Singularities of A Certain Class of Toric Varieties

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ABSTRACT OF DISSERTATION

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Abstract

Hibi considered the class of algebras $k[\mathcal{L}] = k[x_\alpha | \alpha \in \mathcal{L}]$ with straightening laws associated to a finite distributive lattice $\mathcal{L}$ in his paper [2]. In that paper he proves that these algebras are normal and integral domains. This result along with the work of Sturmfels and Eisenbud [7] on binomial prime ideals implies that the affine varieties associated to the algebra $k[\mathcal{L}]$ are normal toric varieties.

In the present work we will consider the toric variety $X(\mathcal{L}) = \text{spec}(k[\mathcal{L}])$, we will give the combinatorial description of the cone $\sigma$ associated to it. The final result will be to give a standard monomial basis for the tangent cone $\widehat{T}_{x_\tau}$ where $x_\tau$ is a singular point associated to a torus orbit $O_\tau$ for the action of the torus $T$, where $\tau$ is a face of the cone $\sigma$. 

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To My Family,

বাবা, মা, বোন এবং পিয়ালীকে,

জীবন মরনের বীমানা ছাড়ায়

ব্যু হে আমার

রয়েছ দীঘায়ে
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Chapter 1

Preface

Hibi studies a class of algebras with straightening laws in [2], which are defined by binomial relations as a quotient of full polynomial algebras. These algebras come up as degenerations of Schubert varieties. My work deals with constructing a standard basis for the tangent cone to these toric varieties at a singular point.
Chapter 2

Introduction

Let $K$ denote the base field which we assume to be algebraically closed of arbitrary characteristic. Given a distributive lattice $\mathcal{L}$, let $X_\mathcal{L}$ denote the Hibi variety - the affine variety in $\mathbb{A}^{\#\mathcal{L}}$ whose vanishing ideal is generated by the binomials $X_\tau X_\varphi - X_{\tau \vee \varphi} X_{\tau \wedge \varphi}$ in the polynomial algebra $K[X_\alpha, \alpha \in \mathcal{L}]$; here, $\tau \vee \varphi$ (resp. $\tau \wedge \varphi$) denotes the join - the smallest element of $\mathcal{L}$ greater than both $\tau, \varphi$ (resp. the meet - the largest element of $\mathcal{L}$ smaller than both $\tau, \varphi$). In the sequel, we shall refer to the quadruple $(\tau, \varphi, \tau \vee \varphi, \tau \wedge \varphi)$ ($(\tau, \varphi)$ being a skew pair) as a diamond. These varieties were extensively studied by Hibi in [13] where Hibi proves that $X_\mathcal{L}$ is a normal variety. On the other hand, Eisenbud-Sturmfels show in [7] that a binomial prime ideal is toric (here, “toric ideal” is in the sense of [17]). Thus one obtains that $X_\mathcal{L}$ is a normal toric variety. We refer to such an $X_\mathcal{L}$ as a Hibi toric variety.

For $\mathcal{L}$ being the Bruhat poset of Schubert varieties in a minuscule $G/P$, it is shown in [9] that $X_\mathcal{L}$ flatly deforms to $\widehat{G/P}$ (the cone over $G/P$), i.e., there exists a flat family over $\mathbb{A}^1$ with $\widehat{G/P}$ as the generic fiber and $X_\mathcal{L}$ as the special fiber. More generally for a Schubert variety $X(w)$ in a minuscule $G/P$, it is shown in
that $X_{L_w}$ flatly deforms to $\widehat{X}(w)$ (here, $L_w$ is the Bruhat poset of Schubert subvarieties of $X(w)$). In a subsequent paper (cf. [11]), the authors of loc.cit., studied the singularities of $X_L$, $L$ being the Bruhat poset of Schubert varieties in the Grassmannian; further, in loc.cit., the authors gave a conjecture giving a necessary and sufficient condition for a point on $X_L$ to be smooth, and proved in loc.cit., the sufficiency part of the conjecture. Subsequently, the necessary part of the conjecture was proved in [4] by Batyrev et al. The toric varieties $X_L$, $L$ being the Bruhat poset of Schubert varieties in the Grassmannian play an important role in the area of mirror symmetry; for more details, see [3, 4]. We refer to such an $X_L$ as a Grassmann-Hibi toric variety.

In this thesis, we first determine an explicit description (cf. §4.1) of the cone $\sigma$ associated to a Hibi toric variety and the dual cone $\sigma^\vee$ (note that if $S_\sigma$ denotes the semigroup of integral points in $\sigma^\vee$, then $K[X_L]$, the $K$-algebra of regular functions on $X_L$ can be identified with the semigroup algebra $K[S_\sigma]$). Using this we determine the cotangent space at any point on $X_L$ very explicitly as described below.

For each face $\tau$ of $\sigma$, there exists a distinguished point $P_\tau$ in the torus orbit $O_\tau$ corresponding to $\tau$; namely, identifying a (closed) point in $X_L$ with a semigroup map $S_\sigma \to K^* \cup \{0\}$, $P_\tau$ corresponds to the semigroup map $f_\tau$ which sends $u \in S_\sigma$ to 1 or 0 according as $u$ is in $\tau^\perp$ or not. We have that $f_\tau$ induces an algebra map $\psi_\tau : K[X_L] \to K$ whose kernel is precisely $M_{P_\tau}$, the maximal ideal corresponding to $P_\tau$. For $\alpha \in L$, we shall denote by $P_\tau(\alpha)$, the co-ordinate of $P_\tau$ (considered as a point of $A^\#L$) corresponding to $\alpha$. Set

$$D_\tau = \{ \alpha \in L \mid P_\tau(\alpha) \neq 0 \}$$

It turns out that $D_\tau$ is an embedded sublattice of $L$; further, $D_\tau$ determines the
2.1 Main Results.

local behavior at $P_\tau$. To make this more precise, identifying $K[X_L]$ as the quotient
of the polynomial algebra $K[X_\alpha, \alpha \in L]$ by the ideal generated by the binomials
$X_\tau X_\varphi - X_{\tau \vee \varphi} X_{\tau \wedge \varphi}$, let $x_\alpha$ denote the image of $X_\alpha$ in $K[X_L]$. For $\alpha \in L$, set

$$F_\alpha = \begin{cases} x_\alpha, & \text{if } \alpha \not\in D_\tau \\ 1 - x_\alpha, & \text{if } \alpha \in D_\tau \end{cases}$$

Then we have that $M_{P_\tau}$ is generated by \{\(F_\alpha, \alpha \in L\}\}. Fix a maximal chain $\Gamma$ in
$D_\tau$. Let $\Lambda_\tau(\Gamma)$ denote the sublattice of $L$ consisting of all maximal chains of $L$
containing $\Gamma$. Let $E_\tau$ denote the set of all $\alpha \in L$ such that there exists a $\beta \in D_\tau$
such that $(\alpha, \beta)$ is a diagonal of a diamond whose other diagonal is contained in
$L \setminus D_\tau$. Let $Y_\tau(\Gamma) = \Lambda_\tau(\Gamma) \setminus E_\tau$. For $F \in M_{P_\tau}$, let $F$ denote the class of $F$
in $M_{P_\tau} / M^2_{P_\tau}$. We define an equivalence relation (cf. §4.2) on $L \setminus D_\tau$ in such a
way that for two elements $\theta, \theta'$ in the same equivalence class, we have $F_\theta = F_{\theta'}$.

Denoting by $G_\tau(\Gamma)$ the set of equivalence classes $[\theta]$, where $\theta \in Y_\tau(\Gamma) \setminus \{\Gamma \cup E_\tau\}$
and $F_{[\theta]}$ is non-zero in $M_{P_\tau} / M^2_{P_\tau}$, we have

2.1 Main Results.

2.1.1 Main Result 1.

Theorem 1: (cf. Theorem 4.3.12) \{\(F_{[\theta]}, [\theta] \in G_\tau(\Gamma)\) $\cup$ \{\(F_{\gamma}, \gamma \in \Gamma\)\}$ is a basis for
the cotangent space $M_{P_\tau} / M^2_{P_\tau}$.

Using the above result, we determine $\text{Sing } X_L$ (cf. Theorem 4.3.15):

Let $S_L = \{\tau < \sigma \mid \#G_\tau(\Gamma) + \#\Gamma > \#L\}$.

Theorem 2: (cf. Theorem 4.3.15) $\text{Sing } X_L = \bigcup_{\tau \in S_L} X(D_\tau)$.

(Here, $X(D_\tau)$ denotes the Hibi variety associated to the distributive lattice $D_\tau$.)
2.1.2 Main Result 2.

We first determine the vanishing ideal $J(\tau)$ of the tangent cone at a singular point $P_\tau$ on the Hibi toric variety $X_\mathcal{L}$ (here, $P_\tau$ is the distinguished point corresponding to a face $\tau$ of the convex polyhedral cone associate to $X_\mathcal{L}$). We then introduce a monomial order $>$ (on the polynomial algebra containing $J(\tau)$), and determine $\text{in } J(\tau)$, the initial ideal of $J(\tau)$. Using Macaulay’s Theorem [6], we obtain a “standard monomial basis” for $TC_{P_\tau}X_\mathcal{L}$ (cf. Theorem 5.2.8).

2.2 Sketch of the proof of the main result 1:

Using our explicit description of the generators for $\sigma, \sigma^\vee$ (cf. §4.1), we first determine explicitly the embedded sublattice associated to a face $\tau$ of $\sigma$. We then analyze the local expression around $P_\tau$ for any $f \in I(X_\mathcal{L})$, the vanishing ideal of $X_\mathcal{L}$, and show the generation of the degree one part of $\text{gr} (R_\mathcal{L}, M_{P_\tau})$ by $\{F_\theta, \theta \in Y_\tau(\Gamma)\}$ (here, $R_\mathcal{L} = K[X_\mathcal{L}]$, and $M_{P_\tau}$ is the maximal ideal in $R_\mathcal{L}$ corresponding to $P_\tau$). We then define the equivalence relation on $\mathcal{L} \setminus D_\tau$. The linear independence of $\{F_{[\theta]} [\theta] \in G_\tau(\Gamma)\} \cup \{F_{[\gamma]}, \gamma \in \Gamma\}$ in $M_{P_\tau} / M^2_{P_\tau}$ is proved using the defining equations of $X_\mathcal{L}$ (as a closed subvariety of $\mathbb{A}^n_\mathcal{L}$), thus proving Theorem 1. Theorem 2 is then deduced from Theorem 1.

2.3 Sketch of proof of main result 2.

Let $\sigma$ be the convex polyhedral cone associate to the toric variety $X_\mathcal{L}$. Let $S_\sigma$ be the semigroup of integral points in $\sigma^\vee$ (the cone dual to $\sigma$), then $K[X_\mathcal{L}]$, the $K$-algebra of regular functions on $X_\mathcal{L}$ can be identified with the semigroup algebra $K[S_\sigma])$. For each face $\tau$ of $\sigma$, there exists a distinguished point $P_\tau$ in the torus
orbit $O_\tau$ corresponding to $\tau$; namely, identifying a (closed) point in $X_L$ with a semigroup map $S_\tau \to K^* \cup \{0\}$, $P_\tau$ corresponds to the semigroup map $f_\tau$ which sends $u \in S_\tau$ to $1$ or $0$ according as $u$ is in $\tau^\perp$ or not. We have that $f_\tau$ induces an algebra map $\psi_\tau : K[X_L] \to K$ whose kernel is precisely $M_{P_\tau}$, the maximal ideal corresponding to $P_\tau$. For $\alpha \in L$, we shall denote by $P_\tau(\alpha)$, the co-ordinate of $P_\tau$ (considered as a point of $\mathbb{A}^{\#L}$) corresponding to $\alpha$. Set

$$D_\tau = \{\alpha \in L \mid P_\tau(\alpha) \neq 0\}$$

It turns out that $D_\tau$ is an embedded sublattice of $L$; further, $D_\tau$ determines the local behavior at $P_\tau$. To make this more precise, identifying $K[X_L]$ as the quotient of the polynomial algebra $K[X_\alpha, \alpha \in L]$ by the ideal generated by the binomials $X_\tau X_\varphi - X_{\tau \vee \varphi} X_{\tau \wedge \varphi}$, let $x_\alpha$ denote the image of $X_\alpha$ in $K[X_L]$. For $\alpha \in L$, set

$$F_\alpha = \begin{cases} x_\alpha, & \text{if } \alpha \notin D_\tau \\ 1 - x_\alpha, & \text{if } \alpha \in D_\tau \end{cases}$$

Then we have that $M_{P_\tau}$ is generated by $\{F_\alpha, \alpha \in L\}$.

Fix a maximal chain $\Gamma$ in $D_\tau$. Let $\Lambda_\tau(\Gamma)$ denote the sublattice of $L$ consisting of all maximal chains of $L$ containing $\Gamma$. This sublattice plays a crucial role in our discussion as explained below:

For $F \in M_\tau$, let $\overline{F}$ denote the class of $F$ in $M_\tau/M_\tau^2$. Then $J(\tau)$ gets identified with the kernel of the surjective map

$$\pi : K[X_\theta, \theta \in L] \to gr(R, M_\tau), \quad X_\theta \mapsto \overline{F_\theta}$$

For $r \in \mathbb{N}$, let $\pi^{(r)}$ be the restriction of $\pi$ to the degree $r$ part of the polynomial algebra $K[X_\theta, \theta \in L]$. Then one of our main results is that for a diamond $D$ with $(\tau, \varphi)$, the diamond relation $f_D := X_\tau X_\varphi - X_{\tau \vee \varphi} X_{\tau \wedge \varphi}$ a diamond relation
is in the ideal generated by $\ker \pi^{(1)}$ and the diamond relations arising from diamonds in $\Lambda_\tau(\Gamma)$. This result together with the description of the cotangent space as determined in [15] reduces our discussion to an analysis of the diamond relations arising from diamonds in the distributive lattice $\Lambda_\tau(\Gamma)$; notice one obvious simplification, namely, by going to $\Lambda_\tau(\Gamma)$, we have “shrunk” $D_\tau$ to just one chain (namely $\Gamma$) in $D_\tau$. The precise analysis of the diamond relations arising from diamonds in $\Lambda_\tau(\Gamma)$ enables us to determine $J(\tau)$ explicitly. The details will be discussed in chapter 5.

The sections are organized as follows: In §2.4, we recall generalities on toric varieties. In §3.1, we recall some basic results on distributive lattices. In §3.2, we introduce the Hibi variety $X(L)$, and recall some basic results on $X(L)$. In §4.1, we determine generators for the cone $\sigma$ (and the dual cone $\sigma^\wedge$); further, for a face $\tau$ of $\sigma$, we introduce $D_\tau$ and derive some properties of $D_\tau$. In §4.2, we present our main results on the tangent and cotangent spaces at the distinguished point $P_\tau$ in the orbit corresponding to $\tau$. In §5.1 we study the diamond relations in view of the local descriptions. tangent cone at point $P_\tau$ in §5.2, we describe the ideal generators and the standard basis for the tangent cone in §5.2.2.

2.4 Generalities on toric varieties

Since our main object of study is a certain affine toric variety, we recall in this section some basic definitions on affine toric varieties. Let $T = (K^*)^m$ be an $m$-dimensional torus.

**Definition 2.4.1.** (cf. [8], [14]) An equivariant affine embedding of a torus $T$ is an affine variety $X \subseteq \mathbb{A}^l$ containing $T$ as an open subset and equipped with a $T$-action $T \times X \to X$ extending the action $T \times T \to T$ given by multiplication of
the group structure of $T$. If in addition $X$ is normal, then $X$ is called an affine toric variety.

2.4.2 Resumé of combinatorics of affine toric varieties

Let $M$ be the character group of $T$. Let $X$ be a toric variety with a torus action by $T$; let $R = K[X]$. We note the following:

- **$M$-gradation:** We have an action of $T$ on $R$, and hence writing $R$ as a sum of $T$ weight spaces, we obtain a $M$-grading for $R$: $R = \bigoplus_{\chi \in M} R_{\chi}$, where $R_{\chi} = \{ f \in R | tf = \chi(t)f, \forall t \in T \}$.

- **The semi group $S$:** Let $S = \{ \chi \in M | R_{\chi} \neq 0 \}$. Then via the multiplication in $R$, $S$ acquires a semi group structure. Thus $S$ is a semi subgroup of $M$, and $R$ gets identified with the semi group algebra $K[S]$.

- **Finite generation of $S$:** In view of the fact that $R$ is a finitely generated $K$-algebra, we obtain that $S$ is a finitely generated semi subgroup of $M$.

- **Generation of $M$ by $S$:** The fact that $T$ and $X$ have the same function field (since $T$ is a dense open subset of $X$) implies that $M$ is generated by $S$.

- **Saturation of $S$:** The normality of $R$ (being identified as the semi group algebra $K[S]$) implies that $S$ is saturated (recall that a semi subgroup $A$ in $M$ is saturated, if for $\chi \in M$, $r\chi \in A \implies \chi \in A$ (where $r$ is an integer $> 1$; equivalently, $A$ is precisely the lattice points in the cone generated by $A$, i.e., $A = \theta \cap A$ where $\theta = \sum a_i x_i, a_i \in \mathbb{R}_+, x_i \in A$)).

Thus given an affine toric variety $X$ (with torus action by $T$), $X$ determines a finitely generated, saturated semi subgroup $S$ of $M$ which generates $M$ in such a way that the semi group algebra is the ring of regular functions on $X$.

Let us also observe that starting with a finitely generated, saturated semi
subgroup $S$ of $M$ which generates $M$, $Spec K[S]$ is an affine toric variety (with torus action by $T$).

This sets up a bijection between \{affine toric varieties with torus action by $T$\} and \{finitely generated, saturated semi subgroups of $M$ which generate $M$\}. See [14] for details.

Let $\theta := \sum a_i x_i, a_i \in \mathbb{R}_+, x_i \in$ be the cone generated by $S$. Then $\theta$ is a rational convex polyhedral cone (i.e., a cone with a (finite) set of lattice points as generators), and $\theta$ is not contained in any hyperplane in $M_\mathbb{R} := M \otimes \mathbb{R}$. Let

$$\sigma := \theta^\vee = \{v \in M_\mathbb{R}^*|v(f) \geq 0, \forall f \in \theta\}$$

(here, $M_\mathbb{R}^*$ is the linear dual of $M_\mathbb{R}$) Then $\sigma$ is a strongly convex rational convex polyhedral cone (one may take this to be the definition of a strongly convex rational convex polyhedral cone, namely, a rational convex polyhedral cone $\tau$ is strongly convex if $\tau^\vee$ is not contained in any hyperplane; equivalently, $\tau$ does not contain any non-zero linear subspace.)

Thus an affine toric variety $X$ (with a torus action by $T$) determines a rational convex polyhedral cone $\sigma$ in such a way that $K[X]$ is the semigroup algebra $K[S_\sigma]$, where $S_\sigma$ is the semi subgroup in $M$ consisting of the set of lattice points in $\sigma^\vee$.

Conversely, starting with a strongly convex rational convex polyhedral cone $\sigma$, $S_\sigma := \sigma^\vee \cap M$ is a finitely generated, saturated semi subgroup of $M$ which generates $M$; and hence determines an affine toric variety $X$ (with a torus action by $T$), namely, $X = Spec K[S_\sigma]$. 
2.4. Generalities on toric varieties

2.4.3 Toric ideals (cf. [17])

Let \( A = \{ \chi_1, \ldots, \chi_l \} \) be a subset of \( \mathbb{Z}^d \). Consider the map
\[
\pi_A : \mathbb{Z}_+^l \to \mathbb{Z}^d, \quad u = (u_1, \ldots, u_l) \mapsto u_1 \chi_1 + \cdots + u_l \chi_l.
\]

Let \( K[x] := K[x_1, \ldots, x_l] \), \( K[t^\pm] := K[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}] \).

The map \( \pi_A \) induces a homomorphism of semigroup algebras
\[
\hat{\pi}_A : K[x] \to K[t^\pm], \quad x_i \mapsto t^{\chi_i}.
\]

**Definition 2.4.4.** (cf. [17]) The kernel of \( \hat{\pi}_A \) is denoted by \( I_A \) and called the toric ideal associated to \( A \).

Note that a toric ideal is prime. We shall now show that we have a bijection between the class of toric ideals and the class of equivariant affine toroidal embeddings.

Consider the action of \( T(= (K^*)^d) \) on \( \mathbb{A}_d \) given by
\[
t(a_1, \cdots, a_l) = (t^{\chi_1}a_1, \cdots, t^{\chi_l}a_l)
\]
Then \( V(I_A) \), the affine variety of the zeroes in \( K^d \) of \( I_A \), is simply the Zariski closure of the \( T \)-orbit through \((1,1,\ldots,1)\). Let \( M_A \) be the \( \mathbb{Z} \)-span of \( A \) (inside \( X(T) = \mathbb{Z}^d \)), and \( d_A \) the rank of \( M_A \). Let \( S_A \) be the sub semigroup of \( \mathbb{Z}^d \) generated by \( A \). Let \( T_A \) be the \( d_A \)-dimensional torus with \( M_A \) as the character group. Then we have

**Proposition 2.4.5.** \( V(I_A) \) is an equivariant affine embedding of \( T_A \) (of dimension \( d_A \)); further, \( K[V(I_A)] \) is the semigroup algebra \( K[S_A] \).

**Remark 2.4.6.** In the above definition, we do not require \( V(I_A) \) to be normal. Note that \( V(I_A) \) is normal if and only if \( S_A \) is saturated. A variety of the form
\( V(I_A) \), is also sometimes called an affine toric variety. But in this paper, by an affine variety, we shall mean a normal toric variety as defined in Definition 2.4.1.

**Remark 2.4.7.** Consider the action of \( T(=(K^*)^d) \) on \( K^l \) given by \( te_i = t^\chi_i e_i \) (here, \( e_i, 1 \leq i \leq l \) are the standard basis vectors of \( K^l \)). Then \( V(I_A) \), the affine variety of the zeroes in \( K^l \) of \( I_A \) is simply the Zariski closure of the \( T \)-orbit through \((1,1,\ldots,1)\). In particular, \( V(I_A) \) is an equivariant affine embedding of \( T = (K^*)^d \).

**The vanishing ideal of \( X \)**

Let us return to the affine toric variety \( X \hookrightarrow \mathbb{A}^l \) with a torus action by \( T(= (K^*)^d) \) as in Definition 2.4.1. Let \( T_l = (K^*)^l \) (an \( l \)-dimensional torus). Consider \( T \) as a subtorus of \( T_l \). The natural action of \( T_l \) on \( \mathbb{A}^l \) induces an action of \( T \) on \( \mathbb{A}^l \), and the embedding \( X \hookrightarrow \mathbb{A}^l \) is \( T \)-equivariant. Hence if we denote the images of \( \{x_1,\ldots,x_l\} \) in \( K[X] \) by \( \{y_1,\ldots,y_l\} \), then \( \{y_1,\ldots,y_l\} \) are \( T \)-weight vectors with weights \( \{\chi_1,\ldots,\chi_l\} \), say. Then \( \{\chi_1,\ldots,\chi_l\} \) is a set of generators for the semi group \( S_\sigma \) (notation being as in §2.4.2, namely, \( K[X] \) equals the semi group algebra \( K[S_\sigma] \)). Denoting, \( \mathcal{A} := \{\chi_1,\ldots,\chi_l\} \), we have that \( I_A \) of Definition 2.4.4 is precisely the vanishing ideal of \( X \). Thus we obtain

**Proposition 2.4.8.** \( I(X) \), the vanishing ideal of \( X \) (for the embedding \( X \hookrightarrow \mathbb{A}^l \)) is a toric ideal.

**Remark 2.4.9.** Note that we also have the saturation property for \( \mathcal{A} \) (since \( S_\sigma \) is a saturated sub semigroup of \( X(T) \))

Recall the following (see [17]).
2.4. Generalities on toric varieties

Proposition 2.4.10. The toric ideal $I_A$ is spanned as a $K$-vector space by the set of binomials

$$\{x^u - x^v \mid u, v \in \mathbb{Z}^n_+ \text{ with } \pi_A(u) = \pi_A(v)\}. \quad (*)$$

(Here, a binomial is a polynomial with at most two terms.)

An example

Let us fix the integers $n_1, \ldots, n_d > 1$, and let $n = \prod_{i=1}^{d} n_i$, $m = \sum_{i=1}^{d} n_i$. Let $e_1^l, \ldots, e_{n_l}^l$ be the unit vectors in $\mathbb{Z}^{n_l}$, for $1 \leq l \leq d$. For $1 \leq \xi_1 \leq n_1, \ldots, 1 \leq \xi_d \leq n_d$, define

$$a_{\xi_1 \ldots \xi_d} = e_{\xi_1}^1 + \cdots + e_{\xi_d}^d \in \mathbb{Z}^{n_1} \oplus \cdots \oplus \mathbb{Z}^{n_d}$$

and let

$$\mathcal{A}_{n_1, \ldots, n_d} = \{a_{\xi_1 \ldots \xi_d} \mid 1 \leq \xi_1 \leq n_1, \ldots, 1 \leq \xi_d \leq n_d\}.$$

The corresponding map

$$\pi_A : \mathbb{Z}^n_+ \rightarrow \mathbb{Z}^{n_1+\cdots+n_d}$$

is defined as follows: for $1 \leq l \leq d$ and $1 \leq i_l \leq n_l$ fixed, the $(n_1 + \cdots + n_{l-1} + i_l)$-th coordinate of $\pi_A(u)$ is given by $\sum u_{\xi_1 \ldots \xi_{l-1} \xi_l \xi_{l+1} \ldots \xi_d}$, the sum being taken over the elements $(\xi_1, \ldots, \xi_{l-1}, \xi_l, \xi_{l+1}, \ldots, \xi_d)$ of $\mathcal{C}(n_1, \ldots, n_d)$ with $\xi_l = i_l$. We call this subset the $l$-th slice of $\mathcal{C}(n_1, \ldots, n_d)$ defined by $i_l$, and denote it by $\{\xi_l = i_l\}$. The components (or entries) of an element $u \in \mathbb{Z}^{n_1\ldots n_d}$ are indexed by the elements $(i_1, \ldots, i_d)$ of $\mathcal{C}(n_1, \ldots, n_d)$. If $(j_1, \ldots, j_d) \in \{\xi_l = i_l\}$, sometimes we also say that $u_{j_1 \ldots j_d}$ itself belongs to the slice $\{\xi_l = i_l\}$.

The map $\pi_A$ induces the map

$$\widehat{\pi_A} : k[x_{11} \ldots 1, \ldots, x_{\xi_1 \xi_2 \ldots \xi_d}, \ldots, x_{n_1 n_2 \ldots n_d}] \rightarrow k[l_{11}, \ldots, l_{1n_1}, \ldots, l_{d1}, \ldots, l_{dn_d}]$$
2.4. Generalities on toric varieties

given by
\[ x_{\xi_1 \cdots \xi_d} \mapsto t_{1\xi_1} \cdots t_{d\xi_d}, \quad \text{for } 1 \leq \xi_1 \leq n_1, \ldots, 1 \leq \xi_d \leq n_d. \]

2.4.11 Varieties defined by binomials

Let \( l \geq 1 \), and let \( X \) be an affine variety in \( \mathbb{A}^l \), not contained in any of the coordinate hyperplanes \( \{ x_i = 0 \} \). Further, let \( X \) be irreducible, and let its defining prime ideal \( I(X) \) be generated by \( l \) binomials
\[
x_1^{a_{i1}} \cdots x_l^{a_{il}} - \lambda_1 x_1^{b_{i1}} \cdots x_l^{b_{il}}, \quad 1 \leq i \leq m.
\]  

Consider the natural action of the torus \( T_l = (\mathbb{K}^*)^l \) on \( \mathbb{A}^l \),
\[
(t_1, \ldots, t_l) \cdot (a_1, \ldots, a_l) = (t_1a_1, \ldots, t_la_l).
\]

Let \( X(T_l) = \text{Hom}(T_l, \mathbb{G}_m) \) be the character group of \( T_l \), and let \( \varepsilon_i \in X(T_l) \) be the character
\[
\varepsilon_i(t_1, \ldots, t_l) = t_i, \quad 1 \leq i \leq l.
\]

For \( 1 \leq i \leq m \), let
\[
\varphi_i = \sum_{t=1}^l (a_{it} - b_{it}) \varepsilon_t.
\]

Set \( T = \cap_{i=1}^m \ker \varphi_i \), and \( X^\circ = \{(x_1, \ldots, x_l) \in X \mid x_i \neq 0 \text{ for all } 1 \leq i \leq l \} \).

**Proposition 2.4.12.** Let notations be as above.

(1) There is a canonical action of \( T \) on \( X \).

(2) \( X^\circ \) is \( T \)-stable. Further, the action of \( T \) on \( X^\circ \) is simple and transitive.

(3) \( T \) is a subtorus of \( T_l \), and \( X \) is an equivariant affine embedding of \( T \).

**Proof 2.4.13.** (1) We consider the (obvious) action of \( T \) on \( \mathbb{A}^l \). Let \( (x_1, \ldots, x_l) \in X \), \( t = (t_1, \ldots, t_l) \in T \), and \( (y_1, \ldots, y_l) = t \cdot (x_1, \ldots, x_l) = (t_1x_1, \ldots, t_lx_l) \). Using
2.4. Generalities on toric varieties

the fact that \((x_1, \ldots, x_l)\) satisfies (*), we obtain

\[ y_1^{a_{i1}} \cdots y_l^{a_{il}} = t_1^{a_{i1}} \cdots t_l^{a_{il}} x_1^{a_{i1}} \cdots x_l^{a_{il}} = \lambda_i t_1^{b_{i1}} \cdots t_l^{b_{il}} x_1^{b_{i1}} \cdots x_l^{b_{il}} = \lambda_i y_1^{b_{i1}} \cdots y_l^{b_{il}}, \]

for all \(1 \leq i \leq m\), i.e. \((y_1, \ldots, y_l) \in X\). Hence \(t \cdot (a_1, \ldots, a_l) \in X\) for all \(t \in T\).

(2) Let \(x = (x_1, \ldots, x_l) \in X^\circ\), and \(t = (t_1, \ldots, t_l) \in T\). Then, clearly \(t \cdot (x_1, \ldots, x_l) \in X^\circ\). Considering \(x\) as a point in \(A^l\), the isotropy subgroup in \(T_l\) at \(x\) is \(\{\text{id}\}\). Hence the isotropy subgroup in \(T\) at \(x\) is also \(\{\text{id}\}\). Thus the action of \(T\) on \(X^\circ\) is simple.

Let \((x_1, \ldots, x_l), (x'_1, \ldots, x'_l) \in X^\circ\). Set \(t = (t_1, \ldots, t_l)\), where \(t_i = x_i/x'_i\). Then, clearly \(t \in T\). Thus \((x_1, \ldots, x_l) = t \cdot (x'_1, \ldots, x'_l)\). Hence the action of \(T\) on \(X^\circ\) is simple and transitive.

(3) Now, fixing a point \(x \in X^\circ\), we obtain from (2) that the orbit map \(t \mapsto t \cdot x\) is in fact an isomorphism of \(T\) onto \(X^\circ\). Also, since \(X\) is not contained in any of the coordinate hyperplanes, the open set \(X_i = \{(x_1, \ldots, x_l) \in X \mid x_i \neq 0\}\) is nonempty for all \(1 \leq i \leq l\). The irreducibility of \(X\) implies that the sets \(X_i, 1 \leq i \leq l\), are open dense in \(X\), and hence their intersection

\[ X^\circ = \bigcap_{i=1}^{l} X_i = \{(x_1, \ldots, x_l) \in X \mid x_i \neq 0 \text{ for any } i\} \]

is an open dense set in \(X\), and thus \(X^\circ\) is irreducible. This implies that \(T\) is irreducible (and hence connected). Thus \(T\) is a subtorus of \(T_l\). The assertion that \(X\) is an equivariant affine embedding of \(T\) follows from (1) and (2).

**Remark 2.4.14.** One can see that the ideal \(I(X)\) is a toric ideal in the sense of Definition 2.4.4; more precisely, \(A = \{\rho_i, 1 \leq i \leq l\}\), where \(\rho_i = \varepsilon_i|_T\) (here \(K[T]\) is identified with \(K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]\), and the character group \(X(T)\) with \(\mathbb{Z}^d\)).
2.4.15 Orbit decomposition in affine toric varieties

We shall preserve the notation of §2.4.2. As in §2.4.2, let $X$ be an affine toric variety with a torus action by $T$. Let $K[X] = K[S_\sigma]$. We shall denote $X$ also by $X_\sigma$.

Let us first recall the definition of faces of a convex polyhedral cone:

**Definition 2.4.16.** A face $\tau$ of $\sigma$ is a convex polyhedral sub cone of $\sigma$ of the form $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$, and is denoted $\tau < \sigma$.

We have that $X_\tau$ is a principal open subset of $X_\sigma$, namely,

$$X_\tau = (X_\sigma)_f$$

Each face $\tau$ determines a (closed) point $P_\tau$ in $X_\sigma$, namely, it is the point corresponding to the maximal ideal in $K[X](= K[S_\sigma])$ given by the kernel of $e_\tau : K[S_\sigma] \to K$, where for $u \in S_\sigma$, we have

$$e_\tau(u) = \begin{cases} 
1, & \text{if } u \in \tau^\perp \\
0, & \text{otherwise}
\end{cases}$$

**Remark 2.4.17.** As a point in $\mathbb{A}^l$, $P_\tau$ may be identified with the $l$-tuple with 1 at the $i$-th place if $\chi_i$ is in $\tau^\perp$, and 0 otherwise (here, as in §2.4.3, $\chi_i$ denotes the weight of the $T$-weight vector $y_i$ - the class of $x_i$ in $K[X_\sigma]$).

Let $O_\tau$ denote the $T$-orbit in $X_\sigma$ through $P_\tau$. We have the following orbit decomposition in $X_\sigma$:

$$X_\sigma = \bigcup_{\theta \leq \sigma} O_\theta$$

$$\overline{O_\tau} = \bigcup_{\theta \geq \tau} O_\theta$$

$$\dim \tau + \dim O_\tau = \dim X_\sigma$$
2.4. Generalities on toric varieties

(here, by dimension of a cone \( \tau \), one means the vector space dimension of the span of \( \tau \)).

See [8], [14] for details.

Thus \( \tau \mapsto \overline{O_\tau} \) defines an order reversing bijection between \{faces of \( \sigma \)\} and \{\( T \)-orbit closures in \( X_\sigma \)\}. In particular, we have the following two extreme cases:

1. If \( \tau \) is the 0-face, then \( P_\tau = (1, \cdots , 1) \) (as a point in \( \mathbb{A}^l \)), and \( O_\tau = T \), and is contained in \( X_\theta, \forall \theta < \sigma \). It is a dense open orbit.

2. If \( \tau = \sigma \), then \( P_\tau \) (as a point in \( \mathbb{A}^l \)) is the \( l \)-tuple with 1 at the \( i \)-th place if \( \chi_i \) is in \( \tau^\perp \), and 0 otherwise, and \( O_\tau = \{ P_\tau \} \), and is the unique closed orbit.

For a face \( \tau \), let us denote by \( N_\tau \) the sublattice of \( N \) (the \( \mathbb{Z} \)-dual of \( M \)) generated by the lattice points of \( \tau \). Let \( N(\tau) = N/N_\tau \), and \( M(\tau) \), the \( \mathbb{Z} \)-dual of \( N(\tau) \). For a face \( \theta \) of \( \sigma \) such that \( \theta \) contains \( \tau \) as a face, set

\[ \theta_\tau := (\theta + (N_\tau)_\mathbb{R})/(N_\tau)_\mathbb{R} \]

(here, for a lattice \( L \), \( L_\mathbb{R} = L \otimes \mathbb{R} \)). Then the collection \( \{ \theta_\tau, \sigma > \theta > \tau \} \) gives the set of faces of the cone \( \sigma_\tau (\subset N(\tau)_\mathbb{R}) \).

**Lemma 2.4.18.** For a face \( \tau < \sigma \), \( \overline{O_\tau} \) gets identified with the toric variety \( \text{Spec} \, K[S_{\sigma_\tau}] \). Further, \( K[\overline{O_\tau}] = K[S_{\sigma} \cap \tau^\perp] \).

**Proof 2.4.19.** The first assertion follows from the description of the orbit decomposition ( and the definition of \( \sigma_\tau \)). For the second assertion, we have

\[ S_{\sigma_\tau} = \sigma_\tau^\vee \cap M(\tau) = \sigma^\vee \cap (\tau^\perp \cap M) = S_\sigma \cap \tau^\perp \]

(note that we have, \( M(\tau) = \tau^\perp \cap M \)).
Chapter 3

Distributive lattices and Toric Varieties

3.1 Generalities on finite distributive lattices

In this section we recall elementary properties of Distributive lattice.

A partially ordered set which sometime also called a poset is defined as a set $P$ with a transitive relation $\geq$. If $x \geq y$, in phrase we call $x$ is greater than $y$ or $y$ is less than $x$. If $x, y \in P$ and $x \geq y$ and $x \neq y$ then we call $x$ is strictly greater than $y$, and write it as $x > y$. An element $x \in P$ is called maximal if $x$ is greater than all $y \in P$. In the case when an element $x \in P$ is less than all the elements $y \in P$ then we $y$ the minimal element of the poset.

**Definition 3.1.1.** A finite poset $P$ is called bounded if there exist both maximal and minimal elements and are unique. For the rest of the text we denote $\hat{1}$ as the unique maximal element of a bounded poset, and $\hat{0}$ for the minimal element of the same bounded poset.
Example Let $S$ be a non empty set with finite elements. Let $\mathcal{P}(S)$ be the set of subsets of $S$, then the inclusion among the subsets define a partial order. $A \geq B$ if $A \subseteq B$. In this poset there is a minimal element $\emptyset$ and the set $S$ considered as a subset of itself is a maximal element which are clearly unique. This gives an example of a bounded poset.

Definition 3.1.2. A totally ordered subset $C$ of a finite poset $P$ is called a chain, and the number $\#C - 1$ is called the length of the chain.

A chain in a bounded poset $p$ is called a maximal chain if it contains both the minimal and the maximal element.

Definition 3.1.3. A bounded poset $P$ is said to be graded (or also ranked) if all maximal chains have equal length (note that $\emptyset$ and $\bigwedge$ belong to any maximal chain).

Example The bounded poset in the above example is also a ranked poset; each maximal chain has length equal to the number of elements in the set $S$.

Definition 3.1.4. Let $P$ be a graded poset. The length of a maximal chain in $P$ is called the rank of $P$.

Definition 3.1.5. In a graded poset $P$ for elements $\lambda, \mu \in P$ with $\lambda \geq \mu$, the poset $\{\tau \in P \mid \mu \leq \tau \leq \lambda\}$ is called the interval from $\mu$ to $\lambda$, and denoted by $[\mu, \lambda]$. The interval is clearly a graded poset. We define the rank of $[\mu, \lambda]$ by $l_{\mu}(\lambda)$; if $\mu = \emptyset$, then we denote $l_{\mu}(\lambda)$ by just $l(\lambda)$.

Theorem 3.1.6. Let $P$ be a graded poset, $x, y \in P$ two element $x \geq y$ then the interval $[y, x]$ is a graded poset.

Proof 3.1.7. Lets call the interval $J$, $J$ gets the poset structure from the poset relation of the larger poset $P$, its also bounded since the element $x$ is the maximal
element where $y$ is the minimal element of the poset. To prove that all the maximal chains have same length lets assume that $M_1, M_2$ are two maximal chains with different lengths. We give an order relation $\geq$ on the totally ordered subsets of the graded poset $P$ given by the inclusion. Under this order any inductively ordered chain must have a maximal element since the set of all totally ordered chains in $P$ is finite. Note that this maximal is a maximal chain of the poset $P$. So we can extend any totally ordered chain of the lattice $P$ to a maximal chain.

**Remark 3.1.8.** In the later part of the work, we refer to $l(\lambda)$ as the level of $\lambda$ in $\mathcal{L}$.

**Definition 3.1.9.** Let $P$ be a graded poset, and $\lambda, \mu \in P$, with $\lambda \geq \mu$. The ordered pair $(\lambda, \mu)$ is called a cover (and we also say that $\lambda$ covers $\mu$) if $l_\mu(\lambda) = 1$.

### 3.1.10 Generalities on distributive lattices

**Definition 3.1.11.** A lattice is a partially ordered set $(\mathcal{L}, \leq)$ such that, for every pair of elements $x, y \in \mathcal{L}$, there exist elements $x \vee y$ and $x \wedge y$, called the join, respectively the meet of $x$ and $y$, defined by:

\[
\begin{align*}
x \vee y &\geq x, \quad x \vee y \geq y, \text{ and if } z \geq x \text{ and } z \geq y, \text{ then } z \geq x \vee y, \\
x \wedge y &\leq x, \quad x \wedge y \leq y, \text{ and if } z \leq x \text{ and } z \leq y, \text{ then } z \leq x \wedge y.
\end{align*}
\]

It is easy to check that the operations $\vee$ and $\wedge$ are commutative and associative.

**Definition 3.1.12.** An element $z \in \mathcal{L}$ is called the zero of $\mathcal{L}$, denoted by $\hat{0}$, if $z \leq x$ for all $x$ in $\mathcal{L}$. An element $z \in \mathcal{L}$ is called the one of $\mathcal{L}$, denoted by $\hat{1}$, if $z \geq x$ for all $x$ in $\mathcal{L}$.
3.1. Generalities on finite distributive lattices

Definition 3.1.13. Given a lattice $\mathcal{L}$, a subset $\mathcal{L}' \subset \mathcal{L}$ is called a sublattice of $\mathcal{L}$ if $x, y \in \mathcal{L}'$ implies $x \land y \in \mathcal{L}'$, $x \lor y \in \mathcal{L}'$.

Definition 3.1.14. Two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic if there exists a bijection $\varphi : \mathcal{L}_1 \to \mathcal{L}_2$ such that, for all $x, y \in \mathcal{L}_1$,

$$\varphi(x \lor y) = \varphi(x) \lor \varphi(y) \text{ and } \varphi(x \land y) = \varphi(x) \land \varphi(y).$$

Definition 3.1.15. A lattice is called distributive if the following identities hold:

$$x \land (y \lor z) = (x \land y) \lor (x \land z) \quad (3.1)$$
$$x \lor (y \land z) = (x \lor y) \land (x \lor z). \quad (3.2)$$

Example of a distributive lattice would be the poset of the powerset of a set $S$. Where the operations $\lor$ and $\land$ are given by union and intersection respectively.

Definition 3.1.16. An element $z$ of a lattice $\mathcal{L}$ is called join-irreducible (resp. meet irreducible) if $z = x \lor y$ (respectively $z = x \land y$) implies $z = x$ or $z = y$. The set of join-irreducible (resp. meet irreducible) elements of $\mathcal{L}$ is denoted by $J_\mathcal{L}$ (resp. $M_\mathcal{L}$), or just by $J$ (resp. $M$) if the reference to the lattice $\mathcal{L}$ is obvious in the context.

Examples: 1. The lattice of all subsets of the set $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{B}(n)$, and called the Boolean algebra of rank $n$.

A good class of example of distributive lattice would be the chain product lattices defined below.

2. Chain product lattice: Given an integer $n \geq 1$, $\mathcal{C}(n)$ will denote the chain $\{1 < \cdots < n\}$ and for $n_1, \ldots, n_d > 1$, $\mathcal{C}(n_1, \ldots, n_d)$ will denote the chain product lattice $\mathcal{C}(n_1) \times \cdots \times \mathcal{C}(n_d)$ consisting of all $d$-tuples
(\(i_1, \ldots, i_d\)), with \(1 \leq i_1 \leq n_1, \ldots, 1 \leq i_d \leq n_d\). For \((i_1, \ldots, i_d), (j_1, \ldots, j_d)\) in \(\mathcal{C}(n_1, \ldots, n_d)\), we define
\[
(i_1, \ldots, i_d) \leq (j_1, \ldots, j_d) \iff i_1 \leq j_1, \ldots, i_d \leq j_d.
\]
We have
\[
(i_1, \ldots, i_d) \lor (j_1, \ldots, j_d) = (\max\{i_1, j_1\}, \ldots, \max\{i_d, j_d\})
\]
\[
(i_1, \ldots, i_d) \land (j_1, \ldots, j_d) = (\min\{i_1, j_1\}, \ldots, \min\{i_d, j_d\}).
\]
\(\mathcal{C}(n_1, \ldots, n_d)\) is a finite distributive lattice, and its zero and one are \((1, \ldots, 1)\), \((n_1, \ldots, n_d)\) respectively.

Note that there is a total order \(<\) on \(\mathcal{C}(n_1, \ldots, n_d)\) extending \(<\), namely the lexicographic order, defined by \((i_1, \ldots, i_d) < (j_1, \ldots, j_d)\) if and only if there exists \(l < d\) such that \(i_1 = j_1, \ldots, i_l = j_l, i_{l+1} < j_{l+1}\). Also note that two elements \((i_1, \ldots, i_d) < (j_1, \ldots, j_d)\) are non-comparable with respect to \(\leq\) if and only if there exists \(1 < h \leq d\) such that \(i_h > j_h\).

Sometimes we denote the elements of \(\mathcal{C}(n_1, n_2, \ldots, n_d)\) by \(x_{i_1\ldots i_d}\), with \(1 \leq i_1 \leq n_1, \ldots, 1 \leq i_d \leq n_d\).

In the case \(x \neq y \in \mathcal{L}\), where \(\mathcal{L}\) is a lattice, when \(x\) is neither larger nor less than \(y\), we call \(x\) and \(y\) non comparable.

One has the following embedding of distributive lattices into sublattices of boolean algebra. (see [1]):

**Theorem 3.1.17.** Any finite distributive lattice is isomorphic to a sublattice of a Boolean algebra of finite rank, and hence, in particular, to a sublattice of a finite chain product lattice.

We will soon concentrate on the maximal chains of a distributive lattice, and in turn we will build up these maximal chains with covers. As a good understanding of the covers the following easy corollary to the definition of a cover will be useful.
Corollary 3.1.18. Let $(\tau, \lambda_1), (\tau, \lambda_2)$ be two covers in a distributive lattice $L$. Then $\lambda_1 \land \lambda_2$ is covered by both $\lambda_1$ and $\lambda_2$.

Definition 3.1.19. A sublattice of the form $\{\tau, \phi, \tau \lor \phi, \tau \land \phi\}$, with $\tau, \phi \in L$ non-comparable is called a diamond, and is denoted by $D\{\tau, \phi, \tau \lor \phi, \tau \land \phi\}$ or also just $D(\tau, \phi)$. The pair $(\tau, \phi)$ (respectively $(\tau \lor \phi, \tau \land \phi)$) is called the skew (respectively main) diagonal of the diamond $D(\tau, \phi)$.

A diamond of the form $\{x \lor y, x, y, x \land y\}$ where $x$ and $y$ are non comparable and $x \lor y$ covers both $x$ and $y$, hence $x$ and $y$ both covers $x \land y$ by the above corollary, will be called a simple diamond.

Definition 3.1.20. A subset $I$ of a poset $P$ is called an ideal of $P$ if for all $x, y \in P$,

$$x \in I \text{ and } y \leq x \text{ imply } y \in I.$$ 

Theorem 3.1.21. (Birkhoff) Let $\mathcal{L}$ be a distributive lattice with $\hat{0}$, and $P$ the poset of its nonzero join-irreducible elements. Then $\mathcal{L}$ is isomorphic to the lattice of finite ideals of $P$, by means of the lattice isomorphism

$$\alpha \mapsto I_\alpha = \{\tau \in P \mid \tau \leq \alpha\}, \quad \alpha \in \mathcal{L}.$$ 

Since in the later part of the work we would be required to collect the set of all non comparable pairs in a particular distributive lattice, $\mathcal{L}$ we set,

$$Q(\mathcal{L}) = \{(\tau, \varphi), \tau, \varphi \text{noncomparable}\}$$

In the sequel, $Q(\mathcal{L})$ will also be denoted by just $Q$ when there is no possible confusion.

We prove the following lemma to give another criterion for the join irreducibles in terms of the language of covers.
3.1. Generalities on finite distributive lattices

Lemma 3.1.22. \( J_L = \{ \tau \in L \mid \text{there exists at most one cover of the form } (\tau, \lambda) \} \).

Proof 3.1.23. The proof is clear from the observation that if \( \tau \in L \) covers more than one element, let us say two of them are \( \alpha_1, \alpha_2 \), then, consider \( \alpha_1 \lor \alpha_2 \) which is larger than both \( \alpha_1 \) and \( \alpha_2 \). But since \( \tau \geq \alpha_1 \) and \( \tau \geq \alpha_2 \), we have \( \tau \geq \alpha_1 \lor \alpha_2 \geq \alpha_1 \) but since \( \tau \) was a cover for \( \alpha_1 \) this means that either \( \alpha_1 = \alpha_1 \lor \alpha_2 \) which in turn implies that \( \alpha_1 \geq \alpha_2 \). Once again the repeated argument considering that \( \tau \) is a cover of \( \alpha_2 \) we must have \( \alpha_1 = \alpha_2 \) contrary to our assumption that these are two different elements. On the other hand if \( \alpha_1 \lor \alpha_2 = \tau \) then \( \tau \) being the join of these two is no longer a join irreducible contradicting our hypothesis. This ends the proof.

For \( \alpha \in L \), let \( I_\alpha \) be the ideal corresponding to \( \alpha \) under the isomorphism in Theorem 3.1.21.

Lemma 3.1.24. Let \( (\tau, \lambda) \) be a cover in \( L \). Then \( I_\tau \) equals \( I_\lambda \cup \{ \beta \} \) for some \( \beta \in J_L \).

Proof 3.1.25. If \( \tau \in J_L \), then \( \lambda \) is the unique element covered by \( \tau \); it is clear that in this case that \( I_\tau = I_\lambda \cup \{ \tau \} \) since by definition \( I_\tau = \{ \alpha, \alpha \leq \tau \} \) which is same as \( \{ \tau \} \cup \{ \alpha \leq \lambda \} = I_\tau \cup \{ \tau \} \).

Let then \( \tau \not\in J_L \). By the previous lemma we know that \( \tau \) must cover more than one element. Let \( \lambda' \) be another element covered by \( \tau \). Let \( \phi = \lambda \land \lambda' \). Now note that \( \phi = \tau \) since \( \tau \) covers \( \lambda \) and we have \( \tau \geq \phi \leq \lambda \) which is possible only when \( \tau = \phi \). Now by induction on the level of \( \tau \), we have, \( I_\lambda = I_\phi \cup \{ \beta \} \), for some \( \beta \in J_L \) (note that \( (\lambda, \phi), (\lambda', \phi) \) are both covers in view of Corollary 3.1.18). Then \( I_\lambda \cup \{ \beta \} \) is an ideal contained in \( I_\tau \) (note that if \( \gamma \in J_L \) is such that \( \gamma < \beta \), then \( \gamma \) is in \( I_\lambda \), and hence in \( I_\tau \)). Hence if \( \theta \) is the element of \( L \) corresponding to
3.2. The variety $X(\mathcal{L})$

the ideal $I_{\lambda'} \cup \{\beta\}$ (under the bijection given by Theorem 3.1.21), then we have, $\lambda' < \theta \leq \tau$. Hence, we obtain that $\theta = \tau$ (since $(\tau, \lambda')$ is a cover).

**Starting point of induction:** Let $\tau$ be an element of least length among all $\theta$’s such that $\theta$ covers an element $\theta'$. Then the above reasoning implies clearly that $\tau \in J_\mathcal{L}$, and therefore, $I_\tau = I_\lambda \cup \{\tau\}$.

Note that as a corollary to the above lemma we get that the length of a maximal chain is less than equal to the number of elements in the join irreducible set. In fact we have the following proposition.

**Proposition 3.1.26.** Any maximal chain in (the ranked poset) $\mathcal{L}$ has cardinality equal to $\#J_\mathcal{L}$.

**Proof 3.1.27.** which follows from the observation $I_1$ equals $J_\mathcal{L}$ along with the lemma 3.1.24

3.2 The variety $X(\mathcal{L})$

Consider the polynomial algebra $K[x, \alpha \in \mathcal{L}]$; let $I(\mathcal{L})$ be the ideal generated by $\{x_\alpha x_\beta - x_{\alpha \lor \beta} x_{\alpha \land \beta}, \alpha, \beta \in \mathcal{L}\}$. Then one knows (cf.[13]) that $K[x, \alpha \in \mathcal{L}] / I(\mathcal{L})$ is a normal integral domain; in particular, we have that $I(\mathcal{L})$ is a prime ideal.

Let $X(\mathcal{L})$ be the affine variety of the zeroes in $K^n$ of $I(\mathcal{L})$ (where, $n = \#\mathcal{L}$). Since the ideal $I(\mathcal{L})$ is a binomial ideal we can say that $X(\mathcal{L})$ is a affine normal variety defined by binomials, and hence by §we know that the variety is a Toric embedding. That is its an affine variety with a dense torus $T$ with an action on $X(\mathcal{L})$ extending the action of $T$ on $T$. We also know that since the toric embedding $X(\mathcal{L})$ is normal it must be a toric variety associated to a normal cone.
3.2. The variety $X(\mathcal{L})$

To discuss the singularity of the variety $X(\mathcal{L})$ we will first calculate the dimension of $X(\mathcal{L})$. Since a toric variety $X$ contains a dense torus $T$ the dimension of $X$ is same as the dimension of the torus $T$. To compute the dimension of the variety $X(\mathcal{L})$ we will compute the dimension of the torus $T$.

Before we calculate the dimension we must define a suitable combinatorial set, which will be essential later. Let

$$I = \{((\tau, \phi, \tau \vee \phi, \tau \wedge \phi)) \mid (\tau, \phi) \in Q\},$$

where

$$Q = \{(\tau, \phi) \mid \tau, \phi \in \mathcal{L} \text{ non-comparable}\}.$$

Let $T_i = (K^*)^l$, $\pi : X(T_i) \to X(T)$ be the canonical map, given by restriction, and for $\chi \in X(T_i)$, denote $\pi(\chi)$ by $\chi$. Let us fix a $\mathbb{Z}$-basis $\{\chi_\tau \mid \tau \in \mathcal{L}\}$ for $X(T_i)$.

For a diamond $D = (\tau, \phi, \tau \vee \phi, \tau \wedge \phi) \in I$, let $\chi_D = \chi_{\tau \vee \phi} + \chi_{\tau \wedge \phi} - \chi_{\tau} - \chi_{\phi}$.

**Lemma 3.2.1.** We have

1. $X(T) \simeq X(T_d)/\ker \pi$.
2. $\ker \pi$ is generated by $\{\chi_D \mid D \in I\}$.

**Proof 3.2.2.** The canonical map $\pi$ is, in fact, surjective, since $T$ is a subtorus of $T_d$. Now (1) follows from this. The assertion (2) follows from the definition of $T$.

Let $X(J_{\mathcal{L}})$ be the $\mathbb{Z}$-span of $\{\chi_\theta, \theta \in J_{\mathcal{L}}\}$. As an immediate consequence of the above Lemma, we have

**Corollary 3.2.3.** $\pi$ maps $X(J_{\mathcal{L}})$ isomorphically onto its image.

**Proof 3.2.4.** The result follows since there cannot be a diamond $D$ contained completely in $J_{\mathcal{L}}$. 

3.2. The variety $X(\mathcal{L})$  

3.2.5 Some more lattice theory

For $\alpha \in \mathcal{L}$, let $I_\alpha$ be the ideal corresponding to $\alpha$ under the isomorphism in Theorem 3.1.21. Let us define

$$\psi_\alpha = \sum_{\theta \in I_\alpha} \chi_\theta.$$ 

Taking a total order on $\mathcal{L}$ extending the partial order, we see easily that the transition matrix expressing $\psi_\alpha$’s in terms of $\chi_\theta$’s is triangular with diagonal entries equal to 1. Let $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ denote the total order on the lattice $\mathcal{L}$ extending the partial order relation on $\mathcal{L}$. Further if we call $\delta_{\alpha,\beta}$ the indicator function whether $x_\beta$ occurs in $\psi_\alpha$. That is $\delta_{\alpha,\beta} = 1$ if and only if $x_\beta$ is a summand of $\psi_\alpha$. Then we have the following matrix:

$$\Psi = \begin{pmatrix} \psi_{\alpha_1} \\ \psi_{\alpha_2} \\ \vdots \\ \psi_{\alpha_n} \end{pmatrix} = (\delta_{\alpha,\beta})$$ 

Since the matrix is upper triangular with diagonal entries equal to 1 we have the following obvious lemma.

**Lemma 3.2.6.** $\{\psi_\alpha, \alpha \in \mathcal{L}\}$ is a $\mathbb{Z}$-basis for $X(T_n)$ the torus $T_n$ being embedded as an open set in the affine space $A^n$.

Clearly Since $\psi_\alpha$ is a basis for the torus $T_n$ and the torus $T$ embedded into it, we get the following lemma.

If we consider the surjective (restriction) map $\pi : X(T_n) \rightarrow X(T)$, and denoting $\pi (\psi_\alpha)$ by $\overline{\psi}_\alpha$, we obtain

**Lemma 3.2.7.** The set $\{\overline{\psi}_\alpha | \alpha \in \mathcal{L}\}$ generates $X(T)$ as a $\mathbb{Z}$-module.
3.2. The variety $X(\mathcal{L})$

Since we know that rank of $T$ is equal to the rank of the character group $X(T)$. Calculating the dimension of the kernel of the surjection map will be sufficient to calculate the dimension of the torus $T$.

**Lemma 3.2.8.** Let $\alpha \in \mathcal{L}$. Then $\overline{\psi}_\alpha$ is in the $\mathbb{Z}$-span of $\{\overline{\chi_\theta}, \theta \leq \alpha, \theta \in J_\mathcal{L}\}$. In particular, $\{\overline{\chi_\theta}, \theta \leq \alpha, \theta \in J\}$ generates $X(T)$ as a $\mathbb{Z}$-module.

**Proof 3.2.9.** (by induction)

**Case 1:** $\alpha \in J_\mathcal{L}$ (may suppose $\alpha \neq 0$, since if $\alpha = 0$, then the result is clear). There exists a unique $\beta \in \mathcal{L}$ covered by $\alpha$. We have

$$\overline{\psi}_\alpha = \overline{\chi_\alpha} + \overline{\psi}_\beta$$

By induction, $\overline{\psi}_\beta$ belongs to the $\mathbb{Z}$-span of $\{\overline{\chi_\theta}, \theta \leq \beta, \theta \in J_\mathcal{L}\}$. The result follows from this.

**Case 2:** $\alpha \notin J_\mathcal{L}$.

Hypothesis implies that there exist $\alpha_1, \alpha_2$ both of which are covered by $\alpha$. Then $\alpha_1, \alpha_2$ are non-comparable since if possible $\alpha \geq \alpha_1 \geq \alpha_2$ will contradict the hypothesis that $\alpha$ covers $\alpha_2$. We have $\alpha_1 \lor \alpha_2 = \alpha$; let $\beta = \alpha_1 \land \alpha_2$. We have

$$\overline{\psi}_\alpha = \overline{\psi}_{\alpha_1} + \overline{\psi}_{\alpha_2} - \overline{\psi}_\beta$$

The result follows by induction.

Combining the above Lemma with Corollary 3.2.3, we obtain

**Proposition 3.2.10.** The set $\{\overline{\chi_\tau} \mid \tau \in J_\mathcal{L}\}$ is a $\mathbb{Z}$-basis for $X(T)$.

Now, since $\dim X(\mathcal{L}) = \dim T$, we obtain

**Theorem 3.2.11.** The dimension of $X(\mathcal{L})$ is equal to $\#J_\mathcal{L}$. 

3.2. The variety $X(\mathcal{L})$

Combining the above theorem with Lemma 3.1.24 and Corollary 3.1.26, we obtain

**Corollary 3.2.12.**

1. The dimension of $X_\mathcal{L}$ is equal to the cardinality of the set of elements in a maximal chain in $\mathcal{L}$.

2. Fix any chain $\beta_1 < \cdots < \beta_d$, $d$ being $\#J_\mathcal{L}$. Let $\gamma_{i+1}$ be the element of $J_\mathcal{L}$ corresponding to the cover $(\beta_{i+1}, \beta_i)$, $i \geq 1$; set $\gamma_1 = \beta_1$. Then $J_\mathcal{L} = \{\gamma_1, \cdots, \gamma_d\}$

**Definition 3.2.13.** For a finite distributive lattice $\mathcal{L}$, we call the cardinality of $J_\mathcal{L}$ the dimension of $\mathcal{L}$, and we denote it by $\dim \mathcal{L}$.

**Lemma 3.2.14.** Let $P = (P_\theta)_{\theta \in \mathcal{L}} \in X(\mathcal{L})$ be such that $P_\tau \neq 0$ for any $\tau \in J_\mathcal{L}$. Then $P_\theta \neq 0$ for any $\theta \in \mathcal{L}$.

**Proof 3.2.15.** (by induction) Let $\theta \in \mathcal{L}$. If $\theta \in J_\mathcal{L}$, there is nothing to check. Let then $\theta \in \mathcal{L} \setminus J_\mathcal{L}$.

Let $\theta$ be a minimal element of $\mathcal{L} \setminus J_\mathcal{L}$. This implies that every $\tau \in \mathcal{L}$ such that $\tau < \theta$ belongs to $J_\mathcal{L}$. The fact that $\theta \in \mathcal{L} \setminus J_\mathcal{L}$ implies that there are at least two elements $\theta_1, \theta_2$ of $\mathcal{L}$ which are covered by $\theta$. Note that $\theta_1, \theta_2$ are not comparable.

We have $\theta_1 \lor \theta_2 = \theta$. Let $\mu = \theta_1 \land \theta_2$. We have $P_\theta P_\mu = P_{\theta_1} P_{\theta_2}$. Now $P_{\theta_1} \neq 0$, $P_{\theta_2} \neq 0$, since $\theta_1, \theta_2 \in J_\mathcal{L}$. Hence we obtain that $P_\theta \neq 0$.

Let now $\phi$ be any element of $\mathcal{L} \setminus J_\mathcal{L}$. Assume, by induction, that $P_\tau \neq 0$ for any $\tau < \phi$. Since $\phi \not\in J_\mathcal{L}$, there are at least two elements $\phi_1, \phi_2$ of $\mathcal{L}$ which are covered by $\phi$. We have $\phi_1 \lor \phi_2 = \phi$, $P_\phi P_\delta = P_{\phi_1} P_{\phi_2}$, where $\delta = \phi_1 \land \phi_2$. Also $P_{\phi_1} \neq 0$, $P_{\phi_2} \neq 0$ (since $\phi_1, \phi_2$ are both $< \phi$). Hence we obtain $P_\phi \neq 0$.

We would like to list all the toric subvarieties of the toric variety $X(\mathcal{L})$, towards that we define an important class of distributive sublattices.
3.2. The variety $X(L)$

**Definition 3.2.16.** A sublattice $L'$ of $L$ is called an embedded sublattice of $L$ if

$$\tau, \phi \in L, \quad \tau \lor \phi, \tau \land \phi \in L' \quad \Rightarrow \quad \tau, \phi \in L'.$$

Given a sublattice $L'$ of $L$, let us consider the variety $X(L')$, and consider the canonical embedding $X(L') \hookrightarrow \mathbb{A}(L') \hookrightarrow \mathbb{A}(L)$ (here $\mathbb{A}(L') = \mathbb{A}^#L'$, $\mathbb{A}(L) = \mathbb{A}^#L$).

In the following lemma we classify all the toric subvarieties of the variety $X(L)$.

**Proposition 3.2.17.** $X(L')$ is a subvariety of $X(L)$ if and only if $L'$ is an embedded sublattice of $L$.

**Proof 3.2.18.** Under the embedding $X(L') \hookrightarrow \mathbb{A}(L)$, $X(L')$ can be identified with

$$\{(x_\tau)_\tau \in L \mid x_\tau = 0 \text{ if } \tau \not\in L', \quad x_\tau x_\phi = x_{\tau \lor \phi} x_{\tau \land \phi} \text{ for } \tau, \phi \in L' \text{ noncomparable }\}.$$ 

Let $\eta'$ be the generic point of $X(L')$. We have $X(L') \subset X(L)$ if and only if $\eta' \in X(L)$.

Assume that $\eta' \in X(L)$. Let $\tau, \phi$ be two noncomparable elements of $L$ such that $\tau \lor \phi, \tau \land \phi$ are both in $L'$. We have to show that $\tau, \phi \in L'$. If possible, let $\tau \not\in L'$.

This implies $\eta'_r = 0$. Hence either $\eta'_{r \lor \phi} = 0$, or $\eta'_{r \land \phi} = 0$, since $\eta' \in X(L)$. But this is not possible (note that $\tau \lor \phi, \tau \land \phi$ are in $L'$, and hence $\eta'_{r \lor \phi}$ and $\eta'_{r \land \phi}$ are both nonzero).

Assume now that $L'$ is an embedded sublattice. We have to show that $\eta' \in X(L)$.

Let $\tau, \phi$ be two non-comparable elements of $L$. The fact that $L'$ is a sublattice implies that if $\eta'_{r \lor \phi}$ or $\eta'_{r \land \phi}$ is zero, then either $\eta'_r$, or $\eta'_\phi$ is zero. Also, the fact that $L'$ is an embedded sublattice implies that if $\eta'_r$ or $\eta'_\phi$ is zero, then either $\eta'_{r \lor \phi}$ or $\eta'_{r \land \phi}$ is zero. Further, when $\tau, \phi, \tau \lor \phi, \tau \land \phi \in L'$,

$$\eta'_r \eta'_\phi = \eta'_{r \lor \phi} \eta'_{r \land \phi}.$$
3.2. The variety $X(\mathcal{L})$

Thus $\eta'$ satisfies the defining equations of $X(\mathcal{L})$, and hence $\eta' \in X(\mathcal{L})$. 
Chapter 4

A Basis for Cotangent Space

4.1 Cone and dual cone of $X_L$:

In this section we shall determine the cone and the dual cone of $X_L$, $L$ being a finite distributive lattice. As in the previous sections, denote the poset of join-irreducibles of $L$ by $J_L$; let $d = \# J_L$. Denote $J_L = \{\beta_1, \cdots, \beta_d\}$. Let $T$ be a $d$-dimensional torus. Identifying $T$ with $(K^*)^d$, let $\{f_{\beta}, \beta \in J_L\}$ denote the standard $\mathbb{Z}$-basis for $X(T)$, namely, for $u \in T$, $u = (u_{\beta}, \beta \in J_L)$, $f_{\beta}(u) = u_{\beta}$. Then $K[X(T)]$ may be identified with $K[u_{\beta_1}^{\pm 1}, \cdots, u_{\beta_d}^{\pm 1}]$, the ring of Laurent polynomials. If $\chi \in X(T)$, say $\chi = \sum a_{\beta} f_{\beta}$, then as an element of $K[X(T)]$, $\chi$ will also be denoted by $\prod u_{\beta}^{a_{\beta}}$.

Denote by $I(J_L)$ the poset of ideals of $J_L$. For $A \in I(J_L)$, set

$$f_A := \sum_{z \in A} f_z$$

Recall (cf. Theorem 3.1.21) the bijection $L \rightarrow I(J_L), \theta \mapsto A_\theta := \{\tau \in J_L \mid \tau \leq \theta\}$.

**Lemma 4.1.1.** $\{f_{A_\theta}, \theta \in J_L\}$ is a $\mathbb{Z}$-basis for $X(T)$. 

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4.1. Cone and dual cone of $X_L$:

**Proof 4.1.2.** Take a total order on $J_L$ extending the partial order. Then the matrix expressing $\{f_{A\theta}, \theta \in J_L\}$ in terms of the basis $\{f_{\theta}, \theta \in J_L\}$ is easily seen to be triangular with diagonal entries equal to 1. The result follows from this.

Let $A = \{f_A, A \in \mathcal{J}(J_L)\}$. Then as a consequence of the above Lemma, we obtain

**Corollary 4.1.3.** $A$ generates $X(T)$; in particular, $d_A = d$, $d_A$ being as in Remark 2.4.7.

For $A \in \mathcal{J}(J_L)$, denote by $m_A$ the monomial:

$$m_A := \prod_{\tau \in A} u_\tau$$

in $K[X(T)]$. If $\alpha$ is the element of $\mathcal{L}$ such that $I_\alpha = A$ (cf. Theorem 3.1.21), then we shall denote $m_A$ also by $m_\alpha$. Consider the surjective algebra map

$$F : K[X_\alpha, \alpha \in \mathcal{L}] \to K[m_A, A \in \mathcal{J}(J_L)] \subset K[X(T)], \quad X_\alpha \mapsto m_A, \quad A = I_\alpha$$

Then with notation as in Definition 2.4.4, we have, $F = \pi_A$, $\ker F = I_A$. Further, Proposition 2.4.7 and Corollary 4.1.3 imply the following

**Proposition 4.1.4.** $V(I_A) = \text{Spec } K[m_A, A \in \mathcal{J}(J_L)]$ and is of dimension $d (= \#J_L)$ (here, $V(I_A)$ is as in Remark 2.4.7)

Let us denote $V(I_A)$ by $Y$.

**Lemma 4.1.5.** Kernel of $F$ is generated by $\{X_\alpha X_\beta - X_{\alpha \lor \beta} X_{\alpha \land \beta}, \alpha, \beta \in \mathcal{L}\}$

**Proof 4.1.6.** We have (in view of Lemma 3.1.24) that if $(\alpha, \lambda)$ is a cover, then $I_\alpha$ equals $I_\lambda \cup \{\beta\}$ for some $\beta \in J_L$. A repeated application of this result implies that if $\gamma \leq \alpha$, and $m = l(\alpha) - l(\gamma)$ (here, $l(\beta)$ denotes the level of $\beta$ (cf. Remark 3.1.8)), then there exist $\alpha_1 \cdots, \alpha_m$ in $J_L$ such that $I_A \setminus I_\gamma = \{\alpha_1 \cdots, \alpha_m\}$. Let
now $\beta, \beta'$ be two non-comparable elements in $L$. Let $\alpha = \beta \lor \beta', \gamma = \beta \land \beta'$. Let $l(\beta) - l(\gamma) = r, l(\beta') - l(\gamma) = s$. Then there exist $\beta_1, \ldots, \beta_r, \beta'_1, \ldots, \beta'_s$ in $J_L$ such that

$$I_\beta \setminus I_\gamma = \{\beta_1, \ldots, \beta_r\}$$
$$I_{\beta'} \setminus I_\gamma = \{\beta'_1, \ldots, \beta'_s\}$$
$$I_\alpha \setminus I_\beta = \{\beta'_1, \ldots, \beta'_s\}$$
$$I_\alpha \setminus I_{\beta'} = \{\beta_1, \ldots, \beta_r\}$$

Hence we obtain

$$m_\beta = m_\gamma u_{\beta_1} \cdots u_{\beta_r}$$
$$m_{\beta'} = m_\gamma u_{\beta'_1} \cdots u_{\beta'_s}$$
$$m_\alpha = m_\beta u_{\beta'_1} \cdots u_{\beta'_s}$$
$$m_\alpha = m_{\beta'} u_{\beta_1} \cdots u_{\beta_r}$$

From this it follows that

$$m_\alpha m_\gamma = m_\beta m_{\beta'}$$

Thus for each diamond in $L$, i.e., a quadruple $(\beta, \beta', \beta \lor \beta', \beta \land \beta')$, we have $X_\beta X_{\beta'} - X_{\beta \lor \beta'} X_{\beta \land \beta'}$ is in the kernel of the surjective map $F$. Hence $F$ factors through $K[X_L]$; hence, we obtain closed immersions (of irreducible varieties):

$$Y \hookrightarrow X_L \hookrightarrow A^{\#L}$$

(Y being $\mathcal{V}(I_A)$). But dimension considerations imply that $Y = X_L$ (note that in view of Proposition 4.1.4, $dim Y = d = dim X_L$ (cf. Theorem 3.2.11)), and the result follows.

As an immediate consequence, we obtain (as seen in the proof of the above Lemma)

**Theorem 4.1.7.** We have an isomorphism $K[X_L] \cong K[m_A, A \in J(J_L)]$. 
4.1. Cone and dual cone of $X_L$:

Denote $M := X(T)$, the character group pf $T$. Let $N = \mathbb{Z}$-dual of $M$, and \{${e_y, y \in J_L}$\} be the basis of $N$ dual to \{${f_z, z \in J_L}$\}. As above, for $A \in \mathcal{J}(J_L)$, let

$$f_A := \sum_{z \in A} f_z$$

Let $V = N_{\mathbb{R}}(= N \otimes_{\mathbb{Z}} \mathbb{R})$. Let $\sigma \subset V$ be the cone such that $X_L = X_{\sigma}$. Let $\sigma^\vee \subset V^*$ be the cone dual to $\sigma$. Let $S_\sigma = \sigma^\vee \cap M$, so that $K[X_L]$ equals the semi group algebra $K[S_\sigma]$.

As an immediate consequence of Theorem 4.1.7, we have

**Proposition 4.1.8.** The semigroup $S_\sigma$ is generated by \{${f_A, A \in \mathcal{J}(J_L)}$\}.

Let $M(J_L)$ be the set of maximal elements in the poset $J_L$. Let $Z(J_L)$ denote the set of all covers in the poset $J_L$ (i.e., ($z, z'$), $z > z'$ in the poset $J_L$, and there is no other element $y \in J_L$ such that $z > y > z'$). For a cover $(y, y') \in Z(J_L)$, denote

$$v_{y,y'} := e_{y'} - e_y$$

**Proposition 4.1.9.** The cone $\sigma$ is generated by \{${e_z, z \in M(J_L), v_{y,y'}, (y, y') \in Z(J_L)}$\}.

**Proof 4.1.10.** Let $\theta$ be the cone generated by \{${e_z, z \in M(J_L), v_{y,y'}, (y, y') \in Z(J_L)}$\}. Then clearly any $u$ in $S_\sigma$ is non-negative on the generators of $\theta$, and hence $\sigma^\vee \subseteq \theta^\vee$. We shall now show that $\theta^\vee \subseteq \sigma^\vee$, equivalently, we shall show that $S_\theta \subseteq S_\sigma$. Let $f \in M$, say, $f = \sum_{z \in J_L} a_z f_z$. Then it is clear that $f$ is in $S_\theta$ if and only if

$$a_z \geq 0, \text{ for } z \text{ maximal in } J_L, \text{ and } a_x \geq a_y, \text{ for } x, y \in J_L, x < y$$

**Claim:** Let $f \in M$. Then $f$ has property $(\ast)$ if and only if $f = \sum_{A \in \mathcal{J}(J_L)} c_A f_A$,.
4.1. Cone and dual cone of $X_L$:

$c_A \in \mathbb{Z}_+$.  

Note that Claim implies that $S_\theta \subseteq S_\sigma$, and the required result follows.  

Proof of Claim: The implication $\Leftarrow$ is clear.  

The implication $\Rightarrow$: Let $f \in S_\theta$, say, $f = \sum_{z \in J_L} a_z f_z$. The hypothesis that $f$ has property (*) implies that $a_z$ being non-negative for $z$ maximal in $J_L$, $a_x$ is non-negative for all $x \in J_L$. Thus $f = \sum_{z \in J_L} a_z f_z, a_z \in \mathbb{Z}_+, z \in J_L$. Further, the property in (*) that $a_x \geq a_y$ if $x < y$ implies that  

$$\{x \mid a_x \neq 0\}$$

is an ideal in $J_L$. Call it $A$; in the sequel we shall denote $A$ also by $I_f$. Let $m = \min\{a_x, x \in A\}$. Then either $f = m f_A$ in which case the Claim follows, or $f = m f_A + f_1$, where $f_1$ is in $S_\theta$ (note that $f_1$ also has property (*)); further, $I_{f_1}$ is a proper subset of $A$. Thus proceeding we arrive at positive integers $m_1, \cdots, m_r$, elements $f_1, \cdots, f_r$ (in $S_\theta$) with $A_1 := I_{f_1}$, an ideal in $J_L$, and a proper subset of $A_i := I_{f_{i-1}}$ such that  

$$f = m f_A + m_1 f_{A_1} + \cdots + m_r f_{A_r}.$$  

(In fact, at the last step, we have, $f_r = m_r f_{A_r}$.) The Claim and hence the required result now follows.  

4.1.11 Analysis of faces of $\sigma$

We shall concern ourselves just with the closed points in $X_L$. So in the sequel, by a point in $X_L$, we shall mean a closed point. Let $\tau$ be a face of $\sigma$. Let $P_\tau$ be the distinguished point of $O_\tau$ with the associated maximal ideal being the kernel
of the map

\[ K[S_\sigma] \to K, \]

\[ u \in S_\sigma, u \mapsto \begin{cases} 
1, & \text{if } u \in \tau^\perp \\
0, & \text{otherwise}
\end{cases} \]

Then for a point \( P \in X_L \) (identified with a point in \( A^l \)), denoting by \( P(\alpha) \), the \( \alpha \)-th co-ordinate of \( P \), we have,

\[ P_\tau(\alpha) = \begin{cases} 
1, & \text{if } f_{I_\alpha} \in \tau^\perp \\
0, & \text{otherwise}
\end{cases} \]

With notation as above, let \( D_\tau = \{ \alpha \in L \mid P_\tau(\alpha) \neq 0 \} \)

We have,

**Lemma 4.1.12.** \( D_\tau \) is an embedded sublattice of \( L \).

**Proof 4.1.13.** Let \( \theta, \delta \) be a pair of non-comparable elements in \( D_\tau \). The fact that \( P_\tau(\theta), P_\tau(\delta) \) are non-zero, together with the diamond relation \( x_\theta x_\delta = x_{\theta \lor \delta} x_{\theta \land \delta} \) implies that

\( P_\tau(\theta \lor \delta), P_\tau(\theta \land \delta) \) are again non-zero, and are equal to 1 (note that any non-zero co-ordinate in \( P_\tau \) is in fact equal to 1). Thus, \( D_\tau \) is a sublattice of \( L \).

The above reasoning implies that if \( \theta, \delta \) in \( D_\tau \) form the main diagonal in a diamond \( D \) in \( L \), then \( P_\tau(\alpha), P_\tau(\beta) \) are non-zero, and are equal to 1, where \( \alpha, \beta \) form the skew diagonal of \( D \). Hence \( \alpha, \beta \) are in \( D \), and hence \( D \subset D_\tau \). Thus we obtain that \( D_\tau \) is an embedded sublattice of \( L \).

Conversely, we have
4.1. Cone and dual cone of $X_L$:

**Lemma 4.1.14.** Let $\mathcal{D}$ be an embedded sublattice in $L$. Then $\mathcal{D}$ determines a unique face $\tau$ of $\sigma$ such that $D_\tau$ equals $\mathcal{D}$.

**Proof 4.1.15.** Denote $P$ to be the point in $A^\#_L$ with 

$$P(\alpha) = \begin{cases} 
1, & \text{if } \alpha \in \mathcal{D} \\
0, & \text{otherwise}
\end{cases}$$

Then $P$ is in $L$ (since $\mathcal{D}$ is an embedded sublattice in $L$, the co-ordinates of $P$ satisfy all the diamond relations). Set 

$$u = \sum_{\alpha \in \mathcal{D}} f_{I_\alpha}, \quad \tau = \sigma \cap u^\perp$$

Then clearly, $P = P_\tau$ and $D_\tau = \mathcal{D}$

Thus in view of the two Lemmas above, we have a bijection 

$$\{ \text{faces of } \sigma \} \xleftrightarrow{\text{bij}} \{ \text{embedded sublattices of } L \}$$

**Proposition 4.1.16.** Let $\tau$ be a face of $\sigma$. Then we have $\overline{O_\tau} = X_{D_\tau}$.

**Proof 4.1.17.** Recall (cf. Lemma 2.4.18) that $K[O_\tau] = K[S_\sigma \cap \tau^\perp]$. From the description of $P_\tau$, we have that $\tau^\perp$ is the span of $\{f_{I_\alpha}, \alpha \in D_\tau\}$; hence 

$$S_\sigma \cap \tau^\perp = \{ \sum_{\alpha \in D_\tau} c_\alpha f_{I_\alpha}, c_\alpha \in \mathbb{Z}^+ \}$$

Thus we obtain 

$$S_\sigma \cap \tau^\perp = S_{\sigma_\tau}$$

(here, $\sigma_\tau$ is as in Lemma 2.4.18). On the other hand, if $\eta$ is the cone associated to the toric variety $X_{D_\tau}$, then by Proposition 4.1.8 we have that $S_\eta$ is the semigroup generated by $\{f_{I_\alpha}, \alpha \in D_\tau\}$. Hence we obtain that $\eta = \sigma_\tau$ (in view of (*) and (**)'). The required result now follows.
4.2 Generation of the cotangent space at $P_\tau$

For $\alpha \in \mathcal{L}$, let us denote the image of $X_\alpha$ in $R_\mathcal{L}$ (under the surjective map

$K[X_\alpha, \alpha \in \mathcal{L}] \rightarrow R_\mathcal{L}$) by $x_\alpha$. Let $R_\tau = K[X(D_\tau)]$, the co-ordinate ring of $X(D_\tau)$, and $\pi_\tau : R_\mathcal{L} \rightarrow R_\tau$ be the canonical surjective map induced by the closed immersion $X(D_\tau) \hookrightarrow X_\mathcal{L}$; clearly, kernel of $\pi_\tau$ is generated by $\{x_\theta, \theta \in \mathcal{L} \setminus D_\tau\}$. Set

$$F_\alpha = \begin{cases} x_\alpha, & \text{if } \alpha \not\in D_\tau \\ 1 - x_\alpha, & \text{if } \alpha \in D_\tau \end{cases}$$

Denoting by $M_\tau$, the maximal ideal in $R_\mathcal{L}$ corresponding to $P_\tau$, we have (cf. §4.1.11)

**Lemma 4.2.1.** The ideal $M_\tau$ is generated by $\{F_\alpha, \alpha \in \mathcal{L}\}$.

**A set of generators for the cotangent space $M_\tau/M^2_\tau$**

For $F \in M_\tau$, let $\overline{F}$ denote the class of $F$ in $M_\tau/M^2_\tau$.

**Lemma 4.2.2.** Fix a maximal chain $\Gamma$ in $D_\tau$. For any $\beta \in D_\tau \setminus \Gamma$, we have that in $M_\tau/M^2_\tau$, $\overline{F_\beta}$ is in the span of $\{\overline{F_\gamma}, \gamma \in \Gamma\}$

**Proof 4.2.3.** Fix a $\beta \in D_\tau \setminus \Gamma$; denote by $l_\tau(\beta)$, the level (cf. Remark 3.1.8) of $\beta$ considered as an element of the distributive lattice $D_\tau$. We shall prove the result by induction on $l_\tau(\beta)$. If $l_\tau(\beta) = 0$, then $\beta$ coincides with the (unique) minimal element of $D_\tau$, and there is nothing to prove. Let then $l_\tau(\beta) \geq 1$. Let $\beta'$ in $D_\tau$ be such that $\beta$ covers $\beta'$. Then $I_\beta \setminus I_{\beta'} = \{\theta\}$, for an unique $\theta \in J_{D_\tau}$ (cf. Lemma 3.1.24). Then there exists a unique cover $(\gamma, \gamma')$ in $\Gamma$ such that $I_\gamma \setminus I_{\gamma'} = \{\theta\}$ (cf. Lemma 3.1.24).

**Claim:** $F_\beta - F_{\beta'} \equiv F_\gamma - F_{\gamma'} (mod M_\tau)$.
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First observe that the fact that $\theta$ belongs to $I_\beta, I_\gamma$ and does not belong to $I_\beta, I_\gamma'$ implies

\[ \gamma' \not\geq \beta, \beta' \not\geq \gamma. \]

We now divide the proof of the Claim into the following two cases.

**Case 1:** $\gamma' < \beta$.

This implies that $\gamma'$ is in fact less than $\beta'$, and $\gamma < \beta$ (since, $I_\beta \setminus I_\beta' = I_\gamma \setminus I_\gamma' = \{\theta\}$, and $\theta \not\in I_\gamma'$); this in turn implies that $\beta' \not\leq \gamma$ (for, otherwise, $\beta' \leq \gamma$ would imply $\gamma' < \beta' < \gamma$, not possible, since, $(\gamma, \gamma')$ is a cover). Let $D$ be the diamond having $(\gamma, \beta')$ as the skew diagonal. Now $\gamma'$ being less than both $\gamma$ and $\beta'$, we get that $\gamma' \leq \beta' \land \gamma$, and in fact equals $\beta' \land \gamma$ (note that the fact that $\gamma' \leq \beta' \land \gamma < \gamma$ together with the fact that $(\gamma, \gamma')$ is a cover in $D_\tau$ implies that $\gamma' = \beta' \land \gamma$). In a similar way, we obtain that $\beta = \beta' \lor \gamma$. Now the diamond relation $x_\beta x_\gamma = x_\beta x_\gamma'$ implies the claim in this case (by definition of $F_\xi$'s).

**Case 2:** $\gamma' \not\leq \beta$.

This implies that $\gamma \not\leq \beta$.

If $\gamma > \beta$, then clearly

\[ \gamma = \gamma' \lor \beta, \beta' = \gamma' \land \beta. \]

Claim in this case follows as in Case 1.

Let then $\gamma \not\geq \beta$. Then we have that $(\gamma, \beta), (\gamma', \beta')$ are non-comparable pairs (note that $\gamma' \leq \beta'$ if and only if $\gamma' < \beta$). Denote $\delta = \gamma \land \beta, \delta' = \gamma' \land \beta'$. The facts that

\[ I_\delta = I_\gamma \cap I_\beta, I_\delta' = I_\gamma' \cap I_\beta', I_\gamma = I_\gamma' \cup \{\theta\}, I_\beta = I_\beta' \cup \{\theta\}. \]

imply that

\[ I_\delta = I_\delta' \cup \{\theta\}. \]
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Hence we obtain
\[ I_{\tau'} \cap I_\delta = I_{\tau'} \cap I_\delta' = I' \; \; ; \; I_{\tau'} \cup I_\delta = I_\gamma \]
\[ I_{\delta'} \cap I_\delta = I_{\delta'} \cap I_\delta' = I' \; \; ; \; I_{\delta'} \cup I_\delta = I_\beta \]

Thus we obtain that
\[ \gamma' \wedge \delta = \delta', \gamma' \vee \delta = \gamma \]
\[ \beta' \wedge \delta = \delta', \beta' \vee \delta = \beta \]

Considering the diamonds with skew diagonals $(\gamma', \delta), (\beta', \delta)$ respectively, the diamond relations
\[ x_{\gamma'} x_\delta = x_{\gamma} x_{\delta'}, x_{\beta'} x_\delta = x_{\beta} x_{\delta'} \]

imply that in $M_\tau/M^2_\tau$, we have the following relations
\[ F_\gamma - F_{\gamma'} = F_\delta - F_{\delta'}; F_\beta - F_{\beta'} = F_\delta - F_{\delta'} \]

Hence we obtain that $F_\beta - F_{\beta'} \equiv F_\gamma - F_{\gamma'} \text{(mod } M_\tau) \text{ as required.}$

This completes the proof of the Claim. Note that Claim implies the required result (by induction on $l_\tau(\beta)$). (Note that when $l_\tau(\beta) = 1$, then $\beta'$ is the (unique) minimal element of $D_\tau$, and hence $\beta' \in \Gamma$. The result in this case follows from the Claim).

Under the (surjective) map $\pi_\tau : R_\mathcal{L} \rightarrow R_\tau$, denote $\pi_\tau(M_\tau)$ by $M'_\tau$; then $\pi_\tau$ induces a surjection $\pi_\tau : M_\tau/M^2_\tau \rightarrow M'_\tau/M'^2_\tau$.

**Corollary 4.2.4.** \{\(\pi_\tau(F_\gamma), \gamma \in \Gamma\}\} is a basis for $M'_\tau/M'^2_\tau$.

**Proof 4.2.5.** We have that $\text{dim } M'_\tau/M'^2_\tau \geq \text{dim } X(D_\tau) = \#\Gamma$ (cf. Corollary 3.2.12(1)). On the other hand, the above Lemma implies that $\text{dim } M'_\tau/M'^2_\tau \leq \#\Gamma$, and the result follows.

**Lemma 4.2.6.** Let $\alpha \in \mathcal{L} \setminus D_\tau$ be such that there exists an element $\beta \in D_\tau$ and a diamond $D$ in $\mathcal{L}$ such that
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(1) \((\alpha, \beta)\) is a diagonal (main or skew) in \( D \).

(2) \( D \cap D_\tau = \{\beta\} \).

Then \( F_\alpha \in M^2_\tau \).

**Proof 4.2.7.** Let us denote the remaining vertices of the diamond by \( \theta, \delta \). Writing the diamond relation \( x_\alpha x_\beta = x_\theta x_\delta \) in terms of the \( F_\xi \)'s, we have,

\[
 x_\alpha x_\beta - x_\theta x_\delta = F_\alpha (1 - F_\beta) - F_\theta F_\delta
\]

Hence in \( R_L \), we have,

\[
 F_\alpha = F_\alpha F_\beta + F_\theta F_\delta
\]

The required result follows from this.

**The set \( E_\tau \)**

Define

\[
 E_\tau := \{\alpha \in L, \alpha \text{ as in Lemma 4.2.6}\}
\]

In the sequel, we shall refer to an element \( \alpha \in E_\tau \) as an \( E_\tau \)-element.

**The equivalence relation:**

For two distinct elements \( \theta, \delta \in L \setminus D_\tau \), we say \( \theta \) is equivalent to \( \delta \) (and denote it as \( \theta \sim \delta \)) if there exists a sequence \( \theta = \theta_1, \cdots, \theta_n = \delta \) in \( L \setminus D_\tau \) such that \((\theta_i, \theta_{i+1})\) forms one side of a diamond in \( L \) whose other side is contained in \( D_\tau \). For \( \theta \in L \setminus D_\tau \), we shall denote by \([\theta]\), the set of all elements of \( L \setminus D_\tau \) equivalent to \( \theta \), if there exists such a diamond as above having \( \theta \) as one vertex; if no such diamond exists, then \([\theta]\) will denote the singleton set \( \{\theta\} \). Clearly, for all \( \theta \) in a given equivalence class, \( F_\theta \) (mod \( M^2_\tau \)) is the same (by consideration of diamond relations), and we shall denote it by \( \overline{F}_{[\theta]} \) or also just \( \overline{F}_{\theta} \). Note also that in view
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of Lemma 4.2.6, $F_\theta = 0$ in $M_\tau/M_\tau^2$, if $[\theta] \cap E_\tau \neq \emptyset$. We shall refer to $[\theta]$ as an $E_\tau$-class or a non $E_\tau$-class according as $[\theta] \cap E_\tau$ is non-empty or empty.

The sublattice $\Lambda_\tau(\Gamma)$: Fix a chain $\Gamma$ in $D_\tau$. Let $\Lambda_\tau(\Gamma)$ denote the union of all the maximal chains in $\mathcal{L}$ containing $\Gamma$. Note that $\alpha$ in $\mathcal{L}$ is in $\Lambda_\tau(\Gamma)$ if and only if $\alpha$ is comparable to every $\gamma \in \Gamma$.

**Lemma 4.2.8.** $\Lambda_\tau(\Gamma)$ is a sublattice of $\mathcal{L}$.

**Proof 4.2.9.** Let $(\theta, \delta)$ be a pair of non-comparable elements in $\Lambda_\tau(\Gamma)$. Then for any $\gamma \in \Gamma$, we have (by definition of $\Lambda_\tau(\Gamma)$) that either $\gamma$ is less than both $\theta$ and $\delta$ or greater than both $\theta$ and $\delta$; in the former case, $\gamma$ is less than both $\theta \vee \delta$ and $\theta \wedge \delta$, and in the latter case, $\gamma$ is greater than both $\theta \vee \delta$ and $\theta \wedge \delta$.

**Remark 4.2.10.** $\Lambda_\tau(\Gamma)$ need not be an embedded sublattice:

Take $\mathcal{L}$ to be the distributive lattice consisting of $\{(i, j), 1 \leq i, j \leq 4\}$ with the partial order $(a_1, a_2) \geq (b_1, b_2) \iff a_r \geq b_r, r = 1, 2$. Take

$$D_\tau := \{(i, j), 2 \leq i, j \leq 3\}, \Gamma := \{(2, 2), (3, 2), (3, 3)\}$$

Then

$$\Lambda_\tau(\Gamma) = \Gamma \cup I_1 \cup I_2$$

where $I_1 = \{(1, 1), (1, 2), (2, 1)\}, I_2 = \{(3, 4), (4, 3), (4, 4)\}$. Consider $x = (2, 4), y = (4, 2)$; then $x \vee y = (4, 4), x \wedge y = (2, 2)$. Now $x \vee y, x \wedge y$ are in $\Lambda_\tau(\Gamma)$, but $x, y$ are not in $\Lambda_\tau(\Gamma)$. Thus $\Lambda_\tau(\Gamma)$ is not an embedded sublattice.

**Proposition 4.2.11.** Let $\theta_0 \in \mathcal{L} \setminus D_\tau$. Then $\theta_0 \sim \mu$, for some $\mu \in \Lambda_\tau(\Gamma) \cup E_\tau$.

**Proof 4.2.12.** Let us denote $\mathcal{L}'_\tau = \mathcal{L} \setminus D_\tau$. By induction, let us suppose that the result holds for all $\theta \in \mathcal{L}'_\tau, \theta > \theta_0$; we shall see that the proof for the case
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when $\theta_0$ is a maximal element in $L'_\tau$ (the starting point of induction) is included in the proof for a general $\theta_0$.

If $\theta_0 \in \Lambda_\tau(\Gamma)$, then there is nothing to prove. Let then $\theta_0 \not\in \Lambda_\tau(\Gamma)$. Fix $\gamma_1$ minimal in $\Gamma$ such that $\theta_0$ and $\gamma_1$ are non-comparable. Let $\xi = \gamma_1 \land \theta_0, \theta_1 = \gamma_1 \lor \theta_0$. We divide the proof into the following two cases.

**Case 1:** $\gamma_1$ is the minimal element of $\Gamma$ (note that $\gamma_1$ is the minimal element of $D_\tau$ also).

We have $\xi \in \Lambda_\tau(\Gamma)$ (since $\xi < \gamma_1$); further, $\xi \not\in D_\tau$ (again, since $\xi < \gamma_1$, the minimal element of $D_\tau$).

**Subcase 1(a):** Let $\theta_1$ be in $D_\tau$. Then considering the diamond with $(\theta_1, \gamma_1), (\theta_0, \xi)$ as opposite sides, we have, $\theta_0 \sim \xi$, and the result follows (note that $\xi \in \Lambda_\tau(\Gamma)$).

**Subcase 1(b):** Let $\theta_1 \not\in D_\tau$. This implies $\theta_0 \in E_\tau$, and the result follows.

**Case 2:** $\gamma_1$ is not the minimal element of $\Gamma$.

Let $\gamma_0 \in \Gamma$ be such that $\gamma_0$ is covered by $\gamma_1$ in $D_\tau$. Then $\xi \geq \gamma_0$ (since, both $\theta_0$ and $\gamma_1$ are $> \gamma_0$).

**Subcase 2(a):** Let $\xi > \gamma_0$. Then the fact that $(\gamma_1, \gamma_0)$ is a cover in $D_\tau$ together with the relation $\gamma_0 < \xi < \gamma_1$ implies that $\xi \in \Lambda_\tau(\Gamma) \setminus \Gamma$; in particular, $\xi \not\in D_\tau$.

Then as in Case 1, we obtain that $\theta_0 \in E_\tau$ if $\theta_1 \not\in D_\tau$, and $\theta_0 \sim \xi$, if $\theta_1 \in D_\tau$; and the result follows (again note that $\xi \in \Lambda_\tau(\Gamma)$).

**Subcase 2(b):** Let $\xi = \gamma_0$. We first note that $\theta_1 \not\in D_\tau$; for $\theta_1 \in D_\tau$ would imply that $\theta_0 \in D_\tau$ (since $D_\tau$ is an embedded sublattice (cf. Lemma 4.1.12)). Hence by induction hypothesis, we obtain that

\[ (*) \quad \theta_1 \sim \eta, \text{ for some } \eta \in \Lambda_\tau(\Gamma) \cup E_\tau \]

Also, considering the diamond with $(\theta_1, \theta_0), (\gamma_1, \gamma_0)$ as opposite sides, we have,
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$\theta_0 \sim \theta_1$. Hence, the result follows in view of (*).

Note that the above proof includes the proof of the starting point of induction, namely, $\theta_0$ is a maximal element in $L'_\tau$. To make this more precise, let $\gamma_1, \xi, \theta_1$ be as above. Proceeding as above, we obtain (by maximality of $\theta_0$) that $\theta_1 \in D_\tau$. Hence Subcases 1(b) and 2(b) do not exist (since in these cases $\theta_1 \not\in D_\tau$). In Subcases 1(a) and 2(a), we have $\xi \in \Lambda_\tau(\Gamma)$, and $\theta_0 \sim \xi$. The result now follows in this case.

**Lemma 4.2.13.** Let $(\theta_1, \delta_1), (\theta_2, \delta_2)$ be two skew pairs in $\Lambda_\tau(\Gamma)$ such that either $\theta_1 \lor \delta_1 = \theta_2 \lor \delta_2$ or $\theta_1 \land \delta_1 = \theta_2 \land \delta_2$, say $\theta_1 \lor \delta_1 = \theta_2 \lor \delta_2$, and $\gamma_i := \theta_i \land \delta_i, i = 1, 2$ are in $\Gamma$. Then $\gamma_1 = \gamma_2$.

**Proof 4.2.14.** Assume $\gamma_1 \neq \gamma_2$. We have $\gamma_1, \gamma_2$ are comparable (since $\Gamma$ is totally ordered), say $\gamma_1 > \gamma_2$. Denote $\alpha := \theta_1 \lor \delta_1 = \theta_2 \lor \delta_2$. The fact that any element of $\Lambda_\tau(\Gamma)$ is comparable to all elements of $\Gamma$ together with the hypothesis that $(\theta_2, \delta_2)$ is a skew pair implies that either $\theta_2, \delta_2$ are both $> \gamma_1$ or are both $< \gamma_1$. If $\theta_2, \delta_2$ are both $> \gamma_1$, then we would obtain $\gamma_2(= \theta_2 \land \delta_2) \geq \gamma_1$ which is not true. Hence we obtain that $\theta_2, \delta_2$ are both $< \gamma_1$; this implies that $\alpha(= \theta_2 \lor \delta_2) < \gamma_1$ which is again not true. Hence our assumption is wrong and the result follows.

Let $Y_\tau(\Gamma) = \Lambda_\tau(\Gamma) \setminus E_\tau$. Let us write

$$Y_\tau(\Gamma) = \Gamma \cup Z_\tau(\Gamma)$$

$$G_\tau(\Gamma) = \{[\theta], \theta \in Z_\tau(\Gamma), [\theta] \text{ is a non } E_\tau-\text{class}\}$$

Combining the above Proposition with Lemmas 4.2.2, 4.2.6, we obtain

**Proposition 4.2.15.** $M_\tau/M^2_\tau$ is generated by $\{\overline{F}_{[\theta]}, [\theta] \in G_\tau(\Gamma)\} \cup \{\overline{F}_\gamma, \gamma \in \Gamma\}$. 
4.3 A basis for the cotangent space at $P_\tau$

In this section, we shall show that $\{\bar{F}_{[\theta]}, [\theta] \in G_\tau(\Gamma)\} \cup \{\bar{F}_\gamma, \gamma \in \Gamma\}$ is in fact a basis for $M_\tau/M_\tau^2$. We first recall some basic facts on tangent cones and tangent spaces.

Let $X = \text{Spec} R \hookrightarrow \mathbb{A}^l$ be an affine variety; let $S$ be the polynomial algebra $K[X_1, \cdots, X_l]$. Let $P \in X$, and let $M_P$ be the maximal ideal in $K[X]$ corresponding to $P$ (we are concerned only with closed points of $X$). Let $A = \mathcal{O}_{X,P}$, the stalk at $P$; denote the unique maximal ideal in $A$ by $M (= M_P R_{M_P})$. Then $\text{Spec} gr(A, M)$, where $gr(A, M) = \bigoplus_{j \in \mathbb{Z}_+} M^j/M^{j+1}$ is the tangent cone to $X$ at $P$, and is denoted $TC_P X$. Note that

$$gr(R, M_P) = gr(A, M)$$

**Tangent cone & tangent space at $P$:** Let $I(X)$ be the vanishing ideal of $X$ for the embedding $X = \text{Spec} R \hookrightarrow \mathbb{A}^l$. Expanding $F \in I(X)$ in terms of the local co-ordinates at $P$, we have the following:

- $T_P(X)$, the tangent space to $X$ at $P$ is the zero locus of the linear forms of $F$, for all $F \in I(X)$, i.e., the degree one part in the (polynomial) local expression for $F$.
- $TC_P(X)$, the tangent cone to $X$ at $P$ is the zero locus of the initial forms (i.e., form of smallest degree) of $F$, for all $F \in I(X)$.

In the sequel, we shall denote the initial form of $F$ by $IN(F)$.

We collect below some well-known facts on singularities of $X$:

**Facts:**
1. $\dim T_P X \geq \dim X$ with equality if and only if $X$ is smooth at $P$.
2. $X$ is smooth at $P$ if and only if $gr(R, M_P)$ is a polynomial algebra.
4.3 A basis for the cotangent space at \( P_\tau \)

4.3.1 Determination of the degree one part of \( J(\tau) \):

We now take \( X = X_L, P = P_\tau \); we shall denote \( I = I(X_L), M_P = M_\tau \). As above, for \( F \in M_\tau \), let \( \bar{F} \) denote the class of \( F \) in \( M_\tau / M_\tau^2 \). Let \( J(\tau) \) be the kernel of the surjective map

\[
\begin{align*}
f_\tau : K[X_\theta, \theta \in \mathcal{L}] & \rightarrow gr(R, M_\tau), \quad X_\theta \mapsto \bar{F}_\theta
\end{align*}
\]

For \( r \in \mathbb{N} \), let \( f^{(r)} \) be the restriction of \( f \) to the degree \( r \) part of the polynomial algebra \( K[X_\theta, \theta \in \mathcal{L}] \). We are interested in

\[
f_\tau^{(1)} : \bigoplus_{\theta \in \mathcal{L}} K X_\theta \rightarrow M_\tau / M_\tau^2
\]

We shall first describe a complement to the kernel of \( f_\tau^{(1)} \), and then deduce a basis for \( M_\tau / M_\tau^2 \) (which will turn out to be the set \( \{ \bar{F}_\theta, [\theta] \in G_\tau(\Gamma) \} \cup \{ \bar{F}_\gamma, \gamma \in \Gamma \} \)).

Let \( V_\tau \) be the span of \( \{ \bar{F}_\beta, \beta \in D_\tau \} \) (in \( M_\tau / M_\tau^2 \)); by Lemmas 4.2.2, we have that \( V_\tau \) is spanned by \( \{ \bar{F}_\gamma, \gamma \in \Gamma \} \). Let \( W_\tau \) be the span of \( \{ \bar{F}_\theta, \theta \notin D_\tau \} \) (in \( M_\tau / M_\tau^2 \)).

Lemma 4.3.2. \( \{ \bar{F}_\gamma, \gamma \in \Gamma \} \) is a basis for \( V_\tau \).

Proof 4.3.3. The surjective map \( \pi_\tau : M_\tau / M_\tau^2 \rightarrow M'_\tau / M'^2_\tau \) induces a surjection \( V_\tau \rightarrow M'_\tau / M'^2_\tau \), and the result follows from Corollary 4.2.4.

Lemma 4.3.4. The sum \( V_\tau + W_\tau \) is direct.

Proof 4.3.5. Let \( v \in V_\tau \); by Lemma 4.3.2, we can write \( v = \sum_{\gamma \in \Gamma} a_\gamma \bar{F}_\gamma \). Now if \( v \in W_\tau \), then under the map

\[
\begin{align*}
\pi_\tau : M_\tau / M_\tau^2 & \rightarrow M'_\tau / M'^2_\tau
\end{align*}
\]

we obtain that \( \pi_\tau(v) = 0 \) (since, \( W_\tau \subseteq ker \pi_\tau \)). Hence we obtain that \( \pi_\tau(\sum_{\gamma \in \Gamma} a_\gamma \bar{F}_\gamma) = 0 \) in \( M'_\tau / M'^2_\tau \); this in turn implies that \( a_\gamma = 0, \forall \gamma \) (cf. Corollary 4.2.4). Hence \( v = 0 \) and the required result follows.
4.3. A basis for the cotangent space at $P_\tau$

Let us write $\bigoplus_{\theta \in L} KX_{\theta} = A_\tau \oplus B_\tau$ where

$$A_\tau = \bigoplus_{\beta \in D_\tau} KX_\beta, \quad B_\tau = \bigoplus_{\theta \notin D_\tau} KX_\theta$$

Write $f_\tau^{(1)} = g_\tau^{(1)} + h_\tau^{(1)}$, where $g_\tau^{(1)}$ (respectively $h_\tau^{(1)}$) is the restriction of $f_\tau^{(1)}$ to $A_\tau$ (respectively $B_\tau$). Note that we have surjections

$$g_\tau^{(1)} : A_\tau \rightarrow V_\tau, \quad h_\tau^{(1)} : B_\tau \rightarrow W_\tau$$

As a consequence of Lemma 4.3.4, we get the following

**Corollary 4.3.6.** $\ker f_\tau^{(1)} = \ker g_\tau^{(1)} \oplus \ker h_\tau^{(1)}$

**Proof 4.3.7.** The inclusion “$\supseteq$” is clear. Let then $v \in \ker f_\tau^{(1)}$; write $v = a + b$, where $a \in A_\tau, b \in B_\tau$. Then denoting $a' = f_\tau^{(1)}(a), b' = f_\tau^{(1)}(b)$, we have, $0 = a' + b'$. Also, we have that $a' \in V_\tau, b' \in W_\tau$. Hence in view of Lemma 4.3.4, we obtain $a' = 0 = b'$. This implies that $a \in \ker g_\tau^{(1)}, b \in \ker h_\tau^{(1)}$, and the result follows.

**Lemma 4.3.8.** The span of $\{X_{\gamma}, \gamma \in \Gamma\}$ is a complement to the kernel of $g_\tau^{(1)}$.

**Proof 4.3.9.** We have, $g_\tau^{(1)}(X_\beta) = F_\beta$. By Lemma 4.3.2, we have that $\{F_\gamma, \gamma \in \Gamma\}$ is a basis for $V_\tau$. The result now follows.

Let $\{\xi_1, \cdots, \xi_r\}$ be a complete set of representatives for $G_\tau(\Gamma)$.

**Lemma 4.3.10.** The span of $\{X_{\xi_1}, \cdots, X_{\xi_r}\}$ is a complement to the kernel of $h_\tau^{(1)}$.

**Proof 4.3.11.** A typical element $F$ of $I(X_L)$ such that $IN(F)$ is in $B_\tau$ is of the form

$$F = F_1 + F_2$$

where $F_1$ is a linear sum of diamond relations arising from diamonds having precisely one vertex in $D_\tau$, and $F_2$ is a linear sum of diamond relations arising
4.3. A basis for the cotangent space at $P_\tau$

from diamonds having precisely one side in $D_\tau$. Note that in a typical term in $F_1$, the linear term is of the form $a_\alpha X_\alpha$, $a_\alpha \in K$ where $\alpha \in E_\tau$; similarly, in a typical term in $F_2$, the linear term is of the form $b_{\theta \delta}(X_\theta - X_\delta)$, $\theta, \delta \not\in D_\tau, \theta \sim \delta, b_{\theta \delta} \in K$. Hence the kernel of $h^{(1)}_\tau$ is generated by $\{X_\alpha, \alpha \in E_\tau\} \cup \{(X_\theta - X_\delta), \theta \sim \delta\}$. The required result now follows in view of Proposition 4.2.11.

**Theorem 4.3.12.** $\{\bar{F}_{[\theta]}, [\theta] \in G_\tau(\Gamma)\} \cup \{\bar{F}_{\gamma}, \gamma \in \Gamma\}$ is a basis for $M_\tau/M^2_\tau$.

**Proof 4.3.13.** In view of Lemmas 4.3.8, 4.3.10 and Corollary 4.3.6, we obtain that

$$\bigoplus_{\gamma \in \Gamma} KX_\gamma \oplus \bigoplus_{1 \leq i \leq r} KX_\xi_i$$

is a complement to the kernel of the surjective map $f^{(1)}_\tau: \bigoplus_{\theta \in \mathcal{L}} KX_\theta \rightarrow M_\tau/M^2_\tau$. The result now follows.

Let $T_\tau X_\mathcal{L}$ denote the tangent space to $X_\mathcal{L}$ at $P_\tau$. As an immediate consequence of the above Theorem, we have the following

**Corollary 4.3.14.** $\dim T_\tau X_\mathcal{L} = \#G_\tau(\Gamma) + \#\Gamma$.

In view of the above Corollary, we obtain that $X_\mathcal{L}$ is singular along $O_\tau$ if and only if $\#G_\tau(\Gamma) + \#\Gamma > \#\mathcal{L}$. Let

$$\mathcal{S}_\mathcal{L} = \{\tau < \sigma \mid \#G_\tau(\Gamma) + \#\Gamma > \#\mathcal{L}\}$$

(here, $\sigma$ is the cone associated to $X_\mathcal{L}$). We obtain from Proposition 4.1.16 the following

**Theorem 4.3.15.** $\text{Sing } X_\mathcal{L} = \bigcup_{\tau \in \mathcal{S}_\mathcal{L}} X(D_\tau)$. 
Chapter 5

Standard Basis for Tangent Cone

5.1 Analysis of diamond relations:

For a diamond $D$ in $\mathcal{L}$, with $(\theta, \delta)$ as the skew diagonal, let

$$f_D = X_\theta X_\delta - X_{\theta \vee \delta} X_{\theta \wedge \delta}$$

We shall refer to $f_D$ as a diamond relation.

The ideal $b(\tau)$: Let $b(\tau)$ be the ideal in $K[X_\alpha, \alpha \in \mathcal{L}]$ generated by $\ker \pi^{(1)}$ and the diamond relations $f_{D'}$ for $D'$ in $\Lambda_\tau(\Gamma)$.

Our main goal in this section is to show that given a diamond $D$ in $\mathcal{L}$, $IN f_D$ is in $b(\tau)$. Our first step is to collect $IN f_D$, for all diamonds $D$ in $\mathcal{L}$. In our discussion in this section, we will be repeatedly using the following two lemmas.

Lemma 5.1.1. Let $D$ be a diamond with diagonals $(\theta_1, \theta_4), (\theta_2, \theta_3)$. Suppose either

$$F_{\theta_1} \equiv F_{\theta_2} \pmod{M_\tau^2}, F_{\theta_3} \equiv F_{\theta_4} \pmod{M_\tau^2}$$

or

$$F_{\theta_1} \equiv F_{\theta_3} \pmod{M_\tau^2}, F_{\theta_2} \equiv F_{\theta_4} \pmod{M_\tau^2}$$
Then \( f_D \in b(\tau) \)

**Proof 5.1.2.** Let us prove the case when \( F_{\theta_1} \equiv F_{\theta_4}(mod\ M^2_\tau), F_{\theta_2} \equiv F_{\theta_3}(mod\ M^2_\tau) \) (proof of the other case being similar). Let \((\theta_2, \theta_3)\) be the skew diagonal. Hypothesis implies that \( X_{\theta_1} - X_{\theta_3}, X_{\theta_2} - X_{\theta_4} \) are in \( ker\ \pi^{(1)} \). The assertion follows by writing

\[
f_D(= \pm(X_{\theta_2}X_{\theta_3} - X_{\theta_1}X_{\theta_4})) = \pm[X_{\theta_3}(X_{\theta_2} - X_{\theta_4}) - X_{\theta_4}(X_{\theta_1} - X_{\theta_3})]
\]

**Lemma 5.1.3.** Let \( D_1, D_2 \) be two diamonds with diagonals

\[\{(\theta_1, \theta_4), (\theta_2, \theta_3)\}, \{(\delta_1, \delta_4), (\delta_2, \delta_3)\}\] respectively; further, let \((\theta_2, \theta_3), (\delta_2, \delta_3)\) be the skew diagonals for \( D_1, D_2 \) respectively. Suppose \( F_{\theta_j} \equiv F_{\delta_j}(mod\ M^2_\tau) \).

Then \( f_{D_1} \equiv f_{D_2}(mod\ b(\tau)) \).

**Proof 5.1.4.** The assertion follows by writing \( f_{D_1}(= X_{\theta_2}X_{\theta_3} - X_{\theta_1}X_{\theta_4}) \) as

\[
f_{D_1} = X_{\theta_3}(X_{\theta_2} - X_{\theta_1}) - X_{\theta_4}(X_{\theta_1} - X_{\delta_1}) - X_{\delta_1}(X_{\theta_4} - X_{\delta_4}) + X_{\delta_2}(X_{\theta_3} - X_{\delta_3}) + f_{D_2}
\]

(note that hypothesis implies that \( X_{\theta_j} - X_{\delta_j}, j = 1, 2, 3, 4 \) are in \( ker\ \pi^{(1)} \)).

As mentioned in the beginning of this section, we want to collect \( INf_D \), for all diamonds \( D \) in \( L \). Since we know the description of \( M_\tau/M^2_\tau \) (and hence \( ker\ \pi^{(1)} \), we shall suppose the following:

(i) In view of Lemma 4.1.12 and Lemma 4.2.4, we may suppose that any diagonal of \( D \) is not contained in \( D_\tau \).

(ii) In view of Lemma 4.2.2, we may suppose that any side of \( D \) is not contained in \( D_\tau \) (note that in this case, in view of that Lemma, \( INf_D \in ker\ \pi^{(1)} \)).

In view of (i), (ii), we may suppose that \( \#\{D \cap D_\tau\} \leq 1 \).

(iii) Let \( \#\{D \cap D_\tau\} = 1 \); let \( \beta \) be the vertex of \( D \) such that \( \beta \in D_\tau \). Let \( \alpha \) be the other vertex on the diagonal of \( D \) through \( \beta \). Let \((\theta, \delta)\) be the other diagonal of \( D \).
5.1. Analysis of diamond relations:

We obtain that $\alpha \in E_\tau$ (note that $\theta, \delta \notin D_\tau$), and hence $IN f_D = X_\alpha (\in ker \pi^{(1)})$ (cf. Lemma 4.2.6).

Thus we may suppose that $D \cap D_\tau \neq \emptyset$.

(iv) In view of Lemma 4.2.6, we may suppose that any side of $D$ is not contained in $E_\tau$ (note that in this case, in view of that Lemma, $f_D \in b(\tau)$). Thus we may suppose that $\# D \cap E_\tau \leq 2$.

For the rest of this section, $D$ will denote a diamond in $\mathcal{L}$ satisfying

$$D \cap D_\tau = \emptyset, \# D \cap E_\tau \leq 2$$

In particular, $f_D$ is homogeneous (and hence $IN f_D = f_D$).

Let $b(\tau)$ be as above (namely, the ideal in $K[X_\alpha, \alpha \in \mathcal{L}]$ generated by $ker \pi^{(1)}$ and the diamond relations $f_{D'}$ for $D'$ in $\Lambda_\tau(\Gamma)$). Let us denote $\Gamma$ by

$$\Gamma = \{\gamma_{n+1} > \cdots > \gamma_1\}$$

where $n$ is the rank of the distributive lattice $D_\tau$. Denote $\gamma_0 := \hat{0}$ (the “zero element” of $\mathcal{L}$). For $0 \leq i \leq n+1$, let

$$\Lambda_i(\Gamma) := \{\theta \in \mathcal{L} \mid (\theta, \gamma_r) \text{ comparable}, r \leq i\}$$

We have

$$\Lambda_i(\Gamma) \supset \Lambda_{i+1}(\Gamma), \Lambda_{n+1}(\Gamma) = \Lambda_\tau(\Gamma), \Lambda_0(\Gamma) = \mathcal{L}$$

(Note that if $\theta \in \mathcal{L} \setminus \Lambda_\tau(\Gamma)$, then $\theta \in \Lambda_i(\Gamma)$ where $i+1$ is the smallest such that $(\theta, \gamma_{i+1})$ is skew.)

**Lemma 5.1.5.** $\Lambda_i(\Gamma)$ is a sublattice of $\mathcal{L}$.

**Proof 5.1.6.** Let $(\tau, \varphi)$ be a skew pair in $\Lambda_i(\Gamma)$. Then for $1 \leq r \leq i$, we have that $\gamma_r$ is comparable to both $\tau, \varphi$. In fact, $\gamma_r$ is either less than both $\tau, \varphi$ or
greater than both \( \tau, \varphi \); hence, \( \tau \lor \varphi, \tau \land \varphi \) are both comparable to \( \gamma_r \), and we obtain that \( \tau \lor \varphi, \tau \land \varphi \) are in \( \Lambda_i(\Gamma) \).

Let \( D \) be a diamond in \( \Lambda_i(\Gamma) \) for some \( i, 0 \leq i \leq n \). Let \( b_{i+1}(\tau) \) be the ideal in \( K[\alpha, \alpha \in L] \) generated by \( \ker \pi^{(1)} \) and the diamond relations \( f_{D'} \) for \( D' \) in \( \Lambda_{i+1}(\Gamma) \).

**Proposition 5.1.7.** \( f_D \) is in \( b_{i+1}(\tau) \).

Before proving the Proposition, let us derive the following important consequence (which is in fact the essence of this section as mentioned above).

**Corollary 5.1.8.** \( f_D \) is in \( b(\tau) \).

**Proof 5.1.9.** This is immediate from Proposition 5.1.7 by decreasing induction on \( i \); the starting point being \( i = n + 1 \) for which the result is clear (note that \( f_D \in b_{n+1}(\tau)(= b(\tau)) \)).

**Proof of Proposition 5.1.7:** Let us denote the main diagonal of \( D \) by \( (\theta_4 \succ \theta_1) \), and the skew diagonal by \( (\theta_2, \theta_3) \). We break our discussion into the two cases: \( D \) contained in \( \Lambda_i(\Gamma) \setminus \Lambda_{i+1}(\Gamma) \) and \( D \) not contained in \( \Lambda_i(\Gamma) \setminus \Lambda_{i+1}(\Gamma) \).

**A.** Let \( D \) be contained in \( \Lambda_i(\Gamma) \setminus \Lambda_{i+1}(\Gamma) \). Denote \( \gamma := \gamma_{i+1} \). We have \( (\theta_j, \gamma), j = 1, 2, 3, 4 \) are skew. Denote

\[
\epsilon_j := \theta_j \lor \gamma, \delta_j := \theta_j \land \gamma, j = 1, 2, 3, 4
\]

By distributivity of join, meet operations, we have

\[
\epsilon_2 \lor \epsilon_3 = \epsilon_4, \ \epsilon_2 \land \epsilon_3 = \epsilon_1
\]
\[
\delta_2 \lor \delta_3 = \delta_4, \ \delta_2 \land \delta_3 = \delta_1
\]
5.1. Analysis of diamond relations:

For simplicity, let us refer to the diamond with $\epsilon_j$ (respectively $\delta_j$) as vertices as the $\epsilon$-diamond (respectively $\delta$-diamond), whenever $(\epsilon_j, \epsilon_k)$ (resp. $(\delta_j, \delta_k)$) is skew. Note that $\epsilon_j, \delta_j, j = 1, 2, 3, 4$ belong to $\Lambda_{i+1}(\Gamma)$; for, $(\theta_j, \gamma_k)$ being skew, $j = 1, 2, 3, 4$, for any $\gamma_r, r \leq i$, we have that either both $\theta_j, \gamma_k$ are $> \gamma_r$ or both $< \gamma_r$. Thus for $r \leq i$, $\epsilon_j, \gamma_r$ (respectively $\delta_j, \gamma_r$) are comparable, and in addition $\epsilon_j, \gamma_k$ (respectively $\delta_j, \gamma_k$) are comparable (note that $\gamma = \gamma_{i+1}$).

Also, the fact that $\theta_j \not\in D_\tau$ implies (by consideration of the diamond having $(\theta_j, \gamma)$ as the skew diagonal):

\[ (*) \quad \text{At most one of } \{\epsilon_j, \delta_j\} \text{ is in } D_\tau, j = 1, 2, 3, 4 \]

(since $D_\tau$ is an embedded sublattice of $\mathcal{L}$).

**Case 1:** Let $D \cap E_\tau = \emptyset$. This implies (by $(*)$) that precisely one of $\{\epsilon_j, \delta_j\}$ is in $D_\tau$ for $j = 1, 2, 3, 4$ (by definition of $E_\tau$). If either one of the pairs $(\epsilon_1, \epsilon_4), (\epsilon_2, \epsilon_3)$ is contained in $D_\tau$, then $\epsilon_j \in D_\tau$ for $j = 1, 2, 3, 4$.

(This is clear if $\epsilon_2, \epsilon_3$ are comparable - for then $\{\epsilon_2, \epsilon_3\} = \{\epsilon_1, \epsilon_4\}$; if $(\epsilon_2, \epsilon_3)$ is skew, then this follows from the fact that $D_\tau$ is an embedded sublattice of $\mathcal{L}$).

As a consequence we obtain (by considering the diamond with $(\theta_j, \gamma)$ as the skew diagonal)

\[ F_{\theta_j} \equiv F_{\delta_j} (\mod M_\tau^2), j = 1, 2, 3, 4 \]

If $\delta_2, \delta_3$ are comparable, say $\delta_2 > \delta_3$, then $\delta_2 = \delta_4, \delta_3 = \delta_1$. Hence we obtain

\[ F_{\theta_2} \equiv F_{\theta_4} (\mod M_\tau^2) F_{\theta_3} \equiv F_{\theta_1} (\mod M_\tau^2) \]

It follows (in view of Lemma 5.1.1) that $f_D \in b(\tau)$.

If $(\delta_2, \delta_3)$ is skew, then we obtain (in view of Lemma 5.1.3) that $f_D \equiv f_{D'} (\mod b(\tau))$ where $D'$ is the $\delta$-diamond ($\subseteq \Lambda_{i+1}$). Hence it follows that $f_D \in b_{i+1}(\tau)$. 


Interchanging the roles of the \( \epsilon \)'s and \( \delta \)'s, we obtain that if either one of the pairs \((\delta_1, \delta_4), (\delta_2, \delta_3)\) is contained in \(D_\tau\), then we obtain the following:

(i) \( f_D \in b(\tau), \) if \( \epsilon_2, \epsilon_3 \) are comparable.

(ii) If \((\epsilon_2, \epsilon_3)\) is skew, then \( f_D \equiv f_{D'}(mod b(\tau)) \) where \( D' \) is the \( \epsilon \)-diamond \((\subseteq \Lambda_{i+1})\), and \( f_D \in b_{i+1}(\tau) \).

To discuss the remaining possibilities, we shall denote the diagonals of \( D \) by \((\theta_j, \theta_m), (\theta_k, \theta_l)\). In view of the fact that precisely one of \( \{\epsilon_h, \delta_h\} \) is in \( D_\tau \) for \( h = j, k, l, m \), we are left to discuss the case when the following hold:

(a) precisely one of the pairs \((\epsilon_j, \epsilon_k), (\epsilon_l, \epsilon_m)\) is contained in \( D_\tau \) and the other pair has empty intersection with \( D_\tau \).

(b) precisely one of the pairs \((\delta_j, \delta_k), (\delta_l, \delta_m)\) is contained in \( D_\tau \) and the other pair has empty intersection with \( D_\tau \).

Let \( \{\epsilon_j, \epsilon_k\} \) be contained in \( D_\tau \). This implies that \( \{\delta_l, \delta_m\} \) is contained in \( D_\tau \). We have

\[
(\dagger) \quad F_{\delta_j} \equiv F_{\delta_k}(mod M^2_{\tau}), \quad F_{\epsilon_l} \equiv F_{\epsilon_m}(mod M^2_{\tau})
\]

(This is clear if \((\epsilon_j, \epsilon_m)\) (resp. \((\delta_j, \delta_m)\)) is non-skew in which case \( \epsilon_j = \epsilon_k, \epsilon_l = \epsilon_m \) (resp. \( \delta_j = \delta_k, \delta_l = \delta_m \); if \((\epsilon_j, \epsilon_m)\) (resp. \((\delta_j, \delta_m)\)) is skew, then this follows by consideration of the \( \epsilon \)-diamond (resp. \( \delta \)-diamond)). Also considering the diamonds with \((\theta_r, \gamma), r = j, k, l, m \) as the skew diagonals, we have

\[
(\ddagger) \quad F_{\theta_h} \equiv F_{\delta_h}(mod M^2_{\tau}), h = j, k \quad F_{\theta_h} \equiv F_{\epsilon_h}(mod M^2_{\tau}), h = l, m
\]

Hence we obtain (from \((\dagger), (\ddagger))

\[
F_{\theta_j} \equiv F_{\theta_k}(mod M^2_{\tau}), \quad F_{\theta_l} \equiv F_{\theta_m}(mod M^2_{\tau})
\]

From this it follows (in view of Lemma 5.1.1) that \( f_D \) is in \( b_{i+1}(\tau) \).
This completes the discussion in the case \( D \cap E_\tau = \emptyset \).

**Case 2:** Let \( \#\{D \cap E_\tau\} = 1 \).

As in Case 1, let us denote the diagonals of \( D \) by \((\theta_j, \theta_m), (\theta_k, \theta_l)\). Let \( \theta_j \in E_\tau \) (and hence the other vertices are not in \( E_\tau \)). As in Case 1, we may suppose that none of the pairs \((\epsilon_k, \epsilon_l), (\epsilon_j, \epsilon_m), (\delta_k, \delta_l), (\delta_j, \delta_m)\) is contained in \( D_\tau \). Also, since \( \theta_h \notin E_\tau, h \neq j \), we obtain that precisely one of \( \{\epsilon_h, \delta_h\}, h \neq j \) is in \( D_\tau \). Hence, precisely one of the pairs in \( \{(\epsilon_k, \epsilon_m), (\epsilon_l, \epsilon_m), (\delta_k, \delta_m), (\delta_l, \delta_m)\} \) is contained in \( D_\tau \), say, \( \{\epsilon_k, \epsilon_m\} \) is contained in \( D_\tau \); this implies

\[
F_{\epsilon_l} \equiv F_{\epsilon_j} \pmod{M^2_\tau}
\]

(This is clear if \( \epsilon_j, \epsilon_m \) are comparable - for then \( \epsilon_j = \epsilon_l \epsilon_k = \epsilon_m \); if \( (\epsilon_j, \epsilon_m) \) is skew, then this follows by the consideration of the \( \epsilon \)-diamond and the fact that \( \epsilon_k, \epsilon_m \) are in \( D_\tau \).) This together with the consideration of the diamonds with \((\theta_l, \gamma), (\theta_j, \gamma)\) as skew diagonals implies

\[
(1) \quad F_{\theta_l} \equiv F_{\epsilon_l} \equiv F_{\epsilon_j} \equiv F_{\theta_j} \pmod{M^2_\tau}
\]

Also, we have, \( \delta_l \in D_\tau \) (since precisely one of \( \{\epsilon_l, \delta_l\} \) is in \( D_\tau \) and \( \epsilon_l \notin D_\tau \)). Hence we obtain the following:

If \( \delta_j \in D_\tau \), then

\[
(2) \quad F_{\theta_k} \equiv F_{\delta_k} \equiv F_{\delta_m} \equiv F_{\theta_m} \pmod{M^2_\tau}
\]

(Note that if \( \delta_j, \delta_m \) are comparable, then \( \delta_k = \delta_m, \delta_j = \delta_l \); if \( (\delta_j, \delta_m) \) is skew then \( F_{\delta_k} \equiv F_{\delta_m} \equiv F_{\theta_m} \pmod{M^2_\tau} \)). It follows from (1), (2) and Lemma 5.1.1 that \( f_D \) is in \( b_{i+1}(\tau) \).

If \( \delta_j \notin D_\tau \), then

\[
\delta_k \in E_\tau, F_{\theta_k} \equiv F_{\delta_k} \pmod{M^2_\tau}
\]
5.1. Analysis of diamond relations:

(note that $\delta_l \in D_{\tau}, \delta_m \not\in D_{\tau}$). We have $X_{\theta_j}, X_{\delta_k}$ are in $\ker \pi^{(1)}$ (since $\theta_j, \delta_k \in E_{\tau}$), and $X_{\theta_k} - X_{\delta_k}$ (and therefore $X_{\theta_k} \in \ker \pi^{(1)}$). Hence it follows that $f_D \in b_{i+1}(\tau).

**Case 3:** Let $\#\{D \cap E_{\tau}\} = 2$.

This implies (in view of (iv) above) that a diagonal of $D$ is contained in $E_{\tau}$. As in Case 1, let us denote the diagonals of $D$ by $(\theta_j, \theta_m), (\theta_k, \theta_l)$, and let us suppose that $\{\theta_j, \theta_m\}$ is contained in $E_{\tau}$. This together with the hypothesis that $\#\{D \cap E_{\tau}\} = 2$ implies that $\theta_k, \theta_l \not\in E_{\tau}$, and hence precisely one of $\{\epsilon_k, \delta_k\}$ and one of $\{\epsilon_l, \delta_l\}$ is in $D_{\tau}$. Also, as in Case 1, we may suppose that none of the pairs $(\epsilon_k, \epsilon_l), (\epsilon_j, \epsilon_m), (\delta_k, \delta_l), (\delta_j, \delta_m)$ is not contained in $D_{\tau}$. Hence we obtain that precisely one of the pairs $(\epsilon_k, \delta_l), (\epsilon_l, \delta_k)$ is contained in $D_{\tau}$ and the other has empty intersection with $D_{\tau}$; say $\{\epsilon_k, \delta_l\}$ is contained in $D_{\tau}$. In view of the fact that at least one of $\{\epsilon_j, \epsilon_m\}$ is not in $D_{\tau}$, we have the following possibilities:

If both $\epsilon_j, \epsilon_m$ are not in $D_{\tau}$, then $(\epsilon_j, \epsilon_m)$ is skew necessarily (since $\epsilon_k \in D_{\tau}$), and we have

$$\epsilon_l \in E_{\tau}, F_{\theta_l} \equiv F_{\epsilon_l} (mod M^2_{\tau})$$

If $\epsilon_m \in D_{\tau}, \epsilon_j \not\in D_{\tau}$, then $F_{\theta_m} \equiv F_{\delta_m} (mod M^2_{\tau})$. Further, we have

if $\delta_j \not\in D_{\tau}$, then $\delta_k \in E_{\tau}, F_{\theta_k} \equiv F_{\delta_k} (mod M^2_{\tau})$

if $\delta_j \in D_{\tau}$, then $F_{\theta_k} \equiv F_{\delta_k} \equiv F_{\delta_m} \equiv F_{\theta_m} (mod M^2_{\tau})$

(Note that if $\delta_j \not\in D_{\tau}$, then $(\delta_j, \delta_m)$ is skew necessarily (since $\delta_m \not\in D_{\tau}, \delta_l \in D_{\tau}$). If $\delta_j \in D_{\tau}$, then $F_{\delta_k} \equiv F_{\delta_m} \equiv F_{\theta_m} (mod M^2_{\tau})$ whether $(\delta_j, \delta_m)$ is skew or not). It follows that $f_D \in b_{i+1}(\tau)$.

The discussion of the case $\epsilon_j \in D_{\tau}, \epsilon_m \not\in D_{\tau}$ is completely analogous (the roles of $j$ and $m$ are interchanged) in the above discussion and we obtain that $f_D \in b_{i+1}(\tau)$.

This finishes the proof of the Proposition in case A.
5.1. Analysis of diamond relations:

**B.** Let $D$ be not contained in $\Lambda_i(\Gamma) \setminus \Lambda_{i+1}(\Gamma)$. As above, denote $\gamma := \gamma_{i+1}$.

Let us denote the skew diagonal of $D$ by $(\theta_2, \theta_3)$, and the main diagonal by $(\theta_4 > \theta_1)$. The hypothesis implies that at least one of $\{\theta_2, \theta_3\} \notin \Lambda_{i+1}(\Gamma)$ (note that if both $\{\theta_2, \theta_3\}$ are in $\Lambda_{i+1}(\Gamma)$, then Lemma 5.1.5 would imply that $D$ is contained in $\Lambda_{i+1}(\Gamma)$, and the result follows). We divide the discussion into two cases:

1. precisely one of $\{\theta_2, \theta_3\}$ is in $\Lambda_{i+1}(\Gamma)$
2. both $\theta_2, \theta_3$ are not in $\Lambda_{i+1}(\Gamma)$.

**Case B1:** Precisely one of $\{\theta_2, \theta_3\}$ is in $\Lambda_{i+1}(\Gamma)$, say, $\theta_3 \in \Lambda_{i+1}(\Gamma)$ (the proof for the case $\theta_2 \in \Lambda_{i+1}(\Gamma)$ being similar). This implies that $\theta_3, \gamma (= \gamma_{i+1})$ are comparable, and the pair $(\theta_2, \gamma)$ is skew. We shall prove the case when $\theta_3 < \gamma$ (the proof for the case $\theta_3 > \gamma$ being similar). We have $\theta_1 < \gamma$; further

$$\delta_3 = \theta_3, \delta_1 = \theta_1, \epsilon_1 = \epsilon_3 = \gamma$$

(by the definition of the $\delta_i$’s and the $\epsilon_i$’s).

We will be using the following four facts:

**Fact 1:** $\epsilon_4 = \epsilon_2$

(This follows, since, $\epsilon_4 = \epsilon_3 \lor \epsilon_2 = \gamma \lor \epsilon_2 = \epsilon_2$.)

**Fact 2:** If $\delta_2, \delta_3$ are comparable, then $\delta_2 < \delta_3 (= \theta_3)$ necessarily (since $\delta_2 < \theta_2$, and $(\theta_2, \theta_3)$ is skew), and we have

$$\delta_4 = \delta_3 (= \theta_3), \delta_2 = \delta_1 (= \theta_1)$$

(This follows since $\delta_4 = \delta_3 \lor \delta_2 = \delta_3$. Similarly we have $\delta_2 = \delta_1$).

**Fact 3:** If $\theta_4, \gamma$ are comparable, then $\gamma < \theta_4$ necessarily, and we have

$$\delta_4 = \gamma, \theta_4 = \epsilon_4$$
5.1. Analysis of diamond relations:

(This follows since $\theta_4 < \gamma$ would imply that $\theta_2 < \gamma$, which is not true.)

**Fact 4:** If $\delta_4 \not\in D_\tau$, then $(\theta_4, \gamma)$ is skew necessarily.

(This follows since $\theta_4, \gamma$ are comparable would imply (cf. Fact 3) that $\delta_4 \in D_\tau$.)

We now divide the discussion into the following two subcases:

**Subcase** $B_1(a)$: Let $\delta_2 \in D_\tau$.

By considering the diamond with $(\theta_2, \gamma)$ as the skew diagonal we obtain that $\epsilon_2 \not\in D_\tau$; hence

\[(*) \quad F_{\theta_2} \equiv F_{\epsilon_2} (mod M^2_\tau)\]

Also Fact 2 implies that $(\delta_2, \delta_3)$ is skew necessarily (since $\delta_1 (= \theta_1) \not\in D_\tau$).

If $\delta_4 \not\in D_\tau$, then considering the $\delta$-diamond, and the diamond with $(\theta_4, \gamma)$ as skew diagonal (note that $(\theta_4, \gamma)$ is skew (cf. Fact 4)), we obtain

\[\theta_3(= \delta_4), \theta_4 \in E_\tau\]

(note that $\delta_2 \in D_\tau$ and therefore $\epsilon_4 (= \epsilon_2) \not\in D_\tau$). It follows that $f_D$ is in $f_D \in b(\tau)$ (since $X_{\theta_3}, X_{\theta_4}$ are in $ker \pi^{(1)}$).

Let then $\delta_4 \in D_\tau$.

**Claim 1:** $F_{\theta_2} \equiv F_{\theta_4} (mod M^2_\tau)$.

If $\theta_4, \gamma$ are comparable, then $\theta_4 = \epsilon_4$ (cf. Fact 3). The Claim follows from this in view of $(*)$ (note that $\epsilon_4 = \epsilon_2$ by Fact 1).

If $(\theta_4, \gamma)$ is skew, then

\[(**) \quad F_{\theta_4} \equiv F_{\epsilon_4} (mod M^2_\tau)\]

(note that $\delta_4 \in D_\tau$). Claim 1 follows from $(*)$and $(**)$ (in view of Fact 1).

Now in view of Fact 2 and the hypothesis that $\delta_4 \in D_\tau$, we have that $(\delta_2, \delta_3)$ is
5.1. Analysis of diamond relations: skew; hence by considering the $\delta$-diamond, we have

\[(***) \quad F_{\theta_1} \equiv F_{\theta_3} (\text{mod } M^2_\tau)\]

(note that $\delta_1 = \theta_1, \delta_3 = \theta_3$, and $\delta_2, \delta_4 \in D_\tau$). It follows from Claim 1, (**), and Lemma 5.1.1 that $f_D \in \mathfrak{b}(\tau)$.

**Subcase** $B_1(\mathfrak{b})$: Let $\delta_2 \not\in D_\tau$.

(i) Let $\epsilon_2 (= \epsilon_4) \in D_\tau$. Then we get

\[(\dagger) \quad F_{\theta_2} \equiv F_{\delta_2} (\text{mod } M^2_\tau)\]

Also, we have (cf. Fact 1)

\[(\dagger\dagger) \quad \delta_1 = \theta_1, \delta_3 = \theta_3\]

The hypothesis that $\epsilon_4 \in D_\tau$ implies that $(\theta_4, \gamma)$ is skew (cf. Fact 3), and hence we obtain

\[(\dagger\dagger\dagger) \quad F_{\theta_4} \equiv F_{\delta_4} (\text{mod } M^2_\tau)\]

If $\delta_2, \delta_3$ are comparable, then $\theta_3 = \delta_4, \theta_1 = \delta_2$ (cf. Fact 2), and we obtain (by \(\dagger\), \(\dagger\dagger\), \(\dagger\dagger\dagger\))

\[F_{\theta_2} \equiv F_{\theta_1} (\text{mod } M^2_\tau), F_{\theta_4} \equiv F_{\theta_3} (\text{mod } M^2_\tau)\]

It follows that $f_D \in \mathfrak{b}(\tau)$ (cf. Lemma 5.1.1).

If $(\delta_2, \delta_3)$ is skew, then by considering the $\delta$-diamond, we obtain in view of \(\dagger\), \(\dagger\dagger\), \(\dagger\dagger\dagger\), and Lemma 5.1.3 that $f_D \in \mathfrak{b}_{i+1}(\tau)$.

(ii) Let $\epsilon_2 (= \epsilon_4) \not\in D_\tau$. This implies (by considering the diamond with $(\theta_2, \gamma)$ as the skew diagonal)

\[(\diamond) \quad \theta_2 \in E_\tau\]
5.1. Analysis of diamond relations:

(note that by hypothesis we have $\delta_2 \notin D_\tau$).

If $\theta_1 \in E_\tau$, then clearly $f_D \in b(\tau)$.

Let then $\theta_1 \notin E_\tau$.

Claim 2: $\theta_4 \in E_\tau$.

If $\delta_2, \delta_3$ are comparable, then $(\theta_4, \gamma)$ is skew necessarily (since in this case $\delta_4 (= \theta_3) \notin D_\tau$ (cf. Fact 2)), and $\theta_4 \in E_\tau$ (by considering the diamond with $(\theta_4, \gamma)$ as skew diagonal, and also noting that by hypothesis $\epsilon_4 \notin D_\tau$).

If $(\delta_2, \delta_3)$ is skew, then by considering the $\delta$-diamond we obtain that $\delta_4 \notin D_\tau$ (since $\delta_1 (= \theta_1) \notin E_\tau$). Hence we obtain that $(\theta_4, \gamma)$ is skew necessarily, and therefore $\theta_4 \in E_\tau$ (note that by hypothesis $\epsilon_4 \notin D_\tau$).

Thus Claim 2 follows. Claim 2 together with $(\diamondsuit)$ implies that $f_D \in b(\tau)$.

This completes the discussion in Case $B_1$.

Case $B_2$: $\theta_2, \theta_3 \notin \Lambda_{i+1}(\Gamma)$.

This implies that at least one of $\{\theta_4, \theta_1\}$ is in $\Lambda_{i+1}(\Gamma)$. We divide the discussion into the following two cases:

(1) precisely one of $\{\theta_4, \theta_1\}$ is in $\Lambda_{i+1}(\Gamma)$

(2) both $\theta_4, \theta_1$ are in $\Lambda_{i+1}(\Gamma)$.

Subcase $B_2(a)$: Precisely one of $\{\theta_4, \theta_1\}$ is in $\Lambda_{i+1}(\Gamma)$, say, $\theta_4 \in \Lambda_{i+1}(\Gamma)$ (the proof for the case $\theta_1 \in \Lambda_{i+1}(\Gamma)$ being similar). Thus $\theta_4, \gamma$ are comparable, and the pair $(\theta_1, \gamma)$ is skew.

We first gather some facts:

Fact 5: We have $\theta_4 > \gamma$ necessarily (since $(\theta_i, \gamma)$ is skew for $i = 2, 3$), and hence $\theta_4 = \epsilon_4, \delta_4 = \gamma(\in D_\tau)$. Hence $\{\epsilon_1, \epsilon_4\}$ is not contained in $D_\tau$. This in turn
5.1. Analysis of diamond relations:

implies that \( \{\epsilon_2, \epsilon_3\} \) is not contained in \( D_{\tau} \) (this is clear if \( \epsilon_2, \epsilon_3 \) are comparable - then \( \{\epsilon_2, \epsilon_3\} = \{\epsilon_1, \epsilon_4\} \); if \( (\epsilon_2, \epsilon_3) \) is skew, this follows from the fact that \( D_{\tau} \) is a sublattice of \( \mathcal{L} \)).

**Fact 6:** \( (\delta_2, \delta_3) \) is skew

(Say, \( \delta_2 > \delta_3 \). We have

\[
\gamma = \delta_4 = \delta_2 \lor \delta_3 = \delta_2
\]

which is not true, since \( (\theta_2, \gamma) \) is skew and \( \delta_2 = \theta_2 \land \gamma \).)

In particular, we get the \( \delta \)-diamond \( (\subseteq \Lambda_{i+1}(\Gamma)) \).

**Fact 7:** If \( \delta_1 \in D_{\tau} \), then the \( \delta \)-diamond is contained in \( D_{\tau} \) (since \( \delta_4(= \gamma) \in D_{\tau} \), and \( D_{\tau} \) is a embedded sublattice), and hence

\[
F_{\theta_j} \equiv F_{\epsilon_j}(\text{mod } M_{2\tau}^2), \ j = 1, 2, 3, 4
\]

If \( \epsilon_2, \epsilon_3 \) are comparable, say, \( \epsilon_2 > \epsilon_3 \), then \( \epsilon_2 = \epsilon_4, \epsilon_3 = \epsilon_1 \), and hence

\[
F_{\theta_2} \equiv F_{\theta_4}(\text{mod } M_{2\tau}^2), F_{\theta_1} \equiv F_{\theta_3}(\text{mod } M_{2\tau}^2)
\]

It follows that \( f_D \in b(\tau) \) (cf. Lemma 5.1.1).

If \( (\epsilon_2, \epsilon_3) \) is skew, then by Lemma 5.1.3, we obtain that \( f_D \equiv f_{D'}(\text{mod } b(\tau)) \), \( D' \)

being the \( \epsilon \)-diamond \( (\subseteq \Lambda_{i+1}(\Gamma)) \). For similar considerations, we may suppose \( \{\delta_2, \delta_3\} \) is not contained in \( D_{\tau} \) (since \( D_{\tau} \) is a sublattice of \( \mathcal{L} \)).

Hence we shall suppose that \( \delta_1 \notin D_{\tau} \), and at least one of \( \{\delta_2, \delta_3\} \) is not in \( D_{\tau} \).

**Fact 8:** In view of Facts 5 & 7, we may suppose that none of the pairs

\( (\epsilon_1, \epsilon_4), (\epsilon_2, \epsilon_3), (\delta_1, \delta_4), (\delta_2, \delta_3) \)

is contained in \( D_{\tau} \).

We now divide our discussion into the following two subcases:

**Subcase** \( B_2(a_1) \): At least one of \( \{\theta_1, \theta_4\} \) is in \( E_{\tau} \). Let us suppose that \( \theta_1 \in E_{\tau} \) (the proof for the case when \( \theta_4 \in E_{\tau} \) being similar). Hence (by our
5.1. Analysis of diamond relations:

general reduction that a side of $D$ is not contained in $E_\tau$), we obtain that $\theta_2, \theta_3$
are not in $E_\tau$. This implies that for $j = 2, 3$ precisely one of $\{\epsilon_j, \delta_j\}$ is in $D_\tau$. From
this we get (in view of Fact 8) that precisely one of $\{\epsilon_2, \delta_3\}, \{\epsilon_3, \delta_2\}$ is contained
in $D_\tau$ and the other has empty intersection with $D_\tau$; let us suppose that $\{\epsilon_2, \delta_3\}$
is contained in $D_\tau$, and $\epsilon_3, \delta_2 \not\in D_\tau$ (proof of the other case being similar). This
implies

$$F_{\theta_2} \equiv F_{\delta_2} (mod \, M^2_\tau)$$
$$F_{\theta_3} \equiv F_{\epsilon_3} (mod \, M^2_\tau)$$
$$F_{\delta_1} \equiv F_{\delta_2} (mod \, M^2_\tau)$$

(by considering the diamonds with skew diagonals $(\theta_2, \gamma), (\theta_3, \gamma), (\delta_2, \delta_3)$ (cf. Fact
6 and the hypothesis that $\theta_2, \theta_3 \not\in \Lambda_{i+1}(\Gamma)$) respectively, and also observing that
$\epsilon_2, \delta_3, \delta_4$ are in $D_\tau$).

If $\epsilon_1 \in D_\tau$, then $F_{\theta_1} \equiv F_{\delta_1} (mod \, M^2_\tau)$, and hence we obtain

$$F_{\theta_1} \equiv F_{\delta_4} (mod \, M^2_\tau)$$

Hence it follows $f_D \in b(\tau)$ (since $X_{\theta_1} - X_{\delta_2}, X_{\theta_1} \in ker \, \pi^{(1)}$, we get $X_{\theta_2} \in ker \, \pi^{(1)}$).

If $\epsilon_1 \not\in D_\tau$, then $\epsilon_3 \in E_\tau$ (by consideration of the $\epsilon$-diamond; note that $(\epsilon_2, \epsilon_3)$ is
skew (for, $\epsilon_2, \epsilon_3$ comparable would imply $\{\epsilon_2, \epsilon_3\} = \{\epsilon_1, \epsilon_4\}$ which is not possible
since $\epsilon_2 \in D_\tau$ while $\epsilon_1, \epsilon_4 (= \theta_4)$ are not in $D_\tau$). This together with the fact that
$F_{\theta_3} \equiv F_{\epsilon_3} (mod \, M^2_\tau)$ implies that $X_{\theta_3} \in ker \, \pi^{(1)}$. Hence we obtain that $f_D \in b(\tau)$
(since by hypothesis, $\theta_1 \in E_\tau$, and hence $X_{\theta_1} \in ker \, \pi^{(1)}$).

**Subsubcase B2(a2):** Both $\theta_1, \theta_4$ are not in $E_\tau$.

This implies that $\epsilon_1 \in D_\tau$ (since $\delta_1 \not\in D_\tau$ (cf. Fact 7) ), and

$$(1) \quad F_{\theta_1} \equiv F_{\delta_1} (mod \, M^2_\tau)$$

(by considering the diamond with $(\theta_1, \gamma)$ as the skew diagonal).
5.1. Analysis of diamond relations:

Further, precisely one of \( \{ \epsilon_2, \epsilon_3 \} \) is in \( D_\tau \); this is clear if \( \epsilon_2, \epsilon_3 \) are comparable (in which case \( \{ \epsilon_2, \epsilon_3 \} = \{ \epsilon_1, \epsilon_4 \} \), and \( \epsilon_4(= \theta_4) \notin D_\tau \), and \( \epsilon_1 \in D_\tau \). If \( (\epsilon_2, \epsilon_3) \) is skew, then in the \( \epsilon \)-diamond, we have that \( \epsilon_4(= \theta_4) \notin E_\tau \), and \( \epsilon_1 \in D_\tau \).

Let us then suppose that \( \epsilon_2 \in D_\tau, \epsilon_3 \notin D_\tau \) (proof of the other case being similar). Hence (by considering the diamond with \( (\theta_2, \gamma) \) as the skew diagonal) we obtain

\[
(2) \quad F_{\theta_2} \equiv F_{\delta_2} (\text{mod } M_\tau^2)
\]

Also,

\[
(3) \quad F_{\epsilon_3} \equiv F_{\theta_4} (\text{mod } M_\tau^2)
\]

(If \( \epsilon_2, \epsilon_3 \) are comparable, then the facts that \( \epsilon_1, \epsilon_2 \in D_\tau, \epsilon_4(= \theta_4) \notin D_\tau \) imply that \( \epsilon_1 = \epsilon_2, \epsilon_3 = \theta_4 \), and hence (3) follows. If \( (\epsilon_2, \epsilon_3) \) is skew, then by considering the \( \epsilon \)-diamond and using the facts that \( \epsilon_1, \epsilon_2 \in D_\tau \), we obtain that \( F_{\epsilon_1} \equiv F_{\epsilon_4} (\text{mod } M_\tau^2) \), and (3) follows since \( \epsilon_4 = \theta_4 \).)

If \( \delta_3 \in D_\tau \), then by considering the \( \delta \)-diamond (note \( (\delta_2, \delta_3) \) is skew (cf. Fact 6))) and the diamond with \( (\theta_3, \gamma) \) as skew diagonal, we obtain (noting that \( \delta_4(= \gamma) \in D_\tau \))

\[
(4) \quad F_{\delta_1} \equiv F_{\delta_2} (\text{mod } M_\tau^2)
\]

\[
(5) \quad F_{\theta_3} \equiv F_{\epsilon_4} (\text{mod } M_\tau^2)
\]

From (1),(2),(3),(4), (5) we obtain

\[
F_{\theta_1} \equiv F_{\theta_2} (\text{mod } M_\tau^2), \quad F_{\theta_3} \equiv F_{\theta_4} (\text{mod } M_\tau^2)
\]

Hence in view of Lemma 5.1.1, we obtain that \( f_D \in b(\tau) \).

If \( \delta_3 \notin D_\tau \), then \( \delta_1 \in E_\tau \) (note that in the \( \delta \)-diamond, \( \delta_1 \notin D_\tau \) (cf. Fact 7), \( \delta_2 \notin D_\tau \) (since \( \epsilon_2 \in D_\tau \)) while \( \delta_4 \in D_\tau \)); similarly, consideration of the diamond
with \((\theta_3, \gamma)\) as the skew diagonal implies \(\theta_3 \in E_\tau\) (note that \(\epsilon_3, \delta_3 \notin D_\tau\)). The facts that \(\delta_1, \theta_3 \in E_\tau\) together with (1) imply that \(X_{\theta_1}, X_{\theta_3}\) are in \(\ker \pi^{(1)}\) and hence \(f_D \in b(\tau)\).

**Subcase B_2(b):** Both \(\theta_1, \theta_4\) are in \(\Lambda_{i+1}(\Gamma)\) (and \(\theta_2, \theta_3\) are not in \(\Lambda_{i+1}(\Gamma)\))

We have \(\theta_1 < \gamma < \theta_4\) necessarily.

**Fact 9:** We have \(\theta_1 = \delta_1, \theta_4 = \epsilon_4, \epsilon_1 = \gamma = \delta_4\).

As in \(B_2(a)\) (cf. Fact 6), we have

**Fact 10:** \((\delta_2, \delta_3), (\epsilon_2, \epsilon_3)\) are skew.

(The proof of the assertion that \((\epsilon_2, \epsilon_3)\) is skew is similar to that of Fact 6; namely, say \(\epsilon_2 > \epsilon_3\). We have

\[\gamma = \epsilon_1 = \epsilon_2 \land \epsilon_3 = \epsilon_3\]

which is not true (since \((\theta_3, \gamma)\) is skew, and \(\epsilon_3 = \theta_3 \lor \gamma\)).

In particular, we get the \(\delta\)-diamond and the \(\epsilon\)-diamond (contained in \(\Lambda_{i+1}(\Gamma)\)).

**Fact 11:** In view of Facts 9 & 10, we have that none of the pairs

\((\epsilon_1, \epsilon_4), (\epsilon_2, \epsilon_3), (\delta_1, \delta_4), (\delta_2, \delta_3)\) is contained in \(D_\tau\).

**Subsubcase B_2(b_1):** At least one of \(\{\theta_1, \theta_4\}\) is in \(E_\tau\).

Let us suppose that \(\theta_1 \in E_\tau\) (the proof for the case when \(\theta_4 \in E_\tau\) being similar).

Hence (by our general reduction that a side of \(D\) is not contained in \(E_\tau\)), we obtain that \(\theta_2, \theta_3\) are not in \(E_\tau\). This implies that for \(j = 2, 3\) precisely one of \(\{\epsilon_j, \delta_j\}\) is in \(D_\tau\). From this we get (in view of Fact 11) that precisely one of \(\{\epsilon_2, \delta_3\}, \{\epsilon_3, \delta_2\}\) is contained in \(D_\tau\) and the other has empty intersection with \(D_\tau\); let us suppose that \(\{\epsilon_2, \delta_3\}\) is contained in \(D_\tau\), and \(\epsilon_3, \delta_2 \notin D_\tau\) (proof of the other
case being similar). This implies

\[
F_{\theta_2} \equiv F_{\delta_2} \pmod{M_\tau^2}
\]
\[
F_{\delta_1} \equiv F_{\delta_2} \pmod{M_\tau^2}
\]
\[
F_{\theta_3} \equiv F_{\epsilon_3} \pmod{M_\tau^2}
\]
\[
F_{\epsilon_3} \equiv F_{\epsilon_4} \pmod{M_\tau^2}
\]

(by considering the diamonds with skew diagonals \((\theta_2, \gamma), (\delta_2, \delta_3), (\theta_3, \gamma), (\epsilon_2, \epsilon_3)\) respectively). This combined with the fact that \(\delta_1 = \theta_1, \epsilon_4 = \theta_4\) implies

\[
F_{\theta_2} \equiv F_{\theta_1} \pmod{M_\tau^2}, F_{\theta_3} \equiv F_{\theta_4} \pmod{M_\tau^2}
\]

Hence we obtain (in view of Lemma 5.1.1) that \(f_D \in \mathfrak{b}(\tau)\).

**Subcase** \(B_2(b_2)\): Both \(\theta_1, \theta_4\) are not in \(E_\tau\).

This implies (in view of Facts 9 & 11) that precisely one of \(\{\epsilon_2, \epsilon_3\}\), and that precisely one of \(\{\delta_2, \delta_3\}\) is in \(D_\tau\) (by considering the \(\epsilon\)-diamond, \(\delta\)-diamond respectively); also, consideration of the diamond with \((\theta_j, \gamma), j = 2, 3\) as the skew diagonal implies that for \(j = 2, 3\) at most one of \(\{\epsilon_j, \delta_j\}\) is in \(D_\tau\). Hence we obtain that precisely one of \(\{\epsilon_2, \delta_3\}, \{\epsilon_3, \delta_2\}\) is contained in \(D_\tau\) and the other has empty intersection with \(D_\tau\). The rest of the proof is as in Subcase \(B_2(b_1)\), and we obtain that \(f_D \in \mathfrak{b}(\tau)\).

This completes the proof of the case \(B_2(b)\) and hence that of Proposition 5.1.7.

## 5.2 Tangent cone at \(P_\tau\)

In this section, we construct “standard monomial basis” for \(TC_{P_\tau}X_\mathcal{L}\), the tangent cone to \(X_\mathcal{L}\) at \(P_\tau\). We first determine \(J(\tau)\). We then introduce a monomial order \(>\) and determine \(in J(\tau)\), the initial ideal of \(J(\tau)\). Using MaCaulay’s Theorem, we obtain a “standard monomial basis” for \(TC_{P_\tau}X_\mathcal{L}\).
Let $X = \text{Spec} R \hookrightarrow \mathbb{A}^l$ be an affine variety. Let $P \in X$, and let $M_P$ be the maximal ideal in $K[X]$ corresponding to $P$ (we are concerned only with closed points of $X$). Let $A = \mathfrak{O}_{X,P}$, the stalk at $P$; denote the unique maximal ideal in $A$ by $M(= M_PR_{M_P})$. Then $\text{Spec gr}(A, M)$, where $\text{gr}(A, M) = \bigoplus_{j \in \mathbb{Z}_+} M^j/M^{j+1}$ is the tangent cone to $X$ at $P$, and is denoted $TC_P X$. Note that $\text{gr}(R, M_P) = \text{gr}(A, M)$

**Remark 5.2.1.** Let $I(X)$ be the vanishing ideal of $X$ for the embedding $X = \text{Spec} R \hookrightarrow \mathbb{A}^l$. Expanding $f \in I(X)$ in terms of the local co-ordinates at $P$, we have the following:

- $T_P(X)$, the tangent space to $X$ at $P$ is the zero locus of the linear forms of $f, f \in I(X)$.
- $TC_P(X)$, the tangent cone to $X$ at $P$ is the zero locus of the initial forms (i.e., form of smallest degree) of $f, f \in I(X)$.

In the sequel, we shall denote the initial form of $f$ by $IF(f)$. We have a natural surjective map

$$\text{Sym}(M_P/M_P^2) \to \text{gr}(R, M_P)$$

(here, $\text{Sym}(M_P/M_P^2)$ is the symmetric algebra of $M_P/M_P^2$). Thus we have a natural embedding

$$TC_P(X) \hookrightarrow T_P(X) (= \mathbb{A}^r)$$

$r$ being the dimension of $T_P(X)$, and $TC_P(X)$ is the zero locus of the ideal $I$ in $\text{Sym}(M_P/M_P^2)$ such that $\text{gr}(R, M_P)$ is the quotient $\text{Sym}(M_P/M_P^2)/I$ (note that $I$ consists of $IF(f), f \in I(X)$).

We collect below some well-known facts on singularities of $X$:

**Facts:** 1. $\dim T_P X \geq \dim X$ with equality if and only if $X$ is smooth at $P$. 

5.2. Tangent cone at $P$.

2. $X$ is smooth at $P$ if and only if $\text{mult}_P X$ equals 1.

3. $X$ is smooth at $P$ if and only if the canonical surjective map $\text{Sym}(M_P/M_P^2) \rightarrow \text{gr}(R, M_P)$ is an isomorphism, i.e., if and only if $\text{gr}(R, M_P)$ is a polynomial algebra.

5.2.2 Determination of $I(\tau)$:

We now take $X = X_L (= \text{Spec } K[S_\sigma])$, and $P = P_\tau$, $\tau$ being a face of $\sigma$ (notation as in §4.1). We shall denote $X_L$ by $X$, $K[X_L]$ by $R$. Let $J(\tau)$ be the kernel of the surjective map

$$\pi : K[X_\theta, \theta \in \mathcal{L}] \rightarrow \text{gr}(R, M_\tau), \ X_\theta \mapsto \overline{F_\theta}$$

Note that $J(\tau)$ consists of the initial forms of elements of $I(X_L)$ ($I(X_L)$ being the ideal in $K[X_\theta, \theta \in \mathcal{L}]$ generated by the diamond relations). For $r \in \mathbb{N}$, let $\pi^{(r)}$ be the restriction of $\pi$ to the degree $r$ part of the polynomial algebra $K[X_\theta, \theta \in \mathcal{L}]$; then $\ker \pi^{(r)} = J(\tau)_r$, the $r$-th graded piece of the homogeneous ideal $J(\tau)$. By [15], we have that $\ker \pi^{(1)} = \ker g_\tau^{(1)} \oplus \ker h_\tau^{(1)}$, $\ker h_\tau^{(1)}$ is generated by $\{X_\alpha, \alpha \in E_\tau\} \cup \{(X_\theta - X_\delta), \theta \sim \delta\}$; further, by Lemma 4.2.11, for any $\theta \in \mathcal{L} \setminus D_\tau$, there exists a $\mu \in \Lambda_\tau(\Gamma) \cup E_\tau$ such that $\theta \sim \mu$. In view of this and Corollary 5.1.8, we may work with $K[X_\theta, \theta \in \Lambda_\tau(\Gamma)]$, and identify $J(\tau)$ with the kernel of the surjective map

$$f_\tau : K[X_\theta, \theta \in \Lambda_\tau(\Gamma)] \rightarrow \text{gr}(R, M_\tau), \ X_\theta \mapsto \overline{F_\theta}$$

Note that $J(\tau)$ consists of the initial forms of elements of $I(X_{\Lambda_\tau(\Gamma)})$ (the ideal in $K[X_\theta, \theta \in \Lambda_\tau(\Gamma)]$ generated by the diamond relations arising from diamonds in $\Lambda_\tau(\Gamma)$).
5.2. Tangent cone at $P_{\tau}$

We shall next describe $\text{ker} \, \pi^{(2)}$, deduce a set of generators for $\text{ker} \, \pi^{(r)}$, $r \in \mathbb{N}$, and then determine a set of generators for $J(\tau)$. We first start with an analysis of the diamond relations in $\Lambda_{\tau}(\Gamma)$.

For a diamond $D$ in $\Lambda_{\tau}(\Gamma)$, with $(\theta, \delta)$ as the skew diagonal, let

$$f_D = X_\theta X_\delta - X_{\theta \lor \delta} X_{\theta \land \delta}$$

**Certain quadratic relations on $TC_{P_{\tau}}X_{L}$:** We have the following three types of elements of $J(\tau)_2$ arising from the diamond relations:

(i) Let $D$ be such that

- $D \cap \Gamma = \emptyset$.
- One vertex, say $\alpha$, is in $E_{\tau}$ ($E_{\tau}$ being as in §4.2). Denote the other vertex of the diagonal through $\alpha$ by $\xi$.
- The diagonal not containing $\alpha$ has empty intersection with $E_{\tau}$. Denote this diagonal by $(\theta, \delta)$.

We have that $f_D = X_\theta X_\delta - X_\alpha X_\xi$ (in local co-ordinates around $P_{\tau}$) and hence $X_\theta X_\delta - X_\alpha X_\xi$ belongs to $J(\tau)_2$. Now $X_\alpha \in J(\tau)_1$ (cf. Lemma 4.2.6). Hence we obtain that $X_\theta X_\delta$ is in $J(\tau)_2$.

(ii) Let $D$ be such that $D \cap \{\Gamma \cup E_{\tau}\} = \emptyset$. Then $f_D$ remains homogeneous (in local co-ordinates around $P_{\tau}$) and belongs to $J(\tau)_2$.

(iii) Let $D_1, \cdots, D_r$ be a set of diamonds such that there exists a pair $(\alpha, \gamma)$ in $\Lambda_{\tau}(\Gamma), \alpha \in E_{\tau}, \gamma \in \Gamma$ such that $(\alpha, \gamma)$ is a diagonal of $D_i$; note that $(\alpha, \gamma)$ is necessarily the main diagonal of $D_i$ (since $\alpha \in \Lambda_{\tau}(\Gamma)$), and the other diagonal of $D_i$ has empty intersection with $\Gamma$ (since any $\gamma \in \Gamma$ can not be on the skew diagonal of any diamond in $\Lambda_{\tau}(\Gamma)$). Further, for each $i$, let the other diagonal of $D_i$ have empty intersection with $E_{\tau}$. Denote this diagonal by $(\theta_i, \delta_i)$; note that
5.2. Tangent cone at $P_\tau$

by Lemma 4.2.13, if $r > 1$, then $\gamma$ is uniquely determined by $\alpha$. If $r > 1$, let us further suppose that $D_1, \cdots, D_r$ are all the diamonds in $\Lambda_\tau(\Gamma)$ having $(\alpha, \gamma)$ as the main diagonal.

(a) If $r = 1$ (i.e., there exists a unique skew pair $(\theta, \delta)$ with $\{\theta \vee \delta, \theta \vee \delta\} = \{\alpha, \gamma\}$), then denoting the corresponding diamond by $D$, we have $IN(f_D) = X_\alpha$ (and $X_\alpha \in J(\tau)_1$).

(b) If $r > 1$, then for a pair $(i, j), i \neq j$, we have, $IF(f_{D_i} - f_{D_j}) \equiv X_{\theta_i}X_{\delta_i} - X_{\theta_j}X_{\delta_j} \pmod{J(\tau)}$; hence we obtain that $X_{\theta_i}X_{\delta_i} - X_{\theta_j}X_{\delta_j} \in J(\tau)_2$.

In the sequel, we shall refer to a $D$ as in (i) (resp.(ii),(iii)) as a Type I (resp. Type II, Type III) diamond. The elements of $J(\tau)$ appearing in (i)(resp. (ii),(iii)(a),(b)) above will be referred to as a Type I-element (resp. Type II, III-elements) of $J(\tau)$.

**Ideal generators for $J(\tau)$:**

Let $a(\tau)$ denote the ideal in the polynomial algebra $K[X_\theta, \theta \in \Lambda_\tau(\Gamma)]$ generated by elements in $J(\tau)$ of Type I,II,III (as described above). Clearly $a(\tau) \subseteq J(\tau)$.

**Theorem 5.2.3.** The inclusion $a(\tau) \subseteq J(\tau)$ is an equality.

**Proof 5.2.4.** Let $h \in I(X_{\Lambda_\tau(\Gamma)})$. We are required to show that $IF(h)$ belongs to $a(\tau)$. We shall first write $h$ as a polynomial expression in the $f_D$’s, and then find $IF(h)$. Now $h$ is a polynomial expression in the $f_D$’s, $D$ being of Type I,II or III of §5.2.2. Let

$$h = H_1 + H_2 + H_3$$

where $H_1, H_2, H_3$ are polynomial expression in the $f_D$’s $D$’s being of Type I,II, III respectively. Let $d$ be the degree of $IF(h)$. Now for any $l$, the degree $l$ forms of $H_1, H_2$ are clearly in $a(\tau)$ (since for $D$ of Type I,II, $f_D$ is in $a(\tau)$). Thus we are
led to analyze the degree $l$ forms in $H_3$ (for any $l$). Let

\[ H_3 = \sum_{\alpha \in \{\Lambda_\tau(\Gamma) \cap E_\tau\}} \sum_i A_{\alpha,i} [X_\alpha(1 - X_\gamma) - X_{\theta_i}X_{\delta_i}] \]

where $(\alpha, \gamma), (\theta_i, \delta_i)$ are the diagonals of diamonds of Type III. Since $X_\alpha \in a(\tau)$ for $\alpha \in E_\tau$, we are led to analyze the degree $d$ forms in a $H_3$ as above, where we may suppose that none of the monomials present in $A_{\alpha,i}$'s is divisible by any $X_{\alpha'}$ for $\alpha' \in E_\tau$. If

\[ \sum_{\alpha \in \{\Lambda_\tau(\Gamma) \cap E_\tau\}} \sum_i A_{\alpha,i}X_\alpha \neq 0 \]

then the result is clear (since, $X_\alpha \in a(\tau)$, for $\alpha \in E_\tau$). Let then

\[ \sum_{\alpha \in \{\Lambda_\tau(\Gamma) \cap E_\tau\}} \sum_i A_{\alpha,i}X_\alpha = 0 \]

Hence we obtain

\[ \sum_i A_{\alpha,i} = 0, \forall \alpha \]

(in view of our assumption that none of the monomials in $A_{\alpha,i}$'s is divisible by any $X_{\alpha'}$ for $\alpha' \in E_\tau$). Hence we obtain

\[ H_3 = -\sum_{\alpha \in \{\Lambda_\tau(\Gamma) \cap E_\tau\}} \sum_i A_{\alpha,i} X_{\theta_i}X_{\delta_i} \]

Now (**) implies that if a certain monomial $m$ occurs with a coefficient $s$ in $\sum_i A_{\alpha,i}$, then $-m$ also occurs with the same coefficient $s$ in $\sum_i A_{\alpha,i}$. Hence denoting by $M_\alpha$ the set of all monomials appearing in the polynomial $\sum_i A_{\alpha,i}$, we have,

\[ H_3 = \sum_{\alpha} \sum_{m \in M_\alpha} m(X_{\theta_i}X_{\delta_i} - X_{\theta_j}X_{\delta_j}) \]

where $(\theta_i, \delta_i), (\theta_j, \delta_j)$ are skew diagonals of diamonds $D_i, D_j$ in $\Lambda_\tau(\Gamma)$ for which $(\alpha, \gamma)$ is the main diagonal (cf. (iii) of §5.2.2). The required result now follows
(note that (*** implies that $H_3$ is in the homogeneous ideal generated by Type III (b) elements of §5.2.2, and hence any homogeneous component - in particular, IN $H_3$ is again in that ideal).

A monomial order in $K[X_\theta, \theta \in \Lambda_r(\Gamma)]$: Take a total order $\succ$ on $\Lambda_r(\Gamma)$ extending the partial order. Define $X_\theta \succ X_\delta$ if $\theta \prec \delta$; a monomial will be written as $X_{\theta_1} \cdots X_{\theta_r}$ where $\theta_1 \geq \cdots \geq \theta_r$. Given two monomials $m_1, m_2$, define $m_1 \succ m_2$ if either $\deg m_1 > \deg m_2$ or $\deg m_1 = \deg m_2$, and $m_1 \succeq m_2$ (under the lexicographic order induced by $\succ$, i.e., if $m_1 = X_{\theta_1} \cdots X_{\theta_r}, m_2 = X_{\delta_1} \cdots X_{\delta_t}$, then there exists a $t < r$ such that $\theta_i = \delta_i, i < t$ and $\theta_t \succ \delta_t$).

Let $in J(\tau)$ be the initial ideal of $J(\tau)$; note that

$$in J(\tau) = \{in f, \ f \in J(\tau)\}$$

where for a polynomial $g \in K[X_\theta, \theta \in \Lambda_r]$, $in g$ is the greatest (under $\succ$) monomial occurring in $g$. Let

$$E_r(\Gamma) = \Lambda_r(\Gamma) \cap E_r$$

Let notation be as in §5.2.2. Further, given an equivalence class $[\theta]$ we shall denote the maximal element (under $\succ$) in $\{\delta, \delta \sim \theta\}$ by $[\theta]_{\max}$

**Proposition 5.2.5.** $in J(\tau)$ is generated by

- $\{X_\theta, \theta \neq [\theta]_{\max}\}$
- $\{X_\alpha, \alpha \in E_r(\Gamma)\}$
- $\{X_\theta X_\delta, (\theta, \delta)$ being a diagonal of a diamond of Type I\}$
- $\{X_\theta X_\delta, (\theta, \delta)$ being the skew diagonal of a diamond of Type II\}$
- $\{X_\theta X_\delta, (\theta, \delta)$ being the skew diagonal of a diamond of Type III with $r > 1$, and $(\theta, \delta) \neq \max\{([\theta_{i}]_{\max}, [\delta_{i}]_{\max})\}\}$
(Note that in the last set \((\theta_i, \delta_i)\) are all the skew diagonals such that 
\(\theta_i \lor \delta_i = \theta \lor \delta, \theta_i \land \delta_i = \theta \land \delta\) with the main diagonal as in Type III.)

**Proof 5.2.6.** Let \(b_\tau\) be the ideal generated by the set of monomials as described in the Proposition; clearly, we have that \(b_\tau \subseteq in J(\tau)\). We have that \(in J(\tau)\) consists of initial monomials in the initial forms of elements of \(I(X_{\Lambda_\tau(\Gamma)})\) (the ideal in \(K[X_\theta, \theta \in \Lambda_\tau(\Gamma)]\) generated by diamond relations arising from diamonds in \(\Lambda_\tau(\Gamma)\)). Now a typical element \(h \in I(X_{\Lambda_\tau(\Gamma)})\) is a polynomial expression in the \(f_D\)'s, \(D\) being of Type I, II or III of §5.2.2. Let 
\[ h = H_1 + H_2 + H_3 \]
where \(H_1, H_2, H_3\) are polynomial expression in the \(f_D\)'s \(D\)'s being of Type I, II, III respectively. Proceeding as in the proof of Proposition 5.2.3, we see that \(in h\) is in \(b_\tau\).

**Definition 5.2.7.** Call a monomial in \(K[X_\theta, \theta \in \Lambda_\tau(\Gamma)]\) *standard* if it is not in the collection as described in the above Proposition.

As an immediate consequence of MaCaulay’s theorem (cf. [6], Theorem 15.3), we have the following

**Theorem 5.2.8.** The standard monomials form a basis for the ring of regular functions on \(TC_p, X(\mathcal{L})\).

**Theorem 5.2.9.** Let \(\tau\) be such that \(E_\tau(\Gamma)\) is empty. Further, for each \(\theta \in \mathcal{L}\), let \([\theta]\) consist of just \(\{\theta\}\). Then \(TC_p, X(\mathcal{L})\) is again a Hibi variety.

**Proof 5.2.10.** Hypothesis implies that \(J(\tau)\) is generated by just Type II elements, namely, the diamond relations in \(\Lambda_\tau(\Gamma)\). Thus we obtain that \(TC_p, X(\mathcal{L})\) is simply the Hibi variety associated to the distributive lattice \(\Lambda_\tau(\Gamma)\).
Bibliography


5.2. Tangent cone at $P$.


5.2. Tangent cone at $P_\tau$.