1.2.2) the complete bipartite graph $K_{m,m}$ (see Section 1.2.1.1) is isomorphic to the 2-dimensional complex cross polytope $\beta^m_2$. In general the power complex $nK_{m,m}$ has $n^{2m}$ vertices, $2m \cdot n^{2m-1}$ edges, and $m^2 \cdot n^{2m-2}$ facets isomorphic to $n \wr 2$. In the power complex $2K_{3,3}$, there are 64 vertices. There are $6 \cdot 2^5$ edges and $9 \cdot 2^4$ facets. Each vertex is on 6 edges and 9 facets (as shown in Figure 2.12). The facets are isomorphic to $2 \wr 2 \cong \{4\}$, the square. Three of the squares in Figure 2.12 have two adjacent edges meeting in $\varepsilon$.

![Figure 2.12: The nine facets around $\varepsilon = (111111)$ in $2K_{3,3}$](image)

The automorphism group is $S_n \wr (S_m \wr S_2) \cong S^{2m}_n \wr (S_m \wr S_2)$. 

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Chapter 3

Twisting

In this section we generalize certain twisting constructions for regular polytopes [32, Chapter 8] to more general incidence complexes.

3.1 Automorphism Action

As a simple illustration of a twisting construction consider the diagram for a Coxeter group $W = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ shown in Figure 3.1 [32, Section 8A]. The symmetry of the diagram provides an involutory group automorphism $\tau$ of $W$ acting on the generators by $\tau \sigma_1 \tau = \sigma_2$ and $\tau \sigma_3 \tau = \sigma_3$.

![Diagram of Coxeter group W]

Figure 3.1: $\tau$ acting on the Coxeter group $W$ with diagram $D$
We have \((τσ_2)^2 = τσ_2τσ_2 = σ_1σ_2\). Therefore, \((τσ_2)^{2p} = (σ_1σ_2)^p = 1\). We also have \((τσ_3)^2 = τσ_3τσ_3 = σ_3^2 = 1\). Then \(Γ := ⟨τ, σ_2, σ_3⟩ = W ∗ C_2\) is the Coxeter group \([2p, q]\), the automorphism group of the regular tessellation group \(\{2p, q\}\).

### 3.2 Constructing \(n^K\) from a Small Flag-Transitive Subgroup

Let \(\mathcal{K}\) be a (regular) incidence complex of rank \(d ≥ 1\) with \(v\) vertices, vertex set \(V\), and base flag \(\{F_{-1}, F_0, F_1, \ldots, F_d\}\). Let the automorphism group of \(\mathcal{K}\) relative to the base flag be given by \(Γ(\mathcal{K}) = ⟨R_0, R_1, \ldots, R_{d-1}⟩ = ⟨R_{-1}, R_0, \ldots, R_d⟩\). We begin by constructing a regular \((d + 1)\)-complex with vertex-figure isomorphic to \(\mathcal{K}\), and later establish isomorphism with \(n^K\).

Consider the direct product \(W\) of \(v\) copies of the cyclic group \(C_n\) of order \(n\), each \(C_n\) with a generator \(σ_F\) indexed by a vertex \(F\) of \(\mathcal{K}\). We have then

\[
W = ⟨σ_F|F ∈ V⟩ = \bigtimes_{F ∈ V} ⟨σ_F⟩ = C_n^v.
\]

We view \(W\) as a subgroup of the semi-direct product \(Γ := W ∗ Γ(\mathcal{K}) = C_n ∗ Γ(\mathcal{K})\), where the conjugation action of \(Γ(\mathcal{K})\) on \(W\) is determined by

\[
τ^{-1}σ_Fτ = σ_{Fτ}, \text{ for } F ∈ V, τ ∈ Γ(\mathcal{K}).
\]

For example, if \(τ ∈ R_j\) with \(j = 1, \ldots, d − 1\), then \(τ\) fixes the base vertex \(F_0\) of \(\mathcal{K}\) so \(τ^{-1}σ_{F_0}τ = σ_{F_0}\).
Define the subgroups $\hat{R}_0, \hat{R}_1, \ldots, \hat{R}_d$ of $\Gamma$ by

$$
\hat{R}_i = \begin{cases} 
\langle \sigma_{F_0} \rangle & \text{if } i = 0, \\
R_{i-1} & \text{if } 1 \leq i \leq d. 
\end{cases}
$$

(3.2.1)

Since $\Gamma (\mathcal{K}) = \langle R_0, \ldots, R_{d-1} \rangle$ acts vertex-transitively on $\mathcal{K}$, each generator $\sigma_F$ of $W$ lies in $\langle \hat{R}_0, \ldots, \hat{R}_d \rangle$; in fact, if $F \in V$ and $F = F_0 \tau$ for some $\tau \in \Gamma (\mathcal{K})$, then $\sigma_F = \tau^{-1} \sigma_{F_0} \tau$. It follows that

$$
\Gamma = \langle \hat{R}_0, \hat{R}_1, \ldots, \hat{R}_d \rangle.
$$

(3.2.2)

Moreover, if $i \geq 2$ and $\tau \in \hat{R}_i = R_{i-1}$, then $F_0 \tau = F_0$ and hence $\sigma_{F_0} \tau = \tau \sigma_{F_0}$; therefore, $\hat{R}_0 \hat{R}_i = \hat{R}_i \hat{R}_0$ (even at the level of elements).

We need to verify the intersection property for $\Gamma$, that is $\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J}$ for $I, J \subset \{0, \ldots, d\}$. First we note that if $0 \in I$ and $I = \{0\} \cup I'$ (say) then

$$
\Gamma_I = \left( \bigtimes_{\tau \in \Gamma_{I'}} \langle \sigma_{F_\tau} \rangle \right) \ltimes \Gamma_{I'}.
$$

If $0 \notin J$ and $J = \{0\} \cup J'$, then

$$
\Gamma_I \cap \Gamma_J = \left( \bigtimes_{\tau \in \Gamma_{I'}} \langle \sigma_{F_\tau} \rangle \ltimes \Gamma_{I'} \right) \cap \left( \bigtimes_{\tau \in \Gamma_{J'}} \langle \sigma_{F_\tau} \rangle \ltimes \Gamma_{J'} \right)
$$

$$
= \left( \bigtimes_{\tau \in \Gamma_{I' \cap J'}} \langle \sigma_{F_\tau} \rangle \ltimes \Gamma_{I' \cap J'} \right) \cap \Gamma_{I' \cap J'}
$$

$$
= \Gamma_{I \cap J},
$$

by the semi-direct product structure of the groups and the intersection property in $\Gamma (\mathcal{K}) = \langle R_0, \ldots, R_d \rangle$. 

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\( \langle \hat{R}_1, \ldots, \hat{R}_d \rangle \). This settles the intersection property in the case \( 0 \in I, J \). The remaining cases are simpler. If \( 0 \notin I, J \) then \( \Gamma_I \) and \( \Gamma_J \) both lie in \( \Gamma(\mathcal{K}) \), so we can directly appeal to the intersection property of \( \Gamma(\mathcal{K}) \). If only one set, \( I \) (say) contains 0, then

\[
\Gamma_I \cap \Gamma_J = \left( \bigotimes_{\tau \in \Gamma'_{\nu}} \langle \sigma_{\tau' \nu} \rangle \ltimes \Gamma'_{\nu} \right) \cap \Gamma_J \\
= \Gamma_{\nu} \cap \Gamma_J = \Gamma_{I \cap J} \\
= \Gamma_{I \cap J},
\]

again by the semi-direct product structure and the intersection property of \( \Gamma(\mathcal{K}) \).

By Theorem 3 of [46] we have that the complex produced by the above (twisting) construction with automorphism group \( \Gamma = C_n \wr \Gamma(\mathcal{K}) \) is isomorphic to \( n^\mathcal{K} \). In fact, \( \Gamma(n^\mathcal{K}) \) has a flag-transitive subgroup isomorphic to \( C_n \wr \Gamma(\mathcal{K}) \) (see Proposition 2.2.5), and this subgroup acts on \( n^\mathcal{K} \) in just the same manner as \( \Gamma \) on the regular \((d + 1)\)-complex associated with \( \Gamma \).

We have proven the following.

**Theorem 3.2.1.** Let \( \mathcal{K} \) be a \( d \)-incidence complex with group \( \langle R_0, \ldots, R_{d-1} \rangle \) and let \( n \geq 2 \). The incidence complex associated with the group \( \Gamma = C_n \wr \Gamma(\mathcal{K}) \) and generated by the \( \hat{R}_i \) as defined by equations (3.2.1) and (3.2.2) is isomorphic to \( n^\mathcal{K} \).

Thus \( n^\mathcal{K} \) can be constructed from a small flag-transitive subgroup, \( C_n \wr \Gamma(\mathcal{K}) \), of the full automorphism group \( \Gamma(n^\mathcal{K}) = S_n \wr \Gamma(\mathcal{K}) \) by a twisting operation.

**Corollary 3.2.2.** The power complex \( n^\{3^{m-1}\} \) is isomorphic to \( \gamma_{m+1}^n \) the complex \((m+1)\)-cube with \( n \)-edges.

**Proof.** Now \( \mathcal{K} = \{3^{m-1}\} \) and \( \Gamma(\mathcal{K}) = [3^{m-1}] = \langle \rho_0, \ldots, \rho_{m-1} \rangle \). We need to show that \( \Gamma = \langle \sigma_{F_0}, \rho_0, \ldots, \rho_{m-1} \rangle \cong n[4]3[2]3 \cdots 3[2]3 \), the unitary symmetry group of \( \gamma_{m+1}^n \); the latter is a flag-transitive subgroup of \( \Gamma(\gamma_{m+1}^n) \).
For $i \geq 1$ we have $\rho_i \sigma_{F_0} \rho_i = \sigma_{F_0}$ so $\rho_i \sigma_{F_0} = \sigma_{F_0} \rho_i$. We also have

$$
\rho_0 \sigma_{F_0} \rho_0 \sigma_{F_0} = \sigma_{F_0} \rho_0 \sigma_{F_0} \\
= \sigma_{F_0} \rho_0 \sigma_{F_0} \\
= \sigma_{F_0} \rho_0 \sigma_{F_0} \rho_0
$$

Thus $\sigma_{F_0}, \sigma_0, \ldots, \sigma_{m-1}$ satisfy the defining relations of the group $n[4]3[2]3\ldots3[2]3$. Since both $\langle \sigma_{F_0}, \rho_0, \ldots, \rho_{m-1} \rangle$ and $n[4]3[2]3\ldots3[2]3$ are isomorphic to $C_n S_{m+1}$, the corresponding complexes must be isomorphic, that is $n^{(g^{m-1})} \cong n\{4\}3\{2\}3\ldots3\{2\}3 = \gamma_{m+1}^n$. □

From this, for any incidence complex $K$ we may regard $n^K$ as a generalization of the complex hypercube.

### 3.3 Generalized Power Complexes

The results in this section are inspired by the twisting construction for the regular polytopes $L^{K,G}$ described in [32, Section 8B]. This construction proceeds from a suitable Coxeter group $W$ on which $\Gamma(K)$ acts as a group of (group) automorphisms, and extends $W$ by $\Gamma(K)$ to find $L^{K,G}$. The Coxeter diagram $W$ depends on the polytopes $K$ and $L$ and contains $G$ as an induced subdiagram. In particular, $\Gamma(L^{K,G}) \cong W \rtimes \Gamma(K)$.

In the context of arbitrary regular complexes, the applicability of a similar technique is severely constrained by the lack of readily available classes of groups $W$ on which the whole groups of a regular complex $K$ can act in a suitable way (see Section 3.3.2 for a fuller discussion). Here we limit ourselves to two special cases for regular complexes $K$ and $L$:

1. $L$ is a universal regular polytope of rank $d \geq 1$ (and then $W$ is a Coxeter group),
2. \( \mathcal{L} \) has rank 1 (and \( \mathcal{W} \) is the direct product of cyclic groups).

In these two cases the twisting operation carries over and gives regular complexes which we denote again by \( \mathcal{L}^{\mathcal{K}, \mathcal{G}} \). The second case was already investigated in Section 3.2; in fact if \( \mathcal{L} \) is an \( n \)-edge \( \{ \} \) (and \( \mathcal{G} \) is the trivial diagram), then \( \mathcal{L}^{\mathcal{K}, \mathcal{G}} \cong n^{\mathcal{K}} \).

### 3.3.1 Basic Construction from Coxeter Groups

Throughout this section, \( \mathcal{L} \) is a universal regular \( d \)-polytope of type \( \{ q_1, \ldots, q_{d-1} \} \) ([32, Section 3D]). Suppose \( \Gamma(\mathcal{L}) = \langle \rho_0, \ldots, \rho_{d-1} \rangle \), where \( \rho_0, \ldots, \rho_{d-1} \) are the distinguished generators. Then \( \Gamma(\mathcal{L}) \) is the Coxeter group with a string diagram on \( d \) nodes, in which the branches are labeled \( q_1, \ldots, q_{d-1} \), respectively.

Now let \( \mathcal{K} \) be a regular complex of rank \( k \geq 1 \) with automorphism group \( \Gamma(\mathcal{K}) = \langle R_0, \ldots, R_{k-1} \rangle \) (since \( k \geq 1 \) we can ignore mentioning \( R_{-1} \) and \( R_k \)). Let \( \mathcal{G} \) be a Coxeter diagram in which the nodes are indexed by the vertices of \( \mathcal{K} \); that is the node set \( V(\mathcal{G}) \) is the vertex set \( V(\mathcal{K}) \) of \( \mathcal{K} \). Now suppose \( \Gamma(\mathcal{K}) \) acts on \( \mathcal{G} \) as a group of diagram symmetries such that \( \Gamma(\mathcal{K}) \) acts transitively on \( V(\mathcal{G}) \), the vertex stabilizer-subgroup \( \langle R_1, \ldots, R_{k-1} \rangle \) of \( \Gamma(\mathcal{K}) \) fixes the node \( F_0 \) (say) of \( \mathcal{G} \) corresponding to the base vertex of \( \mathcal{K} \) (it may fix more than one node), and the action of \( \Gamma(\mathcal{K}) \) on \( \mathcal{G} \) respects the following intersection property for the generating subgroups \( R_0, \ldots, R_{k-1} \) of \( \Gamma(\mathcal{K}) \). The latter means that, if \( V(\mathcal{G}, I) \) denotes the set of images of \( F_0 \) under the subgroup \( \langle R_i \mid i \in I \rangle \) for \( I \subseteq \{0, \ldots, k-1\} \), then

\[
V(\mathcal{G}, I) \cap V(\mathcal{G}, J) = V(\mathcal{G}, I \cap J) \quad (I, J \subseteq \{0, \ldots, k-1\}) \tag{3.3.1}
\]

In applications the action of \( \Gamma(\mathcal{K}) \) on the nodes of \( \mathcal{G} \) is just the standard action of \( \Gamma(\mathcal{K}) \) on the vertices of \( \mathcal{K} \). In this case the intersection condition (3.3.1) is satisfied if \( \mathcal{K} \) is a lattice. We will make this assumption from now on.
Given $\mathcal{L}$ and $\mathcal{K}$ as above we now merge the string Coxeter diagram of $\mathcal{L}$ with the Coxeter diagram $\mathcal{G}$ to obtain a larger diagram $\mathcal{D}$, which also admits an action of $\Gamma(\mathcal{K})$ as a group of diagram symmetries. More precisely, we extend the diagram for $\mathcal{L}$ by the diagram $\mathcal{G}$ to a diagram $\mathcal{D}$ by identifying the node $F_0$ of $\mathcal{G}$ with the node $d-1$ of the diagram of $\mathcal{L}$ and adding, for each $G \in V(\mathcal{G})$, a node labeled $G$ and a branch marked $q_{d-1}$ between node $d-2$ and $G$ (see Figure 3.2). In addition, any branches and labels from $\mathcal{G}$ are included in $\mathcal{D}$ (Figure 3.2 suppresses any such branches). The vertex set of $\mathcal{D}$ is

$$V(\mathcal{D}) := V(\mathcal{G}) \cup \{0, \ldots, d-2\}.$$ 

![Figure 3.2: Twisting operation diagram $\mathcal{D}$](image)

Throughout the remainder of this discussion we will take $\mathcal{G}$ to be the trivial diagram with nodes the vertices of $\mathcal{K}$. In other words, the Coxeter group with diagram $\mathcal{G}$ is $C_v^2$, where $v$ is the number of vertices of $\mathcal{K}$.

The group $\Gamma(\mathcal{K})$ acts on $\mathcal{D}$ as a group of diagram symmetries and also on the Coxeter group

$$W := \langle \sigma_H \mid H \in V(\mathcal{D}) \rangle$$

as a group of group automorphisms permuting the generators and fixing $\sigma_0, \ldots, \sigma_{d-2}$. We note that $W$ contains $\Gamma(\mathcal{L})$ as a subgroup.
We generate the desired \((d + k)\)-incidence complex from the subgroup

\[
\Gamma := \langle \hat{R}_0, \hat{R}_1, \ldots, \hat{R}_{d+k-1} \rangle
\]

of \(W \rtimes \Gamma(K)\), when

\[
\hat{R}_i := \begin{cases} 
\langle \rho_i \rangle & i = 0, \ldots, d-1 \\
R_{i-d} & i = d, d+1, \ldots, d+k-1.
\end{cases}
\]

**Theorem 3.3.1.** The group \(\Gamma\) is a flag-transitive subgroup of the automorphism group of a regular incidence complex of rank \(d + k\), and \(\Gamma = W \rtimes \Gamma(K)\).

**Proof.** Since \(\Gamma(K) = \langle R_0, \ldots, R_{d-1} \rangle\) acts transitively on the vertices of \(K\), each generator \(\sigma_H\) of \(W\) lies in \(\Gamma\). In fact, if \(F \in V(K)\) then there is a \(\tau \in \Gamma(K)\) such that \(F = F_0\tau\), and then \(\sigma_F = \tau^{-1}\sigma_{F_0}\tau = \tau^{-1}\rho_{d-1}\tau\). It then follows that \(\Gamma = W \rtimes \Gamma(K)\).

Since \(\Gamma(K)\) fixes \(L_i\), the nodes \(0, \ldots, d - 2\) of \(D\), it centralizes the generators \(\rho_0, \ldots, \rho_{d-2}\) of \(\Gamma\) and \(\langle \rho_i \rangle R_j = R_j \langle \rho_i \rangle\) for \(0 \leq i \leq d - 2, 0 \leq j \leq k - 1\). Thus, if \(|i - j| \geq 2\), \(\hat{R}_i \hat{R}_j = \hat{R}_j \hat{R}_i\).

To see that \(\Gamma\) satisfies the intersection property, let \(I, J \subseteq \{0, \ldots, d + k - 1\}\). We need to show that \(\Gamma_{I \cap J} = \Gamma_I \cap \Gamma_J\). We have \(I = I_1 \cup I_2\) with \(I_1 \subseteq \{0, \ldots, d - 1\}\) and \(I_2 \subseteq \{d, d+1, \ldots, d+k-1\}\). Then

\[
\Gamma_I = \langle \hat{R}_i \mid i \in I \rangle
= \langle \rho_i, R_{j-d} \mid i \in I_1, j \in I_2 \rangle
= \begin{cases} 
\langle \rho_i \rangle & i \in I_2 \\
\langle \rho_i \rangle \langle R_{j-d} \rangle & j \in I_2 \text{ if } d - 1 \notin I_1 \\
\langle \rho_i, \sigma_H \rangle & i \in I_2, H \in V(G, I_2 - d) \\
\langle R_{j-d} \rangle & j \in I_2 \text{ if } d - 1 \in I_1
\end{cases}
\]

(3.3.2)

When \(d - 1\) is not in \(I_1\) the product in (3.3.2) is a direct product. If \(d - 1\) is in \(I_1\) the product is a semi-direct product. In either case, the first factor is a subgroup of \(W\) and the second factor is a subgroup of \(\Gamma(K)\). For \(J \subseteq \{0, \ldots, d + k - 1\}\) we can similarly define \(J_1\) and \(J_2\).
so that \( J = J_1 \cup J_2 \). Then

\[
I \cap J = (I_1 \cup I_2) \cap (J_1 \cup J_2) = (I_1 \cap J_1) \cup (I_2 \cap J_2).
\]

By the intersection property on \( W \) and \( \Gamma(\mathcal{K}) \) we have

\[
\Gamma_{I_1 \cap J_1} = \Gamma_{I_1} \cap \Gamma_{J_1}, \tag{3.3.3}
\]

\[
\Gamma_{I_2 \cap J_2} = \Gamma_{I_2} \cap \Gamma_{J_2} \tag{3.3.4}
\]

and \( \Gamma_{I \cap J} = \Gamma_I \cap \Gamma_J \). Hence \( \Gamma \) satisfies the intersection property. \( \square \)

Note that if \( \{F_{-1}, F_0, \ldots, F_k\} \) is the base flag of \( \mathcal{K} \), then for \( d \leq j \leq d + k - 1 \) we have

\[
\langle \hat{R}_0, \ldots, \hat{R}_j \rangle = \langle \rho_k, \sigma_H \mid 0 \leq k \leq d - 1, H \in V(\mathcal{G}, \{0, \ldots, j - d\}) \rangle \langle R_0, \ldots, R_{j-d} \rangle \tag{3.3.5}
\]

where

\[
V(\mathcal{G}, \{0, \ldots, j - d\}) = \{F \in V(\mathcal{K}) \mid F \leq F_{j-d+1}\}.
\]

**Proposition 3.3.2.** Let \( \mathcal{L} \) be a universal regular \( d \)-polytope \( \{q_1, \ldots, q_{d-1}\} \) and \( \mathcal{K} \) a regular \( k \)-incidence complex. For \( 1 \leq j \leq k \) let \( \mathcal{K}_i \) be an \( i \)-face of \( \mathcal{K} \) and \( \mathcal{G}_i \) the induced subdiagram of \( \mathcal{G} \) on the subset \( V(\mathcal{G}, \{0, \ldots, i-1\}) \) of \( V(\mathcal{G}) \). Then the \((d+i)\)-faces of \( \mathcal{L}^{\mathcal{K},\mathcal{G}} \) are isomorphic to \( \mathcal{L}_i^{\mathcal{K}_i,\mathcal{G}_i} \). Additionally, if \( 0 \leq i \leq d - 2 \) then the co-faces at \( i \)-faces of \( \mathcal{L}^{\mathcal{K},\mathcal{G}} \) are isomorphic to \( \mathcal{L}_i^{\mathcal{K},\mathcal{G}} \), where \( \mathcal{L}_i \) is the universal regular polytope \( \{q_{i+2}, \ldots, q_{d-1}\} \), the co-face at an \( i \)-face of \( \mathcal{L} \).

**Proof.** For the first part, let \( j = d + i - 1 \), then apply (3.3.5) to obtain

\[
\langle \hat{R}_0, \ldots, \hat{R}_j \rangle = \langle \rho_k, \sigma_H \mid 0 \leq k \leq d - 1, H \in V(\mathcal{G}, \{0, \ldots, i - 1\}) \rangle \langle R_0, \ldots, R_{i-1} \rangle.
\]
For the second part, the construction using the induced diagram on \( V(\mathcal{G}) \cup \{i+1, \ldots, d-2\} \) is the same as the construction using \( \mathcal{L}_i, \mathcal{K} \) and \( \mathcal{G} \).

Proposition 3.3.2 says that the automorphism group of the facet and vertex-figure of a twisted complex is what we would expect. For the facet, the automorphism group is \( \langle \widehat{R}_0, \ldots, \widehat{R}_{d+k-2} \rangle \), for the vertex-figure the automorphism group is \( \langle \widehat{R}_1, \ldots, \widehat{R}_{d+k-1} \rangle \).

Remark 3.3.3. In the construction of the complexes \( \mathcal{L}^{K, \mathcal{G}} \) the automorphism group \( \Gamma(K) \) of \( K \) can be replaced by any flag-transitive subgroup, \( \Lambda \) (say), of \( \Gamma(K) \), in the sense that the resulting regular complex obtained from \( W \ltimes \Lambda \) (with the same \( W \)) is isomorphic to \( \mathcal{L}^{K, \mathcal{G}} \). The group \( W \ltimes \Lambda \) is a flag-transitive subgroup of \( \Gamma(K) \). The distinguished generator subgroups of \( W \ltimes \Gamma \) are given by \( \langle \rho_0, \ldots, \rho_{d-1} \rangle, R_0^\Lambda, \ldots, R_{k-1}^\Lambda \), where \( R_0^\Lambda, \ldots, R_{k-1}^\Lambda \) are the distinguished generator subgroups of \( \Lambda \).

As we noted earlier, the diagram of \( \mathcal{G} \) involved in the construction of the complex \( \mathcal{L}^{K, \mathcal{G}} \) was the trivial diagram on the vertices of \( K \). From now on we use the simpler notation \( \mathcal{L}^{K} \) in place of \( \mathcal{L}^{K, \mathcal{G}} \) (whenever \( \mathcal{G} \) is trivial).

We now introduce a class of diagrams that typically arises in the twisting construction when using simplices. From [32], for \( d, k \geq 1 \), let \( \mathcal{D}_{d,k} \) denote the Coxeter diagram with \( d+k \) nodes and \( d+k-1 \) unmarked branches depicted in Figure 3.3. In particular, \( \mathcal{D}_{0,1} \) is the trivial diagram, and \( \mathcal{D}_{d,0} \) is the string diagram for the \( d \)-simplex.

**Proposition 3.3.4.** [32, Corollary 8B10] Let \( d, k \geq 1 \). Let \( \mathcal{D}_{d-1,k+1} \) be as in Figure 3.3, and let \( W(\mathcal{D}_{d-1,k+1}) \) be the corresponding Coxeter group. Then

\[
\{3^{d-1}\}^{3^{k-1}} = \{3^{d-1}, 4, 3^{k-1}\},
\]

with group

\[
[3^{d-1}, 4, 3^{k-1}] = W(\mathcal{D}_{d-1,k+1}) \rtimes S_{k+1}.
\]
Corollary 3.3.5. By Proposition 3.3.4, the group \([3, 4, 3] = W(\mathcal{D}_{1,3}) \rtimes S_3\).

### 3.3.2 Remarks

The twisting operation described in the previous section generalizes to regular complexes \(\mathcal{K}\) and \(\mathcal{L}\) for which a suitable group \(W\) can be found.

As before let \(\mathcal{L}\) and \(\mathcal{K}\) have ranks \(d\) and \(k\) respectively, and let \(\Gamma(\mathcal{L}) = \langle L_0, \ldots, L_{d-1} \rangle\) and \(\Gamma(\mathcal{K}) = \langle R_0, \ldots, R_{k-1} \rangle\). Unlike in the previous section we are not assuming that \(\mathcal{L}\) is a polytope. Now suppose there exists a group \(W\) generated by a distinguished family of subgroups, such that \(\Gamma(\mathcal{K})\) suitably acts on \(W\) as a group of (group) automorphisms permuting the subgroups in this distinguished family.

More precisely, suppose that \(\Gamma(\mathcal{L})\) is a subgroup of \(W\) and that the distinguished family of subgroups of \(W\) is given by the generating subgroups \(L_0, \ldots, L_{d-1}\) of \(\Gamma(\mathcal{L})\) and by subgroups \(S_H, H \in V(\mathcal{K})\). Further assume that the subgroup \(S_{F_0}\) associated with the base vertex \(F_0\) of \(\mathcal{K}\) coincides with the subgroup \(L_{d-1}\) of \(\Gamma(\mathcal{L})\) and that the action of \(\Gamma(\mathcal{K})\) on \(W\) resembles that of the previous section. In other words, the elements \(\tau \in \Gamma(\mathcal{K})\) leave each subgroup \(L_k\) with \(k \leq d - 2\) invariant and take a subgroup \(S_H\) to \(S_H \tau\) for \(H \in V(\mathcal{K})\), and the vertex stabilizer subgroup \(\langle R_1, \ldots, R_{k-1}\rangle\) of \(\Gamma(\mathcal{K})\) leaves the subgroup \(S_{F_0}\) of \(W\) invariant.
This setup already allows us to construct the semi-direct product

\[ \Gamma := W \rtimes \Gamma(K) = \langle \hat{R}_0, \ldots, \hat{R}_{k+d-1} \rangle, \]

where

\[ \hat{R}_i := \begin{cases} L_i & i = 0, \ldots, d - 1 \\ R_{i-d} & i = d, d + 1, \ldots, d + k - 1. \end{cases} \]

By our assumption on the action of the vertex stabilizer subgroup \( \langle R_1, \ldots, R_{k-1} \rangle \) on \( W \) we have the desired commutativity property \( \hat{R}_i \hat{R}_j = \hat{R}_j \hat{R}_i \) for \( |i - j| \geq 2 \). However, the crucial condition is the intersection property for \( \Gamma \), and this can usually only be guaranteed if \( W \) already has the intersection property with respect to its distinguished family of generator subgroups. This severely restricts the possibilities and explains why we have focused on the cases outlined in the beginning of Section 3.3.

We should add that in the case where \( \mathcal{L} \) is a polytope and the distinguished generator subgroups of \( W \) are generated by involutions, the intersection property of \( W \) is satisfied if and only if \( W \) is a C-group (in the sense of [32, Section 2E]).

### 3.4 Skeleton and Coskeleton

We start with the generalization of Lemma 2.1.7 and use the notation of Section 3.3.1

**Theorem 3.4.1.** Let \( \mathcal{L} \) be a universal regular \( d \)-polytope, and \( \mathcal{K} \) a regular \( k \)-incidence complex. For \( j \leq k-1 \), \( \text{skel}_{d+j}(\mathcal{L}^\mathcal{K}) = \mathcal{L}^{\text{skel}_j(\mathcal{K})} \).

**Proof.** Let \( \Gamma(\mathcal{L}) = \langle \rho_0, \ldots, \rho_{d-1} \rangle \), \( \Gamma(\mathcal{K}) = \langle R_0, \ldots, R_{k-1} \rangle \) and \( W = \langle \sigma_H | H \in V(\mathcal{D}) \rangle \), and
let \( \Gamma := W \ltimes \Gamma(K) = \langle \hat{R}_0, \ldots, \hat{R}_{d+k-1} \rangle \) with
\[
\hat{R}_i := \begin{cases} 
\langle \rho_i \rangle & i = 0, \ldots, d - 1 \\
R_i & i = d, d + 1, \ldots, d + k - 1.
\end{cases}
\]

Recall that \( \Gamma \) is flag-transitive subgroup of \( \Gamma(L^K) \).

The \( j \)-skeleton of \( \text{skel}_j(K) \) of \( K \) is a regular \((j + 1)\)-complex on which \( \Gamma(K) \) acts flag-transitively as a group of automorphisms. Relative to \( \text{skel}_j(K) \), the distinguished generator subgroups of \( \Gamma(K) \) are given by \( R_0, \ldots, R_{j-1}, R \), where \( R = \langle R_j, \ldots, R_{k-1} \rangle \).

Now bear in mind Remark 3.3.3. Then it is clear that the regular complex \( L^\text{skel}_j(K) \) can be constructed from the flag-transitive subgroup \( \Gamma(K) \) of \( \Gamma(\text{skel}_j(K)) \), rather than from \( \Gamma(\text{skel}_j(K)) \) itself. The resulting flag-transitive subgroup of \( \Gamma(L^\text{skel}_j(K)) \) is \( W \ltimes \Gamma(K) \), with \( W \) as in the construction of \( L^K \), and the distinguished generators of \( W \ltimes \Gamma(K) \) relative to \( L^K \), and the distinguished generators of \( W \ltimes \Gamma(K) \) relative to \( L^\text{skel}_j(K) \) are given by \( \hat{R}_0, \ldots, \hat{R}_{d-1}, R_0, \ldots, R_{j-1}, R \).

On the other hand, the \((d + j)\)-skeleton of \( L^K \) admits the subgroup \( \langle \hat{R}_0, \ldots, \hat{R}_{d+j-1}, \hat{R} \rangle \), with \( \hat{R} := \langle \hat{R}_{d+j}, \ldots, \hat{R}_{d+k-1} \rangle \), as a flag-transitive subgroup. But \( \hat{R} = R \) and \( \langle \hat{R}_0, \ldots, \hat{R}_{d+j-1}, \hat{R} \rangle = \langle \hat{R}_0, \ldots, \hat{R}_{d-1}, R_0, \ldots, R_{j-1}, R \rangle = W \ltimes \Gamma(K) \). As the reconstruction of regular complexes from flag-transitive subgroups is unique, we must have \( \text{skel}_j(L^K) = L^\text{skel}_j(K) \).

\[\square\]

**Corollary 3.4.2.** \( \{3\}^\{\{3\}\} \cong \text{skel}_2(\{3, 4, 3\}) \).

**Proof.** The 3-edge, \( \{\} \), is the 0-skeleton of \( \{3\} = L \). Therefore, \( \text{skel}_2(\{3, 4, 3\}) \cong \text{skel}_2(\{3\}^\{\{3\}\}) \cong \{3\}^\{\text{skel}_0(\{3\})\} \cong \{3\}^\{\{3\}\} \).

\[\square\]

The \( j \)-coskeleton \( \text{coskel}_j(K) \) of a \( k \)-incidence-complex \( K \) is the incidence complex of rank \( k - j \) obtained as follows. The faces of \( \text{coskel}_j(K) \) of non-negative rank \( i - j \) are the \( i \)-faces of
\(K\) with \(i \geq j\), and the face of rank \((-1)\)-face of \(K\); the partial order of \(\text{coskel}_j(K)\) is inherited from \(K\).

Now suppose that \(L\) is a universal regular \(d\)-polytope \(\{q_1, \ldots, q_{d-1}\}\), and that \(K\) is a regular \(k\)-complex. Let \(\Gamma(L), \Gamma(K), W, \Gamma\) and \(\widehat{R}_0, \ldots, \widehat{R}_{d+k-1}\) be as in the proof of the previous theorem. Further suppose that \(0 \leq j \leq d - 1\). Then \(\text{coskel}_j(L)\) is no longer a (universal regular) polytope if \(j > 0\), so \(a\ priori\ \text{coskel}_j(L)^K\) is not defined in this case. (Of course, \(\text{coskel}_j(L) = L\) if \(j = 0\)). However, this is an instance when the remarks of Section 3.3.2 apply if \(j \leq d - 2\), and permit the construction of a regular complex which one might denote \(\text{coskel}_j(L)^K\). This can be seen as follows.

First note that \(\text{coskel}_j(L)\) is a regular complex of rank \(d - j\), on which \(\Gamma(L)\) acts flag-transitively as a group of automorphisms. Relative to \(\text{coskel}_j(L)\), the distinguished generator subgroups of \(\Gamma(L)\) are given by \(\widehat{R}, \widehat{R}_{j+1}, \ldots, \widehat{R}_{d-1}\), where \(\widehat{R} = \langle \widehat{R}_0, \ldots, \widehat{R}_j \rangle = \langle \rho_0, \ldots, \rho_j \rangle\). Observe that all distinguished generator subgroups save \(\widehat{R}\) are generated by involutions, namely \(\rho_{j+1}, \ldots, \rho_{d-1}\), and that there is at least one such generator subgroup since \(j \leq d - 2\). Hence we can reinterpret the action of \(\Gamma(K)\) on the generators \(\rho_0, \ldots, \rho_{d-1}\) and \(\sigma_H\), \(H \in V(K)\), of the Coxeter group \(W\) as an action on the generator subgroups, respectively generators, \(\widehat{R}, \widehat{R}_{j+1}, \ldots, \widehat{R}_{d-1}\) and \(\sigma_H\), \(H \in V(K)\), of (the same group) \(W\). In particular, the subgroups \(\widehat{R}, \widehat{R}_{j+1}, \ldots, \widehat{R}_{d-2}\) are invariant under \(\Gamma(K)\). The intersection property of \(W\) carries over to this new system of generator subgroups and generators.

Hence, with appropriate interpretation as described we arrive at the equation

\[
\text{coskel}_j(L^K) = \text{coskel}_j(L)^K \quad (0 \leq j \leq d - 2)
\]
3.5 Twisting Examples

Example 3.5.1. \{3\}_3^{13}

Consider the 2-simplex, \{3\}, with automorphism group \(\langle \rho_0, \rho_1 \rangle\) and the 3-edge, \{\}\_3, with automorphism group \(S_3 = \langle \tau_1, \tau_2 \rangle\) permuting the three vertices. The regular 3-complex \{3\}_3^{13}, see Figure 3.4, has group \(\langle \rho_0, \rho_1, S_3 \rangle\). The group \(\langle \rho_0, \rho_1, S_3 \rangle\) is isomorphic to \(W(D_{1,3}) \rtimes S_3 = [3, 4, 3]\).

![Figure 3.4: Twisting operation for \{3\}^{13}_3](image)

The vertex-figure has group \(\langle \rho_1, S_3 \rangle\) and is isomorphic to 2\text{?}3, the 1-skeleton of the 3-cube. The facets are 2-simplices \{3\}.

As shown in Corollary 3.4.2, \{3\}_3^{13} is the 2-skeleton of the 24-cell \{3, 4, 3\}.

Example 3.5.2. \{3^{d-2}\}_k

The \(k\)-edge is the 0-skeleton of the \((k - 1)\)-simplex \{3^{k-2}\} and by Proposition 3.3.4 and Theorem 3.4.1, the \(d\)-complex \{3^{d-2}\}_k^{1k} is the \((d - 1)\)-skeleton of the regular \((d + k - 2)\)-polytope \{3^{d-2}, 4, 3^{k-2}\}. In fact,

\[
\{3^{d-2}\}_k^{1k} = \{3^{d-2}\}_{\text{ske}l_0(3^{k-2})} = \text{ske}l_{d-1}(\{3^{d-2}\})_{3^{k-2}} = \text{ske}l_{d-1}(\{3^{d-2}, 4, 3^{k-2}\}).
\]
Example 3.5.3. $3\{3\}3^\{\}^\{\}$

This example employs the generalization of our previous twisting method described in Section 3.3.2. The complexes involved are the complex regular polygon $\mathcal{L} = 3\{3\}3$ of rank 2 and the 2-edge $\mathcal{K} = \{\} = \{\}$ of rank 1. The construction proceeds from the symmetry group $W = 3[3][3][3]3$ of the regular complex 3-polytope and exploits the self-duality of the complex 3-polytope. Recall that $W$ is generated by unitary reflections $\tau_0, \tau_1, \tau_2$ of period 3 and admits the presentation

$$
\tau_0^3 = \tau_1^3 = \tau_2^3 = 1,
\tau_0 \tau_2 = \tau_2 \tau_0, \tau_0 \tau_1 \tau_0 = \tau_1 \tau_0 \tau_1, \tau_1 \tau_2 \tau_1 = \tau_2 \tau_1 \tau_2.
$$

By the symmetry of the presentation, the mapping $\tau_0 \rightarrow \tau_2, \tau_2 \rightarrow \tau_0, \tau_1 \rightarrow \tau_1$ extends to an involutory group automorphism, $\tau$ (say), of $W$. Hence, to reconnect with our twisting method, we can take the complex reflection group $W$ as the underlying group on which $\Gamma(\mathcal{K})$, here isomorphic to $C_2$ and considered to be generated by $\tau$, acts in a suitable way. The twisting operation is indicated in Figure 3.5.

![Figure 3.5: Twisting operation for $3\{3\}3^\{\}$](image)

Thus from Section 3.3.2 we have a resulting regular 3-complex $3\{3\}3^\{\}$, on which the group $\Gamma := W \rtimes C_2$ of order 1296 acts as a flag-transitive subgroup. The complex has 72 vertices and 54 facets. The facets are isomorphic to $3\{3\}3$, and the vertex-figures are isomorphic to $\{\}^\{\}^\{\}$.
We want to show that this is the complex 3-polytope $3\{3\}3\{4\}2$. As generators for the group $\Gamma$ we can take $\tau_1, \tau_0, \tau$ giving the distinguished generating subgroups $\langle \tau_1 \rangle, \langle \tau_0 \rangle, \langle \tau \rangle$ of $\Gamma$. These generators $\tau_1, \tau_0, \tau$ satisfy the defining relations for the distinguished generators of the symmetry group $3[3]3[4]2$ of the complex 3-polytope $3\{3\}3\{4\}2$. In fact, since $\tau \tau_1 \tau = \tau_1$ and $\tau \tau_0 \tau = \tau_2$ we have

$$\tau_1^3 = \tau_0^3 = \tau^2 = 1, \tau \tau_1 = \tau_1 \tau,$$

$$\tau_1 \tau_0 \tau_1 = \tau_0 \tau_1 \tau_0, \tau_0 \tau \tau_0 \tau = \tau \tau_0 \tau \tau_0.$$

Thus $\Gamma$ is a quotient of $3[3]3[4]2$, and since both groups have the same order, $\Gamma$ must be isomorphic to $3[3]3[4]2$. But then the incidence complex with group $\Gamma$ must be isomorphic to the complex polytope $3\{3\}3\{4\}2$, since both can be constructed in the same canonical way from the generating subgroups $\langle \tau_1 \rangle, \langle \tau_0 \rangle, \langle \tau \rangle$ of $\Gamma$.

The complex $3\{3\}3\{4\}2$ and complex hypercube $n\{4\}3\{2\}3\cdots3\{2\}3 = n^{\{3,\ldots,3\}}$ are the only complex polyhedra that can be constructed by twisting.

**Example 3.5.4.** $\{3\}^{PG(2,2)}$

We have $\Gamma(\{3\}) = \langle \rho_0, \rho_1 \rangle$ and $\Gamma(PG(2, 2)) = \langle R_0, R_1 \rangle = PGL(3, 2)$. The regular 4-complex $\{3\}^{PG(2,2)}$ has the infinite group $\Gamma \cong W(D_{1,7}) \rtimes PGL(3, 2)$ as a flag-transitive subgroup of the automorphism group.

The vertex-figure stabilizer subgroup in $\Gamma$ is $\langle \rho_1, R_0, R_1 \rangle$. That is, the vertex-figure is isomorphic to the 3-complex $2^{PG(2,2)}$ (see Example 2.3.4). The facet stabilizer subgroup in $\Gamma$ is $\langle \rho_0, \rho_1, R_0 \rangle \cong W(D_{1,3}) \rtimes \langle R_0 \rangle$ and the facet itself is isomorphic to $\{3\}^{\{3\}}$ (see Example 3.5), the 2-skeleton of $\{3, 4, 3\}$. 

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3.6 Applications

3.6.1 Abstract Polytopes

Throughout this section we discuss complexes $2^K$, not necessarily regular, with $K$ an abstract polytope. These are themselves abstract polytopes.

Suppose $K = \{q\}$ is a $q$-gon with $3 \leq q < \infty$. It was shown in [32, Chapter 8D] that $2^\{q\}$ is isomorphic to Coxeter’s regular map $\{4, q|4^{\lfloor q/2\rfloor-1}\}$ [12] in the 2-skeleton of the ordinary $q$-cube $\gamma_q = \{4, 3^{q-2}\}$, whose edge-graph coincides with that of the cube.

The method of construction directly produces a realization of $2^\{q\}$ in the 2-skeleton of $\gamma_q$.

The case when $K$ is an (abstract) regular polytope has inspired a number of generalizations of the $2^K$ construction that have proved important in the study of universality questions and extensions of regular polytopes (see [32, Chapter 8] and [38]). A versatile generalization is to polytopes $2^{K,D}$ where $K$ is a (vertex-describable) regular $k$-polytope with $v$ vertices and $D$ is a Coxeter diagram on $v$ nodes admitting a suitable action of $\Gamma(K)$ as a group of
diagram symmetries. The corresponding group $W(\mathcal{D})$ can be extended by $\Gamma(\mathcal{K})$ to obtain
the automorphism group $W(\mathcal{D}) \times \Gamma(\mathcal{K})$ of a regular $(k + 1)$-polytope denoted $2^{\mathcal{K},\mathcal{D}}$. When
$\mathcal{D}$ is a trivial diagram on the vertex set of $\mathcal{K}$ the polytope $2^{\mathcal{K},\mathcal{D}}$ is isomorphic to the power
complex $2^\mathcal{K}$ of Section 3.2.

The polytopes $2^{\mathcal{K},\mathcal{D}}$ are very useful in the study of universal regular polytopes. Let $\mathcal{K}$ be
a regular map of type $\{3, r\}$ on a surface. For instance, a torus map $\{3, 6\}_{(b,0)}$ or $\{3, 6\}_{(b,b)}$
(see Figure 3.6).

Suppose we wish to investigate regular 4-polytopes with cubes $\{4, 3\}$ as facets and copies of
$\mathcal{K}$ as vertex-figures. In particular, this would involve determining when the universal such
structure, denoted

$$U := \{\{4, 3\}, \mathcal{K}\},$$

is a finite polytope. It turns out that this universal polytope $U$ always exists for any $\mathcal{K}$, and
that $U = 2^{\mathcal{K},\mathcal{D}}$ for a certain Coxeter diagram $\mathcal{D}$ depending on $\mathcal{K}$ (see [32, Theorem 8E10]). In
particular, $U$ is finite if and only if $\mathcal{K}$ is neighborly. In this case $U = 2^\mathcal{K}$ and $\Gamma(U) = C_2^\omega \times \Gamma(\mathcal{K})$
(and $\mathcal{D}$ is trivial). For example if $\mathcal{K}$ is the hemi-icosahedron $\{3, 5\}_5$ (with group $[3, 5]_5$) then

$$U = \{\{4, 3\}, \{3, 5\}_5\} = 2^{\{3,5\}_5}$$
\[ \Gamma(\mathcal{U}) = C_2 \wr [3, 5]_5 = C_2^6 \rtimes [3, 5]_5. \]

### 3.6.2 Incidence Complexes

In communication networks complex hypercubes offer advantages over binary hypercubes in redundant routing and reduced diameters for an equivalent number of nodes. Typically the structures used are not identified as complex hypercubes. For example Dally [14] identifies the complex hypercube as a \( k \)-ary \( n \)-cube; this is \( \gamma^k_n \). Using the power complex construction results in easier combinatorial analysis of the network. The **diameter** of an incidence complex is the number of vertices in its longest edge path. For a power complex \( n^\mathcal{K} \), with \( \mathcal{K} \) any complex, the distance between vertices \( \varepsilon \) and \( \varepsilon' \) is equal to the number of indices \( j \) such that \( \varepsilon_j \neq \varepsilon'_j \). The diameter can then be seen to be the number of vertices of \( \mathcal{K} \).

A **Latin square** is an \( n \times n \) array filled with \( n \) different symbols such that any one symbol only occurs once in each row and column. Dougherty and Szczepanski [17] examine the complex hypercube \( \gamma^k_n \) as a graph with labeled vertices as a generalization of a Latin Square, their Latin \( k \)-hypercube. They use the construct to prove existence theorems for maximum distance separable codes.

Complex reflection groups have also been of recent interest in coding theory. The sporadic complex reflection groups identified by Shephard and Todd [52] have properties that are ideal for group codes. A group code is determined by a group acting on a vector space and an initial vector in the vector space. The code is the orbit of the point under the group, and individual codewords are the images of the initial vector under elements of the group. Recent work (see [41], [28]) has examined using the case where the group is a complex reflection groups or a wreath product of complex reflection groups. Refinements in decoding using sequences of subgroups that are themselves reflection groups. The complex reflection groups and wreath products of complex reflection groups have been shown [37] [28] to have ideal
properties for decoding. The wreath products studied have been of the type $G \wr S_n$, where $G$ is a complex reflection group, and are used to develop large block matrices. The power complex wreath product is of the type $S_n \wr G$. The usefulness of the reflection groups of power complexes for group coding has not been examined.
Chapter 4

Covering

4.1 General Covering Results

Throughout this section all incidence complexes are assumed to be vertex describable, meaning that every face is uniquely determined by its vertex set. In discussing coverings we follow [18].

4.1.1 Coverings $n^K \rightarrow m^L$

When $n \geq m$ coverings between incidence-complexes naturally induce either coverings or weak coverings between the corresponding power complexes.

**Theorem 4.1.1.** Let $K$ and $L$ be finite vertex-describable incidence complexes of rank $k$, and let $\gamma : K \rightarrow L$ be a covering. Moreover, let $n \geq m \geq 2$ and $f : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\}$ be a surjective mapping. Then $\gamma$ and $f$ induce a weak covering $\pi_{\gamma,f} : n^K \rightarrow m^L$ between the power complexes $n^K$ and $m^L$. Moreover, $\pi_{\gamma,f}$ is a covering if and only if $f$ is a bijection (and $m = n$).
Proof. Suppose $V(K) := \{1, \ldots, v(K)\}$ and $V(L) = \{1, \ldots, v(L)\}$ are the vertex sets of $K$ and $L$ respectively. Then $v(L) \leq v(K)$ since there is a covering from $K$ to $L$. Define $N := \{1, \ldots, n\}$ and $M := \{1, \ldots, m\}$.

First note that a typical flag of $n^K$ has the form

$$\Phi(\varepsilon) := \{\emptyset, F_0(\varepsilon), \ldots, F_k(\varepsilon)\},$$

where $\varepsilon$ is a vector in $N^{v(K)}$ and $\Phi = \{F_{-1}, F_0, \ldots, F_k\}$ is a flag of $K$. Clearly, if $r \geq 1$ and $\Phi, \Phi'$ are $(r - 1)$-adjacent flags of $K$, then $\Phi(\varepsilon), \Phi'(\varepsilon)$ are $r$-adjacent flags of $n^K$. Similar statements also hold for $m^L$.

Now consider the given map $\gamma: K \to L$. For a vertex $j$ of $K$ write $\bar{j} := j\gamma$, so $\bar{j}$ is a vertex of $L$. Since $\gamma$ is surjective, we may assume that the vertex labeling for $K$ and $L$ is such that the vertices $1, 2, \ldots, v(L)$ comprise all the vertices of $L$, and in particular that $\bar{j} = j$ for each $j = 1, \ldots, v(L)$. Now define the mapping

$$\pi_{\gamma,f}: n^K \to m^L$$

(4.1.1)

$$F(\varepsilon) \to (F\gamma)(\varepsilon_f),$$

(4.1.2)

where as usual $F$ denotes a face of $K$ and $\varepsilon$ a vector in $N^{v(K)}$, and

$$\varepsilon_f := (\varepsilon_1 f, \ldots, \varepsilon_{v(L)} f)$$

is the vector in $M^{v(L)}$ given by the images under $f$ of the first $v(L)$ components of $\varepsilon$. We claim that $\pi := \pi_{\gamma,f}$ is a well-defined weak covering.

First we prove that $\pi$ is well-defined. For a face $F$ of a complex we let $V(F)$ denote its vertex set. Suppose that we have $F(\varepsilon) = F'(\varepsilon')$ in $n^K$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{v(K)})$ and $\varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_{v(K)})$ belong to $N^{v(K)}$ and $F$ and $F'$ are faces of $K$. By Lemma 2.1.2 we have
\( F = F' \) and \( \varepsilon_i = \varepsilon'_i \) for each \( i \notin V(F) = V(F') \). Therefore \( F\gamma = F'\gamma \) and \( \varepsilon_f = \varepsilon'_f \) agree on all components indexed by vertices \( i \) with \( i \notin V(F) \). All other components of \( \varepsilon_f \) and \( \varepsilon'_f \) are indexed by a vertex \( i \) of \( F \); but if \( i \in V(F) \) then \( \tilde{i} = (i)\gamma \in V(F\gamma) = V(F'\gamma) \), and hence \( i \) indexes a component where entries are allowed to range freely over \( M = (N)f \). Therefore, \((F\gamma)(\varepsilon_f) = (F'\gamma)(\varepsilon'_f)\). Thus \( \pi \) is well-defined.

Clearly, \( \pi \) is a homomorphism since this is true for \( \gamma \). For the same reason \( \pi \) is rank-preserving and surjective.

It remains to show that \( \pi \) is weakly adjacency preserving. To this end, let

\[
\Phi(\varepsilon) := \{\emptyset, \varepsilon, F_0(\varepsilon), \ldots, F_k(\varepsilon)\}
\]
\[
\Phi'(\varepsilon') := \{\emptyset, \varepsilon', F'_0(\varepsilon'), \ldots, F'_k(\varepsilon')\}
\]

be flags of \( n^K \), where

\[
\Phi := \{F_{-1}, F_0, \ldots, F_k\}, \quad \Phi' := \{F'_{-1}, F'_0, \ldots, F'_k\}
\]

are flags of \( K \) and \( \varepsilon, \varepsilon' \) are vectors in \( N^{v(K)} \). Suppose \( \Phi(\varepsilon) \) and \( \Phi'(\varepsilon') \) are \( r \)-adjacent for some \( r \geq 0 \). Then two possibilities can arise.

If \( r > 0 \) then \( \varepsilon = \varepsilon' \) and \( \Phi, \Phi' \) must be \((r - 1)\)-adjacent flags of \( K \). It follows that \( \varepsilon_f = \varepsilon'_f \) and that \( \Phi\gamma, \Phi'\gamma \) are \((r - 1)\)-adjacent flags of \( L \) since \( \gamma \) is adjacency preserving. Hence the image flags of \( \Phi(\varepsilon) \) and \( \Phi'(\varepsilon') \) under \( \pi \), which are given by

\[
\Phi(\varepsilon)\pi = \{\emptyset, \varepsilon_f, (F_0\gamma)(\varepsilon_f), \ldots, (F_k\gamma)(\varepsilon_f)\}
\]

and

\[
\Phi'(\varepsilon')\pi = \{\emptyset, \varepsilon'_f, (F'_0\gamma)(\varepsilon'_f), \ldots, (F'_k\gamma)(\varepsilon'_f)\}
\]

respectively are also \( r \)-adjacent. Thus, when \( r > 0 \), the map \( \pi \) takes \( r \)-adjacent flags of \( n^K \)
to $r$-adjacent flags of $m^\mathcal{L}$.

Now suppose $r = 0$. Then $\Phi = \Phi'$ (but $\varepsilon \neq \varepsilon'$), since the faces $F_s$ and $F'_s$ of $\mathcal{K}$ must have the same vertex sets for each $s \geq 0$. Moreover, since $F_0 = F'_0$ and $r \neq 1$, we have $F_0(\varepsilon) = F'_0(\varepsilon') = F_0(\varepsilon')$, so $\varepsilon_i = \varepsilon'_i$ for each vertex $i$ of $\mathcal{K}$ distinct from $i_0 := F_0$; hence $\varepsilon$ and $\varepsilon'$ differ in the position indexed by $i_0$. Then we certainly have $(F_s\gamma)(\varepsilon_f) = (F'_s\gamma)(\varepsilon'_f)$ for all $s \geq 0$. Hence $(\Phi(\varepsilon))\pi$ and $(\Phi'(\varepsilon'))\pi$ are either 0-adjacent or identical.

At this point we know that $\pi : n^\mathcal{K} \to m^\mathcal{L}$ is weakly adjacency preserving, that is $\pi$ is a weak covering. This proves the first part of the theorem.

Since $\varepsilon$ and $\varepsilon'$ differ only in the position indexed by $i_0$, the corresponding shortened vectors $(\varepsilon_1, \ldots, \varepsilon_{v(\mathcal{L})})$ and $(\varepsilon'_1, \ldots, \varepsilon'_{v(\mathcal{L})})$ in $N^{v(\mathcal{L})}$ also differ only in the position indexed by $i_0$; note here that $\tilde{i}_0 = i_0$, by our labeling of the vertices in $\mathcal{K}$ and $\mathcal{L}$. Hence the two vertices $\varepsilon_f = (\varepsilon_1 f, \ldots, \varepsilon_{v(\mathcal{L})} f)$ and $\varepsilon'_f = (\varepsilon'_1 f, \ldots, \varepsilon'_{v(\mathcal{L})} f)$ of $m^\mathcal{L}$ in $(\Phi(\varepsilon))\pi$ and $(\Phi'(\varepsilon'))\pi$, respectively, either coincide or differ in a single position, indexed by $i_0$; the former occurs precisely when $\varepsilon_{i_0} f = \varepsilon'_{i_0} f$. Therefore, since $\varepsilon_{i_0}$ and $\varepsilon'_{i_0}$ can take any value in $N$, the mapping $\pi$ is a covering if and only if $f$ is a bijection. This completes the proof.

The previous Theorem 4.1.1 describes quite general circumstances under which coverings or weak coverings between complexes $n^\mathcal{K}$ and $m^\mathcal{L}$ are guaranteed to exist. Under the basic assumption that $n \geq m$ this generally leads to a host of possible weak covering maps. We will now work with coverings or weak coverings between power complexes in situations where certain well-behaved (equifibred) coverings between the original complexes $\mathcal{K}$ and $\mathcal{L}$ exist. This also permits examples with $n \leq m$.

Let $\mathcal{K}$ and $\mathcal{L}$ be vertex-describable complexes of rank $k$, and let $V(\mathcal{K}) := \{1, \ldots, v(\mathcal{K})\}$ and $V(\mathcal{L}) := \{1, \ldots, v(\mathcal{L})\}$, respectively, denote their vertex sets. Suppose there is a covering $\gamma : \mathcal{K} \to \mathcal{L}$ that is equifibred (with respect to the vertices), meaning that the fibres $\gamma^{-1}(j)$ of the vertices $j$ of $\mathcal{L}$ under $\gamma$ all have the same cardinality, $l$. In other words, the restriction
of $\gamma$ to the vertex sets of $K$ and $L$ is $l : 1$, so in particular $v(K) = l \cdot v(L)$.

Examples are given by the regular $k$-polytopes $K$ that are (properly) centrally symmetric, in the sense that the group $\Gamma(K)$ contains a central involution that does not fix any of the vertices [32, p. 255]; any such involution $\alpha$ pairs up the vertices of $K$ and naturally determines an equifibred covering $K \to K/\langle \alpha \rangle$, of $K$ onto its quotient $K/\langle \alpha \rangle$, satisfying the desired property with $l = 2$.

**Theorem 4.1.2.** Let $K$ and $L$ be vertex-describable incidence complexes of rank $k$, and let $\gamma : K \to L$ be a covering and $m, n \geq 2$. Suppose that $\gamma$ is equifibred with fibres of cardinality $l \geq 1$, and $g : \{1, \ldots, n\}^l \to \{1, \ldots, m\}$ is a surjective mapping (hence $m \leq n^l$). Then $\gamma$ and $g$ induce a weak covering $\pi^{\gamma,g} : n^K \to m^L$ between the power complexes $n^K$ and $m^L$. Moreover, $\pi^{\gamma,g}$ is a covering if and only if $g$ is a bijection (and $m = n^l$); in this case $\pi^{\gamma,g}$ is one-to-one on the vertices.

**Proof.** Let $n, m \geq 2$ and $l$ be as above. Define $N := \{1, \ldots, n\}$, $M := \{1, \ldots, m\}$ and $L := \{1, \ldots, l\}$. We wish to describe coverings $n^K \to m^L$ that can balance the effect of $\gamma$ as an $l : 1$ mapping on the vertex sets. Assume that the vertices of $K$ and $L$ are labeled in such a way that

$$\gamma^{-1}(j) = L_j := \{(j - 1)l + 1, \ldots, (j - 1)l + l \} \text{ (for } j \in V(L)),$$

Thus for each $j$, the map $\gamma$ takes the vertices of $K$ in $L_j$ to the vertex $j$ in $L$. With a slight abuse of notation we can write a vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{v(K)})$ in $N^{v(K)} = (N^l)^{v(L)}$ in the form $\varepsilon = (\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_{v(L)})$, where

$$\hat{\varepsilon}_j := (\varepsilon_{(j-1)l+1}, \ldots, \varepsilon_{(j-1)l+l})$$

lies in $N^l$ for each $j = 1, \ldots, v(L)$.

Now suppose that, in addition to $\gamma$, we also have a surjective mapping $g : N^l \to M$ (and
hence \( m \leq n^l \). Then \( \gamma \) and \( g \) determine a mapping

\[
\pi_{\gamma,g} : n^K \rightarrow m^L
\]  
(4.1.3)

\[
F(\varepsilon) \rightarrow (F\gamma)(\varepsilon_g)
\]  
(4.1.4)

where again \( F \) denotes a face of \( K \) and \( \varepsilon \) a vector in \( N^{v(K)} \), and

\[
\varepsilon_g := (\hat{\varepsilon}_1 g, \ldots, \hat{\varepsilon}_v(L) g)
\]

is the vector in \( M^{v(L)} \) given be the images under \( g \) of the components of \( \varepsilon \) in its representation as \((\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_v(L))\). We must prove that \( \pi := \pi_{\gamma,g} \) is a covering.

First we must show that \( \pi \) is well defined. Suppose we have \( F(\varepsilon) = F'(\varepsilon') \) in \( n^K \), where \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_{v(K)}) \) and \( \varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_{v(K)}) \) belong to \( N^{v(K)} \) and \( F, F' \) are faces of \( K \). As in the proof of Theorem 4.1.1, \( F = F' \), \( F\gamma = F'\gamma \) and \( \varepsilon_i = \varepsilon'_i \) for \( i \notin V(F) = V(F') \). Since \( \gamma \) is a covering, if \( i \in V(F) \) then \( (i)\gamma \in V(F\gamma) \); or equivalently, if \( j \notin V(F\gamma) \) then \( V(F) \cap L_j = \emptyset \). It follows that, if \( j \notin V(F\gamma) \) then \( \varepsilon_i = \varepsilon'_i \) for every \( i \) in \( L_j \), and therefore \( \hat{\varepsilon}_j = \hat{\varepsilon}'_j \) and \( \hat{\varepsilon}_j g = \hat{\varepsilon}'_j g \). Hence \( \varepsilon_g \) and \( \varepsilon'_g \) agree on every component represented by vertices of \( L \) outside \( F\gamma = F'\gamma \). As the remaining components are allowed to take on any value in \( M \), we conclude that \((F\gamma)(\varepsilon_g) = (F'\gamma)(\varepsilon'_g) \). Thus \( \pi \) is well-defined.

It is straightforward to verify that \( \pi \) is a rank-preserving surjective homomorphism. To show that \( \pi \) is also weakly adjacency preserving, Let

\[
\Phi(\varepsilon) := \{ \emptyset, \varepsilon, F_0(\varepsilon), \ldots, F_k(\varepsilon) \}
\]

\[
\Phi'(\varepsilon') := \{ \emptyset, \varepsilon', F_0'(\varepsilon'), \ldots, F_k'(\varepsilon') \}
\]
be \( r \)-adjacent flags of \( n^K \), where

\[
\Phi := \{ F_{-1}, F_0, \ldots, F_k \}
\]

\[
\Phi' := \{ F'_{-1}, F'_0, \ldots, F'_k \}
\]

are flags of \( n^K \) and \( \varepsilon, \varepsilon' \) lie in \( N^v(K) \). Again, two possibilities arise. First if \( r > 0 \) then \( \varepsilon = \varepsilon' \) and \( \Phi, \Phi' \) are \( (r - 1) \)-adjacent in \( K \). Hence \( \varepsilon_g = \varepsilon'_g \) and \( \Phi_\gamma, \Phi'_\gamma \) are \( (r - 1) \)-adjacent in \( L \).

It follows that the two image flags under \( \pi \),

\[
(\Phi(\varepsilon))\pi = \{ \emptyset, \varepsilon_g, (F_0\gamma)(\varepsilon_g), \ldots, (F_k\gamma)(\varepsilon_g) \},
\]

\[
(\Phi'(\varepsilon'))\pi = \{ \emptyset, \varepsilon'_g, (F'_0\gamma)(\varepsilon'_g), \ldots, (F'_k\gamma)(\varepsilon'_g) \},
\]

are also \( r \)-adjacent. Now if \( r = 0 \) then \( \Phi = \Phi' \) (but \( \varepsilon \neq \varepsilon' \)); in fact, \( V(F_s) = V(F'_s) \) and hence \( F_s = F'_s \) for each \( s \geq 0 \). When \( s = 0 \) this gives \( F_0(\varepsilon) = F'_0(\varepsilon') = F_0(\varepsilon') \) (since \( r \neq 1 \)), and therefore \( \varepsilon_i = \varepsilon'_i \) for each vertex \( i \) of \( K \) distinct from \( i_0 := F_0 \); hence \( \varepsilon \) and \( \varepsilon' \) only differ in the position indexed by \( i_0 \). This already implies that \( (F_s\gamma)(\varepsilon_g) = (F'_s\gamma)(\varepsilon'_g) \) for all \( s \geq 0 \), and hence that \( (\Phi(\varepsilon))\pi \) and \( (\Phi'(\varepsilon'))\pi \) are weakly \( 0 \)-adjacent flags of \( m^L \). Thus \( \pi \) is a weak covering.

Moreover, since \( \varepsilon_i = \varepsilon'_i \) if and only if \( i \neq i_0 \), we also know that \( \widehat{\varepsilon}_j = \widehat{\varepsilon}'_j \) if and only if \( j \neq j_0 := (i_0)\gamma \). Hence the two vertices

\[
\varepsilon_g := (\widehat{\varepsilon}_1g, \ldots, \widehat{\varepsilon}_vg),
\]

\[
\varepsilon'_g := (\widehat{\varepsilon}'_1g, \ldots, \widehat{\varepsilon}'_vg)
\]

of \( m^L \) lying in \( (\Phi(\varepsilon))\pi \) and \( (\Phi'(\varepsilon'))\pi \), respectively, either coincide or differ in a single position, indexed by \( j_0 \); the former occurs precisely when \( \widehat{\varepsilon}_{j_0}g = \widehat{\varepsilon}'_{j_0}g \). Since \( \widehat{\varepsilon}_{j_0} \) can take any value in \( N^l \), the mapping \( \pi \) is a covering if and only if \( g \) is a bijection.
Finally, suppose \( g \) is a bijection, so in particular \( m = n^l \). Then \( n^K \) and \( m^L \) must have the same number of vertices,

\[
n^v(K) = n^{l \cdot v(L)} = m^{v(L)},
\]

and hence \( \pi \) must be a covering that is one-to-one on the vertices.

As an example consider finite regular polygons \( K = \{2p\} \) and \( L = \{p\} \), with \( 2p \) or \( p \) vertices, respectively for some \( p \geq 2 \). The central symmetry of \( K \) gives an obvious equifibred covering \( \gamma: K \to L \) between \( K \) and \( L \) with fibres of size \( l = 2 \). Now choose \( m = n^2 \) and pick any bijection \( g : \{1, \ldots, n\}^2 \to \{1, \ldots, n^2\} \). Then

\[
\pi^{\gamma,g} : n^\{2p\} \to (n^2)^\{p\}
\]

is a covering. Either complex has \( n^{2p} \) vertices, and \( \pi^{\gamma,g} \) is one-to-one on the vertices. For example, when \( n = 2 \) we obtain a covering

\[
\pi^{\gamma,g} : 2^\{2p\} \to 4^\{p\}.
\]

Here, \( 2^\{2p\} \) is Coxeter’s regular map \( \{4, 2p | 4^{|p| - 1}\} \) described in Chapter 2.
Bibliography


