Negative Refraction in Photonic Crystals and Metamaterials for Transformation Optics

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ABSTRACT OF DISSERTATION

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Abstract

This thesis is about metamaterials and its application in negative refraction, slow light and transformation optics. Metamaterials have properties not available in nature and are typically man-made. They have exciting applications such as invisible cloaking, modulators, band-pass filters, perfect lenses, subwavelength resolution, beam compressors, slow-light devices and so on. We demonstrated one-dimensional, two-dimensional photonic crystals negative lenses, prisms, flat lens theoretically and experimentally at both microwave and optical frequency. We proposed a new mechanism on surface grating to realize negative refraction in photonics crystals. This approach can be used to image large and/or far way objects. We studied the slow light effect in extremely anisotropic nano-wire and planar waveguide consisting of conventional dielectric cladding with single-negative materials. Super-resolution imaging in three-dimensional metamaterial nanolens consisting of nanowires template is also demonstrated.

We also studied the optical transmission at the interface of two transformation optics media or between a transformation optics medium and free space. We derived the generalized reflectionless boundary condition for the interface, which makes possible to guide and focus light without reflection. Flat lens and non-magnetic beam concentrator are two examples of this new design. They showed great advantage to previous research results in literature.
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Introduction

This thesis is about metamaterials and its application in negative refraction, slow light and transformation optics. The world “meta” means “beyond” in Greek, and in this sense, the name “metamaterials” refers to “beyond conventional materials”. Metamaterials have properties not available in nature and are typically man-made. All known materials have a positive index of refraction; whereas artificially engineered metamaterial can exhibit a negative index of refraction. They utilize both field components: electric and magnetic field, enabling entirely new optical properties and exciting applications such as invisible cloaking, modulators, band-pass filters, perfect lenses, subwavelength resolution, beam compressors, slow-light, supercollimation etc. This gives particular advantage over conventional materials.

Metamaterial consist of periodic structures. There are two methods to achieve metamaterials. One way is to use composite structures made of repeated unit cells of split ring resonator (SRR) and wire; and the other way is to photonic crystal (PhC) structures whose dielectric constant is periodically modulated.

One of the applications of metamaterials is to create negative refractive index structures that are not found in nature. Almost all materials encountered in optics, such as glass or water, have positive values for both permittivity $\varepsilon$ and permeability $\mu$. However, many metals (such as silver and gold) have negative $\varepsilon$ at visible wavelengths. In the optical frequencies, metal could be described in Drude model. $\varepsilon(\omega) = \varepsilon_{\infty} - \frac{\omega_p^2}{\omega(\omega + i\gamma)}$, where $\omega_p$ is the plasma frequency, and $\omega, \gamma$
are the radiation and scattering frequencies, respectively. A material having either (but not both) \( \varepsilon \) or \( \mu \) negative is opaque to electromagnetic waves.

Although the optical properties of a material are fully specified by the parameters \( \varepsilon \) and \( \mu \), refractive index \( n \) is often used in practice, which can be determined by \( n = \pm \sqrt[+]{\varepsilon \mu} \). All known non-metamaterial materials have positive \( \varepsilon \) and \( \mu \). By convention the positive square root is used for \( n \). However, some engineered metamaterials have \( \varepsilon < 0 \) and \( \mu < 0 \). Because the product \( \varepsilon \mu \) is positive, \( n \) is real. This kind of material – is also called as “Left-Handed Materials” (LHM). The relation of \( E, H \) and \( k \) in these materials form a “left-handed” system. Under such circumstances, it is necessary to take the negative square root for \( n \). Physicist Victor Veselago proved that such substances can transmit light and have negative refractive index.[1].

In the Chapter 1, Sec I, we demonstrated one-dimensional (1D) dielectric photonic crystals (PCs) at microwave frequencies. Focusing by planoconcave lens made of 1D PCs due to negative refraction is also demonstrated. The frequency-dependent negative refractive indices, calculated from the experimental data, match very well with those determined from band structure calculations. In Sec II, we have experimentally realized an ultra short focal length planoconcave microlens in an InP/InGaAsP semiconductor two-dimensional (2D) photonic crystal with negative index of refraction (\( n_{\text{eff}} = -0.7 \)) at \( \lambda = 1.5 \) \( \mu \)m, the lens exhibits ultra short focal lengths of 12 \( \mu \)m (\( \sim 8\lambda \)) and numerical aperture close to unity. The focused beam has a near diffraction-limited spot size of 1.05 \( \mu \)m (\( \sim 0.68\lambda \)) at full width at half maximum. The negative refractive index and focusing properties of the microlens are verified by 2D finite-difference time-domain (FDTD) simulations. Such ultra refractive negative-index nano-optical
microlenses can be integrated into existing semiconductor hetero-structure platforms for next-generation optoelectronic applications. In Sec III, we compared the imaging performance of three different types of lenses. In Sec IV, we demonstrated experimentally negative refraction by a photonic crystal prism and imaging of a point source by a photonic crystal slab at 1.5 µm wavelength. The photonic crystal structures were nanofabricated in an InP/InGaAsP heterostructure platform, and optical characterization was performed using a near-field scanning optical microscope. By designing a suitable lens surface termination, an image spot size of $0.122\lambda^2$ was achieved, demonstrating superlens imaging with subwavelength resolution well below Abbe’s diffraction limit of $0.52\lambda$.

Refraction at a smooth interface is accompanied by momentum transfer normal to the interface. In Chapter 2, we show that corrugating an initially smooth, totally reflecting, non-metallic interface provides a momentum kick parallel to the surface, which can be used to refract light negatively or positively. This new mechanism of negative refraction is demonstrated by visible light and microwave experiments on grisms (grating-prisms). Single-beam all-angle-negative-refraction (AANR) is achieved by incorporating a surface grating on a flat multilayered material. This negative refraction mechanism is used to create a new optical device, a grating lens. A plano-concave grating lens is demonstrated to focus plane microwaves to a point image. These results show that customized surface engineering can be used to achieve negative refraction even though the bulk material has positive refractive index. The surface periodicity provides a tunable parameter to control beam propagation leading to novel optical and microwave devices. In section II, we show that with an appropriate surface modification, a slab
of photonic crystal can be made to allow wave transmission within the photonic band gap. Furthermore, AANR can be achieved by this surface modification in frequency windows that were not realized before in two-dimensional photonic crystals [3]. This approach to AANR leads to different applications in flat lens imaging. Previous flat lens using photonic crystals requires object-image distance \( u+v \) less than or equal to the lens thickness \( d \), \( u+v \ll d \). Our approach can be used to design a flat lens with \( u+v=\sigma d \) with \( \sigma \gg 1 \), thus being able to image large and/or far away objects. We use FDTD to confirm our results. In Sec III, we show that a binary-staircase optical element can be engineered to exhibit an effective negative index of refraction, thereby expanding the range of optical properties to be theoretically available for future optoelectronic devices. The mechanism for achieving a negative-index lens is based on the periodicity of the surface corrugation. By designing and nanofabricating a planoconcave binary-staircase lens in the InP/InGaAsP platform, we have experimentally demonstrated at 1.55 \( \mu m \) that such negative index concave lenses can focus plane waves. The beam propagation in the lens was studied experimentally and was in excellent agreement with the three-dimensional FDTD simulations.

In Chapter 3, we studied the exact solutions for all propagation modes along an extremely anisotropic cylindrical wave-guide. Closed-form expressions for the energy flow on the waveguide are also derived. For extremely anisotropic waveguide where the transverse permittivity is negative \( (\varepsilon_\perp < 0) \) while the longitudinal permittivity is positive \( (\varepsilon_\parallel > 0) \), only transverse magnetic TM and hybrid modes will propagate on the waveguide. At any given frequency the waveguide supports an infinite number of eigenmodes. Among the TM modes, there is at most one forward mode, the rest of them are backward waves which can have very
large effective index. At a critical radius, the waveguide supports degenerate forward- and backward-wave modes with zero group velocity. These waveguides can be used as phase shifters and filters, and as optical buffers to slow down and trap light.

In Chapter 4, we show that a non-resonant planar waveguide consisting of conventional dielectric cladding with single-negative materials supports degenerate propagating modes for which the group velocity and total energy flow can be zero if the media are lossless. Absorptive losses will destroy the zero-group velocity condition for real frequency/complex wave vector modes. We show that by incorporating gain $G$ into the core dielectric, there exists a critical gain value $G_c$ at which we can recover the condition of zero group velocity, so that light pulses can be stopped and stored. This structure is simpler to achieve than double-negative metamaterials, has small footprint, and can be incorporated into ultra-compact on-chip optoelectronics.

Super-resolution imaging beyond Abbe’s diffraction limit can be achieved by utilizing an optical medium or “metamaterial” that can either amplify or transport the decaying near-field evanescent waves that carry subwavelength features of objects. Earlier approaches at optical frequencies mostly utilized the amplification of evanescent waves in thin metallic films or metal-dielectric multilayer structures, but were restricted to very small thicknesses ($<<\lambda$, wavelength) and accordingly short object-image distances, due to losses in the material. In Chapter 5, we present an experimental demonstration of super-resolution imaging by a low-loss three-dimensional metamaterial nanolens consisting of aligned gold nanowires embedded in a porous alumina matrix. This composite medium possesses strongly anisotropic optical properties with negative permittivity in the nanowire axis direction, which enables the transport of both far-field
and near-field components with low-loss over significantly longer distances (>6\(\lambda\)), and over a broad spectral range. We demonstrated imaging of large objects, having subwavelength features, with a resolution of at least \(\lambda/4\) at near-infrared wavelengths. The results are in good agreement with a theoretical model of wave propagation in anisotropic media.

In chapter 6, we studied the optical transmission at the interface of two transformation optics media or between a transformation optics medium and free space. We derived a generalized reflectionless boundary condition for the interface, which makes guiding and focusing light without reflection possible. Two examples are analyzed: (i) a flat lens, which can focus plane waves to a single point, and (ii) a non-magnetic beam concentrator, which can concentrate an incident beam smoothly to a narrower beam. The performance of these structures is confirmed by 2D full-wave simulations.
Chapter 1 Negative Refraction in Photonic Crystals

I. Negative Refraction and Plano-Concave Lens Focusing in One-Dimensional Photonic Crystals

Metamaterials are artificial materials engineered to provide properties which “may not be readily available in nature”. These materials usually gain their properties from structure rather than composition, using the inclusion of small inhomogeneities to enact effective macroscopic behavior.[1][2][3]

The primary research in metamaterials investigates materials with negative refractive index.[4][5][6] Negative refractive index materials appear to permit the creation of 'superlenses' which can have a spatial resolution below that of the wavelength, and a form of 'invisibility' has been demonstrated at least over a narrow wave band. Although the first metamaterials were electromagnetic,[4] acoustic[7] and seismic metamaterials[4] are also areas of active research.

Negative refraction and related phenomena such as flat lens imaging [2][3][4] and plano-concave lens focusing[5] have attracted a lot of attention in physics and engineering. Negative refraction allows subwavelength imaging[6] and focusing of far field radiation by concave rather than convex surfaces with the advantage of reduced aberration[7][8] for the same radius of curvature. Negative refraction has been realized in two- and three-dimensional structures in metamaterials [9][10] and photonic crystals [11][12]. But negative refraction has not yet been demonstrated in one-dimensional (1D) photonic crystals (PCs).
In this chapter we present a study of left-handed electromagnetism in 1D PCs at microwave frequencies. Negative refraction is achieved in the second band of the 1D PCs. Focusing of plane wave radiation by plano-concave lenses made of 1D PCs is also demonstrated. The inverse experiment, in which the lens produces plane waves from a point source placed at the focal length, at the same frequency of operation, is confirmed as well. The frequency-dependent negative refractive indices, calculated from the experimental data, match very well with those determined from band structure calculations. The easy fabrication of 1D PCs may open the door for microwave and optical applications.

![Fig. 1-1 Schematic diagram of the microwave experimental setup and negative refraction of plane waves by the 1D PC1 prism. Angle of incidence is 45° and angle of refraction is −36.9° resulting in negative refraction with refractive index \( n_p = -0.85 \) at 10.55 GHz. The real part of \( S_{21} \) scale: on the left side varies from −0.015 to 0.015, on the right side from −0.006 to 0.006.](image)

The experiments were carried out in a parallel-plate waveguide of height \( h = 1.25 \) cm and size of \( 3 \times 6 \) ft\(^2\). For frequency below 12 GHz, the excitation in these quasi-two dimensional system is the transverse magnetic (TM) modes with the electric field in the vertical direction. The electric field of the microwaves is scanned using a monopole antenna attached to an X-Y robot in the frequency window of 3–11.5 GHz. An HP-8510C network analyzer is used for...
measuring the transmission characteristics. A schematic diagram of the experimental setup is shown in Fig. 1-1 and Fig. 1-2.

Alumina bars with permittivity $\varepsilon = 8.9$ were laid out to form prisms of right angle triangles. The bars have a height $h = 1.25$ cm and width $d = 0.5$ cm. The refraction experiments were performed on two PC prisms. The first prism PC1 has a lattice constant $a = 1$ cm and incident angle of $45^\circ$ while the second one PC2 has $a = 0.8$ cm and angle of $51^\circ$. All bars have a perpendicular cut instead of a slanted one, as numerical simulations show that a perpendicular cut reduces the modulations of the outgoing waves, partly due to the absence of sharp corners. A plane wave incident normally to one surface is refracted by the hypotenuse of the 1D PC prism as shown in Fig. 1-1, defined as the surface of refraction.
The 1D PC is a model that is exactly soluble [13]. For the TM modes, it is just the Kronig-Penney model [14] with an energy dependent potential. The equi-frequency and band structure for the TM modes for the filling factor \( d / a = 0.5 \) are shown in Fig. 1-3(a)–(c). A band gap for normal incidence is located between 5.55 and 8.9 GHz. The second pass-band is between 8.9 and 12.7 GHz and has negative refractive index. The refraction of a microwave beam by the PC1 prism at 10.55 GHz is shown in Fig. 1-1. By fitting the outgoing beam with a plane wave, the wave front is determined and an effective index \( n_p = -0.85 \) is obtained using Snell’s law.

Fig. 1-2 (a) and (c) The Geometry of 1D PC and incident plane wave, (b) Band structure the TM modes of the 1D PC with \( a = 1 \text{cm}, \ d = 0.5 \text{cm}, \ \text{and} \ \varepsilon = 9 \). (d) the photographs of the samples used in experimental setup.
Although positive refraction is predicted for both PC prisms in the first band, the incident angle for each PC prism exceeds the corresponding critical angle, resulting in total internal refraction. The average power of the scanned points is plotted as a function of frequency in Fig. 1-3(d). While the outgoing signal is very weak for the total internal reflection and band gap frequency regions, an interesting observation is that there is a strong leaking from the prism near the edges of the band gap.
The same alumina bars were used to form a plano-concave lens of 1D PCs, with $a = 1$ cm. The concave radius of the lens is $R = 18$ cm. A sharp focal point is located at 6.15 cm away from the curved surface when a microwave beams incidents at frequency of 9.5 GHz. From left to right in Fig. 1-4, the incoming plane wave, a real picture of the PC lens and the emerging mapped field are shown. Clear focusing is observed in the frequency range of 9.2–11.5 GHz.
An inverse experiment in which a point source is placed at the observed focal point of the lens at a single frequency is also carried out. As shown in Fig. 1-5, a circular wave front from the point source after passing through the lens emerges as a plane wave. These two remarkable results validate the behavior of a left-handed plano-concave lens. The photograph of the experimental setup is shown in Fig. 1-6.

Fig. 1-5 Field maps of the incident source and the emerging plane wave. Scale: on the left side intensity varies from 0 to 0.14, on the right side the real part of $S_{21}$ from $-0.03$ to 0.02. The source is placed at the focal length 6.15 cm and plane wave is observed at 9.55 GHz.
The refractive index of the lens can be estimated using the lens equation \( n = 1 - R / f \) which is valid for thin planoconcave and planoconvex lenses in the geometric optics limit. Here \( R \) is the radius of curvature and \( f \) is the focal length. Using this equation we get \( n = -0.85 \) at 9.5 GHz for the lens shown in Fig. 1-4. A real focus by a planoconvex lens is achieved with \( n > 1 \) and \( R < 0 \) while for the planoconcave lens with \( n < 1 \) and \( R > 0 \).

Fig. 1-6 Photographs of the samples used in the experimental setups. (a) A plane wave hits the 1D PhC lens after traveling for 4.5ft from the source. (b) Photograph of the inverse setup.
The refractive indices $n_p$ determined from the prism refraction experiments (PC1 and PC2) and the plano-concave lens one are shown in Fig. 1-7. Very good agreement with those calculated from the band structure is observed. The index $n_p$ determined from the focusing experiment fits better with theoretical results as the frequency is increased. This may be due to the reduced finite-size effect and aberration at higher frequency.

The nature of the left-handed electromagnetism and focusing can be understood from the dispersion characteristics of the 1D PCs. From the equi-frequency surfaces shown in Fig. 1-3(b), it can be deduced that in the second band the wave vector is in the opposite direction to group velocity, $\vec{v}_g \cdot \vec{k} < 0$, resulting in negative refraction in the second band and correspondingly negative refractive indices.
The bandwidth for obtaining a sharp focus point is a crucial parameter for applications of the left-handed lenses. Due to the resonant nature of the metamaterial the bandwidth is usually restricted to a narrow region and the dispersion is strong.[15] The PC1 reveals a wide bandwidth of 3.8GHz, which is 35% at the current operating frequencies. The weaker dispersion in the PC makes it a better candidate for focusing a pulse or broadband radiation.

The present PC lens with negative refraction has several advantages when compared to the one with positive refraction. Lenses with reduced geometric aberrations produce sharper image with enhanced resolution and find numerous applications. Larger radius of curvature gives the advantage of reduced aberration in the image formed. A PC lens having the same focal length as that of a conventional lens weighs far less, and is attractive to space applications. The tailor made refractive index achievable in PCs (Ref. [16]) allows further control on the focal length and thereby helps to reduce the size of the optical systems.

In conclusion the feasibility of designing a 1D broadband left-handed PC lens is experimentally demonstrated. Negative refraction of plane waves and plano-concave lens focusing is achieved in 1D PCs. The focal length follows the standard laws of geometrical optics combined with negative refraction. The measured values of refractive indices of the lens from both refraction and focusing experiments are in excellent agreement with those determined from band structure calculations. Earlier works have shown that 1D PCs can be used as omni-directional reflectors [17][18] The observed negative refraction in 1D PCs reported here, adds diversity to these simple systems.

II. Optical Experiment of 2D PC Plano-concave Lens
The pressing need for miniaturization in the microelectronics industry and the advent of fiber optics for communications purposes have led to the emergence of microlenses\cite{19} of dimensions less than a millimeter down to a few microns. Microlens arrays are used to increase the light collection efficiency, particularly in the infrared spectrum, in charge coupled devices (CCDs) that form the backbone of modern digital cameras and other sensor-type devices. As the demand for higher resolution digital cameras continues to rise, the pixel size of CCDs will shrink and lead to a need for much smaller microlenses having shorter focal lengths, less spherical aberrations, larger numerical aperture, and diffraction-limited beam spot size.

Parallel to the evolution of the micro-optics area, theoretical investigations on negative refractive index metamaterials\cite{20}\cite{21}\cite{22} and photonic crystals \cite{23}\cite{24}\cite{25}\cite{26}\cite{27} (PhCs) have resulted in a new breed of artificially engineered metamaterials, \cite{28}\cite{29}\cite{30}\cite{31} exhibiting a myriad of interesting properties \cite{33}\cite{34}\cite{35}\cite{36}\cite{37}\cite{38} Because the band structure and isofrequency contours of PhCs are so different from the circles or ellipsoids of a homogeneous dielectric medium, a number of exotic effects can be obtained. By appropriately designing the PhC, one can emulate the behavior of a negative refractive index homogeneous metamaterial,
where the negative index arises when both $\varepsilon$ and $\mu$ are negative. For strongly modulated PhCs in the vicinity of the photonic bandgap, an effective phase refractive index can be defined, which can be controlled by the band structure. This effective refractive index can be less than unity and can also be negative.

When the refractive index $n$ is negative, focusing can be achieved using planoconcave lenses, in contrast to convex lenses necessary for ordinary positive index materials. Negative-index planoconcave PhC lenses can be superior to planoconvex microlenses[39] as they possess a larger numerical aperture (close to unity), diffraction-limited spot size, display less spherical aberrations [40] and have shorter focal lengths. These properties make negative-index nano-optical microlenses ideal candidates for integration in future CCDs and other next-generation optoelectronic applications. Preliminary negative refraction experiments on two-dimensional (2D) planoconcave PhCs have been realized earlier, in the microwave spectrum, by our group [41] and others.[42] However, the PhCs used by our group in the microwave regime consisted of an assembly of commercial alumina rods, placed in a parallel plate waveguide, which are unsuitable for nano-optics applications.

In this section, we report the first experimental realization of an ultrarefractive (refractive index less than 1 and close to zero) negative-index 2D PhC microlens at optics, nanofabricated in an InP/InGaAsP semiconductor heterostructure platform. I would like to mention Dr. Bernard Didier F. Casse did most of the experiment measurements and I did all the theoretical calculation and simulations. Dispersion engineering principles were used to design the PhC microlens with an effective phase refractive index $n_p \sim 0.7$. In the case described here, the 2D PhCs behave like
an isotropic dielectric medium, where refraction can be described by Snell’s law. The ultrarefractive 2D PhC microlens was illuminated with near-infrared light at normal incidence. The wave field at the focal point was mapped with a near-field scanning optical microscope (NSOM). The study of the beam propagation in the PhC microlens was carried out using 2D finite-difference time-domain (FDTD) simulations. The numerical simulations were found to be in excellent agreement with the experimental results.

To illustrate this, we consider a 2D PhC consisting of a square lattice of air holes, with diameter of 295 nm and lattice spacing of 470 nm, in an InP/InGaAsP heterostructure dielectric medium, as shown in Fig. 1-8(a). This heterostructure platform is made up of a 400 nm InGaAsP core layer sandwiched between a 200 nm InP top cladding layer and a 300 µm bottom cladding InP substrate layer, as shown in Fig. 1-8(b). The waves are trapped and propagated within the core layer plane with an effective refractive index of 3.231 transverse electric (TE) modes and 3.216 transverse magnetic (TM) modes. The dielectric medium is assumed to be nonmagnetic and thus $\mu = 1$. The band structure of this lattice, which shows the set of available states $\omega(k)$ around the boundaries of the Brillouin zone, is illustrated in Fig. 1-9(a), where $k$ is the Bloch wave vector and $\omega$ is the frequency of the propagating waves. Fig. 1-9(a) reveals that this 2D PhCs will exhibit an effective negative phase refractive index in the second band close to the photonic bandgap. To show this more clearly, we make a contour plot (or isofrequency diagram) of $\omega(k_x,k_y)$ in the $(k_x,k_y)$ plane, showing the periodic curves of constant $\omega$. Given the isofrequency diagram illustrated in Fig. 1-9(b), one can determine the number of refracted waves (if any), and in what states or directions they propagate.
The propagation direction is determined by the group velocity $\mathbf{v}_g = \nabla_k \omega$, which is perpendicular to the $\omega$ contours and points in the direction of increasing $\omega$. Here we observe that the isofrequency contours in the second band move inward with increasing $\omega$, indicating that $\mathbf{v}_g \cdot \mathbf{k} < 0$. This implies that the phase refractive index $n_p$ is negative. From band structure calculations, $n_p \sim -0.7$.

The propagation direction is determined by the group velocity $\mathbf{v}_g = \nabla_k \omega$, which is perpendicular to the $\omega$ contours and points in the direction of increasing $\omega$. Here we observe that the isofrequency contours in the second band move inward with increasing $\omega$, indicating that $\mathbf{v}_g \cdot \mathbf{k} < 0$. Since the phase refractive index $n_p$ adopts the same sign as $\mathbf{v}_g \cdot \mathbf{k}$, then $n_p < 0$. Fig. 1-9(b) also reveals that the isofrequency contours are circular near the bandgap, which means that we can define an effective refractive index from the radius of the isofrequency contours using Snell’s law, i.e., the formula $|\mathbf{k}| = |n_p| \omega/c$ applies in this case. Band structure calculations indicate that $n_p \sim -0.7$ around $\lambda = 1.5 \mu$m wavelengths.
Interestingly, ultrarefractive planoconcave lenses have less spherical aberration than their planoconvex analogs in micro-optics. This can be deduced by deriving an analytical formula to quantify the extent of spherical aberrations of a lens. Spherical aberrations originate from the angular dependence $\theta$ on the focal length $f(\theta)$. It implies that if $f(\theta)/f(0)$ is relatively flat for all angles, then the aberrations are very small. For both planoconvex and planoconcave lenses,

$$\frac{f(\theta)}{f(0)} = 1 - \frac{n}{1+n} \left[ n(1 - \cos \theta) + 1 - \sqrt{1 - n^2 \sin^2 \theta} \right]$$  \hspace{1cm} (1-1)
From Eq.(1-1), if \( n \sim 0 \), the spherical aberrations are small. The aberrations will vanish if \( n = 0 \). Negative-index planoconcave lenses also have shorter focal lengths than their planoconvex counterpart. This can be seen from the geometrical optics formula 
\[ f = \frac{R}{n - 1}, \forall n \]. In our design, with \( R = 20 \, \mu m \), the focal point is located around \( \sim 12 \, \mu m \) away from the concave face.

Since the focal point is inward and very close to the concave face for negative-index planoconcave designs, then the opening angle \( 2\alpha \to \pi \). Thus the numerical aperture ( \( NA = n_{air} \sin \alpha \) ) is close to 1.

For optical characterization purposes, the final structure design consisted of the following three subcomponents [shown in Fig. 1-10(a)]. (i) A 0.501 mm long waveguide, having 5 \( \mu m \) wide trenches on each side, for transporting light to the microlens; the waveguide was inversely tapered 100 \( \mu m \) from the end, with a tapered core width varying from 5 to 40 \( \mu m \). (ii) The negative-index 2D PhC ultrarefractive microlens consists of air holes with a diameter of 295 nm and lattice spacing of 480 nm [shown in Fig. 1-10(b)]. (iii) Finally an open cavity (semicircle juxtaposed to a \( 30 \times 70 \, \mu m^2 \) rectangle) at the end of the ultrarefractive microlens. The design was patterned on polymethylmethacrylate (PMMA) polymer resist using electron beam lithography. The etching in the InP/InGaAsP heterostructure layers was a two-step process. (a) The pattern defined in the PMMA polymer was first transferred to a silicon nitride hard mask. (b) The pattern was then subsequently transferred to the InP/InGaAsP, both using reactive ion etching processes.
In the characterization experiment, light is butt coupled into the end of the waveguide, at the cleaved facet, with a monomode optical fiber mounted on a $xyz$ positioning stage. The light in the optical fiber originates from a continuous wave tunable semiconductor laser (1550–1580 nm) through a series of optical components. In order to reduce spatial misalignment in the butt coupling process, an infrared camera connected to a microscope port was used as a visual aid. The beam travels in the 5 $\mu$m wide integrated waveguide for a distance of 0.5 mm before expanding in the inverse taper and reaching the planoconcave microlens with a transverse size of 40 $\mu$m. After emerging from the 2D PhC microlens, the planar wave front is expected to focus in the air cavity, at a distance of 12 $\mu$m away from the concave face. The optical measurement was performed with a near field optical microscope (Nanonics MultiView 2000™) where a tapered fiber probe, with an aperture diameter of 150 nm, was used in collection mode. The fiber tip was raster scanned at a constant height of about 500 nm above the sample surface, allowing us to map

Fig. 1-11 (a) NSOM reconstructed image of the transmitted field in the air cavity at 1530 nm. Focusing is observed $\sim$ 12 $\mu$m away from the concave face. The FWHM of the beam spot size is in the order of $\sim$ 0.68$\lambda$ (shown in the inset picture). The white dashed lines indicate the beam path. Note that (a) is a composite picture consisting of a SEM image of the lens superposed on a NSOM scan. (b) Electric field intensity plot of FDTD simulations results of the planoconcave microlens. For better visual illustration the radius of the concave face in the Z plane has been truncated to 14 $\mu$m, while the transverse size (X plane) is still maintained at 40 $\mu$m. The focusing effect occurs 12 $\mu$m away from the concave face matching the experimental results. The FWHM of the focused beam is 0.67$\lambda$. 

In the characterization experiment, light is butt coupled into the end of the waveguide, at the cleaved facet, with a monomode optical fiber mounted on a $xyz$ positioning stage. The light in the optical fiber originates from a continuous wave tunable semiconductor laser (1550–1580 nm) through a series of optical components. In order to reduce spatial misalignment in the butt coupling process, an infrared camera connected to a microscope port was used as a visual aid. The beam travels in the 5 $\mu$m wide integrated waveguide for a distance of 0.5 mm before expanding in the inverse taper and reaching the planoconcave microlens with a transverse size of 40 $\mu$m. After emerging from the 2D PhC microlens, the planar wave front is expected to focus in the air cavity, at a distance of 12 $\mu$m away from the concave face. The optical measurement was performed with a near field optical microscope (Nanonics MultiView 2000™) where a tapered fiber probe, with an aperture diameter of 150 nm, was used in collection mode. The fiber tip was raster scanned at a constant height of about 500 nm above the sample surface, allowing us to map

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the optical intensity distribution over a grid of 256 × 256 points spanning an area of 30 × 30 µm².

The output end of the fiber probe was connected to a nitrogen cooled germanium detector (North Coast Scientific Corp., model No. EO-817L). Additionally, a typical lock-in amplifier and optical chopper were utilized to minimize the signal-to-noise ratio. The reconstructed image, at a wavelength of 1530 nm, is shown in Fig. 1-11(a). Note that Fig. 1-11(a) is a composite picture consisting of a scanning electron microscopy (SEM) image of the lens superposed on a NSOM scan. The high field intensity distribution near the cavity center clearly demonstrates the focusing property of the planoconcave microlens. The focused beam spot has an experimental full width at half maximum (FWHM) of ~0.68 λ, as shown in the inset picture in Fig. 1-11(a). Identical focusing was observed over a range of wavelengths varying from 1510 to 1560 nm.

The results were further confirmed using 2D FDTD simulations. A 40 µm wide plane parallel TE-polarized Gaussian beam was chosen as incident field excitation for the microlens in order to match the experimental conditions. The FDTD simulation results of the microlens for the electric field intensity are shown in Fig. 1-11(b). The calculated FWHM from simulations is 0.67 λ.

We have experimentally realized an ultrarefractive 2D PhC microlens, exhibiting an effective phase index of refraction of −0.7 and a numerical aperture close to 1, in an InP/InGaAsP platform. The effective refractive index of the PhC lenses can be tuned by properly designing the crystal lattice. Near-field scanning microscopy experiments at infrared wavelengths revealed that the lens focuses plane waves down to a spot size of 0.68 λ (FWHM) with a focal length of 12 µm (~8λ). The type of microlens described in this section is a purely
dielectric system and therefore has a low intrinsic material loss, unlike the traditional metallic resonator metamaterials that suffer from high losses due to the inherent imaginary part of the metal’s permittivity.[42] The extrinsic losses that are expected in this system only originate from the imperfections in the nanofabrication, coupling losses, and interface losses.

We have thus demonstrated a superior negative-index planoconcave microlens having compact footprint, ultra short focal length, diffraction-limited spot size, larger numerical aperture, and reduced spherical aberrations compared to a conventional positive index planoconvex lens. With the high level of design flexibility of PhCs and low-loss dielectric medium, the microlens can be easily scaled to operate in any frequency region.

III. Subwavelength Imaging by 1D Photonic Crystals

In this section, we directly compare three types of lenses: (a) aspherical lens (b) spherical lens and (c) photonic crystals lens.

The geometry of aspherical lens is shown in Fig. 1-12(a). The coordinate of the aspherical lens surface following the equation

\[ z = D - \sqrt{f^2 + x^2 \frac{n+1}{n-1}} - f \]  

(1-2)

Here \( f \) is the focal length and \( n \) is the refractive index. We use \( f = 12 \mu m \) and \( n = 3.3 \) for both lenses.
Fig. 1-12 The geometry of (a) aspherical lens and (b) spherical lens. The focal length is 12 µm and the refractive index is $n = 3.3$ for both lenses.
The geometry of photonic crystals lens. The photonic crystal is made by two materials: air and silica which have $n = 3.3$. The lattice constant is $a = 760$ nm, the thickness of each layer is $h_1 = 228$ nm and $h_2 = 532$ nm. The focal length of this photonic crystals lens is $12 \, \mu m$ and have an effective negative refractive index is $n_e = -0.46$. 

$$f = (1+n_e)R$$

$$z = \frac{f - x}{1 + \frac{n_e}{n_e - 1} \left( \frac{n_e + 1}{n_e + 1} \right)}$$
The geometry of photonic crystals lens is shown in Fig. 1-13. The photonic crystals is made by two materials, one is typical silica which have a refractive index $n_1 = 3.3$ in near IR frequency, the other one is air which have refractive index $n_2 = 1$. The lattice constant is $a_s = 760$ nm, the thickness of each layer is $h_1 = 228$ nm and $h_2 = 532$ nm. The band structure of this 1D PhC is shown in Fig. 1-14. The effective refractive index of this photonic crystal is $n_e = -0.46$. The focal length of this PhC lens is $f = (1 + n_e)R = 12 \, \mu m$, here $R = 22.2 \, \mu m$.

Now we can use FDTD simulations to compare of these three lenses, all of them have same focal length 12 \, \mu m. The intensity distribution is shown in Fig. 1-15.
Fig. 1-15 The intensity of $E_z$ of three lenses.
The FWHM of the focal spot size is compared in Fig. 1-16, in the $x$-direction, FWHM of spherical, aspherical and PhC lens are 0.651$\lambda$, 0.560$\lambda$ and 0.434$\lambda$. In $z$-direction, the FWHM of each lens are 3.86$\lambda$, 2.45$\lambda$ and 0.9$\lambda$. The PhC lens has the best focusing performance.

Fig. 1-16 The intensity of $E_z$ of three lenses. These two figures shows the FWHM of each lenses in $x$ and $z$ directions.
IV. Subwavelength Imaging by Superlens at Infrared Wavelengths

Fig. 1-17 (a) Band structure of a square PhC consisting of air holes with $r/a=0.31$ in $\varepsilon_{\text{eff}}=10.44$ for the TE modes (solid curves) and $\varepsilon_{\text{eff}}=10.34$ for the TM modes (dashed curves). The horizontal dashed line is at $\lambda=1.52 \mu m$ for $a=470 \text{ nm}$. (b) FDTD simulation of TE polarized light beam showing negative refraction at the PhC prism–air interface. The incident beam with $k_i$ is in the dense medium with $\varepsilon_{\text{eff}}$. (c) Image of the right angle PhC prism captured with a 50× long-focal-distance microscope objective. The arrow indicates the direction of input light from the waveguide. The illuminated region inside the PhC prism was imaged with a high-sensitivity near-IR camera, and (d) is a NSOM area scan ($30 \mu m \times 30 \mu m$)

The most celebrated consequence of negative refraction (NR) [43] is the Veselago–Pendry perfect lens [44]. This perfect lens, made of a homogeneous medium, operates at a fixed frequency with $n = \varepsilon = \mu = -1$ and can resolve subwavelength geometrical details of an object by amplification of evanescent waves, theoretically with infinite spatial resolution. However, NR
media constructed with realistic materials, especially metallic-based metamaterials [45][46], are intrinsically lossy (due to the imaginary part of $\varepsilon$ and $\mu$), which limits the resolution of the lens [47]. Nevertheless, subwavelength imaging is still possible to some degree with these imperfect lenses, which we henceforth refer to as “superlenses.” The fundamental challenge of dealing with material losses led some researchers to investigate low-loss dielectric photonic crystals (PhCs) as NR media, in which the behavior of a negative-refractive-index homogeneous metamaterial can be emulated [48][49] by appropriately designing the PhC lattice. In the microwave spectral range, both NR and subwavelength resolution imaging have been clearly demonstrated, while in optics only partial concepts of NR have been shown [50][51][52][53][54][55] mostly with the absence of subwavelength resolution. Notable success has been achieved in constructing the so-called “poor man’s” superlens [44][56], which, however, do not achieve actual focusing of propagating waves, which is required for full imaging.

In this work, we utilize band structure calculations to design and fabricate (i) a negative-index PhC prism and (ii) a flat lens in InGaAsP/InP heterostructure at telecommunication frequencies. Optical experiments were carried out confirming NR by the PhC prism and imaging of a point source by the PhC flat lens. Most importantly, the imaging experiments show that in addition to image formation by propagating waves, reconstruction of evanescent waves (superlensing effect) was also achieved by appropriate modification of the lens surface, leading to a spot size of $0.12\lambda^2$, which is below the diffraction limit ($0.5\lambda^2$).
The fabrication platform consisted of a 400 nm InGaAsP core layer on an InP substrate with a 200 nm InP top cladding layer. The waves are trapped and propagate within the core layer plane with an effective permittivity. At 1.52 μm, the effective permittivity is $\varepsilon_{\text{eff,TE}} = 10.44$ for the TE waves, while $\varepsilon_{\text{eff,TM}} = 10.34$ for the TM waves. Here, we use $n_{\text{InP}} = 3.17$ and $n_{\text{core}} = 3.35$. The band structure of the quasi-2D PhC is then calculated as 2D PhC with a constant effective permittivity by employing a standard plane-wave expansion method. The 2D band structure of a square lattice PhC with air holes of 290 nm diameters and a lattice spacing of 470 nm is shown in Fig. 1-17(a).

We demonstrate NR by a PhC prism created from a square lattice with geometrical dimensions stated above forming a right-angle triangle with sides of equal length of 100 μm. The final structure for optical characterization of the PhC prism includes (i) an array of 0.5-μm-wide, 0.5-mm-long waveguides with 10-μm-wide trenches on each side, and (ii) an air cavity at the hypotenuse interface of the PhC as shown in Fig. 1-17(c). IR light at 1.52 μm was coupled into the single-mode ridge waveguide that interconnects the right-angle PhC prism through a single-mode lensed fiber. The out-of-plane radiated light was imaged from above with a high-sensitivity near-IR camera.

The incident light path due to the radiation from the PhC is clearly seen in the IR image shown in Fig. 1-17(c). To probe the outgoing radiation at the hypotenuse face of the PhC prism–air interface, a 2-μm-deep triangular-shaped air cavity was etched next to the PhC prism. The emerging light inside the air cavity was collected with a 250 nm metalized aperture of a single-mode optical fiber probe of the near field scanning optical microscope (NSOM).
In the NSOM image shown in Fig. 1-17(d), the direction of the emerging beam is clearly negative. The refracted beam angle was consistently found to be about $\sim 30^\circ$ for all waveguide inputs. The incidence angle at the second interface is equal to the prism wedge angle $45^\circ$. The corresponding phase refractive index obtained from Snell’s law is approximately $n_p = -0.71$, which closely matches with the band structure calculation of $n_p = -0.67$. The experimental results were further confirmed by 2D finite-difference time-domain (FDTD) simulations. The FDTD simulation of a TE beam with $\lambda=1.52 \, \mu m$ incident on a PhC prism clearly shows NR in Fig. 1-17(b).

Next, we demonstrate flat lens imaging by the PhC and the superlensing effect by appropriate modification of surface terminations for TE-polarized light. Imaging of a point source is achieved using a PhC flat lens fabricated by patterning a square lattice of air holes on InGaAsP/InP. The PhC flat lens has the same lattice geometrical configuration as the PhC prism used in NR. To facilitate the NSOM measurements, a $10 \, \mu m \times 20 \, \mu m$ rectangular air cavity is etched next to the PhC slab. To avoid overexposing the PhC slab with the electron beam during the fabrication process, a dense medium gap of $\sim 0.7 \, \mu m$ (0.9 \, \mu m) is maintained between the first
(second) air–PhC interface as shown in Fig. 1-18. The dense medium gaps can also serve to localize electromagnetic modes (known as surface states) which may be employed to achieve amplification of evanescent waves [49]. The width of the PhC slab is 7 µm. The point source for the flat slab is realized by fabricating a 5-µm-wide waveguide tapered to 350 nm and having 3-µm-wide trenches on both sides (see Fig. 1-18). For evanescent coupling of the source, a 500 nm air-gap is created between the waveguide tip and the front edge of the PhC slab. The in-plane electromagnetic radiation into air is expected to converge and form an image at the far side of the flat lens.

Fig. 1-19 (a)–(c) show the SEM images of the various designs overlaid with the corresponding NSOM scans at λ=1.52 µm at the object and image planes. (a) Original design A with extra padding dielectric layers produced a focused spot of 1.4λ at FWHM. (b) Design B with a cut through the middle of the air holes at the surface termination of the object plane yielded a focused spot of 1.0λ. (c) Optimized design C with cuts through the middle of the air holes on both sides and waveguide tapered to 50 nm. The focused spot achieved by this design is 0.4 λ. The spatial profile of the focused spot size in the (d) transverse (T) and (e) longitudinal (L) directions for the three designs are presented and compared with a theoretical diffraction limited (DL) profile (λ/2 for T and λ for L). Note that the NSOM

In Fig. 1-18, the IR camera image shows the radiating point source in front of the PhC slab and the corresponding focused image formed ~2.0 µm away from the trailing edge of the flat lens. Here, the PhC images TE-polarized light and spatially disperses any small amount of TM-
polarized light that may be radiated by the waveguide tip. The FWHM of the point source, from NSOM measurements [shown in Fig. 1-19(a)], is $1.4\lambda$.

The PhC flat lens slab also supports surface waves [7] (if surface terminations are properly designed), which can enhance evanescent fields, thus leading to reconstruction of subwavelength details of an object at the image plane. From an effective medium approximation of the PhC[57], different surface terminations can be shown to lead to different excitations of surface modes.

For the lens shown in Fig. 1-19(a) the reconstruction of evanescent waves was not observed mainly due to the weak excitation of surface modes, and, on a minor note, because of the non-ideal point source (350 nm-wide waveguide). We have optimized the design of the superlens to reflect subwavelength resolution enhancement by (i) modification of the surface terminations of the lens, thereby enhancing the excitation of surface modes, and (ii) by fabricating a waveguide tip that can approximate better the radiation field characteristics of an ideal point source (i.e., tapering the waveguide down to 50 nm). In design B [shown in Fig. 1-19(b)], we show that a cut through the middle of the air holes at the surface termination of the object plane leads to a focused spot size with FWHM of $1.0\lambda$. For the optimized design C [shown in Fig. 1-19(c)], we removed the dielectric padding layers and terminated the lens surface by cutting through the middle of the air holes on both sides. Additionally, the waveguide was tapered down to 50 nm to exhibit radiation field patterns that are closer (compared with the 350 nm waveguide tip) to those of an ideal point source. The dramatic reduction in spot size from design A to design C is evident from Fig. 1-19.
The spatial profile of the focused spot size in the transverse (T) and longitudinal (L) directions for the three designs are presented and compared with a theoretical diffraction limited profile (DL) in Fig. 1-19(d) and 3(e), respectively. The spot size shrinks from $1.4\lambda$ (design A) to $0.4\lambda$ (design C). The spot area for the optimized design C ($0.12\lambda^2$) is four times smaller than the diffraction limited spot area $0.5\lambda^2$. This subwavelength image can be mapped by the NSOM only if reconstruction of evanescent waves takes place, along with the propagating waves required for focusing. The images contain some distortions, as observed from their asymmetric nature, since the evanescent components cannot be uniformly enhanced by the surface mode for all wave vectors [49].

In this section, we experimentally demonstrated negative refraction and flat lens imaging in InP/InGaAsP PhC nanostructures at 1.5 µm wavelength. The experimental negative-phase refractive index, calculated from the measured angle of refraction in the PhC prism experiment, is in excellent agreement with the one derived from band structure calculations. Also, the PhC flat lens imaging properties are consistent with the theory of generalized superlens imaging[58]. Most importantly, we have shown experimentally that the PhC lens is more than just a flat lens and that enhancement of evanescent fields (superlensing effect) can be engineered by appropriate truncation of the surface terminations of the lens. Subwavelength resolution of $0.12\lambda^2$ was achieved, which clearly overcomes Abbe’s diffraction limit ($0.5\lambda^2$). This work comes closer than ever before to the realization of a true superlens in optics that recovers evanescent waves on top of the propagating waves required to achieve a real focus or image.

Appendix: FDTD Simulation
Finite-difference time-domain method (FDTD) is a favorite computational electrodynamics modeling technique. It belongs in the general class of grid-based differential time-domain numerical modeling methods. The time-dependent Maxwell's equations (in partial differential form) are discretized using central-difference approximations to the space and time partial derivatives. The resulting finite-difference equations are solved in a leapfrog manner: the electric field vector components in a volume of space are solved at a given instant in time; then the magnetic field vector components in the same spatial volume are solved at the next instant in time; and the process is repeated over and over again until the desired transient or steady-state electromagnetic field behavior is fully evolved.

The basic FDTD space grid and time-stepping algorithm trace back to a seminal 1966 paper by Kane Yee in IEEE Transactions on Antennas and Propagation.[59] Since about 1990, FDTD techniques have emerged as primary means to computationally model many scientific and engineering problems dealing with electromagnetic wave interactions with material structures. Current FDTD modeling applications range from near-DC (ultralow-frequency geophysics involving the entire Earth-ionosphere waveguide) through microwaves (radar signature technology, antennas, wireless communications devices, digital interconnects, biomedical imaging/treatment) to visible light (photonic crystals, nanoscale plasmonics, solitons, and biophotonics).[60]

Here we use both commercial software (Rsoft) and our self-developed matlab code to do the FDTD simulations.
Chapter 2 New Mechanism of Negative Refraction

I. A New Mechanism for Negative Refraction and Focusing using Selective Diffraction from Surface Corrugation

A. Introduction

Negative refraction (NR) has been theoretically predicted and experimentally realized in two types of materials. One consists of simultaneously double negative permittivity and permeability material [67][68][69][70], leading to negative refractive index for the medium. The other consists of a photonic crystal (PC) [71][72][73], which is a periodic arrangement of scatters in which the group and phase velocities can be in different directions leading to NR. In both cases, the bulk properties of the medium, which is inherently inhomogeneous, can be described as having an effective negative refractive index. These new types of materials lead to new applications such as perfect lens [74] and subwavelength imaging [76].

In this section we describe a new mechanism for achieving NR that utilizes surface corrugation at an initially smooth interface that is totally reflecting. Consider a plane wave incident on the corrugated surface of a material with bulk refractive index \( n > 1 \) at an incident angle \( \theta \) (see Fig. 2-1(a)). A single beam of light can be refracted negatively or positively by (i) making the incident angle greater than the critical angle \( \theta > \theta_c = \sin^{-1}(1/n) \) which suppresses the zeroth and all positive orders, and then (ii) tuning the corrugation wave vector to select the
order. We demonstrate this concept experimentally using grisms (grating prisms) at visible and microwave frequencies. We further generalize this concept to eliminate the critical angle restriction and achieve all-angle negative refraction (AANR), which is demonstrated by negative lateral shift of an incident microwave beam by a flat multilayered structure with a surface grating. The presence of the surface grating changes the wave vector in the bulk medium and gives a new handle to control the emerging light from the interface. The corrugation provides a momentum kick to the incident light, enabling it to cross the interface and emerge refractively at angles that can be controlled. We further use this mechanism to create a new optical device: a transmission grating lens. As a prototype, we demonstrate focusing of plane waves by a plano-concave lens with a grating on the curved surface.

B. Negative refraction with visible light and microwave

The parallel component of the incident wave vector \( \vec{k} \) along the surface is \( k_\| = nk_0 \sin \theta \).

Here \( k_0 = 2\pi / \lambda \) is the wave number and \( \lambda \) the wavelength in free space. Due to the surface corrugation of periodicity as, the wave vector along the grating surface is not conserved. The parallel components of the transmitted and reflected wave vectors along the grating surface are

\[
k_{\|/m} = nk_0 \sin \theta + 2m\pi / a_s,
\]

according to the Floquet theorem [77]. Here \( m \) is the order of the so-called Bragg waves. The radiating Bragg waves into the air have 

\[
-1 < n \sin \theta + m\lambda / a_s < 1.
\]

Otherwise, they will be evanescent, constituting the surface waves. For incident angles larger than the critical angle, all the radiating Bragg waves have negative orders, \( m < 0 \). Within the wavelength range

\[
a_s(1 + n \sin \theta) / 2 < \lambda < a_s(1 + n \sin \theta),
\]

only the \( m = -1 \) Bragg wave will radiate.
from the grating surface into the air, which we call the refracted beam with wave vector $\vec{k}_f$ and $k_{f//} = n k_0 \sin \theta - 2\pi / a_s$. For light within this range, an effective refractive index can be defined as

$$n_{eff} = n - \frac{\lambda}{(a_s \sin \theta)}$$  \hspace{1cm} (2-1)

And Snell’s law applies. If $a_s (1 + n \sin \theta) / 2 < \lambda < n a_s \sin \theta$, $n_{eff} > 0$ the refraction will be positive while for, $n a_s \sin \theta < \lambda < a_s (1 + n \sin \theta)$, $n_{eff} > 0$ the refraction will be negative. This is illustrated in Fig. 2-1(a).

Fig. 2-1. (a) Wave vector diagram for NR from corrugated surface with grating periods. The red semicircle is the equal-frequency surface (EFS) in the air while the blue circle is the EFS in the dielectric. Here $k_i$ is the incident wave vector in the glass, $\vec{k}_f$ is the refracted one of minus one order in the air. $k_{r,0}$ and $k_{r,-1}$ are the reflected wave vectors in the dielectric of the zero-th and –1 order, respectively. (b) Optical experiment demonstrating NR using a grism of size 2 cm with a grating density of 2400 lines/mm on the upper short surface. The He-Ne laser beam is normally incident to the hypotenuse. To visualize the beam path in air, the grism was placed inside a glass enclosure that was sparsely filled with smoke. The solid lines with arrows indicate the propagation of the beams inside the grism.

An experimental demonstration of NR at visible light using this mechanism is shown in Fig. 2-1(b). Here Dr. Ravinder Banyal did the optical experiments and I did the microwave experiments. A holographic transmission grating with ruling density 2400 lines/mm and estimated groove depth $h \sim 130$ nm was replicated on one of the sides of an equilateral right-
angle BK7 prism of size 2 cm. A collimated laser beam is incident on the hypotenuse and passes through the grating surface of the grism. The incident angle at the grating is the prism angle \( \theta = \pi / 4 \) which is greater than the critical angle for the BK7 glass. Theoretical analysis indicates that only the \( m = -1 \) order beam would radiate into the air at negative angles if the incident light is within 435-860 nm wavelength range. Photographs of the experiments clearly indicate that the incident He-Ne (632.8 nm) laser beam “refRACTs” negatively with angle \( \varphi = 27^\circ \) (see Fig. 2-1(b)). Indeed this is indistinguishable from refraction by a prism made of a negative refractive index material with \( n_{\text{eff}} = -0.63 \) for the red light. A sketch of the main beam trajectories inside the glass is shown in Fig. 2-1(b). All the beams can and have been explained by diffraction theory.

NR was also observed on this grism for the green laser (532 nm) in which case \( n_{\text{eff}} = -0.29 \).

The above experiments were repeated on an 1800 lines/mm grism with an estimated groove depth \( h \sim 150 \text{nm} \). In the case of the red light with normal incidence to the hypotenuse, only the \( m = -1 \) beam will emerge negatively from the grating surface, whereas for the green light the \( m = -1 \) order is diffracted positively with additional appearance of \( m = -2 \) order. Control experiments performed on regular prisms (without the grating) show positive refraction or complete reflection without any transmitted beam, depending on the incident angle, as is to be expected.
For a fixed wavelength and groove geometry, the fraction of light diffracted into the $m = -1$ order depends strongly on the polarization state of the incident light. The intensity
transmission efficiency \( \eta = I_{-1} / I_{in} \) for \( m = -1 \) order was measured at different polarization orientations of the incident light. A half-wave plate inserted between the grism and a polaroid is used to rotate the orientation of the linearly polarized light. For the P-polarization, the electric-field vector was parallel to the groove whereas for the S-polarization it was perpendicular to the grooves. A maximum efficiency of 25\% is attained for the 2400 lines/mm grism with the P-polarized green light. The detailed transmission curves for different polarizations are shown in Fig. 2-2(a).

For practical applications, the efficiency of the power transmission is of primary concern. Since the direction of the refracted beam is determined only by the surface periodicity and not the detailed geometry of the grating, there is considerable freedom to design the grating surface to optimize the transmission. To exploit this freedom for transmission enhancement, we consider a specific grating, the lamellar grating. The transmission and reflection of waves were calculated using Bloch wave expansion method [78]. The transmission efficiency is plotted as a function of the groove depth \( h \) and the filling ratio for both the S- and P-polarizations at \( \lambda = 532 \) nm as shown in Fig. 2-2(b) and Fig. 2-2(c), respectively. For groove depth \( h < 50 \mu m \), the P-polarization can reach efficiency over 60\%, for \( h \sim 1 \mu m \) and filling ratio 0.34. For \( h \sim 0.4 \mu m \) and filling ratio is 0.4, the efficiency is 50\%.
The concept of NR by using selective diffraction is applicable over a wide range of frequencies. This is further exemplified by the grism experiments performed using microwaves as shown in Fig. 2-3. The grism consists of a right-angled polystyrene (\(\varepsilon = 2.56\)) prism with a surface grating of alumina rods next to the hypotenuse. The alumina rods have diameter 0.635 cm with grating periodicity \(a_s = 2\) cm. The experiments were carried out in a parallel-plate waveguide. The distance between the two plates is 1.26 cm. The excitation in the parallel-plate waveguide is a transverse magnetic (TM) mode up to 12 GHz such that the electric field is vertical and the magnetic field is within the plane. The collimated microwave beam incidents...
normally to the shortest side of the prism and hits the hypotenuse with an incident angle \( \theta = \pi / 3 \). A dipole antenna attached to an X-Y robot maps the electric field. As shown in Fig. 2-3 at 9GHz, the beam emerges as if it was refracted negatively at an angle \( \varphi = -16^\circ \), leading to \( n_{\text{eff}} = -0.32 \). NR was observed between 6.3-10.8GHz. Experimental data are in excellent agreement with theory and numerical simulations. For all the microwave experiments and the corresponding numerical simulations carried out in this section, only the TM modes are considered.

**C. Focusing by a plano-concave grating lens**

A unique feature of a negative index material is that it leads to focusing by a plano-concave lens [79][80]. We show that the NR mechanism demonstrated above can be used to design a plano-concave grating lens. For a plano-concave lens with circular curved surface of radius \( R \), if the grating is placed such that the groove distance along the optical axis is a fixed number \( a \), the surface periodicity will be \( a_s = a / \sin \theta \). Here the angle \( \theta \) is the incident angle toward the curved surface. The effective refractive index is

\[
    n_{\text{eff}} = n - \frac{\lambda}{a}
\]

which is independent of \( \theta \).
A focus is expected with a focal length \( f(\theta) = R[1 + \sin \varphi / \sin(\theta - \varphi)] \). The focal length depends on the angle \( \theta \), leading to aberration, which is present even in conventional lenses. The image quality is mainly impacted by a) the variation of the focal length for large incident angle \( \theta \) and b) the zero-th order diffraction which is present when \( \theta < \sin^{-1}(1/n) \). The strategy to improve the image quality is discussed next.

Fig. 2-4 Demonstration of plano-concave grating lens focusing. (a) Composite figure of the microwave focusing experiment at 8.4 GHz using a plano-concave grating lens made of alumina with a grating on the curved surface. On the left the electric field of the incident beam measured without the presence of the grating lens is plotted. On the right the intensity of the electric field is plotted. In the middle is a photo of the lens. The grating lens behaves like a smooth plano-concave lens made of negative index material with \( n_{\text{eff}} = -0.57 \) at 8.4 GHz. (b) FDTD simulations at plano-concave lens without aberration made with \( n = 3, R = 15 \text{ cm}, \) and \( a = 1 \text{ cm} \) at 8.5 GHz. Plotted is the electric field. The size of the system is in cm. (c) Details of the plano-concave lens (shown half). The dashed curve is an ellipse with semimajor 15 cm and semiminor 12.73 cm. The horizontal distance of the grooves is 1 cm.
A good quality focus can be observed for the plano-concave grating lens with circular surface if \( \lambda / a \sim n, n_{\text{eff}} \sim 0 \), in which case the focal length \( f(\theta) \) is flat. For \( |n_{\text{eff}}| < 1 \), one can use a noncircular curve instead of a circular one to minimize spherical aberration. This curve assumes an elliptical form \( \frac{y^2}{b^2} + x^2 = R^2 \) with \( b = (1 - n_{\text{eff}})^{1/2} \) and \( f = R(1 + |n_{\text{eff}}|) \) being the desired focal length. On this elliptical curve one places the grating such that the distance along the optical axis is a constant \( a \) as in the circular case. In order to eliminate the diverging beam around the optical axis of the plano-concave lens due to the zero-th order diffraction, one can simply block this part of the lens. Even if the interference from the zero-th order diffraction could not be eliminated, it can be reduced. For plano-concave lenses with higher refractive indices, this effect is smaller. For certain gratings on the plano-concave lens, the zero-th order diffraction can also be suppressed. For example for the staggered cut as in the one-dimensional (1D) PC, the part of the lens around the optical axis is flat as shown in the Fig. 2-4. For this grating one can choose the thickness of the lens around the optical axis such that the transmission through this part is a minimum. This is confirmed by numerical simulation.

An elliptical plano-concave grating lens made of alumina with semimajor axis \( R = 15 \) cm and semiminor axis 12.7 cm is shown in Fig. 2-4(c). The grating is made by staggered cuts on the concave surface, such that the horizontal distance of consecutive cuts is 1 cm. The effective index is \( n_{\text{eff}} = -0.53 \) at 8.5 GHz, respectively. A high quality focus of a microwave beam is observed at 8.4 GHz as shown in Fig. 2-4(a). The inverse experiment was also performed in which a point source placed at the focal point will radiate a plane wave beam at 8.4 GHz. The
planoconcave grating lens was placed inside the parallel-plate waveguide [79]. Numerical simulation (Fig. 2-4(b)) verifies both of the above mentioned focusing experiments, at 8.5 GHz.

D. All-angle negative refraction and negative lateral shift through grating multilayered structure

Fig. 2-5 AANR using surface grating. (a) A slab of 1D multilayer PC of lattice spacing $a$, with surface grating as on both surfaces. The surface grating gives rise to NR for the 1D PC operating within the band gap. (b) Illustration of NR using surface grating. The EFS of E-polarized modes in a 1D PC made of alumina bars with lattice spacing $a = 0.9$ cm, bar thickness $d = 0.5$ cm at 6.85 GHz is shown as the blue curves. The green semicircle is the corresponding EFS in the air.

So far we have demonstrated NR through the combination of total internal reflection and selective negative diffraction by combining a surface grating with a homogeneous isotropic bulk material. The same NR mechanism introduced above is further applied in this section to achieve AANR [73] through a flat corrugated interface between air and an anisotropic medium.

To illustrate this principle to achieve AANR, we consider a multilayered structure which behaves as a one-dimensional photonic crystal (1D PC). For a 1D PC as shown in Fig. 2-5(a),
there will be a band gap for normal incident plane waves within a certain frequency range. For these frequencies, transmission may be allowed for oblique incident angles. For example for the equifrequency surface (EFS) of the 1D PC shown in Fig. 2-5(b), waves with incident angle $\theta$, such as $k_a < k_0 \sin \theta < k_b$ will propagate. If for some frequencies, $k_0 < k_a$, then for all the incident plane waves, there is total external reflection. So this 1D PC behaves as an omni-directional mirror [81] for these E-polarized modes. If a grating with period $a_s$ is introduced on the flat surface of the 1D PC, for example with $2\pi / a_s = k_a$, then a plane wave with an incident angle $\theta$ will get a positive momentum kick along the surface. Thus the incident wave will couple to the Bloch wave with $k_y = k_0 \sin \theta + 2\pi / a_s$ and propagate inside the 1D PC. However if $2\pi / a_s = k_b$, the incident wave will receive a negative momentum kick along the surface and couple to the Bloch wave with $k_y = k_0 \sin \theta - 2\pi / a_s$. And with proper design, it is possible that only the Bloch wave with $k_y = k_0 \sin \theta - 2\pi / a_s$ will propagate inside the 1D PC. This refraction is well-defined and negative. Furthermore if ..., all the propagating waves will be transmitted into the 1D PC. Since $k_y$ of the Bloch wave is negative for every positive incident angle $\theta$ this leads to a single-beam AANR. In this case, both the wave vector and group velocity refraction is negative. This scenario for NR is illustrated in Fig. 2-5(b). For 1D PCs different $k_y$ correspond to different modes. With the introduction of surface grating, Bloch states with $k_y$ and $k_y + 2m\pi / a_s$ are identical. Thus the 1D PC effectively becomes a 2D PC, resulting in a finite-sized first Brillouin zone of a rectangle shape. This is the simplest all-dielectric structure to achieve AANR.
Fig. 2-6 (a) Experimental demonstration of negative lateral shift by a 1D PC with a surface grating, at 6.96 GHz. A 5.6 cm negative lateral shift was observed. The 1D PC is made of 6 layers of alumina bars with width $d = 0.5$ cm and spacing $a = 1$ cm. The surface grating was formed by rods of the same material, alumina, with diameter 0.63 cm and spacing $a_z = 1.8$ cm. The width of the incident beam is 10 cm and the incident angle is 13.5$^\circ$. The incident and outgoing beams are plotted as the real part of the measured transmission coefficient $S_{21}$. (b) Positive lateral shift for a microwave beam at 6.96 GHz by a slab of polystyrene with thickness 7.5 cm.
A microwave experiment carried out in parallel-plate waveguide confirms the above mechanism for NR. Negative lateral shift was observed experimentally from 6.65-7.74 GHz for a grating multilayered structure. The 1D PC is made of 6 layers of alumina bars with thickness \( d = 0.5 \text{cm} \), lattice spacing \( a = 0.9 \text{ cm} \) and surface grating \( a_s = 1.8 \text{ cm} \) (see Fig. 2-6). The incident angle of the 10cm wide microwave beam was 13.5°. A 5.6cm negative lateral shift is observed at 6.96 GHz, as shown in Fig. 2-6. Numerical simulations confirm AANR and negative lateral shift for a large range of incident angles for frequencies around 6.85GHz as shown in Fig. 2-7.

**E. Discussion and conclusion**

This work provides a new perspective to the phenomenon of negative refraction. The NR can be seen as attributed to the bulk modification of the EFS and the surface grating. Surface periodicity alone is sufficient to achieve NR even with homogeneous positive index materials, as
we have demonstrated. For 2D or higher dimension PC, the bulk periodic structure naturally introduces a surface periodicity at the interface. Improper surface modification of the PC may suppress or even diminish NR. Here we have shown that NR can be achieved by combining a surface grating with a multilayer 1D PC structure that is relatively easier to fabricate. Previous approaches to create NR materials with multilayered structures required the use of alternating layers of negative permittivity and negative permeability materials [82][83] and have not yet been realized experimentally. The realization of AANR in 1D PCs with surface corrugation opens a whole new realm of NR applications. Many structures such as photonic band gap materials currently used to guide waves or form cavities as photon insulators [84][85][86][87][88] can be modified to have NR and AANR through surface engineering [89].

Although the phenomena presented in this section are due to diffraction, ray optics does apply as we have shown in the design of the plano-concave grating lens (see Fig. 2-4). The mechanism of plano-concave lens focusing is different from that of the zone-plate where concentric rings are carved to give each ray the corrected phase and ray optics does not apply. Previous use of diffraction optics has been limited to reflection gratings. The grism used in astronomy [90] is under the condition \( n \sin \theta < 1 \) which allows the zero-th order diffraction. Our approach is different from the suppression of zero-th and enhancement of -1 order transmission through surface grating depth modification [91]. By removing the zero-th order Bragg diffraction completely, our work opens the door for new phenomena and applications such as plano-concave lens focusing and flat lens imaging.
Surface engineering provides us a new dimension to manipulate waves. Refraction is an interfacial phenomenon, and this work shows that by controlled engineering of the interface, a totally reflecting surface can be made to refract negatively or positively, even though the materials utilized do not possess bulk negative refractive indices. The importance of surface modification has been previously recognized [92][93] but has not been used as a mechanism to achieve negative refraction.

The concepts discussed in this section are particularly suitable for integrated optical circuits, where the device dimension is about the size of the free space wavelength. Our work provides key ideas to harness diffraction to produce focusing devices. Thus new types of optical elements can be produced by using the above mechanism as a principle of design.

II. Alternative Approach to All-angle Negative Refraction in Two-dimensional Photonic Crystals

We show that with an appropriate surface modification, a slab of photonic crystal can be made to allow wave transmission within the photonic band gap. Furthermore, negative refraction and all-angle negative refraction (AANR) can be achieved by this surface modification in frequency windows that were not realized before in two-dimensional photonic crystals [3]. This approach to AANR leads to different applications in flat lens imaging. Previous flat lens using photonic crystals requires object-image distance $u+v$ less than or equal to the lens thickness $d$, $u+v \sim d$. Our approach can be used to design a flat lens with $u + v = \sigma d$ with $\sigma >> 1$, thus being able to image large and/or far away objects. Our results are confirmed by finite-difference time-domain simulations.
A. Introduction

Negative refraction (NR) was proposed theoretically a long time ago [94]. It was realized only recently in two classes of materials. One type is the so-called metamaterials [95][96][97], consisting of wires and split-ring resonators. The second type is photonic crystals [98], which have periodic permittivity and/or permeability.

In these artificial materials, there are up to now two mechanisms to achieve NR. One is to use the anti-parallelism between the wave vector and the group velocity [99]. This can be realized in isotropic metamaterials [95] and in the second or even higher band around the Γ point in photonic crystals (PCs) [99]. In this case, the phase refractive index \( n_p \) is negative.

The second mechanism is to use the anisotropy and the concavity of the equifrequency surface (EFS) such as the EFS around the M point in the first band of a square lattice PC. In this case though the phase refractive index \( n_p \) is positive, the group refractive index \( n_g \) is negative [100]. In order to show negative lateral shift or all-angle negative refraction (AANR), one has to orient the lattice such that the ΓM direction is along the surface normal. NR and flat lens imaging [101] using both mechanisms have been observed in microwave [102][103][104][105][106][107][108] and near infrared experiments [109].

Recently, we proposed a different mechanism for NR using surface grating [110]. This mechanism combines photonic band gap with surface grating to achieve NR and AANR. Negative lateral shift and AANR have been demonstrated in a multilayered structure with surface grating.
In this section, we obtain windows of AANR in two dimensional (2D) PCs using this mechanism of surface modification. We also show that flat lens made of photonic crystal with surface grating can have \( u + v = \sigma d \) with \( \sigma >> 1 \) while for the Veselago-Pendry flat lens [101] \( \sigma = 1 \). Here \( d \) is the thickness of the flat lens, and \( u \) and \( v \) are the distance from the lens to the object and the image, respectively. Thus a flat lens can focus large and far away objects.

**B. Another Approach to AANR**

In a pioneering paper [100], Luo et al. showed that within certain frequency window in the first band of a PC, AANR can be achieved. Specifically, within the first band, AANR is possible along the \( \Gamma \text{M} \) direction for a square lattice PC. We will show that with appropriate surface grating, NR and AANR is also possible along the \( \Gamma \text{X} \) direction in the first band of a square lattice PC.
In the main text of this section, we only consider the transverse magnetic (TM) modes of a square lattice PC of square rods. Square lattice of circular rods or even rhombus rods can be treated similarly. The generalization to transverse electric (TE) modes and lattice structures other than square lattice is straightforward. As a specific example, we consider a square lattice of square rods with size $b/a=0.7$, thus a filling ratio of 0.49. The EFS of this PC is calculated by
using the plane-wave expansion method [98] with 5041 plane waves. The EFS of the first band is shown in Fig. 2-8. The frequencies at the X point and the M point are $\omega_X = 0.1943 \times 2\pi c / a$ and $\omega_M = 0.2446 \times 2\pi c / a$, respectively.

Consider a slab of this PC with surface normal along the $\Gamma X$ direction. If one increases the frequency, $\omega > \omega_X$, there will be a partial band gap for waves incident to the air-PC interface since the Bloch waves have $k_y \geq k_a$ and the incident plane wave with $k_y < k_a$ will be completely reflected. Here $k_a$ is the $k_y$ value of the crossing point of the EFS with the XM boundary of the first Brillouin zone, as shown in Fig. 2-8 and Fig. 2-9. For certain frequencies $\omega_l < \omega < \omega_M$ with

![Image](image-url)
$\omega_l = 0.2089 \times 2\pi c / a$ when $k_a = \omega / c$, a flat slab of such PC is an omni-directional reflector [112]. For example for $\omega = 0.219 \times 2\pi c / a$ as shown in Fig. 2-9, there will be total external reflection for any incident plane wave. However for these frequencies, a surface grating with period

$$a_s = 2a$$

(2-3)

![Surface grating diagram](image)

**Fig. 2-10** Details of slab made of a square lattice PC with surface grating $a_s = 2a$. For simplicity, the thickness $d$ of the slab is defined as the distance from the first surface to the last surface of the structure.

will give a momentum boost along the surface to the incident plane wave with incident angle such that it will be coupled to the Bloch waves inside the PC with transverse momentum $k_y = \pi / a + (\omega / c) \sin \theta$ if $\theta$ is negative and $k_y = -\pi / a + (\omega / c) \sin \theta$ if $\theta$ is positive. The refracted wave will propagate on the opposite side of the surface normal with respect to the incident beam. Thus NR is achieved. This is illustrated in Fig. 2-9. The effect of this surface grating is
equivalent to bringing down the EFS around the M point to the X point for \( \omega_L < \omega < \omega_M \). As we pointed out in Ref. [110], it is the surface periodicity which determines the size of the EFS and the folding of the band structure. Furthermore, if \( \pi / a - k_a \geq \omega / c \), AANR can be achieved. The upper limit for AANR is \( \omega_u = 0.2192 \times 2\pi c / a \). Thus we obtained a 4.7% AANR around \( \omega_u \).

The above approach to NR and AANR is confirmed in our numeric simulations using finite-difference time-domain (FDTD) method [113]. Here we consider the lateral shift of an incident beam by a slab made of a square lattice PC. The detail of the slab is shown in Fig. 2-10. Negative lateral shifts are observed for different incident angles as shown in Fig. 2-11 for beams at \( \omega = 0.219 \times 2\pi c / a \). It can be verified that for this slab, AANR can be achieved for \( 0.2089 \leq \omega a / 2\pi c \leq 0.2192 \). Note that the details of the surface grating are not essential except its period \( a_s = 2a \). The grating can be holographic or an array of circular rods as long as it is not too thick. For the specific surface grating shown in Fig. 2-10, the energy transmissions are 99.2% and 4.7% for plane waves with incident angles 15° and 30°, respectively.

### C. Flat Lens with Large \( \sigma \)

One prominent application of NR is the Veselago-Pendry perfect lens [101]. A flat slab of thickness \( d \) can focus an object with distance \( u \) on one side to a distance \( v \) on the other side with \( u + v = d \) if the refractive index \( n = -1 \). For a generalized flat lens without optical axis [111], the lens equation takes the form

\[
 u + v = \sigma d
\]  

(2-4)
with $\sigma$ a material property, depending on the dispersion characteristics of the flat lens. This is illustrated in Fig. 2-12. This lens equation requires the following form of the EFS at the operating frequency:

$$k_{rx} = \kappa - \sigma \sqrt{\frac{\omega^2}{c^2} - k_y^2}$$  \hspace{1cm} (2-5)$$

The lens surface is in the $y$ direction. The surface normal is along the $x$ axis. Here $k_{rx}$ is the longitudinal component of the wave vector in the lens medium and $\kappa$ is the center of the EFS ellipse [111].
Even though AANR can be realized in the first band along the $\Gamma M$ direction for a square lattice PC [100], the EFS is very flat around the lens normal. Thus one has $\sigma << 1$[111]. Though $\sigma \sim 1$ has been reported in PCs with other structures [114][115], the focusing is still limited to the vicinity of the lens surface [116]. For practical applications, we need large $\sigma$ so that the object and image can be far away from the lens. We will show that with our new mechanism for AANR, large $\sigma$ can be achieved.

Fig. 2-12  A flat lens with lens equation $u + v = \sigma d$. 
Fig. 2-13  Fitting of EFS of the TM modes at $\omega a/2\pi c = 0.219$ by Eq. (2-5). Here $\sigma_0 = 4$ and $\kappa a/2\pi c = 1.16$ Note that the center of EFS is shifted from the M point to the X point due to the surface grating $a_s = 2a$.

Fig. 2-14  FDTD simulation of flat lens focusing of a point source. For better contrast effect, the field intensity at the point source is suppressed. The detail of the lens is given in Fig. 2-10. The distance is measured in the unit of lattice spacing $a$. 
As we have stated in the previous section, the first Brillouin zone of a square lattice PC with surface grating $a_s = 2a$ takes the shape of a rectangle instead of a square and its vertical size is reduced to $-\pi / a \leq k_y \leq \pi / a$. The center of the EFS for $\omega_i \leq \omega \leq \omega_m$ of the original PC is moved from the M point to the X point. The fitting of the modified EFS for this frequency by Eq.(2-5) will give the lens property $\sigma$. An inspection of the band structure reveals that the EFS is not elliptical. This results in incident angle dependent $\sigma$ [111]. Nevertheless, the EFS can be fitted well with a constant $\sigma_0$ for small $k_y$ as shown in Fig. 2-13. For the square lattice PC we have designed (Fig. 2-10), one has $\sigma_0 \sim 4$ for $\omega_i \leq \omega \leq \omega_u$.

![Graph](image)

*Fig. 2-15* The ratio of the object-image distance to the slab thickness $(u+v)/d$ vs the object distance $u$ for the flat lens shown in Fig. 2-10 at the operating frequency $\omega = 0.219 \times 2\pi c / a$.

Focusing by such a flat lens is shown in Fig. 2-12 For a point source with $u = 13.6a$, a clear focused image is obtained at $v = 12.4a$ for the operating frequency $\omega = 0.219 \times 2\pi c / a$, which is consistent with the lens equation $u + v = \sigma_{eff} d$ with $\sigma_{eff} = 3.5$ and $d = 7.4a$. There are two reasons...
for $\sigma_{eff} < \sigma_0$. First that the EFS is elliptical only for small $k_y = (\omega / c) \sin \theta$ and $\sigma = -\frac{d k_x}{d k_z}$ decreases with increasing incident angle $\theta$. The effective $\sigma_{eff}$ is an average and thus smaller than $\sigma_0 = -\frac{d k_x}{d k_z} |_{k_z=0}$. Second, the thickness of a PC slab is not a well-defined quantity. Here we simply define the lens thickness as the distance from the first surface to the last surface as shown in Fig. 2-10. This may overestimate the effective thickness of the lens.

To further check the performance of this flat lens, we vary the object distance $u$. In Fig. 2-15 we show that the ratio $(u + v)/d$ is almost constant and very close to $\sigma_0$ for different object distance $u$.

![Graph showing transmission coefficient](image)

Fig. 2-16 The transmission coefficient for plane wave incident on the flat lens shown in Fig. 2-10.
The primary concern of practical application is the power transmission through the lens. As expected, the transmission through the flat lens is low due to the impedance mismatch. However, since the details of the grating on the PC will not alter the scenario for NR, the power transmission can be enhanced through careful engineering of the grating. In our simulation, we find that the grating on the 2D PC with large dielectric constant has strong power transmission. The parameters of the surface grating shown in Fig. 2-10 are not optimized. Further improvement of transmission may be possible. For the grating parameters given in Fig. 2-10, the transmission coefficient is calculated and plotted in Fig. 2-16.

D. Discussion and Conclusion

In this section, we have achieved NR using a different approach: Photonic band gap with surface grating. This approach to NR gives another window for AANR in 2D PCs. This approach also gives a flat lens made of these PCs with a large object-image distance. Thus these flat lenses are able to image large and far away objects.

Our approach gives a much larger window of AANR than previous realized. For example, for square lattice air holes in $\varepsilon = 12$ with $r / a = 0.35$ studied in Ref. [100], our approach gives a lower limit $\omega_l = 0.183 \times 2\pi c / a$ and an upper limit $\omega_u = 0.206 \times 2\pi c / a$; hence a fraction of AANR frequency range of 11% around $0.206 \times 2\pi c / a$. This range is much larger than the 6.1% AANR range along the $\Gamma M$ direction for the TE modes. This window of AANR is easier to locate. The two limits are obtained from the crossing of the band with the light lines around X point and M point, respectively. For the determination of the lower limit $\omega_l$, there is no need to
compute the frequency at which the radius of curvature of the contours along ΓM diverges [100].

Our approach to AANR can also be extended to three dimensional photonic crystals.

III. Nanoengineering of a Negative-index Binary-staircase Lens for the Optics Regime

We show that a binary-staircase optical element can be engineered to exhibit an effective negative index of refraction, thereby expanding the range of optical properties theoretically available for future optoelectronic devices. Here Dr. Bernard Didier F. Casse and Dr. Ravinder Banyal did the optical experiments and I did the theoretical calculation and numerical simulations. The mechanism for achieving a negative-index lens is based on exploiting the periodicity of the surface corrugation. By designing and nanofabricating a planoconcave binary-staircase lens in the InP/InGaAsP platform, we have experimentally demonstrated at 1.55 µm that such negative-index concave lenses can focus plane waves. The beam propagation in the lens was studied experimentally and was in excellent agreement with the three-dimensional finite-difference time-domain numerical simulations.

The field of metaphotonics,[117][118][119][120][121][122][123][124][125] or the merging of metamaterials [126] and photonics,[127][128][129] has opened doors to a plethora of unusual electromagnetic properties, such as negative refraction, [130][131] cloaking, [132] and optical data storage, [133] which cannot be obtained with naturally occurring materials. The holy grail of manufacturing these artificial photonic metamaterials structures is to manipulate light at the nano-scale level for optical information processing and high resolution imaging. In this section we demonstrate how a binary-staircase optical element can be tailor-made to have an effective
negative refractive index, thus bringing an alternative approach to negative-index optical elements.

Here we consider a binary-staircase type of lens,[134] which consists of a sequence of zones configured as flat parallel steps each having an annular shape. The binary-staircase lens is a planoconcave lens. Focusing by planoconcave lenses [135][136] was realized in two-dimensional and one-dimensional photonic crystals.[137][138] Proof-of-concept experiments that demonstrate negative refraction in a planoconcave grating lens have been realized earlier by our group in the microwave regime.[139] However the planoconcave lens used in the microwave range consisted of an assembly of commercial alumina bars, placed in a parallel-plate waveguide, which are not suitable for integration in optoelectronic circuits.

![Fig. 2-17 Sketch of a planoconcave grating lens. The lens is made of a medium with \( n_{\text{med}} \). The horizontal step size \( a \) is smaller than the free space wavelength \( \lambda \) with \( a \approx \lambda/n_{\text{med}} \).](image)

Geometrical parameters of the binary-staircase lens were determined by considering the transverse size of the lens, the focal length, wavelength of the incoming radiation, the index of
the material used to fabricate the lens itself, and mainly the surface periodicity. The actual lens has been nanofabricated by a combination of electron beam lithography and reactive ion etching in an InP/InGaAsP heterostructure. Subsequently, the focusing properties of the device were experimentally verified using a scanning probe optical technique. Three-dimensional (3D) finite-difference time-domain (FDTD) simulations have been used to further study beam propagation in the lens. The FDTD simulations were in excellent agreement with the experimental results.

We use a surface modification scheme to alter the index of refraction of the medium.[139][140] An incident wave impinging on a smooth surface with an incident angle larger than the critical angle will be totally reflected. However a proper surface grating will allow the wave to be transmitted. This is equivalent to giving the incident wave a transverse momentum kick. In the case that the grating period is much smaller than the incident wavelength, an effective refractive index $n_{\text{eff}}$ can be used to describe the refraction at the modified surface. For a binary-staircase lens with a planoconcave shape, as shown in Fig. 2-17, this effective index is related to the bulk refractive index of the medium $n_{\text{med}}$ by [139]

$$n_{\text{eff}} = n_{\text{med}} - \frac{\lambda}{a} \tag{2-6}$$

where $a$ is a fixed step length along the optical axis (or surface periodicity) and $\lambda$ is the free space wavelength (with $a < \lambda$). The number of steps $N_{\text{steps}}$ or zones is then $R/a$, where $2R$ is the transverse size of the binary-staircase lens. The focal length $f$ is calculated by using the formula $f = R/(1-n_{\text{eff}})$. To obtain a good focus, $a \sim \lambda / n_{\text{med}}$ (Abbe’s diffraction limit). In the present case, $\lambda = 1550$ nm and $a$ was chosen as 450 nm with $n_{\text{med}} = 3.231$ for the transverse electric (TE) modes.
and $n_{\text{med}} = 3.216$ for the transverse magnetic (TM) modes. $a$ has been chosen as an arbitrary value in the vicinity of $\lambda/n_{\text{med}}$. $N_{\text{steps}} = 11$, so that $2R$ reads as 10 $\mu$m. Thus $n_{\text{eff}}$ is $-0.2133$ and $-0.2889$ for TE and TM modes, respectively.

The fabrication platform consisted of a 400nm InGaAsP core layer on an InP substrate with a 200nm InP top cladding layer. The waves are trapped and propagate within the core layer plane with an effective permittivity of 3.231 (TE modes) and 3.216 (TM modes). The final structure for optical measurements consisted of three subunits (shown in Fig. 2-18(a)): (i) A 0.5 mm long waveguide, laterally tapered, having 5 $\mu$m wide trenches on each side. The taper starts at a distance of 100 $\mu$m from the edge of the waveguide, with a core width varying from 5 to 10 $\mu$m.

Fig. 2-18 (a) Bird’s eye view of the tapered waveguide and the binary-staircase lens. (b) Close-up view of the binary-staircase lens.

The fabrication platform consisted of a 400nm InGaAsP core layer on an InP substrate with a 200nm InP top cladding layer. The waves are trapped and propagate within the core layer plane with an effective permittivity of 3.231 (TE modes) and 3.216 (TM modes). The final structure for optical measurements consisted of three subunits (shown in Fig. 2-18(a)): (i) A 0.5 mm long waveguide, laterally tapered, having 5 $\mu$m wide trenches on each side. The taper starts at a distance of 100 $\mu$m from the edge of the waveguide, with a core width varying from 5 to 10 $\mu$m.
(ii) Binary-staircase planoconcave lens with ten zones on the optical axis, having a step height of 450 nm and a transverse size of 10 µm, located at a distance of 5 µm from the tapered end of the waveguide (shown in Fig. 2-18(b)). (iii) Finally, an open cavity (semicircle juxtaposed to a 20x20 µm² square) at the end of the binary-staircase lens.

An analog structure, having the same geometrical dimensions but bearing no steps (or zones), was fabricated. The purpose of the analogous design was to prove that the periodicity of the steps is decisive structure elements to realize a negative-index prototype. The structures were written using electron beam lithography on a polymethylmethacrylate resist. Pattern transfers to a silicon nitride working mask and subsequently to the InP/InGaAsP layers were achieved with a reactive ion etching method.

In the characterization experiment, a continuous wave tunable semiconductor laser (1550–1580 nm) was used as the input light source. The laser light was coupled into the cleaved end of the input waveguides using a monomode lensed fiber (working distance ~14 µm and full width at half maximum ~2.5 µm in air) mounted on a five-axis positioning stage. An infrared (IR) camera (Hamamatsu Model C2741) connected to a microscope port aids the initial alignment to optimize the IR light coupling from the optical fiber to the waveguide. In the FDTD simulation, a 10 µm wide plane parallel, Gaussian beam was chosen as incident field for the grating lens. In the actual sample, the 5 µm wide input facet of the waveguide was inversely tapered to 10 µm width (see Fig. 2-18(a)) so that the propagating Gaussian beam is expanded sufficiently inside the guiding channel before reaching the device end. The planar wave front after emerging from the binary-staircase lens is expected to focus in the air cavity.
A tapered fiber probe (250 nm aperture diameter) metalized with a thin chromium and gold layer was raster scanned just above the sample surface. The output end of the fiber probe was connected to a nitrogen cooled germanium detector (North Coast Scientific Corp. Model EO-817L). Additionally, a typical lock-in amplifier was utilized to optimize the detection scheme. Scanning the fiber tip at a constant height about 500 nm above the sample surface allowed us to probe the optical intensity distribution over a grid of 256×256 points spanning a 15×15 µm² area. The reconstructed image is shown in Fig. 2-19(a).

Intensity distribution near the cavity center clearly shows the light focusing from the binary-staircase lens. Identical focusing fingerprints were observed when the experiment was repeated over a range of wavelengths varying from 1510 to 1580 nm. Another controlled experiment was performed where the binary-staircase lens was replaced by an analogous structure (having the same geometrical features) with no steps. In the latter case, as shown in Fig. 2-19(b), no beam...
focusing was observed. Nevertheless we can distinguish a bright spot near the device’s edge, which is attributed to a sudden beam divergence as it propagates into open space from initial confinement in the InGaAsP core waveguide layer (diffraction).

The 3D FDTD simulations were performed using perfectly matched layer boundary conditions that minimize reflections at the edges. The chosen input field excitation for the FDTD simulation was a TE polarized Gaussian beam that closely resembles the beam shape of the fiber source in an actual experiment. The energy density of the propagating H-field was mapped at different plane heights. Fig. 2-20(a) and Fig. 2-20(b) show the simulated H-field density of the binary-staircase lens and the analog structure at about 800 nm above the center of the core layer, respectively.

Fig. 2-20 (a) 3D FDTD simulation of the planoconcave binary-staircase lens and (b) 3D FDTD simulation of the lens having the same geometrical dimensions as the binary-staircase one, but bearing no steps (or zones).

The traditional metamaterial structures are composed of arrays of split ring resonators and metal wires. This type of metallic structures, which operates under resonance, becomes lossy at
optical frequencies due to the inherent imaginary part of the metal’s permittivity. The purely
dielectric system, such as the one mentioned in this section, is free from these drawbacks and
thus has low intrinsic material loss, which is a clear-cut advantage for optical frequency
operations. Extrinsic losses in the binary-staircase dielectric structure itself arise solely from the
imperfections in the fabrication (e.g., surface and sidewall roughness).

We have experimentally designed a binary-staircase optical element having an effective
negative index of refraction, whereby the surface periodicity of the structure acted as the tunable
parameter for controlling the sign change of the refractive index. The beam propagation in the
planoconcave lens was simulated using in-house 3D FDTD codes. Based on the design and
simulations, we have nanoengineered a prototype structure in an InP/InGaAsP heterostructure
tailored for the 1.55 µm wavelength, where indium phosphide (InP) is a natural starting
fabrication platform for wholesale integration of passive and active devices for a complete
system on a chip at this frequency. Characterization of the prototype with a near-field scanning
optical microscope revealed that the planoconcave binary-staircase lens can act as a convex lens
and thereby focusing plane waves. No focusing is achieved if the zones are removed; reinforcing
the fact the steps are the decisive structure elements. A notable aspect of our work is the
extension of electromagnetic properties (which are theoretically available) of optical elements
for possible integration in optoelectronic circuits.
Chapter 3 Slow Light of Extremely Anisotropic Nanowire

I. Introduction

Since the realization of negative refraction 0 in microwaves [142], there is renewed and intense interest in electromagnetic metamaterials. Negative refraction has added a new arena to physics, leading to new concepts such as perfect lens [143][144], superlens [143][145][146], and focusing by plano-concave lens [147][148]. Negative refraction has subsequently been realized in microwaves [149][150][151][152][153], THz waves, and optical wavelengths [154][155][156], in metamaterials made of wire and split ring resonators [157] or photonic crystals [158][159][160].

Metamaterials are artificially fabricated structures possessing certain desirable properties which are not available in natural materials. Metamaterials can have double negative index [161] or single negative index. Metamaterials can be periodic, such as photonic crystals [162]. They can also be non-periodic, such as the materials for cloaking [163]. They can also be made to be anisotropic and have indefinite index [164][165][166]. Indefinite index metamaterials can be used to make hyperlens [167][168]. This range of properties opens infinite possibilities to use metamaterials in frequencies from microwave all the way up to the visible.

Wave propagation in waveguide of nanometer size [169] has unique properties. In this section, we consider wave propagation along anisotropic nanowires. In the case where the transverse permittivity is negative while the longitudinal one is positive ($\varepsilon_\perp<0$, $\varepsilon_\parallel<0$), these
indefinite index waveguides can support both forward and backward waves. High effective index can be obtained for these modes. These waveguides can also support degenerate modes which can be used to slow down and trap light.

In Sec. II, we derive the formulas for all the modes on the anisotropic cylinders. Exact solutions for all the modes and closed-form expressions for the energy flow will be obtained. Possible zero net-energy flow modes will also be discussed. The situation for trapping light is presented in Sec. III. In Sec. IV, we propose the realization of nanowires made of indefinite index medium, which is confirmed in finite difference time-domain simulations. We conclude in Sec. V with possible applications for these anisotropic nanowires.

II. Wave Propagation and Energy Flow on Anisotropic Cylindrical Waveguides

We consider wave propagation on a cylindrical waveguide. The axis of the waveguide is along the z direction as shown in Fig. 3-1. The waveguide is nonmagnetic and has an anisotropic optical property

\[ \varepsilon_x = \varepsilon_y = \varepsilon_i \neq \varepsilon_z \]  

(3-1)

The waves will propagate along the cylinder axis with

\[ E = E_0 e^{i(\beta z - \omega t)}, H = H_0 e^{i(\beta z - \omega t)} \]  

(3-2)

Here \( \beta \) is the propagation wave number along the waveguide.

Due to the symmetry of the waveguide, all of the field components can be expressed in terms of the longitudinal components \( E_z \) and \( H_z \). In the polar coordinate system, one has for the fields inside the waveguide \( r < a \) with \( a \) the radius,
Here $k_0$ is the wave number in the vacuum.

Fig. 3-1 An anisotropic cylindrical waveguide with axis along the $z$ axis. The longitudinal permittivity is positive while the transverse permittivity is negative.

The wave equations for the longitudinal components inside the waveguide are

\[(\partial_x^2 + \partial_y^2)E_z + \varepsilon_z(k_0^2 - \beta^2 / \varepsilon_z)E_z = 0,\]
\[(\partial_x^2 + \partial_y^2)H_z + (\varepsilon_z k_0^2 - \beta^2)H_z = 0.\]
The waveguide is free standing in air, so the wave equations for \( r > a \) are given by the above equations with the permittivity replaced by unity. The solutions are expressed in terms of the Bessel functions of various kinds

\[
E_z = AJ_n(Kr)e^{in\phi}, \quad r < a, \\
= CK_n(\kappa_0 r)e^{in\phi}, \quad r > a
\] (3-5)

and

\[
H_z = BI_n(\kappa r)e^{in\phi}, \quad r < a, \\
= DK_n(\kappa_0 r)e^{in\phi}, \quad r > a
\] (3-6)

The coefficients will be determined by matching the boundary conditions. Here

\[
K = \sqrt{\varepsilon_z \sqrt{k_0^2 - \beta^2} / \varepsilon_i}, \\
\kappa = \sqrt{\beta^2 - \varepsilon_i k_0^2}, \\
\kappa_0 = \sqrt{\beta^2 - k_0^2}.
\] (3-7)

We only consider the extremely anisotropic case such that the longitudinal permittivity is positive while the transverse permittivity is negative,

\[
\varepsilon_z > 0, \quad \varepsilon_i < 0
\] (3-8)

One can see that due to the anisotropic nature of the waveguide, \( H_z \) and \( E_z \) inside the waveguide will have completely different behaviors.

The continuity of \( E_z \) and \( H_z \) at the interface \( r = a \) gives

\[
\frac{C}{A} = \frac{J_n(Ka)}{K_n(\kappa_0 a)}, \quad \frac{D}{B} = \frac{I_n(\kappa a)}{K_n(\kappa_0 a)}.
\] (3-9)

The continuity of \( E_\phi \) at the interface gives
\[
\begin{align*}
\text{in} \beta \frac{\kappa_0^2 - \kappa^2}{k_0 a^2 \kappa_0^2 \kappa^2} &= \frac{B}{A} \frac{I_n(\kappa a)}{J_n(Ka)} [g_n(\kappa_0 a) + h_n(\kappa a)] \\
(3-10)
\end{align*}
\]

with the following defined functions

\[
\begin{align*}
g_n(x) &= -\frac{K_n'(x)}{xK_n(x)} = \frac{K_{n-1}(x)}{xK_n(x)} + \frac{n}{x^2}, \\
& \quad \text{(3-11)} \\
h_n(x) &= \frac{I_n'(x)}{xI_n(x)} = \frac{I_{n-1}(x)}{xI_n(x)} - \frac{n}{x^2}.
\end{align*}
\]

The continuity of \(H_\Phi\) at the interface gives

\[
\begin{align*}
\text{in} \beta \frac{\kappa^2 - \kappa_0^2}{k_0 a^2 \kappa_0^2 \kappa^2} &= \frac{A}{B} \frac{J_n(Ka)}{I_n(\kappa a)} [g_n(\kappa_0 a) - \varepsilon_z f_n(Ka)] \\
(3-12)
\end{align*}
\]

with

\[
\begin{align*}
f_n(x) &= \frac{J_n'(x)}{xJ_n(x)} = \frac{J_{n-1}(x)}{xJ_n(x)} - \frac{n}{x^2}. \\
& \quad \text{(3-13)}
\end{align*}
\]

Thus we obtain the equation for all of the modes

\[
\begin{align*}
[g_n(\kappa_0 a) - \varepsilon_z f_n(Ka)][g_n(\kappa_0 a) + h_n(\kappa a)] \\
= n^2[(\kappa_0 a)^2 - (\kappa a)^2][(\kappa_0 a)^2 - \varepsilon_z(\kappa a)^2]. \\
(3-14)
\end{align*}
\]

For wave propagation on the cylindrical waveguide, the components of the Poynting vector are

\[
\begin{align*}
S_z &= \frac{1}{4\pi} (E_z H_\Phi^* - E_\Phi H_z^*), \\
S_r &= -\frac{1}{4\pi} (E_r H_\Phi^* - E_\Phi H_r^*), \\
S_\Phi &= \frac{1}{4\pi} (E_\Phi H_r^* - E_r H_\Phi^*). \\
(3-15)
\end{align*}
\]

The physical Poynting vector is given by \text{Re} S.
The total energy flow along the waveguide is the sum of energy flow inside and outside the waveguide

\[ P_z = P_z^{\text{in}} + P_z^{\text{out}} \]  \hspace{1cm} (3-16)

with

\[ P_z^{\text{in}} = 2\pi \int_0^a S_z \, r \, dr, \quad P_z^{\text{out}} = 2\pi \int_a^\infty S_z \, r \, dr. \]  \hspace{1cm} (3-17)

Following Ref. [170], the total energy flow is normalized as

\[ < P_z > = \frac{P_z^{\text{in}} + P_z^{\text{out}}}{\left| P_z^{\text{in}} \right| + \left| P_z^{\text{out}} \right|} \]  \hspace{1cm} (3-18)

Thus one has \(-1 < P_z > < 1\).

In the following, we discuss different modes in detail.

**A. TE modes**

For the transverse electric (TE) modes, \( E_z = 0 \). The longitudinal magnetic field is given by Eq.(3-6). One has

\[ E_\phi = i \frac{k_0}{K^2} K_1(\kappa r), \quad H_z = -i \frac{k_0}{K} B_1(\kappa r), \quad r < a, \]  \hspace{1cm} (3-19)

\[ E_\phi = i \frac{k_0}{K^2} K_1(\kappa r), \quad H_z = -i \frac{k_0}{K_0} D_1(\kappa r), \quad r > a. \]

The continuity of \( E_\phi \) at the interface requires that

\[ h_0(\kappa a) + g_0(\kappa_0 a) = 0. \]  \hspace{1cm} (3-20)
For materials without loss, each term on the left-hand side is positive, thus there is no solution. The waveguide does not support TE modes. This is exactly like that of a metallic wire which does not support TE surface waves since current must flow along the waveguide.

Only when $\varepsilon_t > 1$, the waveguide will support TE modes, like an ordinary dielectric fiber.

**B. TM modes**

For the transverse magnetic (TM) modes, $H_z = 0$. The longitudinal electric field is given by Eq.(3-5). One has

$$H_\varphi = -i\varepsilon_z \frac{k_0}{\kappa} A_1(Kr), \quad r < a,$$

$$= i \frac{k_0}{\kappa_0} C_{11}(\kappa_0 r), \quad r > a. \quad (3-21)$$

The continuity of $H_\varphi$ leads to the equation

$$\varepsilon_z f_0(Ka) = g_0(\kappa_0 a) \quad (3-22)$$

The solutions to this equation give all the TM modes.

**1. Band structure of TM modes**

We first consider the solutions for fixed and real values of $\varepsilon_z$ and $\varepsilon_t$. This is normally associated with a fixed $k_0$. It is convenient to consider solution in the form of $K a$ or the reduced radius $k_0 a$ as a function of $\kappa_0 a$. The wave number along the waveguide can be obtained through $\beta = (k_0^2 + \kappa_0^2)^{1/2}$. Before we seek general solutions, it is better to consider the solutions in certain limits to reveal some important features of the TM modes on the anisotropic waveguide.

For the TM modes close to the light line, $\kappa_0 a \to 0$, one has
Here we have used $K_0 x = -\ln(x/2) - \gamma$ for small argument with $\gamma$ the Euler constant. For complex $\varepsilon$ with $\text{Re}(\varepsilon) > 0$ and $\text{Im}(\varepsilon) < 0$, the real and imaginary parts of $\beta$ of the allowed modes will have the same signs. These modes are forward waves, similar to that of an ordinary optical fiber. We note that close to the light line, the property of the TM modes of the anisotropic waveguide is similar to that of an isotropic fiber with $\varepsilon = 1 + \varepsilon_z (1 - \varepsilon_t^{-1})$.

In the limit of long wavelength or small waveguide radius, $k_0 a \ll 1$. Eq.(3-22) is reduced to

$$\varepsilon_z f_0 (\kappa_0 a / \eta) = g_0 (\kappa_0 a)$$

(3-23)

Fig. 3-2 Band structure of the guided TM modes on an anisotropic waveguide of radius $a$ with $\varepsilon_t = -3$ and $\varepsilon_z = 2$. Open circles denote the degenerate points of forward-wave and backward-wave modes. The dashed lines are for a dielectric waveguide with $\varepsilon = 1 + \varepsilon_z (1 - \varepsilon_t^{-1})$. 

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With \( \eta = \sqrt{-\varepsilon_t / \varepsilon_z} \). This equation gives an infinite number of solutions \( \kappa_0a = \xi_{0,m} \) with \( m = 1,2,3,.... \). This indicates that the anisotropic waveguide supports infinite number of propagating modes, no matter how thin the waveguide is. For \( \kappa_0a \to \infty \), since \( g_0(\kappa_0a) = (\kappa_0a)^{-1} \to 0 \), one has \( \xi_{0,m} = \eta x_{1,m} \). Here \( x_{n,m} \) is the \( m \)-th zero of \( J_m(x) \) away from the origin. For the \( m \)-th TM band, one has \( 0 \leq \kappa_0a \leq \xi_{0,m} \). The \( m \)-th band starts with \( k_0a = x_{0,m} / \sqrt{\varepsilon_z - \varepsilon_z / \varepsilon_t} \) when \( \kappa_0a = 0 \) and ends at \( k_0a = 0 \) when \( \kappa_0a = \xi_{0,m} \). The modes with \( \kappa_0 \gg k_0 \) have \( d(k_0a) / d(\kappa_0a) < 0 \) and are backward waves. It will be obvious if we include a small imaginary part in \( \varepsilon_t \) with \( \text{Im} \varepsilon_t > 0 \). The equation

![Graph showing Effective index \( n_{eff} \) as a function of the reduced radius \( k_0a \) for the first three bands of TM modes on an anisotropic waveguide with \( \varepsilon_t = -3 \) and \( \varepsilon_z = 2 \). Open circles denote the degenerate points of forward-wave and backward-wave modes.](image_url)
will give $\beta$ with the real and imaginary parts having opposite signs. The energy flow is opposite to the phase velocity, which will be discussed later.

For arbitrary values of $\kappa_0 a$, the solution must be sought numerically. Since the right-hand side of Eq.(3-22) is always positive, the solution requires that $J_1(Ka)$ and $J_0(Ka)$ have different signs. For the $m$-th band, since $0 \leq \kappa_0 a \leq \xi_{0,m}$ with $\xi_{0,m}$ the solutions of Eq.(3-23), one has $x_{0,m} \leq Ka \leq \xi_{0,m} / \eta < x_{1,m}$. For each $\kappa_0 a$ value, the $Ka$ value can be searched within $[x_{0,m}, x_{1,m}]$ to satisfy Eq.(3-22). Once the corresponding $Ka$ is found, the reduced radius can be obtained as

$$k_0 a = \sqrt{\frac{(-\varepsilon_i / \varepsilon_z)(Ka)^2 - (\kappa_0 a)^2}{1 - \varepsilon_i}}.$$  

For the $m$-th band, the corresponding transverse electric field $E_z$ will have $m$ nodes. The band structure for a waveguide with $\varepsilon_i = -3$ and $\varepsilon_z = 2$ is shown in Fig. 3-2. The effective index of the waveguide $n_{\text{eff}} \equiv \beta / k_0$ is also evaluated and plotted in Fig. 3-3.

Unlike an ordinary fiber where for each band, $d(k_0 a) / d(\kappa_0 a) > 0$, each TM band of the anisotropic waveguide starts with $d(k_0 a) / d(\kappa_0 a) > 0$ for small $\kappa_0 a$ or near the light line. At certain value of $\kappa_0 a$ or $k_0 a$ which is marked in Fig. 3-2, $d(k_0 a) / d(\kappa_0 a) = 0$. Further increasing $\kappa_0 a$ results in $d(k_0 a) / d(\kappa_0 a) < 0$. The band ends at a finite $\kappa_0 a = \xi_{0,m}$. Immediately below the point $k_0 a$ where $d(k_0 a) / d(\kappa_0 a) = 0$, each band has two modes with opposite signs of $d(k_0 a) / d(\kappa_0 a)$. One mode is forward wave and the other backward wave. This will be discussed later in the chapter.
We next consider a waveguide of a fixed radius $a$ with $\varepsilon_t$ and $\varepsilon_z$ given by Eq. (3-24) with $a=10/k_p$ and $\varepsilon_a = 2.25$.

Here $\varepsilon_a$ and $k_p$ are positive constants. The realization of this property will be discussed later in Sec. IV. If $k_0 < k_p / (1 + \varepsilon_z)^{1/2}$, one has $\varepsilon_t < 0$ and $\varepsilon_z > 0$. The band structure of the TM modes on this waveguide is obtained by numeric means and plotted in Fig. 3-4 with the corresponding effective index in Fig. 3-5. For this waveguide, there is no cutoff of $\kappa_0 a$ for each band. This is
because as \( k_0 \to 0 \), \( \varepsilon_z = 2\varepsilon_a \) and \( \varepsilon_t \to -\infty \), thus \( \eta = \sqrt{-\varepsilon_t / \varepsilon_z} \to \infty \). The cutoff
\[
\xi_{0,m} = \eta \chi_{1,m} \to \infty .
\]

Fig. 3-5 Effective index \( n_{\text{eff}} \) as a function of the reduced wave number \( k_0a \) for the first three bands of TM modes on an anisotropic waveguide with \( \varepsilon_t \) and \( \varepsilon_z \) given by Eq.(3-24) with \( a=10/k_p \) and \( \varepsilon_a = 2.25 \).

We point out that the Padé approximant for the function \( f_0(x) \equiv -x^{-1}J_1(x) / J_0(x) \) can be used to obtain a good estimate of the solutions. This will be discussed in the Appendix.

If \( \varepsilon_t > 0 \) and \( \varepsilon_z < 0 \), the waveguide also supports TM modes. The details will not be presented here.

2. Energy flow of TM modes
Within the waveguide, one has \( E_z = AJ_0(Kr) \), \( E_r = i(\varepsilon_0 \beta / \varepsilon)AJ_1(Kr) \), and \( H_\phi = i(\varepsilon_0 \beta / K)AJ_1(Kr) \). So the Poynting vector component along the axis of the waveguide is

\[
S_z = \frac{\beta}{\varepsilon k_0 a^2} \left| \frac{J_1(Kr)}{J_1(Ka)} \right|^2
\]  

(3-25)

Here we set the coefficient \( A = K / [\varepsilon_0 k_0 a J_1(Ka)] \). Since \( \varepsilon < 0 \), the energy flow inside the nanowire is always opposite to the phase velocity.

For the field in the air \( r > a \), one has \( E_z = CK_0(\kappa_0 r), E_r = i(\beta \kappa_0 r)CK_1(\kappa_0 r) \), and \( H_\phi = i(k_0 / \kappa_0)CK_1(\kappa_0 r) \), thus

\[
S_z = \frac{\beta}{k_0 a^2} \left| \frac{K_1(\kappa_0 r)}{K_1(\kappa_0 a)} \right|^2
\]  

(3-26)

Here the coefficient \( C = -\kappa_0 / [k_0 a K_1(\kappa_0 a)] \).

For the TM modes, one has

\[
P_{z\text{in}} = \frac{\beta}{2\varepsilon_0 k_0 a^2} \int_0^a \left| \frac{J_1(Kr)}{J_1(Ka)} \right|^2 r dr,
\]  

\[
P_{z\text{out}} = \frac{\beta}{2k_0 a^2} \int_0^a \left| \frac{K_1(\kappa_0 r)}{K_1(\kappa_0 a)} \right|^2 r dr.
\]  

(3-27)

The above integrals can be carried out and more compact expressions for the energy flow can be obtained as
\[ P_z^{\text{in}} = \frac{\beta}{4\varepsilon_0 k_0 (Ka)^2} \left( \frac{1}{f_0^2(Ka)} + \frac{2}{f_0(Ka)} + (Ka)^2 \right) \]

\[ = -\frac{\beta}{4k_0\varepsilon_0^2 f_0^2(Ka)} \frac{\varepsilon_0^2 f_0'(Ka)}{\varepsilon_0 Ka}, \]  

\[ P_z^{\text{out}} = \frac{\beta}{4k_0 (\kappa_0 a)^2} \left( \frac{1}{g_0^2(Ka)} + \frac{2}{g_0(Ka)} - (\kappa_0 a)^2 \right) \]

\[ = -\frac{\beta}{4k_0 g_0^2(\kappa_0 a)} \frac{g_0'(\kappa_0 a)}{\kappa_0 a}. \]  

(3-28)

Here \( g_0'(x) \) and \( f_0'(x) \) are the derivatives of \( g_0(x) \) and \( f_0(x) \), respectively.

For convenience, we set \( \beta > 0 \) throughout the chapter. Since \( g_0'(x) < 0 \) and \( f_0'(x) < 0 \), one has \( P_z^{\text{in}} < 0 \) and \( P_z^{\text{out}} > 0 \). In this convention, if \( \{P_z\} > 0 \), this indicates that the energy flow and the phase propagation are in the same directions and the mode is a forward-wave mode. Otherwise \( \{P_z\} < 0 \), the group velocity and the phase velocity are in the opposite direction and the mode is a backward-wave mode. The normalized energy flow for TM modes on a waveguide with \( \varepsilon_i = -3 \) and \( \varepsilon_z = 2 \) is shown in Fig. 3-6. We note that for some portion of the bands the value of \( \{P_z\} \) is negative and thus these modes are backward waves.
3. Forward-wave and backward-wave TM modes

There are three ways to determine whether a mode is a forward wave or a backward wave. One is through the sign of the derivative $d(k_0a)/d(\kappa_0a)$. From the band structure, we have already noticed that for the modes near the light line, $d(k_0a)/d(\kappa_0a) > 0$. These modes are forward waves. For large $\kappa_0a$ or small $k_0a$, one has $d(k_0a)/d(\kappa_0a) < 0$, these modes are backward waves. From the band structure shown in Fig. 3-2, the majority of the modes are backward waves.
The second way is through the sign of $\langle P_z \rangle$. For the TM modes when $\kappa_0 a \to \infty$, one has $f_0(Ka) \to 0$ since $g_0(\kappa_0 a) \to 0$. One thus has $Ka \to x_{0,m}$. This solution leads to the divergence of $P_z^{in}$ which is negative and the vanishing of $P_z^{out}$ which is positive, subsequently $\langle P_z \rangle \to -1$, these modes are all backward waves. Correspondingly, one has $d(k_0 a) / d(\kappa_0 a) \to -\infty$ for $\kappa_0 a \to \infty$. This is evident from the band structure shown in Fig. 3-2.

In the following, we prove that $d(k_0 a) / d(\kappa_0 a) \geq 0$ leads to $\langle P_z \rangle \geq 0$ and vice versa. We consider the derivative

$$\frac{d(ka)}{d(\kappa_0 a)} = -\frac{\varepsilon_z}{\varepsilon_Ka} \left( \kappa_0 a + (1 - \varepsilon_\varepsilon)k_0 a \frac{d(k_0 a)}{d(\kappa_0 a)} \right)$$

Thus one has

$$\frac{d(Ka)}{d(\kappa_0 a)} + \frac{\varepsilon_\varepsilon \kappa_0 a}{\varepsilon_Ka} \geq 0 \quad \text{if} \quad \frac{d(k_0 a)}{d(\kappa_0 a)} \geq 0$$

On the other hand, according to the eigenequation $\varepsilon_z f_0(Ka) = g_0(\kappa_0 a)$, one has

$$\frac{d(Ka)}{d(\kappa_0 a)} = \frac{g_0'(\kappa_0 a)}{\varepsilon_z f_0'(Ka)}$$

Making use of the above expression, one arrives at the inequality

$$-\frac{\varepsilon_z f_0'(Ka)}{\varepsilon_Ka} - \frac{g_0'(\kappa_0 a)}{\kappa_0 a} \geq 0,$$

and subsequently

$$P_z^{in} + P_z^{out} \geq 0 \quad \text{if} \quad \frac{d(k_0 a)}{d(\kappa_0 a)} \geq 0 \quad (3-29)$$
Similarly one has

\[ P^\text{in}_z + P^\text{out}_z \leq 0 \quad \text{if} \quad \frac{d(k_0 a)}{d(\kappa_a)} \leq 0 \]  \hspace{1cm} (3-30)

So the condition for \( P_z = 0 \) can be allocated from the band structures as shown in Fig. 3-2 and Fig. 3-4 when \( \frac{d(k_0 a)}{d(\kappa_0 a)} = 0 \).
The third way to determine whether a mode is a forward or backward wave is through the relative sign of the real and imaginary parts of $\beta$ if dissipation is included. For example, we consider $\varepsilon_t = -3 + 0.05i$ and $\varepsilon_z = 2$. At $k_0a = 1.6$, the wave numbers of the first three eigenmodes are $\beta a = \pm (1.7112 + 0.0067i)$, $\pm (2.7250 - 0.0397i)$, $\pm (7.5756 - 0.0676i)$. Since the free space wavelength is $\lambda = 3.927a$, this is a subwavelength waveguide. For the TM modes, except for the first mode, all the other modes are backward-wave modes since for those modes $\text{Re} \beta$ and $\text{Im} \beta$ have different signs. The normalized energy flow is $\langle P_z \rangle = 0.5151 - 0.0020i$, $-0.4002 - 0.0058i$, $-0.8760 - 0.0078i$ for the above three modes, respectively. Here we set $\text{Re} \beta > 0$. The field and Poynting vector profiles are plotted in Fig. 3-7.

There is an interesting feature of the modes on the anisotropic waveguide.

At a fixed $k_0$, for $a < a_m = \sqrt{-\varepsilon_t / \varepsilon_z (1 - \varepsilon_t)} x_{0,m} / k_0$, the $m$-th band TM modes are backward waves. If the radius $a > a_m$, the waveguide supports two TM modes for the $m$-th band, one forward and one backward. At $a = a_c$, these two modes become degenerate and the total energy flow is zero. This can be seen in Fig. 3-2, Fig. 3-3, and Fig. 3-6 where degenerate points are marked. Further increasing the radius, the waveguide will no longer support the $m$-th band. The critical radius $a_c$ is located such that $\langle P_z \rangle = 0$, $d(k_0a) / d(\kappa_0a) = 0$ or $dn_{\text{eff}} / da = \infty$. These degenerate modes can be used to slow down and even trap light. This will be discussed in the next section.

C. Hybrid modes
The modes with both \( E_z \neq 0 \) and \( H_z \neq 0 \) are called hybrid modes. Their dispersions are contained in the solutions of Eq. (3-14) with \( n \neq 0 \). We recast the equation in the following form

\[
\varepsilon_z f_n(Ka) = g_n(\xi) - \frac{n^2(\xi^{-2} - y^{-2})(\xi^{-2} - \varepsilon_z y^{-2})}{g_n(\xi) + h_n(y)}. \tag{3-31}
\]

Here we use the notation \( \xi = \kappa_0a \) and \( y = \kappa a \). Note that \( Ka = y/\eta \) with \( \eta = \sqrt{-\varepsilon_y/\varepsilon_z} \).

1. Band structure of hybrid modes

At a fixed wavelength \( \lambda \) or wave number \( k_0 \), \( \varepsilon_y, \varepsilon_z \) are constant. Since \( Ka = y/\eta \), if we use \( \xi = \kappa_0a \) as a free parameter, the eigenequation gives a single value of \( Ka \) or \( y \) for each \( \xi \). Since \( y = \sqrt{\xi^2 + (1 - \varepsilon_y)(k_0a)^2} \), so the eigenequation actually gives the reduced radius \( k_0a \) for each \( \xi = \kappa_0a \).

In the limit of long wavelength or small waveguide radius, \( k_0a \to 0 \), Eq.(3-31) is reduced to

\[
\varepsilon_z f_n(\xi/\eta) = g_n(\xi) \tag{3-32}
\]

This will give a discrete set of solutions \( \kappa_0a = \xi_{n,m} \) for each \( n \). For \( \eta \) and \( \varepsilon_z \) not very small, the values \( \xi_{n,m} \) can be obtained approximately by making use of the asymptotic behavior \( g_n(x) \sim x^{-1} \) for \( x \to \infty \) and the Padé approximant of \( f_n(x) \), which will be discussed in the Appendix. The anisotropic waveguide supports infinite number of hybrid modes, no matter how thin the waveguide is.

Close to the light line, \( \xi \to 0 \), the eigen equation can be simplified. We consider the hybrid modes with different \( n \) separately.
For \( n = 1 \), one has \( g_1(x) = x^{-2} - \ln(x/2) - \gamma \) for \( x \to 0 \), thus Eq.(3-31) becomes

\[
\varepsilon_i f_1(Ka) = -2\left( \ln \frac{x}{\xi} + \gamma \right) + h_1(y) + \frac{1 + \varepsilon_i}{y^2}. \tag{3-33}
\]

with \( \xi \ll y \). One has the solution \( Ka \to x_{1,m} \) for \( \xi \to 0 \). Note that throughout this chapter, we use \( x_{n,m} \) to denote the \( m \)-th zero of \( J_n(x) \). Also that \( x_{n,0} = 0 \) for \( n \geq 1 \). Since \( x_{1,0} = 0 \), special care should be taken for solutions \( Ka \approx 0 \). Making use of the asymptotic behaviors

\[
h_n(x) = nx^{-2} + 1/[2(n + 1)] \quad \text{and} \quad f_n(x) = nx^{-2} - 1/[2(n + 1)] \quad \text{for} \quad x \to 0 \quad \text{with} \quad n \geq 1,
\]

one gets

\[
Ka = \frac{1}{\eta} \sqrt{\frac{1 + \varepsilon_i}{\ln(\xi/2) + \gamma - (\varepsilon_i + 1)/8}} \tag{3-34}
\]

From this expression, one can see that only if \( \varepsilon_i < -1 \), there will be solution for \( Ka \approx 0 \) when \( \xi \to 0 \). The above expression gives the dispersion of the first band with \( n = 1 \). The \( m \)-th band will start with \( Ka = x_{1,m-1} \) and end with \( Ka = \xi_{1,m} / \eta \). One has \( \xi_{1,1} < \eta x_{1,1} \). However if \(-1 < \varepsilon_i < 0 \), there will be no solution for \( Ka < x_{1,1} \). All the allowed modes will have finite \( Ka \). For the \( m \)-th band, one has \( x_{1,m} \leq Ka \leq \xi_{1,m} / \eta \) with \( \xi_{1,1} > \eta x_{1,1} \). The first band starts with \( Ka = x_{1,1} \). For any \( x_{1,m} > 0 \), one has

\[
Ka = x_{1,m} - \frac{\varepsilon_i x_{1,m}}{(\eta x_{1,m})^2 [-2(\ln(\xi/2) + \gamma) + h_1(\eta x_{1,m})] + \varepsilon_i + 1}. \tag{3-35}
\]

The hybrid modes near the light line with \( n = 1 \) are all forward waves. For \( n \geq 3 \), one has the following asymptotic behavior of \( g_n(x) \) for \( x \to 0 \),
The eigenmodes on the light line are given by the equation $c_0 = 0$. The existence of solution requires that $c_0 \geq 0$. Only the positive root with small magnitude of the above cubic polynomial will give the dispersion for the modes near the light line. Since $\xi$ is small, the physical solution can be approximated as $\xi^2 = -c_0 / c_1 - c_2 c_0 / c_1 - a^2 c_0 / c_1$. Those modes all have $K\alpha > x_{n,m}$ for the $m$-th band.

For $n = 2$, the asymptotic behavior of $g_n(x)$ is still given by Eq.(3-36), but with a non-constant coefficient $\alpha = [\ln (x / 2) + \gamma] / 4$. The eigenequation near the light line is reduced further from Eq.(3-37) to

$$(\kappa_0 \alpha)^2 \left( \ln \frac{\kappa_0 \alpha}{2} + \gamma \right) + c_0 = 0$$

(3-39)

with $c_0$ given in Eq.(3-38) with $n = 2$.  

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For the allowed eigenmodes of the \( m \)-th hybrid band, one has the range \( 0 \leq \kappa_0 a < \xi_{n,m} \) with \( \xi_{n,m} \) the solutions of Eq.(3-32). The solutions near both ends of the above range can be obtained analytically as we have done. For arbitrary \( \xi \) within this range, the solution must be obtained numerically. However only when \( \varepsilon_i < -1 \), one can have eigenmodes with \( 0 < Ka < x_{n,1} \). For the \( m \)-th band, one has \( x_{n,m} < Ka \leq \xi_{n,m} / \eta \). Otherwise, the solution for the first band requires \( x_{n,1} < Ka \leq \xi_{n,1} / \eta \) with \( \xi_{n,1} > \eta x_{n,1} \). The band structure for hybrid modes on a waveguide with \( \varepsilon_i = -3 \) and \( \varepsilon_z = 2 \) is shown in Fig. 3-8. The effective index of the waveguide \( n_{\text{eff}} \equiv \beta / k_0 \) is also evaluated and plotted in Fig. 3-9.

Fig. 3-8 Band structure of the guided hybrid modes with \( n = 1 \) on an anisotropic waveguide with \( \varepsilon_i = -3 \) and \( \varepsilon_z = 2 \). Open circles denote the degeneracy of forward-wave and backward-wave modes.
2. Energy flow of hybrid modes

The energy flow can also be evaluated for hybrid modes. The expression for $S_z$ is much more complex than that of the TM modes. However the final expression for $P_z$ is much simpler than expected, once the integrals are all carried out. One has

$$\begin{align*}
P_{z}^{in} &= \frac{\beta}{4k_0} \left\{ -\frac{\varepsilon_z^2}{\varepsilon_i Ka} (g_n + h_n) f_n'' - \frac{1}{\kappa a} (g_n - \varepsilon_z f_n) h_n'' + \frac{2n^2}{(\kappa a)^4} \left[ \frac{1 + \varepsilon_i}{(\kappa a)^2} - \frac{2\varepsilon_i}{(\kappa a)^2} \right] \right\}, \\
P_{z}^{out} &= \frac{\beta}{4k_0} \left\{ -\frac{1}{\kappa_0 a} (2g_n + h_n - \varepsilon_z f_n) g_n'' - \frac{2n^2}{(\kappa_0 a)^4} \left[ \frac{2}{(\kappa_0 a)^2} - \frac{1 + \varepsilon_i}{(\kappa a)^2} \right] \right\}. \quad (3-40)\end{align*}$$

Fig. 3-9  Effective index $n_{eff}$ as a function of the reduced radius $k_0a$ for the first three bands of hybrid modes with $n = 1$ on an anisotropic waveguide with $\varepsilon_r = -3$ and $\varepsilon_z = 2$. Open circles denote the degeneracy of forward-wave and backward-wave modes.
Here \( f'_n, g'_n, \) and \( h'_n \) are the derivatives of \( f_n, g_n, \) and \( h_n, \) respectively. In this derivation, we assume \( \varepsilon_i \) and \( \varepsilon_z \) are real, thus the arguments of the Bessel functions are all real. We set
\[ A = \sqrt{g_n + h_n} / (k_o \omega J_n). \]
Use has also been made of the following integrals which cannot be found in any mathematics manual
\[
\int_0^x \left( J_n^2(x) + \frac{n^2}{x^2} J_n^2(x) \right) dx = -\frac{x^3}{2} J_n^2(x) f'_n(x),
\]
\[
\int_0^x \left( I_n^2(x) + \frac{n^2}{x^2} I_n^2(x) \right) dx = -\frac{x^3}{2} I_n^2(x) h'_n(x),
\]
\[
\int_x^\infty \left( K_n^2(x) + \frac{n^2}{x^2} K_n^2(x) \right) dx = -\frac{x^3}{2} K_n^2(x) g'_n(x).
\]
Explicitly, one has
\[
f'_{n}(x) = -xf_{n}(x) - \frac{2}{x} f_{n}(x) + \frac{n^2}{x^3} - \frac{1}{x},
\]
\[
h'_{n}(x) = -xh_{n}(x) - \frac{2}{x} h_{n}(x) + \frac{n^2}{y^3} + \frac{1}{x},
\]
\[
g'_{n}(x) = xg_{n}(x) - \frac{2}{x} g_{n}(x) - \frac{n^2}{x^3} - \frac{1}{x}.
\]
We point out that the above expressions for \( P_z \) can be readily modified for dielectric or metallic cylindrical waveguide with the exchange of \( f_n(ix) = -h_n(x) \) and \( ixf'_n(ix) = -xh'_n(x). \)
The normalized energy flow on a waveguide with \( \varepsilon_i = -3 \) and \( \varepsilon_z = 2 \) for the hybrid modes with \( n = 1, 2, 3 \) is plotted in Fig. 3-10.
3. Backward-wave and forward-wave hybrid modes

Fig. 3-10 Normalized energy flow $\langle P_z \rangle$ for the first few bands of the hybrid modes with $n=1,2,3$ on an anisotropic waveguide with $\varepsilon_t = -3$ and $\varepsilon_z = 2$. Open circles denote the degeneracy points where $\langle P_z \rangle = 0$. 
The hybrid modes have similar features as the TM modes that both forward waves and backward waves can co-exist within the same band. For \( k_0 a \to 0, \kappa = \kappa_0 \), the hybrid modes have \( d(k_0 a) / d(\kappa_0 a) < 0 \) and are all backward waves. Only the modes near the light line can be forward waves. Making use of the expressions for \( P_z \) in Eq.(3-40), one can prove that \( d(k_0 a) / d(\kappa_0 a) \geq 0 \) leads \( P_z \geq 0 \) and vice versa. The degeneracy of forward- and backward-wave modes is located at \( d(k_0 a) / d(\kappa_0 a) = 0 \) or \( \langle P_z \rangle = 0 \).

For the hybrid modes with \( n = 1, 2 \), the modes near the light line are forward waves since \( d(k_0 a) / d(\kappa_0 a) > 0 \). However as can be seen in Fig. 3-10, the whole first band of the hybrid modes with \( n = 3 \) are backward waves. Hybrid modes with higher \( n \) will have more bands to be all backward waves. If one further increases the angular index \( n \) of the hybrid modes, more hybrid mode bands will be all backward waves.

**III. Slow and Trapped Light**

Recently Tsakmakidis et al. [171] proposed to trap light in a tapered waveguide with double negative index. The indefinite index waveguide we have studied so far in this chapter can also be used to slow down and trap light. These waveguides can thus be used as optical buffers [172]. The reason is that unlike the ordinary optical fiber, these waveguides support both forward and backward waves.

For the anisotropic waveguide we have considered, \( \varepsilon_i < 0 \), one has \( P_z^{\text{in}} < 0 \) and \( P_z^{\text{out}} > 0 \) if one sets \( \beta > 0 \). If \( P_z = P_z^{\text{in}} + P_z^{\text{out}} < 0 \), the mode is a backward mode since the total energy flow is
opposite to the phase velocity. Otherwise, the mode is a forward mode. At the critical radius $a_c$, the backward and forward modes become degenerate; the energy flow inside the waveguide cancels out that in the air. One can prove that at the critical radius $a_c$ where $P_z = 0$, the group velocity is indeed zero. One does not need to know the material dispersion to locate the zero group velocity point. This is due to the fact that for these waveguides, the dispersion due to geometric confinement dominates the material dispersion at and around the critical radius.

![A sketch of the slow light waveguide.](image)

Fig. 3-11 A sketch of the slow light waveguide.

The unique properties of the modes on anisotropic waveguide can be used to slow down and even trap light. Even though the waveguide supports infinite number of both TM and hybrid modes at any fixed radius and frequency, with appropriate laser coupling, the excitation of the hybrid modes in the waveguide can be suppressed or even eliminated. Among the TM modes, the first TM mode will be more favorably excited. Furthermore, due to the material dissipation, the first TM mode will propagate the longest distance. The rest of the TM modes will all decay out at about half the decay length of the first TM mode. It is the first TM band which can be used
for slow light application. Unlike the double negative waveguide [171], the anisotropic waveguide will slow down and trap light if one increases the radius to the critical radius. A sketch of a slow light waveguide is shown in Fig. 3-11.

IV. Realization of Extremely Anisotropic Nanowires

These extremely anisotropic media can be realized in metamaterials. According to the effective medium theory [166][173], one has for multilayered structure of dielectric \( \varepsilon_a \) and metal \( \varepsilon_m \), the effective permittivity’s are

\[
\begin{align*}
\varepsilon_x &= f \varepsilon_m + (1-f) \varepsilon_a, \\
\varepsilon_z &= \frac{\varepsilon_a \varepsilon_m}{f \varepsilon_a + (1-f) \varepsilon_m}.
\end{align*}
\]  
(3-43)

Fig. 3-12 A sketch of the realization of an anisotropic nanowire made of alternative disks of metal and dielectric.
Here \( f \) is the filling ratio of the metal. For \( f > f_{\min} \equiv c_a / (\varepsilon_a - \Re \varepsilon) \), one has \( \Re \varepsilon < 0 \). A realization of the anisotropic nanowire is shown in Fig. 3-12.

We first consider a metamaterial waveguide at a fixed wavelength. For silver at \( \lambda = 488 \) nm, one has \( \varepsilon_m = -9.121 + 0.304i \) [174]. A nanowire made of alternative disks of silver and glass (\( \varepsilon_a = 2.25 \)) of equal thickness will have \( \varepsilon_r = -3.436 + 0.152i \) and \( \varepsilon_z = 5.971 + 0.065i \) by using the effective medium theory. Here the disk thickness is 10 nm for both materials. For example if one sets \( a = 60 \) nm, one has \( k_0a = 0.7725 \). The first three TM modes will have \( \beta a = 2.2525 - 0.0747i, 4.9318 - 0.1419i, 7.4031 - 0.2088i \). Thus one has \( \lambda_\beta \sim 167 \) nm and phase refractive index \( n_{\text{eff}} = 2.92 \) for the first TM mode. The decay length is 803, 423, and 287 nm, respectively. After traveling about 420 nm along the nanowire, only the first one will survive.

Finite-difference time-domain (FDTD) simulations [175] were performed to obtain the effective index \( n_{\text{eff}} \) of the metamaterial nanowire. The procedure is the following. We illuminate the free-standing nanowire of finite length with a Gaussian beam, and then get \( E_z \) after the termination of the simulation. The length of the waveguide is set to be larger than the decay length of the first TM mode. We get the phase from \( E_z \), then determine \( \beta \). Though the waveguide supports infinite number of modes including TM and hybrid modes, our method is legitimate due to the following two reasons. First, that the excitation of hybrid modes is small due to the profile of the incident Gaussian beam. So mainly the TM modes are excited. Second that due to the dissipation in the metal, after certain distance, only the first TM mode will
survive. Thus the phase propagation is mainly due to the first TM mode. The amplitude and phase propagation of $E_z$ along the above metamaterial nanowire is shown in Fig. 3-13.

![Diagram of Abs($E_z$) and Phase of $E_z$](image)

**Fig. 3-13** FDTD simulation of the amplitude and phase propagation of the longitudinal electric field $E_z$ along the nanowire with radius $a = 60$ nm at $\lambda = 488$ nm. The metamaterial nanowire consists of alternative disks of silver and glass disks of thickness 10 nm.

The relation between the effective index $n_{\text{eff}}$ and the nanowire radius $a$ is shown in Fig. 3-14. Very good agreement between FDTD simulations and analytical results has been obtained. However for small radius, there is noticeable discrepancy. This is expected since when the radius is comparable with the lattice spacing of the multilayered metamaterial, the effective medium theory will fail. We have also performed FDTD simulations for the nanowire with smaller lattice spacing. Better agreement is indeed obtained.
We also consider the band structure for different frequencies. The permittivity given by Eq.(3-24) can be realized through the multilayered heterostructure with Drude metal with 

$$\varepsilon_m = 1 - k_p^2 / k_0$$

and dielectric \(\varepsilon_a\). The nanowire is made of alternative disks of a Drude metal and a dielectric. The band structure and the effective index \(n_{\text{eff}}\) of the TM modes are shown in Fig. 3-4 and Fig. 3-5, respectively. One noticeable feature of these bands is the flatness of each band, which indicates small group velocity. We have performed the FDTD simulation for different

Fig. 3-14 The effective index \(n_{\text{eff}}\) of the first TM band on a nanowire with different radius at \(\lambda = 488\) nm. The nanowire is made of alternative disks of silver and glass (see Fig. 3-12). The disk thickness is 10 nm. Filled circle (green) is obtained from FDTD simulations. The dashed line (blue) is the fitting of simulation data. The solid line (red) is calculated from band equation with effective index \(\varepsilon_t = -3.436+0.152i\) and \(\varepsilon_z = 5.971+0.065i\).
frequencies for nanowire with a fixed radius. The results are shown in Fig. 3-15. Again good agreement between FDTD simulations and analytical results is achieved.

![Graph showing effective index vs. wavelength](image)

Fig. 3-15 The effective index $n_{\text{eff}}$ for the TM modes on a nanowire with radius $a = 40$ nm. The nanowire is a stack of equally thick alternative disks made of a Drude metal $\varepsilon_m = 1 - \frac{k_p^2}{k_0(k_0 + i\Gamma)}$ and glass. Here $k_p a = 1.64$ and $\Gamma a = 0.0155$. The analytical curves (solid) are calculated by using real $\varepsilon_m$.

V. Conclusions

Indefinite index materials can be used to achieve negative refraction [165] and hyperlensing [167][168]. They can also be used as superlens [166]. In this chapter, we consider the wave propagation along a cylindrical waveguide with anisotropic optical constant. We have derived the eigenmodes equation and obtained the solutions for all the propagation modes. The field
profiles and the energy flow on the waveguide are also analyzed. Closed-form expressions for the energy flow for all the modes are derived. For extremely anisotropic cylinder where the transverse component of the permittivity is negative and the longitudinal is positive \( \varepsilon_t < 0, \varepsilon_z > 0 \), the waveguide supports TM and hybrid modes but not the TE modes. Among the supported TM modes, at most only one mode can be forward wave. The rest of them are backward waves.

The case that \( \varepsilon_t > 0 \) and \( \varepsilon_z < 0 \) can be discussed similarly. Anisotropic waveguides of cross section other than circle were also considered. The results will be published elsewhere.

Possible realization of these extremely anisotropic nanowires is proposed. Extensive FDTD simulations have been performed and confirmed our analytical results.

Two unique properties have been revealed for the modes on nanowire waveguides made of indefinite index metamaterials. The first is that the backward-wave modes can have very large effective index. These nanowires can be used as phase shifters and filters in optics and telecommunication. The second is that the waveguide supports modes of zero group velocity. This is due to the fact that the waveguide can support both forward and backward waves at a fixed radius. If the waveguide is tapered, at certain critical radius, the two modes will be degenerate and carry zero net energy flow. At other radii, these waveguides support modes with small group velocity. These waveguides can also be used as ultra-compact optical buffer [172] in integrated optical circuits.

**Appendix: PADÉ Approximant of \( f_n(x) \)**
Consider the function \( f_n(x) = J_n'(x) / [xJ_n(x)] \). Let \( x_{n,m} \) be the \( m \)-th zero of the Bessel function \( J_n(x) \). We also denote \( x_{n,m}' \) as the zeros of \( J_n'(x) \). In order to get the eigenmodes on the anisotropic waveguide easily, we may need the inverse function \( f_n^{-1}(x) \) in the interval \([x_{n,m}, x_{n,m}']\). We consider the Padé approximant to the function \( f_n(x) \),

\[
f_n(x) = \frac{(x - x_{n,m}')(x - x_{n,m} - b)}{c(x - x_{n,m})(x - x_{n,m} - a)}. \tag{3-44}
\]

Instead of fixing the three unknowns through the coefficients of the Taylor expansion of \( f_n(x) \), here we determine them by the exact values of \( f_n(x) \) at some points. Since there are three unknowns, we only need the value of \( f_n(x) \) at three points. For simplicity, we evaluate \( f_n(x) \) at three evenly spaced points

\[
f_{n,j} = f_n(x_{n,m} + j\Delta / 4) \tag{3-45}
\]

for \( j = 1, 2, 3 \). Here \( \Delta = x_{n,m}' - x_{n,m} \).

One thus has \( f_{n,j} = (4 - j)(j\Delta - 4b) / [jc(j\Delta - 4a)] \). We further define

\[
\sigma_1 = 3f_{n,2} / f_{n,1}, \quad \sigma_2 = 3f_{n,3} / f_{n,2}. \tag{3-46}
\]

After some manipulations of the algebra, one obtains

\[
a = \frac{\Delta}{4} \left(2 + \frac{\sigma_2 - \sigma_1^{-1}}{\sigma_2 + \sigma_1^{-1} - 2}\right),
\]

\[
b = \frac{\Delta}{4} \left(2 - \frac{\sigma_1 - \sigma_2^{-1}}{\sigma_1 + \sigma_2^{-1} - 2}\right), \tag{3-47}
\]

\[
c = \frac{1}{f_2} \frac{\Delta - 2b}{\Delta - 2a}.
\]

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The inverse of the function \( f_n(x) \) can be obtained as one of the roots of a quadratic equation. As it turns out, the expression obtained in this way gives very good approximation to \( f_n^{-1}(x) \).

Once the inverse function is obtained, one has \( Ka = f_0^{-1}(y) \) with \( y = g_0(\kappa_0 a) / \varepsilon_z \) for the TM modes. Together with the asymptotic expressions of \( g_n(x) \) and \( h_n(x) \), Eq.(3-44) can also be used to obtain approximate solutions of the hybrid modes.
Chapter 4 Storing Light in Active Optical Waveguides with Single-Negative Metamaterials

I. Introduction

There is a strong interest in controlling the speed of light for a variety of optical applications. Slow light has been realized in atomic gas and other systems using phenomena such as electromagnetically induced transparency. Slow light by geometric confinement is more suitable for integrated optical circuits. Various waveguide structures have been proposed and some have been realized, such as the coupled resonator optical waveguide (CROW) and the line-defect photonic crystal waveguide. While low loss can be achieved in these waveguide structures, device applications are hindered by the narrowness of the bandwidth due to the resonant nature of these structures, difficulty of fabrication, or the need to excite surface waves. A figure of merit for these pulse delay devices is the delay bandwidth product (DBP). So far the achieved DBP is about 80. Though very large DBP ~10^3-4 has been achieved by using light-matter interactions in Bose-Einstein condensates, they are not ideal for integrated optics.

Recently Tsakmakidis et al. proposed a novel planar waveguide to stop light by using a double-negative metamaterial (DNM) as core layer cladded with positive index material. Since light experiences negative Goos-Hänchen lateral shifts between positive and negative index materials, different wavelengths will be stopped at different waveguide thickness, thus
forming a “trapped rainbow”. One difficulty is that it is not easy to realize DNMs at optical wavelengths[193]. A fundamental limitation [194][195] is that absorptive losses, which are always present in real materials when a gain mechanism is absent[196][197], will destroy the zero group velocity condition[198] and severely limit achievable performance for parameters such as DBP. The presence of loss means that light cannot be stored indefinitely in the waveguide structure.

II. Waveguide geometry and transverse wave modes

Fig. 4-1 Waveguide geometry and transverse magnetic modes. a, Symmetric planar waveguide made of dielectric $\varepsilon_D$ cladded with a single negative material $\varepsilon_M$ with $\text{Re}(\varepsilon_M / \varepsilon_D) < 0$ for stopping the TM modes. b, Parameter range of cladding layer $\varepsilon_M$ vs. core layer $\varepsilon_D$ for which zero group velocity is possible (shaded area). c, The effective indices and d, the normalized energy flow of the $\text{TM}_m$ modes in a planar dielectric waveguide $\varepsilon_D=12.12$ cladded with a metal with $\varepsilon_M = -1$ at a fixed wavelength.
Here we propose a slow light waveguide consisting of an active dielectric core layer cladded by single-negative ($\varepsilon < 0$ or $\mu < 0$) metamaterial [199]. We have discovered a specific window of material parameter conditions in which this waveguide will support degenerate forward- and backward-wave modes and achieve zero group velocity. Most importantly, in this structure, gain can be incorporated into the core layer which can lead to dramatic improvements in device parameters. We show that at a critical gain value the condition $v_g = 0$ can be recovered, compensating for the limitations imposed by absorptive losses. Our results hold even in the presence of gain pinning.

We first analyze the basic structure consisting of a symmetric planar waveguide of thickness $d$ made of a dielectric core layer $\varepsilon_D > 0$ cladded with a plasmonic metal (Fig. 4-1). The waveguide supports both transverse electric (TE) modes and transverse magnetic (TM) modes. An important quantity for the waveguide is the ratio $\sigma = -\varepsilon_D / \varepsilon_M$ which depends on the operating wavelength $\lambda$. Here, for simplicity of analytical discussion, we assume that $\sigma$ is real and also ignore the dispersion of the dielectric. At low frequencies, the metal can be treated as a perfect electronic conductor (PEC), thus $\sigma = \infty$. In the near infrared and the visible, the permittivity of metal $\varepsilon_M$ is complex with $\text{Re} \ \varepsilon_M < 0$. For TM modes, the energy flow inside the waveguide has the opposite sign of the energy flow in the cladding due to the negative permittivity of the cladding layer. As the frequency increases, the ratio $\sigma$ becomes smaller and the field has more penetration into the metal. For some modes, the energy flow in the cladding can be increased to completely compensate that in the core layer, resulting in zero net energy flow, thus zero group velocity. This is also due to the negative Goos-Hänchen lateral shifts at the
interface between single-negative medium and positive index medium [200]. Upon further increase of the frequency, the waveguide may support both forward and backward TM waves [201].

For the TM modes in a symmetric planar waveguide we considered in the chapter, the eigenmode equation is \( \sqrt{\varepsilon_D} k_0 d = f_{TM}(\xi) \) with \( f_{TM}(\xi) = \sqrt{1 + \xi^2 \left( m\pi - 2 \arctan(\xi / \sigma) \right) / \sqrt{1 + \sigma^2}} \).

Here \( k_0 \) is the wave number in the vacuum and \( \xi \) is a free parameter with range \( \xi \geq \sqrt{\sigma} \) for supporting guided modes in the waveguide. As illustrated in Fig. 4-1, the modes propagate along the \( z \)-direction with longitudinal wave-vector component \( k_z \). The condition for the waveguide to support degenerate TM\(_m\) modes can be shown to be

\[
\frac{1}{\sqrt{\sigma}} + \arctan \left( \frac{1}{\sqrt{\sigma}} \right) \geq m\pi / 2
\]  

(4-1)

One has \( \sigma < 1.3510 \) for TM\(_1\) modes and \( \sigma < 0.2430 \) for TM\(_2\) modes. The TM\(_1\) modes can exist in ultra-thin waveguide[202]. Using the parameter \( \xi \), one has the phase refractive index \( n_p = k_z / k_0 = \sqrt{\varepsilon_D} \sqrt{\xi^2 - \sigma} / \sqrt{1 + \xi^2} \). The group refractive index is

\[
n_g = n_p - \frac{1}{2(1 + \xi^2)} \frac{\varepsilon_D}{n_p} \frac{d\varepsilon_D}{dk_0} + \frac{\xi(n_p - n_p^2)}{n_p(1 + \xi^2)} \left( f_{TM} - \partial_\sigma f_{TM} k_0 \frac{d\sigma}{dk_0} \right) / \partial_\xi f_{TM}
\]  

(4-2)

Thus the vanishing of \( \partial_\xi f_{TM} \) will lead to the divergence of group index \( n_g \). Here we will only consider the TM\(_2\) mode as a specific example. If the condition \( \sigma < 0.2430 \) is fulfilled, there exists a critical thickness \( d_c \) for the TM\(_2\) modes such that for if \( d > d_c \), the waveguide support two modes of the same symmetry, one is a forward-wave and the other one is a backward-wave mode, which carry total energy flow with opposite signs. At the critical thickness, these two
modes become a single one, the light trajectory forms a double light cone [183], the total energy flow is zero, resulting in zero group velocity $v_g = 0$ and infinite group index $n_g = \infty$.

As a specific example, we consider a planar waveguide with $\varepsilon_D = 12.12$ and $\varepsilon_M = -1$ at $\lambda = 1.55 \, \mu m$. Dispersion and loss were incorporated by using a Drude model everywhere in this chapter. At $\lambda = 1.55 \, \mu m$, $d_c = 260 \, \text{nm}$. The phase indices of the guided TM modes are shown in Fig. 4-2. For the first branch of the TM modes which have odd-parity with $m = 1$, these modes are all backward waves with negative slope of $n_p$ with respect to waveguide thickness $d$. For $m > 3$, all the TM modes are forward waves. However for the second branch with $m = 2$ and 3, the phase index $n_p$ is not a single-valued function of the waveguide thickness. The corresponding normalized total energy flow [203] is also plotted in Fig. 4-1. The sign (+) or (−) of the derivative of $n_p$ with respect to the thickness determines if the wave is a forward- or backward-wave mode, respectively.
Finite-difference time-domain (FDTD) simulations show (Fig. 4-2) that light is stopped in a tapered waveguide, whose initial width is 500 nm and is tapered down to 220 nm over the distance of 8 µm. At the width of around 260 nm of the waveguide, we observe a strong yet finite field accumulation. No propagating mode is allowed for $d < d_c$. The magnetic field

Fig. 4-2 Stopped light in a tapered waveguide. a, FDTD simulations of the magnetic field $H_y$ of a continuous wave propagating through a linearly tapered waveguide at $\lambda = 1.55$ µm. The dielectric core layer with $\varepsilon_D = 12.12$ is tapered from 500 nm down to 220 nm over a length of 8 µm, which is indicated by the dashed lines. The cladding metal has $\varepsilon_M = -1 + 0.001i$. b, The complex phase refractive index $n_p$ and group refractive index $n_g$ for the forward-wave TM2 modes.

Finite-difference time-domain (FDTD) simulations show (Fig. 4-2) that light is stopped in a tapered waveguide, whose initial width is 500 nm and is tapered down to 220 nm over the distance of 8 µm. At the width of around 260 nm of the waveguide, we observe a strong yet finite field accumulation. No propagating mode is allowed for $d < d_c$. The magnetic field
distribution for the tapered waveguide is shown in Fig. 4-2(a). The figure also shows that in this structure light can be focused down to very small regions at high intensities. The phase and group index are calculated and plotted in Fig. 4-2(b) for the forward-wave modes. We have also simulated the arrival time of a wave packet through a quadratically tapered waveguide (not shown here). Excellent agreement is obtained between analytical results and numerical simulations.

![Graphs showing phase index and group velocity](image)

Fig. 4-3 **The effect of loss and gain on the phase index and group velocity.**

- **a,** Without loss, the forward- ($TM_2^f$) and backward-wave ($TM_2^b$) modes merge at the critical thickness with the same phase index, thus **d,** the zero group velocity.

- **b,** When the loss is present, the $TM_2^f$ and $TM_2^b$ modes are separated, leading to **e,** finite group velocity.

- **c,** When gain $G = -4\pi \Im \sqrt{\varepsilon_{11}} / \lambda$ is introduced and tuned to the critical value, the $TM_2^f$ and $TM_2^b$ modes merge again at the critical thickness, leading to **f,** the resorting of zero group velocity. The unit of the gain is cm$^{-1}$. The critical gain value is $G_c = 197.07$ cm$^{-1}$. 

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For a fixed thickness of the waveguide, there is a critical wavelength $\lambda_c$ such that group index will be infinite $n_g = \infty$ if loss is absent. For waveguide made of real materials, loss is unavoidable. For our waveguide, the loss comes primarily from the ohmic loss $\text{Im} \varepsilon_M$ in the cladding since the loss in the dielectric core layer can be negligible. The presence of non-zero loss splits the mode structure as shown in Fig. 4-3(a), prevents the group velocity from approaching zero, and hence in any real structure, light cannot be stopped indefinitely [194]. The effect of loss on the maximum group index for different loss in the cladding layer is shown in Fig. 4-3, where the group index is shown to follow a power law with the loss $\text{Im} \varepsilon_M$ in the cladding.

Though it may be difficult to directly reduce the loss in the single-negative metamaterial, we can introduce gain in the dielectric core layer realized by using active medium such as a semiconductor optical amplifier[204]. Recent progress has also been made to use gain to boost the propagation of surface plasmon polaritons [205][206].

The introduction of gain in the core layer greatly enhances the group index at and around the critical wavelength. In the lossless case, the backward- and forward-wave modes join together at the critical point and the group index diverges. However the gain $G = -4\pi \text{Im} \sqrt{\varepsilon_D} / \lambda$ in the core layer can compensate for the loss for modes and can furthermore make the phase index $n_p$ become real for the propagating modes, thus forcing the forward- and backward-wave modes to recombine. This is evident in Fig. 4-3(a-c). For a fixed amount of loss $\text{Im} \varepsilon_M$ in the cladding layer, there exists a critical gain $G_c$ such that $v_g \propto |G - G_c|^{1/2} \text{ or } n_g \sim |G - G_c|^{1/2}$. The
effect of gain on the phase index and group velocity is shown in Fig. 4-4(a), which shows that *the fine-tuning of the core layer gain can recover zero group velocity* $v_g = 0$ *in the composite waveguide structure.* This implies that the double light cone which is destroyed by losses can be recovered. For different losses in the cladding, different gain in the core layer is required to compensate the losses and maintain a divergent group index. The relation between the losses in the cladding and gain in the core layer to achieve $v_g \sim 300$ m/s or $n_g \sim 10^6$ at the critical wavelength is shown in Fig. 4-4(b).
The introduction of gain can also enhance the propagation length of modes whose wavelengths are away from the critical wavelength. Here, we consider a specific case for a waveguide of thickness $d = 260$ nm. The group index is shown in Fig. 4-4(c) for a band away from the critical wavelength. (a), The effect of gain on the group velocity of the TM$_2$ mode in a uniform waveguide of thickness $d = 260$ nm at $\lambda = 1.55$ µm. The waveguide core layer has $\text{Re}(\varepsilon_D) = 12.12$ and the cladding has $\varepsilon_M = -1 + 0.01i$. (b), The required minimum gain to have $v_g \sim 300$ m/s or $n_g \sim 10^6$ at the critical wavelength for different loss in the cladding. (c), The group index of TM$_2$ mode in a uniform waveguide for different wavelength. (d), The gain required to have $\text{Im}(n_p) \sim 10^{-6}$ or loss of about 0.5dB/cm for different wavelengths. The waveguide has $\varepsilon_D = 12.12$ and the cladding Drude metal has $\varepsilon_M = -1 + 0.01i$ at $\lambda = 1.55$ µm. (e), $F(z)/I(z)/I_0$ vs position along waveguide $z$. The results demonstrate that gain pinning does not affect the slowing of light discussed in this chapter.

The introduction of gain can also enhance the propagation length of modes whose wavelengths are away from the critical wavelength. Here, we consider a specific case for a waveguide of thickness $d = 260$ nm. The group index is shown in Fig. 4-4(c) for a band away
from the critical wavelength $\lambda_c \sim 1.55 \, \mu m$. The real parts of $n_g$ and $n_p$ are not sensitive to the amount of gain in the core layer for wavelength away from the critical wavelength. However, the imaginary part of the phase index $n_p$ is sensitive to the gain, which determines the decay length. For a CROW, the typical loss is $0.5\text{dB/cm}$. This requires $\text{Im} \, n_p \approx 10^{-6}$, for which the minimum gain required is shown in Fig. 4-4(b) for different wavelength. For our waveguide, one has $\Delta n \sim 1$, corresponding to $L = 12\text{cm}$, thus one can have $\text{DBP} \sim 79,600$. For $L = 1.2\text{cm}$, $\text{DBP} \sim 8000$. Thus very large DBP can be obtained for our waveguide.

**A. Delay Bandwidth Product**

Though one can obtain $v_g = 0$ at the critical wavelength for a specific waveguide, the bandwidth is actually zero. Thus to make use of this waveguide, one must use wavelengths away from the critical wavelength. Very large $n_g$ can be obtained for a very wide bandwidth in the present device, which is much larger than that of CROW optical buffers. The delay time can be very long if these modes can have a very long decay length, which in turn is determined by the imaginary part of the phase index $n_p$. For our waveguide one has the DBP

$$\Delta t \Delta f = \frac{L \Delta n}{\lambda} = \frac{\Delta n}{4\pi \text{Im} \, n_p}$$

(4-3)

Here $\Delta n$ is the difference of $n_p$ within the bandwidth. The maximum waveguide length $L$ is the inverse of the absorption and thus related to the imaginary part of $n_p$.

**B. Effect of gain saturation**
We next show that gain pinning does not affect the performance of the structure. At the entrance of the slow-light waveguide $z = 0$, assume that the gain is $G_0$. As the wave propagates along the waveguide, one has the following coupled equation between the local gain and the intensity

\[
\frac{1}{I} \frac{dI}{dz} = G + \frac{1}{F} \frac{dF}{dz}
\]

(4-4)

\[
G = \frac{G_0}{1 + \frac{I}{I_{sat}}}
\]

(4-5)

Here the change of intensity is due to the gain and geometry confinement which we assume that it can be factorized as $I(z) = F(z)I_{gain}(z)$ with $d \ln I_{gain} / dz = G$. Here $F(z)$ is the change of light intensity only due to the geometry confinement of the slow-light waveguide and can be calculated analytically under adiabatic approximation. Even in the case of lossless slow-light waveguide $F(z)$ is finite at location of zero group velocity, as also shown in Ref. [183]. If loss is present in the metal cladding of our waveguide, $\frac{d(\ln F)}{dz}$ will be even smaller. In any case, $F(z)$ is always finite and hence the intensity never diverges for small incident signal. Explicit calculations confirm this result (Fig. 4-4(e)) and will be shown in Appendix.

**C. TE modes**

Similar results are also obtained for the stopping of TE modes if we consider *negative permeability* metamaterials ($\mu_M < 0, \varepsilon_M > 0$) for cladding of a dielectric core layer. This waveguide
is characterized by two parameters $\sigma = -\varepsilon_M\mu_M / \varepsilon_D$ and $\rho = -\mu_M$. The condition on $\sigma$ and $\rho$ for the waveguide to support degenerate TE modes as

$$\rho(1 + \sigma) / [\sqrt{\sigma(\sigma + \rho^2)}] + \arctan(\sqrt{\sigma} / \rho) \geq m\pi / 2 \quad (4-6)$$

If $\sigma = \rho$, one gets the condition $\text{Re}\sigma < 0.2430$, as the case for the TM$_2$ modes. For anisotropic negative-$\mu$ metamaterials cladding, one has $\sigma = -\varepsilon_M\mu_M / \varepsilon_D$ and $\rho = -\sqrt{\mu_M\mu_M}$. For stopping the TE modes, only negative permeability is necessary.

Due to the ability to stop light with nonzero phase index, the effective impedance of the waveguide is finite. Thus light can be easily coupled to and from the slow-light waveguide. In contrast, for the more common case of waveguide cladded by PEC, the group velocity is zero only when the phase index is zero, resulting in complete reflection at the cutoff wavelength no matter how slowly the waveguide is tapered due to the vanishing or divergence of the effective impedance. Thus the present structure as well as that in Ref.[183] has the significant advantage of complete coupling of light with minimal reflections.

### III. Realizability of the waveguides

We note that results similar to those discussed in this chapter can be obtained if the cladding layer is made of DNMs, a structure complementary to that of Ref.[183]. However here we focus on the use of single-negative metamaterials since they have the benefit of easier realization and smaller loss.
Since only single-negative metamaterials are needed, our proposed scheme to slow and stop light can be realized for a broad range of frequencies. For cladding with metallic materials, the key is to have \( \text{Re}(-\varepsilon_m / \varepsilon_D) < 0.243 \) to stop TM modes.

The infrared, one can use silicon as the core layer and various polar materials with negative permittivity for cladding, such as silicon carbide [207] which has \( \varepsilon_{\text{SiC}} = -1.15 + 0.13i \) at \( \lambda = 10.6 \mu\text{m} \). In the visible, we can use noble metals for the cladding layer. The challenge is to have materials with high dielectric constant. In the UV range, most materials are nonmagnetic, so one may only be able to use negative permittivity materials for cladding.

Many metamaterials can be engineered to exhibit negative permittivity, such as a metallic wire mesh structure [208] or other resonant structures [209]. Negative permeability materials are available such as ferrites or metamaterials such as split-ring resonator arrays [210] in the GHz range. Negative permeability has also been realized in high dielectric systems [211]. Recently realized negative permeability metamaterials [212][213][214] can also be used.

**IV. Conclusion**

In this chapter, we have provided a complete framework for the analysis of the modes in a planar active dielectric waveguide cladded by single-negative metamaterials. We show that under certain parameter conditions that are derived here, the waveguide supports degenerate TM modes when cladded by negative permittivity materials and degenerate TE modes when cladded by negative permeability materials, for which the group velocity and total energy flow for these modes can be zero if the media are lossless. While the proposed waveguide structure will have
much lower loss than that made of DNMs, we also show that the limitations due to loss in the cladding can be easily overcome by including gain in the conventional core layer. Stopped light can be recovered at a critical gain value in the core layer. Since the critical waveguide thickness for stopping varies with frequency, trapped rainbow can be achieved in tapered structures without the use of DNMs but with available single-negative metamaterials. Analytical results are confirmed by FDTD simulations. The relaxation of the requirement of DNMs will greatly enhance the feasibility of realization and improve the performance. Our results can be readily tested in experiments. The proposed ideas here have important potential for applications in future generation optoelectronics.

Appendix I. Methods

WAVEGUIDE EQUATIONS

For the TM or the P-polarized modes in a symmetric planar waveguide we considered in the main text, the eigenmode equation is \( \sqrt{\varepsilon_D} k_0 d = f_{TM}(\xi) \) with

\[
f_{TM}(\xi) = \sqrt{1 + \xi^2 [m\pi - 2\arctan(\xi / \sigma)]}/\sqrt{1 + \sigma}
\]

Here \( \xi \) is a free parameter with range \( \xi \geq \sqrt{\sigma} \) for supporting guided modes in the waveguide. As illustrated in Fig. 4-1, the modes propagate along the z-direction with longitudinal wave vector \( k_z \). The transverse wave vector in the core layer and in the cladding are expressed through \( \xi \) as \( k_{2x} = \sqrt{\varepsilon_D - \varepsilon_M} k_0 / \sqrt{1 + \xi^2} \) and \( k_{1x} = i\sqrt{\varepsilon_D - \varepsilon_M} k_0 \xi / \sqrt{1 + \xi^2} \). The condition for the waveguide to support degenerate TM modes requires that \( \partial_\xi f_{TM}(\xi_{\text{min}}) \leq 0 \).
which leads to the condition Eq.(4-1) for $\sigma$. Using the parameter $\xi$, one has the phase refractive index

$$n_p = k_z / k_0 = \sqrt{\varepsilon_D \sqrt{\varepsilon^2 - \sigma} / \sqrt{1 + \varepsilon^2}}$$  

(4-7)

The group refractive index is

$$n_g = n_p + k_0 \frac{dn_p}{dk_0}$$  

(4-8)

From the dependence of $n_p$ on the frequency, Eq.(4-7), one has:

$$\frac{dn_p}{dk_0} = \frac{\hat{\partial} n_p}{\hat{\partial} \sigma} \frac{d\sigma}{dk_0} + \frac{\hat{\partial} n_p}{\hat{\partial} \xi} \frac{d\xi}{dk_0}$$

One has the following partial derivatives

$$\frac{\hat{\partial} n_p}{\hat{\partial} \sigma} = -\frac{\varepsilon_D}{2(1 + \varepsilon^2)n_p}$$

$$\frac{\hat{\partial} n_p}{\hat{\partial} \xi} = \frac{\xi}{1 + \varepsilon^2} \left( \frac{\varepsilon_D - n_p}{n_p} \right)$$

We further assume a Drude model dispersion for the negative permittivity metamaterial. Thus one has

$$\frac{d\sigma}{dk_0} = -\frac{1}{\varepsilon_D} \frac{d\varepsilon_M}{dk_0} = -\frac{1}{\varepsilon_D k_0} \left[ 2(1 - \varepsilon_M) - i \text{Im} \varepsilon_M \frac{1 - \varepsilon_M}{1 - \varepsilon_M^2} \right]$$

One also has

$$\frac{d\xi}{dk_0} = \left( \sqrt{\varepsilon_D d - \hat{\partial}_\sigma f_{TM}} \frac{d\sigma}{dk_0} \right) \frac{1}{\hat{\partial}_\xi f_{TM}}$$

With
\[ \partial_\sigma f_{TM} = \frac{f_{TM}}{2\sigma(1+\sigma)} - \frac{2\xi\sqrt{1+\varepsilon^2}}{\sqrt{1+\sigma(\sigma^2 + \xi^2)}}. \]

After putting all the above expressions together, we finally obtain the expression for the group velocity

\[ n_g = n_p - \frac{1}{2(1+\varepsilon^2)} \frac{\varepsilon_D}{d} \frac{d\sigma}{dk_0} + \frac{\xi(\varepsilon_D - n_p^2)}{n_p(1+\varepsilon^2)} \left( f_{TM} - \frac{\partial_\sigma f_{TM} k_0}{d\sigma} \right) / \partial_\xi f_{TM} \]

Thus the vanishing of \( \partial_\xi f_{TM} \) will lead to the divergence of group index \( n_g \).

For the TE or the S-polarized modes, the eigenmode equation is \( \sqrt{\varepsilon_D} k_0 d = f_{TE}(\xi) \) with \( f_{TE}(\xi) = \sqrt{1 + \varepsilon^2 [m\pi - 2\arctan(\xi/\rho)]} / \sqrt{1 + \sigma} \).

The condition for the waveguide to support degenerate TE modes requires that \( \partial_\xi f_{TE}(\xi_{\text{min}}) \leq 0 \), which leads to the condition Eq.(4-6) for \( \sigma \) and \( \rho \).

### Appendix II. Calculation of \( F(z) \)

The field of each mode is:

\[ H_y = e^{ikz - j\omega t} \begin{cases} 
  e^{-\kappa x} & (x > d / 2) \\
  Ce^{ik_2z + x} + C'e^{-ik_2z - x} & (-d / 2 < x < d / 2) \\
  C''e^{\kappa x} & (x < -d / 2) 
\end{cases} \] \hspace{1cm} (4-9)

Here \( k_{2z} = \sqrt{\varepsilon_D k_0^2 - k_\xi^2} \) and \( \kappa = \sqrt{k_\xi^2 - \varepsilon_M k_0^2} \).

For even modes: \( C'' = 1, \frac{\varepsilon_D \kappa}{\varepsilon_M k_{2z}} = \tan(k_{2z}d / 2), C = C' \frac{e^{-\kappa d/2}}{2\cos(k_{2z}d / 2)} \)

So:
\[ H_y = e^{ik_z z - i\omega t} \begin{cases} e^{-k_x x} & (x > d / 2) \\ 2C \cos(k_{2x} x) & (-d / 2 < x < d / 2) \\ e^{k_x x} & (x < -d / 2) \end{cases} \quad (4-10) \]

And:
\[ E_x = \frac{k_z}{\omega \varepsilon_x} H_y = \frac{k_z}{\omega} e^{ik_z z - i\omega t} \begin{cases} e^{-k_x x} / \varepsilon_M & \\ 2C \cos(k_{2x} x) / \varepsilon_D & \\ e^{k_x x} / \varepsilon_M \end{cases} \quad (4-11) \]

The Poynting vector of each layer is:
\[ \tilde{S} = E \times H = \frac{1}{4} (E \times H^* + c.c) = \frac{1}{2} \operatorname{Re}(E \times H^*) = \frac{1}{2} \operatorname{Re} H_y^*(E_x \bar{k} - E_z i) \]
\[ S_z = \frac{1}{2} \operatorname{Re}(E_x H_y^*) \]

In the region \((x > d/2)\) and \((x < -d/2)\), \(k_1 x = i \kappa\). So:
\[ S_z = \frac{1}{2} \operatorname{Re}(E_x H_y^*) = \operatorname{Re} \frac{k_z}{2\omega} e^{-2\Im(k_x) x} \begin{cases} e^{-k_x x} e^{-\kappa z} / \varepsilon_M \\ C \cos^2(e^{ik_{2x} x} + e^{-ik_{2x} x})(e^{-ik_{2x} x} + e^{-ik_{2x} x}) / \varepsilon_D \\ e^{k_x x} e^{-\kappa z} / \varepsilon_M \end{cases} \quad (4-13) \]

Take the integration of \(x\) from \(-\infty\) to \(\infty\), we get total energy flow in \(z\)-direction:
\[ P_z = e^{-2\Im(k_z) z} \frac{\varepsilon_D}{\omega} \operatorname{Re} k_z \left[ \frac{e^{-\Re(k_x) d}}{2 \Re(k_x) \varepsilon_M} + \frac{C^2}{\varepsilon_D} \left( \frac{\sin(\Re(k_{2x})d)}{\Re(k_{2x})} + \frac{\sinh(\Im(k_{2x})d)}{\Im(k_{2x})} \right) \right] \quad (4-14) \]

For lossless, no gain material, where \(\varepsilon_D\) and \(\varepsilon_M\) are real, so \(k_z, k_{2x}, \kappa\) also real, we have:
\[ P_z = \frac{k_z}{\omega} \left[ \frac{e^{-kd}}{2\kappa \varepsilon_M} + \frac{C^2}{\varepsilon_D} \left( \frac{\sin(k_{2x}d)}{k_{2x}} + d \right) \right] \quad (4-15) \]

When \(d\) changed slowly (adiabatic), the total time-averaged power flow is conserved:
\[ P_{\text{tot}}^{\text{con}} = |P_1^z| + |P_2^z| + |P_3^z| \], here 1, 2, 3 means different layer, 1, 3 are the cladding layers and 2 is the core layer. For different location of \( z \), we define \( F(d) \) as:

\[
H_y = \sqrt{F} e^{ik_z z - \phi} \begin{cases} e^{-\kappa z^2 / 2C} \\ \cos(k_{2x} x) \\ e^{\kappa z^2 / 2C} \end{cases}
\]

(4-16)

Here \( C = \frac{e^{-k_0 d^2 / 2}}{2 \cos(k_{2x} d / 2)} \), so:

\[
P_{\text{tot}}^{\text{con}} = F \left[ \frac{k_z e^{-k_0 d}}{4C^2 \omega^2 2 \kappa M} + \frac{k_z 1}{4 \omega \varepsilon_D} \frac{\sin(k_{2x} d)}{k_{2x}} + d \right]
\]

\[
= \frac{F k_z}{\omega} \left[ \frac{\cos^2(k_{2x} d / 2)}{(-2 \kappa M)} + \frac{\sin(k_{2x} d) + k_{2x} d}{4k_{2x} \varepsilon_D} \right]
\]

More general:

\[
P_{\text{tot}}^{\text{con}} = \frac{F}{4|C|^2} \omega \left[ \text{Re} \left( \frac{k_z e^{-\text{Re}(k) d}}{2 \text{Re}(\kappa) M} \right) + \text{Re} \left( \frac{k_z^2 |C|^2}{\varepsilon_D} \left( \frac{\sin(\text{Re}(k) d)}{\text{Re}(k) d} + \frac{\sinh(\text{Im}(k) d)}{\text{Im}(k) d} \right) \right) \right]
\]

\[
= \frac{F}{\omega} e^{-2 \text{Im}(k_z) z} \left[ \cos^2(k_{2x} d / 2) \text{Re} \left( \frac{k_z}{2 \text{Re}(\kappa) M} \right) + \frac{1}{4} \text{Re} \left( \frac{k_z^2 \varepsilon_D}{\text{Re}(k) d} \left( \frac{\sin(\text{Re}(k) d)}{\text{Re}(k) d} + \frac{\sinh(\text{Im}(k) d)}{\text{Im}(k) d} \right) \right) \right]
\]

So: \( F = \frac{\omega P_{\text{tot}}^{\text{con}}}{k_z} \left[ \frac{\cos^2(k_{2x} d / 2)}{(-2 \kappa M)} + \frac{\sin(k_{2x} d) + k_{2x} d}{4k_{2x} \varepsilon_D} \right]^{-1} \)

Or:

\[
F = P_{\text{tot}}^{\text{con}} \omega e^{-2 \text{Im}(k_z) z} \left[ \cos^2(k_{2x} d / 2) \text{Re} \left( \frac{k_z}{2 \text{Re}(\kappa) M} \right) + \frac{1}{4} \text{Re} \left( \frac{k_z^2 \varepsilon_D}{\text{Re}(k) d} \left( \frac{\sin(\text{Re}(k) d)}{\text{Re}(k) d} + \frac{\sinh(\text{Im}(k) d)}{\text{Im}(k) d} \right) \right) \right]^{-1}
\]

While all the values are positive, we can easily see \( F(z) \) is finite.
The field intensity at \( x = 0 \) is:

\[
I = H_y^2(x = 0) = F |e^{ik_yz}|^2 = Fe^{-2\Im(k_z)z} =
\]

\[
P_{\text{tot}} = \left[ \frac{1}{2} \cos^2(k_{2s}d / 2) \Re \left( \frac{k_z}{\Re(\kappa)\epsilon_M} \right) \right] + \frac{1}{4} \Re \left( \frac{k_z}{\epsilon_D} \right) \left( \frac{\sin(\Re(\kappa)\epsilon_M) + \sinh(\Im(\kappa)\epsilon_M)}{\Re(\kappa)\epsilon_M} \right) \right]^{-1}
\]

For lossless material, we have:

\[
I = H_y^2(x = 0) = F = \frac{4\omega P_{\text{tot}}^\text{con}(-\epsilon_M)k_{2s} / k_z}{k_z [1 + \cos(k_{2s}d)] / \kappa + \sigma \sin(k_{2s}d) + k_{2s}d}
\]

Example of \( F(z) \): we take \( \epsilon_D = 12.12, \epsilon_M = -1 \).

The waveguide is tapered from 0.4 \( \mu \text{m} \) to 0.15 \( \mu \text{m} \). The length of waveguide is 100 \( \mu \text{m} \). The critical distance is \( d_c = 0.26 \mu \text{m} \). The corresponding distance is \( z_c = 56 \mu \text{m} \). Result of phase index, group index, \( F(z) \) and relative field amplitude at \( x = 0 \) are showed in following Fig. 4-5:

![Fig. 4-5](image_url)

Fig. 4-5 \( F(z) \) vs. position \( z \) along a tapered waveguide. The waveguide core layer has \( \epsilon_D = 12.12 \) and the cladding has \( \epsilon_M = -1 \). The waveguide is tapered from 0.4 \( \mu \text{m} \) to 0.15 \( \mu \text{m} \). The length of waveguide is 100 \( \mu \text{m} \). The results demonstrate that gain pinning does not affect the slowing of light discussed in this chapter.

Take gain inside, where \( I(z) = F(z)I_{\text{gain}}(z) \) and

\[
\frac{1}{I} \frac{dI}{dz} = \frac{\alpha_0}{1 + I/I_{\text{sat}}} + \frac{1}{F} \frac{dF}{dz}
\]
If we take the number: $\alpha_0 = 100 \, cm^{-1} = 0.01 \, \mu m^{-1}$ and using different input $I_0$, we get $I(z)/I_s$ and $F(z)I_0/I_s$:

![Graph](image)

Fig. 4-6 $F(z)I(z)/I_0$ vs. position along waveguide $z$. The waveguide core layer has $\varepsilon_D = 12.12$ and the cladding has $\varepsilon_M = -1$. The waveguide is tapered from 0.4 µm to 0.15 µm. The length of waveguide is 100 µm. The results demonstrate that gain pinning does not affect the slowing of light discussed in this chapter.

We can clearly see that $I(z)/I_s$ will not go to infinity when $z$ close to critical distance. So if the input intensity is relative small, gain will not go to depletion region.
Chapter 5 Superresolution Imaging Using a 3D Metamaterials Nanolens

I. Introduction

Superresolution imaging beyond Abbe’s diffraction limit can be achieved by utilizing an optical medium or ‘metamaterial’ that can either amplifies or transport the decaying near-field evanescent waves that carry subwavelength features of objects. Earlier approaches at optical frequencies mostly utilized the amplification of evanescent waves in thin metallic films or metal-dielectric multilayer, but were restricted to very small thicknesses (<<\(\lambda\), wavelength) and accordingly short object-image distances, due to losses in the material. Here, we present an experimental demonstration of superresolution imaging by a low-loss three-dimensional metamaterial nanolens consisting of aligned gold nanowires embedded in a porous alumina matrix. This composite medium possesses strongly anisotropic optical properties with negative permittivity in the nanowire axis direction, which enables the transport of both far-field and near-field components with low-loss over significant distances (> 6\(\lambda\)), and over a broad spectral range. We demonstrate the imaging of large objects, having subwavelength features, with a resolution of at least \(\lambda/4\) at near-infrared wavelengths. The results are in good agreement with a theoretical model of wave propagation in anisotropic media.

Image formation by a lens is subject to a fundamental limit of resolution of any optical system known as “Abbe’s diffraction limit”— features of an object smaller than \(\lambda/2\) (\(\lambda\), wavelength) cannot be imaged by conventional optics [215]. Abbe’s diffraction limit arises
because evanescent waves which carry sub-\(\lambda\) information decay exponentially, leaving only the propagating waves that carry low frequency spatial components (coarser details) to be collected at the image plane. Almost 10 years ago, Pendry discussed the possibility of restoring these evanescent waves using the amplifying property of a slab of “metamaterial” which has a negative refractive index [216]. The mechanism behind amplification of the evanescent waves in this perfect lens is the resonant excitation of surface waves supported at the interfaces of the metamaterials slab. Several approaches, based on Pendry’s archetype, to creating optical components with negative-index metamaterials for superresolution imaging have been investigated[217][218][219][220]. However, at optical frequencies, negative-index metamaterials[221][222][223] constructed from realistic materials (in particular resonance-based metallic elements) are intrinsically associated with loss (imaginary part of permittivity \(\varepsilon\) or permeability \(\mu\)), which in turn hinders the resolution of the optical components[224]. Besides the severe limitations of material loss, the predominant shortcomings of these superresolution lenses are the narrow spectral bandwidth at which they operate, the small thickness of the lens (poor man’s lens[225][226]), image distortions (photonic crystals[227]), limited imaging area (hyperlens[228]), lack of support for propagating waves (poor man’s lens/ subwavelength plates[229]) and the fact that information needs to be collected from multiple measurements (far-field grating superlens [230]).

An innovative approach, which makes use of arrays of high aspect ratio metallic wires[231][232], has been theoretically suggested to mitigate the aforementioned drawbacks. The underlying mechanism for restoring evanescent waves in such metallic wires medium, is not
based on resonances of materials parameters (i.e. there is no amplification machinery from excitation of surface waves at interfaces), but is rather based on the transport of evanescent waves when the medium exhibits relatively flat equi-frequency dispersion characteristics. In this case the medium can reconstruct sub-wavelength details of an object not only with low-loss but also over a broad frequency range. Specifically, it was proposed that aligned metallic nanowires embedded in dielectric matrices possess strongly anisotropic optical properties with negative permittivity $\Re \varepsilon_z < 0$ along the nanowire axis and positive permittivity $\Re \varepsilon_{x,y} > 0$ in the transverse plane, that can be used to achieve broadband all-angle negative refraction (AANR) and superlens imaging[233]. This medium is an example of so-called indefinite media[234]. To date, sub-$\lambda$ imaging experiments, by means of such wire array media, have been restricted to microwave frequencies[235][236], mainly due to the challenge in manufacturing extreme high aspect ratio metallic wires at a nano scale level.

II. Subwavelength imaging with 3D nanolens
In this chapter, we report superresolution imaging by a low-loss 3D metamaterial nanolens, with a resolution of at least $\frac{\lambda}{4}$ at near-infrared wavelengths and over a significant distance (or thickness) of $> 6\lambda$ (Fig. 5-1). Dr. Bernard Didier F. Casse did most of the experimental work and I did most of the theoretical calculation here. This metamaterials nanolens consists of bulk high aspect ratio aligned gold nanowires embedded in a porous alumina host matrix of thickness
10 µm. The composite medium is fabricated by a combination of bottom-up self assembly and electrochemical process[237]. Earlier optical measurements have shown that for long wavelengths, in the infrared[237], the material has strongly anisotropic optical properties with negative permittivity $\text{Re}\varepsilon_z < 0$ along the nanowire (z) axis and positive permittivity $\text{Re}\varepsilon_{x,y} > 0$ in the transverse ($x,y$) plane (Fig. 5-2). Optical negative refraction in these bulk metamaterials of metallic nanowires has been recently reported[238]. The lens can be manufactured on a large scale, and a representative sample 3/4 of the size of a U.S. penny is shown in Fig. 5-2(a). A 3D oblique view illustration of a section of the nanolens, portraying the metamaterial nano-structural design is shown in Fig. 5-2(b). In the scanning electron microscope (SEM) picture representing the top-view of the nanolens (shown in Fig. 5-2(c)), the tips of the nanowires can be seen on the surface. An SEM picture of a cross-section of the 10 µm thick porous alumina template is shown in Fig. 5-2(d). The nanopores are 12 nm in diameter and have an average spacing of 25 nm.

The schematic of imaging with the metamaterials nanolens is depicted in Fig. 5-1(a). In our experiment, the object (void pattern) with subwavelength features is illuminated with a continuous wave (CW) laser at 1550 nm. The void pattern was milled by focused ion beam lithography on a 100 nm thick metallic film deposited on 0.1 mm thick glass substrate. After passing through the nanolens, the transmitted beam is then mapped with a near-field scanning optical microscope (NSOM) probe above the nanolens surface. The representative object imaged consists of the letters ‘N E U’ (acronym for Northeastern University) with 600 nm wide (0.4λ) arms, as shown in the SEM) picture in Fig. 5-1(b). The NSOM scan of this source object illuminated by the laser light is shown in Fig. 5-1(c). NSOM scans were carried out at the far end
of the metamaterial nanolens (image plane). As illustrated in Fig. 5-1(d), the subwavelength letters were clearly resolved with very low distortions.

Fig. 5-2 (a) Bulk metamaterial (pink circle) manufactured in large scale (> 10 mm in diameter or almost 3/4 the size of a U.S. penny). (b) 3D illustration of the nanoscale architecture of the nanolens. (c) Top view: SEM image showing the tips of the gold nanowires. The metamaterial nanolens consists of aligned gold nanowires, with 12 nm diameters and lattice spacing of 25 nm, embedded in porous alumina template matrix. (d) Side view: SEM image of the cross-section of the 10 µm thick nanoporous alumina template without the gold nanowires. (e) Anisotropic optical property of the metamaterial: Negative permittivity in the nanowire axis z direction (Re(ε_z) < 0) and positive permittivity in the (x – y) plane (Re(ε_x,y) > 0)).

For a complete set of control experiments, we have also attempted to image the “NEU” letters with (a) porous alumina template without the gold nanowires (essentially a pure dielectric) and (b) 10 µm above the surface of the object in air without the alumina template (not shown).
For both cases, the images were almost unrecognizable in the far-field and superresolution imaging was not achieved, demonstrating that the composite metamaterial behaves as a superresolution medium. Furthermore, our experiments corroborate that a 100 nm thick gold film almost completely blocks infrared light (thickness being larger than the skin depth of the metal), so that bulk gold of thickness comparable to the nanowire material (~10µm) could not possibly yield the present imaging results.

To demonstrate the broadband aspect of the lens, imaging experiments were performed within the tunable wavelength range of our near-infrared laser (i.e. 1510 nm–1580 nm) (not shown). No significant distortions were observed when the wavelength was varied from 1510 nm to 1580 nm, indicating that some degree of broadband imaging (~5%) is also realizable with this nanolens. Theoretically, the nanolens will also operate at longer wavelengths, up to 21 microns[233]. In the microwave spectral range, a bandwidth of 18% was reported [236].

The metamaterial nanolens can also image very large objects with subwavelength features and the subwavelength features are retained even for large objects, although distortions arising from large scale inhomogeneities may be evident.
In order to assess the resolving capability (or resolution) of the lens, imaging of a subwavelength two-slit object was carried out. The two-slit object in this case is composed of two 600 nm (~0.4λ) slits spaced 400 nm (0.26λ) apart (inset in Fig. 5-3(c)). Fig. 5-3(a) and (b) show the NSOM map of the source object and image, respectively. Experimentally, we find that the slits can be clearly resolved by the bulk nanolens even though their critical dimensions are

![Image](image-url)
much smaller than the wavelength (Fig. 5-3(b)). The smallest length scale, i.e. the edge-to-edge
distance of 400 nm, is clearly distinguished indicating that the lens has a resolution capability of
at least $\lambda/4$ for the near infrared spectral range (Fig. 5-3(c)). The curve for a conventional optical
microscope is also shown, demonstrating that the two individual slits cannot be resolved by a
diffraction-limited optical system.

The above experiments clearly establish that the metamaterial nanolens is a superresolution
medium which can transport, with low-loss, object details down to $\lambda/4$ length scales, over large
distances $> 6\lambda$. Interestingly, attenuation by the metamaterial nanolens was recorded to be less
than 1 dB/cm. Furthermore, the Figure-of-merit (FOM) of the nanolens is given by
$\text{FOM} = -\frac{\text{Re}(n)}{\text{Im}(n)} \sim 12$, which is $4 \times$ higher than the best fabricated metallic-based
metamaterial at 1.5 μm[239].

The mechanism behind superresolution reconstruction in this metamaterial is the transport
of evanescent waves by guided modes through the engineered media. Imaging beyond Abbe’s
diffraction limit requires guided modes with flat isofrequency contours, where the longitudinal
wave vector $k_z$ ( $z$ is in the direction of the nanowires) is nearly independent of the transverse
wave vector $k_{x,y}$. In arrays of metallic rods medium, in which the permittivity of nanowires is
sufficiently negative ($\text{Re} \varepsilon_z < 0$), a quasi-TEM (transverse electromagnetic) nearly dispersionless
mode capable of guiding evanescent waves is supported [232]. From both optical
characterization [237] and analytical formula [233], we have shown that the medium is strongly
anisotropic ($\text{Re} \varepsilon_z < 0$ and $\text{Re} \varepsilon_{x,y} > 0$), corresponding to the situation where polarized
electromagnetic modes can be excited. In this composite medium, the guided quasi-TEM mode does not effectively penetrate the metallic nanowires and thus most of the energy propagates in the host porous alumina dielectric medium in between the nanowires, so that the influence of the metallic wires losses are negligible. The mechanism is different from that in ref.[240], where Ono et al. exploit the excitation and propagation of surface plasmon polaritons (SPPs) to achieve superresolution imaging. Full-wave simulations and analytical derivations confirm that subwavelength details of an object can be transported along such wire arrays over long distances[232][241][242].

III. Subwavelength resolution of rod medium slab

Calculations of the wave propagation the nanowire medium are based on the effective permittivity model developed by Silveirinha et al.[232]. Relevant equations used for the calculations are provided below for the reader’s convenience.

For long wavelength, the fields in the periodic structure can be characterized using a nonlocal permittivity as:

\[
\varepsilon(\omega, k_z) = \varepsilon_h \left[ \varepsilon_i(\mathbf{x} \mathbf{x} + \mathbf{y} \mathbf{y}) + \varepsilon_z(\omega, k_z) \mathbf{z} \mathbf{z} \right]
\]

\[
\varepsilon_i = 1 + \frac{2}{\frac{1}{f_Y} \varepsilon_m - \varepsilon_h} - 1
\]

\[
\varepsilon_z(\omega, k_z) = 1 + \frac{1}{\frac{1}{f_Y} \varepsilon_m - \varepsilon_h} \frac{\beta^2 - k_z^2}{\beta_p^2} = 1 + \frac{\beta_p^2}{k_z^2 + \beta_c^2 - \beta^2}
\]

(5-1)
Where \( f_v = \frac{\pi R^2}{a^2} \) is the volume filling factor of the rods, \( \varepsilon_m, \varepsilon_h \) are the permittivity of metal rods and host material, \( k_z \) is the \( z \) component of the wave vector, \( \beta = \sqrt{\varepsilon_h k_0} = \omega \sqrt{\varepsilon_h / c} \) is the wave number in the host medium, and \( \beta_p \) is the plasma wave number for rods given by:

\[
(\beta_p a)^2 = \frac{2\pi}{\ln\left(\frac{a}{2\pi R}\right) + 0.5275} \tag{5-2}
\]

And

\[
\beta_c^2 = -\frac{\varepsilon_h \beta_p^2}{(\varepsilon_m - \varepsilon_h) f_v} \tag{5-3}
\]

We consider that the thickness of the slab is \( L \) and that for \( z < 0 \) and \( z > L \) the medium is air. A plane wave with transverse wave vector \( k = (k_x, k_y, 0) \) illuminates the slab. The incident wave excites two TM-polarized electromagnetic modes in the rod medium with propagation constants \( k_z^{(1)} \) and \( k_z^{(2)} \), respectively. The TM modes satisfy the characteristic equation,

\[
\frac{k_t^2}{\varepsilon_z} + \frac{k_i^2}{\varepsilon_i} = k_0^2 \tag{5-4}
\]

From expression of \( \varepsilon_z \) in Eq.(5-1) and Eq.(5-4), we can solve \( k_z^{(1)} \) and \( k_z^{(2)} \),

\[
(k_z^{(1,2)})^2 = \frac{1}{2} \left\{ \varepsilon_i (\beta^2 - k_i^2) + (\beta^2 + \beta_c^2 - \beta_p^2) \right\} \\
\pm \sqrt{\left[ \varepsilon_i (\beta^2 - k_i^2) - (\beta^2 + \beta_c^2 - \beta_p^2) \right]^2 + 4 \varepsilon_i k_i^2 \beta_p^2} \tag{5-5}
\]
Since $\varepsilon_h E_n$ is continuous at the interface, we can now study the transmission properties of the rod medium slab. Assuming that the incident magnetic field is along the $y$ direction and has amplitude $H_0$, we can write that:

$$H_y = H_0 e^{-jk_z z} \begin{cases} e^{-jk_z z} + \rho e^{jk_z z} & z < 0 \\ A_1 e^{-jk_z z} + A_2 e^{jk_z z} + B_1 e^{-jk_z z^2} + B_2 e^{jk_z z^2} & 0 < z < L \\ T e^{-jk_z (z-L)} & z > L \end{cases}$$  \tag{5-6}

The boundary condition gives:

$$\begin{bmatrix} -1 & 1 & 1 & 1 & 1 & 0 \\ -k_z^{(a)} & -k_z^{(1)} & k_z^{(1)} & -k_z^{(2)} & k_z^{(2)} & 0 \\ -(k_z^{(b)})^2 & (k_z^{(1)})^2 & (k_z^{(2)})^2 & (k_z^{(2)})^2 & 0 \\ 0 & e^{-jk_z^2 L} & e^{jk_z^2 L} & e^{-jk_z^2 L} & e^{jk_z^2 L} & -1 \\ 0 & k_z^{(1)} e^{-jk_z^2 L} & k_z^{(1)} e^{jk_z^2 L} & k_z^{(2)} e^{-jk_z^2 L} & k_z^{(2)} e^{jk_z^2 L} & k_z^{(a)} \\ 0 & (k_z^{(1)})^2 e^{-jk_z^2 L} & (k_z^{(2)})^2 e^{jk_z^2 L} & (k_z^{(2)})^2 e^{-jk_z^2 L} & (k_z^{(2)})^2 e^{jk_z^2 L} & -(k_z^{(b)})^2 \end{bmatrix} \begin{bmatrix} \rho \\ A_1 \\ A_2 \\ B_1 \\ B_2 \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$  \tag{5-7}

Where we define $k_z^{(a)} = -j\sqrt{k_t^2 - k_0^2}$, and $k_z^{(b)} = -j\sqrt{k_t^2 - \varepsilon_h k_0^2} = -j\sqrt{k_t^2 - \beta^2}$. In general, the solution of Eq.(5-7) must be obtained numerically.

One typical example is (at 30THz, where wavelength is $\lambda = 10 \mu$m): $a = 215$ nm, $R = 21.5$ nm, $L = 5.93$ $\mu$m, $\varepsilon_i = 2.2$, $\varepsilon_m = -5143 -746i$. We got the transmission as:
Another example is (at \( f = 193.41 \text{THz} \), where wavelength is \( \lambda = 1.55 \, \mu\text{m} \)): \( a = 27 \, \text{nm}, \; R = 6 \, \text{nm}, \; L = 10 \, \mu\text{m}, \; \varepsilon_h = 3.2, \; \varepsilon_m = -115 -8.17i \). The transmission is:

Fig. 5-5 Transmission of a nano-wire template slab at \( \lambda = 1.55 \, \mu\text{m} \). The geometry is : \( a = 27 \, \text{nm}, \; R = 6 \, \text{nm}, \; L = 10 \, \mu\text{m}, \; \varepsilon_h = 3.2, \; \varepsilon_m = -115 -8.17i \).
Now we calculate the image field. Assuming we have two spikes source at location: 
\[ z_o = -u, x_o = 0 \]. The field of this source is:

\[ H_y(x, z) = \int_{-\infty}^{\infty} v(k_x) e^{-jk_x x} e^{-jk_z (z+u)} dk_x = 2\int_{0}^{\infty} v(k_x) \cos(k_x x) e^{-jk_z (z+u)} dk_x \] (5-8)

Where

\[ v(k_x) = \left( \frac{2}{\pi k_x} \right) \sin(k_x a) \cos(k_x b) \] (5-9)

Here \( 2b \) is the distance between the two spikes and \( 2a \) is the individual spike width. The spectrum of the source at the front plane (\( z = 0 \)) of the slab is:

\[ s(k_x) = v(k_x) e^{-jk_z u} \] (5-10)

The field at the output plane of the slab (\( z = L \)) is:

\[ H_y(x, z = L) = \int_{-\infty}^{\infty} T(k_x) s(k_x) e^{-jk_x x} dk_x \] (5-11)

The field in the input region (-\( u < z < 0 \)) is:

\[ H_y(x, z = L) = \int_{-\infty}^{\infty} e^{-jk_z (z-d)} + \rho(k_x) e^{jk_x x} ]s(k_x) e^{-jk_x x} dk_x \] (5-12)

The field outside the slab (\( z > L \)) is:

\[ H_y(x, z) = \int_{-\infty}^{\infty} e^{jk_z (z-d)} T(k_x) s(k_x) e^{jk_x x} dk_x \] (5-13)

The field inside the slab is:

\[ H_y(x, z) = \int_{-\infty}^{\infty} \left[ A_1(k_x) e^{-jk_z x} + A_2(k_x) e^{jk_z x} + \ldots \right. \]
\[ \left. \ldots + B_1(k_x) e^{-jk_z x} + B_2(k_x) e^{jk_z x} \right] s(k_x) e^{-jk_x x} dk_x \] (5-14)

For (\( u = 0.1 \mu m \), \( a = 0.4 \mu m \), \( b = 0.6 \mu m \), so the distance of two edge is: \( 2(b-a) = 0.4 \mu m \), close to \( \lambda/4 \)), we show the total field intensity (\( |H_y|^2 \)) distribution:
Fig. 5-6 Intensity distribution of two point spikes put a distance $u = 0.1 \ \mu m$ away from a nano-wire template at $\lambda = 1.55 \ \mu m$. The geometry of the spikes are: $a = 0.4 \ \mu m$, $b = 0.6 \ \mu m$. The geometry of the slab is: $a = 27 \ \text{nm}$, $R = 6 \ \text{nm}$, $L = 10 \ \mu m$, $\varepsilon_h = 3.2$, $\varepsilon_m = -115 - 8.17 i$. 
We can clearly see that at $v$ near 0.7 $\mu$m, two spikes are clearly separated.

IV. Ray Tracing

For an indefinite medium, it is appropriate to discuss the energy flow. The group velocity refraction is governed by $\tan \phi = -\sigma \tan \theta$ [58]. Here, $\theta$ is the angle for the incident group velocity and $\phi$ is that for the refracted group velocity as shown in Fig. 5-8. The material property $\sigma$ is related to the effective permittivity of the indefinite medium as

$$\sigma \equiv -\frac{dk_z}{dk_{0z}} = -\frac{\sqrt{\varepsilon_x}}{\varepsilon_z} \frac{\sqrt{k_0^2 - k_x^2}}{\sqrt{k_0^2 - k_z^2 / \varepsilon_z} / \varepsilon_z}$$

(5-15)
Where $k_{0z} = \sqrt{k_{0x}^2 - k_z^2}$ and the direction $z$ is parallel to the nanowires. For an ideal flat lens, the medium obeys elliptic dispersion and $\sigma$ is independent of angle. In this case imaging by the flat lens obeys the equation $u + v = \sigma d$ where $u$ and $v$ are the object and image distances from the lens and $d$ is the lens thickness \[58\].

To reveal the significant role of the anisotropic permittivities in the lensing effect, we constructed a 3D ray diagram (shown in Fig. 5-9) using in-house codes. In the ray diagram, the “N E U” object is created by multiple point sources.

The light rays emanating from the object are refracted inside the material and their direction is governed by the group velocity given by:

$$\tan \phi_{r,S} = -\sigma \tan \theta = \frac{\sqrt{\varepsilon_{x,y} \sin \theta}}{\sqrt{\varepsilon_z \mu - \sin^2 \theta}}$$ \hspace{1cm} (5-16)

Fig. 5-8 Refractive angle $\theta$ vs. incident angle $\phi$. 

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Here $\phi_{r,S}$ represents the angle of the Poynting vector in the metamaterial with respect to the $z$-axis. From Eq.(5-16) we can see that if $\varepsilon_z$ is negative, the refracted rays will have a different sign from the incident rays.

If we take

$$k_z = \sqrt{\varepsilon_{x,y}(\mu k_0^2 - \frac{k_{x,y}^2}{\varepsilon_z})} = k_0 \sqrt{\varepsilon_{x,y}(\mu - \sin^2 \theta)}$$

$$k_{0z} = \sqrt{k_0^2 - k_{x,y}^2} = k_0 \cos \theta$$

From Eq.(5-15), $\sigma$ reads
\[
\sigma = -\frac{dk_x}{dk_{0z}} = -\frac{\sqrt{\varepsilon_{x,y}} \cos \theta}{\varepsilon_z \sqrt{\mu - \frac{\sin^2 \theta}{\varepsilon_z}}} \tag{5-18}
\]

The permittivities used for the indefinite material are \( \varepsilon_x = \varepsilon_y = 4.4 \), \( \varepsilon_z = -14.76 \) and \( \mu = 1 \). The refracted rays focus at the other end of the slab.

The result of ray tracing for a lens of thickness \( d = 10 \, \mu m \), is shown in Fig. 5-9(a). The resulting images at various planes \( Z \) above the indefinite-index media (metamaterials nanolens) is shown in Fig. 5-9(b). The cuts reveal that there exists a focal point (\( Z = 10.5 \, \mu m \)) at which a perfect focus can be obtained, i.e. 0.05\( d \) above the surface of the lens (\( Z = 10 \, \mu m \)), matching experimental data.

Following the same arguments, the refraction angle of the wave vector is positive and can be expressed as

\[
\tan \phi_{r,k} = \frac{k_{x,y}}{k_z} = -\frac{\sin \theta}{\sqrt{\varepsilon_{x,y} \left( \mu - \frac{\sin^2 \theta}{\varepsilon_z} \right)}} \tag{5-19}
\]

leading to the plot in Fig. 5-8.

V. Conclusion

The metamaterial nanolens has noteworthy advantages over other currently available metamaterials-based approaches to superresolution imaging— Earlier superlenses have been restricted to very small thicknesses (\( \ll \lambda \)) [225] or to 2-dimensions [227] so that some degree of
resonant enhancement of the evanescent waves can be obtained without being buried by material loss. In contrast to a grating far-field superlens, it needs only a single measurement to obtain a very large bandwidth in Fourier space to reconstruct superresolution details of an object. And, in comparison to the poor man’s superlens (thin film $\sim 60$ nm of homogeneous negative permittivity $\varepsilon < 0$), the metamaterial nanolens support both propagating and evanescent waves required for full imaging in a bulk medium ($\sim 10$ $\mu$m). In addition, the poor man’s superlens has a much narrower spectral bandwidth than the nanolens. Also, the metamaterial nanolens has theoretically no limitations on the imaging area (unlike the hyperlens) and the concepts described here are applicable over a broad spectral range, from far infrared up to UV frequencies, by suitable choice of material and filling factor, enabling broadband color imaging. Furthermore, the nanolens can be nanomanufactured in large scale and does not require slow serial nanolithography techniques. The superior optical properties shown in this chapter, and the ability to manufacture these artificial metamaterials in large scale, offers the potential for numerous applications in transformation optics, optical storage devices, nanolithography and biomedical imaging.
Chapter 6 Generalized Reflectionless Conditions for 2D Transformation Optics

I. Introduction

In the recent years, the transformation optics has attracted much interest due to its unprecedented ability in manipulating and controlling the behavior of electromagnetic waves to produce functions. The device design is obtained via a particular spatial coordinate transformation from original flat space to the transformed space\([244][245]\). The coordinate transformation is then applied to the permittivity and permeability tensor to yield the specification for a complex medium with desired functionality. Although such media would be very difficult to fabricate using conventional materials, the rapid advances in the engineering of artificially structured metamaterials\([246][247][248][249][250][251][252][253]\) with controllable anisotropy and spatial inhomogeneity offer a practical approach for the realization of transformation design.

One of the most striking applications of transformation optics to date is the invisibility cloaking\([254][255][256][257][258][259][260][261][262][263][264]\), a device creates a coordinate transformation that carries a hole for hiding objects. In a typical cloak design, a specific region of free space is transformed into a shell-type region that has the same external boundary as the original one. Incident external wave is guided smoothly around the cloaked region such that the output wave from the cloaking area is exactly the same as if it had just passed through free space. Besides the invisible cloaks, other interesting devices with exotic EM behaviors have been created, such as EM wave concentrators [265] and rotators[266].
However, the electromagnetic behaviors of incident waves are only altered within restricted region in the transformation optical device. The change of incoming fields cannot be transferred to another medium or to free space thus remains a local phenomenon. This means, inside the structures, there is no reflections. The external surface of the cloaking structure is same as the original space. In this case, there is no reflection on surface. Since the material, due to the inherently reflectionless property of the transformation optics, the whole system will leave the transmitted wave undisturbed. However, in many other cases, such as lenses or concentrators, when we need to focus or concentrate light as desired, the transformed space will not always share the same external boundary as the original space, especially when the material is finite. For the purpose of changing the position and direction of output waves, normally, the external boundary of the transformed region in transformed space is not same as the original region in original space.

In Ref.[267], Rahm et al. proposed finite embedded coordinate transformation and designed a reflectionless parallel-beam shifter and a beam splitter. Unlike in cloak designs, discontinuities in the transformations along the domain boundaries were allowed. This class of transformations adds flexibility to the design of complex medium and enables the transformed waves exit from the device to the surrounding normal medium without reflections.

When the material size is finite, embedded coordinate transformations are applied, although it is inherently reflectionless inside the material, the discontinuities of two-coordinate systems will increase the reflection at interface.
In this paper, we studied the behaviors of field in two transformed region and derived the generalized reflectionless boundary conditions between two transformed medium. We call this condition as “extended slippery boundary condition” because in very special case, the condition reduced to Rahm’s result where slippery boundary condition can be applied (which means the unit length at each boundary is same). We propose two designs to confirm this condition. A flat lens and a non-magnetic beam concentrator are analyzed as two examples. The performance of these structures is confirmed by 2D full-wave simulations and also compared with previous research results. TM modes and three-dimensional condition can also be derived based on the same proposed method.

II. General reflectionless condition for 2D transformation optics

For a transformation optics media which transformed from original space by:

\[ x' = f(x, y), \quad y' = g(x, y), \quad z' = z \] \hspace{1cm} (6-1)

If the electromagnetic parameters transformed as:

\[ \bar{\varepsilon'} = \frac{J \varepsilon J^T}{\det(J)}, \quad \bar{\mu'} = \frac{J \mu J^T}{\det(J)} \] \hspace{1cm} (6-2)

where \( J \) is the Jacobian matrix, then the Maxwell equation in transformed space \((x', y')\) will keep the same form as:

\[
\nabla' \times \bar{H}' = -j\omega \bar{\varepsilon}' \bar{E}'
\]

\[
\nabla' \times \bar{E}' = j\omega \bar{\mu}' \bar{H}'
\] \hspace{1cm} (6-3)

and the field parameters will be transformed as:

\[ \bar{H}' = (J^{-1})^T \bar{H}, \bar{E}' = (J^{-1})^T \bar{E} \] \hspace{1cm} (6-4)
For simplicity, we only consider TE mode, where the magnetic field is only in \(z\) direction:

\[ H_x = 0, H_y = 0, E_z = 0. \]

Define \( A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \), here \( A \) is the 2D Jacobian matrix. We will still use \( E, D, E', D' \) to represent 2D vectors instead of 3D vectors, so the Jacobian matrix is:

\[
J = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.
\]

We define \( \varepsilon_{2D}' = \frac{A \varepsilon_{2D}' A^T}{\det(A)} \), then \( \varepsilon' = \begin{pmatrix} \varepsilon_{2D}' \\ 0 \\ \frac{1}{\det(A)} \end{pmatrix} \), \( \mu'_z = \frac{1}{\det(A)} \).

In the 2D TE mode, \( \varepsilon_z \) is not used, it can be arbitrary number, so we only consider \( \varepsilon_{2D}' \). The Maxwell equation in prime space is:

\[
(\partial_y, -\partial_x)\varepsilon_{2D}^{-1} \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} H_z + \mu'_z k_0^2 H_z = 0 \quad (6-4)
\]

Put \( \varepsilon' \) and \( \mu'_z \) inside Eq.(6-4), after some algebra calculation, we get:

\[
(\partial_y, -\partial_x)\varepsilon_{2D}^{-1} \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} H_z + \mu k_0^2 H_z = 0 \quad (6-5)
\]

This returns to the Maxwell equations in original \((x, y)\) space. If \( \varepsilon_{2D}' \) and \( \mu_z \) are constant, the eigenfunction of above equation is:

\[
H'_z = H_z = H_0 e^{jk_x x + jk_y y} = H_0 e^{jk' (x', y') + jk_z z'} \quad (6-6)
\]

And

\[
E' = (A^{-1})^T E = \frac{i}{\omega} (A^{-1})^T \varepsilon_{2D}^{-1} \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} H_z = \frac{-1}{\omega} (A^{-1})^T \varepsilon_{2D}^{-1} \begin{pmatrix} k_y \\ -k_x \end{pmatrix} H_z \quad (6-7)
\]

\[
D' = \varepsilon_{2D}' E' = \frac{-1}{\omega} \varepsilon_{2D}' (A^{-1})^T \varepsilon_{2D}^{-1} \begin{pmatrix} k_y \\ -k_x \end{pmatrix} H_z = \frac{-1}{\omega \det(A)} A \begin{pmatrix} k_y \\ -k_x \end{pmatrix} H_z \quad (6-8)
\]
here \( k_x \) and \( k_y \) satisfy the constitution equation:

\[
(k_y, -k_x)\mathbf{\varepsilon}_{2D}^{-1}\begin{pmatrix} k_y \\ -k_x \end{pmatrix} = \mu k_0^2
\]  

(6-9)

Since there is no reflection inside the material, the incoming beam only reflected at the interface between two media. Assuming \( H'_i, H'_s, H'_f \) represent incoming, scattering and output beam, then the general boundary condition between two media is:

\[
H'_i + H'_s = H'_f \\
\vec{n} \cdot (\vec{D}'_f + \vec{D}'_s) = \vec{n} \cdot \vec{D}'_f \\
(\vec{n} \times (\vec{E}'_f + \vec{E}'_s))_z = (\vec{n} \times \vec{E}'_f)_z
\]  

(6-10)

The reflectionless condition requires there is no scattering beam, all the scattered field are zero: \( H'_s = 0, \vec{D}'_s = 0, \vec{E}'_s = 0 \). So \( H'_z = H_z \) and \( \frac{\vec{n} \cdot \vec{D}'}{(\vec{n} \times \vec{E}')_z} \) should be continuous. Since:

\[
\begin{pmatrix} \vec{n} \cdot \vec{D}' \\ (\vec{n} \times \vec{E}')_z \end{pmatrix} = \begin{pmatrix} (n_x, n_y) & (D'_x) \\ (n_x, n_y) & (D'_y) \end{pmatrix} = \frac{-H_z}{\omega} \begin{pmatrix} (n_x, n_y) \frac{A}{\Delta} \\ \frac{(n_x, n_y)}{(A^{-1})^T} \mathbf{\varepsilon}_{2D}^{-1} \end{pmatrix} \begin{pmatrix} k_y \\ -k_x \end{pmatrix}
\]  

(6-11)

Define: \( \vec{\eta} = \mathbf{\varepsilon}_{2D}^{-1}, \ F = \frac{-H_z}{\omega} \begin{pmatrix} (n_x, n_y) \frac{A}{\Delta} \\ \frac{(n_x, n_y)}{(A^{-1})^T} \mathbf{\varepsilon}_{2D}^{-1} \end{pmatrix}, \) and \( \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left( \frac{\vec{n} \cdot \vec{D}'}{(\vec{n} \times \vec{E}')_z} \right) \), so the above equation can be written as:

\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = F_1 \begin{pmatrix} k_y \\ -k_x \end{pmatrix} = F_2 \begin{pmatrix} k_fy \\ -k_fs \end{pmatrix}
\]  

(6-12)

The constitution equations of incident media and output media are given by:
\[(k_{iy} - k_{ix})\eta_i (k_{iy} - k_{ix})^T = k_0^2 \mu_z\]
\[(k_{ix} - k_{ix})\eta_i (k_{ix} - k_{ix})^T = k_0^2 \mu_{ix}\]  

(6-13)

Putting Eq.(6-12) into Eq.(6-13), we have:

\[(\alpha, \beta) \left[ \frac{(F_f)^{-1}\eta_i (F_f)^{-1}}{\mu_z} \right] \alpha = k_0^2, \quad \quad (\alpha, \beta) \left[ \frac{(F_f)^{-1}\eta_i (F_f)^{-1}}{\mu_{ix}} \right] \beta = k_0^2. \]  

(6-14)

The reflectionless condition holds for any incident angle. If we define \(M\) is the middle matrix of above equations, Eq.(6-14) holds for arbitrary incoming, which require all the coefficients of \(\alpha^2, \beta^2\) and \(\alpha\beta\) in these two equations be same. This means

\[(M_{11})_i = (M_{11})_f, (M_{22})_i = (M_{22})_f, (M_{12} + M_{21})_i = (M_{12} + M_{21})_f \]  

(6-15)

Since the transformation equation can be arbitrary, so let's first assume \(x'\)-direction is the normal direction of the interface without losing the generality, then the above equation can be written as:

\[
\begin{pmatrix}
\tilde{n} \cdot \tilde{D}' \\
(\tilde{n} \times \tilde{E}')_z
\end{pmatrix} =
\begin{pmatrix}
D'_{x} \\
E'_{y}
\end{pmatrix} =
\begin{pmatrix}
\epsilon'_{xx} & \epsilon'_{xy} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
E'_{x} \\
E'_{y}
\end{pmatrix} =
\begin{pmatrix}
-H_x \omega & \epsilon'_{xx} & \epsilon'_{xy} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \omega \\
\eta'_{yy}
\end{pmatrix}
\begin{pmatrix}
A \\
-\Delta
\end{pmatrix}
\begin{pmatrix}
k_x \\
-k_y
\end{pmatrix}
\]  

(6-16)

So:

\[
F =
\begin{pmatrix}
\epsilon'_{xx} & \epsilon'_{xy} \\
0 & 1
\end{pmatrix}
(A^{-1})^{T^2D^{-1}} =
\begin{pmatrix}
1 & 0 \\
\eta'_{yy} & \Delta
\end{pmatrix}
\begin{pmatrix}
A \\
\Delta
\end{pmatrix}
\]  

(6-17)

Here \(\eta' = \epsilon'^{-1} = \frac{1}{\det(\epsilon')} \begin{pmatrix}
\epsilon'_{yy} & -\epsilon'_{xy} \\
-\epsilon'_{yx} & \epsilon'_{xx}
\end{pmatrix}\), now:
\[
\left( F^T \right)^{-1} \eta F^{-1} = \frac{\Delta}{\mu_z \eta_{yy}} \begin{pmatrix}
\eta_{yy}' & -\eta_{yx}' \\
0 & 1
\end{pmatrix} \left( A^{-1} \right)^T \eta_{xx} A^T \begin{pmatrix}
\epsilon_{xx}' & \epsilon_{xy}' \\
0 & 1
\end{pmatrix}^{-1} \\
= \frac{\Delta}{\mu_z \eta_{yy}} \begin{pmatrix}
\eta_{yy}' & -\eta_{yx}' \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\epsilon_{xx}' & \epsilon_{xy}' \\
0 & 1
\end{pmatrix}^{-1} = \frac{\Delta}{\mu_z \eta_{yy} \det(\epsilon')} \begin{pmatrix}1 & \epsilon_{yx}' - \epsilon_{xy}' \\
0 & \Delta_e
\end{pmatrix}
\]

(6-18)

If $\epsilon'$ is a symmetric matrix, from $\epsilon' = A \epsilon A^T / \Delta_e$, $\epsilon'$ is also a symmetric matrix, so $\epsilon_{xy}' = \epsilon_{yx}'$.

Put Eq.(6-18) into Eq.(6-15), we get the reflectionless boundary conditions:

\[
[\det(\bar{\epsilon}')] = [\det(\bar{\epsilon'})]_f, \quad \left[ \mu_z \eta_{yy}' / \Delta \right] = \left[ \mu_z \eta_{yy}' / \Delta \right]_f.
\]

(6-19)

Since $\det(\epsilon_{xx}') = \det(A \epsilon A^T / \Delta) = \det(\tau)$, and more generalized, we using $n, t$ to represent the normal direction and tangential direction of the interface, we get:

\[
\det(\bar{\epsilon}) = \det(\bar{\epsilon}_f), \quad \frac{\mu_z (\epsilon')_{nn}}{\Delta_i} = \frac{\mu_z (\epsilon')_{nn}}{\Delta_f}
\]

(6-20)

Here $\epsilon'_{nn} = n^2 \epsilon_{xx}' + 2n_x n_y \epsilon_{xy}' + n^2 \epsilon_{yy}' = \epsilon_{xx}' \cos^2 \theta' + 2 \epsilon_{xy}' \sin \theta' \cos \theta' + \epsilon_{yy}' \sin^2 \theta'$

(6-21)

where $\theta'$ is the angle of normal direction at the interface in transformed space. Eq.(6-20) is the generalized reflectionless boundary condition for two transformed media in TE modes. Here $\theta'$ is the angle of normal direction of the interface, $ds'$ is the small distance on boundary, we have:

\[
\sin \theta' = -dx'/ds', \quad \cos \theta' = dy'/ds'
\]

(6-22)
Then,

\[
\varepsilon'''_{mn} = \frac{\varepsilon'_x dy'^2 - 2\varepsilon'_{xy} dx' dy' + \varepsilon'_y dx'^2}{ds'^2} = \frac{1}{ds'^2} \left( dx' \quad dy' \right) \left( \begin{array}{cc} \varepsilon'_x & -\varepsilon'_y \\ -\varepsilon'_y & \varepsilon'_y \end{array} \right) \left( dx' \quad dy' \right)
\]

\[
= \frac{1}{ds'^2} \left( dx' \quad dy' \right) \left[ \det(\varepsilon')(\varepsilon'^{-1}) \right] \left( dx' \quad dy' \right) = \frac{1}{ds'^2} (dx \quad dy) A^T \left[ \det(\varepsilon')(A^T)^{-1} \varepsilon^{-1} A^{-1} \right] A (dx \quad dy)
\]

\[
= \frac{\Delta}{ds'^2} (dx \quad dy) \left[ \det(\varepsilon)\varepsilon^{-1} \right] \left( dx \quad dy \right) = \frac{\Delta}{ds'^2} (dx \quad dy) \left( \begin{array}{cc} \varepsilon_{yy} & -\varepsilon_{xy} \\ -\varepsilon_{xy} & \varepsilon_{xx} \end{array} \right) \left( dx \quad dy \right)
\]

Put \( \varepsilon'''_{mn} \) to Eq.(6-20), we get:

\[
\det(\varepsilon_i) = \det(\varepsilon_J), \quad \frac{\mu_e}{ds'^2} (dx_i \quad dy_i) \varepsilon_i^{-1} \left( \begin{array}{c} dx_i \\ dy_i \end{array} \right) = \frac{\mu_e}{ds'^2} (dx_f \quad dy_f) \varepsilon_f^{-1} \left( \begin{array}{c} dx_f \\ dy_f \end{array} \right)
\]

(6-23)

This is the general reflectionless boundary condition. We call it as “extended slippery boundary condition”. If the incoming and output original space are both isotropic, we get the condition:

\[
\epsilon_i = \epsilon_f, \quad \mu_e \frac{ds_i^2}{ds'^2} = \mu_f \frac{ds_f^2}{ds'^2}
\]

(6-24)

Or:

\[
\epsilon_i = \epsilon_f, \quad \mu_e ds_i^2 = \mu_f ds_f^2
\]

(6-25)

If the transformed region is embedded in isotropic media, which means the output space have no transformation. In this case, \( ds_f = ds' \). We have:

\[
ds = \sqrt{\frac{\mu_e}{\mu_e}} ds'
\]

(6-26)

We will take “flat lens” and “beam concentrator” as two examples of how this condition works.
III. Flat lens

Fig. 6-1 Spatial coordinate transformation used for 2D flat lens design by Eq. (6-27): (a) any point with the coordinate \((x, y)\) in original coordinate space will transformed to point \((x', y')\) in (b) the transformed space.

We use “Flat lens” as our first example and compare two different transformations. Fig. 6-1 shows the region which we want transformed. One possible transformation was discussed by Kwon[268]:

\[
\begin{align*}
    x_1 &= (a + b) - \sqrt{R^2 - y'^2}, \quad y_1 = y' \\
    x_2 &= a, \quad y_2 = y' \\
    x &= x_1 + \eta(x_2 - x_1), \quad y = y' \quad (\eta = \frac{x'}{a})
\end{align*}
\]  

(6-27)

In this transformation, the \(y\) coordinate doesn’t change while the \(x\) coordinate transformed proportional to the relative position in each horizontal segment (see Fig. 6-1) Only the output interface matches the reflectionless condition \(ds = ds'\) while the incoming interface doesn’t. Fig. 6-2 shows the simulation result. We can see that the wavefront of incoming planewave has been
transformed to a cylindrical wave and has a focus even though the quality of the focusing is not so good. There is a slightly reflection at the incoming interface.

Here we take a new transformation where both the incoming and output interface of the material match the reflectionless boundary condition Eq.(6-26):

\[
x_1 = (a + b) - R \cos \left( \frac{y'}{R} \right), \quad y_1 = R \sin \left( \frac{y'}{R} \right)
\]

\[
x_2 = a, \quad y_2 = y'
\]

\[x = x_1 + \eta(x_2 - x_1), \quad y = y_1 + \eta(y_2 - y_1) \quad (\eta = \frac{x'}{c}) \tag{6-28}
\]

In this transformation, since there is no reflection on all sides (even the top and bottom side), We can expect a perfect focusing. The relation between parameters is:

\[R = \sqrt{L^2 + (a + b)^2} \quad c = \sqrt{(R \cos \frac{L}{R} - b)^2 + (R \sin \frac{L}{R} - L)^2} \tag{6-29}
\]

Fig. 6-2 Snapshots of (a) the total magnetic field and (b) intensity distribution due to a normally incident Gaussian beam pass through 2D flat lens. The transformation is described by Eq.(6-27) The geometry of the above design is: \(a = 1\) m, \(b = 3\) m, \(L = 4\) m, \(R = 5\) m and \(c = 1\) m. The wavelength of incoming beam is \(\lambda = 0.3\) m. The wavelength of incoming beam is \(\lambda = 0.3\) m. There is reflection at output surface.
The geometry of this transformation is showed in Fig. 6-3 and the full wave simulation result is showed in Fig. 6-4. It is observed an incoming plane wave is focused by the flat lens. Compare to the result of old transformation, this new one has better focusing quality and have no reflections anywhere.

Fig. 6-3  Spatial coordinate transformation used for the new 2D flat lens design by Eq.(6.28): (a) any point with the coordinate \((x,y)\) in original coordinate space will transformed to point \((x',y')\) in (b) the transformed space.
Considering that this far-zone flat lens converts planar waves into cylindrical waves and form an image at the focal point, the reverse operation should also hold true. In other words, the same lens has the ability to convert a cylindrical wave coming from a line source to planar waves. The above observation lets us design a near-zone flat lens which forms an image of a line source to the other side. Specifically, if two far-zone focusing lenses are arranged in a back-to-back configuration, the first lens will transform the diverging cylindrical waves into planar waves, and the second lens will convert the planar waves into converging cylindrical waves to form an image in the near field.

The performance of the near-zone focusing lenses is shown in Fig. 6-5 and Fig. 6-6 for two different configurations. One line source is located at a distance  \( d_{object} = b_1 = 1 \) m from the left interface of first lens. This lens has transformation parameters of  \( a_1 = 1 \) m,  \( b_1 = 1 \) m,  \( L = 4 \) m and the right media has  \( a_2 = 0.707 \) m,  \( b_2 = 2 \) m,  \( L = 4 \) m. From Eq. (6-29), the thickness of each transformed
media is: $c_1 = 1.871m, c_2 = 1.34m$. The output wave focused at $d_{\text{image}} = b_2 = 2m$ from the right interface of second lens. In Fig. 6-5, these two lenses are placed directly without separation and in Fig. 6-6 they have a separation of 1m.

![Diagram](image_url)

Fig. 6-5 Near-zone focusing lens designs without reflections realized by combining two far-zone focusing lens in a back-to-back configuration: a line source is located at a distance $d_{\text{object}} = 1m$ to the left interface of the first media, a focus in located at a distance $d_{\text{image}} = 2m$ from the right interface of the second media. The geometry of the first media is: $a_1 = 1m$, $b_1 = 1m$, $L = 4m$ and the right media is: $a_2 = 0.707m$, $b_2 = 2m$, $L = 4m$. The thickness of each transformed media is: $c_1 = 1.871m$, $c_2 = 1.34m$. The wavelength of the line source is $\lambda = 0.3m$. 

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IV. Beam concentrator

Fig. 6-6 Near-zone focusing lens designs without reflections realized by combining two far-zone focusing lens in a back-to-back configuration: a line source is located at a distance $d_{\text{object}} = 1\text{m}$ to the left interface of the first media, a focus in located at a distance $d_{\text{image}} = 2\text{m}$ from the right interface of the second media. The geometry of the first media is: $a_1 = 1\text{m}, b_1 = 1\text{m}, L = 4\text{m}$ and the right media is: $a_2 = 0.707\text{m}, b_2 = 2\text{m}, L = 4\text{m}$. The thickness of each transformed media is: $c_1 = 1.871\text{m}, c_2 = 1.34\text{m}$. The wavelength of the line source is $\lambda = 0.3\text{m}$.

Fig. 6-7 Spatial coordinate transformation used for the 2D concentrator design by Eq.(6-30): (a) any point with the coordinate $(x,y)$ in original coordinate space will transformed to point $(x',y')$ in (b) the transformed space.

Now we take beam concentrator as our second example. If we want compress the incoming beam, Fig. 6-7 shows the geometry of a very convenient transformation (discussed by Xu[269]):
\[ x' = x, \quad y' = y[1 - (1 - \eta)\frac{x}{a}] \] (6-30)

Here \( \eta \) is the compression ratio. However, this transformation ignore the reflectionless condition of the output interface so the result is not perfect. Fig. 6-8 shows the field and intensity distribution of this transformation, the reflection is exist at output boundary.
Here we propose a new transformation, which will compress the incoming beam without any loss due to reflection and also have some other advantage. Fig. 6-9 shows the coordinates transformation:

\[
\begin{align*}
\cos \theta & = - \frac{b}{r} \\
\sin \theta & = \frac{a}{r} \\
x' & = \frac{c(b-r)}{b-a} \\
\gamma y' & = r \theta 
\end{align*}
\]  

(6-31)

In this transformation, the original space is full with isotropic magnetic material where the permeability is \(1/\gamma^2\), here \(\gamma\) could be any value. The transformed region is embedded in free space. We should use the boundary condition Eq.(6-26) to match the reflectionless requirement. Each small length \(ds\) at boundary of original space is transformed correspondingly to a distance \(ds' = ds / \gamma\) at boundary in the transformed space. The permittivity and permeability inside the transformed region now is:

Fig. 6-9  Spatial coordinate transformation used for the 2D concentrator design by Eq.(6-31): (a) any point with the coordinate \((x,y)\) in original coordinate space will transformed to point \((x',y')\) in (b) the transformed space.
\[
\varepsilon' = \begin{bmatrix}
\frac{\gamma}{\beta} & -\theta \\
-\theta & \frac{\beta}{\gamma} (1 + \theta^2)
\end{bmatrix}, \mu' = \frac{\beta}{\gamma}
\] (6-32)

Here \( \beta = \frac{b-a}{c} \) is a constant and \( \theta = \frac{\gamma y'}{r} = \frac{\gamma y'}{b - \beta x'} \) is a function of \( x', y' \). The compression ratio is \( \eta = \frac{a}{b} = 1 - \frac{\beta c}{b} \). For more simplicity, we take the condition of \( \beta = \gamma \), then the above permittivity distribution reduced to:

\[
\varepsilon' = \begin{bmatrix}
1 & -\theta \\
-\theta & (1 + \theta^2)
\end{bmatrix}, \mu' = 1
\] (6-33)

It is an interesting distribution where the permeability in transformed region is one so no magnetic material is needed to fabricate such structure. It is a non-magnetic concentrator. We can use the dielectric material to adjust the distribution of permittivity by effective medium theory. Fig. 6-10 shows the performance of such design. An incoming Gaussian beam is perpendicularly incident on the slab with a width of \( w = 6\lambda_0 \). The length and width of the slab is \( L = 4m, c = 1m \). The compression ratio is 75%.
The reflectionless characteristics of the proposed structure enable us to obtain a large compression ratio of the incoming beam by multiple slabs. Fig. 6-11 shows an example of beam compression through four slabs, resulting in more compressions. The total compression ratio is 31.6%.

Fig. 6-10  (a) The transverse magnetic field distribution and (b) the power density distribution for a non-magnetic beam concentrator. The transformation is described by Eq.(6-31). The wavelength of incoming beam is $\lambda = 0.3\text{m}$. There is no reflection on incoming and output interface. The compression ratio of this transformation is 75%.

Fig. 6-11  (a) The transverse magnetic field distribution and (b) the power density distribution for non-magnetic beam concentrators compressed by four slabs. The transformation is described by Eq.(6-31). The wavelength of incoming beam is $\lambda = 0.3\text{m}$. There is no reflection on incoming and output interface. The total compression ratio is 31.6%.
Since the permittivity of each slab is limited by the maximum value of \( \theta \), this multi-slabs structure can reduce the difficulties of making large anisotropic material.

V. Conclusions

In this chapter, we have successfully derived the general reflectionless condition of any transformation optics. There are unlimited coordinate transformations we could use in designing while this reflectionless boundary condition gives us a limitation in choosing these transformations. We show two types of examples to describes how this condition works in designing transformation optical structures. Both the EM field distribution and power density flows though 2D full-wave numerical simulations have confirmed the results.
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