Electronic Enhancements in the Detection of Gravitational Waves by Metallic Antennae

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Abstract
For mechanical Weber gravitational wave antennae, it is thought that gravity waves are weakly converted into acoustic vibrations. Acoustic vibrations in metals (such as Aluminum) are experimentally known to be attenuated by the creation of electron-hole pairs described via the electronic viscosity. These final state electronic excitations give rise to gravitational wave absorption cross sections which are considerably larger (by four orders of magnitude) than those in previous theories which have not explicitly considered electronic excitations.

1 Introduction
Weber gravitational wave antennae are designed to weakly convert gravitational waves into acoustic vibrations[1, 2]. The damping of acoustic vibrations ultimately heat the detector accounting for the energy lost by the incident gravitational disturbance. The total cross section for the absorption of the gravitational wave is (of course) very small due to the weak nature of the gravitational forces. If \( g \) denotes an incident graviton which is converted into a phonon \( \phi \), then the first stage of graviton absorption is thought to involve the conversion

\[
g \rightarrow \phi.
\]  (1)

In the second stage of absorption, the phonon decays. In metallic samples (such as Aluminum) at not too high a temperature, the dominant mode of decay of
the phonon is into an electron-hole pair

\[ \phi \rightarrow e^- + e^+ . \]  

(2)

The decay in Eq.(2) is known to be true in virtue of experimental work on sound wave attenuation in metals. The final electronic excited states are described in terms of an effective “electronic viscosity”. Taking Eqs.(1) and (2) together, one finds that the gravitational wave decays into electronic particle hole pairs via the process

\[ g \rightarrow \phi \rightarrow e^- + e^+ \]  

(3)

wherein the acoustic phonon \( \phi \) enters into the absorption process as a virtual (resonant) intermediate state. Also possible is the direct conversion of the graviton into an electron hole pair

\[ g \rightarrow e^- + e^+ . \]  

(4)

At first glance, it might appear that the electron-graviton coupling would be much weaker than the phonon-graviton coupling since the mass of the vibrating nuclei is much larger than the mass of the electrons. However, as will be discussed in detail below, the graviton couples into the pressure and not directly into the energy density. The spatial (pressure) part of the space-time stress tensor \( T_{ij} = P_{ij} \) directly couples into a gravitational wave while the time (energy density) part of the space-time stress tensor \( T_{00} = \epsilon \) couples into the Newtonian part of gravity.

The electron contribution to an earthly (ordinary) condensed matter pressure tensor \( P_{ij} \) dominates the nuclei contribution. The absorption rate of gravitational disturbances are considerably enhanced by the large kinetic energy and high density of final electronic states. A similar view has been discussed in some previous work. Nevertheless, the conventional wisdom is that the nature of the acoustic wave decay rate has little to do with the total absorption rate of the gravitational wave, provided that the former is much larger than the latter. Our purpose is to exhibit the flaw in the above conventional argument by providing a counter example. The fact that the electrons dominate the coupling to both the phonons and the gravitons considerably enhances the total cross section in metallic antennae (such as Aluminum).

In Sec.2, the gravitational wave propagator will be discussed. It will be shown that the “self energy” part of the propagator (as a fourth rank tensor “stress to strain ratio”) can be viewed as a dynamical elastic tensor Young’s modulus for distorting space-time. In Sec.3, the gravitational wave will be treated as a weak curvature strain on a flat space-time background. The dynamical elastic moduli of metallic antennae will be explored. In Sec.4, the amplitude for scattering a gravitational wave off a metallic antenna will be computed in the mass quadrupole approximation. For solid metals, the final total cross section will be described in terms of the dynamical Lamé elastic coefficient \( \mu \). In Sec.5, the microscopic quantum mechanical expression for the gravitational wave total cross section will be exhibited. The mathematical derivations rely

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on the Kubo formulae (in condensed matter physics) for the transport coefficients associated with phonon damping. In Sec. 6, the dispersion relations for the dynamical elastic coefficients will be discussed. The results will be applied to the subtracted dispersion relations for the dynamic Lamé coefficient and the viscosity. The electronic viscosity enhancement of the elastic Lamé coefficient will be explicitly exhibited. In Sec. 7, the Feynman diagrams for the absorption of a graviton will be explored and the enhancement of the absorption due to electronic coupling will be computed. In the concluding Sec. 8, we present a brief summary of our results as well as directions for future work.

2 Gravitational Propagators

In relativity one considers (in general) a background curved space-time metric for the local proper time interval

$$-c^2 dt^2 = g_{\mu\nu} dx^\mu dx^\nu$$

which arises from a solution of the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \left( \frac{8\pi G}{c^4} \right) T_{\mu\nu}. \quad (6)$$

Upon varying by $\delta T_{\mu\nu}$ the “stress” applied to space-time

$$T_{\mu\nu} \rightarrow T_{\mu\nu} + \delta T_{\mu\nu}$$

one finds a response in the distortion (i.e. “strain” $(1/2)\delta g_{\mu\nu}$) of space-time by requiring a neighboring solution to Eq. (6), i.e.

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}. \quad (8)$$

The response in the strain due to the application of a stress defines the gravitational wave propagator $D_{\mu\nu\lambda\sigma}$ via

$$\delta g_{\mu\nu}(x) = \int D_{\mu\nu\lambda\sigma}(x, x') \delta T^{\lambda\sigma}(x') d\Omega' \quad (9)$$

wherein $d\Omega = \sqrt{-g} d^4 x$.

Let us decompose the stress into a source $\delta T^{\lambda\sigma}_{\text{ext}}$ external to the antenna and a stress induced in the antenna $\delta T^{\lambda\sigma}_{\text{ind}}$ by the strain $(1/2)\delta g_{\mu\nu}$. In total

$$\delta T^{\lambda\sigma} = \delta T^{\lambda\sigma}_{\text{ext}} + \delta T^{\lambda\sigma}_{\text{ind}}. \quad (10)$$

The full propagator describes the response in the strain to an “external” stress

$$\delta g_{\mu\nu}(x) = \int \Delta_{\mu\nu\lambda\sigma}(x, x') \delta T^{\lambda\sigma}_{\text{ext}}(x') d\Omega', \quad (11)$$
while the “induced” stress responds to the strain via the elastic response function

$$\delta T^{\lambda\sigma}_{\text{ind}}(x') = \int \Pi^{\lambda\alpha\beta}(x', x'') \delta g_{\alpha\beta}(x'') d\Omega''.$$  \hspace{1cm} (12)

We note in passing that Eqs. (9)–(12) imply Dyson’s Eq. (13) below; i.e.

$$\Delta_{\mu\nu\lambda\sigma}(x, x'') = D_{\mu\nu\lambda\sigma}(x, x'') + \int \int D_{\mu\nu\alpha\beta}(x, x') \Pi^{\alpha\beta\xi\eta}(x', x'') \Delta_{\xi\eta\lambda\sigma}(x'', x'''') d\Omega' d\Omega'''$$  \hspace{1cm} (13)

wherein the dynamic elastic coefficients $\Pi^{\alpha\beta\xi\eta}$ play the role of the gravitational wave propagator self energy.

Since the external stress radiates the incoming strain via the relation

$$\delta g_{\mu\nu}(x) = \int D_{\mu\nu\lambda\sigma}(x, x') \delta T^{\lambda\sigma}_{\text{ext}}(x') d\Omega',$$  \hspace{1cm} (14)

it follows from Eqs. (11), (13) and (14) that the gravitational wave scattering equation takes the form

$$\delta g_{\mu\nu}(x) = \delta g_{\mu\nu}^{(in)}(x) + \int \int D_{\mu\nu\alpha\beta}(x, x') \Pi^{\alpha\beta\lambda\sigma}(x', x'') \delta g_{\lambda\sigma}(x''') d\Omega' d\Omega'''.$$  \hspace{1cm} (15)

In Eq. (15), the total gravitational wave is the sum of an incoming gravitational wave and an “elastic” scattered gravitational wave

$$\delta g_{\mu\nu}(x) = \delta g_{\mu\nu}^{(in)}(x) + \delta g_{\mu\nu}^{(el)}(x)$$
$$\delta g_{\mu\nu}^{(el)}(x) = \int \int D_{\mu\nu\alpha\beta}(x, x') \Pi^{\alpha\beta\lambda\sigma}(x', x'') \delta g_{\lambda\sigma}(x''') d\Omega' d\Omega'''.$$  \hspace{1cm} (16)

From the imaginary part of the forward elastic scattering amplitude one may compute the total cross section for the wave to scatter off the antenna. Since the propagator $D_{\mu\nu\alpha\beta}$ contains the weak gravitational coupling strength $G$ to first order, it is sufficient to employ the lowest Born amplitude for the scattered gravitational wave

$$\delta g_{\mu\nu}^{(el,\text{Born})}(x) = \int \int D_{\mu\nu\alpha\beta}(x, x') \Pi^{\alpha\beta\lambda\sigma}(x', x'') \delta g_{\lambda\sigma}^{(in)}(x''') d\Omega' d\Omega'''.$$  \hspace{1cm} (17)

To compute the scattered wave to lowest order in $G$ one needs to know (i) the incoming gravitational wave $\delta g_{\lambda\sigma}^{(in)}$, (ii) the dynamic elastic coefficients $\Pi^{\alpha\beta\lambda\sigma}$ of the antenna and (iii) the unperturbed (by the antenna) gravitational wave propagator $D_{\mu\nu\alpha\beta}$. Let us consider this in more detail.

### 3 Gravitational Wave Antennae

Consider a gravitational wave moving through a flat space-time background in a reference frame in which an antenna is at rest. In the transverse traceless gauge
one may describe the gravitational wave by a spatial strain tensor \( u_{ij}(r, t) \) which enters into the proper time as follows

\[
c^2 d\tau^2 = c^2 dt^2 - ds^2
\]

\[
ds^2 = |dr|^2 + 2dr \cdot u \cdot dr.
\]  

The definition of “strain” in terms of the spatial metric in \( ds^2 \) conforms to the standard usage in elasticity theory\[10\]. In terms of the traceless part of the spatial pressure tensor \( p_{ij}(r, t) \) in the antenna,

\[
p = P - \frac{1}{3} (trP),
\]  

the gravitational wave equation (with a pressure source) reads

\[
\left\{ \frac{1}{c^2} \left( \frac{\partial}{\partial t} \right)^2 - \Delta \right\} u(r, t) = \left( \frac{8\pi G}{c^4} \right) p(r, t).
\]  

The general solution of Eq. \( \text{(20)} \) may be written

\[
u(r, t) = u^{(m)}(r, t) + \left( \frac{2G}{c^4} \right) \int \frac{p(r', t - R/c)}{R} d^3 r'
\]  

where \( R = |r - r'| \). The incoming gravitational wave obeys

\[
\left\{ \frac{1}{c^2} \left( \frac{\partial}{\partial t} \right)^2 - \Delta \right\} u^{(m)}(r, t) = 0.
\]  

For a dynamical elastic antenna, the linear causal response in the pressure \( p \) to a material spatial strain \( u \) is described by a non-local (in space and time) expression of the form

\[
p_{ij}(r, t) = - \int \int_{0}^{\infty} G_{ijkl}(r, r', \tilde{t}) u_{kl}(r', t - \tilde{t}) d\tilde{t} d^3 r'.
\]  

If the strain is turned on at a complex frequency \( \zeta \) with \( \Im \zeta > 0 \) and if

\[
u(r, t) = \Re \left\{ e^{-i\zeta t} u(r; \zeta) \right\},
\]  

then the pressure response

\[
p(r, t) = \Re \left\{ e^{-i\zeta t} p(r; \zeta) \right\}
\]  

is described by a frequency dependent Young’s modulus \( Y \). The explicit expression for the pressure is

\[
p_{ij}(r; \zeta) = - \int Y_{ijkl}(r, r'; \zeta) u_{kl}(r'; \zeta) d^3 r'.
\]
where the dynamical Young’s modulus obeys

$$\mathcal{Y}_{ijkl}(r, r'; \zeta) = \int_0^\infty e^{i\zeta t} G_{ijkl}(r, r', t) dt.$$  \hspace{1cm} (27)

The scattering equation follows from Eqs. (21) and (24)-(26); it is

$$u_{ij}(r; \zeta) = u_{ij}^{(in)}(r; \zeta) + u_{ij}^{(el)}(r; \zeta)$$

$$u_{ij}^{(el)}(r; \zeta) = - \frac{2G}{c^4} \int \int \left( \frac{e^{i\zeta R/c}}{R} \right) \mathcal{Y}_{ijkl}(r', r''; \zeta) u_{kl}(r''; \zeta) d^3 r' d^3 r''$$  \hspace{1cm} (28)

where $R = |r - r'|$.

Note that Eq. (28) is the flat space-time limit of Eq. (16) in thinly disguised form. Lowest order perturbation in gravitational coupling constant $G$ yields

$$u_{ij}^{(el, Born)}(r; \zeta) = - \frac{2G}{c^4} \int \int \left( \frac{e^{i\zeta R/c}}{R} \right) \mathcal{Y}_{ijkl}(r', r''; \zeta) u_{kl}^{(in)}(r''; \zeta) d^3 r' d^3 r''$$  \hspace{1cm} (29)

as the flat space-time limit of Eq. (17). Far away from the antenna, the scattered gravitational wave for real frequency obeys

$$u_{ij}^{(el, Born)}(r; \omega) \to F_{ij}(\omega) e^{i\omega r/c} r \quad \text{as} \quad r \to \infty$$  \hspace{1cm} (30)

where

$$F_{ij}(\omega) = - \left( \frac{2G}{c^4} \right) \int \int e^{-ik_f \cdot r} \mathcal{Y}_{ijkl}(r, r'; \omega + i0^+) u_{kl}^{(in)}(r'; \omega) d^3 r d^3 r'$$  \hspace{1cm} (31)

with the outgoing wave vector $k_f = (\omega r/cr) = (\omega \hat{r}/c)$. If the incoming wave has the form

$$u_{ij}^{(in)}(r; \omega) = (\epsilon_{ij})_t e^{i\epsilon_{ij} \cdot \hat{r}},$$  \hspace{1cm} (32)

then the elastic scattering amplitude

$$(f | \mathcal{F}(\omega)| i) = (\epsilon_{ij})^*_t F_{ij}(\omega)$$  \hspace{1cm} (33)

follows from Eq. (31) and (32) to be

$$(f | \mathcal{F}(\omega)| i) = - \left( \frac{2G}{c^4} \right) \int \int d^3 r d^3 r' \times$$

$$\left\{ e^{-ik_f \cdot r} (\epsilon_{ij})^*_t \mathcal{Y}_{ijkl}(r, r'; \omega + i0^+)(\epsilon_{kl})_t e^{i\epsilon_{kl} \cdot r'} \right\}.$$  \hspace{1cm} (34)

The total cross section $\sigma_i$ for a graviton (in state $i$) to be absorbed by the antenna is determined by the imaginary part of the forward scattering amplitude

$$\sigma_i(\omega) = \left( \frac{4\pi c}{\omega} \right) \Im(m(i|\mathcal{F}(\omega)|i)).$$  \hspace{1cm} (35)

Eqs. (34) and (35) constitute the central result of this section. The total cross section $\sigma$ is determined by the dynamical elastic Young’s modulus $\mathcal{Y}$ of the antenna.
4 Mass Quadrupole Antennae

If the wavelength of the incident graviton is large on the scale of the size $L$ of the antenna,

$$(\omega/c) << (1/L),$$

then one may ignore the wave vectors $k_i,f$ in the integrals of Eq.(34). This constitutes the mass quadrupole approximation. For an antenna of volume $\Omega$ and with

$$\Omega \chi_{ijkl}(\zeta; \Omega) = \int_\Omega \int_\Omega \mathcal{V}_{ijkl}(r, r'; \zeta) d^3r d^3r',$$

the mass quadrupole absorption cross section is

$$\sigma_i(\omega) \approx -\left( \frac{8\pi G \Omega}{c^3 \omega} \right) \Im \{ (\epsilon_{ij})^*_i \chi_{ijkl}(\omega + i0^+; \Omega)(\epsilon_{kl})_i \}.$$ (38)

Averaging over the initial gravitational wave polarizations (i.e. helicity),

$$\sigma(\omega) = \frac{1}{2} \sum_{i=\pm} \sigma_i(\omega),$$ (39)

and defining the transverse dynamical elastic Lamé coefficient as

$$\mu(\zeta, n; \Omega) = \frac{1}{4} \sum_{i'=\pm} \{(\epsilon_{ij})^*_i \chi_{ijkl}(\zeta; \Omega)(\epsilon_{kl})_l \}$$ (40)

yields the total cross section

$$\sigma(\omega, n; \Omega) = -\left( \frac{16\pi GM}{c^2 \omega} \right) \Im \mu(\omega + i0^+, n; \Omega),$$ (41)

where $n$ denotes a unit vector along the incident propagation direction of the gravitational wave. We note in passing that the dynamical Lamé coefficient $\mu(\zeta, n; \Omega)$ for a finite volume system may have frequency $\zeta$-poles corresponding to both transverse and longitudinal acoustic modes. The reason is that when a transverse sound wave reflects off the boundary surface, a longitudinal sound wave may be produced with finite amplitude and vice-versa.

We now presume that a large antenna can be represented as a poly crystalline “isotropic” elastic system. The absorption cross section shall nevertheless depend on the incident propagation direction of the gravitational wave if the shape of the antenna is not spherical.

The imaginary part of the Lamé coefficient may also be expressed in terms of the real part of the viscosity coefficient according to

$$-\Im \{ \mu(\omega + i0^+, n; \Omega) \} = \omega \Re \{ \eta(\omega + i0^+, n; \Omega) \}.$$ (42)

Thus, with the mass of the antenna denoted by $M = \bar{\rho} \Omega$, Eq.(41) reads

$$\sigma(\omega, n; \Omega) = \left( \frac{16\pi GM}{c^2} \right) \frac{\Re \{ \eta(\omega + i0^+, n; \Omega) \}}{\bar{\rho}}.$$ (43)
Eq. 43 is the central result of this section. The absorption cross section for a graviton is directly related to the “kinematic viscosity” \( \eta/\bar{\rho} \) whose value determines the life time of the converted transverse phonon.

## 5 Microscopic Expressions

If one writes the microscopic dynamical Young’s modulus as

\[
Y_{ijkl}(rr'; \zeta) = Y_{ijkl}(rr'; 0) - i\zeta H_{ijkl}(rr'; \zeta),
\]

(44)

then the Kubo formula for the non-local (in space) and frequency dependent viscosity response function is given by

\[
H_{ijkl}(rr'; \zeta) = \frac{1}{\hbar} \int_0^\beta \left\{ \int_0^\infty e^{i\zeta t} \langle p_{kl}(r', -i\lambda)p_{ij}(r, t) \rangle \, dt \right\} \, d\lambda,
\]

(45)

where

\[
\beta = \left( \frac{\hbar}{k_B T} \right).
\]

(46)

The viscosity is then

\[
\Omega \eta(\zeta, n; \Omega) = \frac{1}{4} \sum_{i'=\pm} \left\{ \int_\Omega \int_\Omega (\epsilon_{ij})_{i'}^* H_{ijkl}(rr'; \zeta)(\epsilon_{kl})_{i'} d^3r d^3r' \right\}
\]

(47)

From Eqs. 49, 49 and 47 it follows that the pressure integral

\[
\int P d^3r = \frac{1}{2} \int rr \text{divdiv} P d^3r.
\]

(48)

is of central importance. Eq. 48 follows from twice integrating by parts. On the other hand, we have from conservation of mass

\[
\frac{\partial \rho}{\partial t} + \text{div} J = 0
\]

(49)

and from conservation of momentum

\[
\frac{\partial J}{\partial t} + \text{div} P = 0
\]

(50)

that

\[
\frac{\partial^2 \rho}{\partial t^2} = \text{divdiv} P.
\]

(51)

From Eqs. 48 and 51 it follows that

\[
\int P(r, t) d^3r = \frac{1}{2} \left( \frac{d}{dt} \right)^2 \int rr \rho(r, t) d^3r.
\]

(52)
For the problem at hand, the mass quadrupole of the antenna may be defined as

\[ D(t) = \int_{\Omega} (3rr - r^2 \mathbf{1}) \rho(r, t) d^3r. \]  

(53)

From Eqs. (19), (52) and (53) it follows that

\[ \int_{\Omega} p(r, t) d^3r = \frac{1}{6} \dot{D}(t), \]  

(54)

while Eqs. (45) and (54) imply

\[ h_{ijkl}(\zeta, n; \Omega) = \frac{1}{\Omega} \int_{\Omega} \int_{\Omega} H_{ijkl}(rr'; \zeta) d^3r d^3r' \]

\[ = \frac{1}{36h\Omega} \int_0^\beta \left\{ \int_0^\infty e^{ict} \langle \ddot{D}_{kl}(-i\lambda) \ddot{D}_{ij}(t) \rangle dt \right\} d\lambda. \]  

(55)

Finally, from Eqs. (47) and (55) it follows that

\[ \eta(\zeta, n; \Omega) = \frac{1}{4} \sum_{i' = \pm} (\epsilon_{ij})^*_{i'} h_{ijkl}(\zeta, n; \Omega)(\epsilon_{kl})_{i'} . \]  

(56)

Eqs. (53), (55) and (56) yield a rigorously exact microscopic expression (to lowest order in $G$) for the total gravitational wave cross section in Eq. (43).

### 6 Dispersion Relations and Sum Rules

The viscosity coefficient obeys the analytic dispersion relation

\[ \eta(\zeta, n; \Omega) = - \left( \frac{2i\zeta}{\pi} \right) \int_0^\infty \frac{\Re \{ \eta(\omega, n; \Omega) \} d\omega}{\omega^2 - \zeta^2} \text{ for } \Im \zeta > 0. \]  

(57)

With a finite viscosity $\eta(0, n; \Omega)$ and shear modulus $\mu(0, n; \Omega)$, the full dynamical Lamé coefficient reads

\[ \mu(\zeta, n; \Omega) = \mu(0, n; \Omega) - i\zeta \eta(\zeta, n; \Omega). \]  

(58)

In the limit $|\zeta| \to \infty$, Eqs. (57) and (58) imply

\[ \frac{2}{\pi} \int_0^\infty \Re \{ \eta(\omega, i0^+, n; \Omega) \} d\omega = (\mu(\infty, n; \Omega) - \mu(0, n; \Omega)). \]  

(59)

The microscopic evaluation of the sum rule follows from Eqs. (55), (56) and (59) via an equal time commutator

\[ \Delta \mu = \mu(\infty, n; \Omega) - \mu(0, n; \Omega) \]

\[ = \frac{1}{2\pi} \sum_{i' = \pm} \Re \left\{ \int_0^\infty (\epsilon_{ij})^*_{i'} h_{ijkl}(\omega, i0^+, n; \Omega)(\epsilon_{kl})_{i'} d\omega \right\} \]

\[ = \frac{1}{72\pi \Omega} \Re \left\{ \langle \frac{i}{\hbar} \sum_{i' = \pm} (\epsilon_{ij})^*_{i'} \left[ \ddot{D}_{ij}, \ddot{D}_{kl} \right] \rangle (\epsilon_{kl})_{i'} \right\}. \]  

(60)
From Eq. (53) the mass quadrupole quantum operator may be written as

\[ D_{ij} = \sum_a M_a \left( 3r_{ai}r_{aj} - \delta_{ij}r_a^2 \right). \]  

(61)

wherein \( M_a \) is the mass and \( r_a \) is the position of the \( a \)th particle in the gravitational antenna. These particles include both the electrons and the nuclei in the condensed matter. If we employ the Hamiltonian \( H = K + U \) which is the sum of the kinetic energy \( K \) and the potential energy \( U \), i.e.

\[ H = \sum_a \left( \frac{p_a^2}{2M_a} \right) + U(r_a, \ldots, r_a, \ldots) \]

(62)

then the time rate of change of the quadrupole operator in Eq. (61) reads

\[ \dot{D}_{ij} = \sum_a \left\{ 3(r_{ai}p_{aj} + p_{ai}r_{aj}) - \delta_{ij}(r_a \cdot p_a + p_a \cdot r_a) \right\}. \]  

(63)

Similarly,

\[ \ddot{D}_{ij} = \sum_a \left\{ \left( \frac{6p_ap_{aj}}{M_a} \right) + 3(r_{ai}f_{aj} + r_{aj}f_{ai}) \right\} \]

\[ -2\delta_{ij} \sum_a \left\{ \left( \frac{p_a^2}{M_a} \right) + (r_a \cdot f_a) \right\}, \]

(64)

where for the case of two particle potentials

\[ \dot{p}_a \equiv f_a = -\nabla_a U = -\nabla_a \sum_{b \neq a} u_{ab}(r_{ab}). \]  

(65)

The equal time commutation relations are computed employing

\[ [p_{ai}, r_{bj}] = -i\hbar\delta_{ab}\delta_{ij}. \]  

(66)

A very tedious computation employing Eqs. (60)-(66) yields the sum rule

\[ \Delta \bar{\mu} = \frac{2}{\pi} \int_0^\infty \text{Re} \bar{\eta}(\omega + i0^+; \Omega) d\omega \]

\[ = \frac{1}{15\Omega} \left\{ 10 \langle K \rangle + 4 \left( \sum_{a<b} r_{ab}u_{ab}(r_{ab}) \right) + \left( \sum_{a<b} r_{ab}^2u_{ab}(r_{ab}) \right) \right\}, \]  

(67)

where \( \bar{\eta}(\Omega; \Omega) \) represents a spherical average over \( n \) in \( \eta(\zeta, n; \Omega) \). For the case of a model Hamiltonian for which the potentials are completely described by Coulomb’s law,

\[ U = e^2 \sum_{a<b} \frac{z_a z_b}{r_{ab}}, \]  

(68)

10
Eqs. (67) and (68) imply
\[ \Delta \bar{\mu} = \frac{2}{\pi} \int_0^\infty \text{Re}\bar{\eta}(\omega + i0^+; \Omega) d\omega = \left( \frac{10 \langle K \rangle - 2 \langle U \rangle}{15\Omega} \right). \] (69)

From the virial theorem for a collection of particles (electrons and nuclei) interacting with pure Coulomb potentials, the thermodynamic pressure obeys
\[ P = \left( \frac{2 \langle K \rangle + \langle U \rangle}{3\Omega} \right) = \left( \frac{\Omega \epsilon + \langle K \rangle}{3\Omega} \right), \] (70)
while the energy per unit volume
\[ \epsilon = \frac{\langle K \rangle + \langle U \rangle}{\Omega}. \] (71)

Eqs. (69)-(71) imply
\[ \Delta \bar{\mu} = \frac{2}{\pi} \int_0^\infty \text{Re}\bar{\eta}(\omega + i0^+; \Omega) d\omega = \left( \frac{36P - 14\epsilon}{15} \right). \] (72)

Eqs. (70)-(72) imply that
\[ \Delta \bar{\mu} = \left( \frac{14}{15} \right) \bar{\epsilon} - \left( \frac{2}{5} \right) P, \] (73)
where the kinetic energy per unit volume
\[ \bar{\epsilon} = \frac{\langle K \rangle}{\Omega}. \] (74)

For pressures of the order of one atmosphere (or less) one finds that \( \bar{\epsilon} >> P \). Thus
\[ \frac{\Delta \bar{\mu}}{\bar{\mu}} \approx \left( \frac{14\bar{\epsilon}}{15\rho v_s^2} \right). \] (75)

wherein \( \rho \) and \( v_s \) represent, respectively, the mass density and the shear sound wave velocity. Note that the dynamic Lamé coefficient \( \mu(\zeta, n; \Omega) \) contains \( \zeta \)-poles at modes determined by both the longitudinal and transverse acoustic phonons. Nevertheless, the zero frequency static Lamé coefficient \( \bar{\mu} = \rho v_s^2 \) depends only on the shear (transverse) wave sound velocity \( v_s \).

The mean kinetic energy per unit volume is dominated by the electrons. If \( n \) represents the number of electrons per unit volume, then a Thomas-Fermi uniform electron distribution estimate of the kinetic energy per unit volume of the electrons reads
\[ \bar{\epsilon} \approx \left( \frac{3(3\pi^2)^{2/3}}{10} \right) \left( \frac{\hbar^2 n^{5/3}}{m} \right). \] (76)

For example, an Aluminum atom has \( Z = 13 \) electrons and an atomic weight of \( A = 27 \) so that the number of electrons per unit volume is given by \( n = (Z\rho/AM) \). Here, \( M \) is the atomic mass unit. Thus, we find the estimate
\[ \left( \frac{\Delta \bar{\mu}}{\bar{\mu}} \right) \approx 84 \text{ (Aluminum)}. \] (77)
Figure 1: Shown above are the following processes: (i) A graviton decays into a particle hole pair \( g \rightarrow e^- + e^+ \). (ii) A phonon decays into a particle hole pair \( \phi \rightarrow e^- + e^+ \). (iii) A graviton again decays into a particle hole pair, but there is a conversion to an intermediate resonant phonon \( g \rightarrow \phi \rightarrow e^- + e^+ \). The conversion coupling strength is \( \bar{\mu} \). (iv) A graviton decays into a particle hole pair, again with a conversion to an intermediate resonant phonon \( g \rightarrow [e^- e^+] \rightarrow \phi \rightarrow e^- + e^+ \) employing an electron loop coupling. The enhanced loop conversion coupling strength is \( \Delta \bar{\mu} \).

The electronic contribution \( \Delta \bar{\mu} \) to the dynamic Lamé coefficient leads to an electronic renormalization of the conversion of a graviton into a phonon as will be discussed directly below.

7 Enhancement Factor for Graviton Absorption

The Feynman diagrams for the physical processes here considered are shown in Fig. 1. The absorption rates are controlled by the material viscosity \( \eta \) which for a metal (at reasonably low temperatures) is determined by the motions of the electrons. (i) If there is one graviton inside of a very large material object of volume \( \Omega \), then the transition rate per unit time for a low frequency graviton to be absorbed is equal to the flux per unit time per unit area times the cross section; i.e.

\[
\Gamma_g = \lim_{\omega \to 0} \left( \frac{c \bar{\sigma}(\omega)}{\Omega} \right)
\]

which, in virtue of Eq. (43), reads

\[
\Gamma_g = \left( \frac{16\pi G}{c^2} \right) \eta \quad \text{for} \quad g \rightarrow e^- + e^+ .
\]

(ii) Similarly, the absorption of a phonon is also controlled by the material transverse and bulk viscosities, respectively, \( \eta \) and \( \eta_b \). For a phonon with wave vector \( \mathbf{k} \) the transition per unit time for absorption by the material is given by

\[
\Gamma_\phi(longitudinal) = \left( \frac{|\mathbf{k}|^2}{\rho} \right) \left( \eta_b + \frac{4\eta}{3} \right) \quad \text{for} \quad \phi \rightarrow e^- + e^+ ,
\]
\[ \Gamma_{\phi(\text{transverse})} = \left( \frac{|k|^2}{\rho} \right) \eta \text{ for } \phi \rightarrow e^- + e^+. \] (80)

(iii) In gravitational wave antennae the absorption of a graviton takes place through the intermediate conversion of a resonant phonon via

\[ g \rightarrow \phi \rightarrow e^- + e^+. \] (81)

The matrix element for the conversion \( g \rightarrow \phi \) is described by the Hamiltonian

\[ H'(g, \phi) = 2\bar{\mu} \int (s : u) d^3r, \] (82)

where (in the antenna) \( s \) is the strain due to the phonon and \( u \) is the strain due to the graviton. Eq. (82) describes the graviton conversions into both longitudinal and transverse phonons. Nevertheless, only the Lamé coefficient \( \mu \) enters into the coupling. The reason is that the graviton is purely transverse and traceless no matter which kind of phonon it produces. (iv) An electronic enhancement of the graviton to resonant phonon conversion process arises from the electron loop process shown in Fig. (1); i.e. the one loop conversion process

\[ g \rightarrow [e^- e^+] \rightarrow \phi \] (83)

determines the coupling strength renormalization

\[ \bar{\mu} \rightarrow \bar{\mu} + \Delta \bar{\mu}, \] (84)

wherein \( \Delta \bar{\mu} \) has been evaluated in Eq. (75).

Since the absorption rate for gravitons is computed from the absolute value squared of the matrix element, the central point of this work is finally evident. The enhancement of the graviton to resonant phonon detection process via electronic loop viscosity is given by

\[ \mathcal{E} = \left| \frac{\bar{\mu} + \Delta \bar{\mu}}{\bar{\mu}} \right|^2. \] (85)

For the case of an Aluminum gravitational wave antenna, Eqs. (77) and (85) imply a large electronic induced detection enhancement; It is

\[ \mathcal{E} \approx 7.2 \times 10^3 \text{ (Aluminum)}. \] (86)

Such an enhancement allows for the experimental feasibility of detecting gravitational radiation from sources within our own galaxy.

8 Conclusions

In this work, we have considered in detail the dynamical response produced in a metallic Weber bar by a gravitational wave. Our treatment differs from all
previous analyses in several major aspects. First, departing from tradition, we
follow the Einstein equations so that a (transverse, traceless) gravitational wave
is directly coupled to the (transverse, traceless) part of the pressure. Secondly,
observing that in a metal (at not too high a temperature) electrons (not the
phonons) provide the bulk of the pressure. Previous estimates of the couplings
based on direct graviton to resonant phonon conversions need a major revision.
In fact, we find that the electron-hole pair loop contributions significantly renor-
malize the relevant couplings. We have computed this using a Thomas-Fermi
model with uniform electron density to estimate the average kinetic energy per
unit volume of the electrons. The enhancement factor is found in this way to be
\[ E \sim 7 \times 10^3. \] The true factor is likely to be somewhat higher since the as-
umption of uniform electron density underestimates the kinetic energy of the
localized core electrons.

Our results can be employed to explain the solution to a theoretical problem
raised by experimental observations with the room temperature resonant detectors of the Maryland and Rome groups during the SN1987A\[11, 12, 13\]. At this
time the Rome and Maryland detectors showed signals with energy of about
100 K correlated with the observation of the neutrino detectors Mont Blanc,
Kamiokande, Baksan and IMB. Using the previous cross-section, a 100 K signal
requires the total conversion into gravitational waves of about one thousand solar masses. This would be impossible for an original star mass of \( M \sim 20M_{\text{sun}} \).
However, with the renormalized detection cross section of this work, each 100 K
signal requires just one tenth of a solar mass (still high but energetically possi-
bly yielding important information on collapse mechanisms. Also the rennor-
malized cross-section yields a sensible astrophysical interpretation of the recent
result concerning experimental coincidences\[14\] between the cryogenic resonant
detectors Explorer (CERN) and Nautilus (Frascati).

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References


