EXACT STABILITY ANALYSIS OF NEUTRAL SYSTEMS WITH CROSS-TALKING DELAYS

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Abstract: The stability of neutral systems with two cross-talking delays is investigated using the method of cluster treatment of characteristic roots (CTCR). There are two main outcomes of this study: (a) Generation of the well-known delay stabilizability condition as a by-product of the CTCR procedure. This is achieved by a small delay stability treatment over the system. We also demonstrate for the delay-stabilizable systems the exact bounds of the stability regions in the domain of the delays. (b) Validation of these stability regions using an alternative point-wise algorithm, which computes the right-most roots of the characteristic quasi-polynomial. Copyright © 2006 IFAC

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1. INTRODUCTION

The method of cluster treatment of characteristic roots (CTCR) has shown to be a powerful procedure for determining the stability picture of a linear time invariant (LTI) time delay system (TDS) in the domain of delays. In (Sipahi and Olgac, 2004, 2005a) the method was applied to retarded systems with multiple delays. (Sipahi and Olgac, 2003), (Olgac and Sipahi, 2004, 2005) address neutral systems with single delay only, and (Sipahi and Olgac, 2006) extends the CTCR procedure to scalar neutral system with multiple delays and cross-talking delay effects in the retarded part. In this paper, we apply CTCR to study the stability of neutral systems with two cross-talking delays in both neutral and retarded parts. The general system studied here is given as follows

\[
\frac{d}{dt}[x(t) - A x(t - \tau_1) - B x(t - \tau_2) - C x(t - \tau_1 - \tau_2)] = D x(t - \tau_1) + F x(t - \tau_2) + G x(t - \tau_1 - \tau_2) + H x(t)
\]

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(\tau = (\tau_1, \tau_2) \in \mathbb{R}^{2+}\) are the delays and \(A, B, C, D, F, G, H\) are the constant coefficient matrices of \(\mathbb{R}^{n \times n}\). The objective is to analyze the stability with respect to time delays in the semi-infinite \((\tau_1, \tau_2) \in \mathbb{R}^{2+}\) quadrant.

The characteristic equation of the system (1)

\[
CE(s, \tau_1, \tau_2) = \det(s I - A e^{-s \tau_1} - B e^{-s \tau_2} - C e^{-s(\tau_1 + \tau_2)}) -
- D e^{-s \tau_1} - F e^{-s \tau_2} - G e^{-s(\tau_1 + \tau_2)} - H = 0
\]

is transcendental with infinitely many roots. Assuming the system matrices being constant, the stability of this system reduces to finding \((\tau_1, \tau_2)\) regions where all the characteristic roots of (2) remain in \(\mathbb{C}^-\), i.e., in the open left half of the complex plane. As a notational selection we represent the right half open plane by \(\mathbb{C}^+\) and the imaginary axis by \(\mathbb{C}^0\).

A well-known necessary condition for stability of any neutral system is the stability of a difference equation associated with (1) (Hale and Verduyn Lunel, 1993, 2002). It claims that, the discrete dynamics

\[
x(t) - A x(t - \tau_1) - B x(t - \tau_2) - C x(t - \tau_1 - \tau_2) = 0
\]

must be stable. Alternatively, all the roots of the corresponding characteristic equation

\[
L(s, \tau_1, \tau_2) = \det(s I - A e^{-s \tau_1} - B e^{-s \tau_2} - C e^{-s(\tau_1 + \tau_2)}) = 0
\]
must lie in $C^-$. A critical feature of the spectrum of a difference equation is that even infinitesimal changes in the delays may cause stability loss with infinitely many roots crossing the stability boundary. To handle this concept of strong stability has been introduced by Hale and Verduyn Lunel, (1993), (2002) which claims that, a difference equation is strongly stable if it remains stable when subjected to small variation in the delays. They also show that the strong stability of a difference equation is independent of the delays. Since at least a part of the spectrum of the neutral system asymptotically approaches to that of its associated difference equation for large $|s|$, strongly stable difference equation is the necessary condition for delay-stabilizability of the neutral system.

The problem of determining the stability crossing curves in the delay domain was recently studied by (Gu, et al. 2005) for a simpler class of systems with two delays but without delay crosstalk. They cleverly use a geometric triangulation property to bound the potential root crossing frequencies. There is also an elegant numerical effort reported in (Breda, et al., 2006) where all the root sets containing at least one pair of imaginary roots can be found. Notice that there are $x^2$ (2-dimensional infinite) candidate points defined by (6) in $\mathbb{R}^{2+}$ resulting in one single imaginary root, $s \in S_{\Omega}$. Those curves corresponding to $j > 0$ and $k > 0$ are called the “offspring curves”, and denoted by $\varphi_{jk} (\tau_1, \tau_2)$ where $j$ and $k$ identify the $j^{th}$ and $k^{th}$ generation offspring in $\tau_1$ and $\tau_2$. Consequently, the complete set of kernel curves and offspring becomes $\varphi (\tau_1, \tau_2) :$

$$\varphi (\tau_1, \tau_2) = \varphi_0 (\tau_1, \tau_2) \cup \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \varphi_{jk} (\tau_1, \tau_2) \quad (8)$$

The root tendency along $\tau_j$, $j=1,2$, axis at the crossing of $s=\omega i$ is defined by

$$RT_{s=\omega i} = \text{sgn} \left( \text{Re} \left( \frac{\partial s}{\partial \tau_j} \right) \right) \bigg|_{s=\omega i} \quad (9)$$

Steps of CTCR:

1. Determine exhaustively the kernel and offspring curves, $\varphi (\tau_1, \tau_2)$.
2. For non-delayed system, $\tau = 0$, evaluate $NU(0)$.
3. Following line segments in $\tau \in \mathbb{R}^{2+}$, which are parallel to the individual coordinates $\tau_1$ and $\tau_2$, connect the origin ($\tau = 0$) to a point of interest $\tau_0$.
4. As this path crosses the kernel and offspring curves, change $NU$ by $+2$ (or $-2$) for $RT = +1$ ($-1$).
5) Exhaustively identify the regions in \((\tau_1, \tau_2)\) space with \(NU = 0\) as "stable" and the others (\(NU > 0\)) "unstable".

The step \#1 is crucial in this procedure. In most of our works it is achieved by applying Rekasius substitution (Rekasius, 1980)

\[
e^{-\tau_j s} = \frac{1 - T_j s}{1 + T_j s}, \quad T_j \in \mathbb{R}, \quad j = 1, 2.
\]

(10)

This representation becomes exact for \(s = \omega i\), provided that relation between \(T_j\) and \(\tau_j\)

\[
\tau_j = \frac{2}{\omega} \tan^{-1}(\omega T_j) + k\pi, \quad k = 0, 1, 2, ...
\]

(11)

holds. It is an asymmetric mapping between \(\tau_j\) and \(T_j\) transforming the transcendental \(\overline{CE}(s, \tau_1, \tau_2)\) given by (2) into a new characteristic equation \(\overline{CE}_2(s, \tau_1, \tau_2) = 0\) which is of fractional polynomial type. Multiplying this equation by \((1 + T_1 s)^n (1 + T_2 s)^n\), one obtains

\[
\overline{CE}(s, \tau_1, \tau_2) = \sum_{j=0}^{3n} p_j(T_1, T_2)s^j = 0.
\]

(12)

An interesting relation between the infinite dimensional equation (2) and the \(3n\) degree equation (12) is that they share the same imaginary root sets completely. That is,

\[
S_{\omega} \{s \in \mathbb{C} \mid \overline{CE}(s, \tau_1, \tau_2) = 0, (\tau_1, \tau_2) \in \mathbb{R}^2 \} \cap C^0 = \bigcup_{m=0}^{m} S_{\omega} \{s \in \mathbb{C} \mid \overline{CE}(s, \tau_1, \tau_2) = 0, (\tau_1, \tau_2) \in \mathbb{R}^2 \} \cap C^0
\]

(13)

The most beneficial point in transforming \(\overline{CE}(s, \tau_1, \tau_2)\) to \(\overline{CE}(s, \tau_1, \tau_2)\) is that of the parametric equation (12) is much easier to study compared with (2). Thus, the kernel curves are obtained by determining their projections in \((\tau_1, \tau_2)\) space, which are called the "core curves".

Let us now present how to determine the core curve of \(\overline{CE}(s, \tau_1, \tau_2)\), i.e., those \((T_1, T_2)\) that give rise to \(s = \omega i \in S\). The easiest procedure to find all the imaginary roots of such characteristic polynomials is the classical Routh-Hurwitz method (Kuo, 1987). From the well-known rules of the Routh’s array of Table 1, the imaginary roots of the equation (12) are found at:

\[
R_j(T_1, T_2) = 0, \text{with cond.} \quad R_{21}(T_1, T_2) > 0
\]

(14)

At every point \((T_1, T_2)\) satisfying (14) we determine a crossing frequency \(\omega = \sqrt{p_0/R_{21}(T_1, T_2)}\).

The next step in CTCR is to numerically map these core curves into kernel curves via (11) and further to offspring in \((\tau_1, \tau_2)\) space via (6). This completes the step 1 in CTCR procedure, i.e., the exhaustive determination of the entire set of kernel and offspring curves, \(\varphi(\tau_1, \tau_2)\).

Table 1. The Routh’s array for \(\overline{CE}(s, \tau_1, \tau_2)\)

<table>
<thead>
<tr>
<th>(s)</th>
<th>(p_0(T_1, T_2))</th>
<th>(p_{3n-3}(T_1, T_2))</th>
<th>(\cdots)</th>
<th>(p_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s^{3n})</td>
<td>(p_{3n-1}(T_1, T_2))</td>
<td>(\cdots)</td>
<td>(p_0)</td>
<td></td>
</tr>
<tr>
<td>(s^{3n-1})</td>
<td>(\cdots)</td>
<td>(\cdots)</td>
<td>(\cdots)</td>
<td></td>
</tr>
<tr>
<td>(s^1)</td>
<td>(R_1(T_1, T_2))</td>
<td>(R_2(T_1, T_2) = p_0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s^0)</td>
<td>(R_0(T_1, T_2) = p_0)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. ASSESSING DELAY- STABILIZABILITY

The contribution presented in this section is a procedure to determine the delay stabilizability of the neutral system (1), which results as a by-product of the CTCR approach following similar steps as in (Olgac and Sipahi, 2005). The procedure is based on checking the stability posture of the associated difference equation (3) as \(\tau(0, 0) \rightarrow (0^+, 0^+)\). Notice that for small delays, equation (11) becomes

\[
\tan(\frac{\tau_j \omega}{2}) = T_j \omega, \quad j = 1, 2 \quad .
\]

(15)

which implies that \(\tau_j : 0 \rightarrow \epsilon_j\) corresponds to \(T_j : 0 \rightarrow \epsilon_j / 2\). In two-dimensional delay domain such a transition can be represented by selecting directional slope \(m\), which is defined as

\[
m = \frac{\tau_2}{\tau_1} = \frac{T_2}{T_1}, \quad m \in (0, \infty)
\]

(16)

Let us point out that "no crossing" requirement should be independent of \(m\).

Consider (16) and taking the limit for \(0 < T_1 < 1\) on the equation (12), by dropping the higher powers of \(T_1\) and favoring the lowest power in the coefficients of \(s\). We obtain

\[
\overline{CE}_1(s, T_1, mT_1) = \frac{\tilde{p}_{3n}(m)T_1^{2n}s^{3n} + \cdots + \tilde{p}_{n+1}(m)T_1^{n+1}s^{n+1} + \tilde{p}_n s^n + \cdots + \tilde{p}_1 s + \tilde{p}_0 = 0}{\tilde{p}_{n+1}(m)T_1^{n+1}s^{n+1} + \cdots + \tilde{p}_1 s + \tilde{p}_0 = 0}
\]

(17)

where the coefficients \(\tilde{p}_j(m)\) are composed of the elements of the neutral part matrices only for \(j = n, 3n, \ldots\), of both neutral and retarded parts for \(j = 1, \ldots, n-1\) and retarded part only for \(j = 0\), see (Olgac et al., 2006, Sipahi et al., 2006).

Building the Routh’s array for (17), the stability condition for the transition \(\tau(0, 0) \rightarrow (0^+, 0^+)\) is that the first column exhibits no sign change independently of \(m\). We state two key observations here deferring the proofs to (Olgac et al., 2006): (a) The first column coefficients corresponding to the rows of powers \(n, 3n, \ldots\) of \(s\) determines the stability of
the difference equation (3), (b) The coefficients corresponding to the rows of powers 0..n of s determines the stability of the delay free system (1), see (Olgac et al., 2006, Sipahi et al., 2006).

Using CTCR, we check the stability of (3) for all possible small variations of the delays at \( \tau_i(0,0) \rightarrow (0^+,0^+) \), i.e., the strong stability of the difference equation. This feature is known to be independent of the delays. Thus the above observation (a) implies the following. (a1) It is sufficient to verify the stability in the transition \( \tau_i(0,0) \rightarrow (0^+,0^+) \) both for the strong stability of the difference equation (3) and for the delay-stabilizability of the neutral system (1). (a2) This finding is a natural by-product of the CTCR.

3.1 Scalar case

Next we take a simpler scalar version of equation (12), n=1, to validate the claims (a1) and (a2). The corresponding equation (12) is

\[
\begin{align*}
CE_i(s, T_i, mT_i) &= (1 + a + b - c) m T_i s^3 + \\
 &\quad (1 + a - b + c + m(m - a + b + c)) T_i s^2 + \\
 &\quad +(1 - a - b - c) s - f - d - g - h
\end{align*}
\] (18)

The Routh’s array for (18) is given in Table 2.

Table 2. Routh’s array of \( CE_i(s, T_i, mT_i) \) given by (18)

<table>
<thead>
<tr>
<th>s^3</th>
<th>(1 + a + b - c) m T_i^2</th>
<th>(1 - a - b - c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^2</td>
<td>(1 + a - b + c + m(m - a + b + c)) T_i</td>
<td>(1 - a - b - c) T_i</td>
</tr>
<tr>
<td>s^1</td>
<td>(1 - a - b - c) T_i</td>
<td>(1 - a - b - c)</td>
</tr>
<tr>
<td>s^0</td>
<td>(1 - a - b - c)</td>
<td>(1 - a - b - c)</td>
</tr>
</tbody>
</table>

Notice that the block No. 1 is fully determined by the difference equation after Rekasius substitution

\[
\bar{T}(s, T_i, mT_i) = (1 + a + b - c) m T_i^2 s^3 + \\
\quad (1 + a - b + c + m(m - a + b + c)) T_i s^2 + \\
\quad +(1 - a - b - c) s - f - d - g - h
\] (19)

while the block No. 2 consists of the coefficients of the characteristic equation for non-delayed system. Stabilizability of the difference equation is then determined by the following inequalities

1. \( 1 + a + b - c > 0 \)
2. \( 1 + a - b + c + m(m - a + b + c) > 0 \)
3. \( 1 - a - b - c > 0 \)

which for \( m \in [0, \infty) \), can be reduced to two inequalities

4. \( 1 + a > \|b - c\| \)
5. \( 1 - a > \|b + c\| \)

defining a tetrahedron as the stability domain in the coordinates of the parameters, \( a, b \) and \( c \). We next compare these results with the existing literature.

The stability of (3) has been solved in Hale and Verduyn Lunel, (1993), p. 287-288, where the strong stability condition is given by the inequality

\[
\gamma_0 < 1, \quad \gamma_0 = \sup \{ r_0 \left| \begin{array}{cc}
A & 0 \\
1 & 0
\end{array} \right| e^{\omega t} + \left| \begin{array}{cc}
B & C \\
0 & 0
\end{array} \right| e^{\omega t} \} 
\] (22)

where \( r_0 \) is the spectral radius. For the scalar case it results in the same conditions as given in (21), except for the typographical errors in sign in the cited monograph. Furthermore, if the difference equation (3) is considered with three independent (and non-crosstalk) delays instead of two that crosstalk, the strong stability condition becomes

\[
|a| + |b| + |c| < 1
\] (23)

This represents, expectedly, a sub-region within the tetrahedron stability domain given by (21).

In (Olgac et al., 2006) we discuss higher degree case, \( n=2 \), more thoroughly with a number of interesting features.

4. Determining Stability Domain via Mapping the Quasi-Polynomial Roots

In order to cross-validate the results of CTCR, we include here a fundamentally different method for determining the regions in the delay domain distinguished by the number of unstable roots \( NU \). This method is based on direct computing the rightmost roots of the characteristic equation at each point of the mesh grid spread over the region of the delays \( [0, \tau_{\text{max}}] \times [0, \tau_{\text{max}}] \). Unlike in (Breda, et al. 2004), where the roots are computed by the method based on approximation of the infinitesimal generator, we use a quasi-polynomial mapping based rootfinder (QPMR) (Vyhildal and Zítek, 2003, 2006).

The basic idea of the QPMR algorithm is as follows. Consider \( s = \beta + i\omega \), the characteristic equation (2) can be split into

\[
R(\beta, \omega) = \Re(CE(\beta + i\omega)) = 0
\] (24)
\[
I(\beta, \omega) = \Im(CE(\beta + i\omega)) = 0
\] (25)

Notice that these equations determine the intersection contours of the surfaces described by \( R(\beta, \omega) \) and \( I(\beta, \omega) \), respectively, with the \( s \)-plane. Mapping these contours on a region \( \mathfrak{D} = [\beta_{\text{max}}, \beta_{\text{max}}] \times [0, \omega_{\text{max}}] \), the roots of (2) are given as the intersection points of contours given by (24) and (25). Obviously, the real roots are given as the intersection points of contours (24) and the real axis. Note that mapping the zero level contours is to be done numerically applying a
level curve tracing algorithm, (e.g. function \textit{contour} in Matlab).

Since the QPMR performs the root finding task on a defined region, it is crucial to determine the boundary values of the region so that the roots of interest lie safely in the region. For determining the boundaries \(\beta, \beta\) and \(\max \omega\) we follow the adaptation rule described in (Olgac et al., 2006).

5. APPLICATION EXAMPLES

5.1 Example 1, delay-stabilizable neutral system

Consider a neutral system \((1)\) of scalar form with \(a=0.5, b=-0.3, c=-0.31, d=-1.5, f=2, g=-2, h=-3\). The first step which has to precede searching for the stability domain in the delays is the delay-stabilizability test, which is given for the scalar case by inequalities \((21)\). Since both the inequalities are satisfied, \((1.5>0.66, 0.5>0.04)\), the difference equation associated with the neutral system is strongly stable and the neutral system is delay stabilizable.

This result is also confirmed by evaluating \((22)\) \((\gamma_0 = 0.637 < 1)\). Applying the algorithm of CTCR, we obtain the stability picture given in Fig. 1. The colour of the curves distinguishes the root tendency with respect to the delay \(\tau\), red \(\tau = \tau_R\), blue \(\tau = \tau_R - 1\). The number of unstable roots in each region, \(NU\), is also shown sparingly. Obviously, in the stable regions, \(NU=0\).

In Fig. 2 we can see the regions with fixed number of unstable roots, which were determined by employing QPMR algorithm. To obtain this figure the algorithm was used at sufficiently dense grid points in \((\tau, \tau)\) plane. As can be seen, the results provided by CTCR and QPMR are identical. In Fig. 3 the spectra of neutral system and its associated difference equation, computed using QPMR, are shown for two selected points in the delay domain. The first point \(P_1\) is at \((\tau_1, \tau_2) = (\pi, 3)\) and the second point \(P_2\) is at \((\tau_1, \tau_2) = (\pi, 4)\), see Fig. 1. As can be seen, the number of unstable roots agree with the result of CTCR given in Fig. 1, i.e. for \(P_1\) \(NU=0\) and for \(P_2\) \(NU=6\).

5.2 Example 2, delay-nonstabilizable neutral system

Consider a scalar neutral equation \((1)\) with \(a=0.88, b=0.3, c=-0.53, d=-1.5, f=2, g=-2, h=-3\). Since the second inequality in \((21)\) is not satisfied \((1.88>0.83, 0.12<0.23)\) the difference equation is not strongly stable and the neutral system is not delay stabilizable (verified also by condition \((22)\) \(\gamma_0 = 1.073 > 1)\).

Thus, the stability analysis stops here. As a demonstration, in Fig. 4, we present distribution of the spectra of this system for \((\tau_1, \tau_2) = (2, 4)\), for which \(NU=10\) and \((\tau_1, \tau_2) = (2.1, 4)\), for which \(NU=\infty\).
Re(\(s\)) −0.3

Fig. 4 Spectra of the neutral system (blue dots) and the difference equation (red asterisks), left \((\tau_1,\tau_2) = (2.4)\), with \(NU=10\),
right \((\tau_1,\tau_2) = (2.1,4)\) with \(NU=\infty\)

As can be seen, a relatively small change of \(\tau_1\) has the consequence that infinitely many roots of both neutral system and its associated difference equation appear in the right half plane.

6. CONCLUSIONS

A general class of neutral systems with two time delays is studied for the stability robustness against delay uncertainties. Note that both delays appear in both neutral and retarded parts, and in cross-talking format. First, the method of cluster treatment of characteristic roots (CTCR) is employed to determine the stability domain in the delays. As the main contribution of the paper, an interesting delay-stabilizability criterion is observed based on checking the transition \(\tau; (0,0) \rightarrow (0^*,0^*)\) as a natural by product of CTCR. As the second contribution, a numerical algorithm is deployed to validate the findings of CTCR. This algorithm is based on computing the rightmost roots.

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Re(\(s\)) 0

Im(\(s\))