Electric Field Effects and the Experimental Value of the Muon g-2 Anomaly

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The electric field corrections to the recently measured muon magnetic moment g-2 anomaly are considered from both the classical (BMT) and the quantum mechanical (Dirac) viewpoints. In both views, we prove that the electric field inducing the horizontal betatron tune does not renormalize the anomaly frequency. With this result kept in mind, the experimental muon magnetic moment anomaly is in closer agreement with standard model predictions than has been previously reported.

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There has been considerable experimental effort in accurately determining the muon magnetic moment anomalous g-factor

\[ \kappa = (g - 2)/2. \] (1)

The measurement is performed via the magnetic field induced anomaly frequency

\[ \omega_\kappa = \frac{e\kappa\mathbf{B}}{MC}. \] (2)

The increased precision of the most recent experimental determinations of \( \kappa \) is mainly due to the technical progress that has been made in producing a uniform magnetic field \( \mathbf{B} \) which is thought to yield a very accurately determined frequency \( \nu = (\omega_\kappa/2\pi) \). It has been claimed that the anomaly \( \nu \) is known to within \( \Delta \nu \sim 5 \times 10^{-2} \) Hz by signal averaging over experimental runs with finite (due to muon decay) time intervals obeying \( \Delta t \sim 7 \times 10^{-3} \) sec. Thus, there is an experimental uncertainty product \( \Delta \nu \Delta t \sim 4 \times 10^{-4} << 1 \). With \( N \sim 3 \times 10^8 \) muon decay events, the reported frequency resolution also obeys \( (\delta \nu/\nu) \sim 10^{-6} < 1/\sqrt{N} \sim 2 \times 10^{-5} \).

For a muon with kinetic energy \( E = Mc^2 \gamma \), it has been proposed from a classical spin viewpoint, that the anomaly frequency Eq.(2) has an additional electric field induced shift \( \delta \omega_\perp \) given by

\[
\frac{\delta \omega_\perp}{\omega_\kappa} = \frac{1}{\kappa(\gamma^2 - 1)} \left\langle \frac{\mathbf{v} \times \mathbf{E}}{c|\mathbf{B}|^2} \right\rangle,
\] (3)

where \( \langle \ldots \rangle \) denotes an average over the cyclotron orbits of the muon beam. To minimize the electric field effect, one attempts to choose the energy so that the term within brackets \( \{\ldots\} \) is as small in absolute magnitude as is experimentally possible.

The reported discrepancies between the standard model and experimental values of \( \kappa \) are in part due to the overestimate of experimental electric field effects and the underestimate of standard model values and errors therein. When \( \delta \omega_\perp \) is properly computed, the experimental determination of \( \kappa \) must be evaluated anew. We find that the experimental value is thereby drawn closer to the standard model value.

As we show in what follows, the contribution to the anomaly frequency \( \delta \omega_\perp \) due to the electric field \( \mathbf{E} \) (perpendicular to \( \mathbf{B} \)) vanishes. The rule \( \delta \omega_\perp = 0 \) is most easily proved employing quantum mechanics via the Dirac equation. However the result may also be understood from a classical viewpoint. Both the quantum and classical arguments will be discussed in what follows.

To evaluate Eq.(3) from a classical viewpoint, let us discuss the muon motion projected on to a plane perpendicular to the uniform magnetic field \( \mathbf{B} \); i.e. let us define the position and velocity, respectively,

\[
\mathbf{r}_\perp = \frac{\mathbf{B} \times (\mathbf{r} \times \mathbf{B})}{|\mathbf{B}|^2} \quad \text{and} \quad \mathbf{v}_\perp = \frac{\mathbf{B} \times (\mathbf{v} \times \mathbf{B})}{|\mathbf{B}|^2}.
\] (4)

Employing the Lorentz force on the muon charge

\[
M \frac{d(\gamma \mathbf{v})}{dt} = e(\mathbf{E} + (\mathbf{v} \times \mathbf{B})/c),
\] (5)

with

\[
\rho = c \frac{\mathbf{B} \times (M \gamma \mathbf{v})}{|\mathbf{B}|^2} \quad \text{and} \quad \mathbf{r}_\perp = \rho + \mathbf{R},
\] (6)

we have in virtue of Eqs.(5) and (6) that

\[
\frac{d\rho}{dt} = \frac{c}{|\mathbf{B}|^2} \left\{ \mathbf{B} \times (\mathbf{E} + (\mathbf{v} \times \mathbf{B})/c) \right\} = \left( \frac{e\mathbf{B} \times \mathbf{E}}{|\mathbf{B}|^2} \right) + \mathbf{v}_\perp
\] (7)

and in virtue of Eqs.(6) and (7) we have the desired result

\[
\frac{d\mathbf{R}}{dt} = \left( \frac{e\mathbf{E} \times \mathbf{B}}{|\mathbf{B}|^2} \right).
\] (8)

As discussed in a more leisurely fashion in previous work, the curvature length \( |ho| = (eM\gamma/|\mathbf{E}|) \) of the muon path is the usual one for experimentally determining the muon transverse momentum. The coordinate \( \mathbf{R} \) locates the center of the cyclotron orbit. The center \( \mathbf{R} \) of the cyclotron orbit can also drift if an electric field acts on the muon charge in accordance with Eq.(8).

In fact, the entire horizontal electric field correction in Eq.(3) is due to the motion of the cyclotron orbit center.
velocity in Eq.(8). The classically predicted frequency shift is exactly given by

$$\frac{\delta \omega_{\perp,k}}{\omega_k} = \frac{1}{e^2} \left\{ \left( \frac{1}{\kappa} \right) - 1 \right\} V_{\perp} \cdot \left( \frac{dR}{dt} \right) \right) \right) .$$

(9)

The velocity $V_{\perp}$ circulates with a cyclotron angular velocity $\omega_k = eB/Mc\gamma$, while the center of the orbit $R$ circulates with an angular velocity $\Omega_{\perp}$ of the horizontal tune; Experimentally [3].

$$\left( \Omega_{\perp}/\omega_c \right) \approx 7.0 \times 10^{-2} .$$

(10)

In any case, the classical time average in Eq.(9) involves the two frequencies $\langle \cos \left( (\omega_k + \Omega_{\perp})t + \Theta \right) \rangle \approx 0$ leading to a vanishing frequency shift $\delta \omega_{\perp,k}/\omega_k \approx 0$.

That the electric field components $E_{\perp}$ perpendicular to $B$ do not induce an anomaly frequency shift can be made abundantly clear by employing quantum theory via the Dirac equation for the muon. For electrostatic $E = -\nabla \Phi$ and magnetostatic $B = curl A$ fields and with the gauge invariant momentum $\Pi = -ie\nabla \Phi - (e/c) A$, the Dirac Hamiltonian is given by

$$H = \alpha \cdot \Pi + \beta Mc^2 + e\Phi + H_k ,$$

(11)

where

$$H_k = -\left( \frac{\hbar \kappa}{2Mc} \right) \left( \beta \Sigma \cdot B - i\beta \gamma_5 \Sigma \cdot E \right) .$$

(12)

The discussion of the Dirac equation will proceed in three steps: (i) We consider the case in which only the magnetic field is present [10]. For motion in a plane perpendicular to the magnetic field, the anomaly frequency in Eq.(2) arises because the operator

$$\Lambda = \left( \frac{\hbar \kappa}{2} \right) \Lambda \Rightarrow \left( \frac{\hbar \kappa}{2} \right) \beta \Sigma \cdot B \Rightarrow \left( \frac{\hbar \kappa}{2} \right) \beta \Sigma_3 .$$

(13)

is conserved. (ii) We now add the electric field components $E_{\perp}$ and prove that $\Lambda$ is again conserved for planar motions. Thus, the frequency shift $\delta \omega_{\perp,k}$ due to $E_{\perp}$ is then strictly zero. This is a rigorous quantum selection rule. (iii) When the field component $E_z$ parallel to the magnetic field is included there will be a frequency shift due to normal (out of the plane) oscillations in the muon beam. The implications of these results for the experimental determination of $\kappa$ will be discussed below.

In zero electric field and a uniform magnetic field with $A = (1/2) B \times r$, the Dirac Hamiltonian reads

$$\mathcal{H}_B(p_z) = \alpha_0 p_z + \alpha_{\perp} \cdot \Pi_{\perp} + \beta Mc^2 - (\hbar \omega_k/2) \Lambda \Rightarrow \mathcal{H}_B(p_z) - (\hbar \omega_k/2) \Lambda .$$

(14)

For motion in the plane, the operator $\Lambda = \beta \Sigma_3$ is conserved

$$[\mathcal{H}_B(p_z = 0), \Lambda] = 0 .$$

(15)

One may diagonalize the “in plane” motion according to $H_B(p_z = 0) \psi_{\pm} = E \psi_{\pm}$ and $\Lambda \psi_{\pm} = \pm \psi_{\pm}$. Thus, the in plane motion energy states obey

$$\mathcal{H}_B(p_z = 0) \psi_{\pm} = \left\{ E \mp (\hbar \omega_k/2) \right\} \psi_{\pm} .$$

(16)

**Theorem I:** For planar motions in a uniform magnetic field, the Bohr transition frequency $\omega_k = (\mathcal{E}_- - \mathcal{E}_+)/\hbar$ for chiral spin rotations is precisely the anomaly frequency of Eq.(2).

Let us now consider the theoretical case where, in addition to the uniform magnetic field $B$, there exists an in plane electric field $E_{\perp} = -\nabla \Phi_\perp(r_{\perp})$. For this case Eq.(14) is modified to read

$$\mathcal{H}_{B.E,\perp}(p_z = 0, \Lambda) = 0 .$$

(18)

Again, one may diagonalize the in plane motion according to $H_{B.E,\perp}(p_z = 0) \psi_{\perp} = \mathcal{E} \psi_{\pm}$ and $\Lambda \psi_{\pm} = \pm \psi_{\pm}$ so that

$$\mathcal{H}_{B.E,\perp}(p_z = 0) \psi_{\perp} = \left\{ \mathcal{E} \mp (\hbar \omega_k/2) \right\} \psi_{\perp} .$$

(19)

**Theorem II:** For planar motions with both a uniform magnetic field and any in plane electric field, the Bohr transition frequency $\omega_k = (\mathcal{E}_--\mathcal{E}_{+})/\hbar$ for chiral spin rotations is precisely the anomaly frequency of Eq.(2) with a strictly zero frequency shift.

The above theoretical quantum mechanical proof of $\delta \omega_{\perp,k} = 0$, employing the conservation law for the operator $\Lambda = \beta \Sigma_3$, is a central result of this work. For the full Hamiltonian in Eqs.(11) and (12), $\Lambda$ is not quite conserved. In detail, $\Lambda = (i/\hbar) \left[ H, \Lambda \right]$ is given by

$$\left[ \mathcal{H}_{B.E,\perp}(p_z = 0), \Lambda \right] = -\left( \frac{2e\kappa}{\hbar} \right) i\beta \gamma_5 + \left( \frac{\kappa eE_z}{Mc} \right) \gamma_5 .$$

(20)

The only frequency shift that can arise is due solely to non-planar motions with momentum $p_z = (\mathbf{p} \cdot \mathbf{B})/|\mathbf{B}|$ out of the plane driven by the electric field component $E_z = (\mathbf{E} \cdot \mathbf{B})/|\mathbf{B}|$. The vertical betatron oscillation correction is thus a bit more subtle.

From a classical relativistic viewpoint, the planar motions of the muons may be considered to be within a circulating “hoop” with an “effective mass”

$$M_{eff}c^2 = \mathcal{E} .$$

(21)
The effective classical Lagrangian for the vertical betatron oscillation in the $z$-direction may then be shown to be

$$L_{||}(v_z, z) = -\hat{\mathcal{E}} \sqrt{1 - \left(\frac{v_z}{c}\right)^2} - e\Phi_{||}(z),$$ (22)

where the electric field $E_z = -(d\Phi_{||}(z)/dz)$ describes the restoring force of the vertical betatron oscillation at frequency

$$\Omega_{||} = \sqrt{n\omega_c}$$ (23)

where $n$ is the electric field index. The betatron oscillator Lagrangian in Eq.(22) renormalizes the anomaly frequency according to the relativistic Doppler frequency shift formula

$$\hbar\omega_n = \left(\langle \hat{\mathcal{E}}_+ \rangle - \langle \hat{\mathcal{E}}_- \rangle \right) \sqrt{1 - \left(\frac{v_z}{c}\right)^2},$$ (24)

i.e. the classical (transverse) Doppler shift for $|v_z| < c$ is given by

$$\frac{\delta\omega_n}{\omega_n} \approx -\frac{1}{2} \left(\langle \frac{v_z}{c}\rangle^2\right) \approx -\left(\frac{\Omega_{||}^2}{2c^2}\right) \langle z^2 \rangle.$$ (25)

An alternative quasi-classical argument for the vertical betatron induced shift reads as follows: (i) The number $N >> 1$ of “quanta” in the vertical betatron oscillation for a total energy $\mathcal{E}$ and transverse energy $\hat{\mathcal{E}}$ is given by the Bohr rule

$$N = \oint \frac{p_z dz}{2\pi\hbar},$$

$$c p_z = \sqrt{(\mathcal{E} - e\Phi_{||}(z))^2 - \hat{\mathcal{E}}^2},$$ (26)

(ii) The velocity $v_z = (\partial\mathcal{E}/\partial p_z)|_{\mathcal{E}, z}$ determines the vertical betatron tune frequency via

$$\frac{\partial N}{\partial \mathcal{E}} = \oint \frac{\partial p_z}{\partial \mathcal{E}} |_{\mathcal{E}, z} \frac{dz}{2\pi\hbar},$$

$$\frac{\partial N}{\partial \hat{\mathcal{E}}} = \oint \frac{dz}{2\pi\hbar v_z},$$

$$\frac{\partial N}{\partial \hat{\mathcal{E}}} = -\oint \frac{\hat{\mathcal{E}}}{\mathcal{E} - e\Phi_{||}(z)} \frac{dz}{2\pi\hbar v_z}.$$ (27)

The maximum vertical betatron amplitude $z_0$ obeys

$$\mathcal{E} = \hat{\mathcal{E}} + e\Phi_{||}(z_0) - e\Phi_{||}(0),$$ (28)

and without loss of generality we may choose $\Phi_{||}(0) = 0$. Eqs.(27) and (28) then imply

$$\frac{\partial \mathcal{E}}{\partial \mathcal{E}} = -\left(\frac{\partial N}{\partial \mathcal{E}}\right) \hat{\mathcal{E}} \frac{\partial \mathcal{E}}{\partial N} \hat{\mathcal{E}},$$

$$\frac{\partial \hat{\mathcal{E}}}{\partial \mathcal{E}} = \left(\frac{\partial N}{\partial \mathcal{E}}\right) \hat{\mathcal{E}} - e\Phi_{||}(z_0) - \Phi_{||}(z),$$ (29)

where the average is over the vertical betatron oscillation. The frequency shift

$$\frac{\delta \omega}{\omega_n} = 1 - \left(\frac{\partial \mathcal{E}}{\partial \mathcal{E}}\right)_N$$ (30)

thereby obeys

$$\frac{\delta \omega}{\omega_n} = -\left(\frac{\Omega_{||}^2}{2c^2}\right) \langle z^2 \rangle.$$ (31)

For an oscillator with $\hat{\mathcal{E}} >> |e\Phi_{||}(z_0)|$ Eq.(31) reads

$$\frac{\delta \omega}{\omega_n} \approx -\left(\frac{n}{2\rho_0^2}\right) \langle z^2 \rangle \approx -0.23 \times 10^{-6}.$$ (32)

since $n \approx 0.137$ and the cyclotron radius $\rho_0 \approx 711.2$ cm.

Shown in Fig.1 are the central values and error ranges for the BNL experimental measurement of $\kappa$ as well as

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$$(10^{10} \kappa \cdot 11659000)$$

FIG. 1: The value of $\kappa = (g - 2)/2$ is shown together with estimated errors. Reading from top to bottom: (i) the experimental BNL data with the “over compensation” of the electric field corrections as reported in [1], (ii) the experimental data as analyzed employing the quantum Dirac equation of this work and thereby including only the vertical betatron tune, (iii) the theoretical standard model calculation as reported in [1], (iv) the theoretical standard model calculation as reported in [12] and (v) the theoretical standard model calculation as reported in [13, 14].
the more recent standard model theory values and error ranges.

The original experimental report over-compensated the electric field correction to \( \omega_\kappa \). The electric field \( \mathbf{E}_\perp \) contributes zero to a frequency shift as shown above employing the Dirac wave function for the muon beam. From a classical viewpoint, when the muons are first injected into the beam line, the “kick” slightly off-centers the orbit setting the center \( \mathbf{R} \) of the cyclotron orbit to rotate at the horizontal betatron frequency \( \Omega_\perp \). This effect on \( \omega_\kappa \) averages away to zero. The Dirac wave function analysis thus yields a lower magnitude of the electric field shift than was previously reported. The resulting \( \kappa \) with the Dirac equation analysis is pictured in Fig.1. This describes the only possible electric field correction to \( \omega_\kappa \) as given by Eq.(33)

The damping of the horizontal betatron mode velocity \( \mathbf{V} = (d\mathbf{R}/dt) \) takes place in an estimated time of \( \sim 168 \) microsec. In this regard, if the damping of the vertical betatron coordinate \( z \) takes place in a comparable time, then the vertical betatron contribution Eq.(33) may also be moot.

The nature and values of the direct vertical betatron measurements have not yet been made available to the physics community. If the vertical tune relaxation times are sufficiently small, then the standard model theory and the \( (g-2) \) experiment would be brought into yet closer agreement than is shown in Fig.1.