Mechanics of Regular, Chiral and Hierarchical Honeycombs

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Abstract

Approaches to obtain analytical closed-form expressions for the macroscopic elastic, plastic collapse, and buckling response of various two-dimensional cellular structures are presented. First, we will provide analytical models to estimate the effective elastic modulus and Poisson's ratio of hierarchical honeycombs using the concepts of mechanics of materials and compare the analytical results with finite element simulations and experiments.

For plastic collapse, we present a numerical minimization procedure to determine the macroscopic ‘plastic collapse strength’ of a tessellated cellular structure under a general stress state. The method is illustrated with sample cellular structures of regular and hierarchical honeycombs. Based on the deformation modes found by minimization of plastic dissipation, closed-form expressions for strength are derived. The work generalizes previous studies on plastic collapse analysis of lattice structures, which are limited to very simple loading conditions.

Finally, the method for calculation of buckling strength is based on classical beam-column end-moment behavior expressed in a matrix form. It is applied to regular, chiral, and hierarchical honeycombs with square, triangular, and hexagonal unit cells to determine their buckling strength under a general macroscopic stress state. The results were verified using finite element eigenvalue analysis.
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Chapter 1: Introduction
1.1 Hierarchical Structures: Applications and Background

Hierarchical structures are ubiquitous in nature and can be observed at many different scales in organic materials and biological systems [1-8]. The hierarchical organization of these systems generally plays a key role in their properties, function and survival [2, 4]. Hierarchy is also important in engineering designs, materials and architecture. Examples range from the Eiffel tower [3] and polymers with micro-level hierarchical structures [3], to sandwich panels with cores made of foams or composite lattice structures [9-13]. There, the hierarchical organization can lead to superior mechanical behavior and tailorable properties, as described recently for sandwich cores with hierarchical structure [9] and for hierarchical corrugated truss structures [10]. The overall mechanical behavior of these structures is governed by the response at different length scales and levels of hierarchy; and increasing levels of structural hierarchy can result in lighter-weight and better-performing structures [2, 3, 10, 14-17].

Honeycombs are two-dimensional cellular structures used in different applications including thermal isolation [18], impact energy absorption and structural protection [19-21], and as the core of lightweight sandwich panels [22-24]. The transverse (i.e., in-plane) stiffness and strength of honeycombs are generally governed by the bending deformation of cell walls, and strongly depend on the honeycomb relative density [25]. Under uniform transverse loading, the maximum bending moment in each cell wall occurs at the honeycomb vertices (i.e., cell wall corners). Thus, moving material from the middle part of each wall closer to the vertices can potentially increase the transverse stiffness and strength [26-28]. Here, we replace the vertices of a regular hexagonal lattice with smaller hexagons (simultaneously reducing the wall thickness to maintain fixed overall density), to achieve a structure with one level of hierarchy. This will be
shown able to exhibit a Young's modulus superior to that of its regular hexagonal counterpart of equal relative density. This replacement procedure for three-edge vertices can be repeated at smaller scales to achieve fractal-appearing honeycombs with higher orders of structural hierarchy. Figure 1-1a shows the evolution of a hexagonal honeycomb cell as structural hierarchy is increased. The structural organization of the honeycomb at each level of hierarchy can be defined by the ratio of the introduced hexagonal edge length ($L_1$ for $1^{\text{st}}$ order hierarchy and $L_2$ for $2^{\text{nd}}$ order hierarchy), to the original hexagon’s edge length, $L_0$, as described in Figure 1-1a (i.e., $\gamma_1 = L_1/L_0$ and $\gamma_2 = L_2/L_0$). For a honeycomb with $1^{\text{st}}$ order hierarchy, $0 \leq L_1 \leq L_0/2$ and thus, $0 \leq \gamma_1 \leq 0.5$, where $\gamma_1 = 0$ denotes the regular honeycomb structure. For a honeycomb with $2^{\text{nd}}$ order hierarchy, there are two geometrical constraints, $0 \leq L_2 \leq L_1$ and $L_2 \leq L_0/2 - L_1$. In terms of the ratio parameters, the constraints are $0 \leq \gamma_2 \leq \gamma_1$ if $\gamma_1 \leq 0.25$ and $0 \leq \gamma_2 \leq (0.5 - \gamma_1)$ if $0.25 \leq \gamma_1 \leq 0.5$. The dimensionless relative density (i.e., area fraction), can be given in terms of $t/a$:

$$\rho = 2/\sqrt{3} \cdot (1 + 2\gamma_1 + 6\gamma_2) \cdot t_0/L_0 \tag{1}$$

where $t_0$ is the thickness of the cell walls, from which the special cases of $\gamma_2$, $\gamma_1 = 0$ can be read off immediately. (For regular honeycomb, $\rho = 2/\sqrt{3} \cdot t_0/L_0$; and for honeycomb with $1^{\text{st}}$ order hierarchy, $\rho = 2/\sqrt{3} \cdot (1 + 2\gamma_1) \cdot t_0/L_0$). This relation clearly shows that $t_0/L_0$ must decrease to maintain fixed relative density as $\gamma_1$, $\gamma_2$ are increased.
Figure 1-1 Hierarchical honeycombs. (a) Regular and hierarchical honeycombs with 1st and 2nd order hierarchy. (b) Images of hierarchical honeycombs fabricated using three-dimensional printing.
1.2 Objectives

In this work, we have studied the effective elastic, plastic collapse, and nonlinear elastic properties of regular, chiral, and hierarchical honeycombs using analytical, numerical, and experimental methods. The hierarchical honeycomb samples are fabricated using 3D printing. In chapter 2 we will provide analytical models to estimate the effective elastic modulus and Poisson's ratio of hierarchical honeycombs using the concepts of mechanics of materials and compare the analytical results with finite element simulations and experiments.

For plastic collapse strength, first in chapter 3 we will use the analytical method of finding different plausible mechanisms of collapse for the unit cell of the hierarchical honeycomb of first order under biaxial state of loading. We will compare the results with FEM results. Next in chapter 4, we present a numerical minimization procedure to determine the macroscopic ‘plastic collapse strength’ of a tessellated cellular structure under a general stress state. The method is illustrated with sample cellular structures of regular and hierarchical honeycombs. Based on the deformation modes found by minimization of plastic dissipation, closed-form expressions for strength are derived. The work generalizes previous studies on plastic collapse analysis of lattice structures, which are limited to very simple loading conditions.

Finally in chapter 5, we will study the nonlinear elastic behavior of regular, chiral and anti-chiral honeycombs. An approach to obtain analytical closed-form expressions for the macroscopic ‘buckling strength’ of various two-dimensional cellular structures is presented. The method is based on classical beam-column end-moment behavior expressed in a matrix form. It is applied to sample honeycombs with square, triangular, and hexagonal unit cells to determine their buckling strength under a general macroscopic stress state. The results are verified using finite element eigenvalue analysis.
Chapter 2: Analytical Expressions of Elastic Moduli for Hierarchical Honeycombs
2.1 Analytical Expressions of Elastic Moduli: Young’s Modulus

For the analytical approach, we used Castigliano's second theorem [29] to determine the uniaxial in-plane deformation of hierarchical honeycombs made of an isotropic elastic material with elastic modulus, $E_z$. It is well known that plane lattices with threefold symmetry will exhibit macroscopically isotropic in-plane elastic behavior [30]. Thus, the macroscopic in-plane linear elastic behavior of hierarchical honeycomb can be characterized by just two constants, $E$ and $\nu$, to be found by whatever loadings are most convenient. We imposed a far field $y$-direction stress, $\sigma_{yy} = -(2/3)F/L_0$, in a vertical direction (perpendicular to the horizontal hexagon edges in Figure 2-1a). This is equivalent to applying a vertical force $F$ at every cut-point of a horizontal line (such as L1 in Figure 2-1a) passing through the mid-points of non-horizontal edges in a row of underlying (i.e., no hierarchy) hexagons.
To understand the analysis, it is helpful to envision the underlying regular hexagonal network as illustrated in Figure 2-1a. Midpoints of various edges have been labeled $MP_1$ to $MP_5$. For the imposed state of stress, no net horizontal or vertical force is transmitted across $L_2$. Yet every horizontal bar is equivalent, allowing us to conclude that each one transmits neither axial nor shear force. Furthermore, they also transmit no bending moment, because it would break the symmetry about horizontal lines. Therefore, the horizontal edges are entirely load-free for this state of stress. Next, considering the edges cut at their midpoints by line $L_1$, it is clear that for the given average stress, each cut bar must sustain a vertical force $F$. Cut at the midpoints as they are, we can conclude that no moment is transmitted across the cut, because that would lead to the bars being bulged 'out' of some hexagons and 'into' others, in a way prohibited by the symmetry of reflecting in $L_1$. The stress state also implies that no net horizontal force is transmitted across $L_1$. Therefore, considering the structure below $L_1$, a leftward force at $MP_3$ balanced by a rightward force at $MP_4$ might be envisaged. But by reflection in a horizontal line through the hexagon center, we would also have to expect a leftward force on the bar above a cut at $MP_4$. The resulting net leftward force on the bars between cuts at $MP_3$ and $MP_4$ is not possible because we already know that the horizontal bars (e.g., on the line through the hexagon center) are tension-free. We can thus conclude that the forces at cut points $MP_1, MP_3,$ and $MP_4$ are purely vertical with magnitude $F$. This 'hexagon midpoint' reasoning is unchanged when structural hierarchy is introduced.
Figure 2-2a shows the free body diagram of a subassembly able to represent an entire honeycomb with 1st order hierarchy subject to $\sigma_{y,y}$ loading (therefore, for this section, to find effective elastic modulus in $y$-direction, we are ignoring the horizontal forces shown at point 2,3, and 4). According to the above arguments, $MP_3$ is load-free, and $MP_4$ is subject only to force $-F$ in the $y$ direction. Since the subassembly is also cut free at points 1 and 2, we need to find the force and moment reactions at those cuts. $y$-direction forces acting on the subassembly are denoted by $N_1$ and $N_2$, and moments are denoted by $M_1$ and $M_2$. There can be no horizontal force at point 1 because of reflection symmetry about the $x$ axis, along with the lack of any third force on point 1 to balance same-direction horizontal inputs from above and below. At point 2, since no other horizontal forces act on the subassembly, we can also be sure that there is no horizontal reaction. So in this problem, $x$-direction equilibrium is trivially satisfied. By applying the $y$-force and moment balance laws to the subassembly, $N_2$ and $M_2$ can therefore be written as linear functions of $N_1$, $M_1$, and $F$. The bending energy stored in the subassembly can be expressed as a sum over all the beams: $U(F, M_1, N_1) = \sum \int \left(\frac{M^2}{2E_sI}\right) ds$, where $M$ is the bending moment at location $s$ along the beam, $E_s$ is the elastic modulus of the cell wall material, and $I$ is the beam's cross sectional area moment of inertia at $s$ (cell walls are considered to have rectangular cross section with thickness, $t$, and unit depth; i.e., $I = t^3/12$). Since the beam resultants are linear in $F$, $M_1$ and $N_1$, $U$ is then a quadratic function of those same quantities. The horizontal beam connecting nodes 2 and 3 can be excluded from the analysis since it is load-free.
Figure 2-2 Free body diagrams of the subassembly of honeycombs with (a) 1st and (b) 2nd order hierarchy used in the analytical estimation. Ni and Mi (i= 1 to 3) denote the reaction vertical forces and moments in the nodes of the subassembly structures as denoted in the pictures.

Since there is zero vertical displacement and zero rotation at point 1 due to symmetry, we can use Castigliano's method to write $\frac{\partial U}{\partial N_1} = 0$, and $\frac{\partial U}{\partial M_1} = 0$. These two relations allow $N_i$ and $M_i$ to be calculated in terms of $F$: $N_1 = F(0.533 + 0.15\gamma_1)$, $M_1 = Fa(0.283\gamma_1 - 0.017)$. At point 4 we can find the displacement $\delta = \frac{\partial U}{\partial F}$, and then the above substitution for $N_1$ and $M_1$ gives $\delta = \sqrt{3}FL_0^3/(72EIf(\gamma_1))$. The effective elastic modulus (to be normalized by beam material modulus, $E_s$) is then defined as the ratio of average stress ($-2F/3L_0$) and average strain, $(-4\delta/L_0\sqrt{3})$:

$$E/E_s = \left(\frac{t_o}{L_0}\right)^3 f(\gamma_1)$$

(2)

where $f(\gamma_1) = \sqrt{3}/(0.75 - 3.525\gamma_1 + 3.6\gamma_1^2 + 2.9\gamma_1^3)$. To find the maximum normalized elastic modulus for structures with first-level hierarchy and constant relative density, we eliminate $(t_o/L_0)$ from Eq. (2) by using the relative density expression of Eq. (1) The resulting
expression for \( E/E_s \) is \( \rho^3 \) times a function of \( \gamma_1 \), and setting \( (\partial(E/E_s)/\partial \gamma_1)\rho = 0 \) gives \( \gamma_1 = 0.32 \). Making this substitution leads to \( E/E_s = 2.97\rho^3 \), a stiffness almost twice the stiffness of the regular honeycomb structure [25], for which \( E_0/E_s = 1.5\rho^3 \). (The regular honeycomb result can be found by letting \( \gamma_1 = 0 \) in Eq. (2), and using Eq. (1) to eliminate \( t/a \)).

The same analytical approach was used to evaluate the in-plane effective Young's modulus of honeycomb with two orders of hierarchy, as a function of hierarchy indices \( \gamma_1 \) and \( \gamma_2 \). Figure 2-2b shows the free body diagram of a subassembly chosen to minimize calculation. As before, the vertical compressive stress \(-2F/3L_0\) is achieved by the external force, \( F \), applied downward at point 5 (a midpoint of the underlying hexagon side), with symmetry arguments showing that no other loads act at that point. Bar 3-4 is again load-free. The same argument applies to point 3 as formerly applied to point 2 for the honeycomb with one order of hierarchy. And, the same argument applies to points 2 and 1 as formerly applied to point 1. Therefore, \( N_1, M_1, N_2, M_2, N_3, \) and \( M_3 \) are the unknown reaction forces and moments at vertices 1, 2, and 3 as shown in Figure 2-2b. One additional step required for analysis of the second-order hierarchy is to determine the beam resultants for the statically indeterminate, complete (small) hexagon of side \( c \) embedded in each subassembly, loaded at nodes 6 and 7 with reactions at node 8. The bending moments along each side of the \( c \)-hexagon are determined from a subsidiary analysis in which it is divided at nodes 6 and 7, and then three compatibility conditions are enforced at each of those nodes. The details of that analysis are omitted for brevity. Similar to honeycombs with first order hierarchy, using the \( y \)-direction and rotational equilibrium equations, \( N_3 \) and \( M_3 \) can be written as a function of \( N_1, M_1, N_2, M_2, \) and \( F \). Therefore, the total energy of the investigated substructure, which is the sum of the bending strain energy of all the beams, can be written as:

\[
U(F, M_1, N_1, M_2, N_2) = \sum \int (M^2/(2E_s I))\,ds.
\]

The following four boundary conditions are
imposed at points 1 and 2 to achieve the zero rotation and zero displacement demanded by symmetry, as shown in Figure 2-2b: \( \partial U / \partial N_1 = 0, \partial U / \partial M_1 = 0, \partial U / \partial N_2 = 0, \) and \( \partial U / \partial M_2 = 0. \) These relations allow us to solve for \( M_i, N_i, M_2, N_2. \) In a similar way as above the effective elastic modulus can be presented as:

\[
E_s / E_s = (t_o / L_0)^3 f(\gamma_1, \xi)
\]

where \( \xi = \gamma_2 / \gamma_1 \) and \( f(\gamma_1, \xi) = N_4(\xi) / (\gamma_1^2 D_7(\xi) + \gamma_1^2 D_6(\xi) + \gamma_1 D_5(\xi) + D_4(\xi)) \)

\[
N_4(\xi) = 29.62 - 54.26 \xi + 31.75 \xi^2 - 4.73 \xi^3 - \xi^4
\]

\[
D_7(\xi) = 49.64 - 609.01 \xi + 862.56 \xi^2 - 195.50 \xi^3 - 270.14 \xi^4 + 159.95 \xi^5 - 18.13 \xi^6 - 2.20 \xi^7
\]

\[
D_6(\xi) = 61.73 + 310.43 \xi - 662.32 \xi^2 + 334.12 \xi^3 + 9.70 \xi^4 - 29.38 \xi^5 - 1.88 \xi^6
\]

\[
D_5(\xi) = 60.43 + 12.80 \xi + 123.22 \xi^2 - 108.06 \xi^3 + 20.50 \xi^4 + 3.90 \xi^5
\]

\[
D_4(\xi) = 12.80 - 23.46 \xi + 13.74 \xi^2 + 2.04 \xi^3 - 0.43 \xi^4
\]

For the 2\textsuperscript{nd} order hierarchical structure, once again eliminating \( (t/a) \) in favor of density, and then differentiating at constant density, \( (\partial (E/E_s) / \partial \gamma_1)_{\rho} = (\partial (E/E_s) / \partial \gamma_2)_{\rho} = 0 \) give \( \gamma_1 = 0.32, \) and \( \gamma_2 = 0.135, \) leading to \( E_s / E_s = 5.26 \rho^3, \) a stiffness almost 3.5 times that of the regular honeycomb.

To validate the theoretical results we simulated the structural response using finite element analysis. Two-dimensional hierarchical honeycombs were modeled using Abaqus 6.10
All models were meshed using the BEAM22 element, which is capable of capturing not only the bending compliance of the above theory, but also the axial and shear deformations which may become significant at greater values of $t_o/L_0$. A rectangular cross section with unit length normal to the plane of loading was assumed for the cell wall beams. The thickness of all the beams was adjusted to control the overall relative density of the structure. The material properties of aluminum, $E_s=70$ GPa, and $\nu_s = 0.3$, were used in this study. We performed the analysis with two different boundary conditions representing our analytical model and experimental tests, respectively. In the first set, we applied periodic boundary conditions to matching nodes on the left and right edges, as if the sample were infinitely wide but free to strain laterally [31]. To model infinitely long cellular structure, all the nodes lying along the dashed symmetry line of Figure 2-2a or 2-2b at the top and bottom of the model were connected to a rigid plate. Those nodes were constrained by symmetry conditions, i.e. free to slide left or right, but all maintaining the same y coordinate, and prevented from rotating. This model represents an infinite cellular structure in both in-plane directions and thus, the mechanical response is not dependant on the model size (i.e. eliminating the size effect). We confirmed the independence of the results from the model size by systematically changing the model size from a single unit cell to structure comprising of $8 \times 8$ unit cells for a honeycomb with one order hierarchy and relative density of 6% and $\gamma_1 = 0.3$. The unit cell was only 0.5% stiffer than the $8 \times 8$ cellular structure (measured by comparing the effective elastic modulus of each system).

In the second set of simulations, those same top and bottom nodes were constrained horizontally by being built into the fixed rigid plates (i.e. tied boundary condition with no rotation and no displacement in the horizontal direction and with equal displacement in the vertical direction for the nodes in contact with the rigid plates), and the side nodes were free as in
the experimental setup. In this case, the effective elastic modulus is strongly dependant on the size of the structure. For a model comprising of $5 \times 5$ unit cells, which is consistent with our experiments, the increase in modulus caused by this constraint in the second kind of simulation ranged from 3% up to a maximum of about 20% depending on the honeycomb relative density (the different is smaller for honeycombs with lower relative density). Here, we show the numerical results from the first set, which matched the boundary conditions of our analytical model (first model described above for simulating the response of an infinitely long and wide cellular structure). The effective elastic modulus of each structure was calculated from the slope of compressive stress-strain response.

Figure 2-3a shows the effective elastic modulus of first order hierarchical honeycombs for all possible values of $\gamma_1$. In this figure, the elastic modulus is normalized by the effective stiffness of the counterpart regular honeycomb with the same relative density, $1.5E_s \rho^3$, allowing us to present results for every density on a single curve. In the finite element simulations, structures with three different relative densities (2%, 6% and 10%) were analyzed. Results show quite good agreement between numerical and theoretical approaches, even though the theoretical analysis ignored the axial and shear deformation of the beams (a good approximation only for low density honeycombs with small beam thickness [31]). We suspect that the numerical incorporation of shear and stretching accounts for the FEA-determined modulus falling somewhat below the theory, particularly as density increases or beam lengths decrease. The FEA results nicely confirm the near-doubling of stiffness for $\gamma_1 = 0.32$. In this figure, experimental results are also plotted which show reasonable agreement with both theory and numerical results. For honeycombs with 2$^{nd}$ order hierarchy, we fixed $\gamma_1 = 0.3$ and plotted the normalized effective elastic modulus for various values of $\gamma_2$. The results again match theory best for low density, and
show that honeycombs with two orders of hierarchy where $\gamma_1 = 0.3$, and $\gamma_2 = 0.135$, have stiffness approximately 3.5 times that of regular hexagonal honeycomb with same relative density (i.e. specific stiffness 3.5 times of the regular hexagonal honeycomb).
Figure 2-3 Stiffness of hierarchical honeycombs. (a) Normalized stiffness for honeycombs with 1st order hierarchy versus $\gamma_1$. (b) Normalized stiffness versus $\gamma_2$, for honeycombs with 2nd order hierarchy and $\gamma_1 = 0.3$. The schematic of the honeycomb unit cells are shown for selected values of $\gamma_1$ and $\gamma_2$ in each plot. The finite element results are shown for honeycombs with three different relative densities. Experimental results for structures with different hierarchy levels are also shown (black circles). The error bars show the results variation. Each experimental point is from 3 tested specimens.

2.2 Analytical Expressions of Elastic Moduli: Poisson's Ratio

To fully characterize the linear elastic behavior of hierarchical honeycombs, we also need to obtain the dependence of Poisson’s ratio, $\nu$, on the dimension ratios. We again used Castigliano's second theorem and considered the same subassemblies under biaxial loading (where the horizontal stress is finally set to zero after differentiating). This is a bending-based approximate analysis that ignores axial and shear deformation of the cell walls.

Temporarily considering horizontal loading only, we apply reasoning similar to that in previous section, to midpoint cut lines such as L2 and L3. This establishes that the horizontal segment aligned with the dotted line is subjected to pure compression (no bending), and that the segment midpoint to the upper right of each subassembly experiences only a horizontal force. There is no horizontal reaction at node 1 for first order hierarchy, or at nodes 1, 2 for second order hierarchy. But node 2 (first order) and node 3 (second order) has a horizontal reaction to balance $2P$, -$P$, and nodes on the dashed horizontal line still require vertical and moment reactions. The composite free body diagrams for both horizontal and vertical stress are shown in Figure 2-2-a is for 1st order hierarchy where the external forces $P$ and $F$ are applied at point 4 in x- and y-directions,
and N1, M1, N2, and M2 are the reaction vertical forces and moments at vertices 1 and 2, respectively. The two non-trivial equations of equilibrium (vertical and angular) allow us to write N2 and M2 as functions of N1, M1, P and F. Therefore, the bending energy stored in the subassembly can be expressed as the summation of bending energy in all beams, $U(F, P, M_1, N_1) = \sum \int (M^2 / 2E_s I) ds$, where M is the bending moment at position s along each beam, $E_s$ is the elastic modulus of the cell wall material, and I is the beam's cross sectional area moment of inertia (cell walls are considered to have rectangular cross section with thickness t and unit depth, i.e., $I = t^3 / 12$). The horizontal beam connecting the nodes 2 and 3 can be excluded from the analysis since it experiences no bending moment. Assuming zero displacement and zero rotation at vertices 1 and 2 due to symmetry, one can write $\partial U / \partial N_1 = 0$, and $\partial U / \partial M_1 = 0$. These two relations allow N1 and M1 to be calculated as a function of P and F. The bending energy stored in the subassembly can be subsequently expressed as $U = U(F, P)$.

When P is zero (free lateral expansion), the x and y displacements of point 4 due to force F can be expressed as follows, respectively: $\delta_X^F = (\partial U / \partial P) \big|_{p=0}$; $\delta_Y^F = (\partial U / \partial F) \big|_{p=0}$. Considering the initial dimensions of the subassembly to be $3a/4$ and $\sqrt{3}a/4$ in X- and Y- dimensions, respectively, the Poisson's ratios in direction Y is obtained as $\nu = \delta_X^F / \sqrt{3} \delta_Y^F$, which gives: $\nu = 1 - \gamma_1^3 / (2.9 \gamma_1^3 + 3.6 \gamma_1^2 - 3.525 \gamma_1 + 0.75)$, which is plotted in Figure 2-4. The value of $\nu$ is $\nu = 1$ at $\gamma_1 = 0$, $\nu = 0.5$ at $\gamma_1 = 1$, with the minimum value 0.37 at $\gamma_1 = 0.4$. Finite element results are also shown which were obtained by calculating the lateral deformation of honeycombs with periodic boundary condition under uniaxial in-plane loading.

For the biaxially loaded 2nd order hierarchical honeycomb subassembly illustrated in Figure 2-2-b. N1, M1, N2, M2, N3, and M3 are the unknown reaction vertical forces and moments. Once
again, vertical and rotational equilibrium equations allow us to write N3 and M3 as functions of N1, M1, N2, M2, P and F. Therefore, the bending energy stored in the subassembly can be expressed as the summation of bending energy in all beams, \( U(F, P, M_1, N_1, M_2, N_2) = \sum \int (M^2/2E_s I) \text{ds} \). The horizontal beam connecting nodes 3 and 4 is again excluded from the analysis. Since symmetry prevents vertical displacement or rotation at vertices 1 and 2, one can write \( \partial U/\partial N_1 = 0 \), \( \partial U/\partial M_1 = 0 \), \( \partial U/\partial N_2 = 0 \), and \( \partial U/\partial M_2 = 0 \). These four relations allow N1, N2, M1 and M2 to be calculated as functions of P and F. The bending energy stored in the subassembly can be subsequently expressed as \( U = U(F, P) \). The x and y displacements of point 5 due to force F can be expressed as: \( \delta_X = (\partial U/\partial P)|_{P=0} \); \( \delta_Y = (\partial U/\partial F)|_{P=0} \). Considering the initial dimensions of the subassembly to be \( 3a/4 \) and \( \sqrt{3}a/4 \) in the x and y directions, the Poisson ratio is obtained as: \( \nu = \delta_X / \sqrt{3}\delta_Y \). The value of \( \nu \) ranges from 0.28 at \( \gamma_1 = \gamma_2 = 0.23 \) to 1.0 at \( \gamma_1 = \gamma_2 = 0 \).

### 2.3 Concluding Remarks

To summarize the behavior of all honeycombs with the investigated hierarchical structures, we have plotted contour maps of the effective (normalized) elastic modulus and Poisson’s ratio of hierarchical honeycombs with second-order hierarchy for all possible values of \( \gamma_1 \) and \( \gamma_2 \), as shown in Figure 5. The x-axis is \( \gamma_1 \) ranges from 0 to 0.5, while \( \gamma_2 \) is limited by the two geometrical constraints, \( \gamma_2 \leq \gamma_1 \) and \( 0 \leq \gamma_2 \leq (0.5 - \gamma_1) \). Hierarchical honeycombs with small to moderate values of \( \gamma_1 \) and \( \gamma_2 \), and especially a simple hexagonal honeycomb, have Poisson ratio near 1.0. This means that the Young’s and Shear moduli, which are controlled by element bending, are far lower than the "Bulk" (really, "Areal") modulus which for those structures is controlled by element stretching.
The results show that a relatively broad range of elastic properties, and thus behavior, can be achieved by tailoring the structural organization of hierarchical honeycombs, and more specifically the two dimension ratios. For example, the hierarchical honeycombs with one order and two orders hierarchy are shown to have specific stiffness up to 2.0 and 3.5 times of the regular hexagonal honeycomb. Increasing the level of hierarchy provides a wider range of achievable properties. Further optimization should be possible by also varying the thickness of the hierarchically introduced cell walls, and thus the relative distribution of the mass, between different hierarchy levels. The proposed work focused only on the elastic properties of hierarchical honeycombs, and the collapse/yield and instability properties of these structures are currently under study.
Figure 2-4 Poisson's ratio of hierarchical honeycombs with one level of hierarchy versus $\gamma_1$. The finite element results are also plotted for honeycomb with relative density 0.06.
Figure 2-5 (a) Contour map of the effective elastic modulus of hierarchical honeycombs with 2\textsuperscript{nd} order hierarchy for all possible geometries (i.e., admissible range of $\gamma_1$ and $\gamma_2$). (b) Contour map for Poisson's ratio.
Chapter 3: Plastic Collapse of Hierarchical Honeycombs under Bi-Axial Stress State
3.1 Basic Unit Cell Relations

In this section, the analytical expressions of plastic collapse strength are derived. First we look at the basic unit cell relations for hierarchical honeycombs. Then, we will derive lower- and upper-bound expressions of strength.

Figure 3-1a shows a schematic of the geometrical substitutions resulting in 1st and 2nd order hierarchical honeycombs. For each level $i$ of hierarchy, two parameters, namely $\gamma_i$ and $\eta_i$, are used to define the substitution geometry. The length ratio, $\gamma_i$, is defined as the ratio of the new hexagon side to the larger (original) hexagon side, $L_o$. The thickness ratio, $\eta_i$, is the ratio of new hexagon wall thickness to the wall thickness of the remaining parts of the original hexagons.

For example, for the first order hierarchy shown in figure 3-1a, $\gamma_1 = L_1/L_o$ and $\eta_1 = t_1/t_o$.

Here, $L_o$ and $t_o$ denote the length and wall thickness of the underlying large hexagons, and $L_1$ and $t_1$ denote the length and wall thickness of the introduced smaller hexagons, respectively. The replacement can be continued to any order of hierarchy, $n$, as long as the inserted hexagons are not so large that they intersect each other or previously created lower order hexagons (i.e. $(\sum_{i=1}^{n} \gamma_i) \leq 0.5$ and $(\sum_{i=j+1}^{n} \gamma_i) \leq \gamma_j$ , $0 < j < n$ and $\gamma_i \leq \gamma_j$ , $i < j$).

Our study has limited our study to the plastic collapse strength of 1st order hierarchical honeycombs. Therefore, for simplicity we name the geometrical parameters of 1st order hierarchical honeycomb $\gamma = \gamma_1$ and $\eta = \eta_1$. The relative density of such honeycombs can be calculated as

$$\frac{\rho}{\rho_s} = \frac{2}{\sqrt{3}} \frac{t_o}{L_o} \left( 1 + 2\gamma(2\eta - 1) \right)$$
where \( \rho \) is the average density of the cellular structure and \( \rho_s \) is the wall material density. The proportions of the wall material distributed in larger and smaller hexagons are equal to \( (1 - 2\gamma)/(1 - 2\gamma + 4\gamma \eta) \) and \( 4\gamma \eta/(1 - 2\gamma + 4\gamma \eta) \), respectively. For a first order hierarchical honeycomb with \( \eta = 1 \) and \( \eta = 0.3 \), 75% of the mass of the structure is allocated to the smaller honeycombs and only 25% is in the remaining parts of the original hexagon.

Figure 3-2a shows a unit cell (circled) of the 1st order hierarchical honeycomb, which is used for plastic analysis of the infinite structure. In-plane biaxial loading is applied in the principal structural directions \( x \) (parallel to a hexagon side) and \( y \) (perpendicular to a hexagon side). This symmetrical loading allows us to consider just half of the unit cell for elastic and plastic analyses. Due to reflection symmetry about the \( x \) axis of loading and geometry, horizontal beams are moment-free. Figure 3-2b shows the free body diagram of the unit cell, where external forces \( F \) at an angle \( \theta \) from the vertical are applied to the tips of the two oblique beams with thickness \( t_o \) and the reaction force \( 2F \sin(\theta) \) is applied to the horizontal beam. Due to 180° rotational symmetry of adjacent unit cells sharing an oblique member, those oblique-beam tips (midpoints of the underlying hexagon edges) are also moment-free (see [32] for further discussion). Since force \( F \) is applied to vertical and horizontal projected unit cell areas \( \sqrt{3}L/2 \) and \( 3L/2 \) respectively, normal stresses in the \( x \) and \( y \) directions, denoted by \( S^x \) and \( S^y \), respectively, can be obtained from

\[
F \sin(\theta) = S^x \frac{\sqrt{3}L}{2} \quad \text{and} \quad F \cos(\theta) = S^y \frac{3L}{2}
\]  

(2)

The angle, \( \theta \), of force \( F \) exerted on point 4 of the unit cell can be found from \( \cot(\theta) = \sqrt{3}S^y / S^x \).
Figure 3-1 Hierarchical honeycombs. (a) Regular and hierarchical honeycombs with 1st and 2nd order hierarchy. (b) Images of hierarchical honeycombs fabricated using three-dimensional printing.
Figure 3-2 (a) A section of 1st order hierarchical honeycomb under biaxial loading, where the unit cell is marked by bold lines. (b) Free body diagram of the unit cell for both elastic and plastic analyses. Only half of the unit cell is analyzed due to symmetry.

3.2 Analytical Modeling

Plastic collapse strength of first order hierarchical honeycomb was evaluated by classical plastic limit analysis. At the collapse load (or limit load) of a structure, plastic deformation can increase indefinitely under a constant load. This behavior presumes that the material exhibits rigid-perfectly plastic behavior with its associated flow rule. For our application, we took the moment versus bend angle at a plastic hinge to be a fixed ‘plastic limit moment’ (Extensional yielding is discounted.) It is also assumed that the loaded structure undergoes small enough displacements that the slope change of structural elements can be neglected in equilibrium calculations. This theory gives no prediction as to whether the actual nonlinear load-displacement curve is concave or convex.

We first develop a ‘lower bound’ plastic collapse analysis of the hierarchical honeycombs. The lower bound is based on finding equilibrium distribution of moments at or
below the collapse moment, which balances the applied load. For this we use the elastic moment distribution of a unit cell, which is outlined in section 3.1. Then, the ‘upper bound’ plastic collapse stress is estimated analytically in section 3.2. This is based on finding the minimum collapse load (as determined by virtual work) among various mechanisms of structural deformation involving different possible locations of the plastic hinges. In our analysis, out-of-plane loads are ignored.

![Diagram](image)

Figure 3-3 (a) Elastic reaction forces and moments exerted on the upper half of unit cell for elastic and lower-band plastic limit analyses. (b) Locations of potential plastic hinges in the upper half of unit cell for upper-band plastic limit analysis.

### 3.2.1 ‘Lower Bound’ Plastic Collapse Analysis

In a frame structure consisting of loaded beams, if bending moments in equilibrium with external loads are less than or equal to the plastic hinge moment of each beam cross section, the structure either will not collapse or will be just at the point of collapse under those external loads [33]. The plastic hinge moment per unit depth of a beam with a rectangular section of thickness
and yield stress $\sigma_{ys}$ is equal to $\sigma_{ys} h^2 / 4$, where the nonlinear contribution of axial force to collapse moment has been neglected. For our lower bound limit analysis, we used the distribution of bending moments found via elastic analysis. The lower bound collapse load was taken as the load sufficient to bring the calculated elastic moment in at least one cross section up to the plastic hinge moment of that cross section. Because the hierarchical structure is not statically determinate, issues of compatibility affect the actual collapse strength. While compatibility is maintained in the purely elastic regime, just one plastic hinge may not permit ongoing plastic deformation. Therefore, in general a lower bound based on elastic analysis is expected to underestimate (i.e., not quite reach) the true collapse strength.

The elastic analysis used to determine the elastic moment distribution in 1st order hierarchical honeycomb under biaxial loading is an extension of that presented in [32] for uniaxial loading with uniform thickness, $\eta = 1$. The description here is abbreviated for reasons of space. Consider the free body diagram of the upper half of structural unit cell as shown in Fig. 3-3a. Due to symmetry about the x axis, only the upper half of the unit cell was modeled, with loading by force $F$ at angle $\theta$. The rotation and vertical displacement of nodes 1, 2 and 3 are zero because of symmetry. The reaction forces and moments acting on nodes 1 and 2 are denoted by $N_1$, $N_2$, $M_1$ and $M_2$. By applying force and moment balance laws to the subassembly, $N_2$ and $M_2$ can be written as linear functions of $N_1$, $M_1$, and $F$. Therefore, the bending energy stored in the subassembly can be expressed as a sum over all the beams: 

$$U(F, M_1, N_1) = \sum \int (M^2 / (2E_s I)) ds,$$

where $M$ is the bending moment at location $s$ along the beam, $E_s$ is the local elastic modulus of the cell wall material, and $I$ is the beam's cross section moment of inertia at location $s$ (cell walls are considered to have rectangular cross section with thickness, $t$, and unit depth; i.e., $I = \ldots$)
The horizontal beam connecting nodes 2 and 3 can be excluded from the analysis since it is moment-free.

Using Castigliano's method to set the displacement and rotation of point 1 to zero, \( \frac{\partial U}{\partial N_1} = 0 \) and \( \frac{\partial U}{\partial M_1} = 0 \), one can obtain the values of reaction forces and moments at point 1, \( N_1 \) and \( M_1 \), in terms of applied force, \( F \)

\[
N_1 = F \sin(\theta)(0.231 - 0.260/\gamma) + F \cos(\theta)(0.533 + 0.150/\gamma) \]
\[
M_1 = Fa \sin(\theta)(0.029 - 0.202\gamma) + Fa \cos(\theta)(0.283\gamma - 0.017) \]  

These results permit calculation of elastic moments at all critical points (i.e. beam ends) in the unit cell; for any given value of \( \theta \), the location of greatest moment can be determined, and thus the lower bound plastic strength as discussed above.

3.2.2 ‘Upper Bound’ Plastic Collapse Analysis

According to the upper bound theorem of plastic limit analysis for frame structures, the structure must collapse if there is a compatible pattern of plastic deformation for which the rate of work done by the external forces equals or exceeds the rate of internal dissipation [33]. Setting boundary work equal to internal dissipation (by the virtual work principle), permits calculation of the necessary boundary load magnitude.

In the case of a structure with straight beams connected and loaded only at their ends (guaranteeing that the maximum bending moment occurs only adjacent to a joint), all compatible deformations of interest involve plastic hinges located where beams join nodes that can make the structure a mechanism. Then the upper bound approach for finding collapse strength is based on
comparing different mechanisms compatible with given boundary displacements, and finding the mechanism or combination of mechanisms that minimizes the required load.

The amount of plastic energy dissipation at each hinge is given by $M_{ph} \times |d\alpha|$, where $d\alpha$ is the change in angle across the plastic hinge, and $M_{ph}$ is the positive plastic hinge (limit) moment of the cross section. A statement of virtual work for the plastically deforming structure is $W_E(F) = \sum M_{ph}^i |d\alpha_i|$, where $W_E(F)$ is the work of external forces and the sum includes dissipation at all plastic hinges. While our lower bound calculations are expected to underestimate the collapse strength, the upper bound calculations are considered likely to be exact. Since straight-beam structures develop their hinges only adjacent to joints, if all possible end-hinged deformations are considered, the actual global minimum of the required load will be found.

Six plastic hinge locations are possible for the upper-half unit cell of a first order hierarchical honeycomb. As shown in figure 3-3b, plastic hinges $i, j, k, l, m$ are at the beam ends of the small hexagon with thickness $t_1$, and the plastic hinge $n$ is at the lower end of the oblique beam with thickness $t_o$. Examining subsets of these six hinges, a total of nine plastic deformation mechanisms having just a single degree of freedom were identified for the half-cell, as given in tables 3-1 and 3-2. The deformed shape and plastic hinges for each mechanism are shown by gray lines and red bullets, respectively. As shown later, mechanisms presented in table 3-1 are the actual deformation modes under uniaxial loading in the x or y direction for all values of $\gamma, \eta$. The mechanisms presented in table 3-2 are observed under various xy biaxial loading states (i.e. $S_c^y/S_c^x \neq 0$ or $\infty$).
Note that out of nine plastic hinge mechanisms presented in tables 3-1 and 3-2 there are only four independent deformation mechanisms, as explained below. The angular deformations at the six possible hinges (i.e., increments of the six relative angles $\alpha_i, \alpha_j, \alpha_k, \alpha_l, \alpha_m, \alpha_n$ in figure 3-3b) form a vector space. These deformation angles are not independent, since the five involved in the closed circuit hexagon are subject to requirements of symmetry (i.e. zero vertical displacement and rotation of node 1 lying on the line of symmetry). The changes in angles $\alpha_i$ to $\alpha_n$, therefore, follow

$$d\alpha_i + d\alpha_j + d\alpha_k + d\alpha_l + d\alpha_m = 0 \quad \text{(sum of angles in a loop)}$$

$$4d\alpha_i + 3d\alpha_j + d\alpha_k + d\alpha_l = 0 \quad \text{(vertical displacement of node 1)}$$

Therefore, for the five variables $d\alpha_i$ to $d\alpha_m$, only three can be chosen independently. (Note that mechanism I is uniquely independent since it alone involves hinge $n$.) For mechanisms I-V the changes in angle at plastic hinges $i$-$n$, represented in the form $[d\alpha_i, d\alpha_j, d\alpha_k, d\alpha_l, d\alpha_m, d\alpha_n]$, are proportional to $[0, 0, 0, 0, 0, 1]$, $[0, 0, 1, -1, 0, 0]$, $[1, 0, 0, -4, 3, 0]$, $[3, -4, 0, 1, 0]$and$[2, -3, 1, 0, 0]$, respectively. Deformed configurations illustrating mechanisms I-IV are shown in figure 3-4. We have selected I-IV as a convenient primary basis for all deformations, and tables 3-1, 3-2 describe each additional mechanism in terms of these.

The single degree of freedom mechanisms VI-IX shown in table 3-2 minimize load only under biaxial loading (i.e. $S_c^V, S_c^H \neq 0$). The change in angle at plastic hinges $i$-$n$ for these mechanisms are proportional to $[0, 1, -3, 0, 2, 0]$, $[1, 0, -4, 0, 3, 0]$, $[0, 1, 0, -3, 2, 0]$ , $[2, -3, 0, 1, 0, 0]$, respectively.
Algebraic expressions for the x direction plastic collapse stress corresponding to each mechanism are also presented in tables 3-1 and 3-2. Parameterizing each mechanism in terms of the plastic hinge angular deformation vector \([d\alpha_i, d\alpha_j, d\alpha_k, d\alpha_l, d\alpha_m, d\alpha_n]\) one can express the displacement of node 4 with respect to node 3 in x and y directions as follows

\[
\begin{align*}
\text{dx} &= -\left[\frac{\sqrt{3}}{4}, \frac{\sqrt{3}(0.5 - \gamma)}{2}, \frac{\sqrt{3}(0.5 - \gamma)}{2}, \frac{\sqrt{3}(0.5 - \gamma)}{2}\right] L_o \cdot [d\alpha_i, d\alpha_j, d\alpha_k, d\alpha_n] \\
\text{dy} &= \left[\frac{1}{4} + \gamma, \frac{1}{4} + \frac{\gamma}{2}, \frac{1}{4} - \frac{\gamma}{2}, \frac{1}{4} - \frac{\gamma}{2}\right] L_o \cdot [d\alpha_i, d\alpha_j, d\alpha_k, d\alpha_n]
\end{align*}
\]

(5)

Then the work of force \(F\) is represented as \(W(F) = -F\sin\theta \text{dx} - F\cos\theta \text{dy}\). Plastic dissipation can be expressed as

\[
\text{PD} = \sigma_{ys} t_o^2 / 4 \times (|d\alpha_i| + |d\alpha_j| + |d\alpha_k| + |d\alpha_l| + |d\alpha_m|) + \sigma_{ys} t_o^2 / 4 \times |d\alpha_n|
\]

(6)

Setting equal the expression of internal dissipation and external work, critical force per unit depth, \(F_c\), required to deform each mechanism can be evaluated.

For example, for mechanism I, parameterized as \([0, 0, 0, 0, 0, 1]\), the required force is obtained as

\[
F_c = \frac{\sigma_{ys} t_o^2 / 4}{L_o \times |\sin(\theta - \pi/6)| \times (0.5 - \gamma)}
\]

(7)

By substituting the expression of density (eq. 1) and the relationship between applied biaxial stresses and load (eq. 2), the components of stress in terms of \(\theta\) are found as
\[
\frac{S_c^x}{\sigma_{ys}} = \frac{1}{4\left(1 - \cot\theta/\sqrt{3}\right) \left(1 + 2\gamma \cdot (2\eta - 1)\right) \left(0.5 - \gamma\right)} \left(\frac{\rho}{\rho_s}\right)^2
\]

\[S_c^y = \cot\theta/\sqrt{3} \cdot S_c^x\]

where \(S_c^x\) and \(S_c^y\) denote the stresses in the x and y direction required for plastic collapse under mechanism I, respectively. These expressions are entered into the first row of table 3-1. Mechanism I is the simple mechanism involved in plastic collapse of a regular honeycomb structure [34].

Note that plastic collapse strength is proportional to the square of the relative density, which is consistent with the classical relationships for strength of bending-dominated cellular structures [34]. By finding the lowest calculated upper bound strength, numerically or algebraically, it is possible to determine which mechanism controls collapse behavior in any specific case. Over the entire admissible range of \(\eta > 0, 0 \leq \gamma \leq 0.5\), mechanisms I-VIII yield the minimum collapse load at certain loading/geometry combinations, and therefore are the dominant mechanisms over certain ranges of \(\eta, \gamma\). (Mechanism IX never yields the minimum collapse load and thus is not dominant under any biaxial loading or honeycomb geometry.) Under uniaxial loading in the x direction, only four mechanisms, I, II, IV and V are dominant. Similarly, for uniaxial loading in y direction, the three dominant mechanisms are I, II and III over entire admissible range of \(\eta\) and \(\gamma\).
Table 3-1 Dominant deformation modes for the plastic collapse of first order hierarchical honeycomb under uniaxial loading in x and y directions. The corresponding plastic collapse loads in terms of geometrical parameters $\gamma$ and $\eta$ are also given for each mode.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Independent Basis Mechanism</th>
<th>Plastic Collapse Load in x Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\frac{S_c^x}{\sigma_{ys}} = \frac{1}{4</td>
<td>1 - \cot\theta/\sqrt{3}</td>
</tr>
<tr>
<td>II</td>
<td>$\frac{S_c^x}{\sigma_{ys}} = \frac{\eta^2}{2</td>
<td>1 - \cot\theta/\sqrt{3}</td>
</tr>
<tr>
<td></td>
<td>$\frac{S_c^x}{\sigma_{ys}} = \frac{3\eta^2}{</td>
<td>\sqrt{3}\cot\theta * (0.25 + \gamma) - 0.75</td>
</tr>
</tbody>
</table>
\[ \frac{|S^x_c|}{\sigma_{ys}} = \frac{3\eta^2}{\sqrt{3}\cot\theta \cdot (0.25 - \gamma) + 6\gamma - 0.75 \cdot (1 + 2\gamma \cdot (2\eta - 1))^2 \left(\frac{\rho}{\rho_s}\right)^2} \]

\[ S^y_c = \cot\theta / \sqrt{3} * S^x_c \]

\[ \frac{|S^x_c|}{\sigma_{ys}} = \frac{3\eta^2}{4\gamma \cdot (1 + 2\gamma \cdot (2\eta - 1))^2 \left(\frac{\rho}{\rho_s}\right)^2} \]

\[ S^y_c = \cot\theta / \sqrt{3} * S^x_c \]
Table 3-2 Dominant deformation modes for plastic collapse of first order hierarchical honeycomb under biaxial loading in x and y directions. The corresponding plastic collapse loads in terms of geometrical parameters \( \gamma \) and \( \eta \) are also given for each mode.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Dominant Deformation</th>
<th>Expression for ( \frac{S_c^x}{\sigma_{ys}} )</th>
<th>Expression for ( S_c^y )</th>
</tr>
</thead>
</table>
| I    | \( \mathbf{VI} \propto 12\mathbf{II} - 3\mathbf{III} + \mathbf{IV} \) | \[
\frac{S_c^x}{\sigma_{ys}} = \frac{9\eta^2}{8\sqrt{3}\cot\theta \cdot (0.25 - \gamma) + 3\gamma/2 - 0.75 \cdot (1 + 2\gamma \cdot (2\eta - 1))} \left( \frac{\rho}{\rho_s} \right)^2
\] | \( S_c^y = \cot\theta / \sqrt{3} \cdot S_c^x \) |
| II   | \( \mathbf{VII} \propto -4\mathbf{II} + \mathbf{III} \) | \[
\frac{S_c^x}{\sigma_{ys}} = \frac{\eta^2}{\sqrt{3}\cot\theta \cdot (0.25 - \gamma) + 2\gamma - 0.75 \cdot (1 + 2\gamma \cdot (2\eta - 1))} \left( \frac{\rho}{\rho_s} \right)^2
\] | \( S_c^y = \cot\theta / \sqrt{3} \cdot S_c^x \) |
| III  | \( \mathbf{VIII} \propto 3\mathbf{II} - \mathbf{III} \) | \[
\frac{S_c^x}{\sigma_{ys}} = \frac{9\eta^2}{4\sqrt{3}\cot\theta \cdot (0.25 + \gamma/2) + 3\gamma/2 - 0.75 \cdot (1 + 2\gamma \cdot (2\eta - 1))} \left( \frac{\rho}{\rho_s} \right)^2
\] | \( S_c^y = \cot\theta / \sqrt{3} \cdot S_c^x \) |
\[
\frac{S_c^x}{\sigma_{ys}} = \frac{9\eta^2 - 4\sqrt{3}\cot\theta \times (0.25 - \gamma/2) + 9\gamma/2 - 0.75 \times (1 + 2\gamma \times (2\eta - 1))}{\left(\frac{\rho}{\rho_s}\right)^2}
\]

\[S_c^y = \cot\theta/\sqrt{3} \times S_c^x\]
Figure 3-4 (a-d) Deformed configurations of plastic collapse for hierarchical honeycomb according to the mechanisms I-IV, involving different plastic hinge locations marked by red bullets.

3.3 Finite Element Simulation of the First and Second Order Honeycomb

Finite element simulations were carried out to confirm the analytical results of the previous section. A unit cell of the structure was modeled and meshed using 2-node cubic beam elements (B23) in the ABAQUS finite element package. Two kinds of study were performed: (a) Pre-selecting locations for plastic hinges according to each specific deformation mechanism,
thereby evaluating its associated collapse load; (b) Simulating plastic collapse of the structure with plastic hinge locations unspecified, and finding a plateau in the load. We limited our simulation to uniaxial loading in either x or y directions. The unit cell was subjected to displacement-controlled compression in the direction of loading, with free expansion in the transverse direction. Strength was defined as the stress associated with the level plateau in the force-displacement curve. In the finite element results presented in this paper, the elastic modulus and Poisson ratio were taken as $E = 200$ GPa and $\nu = 0.3$, respectively. Each beam was meshed with 100 elements.

3.3.1 FE Simulation of Individual Plastic Collapse Mechanisms

To model each individual plastic collapse mechanism we constrained the structure to plastically deform at certain hinges specified for that mechanism only. To achieve this, individual elements at the desired hinge locations (just one or two out of 100 elements along that beam) were given elastic-perfectly plastic bending relations while the remaining elements of that beam were specified as linear elastic with the same modulus. By increasing the elastic modulus of the wall material, we approximated rigid-plastic beam bending behavior. This permitted direct comparison to the upper bound analytical result for that mechanism.

3.3.2 FE Simulation of the Cellular Structure Plastic Collapse

Computed deformation (without mode pre-selection) was also obtained by finite element analysis. The simulations were analogous to the simulations described in the previous section except that elastic / perfectly-plastic bending behavior was assigned to all elements.

3.4 Results: Plastic Collapse Under Uniaxial x, y Loading
The above-described analytical and numerical models were used to estimate the plastic collapse strength under uniaxial loading in the x and y directions. The values of collapse strength in each direction, after being normalized by the collapse strength of regular honeycomb of equal density, are denoted by $S_{c x}$ and $S_{c y}$. (Note that both the x and y collapse strengths of regular honeycomb equal $S_c = 0.5 \sigma_{ys} (\rho / \rho_s)^2$ [34].)

The normalized collapse strength of the uniform thickness ($\eta = 1$) first order hierarchical honeycomb structure in x and y directions is shown as a function of $\gamma$ in figures 3-5a and 3-5b, respectively. These graphs present the results of four kinds of analysis:

- The solid lines show the analytical upper bound strengths of each named deformation mechanism shown in tables 3-1 and 3-2 (i.e. plots of the strength expression derived for each table entry.)
- The square points represent constrained FEA (section 4.1), and fall perfectly on the solid analytical lines of the modes they represent. This agreement of manual analysis and FEA for the same plastic hinges implies that both approaches were carried out without error.
- The dashed line shows the analytical lower bound strength based on an elastic moment distribution. It is nowhere above a solid upper bound curve. Over the range of $0 \leq \gamma < 0.36$ for x loading, and $0 \leq \gamma < 0.32$ for y loading it falls exactly on the upper bound curve for deformation mode I – such equality of upper and lower bound calculations implies an ‘exact’ calculation of strength. But relative to deformation modes III and VII, the lower bound result is well below the upper bound result. As explained earlier, we anticipate that when lower and upper bound disagree, the upper bound is more likely correct.
Circular points represent ‘unconstrained’ finite element results from simulating the actual structural response (section 4.2). It may be observed that these numerical results fall precisely on the lowest upper bound curves.

In summary, we take both the analytical upper bound method (when the deformation mode with least required load for the given boundary displacement is selected), and the unconstrained finite element analysis, as yielding equivalent accurate collapse strength determinations.

These results show that when \( \eta = 1 \), the structure fails according to deformation mode I over the range \( 0 \leq \gamma < 0.375 \) for x loading and \( 0 \leq \gamma < 0.35 \) for y loading. For greater \( \gamma \) values, mode IV is controlling for x loading, and III for y loading. Note that the values of collapse strength according to some of the mechanisms in tables 3-1 and 3-2 are too great to appear in the figures as currently scaled. That is because the collapse strengths according to mechanisms I and II approach infinity as \( \gamma \) approaches 0.5, since in each case, the displacement of the point on which the external force \( F \) acts (see figure 3-3b) approaches zero in the relevant (horizontal or vertical) direction. For other mechanisms, the computed strength approaches infinity at different values of \( \gamma \).

Compared to regular honeycomb, the plastic collapse strength of 1\textsuperscript{st} order hierarchical honeycomb of uniform thickness shows a maximum normalized value of 1.3 and 1.15 in uniaxial loading in x and y directions, occurring at \( \gamma = 0.375 \) and \( \gamma = 0.35 \), respectively.

Figure 3-6 introduces the effects of thickness variation (i.e. varying \( \eta \)) on uniaxial x and y strength, as calculated by upper bound analysis. The plots represent the lower envelope of
strength for the previously defined mechanisms, over thickness ratio \(0 < \eta \leq 2\) and length ratio \(0 \leq \gamma \leq 0.5\). The strength is again normalized by that of regular honeycomb of equal density.

Under uniaxial loading in the x direction (figure 3-6a), the four mechanisms I, II, IV and V are the dominant mechanisms over the entire range of \(\eta\) and \(\gamma\) – this was observed by plotting the upper bound strength expressions in the finite domain, and then was proved by algebraic comparison of the derived strength expressions. The five boundary segments denoted by A, B, C, D and E bordering the five regions are found by equating the strength expressions of adjoining deformation modes to give:

\[
\begin{align*}
A: & \quad \gamma = 3/10 \quad 0 < \eta < 1/\sqrt{2} \\
B: & \quad \eta = 1/\sqrt{2} \quad 0 < \gamma < 3/10 \\
C: & \quad \eta = \left(\frac{2\gamma}{3-6\gamma}\right)^{0.5} \quad 3/10 < \gamma < 3/8 \\
D: & \quad \gamma = 3/8 \quad 0 < \eta < 1 \\
E: & \quad \eta = \left(\frac{8\gamma-1}{8-16\gamma}\right)^{0.5} \quad 5/16 < \gamma < 0.5
\end{align*}
\]

The maximum normalized plastic collapse load, \(\tilde{S}_c^x = 1.6\), occurs at the optimal values of \(\gamma = 0.3\) and \(\eta = 1/\sqrt{2} \approx 0.71\), where curves A, B and C join. Along the \(\eta = 0.71\) line, the normalized plastic collapse load of structure is greater than 1 over a wide range of \(0 < \gamma < 0.4\). This contrasts with the uniform thickness structure (figure 3-5a) where for \(0 < \gamma < 0.3\) and \(0.42 < \gamma < 0.5\) the collapse load of the structure is below 1.
For y direction uniaxial loading (figure 3-6b), the dominant mechanisms of plastic collapse are I, II and III. The relationships defining the three border lines of the dominant mechanisms, as denoted by F, G, and H, can be derived as:

F: \[ \gamma = 0.25 \quad 0 < \eta < 1/\sqrt{2} \]

G: \[ \eta = 1/\sqrt{2} \quad 0 < \gamma < 0.25 \]

H: \[ \eta = \left(\frac{4\gamma + 1}{-16\gamma + 8}\right)^{1/2} \quad 0.25 < \gamma < 0.5 \]

The maximum normalized plastic collapse load, \( S_c^y = 1.37 \), occurs at the optimal values of \( \gamma = 0.25 \) and \( \eta = 0.71 \) at the meeting point of curves F, G and H.

Along the \( \eta = 0.71 \) line, the normalized y-direction plastic collapse strength exceeds 1 for honeycombs with \( 0 < \gamma < 0.35 \), compared to the uniform thickness structure (i.e. \( \eta = 1 \)) where in the range of \( 0 < \gamma < 0.3 \) and \( 0.39 < \gamma < 0.5 \) the normalized plastic collapse strength of the structure is less than 1.

These x, y strength results show that the collapse behavior of the hierarchical honeycomb can be improved by introducing different thickness ratios (i.e. departing from \( \eta = 1 \)). The collapse strength of hierarchical honeycombs with thickness variation in the x and y directions can be increased almost 23 and 22 percent, respectively, compared to the hierarchical honeycomb of uniform thickness.
Figure 3-5 Normalized plastic collapse strength of uniform thickness hierarchical honeycomb under uniaxial loading in (a) x and (b) y directions as a function of length ratio, $\gamma$. The plastic collapse load is normalized by that of regular honeycomb of same density. The actual strength is on the curves marked by circles (where least upper bound happens to match unconstrained FEA).

Figure 3-6 Normalized plastic collapse strength for 1st order hierarchical honeycomb under uniaxial loading in (a) x and (b) y directions, as determined analytically from upper bound
analysis. Strength is normalized by that of regular honeycomb of same density and plotted as a function of length ratio, $\gamma$, and thickness ratio, $\eta$. The numbered areas correspond to different (least) upper bound mechanisms from table 3-1.

3.5 **Results: Plastic Collapse Under Biaxial x, y Loading**

The foregoing results were for uniaxial stress in either the x or y direction. However our analytical upper-bound calculations (which we take to be exact) also encompass simultaneous application of differing $S^x$ and $S^y$. As before, the values of biaxial collapse strength in x and y directions are normalized by the x and y uniaxial collapse strength of regular honeycomb of equal density, respectively.

For particular values of $\gamma$, $\eta$ the least analytical upper bound strength can be plotted on a failure locus in $S^x$, $S^y$ space. Figure 3-7 gives the convex normalized collapse boundary for 1st order hierarchical structure, which are found by varying $\theta$ in the equations given in table 3-1, and constructing the innermost boundary of their union. Curves are given for several values of $\gamma$ (0 = no hierarchy, 0.25, 0.31, 0.5) and two values of $\eta$ (1 = uniform thickness and 0.71.)

The plastic collapse surface of regular honeycomb structure ($\gamma = 0$) forms two parallel 45° lines, because only the difference in $S^x$, $S^y$ causes bending of the beams. As expected, at $\eta = 0.71$ the plastic collapse surface is wider in all directions than the case of $\eta = 1$. Each facet of the plastic collapse “polygon” for any given value of $\gamma$ corresponds to a mechanism from table 3-1, but to minimize clutter these have not been indicated. (For example, for the first order hierarchical honeycomb with $\gamma = 0.25$ and $\eta = 1$, the mechanisms [I,VI, VIII, I, VI, VIII, I] are dominant over entire range of biaxial loading, starting from the horizontal axis intercept and continuing counter clock-wise around the polygon to reach the starting point.)
Hierarchical honeycombs demonstrate a bending dominated behavior under any biaxial loading. This is obvious from a simple geometrical argument: In a hierarchical honeycomb, every other vertex of each introduced hexagon is a junction of only two non-collinear beams, which are unable to transmit any load without bending. A triangular rather than hexagonal refinement scheme would improve the strength of regular honeycomb, without introducing such additional weaknesses.

Figure 3-7 Plastic collapse surface of 1st order hierarchical honeycomb under biaxial loading in horizontal and vertical directions for different values of $\gamma$ and (a) $\eta = 1$ and (b) $\eta = 0.71$. The plastic collapse load is normalized by that of regular honeycomb of same density.

3.6 Results: Stiffness-Uniaxial Collapse Strength

Using the elastic analysis method described briefly in 3.1, the effective Young’s Modulus was evaluated for first order hierarchy including thickness variation. In Fig. 3-8, we have plotted analytically determined (upper bound calculation) plastic collapse strength versus modulus.
Plastic collapse strength and stiffness were normalized by the plastic collapse strength and stiffness of regular honeycomb of equal density. The plots are formed by superposing a large number of space curves with coordinates $E$ and $S^x$ or $S^y$, each representing a different value of $\eta$ while $\gamma$ is varied over its entire feasible range ($0 \leq \gamma \leq 0.5$). Figure 3-8a shows the $E$, $S^x$ view while figure 3-9b shows the $E$, $S^y$ view. Note that when $\gamma$ reaches its limits, there is no influence of $\eta$, so all the different space curves emerge from points representing $\gamma = 0$ and 0.5. As is seen most clearly for figure 3-8b, the curve for $\eta=1$ emerges from $\gamma=0$, switches to a different mode at $\gamma=0.35$, and ends at $\gamma=0.5$.

The highest achievable value of normalized modulus (observable in both figures) is 2.15, obtained at $\gamma \equiv 0.30$ and $\eta = 0.76$. At that point the x and y normalized strengths are 1.46 and 1.21, respectively. The greatest x normalized uniaxial strength is 1.60, and at that point the normalized modulus is 2.14, and the y normalized strength is 1.17. In contrast the greatest y normalized strength is 1.37, and at that point the modulus is 1.93 and the x strength is 1.37. Therefore, if we ask for equal x and y strength, the greatest value seems to be 1.37, where modulus is 1.93. The presented graphs show that a wide range of in-plane stiffness and uniaxial strength values can be obtained for first order hierarchical honeycombs by varying the geometrical parameters, $\gamma$ and $\eta$. 
3.7 Conclusions

By substituting smaller hexagons at all three-edge nodes parallel to the underlying grid of a regular hexagonal honeycomb structure, a novel class of hierarchical honeycombs is created. Hierarchical honeycombs with uniform wall thickness were recently shown to possess enhanced specific stiffness (up to two times considering uniform beam thickness, \( \eta = 1 \)) with respect to regular honeycomb [5]. To understand strength / stiffness tradeoffs in introducing structural hierarchy, in the current study the plastic strength of 1st order hierarchical honeycomb was quantified as a function of relative density and the substitution parameters using analytical models. Finite element simulations were also carried out to establish the validity of the proposed analytical models. Depending on the geometry, the resulting structure can exhibit many possible
relative stiffness-strength properties compared to the regular honeycomb of same mass, including (stiffer/stronger), (stiffer/weaker), and (more compliant/weaker) properties. A significant unfinished step in this investigation is to expand it to arbitrary in-plane loadings, i.e. including \( \tau_{xy} \) as well as \( \sigma_x \) and \( \sigma_y \), and determining the resulting 3-D plastic collapse surface. Exploring honeycombs with higher order hierarchy may expand the range of material properties that can be achieved by tailoring their topology.

Recent advancement in active materials has provided new opportunities to create highly-responsive structural systems that can be used in a wide range of applications ranging from robotics to surgery [35-39]. Cellular structures capable of altering their topology based on the loading condition and external stimuli would allow development of new generation of active materials. The findings of this study suggest new avenues for the development of novel materials and structures with desirable and perhaps actively tailorable properties.
Chapter 4: Plastic Collapse of Honeycombs under a General Stress

State
4.1 Introduction

The strength of low density cellular materials, as used in sandwich panels, is a subject of continuing interest. The current state of the literature on plastic collapse of cellular materials is limited to structures subjected to simplified loading conditions such as uniaxial, biaxial or shear loading applied at special orientations [40-45]. The method presented in this article allows calculation of plastic collapse strength under arbitrary states of stress, by numerical and algebraic analysis of a structural unit cell. Although the proposed approach is applicable to both 2D and 3D tessellated cellular structures, here we present the approach only in a 2D form for a space-filling regular network of beams. To illustrate the method, a hexagonal network (honeycomb) with six-fold rotational symmetry is considered. We also analyze the first iteration of the honeycomb structure in a hierarchical refinement scheme in which all three-edge nodes are replaced with smaller, parallel hexagons defined by size ratio $\gamma$. The resulting structure, also maintaining a microscopic six-fold symmetry, is called first order hierarchical honeycomb [42, 46], see figure 4-1. We begin our analysis with 3-fold symmetric definitions of both microscopic and macroscopic stress and strain. We then give a straightforward procedure for minimizing plastic work of our rigid-plastic beam structure, in the cases both of imposed strain and imposed stress. Lastly we exploit the observed unit-cell deformation patterns to derive analytical expressions for strength, to permit efficient computation and plotting.
To carry out the analyses for regular or hierarchical honeycombs, we select a unit cell (with associated tractions and displacements) which tiles the plane to represent the loaded lattice structure. The structural unit cell for our hexagonal-based patterns encompasses one vertex of the original honeycomb network, out to the midpoints of the original hexagon sides (a distance $L_0/2$), see figure 4-2. The area associated with the unit cell is a triangle joining the three hexagon center-points that surround this vertex, with area $3\sqrt{3}L_0^2/4$. The general state of stress is expressed in terms of its normal components in the three material directions $a = 0^\circ$, $b = 120^\circ$
and \( c = 240^\circ \): \( \sigma_{aa}, \sigma_{bb}, \sigma_{cc} \). Given those three normal components, the \( xy \) stress tensor can be written:

\[
\begin{bmatrix}
\sigma_{xx} & \tau_{xy} \\
\tau_{xy} & \sigma_{yy}
\end{bmatrix} =
\begin{bmatrix}
\sigma_{aa} & \frac{\sigma_{cc} - \frac{\sigma_{bb}}{\sqrt{3}}}{\frac{2}{3}} \\
\frac{\sigma_{cc}}{\sqrt{3}} & \sigma_{bb} - \frac{\sigma_{aa}}{3}
\end{bmatrix}
\]

where \( x \) and \( y \) axes are taken along the so-called armchair (or ribbon) and zigzag (or transverse) directions. The relation between \( xy \) stress components and points in \( \sigma_{aa}, \sigma_{bb}, \sigma_{cc} \) space will be described geometrically when discussing the results.

Figure 4-2 (a) Schematic of hierarchical honeycomb where a unit cell of the structure is marked by red lines. (b) Free body diagram of the structural unit cell.

We then examine the allowable external loads at numbered points 1, 4, 7 of the unit cell. First we argue that there are no moments applied at these points: the \( 180^\circ \) rotational symmetry of the tessellated structure (and trivially the components of microscopic stress) means that any upwards curvature at such a point must become downwards curvature after the rotation, and the only way these can be equal is to have the value zero. Then, using the vertical cut line \( \Delta_a \) which intersects horizontal sides with a spacing \( L_0\sqrt{3} \), we deduce the value of radial force \( F_a \) to be

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\[ \sigma_{aa} L_0 \sqrt{3}, \text{ and similarly for radial directions } b, c. \] Note also that the arbitrary radial forces \( F_a, F_b, \) \( F_c \) will not be in equilibrium so there must be transverse forces \( G_a, G_b, G_c, \) defined as positive counterclockwise about the origin. Successively taking moments of forces about pairwise intersections of \( G_a, G_b, G_c, \) we find \( G_a = (F_c - F_b)/\sqrt{3}, \) and cyclically.

Next, we consider relations between macroscopic strain and relative displacements of the unit cell boundary points. Given arbitrary radial and tangential displacements of points 1, 4, 7, we can use rigid body displacements and rotation to place the deformed unit cell uniquely in a canonical position with the boundary points 1, 4, 7 still on the \( a, b, c \) lines. In that unique placement, the boundary point canonical radial displacements along the \( a, b, c \) lines are named \( \delta_a, \delta_b, \delta_c, \) where the segments 1-2, 4-5, 8-7 are generally no longer parallel to those lines. Since the boundary loads are in equilibrium, the introduced rigid body displacements and rotations do not affect the network.

Given \( \delta_a \) alone, the strain is uniaxial along \( a. \) Its magnitude is the change in the unit cell \( x \) dimension divided by the original unit cell \( x \) dimension \( 3L_0/4, \) in other words \( \varepsilon_a = 4\delta_a/3L_0. \) Purely uniaxial strains in all three directions can be superposed to define a general \( xy \) strain tensor:

\[
\begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} \\
\varepsilon_{xy} & \varepsilon_{yy}
\end{bmatrix} =
\begin{bmatrix}
(4\delta_a + \delta_b + \delta_c)/3L_0 & \sqrt{3}(\delta_c - \delta_b)/3L_0 \\
\sqrt{3}(\delta_c - \delta_b)/3L_0 & (\delta_b + \delta_c)/L_0
\end{bmatrix}
\]

With stress based on arbitrary \( F_a, F_b, F_c \) (which require appropriate associated transverse forces \( G_a, G_b, G_c \) for equilibrium), and strain based on arbitrary \( \delta_a, \delta_b, \delta_c, \) the contraction of stress with strain, times unit cell area, equals \( F_a\delta_a + F_b\delta_b + F_c\delta_c, \) namely the work done on the unit cell as it deforms. The transverse boundary forces do no network since transverse boundary displacements are zero.
4.2 Numerical Minimization of Plastic Work

According to the upper bound theory of plasticity, the actual internal deformations of a rigid-plastic structure subjected to given external displacements must minimize the internal plastic work \[33\]. To perform this minimization and thereby find the strength and the deformation pattern, a plastic work minimization code for frame structures of uniform, straight beams was created using MATLAB. End-loaded beams have the maximum bending moment, and therefore the possibility of a plastic hinge, only at their ends. Thus, for a rigid-plastic network of beams the degrees of freedom of the structure may be taken as the displacements \(x_i, y_i\) and rotations \(N_i\) of each node \(i\). The rotation of each beam can be defined uniquely from its end displacements as \(R_{ij} = \hat{u}_{ij} \times \vec{D}_{ij}\), where \(\vec{D}_{ij}\) is the vector displacement of node \(j\) relative to node \(i\), and \(\hat{u}_{ij}\) is the beam unit vector pointing from node \(i\) to node \(j\). Stretching is determined as \(\hat{u}_{ij} \cdot \vec{D}_{ij}\).

The stretching work per unit depth for a beam of thickness \(t\) and length \(L\) connecting nodes \(i\) and \(j\) is calculated as \(\sigma_Y t |\hat{u}_{ij} \cdot \vec{D}_{ij}|\). We suppress axial strain in our structure by elevating the axial collapse strength, \(\sigma_Y t\), to about \(1E6\) times the plastic limit moment divided by beam length, \(\sigma_Y t^2/4L\). This represents a limiting condition where the thicknesses of the beams are small compared to their length. Therefore, the dissipation of interest is that at plastic hinges, which exist wherever a beam rotates differently from an adjoining node. If node \(i\) is the junction of three beams with rotations \(R_{ij}, R_{ik}\) and \(R_{il}\) (as at nodes 2, 5 and 8), the plastic dissipation at node \(i\) can be calculated as \(W_i = M_o \times \left( |N_i - R_{ij}| + |N_i - R_{ik}| + |N_i - R_{il}| \right)\) where \(M_o\) is the plastic limit moment of the beam cross section. Fortunately, the nodal rotations minimizing dissipation at a three-beam joint of equal plastic hinge moment can be determined locally as

\[\hat{u}_{ij}\]
\[ N_i = median(R_{ij}, R_{ik}, R_{ii}) \]. For nodes that are junctions of just two beams of equal plastic hinge moment with rotations \( R_{ij}, R_{ik} \) (as for nodes 3, 6 and 9), the rotation of the node that minimizes the plastic dissipation is equal to the rotation of either beam (or to any rotation between those values), and \( W_i = M_o |R_{ij} - R_{ik}| \). Rotations of nodes 1, 4 and 7 are equal to rotations of beam 1-2, 4-5 and 7-8, respectively since there is no bending moment.

After eliminating nodal rotations as variables, for conventional honeycomb the unit cell with fixed radial displacements \( \delta_a, \delta_b, \delta_c \), has only two degrees of freedom: the x and y displacements of the center node. When a first order of hierarchy is added, the twelve \( x_i, y_i \) displacements of six internal nodes suffice to compute the plastic work. Collapse-strength analysis proceeds as follows: define three radial boundary displacements \( \delta_a, \delta_b, \delta_c \). Minimize the total dissipation over the internal degrees of freedom to find \( W_{min} \). Since \( W_{min} = F_a \delta_a + F_b \delta_b + F_c \delta_c \), (where the displacements are known), possible failure loadings \( (F_a, F_b, F_c) \) are points on a plane in \((F_a, F_b, F_c)\) space, with intercepts \( F_a = W_{min} / \delta_a \), etc. The envelope of sufficiently many of these planes constitutes the convex failure surface. The above statements apply equally to the stress / strain components \( \sigma_{aa} \) and \( \varepsilon_{aa} \), etc., since these are proportional to \( F_a \) and \( \delta_a \), etc..

Unfortunately this displacement-based approach is not well adapted to the somewhat different problem of defining a ratio between load components – a desired direction in radial force or normal stress space – and finding the loading magnitude along that direction that causes collapse. For determining the strength where normal stresses (or equivalently, \( F_a, F_b, F_c \)) are applied to the structural unit cell, a different approach may be employed. Consider that forces \( F_a, F_b, F_c \), or more accurately forces exactly proportional to these (i.e., \( \lambda F_a, \lambda F_b, \lambda F_c \), where \( \lambda \) is a
dimensionless multiplier) are applied, and the value of $\lambda$ is sought at which failure occurs. This represents a force vector of known direction but indeterminate magnitude in $abc$ space, and the task is to determine the value of the multiplier that will put the vector tip on the failure surface. If we explored all choices for $\delta_a, \delta_b, \delta_c$, and used the linear relations of the previous paragraph to solve for $\lambda$ from each $W_{\text{min}}^P$, i.e. finding the $\lambda$ value putting the force vector on various failure-surface tangent planes, it is clear that the correct value of $\lambda$ must be the least to be found by this approach. That is because the minimum length of the vector along $F = (F_a, F_b, F_c)$, ending on a tangent plane, occurs when that plane is tangent at the intersection of $\lambda F$ with the failure surface. We therefore seek $\lambda_{\text{min}} = \min\left(\frac{W}{F_a \delta_a + F_b \delta_b + F_c \delta_c}\right)$, where the minimization is not only over the twelve internal variables $x_i, y_i$ as previously, but also over the additional variables $\delta_a, \delta_b, \delta_c$. Note, however, that the minimized quotient is homogeneous of zero order in the $\delta$ vector, so a unique answer requires some kind of normalization (for simplicity, we take $\delta_a = 1$). Once $\lambda_{\text{min}}$ is found, $\lambda_{\text{min}} F$ may be taken as a point on the failure surface.

To perform this minimization in MATLAB, where both the twelve interior displacements and the three external radial displacements are variables, the $fminsearch$ subroutine was used. Since the absolute value function $|f|$ (essential to the evaluation of plastic work) has slope discontinuities making convergence difficult, we adopted the expedient of approximating it by a sequence of smooth functions $|f|^\alpha$, where $\alpha$ is reduced incrementally from 2 to 1 in a hundred steps, and the result of each step becomes starting point for the next.

While the numerical results can be used for direct construction of the failure surface, in the following section we use the observed deformation mode to derive analytical strength formulas. These allow the entire failure surface to be constructed efficiently.
4.3 Closed-form Expressions of Plastic Collapse Strength

To develop analytical formulae for strength, the modes of deformation found numerically for different geometries and many loadings were examined. Each turned out to be a single degree of freedom bar-hinge mechanism, with only enough hinges to permit deformation. With an expectation that these modes are the only ones to be activated by any loading, we algebraically calculated the external work for each deformation mode under arbitrary loading, and used the equality of external and internal work to give a relation between load components at failure. The inner envelope of these algebraic relations provides the failure surface.

For a regular honeycomb structure the plastic deformation mechanisms involve one plastic hinge at the inner end of any of the three beams constituting the unit cell. Inset of figure 4-3a shows the deformed shape of a unit cell with one plastic hinge at the inner end of the beam along $0^\circ$. By equating the expression of plastic dissipation at a hinge, $\sigma_Y t^2/4. d\theta$, to external work, $|F_c - F_b|/\sqrt{3}.L_0/2. d\theta$, the external forces causing collapse must satisfy $|F_c - F_b| = \sqrt{3}/2. \sigma_Y. bt^2/L_0$. Repeating this procedure for the two other beams, the set of equations for plastic collapse of the regular honeycomb structure is that any one of $|F_a - F_b|$, $|F_a - F_c|$, $|F_b - F_c|$ equals $\sqrt{3}/2. \sigma_Y. bt^2/L_0$, which describe a hexagonal prism in the $abc$ stress space – see figure 4-3a. The infinite extent of the failure prism along the $(1,1,1)$ direction implies that the regular honeycomb structure will not collapse under equi-biaxial loading. This is reasonable, since regular hexagonal honeycomb can sustain such loading with purely axial forces, and axial failure of the beams is neglected in our analysis.
Figure 4-3 Plastic collapse surface in the abc stress space for (a) regular, and (b) hierarchical honeycomb ($\gamma_1 = 0.5$) obtained from upper-bound analysis. Stresses are normalized according to $\bar{\sigma} = (\bar{\sigma}/\sigma_y)/(t/L)^2$. The inset in (a) shows the plastic deformation mechanism for the unit cell of a regular honeycomb. The solid line ($\bar{\sigma}_a = \bar{\sigma}_b = \bar{\sigma}_c$) represents the equi-biaxial state of stress in the abc space.

The supplementary materials detail the numerically determined locations of plastic hinges in first order hierarchical honeycomb for all different loading conditions and geometrical parameters. Isometric views of the failure surfaces for zero and first order hierarchy, derived as the inner envelope of analytical strength criteria, are plotted in figure 4-3. The figure shows the collapse surfaces for regular ($\gamma = 0$) and hierarchical honeycombs ($\gamma = 0.5$) in the $\bar{\sigma}_{aa}, \bar{\sigma}_{bb}, \bar{\sigma}_{cc}$ space, where stress values are normalized according to $\bar{\sigma} = (\sigma/\sigma_y)/(t/L)^2$.

The geometry of 2D stress states in this space deserves mention. It is well known that the principal stresses $S_1 > S_2$ of arbitrary orientation, $\theta$, map onto Mohr’s circle in the $\sigma_{xx} - \sigma_{yy} - \tau_{xy}$ space. Similarly, they map onto a circle parameterized by $\theta$ in the $\sigma_{aa} - \sigma_{bb} - \sigma_{cc}$ space.
The origin of this circle is a distance $\sqrt{3}(S_1 + S_2)/2$ along the (1,1,1) hydrostatic axis. The circle lies in a deviatoric plane perpendicular to (1,1,1) direction and has a radius of $\sqrt{3}(S_1 - S_2)/\sqrt{8}$. When hydrostatic loading is tensile, the angle defined by three points: 1) the mapped point on the circle corresponding to state of stress; 2) the origin of the circle; and 3) the closest point on the circle to the positive $\sigma_{aa}$ axis, is equal to $-2\theta$, where $\theta$ is the angle in $xy$ space between the greatest principal stress ($S_1$) and $x$ (or $a$) axis. In general any fixed proportion between principal stresses $S_1, S_2$ (e.g., $S_2/S_1 = 0$ implies uniaxial stress) represents a cone from the origin, centered along the (1,1,1) axis with a vertex half angle of $\arctan(3/2\sqrt{2} \star |(S_2 - S_1)/(S_2 + S_1)|$).

Given the bounding algebraic relations between $abc$ stress components at failure, the failure condition for any desired state of loading is easily found. Figure 4-4 illustrates the strength as a function of loading direction of three different honeycomb structures under different ratios of principal stresses. The upper bound and lower bound estimates of strength are shown by red and blue lines, respectively. The inner envelope of the upper bound estimates of strength is marked by a black dashed line, which matches the results from numerical analysis. A lower bound strength is derived by elastic beam analysis, in which the maximum elastic moment reaches the collapse moment. This is a relatively weak bound because the moment distribution with collapse at multiple points usually doesn’t match the elastic moment distribution. The lower bound calculation of strength involving elastic analysis of loaded structure is not the scope of this paper and is further discussed in the appendix.

For plotting the figures, the intersections of cones of fixed principal stress ratio in the $abc$ space with the failure surface from each mechanism and the numerical method were obtained as
closed curves. The loading intensity at each point on the curve, corresponding to a loading
direction, is proportional to its distance from cone vertex (origin).

The numerical method can be used to obtain a full map of plastic collapse describing the
plastic collapse of the lattice structure under various loading conditions. Figure 4-5 shows
contours for normalized value of first principal stress causing plastic collapse in regular and
hierarchical honeycombs ($\gamma = 0.5$) as a function of loading direction, $\theta$, and ratio of principal
stresses, $S_2/S_1$, obtained from upper bound analysis. Taking into account the reflection and
rotational symmetries a stress orientation range of is sufficient for numerical trials. As the value
of $S_2/S_1$ approaches 1 (equi-biaxial loading) the collapse strength of regular honeycomb
increases by orders of magnitude since it is stretching-dominated.
\[ \sigma_1 = \left( \frac{\sigma_1}{\sigma_{33}} \right) / \left( t / L \right)^2 \]

\[ \gamma = 0 \quad \gamma = 0.4 \quad \gamma = 0.5 \]

\[ \sigma_2 = 0 \quad \text{uniaxial} \]

\[ \sigma_2 = 1/2 \sigma_1 \]

\[ \sigma_2 = \sigma_1 \quad \text{equi-biaxial} \]

\[ \sigma_2 = -1/2 \sigma_1 \]

\[ \sigma_2 = -\sigma_1 \quad \text{pure shear} \]
Figure 4-4 Plastic collapse maps of regular \((\gamma = 0)\) and hierarchical honeycombs \((\gamma = 0.4, 0.5)\) for different ratios of principal stresses. For each polar plot, the radial value denotes the magnitude of normalized first principal stress, \(\bar{\sigma}_1\), for plastic collapse and the angular value represents the first principal stress angle, \(\theta\).

![Figure 4-4](image)

Figure 4-5 Contours showing the magnitude of first principal stress causing plastic collapse in a (a) regular, and (b) hierarchical honeycomb \((\gamma = 0.5)\) for various ratios of principal stresses, \(\bar{S}_2/\bar{S}_1\), and loading direction, \(\theta\). Stresses are normalized according to \(\bar{\sigma} = (\bar{\sigma}/\sigma_y)/(t/L)^2\).

### 4.4 Summary and Discussion

A numerical scheme is proposed to obtain the plastic collapse mechanism and strength of cellular structures under arbitrary loading conditions. The method minimizes the plastic dissipation inside the unit cell of the structure under given state of loading, and is illustrated for regular and hierarchical honeycombs. The proposed method is also used to obtain closed form formulas of plastic collapse strength, enabling us to obtain a comprehensive plastic collapse surface for the lattice structure. Based on an underlying hexagonal network, regular and hierarchical honeycombs have six-fold symmetry in their properties. For linear properties (such as linear elastic behavior, or thermal conduction along network segments) this means isotropy [47]. But for nonlinear properties, it simply means six-fold symmetry. Taking into account the
reflection symmetry a stress orientation range of 30 degrees is sufficient for experiments or numerical trials. For the cases of γ = 0 and γ = 0.5, the failure surface of the material plotted in the abc stress state happens to exhibit six-fold symmetry; this results in a twelve-fold symmetry in plots of strength versus loading angle. According to the presented results, for regular honeycomb structure the uniaxial collapse strength at angle θ can be obtained by

\[
\left( \frac{\sigma_c}{\sigma_y} \right)_\theta = \frac{\sqrt{3}}{3 \cos \left( 2\frac{\theta'}{6} \right)} \left( \frac{t}{L} \right)^2
\]

where \(\theta'\) is the remainder (positive) of division of \(\theta\) by \(\pi/6\). The shear collapse strength of regular honeycomb structure can be expressed by

\[
\left( \frac{\tau_c}{\sigma_y} \right)_\theta = \frac{\sqrt{3}}{6 \cos \left( 2\frac{\theta''}{6} \right)} \left( \frac{t}{L} \right)^2
\]

where \(\theta''\) is the remainder (positive) of division of \((\theta - \pi/12)\) by \(\pi/6\).

The proposed method provides a simple and efficient way to estimate the collapse strength of tessellated structures under general loading condition, and thus have far reaching implications in designing lightweight and multifunctional structures. It should be noted that the results presented in this article take advantage of a three-fold symmetry which is inherent in regular and hierarchical honeycombs. If a lower-symmetry material is investigated (for example a vertically squashed hexagonal network), the same plotting conventions will be informative, although the symmetry will be lost.
Chapter 5: Instability of Regular, Chiral, and Hierarchical Honeycombs
5.1 Introduction

Low density cellular materials have found widespread application for energy absorption, structural protection, and as the core of lightweight sandwich panels. However, excessive compressive loads cause the cell walls to buckle, which limits the strength. Buckling in cellular structures becomes the dominant failure mechanism as the average density decreases [34]. On the other hand, microscopic instability patterns could be deliberately used as a technique for instant modification or induction of microscopic, periodicity-dependant structural or surface properties such as chirality [48-50], phononic and wave propagation [51-58], optical characteristics [59-62], modulated nano patterns [63], hydrophobicity [64-67], or generating macroscopic responses [68, 69] in periodic solids. We thus see value in predicting the instability of regular cellular structures under a general macroscopic state of stress.

Previous studies on the buckling of periodic cellular solids were largely numerical or experimental. Ohno et al. [70] suggested a numerical method to study the buckling of cellular solids subjected to macroscopically uniform compression using a homogenization framework of updated Lagrangian type. Triantafyllidis and Schraad [71] studied the onset of failure in honeycombs under general in-plane loading using finite element discretization of the Block wave theory. Abeyaratne and Triantafyllidis [72] studied the macroscopic instability of periodic solids associated with the loss of ellipticity in the incremental response of homogenized deformation behavior. A full-scale finite element study of intact and damaged regular honeycombs has been presented by Guo and Gibson [73]. Several experimental studies have concerned the buckling of different cellular structures including regular and circular honeycombs [45, 74-79].
The present analytical method is inspired by the early research of Gibson et al. [80] on the stability of regular hexagonal honeycombs under biaxial in-plane macroscopic stress using the beam-column solution [81]. Here in section 2, we make use of the beam-column method in a different approach, leading to analytical closed-form expressions of the non-localized microscopic buckling strength for 2D and 3D beam structures under a general in-plane loading. The method is a matrix representation of the beam-column relations, allowing simplified systematic calculation of the macroscopic buckling strength from the loads applied to the unit cell of the cellular lattice. In sections 3, the introduced approach is illustrated for the microscopic buckling prediction of five sample 2D periodic structures shown in figure 5-1. Among these, the periodic square grid is shown to buckle according to long-wave macroscopic buckling patterns in addition to microscopic patterns under a certain boundary condition [82], a phenomenon also observed in some 3D foams [83]. Therefore, the beam-column method is used in this case to calculate the limit of long-wave buckling strength of the square grid as the buckling wave-length approaches to infinity under an arbitrary loading condition. Conclusions are drawn in section 4.

**5.2 Methods**

A unit cell, also known as “a primitive cell” in classical physics, is the smallest structural unit by assembling which the undeformed geometrical pattern of the tessellated solid is recreated. The study by Geymond et al. [84] showed that eigenmodes of microscopic bifurcation in an infinite periodic solid can be repeated at wavelengths which are longer than a unit cell of the tessellated solid. These repeating patterns in the buckled structure are called the *Representative Volume Elements* (RVEs), which may be different under various macroscopic loading conditions applied to the structure [85]. In order to obtain the periodicity of a buckled
tessellated structure and the size of the RVE, various approaches including Bloch wave analysis [71, 86, 87], block-diagonalization method [88], eigenvalue analysis on RVEs of progressively increasing size [86, 89], full-scale finite element analysis [71, 73, 77, 90], and experimental investigations [45, 75, 76, 79] have been used.

The methods proposed here for obtaining closed-form expressions of macroscopic buckling strength are based on ‘assumed’ buckling modes, or simply, the overall buckled geometry of the RVE of the periodic solid through a pattern of nodal rotations and corresponding beam curvatures. Fortunately, the number of different buckling modes observed for a cellular structure under different macroscopic loadings is usually small. For instance, just two microscopic buckling patterns are found in the literature for square, triangular, and hexagonal honeycombs under various loading conditions.

**Beam-column End Moments**

The beam-column formula is a classical approach linking the end rotations of an axially loaded beam to its end moments. This approach has been used to obtain the buckling strength of cellular structures under simplified loading conditions including uniaxial and biaxial loadings [80, 91]. However, complexity arising from an arbitrary stress state impedes solving the beam-column equations, especially for larger RVEs with higher nodal connectivity. In this section, we present the characteristic nonlinear beam-column equations in a matrix form, allowing a more systematic calculation of the buckling strength. The *symbolic* calculation tool in MATLAB is then used to obtain closed-form expressions of buckling strength.
Figure 5-1 (a) Types of lattice structures analyzed by the beam-column matrix method: (I) Square grid (Section 3.1); (II) Triangular grid (Isogrid – Section 3.2). The angle $\theta$ gives the orientation of straight walls in the tri-chiral lattice; (III) Hexagonal honeycomb (Section 3.3); (IV) Hierarchical hexagonal honeycomb (Appendix C); and (V) Tri-Chiral honeycomb (Appendix D). (b) The angles of rotation and moments of the ends (positive when counter clock-wise) in a beam-column.

For a single beam connecting nodes $a$ and $b$ under axial compressive force $P$ and subjected to the two counter clock-wise end couples $M_a$ and $M_b$, the end rotations $\theta_a$ and $\theta_b$ (positive when counter clock-wise) can be obtained through the beam-column relations as [81]:

\[ \theta_a, \theta_b \]
\[ \theta_a = \frac{M_a l}{3EI} \psi(u) - \frac{M_b l}{6EI} \phi(u) \]

\[ \theta_b = \frac{M_b l}{3EI} \psi(u) - \frac{M_a l}{6EI} \phi(u). \]

Here, \( \phi(u) = 3(1/sin2u - 1/2u)/u \) and \( \psi(u) = 3(1/2u - 1/tan2u)/2u \) are nonlinear functions of the non-dimensional loading parameter \( u \) defined as

\[ u = \frac{l}{2} \sqrt{\frac{P}{EI}} \]

where \( P \) is the beam axial force, \( l \) is the beam length, and \( EI \) is the flexible rigidity of the beam in the plane of bending. (Note that \( u = \pi/2 \) corresponds to the pinned-pinned Euler buckling load. The functions \( \varphi \) and \( \psi \) can be approximated by even-order polynomials, climbing monotonically from 1 when \( u = 0 \) to about 1.80 and 1.44 respectively, when \( u = 1 \).) Denoting the rigid body rotation of the beam, \( \beta \), as an additional degree of freedom, and also taking into account three additional boundary conditions on the beam ends, the set of equation for the buckling of a single beam under axial loading can be expressed in the following matrix form

\[ \begin{bmatrix} A & B \end{bmatrix}_{5\times5} \begin{bmatrix} \frac{M_a l}{EI} \\ \frac{M_b l}{EI} \\ \theta_a \\ \theta_b \\ \beta_{ab} \end{bmatrix}_{5\times1} + \begin{bmatrix} B \end{bmatrix}_{5\times1} = 0 \]

Here, matrix \( A \) is termed the system’s characteristic matrix, and the condition \(|A| = 0\) gives the critical buckling load \( P_c \). Matrix \( B \) contains all algebraic terms appearing in the problem which do not explicitly include any beam end moments, nodal or beam rotations, and
thus could not be incorporated in matrix A. From the physical point of view, components of matrix B are ‘pre-buckling’ static terms which only result in a static deflection without buckling (e.g. an external transverse load applied to the tip of a simply supported beam). Matrix B does not affect the magnitude of the critical loads that correspond to a bifurcation.

For the case of a cellular structure’s RVE consisting of several beams and connecting nodes, the relationship between nodal rotations and end moments for each beam and also the boundary conditions can be assembled in the following general matrix form:

\[
\begin{bmatrix}
\mathbf{A} & \begin{bmatrix}
\frac{Ml}{EI} \\
\theta \\
\beta \\
\vdots
\end{bmatrix}
\end{bmatrix}_{n \times n} + \begin{bmatrix}
\mathbf{B}
\end{bmatrix}_{n \times 1} = 0
\]

where the condition \(|\mathbf{A}| = 0\) would give the relationship between the magnitudes of axial load for the beams inside the RVE of structure that cause it to become instable. Mathematically, this instability condition translates to the possibility of unlimited increase in values of nodal or beam rotations or beam end moments in the structure’s RVE at finite, fixed values of beam axial loads.

The presented method would be applicable for RVEs of the arbitrary volume; however, increasing the number of beams would increase the size of the characteristic matrix, which in turn makes it more complicated.

The following section shows how this matrix representation can be used for solving standard single beam buckling problems.

**Matrix Representations of Axially Loaded Beams**
In this section the matrix representation of the beam-deflection relations [81] in finding the buckling load of in different cases of an axially loaded beam are illustrated by examples. Figure 5-2 shows axially loaded beams with three different boundary conditions considered.

The beam-deflection relations (Equations (12) of manuscript) can be presented in the following matrix forms for each case:

Case A –

\[
\begin{bmatrix}
-\psi(u) & -\phi(u) & 1 & 0 & 0 \\
3 & 6 & 0 & 1 & 0 \\
-\phi(u) & -\psi(u) & 0 & 1 & 0 \\
6 & 3 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -4u^2 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
M_aL \\
EI \\
M_bL \\
EI \\
\theta_a \\
\theta_b \\
\beta \\
\end{bmatrix} + [B] = 0 , \quad (B = 0)
\]

where the first two rows correspond to beam-deflection relations for the beam-column, the third row satisfies the equilibrium of moments, and the last two rows satisfy the boundary conditions applied on the column. Setting \(|A| = 0\) yields \(1 - 4u^2\psi(u)/3 = 0\), leading to a critical buckling load of \(\pi^2EI/(2L)^2\).

Case B –

\[
\begin{bmatrix}
-\psi(u) & -\phi(u) & 1 & 0 & 0 \\
3 & 6 & 0 & 1 & 0 \\
-\phi(u) & -\psi(u) & 0 & 1 & 0 \\
6 & 3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
M_aL \\
EI \\
M_bL \\
EI \\
\theta_a \\
\theta_b \\
\beta \\
\end{bmatrix} + [B] = 0 , \quad (B = 0)
\]

where the first two rows correspond to beam-deflection relations for the beam-column, and the last three rows are conditions applied on the boundaries of the beam-column. Setting \(|A| = 0\) yields \(\psi = 0\), leading to a critical buckling load of \(20.18EI/L^2\).
Case C –

\[
\begin{bmatrix}
-\psi(u) & \phi(u) & 1 & 0 & 0 \\
3 & 6 & 0 & 1 & 0 \\
\phi(u) & -\psi(u) & 0 & 1 & 0 \\
6 & 3 & 0 & 0 & -4u^2 \\
1 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
M_aL \\
\frac{EI}{M_aL} \\
\frac{EI}{M_bL} \\
\theta_a \\
\theta_b \\
\beta \\
\end{bmatrix}
\]

+ \[B]\] = 0, \quad (B = 0)

where the first two rows correspond to beam-deflection relations on the beam-column, the third row satisfies the moment equilibrium, and the last two rows are conditions applied on the boundaries of the beam-column. Setting \(|A| = 0\) yields \(1 - 2u^2(\psi - \varnothing/2)/3 = 0\), leading to a critical buckling load of \(\pi^2EI/L^2\).

The critical buckling loads found above for cases A, B, and C are in agreement with the buckling load of an axially loaded column, \(F_{cr} = \pi^2EI/l_{eff}^2\), with the effective length \(l_{eff} = 2l, 0.7l\) and \(l\) for cases A, B and C, respectively.
Figure 5-2 Cases of an axially loaded beam subjected to three different boundary conditions.

5.3 Buckling of Regular, Chiral, and Hierarchical Honeycombs

As examples of the method outlined in section 2.1, the buckling stresses of some 2D cellular structures including regular, hierarchical, and chiral hexagonal honeycombs, square and triangular grids (see figure 5-1) are computed.

5.3.1 Buckling of Square Honeycomb

Wah [92] studied the stability of finite size rectangular gridworks for both in-plane and out-of-plane loadings. Under uniaxial compressive loading parallel to cell walls, the microscopic buckling strength of a square grid corresponding to the swaying mode is estimated as $S_c/E_s =$
\((\pi^2 / 12) * (t/L)^3\) by treating cell walls of length L as side-swaying columns with fixed slope at both ends [93]. This is an upper-bound estimate of the actual buckling strength of square honeycomb structure since it ignores structural rotational compliance that actually permits slope change. Fan et al. [91] calculated the uniaxial buckling strength of square honeycombs for the two numerically observed microscopic buckling patterns (non-swaying = mode I, swaying = mode II) using the beam-column method - see figure 5-3 for notation. They expressed the uniaxial buckling strength of the structure as 

\[ S_c / E_s = (1.292 \times \pi^2 / 12) * (t/L)^3 \]

and

\[ S_c / E_s = (0.76 \times \pi^2 / 12) * (t/L)^3 \]

for non-swaying and swaying modes, respectively, with the swaying strength being less than the upper bound solution from [93]. Also, the long-wave bifurcation in square honeycombs under in-plane loading has been previously studied through a two-scale theory of the updated Lagrangian type by Ohno et al. [82]. They showed that unlike hexagonal honeycombs, the long-wave buckling patterns in periodic square honeycombs could occur at a significantly lower load than the critical load for the microscopic buckling patterns. According to the analysis presented by Ohno et al. by increasing the wave-length of the buckling to infinity, the buckling strength under either uniaxial or biaxial (the structures buckles for the higher principal stress and ignores the lower) state of loading would approach \( S_c / E_s = (1/2) * (t/L)^3 \). The buckling wavelength for long-wave buckling patterns is shown to be dependent on the size of the finite structure, and also to the boundary conditions applied to a finite structure [82].

Here, for the first time we derive closed form relations for the microscopic buckling patterns in the square grid as well as the long-wave buckling patterns under general in-plane loading condition. For example, we will show that the equi-biaxial \( (S^x = S^y) \) the macroscopic
buckling strength \( S^x \) of square grid is \( S^x_c/E_s = 0.453 \times (t/L)^3 \); for pure shear \( (S^x = -S^y, S^x > 0) \) it is \( S^x_c/E_s = 0.493 \times (t/L)^3 \).

**Mode I (swaying):**

By making cuts through cell wall midpoints, internal reaction forces per unit depth in the beams’ axial and transverse directions, denoted \( P_x, P_y, \) and \( P_{xy} \) in figure 5-3, are obtained for a square grid of beam length \( L \) [34]:

\[
\begin{bmatrix}
P_x \\
P_y \\
P_{xy}
\end{bmatrix} = L \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\tau_{xy}
\end{bmatrix}
\]

(5)

Consequently, the beam non-dimensional axial loading parameters \( u_x \) and \( u_y \) are obtained as \( u_x = L/4 \times \sqrt{P_x/(EI)} = L/2 \times \sqrt{3\sigma_{xx}} \) and \( u_y = L/4 \times \sqrt{P_y/(EI)} = L/2 \times \sqrt{3\sigma_{yy}}, \)

where stresses are normalized according to \( \bar{\sigma} = (\sigma/E)/(\frac{t^3}{L^3}) \).

For the swaying mode of buckling due primarily to compressive \( y \) stress, the representative volume element consists of two different beam types OA and OB, and is shown in figure 5-3a. These beams experience end moments \( M_{OA} \) and \( M_{OB} \) at node O, respectively, and because of 180° rotational symmetry are moment free at outer nodes A and B. The set of beam-column and equilibrium relations of the different bar types OA and OB can be expressed in the matrix form given below. The first row expresses the beam-column relation for end O of OA, where there is a moment at O only, and beam rotation is zero. The second row, for beam OB again has a moment at O only, and a nonzero rotation \( \beta \). The third row corresponds to the
moment equilibrium of node O, and the last row expresses the moment equilibrium of beam OB about point O:

\[
\begin{bmatrix}
-\psi(u_x)/6 & 0 & 1 & 0 \\
0 & -\psi(u_y)/6 & 1 & -1 \\
1 & 1 & 0 & 0 \\
0 & -1 & 0 & -8u_y^2
\end{bmatrix}
\begin{bmatrix}
M_{OA} L/(EI) \\
M_{OB} L/(EI) \\
\alpha \\
\beta \\
P_{xy}L^2/(2EI)
\end{bmatrix}
= 0
\]

(6)

For buckling to occur, i.e. for \(\alpha\) and \(\beta\) to spontaneously taken on any value, the determinant of matrix \(A\) must vanish due to its dependence on the loads, i.e. \(u_x\) and \(u_y\). The relation for the threshold of in-plane instability in a square grid according to the swaying mode (mode I) of buckling is therefore:

\[
u_y^2 \left(1 - \frac{2u_x}{\tan(2u_x)}\right) - u_x^2 \left(\frac{2u_y}{\tan(2u_y)}\right) = 0
\]

(7a)

This relation is plotted in figure 5-4a, where it can be seen that x tension is very slightly protective. A second relation is required for the sway of structure due primarily to x-direction compression:

\[
u_x^2 \left(1 - \frac{2u_y}{\tan(2u_y)}\right) - u_y^2 \left(\frac{2u_x}{\tan(2u_x)}\right) = 0
\]

(7b)

Note that \(P_{xy}\) in Eq. (6) is irrelevant to the question of buckling. It could be treated as a forcing term, which leads to unbounded \(\alpha\) and \(\beta\) as matrix A approaches singularity.
Figure 5-3 (a) the microscopic buckling modes for the square grid. The representative volume element (RVE) in each mode is denoted by red lines. (b) notations for the beam and nodal rotation in the RVE. (c) free-body diagrams of the RVE beam-elements in modes I, II, and III.
Figure 5-4 (a) Buckling of square honeycomb under x-y biaxial loading according to swaying and non-swaying modes of buckling. (b) The buckling collapse surface for the square grid in the $\sigma_{xx}$-$\sigma_{yy}$-$\tau_{xy}$ stress space, allowing prediction of buckling strength under a general in-plane state of stress.

**Mode II (non-swaying):**

The same definitions of $P_x$, $P_y$, $P_{xy}$, $u_x$ and $u_y$ are used as given above. Figure 5-3b shows the non-swaying mode (mode II) with the structural RVE indicated by red lines, as well as the free body diagram of the RVE. Now the beam midpoints are not moment-free, so both $M_{OA}$ and $M_{AO}$ come into play. The set of beam-column relations can be expressed in the following matrix form. In this matrix, the first two rows express the moment equilibrium of beams OA and OB about point O. Rows three to six represent the beam-column relations for beams OA and OB. The last row expresses the moment equilibrium of node O.
Setting the determinant of the characteristic matrix to zero gives

\[ u_x \tan(2u_x) + u_y \cot(2u_y) = 0 \quad (9) \]

This condition implies either nonzero deformation with no forcing term, or unbounded deformation with nonzero forcing. Since it is symmetric in \( x \) and \( y \) (see figure 5-3b), no additional expression is needed.

**Mode III (long-wave):**

The long-wave mode of buckling was previously studied by Ohno and co-workers through finite element discretization of a two-scale theory of the updated Lagrangian type [82, 94, 95]. Ohno et al. evaluated the uniaxial onset stress of long-wave buckling for the square grid as \( \sigma_{bl}^{lw} / E_s = 0.5 \ast (t/L)^3 \). Here, the long wave buckling strength of a square grid under a general stress state is sought as the wavelength approaches infinity, using the beam-column method. Figure 5-3c shows a deformed structure according to this buckling pattern, where the RVE is defined as a cross-shaped unit connecting four adjacent beam centers. Note that under the long-wave mode shown in this figure all vertical lines equally deform. Also, the horizontal lines

\[
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & -8u_x^2 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & -8u_y^2 \\
\phi(u_x) & \psi(u_x) & 0 & 0 & 0 & -1 & 0 \\
\frac{12}{6} & \frac{\phi(u_x)}{6} & 0 & 0 & -1 & 1 & 0 \\
\frac{6}{12} & \frac{\phi(u_y)}{6} & 0 & 0 & -1 & 0 & 1 \\
1 & 0 & \frac{6}{12} & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
M_{OA} L/(EI) \\
M_{AO} L/(EI) \\
M_{OB} L/(EI) \\
M_{BO} L/(EI) \\
\alpha \\
\beta \\
\gamma
\end{bmatrix}
+ [B] = 0 \quad (8)
\]
deform periodically over \( x \) with the deflections at the mid-points of each segments of length \( L \) equal to zero. As a result, each vertical beam can be independently analyzed as an axially loaded vertical beam supported by simply supported (but free to translate horizontally) horizontal beams of length \( L \) which are welded to the vertical beam at their centers with a spacing of \( L \) along the height of the beam. As the buckling wavelength (along \( y \)) approaches infinity the distance between horizontal beams become smaller with respect to the wavelength, and thus, the effect of horizontal beams can be estimated through the beam’s analogy to an axially loaded beam on a foundation of distributed rotational springs. The following section details the differential relations governing the instability of an axially loaded beam on a distributed rotational spring foundation of intensity \( K_t \) per unit length (with dimension \([N]\)), where it is shown that the critical compressive buckling load equals

\[
P_{cr} = K_t
\]  

The goal here is therefore to obtain an equivalent rotational stiffness for the horizontal segments which are supporting vertical beams using the beam-column method. Once the effective rotational stiffness, \( K_t \), is obtained, the buckling load can be calculated using Eq. (10). Figure 5-3c shows a free body diagram of the RVE, where the central node is rotated by angle \( \beta \) and the vertical beam segment of length \( L \) has an equivalent rotation of \( \alpha \), as resisted by the moment \( M \) in the middle as shown in the figure. Since each half-beam has a moment-free end, the set of beam-column relations for the RVE is

\[
\alpha - \beta = \frac{ML}{12EI}\psi(u_y)
\]  

(11)
\[ \beta = \frac{ML}{12EI} \psi(u_x) \]

where \( u_x = L/4 \cdot \sqrt{P_x/EI} \) and \( u_y = L/4 \cdot \sqrt{P_y/EI} \). The equivalent rotational stiffness of the horizontal segments due to the swaying angle \( \alpha \) can therefore be calculated as

\[ k_t = \frac{M}{L\alpha} = \frac{12EI}{L^2} \frac{1}{\psi(u_x) + \psi(u_y)} \quad (12) \]

Substituting from Eq. (10) the relations between parameters \( u_x \) and \( u_y \) needed for the long wave instability of square grid based on the two variations along \( x \) and \( y \) are

\[ u_x^2 \left( \psi(u_x) + \psi(u_y) \right) = 3/4 \quad (13) \]

\[ u_y^2 \left( \psi(u_x) + \psi(u_y) \right) = 3/4 \]

These are mathematically identical to the relations obtained in Eqs. (7) for buckling of the square grid according to the \( x \) or \( y \) swaying mode of instability. This approach of the long-wave buckling strength to the swaying mode can be justified from the physical point of view. In fact by increasing the buckling wavelength in a structure with long-wave instable shape, shown in figure 5-3c, the deformation field corresponding to an arbitrary volume of \( n \times n \) cells \( (n < \infty) \) will be approach the uniform deformation field observed in the swaying mode of buckling.

**Analysis of a Beam on Rotational Spring Foundation**

In this section, the governing differential equation for an axially loaded beam on a foundation of rotational elastic springs is derived. Let’s assume that springs of rotational stiffness \( K \) [N.m] are repeated over length spans of \( \Delta x \) [m]. Note that in this case it is assumed that each
rotational spring only resists a change in the angle of the beam (from the horizon) and does not resist against the vertical or horizontal deflection of the beam. When the springs are considered close enough, their effect on the beam can be estimated as a distributed rotational spring over the length of the beam with the coefficient \( K_t = K/\Delta x \) [N]. Figure 5-5 shows the free body diagram of such a beam where vertical and horizontal reaction forces and also moments are applied to beam ends as shown in the figure. The equilibrium of moments is satisfied through the following equation

\[
M_0 + R_0 x - P_0 v(x) + \int_0^x k_t \frac{dv(x)}{dx} \, dx = M = EI \frac{d^2v(x)}{dx^2}
\]

The second derivative of the above equation leads to the following differential equation governing the beam deformation

\[
EI \frac{d^4v(x)}{dx^4} + (P - k_t) \frac{d^2v(x)}{dx^2} = 0
\]

The analogy of the above relation to the well-known relation governing the instability of an axially loaded beam with compressive force \( P \) (i.e. \( EI \frac{d^4v}{dx^4} + P \frac{d^2v}{dx^2} = 0 \)) implies that the effect of distributed rotational spring can be regarded as an axial tensional force of magnitude \( k_t \) superposed to the loaded beam. As a result, the critical load for the instability of a beam on a distributed rotational spring foundation of intensity \( k_t \) is equal to \( P_{cr} = k_t \) as the length of beam approaches infinity.
Figure 5-5 Free body diagram of an axially loaded beam on a rotational spring foundation.

**Result**

Figure 5-4a shows the stress curves corresponding to swaying, non-swaying, and long-wave modes of buckling in a square honeycomb under biaxial state of loading, where horizontal and vertical axes correspond to normalized normal stresses in $x$ and $y$, respectively, according to $\bar{\sigma} = (\sigma/E) / (t/L)^3$. The green curve denotes the non-swaying mode of buckling. The red lines correspond to the swaying microscopic buckling mode as well as the long-wave buckling mode. The equi-biaxial buckling strength of a square grid is estimated as $\bar{\sigma}_{bl}^x = \bar{\sigma}_{bl}^y = 0.453$. The results are verified by the FE eigenvalue analysis performed at full as well as RVE scales.

In figure 5-4b, the instability surface of the square honeycomb is presented, which is described by the inner envelope of buckling stresses corresponding to the two rotational variations of the swaying mode of buckling given by Eqs. (7). The presented plot allows prediction of strength according to microscopic and macroscopic modes of buckling for the structure under a general state of loading.

### 5.4 Buckling of Triangular Honeycomb
Wang and McDowell [93] estimated the buckling strength of a series of common cellular structure with a simplistic approach involving the equivalent beam length for cell walls of different periodic structures. They estimated the uniaxial buckling strength of triangular grid along any of the three cell wall directions to be \( \sigma_{bl}/E_s = \left( 2 \pi^2/3\sqrt{3} \right) \times (t/L)^3 \). Similar to the case of a square honeycomb, this is an upper-bound estimate of the actual buckling strength since it suppresses the rotations of the end nodes of the cell walls during buckling. Fan et al. [91] obtained a more precise estimation of uniaxial buckling strength of triangular honeycombs using the beam-column approach which allows for the rotation of the end nodes of cell wall during buckling. They expressed the uniaxial buckling strength along the cell wall direction \((x)\) and the perpendicular to cell walls \((y)\) as \( \sigma_{bl}/E_s = 2.543 \times (t/L)^3 \) and \( \sigma_{bl}/E_s = 2.876 \times (t/L)^3 \), respectively.

Here, for the first time we will provide a simple formula for the in-plane buckling of triangular grid under a general stress state for the first time.

**Mode 1**

For the sake of simplicity and also symmetrical results, the general state of in-plane macroscopic stress is expressed in terms of its normal components, \( \sigma_{aa}, \sigma_{bb}, \sigma_{cc} \), in the three in-plane material directions \( a = 0^\circ, b = 120^\circ \) and \( c = 240^\circ \) from the \( x \) axis as shown in figure 5-1. Given the normal components of stress in \( a \), \( b \) and \( c \) directions, the macroscopic \( xy \) stress tensor can be written as (see previous chapter)
\[
\begin{bmatrix}
\sigma_{xx} & \tau_{xy} \\
\tau_{xy} & \sigma_{yy}
\end{bmatrix} = \begin{bmatrix}
\frac{\sigma_{aa} - \sigma_{bb}}{\sqrt{3}} & \frac{\sigma_{cc} - \sigma_{bb}}{\sqrt{3}} \\
\frac{\sigma_{cc} - \sigma_{bb}}{\sqrt{3}} & \frac{2\sigma_{cc} + 2\sigma_{bb} - \sigma_{aa}}{3}
\end{bmatrix}
\] (14)

According to finite element analysis, the two buckling modes as indicated by modes I and II in figure 5-6 appear in the triangular grid structure under various loading conditions. Mode I of buckling is characterized by the equal rotation of all nodes in each row (i.e., along \(a\)), while adjacent rows have opposite rotations. Mode II of buckling is distinguished by zero rotation of nodes and beams in every other row of structure and the rotation of adjacent nodes in the remaining rows equally but in opposite directions. Note that mode shapes I and II shown in figure 5-6 are unique with respect to direction \(a\) and identical relative to \(b\) and \(c\) directions, so the entire collapse surface is defined by a three sets of mode shapes along the \(a\), \(b\) and \(c\) directions. Wang and McDowell [96] provided the internal forces per unit depth of beams in the equilateral triangular cell (isogrid) honeycomb of beam length \(L\)

\[
\begin{bmatrix}
F_a \\
F_b \\
F_c
\end{bmatrix} = L \begin{bmatrix}
\sqrt{3}/2 & -\sqrt{3}/6 & 0 \\
0 & 1/\sqrt{3} & -1 \\
0 & 1/\sqrt{3} & 1
\end{bmatrix} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\tau_{xy}
\end{bmatrix}
\] (15)

Then, using Eqs. (2),(14),(15) the following non-dimensional axial loading parameters \(u_a, u_b, u_c\) for the beams oriented along the \(a\), \(b\) and \(c\) directions

\[
u_a = \frac{\sqrt{5\bar{\sigma}_{aa} - \bar{\sigma}_{bb} - \bar{\sigma}_{cc}}}{3^{1/4}}, \quad u_b = \frac{\sqrt{5\bar{\sigma}_{bb} - \bar{\sigma}_{aa} - \bar{\sigma}_{cc}}}{3^{1/4}}, \quad u_c = \frac{\sqrt{5\bar{\sigma}_{cc} - \bar{\sigma}_{aa} - \bar{\sigma}_{bb}}}{3^{1/4}}
\] (16)

Here, stresses are normalized according to \(\bar{\sigma} = (\sigma/E)/(\frac{L}{L})^3\).
The free body diagram of an RVE of structure for mode I of buckling in a triangular grid is shown in figure 5-6a. There are no transverse forces acting on the beam ends since the structure is stretching dominated under the general stress state. Considering the beam-column relation for all three types of beam, the matrix form of the beam-column and equilibrium relations can be expressed as follows, where the first three rows are the beam-column relations for beams OA, OB, and OC, and the last row is to satisfy the moment equilibrium for central node O:

\[
\begin{bmatrix}
1 - \left(\psi(u_a) - \frac{1}{2}\phi(u_a)\right) & 0 & 0 \\
1 & 0 & -\left(\psi(u_b) + \frac{1}{2}\phi(u_b)\right) \\
1 & 0 & -\left(\psi(u_c) + \frac{1}{2}\phi(u_c)\right) \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\frac{M_A L}{3EI} \\
\frac{M_B L}{3EI} \\
\frac{M_C L}{3EI}
\end{bmatrix}
\]

+ \[B\] = 0 \quad (17)

Equating the determinant of the characteristics matrix to zero, the relation expressing the buckling of structure according to mode I of buckling can be obtained as

\[
u_b \cot u_b + u_c \cot u_c + \frac{u_a^2}{1 - u_a \cot u_a} = 0
\]

and considering the two other variations of this mode along \(b\) and \(c\) directions required to fully describe the buckling of triangular grid under a general loading
\begin{equation}
uc \cot \uc + ua \cot ua + \frac{u_b^2}{1 - u_b \cot ub} = 0
\end{equation}

(18b)

\begin{equation}
ua \cot ua + ub \cot ub + \frac{u_c^2}{1 - u_c \cot uc} = 0
\end{equation}

(18c)

where parameters \(ua\), \(ub\) and \(uc\) are given in Eq. (28).

Figure 5-6 (a) Modes I and II of buckling in triangular honeycomb, respectively. In mode I, beam types OB and OC have opposite moments at their ends while beam types OA have the same moments at their ends. In mode II, beam types OB and OC have zero rotation at one end while beam types OA have opposite moments at their ends. For each mode, the representative volume element (RVE) is denoted by red lines. (b) Notations for the beam and nodal rotation in the RVE. Bottom: Free-body diagrams of the RVE beam-elements in modes I and II.

Mode II
Free body diagram of RVE in mode II of buckling is shown in figure 5-6b. Under this mode the set of beam-column and equilibrium equation can be written in the following matrix form, where the first six rows represent the beam-column relations for beams OA, OB and OC, and the last row satisfies the condition for the equilibrium of node O

\[
\begin{bmatrix}
1 & -\psi(u_a) - \frac{1}{2}\phi(u_a) & 0 & 0 & 0 & 0 \\
1 & 0 & -\psi(u_b) & -\frac{1}{2}\phi(u_b) & 0 & 0 \\
0 & 0 & \frac{1}{2}\phi(u_b) & \psi(u_b) & 0 & 0 \\
1 & 0 & 0 & 0 & -\psi(u_c) & -\frac{1}{2}\phi(u_c) \\
0 & 0 & 0 & 0 & \frac{1}{2}\phi(u_c) & \psi(u_c) \\
0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\frac{M_{OA}L}{3EI} \\
\frac{M_{OB}L}{3EI} \\
\frac{M_{BO}L}{3EI} \\
\frac{M_{OC}L}{3EI} \\
\frac{M_{CO}L}{3EI}
\end{bmatrix}
= [B] + 0
\]

Equating the determinant of characteristics matrix to zero, the relation between components of stress for mode II of buckling equals

\[
u_a\cot u_a + \frac{u_b\cot u_b}{2} \left(1 + \frac{u_b\tan u_b}{1 - u_b \cot u_b}\right) + \frac{u_c\cot u_c}{2} \left(1 + \frac{u_c\tan u_c}{1 - u_c \cot u_c}\right) = 0
\]  

(20a)

and considering the two other variations this mode along \(b\) and \(c\) directions

\[
u_b\cot u_b + \frac{u_c\cot u_c}{2} \left(1 + \frac{u_c\tan u_c}{1 - u_c \cot u_c}\right) + \frac{u_a\cot u_a}{2} \left(1 + \frac{u_a\tan u_a}{1 - u_a \cot u_a}\right) = 0
\]  

(20b)
\[ u_c \cot u_c + \frac{u_a \cot u_a}{2} \left(1 + \frac{u_a \tan u_a}{1 - u_a \cot u_a}\right) + \frac{u_b \cot u_b}{2} \left(1 + \frac{u_b \tan u_b}{1 - u_b \cot u_b}\right) = 0 \]  

(20c)

Similar to the case of regular honeycomb lattice, the none post-buckling curvature in a series of horizontal beams in mode II of buckling requires, in effect, the moment throughout those beam to be small (i.e. \( M_{bo} \approx M_{co} \)). This condition is only satisfied when the characteristic matrix is symmetrical with respect to \( b \) and \( c \) direction (i.e. \( u_b = u_c \)), translating to a biaxial state of loading in the principal material directions \( x \) and \( y \). This is also supported by full-scale finite element analysis, where the second mode of buckling is observed under biaxial loading and is rapidly suppressed by increasing the \( xy \) shear stress component.

Result

Under a general state of loading, the macroscopic stresses required for the second mode of buckling given in Eqs. (30) are equal or greater than those for first buckling mode given in Eqs. (32). Under biaxial state of loading along material principal directions \( x \) and \( y \), relation (32a) can be simplified to relation (30a). As a result, under a \( xy \) biaxial macroscopic loading with first principal stress along \( y \) (i.e. \( \sigma_{xx} < \sigma_{yy} \)), the microscopic instability in triangular cells could be formed according to either first or second mode of buckling. In full-scale numerical trials, the choice of buckling mode under this loading condition is determined by the far field boundary conditions applied to the finite-size specimen. For the case of bi-axial loading with the first principal stress along \( x \) (\( \sigma_{xx} > \sigma_{yy} \)), the uniaxial mode of buckling is the preferred mode from comparing Eqs. (30) and (32). Figure 5-7a shows the curves corresponding to first and second modes of buckling in a triangular honeycomb under biaxial state of loading, where
horizontal and vertical axes correspond to normalized normal stresses in $x$ and $y$ directions, respectively according to $\tilde{\sigma} = (\sigma/E)/(t/L)^3$. The red line denotes the stress state corresponding to mode I of buckling along $a$. The green line corresponds to mode II as well as the two variations of the uniaxial mode occurring along $b$ or $c$ directions. The green and red lines intersect at the equi-biaxial state of macroscopic loading with the principal stresses equal to $\tilde{\sigma}_{xx} = \tilde{\sigma}_{yy} = -2.395$. Under a state of pure-shear described by compression along $x$ and tension along $y$ ($S_x = -S_y$, $S_x < 0$), the buckling strength ($S_x$) of triangular grid is given by $\sigma_{bl}^x/E_s = -2.099 \times (t/L)^3$. Also, under a state of pure-shear described by compression along $y$ and tension along $x$ ($S_x = -S_y$, $S_y < 0$), the buckling strength ($S_y$) of triangular grid is $\sigma_{bl}^y/E_s = -3.213 \times (t/L)^3$. The results are verified by the FE eigenvalue analysis preformed at the RVE scale which are in agreement with full-scale finite element results. In figure 5-7b, the instability surface corresponding to the triangular grid is presented, described by the inner envelope of buckling stresses corresponding to the three rotational variations of mode I of buckling given by Eqs. (30). The three edges observed in this figure correspond to the three biaxial states of loading along $x$-$y$, $x$-$z$ and $y$-$z$ directions, where the mode II of buckling is also possible. The presented 3D plot allows prediction of macroscopic buckling stresses for the structure under an arbitrary stress state.
Figure 5-7 (a) Buckling of the triangular grid under $xy$ biaxial loading according to modes I and II of buckling. (b) The buckling collapse surface of the triangular grid in the $abc$ stress space, allowing prediction of buckling strength under a general in-plane state of stress.

5.5 Buckling of Regular Hexagonal Honeycomb Structure

The nonlinear elastic response of regular hexagonal honeycomb has been previously studied by different researchers [45, 70, 71, 80, 97-99]. Gibson and Ashby [85] estimated the uniaxial buckling strength of the regular honeycomb structure in the cell wall direction ($x$ in figure 1) as $\sigma_{bl}/E_s = 0.22 \times (t/L)^3$ using the rotational spring approach. Later, Gibson et al. used the beam-column relations to obtain the buckling strength of the hexagonal honeycomb structure under biaxial state of loading along $x$ and $y$ directions [80]. They recognized two modes buckling, commonly referred to as uniaxial and biaxial, for elastic bifurcation of hexagonal honeycomb structure - see figures 5-8a and 5-8b. The large deformation of cell edges before elastic buckling was taken in account by Zhang and Ashby [97] to analyze the biaxial in-plane buckling of hexagonal honeycombs. Ohno et al. numerically analyzed the in-plane biaxial buckling of the regular hexagonal honeycomb using a homogenization framework of updated
Lagrangian type [70]. Triantafyllidis and Schraad [71] studied the onset of bifurcation in hexagonal honeycombs under general in-plane loading using finite element discretization of the Bloch wave theory. A third, more complex, flower-like mode, shown in figure 5-8c, is suggested for the buckling of regular hexagonal honeycomb structure [70]. This mode of buckling has been previously observed in experimental and numerical trials for hexagonal honeycombs with circular cells under equi-biaxial loading condition [75, 90]. In this mode of buckling, groups of six highly deformed cells surround almost intact central cells, and the central cells rotate uniformly in either the clockwise or the counter clockwise direction. Okumera et al. later showed that the flower-like mode does not occur as the first bifurcation under macroscopic biaxial compression control [99]. Here, for the first time we will obtain expressions of buckling strength for the uniaxial, biaxial, and flower-like modes of buckling under a general stress states.

**Mode I (uniaxial):**

Because of the minimum 3 fold symmetry of regular honeycomb structure, the general state of in-plane macroscopic stress is expressed in terms of its normal components, \(\sigma_{aa}, \sigma_{bb}, \sigma_{cc}\), in the three in-plane material directions \(a = 0^\circ, b = 120^\circ, c = 240^\circ\) from the \(x\) axis as shown in figure 5-1. The macroscopic \(xy\) stress tensor can be expressed in terms of the normal components of stress in \(a, b\) and \(c\) directions according to Eq. (14). Also the axial forces in the cell walls (per unit depth) oriented along \(a, b, c\) directions can be expressed as [34]

\[
\begin{bmatrix}
F_a \\
F_b \\
F_c
\end{bmatrix} = l \begin{bmatrix}
\sqrt{3} & 0 & 0 \\
0 & \sqrt{3} & 0 \\
0 & 0 & \sqrt{3}
\end{bmatrix} \begin{bmatrix}
\sigma_{aa} \\
\sigma_{bb} \\
\sigma_{cc}
\end{bmatrix}
\]  

(21)
where $l$ is the size of the hexagon side. The transverse forces in the cell walls of the hexagonal honeycomb can be obtained according to (see previous chapter)

\[
\begin{bmatrix}
G_a \\
G_b \\
G_c \\
\end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0 \\
\end{bmatrix} \begin{bmatrix}
F_a \\
F_b \\
F_c \\
\end{bmatrix}
\]  
(22)

For mode I (uniaxial) mode of buckling the matrix method discussed is section 2.1 is used. Consider the structural RVE consisting of the three beams OA, OB and OC oriented along $a$, $b$ and $c$ directions, respectively. The pre-buckling and post buckling configuration of the RVE are shown by red dashed lines and solid black lines, respectively. Taking into account the symmetry requirements in the buckled configuration of the structure, beam OA is under equal end moments, denoted by $M_a$, and beams OB and OC each are under opposite (with equal magnitude) end moments denoted by $M_b$ and $M_c$, respectively, as shown in figure 5-8a. The set of beam-column and equilibrium relations of the three different bar types OA, OB, and OC can be expressed in the following matrix form, where the first three matrix rows represent the beam-column relations on beams OA, OB, and OC, the fourth line corresponds to equilibrium of node O, and the last relation satisfies the moment equilibrium in beam OA.

\[
\begin{bmatrix}
-\psi(u_a) + \phi(u_a) \\
0 \\
0 \\
1 \\
\end{bmatrix} \begin{bmatrix}
\frac{3}{3} \\
\frac{6}{3} \\
\frac{6}{3} \\
\frac{6}{3} \\
\end{bmatrix} + \begin{bmatrix}
-\psi(u_b) \\
-\phi(u_b) \\
-\psi(u_c) \\
-\phi(u_c) \\
\end{bmatrix} \begin{bmatrix}
0 \\
1 \\
0 \\
1 \\
\end{bmatrix} \begin{bmatrix}
M_a L/EI \\
M_b L/EI \\
M_c L/EI \\
\alpha \\
\beta \\
\end{bmatrix} + [B] = 0
\]  
(23)
Using the symbolic toolbox in MATLAB to set \(|A| = 0\), the relation between \(u_a\), \(u_b\), and \(u_c\) expressing the instability of regular honeycomb structure according to uniaxial mode under a general loading is

\[
u_a \tan(u_a) - u_b \cot(u_b) - u_c \cot(u_c) = 0
\]

or equivalently in the \(abc\) stress space

\[
\sqrt{\sigma_{aa}} \tanh\left(\sqrt{3\sqrt{3}\sigma_{aa}}\right) + \sqrt{\sigma_{bb}} \coth\left(\sqrt{3\sqrt{3}\sigma_{bb}}\right) + \sqrt{\sigma_{cc}} \coth\left(\sqrt{3\sqrt{3}\sigma_{cc}}\right) = 0
\]

(25a)

where the bar on stresses means they are normalized according to \(\bar{\sigma} = (\sigma/E)/(t/L)^3\). Note that the uniaxial mode of buckling shown in figure 5-8a does not possess the three-fold symmetry observed in the honeycomb lattice (i.e. the rotational symmetry with respect to the three directions \(a\), \(b\) and \(c\)), and therefore the following two relations are additionally needed to describe the buckling strength of a regular honeycomb according to uniaxial modes of buckling along \(b\) and \(c\) directions

\[
\sqrt{\sigma_{bb}} \tanh\left(\sqrt{3\sqrt{3}\sigma_{bb}}\right) + \sqrt{\sigma_{cc}} \coth\left(\sqrt{3\sqrt{3}\sigma_{cc}}\right) + \sqrt{\sigma_{aa}} \coth\left(\sqrt{3\sqrt{3}\sigma_{bb}}\right) = 0
\]

(25b)

\[
\sqrt{\sigma_{cc}} \tanh\left(\sqrt{3\sqrt{3}\sigma_{cc}}\right) + \sqrt{\sigma_{aa}} \coth\left(\sqrt{3\sqrt{3}\sigma_{aa}}\right) + \sqrt{\sigma_{bb}} \coth\left(\sqrt{3\sqrt{3}\sigma_{bb}}\right) = 0
\]

(25c)
Figure 5-8 (a) Modes I, II and III of buckling observed in regular honeycomb, respectively. Modes I and II are characterized by a zig-zag collapse of cells due to compression along x-direction and an alternating cell collapse due to compression perpendicular to x-direction, respectively. Mode III is a chiral cell configuration where groups of six highly deformed cells surround almost intact central cells. The representative volume element (RVE) for each mode is
denoted by red lines. (b) Notations for the beam and nodal rotation in the RVE. (c) Free-body diagrams of the RVE beam-elements in modes I, II and III, respectively.

**Mode II (biaxial):**

The biaxial mode of buckling and the associated RVE are shown in figure 5-8b. According to symmetry requirements the beam OA is under opposite (with equal magnitude) moments, denoted by $M_a$, at the ends. Beams OB and OC are subjected to end moments $M_{ob}$, $M_{bo}$, $M_{oc}$, $M_{co}$, as shown in the figure. The set of beam-column and equilibrium relations of the three different bar types OA, OB, and OC can be expressed in the following matrix form, where the first five matrix rows represent the beam-column relations on beams OA, OB, and OC, the sixth line corresponds to equilibrium of node O, and the last relations satisfy the moment equilibrium in beams OB and OC.

\[
\begin{bmatrix}
-\phi(u_a) & \psi(u_a) & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -\psi(u_b) & -\phi(u_b) & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -\phi(u_b) & -\psi(u_b) & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -\psi(u_c) & -\phi(u_c) & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -\phi(u_c) & -\psi(u_c) & 1 & 0 & -1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 4u_b^2 & -4u_b & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 4u_c^2 & 0 & 4u_c^2 \\
\end{bmatrix}
\begin{bmatrix}
M_a L/(EI) \\
M_{ob} L/(EI) \\
M_{bo} L/(EI) \\
M_{oc} L/(EI) \\
M_{co} L/(EI) \\
\alpha \\
\theta_b \\
\theta_c \\
\end{bmatrix} + [B] = 0 \tag{26}
\]

Using the symbolic toolbox in MATLAB to set $|A| = 0$, the relation between $u_a$, $u_b$, and $u_c$ expressing the instability of regular honeycomb structure according to biaxial mode under a general loading is

\[
u_a \cot(u_a) + u_b \cot(2u_b) + u_c \cot(2u_c) = 0 \tag{27}
\]
or considering all three rotations corresponding to this mode in the $abc$ stress space

\[
\sqrt{\sigma_{aa}}\coth\left(\sqrt{3\sqrt{3}\sigma_{aa}}\right) + \sqrt{\sigma_{bb}}\coth\left(2\sqrt{3\sqrt{3}\sigma_{bb}}\right) + \sqrt{\sigma_{cc}}\coth\left(2\sqrt{3\sqrt{3}\sigma_{cc}}\right) = 0 \quad (28a)
\]

\[
\sqrt{\sigma_{bb}}\coth\left(\sqrt{3\sqrt{3}\sigma_{aa}}\right) + \sqrt{\sigma_{cc}}\coth\left(2\sqrt{3\sqrt{3}\sigma_{bb}}\right) + \sqrt{\sigma_{aa}}\coth\left(2\sqrt{3\sqrt{3}\sigma_{cc}}\right) = 0 \quad (28b)
\]

\[
\sqrt{\sigma_{cc}}\coth\left(\sqrt{3\sqrt{3}\sigma_{aa}}\right) + \sqrt{\sigma_{aa}}\coth\left(2\sqrt{3\sqrt{3}\sigma_{bb}}\right) + \sqrt{\sigma_{bb}}\coth\left(2\sqrt{3\sqrt{3}\sigma_{cc}}\right) = 0 \quad (28c)
\]

The zero post-buckling curvature in a series of horizontal beams that connect adjacent RVEs in the biaxial buckling mode requires, in effect, the moment throughout those beam to be small (i.e. $M_{bo} \approx M_{co}$). This condition is only satisfied when the characteristic matrix is symmetrical with respect to $b$ and $c$ direction, which translates to a biaxial state of loading in the principal material directions $x$ and $y$. This is supported by full-scale finite element analysis, where the second mode of buckling is observed under biaxial loading, and is rapidly suppressed by increasing the magnitude of macroscopic $xy$ shear stress component applied to the structure. Also, note that the relations given in (23a) and (26a) obtained for a general state of in-plane stress and can be simplified to the result obtained in [80] for instability of regular honeycomb biaxial loading condition.

**Mode III (flower-like):**

For flower-like mode of buckling shown in figure 5-8c the set of beam-column and equilibrium relations of three different bar types OA, OB, and OC can be expressed in the following matrix form, where the first and second rows satisfy the moment equilibrium of node
O and the intact central hexagon, respectively, rows three to five satisfy the equilibrium of beams OA, OB and OC, and rows six to eleven represent the beam-column relations for beams OA, OB and OC.

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & -4(u_a^2 + u_b^2 + u_c^2) & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4u_a^2 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4u_b^2 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 4u_c^2 \\
\end{bmatrix}
\begin{bmatrix}
\psi(u_a) \\
\psi(u_b) \\
\psi(u_c) \\
\phi(u_a) \\
\phi(u_b) \\
\phi(u_c) \\
\end{bmatrix} = 0
\]

Equating the determinant of the characteristics matrix results in the following relation between \(u_a\), \(u_b\) and \(u_c\) that cause instability of regular hexagonal honeycomb according to the flower-like buckling shape

\[
\frac{1}{u_a^2 + u_b^2 + u_c^2} + \frac{1}{u_a \tan(u_a) + u_b \tan(u_b) + u_c \tan(u_c)} = \frac{1}{u_a \cot(u_a) + u_b \cot(u_b) + u_c \cot(u_c)}
\]

where \(u_a = \sqrt{3\sqrt{3} \sigma_{aa}}\) and cyclically for \(u_b\) and \(u_c\). In the characteristics matrix, the first and second rows correspond to the equilibrium of node O and intact, central hexagon,
respectively, the third, fourth, and fifth rows satisfy the moment equilibrium in beams OA, OB and OC, and rows six to eleven represent the beam-column relations on beams OA, OB, and OC.

**Result**

Under all loading directions, each defined by a ratio between components of macroscopic stress, the stresses required for biaxial mode of buckling according to Eqs. (26) are equal or greater than those needed for the uniaxial mode of buckling given by Eqs. (23). Under in-plane \( xy \) biaxial loading (i.e. \( \tau_{xy} = 0 \) or \( \bar{\sigma}_{bb} = \bar{\sigma}_{cc} \)), Eqs. (23b,c) can be simplified to Eq. (26a) using the trigonometric identity \( \coth(2x) = \frac{\coth(x) + \tanh(x)}{2} \). As a result, for a regular honeycomb structure under bi-axial macroscopic loading with the first principal stress along \( x \) (i.e. \( \sigma_{xx} > \sigma_{yy} \)), the microscopic instability of honeycomb cells could be formed according to either uniaxial or biaxial mode of buckling. In full-scale numerical trials, the choice of buckling mode under this loading condition is determined by the far field boundary conditions applied to the finite-size finite element model. For the case of bi-axial loading with the first principal stress along \( y \) (\( \sigma_{xx} < \sigma_{yy} \)), the uniaxial mode of buckling is the preferred mode. Figure 5-9a shows the curves corresponding to uniaxial, biaxial, and flower-like modes of buckling in a regular honeycomb under biaxial state of loading, where horizontal and vertical axes correspond to normalized normal stresses in \( x \) and \( y \) directions, respectively, according to \( \bar{\sigma} = (\sigma/E)/(t/L)^3 \). The green and red lines denote the uniaxial mode of buckling along \( a \) (or \( x \)) and the flower-like modes, respectively. The blue line corresponds to the biaxial mode as well as the two variations of the uniaxial mode occurring along \( b \) or \( c \) directions. The green and blue lines intersect at the equi-biaxial state of macroscopic loading with the principal stresses equal to \( \bar{\sigma}_{xx} = \bar{\sigma}_{yy} = -0.175 \). The flower-like mode is not a dominant mode under any loading condition. However,
the macroscopic stresses needed for this mode become relatively close to those of uniaxial and biaxial modes under equi-biaxial loading with the macroscopic stresses \( \tilde{\sigma}_{xx} = \tilde{\sigma}_{yy} = -0.198 \).

The results are verified by the FE eigenvalue analyses performed at the RVE scale, which are in good agreement with full-scale finite element results. In figure 5-9b, the instability surface corresponding to the regular hexagonal honeycomb structure is presented. The surface is mathematically described by the inner envelope of buckling stresses corresponding to the three rotational variations of the uniaxial mode of buckling given by Eqs. 23. The three edges observed in this figure correspond to the three biaxial states of loading along \( x \)-\( y \), \( x \)-\( z \) and \( y \)-\( z \) directions, where the biaxial mode (mode II) of buckling is also possible. The presented 3D plot allows prediction of macroscopic buckling stresses for the regular honeycomb structure under an arbitrary stress state.

Figure 5-9 (a) Biaxial plastic collapse of regular hexagonal honeycomb under biaxial loading along \( x \) (the so-called armchair or ribbon direction) and \( y \) (the so-called zigzag or transverse direction) according to uniaxial, biaxial, and flower-like modes of buckling. (b) The buckling
collapse surface of the regular honeycomb structure in the \( abc \) stress space, allowing prediction of buckling strength under a general in-plane state of stress.

### 5.6 Buckling of Hierarchical Honeycomb Structure

Hierarchical honeycombs were recently shown to allow significant enhancements in stiffness and plastic collapse strength compared to regular honeycomb of the same density [32]. A first order hierarchical honeycomb is shown in figure 5-10. This structure is obtained by the first iteration of a hierarchical refinement scheme in which all three-edge nodes are replaced with smaller, parallel hexagons at each refinement level. The length ratio, \( \gamma \), is defined as the side of the newly added hexagons to the side of the original hexagonal network in a first order honeycomb, and is geometrically bound to the range \( 0 \leq \gamma \leq 0.5 \). However, in the buckling analysis presented here the smaller hexagons in the hierarchical lattice are considered small enough to be regarded as rigid parts. The coordinate system of choice is the \( abc \) coordinate.

Figure 5-10a shows the post-buckling free body diagram of the RVE of the hierarchical lattice according to mode I, where the rigid small hexagon at the center of the RVE rotates by the angle \( \alpha \) during buckling. The set of beam-column and equilibrium relations are expressed in the following matrix form

\[
\begin{bmatrix}
-1 & 1 & 1 & -4(u_a^2 + u_b^2 + u_c^2)\frac{\gamma}{1-2\gamma} & 0 \\
-1 & 0 & 0 & 2u_a^2 & 2u_b^2 \\
0 & \frac{\psi(u_a)}{3} + \frac{\phi(u_c)}{6} & 0 & 0 & -1 \\
0 & 0 & \frac{\psi(u_b)}{3} + \frac{\phi(u_c)}{6} & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
M_a \\
M_b \\
M_c \\
\psi(u_a) \\
\phi(u_c)
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
M_aL(1-2\gamma) \\
M_bL(1-2\gamma) \\
M_cL(1-2\gamma) \\
\frac{EI}{\alpha} \\
\frac{EI}{\beta}
\end{bmatrix} + [B]
\]
where the first row expresses the moment equilibrium of central node (hexagon) O, the second row satisfies the moment equilibrium of beam OA, and the three last rows correspond to beam-column relations for beam OA, OB, and OC. Equating the determinant of the characteristics matrix equal to zero leads to the following relation for the buckling of hierarchical lattice of length ratio $\gamma$ according to mode I of buckling

$$\frac{2\gamma}{1-2\gamma}(u_a^2 + u_b^2 + u_c^2) + u_a\tan(u_a) - u_b\cot(u_b) - u_c\cot(u_c) = 0$$  \hspace{1cm} (32)

or in the $abc$ stress space, considering all three $2\pi/3$ rotations corresponding to this mode

$$-(\sigma_{aa} + \sigma_{bb} + \sigma_{cc}) \frac{6\sqrt{3}\gamma}{1-2\gamma} + \sqrt{\sigma_{aa}}\tanh\left(\sqrt{3\sqrt{3}\sigma_{aa}}\right) + \sqrt{\sigma_{bb}}\coth\left(\sqrt{3\sqrt{3}\sigma_{bb}}\right)$$

$$+ \sqrt{\sigma_{cc}}\coth\left(\sqrt{3\sqrt{3}\sigma_{cc}}\right) = 0$$  \hspace{1cm} (33a)

$$-(\sigma_{aa} + \sigma_{bb} + \sigma_{cc}) \frac{6\sqrt{3}\gamma}{1-2\gamma} + \sqrt{\sigma_{aa}}\coth\left(\sqrt{3\sqrt{3}\sigma_{aa}}\right) + \sqrt{\sigma_{bb}}\tanh\left(\sqrt{3\sqrt{3}\sigma_{bb}}\right)$$

$$+ \sqrt{\sigma_{cc}}\coth\left(\sqrt{3\sqrt{3}\sigma_{cc}}\right) = 0$$  \hspace{1cm} (33b)

$$-(\sigma_{aa} + \sigma_{bb} + \sigma_{cc}) \frac{6\sqrt{3}\gamma}{1-2\gamma} + \sqrt{\sigma_{aa}}\coth\left(\sqrt{3\sqrt{3}\sigma_{aa}}\right) + \sqrt{\sigma_{bb}}\coth\left(\sqrt{3\sqrt{3}\sigma_{bb}}\right)$$

$$+ \sqrt{\sigma_{cc}}\tanh\left(\sqrt{3\sqrt{3}\sigma_{cc}}\right) = 0$$  \hspace{1cm} (33c)

Similar to regular honeycomb lattice, the set of beam-column and equilibrium relations for RVE in mode II of buckling can be written in the following matrix form
where the first five rows are the beam-column relations on beams OA, OB, and OC, the sixth line corresponds to equilibrium of node (hexagon) O, and the last two relations satisfy the moment equilibrium in beams OB and OC. Using the symbolic toolbox in MATLAB software to set \(|A| = 0\), the relation expressing the instability of hierarchical honeycomb lattice according to biaxial mode under a general loading is

\[
\begin{bmatrix}
-\phi(u_a) + \frac{-\psi(u_a)}{6} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & \frac{-\psi(u_b)}{3} & \frac{-\phi(u_b)}{6} & 0 & 0 & 0 & 1 & 0 \\
0 & \frac{-\phi(u_b)}{6} & \frac{-\psi(u_b)}{3} & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & \frac{-\psi(u_c)}{3} & \frac{-\phi(u_c)}{6} & 0 & 0 & 1 \\
0 & 0 & 0 & \frac{-\phi(u_b)}{6} & \frac{-\psi(u_c)}{3} & 1 & 0 & -1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & \frac{-4\gamma}{1-2\gamma} \left(u_a^2 + u_b^2 + u_c^2\right) & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 4u_b^2 & -4u_b^2 & 0 \\
0 & 0 & 0 & 1 & -1 & 4u_c^2 & 0 & 4u_c^2 \\
\end{bmatrix} \cdot \begin{bmatrix}
M_{\alpha L} \\
E \alpha \\
\theta_b \\
\theta_c \\
\end{bmatrix}
\]

\( + [B] = 0 \)

\( (34) \)

Similar to regular honeycomb structure, the corresponding macroscopic state to this mode of buckling is equal to that of uniaxial mode under \(xy\) bi-axial loading, and is not dominant under any other loading condition.
5.7 Buckling of Tri-chiral Honeycomb Structure

Chiral honeycombs have attracted a great deal of attention in recent years due to their auxetic properties [100-104], and are suggested for design of compliant structures featuring large multi-axial deformations under targeted loads including micro electro-mechanical systems (MEMS) [105, 106], aircraft morphing components [104, 107-109], etc. According to full-scale finite element analysis on the tri-chiral honeycomb buckling patterns similar to the uniaxial modes in regular honeycomb is developed under different states of in-plane loading - see figure 5-11a. Here, we consider the circular elements in the structure of tri-chiral lattice as rigid parts. This assumption is rather reasonable when the radius $r$ of circular elements is small enough relative to the length $L$ of straight beams. The coordinate system of choice is the $a'b'c'$ coordinate system oriented along the three beam directions in the chiral lattice, and is obtained by rotating the $abc$ coordinate system by $\theta = \tan^{-1}(2r/L)$ in the counter-clock-wise direction.
Figure 11b shows the free body diagram of the RVE of the structure, where the rigid circular element O at the center of the RVE rotates by the angle $\alpha$ during buckling. Compared to the regular hexagonal honeycomb, the non-central forces in the tri-chiral lattice will affect the equilibrium requirements on the RVE central node (here rigid circle) O, so that the set of beam-column and equilibrium relations for instability of tri-chiral honeycomb by uniaxial mode of buckling is expressed in the following matrix form

$$
\begin{bmatrix}
-1 & 1 & 1 & -8(u_{a'}^2 + u_{b'}^2 + u_{c'}^2) \left( \frac{r}{L} \right)^2 & 0 \\
-1 & 0 & 0 & 2u_{a'}^2 & 2u_{a'}^2 \\
\frac{\psi(u_{a'})}{3} - \frac{\phi(u_{a'})}{6} & 0 & 0 & 0 & -1 \\
0 & 0 & \frac{\psi(u_{b'})}{3} + \frac{\phi(u_{b'})}{6} & 0 & -1 \\
0 & 0 & 0 & \frac{\psi(u_{c'})}{3} + \frac{\phi(u_{c'})}{6} & -1 \\
\end{bmatrix}
= \begin{bmatrix}
M_{a'} L/EI \\
M_{b'} L/EI \\
M_{c'} L/EI \\
\alpha \\
\beta \\
\end{bmatrix}
+ [B] = 0
$$

(36)

where the first row expresses the moment equilibrium of central node O, the second row satisfies the moment equilibrium of beam OA, and the three last rows correspond to beam-column relations for beam OA, OB, and OC. Setting the determinant of characteristics matrix equal to zero leads to the following relation for the instability of tri-chiral lattice

$$
\left( \frac{2r}{L} \right)^2 \left( u_{a'}^2 + u_{b'}^2 + u_{c'}^2 \right) + u_{a'} \tan(u_{a'}) - u_{b'} \cot(u_{b'}) - u_{c'} \cot(u_{c'}) = 0
$$

(37)

or in terms of normalized stress components in $a'b'c'$ stress space, considering all three $(2\pi / 3)$ rotations corresponding to this mode.
\[-3\sqrt{3}(\overline{\sigma_{a'a'}} + \overline{\sigma_{b'b'}} + \overline{\sigma_{c'c'}}) \tan^2 \theta + \sqrt{\overline{\sigma_{a'a'}}} \tanh \left( \sqrt{3} \overline{\sigma_{a'a'}} \right) \]
\[+ \sqrt{\overline{\sigma_{b'b'}}} \coth \left( \sqrt{3} \overline{\sigma_{b'b'}} \right) + \sqrt{\overline{\sigma_{c'c'}}} \coth \left( \sqrt{3} \overline{\sigma_{c'c'}} \right) = 0 \]  
\text{(38a)}

\[-3\sqrt{3}(\overline{\sigma_{a'a'}} + \overline{\sigma_{b'b'}} + \overline{\sigma_{c'c'}}) \tan^2 \theta + \sqrt{\overline{\sigma_{a'a'}}} \coth \left( \sqrt{3} \overline{\sigma_{a'a'}} \right) \]
\[+ \sqrt{\overline{\sigma_{b'b'}}} \tanh \left( \sqrt{3} \overline{\sigma_{b'b'}} \right) + \sqrt{\overline{\sigma_{c'c'}}} \coth \left( \sqrt{3} \overline{\sigma_{c'c'}} \right) = 0 \]  
\text{(38b)}

\[-3\sqrt{3}(\overline{\sigma_{a'a'}} + \overline{\sigma_{b'b'}} + \overline{\sigma_{c'c'}}) \tan^2 \theta + \sqrt{\overline{\sigma_{a'a'}}} \coth \left( \sqrt{3} \overline{\sigma_{a'a'}} \right) \]
\[+ \sqrt{\overline{\sigma_{b'b'}}} \coth \left( \sqrt{3} \overline{\sigma_{b'b'}} \right) + \sqrt{\overline{\sigma_{c'c'}}} \tanh \left( \sqrt{3} \overline{\sigma_{c'c'}} \right) = 0 \]  
\text{(38c)}

The results using Eq. (32) are in agreement with results from FE eigenvalue analysis.
Two analytical approaches, the characteristics matrix method and an energy method based on beam-column relations, were introduced for predicting the buckling loads of 2D and 3D cellular beam structures. The first approach uses the characteristic matrix of the post-buckling beam moments and nodal rotations in the cellular structure RVE to predict the buckling load of the periodic structure. For structure with relatively large RVE with high number of nodes an energy method based on the beam-column theorem is offered which significantly reduces the size of the characteristics matrix. The introduced methods were applied to obtain the buckling loads for regular, chiral, and hierarchical honeycombs with hexagonal unit cells and triangular and square honeycombs. Results from finite element eigenvalue simulations were used to
validate the analytical methods. Closed-form relations of macroscopic buckling stress describing
the instability surface of structure under arbitrary in-plane stress were derived and are given in
table 5-1.

An interesting feature in buckling of hexagonal and triangular honeycombs is a secondary
mode of buckling which is statically possible, and numerically observed, only under the biaxial
loading condition in the $xy$ plane. These modes of buckling, although different in shape, were
shown to occur at the same macroscopic stress state as that of the primary modes of buckling in
these structures. The occurrence of these secondary buckling modes is dependent on the
boundary conditions applied to numerical and experimental trials. Similarly, there was no
difference shown for the square grid between the strengths corresponding to the swaying mode
and the long-wave buckling pattern as the wavelength becomes long enough.

The use of beam-column approach for calculation of buckling strength in cellular
structures should be considered with a caveat with regard to the effect of lateral loads and cell
walls deflections on a stable, non-buckling deformation of the structure. As shown for the case of
regular honeycomb structure under uniaxial loading along $y$ direction [34], the lateral loads on
the oblique beams cause the honeycombs to fold up in a stable way. This would suppress the
sudden collapse essentially observed during the buckling and its associated bifurcation in the
macroscopic load-displacement curve. The suppression of buckling could also happen in the
square grid under large macroscopic $xy$ shear stress, where the characteristics matrix was shown
to be independent of lateral loads applied to the vertical and horizontal beams, and thus of the
amount of shear stress. As the value of shear stress increases compared to axial stresses,
approaching a square grid under pure $xy$ shear loading, the beams in the grid simply deflect
statically until the pre-buckling deformations become so large that suppress the buckling phenomenon. For the triangular grid with a stretching dominated behavior, the lateral reaction forces inside cell walls are essentially zero. As a result, the cell walls in the structure will not undergo a pre-buckling bending deformation, and bifurcation of macroscopic load-displacement curve is observed under the general stress state.
Table 5-1- Relations describing the instability stress surface for the structures studied.

<table>
<thead>
<tr>
<th>Honeycomb Type</th>
<th>Instability Stress Surface ( (\bar{\sigma} = (\sigma/E)/(t/L)^3) )</th>
</tr>
</thead>
</table>
| Square grid    | \[
\begin{align*}
\bar{\sigma}_{yy} \left(1 - 3\bar{\sigma}_{xx} \coth(3\bar{\sigma}_{xx})\right) - \bar{\sigma}_{xx} \left(1 - 3\bar{\sigma}_{yy} \coth(3\bar{\sigma}_{yy})\right) &= 0 \\
\bar{\sigma}_{xx} \left(1 - 3\bar{\sigma}_{yy} \coth(3\bar{\sigma}_{xx})\right) - \bar{\sigma}_{yy} \left(1 - 3\bar{\sigma}_{xx} \coth(3\bar{\sigma}_{xx})\right) &= 0
\end{align*}
\] |
| Triangular grid| \[
\begin{align*}
 u_b \cot u_b + u_c \cot u_c + \frac{u_a^2}{1 - u_a \cot u_a} &= 0 \\
 u_c \cot u_c + u_a \cot u_a + \frac{u_b^2}{1 - u_b \cot u_b} &= 0 \\
 u_a \cot u_a + u_b \cot u_b + \frac{u_c^2}{1 - u_c \cot u_c} &= 0
\end{align*}
\] \[
\begin{align*}
 u_a &= \sqrt[3/4]{5\bar{\sigma}_{aa} - \bar{\sigma}_{bb} - \bar{\sigma}_{cc}} \\
 u_b &= \sqrt[3/4]{5\bar{\sigma}_{bb} - \bar{\sigma}_{aa} - \bar{\sigma}_{cc}} \\
 u_c &= \sqrt[3/4]{5\bar{\sigma}_{cc} - \bar{\sigma}_{aa} - \bar{\sigma}_{bb}}
\end{align*}
\] |
| Hexagon based  | \[
\Delta_a = \sqrt{\bar{\sigma}_{aa}} \tanh \left(3\sqrt[3]{3\bar{\sigma}_{aa}}\right) + \sqrt{\bar{\sigma}_{bb}} \coth \left(3\sqrt[3]{3\bar{\sigma}_{bb}}\right) + \sqrt{\bar{\sigma}_{cc}} \coth \left(3\sqrt[3]{3\bar{\sigma}_{cc}}\right)
\] |
\[ \Delta_b \equiv \sqrt{\bar{\sigma}_{bb}} \tanh \left( \sqrt{3\sqrt{3}\bar{\sigma}_{bb}} \right) + \sqrt{\bar{\sigma}_{cc}} \coth \left( \sqrt{3\sqrt{3}\bar{\sigma}_{aa}} \right) \]

\[ + \sqrt{\bar{\sigma}_{aa}} \coth \left( \sqrt{3\sqrt{3}\bar{\sigma}_{bb}} \right) \]

\[ \Delta_c \equiv \sqrt{\bar{\sigma}_{cc}} \tanh \left( \sqrt{3\sqrt{3}\bar{\sigma}_{cc}} \right) + \sqrt{\bar{\sigma}_{aa}} \coth \left( \sqrt{3\sqrt{3}\bar{\sigma}_{aa}} \right) \]

\[ + \sqrt{\bar{\sigma}_{bb}} \coth \left( \sqrt{3\sqrt{3}\bar{\sigma}_{bb}} \right) \]

<table>
<thead>
<tr>
<th>Type</th>
<th>Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Regular</strong></td>
<td>( \Delta_a = 0 ), ( \Delta_b = 0 ), ( \Delta_c = 0 )</td>
</tr>
</tbody>
</table>
| **Tri-Chiral**     | \( \sqrt{3\sqrt{3}(\bar{\sigma}_{a'a'} + \bar{\sigma}_{b'b'} + \bar{\sigma}_{c'c'}) \tan^2 \theta + \Delta_{a'} = 0} \)
|                    | \( \sqrt{3\sqrt{3}(\bar{\sigma}_{a'a'} + \bar{\sigma}_{b'b'} + \bar{\sigma}_{c'c'}) \tan^2 \theta + \Delta_{b'} = 0} \)
|                    | \( \sqrt{3\sqrt{3}(\bar{\sigma}_{a'a'} + \bar{\sigma}_{b'b'} + \bar{\sigma}_{c'c'}) \tan^2 \theta + \Delta_{c'} = 0} \)
| **Hierarchical**   | \( \left( \bar{\sigma}_{aa} + \bar{\sigma}_{bb} + \bar{\sigma}_{cc} \right) \frac{2\sqrt{3\sqrt{3} \gamma}}{1 - 2\gamma} + \Delta_a = 0 \)
|                    | \( \left( \bar{\sigma}_{aa} + \bar{\sigma}_{bb} + \bar{\sigma}_{cc} \right) \frac{2\sqrt{3\sqrt{3} \gamma}}{1 - 2\gamma} + \Delta_b = 0 \)
|                    | \( \left( \bar{\sigma}_{aa} + \bar{\sigma}_{bb} + \bar{\sigma}_{cc} \right) \frac{2\sqrt{3\sqrt{3} \gamma}}{1 - 2\gamma} + \Delta_c = 0 \) |
Chapter 6: References


