I. INTRODUCTION

There has been considerable interest in precision measurements\(^1\,^2\,^3\,^4\,^5\,^6\,^7\) of the muon anomalous magnetic moment factor
\[ g = 2(1 + \kappa). \] (1)

The anomaly factor is measured from the chiral asymmetry rotation frequency
\[ \omega_\kappa = \kappa \left( \frac{|eB|}{Mc} \right). \] (2)

Largely due to experimental improvements in creating an extremely homogeneous magnetic field \(B\), there have been recent improvements\(^8\,^9\) in the accuracy of \(\kappa\) determinations.

In addition to the anomaly frequency of Eq.\,(2), there are three other frequency scales of interest in the experiments. The cyclotron frequency is given by
\[ \omega_c = \left( \frac{c|e|B}{\mathcal{E}} \right) = \left( \frac{|eB|}{\gamma Mc} \right), \] (3)

which is also determined by the homogeneous magnetic field. Due to electrostatic focusing capacitors, there are two more frequencies having to do with collective oscillations of the beam as a whole. These are the vertical beta-tune and horizontal tune. These are the vertical beta-tune and horizontal tune.

 Previous theoretical considerations\(^8\,^9\) lead to a horizontal tune of
\[ \left( \frac{\Omega_\perp}{\omega_c} \right)_{\text{expected}} = \sqrt{1 - n} \approx 0.929. \] (6)

The horizontal tune has been measured by the muon decay oscillations themselves. (See \(^6\) and \(^7\) wherein the notation \(\omega_b\) is used for \(\Omega_\perp\))
\[ \left( \frac{\Omega_\perp}{\omega_c} \right)_{\text{experiment}} \approx 0.0702. \] (7)

The discrepancy between the previous theoretical Eq.(6) and the experimental Eq.(7) is evident.

One purpose of our work is to show that the previous theory is at fault. The index \(n\) describes the electric field gradient inside the electrostatic focusing capacitors; It is
\[ n = \left( \frac{\rho_0}{|B|} \right) \left| \left( \frac{c}{|\nabla\gamma M\mathbf{E}|} \right) \right|, \] (8)

where \(\rho_0 \approx 711.2\) cm is the radius of the cyclotron, \(\mathbf{v} = (d\mathbf{r}/dt)\) is the muon velocity, \(E_{||} = (\mathbf{E} \cdot \mathbf{B}/|\mathbf{B}|)\) is the component of the electric field parallel to the magnetic field and \(r_{||} = (\mathbf{r} \cdot \mathbf{B}/|\mathbf{B}|)\). The averaging is over a cyclotron orbit.

Some insight into the error of the previous theoretical Eq.(6) becomes evident from the fact that when the electric field vanishes, \(\lim_{n \to 0}(\Omega_\perp/\omega_c)_{\text{expected}} = 1\), the horizontal tune is unity. Such “unit horizontal tunes” are in reality not collective modes at all. A charged particle in a uniform magnetic field (and zero electric field) moves in a circle with a fixed center. If the center of circular orbit is not quite equal to the center of the cyclotron machine (which is always the case), then the above corresponds to what has been called a unit horizontal tune.

However, the true nature of the horizontal betatron oscillation is that the center of the cyclotron orbit also rotates if and only if a non-zero electric field is applied. A central result of this work is that the center of the cyclotron orbit also moves in a circle at a horizontal tune of
\[ \left( \frac{\Omega_\perp}{\omega_c} \right) = \frac{1}{2} \left( \frac{\Omega_\perp}{\omega_c} \right)^2 = \left( \frac{n}{2} \right) \approx 0.0685. \] (9)

Our theoretical Eq.(9) is in satisfactory agreement with the experimental Eq.(7).

In order to understand the nature of the horizontal betatron tune, consider a toy “hula hoop” popular with
children near the middle of the last century. Pretend there are particles rotating within (and relative to) the hoop with angular velocity \( \omega_c \). Further imagine the hoop rotating as a whole “off center” with an angular velocity \( \Omega \). Now remove the hoop since it is imaginary anyway. The remaining particle motions constitute a beam with a horizontal tune.

In Sec. II, the dynamics of the muon motion will be discussed starting from the Hamiltonian

\[
H = \sqrt{c^2(p - (e/c)A)^2 + c^4M^2 + e\Phi},
\]

where the magnetostatic and electrostatic fields are given, respectively, by

\[
B = \text{curl} A \text{ and } E = -\text{grad} \Phi. \tag{11}
\]

For simplicity of presentation, we shall employ quantum mechanics for describing the beam dynamics but we shall not employ (for now) the spin dynamics which must be treated with a Dirac Hamiltonian and requires somewhat long and delicate spinor algebra\[10, 11\]. The full quantum Dirac spinor algebra will be discussed elsewhere and should serve as a more accurate quantum replacement for the classical BMT equations\[12\].

In Sec. III, the coherent betatron frequencies will be derived from the Hamiltonian Eq.(10). A physical description of the coherent betatron oscillations will be provided. In the concluding Sec. IV, the consequences for the \((g - 2)\) analyses due to the corrected horizontal tune theory will be explored.

II. QUANTUM BEAM DYNAMICS

The Hamiltonian in Eq.(10) may be considered to be an operator by employing the canonical commutation relations

\[
[p_i, r_j] = -i\hbar \delta_{ij}. \tag{12}
\]

We also consider a coordinate system in the laboratory frame in which the uniform magnetic field vector is along the \( z = r_{||} \) axis; i.e.

\[
A = (A_x, A_y, 0) \text{ and } B = (0, 0, B). \tag{13}
\]

From Eqs.(11)-(13) it follows that

\[
[p_x - \frac{e}{c}A_x, p_y - \frac{e}{c}A_y] = \left( \frac{i\hbar B}{c} \right). \tag{14}
\]

The cyclotron radius vector in the horizontal plane

\[
\rho = \rho_x, \rho_y, 0
\]

points from the cyclotron orbit center to the charged particle position. In mathematical terms

\[
\rho = \left( \frac{c}{eB} \right) B \times \left( p - \frac{e}{c}A \right) \tag{16}
\]

The components of the cyclotron radius vector in Eq.(15) do not commute. From Eqs.(14)-(16)

\[
[r_x, r_y] = \left( \frac{i\hbar c}{eB} \right). \tag{17}
\]

The vector giving the position in the horizontal plane of the cyclotron orbit center

\[
R = (R_x, R_y, 0) \tag{18}
\]

is defined so that

\[
r = r_{||} + r_{\perp} \text{ and } r_{\perp} = \rho + R. \tag{19}
\]

The components of the cyclotron orbit center position also do not commute; i.e.

\[
[R_x, R_y] = -\left( \frac{i\hbar e}{eB} \right). \tag{20}
\]

We note en passant as an example of a noncommutative geometry\[13\] that the center coordinates \( R \) of the cyclotron orbit live in the noncommutative geometric plane in virtue of Eq.(20). Finally, we have the vanishing commutators:

\[
[r, R] = 0, \ [r, z] = [R, z] = 0, \tag{21}
\]

and

\[
[r, p_z] = [R, p_z] = 0. \tag{22}
\]

FIG. 1: The electrostatic focusing field pushes on a particle in a cyclotron orbit of radius vector \( \rho \). The force causes the instantaneous center of the cyclotron orbit to rotate at angular velocity \( \Omega \) with a radius \( R \). As schematically shown above, the resulting wobble in the particle orbit represents the horizontal coherent betatron oscillation.
The Hamiltonian of Eq.(10) now reads
\[ H = \sqrt{c^2p_z^2 + (eB)^2\rho^2 + c^4M^2} + e\Phi(\rho + R, z). \]  
(23)

To calculate the rate at which \( R \) changes with time, one may use
\[ \dot{R} = \frac{i}{\hbar} [H, R] = \left( \frac{ie}{\hbar} \right) [\Phi(\rho + R, z), R]. \]  
(24)

Thus, the center of the cyclotron orbit drifts with a velocity which depends only on the electromagnetic fields acting on the particle; i.e.
\[ V = \dot{R} = e \left( \frac{E \times B}{B^2} \right). \]  
(25)

The method of deriving Eq.(25) using quantum mechanics is due to Schwinger[14].

In the direction parallel to the magnetic field,
\[ v_z = \frac{i}{\hbar} [H, z] = \left( \frac{e^2p_z}{\mathcal{E}} \right), \]  
(26)

where
\[ \mathcal{E} = \sqrt{c^2p_z^2 + (eB)^2\rho^2 + c^4M^2} = \gamma Mc^2. \]  
(27)

Furthermore,
\[ \dot{p}_z = \frac{i}{\hbar} [H, p_z] = \left( \frac{ie}{\hbar} \right) [\Phi(\rho + R, z), p_z]. \]  
(28)

Eqs.(27) and (28) imply
\[ M \frac{d(\gamma v_z)}{dt} = eE_z. \]  
(29)

If the Hamiltonian in Eqs.(10) and (23) were treated using classical mechanics, then the operator equations of motion would turn into equations for particle orbits. There is sometimes simplicity in orbital pictures. Note that the classical orbital equations are recovered by replacing commutators with Poisson brackets:
\[ \text{(quantum)} \quad \frac{i}{\hbar} [a, b] \rightarrow \text{(classical)} \quad \{ a, b \} \]  
(30)

where
\[ \{ a, b \} = \left( \frac{\partial a}{\partial \rho} \right) \cdot \left( \frac{\partial b}{\partial \rho} \right) - \left( \frac{\partial b}{\partial \rho} \right) \cdot \left( \frac{\partial a}{\partial \rho} \right). \]  
(31)

In the classical orbit view wherein \( \dot{\gamma} = \{ H, \gamma \} \), one finds
\[ Mc^2 \left( \frac{d\gamma}{dt} \right) = ev \cdot E. \]  
(32)

The quantum length scale for which the classical orbit picture breaks down can be deduced from the noncommutative geometry of the cyclotron center coordinate \( R \). From Eq.(20), the quantum uncertainties obey
\[ \Delta R_x \Delta R_y \geq (L^2/2), \]  
(33)

where
\[ L = \sqrt{(\hbar c/|eB|)}. \]  
(34)

For the muon experiments of interest \( L \approx 2.0 \times 10^{-6} \) cm, which allows for a classical orbit picture of high accuracy. Even in the classical case, the decomposition \( R_\perp = \rho + R \) of the position in the plane is crucial for understanding the horizontal betatron oscillation mode.

### III. COHERENT BETATRON OSCILLATIONS

To compute the betatron vertical and horizontal tunes, one may expand the electric field \( E(\rho + R, z) \) in powers of the small displacements \( z \) and \( R \). Inside the electrostatic focusing capacitors
\[ d\mathbf{E} = \left( \frac{\partial E_z}{\partial z} \right) + \left( \frac{\partial}{\partial R} \cdot \mathbf{E}_\perp \right) = 0. \]  
(35)

Defining
\[ Q = -e \left( \frac{\partial E_z}{\partial z} \right) = e \left( \frac{\partial}{\partial R} \cdot \mathbf{E}_\perp \right), \]  
(36)

and taking only first powers of the displacement \( z \) in Eq.(29) yields
\[ M \gamma \left( \frac{d}{dt} \right)^2 z = -Qz. \]  
(37)

To this accuracy, the vertical betatron oscillation frequency is given by
\[ M \gamma \Omega_\gamma^2 = Q \]  
(38)

in agreement with Eqs.(4) and (8).

Expanding the electric field in Eq.(25) to linear order in \( R \) yields
\[ \left( \frac{d\mathbf{R}}{dt} \right) = \left( \frac{eQ}{2eB^2} \right) (\mathbf{R} \times \mathbf{B}) \]  
(39)

where Eq.(36) has been invoked. In the \( z = 0 \) plane, the vector \( \mathbf{R} \) rotates with angular velocity whose magnitude is given by
\[ \Omega_\perp = \left| \frac{eQ}{2eB} \right|, \]  
(40)

in agreement with Eqs.(8) and (9); i.e. from Eqs.(3), (38) and (40) it follows that
\[ \Omega_\perp^2 = 2\Omega_\perp \omega_c. \]  
(41)

The horizontal oscillation is depicted in FIG. 1. If a particle is injected into the cyclotron at a somewhat skewed angle, then the center of the cyclotron orbit will be somewhat removed from the center of the cyclotron machine. The instantaneous center of the cyclotron orbit
will itself move in a small circle at an angular velocity $\Omega_\perp$. The small circular rotations of $\mathbf{R}$ induce a \textit{wobble} in the particle’s (almost) circular orbit, at the same frequency $\Omega_\perp$. If a particle is injected into the cyclotron with a small velocity component $v_z$ normal to the plane, then the height $z$ of the particle undergoes oscillations with frequency $\Omega_\perp$. Since, in virtue of Eq.\,(36), there is but one electric field gradient parameter $Q$, the two betatron frequencies are not independent. They are related by Eq.\,(41).

IV. CONCLUSION

On a \textit{phenomenological} basis, the chiral asymmetry of the muon decay is fit to experimental\,[7] data by a function of the product form

$$P_{\text{tot}}(t) = P_\kappa(t)P_{\text{betatron}}(t)P_{\text{loss}}(t),$$

where the chiral rotation function is parameterized by

$$P_\kappa(t) = e^{-t/\gamma_\perp}(1 + A_\kappa \sin(\omega_\perp t + \phi_\kappa)), \quad (43)$$

the coherent horizontal betatron oscillation is parameterized by

$$P_{\text{betatron}}(t) = e^{-t^2/\tau_\perp^2}(1 + A_\kappa \cos(\Omega_\perp t + \phi_\kappa)), \quad (44)$$

and the extra loss of muons from unknown physical processes are parameterized by

$$P_{\text{loss}}(t) = 1 + n_t e^{-t/\gamma_t}. \quad (45)$$

The agreement between the experimental best fit\,[7]

$$\left(\frac{\Omega_\perp}{\omega_\kappa}\right)_{\text{experiment}} \approx 2.05, \quad (46)$$

and our theoretical prediction based on Eqs.\,(8) and \,(9)

$$\left(\frac{\Omega_\perp}{\omega_\kappa}\right)_{\text{theory}} \approx 2.01, \quad (47)$$

gives us confidence that we have properly identified the source of horizontal betatron oscillations. It is something of a mystery to us why the experiment was designed so that $\Omega_\perp$ is so very close to being a “second harmonic” of the anomaly frequency $\omega_\kappa$.

If one wishes to go beyond the phenomenological view of how the electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ affects the coupling $g = 2(1 + \kappa)$, one must employ the Dirac-Schwinger equation

$$\left\{-i\hbar \gamma^\mu d_\mu + Mc - \left(\frac{\hbar e}{4Mc^2}\right) \sigma^{\mu\nu} F_{\mu\nu}\right\} \psi = 0, \quad (48)$$

where $d_\mu = (\partial_\mu - (ie/hc)A_\mu)$. The interaction between the electromagnetic field and the anomalous magnetic moment is thereby

$$\frac{\kappa}{2} \sigma_{\mu\nu} F^{\mu\nu} = \kappa (\mathbf{B} \cdot \sigma - i\gamma_5 \mathbf{E} \cdot \sigma). \quad (49)$$

The electric field coupling of Eq.\,(49) via $\mathbf{E}(\rho + \mathbf{R}, z)$ contains those degrees of freedom ($p_z, z, p_x, \rho_y$) which enter into the kinetic energy $\mathcal{E} = Mc^2 \gamma_\perp$ in Eq.\,(27) and also the cyclotron center coordinates $(R_x, R_y)$ which rotate as in Eq.\,(39).

The experimentally observed coupling of the chiral rotation frequency $\omega_\kappa$ to the horizontal betatron frequency $\Omega_\perp$ is mainly due to the variation in $\mathbf{R}$. This coupling to $\mathbf{R}$ dominates the electric field corrections to $\omega_\kappa$ due to $\rho$.

In assessing the electric field corrections to $\omega_\kappa$\,[7], due to the variations in the radius ($\mathbf{r}_\perp = \rho + \mathbf{R}$) of the beam orbit, the effect was doubly counted: (i) The time variations in $\mathbf{r}_\perp$ were first counted in the function $P_{\text{betatron}}(t)$. (ii) The time variations in $\mathbf{r}_\perp$ were counted yet again as a BMT electric field correction. Leaving out the over counting of the electric field corrections makes the experimental need for non-standard model corrections to the muon ($g - 2$) less compelling.

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\[\text{References}\]