INTRODUCING NONLINEAR PROBLEMS TO UNDERGRADUATE ENGINEERING STUDENTS

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ABSTRACT

It is argued that preparing good engineers to the present day challenging industry should involve the extension of homework problems to include realistic complications of nonlinearity and extensive use of the available computing facilities. This point of view has been illustrated through a simple Faraday's law problem of an electromechanical system that exhibits nonlinear effects if the resistance of the rails is taken into account. The procedure that is employed in this paper to solve the nonlinear equation is simple enough to be introduced at an undergraduate level for the engineering students.

INTRODUCTION

It is argued that preparing good engineers and applied physicists to the present-day challenging industry should involve the extension of homework problems to include realistic complications of nonlinearity and extensive use of the available computing facilities. Wherever possible, simple systems for which differential equations to be solved are linear may be presented to the undergraduate engineering students at first, and then these may be extended to become nonlinear as a realistic and easily visualized complication. While the solution of nonlinear
differential equations is in general very difficult and such methods of solution as the method of perturbations [1] and numerical procedures [2] are usually introduced only at a graduate level, simpler solution techniques should be explored wherever possible. This point of view has been illustrated here through a simple Faraday's law problem of an electromechanical system that exhibits nonlinear effects.

THE ELECTROMECHANICAL SYSTEM UNDER CONSIDERATION

The motion of a crossbar of Mass $M_0$ and resistance $R_0$, which accelerates without friction on a pair of parallel rails is to be analyzed. A battery having potential difference $V_0$, the two rails, and the crossbar constitute a closed electrical circuit, as shown in Figure 1. When a constant external magnetic induction field $B_0$ directed perpendicular to the circuit loop is present and whenever a current is flowing in the circuit, the crossbar is accelerated by the Lorentz force. The motion of the crossbar changes the magnetic flux $\Phi$ linked by the circuit, and a back electromotive-force is induced according to Faraday's law. For simplicity and convenience of analysis, all other inductive, capacitive, radiative, and frictional effects are neglected, except for the ohmic resistance within the circuit. The solution for the time dependence of the motion of the crossbar and the current in the circuit when the switch $S$ is closed is a well-known undergraduate physics problem [3].
LINEAR ANALYSIS

As simple as this problem may appear, the differential equation of motion to be solved is linear only if the resistance of the rails may be neglected. Practical design may often justify linearization to obtain an approximate solution. In such a case, the application of Kirchoff's second law gives for the circuit loop:

\[ V = I R + \frac{d\phi}{dt} \]  

where \[ \phi = B \cdot y \cdot x \]  

Newton's second law yields

\[ B \cdot y \cdot I = M \cdot \ddot{x} \]  

for the motion of the crossbar, where the left-hand side of Eq. (3) is the Lorentz force on the current. Substitution of Eqs. (2) and (3) into Eq. (1) leads to the linear differential equation

\[ \dddot{x} + (B^2 y^2 / M R_0) x = \frac{V}{R_0} \frac{y}{M R_0} \]  

Assuming the initial conditions

\[ x(0) = x_0 \]  
\[ \dot{x}(0) = 0 \]  
\[ I(0^+) = \frac{V}{R_0} \]  

the solution of Eq. (4) for \( t > 0 \) is given by

\[ x = x_0 + V_f t - V_f \tau \left[ 1 - \exp \left( -\frac{t}{\tau} \right) \right] \]  
\[ \dot{x} = V_f \left[ 1 - \exp \left( -\frac{t}{\tau} \right) \right] \]  
\[ \ddot{x} = \frac{V_f}{\tau} \exp \left( -\frac{t}{\tau} \right) \]
and \( I = \left( \frac{V_o}{R_o} \right) \exp \left( -\frac{t}{\tau} \right) \) \hspace{1cm} (7)

where \( \tau = \frac{M_o Y_o}{B_o^2 V_0^2} \) \hspace{1cm} (8)

is the time constant for \( I \) and \( \dot{x} \) to decay to zero.

and \( V_f = \frac{V_0}{B_0 Y_0} \) \hspace{1cm} (9)

is the final velocity of the crossbar. The direction of motion of the crossbar is determined by the relative signs of \( V_0 \) and \( B_0 \), as per Eq. (9). Plots of the time dependence of the position, velocity and acceleration of the crossbar, based on Eq. (6), can easily be obtained [3]. It can also be shown that the energy supplied by the battery is equal to the sum of the kinetic energy of the crossbar and the heat dissipated in the crossbar.

**NONLINEAR PROBLEM**

Eq. (4) becomes a nonlinear differential equation, if any of the parameters \( B_o, Y_o, M_o \), or \( R_o \) becomes explicitly or implicitly time dependent. For undergraduates the extension of the model to account for the resistance of the rails is both a realistic and easily visualized complication. In such a case, \( R_o \) in Eq. (1) is to be replaced by \( (R_o + 2rx) \), where \( r \) is the resistance per unit length of the rails.

Eq. (4) may then be rewritten as

\[
\ddot{x} + P(x-x_0)\dot{x} + \left( \frac{1}{\tau} \right)x - \left( \frac{V_f}{\tau} \right) = 0
\]  \hspace{1cm} (10)

where \( \tau = \frac{M_o (R_o + 2rx_0)}{B_o^2 Y_0^2} \) \hspace{1cm} (11)

and \( P = 2r/(R_o + 2rx_0) \) \hspace{1cm} (12)
The nonlinear equation (10) has been solved [3] by the usual perturbation procedure [2]. However, it can be much more easily solved by integrating once to obtain a first-order differential equation and then developing a simple Fortran program to obtain the solution on a digital computer. Such an approach can be better appreciated by an undergraduate student, as it would be more consistent with his academic background.

**METHOD OF SOLUTION TO THE NONLINEAR EQUATION**

The second order nonlinear differential equation (10) will now be transformed into a first order differential equation. For convenience one may define new variables:

\[ \xi = 1 + P(x - x_0) \]  

(13a)

from which it follows that

\[ \xi = P \)  

(13b)

and

\[ \xi = P \]  

(13c)

Also let

\[ \eta = \dot{\xi} \]  

(14a)

from which it follows that

\[ \ddot{\xi} = \eta = \frac{dn}{dc} \]  

\[ \xi = \eta \]  

(14b)

Using the new variables as defined by Eqs. (13) and (14), one may rewrite Eq. (10) as follows:

\[ \frac{\eta}{P} \frac{dn}{dc} + \frac{r}{P^2} + \frac{V_f}{r} = 0 \]  

(15a)

which may be rearranged as

\[ \frac{\eta}{P} \frac{dn}{dc} = \frac{d\xi}{\xi} \]  

(15b)
Integrating both the sides of Eq. (15b), one obtains
\[ -\tau' \eta - \tau' PV_f \ln \left( \frac{\eta}{\tau} - \frac{\eta_0}{\tau} \right) = \ln \xi + K \]  \hspace{1cm} (16)

where \( K \) is a constant to be evaluated from the initial conditions given by
\[ \eta = \eta_0 \text{ at } t = 0, \quad x = x_0 \text{ and } \dot{x} = 0 \] \hspace{1cm} (17a)
\[ \xi_0 = 1 \text{ and } \eta_0 = 0 \] \hspace{1cm} (17b)

The constant \( K \) may be found to be
\[ K = -\tau' PV_f \ln \frac{\eta}{\tau} \] \hspace{1cm} (18)

On substituting the value of \( K \) and rearranging terms, one obtains
\[ -\tau' \eta - \tau' PV_f \ln(1 - \frac{\eta}{\eta_0}) = \ln \xi \] \hspace{1cm} (19)

Using the definitions given by Eqs. (13) and (14), replacing \( \xi \) and \( \eta \) in terms of the original variables \( x \) and \( t \), the following equation can be written:
\[ PV_{f} \dot{x} + PV_{f} \ln(1 - \frac{\eta}{\eta_0}) + \ln(1 + P(x - x_0)) = 0 \] \hspace{1cm} (20)

Another change of variables will be made for convenience as follows:
\[ \tau = PV_{f} \tau' \] \hspace{1cm} (21a)
\[ x = \frac{x}{\tau} \] \hspace{1cm} (21b)
\[ x_0 = \frac{x_0}{\tau} \] \hspace{1cm} (21c)
\[ t = \frac{t}{\tau} \] \hspace{1cm} (21d)

Using the above, Eq. (20) may be rewritten as
\[ \Gamma \frac{dx}{dt} + \ln[1 - \frac{dx}{dt}] \Gamma + \ln[1 + \Gamma(x-x_0)] = 0 \quad (22) \]

A simple digital-computer program in Fortran language is written to yield an iterative solution of the Eq. (22), while \((x-x_0)\) goes from 0 to 10 in 100 equal steps, and the corresponding time step is estimated as

\[ T_{n+1} = T_n + (\Delta T)_n \quad (23a) \]

where \(\Delta T\) is approximately given by

\[ \Delta T = \frac{\Delta x}{\frac{dx}{dt}} = \frac{0.01}{(dx/dt)} \quad (23b) \]

Plots of time dependence of the position of the crossbar \((x-x_0)\) VS \(T\) and plots of time dependence of the velocity of the crossbar \((\frac{dx}{dt})\) VS \(T\) for different values of \(\frac{1}{0}, 0.0, 1.0,\) and 10.0) are shown in Figures 2 and 3.

It may be finally recalled that

\[ \Gamma = PV_{(f)} = \frac{2M_0V_f}{(B_0Y_0)^2} \cdot r \quad (24a) \]

\[ x = \frac{M_0(R_0 + 2xx_0)}{(B_0Y_0)^2} \quad (24b) \]

and

\[ \dot{x} = V_f \frac{dx}{dt} \quad (24c) \]
DISCUSSION OF THE RESULTS AND CONCLUSIONS

The normalized velocity \( \frac{dx}{dt} \) increased monotonically from 0 to 1, as expected, as can be seen from Fig. (3). As the value of \( \gamma \) is made to increase, the velocity that is reached in a given time decreases. This behaviour can be predicted by the dependence of \( \Gamma \) on the resistance \( r \) per unit length of the rails given by Eq. (24a).

A particular feature of the computer program may be mentioned here: As \( (x-x_0) \) and \( \frac{dx}{dt} \) appear in Eq. (22) to be solved, steps were chosen for the normalized distance \( (x-x_0) \) conveniently instead of for the normalized time \( T \).

The nonlinear problem described in Fig. 1 has been solved through a change of variables, a single integration, and a simple digital-computer-program in Fortran language. While the nonlinear equation (10) can be solved by the method of perturbations, or by numerical methods, or by some other sophisticated procedure normally used to solve the nonlinear equations, it has been successfully solved here much more easily.

The procedure that is employed in this paper to solve the nonlinear equation is simple enough to be introduced at an undergraduate level for the engineering students. The extension of homework problems to include realistic complications of nonlinearity and extensive use of the available computing facilities would indeed better prepare good engineers to the present-day challenging industry. This point of view has been illustrated through a simple Faraday's law problem of an electromechanical system that exhibits nonlinear effects if the resistance of the rails is taken into account.
REFERENCES


Fig. 2 Plot of the time dependence of the position of the crossbar.
FIG. 3. Plot of the time dependence of the velocity of the crossbar.