MIDDLE SCHOOL MATHEMATICS TEACHERS’ USE OF ADVANCED MATHEMATICAL KNOWLEDGE IN PRACTICE: AN INTERPRETATIVE PHENOMENOLOGICAL ANALYSIS

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Abstract

Current policy dictates mathematics teacher must have a deep understanding of the subject to teach it. Many states require a major in mathematics as part of the requirements for certification. There is a lack of research regarding how mathematics learned at the tertiary level is utilized in practice. Utilizing Ball, Thames, and Phelps (2008) Domains of Mathematical Knowledge for Teaching for specificity, this interpretative phenomenological analysis (IPA) documents how three purposefully selected middle school mathematics teachers understood tertiary mathematical knowledge to materialize in practice both directly and pedagogically. Defining “Advanced Mathematical Knowledge” (AMK), as knowledge gained from a major in mathematics, analysis of the data collected through interviews revealed AMK content as not implemented directly due to its advanced nature. Teachers described AMK as categorically different from the mathematics they teach. Teachers’ coursework was heavily weighted in pure content courses with little or no courses to develop pedagogical content knowledge. However, AMK was found to affect teachers’ specialized and horizon knowledge. Teachers found the knowledge gained from pedagogical content courses as relevant and useful in practice. Theory regarding Advanced Mathematical Thinking (AMT) was utilized to understand teachers’ perceptions. These findings are significant for policy makers, teacher preparation program designers, as well as current and prospective mathematics teachers. This study advocates for the formal inclusion of AMT theory into teachers’ preparation to aid their understanding of it; the study also advocates for more opportunities to formally develop pedagogical content knowledge.

Keywords: mathematical knowledge for teaching, pedagogical content knowledge, mathematics teacher preparation, interpretative phenomenological analysis, advanced mathematical thinking
Acknowledgements

Being a runner all my life, one goal I have always had is to complete a marathon. Starting this doctoral journey, I heard others compare the process to the long race, however, didn’t truly grasp what it meant. Having completed this journey, I now feel why and am very thankful to “the village” who has cheered me along the way all the way to the end. My children, being born during the journey, were both my motivation and inspiration. I hope that through me they can see that if they set their heart on it, anything is possible. To my husband, I am thankful for his “you started this, you’re going to finish it” as well as all the support, time, and effort on the way. To our parents, I will forever be thankful for helping out with the kids, life, and balancing it all. To my aunt and uncle, I am very thankful you went through this writing journey with me; your time, effort, expertise, insights, and encouragement kept me going and motivated to keep putting one foot in front of the other. To my committee, I am thankful for your time and high standards; together your expertise strengthened my knowledge, my writing, and ultimately this thesis. While this may have been the first “marathon” I have completed, I look forward to where it will take me next, and a new journey that is beginning.
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Chapter 1: Introduction

Statement of the Problem

Teacher preparation programs in the U.S. commonly require of those intending to teach middle school mathematics a major in that subject, coursework in education, and a practicum experience (Darling-Hammond, 2000). These requirements derive from policy and are intended to increase teacher quality (Ball, Lubienski, & Mewborn, 2001; Cochran-Smith, 2005). However, middle school mathematics teachers asked whether they employed the mathematics learned as a requirement of their preparation while teaching commonly answered, “No.” This circumstance raises several issues. First, since a mathematics major is part of current policy and required of many who teach middle school mathematics, that quick answer is unnerving. Second, a review of the literature suggests that swift reply might over-simplify a matter more complex than it initially seems. Third, if current policy aims to improve teacher quality, deeper understanding of that policy’s consequences through thorough investigation is warranted. Finally, the increasing pressure applied to mathematics teachers and schools to better student achievement in mathematics makes this matter vital.

This study aimed to investigate two issues concerned with mathematics content knowledge teacher preparation and its translation into practice. The first involved qualifying the pure content necessary to optimize teaching, intending specifically to understand how pure content appears in practice. To consider that content, this study defined tertiary level mathematics as “advanced mathematical knowledge” (AMK) (Zazkis & Leikin, 2010). Second, it investigated the translation of pure content knowledge into pedagogical content knowledge in practice while teaching. This study employed the terms “pedagogical content knowledge
(PCK)” and “knowledge for teaching” interchangeably to describe the knowledge that teachers need to elicit student understanding of the pure content knowledge being taught.

To improve understanding of these factors, promote further research, and inform future policy, the study documented the lived experiences of middle school mathematics teachers. Their close examination provided insight into how AMK gained at the tertiary level actually translates into practice. A grasp of the current policy behind the design of mathematics teacher preparation was fundamental to that process.

The number and type of content knowledge courses required of secondary mathematics teachers currently differ from state to state and among tertiary institutions within a state. Many of these differences reflect policy changes resulting from the No Child Left Behind (NCLB) Act, which required states to create a plan to make all teachers “highly qualified” (Flores, Patterson, Shippen, Hinton, & Franklin, 2010). To be deemed such, NCLB required that teachers be fully certified in the subject they teach (E. Smith, 2008). To enact this requirement, some states require teachers to pass a content knowledge test, major in the subject area they teach, or sometimes do both (Darling-Hammond, 2000; Flores et al., 2010).

At the institutional level, the types of courses required for the major also differ. With pure content courses fundamental, programs may or may not include pedagogical content knowledge courses as part of their major requirements. Tertiary level pure content knowledge courses delve into mathematical topics advanced beyond that taught at the secondary level (Robert & Schwarzenberger, 1991). If pedagogical content courses are offered, their design may also vary from program to program. Focused on mathematics and teaching mathematics, the PCK course may integrate secondary level topics, tertiary, or a combination of both, to examine associated instructional practices. It is also an important consideration that, of the two types of
courses, pedagogical content courses are many times taught by education faculty, whereas pure content courses are taught by mathematics faculty (Hill, Rowan, & Ball, 2005).

Variations inherent in the K-12 mathematics curriculum content further complicate understanding the pure mathematical knowledge requirements of mathematics teachers. Course requirements are a function of the grade level being taught, with those of elementary and secondary teachers differing. Secondary level preparation, which many middle school teachers receive, commonly requires a major in the subject to be taught (Schmidt et al., 2007). According to the literature, this requirement could be considered problematic if that major features mainly pure content knowledge courses. For example, Monk (1994) determined that the positive relationship between the number of pure content knowledge courses taken and student achievement plateaus after five courses. Researchers classify the advanced mathematics learned at the tertiary level as categorically different from that at the secondary level (Edwards, Dubinsky, & McDonald, 2005; Robert & Schwarzenberger, 1991). The dissimilarity between tertiary level mathematics and the K-12 curriculum necessitates scrutiny to determine whether, and how, tertiary level mathematics prepares a teacher to teach that content.

When considering knowledge for teaching, current research can help inform teacher preparation program design and policy regarding teacher certification. Monk (1994) concluded that pedagogical content courses exert a more positive impact on student achievement than pure content courses. Yet policy and design of secondary teacher preparation in the United States currently weight course requirements heavily with pure mathematics, less with pedagogy (Schmidt et al., 2007). Thus, it is important to discern how knowledge gained from teachers’ learning experience in the current design translates into knowledge for teaching in practice.
Schmidt et al. (2007) documented a gap in research into the issues described above: “the definitive study of the relationship of teacher preparation to classroom instruction and student achievement has yet to be done” (p. 4). Research into secondary teachers’ content knowledge preparation has focused on their acquisition of knowledge, with little research or understanding regarding how that learning is utilized in practice (Zazkis & Leikin, 2010, p. 279). To extend our understanding of this important question, this study focused on the lived experiences of practicing teachers regarding how AMK is utilized. Teachers’ perceptions helped explain how AMK learned at the tertiary level interacts with, and is implemented in, the middle school curriculum. They also offered insight into how teachers’ mathematical content knowledge preparation translates into knowledge for teaching in practice. The knowledge thus gained suggests further research into the links among teacher preparation, mathematical knowledge for teaching, and tertiary level mathematics’ contribution to teaching practice.

The current content knowledge preparation practices for secondary mathematics teachers and the research examining those practices create a potential policy – knowledge conflict (Clark, Guba, & Smith, 1977). This conflict is heightened by the discrepancy between the content learned at the tertiary level and the content taught at the secondary level (Ball, Thames, & Phelps, 2008; Moreira & David, 2008; Zazkis & Leikin, 2010). The complex interplay among the design of teacher content knowledge preparation, research regarding that preparation, and uncertainty regarding how the knowledge gained applies to teaching the K-12 curriculum warrants examination.

The different levels within the K-12 curriculum as well as the distinct mathematics taught at each make it imperative to restrict investigation to one. Past research into the subject at the secondary level spanned grades 8 – 12, combining both middle and high school grades (Zazkis &
Leikin, 2010). This study narrowed the focus to middle school teachers’ perceptions to gain insight into AMK’s implementation at the middle school level to aid and inform change there. Though this study premised the need for some tertiary mathematics preparation to prepare middle mathematics teachers adequately, the number of these courses, the content that would optimize their preparatory value, and their impact on practice, all remain unclear. This study focused on the last of these listed components to help understand the phenomenon and further inquiry into it.

**Significance of the Problem**

The mathematical skills with which K-12 education must equip students make mathematics teacher quality of utmost importance. The 21st century’s technological advances are increasing the level of mathematical skill necessary to enter the workforce; consequently, the demand for students to be better prepared in mathematics is growing. At the same time, colleges and industry are concerned that incoming students are not equipped with the mathematics necessary for success (Conference Board of the Mathematical Sciences [CBMS], 2001, p. 4). Pupils in other countries are also outscoring U.S. students on international indicators of mathematical knowledge. In the 2007 Trends International Mathematics and Science Study, the average achievement in mathematics by U.S. fourth graders was significantly lower than in eight other countries, while U.S. eighth graders were significantly outscored by five (Mullis et al., 2008).

Well-trained teachers have a greater impact on student achievement than socioeconomic background or race (Darling-Hammond, 2000, p. 33). Mathematics teacher preparation, specifically its translation into practice, matters, not only because demand for higher mathematical skills is growing, but also because, next to reading and writing, mathematics
provides the best indicator of college and career readiness and success (CBMS, 2001, p.3). Research into the translation of teacher content preparation into practice is necessary to shape the design of such programs optimally.

It has plagued researchers into mathematics teacher preparation that, while it is agreed that teaching mathematics requires mathematical content knowledge, exactly how much and how it influences future teaching remain uncertain (Ball et al., 2001; Darling-Hammond, 2000; Krauss, Baumert, & Blum, 2008; Zazkis & Leikin, 2010). Many researchers and national policy organizations utilize the term “deep” when describing and studying the level of mathematical knowledge required to teach mathematics. (see, e.g., Ball et al., 2001; Davis & Simmt, 2006; Grossman, Schoenfeld, & Lee, 2005; Krauss, Baumert, et al., 2008; Krauss, Brunner, et al., 2008; Ma, 1999; Matthews, Rech, & Grandgenett, 2010; Shulman, 1986; Tchoshanov, 2011; Zazkis & Leikin, 2010). It is generally accepted, however, that knowledge of pure mathematics is neither a sufficient nor sole prerequisite (Ball et al., 2008; Davis & Simmt, 2006; Ferguson & T. Womack, 1993; Hill et al., 2005; Philipp et al., 2007; Shulman, 1986; E. Smith, 2008). Simply taking more advanced mathematics classes is unlikely to prepare a teacher with content knowledge adequate to teach mathematics (Ball et al., 2008).

While mathematics teachers require pure content knowledge, they also need pedagogical content knowledge that enables them to break down the content of the concepts to be taught (Ball et al., 2001; Ball et al., 2008; Shulman, 1986). Yet recognition of the need for such knowledge co-exists with lack of understanding of, and appreciation for, the complexity and scholarship necessary to teach the content taught in K-12 mathematics courses (CBMS, 2001). While an assumption may exist that pure knowledge segues into pedagogy during practice, “this does not happen easily, and often does not happen at all” (Ball & Bass, 2000).
This obscurity regarding the effects of current policy and practice affects current as well as prospective mathematics teachers. The former experience the effects as they integrate into the profession, while the latter trust institutions to equip them with the knowledge necessary for professional success and longevity. Yet, research concerning how secondary level teachers use the mathematical knowledge learned from undergraduate mathematics courses remains scarce (Zazkis & Leikin, 2010), inconsistent, and unclear regarding how much pure content knowledge is necessary. Researchers have studied the number of courses taken as one way to represent mathematics teacher knowledge (Ball et al., 2001). Monk (1994) found a plateaued relationship between number of courses taken and student achievement, while in a meta-analysis Begle (1979) documented no relationship between student achievement and the number of mathematics courses taken. Other research demonstrated a positive relationship between the number of courses taken and student achievement (Betts, Zau, & Rice, 2003; Goldhaber & Brewer, 1999). Goldhaber and Brewer (1999) observed that students whose teachers held a degree in mathematics outperformed those whose did not.

Such conflicting results gauging the impact of courses taken on a teacher’s knowledge for teaching in practice compel additional scrutiny. Such research could influence both teacher preparation programs for prospective teachers and professional development programs for those currently practicing. The results could also impact mathematics teacher readiness, effectiveness, and student achievement. With pressure on twenty-first century teachers to better prepare students for college and career growing, it is imperative that those teachers be equipped with the knowledge necessary to do so. Measuring teachers’ pedagogical content knowledge by the number of courses taken may not suffice (Zazkis & Leikin, 2010, p. 265), yet understanding
those courses’ effects on content taught and on the pedagogical content knowledge used to teach it, is vital.

Despite uncertainty regarding AMK’s interactions with content to be taught as well as teachers’ use of it in teaching, AMK is still relied on to integrate “deep” understanding of the mathematics knowledge that teachers require. Lawmakers set policy in that knowledge vacuum (Ball et al., 2008), while preparation programs base subject-matter requirements on “common sense” (Moreira & David, 2008). Since policy potentially impacts teacher effectiveness (Darling-Hammond, 2000), however, policy makers need reliable research results. Ball et al. (2008) concluded that researchers can contribute detail and specificity to the requirements policy-makers presently base on common sense (p.394). Practicing mathematics teachers’ understandings of the knowledge required to teach, and that knowledge’s continuing evolution in practice, provide one way to help pinpoint this elusive phenomenon (Davis & Simmt, 2006; Zazkis & Leikin, 2010).

Attempts to grasp the intricate connections between subject matter knowledge and instruction have focused on deficits and affordances of teacher knowledge, many on one mathematical topic or a single teacher (Hill, Blunk, et al., 2008). Policy makers and teacher preparation programs have leaned heavily on the belief that results gleaned from such constricted studies generalize (Hill, Blunk, et al., 2008). As a result, whether the preparation experiences required optimize the union of teachers’ knowledge of mathematics with knowledge of how to unpack it remains unknown. This concept becomes even more complicated when one considers the range of content in the secondary curriculum. For example, a significant gap surely exists between topics such as fractions and trigonometric functions, both taught at the “secondary” level, the former in middle school, the latter in high school. Yet many middle school teachers’
preparation is largely identical to that of high school teachers. This study will address how the AMK learned is used specifically at the middle school level.

In conclusion, uncertainty regarding AMK’s effects on the teaching of secondary mathematics affects the majority of pre-service and practicing teachers in the U.S. This study explored middle school, local level, mathematics teachers’ lived experiences of AMK’s links to content, both direct and pedagogic, to help clarify this important issue, and influence future research and teacher preparation design.

**Positionality Statement**

The researcher’s primary perspective is that of a constructivist-interpretivist. According to Creswell (2009), social constructivists aim to understand and interpret the meanings that individuals subjectively experience, embracing the complexity that multiple perspectives contribute to meaning (p. 8). Researchers utilizing an interpretivist perspective seek to tell a story accurately, to better understand a phenomenon (Butin, 2010, p. 60). In this study, the perspectives of practicing mathematics teachers are detailed to shed light on the murky question of AMK’s utility, direct and pedagogic, in the practice of teaching.

With constructivism-interpretivism this researcher’s primary perspective, the potential biases that my own experiences have introduced require explanation. Interpretivist researchers maintain that describing and documenting a story make them participants in it (Butin, 2010, p. 60). As a high school mathematics teacher who formerly taught at the middle school level, I have often pondered the use and utility of the mathematics learned from my mathematics teacher preparation experience in my own teaching. Superficially, it often occurred to me that I do not seem to directly utilize much of the knowledge I gleaned from the undergraduate major in mathematics required to become a certified secondary mathematics school teacher.
For that reason, I have often wondered whether the tertiary school time of mathematics teachers prepared like myself might better focus on how to teach secondary mathematics rather than on advanced tertiary content. I have watched mathematics teachers lacking exemplar methods of teaching specific content improvise ways to do so. Since many ways exist to teach mathematical content, perhaps learning how students best learn a particular idea and which methods of presentation are most efficient and effective might prove more valuable than gaining knowledge for knowledge’s sake. Commencing this study with these potential biases, as a constructivist researcher I acknowledged that prior perspective may influence interpretation (Creswell, 2009, p. 6), remained receptive to multiple perspectives and realities, and aimed to document them to further our understanding of this complex issue.

**Research Questions**

Two central questions guided this study. The first aimed to investigate AMK’s direct impact on curriculum concepts at the middle school level, while the second sought to explore how middle school mathematics teachers perceive AMK to aid in their knowledge for teaching mathematics.

- What are middle school mathematics teachers’ perceptions of AMK’s contribution to their *specialized content knowledge*?
  - How do mathematics teachers perceive and describe the extent of their use of AMK in middle school mathematics content?
  - How do mathematics teachers note and relate ways in which their AMK is implemented in middle school mathematics content?
  - What specific examples can teachers provide to illustrate their use of AMK content in middle school mathematics?
• What are middle school mathematics teachers’ perceptions of AMK’s relationship to their knowledge for teaching mathematics?
  
  o How do mathematics teachers perceive and describe AMK’s relationship to their knowledge of middle school mathematics content and students?
  
  o How do mathematics teachers perceive and describe AMK’s bearing on their knowledge of middle school mathematics content and teaching?
  
  o What specific examples do teachers provide that illustrate AMK’s usefulness in mathematical pedagogical tasks?

Within this qualitative research design, these open-ended process questions explored teachers’ experiences of their use of AMK in the everyday practice of teaching. Adapted from Zazkis and Leikin’s (2010) study and utilizing terms from Ball, Thames and Phelps (2008) Domains of Mathematical Knowledge for Teaching, they were also constructed to understand, in depth, each teacher’s content knowledge preparation in terms of pure and pedagogical content knowledge relative to courses required and offered, and to permit analysis of teachers’ convictions regarding AMK’s practical relevance.

Additionally, these questions align with this study’s intellectual and practical goals. According to Maxwell (2005), qualitative studies fit intellectual goals that include understanding participants’ perspectives along with context’s influence on them (p. 22). One of this study’s goals was to determine how, and to what degree, teachers’ experiences in their classroom settings reflect their tertiary preparation, thus improving our grasp of the ways that AMK relates to content taught and the ways to teach it. This study was designed to use teachers’ perspectives to make these complex interactions less obscure, potentially inspiring further research and informing future policy. This goal was consistent with Bolster’s call (1983) to connect research
with teachers’ perspectives on their real classroom experience, to maximize impact on practice
(as cited in Maxwell, 2005, p. 24).

**Theoretical Framework**

The next section details the theoretical scaffolding employed to examine how middle school mathematics teachers integrate their tertiary content knowledge preparation into practice, both directly and pedagogically. A theoretical framework focused on mathematics teachers’ knowledge provided an appropriate tool for this particular study. Of the many such constructs available, Ball, Thames and Phelps’ (2008) Domains of Mathematical Knowledge for Teaching was utilized to explore AMK’s utility in teaching middle school mathematics; the following paragraphs describe and validate its employment here. As a first step in this endeavor, Shulman’s (1986) Teacher Knowledge Construct is introduced as the foundational framework for Ball et al.’s (2008) paradigm. Next, Ball et al.’s (2008) Domains of Mathematical Knowledge for Teaching will be described in detail to frame this study’s examination of the knowledge required of mathematics teachers.

**Background: Shulman’s (1986) teacher knowledge construct.** Shulman (1986) posed questions concerning the development of teacher content knowledge, as well as the source of teachers’ explanations and representations of content knowledge, that apply to this current study: “What are the sources of teacher knowledge? What does a teacher know and when did he or she come to know it? How is new knowledge acquired, old knowledge retrieved, and both combined to form a new knowledge base?” (p. 8). This study premises that the knowledge necessary to teach middle school mathematics, and the effect of current knowledge preparation on middle school teaching practice, remain unclear. These questions posed over 25 years ago thus remain relevant today regarding middle school mathematics teaching.
To address the questions he had posed, Shulman’s (1986) domains of “content knowledge in teaching” separated content knowledge into subject matter content knowledge, pedagogical content knowledge and curricular knowledge (p. 9). *Subject matter content knowledge* described the quantity and structure of a teacher’s content knowledge. *Pedagogical content knowledge* referred to the subject matter knowledge necessary to make the content clear to others; in other words, to “subject matter knowledge for teaching” (p. 9). Shulman (1986) further defined the concept of pedagogical content knowledge as knowledge of the content most frequently taught, including an understanding of student preconceptions and misconceptions as well as tools to overcome them (p. 9-10). *Curricular knowledge* referred to knowledge of materials, methods, and programs available for teaching the content, including knowledge of interdisciplinary content (p. 10). Identifying and defining these three domains shed light on the complexity of the knowledge teachers must have. For this study, the particular type of knowledge that mathematics teachers must have matters even more. Shulman’s (1986) construct laid a foundation for understanding teacher knowledge; the following builds upon Shulman’s (1986) notions and focuses specifically on the knowledge mathematics teachers require to teach.

**Ball et al.’s (2008) Domains of Mathematical Knowledge for Teaching.** Ball et al. (2008) applied Shulman’s (1986) construct directly to mathematics teachers, refining the *subject matter content knowledge* domain and the *pedagogical content knowledge* into subdomains relevant to mathematics teachers. This study utilized the domains described by Ball et al.’s (2008) framework to conceptualize how middle school mathematics teachers employ AMK both directly and pedagogically in practice. Each domain defined below provides clarity and specificity to the types of knowledge that teachers described, and how they utilize that knowledge in practice.
**Subject matter domains.** By “unpacking” the subject matter knowledge domain, Ball et al. (2008) provided a lens useful for analyzing mathematics teachers’ perceptions of their use of AMK in the content taught at the middle school level. The first of these domains, *common content knowledge*, is defined as mathematical content and skills utilized in places other than teaching -- knowledge that individuals other than teachers would have.

*Specialized content knowledge*, the second subject matter knowledge domain, is the unique mathematical content needed for teaching. Simply put, it is imperative that teachers be expert at the content being taught. When teachers make mistakes or are unsure of mathematical content, ineffective teaching results (Ball et al., 2008, p. 399). Teachers must be able to “unpack” and “decompress” mathematical content in a manner that goes beyond understanding it (p. 400); to make a concept understandable to students, teachers must understand the “why” behind it. Beyond knowing the steps to solve problems, to forestall error in student perceptions they must understand their reasoning; this concept is consistent with CBMS’s (2001) recommendation that teachers need an understanding of the process by which mathematics is learned (p. 8). Thus, teachers require knowledge beyond the curriculum to be taught: they must understand representations, approaches, and interpretations (CBMS, 2001, p. 8) that students do not need to (Ball, 2008, p. 400).

Ball et al. (2008) provisionally identified a third subject matter domain, *horizon knowledge*, an awareness of the span of student learning that enables teachers to make connections to prior learning while setting the foundation for future progress, but cited a need for more research to solidify its scope. Ball et al. (2008) cautioned that while such knowledge is important for teachers, whether this category is part of the content knowledge domain itself, or overlaps with other domains, remains uncertain.
Because these domains permit specificity in identifying AMK’s contribution to teaching middle school mathematical content, this study applied Ball et al.’s (2008) framework to its first major question, teachers’ perceptions of AMK’s contribution to their specialized content knowledge. Here, it assisted analysis of those perceptions not only to discern how AMK underpins the middle school mathematics content, directly or as a foundation, but to find actual instances, if any, of its use. This examination also provided an opportunity to consider whether and how the knowledge that mathematics teachers describe overlaps between domains.

**Pedagogical content knowledge domains.** Building upon Shulman’s (1986) pedagogical content knowledge construct, Ball et al. (2008) next particularized the pedagogical knowledge domains that mathematics teachers need. These specific pedagogical content knowledge domains focused this study’s analysis of how teachers’ knowledge for teaching incorporates AMK. The first, **knowledge of content and students**, referred to a teachers’ grasp of student preconceptions, nascent conceptions and possible misconceptions during teaching (p. 401). This domain combines the knowledge teachers must have regarding their students and the understanding they must have of mathematics. Teachers introducing a new concept must select from possible instructional strategies those that anticipate common student misunderstandings, motivations, interpretations and misinterpretations. They must also recognize when student thought emerges (Ball et al., 2008). This construct blends mathematical content knowledge with teachers’ understanding of how students learn.

Ball et al. (2008) also refined from Shulman’s pedagogical category the domain of **knowledge of content and teaching.** This domain referred to the knowledge teachers must have to determine which teaching methods are most appropriate to a particular subject (p. 401). While the previous domain blends student learning with mathematical knowledge, this one combines a
teacher’s teaching and mathematical knowledge. It encompasses selecting instructional methods that consider the specific mathematical concept to be imparted, and understanding when to make adjustments while actually teaching. For example, as the lesson progresses teachers must know when to ask questions that elicit student feedback, and must also recognize when students need more clarification (p. 402), both capabilities essential to the practice of teaching.

This study’s second research question employs these pedagogical content knowledge subdomains to ascertain AMK’s contribution to teachers’ knowledge for teaching. The domains separate the different kinds of pedagogical content knowledge into diverse entities to be studied and described. This study aimed to have teachers describe whether and how AMK influences their understanding of their students’ content needs as well as the instructional decisions they make to teach the content.

**Ball et al.’s (2008) framework: a catalyst for further research.** The last adjustment Ball et al. (2008) made to Shulman’s original framework was to posit the notion of curricular knowledge as a distinct domain within the domain of pedagogical content knowledge, but stipulate that whether it exists as its own domain or is included in others remains uncertain. Ball et al. (2008) cautioned that the domains identified may not be exhaustive, need further research and refinement (p. 403), and have limitations. First, the authors warned, the domains were derived from the inherently situational practice of teaching, and individual teachers may employ different knowledge in diverse ways to address a similar situation. Second, the domains identify distinct types of knowledge, but not how any is applied. Supporting this present study, the authors called for further research into how respective domains actually come into play in the act of teaching. Next, they cautioned that the domains lack clear boundaries or definitions and may
even overlap at certain points, and last, that the impact of teachers’ culture or style of teaching is unknown (p. 404).

Ball et al.’s (2008) construct provided specificity, even with soft boundaries, to complicated domains. It provided an organized way to categorize and break down mathematics teacher knowledge when studying the use of AMK in practice. In reviewing the literature, Ball et al.’s (2008) framework supplied an appropriate lens through which to understand the focus of the literature as well as trends that have been studied. Overarching key words utilized to hone in on relevant material included mathematics knowledge, content knowledge, pedagogical content knowledge, and mathematical content knowledge for teaching. These terms were employed to explore the broad aspects of the problem of practice, starting with how the literature has investigated and defined mathematics knowledge. These concepts were then synthesized in relation to teacher preparation and its implementation in practice.

In the methodology section, Ball et al.’s (2008) framework was employed to develop the interview protocols, and to construct the questions for both interviews. Once the initial data had been collected, the framework was utilized to reflexively create the second interview protocol to make sure the data reflected the study’s original goals. In the analysis stage, it enabled organization of participants’ responses, permitting the researcher to “unpack” them and to identify superordinate themes emanating from the participant’s lived experiences. The framework enabled this in-depth investigation into AMK’s use by middle school mathematics teachers to illuminate how content preparation translates into actual teaching practice.

In conclusion, Ball et al.’s (2008) framework is relevant to this problem of practice because it provided structure and specificity to the design of the study’s research questions, its
review of the literature, and its methodology. Those qualities provided an opportunity to fill in persistent gaps regarding how mathematics teacher knowledge is utilized in practice.

**Organization of this Document**

The remaining four chapters of this document begin by surveying the literature germane to this problem of practice. Next the research design is detailed, followed by an analysis of the major findings. Finally, the findings are discussed in relation to the extant literature in order to motivate further research and inform future practice.
Chapter 2: Literature Review

This study investigated how middle school mathematics teachers perceive the use of mathematical knowledge acquired at the tertiary level in their practice. Findings of this study aimed to (a) increase understanding of the training teachers need, (b) further research in the area of teacher preparation, and (c) facilitate the development of effective policy.

This literature review is presented in three sections: Mathematical Knowledge, Teacher Knowledge, and Mathematics Teacher Knowledge. Viewing mathematical knowledge through the lens of Advanced Mathematical Thinking (AMT) facilitates understanding of the nuances of the mathematics at various levels, their resulting implications for teacher preparation, and their effects on this study. The investigation of teacher knowledge includes a historical perspective regarding policy’s role in this problem of practice. Literature regarding mathematics teacher knowledge is then synthesized to understand theory regarding its use in practice. Ball et al.’s (2008) Domains of Mathematical Knowledge for Teaching framework is used where applicable. This analysis reveals a lack of research regarding the precise knowledge teachers need, and a lack of understanding of the current preparation knowledge’s contribution to the development of mathematical knowledge for teaching. This review concludes by employing such persistent themes to ground this study.

Mathematical Knowledge

Definition of Advanced Mathematical Thinking. To grasp how middle school teachers utilize mathematics from the tertiary level in practice, it is important to understand the nuances of the knowledge gained at each of the levels. It is also important to frame the concept, definition, and theoretical understanding of the mathematical knowledge to be considered.
Motivated by a phenomenon that occurred during Zazkis and Leikin’s (2010) study, this review first explores how mathematical knowledge was framed within it.

To examine secondary teachers’ perceptions regarding their use of AMK in practice, Zazkis and Leikin (2010) utilized Advanced Mathematical Thinking (AMT) to examine the mathematical knowledge being studied. Critical to understanding the concept of AMT is its definition(s). Some researchers focus on tertiary level advanced mathematical topics when considering AMT. A second view of AMT broadens its application to include lower level topics considered in an advanced manner (Harel & Sowder, 2005), while a third involves broadening the mathematics at hand while simultaneously creating new mathematical understanding through a real-life or mathematical problem (Selden & Selden, 2005, p.9).

Prospective mathematics teachers’ programs may include different combinations of these perspectives contributing to the complexity of understanding tertiary level’s application in practice. Pure content knowledge courses, some of which Ball et al.’s (2008) framework would classify as “common content knowledge,” supply knowledge beyond the K-12 curriculum. This type of knowledge differs from the “specialized content knowledge” that focuses on the content knowledge necessary to teach a subject. Pedagogical content knowledge courses might focus on advanced knowledge or on K-12 curriculum concepts.

For instance, the researcher’s experience included a pure content course in abstract algebra. Such a course would typically investigate group theory, its fundamental axioms, and how those axioms apply to mathematics beyond the K-12 curriculum. The researcher later took a pedagogical content knowledge course which utilized the axioms of group theory as a framework to focus on how to present number theory as it is learned in the K-12 curriculum. This course experience created a bridge between the advanced mathematics and the K-12
curriculum. Because of such varying experiences, and to extend our grasp of AMT and its
effects on teachers’ use of tertiary knowledge in practice, a closer investigation into the
theoretical perspectives of AMT follows.

**Perspectives of advanced mathematical thinking.** Since release of *Advanced
Mathematical Thinking* in 1991, edited by David Tall, the relationship between the concept of
Advanced Mathematical Thinking and mathematics education has interested researchers, who
have focused on defining and identifying what comprises AMT, and how to develop it (Zazkis &
Leikin, 2010). According to Tall (1991), learning advanced mathematics is a complex process
that requires learners to transition to the thinking of advanced mathematicians. Edwards,
Dubinsky and McDonald (2005) posit that advanced thinking involves considering mathematical
notions in a way inaccessible to the five senses. Edwards et al. (2008) contend that this
transition occurs at the tertiary level. Advanced mathematics requires more deduction,
formalization, complex abstraction, and acceptance of the need for independent study, than does
the mathematics that precedes it (Robert & Schwarzenberger, 1991). Robert and
Schwarzenberger (1991) conclude that the mental processes required make tertiary mathematics
differ categorically from that which precedes it (p. 133), a distinction that presents an
opportunity to understand how those differences affect teachers’ use of tertiary level
mathematics in teaching the K-12 curriculum.

While carefully considering this first definition of AMT, researchers have also supported
others. Harel and Sowder (2005) highlighted the divergence between two such interpretations of
AMT. Though AMT has traditionally been defined as “thinking in advanced mathematics”
(Harel & Sowder, 2005, p. 27), these researchers suggested that this concept may also encompass
“advanced thinking in mathematics *(advanced mathematical-thinking)*” (p. 27). The first
approach focuses on advanced mathematical topics, the second on advanced mathematical-thinking even with elementary level mathematical concepts. A third interpretation identifies AMT as a process that includes utilizing real-life or mathematics problems to develop new mathematical understandings (Selden & Selden, 2005).

Tall (2008) found AMT to consist of “three worlds of mathematics”; these include “conceptual embodiment, proceptual symbolism, and axiomatic formalism.” The first involves learning through experiences, being able to experience objects and visualize them in the mind. The second builds upon this knowledge and involves conceptualizing symbols and processes from those experiences, while the third shifts from defining mathematics from experiences to defining concepts from theoretical ideas (Tall, 2008). Though Tall (2008) did not relate the worlds to specific learning levels, he regarded tertiary level mathematics as primarily encompassing the third. According to Tall (2008), to transition to tertiary mathematics learners must call upon previous knowledge built upon tangible processes and symbols and extend them to the process of proof in a manner not previously required.

Mathematics teacher content knowledge preparation. These views shed light on the ways that content knowledge preparation has the potential to influence teaching of the K-12 curriculum. Depending on their design, tertiary programs for future mathematics teachers may reflect different perspectives of AMT, and institutions offering both pure and pedagogical content courses may combine these perspectives distinctively. Teachers’ tertiary experience may include (a) learning advanced mathematical notions, (b) considering less advanced notions in an advanced manner and (c) “structuring real world and mathematical problems” (Selden & Selden, 2005, p.8) It is important to understand these experiences’ respective contributions to the practice of teaching.
Robert and Schwarzenberger (1991) posit that many of the processes associated with learning advanced mathematics also apply to lower levels. According to Tall and Vinner (1981), for instance, as students learn mathematics they form a “concept image,” defined as “the total cognitive structure that is associated with a concept, which includes all the mental pictures, and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures” (as cited in Tall, 1991, p. 7). Further, Tall (2008) described learning mathematics as a process that begins with repetition to learn and perform a procedure. Ideally, this evolves into understanding of the concept underlying that procedure, and an ability to utilize and categorize it. Thus, though the mathematics learned at succeeding levels may vary, the manner in which students learn might not. This presents the opportunity to study whether teachers’ experiences learning advanced tertiary concepts contribute to their knowledge for teaching mathematics, even if that knowledge is not directly addressed in the K-12 curriculum. This last possibility advocates for better understanding of how advanced mathematics is utilized by practicing mathematics teachers.

**AMT reflected in the literature.** For this study, AMT offers a useful tool for analyzing existing literature in terms of the type of mathematical knowledge commonly studied. Though studies employ different words to classify that knowledge, use of this term helps identify tertiary level mathematics as a common focus (Davis & Simmt, 2006; Krauss, Brunner, et al., 2008; Monk, 1994; Moreira & David, 2008; Wilburne & Long, 2010; Yow, 2009; Zazkis & Leikin, 2010). Zazkis and Liekin (2010) clearly defined the mathematics being addressed as those concepts related to tertiary mathematics, denoting it “advanced mathematical knowledge” or “AMK” (p. 264). The authors utilized the term AMK, differentiating it from AMT because of the latter’s complexity.
Similarly, Moriera and David (2008) focused on tertiary mathematics, designated “academic mathematics,” in a study of the mathematical knowledge needed in teaching practice (p. 24). Moriera and David (2008) investigated the application of specific content offered at the tertiary level in teaching at lower levels. One finding of Moriera and David (2008) pertinent to this study was that knowledge presented at the tertiary level at times conflicts with its presentation at lower levels.

Another research commonality was its focus on teachers’ advanced thinking in mathematics regardless of content level (Isiksal & Cakiroglu, 2011; Tchoshanov, 2011; Yusof & Zakaria, 2010). For instance, in an effort to understand the effects of tertiary mathematics courses, Davis and Simmt (2006) studied “mathematics-for-teaching” by investigating how teachers learn mathematics. “Mathematics-for-teaching” was posited to describe a potentially distinct branch of mathematics, entailing a discrete type of knowledge (p. 294). Davis and Simmt (2006) used concepts like, “What is multiplication?” to study the evolution in teachers’ learning as a result of in-service sessions. In this manner, Davis and Simmt (2006) considered elementary concepts in an advanced way. Such a practice is consistent with the interpretation that AMT can involve more nuanced notions of elementary mathematical concepts.

Similarly, Tchoshanov (2011) researched the relationship between teachers’ “cognitive type of teacher content knowledge” and student achievement. Tchoshanov (2011) described such knowledge as the kind needed to sustain students’ thinking as well as choose and structure instructional strategies and tasks. Isiksal and Cakiroglu (2011) focused on the advanced knowledge of multiplication of fractions that teachers require to identify and rectify common student misconceptions and difficulties. While these studies focused on the mathematical
knowledge teachers must have, their emphasis was not on tertiary “advanced” mathematical proficiency, but on advanced knowledge of the content being taught at the K-12 level.

**Implications for this study.** This analysis was conducted to understand and define the mathematics involved in this study. Their respective colleges or universities required varied content knowledge curricula of its middle school teacher participants. However, this investigation’s primary interest was the relationship between the tertiary level mathematical content and its application in teaching middle school mathematics. Since tertiary courses may provide coursework in advanced mathematical topics as well as opportunities to delve into advanced mathematical-thinking with less advanced topics, in this investigation tertiary knowledge was understood to possibly consist of a combination of the two notions of AMT. Thus, this study employed Zazkis and Leikin’s (2010) term, “advanced mathematical knowledge,” defined as the mathematical content knowledge learned as the result of undergraduate coursework from colleges or universities as part of a mathematics teacher preparation program (p. 265). This notion was utilized to examine how teachers perceive their usage of tertiary concepts while teaching at the middle school level.

Having defined the mathematics knowledge this study considers, this review will now analyze the types of knowledge provided on the undergraduate level and their impact on teachers’ practice. The next section contemplates general teacher knowledge and utilizes the literature to provide insight into the knowledge needed to teach and the role it plays in teaching.

**Teacher Knowledge**

**Contributions of qualitative and quantitative research: an overview.** The learning material presented in teacher preparation programs is a function of interplay among policy, research, and practice (Cochran-Smith, 2005). Research regarding teachers and variables that
indicate their effectiveness and quality informs policymakers, who then influence teacher qualification standards (Darling-Hammond, 2000). To grasp the design and motivating policy of current preparation programs, the following section analyzes the research surrounding teacher knowledge, then puts the policy-research-practice dialectic into historical context.

Qualitative inquiry into pre-service and practicing teachers has gradually expanded the term “teacher knowledge,” now considered to encompass curricular, pure, and pedagogical content knowledge, along with multiculturalism and societal issues (Ben-Peretz, 2011). Qualitative research has largely provided the theoretical basis for the foundation of teacher knowledge (Hill et al., 2005), and for clarifying the complexity of the knowledge required by teachers it has been imperative. Quantitative research can then measure the impact on student achievement of that knowledge’s components (Hill et al., 2005).

Empirical research that focuses on teacher knowledge in relation to student achievement falls into two distinct categories: process-product literature and production-function literature (Hill et al., 2005). Process-product studies examine the link between teachers’ instructional strategies and resulting student achievement (Hill et al., 2005). Teachers’ ability to begin and end lessons, to communicate with students and address their needs all correlate positively with student achievement (Ferguson & Womack, 1993, p. 56). One limitation of such studies has been the scant attention paid to subject matter’s role in their findings (Hill et al., 2005; Shulman, 1986). Examining AMK’s function in content taught as well as pedagogical strategies utilized, this study helps fill that gap.

Production-function studies probe the impact of educational resources and environmental factors on student achievement (Hill et al., 2005). Teacher ability, intelligence, subject-matter knowledge, pedagogical content knowledge, teaching experience, and certification status have all
been investigated to predict teacher quality and effectiveness (Darling-Hammond, 2000; Hill et al., 2005). Results highlighted the complex nature of teacher quality, however. Cochran-Smith (2005) emphasized that teacher attributes, student performance, and school culture are among the many factors affecting it. Hill et al. (2005) cautioned against policy formulations based on the vague definition of teacher quality, and correspondingly amorphous attempts to measure it.

To date, little or no relationship has been empirically established between overall teaching ability and intelligence (Darling-Hammond, 2000), and findings regarding subject matter knowledge remain inconsistent (Begle, 1979; Betts et al., 2003; Darling-Hammond, 2000; Ferguson & T. Womack, 1993; Goldhaber & Brewer, 1999; Monk, 1994). Teaching experience at first seemed predictive, but that effect leveled off after a few years, possibly suggesting no further growth (Darling-Hammond, 2000). Teachers fully certified, “highly-qualified,” who also majored in their subject area have a stronger effect on student achievement than the number of degrees attained (Darling-Hammond, 2000). Education coursework has also consistently shown itself a positive indicator of teacher effectiveness (Darling-Hammond, 2000; Ferguson & T. Womack, 1993). Darling-Hammond (2000) posited that “highly-qualified” may be such a positive predictor because subject matter knowledge and knowledge for teaching gained from education coursework rely on each other. Ferguson and Womack (1993) questioned the effect of pedagogical content courses and called for further examination of their impact. This study probed teachers’ experiences qualitatively, to build upon the theoretical foundation research has established.

A historical perspective. Considering the concept of teacher knowledge historically helps clarify current practices. Ong’s (1958) description of medieval universities, where a “master” or “doctor” of a subject had to defend their knowledge by teaching it, testifies to the
importance historically assigned to subject-matter knowledge (as cited in Shulman, 1986). Over time, the teacher knowledge desired has alternated between subject matter and pedagogical (Cochran-Smith, 2005). Shulman (1986) found teacher tests over a century old to be concerned mainly with pure content knowledge, with few questions devoted to knowledge for teaching. Contrasting these tests with those of the 1980s, Shulman (1986) concluded that the pendulum had swung toward basic and pedagogical knowledge.

This documented shift from pure content knowledge to basic and pedagogical knowledge resulted from process-product research (Cochran-Smith, 2005; Shulman, 1986). Content receded in importance as attention to manner of delivery and factors influencing it increased. However, in 1983, the report *A Nation At Risk* examined current education in the United States, found that it lacked conditions essential to equip students with an adequate education, and stressed a need for change (National Commission on Excellence in Education [NCEE], 1983). Specifically, the report called for an educational system whose high standards demand excellence, and a society committed to life-long learning. The report criticized teacher preparation programs, claiming that excessive time was being spent in teaching methods courses, not enough in those dealing with the content to be taught (NCEE, 1983). The report catalyzed researchers to take a closer look at current practices. Shulman’s (1986) documentation of a lack of attention to teachers’ subject matter knowledge, as opposed to pedagogical, identified the two as distinct and initiated research into the notion of pedagogical content knowledge (Thanheiser, Browning, Moss, Watanabe, & Garza-Kling, 2010). Shulman (1986) cautioned against heightened focus on pedagogical knowledge at the time, particularly given the lack of understanding of the relationship between subject-matter and pedagogical knowledge. A call for further research into the transformation of teachers’ subject matter knowledge into knowledge-for-teaching resulted.
With influences like the Higher Education Act’s re-authorization in 1998, the structure of teacher education in the 1990s reflected the assumptions that public policy drives reform and preparation programs should be evidence-based and research-driven (Cochran-Smith, 2005). In 2001, the No Child Left Behind Act (NCLB) substantiated policy’s influence on the design of teacher preparation programs as well as on the data that drives them. NCLB required that “highly qualified” teachers must be fully certified in the subject they teach (E. Smith, 2008). To enact this requirement, some states require teachers to pass a content knowledge test, major in the subject area they teach, or both (Darling-Hammond, 2000; Flores et al., 2010). Such policy refocused teacher knowledge on pure subject-matter.

Since Shulman (1986) initiated research into the notion of separate domains of knowledge needed for teaching, a growing body of research has investigated such knowledge relative to mathematics (Ball & Bass, 2000; Ball et al., 2001; Ball et al., 2008; Ferguson & T. Womack, 1993; Hill, Blunk, et al., 2008; Isiksal & Cakiroglu, 2011; Kinach, 2002; Ma, 1999; Matthews et al., 2010; Rowland, 2012; Thanheiser et al., 2010; Van Bommel, 2012; Yusof & Zakaria, 2010). To advance our understanding of such knowledge’s application in practice, the following section will analyze literature that addresses mathematics teacher knowledge.

**Mathematics Teacher Knowledge**

Number of mathematical courses, number and type of degrees attained, and certification level have failed to confirm a relationship between student achievement and teachers’ mathematical knowledge, so these indicators do not accurately gauge teacher knowledge (Begle, 1979; Darling-Hammond, 2000; Hill & Ball; Monk, 1994; Zazkis & Leikin, 2010). Yet a relationship between mathematical courses taken and the development of mathematical knowledge for teaching has been documented (Matthews et al., 2010). To extend understanding
of the type and level of knowledge needed, researchers have consequently dissected mathematics knowledge for teaching.

Influence of teacher knowledge frameworks. Shulman’s (1986) framework led researchers to investigate the ways other types of knowledge impact teaching (Hill & Ball, 2004, p.332), specifically, how mathematics relates to teaching mathematics. Hill and Ball (2004) concluded that teachers must not only understand mathematics, but be able to utilize that knowledge to implement curriculum materials, encourage student participation, and ready students for high-stakes assessments (p. 330). Teachers of mathematics must apply and unpack mathematical ideas for students (Hill & Ball, 2004, p. 331), though their preparation might not have explicitly imparted how to do so (Davis & Simmt, 2006).

Scholars have studied the impact of pedagogical content knowledge on teaching through this lens, too. With respect to teacher preparation, for instance, Monk (1994) concluded that pedagogical content courses exert greater impact on student achievement than pure content courses (p. 130). Matthews et al. (2010) documented that subject matter courses focused on the content to be taught have potential to impact teachers’ mathematical knowledge for teaching positively. Such studies highlight the need to understand the interplay between tertiary content and mathematical knowledge for teaching.

Applying Shulman’s (1986) notions directly to mathematics to further differentiate the knowledge that mathematics teachers need, Ball et al. (2008) enabled researchers to study this phenomenon with greater specificity. Ball et al. (2008) approached the issue of pedagogical content knowledge through separate, somewhat intertwined, subdomains related to knowledge of content, students, and teaching. This framework has been utilized to design measures teachers’ mathematical knowledge for teaching (Hill, Ball, & Schilling, 2008; Hill et al., 2005), enabling
close examination of the level and type of knowledge teachers have. For instance, the framework provided Hill, Blunk, et al.’s (2008) theoretical basis for measuring the relationship between mathematical knowledge for teaching and quality of mathematical instruction.

This theoretical structure has also helped researchers approach the issue from other perspectives. Its application to the design of mathematics teacher preparation programs enabled Ball, Sleep Boerst, and Bass (2009) to begin developing a professional curriculum for teacher educators. Olanoff (2011) utilized this conceptual framework, with others, to improve understanding of the mathematics knowledge prerequisite for teachers. In a study related to design principles for pre-service teacher courses it helped Thanheiser et al. (2010) describe and delineate the types of knowledge teachers should have. Because it renders specific the general term “mathematical knowledge for teaching” this paradigm has repeatedly proved helpful to research. This study employed it to improve understanding of how teachers’ AMK preparation contributes to the knowledge they utilize in practice.

Though its originators and other researchers have documented potential limitations due to this paradigm’s overlapping domains, it has helped extend and deepen the literature examining mathematical knowledge for teaching. Moreira and David (2008) criticized an early version of the framework developed by Ball, Bass, Sleep and Thames (2005), and others like it, for persistent obscurity regarding tertiary mathematics’ role in the domains. While the framework describes the type of knowledge teachers need and use, it fails to specify AMK’s contribution to it (Moreira & David, 2008), a criticism that still applies to Ball et al.’s (2008) mathematical knowledge for teaching framework. The present study provides an opportunity to understand how AMK contributes to the domains of mathematical knowledge necessary for teaching identified within Ball et al.’s (2008) framework. This review next considers the preparation
required of middle school mathematics teachers, then combines this information with the 
literature regarding mathematics teacher knowledge.

**Middle school mathematics teacher preparation requirements.** Though many middle 
school mathematics teachers currently complete an undergraduate major in mathematics as part 
of their teacher preparation program requirements (Schmidt et al., 2007), before 1900 
mathematics teachers’ content knowledge preparation extended at best two years beyond the 
eight primary grades. Once high schools became universal, secondary teachers began to major in 
their subject area, while elementary teachers’ academic preparation focused on developmental 
aspects of the child (CBMS, 2001). With middle school teachers not explicitly part of either 
group, it is important to consider the implications of their middle position. Until recently it was 
optimistically assumed that elementary and middle school teachers learned the subject matter 
knowledge needed to teach at these levels during their own educational experiences (CBMS, 
2001). Such an assumption is controversial considering college preparation often provides little, 
if any, coursework devoted to how to impart the content taught at these levels (CBMS, 2001).

Schmidt et al. (2007) found a wide range in the preparation of middle school teachers in 
six countries. Some United States institutions included middle school preparation in the 
elementary program, some in the secondary, while others had a separate program for middle 
school. Those prepared in an elementary program had significantly less subject matter 
knowledge courses than those prepared in middle and secondary programs because the former 
included significantly more pedagogical coursework. Those prepared in middle school programs 
received more subject matter knowledge preparation, but still 20 – 35% less than those prepared 
in secondary programs. Increasing subject-matter preparation correspondingly decreased 
pedagogical coursework opportunities. Middle school teachers prepared in secondary programs
were heavily weighted with pure mathematics coursework, leaving little opportunity for pedagogical experiences (Schmidt et al., 2007). Schmidt et al. (2007) further documented that countries providing middle school mathematics teachers with a thorough training in mathematics as well as experiences and coursework in teaching middle school mathematics consistently outperformed those that did not. The present study examined teachers’ teaching experiences in combination with their unique preparation experiences to provide insight into this finding.

Hill et al. (2005) emphasized that effective teaching is measured not by how much knowledge a teacher has, but how that teacher utilizes it to elicit learning. Using the concept of AMT as a lens, Harel and Sowder (2005) called upon educators to foster advanced mathematical-thinking, pushing students’ understanding of concepts to continually increasing sophistication. This recommendation is consistent with the National Council of Teachers of Mathematics’ (NCTM’s) (2000) call for mathematical instruction that elevates student content understanding beyond the procedural level. Such a recommendation requires that students not only perform mathematical procedures, but grasp them conceptually (Thanheiser et al., 2010).

With AMK’s contribution to teachers’ ability to elicit such conceptual competence unclear, this study used teachers’ perceptions to help clarify AMK’s impact on their specialized knowledge of the content and their knowledge for teaching. Doing so promotes our understanding of AMK’s contribution to the content knowledge necessary to elicit student understanding and its influence on pedagogical content knowledge. Utilizing Ball et al.’s (2008) framework, the study specifically investigated teachers’ perceptions of how AMK assists their specialized content knowledge, knowledge of content and students, and their knowledge of content and teaching.
Research regarding mathematics teacher knowledge. While the present study’s design did not quantitatively measure teachers’ knowledge or its relationship to student achievement, it was built on both theoretical and empirical results in the existing literature. This section begins by surveying empirical studies of mathematics teacher knowledge, then synthesizes researchers’ approaches to this problem, demonstrating this study’s relevance.

Empirical investigations of mathematics teacher knowledge. Studies have shown that variables like number of courses and level of degree do not reliably assess teachers’ mathematical knowledge for teaching (Begle, 1979; Darling-Hammond, 2000; Monk, 1994). Hill et al. (2005) contended that measurements of teacher knowledge do so more accurately (p. 400). Researchers have employed varied means to determine teachers’ mathematical knowledge. Analysis of those mechanisms, the type of knowledge measured, the very language that describes them, reveals the complexity inherent in their design and use and in the knowledge tested. For instance, having developed an instrument to analyze the connection between different kinds of teacher content knowledge, also the latter’s bearing on student achievement, Tchoshanov (2011) found that teachers’ knowledge of facts and procedures had no significant effect on student results in standardized tests (p. 151). Tchoshanov (2011) used the same tool to study the effects of teachers’ knowledge of concepts and connections and knowledge of models and generalizations. Choosing different terminology to describe the knowledge being tested, Hill et al. (2005) constructed a mechanism to assess teachers’ mathematical knowledge for teaching and found that such knowledge positively affected student achievement in mathematics (p. 399).

Hill et al. (2005) developed an instrument to measure mathematics teachers’ knowledge for teaching that other researchers have subsequently used (Delaney, Ball, Hill Schilling & Zopf, 2008; Hill, Blunk et al. 2008). Delany et al. (2008) adapted this tool’s components to evaluate
Irish teachers’ mathematical knowledge, cautioning that such measures must accurately reflect school curriculum. Hill, Blunk et al. (2008) employed it in a correlational study of teachers’ mathematical knowledge for teaching and the quality of their instruction. Warning that policy makers and preparation program designers have utilized research of inherently limited generalizability to formulate requirements for teachers, they stressed the importance of studying the relationship between teachers’ mathematical knowledge and their instruction in greater detail to aid policy and preparation design.

Approaching the issue differently, Ball et al. (2008) noted a lack of empirical proof that pure content and pedagogical content knowledge constitute distinct domains (p. 389). Literature where this notion was investigated empirically reveals its complexity. Krauss et al. (2008) tested both kinds of knowledge and found them distinct, noting, however, that as teachers’ content knowledge increases, the two integrate. Krauss et al. (2008) tentatively concluded that this result may provide the first indication that content knowledge supports the development of pedagogical content knowledge (p. 724). These results emphasize the importance of furthering our understanding of the course content experiences that optimize the knowledge teachers need for the practice of teaching. It also draws attention to the importance of understanding mathematical knowledge for teaching and its evolution. This study aimed to address how tertiary mathematics, which may include both pure and pedagogical experiences, contributed to the knowledge teachers use in practice.

**Narrowed vs. broad focus.** Analysis of the literature using Ball et al.’s (2008) paradigm reveals that to explore the complexity of pure content knowledge, researchers have zeroed in on the specialized content knowledge necessary to teach, the pure mathematical knowledge relevant to math teachers. Reasoning that it would be impossible to study the entire curriculum (Moreira
& David, 2008; Leinhardt & A. Smith, 1985), some researchers have focused on specific mathematical topics, probing the knowledge needed to teach these (Isiksal & Cakiroglu, 2011; Leinhardt & A. Smith, 1985; Moreira & David, 2008; Olanoff, 2011). Others have considered the broader category of mathematical knowledge (Hill, Blunk, et al., 2008; Zazkis & Leikin, 2010). The following paragraphs discuss the outcomes of these different approaches, validating this investigation’s focus.

Typifying specific-concept research, Moreira and David (2008) considered “number systems” to analyze the relationship between tertiary level mathematics and the mathematical knowledge needed for teaching. Moreira and David (2008) found that aspects of the tertiary level mathematics presented to teachers actually conflict with the knowledge needed in practice to teach it, with comprehending such clashes an important part of understanding the contribution of tertiary level mathematics to knowledge needed by teachers.

In another example, Davis and Simmt (2006) utilized the concept of “multiplication” to investigate and describe aspects of teachers’ mathematical knowledge for teaching. Positing that the mathematical knowledge of math teachers typically differs from the knowledge taught to students, Davis and Simmt (2006) called for more research into “the embodied and enacted understandings – of experienced teachers” in order to better understand mathematical knowledge-for-teaching (p. 315-316). Studying the transformation of pure content knowledge into pedagogical content knowledge regarding integer subtraction in pre-service teachers, Kinach (2002) expressed the same notion. Kinach (2002) deduced that this transformation process is neither simple nor unidirectional, and that teachers’ pedagogical content knowledge evolves with their understanding of pure content knowledge. Studying practicing teachers’ knowledge of fractions, Leinhardt and A. Smith (1985) found that their participants’ understanding varied
widely, from procedural to highly conceptual, and the more procedural a teachers’ knowledge, the more likely student misconceptions and misunderstandings became. Restricting themselves to a single concept enabled these authors to unpack and demonstrate the complexity of teaching it, information essential for teacher preparation program designers and policy makers. These three examples demonstrate how research limited to one concept has provided detail and specificity regarding the type of knowledge required by mathematics teachers. They also highlight the complexity and obscurity of the interplay between tertiary level mathematics and the mathematical knowledge for teaching needed for the K-12 curriculum.

Hill, Blunk et al. (2008) acknowledged the value of such specific studies in discerning the links between knowledge and instruction, but questioned both the generalizability of their results and their influence on student achievement. Building upon relationships identified by these researchers and their studies’ conclusions, Hill, Blunk et al. (2008) adopted a general approach to measure teachers’ mathematical knowledge for teaching and compare it with the quality of their mathematical instruction. The assessment the authors utilized to measure mathematical knowledge for teaching had been validated in the past as well as linked to student achievement gains. Aiming to identify how teachers apply mathematical knowledge for teaching in practice, Hill, Blunk et al. (2008) found that increased knowledge led to more rigorous instruction with fewer errors, though other factors, including teachers’ views about mathematics, how it is learned, best taught, and supported through curriculum materials, might also affect the quality of instruction. However, the authors carefully noted that these, too, were a function of the teachers’ knowledge.

This study took an approach similar to Hill, Blunk et al. (2008) to understand AMK’s implementation in practice. Rather than isolating a specific concept for examination it explored
teachers’ general experiences, and their perceptions of those experiences, to discern how knowledge contributes to practices while teaching.

**Level of teaching studied.** The study of teacher knowledge has shown the subject to be multifaceted, with more than subject knowledge needed to be able to teach. K-12 curriculum complexity complicates research into this notion; to manage this complexity researchers have concentrated on a particular level of the K-12 curriculum, with a majority of the studies reviewed restricting themselves to elementary mathematics (e.g. Ball & Bass, 2000; Ma, 1999; Marks, 1990; Matthews et al., 2010; Ozmantar, 2011; Philipp et al., 2007; A. Stylianides & Ball, 2008; G. J. Stylianides & Stylianides, 2010; Thanheiser et al., 2010). Research on secondary mathematics teachers has increased in recent years (e.g. Ferguson & T. Womack, 1993; Krauss, Brunner, et al., 2008; Wilburne & Long, 2010; Yow, 2009; Yusof & Zakaria, 2010), with less attention given to middle school (e.g. Schmidt et al., 2007; Shechtman, 2010; Tchoshanov, 2011). Valid recommendations for policy and practice at any level require understanding the content knowledge needs of teachers at that level, yet many studies simply lump the middle school level with either the elementary or secondary. Because of the dearth of research specific to teaching mathematics on the middle school level, this study focused there.

**Pre-service teachers vs. practicing teachers.** Existing research singles out those preparing to be teachers or those in practice, with studies of pre-service teachers mainly investigating how to develop mathematical knowledge for teaching in either pure content courses, pedagogical content courses, or a combination of the two (e.g., Kinach, 2002; Philipp et al., 2007; G. J. Stylianides & Stylianides, 2010; Thanheiser, 2010; Wilburne & Long, 2010). Kinach (2002) focused on the development of pedagogical content knowledge in pre-service teachers. Philipp et al. (2007) studied the effects of field experiences on pre-service teachers and
their mathematics content knowledge, concluding that field experiences can increase pre-service teachers’ motivation to master mathematics needed for teaching. These investigations concentrating on pre-service teachers provide information regarding the development of mathematical knowledge for teaching and argue for understanding that knowledge’s translation into practice.

Researchers have also invested in clarifying the issue as it pertains to practicing teachers (e.g., Hill, Blunk, et al., 2008; Hill et al., 2005; Leinhardt & A. Smith, 1985; Olanoff, 2011; Schetman, 2010; A. Styliandies & Ball, 2008; Zazkis & Leikin, 2010). Particularly germane to the present study, Zazkis and Leikin (2010) interviewed practicing secondary mathematics teachers regarding their use of AMK in practice. To determine AMK’s contribution to the specialized knowledge needed to teach at the secondary level, Zazkis and Leikin (2010) utilized a written questionnaire, though allowing some participants to verbalize their answers in semi-structured interviews. Some using this latter format changed their responses when asked first how much they utilized AMK while teaching. “Rarely,” “very little,” “it depends,” and “all the time” typified their wide-ranging replies (p. 268), but some teachers responding verbally “may have changed their minds several times” (p. 268). These inconsistent responses Zazkis and Leikin (2010) excluded. This present study delved deeper into this particular issue, focusing on AMK’s contribution not only to practicing teachers’ specialized knowledge, but to their knowledge for teaching.

Declaring that the effects of a teacher’s knowledge on the quality of their instruction remained uncertain, Hill, Blunk, et al. (2008) measured practicing teachers’ mathematical knowledge for teaching and mathematical quality of instruction quantitatively and qualitatively. The authors defined mathematical quality of instruction as “a composite of several dimensions
that characterize the rigor and richness of the mathematics of the lesson, including the presence or absence of mathematical errors, mathematical explanation and justification, mathematical representation, and related observables” (p. 431). The authors concluded that increased mathematical knowledge leads to more precise and rigorous instruction. To gauge the impact that a teacher’s range and type of knowledge exert on their ability to provide students with instruction conducive to conceptual understanding, Leindhardt and A. Smith (1985) compared the knowledge of expert and novice teachers regarding equivalent fractions. These two studies highlight the effects of a teacher’s level of knowledge on their specialized knowledge and knowledge for teaching. They also demonstrate a teacher’s need for conceptual rather than procedural understanding. Both concluded that increased mathematical knowledge improves conditions for teaching.

The above review of research into the complex nature of the development and understanding of mathematics teachers’ knowledge will next be utilized to advocate for this study’s examination of AMK’s use in teaching middle school mathematics.

**Mathematical Knowledge for Teaching in Practice**

Current research and policy recommend that mathematics teachers assist students to further their conceptual grasp of mathematical abstractions (NCTM, 2000; CBMS, 2001; Harel & Sowder, 2005). However, how knowledge that teachers gain during their tertiary mathematics preparation contributes to realizing this recommendation in their practice is not understood. Researchers have made significant strides toward defining the type of knowledge that teachers need, but more work is required to determine how that knowledge is developed and utilized. Depending on the program, tertiary level mathematics may provide prospective teachers with
pure content knowledge consisting of advanced mathematical theory, or PCK consisting of advanced thinking pertaining to K-12 concepts, or a combination of both.

If the two distinct forms of teacher knowledge merge into one domain as pure content knowledge increases (Krauss, Brunner et al., 2008), further research may discern the point where that integration occurs. Identifying that locus and the learning experiences that reach it efficiently would help optimize the design of mathematics teacher preparation and the process of mathematics teacher certification. While this study did not intend to find that intersection, it did expand research into this complex issue by clarifying it. To accomplish this, this study used Ball et al.’s (2008) Domains of Mathematical Knowledge for Teaching as a lens for regarding the type of knowledge needed by mathematics teachers, aware that some domains likely overlap. Via this framework, AMK’s contribution to teachers’ specialized content knowledge and knowledge for teaching mathematics at the middle school level was explored. Doing so advanced this study’s goals and responded to Ball et al.’s (2008) call to further investigate the relationship among the domains.

Although such paradigms further understanding of the knowledge that mathematics teachers need, inability to clarify AMK’s role in the domains further justifies this study. The fuzzy assumption that AMK helps to develop teachers’ knowledge for teaching should not provide the foundation for policy or practice; specificity regarding whether and how this happens is required (Moreira and David, 2008). Investigating how to develop pre-service teachers’ mathematical knowledge for teaching, Van Bommel (2012) posited common content knowledge as a factor determining knowledge growth in the other domains within Ball et al.’s (2008) framework. Van Bommel’s (2012) insights firmly grounded this study; if AMK develops a teacher’s common content knowledge, this investigation’s goal to understand how that
knowledge contributed to the practice of teaching supplements Van Bommell’s (2012) perceptions.

This review thoroughly examined extant research regarding notions germane to this study. Through teachers’ lived experiences and perceptions, this study investigated how knowledge gained at the tertiary level is expressed in the practice of teaching at the middle school level. It explored how AMK is reflected in the content taught as well as in the knowledge needed to teach that content. The literature review illuminated how researchers have framed this problem and those associated with it. With that insight, the researcher consulted this study’s goals, its research questions, and theory regarding different methodologies to design a study that enabled teachers’ perceptions to be analyzed for common themes and meanings. Building on this review of the literature, the following chapter details this study’s design.
Chapter 3: Methodology

Research Questions

Motivated by an absence of data regarding the utilization of advanced mathematical knowledge (AMK) in practice, this study aimed to explore how middle school mathematics teachers utilize AMK directly and pedagogically. To do so, the researcher investigated middle school mathematics teachers’ experiences regarding AMK’s usefulness in practice. Two central questions guided this study and underpinned its design (Ball et. al, 2008; Zazkis & Leikin, 2010):

- What are middle school mathematics teachers’ perceptions of AMK’s contribution to their specialized content knowledge?
  - How do mathematics teachers identify and describe their use of AMK in middle school mathematics content?
  - How do mathematics teachers note and describe ways in which their AMK is implemented in middle school mathematics content?
  - What specific examples can teachers provide to illustrate their use of AMK content in middle school mathematics?

- What are middle school mathematics teachers’ perceptions regarding AMK’s relationship to their knowledge for teaching mathematics?
  - How do mathematics teachers identify and describe AMK’s influence on their knowledge of middle school mathematics content and students?
  - How do mathematics teachers perceive and describe AMK’s bearing on their knowledge of middle school mathematics content and teaching?
  - What specific examples do teachers provide that illustrate AMK’s assistance in mathematical pedagogical tasks?
According to Maxwell (2005), research questions not only focus the study, but are integral to its design (p. 67). Refining the teaching experiences to be explored, Ball et al.’s (2008) Domains of Mathematical Knowledge for Teaching lend structure and specificity to the questions and their associated sub-questions. The first focused on AMK’s contribution to the mathematical knowledge utilized to teach middle school content, teachers’ specialized content knowledge. The second focused on how they perceive AMK to be implemented in their knowledge for teaching, their pedagogical content knowledge (PCK). Delving into teachers’ lived experiences, these questions provided an opportunity to understand this phenomenon from the point of view of those who have gone through it. Information regarding AMK’s use in teaching has the potential to inform policy, the design of teacher preparation programs, and professional development programs for practicing teachers.

**Research Design**

The study’s open-ended process questions lent themselves to its qualitative design. Process questions focus on the “meaning of events and activities to the people involved” as well as the “process by which these events and activities and their outcomes occurred” (Maxwell, 2005, p. 75). Qualitative research is uniquely suited to studying complex phenomena through exploration of participants’ perspectives (Creswell, 2009, p. 129). Reflecting these notions, this study explored individual teachers’ interpretations of AMK’s implementation in practice. Doing so permitted collecting richly detailed data from teachers’ lived experiences. Close attention was paid to their perspectives to (a) understand the experiences they described and (b) derive from them meaning related to this study’s goals.

Embracing the notion of “multiple realities,” qualitative researchers strive to accurately depict these. (Creswell, 2007, pp. 17-18). This “holistic account” identifies the many
complexities involved in a situation and generates understanding of “the larger picture that emerges” (Creswell, 2009, p. 176). To this end, the participants’ unique perspectives were described and analyzed in an attempt to discern common meanings and themes. The researcher sought to utilize teachers’ varied perceptions and descriptions to deepen the understanding of AMK’s use in daily practice.

Another integral component to qualitative research is a reflexive process. Qualitative research, by definition “inductive” and emergent (Creswell, 2007, p.19), requires a strategy open to modification. Qualitative studies are “reflexive,” i.e., new developments or threats to validity disclosed as data is collected and analyzed may require restructuring of the study’s components (Maxwell, 2005, p. 2). Qualitative research consequently permitted adaptation by the researcher as teachers’ perceptions disclosed AMK’s applications in practice. Necessary alterations in research design reflected the direction of the data, preserving equilibrium between the study’s components and its goals. Such a design maximized the study’s exploration of AMK’s use through participants’ perceptions (Creswell, 2009, p. 16).

**Research Method**

This study was designed to consult middle school mathematics teachers’ life experiences to understand AMK’s application to practice. Teachers were asked how they perceived and described their use of AMK directly and pedagogically. Formerly a middle school mathematics teacher, the researcher had substantial hands-on experience with the concepts studied. Considering this factor, Interpretative Phenomenological Analysis (IPA) optimized the study’s ability to achieve its goals. The following details IPA and its application in this study.

IPA’s philosophical foundation, Phenomenology, attempts to capture “what all participants have in common as they experience the phenomenon” (Creswell, 2007, p. 58). It
aims “to determine what an experience means for the persons who have had the experience and are able to provide a comprehensive description of it” (Moustakas, 1994, p. 13). Phenomenology also requires descriptions of an experience’s “what” and “how” (Moustakas, 1994). Integration of these textural and structural dimensions enables meanings to unfold and the core of participants’ experiences to be revealed (Moustakas, 1994, p.32). To permit both, Transcendental Phenomenology requires the researcher to set aside their own experiences and biases, a process called Epoche (Moustakas, 1994). Moustakas (1994) reflects that this process is not easily achieved, as experiences are sometimes deeply ingrained. In this instance, the researcher’s seven years teaching middle school mathematics made achieving Epoche, and thus, phenomenology, unrealistic for this study.

Rooted in phenomenology, IPA by contrast permits a researcher’s experiences to play a role in the study (J. A. Smith & Osborn, 2008) and requires “open-mindedness; flexibility; patience, empathy; and a willingness to enter into, and respond to the participant’s world” (J. A. Smith, Flowers, & Larkin, 2009, p. 55). Specifically, the researcher utilizes their own experience in an interpretative manner, to help “make sense of the participants trying to make sense of their world” (J. A. Smith & Osborn, 2003, p. 53). Embracing multiple realities, IPA permitted the researcher’s own experiences of AMK’s utility in teaching middle school mathematics to actively assist in interpreting and understanding those of study participants (J. A. Smith & Osborn, 2003). With biases explicitly acknowledged, the researcher’s professional practice and grasp of germane issues provided insight into when to probe participants’ responses more fully (J. A. Smith & Osborn, 2003).

Embedded in participants’ “lifeworld,” IPA “is concerned with trying to understand what it is like, from the point of view of the participants, to take their side” (J. A. Smith & Osborn,
Concerned with participants’ perceptions of AMK’s direct and pedagogical implementation, the study carefully considered how their experiences shaped those perceptions. IPA provided a means to understand AMK’s use by these particular participants, in their particular context, as a result of their particular content knowledge preparation process. Documenting these teachers’ unique experiences and perceptions critically and meticulously yielded insight into AMK’s use in practice, furthering research and understanding of this phenomenon.

**Recruitment and Access**

Preparatory to the study, permission to recruit participants and to conduct it was obtained from the school’s district superintendent. Access was facilitated by the researcher’s status as a former teacher at the school and her professional affiliation as a teacher within the school’s district. Upon IRB approval, the participants were identified through the help of a gatekeeper. The mathematics department chair was contacted for purposes of obtaining the email addresses of potential participants.

Purposeful sampling was employed, consistent with IPA’s utilization (J. A. Smith et al., 2009; J. A. Smith & Osborn, 2003). To maximize obtainable information the researcher can choose a site, and individuals from it, most relevant to the study’s questions (Creswell, 2007, p. 125). The first consideration for this study was the school’s location. Experience teaching in the middle school selected for this study facilitated access to the site. However, the school’s current structure, design and setting also presented a unique opportunity to study this problem. The school loops 7th and 8th grade students so that the students have the same teacher for both grades. The study participants’ knowledge of teaching at both grade levels contributed to the richness of the data.
The participants’ routes to certification were also a consideration. As the literature review described, no one route to middle school mathematics teacher certification exists. The study’s middle school, comprised of sixth, seventh, and eighth grades, is located in a state that offers two traditional options for middle school certification. One mandates a major in mathematics and enables teachers to teach grades 7 – 12, while the other, which does not, permits teachers to teach grades 6-8. Consistent with the study’s goals and to control for the type of AMK experience the teachers had, potential participants were limited to those who completed a mathematics major for 7-12 certification.

The small number of participants was also preset to enable thick descriptions (Creswell, 2009). J. A. Smith and Osborn (2003) recommend that researchers new to IPA limit participants to three to yield adequate in-depth data without overwhelming the researcher (p. 57). Heeding this advice, this study sought out three teachers who majored in mathematics with 7-12 certifications. The small number provided an opportunity for deep analysis into how their specific AMK preparation was reflected in their teaching of middle school mathematics. This sample of homogeneous participants’ perspectives would enable understanding of this phenomenon in their particular context (J. A. Smith et al., 2009).

**Participants**

Sufficient background information was essential to determine how participants’ perspectives reflected the “underlying conditions, precipitating factors, and structural determinants” (Moustakas, 1994, p. 60) relative to their experiences overall. The table below summarizes the participant’s discrete preparations. Following it, a brief description of each participant establishes the context for the findings described in Chapter Four. For confidentiality, participants were assigned culturally relevant pseudonyms.
Table 1

*Summary of Participant Profiles: Participants’ Teaching and Tertiary Experiences*

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Beth</th>
<th>Tom</th>
<th>Jill</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teaching Experience</td>
<td>15</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>Tertiary Experience</td>
<td>State College</td>
<td>State College</td>
<td>University</td>
</tr>
<tr>
<td>Master's Degree</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>PCK Course Experience</td>
<td>1(^a)</td>
<td>1</td>
<td>1(^b)</td>
</tr>
<tr>
<td>Required PCK Courses</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

*Note.* PCK = Pedagogical Content Knowledge. \(^a\)Course was taken during as part of Master’s degree. \(^b\)Course was geared toward teaching elementary level mathematics.

**Beth.** Beth graduated from a state college with a major in mathematics and a minor in secondary education. Initially, she intended to teach elementary school, and was minoring in mathematics. However, her pure courses’ advanced content made her decide to shift to teaching secondary mathematics. Beth’s undergraduate program included only pure content knowledge courses, no PCK courses. As part of her undergraduate experience, she completed all course requirements for 7-12 certification. Beth began completing courses toward her master’s degree in Educational Leadership in her third year teaching, electing to take one pedagogical content knowledge course that year. At the time of the study, she was in the midst of her fifteenth year teaching. She taught at the high school level until an opportunity arose to apply for a department chair position at her district’s middle school. Obtaining that position, she began teaching 7\(^{th}\) and 8\(^{th}\) grade mathematics. Beth’s experience teaching at both the middle school and high school levels both helped and impeded this study’s data collection. Asked to reflect upon the use of AMK in practice, her first instinct many times was to begin talking about her high school
experience. Yet, her varied experience yielded insights regarding how AMK’s use compared between the levels.

**Tom.** Tom obtained his undergraduate degree in mathematics from a state college. Tom declared his major right away and completed all requirements toward 7-12 certification. In addition to its pure requirements, his coursework included one designated PCK course. Tom was in the middle of his 2nd year teaching at the time of the study. With his degree completed only three years earlier, Tom’s status as a beginning teacher meant that his preparation experience reflected current program design, and that his AMK experience was fresh in his memory. Tom did not have a master’s degree, nor had he started one at this point. His coursework was limited to the courses he took as part of his undergraduate program. His student teaching experience included Pre-Calculus and Algebra 2 at the high school level.

**Jill.** Jill earned her major in mathematics from a private university. Undecided regarding her future at the start of her college career, she took a few courses in addition to or in lieu of those required of her. Compared to the other two participants, her major required a considerable number of pure courses. Also, her program did not include a PCK course, though she took one prior to deciding to major in mathematics. The focus of that PCK course was elementary school mathematics, not middle school. Jill began teaching at the current middle school and was in her seventh year of teaching at the time of the study. Jill shared that she might someday want to teach at the high school level but was not ready to make such a change.

**Data Collection**

The researcher served as the fundamental instrument to gather data (Creswell, 2009). Consistent with qualitative study criteria, data collection took place in participants’ natural settings, in this case their classrooms (Creswell, 2009, p. 175). The researcher employed semi-
structured interviews to collect information. Secondary sources of data included a list of the
courses participants took as part of their mathematics major. The lists were generated from the
current course program requirements at the tertiary institution the participants attended.
Supplied with the list, participants were asked to review it for accuracy, adding information
where necessary. The lists were analyzed for the type of courses taken. Since this study’s
concern was not measuring participants’ knowledge, but their descriptions of how they utilize
knowledge gained in their tertiary courses, course grades were not collected, a decision of which
participants were informed during recruitment.

Data collection from multiple sources is consistent with qualitative research (Creswell, 2009). The interviews and course lists furnished not only a variety of data, but a means to
triangulate it (Lincoln & Guba, 1985). The list of courses documented the type of AMK
experience participants had in their mathematical content knowledge preparation program. This
information was utilized to structure the interviews and, during the analysis stage, identify
themes. In combination, these two sources enabled an inductive approach built thematically
upon the foundation the data supplied (Creswell, 2009, p. 175).

The interview procedure was carefully planned to document participants’ experiences and
explore their perceptions of AMK’s implementation in the teaching of middle school
mathematics. Semi-structured interviews suited this IPA study because while providing form
they situated the participants as “experiential experts” who shared responsibility with the
researcher for the interview’s direction (J. A. Smith & Osborn, 2003, p. 59).

To protect study participants, the researcher ensured the confidentiality of the data
collected. Interviews were audio-recorded and subsequently transcribed by a third party.
Participants’ names were not used anywhere in the recordings or transcriptions. When the
transcriptions were made, the participants were designated P1, P2, and P3. Later, during the analysis phase they were assigned pseudonyms (Creswell, 2007, p.143). The audio and text files created were stored on a password-protected computer in a password-protected file database. Data collected will not be utilized for any purpose except this study.

The study employed an adaptation of Siedman’s (2006) three series phenomenological interview protocol. The first and second interviews were combined in this case because the course lists enabled understanding of the context in which participants’ AMK experiences took place. The researcher’s experience as a teacher in the same school and her prior knowledge of two of the participants also aided contextual understanding.

The first interview focused on the participants’ experiences of AMK’s use in practice. The second interview focused on the meanings of those experiences and also provided an opportunity to delve into and clarify ideas produced by the first. Consistent with the study’s goals and design, this enabled exploration of complex issues through “the concrete experience of people in that area and the meaning their experience had for them” (Siedman, 2006, p. 16). These semi-structured sessions were conducted individually in the participants’ classrooms and limited to 90 minutes so as not to exhaust them. As the primary instrument, the researcher was flexible and met with the participants at a time convenient for them. The participants were interviewed individually so that the responses of one did not lead, influence, or quell those of another. As data collection proceeded, the researcher reflexively revisited the study’s original design, consulting research questions, theoretical framework, and methodology to make sure that the data collected and analysis conducted were meeting study goals.

During the interview, the researcher used a modified version of the “standardized open-ended interview” approach (Fraenkel, Wallen, & Hyun, 2012, p. 452). All subjects were asked
the same set of base questions, ensuring that the same material was covered. This uniformity both increased the sameness of the information provided and permitted response comparability (Frankel et al., 2012, p. 452). In this semi-structured format the researcher reflexively delved deeper into relevant concepts to expand understanding, when necessary. Such flexibility increases an interview’s situational quality, opening it to modification, but can also decrease the consistency of concepts covered (Frankel et al., 2012). The researcher remained cognizant of this potential pitfall as the interviews were conducted and subsequently analyzed. Since each subject had more than one interview, the researcher was able to rectify any inconsistencies.

To maximize sameness and ensure the relevance of the interview questions, Ball et al.’s 2008 Mathematical Knowledge for Teaching Framework was employed. An unanticipated occurrence during Zazkis and Leikin’s (2010) data collection also prompts reference to their interview questions. Zazkis and Leikin’s (2010) study of secondary mathematics teachers’ perceptions of AMK’s usage in teaching involved many participants. A small subset were given the opportunity to answer questions in a semi-structured interview. All other participants were asked to write their replies and could not be probed further. Because some of those interviewed changed their responses when asked follow-up questions, the authors excluded interview-derived data from a portion of their analysis.

As in Zazkis and Leikin’s (2010) study, this study’s participants were asked a base of identical pre-formulated questions. In a departure from Zazkis and Leikin’s (2010) protocol, however, all data pertaining to the questions were gleaned through semi-structured interviews. This study’s deliberately small sample size permitted participants’ responses to be probed in depth. The researcher constructed prompts for questions in the event that participants found delving deeply into a topic difficult. The questions were open, neutral and not leading (Roulston,
2010), constructed so as to not to intimidate the participant with technical terms (J. A Smith & Osborn, 2003). The following section describes the procedure for analyzing the data collected.

**Data Analysis**

IPA methodology flexibly applies a set of common processes to document participants’ sense making of their experiences (J. A. Smith et al., 2009). IPA’s underpinnings support close analysis of the participants’ experiences to gain understanding of convergent and divergent themes whose relationships are uncovered through the structure they create (J. A. Smith et al., 2009, p. 79). According to Creswell (2007) the foremost step in data analysis is its careful organization. The computer program MAXqda was utilized to organize and analyze the data. Computer programs enable the researcher to organize raw data, establish codes and develop themes (Creswell, 2007, p. 169). The map tool was integral to developing the superordinate themes documented in the findings section.

IPA is premised on the notion that the researcher’s experiences help capture, understand, and describe the participants’ (J. A. Smith et al., 2009; J. A. Smith, & Osborn, 2003). J.A. Smith et al. (2009) recommend a sequential process to analyze data in an IPA study. Adhering to that advice, the researcher began by carefully listening to the audio recordings of each interview and checking the transcripts for accuracy. The transcripts were then read repeatedly to increase understanding line by line. Next, the researcher created descriptive, linguistic, and conceptual memos to reflect the content included in the data. These “exploratory” comments provided a means to enter into and engage in a participant’s experience (J. A. Smith et al., 2009, p. 84). Careful attention was given to the transcripts’ meanings to the researcher as the process of making sense of the participant’s own interpretations of their experiences began (J. A. Smith et al., 2009, p. 84).
To discern superordinate themes, the researcher mined the memos for commonalities. Emergent themes “reflect a synergistic process of description and interpretation” (J. A. Smith et al., 2009, p. 92). Codes were created by words or phrases spoken by the participants during the interviews when possible and listed chronologically as they emerged. This process was completed for each of the interviews.

Once the first three interviews were analyzed, these codes were employed to find connections, build structure, and ultimately map the understandings gleaned from that process (J. A. Smith et al., 2009, p. 92). The memo and coding process, in conjunction with Ball et al’s. (2008) framework, enabled the researcher to generate questions for the second round of interviews. These reflected the direction of the emerging themes and gave the researcher an opportunity to delve deeper into the meanings of participants’ experiences. The analysis process was repeated after the second interview.

Once this scrutiny was completed, the researcher read through each transcript again, for themes that had emerged during the second round of interviews. In this way, the researcher began analyzing the totality of the data for cross case connections (J. A. Smith et al., 2009). Throughout this process, code maps helped tease out superordinate themes. As the domains of knowledge defined in this study are understood to overlap, use of the code maps enabled the researcher to look at the data closely, utilizing both a macro and micro-perspective. Close attention was paid to individual and group idiosyncrasies that led to higher order connections (J. A. Smith et al., 2009). During the analysis stage, the researcher kept a process log to brainstorm, document progress, and draft emerging superordinate themes.

The researcher then began the process of documenting the study’s findings, with verbatim statements validating the superordinate themes generated. At this point, member
checking was utilized to increase the study’s credibility (Lincoln & Guba, 1985). Each participant was asked to read through and evaluate for accuracy a summary of the meanings the researcher generated from their described experiences.

**Protection of Human Subjects**

This study conformed to the ethical principles of beneficence, respect, and justice as well as the six norms that should guide research, as defined in *The Belmont Report* (as cited in Mertens, 2005, p. 33). Since this particular study collected information about participants’ lived experiences, treating them with respect, courtesy, and consideration was imperative. Participants were invited to contribute; those that volunteered were made fully aware of the informed consent process. Face to face, the researcher detailed the purpose, procedures, and potential risks associated with the study. The participants were given a letter detailing such, as well as an explanation of the participants’ rights and confidentiality (See Appendix A). Details were also provided about the interview process. Since IPA methodology leans heavily on the use of participants’ verbatim reports, participants were made fully aware that these would be included in the published findings (J. A. Smith et al., 2009). To create an atmosphere of respect, participants were encouraged to ask questions at any point.

With transparency essential to establishing trust, the researcher described her dual role as an instrument of the study and, as a former middle school mathematics teacher at the school, their peer. Participants were assured that their experiences were objectively collected and described. Open to multiple perspectives, the researcher aimed to understand individual experiences that might closely resemble or differ significantly from the researcher’s. The researcher sought to utilize her own experience to understand and document the participants’
point of view (J. A. Smith & Osborn, 2003). These concepts were conveyed to encourage open, receptive, non-judgmental conversations.

With regard of the principle of beneficence, effort was made to maximize the benefit of the study and minimize adverse consequences. In this instance, the safeguards put in place to preserve the participants’ anonymity made the risks of taking part seem minimal. The autonomous adults who volunteered for this study were supplied with ample information about this study’s goals, procedures, and design. With that information they could freely determine whether they were interested in contributing their subjective experiences and perceptions (Mertens, 2005, p. 34).

As for the final principle, justice, participants were ensured an equal opportunity for their lived experiences to be heard and not exploited (Mertens, 2005, p. 33). To this end, participants’ identities remained confidential and member checking was also utilized to assure that their experiences were accurately depicted. Once their contribution was analyzed, participants were provided with a summary of their experiences’ contribution to the emergent superordinate themes, and given an opportunity to append or delete information if they felt it necessary.

**Limitations**

This study’s goal was to describe the participants’ life experiences effectively to understand how AMK was utilized and reflected in their teaching. IPA studies are idiographic studies in that they are concerned with individuals and particulars (J. A. Smith & Osborn, 2003). Although this study’s small sample size enabled an in-depth understanding of the participants’ experiences, that size was also a limitation. The researcher carefully provided rich detail to enable others to benefit from its particulars (Creswell, 2007). While participants’ perspectives may not represent all middle school mathematics teachers, they did provide a picture of how the
current policy and design of mathematics teacher content knowledge preparation related to their own teaching practice.

The data’s reliance on memory is another limiting factor in this study, whose participants had been practicing teaching and away from their AMK experience fifteen, seven, and three years. When interpreting the findings, their ability to recall or accurately depict the way that preparation was utilized must be considered. To support their recollections, the researcher provided the participants with a list of the courses each took. Additionally, the researcher was reflexive during the interview process to permit time for their experiences to be fully portrayed. Follow-up questions helped recall when needed.

The lists of courses enabled the research to individualize interview questions, too, further probing participants’ individual experiences. Knowledge of each participants’ program design also provided a means to triangulate the participants’ experiences, adding to the study’s credibility (Lincoln & Guba, 1985). The lists assisted participants’ recollections and aided the process of understanding their experiences.

Another potential limitation to the study’s results was the possible overlap among Ball et al.’s (2008) domains. To mitigate this concern, during data analysis the researcher took care to consider the participants’ reports from each domain of focus. This study aimed to understand AMK’s utility in teachers’ mathematical knowledge both directly and pedagogically. Ball et al.’s domains, however overlapping, provided a means to focus on the nuances of AMK’s application. Their framework’s sub-domains of specialized content knowledge, knowledge of content and teaching, and knowledge of content and students, as well as their definitions, partitioned the study’s data.


**Trustworthiness**

To evaluate the quality of IPA studies, J. A. Smith et al. (2009) cite Yardley’s (2000) method as accessible, broad, and generally applicable (p. 179). Yardley’s (2000) approach is based on four criteria (as cited in J. A. Smith et al., 2009). First, the study must establish “sensitivity to context” (J. A. Smith et al., 2009, p. 180). This study’s findings were grounded in and rest on context. At every stage of the research process, participants’ profiles, undergraduate program design, and AMK experiences were carefully considered. The findings included “verbatim extracts from the participants’ material to support the argument being made, thus giving the participants a voice in the project and allowing the reader to check the interpretation being made” (J. A. Smith et al., 2009, p. 180-181).

This study’s interview process and the design of its analysis satisfied Yardley’s (2000) second requirement, “commitment and rigor” (as cited in J. A. Smith, 2009, p. 181). Meticulously analyzing its data, the researcher adhered closely to IPA methodology recommendations. Scrutiny of each participant’s experience carefully respected the individual meanings gleaned. The sample contributed to the rigor of this study aiming to better understand the applicability of AMK in middle school teaching; in that they majored in mathematics as part of a requirement toward their teacher certification, those selected to participate resembled each other. This focused sample limited to three participants permitted in-depth study of the phenomenon and rich, deep analysis of each participant’s experiences.

Yardley’s (2000) third prerequisite, “transparence and coherence” (as cited in J. A. Smith et al., 2009, p. 182) is evident throughout this study’s design. Each stage of the research process, from selection of the participants, to the interview and analysis, was carefully detailed
(J. A. Smith et al., 2009). Documentation of the researcher’s own bias also contributed to study transparency; others can consider it when interpreting the findings and evaluating conclusions.

Regarding coherence, the researcher frequently ascertained whether the study addressed its research questions and implemented its theoretical framework. These concerns were instrumental during the interview process, especially in reflexively generating questions for the second interview. Reflection assured that the study answered its original research questions, and in so doing, reflected its original goals (J. A. Smith et al., 2009).

According to Yardley (2000) research’s final criterion is its “impact and importance” (as cited in J. A. Smith, 2009, p. 183). This study was based on the premise that content knowledge preparation is essential for a mathematics teacher (Moreira & David, 2008; Zazkis & Leikin, 2010), but that such preparation’s impact is unclear. This study of teachers’ use of AMK in the actual practice of teaching shed light on this previously obscure hypothesis. Its results potentially impact practicing teachers through professional development, and prospective teachers through the design of their educational preparation. Finally, this study aimed to increase understanding of this complex issue, and motivate further research.
Chapter 4: Analysis and Findings

This study examined how middle school mathematics teachers perceived and described their use of Advanced Mathematical Knowledge (AMK) in practice, directly and pedagogically. Concerning its direct employment, this study aimed to determine AMK’s application relative to teachers’ specialized content knowledge (SCK). SCK is the knowledge mathematics teachers require that others who understand and can do mathematics do not, i.e., knowledge that unpacks mathematics for their students. SCK is the “mathematical knowledge, skill, habits of mind, and insight” that teachers use in order to teach (Ball et al., 2008, p.398). Regarding pedagogical content knowledge (PCK), this study intended to understand how AMK’s application related to teachers’ knowledge of content and students and their knowledge of content and teaching (Ball et al., 2008), that is, AMK’s influence on the strategic choices middle school mathematics teachers made to further their students’ understanding of mathematics.

The descriptions that participants in this study provided of their SCK and PCK experiences overlapped significantly. Mindful of this, the research questions, and this study’s goals, the researcher carefully employed a code map to tease out two superordinate themes. The first, “AMK’s influence on specialized content knowledge: It’s not so much the content, but the idea of the content” found AMK’s direct implementation minimal, and investigated indirect emergences described by participants. The second, “AMK’s influence on pedagogical content knowledge: minimal, if any, required and dedicated coursework” documented a lack of formal coursework to develop PCK. It also examined how participants’ limited PCK experiences were both beneficial and different from their pure courses. An overview of these, along with their subthemes, are summarized in Figure 1 below.
Figure 1. Summary of superordinate themes and subthemes regarding AMK’s use in teaching middle school mathematics. Advanced Mathematical Knowledge (AMK) is defined as the mathematical knowledge gained from an undergraduate major in mathematics.

The following chapter delves into these themes, first exploring AMK’s influence on SCK, then the meanings participants conveyed regarding its application in their PCK.

**AMK’s Influence on Specialized Content Knowledge: It’s Not So Much the Content, but the Idea of the Content**

To elucidate AMK’s relation to SCK, the three participants were asked where they perceived AMK’s content to arise, directly and indirectly, while teaching. An analysis of the participants’ experiences and respective assigned meanings revealed that (a) post-secondary mathematical topics were *directly* used minimally, if at all, (b) the math itself was of a different nature, and (c) their unique set of experiences contributed to their current mathematical insights and habits of mind.
The first subtheme to illustrate these ideas, “AMK: directly absent, indirectly provides structure,” depicts each participant’s experience of how AMK related to the middle school mathematics curriculum taught. The next, “different levels, different beasts,” illustrates their perception of how AMK’s nature differed from middle school mathematics. The last, “different experiences, different perspectives,” reveals how participants’ habits of mind and insights were a function of their unique experiences. To better understand participants’ AMK experiences and their SCK, the following paragraphs investigate these three elements.

**AMK: Directly absent, indirectly provides structure.** To understand AMK’s use in middle school, participants were asked to describe instances when it appeared explicitly in their teaching. Though all three answered this question, analysis revealed that none had actually demonstrated direct use of tertiary level mathematics. Instead, their explanations demonstrated understanding of middle school content’s connection to advanced topics. These perspectives revealed such knowledge can emerge as specialized content knowledge, that is, as understanding of the mathematics important to teach as well as how that mathematics needs to be unpacked.

When asked to describe the extent to which she used AMK in her teaching, Jill replied, “limited.” Requested to expand on that terse assessment, she described how she might utilize it when coming up with real world instances. She cited her Business Calculus course, helpful because of its real-life applications, “but that would probably be the only time.” In response to further probing she explained, “because the math content that I learned in college is not what I am teaching now.” Though Jill seemed to answer the question of AMK’s use with a resounding negative, further analysis pointed to a more subtle conclusion: AMK provided an ingrained understanding of a “bigger picture” of mathematics. Both of these concepts are detailed in the paragraphs that follow.
Lack of direct instances. Asked the same question regarding AMK’s direct use, both Beth and Tom attempted to identify instances. Though they named tertiary courses linked to middle school content, both struggled to describe the advanced mathematics being directly implemented. These two participants quickly cited Linear Algebra as a course related to the middle school level; however, Beth then described a high school experience, while Tom talked about mathematical processes, not the advanced content of Linear Algebra. Both made connections between the mathematics at the different levels, describing linear equations and systems as content at the middle school level that related to Linear Algebra. Thus, though hard-pressed to pinpoint tertiary content’s direct appearance, they exhibited knowledge of its connections with material they taught.

Focusing on her prior experience at the high school level, Beth stated that Linear Algebra helped in Algebra II, and that her understanding of differentiation was useful in Pre-Calculus. Asked to think in terms of the middle school curriculum, she hesitated to cite links with the tertiary content, stating, “Very limited topics have been used in an average seventh or eighth grade course.” She noted further, “The topics that are taught in your pure content were just way too advanced, um, to even present to students even on a watered down level for our curriculum.”

To explain their views regarding AMK’s utility, all three participants referred to the Advanced Geometry they took. Beth denied any direct link to the content she taught: “I took Analytical Geometry, um, that was way over and beyond what I’ve ever needed or used.” Tom described his Modern Geometries as “a little bit more advanced,” including concepts of translations, rotations, and parallel lines necessary for understanding advanced content. He saw middle school content as a foundation needed to complete the course.
Jill also brought up Geometry: “When we have a small portion of Geometry, that comes into play, the knowledge of it.” Asked whether the actual advanced content was taught she answered: “No. So, most of the classes that I took at, in college are too advanced for the content that I am teaching my students in middle school.” While rejecting the advanced content per se in her teaching on the lower level, she was nevertheless implicitly acknowledging the existence of links between the two. Thus, both Tom and Jill affirmed a connection between the levels, with geometry concepts at the middle school a prerequisite.

While Jill and Beth openly hesitated to make direct connections, Tom persisted; he cited Discrete Structures and Number Theory as useful. However, he did not describe direct implementation of AMK concepts. Instead, he focused on his conviction that Pre-Algebra related to those advanced courses. He described how his understanding of advanced concepts helped him to understand “the backstage of mathematics:”

I mean number theory, just the concepts of being able to go through and doing a proof. I mean when you’re solving a linear equation, when you do your check aspect of you basically just doing a proof, just some really tiny proof saying, yes I can prove that this is the case, that this is my solution so that’s… that also helps me sort of reorient … how I think about that check process. As a teacher, I sort of value that a little bit more than I did when I was a student, you know, because as a student, yeah, I mean I didn’t wanna do it because it’s actual work, but I also didn’t think it was that important ‘cuz it’s like ‘Okay, I got the answer.’ As a teacher though I recognize now through, you know, all that work with number theory that’s very important to be able to show, hey, this is the answer. I am 100% certain this is the answer and I can prove it, not just being like okay, look, here’s some work that’s why I think it’s my answer, there you go. So … and that’s something
that like scaffolds throughout all of math. You constantly have to be able to back up your argument of why this is my answer.

When asked to compare the mathematics at the two levels Tom stated: “In terms of pure content, they’re nothing alike…but, again, my personal philosophy towards education is there is nothing that is not worth knowing because you never know how, like, knowing one thing might allow you to better understand another thing.” Tom credited AMK with developing the processes he considered evident throughout all of mathematics:

It expanded my ability to think logically which all math does, so even at the higher levels, it made me learn to think in a little bit more of a logical process when it comes to things, which helps when it comes to this level where you’re teaching the kids.

Though his explanations did not demonstrate the content to be directly implemented, Tom believed that the advanced content “solidified” his understanding of mathematics in general. It was that understanding he saw being implemented in his teaching. This indirect application is discussed for all three participants next.

**Ingrained: Knowledge of the big picture.** Both Tom and Jill affirmed AMK’s usefulness in connecting concepts between levels. Though Beth was adamant that actual advanced knowledge appeared minimally, her ideas regarding whether or not it did converged obliquely with Tom’s and Jill’s. When questioned regarding its implicit presence she replied:

Maybe sometimes, um, I would say it was more subconsciously. You know there are times that I’m teaching something to students, um, in the algebra class that I’ll say, you’ll learn that later but if … if you do this now, it will make your life easier later and I’m trying to think of an example on that. Um, I don’t know maybe systems of equations and, uh, you know, but … you know you start in algebra one with, you know, one variable and
then you go into two variables and then you go into systems and, you know, graph them and do several methods. And I think knowing what’s coming up in three dimensional and the types of ways you need to solve those, I think I tried to steer them towards good habits and good decision making, um, and showing out their steps because it’s only gonna get harder and if they don’t start those good habits now, they’re gonna be screwed later, you know, that kind of thing. Um, but in terms of specifics, I can’t say that there are specifics.

Beth documented how AMK is connected to prior foundational knowledge. She was also describing how AMK might “subconsciously” inform her of how to “steer them towards good habits and good decision making.” While not specifying courses, knowledge, or topics, she was acknowledging that her experience might influence her insights and habits of thought.

“Subconsciously” applied to some of what the other participants conveyed, too: AMK content did not directly appear in the curriculum, but its indirect influences, hard to pinpoint, understand, or describe, were nevertheless real. The participants noted AMK might be exerting some indirect impact through knowledge ingrained or embedded as a result of their experience of it. When asked whether her AMK was there in the background, implicitly, Jill replied “consciously, no…Having the knowledge, I guess, there…and trying to figure out other ways of presenting it, I guess helps me teach the students, but I don’t think I’m actually aware of it.”

The researcher detected evidence of this notion in other portions of the interview. Jill’s description of how her experience helped her to understand her students’ learning revealed insights she brought, perhaps subconsciously, to her teaching:

I think that because I've had to go through a challenging course load in college and I've had to learn um, you know, advanced mathematics, I know the work that it takes and so I
know the, the thought process that I had to go through and, and work through and so I think that helps me understand what my students need to go through.

Understanding “what [her] students need to go through,” knowledge regarding the mathematics that needed to be focused on, also underlay her consideration of how to create the experience that they required:

I think because I had to work so hard getting through the material in college, especially towards my junior and senior year with the more advanced topics, that I, I, I can, I try to break it down for my students more so that they have a, uh, solid foundation of the basic um, skills and computation and are going to be able to apply it once they get up into high school.

Jill also documented utilizing her AMK experience to help her students understand her methods and requirements:

I just want to um, get across that, you know, each year as they progress through the math curriculum, um, you know, in middle school and in high school, that the level of math that they are going to be taking is going to get increasingly more challenging and um, that even though they might be able to do some of the problems, you know, all in their head, because it's elementary and it came quick to them, I'm trying to groom them to be better mathematicians and show their work and just prepare them for more advanced topics.

Jill was demonstrating how her teaching called upon the totality of her experience, which she credited with endowing her with a better understanding of the “bigger picture” of mathematics. This included knowing where her students were going mathematically, and using that knowledge to get them to the next step:
I think because I went through, um, you know several math courses throughout college and I know what I don't use. Um, that's when teaching my students I make sure to, I teach what they need to know and you know I might teach beyond that just to kind of like pique their curiosity and get them thinking to the next level and, you know this is what you have to look forward to in high school kind of thing but um, you know I don't want to confuse them so I kind of just stay um within the, the areas that they're going to understand that they're going to use, that they're going to need to go on to the next level.

“I know what I don’t use” echoed Jill’s contention that her AMK did not appear in the content that she taught. Still, she acknowledged the presence of the sum of her experience in her teaching, even if it was not easily described or quantified.

Similarly, Tom expressed difficulty gauging whether AMK helped inform the content he taught because it was “ingrained into [his] thought process already.” He explained that when making choices about teaching mathematical content it was impossible for him to separate out understanding he had derived from the sum of his experiences. Although AMK content might not find direct expression, it helped solidify his overall knowledge of mathematics, and that influenced how and what he taught:

I mean a lot of the higher level mathematics, I mean Calculus, you can’t really use calculus too much in the middle school. What you can use is like that connectedness from understanding, okay how does calculus connect down to trigonometry, how does trigonometry connect down to algebra two? And so you can sort of start at the end and work backwards to the sort of thing … sort of concepts that they need to know by the time they get there and sort of start building a better foundation.
Here Tom demonstrated his understanding of what Jill called the “bigger picture” of mathematics, and Tom himself termed the “backstage of mathematics.” Tom emphasized that the advanced classes helped him to solidify his knowledge and fill in/connect any gaps. They helped him to realize notions and processes evident through all levels of mathematics, like proofs and properties: “The properties stay there. They’re with us throughout all of math.” Tom also believed that the challenge of advanced mathematics helped shape him as a teacher. “Not so much the content, but the idea of the content” reminded him that mathematics could provide challenges no matter what level; this also provided insights into how to approach presenting mathematical ideas.

All three participants struggled to come up with instances where AMK was directly utilized in the middle school curriculum. All three explicitly stated they directly incorporated few, if any, topics from their AMK experience into their teaching. All three agreed that if they were using their AMK, that process was occurring implicitly, subconsciously, and was something they did not understand, and could not gauge or describe. Even so, they also demonstrated how the sum of their AMK experience helped them make decisions about how to present mathematical content and what to focus on.

Their perspectives lent insight into the reasons for AMK’s absence from the level on which they taught and suggested that AMK was not seen in the curriculum because it was of a different nature from the middle school curriculum they taught. These notions are explored further in the next section.

**Different levels, different beasts.** Once all three participants had indicated that the mathematics at the two levels differed, they were asked to describe that dissimilarity. Quantitatively, all three described tertiary level mathematics as significantly more difficult, using
terms like “extremely hard,” “difficult,” and “challenging,” to describe tertiary level math’s pure content. Qualitatively, each participant posited a categorical difference between mathematics at the two levels, which they considered the opposite ends of a spectrum. Beth, Tom, and Jill respectively used words and phrases such as “way too advanced,” “not developmentally appropriate,” and “on completely different levels.” When pressed for explanations, all three consulted their teaching experiences for illustrative examples.

Though Beth used the words “way too advanced,” it was not in reference to her own understanding and learning of mathematics; she was describing AMK’s relation to middle school’s content. To compare the content at the two levels, she spoke of her ability to teach a subject. Despite feeling that she had been always been strong in mathematics, she felt that she would not be able to teach tertiary level material. To illustrate, she used the example of teaching multi-variable calculus as opposed to seventh grade math. She described the former as “extremely difficult” and the latter as “not extremely difficult.” When asked to elaborate, she cast about, eventually settling on capacity to relate to the material to measure difficulty.

You know I think it’s fine in algebra one. I mean obviously sixth, seventh, eighth grade is you know, obviously here I’ve been doing positives and negatives and, you know, area and all that stuff and everyone, of course, understands that. And even in algebra one when you talk about linear equations and costs and, um, you know, slopes and you know, rates of change, even that you can relate. But, you know there comes a time where in algebra two I would say probably is … probably the first time that a student says, “Where am I ever gonna need this?” And though … I think those are the times that I struggle and go, well, if you’re gonna be a math teacher later in life, if you want to be an engineer, you know, I like to be able to answer those questions with honest answers that no matter who
you are, you’re gonna use that stuff and that’s what I mean about difficult versus I guess more relevant to the “average human being.”

Beth’s viewpoint rested on the intertwined notions that, from a student’s perspective, the ability to relate facilitated learning mathematics, while, from a teacher’s perspective, that attribute’s absence made mathematics content harder to teach. In describing her AMK experience she noted, “When I took those classes, that was me. I was sitting on those chairs, going, ‘Really? I need to know this to do what I wanna do? Really?’”

Beth felt that as mathematics increased in difficulty it eventually became less relatable. Consulting her own experience, though confident she could teach it without preparation, she considered Algebra II the point where she would begin to struggle to relate content to her students. With a little preparation, she felt she would even be able to teach up to Calculus. Beth’s salient point here was that, in her experience, the ability to relate to mathematics declined as its abstract complexity increased. To Beth, the categorical difference between the levels was the ability to connect learning to a real-life experience and to answer the question, “Why?”

It all comes back to my … my motto that I live by in teaching, and is it’s the why. If I started teaching a child, a student, multivariable calculus and they said to me, “Why do I need this?” I don’t have any answer for them. I cannot show them where on a regular basis (laughs) they would ever need to use one topic from that course. Whereas the course that I teach, in all the courses, I think the higher level math classes you teach, the more difficult math classes you teach I think unfortunately, there are less places that you can show a student where to use in the everyday real life.”

Throughout the interview, Beth emphatically expressed her aversion to producing “recipe children” who could do mathematics but did not understand “the why” behind the math. She
traced this to her own incoherent understanding of “the why” behind the more advanced concepts she took as part of her AMK experience, describing her inability then to discern any connection between what she was learning and real life. Although the classes made her, in her words, “a critical thinker” and “well-rounded in math,” she saw them primarily as a challenging set of hurdles that she must jump to get her degree and be able to teach. Beth’s experience of a lack of relatable connections made her strive to include connections for her students.

Like Beth, Jill stated unequivocally that the mathematics she was teaching did not include the mathematics she was taught in college. She considered AMK to be more about proving her ability than preparing her for the mathematics she would be teaching.

Basically everything that I learned in college was more for an engineering track or more advanced, um, professions than teaching middle school math… It was making sure that I could, um, making sure that I could learn advanced material and not necessarily teach it.

Again resembling Beth, Jill’s view of the distinct nature of AMK derived from its lack of connections and her inability to relate to the majority of her advanced courses. Reflecting upon the knowledge she drew upon when presenting middle school content she explained:

I draw from content that my students would understand, right? I draw from, um, experiences that they’ve had, that they can relate to. If I talk about a math course that I took in college, that's going to be completely over their head because it has no relation. They're not, they cannot relate to that at all because it is not what I am teaching now. Jill emphasized that not only was tertiary mathematics missing from the mathematics she taught, but she considered it out of her students’ reach. She had found it hard to relate to because of its dearth of connections to real life; she did not think that her students could relate to it, either.
Most of the pure content courses in her experience lacked the connections for which Jill now strove. However, her major required that she take “Mathematical Modeling,” a course that used AMK to solve complex real life problems. “I feel like even though the content is definitely more advanced, what I was learning, I feel like that is something that, you know, is being used in teaching middle school math.” Although not incorporating the actual mathematics from that course in her teaching, she described striving to connect mathematics to real life. She went on to note that if some of her other pure content classes (namely Modern Algebra and Real Analysis) had resembled that course, she might have struggled less.

Like Beth and Jill, Tom rated tertiary mathematics too advanced for his current students, too abstract, difficult to connect to:

Partially because they don't know, really, equations yet, and partially because that's a much more abstract thing. That's hard to put, like, real world examples to with derivatives. I mean, you can, and there are definitely plenty, but even then, a lot of the real world examples are dealing with things like physics and other such things that they haven't really done yet or don't really care too much about.

While he did not say so explicitly, Tom documented a need to relate the mathematics to his students. The abstractness of tertiary mathematics made that hard, while middle school mathematics could be made relatable. Tom described middle school mathematics as “concrete.”

Throughout the transcript it was evident that Tom believed his AMK experience challenged him to think in a more advanced manner. In his words, it necessitated “fluid processing.” Tom described fluid processing as the ability to see mathematics from a higher viewpoint and analyze how the properties of mathematics mattered in understanding, manipulating, and solving problems. Tom would not expect his middle school students to be
able to use previously acquired mathematics knowledge to understand a newly presented situation or problem not necessarily similar. “In college I learned to do that sort of fluid processing a lot better. So it’s something that I try to not necessarily focus on because that is a more sort of advanced thing.”

Yet, while not requiring such thinking from his students, this ability was evident in his reflective teaching and approach. “It's more or less, it gave me a better understanding of what they, what parts of math you need to be able to do other parts of math, and in order to do that at the higher level.” He described its utility in helping his students relate and understand why what they are learning was important. He shared that when he experienced difficulty relating the material to the real world, his overall understanding of mathematics helped students make connections to future learning:

What we're doing right now is we're sort of building a toolbox of math skills. Um, you might not use this skill directly in your life, but you need to know it to learn another skill that you will, and having that understanding of what math flows into what math, um, gives me a better idea of which tools in that toolbox are really important for them to know, and which ones aren't as important.

In this way, all three participants shared that it was vital to their teaching to relate the material to the students. Beth and Jill related how their AMK’s lack of relatedness made connecting to the material hard for them. All three participants documented how mathematics more advanced and not relatable would be difficult to present to middle school students.

Taken together, two notions described in previous sections seemed almost contradictory: AMK influenced decisions regarding teaching mathematical content at the middle school level, yet differed categorically from the mathematics taught at that level. When distinguishing
between the mathematics on these two levels Tom focused on mathematics itself, while Beth and Jill opted for a different perspective, namely, their ability to connect to its content. The next subtheme examines the participants’ experiences in detail, to better understand their views regarding AMK’s use, and how their individual experiences shaped those views.

**Different experiences, different perspectives.** Analysis of the data to date revealed a need for further examination of the interplay between AMK and the teaching of middle school mathematics. Regarding the mathematics at the respective levels as qualitatively different, the teachers noted the absence of AMK’s direct content from the middle school level. Simultaneously, the participants found that their AMK experience implicitly informed decisions made while teaching, also their overall mathematical habits of mind.

To discern the meaning of the participants’ experiences, it was important to consider that while each participant majored in mathematics as part of a teacher preparation program, their interactions remained specific. An analysis of the ways in which each participant’s unique experiences contributed to their perceptions and current habits of mind follows.

When considering each participant’s requirements, and despite the 12 years separating Beth’s completion of the major from Toms, all three AMK cores featured important commonalities. Each program required three courses of Calculus including Multi-variable Calculus, along with Linear Algebra, tertiary level Geometry, Abstract Algebra, an Analysis course, and an advanced Statistics course. Further analysis revealed significant variations in the other courses to which participants were exposed, some a function of the unique path each took, others of program design.

In general, participants’ tertiary AMK experiences varied. They attended different higher education institutions at different times, and their programs’ requirements diverged in focus and
inclusion of PCK courses. Years of teaching experience and completion of additional coursework compounded these variables: two years versus fifteen demarcate a beginning teacher from a veteran, and two of the three participants also completed a master’s degree. Lacking formal PCK courses in her undergraduate program, Beth was heavily influenced by the one she included in her master’s. To understand their perspectives, the participants’ divergent sequencings and other variations in experience, blatant and subtle, require exploration. The following details each participant’s experience, beginning with Jill, moving to Beth, and concluding with Tom. Next, an analysis of their paths’ commonalities and divergences helps explain how the participants’ perspectives and the meanings they extracted reflect both.

**Jill: Math-minded but minimal real-life connections.** The few math courses that she took before declaring her major, among them Business Calculus and one designed for future elementary school teachers, distinguished Jill’s AMK experience. Ironically, she ranked those two courses among those most relevant to teaching. Otherwise, as noted above, Jill stated very clearly that, in her view, the mathematics she was teaching was not the mathematics of the advanced courses she had taken.

She appreciated her Business Calculus class’s real-world connections, and described the connections applicable in her teaching. She also found the elementary mathematics course, though not required of her major, “beneficial because it, it actually, it modeled, you had to model everything. You know, you had to really get down to the students' level.” She went on to describe how “you had to use manipulatives like, you know, tiles and blocks and pictures and so you had to make sure that you were presenting the material to an elementary student.” She appreciated the course’s multiple approaches to presenting mathematical concepts at the elementary level.
This study regarded Jill’s elementary mathematics course as a PCK course because of the content she described. Though not required for her major in mathematics, it also contributed to her AMK. Citing its potential usefulness, she acknowledged that she did not regularly use its concepts because it was geared toward elementary mathematics. Still, she mentioned that similar courses, aimed at the content she taught, might have proved beneficial.

Among her required courses, Jill selected “Mathematical Modeling” as a course whose concepts related to her teaching. The course applied advanced tertiary level concepts learned in other courses to real world situations. While Jill considered the material extremely challenging because of the intensity of the real world problems/solutions, she appreciated this opportunity to see the applications of advanced mathematics. Though she was not using tertiary mathematics directly in her teaching, she did regularly connect math to concrete, real-life experiences.

In her senior year Jill’s mathematics program also stipulated that she complete a project that made connections between advanced mathematics and its applications. She then had to present her findings to an audience of her peers and faculty. Though she found the advanced level of the mathematics involved intimidating, she considered the overall experience something still helpful in her teaching:

I think the overall um, concept of the project you know is a good idea to make sure that you know students are able to take material that we are teaching them and connect it to something you know outside of the classroom and so that is, that was the, the whole idea of it and that's what we tried to do as middle school teachers is take material that we are teaching our student and um, help them make connections to real world problems um, you know through projects, through activities.
When considering her other pure courses, through both her interviews, Jill focused her explanations and perceptions on the degree to which she could connect to them:

I really just think that I had the mindset in college that these are the courses that I'm never going to teach to my middle school or high school students and this knowledge, like they’re, this is not what they are going to need. And, even relating it, like even as a student in college, I was like when am I ever going to use this um, in the real world? When am I ever going to use this in my profession? I had no connection to it so I was just going through the motions to learn the material and be successful as a student then.

The phrase “going through the motions” encapsulated Jill’s view of her tertiary AMK primarily as an obstacle, rather than as knowledge she needed. However, she also noted it was now so ingrained she found it hard to discern whether it had changed her. She noted that she could “provide [her] students maybe with more of uh an in depth explanation of things just because [she had] an extensive background of math knowledge.” She accepted that her teaching relied on that background, yet questioned its usefulness:

Well, now that I have experienced all of those the challenging courses in college and now like the highest math that I'm teaching is, is algebra one and the common question that I get is "When will I ever use this?" Um, You know when I'm teaching algebra which is a very useful math in the real world you know the other courses that I took in college I don't think necessarily are that useful. Um, so you know I think back and I'm like I've took four years and being a middle school teacher I'm using none of it.

Prompted to explain further, she reverted again to the question of transferability:
The math that I teach I feel that um, students might see that more in the real world, that they might um use this knowledge more on a daily basis. Whereas the math that I took in college is something that I, I never use.

Jill’s evaluation of her AMK’s usefulness focused on advanced mathematical content’s lack of connection to her learning and teaching. When asked whether it might have helped her develop mathematical knowledge needed for teaching she proved reluctant to give it much credit. She stated that her upper level courses did not help her “to present the material or teach them anything other than me knowing it's really hard and if they don't get like, for example, their, their basic skills down or if they're not showing their work on all the problems, they're not going to be prepared for what's ahead of them.” However, she did note that her experience had changed her, making her a more math-minded person in her everyday thinking processes. Her proclivity to think in a logical, step-by-step manner she credited to her overall experience and “extensive background.”

Exposure to advanced content provided an understanding of the skills and knowledge her students needed in order to someday grasp the content themselves. She also believed that teaching many of the advanced concepts to herself had helped her develop the mathematics knowledge necessary to teach others. She recalled:

Once you left the classroom, you had to, you had to go back and look at examples in your notes or go back and look at anything in the textbook, um so in a sense, you were a teacher at that point. You were going back and teaching the material to yourself.

In summary, Jill considered AMK’s direct use in the classroom minimal, its indirect influence hard to assay, and the content on the two levels, completely different. She did note, however, that it affected her understanding of the process of learning mathematics, its
challenges, and the skills it required for success. She also felt that her experience of AMK had made her a more “math-minded person.” When asked what that meant to her she spoke about her patterns of thought and overall approach, her appreciation of how “[in life] things like flow like kind of like a math problem. It's just like broken down like step-by-step I, I feel like that's kind of how I, I um deal with most things.”

Jill’s experiences blended traditional pure content courses with electives that presented greater opportunity to connect with their mathematics. Their divergence taught her to value “connectedness” she could perceive, and struggle for similar connection in less relatable courses. Knowledge of Jill’s specific experiences and their impact on her helped in understanding her interpretation of AMK’s use. She looked for ways to help her students connect, and understood which processes they needed to succeed when mathematics became more advanced.

Beth: Well-rounded and missing real-life connections. Beth, too, recalled a disconnect between what she was learning at the tertiary level and what she would be teaching. For Beth, the classes she took as part of her AMK experience were merely hurdles to get over; they lacked coherence and connections, and seemed unrelated and disjointed. “I spent hundreds of hours taking those difficult courses and doing my best to be successful in them, but I don’t think that they were required to do what I’m doing on a daily basis.” The following paragraphs decipher Beth’s experiences and link them to her current perceptions.

At the start of the interview process, Beth seemed hesitant to express her opinions regarding AMK’s utility in the middle school classroom. Having taught at both the middle and high school level, she frequently referred to her high school experience when addressing questions about AMK and its utility. Over that first session and the next her comfort level increased, as did her expressed conviction that she used AMK minimally, if ever, in her teaching:
“I think if I had failed all of those courses, I still would be the teacher I am today.” After this statement, she laughed quickly, adding, “Which I didn’t, by the way (laughing). I didn’t fail…Or else I wouldn’t be here.” Asked late in the second interview how the math she learned in college supported her understanding of the middle school mathematics curriculum, she repeated that perception: “I don’t understand why I had to take what I took. I took it because it’s part of the game and you play the game and you graduate and you get a job.” These statements summarized her feeling that college level math courses had presented a hoop she had to jump through rather than providing knowledge she needed.

Beth’s time away from her AMK experience is also important to consider. AP Calculus is the lowest level of AMK and the highest level of secondary mathematics. Open about her general comfort level teaching, she conveyed that at this point she would need some preparation to teach upper level secondary courses. She offered, “I don’t think I could teach AP Calc without a little background information myself and, you know, looking through the material.” Summarizing, she stated: “You know, when you don’t use it, you lose it, and I, and I really feel that way.” The time lapse since her AMK experience, and its absence from her current teaching, meant that she would need to review the Calculus material.

Viewing her conclusion from a different angle, however, one result of her AMK experience was confidence that even after 15 years she would be able to teach the course. Beth seemed to see the AMK experience as a security blanket that ensured that anyone who completed it could teach any secondary mathematics subject. Given that Beth’s preparation program helped her to earn her 7-12th grade certification, this notion made sense. Her point, however, was that though her AMK background left her confident that she had adequate mathematics knowledge to
teach upper level secondary courses, it did not explicitly help her to teach in middle school. When asked to compare the mathematics at the different levels she stated:

It cannot be compared. It’s not a fair comparison. I teach currently a 7th grade math class and a pre-algebra class, shall we say. It’s an accelerated class. And sh-, I don’t even hit systems this year. Next year, I’m teaching 8th grade, I’ll hit systems. But, you know, what I was put through as an undergrad made more sense to me if I was going to be an engineer or I guess a math college professor, but I think my time could have been better spent on math topics that were more appropriate to this level and trying to understand how to teach something like a system with the best way to teach it, you know.

Beth expressed difficulty identifying instances of AMK’s indirect use in the classroom, and attributed most of the mathematical knowledge that she used to her 15 years of experience teaching. However, she credited her AMK experience with furthering her knowledge and understanding of mathematics: “I absolutely give it credit for that. I mean … other than teaching math, that I’ve never used a derivative in real life, but I can take one. Um, so I think it’s made me well-rounded in math.” Yet, she noted, she could not come up with concrete examples of how it helped her at the middle school level. “Subconsciously,” it helped her understand the bigger picture of the topic she was teaching, and prepare her students optimally.

Beth’s undergraduate requirements did not include pedagogical content courses. Asked whether connections existed between the mathematics classes she took and her pure education courses, Beth replied that the experiences were discrete: she learned pure math in the math courses and pure educational concepts in the education courses. There was little to no overlap and no formal attempt to bridge the gap between the mathematics she was learning at the tertiary level and the mathematical content she was preparing to teach at the secondary level. Other
experiences, among them student teaching and another non-required course, left her with knowledge that still resonated in her teaching:

- It was you need to teach them the why. You can’t just tell them, “This is what it is and this is how you do it,” because they’ll do it. They wanna do it. They wanna get a good grade, they wanna make their parents happy, and they wanna get out of here. But unless you teach them the why and try to make those connections and break them down so they make sense so they can see the big, the bigger picture, it’s not gonna work.

Beth’s traditional pure content courses lacked obvious and evident links to her life or future teaching, a disconnect that sharpened her awareness of her students’ frustration when they questioned the utility of the mathematics they were learning. Speaking about her major’s pure math classes she explained her perspective:

- When I took those classes, that was me. I was sitting on those chairs, going, “Really? I need to know this to do what I wanna do? Really?” And so I sympathize with them because I, I went through the same thing, and it wasn’t fun. It really wasn’t.

Beth’s preparation included the same core courses as the other participants and experiences unique to her. Like Jill, Beth did not declare her mathematics major right away. Entering college intending to teach at the elementary school level, she also began with a minor in mathematics. Once she started taking courses toward that minor, she concluded that if required to master such advanced mathematics she might as well strive for teaching more advanced content.

One course in particular prompted her to change her major to mathematics and teach at the secondary level. She enjoyed the content of the course but appreciated even more the way it was taught. Describing it, she said, “I had an introduction to math course, um, as a freshman that
actually was one of the best courses I ever took and it was interesting, fun facts.” Her
description of that course was markedly warmer than her account of others. Though its content
was not advanced, she appreciated the ways the professor linked every topic to real world
experiences and worked hard to make the material relatable for his students. When describing
other more advanced courses, she exhibited frustration.

The reason why I liked that course a lot was the relevance that it had the real world, the
examples that the teacher, um, used to get his point across. Um, you know, was … it was
the traditional lecture, you know, it wasn’t a group setting, it wasn’t pairs or anything, it
was all the materials that he taught, he really could relate it to why it wasn’t for and why
you’re gonna use it, where you’re gonna see it. Um, whereas let’s be honest when you
take a multivariable calculus class, I don’t see that in my everyday life here nor when I
was at the high school, you know, um, and I enjoyed that piece of it.

Her enthusiastic description explained Beth’s reasons for liking that course and her need
to convey the “why” to her students. Though not a required part of her AMK experience,
knowledge gained in that introductory course influenced her teaching, indirectly but powerfully.
As described above, Beth’s undergraduate experience led her to conclude that mathematics
teachers needed to make connections for their students.

The many experiences subsequent to her undergraduate program that affected this veteran
mathematics teacher included a PCK course that she elected to take toward a Master’s in
Educational Leadership. Though that degree did not directly relate to teaching, this course, taken
during her second year of teaching high school, presented methods for teaching middle school
mathematics. She described how this experience, which brought together for the first time her
knowledge of mathematics and of students’ learning, helped her directly with her struggling high school students, and later, when she switched to teaching middle school:

At the middle school, that if you don’t understand that every time you hit 10 you change in mathematics, if you don’t understand that concept, that 10 ones becomes the 10, and 10 tens becomes a hundred and 10 hundreds becomes a thousand. If you don’t have that mathematical knowledge and understanding, you’re not gonna be a successful math student. And it was something that was presented that had never been presented before to me was extremely interesting and I had a student that didn’t understand the difference between pace … place value and the whole base ten system. Um, you know, and he had made it all the way to the eighth grade. Um, I had struggled with this the whole time.

And, you know, this course talked about, um, how to work with those students and how to provide modifications and what types of things you can do to help them and, um, you know, was much more specific in terms of teaching math than any other course I had taken.

Beth detailed her integration of that course’s concepts into her teaching. Though not part of her undergraduate experience, its contributions to her specialized content knowledge for teaching helped explain her contention that the pure mathematics courses she took as an undergraduate differed in kind. AMK topics appeared minimally in her teaching, in marked contrast to the applicability of this methods course’s concepts:

Um, you know, when you’re going to school to be a math teacher, you take classes that are pure content and, um, it’s not discussed how students make mistakes or what they do or what they’re thinking or other ways you can present this. It’s presented one way and you either get it or you don’t and you move on. And I think that that’s something I’ve
struggled with my whole time, you know, and you … there are so many students that have so many different needs and so many different backgrounds that, you know, I don’t think that that’s … that’s discussed or dealt with when you take a pure content math course, um, you know, required to be a math major. I don’t think that math teaching is taken into consideration in those courses.

The methods course provided Beth with a perspective that she might not have acquired otherwise. She wished that her undergraduate experience had included similar encounters, and wondered aloud whether she would have been better off had she been exposed to concepts it included that she had to figure out on her own: “I think I could have been a better teacher from the get-go if I had been taught else-wise, el- … you know, differently as an undergrad.”

In summary, while Beth denied the direct application of AMK concepts in her teaching, she acknowledged AMK’s contribution to her overall understanding of mathematics. The lack of connection between her pure mathematics courses and knowledge she found relevant to her teaching fundamentally affected her teaching mindset. She felt that the mathematics of her AMK courses was missing in connections to real-life, and wished for more explicit examples why its material mattered for her own future teaching. Her contrasting experience years later with the PCK course led her to conclusions regarding which type of knowledge was more beneficial to her teaching. Her experiences and the meanings she took away from them facilitate understanding her need to present her students with “the why,” also her adamant view that AMK is missing directly from her teaching.

**Tom: Logical thinking and the backstage of mathematics.** As with Jill and Beth, analyzing Tom’s unique AMK experience yields insight into his current perspectives, insights, and habits of mind. Tom described his AMK experience as the first time he struggled with
mathematics, which had always come easily for him. Asked to describe how tertiary mathematics helped him to teach middle school, he responded by illustrating the problems inherent in that question: “It’s difficult to make comparisons when you go from doing derivatives to teaching kids to multiply fractions.” He went on to claim that it was not the mathematics per se that helped, but the process he went through learning the mathematics:

Knowing how to do a derivative and integral doesn't really help me too much in the classroom, but the process of learning how to do the derivatives and the integrals isn't much different than the process of learning how to solve a two-step equation. It boils down to, I don't know how to do this. I view this as something as tough and challenging. Same thing for the kids. They don't know how to solve a two-step equation. They view that as tough and challenging, so by going through the process of learning how to do that two-step, er, learning how to do that integral or learning how to do that, um, you know, derivative, I can relate to the kids when it comes time to, like, wh-, wh-, when they're having problems with it.

The challenge of mathematics helped Tom relate to students learning something new. He believed that his reactions during freshman Calculus resembled his seventh grade students’ when learning fractions: “It’s not that it’s an easy or difficult concept, it’s that it’s a concept they haven’t approached before so it’s just taking those higher level courses for me helps me keep in mind okay this math isn’t necessarily easy.”

Tom expanded on his observation that tertiary level mathematics’ difference lay in its difficulty. Like Beth, Tom described mathematics as an area of strength for him through middle and high school. His initial experiences with difficulty at the tertiary level created his current perception that “no matter how smart you are, you’re gonna hit a point where the math is just
tough.” In high school he had never had needed help from his teachers; in college he frequently sought their support. Tom felt that tertiary math’s difficulty helped solidify his understanding, because to succeed he had to continually utilize mathematics he learned at the secondary level.

When asked to identify instances where he used AMK directly in his teaching Tom described indirect ones. According to Tom, courses like Linear Algebra and Number Theory contained advanced mathematics that could be directly linked to knowledge learned in middle school mathematics. As an example, Tom explained that in pre-algebra the students began to learn about linear concepts, a concept fundamental to Linear Algebra. Students were also taught to check their solutions for accuracy, referred to by Tom as the “check aspect,” a notion underlying proofs in Number Theory. Tom also felt that a solid understanding of properties of mathematics was essential for future success, helping to develop the “fluid processing” he considered necessary to advanced mathematics. These links, which Tom readily made, evidenced understanding of what he termed “the backstage of mathematics.” Tom saw mathematics as containing processes that recur through all its levels, and documented his understanding of those connections. This unifying concept he attributed directly to his AMK experience.

One experience stood out for Tom in helping him formulate this perspective. His major required that he take a course called Senior Seminar in Mathematics. To partially fulfill the course requirements he had to make a portfolio demonstrating his understanding of certain mathematical concepts. It involved:

Going back through your work and finding pieces that demonstrated your understanding and ability to do certain things in math that are almost sort of treated like background
things when it comes to middle school and high school math, [like] the ability to go through and find where you went wrong.

In combination with his whole AMK experience, this process changed Tom’s outlook on mathematics. As he explained:

One of the things in the portfolio was the ability to go through and show step by step how to solve the problem, not just the understanding of what to do, not just the understanding of what my answer was, but the understanding of in order to do this step, I need to do this step first. In order to do this step, I need to do this step first. Before I went to college, I would say I would have probably answered the question, what is the point of math, by saying, "Well, it's … the point of math is for you to be able to solve problems that come up in your life." If someone were to ask me now what I think the point of math is, I would say the point of math is to make it … to learn how to think logically, to learn how to think in a step by step process, cause and effect. If I do this to one side of the equation, what has to happen to the other side of the equation? If I want to save up this much money, what do I have to do in order to get that? Not what equation do I need to write and solve to get it, but what is it that needs to happen for me to do that?

For Tom, understanding relationships among mathematical concepts mattered as much as linking mathematics to the real world. He believed that that making such mathematical connections promoted logical thinking, which in turn developed thinking necessary for real life. That senior course portfolio required him to make connections with advanced mathematics and with its overarching processes, facilitating his current focus on those mathematical processes, not solely on real world connections. When asked how his experience helped him to connect students’ learning to the real world he responded, in part:
I would say part of it is going back to the toolbox. This particular, and I'm blanking on a topic right now, but this particular topic, sure, you might not really be able to connect this to the real world in a very meaningful way, but you're going to need to know it for this, and this is going to connect to the real world in a meaningful way.

Here “toolbox” referred to the mathematical skills and concepts required to grasp more advanced concepts. He felt that relating mathematics to future mathematics was as important as relating mathematics to real life.

Tom demonstrated how his experience led him to conclude that overarching mathematical processes existed, resonating through all of mathematics. Taking into account the thinking they required as well as their intrinsic difficulty, however, he also identified differences between the levels. Tom described middle school mathematics as more concrete, while tertiary mathematics required what he termed “fluid processing.” He subsequently noted that his ability to think in such an advanced manner helped him make decisions about students’ learning that would help them understand “the why.” Tom believed “the why” of mathematics to be as essential as its “how.”

And that higher level math sort of helped embody that in me and wow, you know, you can't really ... It's difficult to do that sort of ... explain that sort of thing to a kid when you're, um, working with them. It, it, I mean, it's different-, why do we have to learn two-step equations? Because down the line, you need to be able to tell where your mistake was in your li- ... like, that's not something that you can really tell to a kid. Um, but to me, when I'm teaching them the math, and I'm teaching them to do those things, it's always in the background of my mind that the important part of it isn't necessarily
learning how to do it, but learning why it happens or how it happens and how it affects the next thing.

Lessons he learned and thinking he developed while taking advanced courses meant that when Tom taught the concept of why remained “always in the background:”

Taking those mathematical courses especially the higher up ones where you’re dealing with more abstract concepts like in abstract algebra or, you know, modern geometries, uh, where you’re forced to think more about, well okay, we’ll … here’s my start, here’s my end. How do I get there? We aren't told the processes, we have to figure it out, sort of help me understand a little bit better the way like the, the backstage of math of why stuff’s happening which helps me convey to my kids why stuff is happening.

Tom’s AMK included one pedagogical content knowledge course, another singularity in his preparation from that of the other two participants. Asked to describe the level of the mathematics in that course, he described it as focusing solely on developing mathematics lessons in general, with most of the topics covered middle and secondary level:

That was the one that was pedagogy based where it was talking about how we would apply math … how we would apply teaching skills to mathematics. We focused on designing lesson plans that were, um, oriented … oriented towards math instead of, just the lesson plan, you know, just like getting us prepared to teach math in our practicum.

The course involved learning how to use technology, e.g., the graphing calculator, to teach mathematics; it also directly addressed teachers’ mathematical knowledge for teaching: “One of the things that we did is we would do practice lessons in classroom. We would come up with a lesson then teach it to the class.” Tom found this process helpful because it challenged him to come up with different explanations to teach a concept. It was also a bit awkward:
I would say it was beneficial not in the sense of presenting it, ‘cuz it had a definite
different feel to it than teaching in [an] actual classroom of people who don’t know what
you’re talking about. Um, I would say it was beneficial in the sense of it helps get you
ready for being in front of a classroom in explaining things. That way you don’t just get
up there and go and then this happens and then this happens and blah. Um, but it was
definitely beneficial in the sense of getting accustomed to that being in front and being
able to collect your thoughts and explain it well. Um, and also due to the fact that you are
in a room with a bunch of other math majors if … that they will call you out so will the
teacher if you start, you know, making a mistake.

This experience let Tom practice the sort of learning and thinking he would need when
actually teaching. Presenting a concept to a class of peers who already understood the material
made him uncomfortable, but forced him to practice his ability when teaching to “collect [his]
thoughts and explain it well.” Teachers must make instantaneous decisions; this course provided
an opportunity for Tom to get a feel for what that was like even prior to his student teaching
experience.

To summarize, Tom’s AMK experience led him to the perception that certain
mathematical processes were evident in all levels of mathematics, and that understanding these
connections in mathematics was equivalent to understanding real life connections. His
pedagogical experience overtly required him to practice utilizing and developing knowledge he
used while teaching: knowledge of multiple representations, of available resources, and of
specific skills needed while in the act of teaching. These experiences helped explain his
teaching’s focus on logical thinking and “the backstage of mathematics.”
Experiences Matter. All three participants agreed that the “different levels [are] different beasts.” All three also agreed that their AMK enabled them to advance student mathematical knowledge toward the goal of being able to do advanced mathematics. However, their motivations and points of focus did not necessarily align. For instance, Beth and Jill documented similar opinions, perspectives, and frustrations regarding their lack of connection to the majority of advanced mathematics, while Tom focused on connections between mathematical processes. The following paragraphs synthesize the respective experiences’ points of convergence.

First, two of the participants took non-required courses that exerted more influence on their teaching than traditional ones. Because Beth and Jill did not declare mathematics as a major right away, both remained free to sign up for mathematics courses outside of that major’s requirements. Beth, who stated that her introduction to mathematics course exerted a lasting impression on the insights she utilized when teaching, relied heavily on real-world examples in her teaching. She also noted a lack of such focus in her pure content classes.

Similarly, Jill described taking a Calculus course for business majors, geared more to real-life situations than traditional Calculus. While she did not use the course’s Calculus content in her teaching, she valued its applied mathematical connections. Jill, too, denied employing most of the concepts from her required pure content courses, with the exception of the “Mathematical Modeling” course. However, she reiterated that it was not that course’s mathematics that she still used, but its theme of connecting mathematics with real life. While expressing her appreciation for that experience Jill also emphasized that her other courses lacked such connections: “I had nothing to relate [the math] to because it just seemed like, I'm like, when am I ever going to use this.”
Another instance of convergence was that at some point all of the participants took a PCK course, whose differential timing, design, and content imprinted on their experiences and perspectives. Jill’s second course outside her major’s requirements exposed her to strategies designed for elementary mathematics teachers. Jill appreciated that this course focused on the mathematics to be taught as well as strategies for teaching it. She denied using specific strategies from this course, but the realization that such courses existed made her wish for similar opportunities geared toward her teaching level.

Similarly, the PCK course Beth encountered after completing her major and preparation program left her wondering why she had never experienced such a course. Tom’s requirements did include a PCK that stipulated he come up with different representations for content and also practice teaching. He found that this experience helped him develop the type of thinking and the knowledge he would need while teaching.

All three participants had a culminating requirement as part of their major, though that experience’s emphasis varied. Jill’s senior project included selecting, researching, and presenting an advanced mathematical topic to her peers and professors. Jill described the project as much like her “Mathematical Modeling” course, in that it required her to connect mathematical topics and processes to one specific application. Jill related how her overall AMK experience made her a more “math-minded” person and also helped explain her current focus on real-life connections.

Tom’s senior seminar course, designed to help connect and record his entire AMK experience, required him to build a portfolio and demonstrate his knowledge of certain overarching mathematical processes such as proofs. Tom reflected on how this experience helped him understand the big picture of mathematics: “[I]t was a class focusing on taking
everything that we had learned and bringing it all together.” He said that it required him to build a portfolio of his work in his other courses that demonstrated his understanding of “mathematical processes” and “the interconnectedness of mathematics.” Understanding that Tom’s preparation program’s requirements included this course assists efforts to grasp this experience’s meaning for him and how that materialized in his teaching. In his words, it helped solidify his understanding of the “backstage of mathematics.”

Beth’s senior experience included building a teaching portfolio that displayed her ability to perform tasks necessary for a future job, e.g., create lesson plans, perform assessments, etc. She called it a “very comprehensive” “big resume,” emphasizing knowledge and skills related to how to teach in general. When asked whether she had a corresponding math course or experience, she said she had not; she did not recollect any experience that required her to pull together her mathematical learning, and conveyed her conclusion that her coursework was unrelated: “There was no rhyme or reason to what I took. There was no order to what I took. It’s not like you needed to take statistics to take linear algebra. It’s not like you needed to take linear algebra to take analytical geometry.” Beth summed up her experience by stating that it had made her “well rounded” in mathematics, but she questioned how it helped her teaching now. Beth’s lack of a required course that forged links between her mathematics coursework and her future teaching, together with her acknowledged inability to relate to the abstract mathematics she learned, explained her current emphasis on such connections for her students.

The phrases, “well-rounded,” “logical thinking,” and “math-minded,” utilized by Beth, Tom, and Jill respectively, summarize the participants’ AMK experience and their perceptions of its after-effects. All three talked about their AMK’s positive, though indirect, impact on their work habits. While they converged on this idea, dissimilarities traceable to their specific
experiences were also evident. Beth and Jill spoke of a lack of connection to the pure mathematics they were learning, about feeling that the experience resembled a “game,” as Beth put it -- that, to become a teacher, they were required to prove they could do advanced mathematics that, in their view, had nothing to do with what they would be teaching. Beth was hesitant to acknowledge any utility for her advanced mathematics beyond understanding how it got advanced, and found it disconnected.

Voicing the same complaint, Jill acknowledged the possibility of such linkages, but considered them beyond anything she would ever utilize in her classroom. However, Jill also recognized the value of her program’s requirement that she make connections, an approach she implemented to this day. She also acknowledged that her AMK experience had given her an understanding of both the overall picture of how mathematics becomes advanced and the type of thinking and skills it requires.

Contrastingly, Tom’s focus was not on higher mathematics’ lack of connections to the real world or his future teaching. When asked to discuss how he felt AMK directly impacted his teaching, Tom emphasized modes of thought as well as his overall understanding of certain mathematical notions and ideas. He neither dwelt on specific examples of content being directly useful, nor went into great detail regarding what was not, instead speaking more generally of connections that persisted across levels of mathematics. He also acknowledged the difficulty of such comparisons across levels because of the differences in content and required thinking. Yet, like Beth and Jill, he stressed that when presenting mathematical ideas he was constantly thinking of and considering “the why.”
Summary of AMK’s influence on specialized content knowledge: It’s not the content, but the idea of the content.

- All three participants claimed that the majority of the AMK mathematics they learned was not directly evident in their teaching. They described AMK as “ingrained,” its indirect influence hard to gauge. All three participants stated that their AMK experience helped solidify their knowledge of mathematics. Beth said it made her “well-rounded” mathematically, Jill, that it left her a more “math-minded person,” Tom, that it opened his eyes to the “backstage of mathematics.” They utilized the understanding they had gained regarding how mathematics became advanced when deciding how to teach mathematics.

- All three participants described the mathematics at the two levels as categorically different. They assessed AMK as challenging, abstract, hard to relate to, and requiring a completely different mode of thinking. Middle school mathematics was described as not challenging, concrete, and easily relatable to real life. All three participants attributed their understanding of those differences to their AMK experience.

- Two participants focused on their own lack of connection to AMK and their perception of it as unrelated to their future teaching; both described the impact of this perception during their tertiary education and its ongoing effect on their teaching. As a result, making connections is a habit of mind and skill they described utilizing to relate mathematical content.

- One participant focused on processes persisting throughout the entire mathematical learning process, from middle school to higher mathematics. He
described how his conviction of conceptual connectedness affected him during his tertiary learning and currently influenced his teaching. His AMK experience helped develop his understanding of the overarching connections that his teaching now stressed.

This theme examined AMK’s contribution to mathematics teachers’ specialized content knowledge from practicing teachers’ perspectives. Study participants could not pinpoint instances of AMK’s direct implementation in teaching, and described AMK as differing categorically from middle school content. Yet, they also considered AMK influential indirectly, part of the totality of their respective learning experiences. Their observations documented that while tertiary mathematics was not directly implemented, it exerted indirect, general effects on the participants and their habits of mind in practice. The contributions that the participants described were shown to result from their unique AMK experience, their program’s design and specific courses within it having imprinted each with distinctive focuses, habits of mind, and insights regarding teaching mathematics. In-depth portrayal of the participants’ experiences revealed the complexity of this phenomenon: though AMK concepts were not implemented directly, the totality of the AMK experience was shown to contribute to teachers’ specialized content knowledge.

**AMK’s Influence on Pedagogical Content Knowledge (PCK): Minimal, if Any, Required and Dedicated Coursework**

The following section will explore participants’ perceptions regarding AMK’s use in the PCK they implement when teaching. This superordinate theme shifts from consideration of mathematics knowledge per se, to examination of the ways that AMK surfaces in teachers’ knowledge for teaching mathematics. It analyzes teachers’ views regarding AMK’s influence on
their knowledge for content and students and their knowledge for content and teaching. The first of these refers to teachers’ understanding of mathematics in terms of their students’ learning process, the second to their instructional strategies to teach mathematics.

In contrast to their pure content course requirements, participants agreed that their preparation had included few, if any, opportunities to develop their PCK. The subthemes “little or no required coursework” and “pedagogical content knowledge courses different and relevant” detail this concept below.

**Little or no required coursework.** When considering the totality of the participants’ experiences, their tertiary mathematics preparation was heavily skewed toward pure content courses; only Tom’s included a single required PCK course that focused specifically on mathematics knowledge for teaching. Reflecting upon the sum of his experience he suggested that, with that sole exception, the knowledge he had gained was more related to his understanding of mathematics in general than to how to teach it:

> In order to get that higher end stuff, I had to become a master of the lower end stuff, which in turn, to a certain degree, minus the whole, "Okay, now how do you explain to a kid," makes me a master of the curriculum that I’m teaching that lower level stuff in.

“Minus the whole, ‘Okay, now how do you explain to a kid’ concretized that his preparation helped him master mathematical content, but not necessarily knowledge for teaching that content. He even contemplated aloud whether he would have found more courses devoted to teaching mathematics specifically worthwhile, or whether taking courses to learn how to teach certain subjects would have been “redundant.” However, his viewpoint changed as he explained:
Like if we had done a thing of … so this semester, you’re gonna take a class on how to teach algebra one. This semester you’re gonna take a class on how to teach algebra two. Like I feel that would have been somewhat redundant ‘cuz I mean I know how to teach it or I shouldn't say I know how to teach, I know algebra one. I know algebra two; really it’s just that teaching aspect of it.

The “I shouldn’t say I know how to teach …” acknowledged that his experience equipped him with mathematical knowledge, but did not adequately address the “teaching aspect of it,” at least not in a way easily described or measured. He went on to describe his appreciation for the one PCK course that incorporated opportunities to understand the teaching of mathematics prior to his student teaching experience and teaching career.

Though both took a PCK course, neither Jill nor Beth was required to do so to complete their undergraduate major. The course Jill took was not related to the middle school level; Beth’s was, but as part of her post-graduate education. Jill provided detail concerning her lack of formal PCK experience. Not only did her program not provide such a course; she considered it lacking in connections between mathematical and educational concepts. When evaluating the potential contribution of her pure education courses to PCK, she hesitated to call her experience helpful. Though such courses required integrating mathematics to create lesson plans and units, the expertise required to do so was left to her because the education faculty lacked expertise in both mathematics and teaching mathematics. Nothing connected her pure content and pure education courses. The elementary PCK course she took looked at the mathematics to be taught from an advanced viewpoint, but was not a requirement of her major, and did not focus on concepts readily implemented at the middle school level. Beth similarly questioned her general lack of PCK experiences:
I do question though sometimes when you … even are gonna be a secondary math teacher which is grade seven through 12, I question the extent which the courses that are required for you to take, um, and instead having more of … of how to teach the math, which I did not have as an undergrad at all.

Understanding that all three participants experienced minimal explicit or required PCK, the following paragraphs delve into how the totality of their AMK materialized in the specific domains of their knowledge of content and students, and next, their knowledge of content and teaching.

Knowledge of content and students. To gauge AMK’s implementation in participants’ knowledge of content and students, they were asked to describe how their experience helped them grasp how students’ understanding evolves, their misconceptions and common errors. Lacking a formal PCK course, Beth and Jill could gauge only whether and how their pure courses helped develop such knowledge, while Tom shared how his experiences of both influenced him.

Beth strongly rejected the idea that her academic preparation helped her develop such expertise, asserting instead that she had acquired the bulk of her knowledge over time through the practice of teaching. Referencing her AMK, she explained, “Students weren’t discussed in the classes … in the pure content that I took … you took … this is how you do calculus, this is how you do abstract algebra. Nobody said this is how students learn.” She described the absence of formal opportunities during her tertiary education to learn about student misconceptions, errors, their mathematical learning process:

When you’re going to school to be a math teacher, you take classes that are pure content and, um, it’s not discussed how students make mistakes or what they do or what they’re
thinking or other ways you can present this. It’s presented one way and you either get it or you don’t and you move on. And I think that that’s something I’ve struggled with my whole time, you know, and you … there are so many students that have so many different needs and so many different backgrounds that, you know, I don’t think that that’s … that’s discussed or dealt with when you take a pure content math course, um, you know, required to be a math major. I don’t think that math teaching is taken into consideration in those courses.

Beth was voicing her disappointment at her undergraduate program’s failure to make such experiences explicitly available. Jill, too, expressed concern that her program did not provide opportunities to develop her knowledge of how students learn mathematics. She, too, attributed her current expertise to her years of teaching and learning mathematics: “I honestly don't think my math knowledge necessarily is um, I don't know, I don't think it plays a role in that, I just think that, I think maybe my personal experiences come into play.” At the same time, she said that, indirectly, AMK helped her gauge when her students would struggle with new concepts because of her own difficulties with challenging material: “As the material gets more challenging you are going to anticipate more, more questions, more confusion especially when the math is starting to look so much different than your, your basic computation.”

Tom described a similar takeaway from his pure coursework: “I would say it, it helps me understand their process of learning fairly well not from the knowledge I gained from it but from the, just aspect of taking those courses.” Here he ascribed AMK’s influence to his global experience of it, not necessarily to its specific content. However, that overall understanding of mathematics and its processes helped him anticipate student misconceptions and errors:
I knew exactly what questions [they]’re gonna have on this because I know exactly what the hard points are in here or being able … having that more in depth understanding of it helps me identify what the problem they’re actually having with it is.

In his teaching Tom made use of his understanding of the “backstage of mathematics” in another way: “I can know my kids are going to have difficulties with multiplying decimals, but if I don't know why they [are] having difficulty multiplying decimals, it's going to be much harder to fix those difficulties.” While Tom attributed his ability to anticipate common misconceptions to his mathematical understanding, he also acknowledged that he was still developing the ability to resolve those misconceptions:

I think it still develops in the sense of I, myself, understood it before I started teaching. If you gave me a problem, I could simplify it. I could solve it. I could do whatever you needed for it, but I would have a harder time explaining why I did it.

Tom’s PCK course mandated that he practice “teaching” secondary level mathematical concepts to mathematics majoring peers who, to replicate a real classroom environment, were to pretend they did not understand the concepts. His experience provided Tom with an opportunity, though simulated, to rehearse how to deal with this real-world situation. Even so, like Beth and Jill, Tom documented that his PCK was continuing to expand through the practice of teaching. Taking a PCK course fostered Tom’s tendency to see his knowledge as developing during his AMK experience, as opposed to the other two participants who had not been exposed to such a course on the undergraduate level.

**Knowledge of content and teaching.** To assess AMK’s contribution to participants’ knowledge of content and teaching, all three were asked to describe how their experience aided in developing knowledge of multiple representations as well as its impact on their decisions
when selecting and sequencing them. Additionally, they were asked how it directed them toward understanding and evaluating the kinds of resources available to them.

Beth rejected the possibility that opportunities to build such knowledge had occurred during her undergraduate AMK experience, attributing the mastery she displayed entirely to her teaching experience:

I’m big into exploration. I’m big into using real-world examples to get that across for months because then they’ll go, “Oh yeah! I remember we did that.” You know, we just, we’re doing a geometry unit, and, um, we just were doing, um, translations and transformations and, you know, I did everything hands-on for that stuff. Um, am I saying everything needs to be hands-on? No. Um, you know, when we did solving equations, we used a balance scale, and we talked about, you know, what happens if we do something to one side and not the other? Um, I want them to see that. You know, when I’m … Even when we go to solving normal equations on the board, I’m still drawing the little mini scales underneath them so they can remember. Um, but again, I think that’s from working with colleagues and 14 years of teaching experience that I’ve gotten to be as strong as I have doing that, those things.

Similarly, Jill contended that her AMK experience did not provide any formal way to acquire knowledge of multiple representations of middle school content. She described her mathematics course on elementary pedagogy as focused on developing concepts for that level, but stressed that nowhere in her course experience was the middle school level specifically addressed:

I wasn't taught like specific strategies in college you know when I, I guess you just basically, you take in um, I mean unconsciously but you're, you're watching how your
math teachers growing up taught you and you're watching how your math professors are teaching you in college. So, I think you're probably taking um, you know certain strategies from them but they aren't specifically, you know saying "this is the best way to teach how to graph a line."

Asked how her tertiary preparation helped her develop familiarity with the tools and resources available to mathematics teachers Jill reflected upon her experience as a whole: “The courses did not make any connection to how we would be able to use it in the classroom, providing us any kind of resources. I don't think that was, that was offered. There was no connection for that.”

Tom, whose program did require him to take a PCK course, an experience the other participants lacked, saw AMK’s influence on his knowledge of developing and selecting multiple representations very differently. The concept of “how to alternately explain things” was a prescribed part of his AMK experiences; his PCK course included situations designed to practice and develop this valuable professional tool:

One of the things that we did is we would do practice lessons in classroom. We would come up with a lesson then teach it to the class. Teaching a math lesson to a class filled with math majors where it’s not like something advanced, it’s like okay today we’re gonna learn how to solve a, you know, linear equation and everyone in the class like, okay we already know how to do this. So we were under sort of instructions that we had to play dumb. We had to pretend we had no idea what they’re talking about and sort of be not belligerent but be like as like I have no idea what the heck you’re saying as we could. That way it was like, okay, well here is how we do it. What … okay, well let’s try to
rearrange what my explanation was to see if I could get that, uh, to explain in different way which is something I try to do in my class.

This tertiary interaction began the process of making this knowledge base his, but Tom described it as ongoing and expanding: “We didn’t really focus on when it’s a good time to alternately explain things and when it isn’t that’s, uh, it’s just something that I’ve been learning on my own.” He explained that through his teaching experience he learned that multiple representations sometimes confuse students.

He related how he had to make a portfolio in which he was expected to show his ability to evaluate instructional resources for their use in teaching:

We had to read … not read through but like look at and … and, um, analyze and sort of evaluate five different textbooks and determine how we thought they were if they were good, if they were bad, what was good about them, what was bad about them. Um, and then the same thing where we had to evaluate five, um, programs that we could find online or at the school.

A course that “had [him] do research into different resources that [teachers] had available to [them] in the teaching community,” specifically geared toward mathematical content, also differentiated Tom’s undergraduate program from the other participants’.

**Pedagogical content courses different and relevant.** The participant’s perceptions regarding AMK’s materialization as PCK, in combination with their unique experiences, warrants further analysis of the nature of PCK courses. While this study sought to understand how mathematical knowledge gained from participant’s undergraduate major was utilized in teaching, not all participants took a PCK course as part of that major. The participants’
perceptions regarding their PCK course, comparison of that course with pure content courses, and PCK’s contribution to their current applied knowledge follows.

The AMK requirements of just one study participant, Tom, included a formal PCK course. Beth and Jill both experienced a PCK course outside of their major’s requirements. Beth’s addressed middle school mathematics, Jill’s the elementary level. Regardless of their PCK course’s timing or content, all three participants described it as differing in kind from their pure courses. The following paragraphs analyze the participants’ comments to document their common conclusion, beginning with Beth and Jill, then moving to Tom.

Beth found the PCK course that she signed up to take after three years of teaching an enlightening experience altogether different from any during her undergraduate preparation. Her major required both pure math and pure education courses, but the math courses lacked any specific and focused PCK, and the diverse majors of enrollees in the pure education courses precluded any specific focus on math. Her PCK course marked “the first time that we had discussed different ways to present things.” Delving into its distinctive features she explained:

One of our textbooks was actually was a sixth grade, um, consumable textbook that we would have to prepare lessons for, that, um, we had to come up with three different ways we could come up with an activity to support a topic. Um, the entire class makeup was, um, people that work in the teaching field of mathematics, you know, as compared to the [Education] classes that it was … it could have been language arts or French or, um, Spanish, special ed. teachers.

Here Beth was describing pronounced differences from the pure content courses to which her tertiary education provided exposure. Her PCK course focused on the middle school level, and required her to come up with multiple representations of content. Even its enrollment,
restricted to “people that work in the teaching field of mathematics” distinguished it from other education courses she had taken. As she described it, the class also addressed topics vital for students to understand in an advanced manner:

Um, it was the first time that we had talked about (clears throat), um, number sense. It was interesting. It talked about how, and I remember this because I had a student a couple years ago that was like this, even at the middle school that if you don’t understand that every time you hit 10 you change in mathematics, if you don’t understand that concept that 10 ones becomes the 10 and 10 tens becomes a hundred and 10 hundreds becomes a thousand. If you don’t have that mathematical knowledge and understanding, you’re not gonna be a successful math student.

The course provided an understanding of the points at which students were likely to flounder, as well as opportunities to develop strategies to combat those difficulties. Her struggle to help a high school student understand place value and the base ten system illustrated her PCK course’s value:

[The] course talked about, um, how to work with those students and how to provide modifications and what types of things you can do to help them and, um, you know, was much more specific in terms of teaching math than any other course I had taken.

Beth reflected further that she would have valued more opportunities for that type of learning before her teaching career began.

Jill’s elementary mathematics methods course, which taught her to use representations like tiles, blocks, and other manipulatives to present mathematical content, left a similar impression. Comparing it to her major’s requirements she surmised:
I mean, even though I took, um, the elementary education class prior to um declaring my major so it wasn't part of my major, I feel like that one was more beneficial because it, it actually, it modeled, you had to model everything. You know, you had to really get down to the students' level.

Beth and Jill’s respective exposures to a PCK course occurred outside of their AMK’s requirements, but Tom’s was integral to his. He, too, assessed it as having contributed knowledge relevant to his current teaching, fitting naturally into the rest of his experience. Having taken many courses that were pure mathematics or purely educational, he described his PCK course as creating a feeling that “at the end you’re taking that class that sort of combines, okay, here’s the math knowledge, here’s your teaching knowledge, let’s combine them together to make math teaching knowledge.” It provided an opportunity, even before his student teaching, to focus on secondary mathematics content and to practice teaching and developing mathematics lessons surrounded by classmates and teachers whose goal was identical.

Tom explained that the course also emphasized other components of teaching mathematics, requiring its enrollees to research instructional resources and teaching them how to utilize technology in the classroom. Excepting only its technology aspect, and that only because of its unavailability where he taught, Tom considered nearly all the course’s topics still relevant.

To sum up, the PCK course experience of the three participants had been positive: all three valued its addressing of less advanced mathematics in an advanced way relevant to teaching. Beth and Tom came up with specific examples of how their teaching still incorporated elements of the knowledge acquired in their PCK courses. Reluctant to say that she used her course’s content per se, Jill attributed her hesitation primarily to its focus on elementary mathematical education; she would have appreciated a similar course focused on middle school
mathematical topics. Their PCK courses had left lasting impressions on all three participants, manifested in their teaching.

**Summary of AMK’s influence on Pedagogical Content Knowledge (PCK): Minimal, if Any, Required and Dedicated Coursework.**

- Only one of the participants had a required pedagogical content knowledge course as part of his AMK program. The other participants also took one course each, but not as a required component of their undergraduate major.
- The advanced pure courses the participants took provided exposure to the big picture of mathematics, including an appreciation for what was challenging and important; to varying degrees, this understanding helped each relate to their students’ learning process.
- His PCK course required enabled Tom to identify specific ways in which his undergraduate AMK program directly provided opportunities to develop his PCK. Lacking such a requirement, the other two could provide few or no such instances, and they expressed a wish for similar experiences.
- The PCK courses were shown to contain elements that developed participants’ knowledge of content and students and their knowledge of content and teaching; all three considered both elements relevant to their current teaching.

In summation, this theme aimed to shed insight into AMK’s contribution to teachers’ pedagogical content knowledge through the experience of practicing teachers. Though each participant’s program was heavily weighted in pure coursework, the participants commonly noted that pure coursework exerted little or no influence on their knowledge for teaching, regarding either students or content. Of the three, only one had a pedagogical content course
required as part of their AMK program. The other two took one course each outside of their program. The participants generally agreed these courses contributed to knowledge for teaching in practice. They described the courses as directly increasing their knowledge of the content taught, student misconceptions and developing understandings, multiple representations, and resources available to aid their teaching. Two expressed a wish that their AMK experience had included more courses devoted to such content. Through the participants’ experiences, pedagogical content knowledge courses were found to contribute to the knowledge for teaching used in practice.

Conclusion

The goal of this study was to contribute to the understanding of tertiary level mathematics’ implementation in the practice of teaching at the middle school level. The study employed Ball et al.’s (2008) Mathematical Knowledge for Teaching framework to focus on how tertiary knowledge contributed to the pure mathematical knowledge that teachers use and to the mathematical knowledge they utilize for teaching. The first of these refers to the mathematical insights and habits of mind teachers utilize while teaching mathematics -- mathematical knowledge that only teachers need. The knowledge for teaching refers to teachers’ knowledge of the continuum of student learning, including ways that students misunderstand concepts. It also refers to teachers’ knowledge of multiple ways to present concepts and the instructional resources available.

Study participants could not describe any direct implementation of pure tertiary level concepts in their teaching. All agreed that tertiary level mathematics differed qualitatively from what preceded it. All believed that the totality of their experience affected their overall understanding of mathematics and their choices, insights, and habits of mind when presenting
mathematical concepts. They credited their experience with helping them understand how learning changes as mathematical content becomes more advanced. Participants’ insights and habits of mind were shown to be functions of their unique set of experiences; their program’s design and required coursework influenced their particular practices. Two participants noted that some of their most influential tertiary concepts originated outside of their program’s requirements.

Though the participants did not credit their pure courses with directly assisting their knowledge needed for teaching, two participants’ programs required many pure content courses, but no pedagogical content knowledge courses, while the third participant’s program required only one of the latter. Yet all agreed that their pedagogical course content courses contributed directly to their knowledge for teaching. Their PCK courses had formalized opportunities to develop their understanding of student learning and misconceptions and knowledge of multiple representations and instructional resources.

Through analysis of the participants’ experiences and the meanings they derived from them, this study examined in depth how tertiary level mathematics contributed to the practice of teaching middle school mathematics. The nature of tertiary mathematics, participants’ educational programs’ designs, the courses required, their inclusion of pedagogical content knowledge courses were all shown to influence participants’ perceptions of AMK’s contribution, as well as the mathematical insights and knowledge for teaching they developed and utilized.

Organized according to the two superordinate themes presented, the following chapter discusses these findings, synthesizing their contribution to and support of the extant literature with their implications for future teachers, tertiary mathematics teacher educators, and the design
of tertiary mathematics programs for teachers. To maximize transparency and instigate further research into this phenomenon, the study’s limitations are then presented.
Chapter 5: Discussion of Research Findings

This interpretative phenomenological analysis (IPA) utilized Ball et al.’s (2008) Domains of Mathematical Knowledge for Teaching to investigate how teachers’ advanced mathematical knowledge (AMK) is implemented directly and pedagogically. Literature regarding the type of knowledge mathematics teachers need is available, but insight regarding how teachers’ content knowledge preparation contributes to this knowledge remains slight (Moreira & David, 2008; Zazkis & Leikin, 2010). This study contributes to understanding this process through the unique experiences of practicing teachers who have lived it.

Teachers’ perceptions were utilized to gain understanding of AMK’s influence on their everyday teaching in terms of their specialized content knowledge, knowledge of content and students, and knowledge of content and teaching (Ball et al., 2008). The first of these domains refers to teachers’ mathematical knowledge, the others to their mathematical knowledge for teaching. Specialized content knowledge (SCK) was defined as the pure mathematical knowledge teachers must understand to teach, also the skills, insights, and habits of mind teachers utilize in order to unpack mathematical ideas. Knowledge of content and students referred to familiarity with the process of students’ learning of mathematical ideas, the ability to understand and anticipate misunderstandings and misconceptions they will have. Knowledge of content and teaching addressed grasp of specific strategies for teaching mathematical ideas and skill in selecting which to employ based on both content and students (Ball et al., 2008).

Ball et al.’s (2008) framework provided organization and specificity to this study of the complex topic of mathematics teachers’ knowledge. The researcher used the specific knowledge domains to discern how teachers’ AMK appeared in and influenced their everyday teaching. Exercising IPA’s double hermeneutic properties, the researcher utilized a meticulous analysis
sequence to gather and interpret meanings present in teachers’ portrayals of their experiences. The researcher’s aim was to obtain an empathetic understanding while questioning further the meanings realized (J. A. Smith & Osborn, 2003). Two prominent themes that emerged summarized how teachers saw themselves utilizing AMK directly and pedagogically in practice.

1) Advanced Mathematical Knowledge’s Influence on Specialized Content Knowledge:
   It’s Not the Content, but the Idea of the Content.

2) Advanced Mathematical Knowledge’s Influence on Pedagogical Content Knowledge:
   Minimal, if any, Required and Dedicated Coursework.

The following discussion explores how these themes support and expand the extant literature. Subsequently, implications for research, policy, and practice will be considered in light of the study’s limitations.

**Advanced Mathematical Knowledge’s Influence on Specialized Content Knowledge: It’s Not the Content, but the Idea of the Content**

According to Ball et al. (2008), SCK is mathematical knowledge unique to mathematics teachers, and specific to the practice of teaching. To lead students through the process of understanding concepts new to them, teachers require mathematical knowledge. To illustrate, consider one participant’s personal take on the process of learning mathematics: “Here’s my start, here’s my end. How do I get there?” Outside of teaching mathematics, getting to that end suffices. But mathematics teachers must also know how to explicate that process so that others can repeat and own it for themselves (Ball et al., 2008, Kinach, 2002). When considering AMK’s role in teaching middle school mathematics, participants described AMK as (a) absent from middle school content, (b) of a different nature, but (c) indirectly influencing their teaching.
The following section discusses these three themes in the context of the existing literature. First, the nature of tertiary level content in terms of its lack of direct implementation by participants is considered. Participants’ experiences are employed to support literature calling for the integration of more relevant experiences. Next, participants’ perceptions of AMK’s qualitatively different nature uphold literature regarding advanced mathematical thinking. And finally, participants’ experiences are utilized to support Ball et al.’s (2008) notion of horizon knowledge as a justifiable domain of teacher knowledge.

**Tertiary level program design.** The first insight participants documented was a lack of direct use of AMK concepts. Analysis revealed that all three participants’ programs shared eight required pure mathematics courses, with additional required courses varying in number and content. Monk (1994) documented a possible plateau effect for number of courses taken and influence on student achievement. The goal of this study was not to evaluate the impact of specific courses, or the relationship between number of courses taken and student achievement, but to gain insight into AMK’s implementation in teaching mathematics at the middle school level. This study applied teachers’ experiences as an indicator of the utility of AMK, providing a different perspective regarding the contribution of AMK courses. This study revealed that the participants’ particular experiences mattered in the development of the habits and skills exercised in practice.

All three participants acknowledged AMK’s contribution to their own mathematical knowledge and understanding. Simultaneously, however, participants could not find instances of its direct use in their teaching and questioned the number of pure courses taken. While researchers agree that mathematics teachers need some level of subject-matter knowledge preparation (Ball et al., 2008; Kinach, 2002; Moreira & David, 2008, Van Brommell, 2012), the
participants’ perspectives validate research claims that pure courses lack formal conceptual descriptions of how to impart the content of earlier levels (Ball et al., 2008; CBMS, 2001; Davis & Simmt, 2006; Moreira & David, 2008).

Consensus exists that simply taking more mathematics courses does not equate to mathematics teacher preparedness (Ball et al., 2008; Kinach, 2002, Moreira & David, 2008). Mathematics teachers require knowledge of how to break down the concepts to promote student understanding (Hill & Ball, 2004; Kinach, 2002; Krauss, Brunner et al., 2008). Research has documented that pedagogical content knowledge is in fact a separate domain of knowledge.

Krauss, Brunner et al. (2008) used Shulman’s (1987) teacher knowledge framework to examine whether the domains of pure and pedagogical content knowledge were disjoint, that framework making their definitions of the knowledge studied differ slightly from this study’s. While adding specificity to Shulman’s (1987) domains, the “soft boundaries” of Ball et al.’s (2008) framework imply that the domain of specialized content knowledge might overlap with Shulman’s two (1987) domains, as Krauss, Brunner et al.’s (2008) study understood them. Thus, Krauss, Brunner et al.’s (2008) findings are tangentially germane to this chapter’s first theme, which addresses AMK’s implementation in teachers’ specialized content knowledge.

Krauss, Brunner et al. (2008) found the two domains to be separate, but noted that as teachers’ pure content knowledge increases, at some point the two integrate into one. As a result, Krauss, Brunner et al. (2008) tentatively concluded that pure content knowledge supports the development of pedagogical content knowledge (p. 724). Consistent with this theory, Van Brommell (2012) posited that knowledge acquisition in the other domains of Ball et al.’s (2008) framework cannot occur without pure subject-matter knowledge. Krauss, Brunner et al. (2008)
called for further research to determine how and where the pure and pedagogical content knowledge that teachers need is acquired during teachers’ preparation.

Reflecting upon the advanced tertiary content they had mastered, participants could not identify instances in their teaching that directly implemented it. However, they did credit the experience with increasing their understanding of mathematics and how learning mathematics changes as its complexity increases. Thus, the participants’ perceptions validate that the domains are distinct yet connected. Using Ball et al.’s (2008) framework for greater specificity, this study documented that teachers’ pure courses impact their specialized content knowledge, the skills, insights and patterns of thought that teachers use to teach. The design of the participants’ particular programs was shown to be influential to their teaching. These findings support Krauss, Brunner et al.’s (2008) notion that pure knowledge contributes to the mathematical knowledge that teachers need. This study documented pure knowledge’s indirect, yet positive, influence on participants’ specialized content knowledge.

Another important finding regarding tertiary content’s impact derived from the mathematical links participants made between tertiary content and middle school content. Asked to describe instances of AMK’s direct use in teaching, participants described its conceptual connectedness to middle school mathematics, rather than content implemented. Experience, not linkages explicitly described in their tertiary courses, had helped them perceive this connectedness. For instance, they described direct links between concepts of Linear Algebra and Advanced Geometry and middle content. The teachers’ experiences lacked formal connections between the tertiary level content and the content taught.

If teachers make these connections without explicit guidance, research suggests this could be problematic. Moreira and David (2008) found that advanced mathematical knowledge can
actually be contradictory to the knowledge needed by mathematics teachers. Moreira and David (2008) compared literature regarding practicing teachers’ needs in making number system concepts understandable with the way those concepts are presented in advanced mathematics, and found the two to be in conflict. Knowledge compressed in advanced mathematics needed to be decompressed for teaching at lower levels; importantly, this conflict was not explicitly acknowledged in advanced mathematics. Moreira and David (2008) posited that clear identification of such instances in advanced courses would result in more relevant professional preparation for mathematics teachers.

When this study’s teachers attempted to link the levels to describe AMK’s implementation, all three stated that experience had taught that AMK was too advanced for their middle school students. Prior to Ball et al.’s (2008) framework and the formalized notion of SCK, Kinach (2002) studied mathematics teachers’ knowledge in terms of its transfer from subject matter knowledge. Kinach (2002) posited that the specific knowledge mathematics teachers need to promote student understanding does not transfer directly and automatically from pure mathematical knowledge; rather, “explaining” and “knowing” mathematics differ for mathematics teachers and non-teachers, a finding consistent with Ball et al.’s (2008) notion of SCK. Kinach theorized that the development of teachers’ knowledge necessitated a reordering of teachers’ “habitual ways of thinking about subject matter and subject-matter teaching” (Kinach, 2002, p. 69).

To amass such knowledge, Kinach (2002) recommended formal practice converting pure knowledge into “relational” knowledge. Participants’ experiences support this recommendation. This suggests the need for additional research to determine what formal experiences enable teachers to apply their pure knowledge in practice while understanding the differences between
the levels. It also supports further research into what types of experiences support teachers’
development of “relational” knowledge.

Though participants acknowledged their pure courses’ role in developing their own
mathematical knowledge, they concluded that those courses lacked direct application in practice
otherwise. Ball et al. (2008) posited that the advanced nature of pure content courses means that
they are “academic in both the best and the worst sense of the word, scholarly and irrelevant,
either way remote from classroom teaching” (p. 404). Researchers have advocated designing
pure content knowledge courses that are professionally and practically relevant for mathematics
teachers (Ball et. al, 2008; CBMS, 2001; Kinach, 2002; Moriera & David, 2008). The lack of
AMK’s direct usefulness that these participants experienced provides impetus to this push.
Schmidt et al. (2007) found that countries whose teachers were prepared with both subject matter
knowledge and explicit practical knowledge related to teaching middle school mathematics had
higher student achievement than those prepared otherwise.

Finally, the focus on real-life connections described by two participants was reactive to
their pure courses’ perceived irrelevance to real life and their future teaching, a disconnectedness
they felt even during the AMK experience. That inapplicability and lack of explicit uses made
both strive to include such experiences while they taught. For both, making mathematical
connections to real-life became an important instructional habit. The third participant also
described making such connections, acknowledging that his formal experience had not helped
him do so.

Ball et al. (2008) posited that teachers must be able to “choose powerful ways of
representing a subject so that it is understandable to students” (p. 404). In this case, all three
participants described choosing real-life representations as a strategy to present content, two of
them as a result of negative tertiary experiences. The two participants noted that AMK concepts’ irrelevance resonated during their college experience and influenced their teaching decisions long afterward. They documented an appreciation of specific AMK experiences that did help them come up with real life examples for their students. From these experiences, the participants derived their shared proclivity for anchoring their students’ learning with real-life experiences.

The teachers in this study considered knowledge of real-life applications vital to their middle school practice, yet all denied that their tertiary preparation had provided formal opportunities to develop such knowledge. The next section will detail how the nature of tertiary mathematics inhibits the occurrence of such opportunities. However, reflecting upon Ball et al.’s (2008) framework, knowledge of real-life applications is not necessarily defined in any of the domains. Thus, the experiences of this study’s participants suggest that further research into the development of such “knowledge of real-life applications” is necessary.

A first consideration is where this knowledge, which may prove as complex as the other defined domains, would fit in Ball et al.’s (2008) framework. To aid student understanding, teachers must have knowledge of an application associated with a mathematical concept, understand how mathematics is used in that application, and choose how to pedagogically integrate it. More research is needed regarding this knowledge’s significance, how and where it is developed, its relationship to the existing domains, and its application in teaching.

**Reflections of Advanced Mathematical Thinking (AMT).** The participants were unable to come up with examples of AMK’s direct utility in their teaching; all three judged AMK too advanced for middle school students. In a similar study of AMK’s use, Zazkis and Leikin (2010) noted a comparable finding, and posited that the gap between secondary and tertiary level content might be the reason. In this present study, focused solely on the middle
school level, that gap yawns even wider. Utilizing AMT theory as a lens helps decipher these findings and support Zazkis and Leikin’s (2010) theory.

To employ AMT theory, a necessary first consideration is its complexity. Though researchers have different ideas as to what constitutes AMT, their criteria commonly stipulate that it (a) is limited to advanced tertiary content, (b) can include less advanced content considered in an advanced manner, and (c) involves the process of using real-life or math problems to conceptualize new understandings of mathematics (Seldon & Seldon, 2005, p. 4). The last two are considered applicable to all levels (Seldon & Seldon, 2005, p. 4).

Study participants documented that their tertiary mathematical courses’ content had lacked real-life connections or association with the content they would teach in the future. Employing the first view of AMT theory clarifies their stance. Focused on thinking required at the tertiary level, the first perspective limits AMT to intangible processes and reasoning “not entirely accessible to us through our five senses” (Edwards et al., 2005, p. 15). Edwards et al. (2005) trace this occurrence’s start to the time when students begin “to deal with abstract concepts and deductive proof,” and previously acquired thinking and skills are no longer helpful (p.16). Edwards et al. (2005) posited that this transition occurs at the tertiary level. During this transition period, mathematics learners find that the skills gained in their pre-tertiary experience are no longer sufficient (Edwards et al., 2005). Interpretation of study participants’ accounts verifies that such a transition occurred; the meanings the participants took away from that experience can provide insight into how to utilize this transition to promote development of the knowledge that teachers need.

Reflecting upon their tertiary experience, two of the participants noted their lack of connectedness to the concepts, the third a demand for a different type of thinking not previously
required, termed “fluid processing.” All three participants voiced concerns regarding their students’ ability to comprehend mathematics at the tertiary level, which they considered more complex, abstract, and simply “different” from the mathematics previously experienced. The thinking required at the tertiary level was described as too advanced to require of middle school level students. These perceptions by teachers who lived it describe the transition to which Edwards et al. (2005) referred. To understand the implication of that sea change on teachers’ knowledge, let us further consider the nature of tertiary mathematics.

Robert and Schwarzenberger (1991) described tertiary mathematics as categorically different, requiring processes of thought distinct from those needed for the mathematics that preceded it. According to Tall (2008), learners transitioning to tertiary mathematics must call upon knowledge built upon tangible processes and symbols and extend it to the process of proof in a manner not previously required. Tall (2008) defined AMT as made up of three domains that encompass mathematical thinking at all levels. In his view, the interrelated “three worlds of mathematics” include “conceptual embodiment, proceptual symbolism, and axiomatic formalism.” While he did not assign each of these worlds to a specific mathematical level, he concluded that the tertiary level involved primarily the third world (p.7). Tall (2008) posited that these three worlds of knowledge grow in distinct ways, the first two relying on definable experiences, while during the last, mathematical properties are deduced based on theoretic definitions (p. 8).

Tall’s (2008) model helps explain the lack of connection participants experienced while transitioning to tertiary level mathematics, and receives support from the notion of “fluid processing.” This study’s participants experienced the transition from pre-tertiary to tertiary mathematics without explicitly understanding it. Tall (2008) postulated that acceptance of these
spheres of mathematical thinking permits informed discussion between mathematicians and their students regarding the transition to tertiary learning processes that the latter must make. This study validates that such conversations will help future mathematics teachers understand how the mathematical knowledge and learning processes of the tertiary level diverge from their previous experiences. Simply acknowledging this transition increases its transparency and promotes conscious awareness that this study’s participants were shown to lack.

Researchers outside of AMT literature have also documented that tertiary level mathematics differs from what precedes it (Ball et al., 2008; Moreria & David, 2008). Though AMT helps explain this necessary transition in thinking, also AMK’s lack of direct implementation in teaching middle school content, the teachers in this study did acknowledge that the totality of their experience contributed to their practice. They described it as helping them understand how learning mathematics changes throughout the levels; this understanding contributed to the decisions they made while teaching. This study’s findings suggest that up-front, explicit information regarding the transition to tertiary mathematics, before beginning it, would prove more beneficial to future teachers than retrospective reflection regarding an experience otherwise difficult and solitary. Additionally, Tall (2008) identified how the worlds of thinking can apply at all levels. Explicit understanding might decrease frustration expressed regarding AMK’s lack of connectedness and the categorical difference between the mathematics.

AMK’s documented demand for “fluid processing” as well as the lack of connection felt with tertiary content by participants reflects the qualitative change AMT theory described. AMT research regarding the nature of mathematics at the tertiary level also lends credence to the participants’ experiences of the abstract nature of tertiary level mathematics, their conviction that concrete connections with tertiary content were not possible. This study supports the notion
that making AMT theory part of their required education would help future teachers understand their students’ acquisition of knowledge, also their own learning process.

It also validates Moreira and David’s (2005) call to incorporate into future mathematics teachers’ preparation more professionally relevant knowledge that explicates conflicts inherent in the levels. In this case, the nature of learning at the tertiary level conflicts with how learning occurs prior to it. Exposure to AMT theory regarding the transition that occurs at the tertiary level would be advantageous for current and future mathematics educators, making possible the conversations advocated by Tall (2008). The study participants’ lack of explicit understanding regarding the categorical difference between pre-tertiary and tertiary mathematical learning suggests that knowledge of that transition would improve their own tertiary learning experience while maximizing its positive impact on their future teaching.

**Evidence to support horizon knowledge domain.** As demonstrated, this study’s participants found direct use of AMK’s concepts in teaching middle school mathematics to be minimal; however, interpretative analysis revealed that knowledge’s implicit use. Acknowledging that their experiences had developed their overall understanding of mathematics, the participants reported that having a sense of the whole mathematical picture helped them make instructional content decisions. The totality of their tertiary experience, rather than its specifics, influenced those decisions, helping them understand the preparation students needed prior to experiencing tertiary level mathematics. That totality of experience had engrained mathematical habits of mind that subtly, indirectly, powerfully, informed their teaching. In sum, their entire journey learning mathematics contributed to the insights they applied in teaching.

Ball et al. (2008) tentatively designated “awareness of how mathematical topics are related over the span of mathematics included in the curriculum” as a separate domain of
mathematical knowledge for teaching. This “horizon knowledge” “includes the vision useful in seeing connections to much later mathematical ideas.” Ball et al. (2008) posited the usefulness of such comprehensive knowledge when deciding how to present mathematical content, and this study’s participants attested to the fundamental importance of that overview in their teaching.

Participants’ comments regarding AMK’s indirect implementation implied understanding of the links between middle school topics and future advanced content. Participants noted their AMK experience improved their knowledge of the “bigger picture” of mathematics. AMK was credited with promoting good intellectual habits; it was also reported to impart an empathetic understanding of how it feels to learn. This “bigger picture” informed participants’ decisions about the process of unpacking mathematical content for middle school students.

Participants’ knowledge of present content’s connections to more advanced content made a difference in their teaching. They chose the depth of content, fostered the habits of thought, most likely to promote their students’ ability to grasp future advanced content. This suggests that their experience of advanced content, though not directly implemented in the middle school content they taught, deepened participants’ understanding of mathematics. Participants thus validated theory regarding pure content knowledge’s capacity to develop the other domains of teachers’ mathematical knowledge (Krauss, Brunner, et al., 2008; Van Bommel, 2012). As the participants saw it, studying mathematics in its totality had helped them understand the discipline’s continuum of learning, including its transition from concrete to abstract; their tertiary level experience had embodied the latter.

The use of “extensive math background,” “well-rounded,” and “math-minded” to describe their current mathematical competence attested to participants’ perception of AMK’s positive contribution. That confidence permitted them to grasp the tertiary level’s categorical
difference from middle school’s, and describe how, even though not implemented directly, their experience of tertiary level mathematics had changed them. It had sharpened their awareness of the skills, habits, and knowledge their students would need to succeed at the tertiary level, and they documented making instructional decisions based on this knowledge. Thus, the participants demonstrated that Ball et al.’s (2008) provisionally defined domain of horizon knowledge was integral to their teaching.

Acknowledging the contributions of their pre-tertiary and AMK experience to their horizon knowledge, participants nevertheless expressed the wish that their AMK course content was more relevant to their current practice. In the previous section, AMT theory verified participants’ perceptions that AMK differed qualitatively from the math learning that preceded it, and recommended that teachers receive explicit exposure to AMT theory. The participants’ incorporation of horizon knowledge into their teaching practice argues further for the inclusion of AMT theory in teachers’ formal tertiary preparation. Tall’s (2008) recommendation that the nature of tertiary mathematics be explained to students of mathematics resonates again with this study’s findings. As future teachers derive horizon knowledge from their academic experience, familiarity with the “three worlds of mathematics” would impart understanding of how tertiary mathematics fits into their knowledge base.

In summary, AMK’s contribution to teachers’ specialized content knowledge was shown to be indirect. Participants ascribed AMK’s lack of direct implementation to its different nature, deriving that conclusion from their experiences. However, the participants’ AMK had imprinted them with skills and habits of mind embodied in their teaching. Additionally, the totality of their educational experience had provided them with understanding of the continuum of mathematical learning, a notion summarized by Ball et al.’s (2008) domain of horizon knowledge. These two
factors enabled study participants to express AMK’s contribution to the pure mathematical knowledge needed to teach middle school mathematics. AMT theory provided a lens useful for understanding participants’ perceptions; it was also posited that its formal inclusion in future mathematics teachers’ preparation would prove beneficial. The next section analyzes this study’s findings regarding AMK’s contribution to teachers’ pedagogical content knowledge.

**Advanced Mathematical Knowledge’s Influence on Pedagogical Content Knowledge:**

**Minimal, if Any, Required and Dedicated Coursework**

The previous section analyzed AMK’s contribution to the pure mathematical knowledge that teachers need. The participant’s question, “Here’s my start, here’s my end. How do I get there?” helped determine the type of pure mathematical knowledge discussed. Considering this question from another angle, unpacking concepts for students requires teachers to have mathematical knowledge, but also knowledge for teaching mathematics, that is, pedagogical content knowledge (PCK) (Ball et al., 2008; Shulman, 1986). Teaching involves not only the ability to do and unpack mathematical ideas, but decisions about presenting and explaining them that take into account the students, their prior learning, and their anticipated struggles (Ball et al., 2008). Additionally, teachers must consider and evaluate the resources and methods available to help relay that content (Ball et al., 2008). These constitute the knowledge for teaching (PCK) that teachers must have.

Policy makers have utilized the term “deep” to describe the mathematical knowledge teachers need (see, e.g., Ball et al., 2001; Davis & Simmt, 2006; Grossman et al., 2005; Krauss, Baumert, et al., 2008; Krauss, Brunner, et al., 2008; Ma, 1999; Matthews et al., 2010; Shulman, 1986; Tchoshanov, 2011; Zazkis & Leikin, 2010). Researchers have agreed that pure mathematical knowledge does not suffice (Ball et al., 2008; Davis & Simmt; Ferguson & T.
Womack, 1993; Hill et al., 2005; Philipp et al., 2007; Shulman, 1986; E. Smith, 2008). This study found pure knowledge’s role in the practice of teaching to be indirect, impacting teachers’ patterns of thought as well as their knowledge of the continuum of learning mathematics.

While study participants showed that advanced pure content contributed to the knowledge teachers utilize in practice, they also established that PCK’s development mattered at least as much. Heavy skewing toward pure courses, omission of any direct application, and dearth of formal PCK made two participants openly question their teacher preparation programs, stating further that more (any) pedagogical content courses would have been beneficial. Deprived of such experiences, these teachers described developing their PCK base themselves, while actually teaching.

Reflecting on their required AMK courses, the same two participants expressed frustration with the absence of any formal pedagogical content knowledge experiences. Each denied that pure coursework had contributed to their knowledge of multiple representations, student understandings, misconceptions, and instructional strategies. Each felt that their program had focused solely on learning advanced mathematics, failing even to consider multiple representations and lower level content. The current recommendation that teachers include formal opportunities for their students to consider multiple representations (NCTM, 2000; Tchoshanov, 2011) renders especially significant the fact that two of the teachers’ formal preparation lacked direct opportunities to build such knowledge.

Extrapolating from these perceptions, teacher preparation programs should carefully consider the PCK courses available to future mathematics teachers. Studies have shown that variables like number of courses and level of degree do not reliably track teachers’ mathematical knowledge for teaching (Begle, 1979; Darling-Hammond, 2000; Monk, 1994). However,
research regarding coursework and student achievement has shown PCK courses to be more beneficial than pure courses (Monk, 1994). The experiences of the participants in this study support this finding.

While two of the participants were not required to take a PCK course, both experienced one outside of their program. Each found it markedly different from their pure courses, its topics relevant to their current practice. They expressed their gratification for the formal opportunities it afforded to delve in-depth into less advanced content, acquiring perspectives previously unknown. Participants appreciated the focus on understanding concepts from the students’ point of view, also the opportunities to develop multiple representations. These PCK courses focused on Ball et al.’s (2008) knowledge of content and students as well as knowledge of content and teaching. With the exception of these single courses, these two participants reported relying for their knowledge base solely on their own experience learning mathematics and teaching.

The third participant’s program was the only one that required a PCK course. Like the other two participants, he appreciated the opportunity it offered to formally present and develop multiple representations of less advanced content. This participant also recalled opportunities to learn about and evaluate available instructional resources. As a required part of the program, the participant saw his PCK course as integrating mathematics learning and the educational theory acquired in pure education courses. This course provided formal opportunities to develop Ball et al.’s (2008) PCK domains inside the participant’s AMK experience.

The participants’ experiences of their PCK courses are consistent with current literature regarding the kinds of opportunities teachers need to develop their PCK. Kinach (2002) recommended creating opportunities that challenge teachers to transition from procedural knowledge to relational understanding. G. J. Stylianides and Stylianides (2010) considered the
knowledge mathematics teachers need as applied mathematics; to develop it they recommended presenting teachers with opportunities to consider mathematical content to be taught. Isiksal and Cakiroglu (2011) advocated providing future teachers with chances to analyze students’ developing understandings and misconceptions. All three recommendations involved dissecting less advanced content in a manner designed to build teachers’ own conceptual understanding.

These studies advocate for preparation programs that include pedagogically related, carefully constructed mathematical tasks that enable teachers to consider in depth content to be taught. Application of AMT theory’s second criterion yields another perspective regarding such recommendations. Viewed through the AMT lens, advanced mathematical thinking can involve thinking about less advanced notions in an advanced manner. The learning situations recommended in G. J. Stylianides and Styliandies (2010), Kinach (2002), and Isiksal and Cakiroglu (2011) would enable teachers to develop, in a structured setting, the conceptual approach associated with AMT’s second perspective. This study’s participants found their opportunities to engage in the mode of thinking that AMT’s second criterion describes germane and beneficial to their teaching. Though the coursework focused on less advanced content, it provided professionally relevant knowledge advocated for by researchers (Ball et al., 2008; Kinach, 2002; Moreira & David, 2005; Styliandies & Styliandies, 2010).

In summary, this study’s three participants all found that their pedagogical content courses contributed directly to their knowledge in practice. They were relevant. The two whose educational programs did not require PCK courses wondered why their academic preparation had not imparted such knowledge. The single participant required to take a pedagogical content course felt that it solidified his AMK experience. And despite the fact that all three participants
considered knowledge gained from their PCK courses more directly relevant to their teaching practice than pure coursework, their programs were all heavily weighted with the latter.

Imlications

Readers of IPA “make links between the findings of an IPA study, their own personal and professional experience, and the claims in the extant literature. The power of an IPA study is judged by the light it sheds within this broader context” (J. A. Smith & Osborn, 2003). This study aimed to determine how, and to what degree, teachers’ experiences in their classroom settings reflect their tertiary preparation directly and pedagogically. This study’s findings add to our understanding of AMK’s role in the development of mathematical knowledge for teaching and its translation to practice. They can help policy makers, mathematics teacher preparation programs, professional development programs, and future and current mathematics teachers gauge the ability of current mathematics teacher education to equip teachers with the knowledge needed in practice.

The study’s design facilitated in-depth, rich, dense description of the participants’ experiences and the meanings they derived from them. Careful analysis yielded a detailed interpretation of the two themes generated regarding AMK’s contribution to teaching middle school mathematical content. To provide readers with insight into their relationship to existing research, these findings were then synthesized with the literature. With lack of understanding regarding this important issue well-documented, this study’s findings provide both significant insight and a foundation for further research into tertiary mathematics’ role in the development of mathematical knowledge for teaching. The experiences of practicing teachers made clear that AMK indirectly influences teachers’ specialized content knowledge and horizon knowledge.
However, their AMK lacked adequate formal opportunities for development of pedagogical content knowledge, whose content was documented to be significant in practice.

**Implications for preparation programs and policy.** Given these notions, mathematics teacher preparation programs that do not include any PCK course requirement need to consider whether they are providing teachers with knowledge useful in practice. Program designers should determine whether current course and program plans formally offer opportunities to develop the types of knowledge that mathematics teachers need. This study does not suggest that conveying such knowledge in pure courses is impossible; however, its participants did not document structural attention to PCK during their pure tertiary experiences. The relevance participants attributed to PCK concepts and the frustration felt by those who did not experience a formal PCK course imply that future mathematics teachers considering teacher preparation programs should carefully consider the opportunities each provides to formally develop their PCK. Teacher educators of both pure and pedagogical content knowledge courses should remain aware of the importance of providing opportunities for teachers to delve in an advanced manner into both the material they will be teaching and the process by which students learn such content.

These findings advocate for preparation programs to analyze the content requirements of future mathematics teachers. All three participants’ programs were heavily weighted with pure courses whose content this study demonstrated lacked direct implementation in practice. Policy requires teachers to have a “deep” understanding of their subject area. The participants’ programs promoted deep understanding of tertiary mathematics. As a result, all three reported that their knowledge of mathematics was significantly increased, also that they had gained understanding of the big picture of learning mathematics through all educational levels. These acquisitions benefited teachers’ horizon knowledge and also contributed to their specialized
content knowledge. However, the significant mathematical knowledge imparted to these future teachers was not used directly in teaching. Further, it was shown that the nature of tertiary mathematical knowledge itself means that such implementation occurs with difficulty, if at all.

This study aimed to provide policy makers with specificity regarding the types of knowledge that teachers need. Its findings suggest that policy makers should consider carefully the nature of the mathematical content required of future mathematics teachers. Participants’ experiences reflected a lack of understanding of the majority of their pure courses’ relevance, purpose, and application in practice. All three participants acknowledged their tertiary experiences’ positive impact on their mathematical knowledge; all three also questioned whether their required courses contributed adequately to the knowledge base needed in teaching practice. Thus, while tertiary level content indirectly benefited teachers, more research is needed to determine whether current requirements can be modified to make teachers’ AMK learning more relevant to their future practice. Teachers benefited from advanced pure knowledge, but they also documented a need for “relational knowledge” (Kinach, 2002).

The lack of AMK concepts’ direct usefulness that this study found supports the need for “re-thinking” the design of pure content courses for prospective mathematics teachers (Ball et al., 2008; CBMS, 2001; Moreira & David, 2008; Schmidt et al., 2007). Teachers and designers of pure content courses for prospective teachers should consider the teacher-educator’s perspective, and identify opportunities to develop “relational” knowledge (Kinach, 2002). Educators teaching pure content courses need to understand the complexity of the knowledge that future teachers must call upon (Rowland, 2012). Van Brommell (2012) found that use of Ball et al.’s (2008) framework to focus a mathematics teacher training course increased the learning of the future teachers and improved the teaching of their teacher educators. The
framework was implemented in a PCK course; further research is required to determine its usefulness in pure content courses.

This study substantiates that developing mathematical knowledge for teaching is a complex task. Just one participant’s tertiary education mandated a pure course that required him to synthesize his learning, an experience documented to have a profound effect on his overall understanding of mathematics and his habits in teaching practice. Additional research regarding optimally beneficial tertiary knowledge and creation of formal opportunities for future teachers to understand the bigger picture of their program’s requirements are needed.

Future research should focus on the contribution each tertiary course makes to the development of teachers’ knowledge. Studies need to determine whether redesigning current pure courses or exchanging them for different pure courses focused on professionally relevant concepts would be more beneficial. One possibility is integrating into pure courses pedagogically related mathematical tasks focusing on the concepts to be taught (G. J. Stylianides & Stylianides, 2010), rather than concentrating exclusively on tertiary content.

This study supported the notion that pure knowledge does not simply segue into knowledge needed for practice (Ball & Bass, 2000). Participants’ descriptions of their experiences should help designers of mathematics teacher preparation curricula gauge the fit between current programs and the needs of future teachers. The study also demonstrated how tertiary content’s distinct nature significantly restricted its implementation in middle school mathematics. Concurrent use of AMT to interpret this phenomenon identified processes characteristic of each level of mathematics. Making AMT theory a prescribed component of the curriculum for future mathematics teachers will assist their formal understanding of the theory, their own learning, and that of their future students.
Implications for future and current teachers. The study’s results can help future teachers determine whether prospective preparation programs present learning opportunities relevant to professional practice. The thick descriptions and discussion should assist their understanding of the sorts of experience they will likely encounter, and whether and how the knowledge that experience imparts will be implemented in practice. Such information will help them acquire objectivity regarding their own development of knowledge. This study highlighted the difference between tertiary pure content and its use in developing teachers’ specialized content knowledge and horizon knowledge; future teachers will enter their experience empowered with such knowledge rather than with the frustration expressed by two participants. In general, future teachers should look for preparation programs that include experiences designed to develop their relational knowledge, pedagogical content knowledge courses, and structured opportunities to synthesize their knowledge.

Developers of professional development programs for practicing teachers should utilize this study’s findings to understand their probable needs. Current teachers have fulfilled their certification requirements and most likely have ample content knowledge. They may not have had adequate opportunity to develop formal mathematical knowledge for teaching, or to consider in depth the content of the level they are teaching. This study advocates professional development maximizing teachers’ relational knowledge of the content taught.

Teachers whose tertiary education lacked pedagogical content knowledge courses had to begin developing that knowledge early in their teaching careers. This study’s teachers found the knowledge such courses imparted particularly relevant to their practice and wished that their environments offered more ways to develop it. This study recommends creating opportunities
for practicing teachers to collaboratively discuss students’ common misconceptions and
developing insights, effective instructional methods, and optimal use of available resources.

**Future research opportunities.** The themes identified in this study as well as its design warrant future research. Coupling AMK’s indirect influence in practice with its lack of direct implementation suggests that better understanding of tertiary mathematics courses’ relationship to mathematical knowledge for teaching would be advantageous. A formal mapping of tertiary courses’ knowledge to prerequisite knowledge as well as to knowledge relevant to the practice of teaching would be beneficial. Moriera and David (2008) advocated “integration” of concepts relevant to teaching into advanced mathematics courses, but cautioned that accomplishing this task might not be easy. Beginning their research effort by considering tertiary mathematics from the perspective of the knowledge practicing teachers need, they found examples where the two sets of knowledge conflicted. Moreira and David (2008) advocated extending such study to more tertiary concepts.

This study’s findings echo this need from the perspective of practicing teachers. To make tertiary mathematics more professionally relevant requires better understanding of how its concepts support and clash with the knowledge teachers require. Two participants felt frustrated with the apparent irrelevance of the content they were required to learn, while the third experienced a course that helped him begin to understand that knowledge’s importance; this study made clear that closer examination of program and course design is warranted. Consideration of tertiary topics from the perspective of knowledge that mathematics teachers need would engender opportunities to capitalize on mutually supporting elements and explicate instances of conflict. Mathematics teachers would still be required to delve deeply into tertiary content to experience the learning process it entails; however, a careful mapping would identify
the content most relevant. Study participants stressed their sense of connectedness to specific tertiary courses; scrutiny of these and other courses inspiring similar responses would benefit mathematics teacher preparation designers, teacher educators and future mathematics teachers.

This IPA study’s very design compels further research; its number of participants was deliberately limited so that a thick detailed analysis could illuminate AMK’s implementation in teaching. Ball et al.’s (2008) framework provided a means to focus in on AMK’s impact on the practice of teaching from these teacher-participants’ perspectives. The domains identified in that framework organized the data collected and enabled the themes generated to focus on the very specific knowledge utilized by mathematics teachers. The conversation these findings begin addresses the need to clarify AMK’s role in developing mathematical knowledge for teaching. A quantitative study that surveys a larger sample, gauging how representative the themes identified are for mathematics teachers, would supplement this study’s findings.

Limitations

Influenced by idiography, IPA studies enable examination of particular participants’ experiences in sufficient detail and depth to decipher the meanings experienced by those unique individuals in their particular context (J. A. Smith et al., 2009). The lived experiences of teachers provided insights into AMK’s contribution to the practice of teaching. Restricting the number of study participants enabled rich data collection; its in-depth analysis and interpretation generated two salient themes. The study’s participants deliberately included only teachers who had completed an undergraduate mathematics major as part of their preparation program. All three taught at the same school with the same curriculum. The small number was requisite for generating the study’s depth of data and analysis. Many verbatim extracts validated the themes generated and interpretations made. Study findings provide specificity and detail to AMK’s
otherwise unclear impact on the practice of teaching. However, while its design facilitated these findings, it may also limit their transferability. Readers of this study’s rich descriptions must judge its findings’ applicability to their own purposes and context.

According to Ball et al. (2001), “studying teachers, although valuable, is incomplete because results deduce teacher’s mathematical demands from teachers’ accounts of what they think or would do” (p.449). Ball et al. (2001) cautioned against studying teachers’ knowledge and reflections without knowing their effectiveness, a warning that should be considered when evaluating this study’s findings. Still, this in-depth study explored real experiences of practicing teachers, and that process instigated more research. Themes common to all three participants yielded insight regarding AMK’s contribution to the participants’ everyday practice of teaching. The study’s teachers were not questioned as to their students’ achievement or their own effectiveness; they were asked to reflect on how their AMK experience helped to contribute to the knowledge they utilized in practice. Thus, their effectiveness relates minimally, if at all, to the phenomenon studied.

In at least two ways, participants’ years of teaching affected this study based on recollection. The more recent the completion of the teacher’s AMK, the greater their ability to illustrate their AMK experience precisely; the longer ago that tertiary education, the easier participants found it to identify which of their AMK experiences had “stuck” with them, contributing in describable ways to their present teaching. Data highly dependent on participants’ recollections require a teacher’s years of experience to be taken into account; however, this study was designed to limit this factor’s effect. The list provided to all participants of the tertiary courses that their undergraduate program included proved helpful in eliciting memories regardless of the passage of time since.
This study also found that, despite its use of Ball et al.’s (2008) domains to delineate the type of knowledge studied, the knowledge described by the teachers overlapped. This confluence was not unexpected: the authors of Ball et al.’s (2008) framework indicated that domains of knowledge have soft boundaries, that knowledge from one may overlap with, and connect to, understanding from another. The researcher embraced this idea while analyzing the data, letting the participants’ experiences speak for themselves regarding the type of knowledge described. In some instances, teachers’ descriptions demonstrated multiple types of knowledge simultaneously. This conceptual framework enabled the researcher to assign a description to the domain(s) most appropriate, while domain overlapping attested to the complexity of the data studied and interpretations made.

Though Ball et al.’s (2008) framework enabled this study’s specificity, its originators cautioned against considering it fixed and static. Current definitions for their domains provided the basis for this study’s findings, which may be affected by future research’s refinement of those definitions. The thick description and interpretation of study participants’ experiences enabled by those present definitions will persist regardless of shifts in domain representations, however.

Finally, it is important to consider the researcher’s personal bias. Chapter 1’s Positionality Statement acknowledged biases present at the study’s start. The researcher is a former middle school mathematics teacher who experienced the phenomenon described in this study. Open to multiple realities, however, the researcher made clear that participants’ experiences would be collected, analyzed, described, and interpreted from their unique perspectives. Consistent with study design, the researcher’s expertise in the area studied positively affected the depth of data interpretation; interpreting meanings generated from people
who have experienced a phenomenon, IPA studies foster that very occurrence. Thus, the researcher’s experiences can be viewed as supporting the richness of study findings.

**Conclusion**

This study aimed to understand tertiary mathematics’ implementation, directly and pedagogically, in the practice of teaching middle school mathematics. Though researchers have made significant strides in defining, specifying, and understanding the kinds of knowledge mathematics teachers need, tertiary mathematics’ role in the development of such knowledge remains unclear (Moreira & David, 2008; Zazkis & Leikin, 2010). This study’s in-depth analysis of practicing teachers who lived that process furthers our understanding of the effects of tertiary mathematics on their knowledge base for practice. Policy makers and mathematics teacher preparation program designers can employ the information, direction, and insight it provides to identify and emphasize educational experiences that will equip future mathematics teachers with professionally relevant knowledge.

The qualitative methodology selected for this study enabled examination of this phenomenon in all its complexity. IPA permitted thorough analysis of participants’ lived experiences, the meanings they brought with them, and the common themes that emerged. Ball et al.’s (2008) mathematical knowledge for teaching framework organized the data collection, allowing specificity and structure to emerge. Their domains assisted in the investigation of tertiary knowledge for its contribution to teachers’ *specialized content knowledge* and *pedagogical content knowledge*, that is, to their *knowledge of content and students* and their *knowledge of content and teaching* (Ball et al., 2008). The first of these refers to the pure mathematical knowledge, habits, skills, and insights teachers utilize to teach. The second refers to their knowledge *for* teaching, to their understanding of the content to be taught in the context
of students’ learning and misconceptions, available resources and instructional strategies, i.e., to knowledge that they actually use.

Study participants’ experiences made clear that tertiary knowledge contributed indirectly to teachers’ specialized content knowledge. The totality of their experience was found to significantly influence the habits of mind and insights utilized in practice. It also contributed to their horizon knowledge, that is, their understanding of the continuum of learning mathematics. However, tertiary content’s direct impact on the practice of teaching middle school mathematics was minimal at most, its contribution to development of teachers’ pedagogical content knowledge, knowledge for teaching, slight. By striking contrast, the few pedagogical content courses these participants had been able to take were found to significantly contribute to knowledge for teaching, with their content directly relevant to their current practice.

This study’s findings and existing research regarding tertiary mathematics’ uncertain relevance to, and implementation in, the practice of teaching middle school mathematics were considered in tandem to determine this research’s contribution to our understanding of its impact. Synthesizing study findings with that literature suggests a pressing need for policy makers to carefully consider the type of knowledge required of future mathematics teachers. While it is generally agreed that mathematics teachers must have a “deep” understanding of mathematics, the complexity of mathematics itself warrants requirement specificity. Advanced Mathematical Thinking theory having revealed the complexity of mathematics at all learning levels, that theory’s formal introduction into future teachers’ knowledge was considered advantageous. Teachers’ need for experiences that translate pure knowledge into relational knowledge was emphasized. This study’s findings help policy makers, mathematics teacher preparation program
and course designers, future teachers, and current teachers evaluate current preparation programs, focusing on the kinds of experiences they offer, foster, and prioritize.

By deciphering how a few individuals made sense of their unique experiences, this study contributes to the qualitative literature regarding tertiary mathematics’ implementation, and pushes understanding of how tertiary mathematics contributes to the knowledge that mathematics teachers need a step farther. It is the researcher’s hope that this examination of the lived experience of practicing teachers reveals this issue’s intricacies, sheds light on the types of experiences mathematics teachers need, and instigates future research aimed at improving tertiary mathematics’ impact on knowledge relevant to professional teachers.
References


teachers. *Journal of Educational Psychology, 100*(3), 716-725. doi: 10.1037/0022-0663.100.3.716


Appendix A: Informed Consent Form

Informed Consent Form

Northeastern University
College of Professional Studies

Name of Investigator(s):
Ronald Brown, Principal Investigator
Kerry Wiley, Student Researcher

Title: Middle School Mathematics Teachers’ Use of Advanced Mathematical Knowledge in Practice: An Interpretative Phenomenological Analysis.

Informed Consent to Participate in a Research Study: Interview

We are inviting you to take part in a research study. This form will tell you about the study, but the researcher will explain it to you first. You may ask this person any questions that you have. When you are ready to make a decision, you may tell the researcher if you want to participate or not. You do not have to participate if you do not want to. If you decide to participate, the researcher will ask you to sign this statement and have provided you a copy to keep.

Why am I being asked to take part in this research study?

We are asking you to be in this study because a middle school mathematics teacher who took part in a traditional teacher preparation program and took mathematics classes as part of that program. Your unique experiences can provide insight into the current project.

Why is this research study being done?

The purpose of this study is to better understand how the mathematics learned in college is used while teaching middle school mathematics. This study will investigate if the mathematics learned is used directly as well as if the mathematics helps to develop knowledge in order to teach middle school mathematics.

What will I be asked to do?

If you decide to take part in this study, we will ask you to discuss your perceptions in two separate interviews. These interviews will be digitally recorded. You will also be asked to provide a list of the courses taken toward your undergraduate degree. The list will be used to understand the type of courses taken. You will also be asked to review a summary in order to make sure your views are accurately described.

Where will this take place and how much of my time will it take?
You will be interviewed twice at a time and place that is convenient for you. If at all possible, the researcher would like the interview to take place in your own classroom. Each interview will take about one hour to ninety minutes.

You will also need to obtain a list of the courses you took for your undergraduate degree. If you provide Kerry Wiley with the name of the institution you graduated from, she can print out a current list of courses required for the major to help you. This process could take up to 20 minutes to verify your courses.

**Will there be any risk or discomfort to me?**

There is no foreseeable risk or discomfort.

**Will I benefit by being in this research?**

There will be no direct benefit to you for taking part in the study. However, the information learned from this study may help us better understand how the mathematics learned in college is used in teaching middle school mathematics. This has the potential to instigate further research and inform policy.

**Who will see the information about me?**

Your part in this study will be confidential. Only the researchers in this study will see the information about you. Your verbatim statements may be used in the final report of this study but no reports or publications will use information that can identify you in any way or any individual as being of this project. You will be given a pseudonym as will your organization in any reports or publications.

Recordings of the interview will be transcribed by a third party transcription company that has confidentiality processes. Only the researchers will have access to the recordings. The recordings and transcriptions will be stored in a password protected computer. The transcriptions will be backed up on a password protected online database.

In rare instances, authorized people may request to see research information about you and other people in this study. This is done only to be sure that the research is done properly. We would only permit people who are authorized by organizations such as the Northeastern University Institutional Review Board to see this information.

**If I do not want to take part in the study, what choices do I have?**

You have the option to choose not to participate.

**What will happen if I suffer any harm from this research?**

No special arrangements will be made for compensation or for payment for treatment solely because of my participation in this research.
Can I stop my participation in this study?

Your participation in this research is completely voluntary. You do not have to participate if you do not want to and you can refuse to answer any question. Even if you begin the study, you may quit at any time. If you do not participate or if you decide to quit, you will not lose any rights, benefits, or services that you would otherwise have.

Who can I contact if I have questions or problems?

If you have any questions about this study, please feel free to contact Kerry Wiley, Kavanagh.k@husky.neu.edu, the person mainly responsible for the research. You can also contact Ronald Brown, Ron.Brown1@neu.edu, the Principal Investigator.

Who can I contact about my rights as a participant?

If you have any questions about your rights in this research, you may contact Nan C. Regina, Director, Human Subject Research Protection, 960 Renaissance Park, Northeastern University, Boston, MA 02115. Tel: 617.373.4588, Email: n.regina@neu.edu. You may call anonymously if you wish.

Will I be paid for my participation?

No.

Will it cost me anything to participate?

No.

Include any pertinent information that may not be stated elsewhere.

You must be at least 18 years old to participate. You must have at least a bachelor’s degree.

You will be asked to verbally affirm your consent to take part in this research at the beginning of the interview.

__________________________________________  ______________________
Signature of person agreeing to take part  Date

__________________________________________
Printed name of person above

__________________________________________  ______________________
Signature of person who explained the study to the participant above and obtained consent  Date

__________________________________________
Printed name of person above
Appendix B: Interview Protocol

Interview 1: Prior to interview:

The teachers will be asked to provide a list of courses taken as part of their undergraduate degree. The list will be utilized to determine if the teacher had pure and pedagogical content courses as part of their preparation program. They will also be utilized during the interview as a reference for the teachers to recall specific experiences and examples. The teachers will be given the definition of the difference between pure and pedagogical content courses to ensure they have a clear understanding of the two. The term “advanced mathematical knowledge” (AMK) will also be defined and explained.

Interview Questions: These are the basic questions to be asked, the participants would be given prompts such as “can you tell me a bit more about that,” as needed as the interview is conducted.

- In general, to what extent are you using AMK in your teaching (Zazkis & Leikin, 2010, p. 265)?

- Tell me about the way(s) AMK topics explicitly arise in the content that you teach?
  - Can you describe an example?
  - Can you think of a specific course you took in which the AMK provided in that course is utilized explicitly in the content you teach?

- Tell me about the way(s) AMK topics implicitly arise in the content that you teach?
  - Can you describe an example?
  - Can you think of content from a specific course you took in which the AMK provided in that course is utilized implicitly in the content you teach?

- Can you tell me about how AMK has helped you to “unpack” mathematics for your students?
  - Can you describe an example?

- How do you think AMK helped you to understand the process of your students’ learning?

- How do you think AMK helps you in your choices to present mathematical ideas?
  - Can you describe an example?

- You took ________(only pure/ a combination of pure and pedagogical content) courses. Tell me about ways the mathematics presented in these courses aids in your teaching.
  - Tell me about how you see the content learned in the pure content courses being utilized in your teaching? Can you describe an example?
  - Tell me about how you see the content learned in the pedagogical content courses being utilized in your teaching? Can you describe an example?
Interview 2 Protocol

- How do you feel your AMK experience shaped who you are as a middle school math teacher?

- In your experience, what is it like to be a middle school mathematics teacher that majored in mathematics?

- How do you feel the math that you learned in college affected your overall view and understanding of mathematics?

- How do you feel your experiences with your instructors influenced you and how you teach?

- How would you compare the math that you teach on a daily basis to the math that you learned in college? (SCK)

- How does the math that you learned in college support your understanding of the middle school math curriculum? (SCK)

- Where would you say your knowledge “to anticipate what students are likely to think and they will find confusing” (Ball, 2008, p.401) comes from? In your experience, what is AMK’s role in that development? (KCS)
  - How about analyzing student errors? (KCS)

- What knowledge do you draw from when selecting between different instructional strategies that might be used to teach a certain topic? In your experience, what is AMK’s role in the development of that knowledge? (KCT)

- How do your AMK experiences aid you in making decisions about the depth of the content you teach? (KCT)

- Math teachers use their knowledge to break concepts down. They have knowledge of different representations for content and insight into the beginnings of student misconceptions/understandings. Where do you feel that knowledge comes from? What role do you feel your AMK experience play in developing that knowledge?