Effects of Extra Space-time Dimensions on the Fermi Constant

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Abstract

Effects of Kaluza-Klein excitations associated with extra dimensions with large radius compactifications on the Fermi constant are explored. It is shown that the current precision determinations of the Fermi constant, of the fine structure constant, and of the $W$ and $Z$ mass put stringent constraints on the compactification radius. The analysis excludes one extra space time dimension below $\sim 1.6$ TeV, and excludes 2, 3 and 4 extra space dimensions opening simultaneously below $\sim 3.5$ TeV, $5.7$ TeV and $7.8$ TeV at the 90\% CL. Implications of these results for future collider experiments are discussed.

The Kaluza-Klein theories have a long and rich history\cite{1,2}. The TeV scale strings provide a new impetus for studying these theories in the context of low energy phenomenology. Till recently much of the string phenomenology has been conducted in the framework of the weakly coupled heterotic string where a rigid relationship exists between the string scale ($M_{\text{str}}$) and the Planck scale $M_{\text{Planck}}$\cite{3}, i.e., $M_{\text{str}} \sim g_{\text{str}} M_{\text{Planck}}$ where $M_{\text{Planck}} = (8\pi G_N)^{-1/2}$, and where $G_N$ is the Newton’s constant and numerically $M_{\text{Planck}} = 1.2 \times 10^{19}$ GeV. However, recently the advent of string dualities has opened the new possibility which relates the strong coupling regime of one string theory to the weak coupling limit of another. Thus it is conjectured that the strongly coupled SO(32) heterotic string compactified to four dimensions is equivalent to a weakly coupled Type I string compactified on four dimensions.
In this context the string scale can be very different\(^4\). While one is very far from generating realistic TeV string models the generic features of models of this type with low values of the string scales can nonetheless be studied\(^5\). Here we adopt the picture that matter resides in D=d+4 dimensions while gravity propagates in the 10 dimensional bulk. In the context of Type I strings one may conjecture that matter resides on a p-brane (p=d+3) with compactification of d dimensions occurring internal to the brane while the compactification of the remaining (6-d) dimensions occurs in directions transverse to the p-brane. We focus on the d compactifications internal to the brane and the compactifications transverse to the brane will not concern us here. We shall work within the framework of an effective field theory, which is a valid approximation in the domain of energy investigated below.

The main motivation of this Letter is to analyse the effects of the Kaluza-Klein excitations associated with the extra dimensions on the Fermi constant \(G_F\) which is one of the most accurately determined quantities in particle physics\(^9\). In our analysis we shall assume that a number \(d\) of extra dimensions open at a low scale each associated with a common radius of compactification \(R=M_R^{-1}\).

We consider the 5D case first. Our starting point is the MSSM Lagrangian in 5D and we exhibit here a few terms to define notation

\[
L_5 = -\frac{1}{4} F_{MN} F^{MN} - (D_M H_i)^\dagger (D^M H_i) - \bar{\psi} i \frac{1}{\xi} \Gamma^M D_M \psi - V(H_i) + ..
\]

(1)

where M,N are the five dimensional indices that run over values 0,1,2,3,5, \(H_i\) (i=1,2) stand for two Higgs hypermultiplets which contain the MSSM Higgs, \(D_M = \partial_M - ig^{(5)} A_M\), where \(g^{(5)}\) are the gauge coupling constants and the \(SU(3) \times SU(2) \times U(1)\) indices are suppressed, and \(V(H_i)\) is the Higgs potential. This theory is constructed to have N=1 supersymmetry in 5D. We compactify this theory on \(S^1/Z_2\) with the radius of compactification \(R\). In our analysis we assume that the gauge fields and the Higgs fields live in the five dimensional bulk while the quarks and leptons are confined to a 4 dimensional wall, i.e, at a \(Z_2\) fixed point\(^10\). After compactification the resulting spectrum contains massless modes with N=1 supersymmetry in 4D, which precisely form the spectrum of MSSM in 4D, and massive
Kaluza-Klein modes which form N=2 multiplets in 4D. The modes in 4D can be further labelled as even or odd under the action of $Z_2$. The MSSM spectrum and their Kaluza-Klein excitations are even under $Z_2$ while the remaining N=1 massive set of Kaluza-Klein fields are odd. We carry out a spontaneous breaking of the electro-weak symmetry in 5D which gives electro-weak masses to the W and Z gauge bosons and to their Kaluza-Klein modes in addition to the compatification masses, i.e., Kaluza-Klein modes have masses $(m_i^2 + n^2 M_R^2)$, $n=1,2,3,..., \infty$, where $m_i^2$ arise from electro-weak symmetry breaking and $n^2 R^2$ arise from compactification of the fifth dimension. In 4D a rescaling of coupling constants is needed, i.e., $g_i^{(5)}/\sqrt{\pi R} = g_i$ where $g_i$ are the gauge coupling constants in four dimensions. The interactions of the fermions to zero modes and to Kaluza-Klein modes after rescaling is given by

$$L_{int} = g_i j^\mu (A_{\mu i} + \sqrt{2} \sum_{n=1}^{\infty} A_{\mu i}^n)$$  \hspace{1cm} (2)

where $A_{\mu i}$ are the zero modes and $A_{\mu i}^n$ are the Kaluza-Klein modes.

The Fermi constant is very accurately known from the weak interaction processes. Its current experimental value obtained from the muon lifetime including the complete 2-loop quantum electromagnetic contributions is

$$G_F = 1.16639(1) \times 10^{-5} \text{ GeV}^{-2}$$  \hspace{1cm} (3)

where the number in the parenthesis is the error. We compare now the experimental result on $G_F$ of Eq.(3) with the predictions of $G_F$ of Eq.(3) in the Standard Model. In the on-shell scheme $G_F^{SM}$ is given by

$$G_F^{SM} = \frac{\pi \alpha}{\sqrt{2} M_W^2 \sin^2 \theta_W (1 - \Delta r)}$$  \hspace{1cm} (4)

where $\sin^2 \theta_W = (1 - M_W^2 / M_Z^2)$ in the on-shell scheme. The fine structure constant $\alpha$ (at $Q^2 = 0$) is very accurately known, i.e., $\alpha^{-1} = 137.0359$, and $\Delta r$ is the radiative correction and is determined to be $\Delta r = 0.0349 \pm 0.0019 \pm 0.0007$ where the first error comes from the error in the mass of the top quark ($m_t = 175 \pm 5 \text{ GeV}$) and the second error comes from
the uncertainty of $\alpha(M_Z)$. With the above one can use the measured values of $M_W$ and $M_Z$ to derive the value of $G_F^{SM}$. The currently measured values of $M_W$ and $M_Z$ are $M_W = 80.39 \pm 0.06$ GeV$^{13}$ and $M_Z = 91.1867 \pm 0.0020$ GeV$^{14}$. Using the above determinations one finds that the Standard Model prediction of $G_F$ is in excellent accord with experiment. Thus we conclude that the contribution of the Kaluza-Klein modes must fall in the error corridors of the Standard Model to be consistent with current experiment. Since the fine structure constant and $M_Z$ are very accurately known most of the error in the evaluation of $G_F^{SM}$ arises from the errors in the measurement of the W mass and in the evaluation of the radiative correction $\Delta r$. To exhibit the relative contribution of the errors from these sources to $G_F$ we find that

$$\Delta G_F/G_F|_{SM} \simeq \sqrt{4(1/\sin^2\theta_W - 2)^2(\delta M_W/M_W)^2 + (\delta \Delta r)^2}$$  

The quantity $2(1/\sin^2\theta_W - 2) \sim 5$ is accidentally large and thus the error in $M_W$ dominates the error in the Standard Model contribution to the Fermi constant. Using the above analysis we find that $G_F^{SM} = 1.16775 \pm 0.0049 \times 10^{-5}$ GeV$^{-2}$. We note that while the error in the measurement of the W mass is less than 0.1% it gets enlarged to around 0.5% due to the enhancement factor of 5. Even so the error corridor of $G_F^{SM}$ is very narrow and places a strong constraint on new physics. Thus we use this corridor to constrain the Kaluza-Klein contributions to the Fermi constant. In our analysis we shall assume that the radiative corrections from the Kaluza-Klein states to $\Delta r$ are small. This assumption will turn out to be a posteriori justified in view of the largeness of the limits on masses of the Kaluza-Klein excitations obtained in this analysis. Under the above assumption we then require that $G_F^{KK}$ be limited by the error in $\Delta G_F^{SM}$, i.e.,

$$\Delta G_F^{KK}/G_F^{SM} \leq \pm 0.82 \times 10^{-2} \quad (90\% \, CL)$$  

For the case of one extra dimension, after compactification and integration over the W boson and its Kaluza-Klein excitations we obtain the effective Fermi constant to leading order in $M_W/M_R$ to be
\[ G_{F}^{\text{eff}} \simeq G_{F}^{\text{SM}} \left(1 + \frac{\pi^2 M_{W}^2}{3 M_{R}^2} \right) \]  

(7)

Defining \( \Delta G_{F}^{KK} = G_{F}^{\text{eff}} - G_{F}^{\text{SM}} \), identifying \( G_{F}^{\text{eff}} \) with the experimental value of the Fermi constant, and using Eq.(6) we find \( M_{R} > 1.6 \text{ TeV} \) (90\% CL). This limit on \( M_{R} \) is stronger than for the case when one has just an extra W recurrence. Thus if one had an extra W recurrence with a mass \( M_{W'} \), the analysis above will give a limit \( M_{W'} > 905 \text{ GeV} \). The stronger limit for the Kaluza-Klein case is due to an enhancement factor of \( \pi/\sqrt{3} \). This factor arises in part due to summation over the tower of Kaluza-Klein states and in part due the coupling of the 4D fermions to the Kaluza-Klein gauge bosons being stronger by a factor of \( \sqrt{2} \) relative to the couplings of the fermions to the zero mode gauge bosons as can be seen from Eq.(2). The current experimental limit on the recurrence of a W is \( M_{W'} > 720 \text{ GeV} \) given by the analysis at the Fermilab Tevatron \( p\bar{p} \) collider data with the D0 detector. Thus our result on \( M_{R} \) is much stronger than the current experimental limit on \( M_{W'} \).

Another independent constraint on \( M_{R} \) can be gotten from atomic parity experiments. The atomic parity violations arise from the Z exchange and the low energy effective interaction governing the violation is

\[ L_{PV}^{\text{SM}} = (G_{F}^{\text{SM}}/\sqrt{2}) \sum_i C_{1i} \bar{e}\gamma_{\mu}\gamma_{5}e\bar{q}_i\gamma^{\mu}q_i \]

In SM the measured quantity is \( Q_{W}^{\text{SM}} = 2[(2Z + N)C_{1u} + (Z + 2N)C_{1d}] \) where \( Z \) is the number of protons and \( N \) is the number of neutron in the atomic nucleus being considered. The exchange of the Kaluza-Klein Z bosons contribute an additional interaction, i.e.,

\[ L_{PV}^{KK} = (\Delta G_{F}^{KK}/\sqrt{2}) \sum_i C_{1i} \bar{e}\gamma_{\mu}\gamma_{5}e\bar{q}_i\gamma^{\mu}q_i \]  

(8)

The most accurate atomic parity experiment is for cesium which gives

\[ Q_{W}^{\exp}(Cs) = -72.41 \pm 0.25 \pm 0.80 \]  

(9)

while the Standard Model gives

\[ Q_{W}^{\text{SM}}(Cs) = -73.12 \pm 0.06 \]  

(10)

The above result gives \( \Delta Q_{W}^{\exp} - Q_{W}^{\text{SM}} = 0.71 \pm 0.84 \) where we have added the errors in quadrature. We define the Kaluza-Klein contribution to \( Q_{W} \) by \( \Delta Q_{W}^{KK} = (\Delta G_{F}^{KK}/G_{F}^{\text{SM}})Q_{W}^{\text{SM}} \) and
require that $\Delta Q_{WW}^{KK}$ be limited by $\Delta Q_W$, i.e., $\Delta Q_{WW}^{KK} \leq \Delta Q_W$. This leads to a constraint on $M_R$ of $M_R > 1.4$ TeV (90% CL) which is less strong than the limit gotten from the analysis of $G_F$ but still quite impressive. The current accuracy of the experimental determinations of other electro-weak quantities produce less stringent constraints on $M_R$.

The above analysis can be extended to d extra dimensions. We consider here only compactifications with common compactification radius $R$ and $Z_2$ type orbifolding. Integration on the $W$ boson and its Kaluza-Klein excitations similar to the 5D case gives the following result for the sum of the Standard Model and Kaluza-Klein mode contributions

$$G_F^{\text{eff}} = G_F^{\text{SM}} K_d(M_W^2/M_R^2)$$

where $K_d$ is the Kaluza-Klein dressing of the Fermi constant

$$K_d(c) = 1 + \sum_{p=1}^{d} (2^p d! C_p^0 C_p(c))$$

Here $d! C_p = d!/(d-p)!$, $C_p(c) = \sum_{k_p} (c/(c+k_p^2))$ where $k_p = (k_1, k_2, ..., k_p)$, and $k_i$ run over the positive integers $1, 2, ..., \infty$. We can express $K_d(c)$ in terms of the Jacobi function

$$K_d(c) = \int_{0}^{\infty} dt e^{-t} (\theta_3(c/t))^d$$

where $\theta_3(z)$ for complex $z$ ($Imz > 0$) is defined by $\theta_3(z) = \sum_{k=-\infty}^{\infty} \exp(i \pi k^2 z)$. For the case of more than one extra dimension both the sum of Eq.(12) and the integral of Eq.(13) diverge. To obtain a convergent result the lower limit on the integral must be replaced by a cutoff. The physical origin of the cutoff is a truncation of the sum over the Kaluza-Klein states when their masses exceed the string scale. With the cutoff we obtain the following approximate expressions for $\Delta G_F^{KK}/G_F^{SM}$

$$\Delta G_F^{KK}/G_F^{SM} \simeq \left( \frac{2 \pi^2}{3} + 2 \pi \ln \left( \frac{M_{str}}{M_R} \right) \right) \left( \frac{M_W}{M_R} \right)^2, \quad d = 2$$

$$\Delta G_F^{KK}/G_F^{SM} \simeq \left( \frac{d}{d-2} \right)^{\frac{d}{2}} \left( \frac{M_{str}}{M_R} \right)^{d-2} \left( \frac{M_W}{M_R} \right)^2, \quad d \geq 3$$

Numerically we find that the approximation of Eq.(15) agrees with the exact result of Eq.(12) to 2-3% while the approximation of Eq.(14) is good only to about 10% because of the slow
convergence of the log function in this case. The quantities $M_R$ and $M_{str}$ are not completely arbitrary but are constrained by the unification of the gauge couplings. Thus in a TeV scale unification the gauge coupling evolution is given by

$$\alpha_i(M_Z)^{-1} = \alpha_U^{-1} + \frac{b_i}{2\pi} \ln \left( \frac{M_R}{M_Z} \right) - \frac{b^K_i}{2\pi} \ln \left( \frac{M_{str}}{M_R} \right) + \Delta_i$$

(16)

Here $\alpha_U$ is the effective GUT coupling constant, $b_i=(-3, 1, 33/5)$ for $SU(3)_C \times SU(2)_L \times U(1)_Y$ describe the evolution of the gauge couplings from the scale $Q$ to the scale $M_R$, $b^K_i = (-6, -3, 3/5)$ are the $b_i$ minus the contribution from the fermion sector which has no Kaluza-Klein excitations, and $\Delta_i$ are the corrections arising from the Kaluza-Klein states. The requirement that $\alpha_i(M_Z)$ be compatible with the LEP data leads to constraints on $M_R$ and $M_{str}$. For a given $d$ and $M_R$ the unification of $\alpha_1$ and $\alpha_2$ thus fixes the ratio $M_{str}/M_R$. In ref. an extension to include two additional multiplets $F_+$ and $F_-$ with $SU(3)_C \times SU(2)_L \times U(1)_Y$ quantum numbers $(1, 1, +2)$ for $F_+$ and $(1, 1, -2)$ for $F_-$ was proposed with a mass scale $M_F$ in the range $M_{EW} \leq M_F \leq M_R$. Inclusion of this multiplet improves the agreement of $\alpha_3$ with experiment but otherwise does not affect our analysis in any substantial way.

The result of our analysis on the lower limit on extra dimensions under the constraints of unification of gauge couplings and under the constraint of Eq.(6) is given in Fig.1. For the case of one extra dimension the analysis of Fig.1 shows that the constraint of Eq.(6) puts a lower limit on $M_R$ of 1.6 TeV. This is the same result that we got previously from Eqs.(6) and (7). This is because in this case the unification of the gauge coupling constant constraint gives a high value of $M_{str}/M_R$ so the finite sum on the Kaluza-Klein states simulates to a very good approximation the sum on the infinite tower of Kaluza-Klein states. For the case of 2, 3 and 4 extra dimensions the analysis of Fig.(1) shows that Eq.(6) produces a lower limit on $M_R$ of 3.5 TeV, 5.7 TeV, and 7.8 TeV. In these cases the truncation of the Kaluza-Klein tower is rather severe as one goes to higher values of $d$, e.g, $M_{str}/M_R \approx 7.6$ for $d=2$ and this ratio becomes smaller for larger values of $d$ (and further the ratio $M_{str}/M_R$ must be progressively fine tuned more severely as one goes to higher values of $d$). The specific nature
of the cutoff imposed introduces an uncertainty in the accuracy of the predictions which is $O(1\%)$ for $d=1$ and gets larger for larger values of $d$ up to $O(10\%)$ for $d=4$. The largeness of the uncertainty for larger values of $d$ arises due to the progressively smaller number of Kaluza-Klein states that are retained in the truncation procedure under the unification of the gauge coupling constants constraint. As pointed out earlier the analysis is very sensitive to the error in the measurement of the $W$ boson mass. Thus if this error on $M_W$ were to decrease by a factor of 2, which is not an unlikely possibility, then the limit on $M_R$ for $d=1$ will increase to 2.1 TeV, and the limits for $d=2$, $d=3$ and $d=4$ will increase to 4.4 TeV, 7.1 TeV and 9.5 TeV as can be seen by the intercepts of the dotted line in Fig.1. These results may be contrasted with the previous limits of $\sim 300$ GeV from an analysis of contact interactions\cite{7}. Thus none of the Kaluza-Klein excitations of the $W$ or $Z$ boson will become visible at the Tevatron. However, Kaluza-Klein excitations may still be accessible at the Large Hadron Colldier (LHC)\cite{16}.

One may ask to what extent our limits are dependent on the assumption of an underlying supersymmetry. Since loop corrections to $\Delta r$ from the Kaluza-Klein states are ignored, which turns out to be a reasonable assumption in view of the largeness of the masses of the Kaluza-Klein states, the essential constraint from supersymmetry arises via the cutoff. For the 5D case this constraint is rather mild but does get severe as one goes to larger number of extra dimensions. Thus our conclusion is that the 5D limit is essentially model independent but the limits deduced for the 6D-8D cases do require the underlying supersymmetric framework and the results here are thus more model dependent.

Finally we note that the limits on extra dimensions are constrained severely by the errors in the Standard Model predictions of $G_F$ which are dominated by the error in the $W$ mass measurement and sub-dominated by the error in the top mass measurement. Thus if future experiments further reduce the errors in these measurements, then one will obtain more stringent bounds on the radius of compactification, or if one sees a deviation from the Standard Model prediction it might be a smoking gun signal for a TeV scale compactification.

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10. We note that the assumption that the quarks and leptons live on the boundary is necessary for them to have non-vanishing couplings to the Kaluza-Klein modes of the gauge bosons for this class of models.


Wood for discussions regarding the experimental data on the W recurrence.


Fig. 1. Plot of the ratio \( \Delta G^K_F / G^SM_F \) as a function of the common scale of extra dimension. The curves correspond to the case \( d=1 \) (solid), \( d=2 \) (dot-dashed), \( d=3 \) (long dashed), and \( d=4 \) (dashed). The horizontal solid line corresponds to the 2\( \sigma \) error on \( \Delta G^SM_F / G^SM_F \) while the horizontal dotted line is what the 2\( \sigma \) error will be if the error in the W boson measurements decreased by a factor of 2.