SEMI-INVARIENTS
OF TUBULAR ALGEBRAS

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ABSTRACT OF DISSERTATION

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Abstract

The focus of this thesis is on the rings of semi-invariants of the representation spaces of tubular algebras. These rings reflect the cyclic nature of the structure of the modules of the tubular algebras. The main theorem gives the generators and relations of the rings of semi-invariants $SI(Q/I, d)$ where $d$ is a dimension vector of a module in the tubes of an algebra where certain conditions hold. The theorem can be directly applied to Euclidean algebras and Tubular algebras since they have separating tubular families. The results reflect the work of Skowronski and Weyman on semi-invariants of Euclidean Algebras. Also, in the case of tubular algebras, applications of shrinking functors to the rings of semi-invariants are explored.
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1 Introduction

The three distinct classes of quiver representations are finite type, tame and wild quivers. For the class of tame algebras, the indecomposable modules occur in each dimension in a finite number of discrete and a finite number of one parameter families. It is an interesting task to find geometric characterizations of the representation type of a quiver. This leads us into the study of semi-invariants of quivers. Given a quiver $Q$ and a dimension vector $d$, to find semi-invariants we study the actions of the corresponding products $G(Q,d)$ of general linear groups on the affine variety $Rep_K(Q,d)$ of representations of a $Q$ of dimension $d$ over an algebraically closed field $K$. In [36], Schofield classified semi-invariants of quivers to be determinantal semi-invariants. In [41], Skowronski and Weyman showed that a finite connected quiver is a Dynkin or Euclidean quiver if and only if the algebra of semi-invariants for any given dimension vector is a complete intersection. Derksen and Weyman in [20] expanded the idea of Schofield’s determinantal semi-invariants to apply to quivers with relations.

Given a Euclidean quiver, we can consider tilted algebras. From a tilted algebra in the preprojective component of the original quiver, we can use several branch extensions to obtain a tubular algebra. These tubular algebras are quasi-titled and have an interesting module structure. The focus of this paper is on the rings of semi-invariants on the representation spaces of tubular algebras. These rings of semi-invariants reflect very nicely the structure of the modules of the tubular algebras.

**Theorem 1.0.1.** \[3.2.1\] Let $\{V_t, E_i^{(j)}\}$ be the simple modules in $R(\Lambda)$ of tubular type $(m_1, \ldots, m_s)$. Let $\beta$ be a regular dimension vector. Then the ring of regular semi-invariants $SI_{reg}(\Lambda, \beta)$ has the generators and relations described as follows.

(a) The non-homogeneous generators are $c^{E_i^{(j)}}_{[i+1, k]}$ where $[i, k]$ is a $\beta$-admissible path on the $j$-th circle.
(b) The homogeneous generators $c^V_t$ which span the space of dimension $p + 1$. We fix a basis $\{c_0, \ldots, c_p\}$ of the weight space $SI_{reg}(\Lambda, \beta)$. 

(c) There are $s$ relations in $SI_{reg}(\Lambda, \beta)$ such that each relation expresses the product of non-homogeneous semi-invariants of index zero on the $j$-th circle as a linear combination of the semi-invariants $c_0, \ldots, c_p$.

More generally, for any algebra whose periodic modules have the structure of a separating tubular family, their rings of semi-invariants will also follow these patterns. So we have the ability to apply this theorem to not only tubular algebras but also to reformulate the results about semi-invariants for Euclidean Quivers.
2 Basic Notions

The preliminaries on quiver representations and semi-invariants included in this section are classical and are included here to define notation and make this note self-contained.

2.1 Definition of Semi-Invariants and the Theorem of Sato and Kimura

Fix an algebraically closed field $K$.

Let $G$ be an algebraic group and let $V$ be its representation. There is an induced action of $G$ on the coordinate ring $K[V]$ of polynomial functions on $V$, namely for $g \in G$, $f \in K[V]$

$$(gf)(x) := f(g^{-1}x).$$

For a character $\chi$ of $G$ we define the space of semi-invariants of weight $\chi$ for the representation $V$ as

$$SI(G,V)_\chi = \{ f \in K[V]| \forall g \in G, g \circ f = \chi(g)f \}.$$ 

The direct sum

$$SI(G,V) := \bigoplus_{\chi \in \text{char}(G)} SI(G,V)_\chi$$

is a ring of semi-invariants for the representation $V$.

**Theorem 2.1.1.** (Sato-Kimura, [34]) Let $G$ be an algebraic group and let $V$ be its representation. Assume that $G$ is connected, and $V$ has a Zariski open $G$-orbit. Then

a) $SI(G,V)$ is a polynomial ring, and the generators $X_1, \ldots, X_d$ can be chosen in the weight spaces $\chi_1, \ldots, \chi_d$. 


b) The weights $\chi_1, \ldots, \chi_d$ are linearly independent in $\text{char}(G)$.

c) The number $d$ and generators $X_1, \ldots, X_d$ have the following interpretation. Let $O$ be a Zariski open $G$-orbit in $V$ and let $Z := V \setminus O$. Decompose $Z$ into irreducible components

$$Z = Z_1 \cup Z_2 \cup \ldots \cup Z_s$$

and reorder them so $Z_1, \ldots, Z_d$ have codimension 1 and $Z_{d+1}, \ldots, Z_s$ have codimension $> 1$. Then $X_j$ can be taken to be an irreducible equation of $Z_j$.

### 2.2 Quivers

We can associate each finite dimensional $K$-algebra to a graphical structure called a quiver. Conversely, each quiver corresponds to an associative $K$-algebra which is finite dimensional under some conditions.

**Definition 2.2.1.** A quiver $Q = (Q_0, Q_1, h, t : Q_1 \to Q_0)$ is given by a set $Q_0$ of vertices $\{1, \ldots, n\}$ as a set of arrows $Q_1$. An arrow $a$ starts at $t(a)$ and ends at $h(a)$.

A quiver $Q$ is said to be finite if $Q_0$ and $Q_1$ are finite sets. The **underlying graph** $\overline{Q}$ of a quiver $Q$ is obtained from $Q$ by forgetting the orientation of the arrows. $Q$ is said to be **connected** if $\overline{Q}$, the underlying graph is a connected graph, that is there is a path between any pair of vertices.

A non trivial path in $Q$ of length $m$ is a sequence $a_m \ldots a_1$ ($m \geq 1$) which satisfies $h(a_i) = t(a_{i+1})$ for each $i$, $1 \leq i \leq m - 1$. This path, $p$, starts at $t(p) = t(a_1)$ and ends at $h(a_m) = h(p)$. The set of all paths of length $l$ ($l \geq 0$) is denoted by $Q_l$. The path of length 1 that starts and ends at $i$ is called the **trivial path**. The trivial path is denoted by $e_i$. A path of length $l \geq 1$ with $h(p) = t(p)$ is called a **cycle**.

The path algebra $KQ$ is the $K$-algebra whose underlying $K$-vector space has as its basis the set of all paths in $Q$ with the product of two paths $p$ and $q$ being defined
by composition if \( h(q) = t(p) \) and as zero otherwise. We associate to each \( x \in Q_0 \) a path \( e_x \) of length zero. For \( x \neq y, x, y \in Q_0 \) we have that \( e_x e_y = e_y e_x = 0 \) and \( e_x e_x = e_x \).

**Definition 2.2.2.** A finite dimensional representation \( V \) of \( Q \) is a set of \( \{V(x) : x \in Q_0\} \) of finite dimensional \( K \)-vector spaces together with a set of \( K \)-linear maps \( \{V(a) : V(ta) \to V(ha) | a \in Q_1\} \).

A morphism \( \phi : V \to V' \) of two representations is a collection of \( K \)-linear maps \( \{\phi(x) : V(x) \to V'(x); x \in Q_0\} \) such that for each \( a \in Q_1 \) we have \( V'(a) \phi(ta) = \phi(ha) V(a) \) i.e. the following diagram commutes:

\[
\begin{array}{ccc}
V(ta) & \xrightarrow{V(a)} & V(ha) \\
\phi(ta) & \downarrow & \phi(ha) \\
V'(ta) & \xrightarrow{V'(a)} & V'(ha)
\end{array}
\]

The representations of a quiver \( Q \) over \( K \) form a category denoted \( Rep_K(Q) \). The dimension vector of a representation \( V \) is the function \( d : Q_0 \to \mathbb{Z}^{Q_0} \) defined by \( d_i := dimV(i) \) for \( i \in Q_0 \).

It is well known that there exists an equivalence of categories

\[
KQ - \text{mod} \cong Rep_K(Q)
\]

**Definition 2.2.3.** If \( V \) and \( W \) are representation of a quiver \( Q \), then we define the direct sum representation \( V \oplus W \) by \( (V \oplus W)(x) = V(x) \oplus W(x) \) for every \( x \in Q_0 \) and for each \( a \in Q_1 \)

\[
(V \oplus W)(a) := \begin{bmatrix} V(a) & 0 \\ 0 & W(a) \end{bmatrix} : V(ta) \oplus W(ta) \to V(ha) \oplus W(ha)
\]
Definition 2.2.4. A representation $V$ is called **decomposable** if $V$ is isomorphic to a direct sum of non-zero representations. Otherwise a representation is **indecomposable**.

The category $Rep_K(Q)$ of representations is hereditary therefore every representation has projective dimension $\leq 1$.

**Definition 2.2.5.** For $V, W \in Rep_K(Q)$ with $\alpha = \dim V$ and $\beta = \dim W$ we can define the **Euler bilinear form** as

$$\langle \alpha, \beta \rangle := \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha)$$

**Proposition 2.2.6.** The spaces $\text{Hom}_Q(V, W)$ and $\text{Ext}^1_Q(V, W)$ are the kernel and cokernel of the following linear map

$$d^V_W : \bigoplus_{x \in Q_0} \text{Hom}(V(x), W(x)) \to \bigoplus_{a \in Q_1} \text{Hom}(V(ta), W(ha))$$

where $d^V_W$ is given by

$$\{\phi(x) | x \in Q_0\} \mapsto \{W(a)\phi(ta) - \phi(ha)V(a) | a \in Q_1\}.$$

It follows from the previous proposition that

$$\langle \alpha, \beta \rangle = \dim \text{Hom}_Q(V, W) - \dim \text{Ext}^1_Q(V, W).$$

The non-isomorphic indecomposable projective modules of $KQ$ are $P_i = e_iKQ$ for each $i \in Q_0$. The non-isomorphic indecomposable injective modules of $KQ$ are
$Q_i = D(KQ_{e_i})$. Every representation $V \in \text{Rep}_K(Q)$ has a canonical resolution:

$$0 \to \bigoplus_{a \in Q_1} V(ta) \otimes P_{ha} \to \bigoplus_{x \in Q_0} V(x) \otimes P_x \to V \to 0$$

**Definition 2.2.7.** For any $K$-algebra $A$ we define the **opposite algebra** $A^{\text{op}}$ of $A$ to be the $K$-algebra whose underlying set and vector space structure are just those of $A$, but the multiplication in $A^{\text{op}}$ is defined by the formula $a \ast b = ba$. The opposite quiver is defined to be $Q^{\text{op}} = (Q_0, Q_1, h', t')$ where for $a \in Q_1$, $t'(a) = h(a)$ and $h'(a) = t(a)$. Furthermore $(KQ)^{\text{op}} = KQ^{\text{op}}$.

**Definition 2.2.8.** Let $A$ be a $K$-algebra. The **Grothendieck group** of $A$ is the abelian group $K_0(A) = F/F'$ where $F$ is the free abelian group having as its basis the set of the isomorphism classes $\tilde{M}$ of modules $M \in \text{mod } A$ and $F'$ is the subgroup of $F$ generated by the elements $\tilde{M} - \tilde{L} - \tilde{N}$ corresponding to all exact sequences

$$0 \to L \to M \to N \to 0$$

in $\text{mod } A$. Denote by $[M]$ the image of the isomorphism class $\tilde{M}$ of the module $M$ under the canonical group epimorphism $F \to F/F'$.

**Definition 2.2.9.** For a finite dimension $K$-algebra $A$, consider the $A$-dual functor

$$(\cdot)^! = \text{Hom}_A(\cdot, A) : \text{mod } A \to \text{mod } A^{\text{op}}.$$ 

If we have a minimal projective presentation of a module $M \in \text{mod } A$,

$$P_1 \xrightarrow{p_0} P_0 \xrightarrow{p_1} M \longrightarrow 0$$
that is that \( p_0, p_1 \) are projective covers. Applying \((-)^t\), we obtain an exact sequence

\[
0 \longrightarrow M^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} P_1^t \longrightarrow \text{coker } p_1^t \longrightarrow 0
\]

Denote \( \text{coker } p_q^t \) by \( \text{Tr } M \) and call it the transpose of \( M \).

**Definition 2.2.10.** Let \( A \) be a finite dimensional \( K \)-algebra. Define the functor

\[
D : \text{mod } A \rightarrow \text{mod } A^{op}
\]

by assigning to each right module \( M \) in \( \text{mod } A \) the dual \( K \)-vector space \( M^* = \text{Hom}_K(M, K) \).

**Definition 2.2.11.** The Auslander-Reiten translation is defined to be the composition of \( D \) with \( \text{Tr} \). In particular, we set \( \tau = D \text{Tr} \) and \( \tau^{-1} = \text{Tr } D \).

**Definition 2.2.12.** An \( A \)-homomorphism is a section (or a retraction) whenever it admits a left inverse (or a right inverse, respectively).

**Definition 2.2.13.** A homomorphism \( f : X \rightarrow Y \) is said to be irreducible provided:

(a) \( f \) is neither a section nor a retraction and

(b) if \( f = f_1 f_2 \) wither \( f_1 \) is a retraction or \( f_2 \) is a section

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{f_2} & & \downarrow{f_1} \\
Z & & \\
\end{array}
\]

**Definition 2.2.14.** Let \( A \) be a finite dimensional \( K \)-algebra, the Auslander-Reiten quiver \( \Lambda_A \) is defined as follows:

1. The points of \( \Lambda \) are the isomorphism classes \([X]\) of indecomposable modules \( X \) in \( \text{mod } A \).
2. Let \([M]\) and \([N]\) be the points in \(\Lambda_A\) corresponding to the indecomposable modules \(M, N\) in \(\text{mod } A\). The arrows \([M] \rightarrow [N]\) are in bijective correspondence with the vectors of a basis of the \(K\)-vectors space of irreducible homomorphism from \(M\) to \(N\).

**Definition 2.2.15.** \(\Lambda_i \subset \Lambda\) is a **component** if its underlying graph is connected.

**Definition 2.2.16.** A connected component \(P\) of \(\Lambda_A\) is called **preprojective** if \(P\) is acyclic (meaning it contains no cycles) and for any indecomposable module \(M\) in \(P\), there exists \(t \geq 0\) and \(a \in Q_0\) such that \(M \cong \tau^{-t}P_a\). An indecomposable module is called preprojective if it belongs to the preprojective component of \(\Lambda\).

**Definition 2.2.17.** A connected component \(Q\) of \(\Lambda_A\) is called **preinjective** if \(Q\) is acyclic and for any indecomposable module \(N\) in \(Q\), there exists \(s \geq 0\) and \(b \in Q_0\) such that \(N \cong \tau^{s}Q_b\). An indecomposable module is called preinjective if it belongs to the preinjective component of \(\Lambda\).

**Definition 2.2.18.** If there exists a path from \([M]\) to \([N]\) in \(\Lambda\) then \([M]\) is a **predecessor** of \([N]\) and \([N]\) is a **successor** of \([M]\). If the path is of length 1, then \([M]\) is a **direct predecessor** of \([N]\) and \([N]\) is a **direct successor** of \([M]\).

**Definition 2.2.19.** Let \(A\) be a finite dimensional \(K\)-algebra with a complete set \(\{e_1, \ldots, e_n\}\) of primitive orthogonal idempotents. The **Cartan matrix** of \(A\) is the \(n \times n\) matrix

\[
C_A = \begin{bmatrix}
c_{11} & \cdots & c_{1n} \\
\vdots & \ddots & \vdots \\
c_{n1} & \cdots & c_{nn}
\end{bmatrix}
\]

where \(c_{ji} = \dim_K e_i A e_j\) for \(i, j = 1, \ldots, n\). The **Euler matrix** of \(A\) is \(E = (C_A^{-1})^t\). The **Coxeter matrix** of \(A\) is \(\Phi_A = -C_A^t C_A^{-1}\).
2.3 Quivers with Relations

**Definition 2.3.1.** Let $Q$ be a finite, connected quiver. The two-sided ideal of the path algebra $KQ$ generated by the arrows of $Q$ is called the **arrow ideal** and is denoted by $R_Q$.

Note that $$R_Q = KQ_1 \oplus KQ_2 \oplus \cdots$$
and that for each $l \geq 1$, $$R^l_Q = \bigoplus_{m \geq l} KQ_m$$
where for each $l \geq 0$, $KQ_l$ is the subspace of $KQ$ generated by the set $Q_l$ of all paths of length $l$.

**Definition 2.3.2.** A two-sided ideal $I$ of $KQ$ is said to be **admissible** if there exists $m \geq 2$ such that $$R^m_Q \subseteq I \subseteq R^2_Q.$$ 

$Q/I$ is called a quiver with relations and has associated path algebra $KQ/I$. It is easily seen that when $Q$ is a finite quiver, every admissible ideal is finitely generated and therefore there exists a finite set of relations $\{\rho_1, \cdots, \rho_p\}$ such that $I = \langle \rho_1, \cdots, \rho_p \rangle$.

A representation $V$ of $Q$ satisfies the relations in $I$ if we have that $V(\rho) = 0$ for all relations $\rho \in I$. Then $V \in Rep_K(Q/I)$.

Let $A = KQ/I$, then the non-isomorphic indecomposable projective modules of $A$ are $P_a = e_aA$. Let $V \in Rep_K(A, \alpha)$, construct the module $\overline{P}_0 = \oplus_{x \in Q_0} V(x) \otimes P_x$. Then for each $V \in Rep_K(A, \alpha)$ we have the presentation

$$0 \rightarrow V(1) \rightarrow \overline{P}_0 \rightarrow V \rightarrow 0$$
the kernel \( V(1) \) has the same dimension vector for each \( V \in \text{Rep}_K(Q, \alpha) \). Define 
\[ \overline{P}_1 = \oplus_{x \in Q_0} V(1)(x) \otimes P_x. \]
Continuing in this fashion we can construct the family of projective resolutions of modules from \( \text{Rep}_K(A, \alpha) \). If \( Q \) has no oriented cycles then the global dimensions of \( A \) is finite so that
\[ \bigoplus_{i \geq 0} (-1)^i \dim \text{Ext}^i_Q(V,W) \]
is finite. For \( V, W \in \text{Rep}_K(A) \) with \( \dim(V) = \alpha, \dim(W) = \beta \), the Euler Form for quivers with relations can be defined as
\[ \langle \langle \alpha, \beta \rangle \rangle = \bigoplus_{i \geq 0} (-1)^i \dim \text{Ext}^i_Q(V,W). \]

### 2.4 Semi-Invariants

Given a dimension vector \( \alpha \), we can define the representation space of \( \alpha \)-dimensional representations of \( Q \) as
\[ \text{Rep}_K(Q, \alpha) = \bigoplus_{a \in Q_1} \text{Hom}_K(K^{\alpha(ta)}, K^{\alpha(ha)}). \]

If \( \text{GL}(\alpha) = \prod_{i \in Q_0} \text{GL}_{\alpha_i}(K) \) then \( \text{GL}(\alpha) \) acts algebraically on \( \text{Rep}(Q, \alpha) \) by simultaneous conjugation, that is, for \( g = (g(i))_{i \in Q_0} \in \text{GL}(\alpha) \) and \( V \in \text{Rep}_K(Q, \alpha) \) we define \( g \cdot V \) by
\[ (g \cdot V)(a) = g(ha)V(a)g(ta)^{-1}, \forall a \in Q_1. \]

We have that \( \text{Rep}_K(Q, \alpha) \) is a representation of \( \text{GL}(\alpha) \) and the \( \text{GL}(\alpha) \) orbits in \( \text{Rep}_K(Q, \alpha) \) are in a one-to-one correspondence with the isomorphism classes of \( \alpha \)-dimensional representations of \( Q \). Since \( Q \) has no oriented cycles, it can be shown that there is only one closed \( \text{GL}(\alpha) \) orbit in \( \text{Rep}(Q, \alpha) \) and therefore the invariant
ring $K[\text{Rep}(Q, \alpha)]^{GL(\alpha)}$ is exactly the base field $K$.

Consider the subgroup $SL(\alpha) \subseteq GL(\alpha)$ defined by $SL(\alpha) = \prod_{i \in Q_0} SL_{\alpha_i}(K)$. The action of $SL(\alpha)$ on $\text{Rep}(Q, \alpha)$ provides us with a non-trivial ring of semi-invariants.

**Definition 2.4.1.** Let $Q$ be a quiver such that $|Q_0| = n$ and let $f$ be a semi-invariant on $\text{Rep}(Q, \alpha)$ and $\chi$ the character of $f$. A **weight of the semi-invariant** $f$ is any vector $\sigma = (\sigma_1, \ldots, \sigma_n)$ for which the character $\chi$ for $f$ can be written as

$$\chi(g) = (\det(g_1))^{\sigma_1} \cdots (\det(g_n))^{\sigma_n}.$$ 

A vector $\sigma \in \mathbb{Z}^n$ will be called a **weight** if it is a weight for some semi-invariant.

**Definition 2.4.2.** We denote by $SI(Q, \alpha)_\chi$ the $K$-vector space of semi-invariants on $\text{Rep}(Q, \alpha)$ with the character $\chi$ and by $SI(Q, \alpha)$ the graded ring

$$SI(Q, \alpha) = \bigoplus_{\chi \in \text{char}(GL(\alpha))} SI(Q, \alpha)_\chi$$

called the **ring of semi-invariants** for the action of $GL(\alpha)$ on $\text{Rep}(Q, \alpha)$.

**Remark 2.4.3.** Let $V \in \text{Rep}_K(Q, \alpha)$ then the codimension of the orbit $O(V)$ of $V$ is equal to $\dim \text{Ext}(V, V)$, in particular when $\text{Ext}(V, V) = 0$, $O(V)$ has codimension 0 therefore $\text{Rep}_K(Q, \alpha)$ has an open $GL(\alpha)$-orbit.

In the case where $\text{Rep}_K(Q, \alpha)$ has an open $GL(\alpha)$-orbit, the theorem of Sato and Kimura [2.1.1] show that $SI(Q, \alpha)$ is a polynomial algebra. Schofield [35] has an effective algorithm for finding the semi-invariants in this case. We call these Schofield semi-invariants $c^V$’s.

**Definition 2.4.4.** For $V \in \text{Rep}_K(Q, \alpha)$, define the vector space homomorphism $(p_M, V) : \text{Hom}(P_0, V) \to \text{Hom}(P_1, V)$ induced by the canonical presentation of $M$.
given in 2.1.1. If we have $V$ such that $\langle \dim V, \alpha \rangle = 0$ then $(p_M, V)$ is a square matrix, then we can define the polynomial function $c^V$ by setting $c^V(M) = \det(p_M, V)$.

**Theorem 2.4.5.** ([15, 37]) Let $Q$ be a quiver and $\alpha \in \mathbb{N}^n$. Then the ring of semi-invariants $SI(Q, \alpha)$ is spanned as a $K$-vector space by the functions $c^V$ for representations $V$ satisfying $\langle \dim V, \alpha \rangle = 0$. Furthermore, the character of the semi-invariant $c^V$ is $\chi_\sigma$ where $\sigma = \langle \dim V, - \rangle$.

**Remark 2.4.6.** Schofield’s method for constructing $c^V$’s is as follows: For $V, W \in \text{Rep}(Q)$, with $\dim V = \alpha, \dim W = \beta$ constructing a non-zero homomorphism from $V$ to $W$ is equivalent to solving a system of homogeneous linear equations

$$V(a)X^{\alpha a} - X^{\alpha a}W(a) = 0 \text{ for all } a \in Q_1$$

where $X^{\alpha}$ is a $\alpha_i \times \beta_i$ matrix of indeterminants. Since $\langle \alpha, \beta \rangle = 0$ we know that the number of indeterminants is equal to the number of equations. If $M$ is the matrix of coefficients of the indeterminants, then the determinant of $M$ is a semi-invariant. Furthermore if $(p_V, W)$ is the matrix of the canonical projective resolution as above, then we have that $(p_V, W) = M$.

### 2.5 Semi-invariants of Quivers with Relations

The purpose of the following section is to outline the results of [20] of the extension of the idea of Schofield’s determinantal semi-invariants to quivers with relations.

Let $V \in \text{Rep}_K(Q/I)$. Let $\tilde{P}_1 \to \tilde{P}_0 \to V \to 0$ be the minimal presentation of $V$ in $\text{Rep}_K(Q/I)$.

**Definition 2.5.1.** Let $W \in \text{Rep}_K(Q/I)$. Then $\tau_V$ can be defined as the determinant of the matrix $\text{Hom}(\tilde{P}_0, W) \to \text{Hom}(\tilde{P}_1, W)$ whenever it is a square matrix.
Note that the representation space $\text{Rep}_K(Q, \alpha)$ does not have to be irreducible, we denote the irreducible components as $\text{Rep}_K(Q, \alpha)_j, j \in \{1, 2, \ldots\}$. The semi-invariant $\tau_V$ is defined on the components $\text{Rep}_K(Q/I, \beta)$ such that, for $W \in \text{Rep}_K(Q/I, \beta)$, $\text{Hom}(P_0, W) = \text{Hom}(P_1, W)$ while $\text{Hom}(V, W) = 0$ for a general $W \in \text{Rep}_K(Q/I, \beta)$.

A component $\text{Rep}_K(Q/I, \beta)_j$ is called **faithful** over $KQ/I$ if the ideal $J = \{x \in KQ \mid x|_{\text{Rep}_K(Q/I, \beta)_j} = 0\}$ is equal to $I$, i.e. there are no additional relations satisfied on $\text{Rep}_K(Q/I, \beta)_j$.

**Theorem 2.5.2.** ([20]) Assume that $\text{char} K = 0$. For each component $\text{Rep}_K(Q/I, \beta)_j$ the semi-invariants $\tau_V$ span $SI(Q/I, \text{Rep}_K(Q/I, \beta)_j)$ for $KQ/I$-modules of projective dimension 1 and of dimension vectors $\alpha$ such that $\langle\langle \alpha, \beta \rangle\rangle = 0$.

### 2.6 $SI(Q, d)$ where Rep$_K(Q, d)$ admits an open orbit

Let $A = KQ/I$ and $\text{Rep}_K(A, d)$ be the set of all representations of dimension $d$. The orbits of $GL(d)$ correspond bijectively to the isomorphism classes of representations of dimension $d$. So therefore when $\text{Rep}_K(A, d)$ admits an open orbit the theorem of Sato and Kimura applies [2.1.1]. In this case, $SI(A, d)$ is a polynomial algebra. In particular, for $Q$ Dynkin, $SI(Q, d)$ is a polynomial algebra. From this point on we will only consider the cases where $\text{Rep}_K(A, d)$ is not irreducible.

### 2.7 Tubes

The purpose of the following section is to give the main properties of algebras whose module structure includes objects called tubes. The main theorem is concerned only with the modules in these tubes.

**Definition 2.7.1.** Let $Q$ be a connected, acyclic quiver. Define an **infinite translation quiver** as follows. The set of points in $\mathbb{Z}Q$ is $(\mathbb{Z}Q)_0 = \mathbb{Z} \times Q_0 = \{(n, x) | n \in \mathbb{Z} \land x \in Q_0\}$. The set of arrows in $\mathbb{Z}Q$ is $(\mathbb{Z}Q)_1 = \{((i, x), (j, y)) | i, j \in \mathbb{Z} \land x, y \in Q_0 \land i = j \pm 1\}$. The relation of composition is defined as $(i, x) \circ ((i, x), (j, y)) = ((i + 1, y), (j, y))$ for $i = j - 1$.
If we have the infinite quiver $A_{\infty}$

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \cdots$$

from this we can have the infinite translation quiver

$$\begin{array}{c}
\cdots \rightarrow (1,1) \rightarrow (1,2) \rightarrow (2,3) \rightarrow \cdots \\
\cdots \rightarrow (0,1) \rightarrow (1,3) \rightarrow \cdots \\
\cdots \rightarrow (-1,1) \rightarrow (0,3) \rightarrow (-1,2) \rightarrow \cdots \\
\end{array}$$

where $\tau(n,i) = (n + 1, i)$ for $n \in \mathbb{Z}$ and $i \geq 1$. Therefore, we have that $\tau$ is an automorphism and hence, so is any power $\tau^r$. For a fixed $r \geq 1$, let $(\tau^r)$ be the infinite cyclic group generated by $\tau^r$ and then $\mathbb{Z}A_{\infty}/(\tau^r)$ is the orbit space of of $\mathbb{Z}A_{\infty}$ under the action of $(\tau^r)$. Thus we are identifying $(n,i)$ with $\tau^r(n,i) = (n + r, i)$ and each arrow $a$ with $\tau^r a : \tau^r x \rightarrow \tau^r y$.

**Definition 2.7.2.** Let $(T, \tau)$ be a translation quiver. Then $(T, \tau)$ is a **stable tube of rank $r$** if there is a isomorphism of translation quivers $T \cong \mathbb{Z}A_{\infty}/(\tau^r)$.
Definition 2.7.3. For a stable tube $T$, the unique $\tau$ orbit formed by the modules having exactly one direct predecessor and exactly one direct successor is called the mouth of $T$.

Definition 2.7.4. Let $T$ be a stable tube.

1. If $r = 1$ we define this to be a homogeneous tube.

2. A sequence $(x_1, \cdots, x_r)$ of $T$ is called a $\tau$-cycle if $\tau x_1 = x_r, \tau x_2 = x_1, \cdots, \tau x_r = x_{r-1}$.

3. Given a point $x$ lying on the mouth of a stable stube, a ray starting at $x$ is defined to be a unique infinite sectional path $x = x[1] \to x[2] \to x[3] \to x[4] \to \cdots$ in the tube $T$.

Let $A$ be a finite dimensional $K$-algebra. A component $C$ of $\text{mod } A$ is said to be a tube provided that $\Lambda_C$ is a tube.

Definition 2.7.5. Let $\mathcal{I}$ be some set, and let us consider families of $T_{\rho}, \rho \in \mathcal{I}$ of pairwise different tubes in $\text{mod } A$. Let $T$ be the module class generated by all $T_{\rho}$. $T$ will be called a tubular family. A tubular family is said to be stable if $T$ does not contain non-zero projective or injective modules.

In the cases we are interested in, the parameter set $\mathcal{I}$ will be the projective line $\mathbb{P}^1 K$.

Let $A$ be any algebra, a brick in $\text{mod } A$ is a module $E$ such that $\text{End } E = K$. Two bricks $E$ and $E'$ are orthogonal if $\text{Hom}_A(E, E') = 0$ and $\text{Hom}_A(E', E) = 0$. If $E_1, \cdots, E_r$ is a family of pairwise orthogonal bricks in $\text{mod } A$ and let $\mathcal{E} = \text{EXT}_A(E_1, \cdots, E_r)$ denote the full subcategory of $\text{mod } A$ (the extension category) whose non-zero objects are all the module $M$ such that there exists a chain of submodules $M = M_0 \supseteq M_1 \cdots \supseteq M_l = 0$ for some $l \geq 1$ with $M_i/M_{i+1}$ isomorphic to
one of the bricks. Thus $\mathcal{E}$ is the smallest additive subcategory of $\text{mod}\ A$ containing the bricks $E_i$ and closed under extensions.

Let the modules at the mouth of a tube $j$ of rank $m_j$ be denoted, $E_0^{(j)}, \ldots, E_{m_j-1}^{(j)}$ of $\dim E_i^{(j)} = e_i^{(j)}$. Since the $E_i^{(j)}$’s are the simple modules at the mouth of each tube we have the following module structure inside the tube of rank $m_j = k + 1$:

Definition 2.7.6. For any indecomposable $A$ module $X$ in a tube of rank $r$, there exists a unique sectional path $X_1 \to X_2 \to \cdots \to X_m = X$ with $X_1$ lying on the mouth of the tube. We call $m$ the regular length of $X$.

Definition 2.7.7. Let $C$ be a component of the Auslander-Reiten quiver $\Lambda_A$ of an algebra $A$. $C$ is defined to be a standard component of $\Lambda_A$ if $K(C)$ is equivalent to the full $K$-subcategory of $\text{mod}\ A$ whose objects are representatives of the isomorphism classes of the indecomposable modules in $C$.

Theorem 2.7.8. Let $A$ be an algebra and $(E_1, \cdots, E_r)$ with $r \geq 1$ be a $\tau$-cycle of pairwise orthogonal bricks in $\text{mod}\ A$ such that $\{E_1, \cdots, E_r\}$ is a self-hereditary family of $\text{mod}\ A$. Then $\mathcal{T}_E$ is a standard stable tube of rank $r$ and the modules $E_1, \cdots, E_r$ form a complete set of modules lying on the mouth of the tube $\mathcal{T}_E$.

Definition 2.7.9. Define a separating tubular family of stable tubes, $\mathcal{T}$, as:
(a) \( \mathcal{T} \) is a family of standard stable tubes

(b) the indecomposable modules not in \( \mathcal{T} \) fall into two classes \( \mathcal{P} \) and \( \mathcal{Q} \) such that

\[
\text{Hom}(\mathcal{P}, \mathcal{Q}) = \text{Hom}(\mathcal{Q}, \mathcal{T}) = \text{Hom}(\mathcal{T}, \mathcal{P}) = 0
\]

(c) for any \( \rho \in \mathcal{I} \), any morphism from \( \mathcal{P} \) to \( \mathcal{Q} \) can be factored through \( T_\rho \).

Remark: We denote the object class of \( \text{add}(X \cup Y) \) by \( X \lor Y \). We call a separating tubular family sincere if it contains a sincere module. A module \( M \) is sincere if \( \dim M \) is a sincere vector. A dimension vector is sincere if it contains only nonzero components.

**Theorem 2.7.10.** Let \( \mathcal{T} \) be a sincere separating tubular family of \( A \) modules separating \( P \) from \( Q \). Then all projective modules belong to \( P \lor T \), all injective modules to \( T \lor Q \). If \( X \in P \), then \( \text{proj dim} X \leq 1 \) if \( Y \in Q \) then \( \text{inj dim} Y \leq 1 \). If in addition \( T \) contains no non-zero injective module, then \( \text{proj dim} T \leq 1 \) for any \( T \in T \) and \( \text{gl dim} A \leq 2 \). Similarly, if \( T \) contains no non-zero projective module then \( \text{inj dim} T \leq 1 \) for any \( T \in T \) and again \( \text{gl dim} A \leq 2 \).

**Definition 2.7.11.** For each tubular family \( \mathcal{T} \) we call \( (n_1, \ldots, n_m) \) the **tubular type** where \( \mathcal{T} \) consists of \( m \) non-homogeneous tubes and the rank of each non-homogeneous tube is \( n_i \), \( (i = 1, \ldots, m) \).

The class of finite dimensional algebras with a separating tubular family of stable tubes contains the tame hereditary algebras, the tame concealed algebras, the tubular algebras, the canonical algebras and the concealed-canonical algebras.
Definition 2.7.12. Consider the quiver $\Delta(n_1, \ldots, n_t)$

with relations $I(x_3, \ldots, x_n)$

$$\alpha_{m_1}^1 \cdots \alpha_1^1 + x_i \alpha_{m_1}^2 \cdots \alpha_1^2 + \alpha_{m_1}^i \cdots \alpha_1^i$$

for $i = 3, \ldots, n$. Then $K\Delta(n_1, \ldots, n_t)/I$ is a canonical algebra of type $(n_1, \ldots, n_t)$. 
3 The Main Theorem

In this section we prove a theorem that gives abstract conditions under which the ring of semi-invariants of certain components of a representation space has similar structure as the rings of semi-invariants of extended Dynkin quivers for regular dimension vectors (see [SW]).

Let $\Lambda = KQ/I$ be the path algebra of the quiver $Q = (Q_0, Q_1)$ with the ideal $I$ of admissible relations.

3.1 The Required Conditions

Definition 3.1.1. We will first define the category of regular modules of index $\gamma$. Let $\Lambda$ be a finite dimensional $K$-algebra. For some index $\gamma \in \mathbb{P}_K$, we have a tubular family. Let the tubular type of $\mathcal{T}_\gamma$ be denoted $(m_1, \ldots, m_s)$. For each $\mathcal{T}_\gamma$, we have a family of homogeneous indecomposable modules $\{V_t\}, t \in \mathbb{P}_K$ of dimension $h_{\gamma}$ which are the homogeneous modules in $\mathcal{T}_\gamma$. We also have a family of nonhomogeneous modules $E_{i,j}^{(i)}$ of dimension vectors $e_{i,j}^{(i)}$, $(1 \leq j \leq s, 0 \leq i \leq m_j - 1)$. The modules $E_{i,j}^{(i)}$ are the modules at the mouth of the tubes. The smallest category $\mathcal{R}(\Lambda)_\gamma$ containing $\{V_t\}$ and $E_{i,j}^{(i)}$ closed under extensions and direct summands we will call the category of regular modules of index $\gamma$.

Definition 3.1.2. Let $\Lambda$ be of tubular type $(m_1, \ldots, m_s)$. For each $j = 1, \ldots, s$ we can associate to the tube of rank $m_j$ a graph $\Gamma(j)$ with $m_j$ vertices $0, 1, \ldots, m_j - 1$ and edges $a_{i,j}^{(j)}$ connecting vertices $i$ and $i + 1$, with $a_{m_j - 1,j}^{(j)}$ connecting $m_j - 1$ and $0$. We label the vertex $i$ with a number $p_{i,j}$. We call a directed path $[i,k]$ on $\Gamma(j)$ a $\beta$-admissible path of index $l$ if $p_{i,j}^{(j)} = p_{k,j}^{(j)} = l$ and $p_{i,j}^{(j)} > l$ for all vertices $t$ inside of $[i,k]$.

Example 3.1.3. This example illustrates admissible paths. Consider a tube of rank
6, with \( p_0 = p_3 = 0, p_1 = p_5 = p_2 = 1 \) and \( p_4 = 2 \). Then we have the following polygon \( \Delta \) with admissible arcs denoted by the dotted lines.

Therefore the admissible paths are \([0, 3], [3, 0]\) of index 0 and \([1, 2]\) of index 1.

**Definition 3.1.4.** The conditions on \( R(\Lambda)_\gamma \):

1. A vector
   \[
   \beta = ph_\gamma + \sum_{j=1}^{s} \sum_{i=0}^{m_j-1} p_i^{(j)} e_i^{(j)}
   \]
   (where we assume that for each \( j = 1, \ldots, s \) we have \( \min \{ p_i^{(j)} ; 1 \leq i \leq m_j-1 \} = 0 \)) is called a **regular dimension vector**. Every dimension vector in \( R(\Lambda)_\gamma \) can be written this way.

2. The dimension vectors \( e_i^{(j)} \) satisfy the relations
   \[
   h_\gamma = \sum_{i=0}^{m_j-1} e_i^{(j)}.
   \]
   for \( j = 1, \ldots, s \), and the dimension of the space spanned by the \( e_i^{(j)} \) equals \( \sum_{j=1}^{s} m_j - s + 1 \),

3. The decomposition of a general vector in \( R(\Lambda)_\gamma \) is given by the same formula as for the extended Dynkin quivers [32]. In particular, for \( V \) of dimension \( \beta \), we have that
   \[
   V = V_{t_1} \oplus \ldots \oplus V_{t_p} \oplus \oplus_{j=1}^{s} \oplus_{[i,k]} \text{ admissible } E_{[i,k]}^{(j)}.
   \]
4. Dimension vectors $e_i^{(j)}$, and the vectors $e_{[i,k]}^{(j)} := e_i^{(j)} + e_{i+1}^{(j)} + \ldots + e_k^{(j)}$ are Schur roots and the generic modules $E_{[i,k]}^{(j)}$ have projective dimension 1 and injective dimension 1 over $\Lambda$.

5. The values of the Euler form are:
   
   a) $\langle \langle e_i^{(j)}, e_k^{(l)} \rangle \rangle = 0$ if $j \neq l$

   b) $\langle \langle e_i^{(j)}, e_k^{(j)} \rangle \rangle = \begin{cases} 
   1 & i = k \\
   -1 & i = k + 1 \\
   0 & \text{otherwise} 
   \end{cases}$

6. The general module in dimension vector $R(\Lambda, h)$ is a 1-parameter family of modules $V_t$ ($t \in K \cup \infty$), which are also of projective dimension 1 and of injective dimension 1.

7. Every indecomposable module $X$ of projective dimension $\leq 1$ orthogonal to $V_t$ (in the sense that $\text{Hom}_\Lambda(X, V_t) = \text{Ext}^1_\Lambda(X, V_t) = 0$ for general $t$) is in the category $\mathcal{R}(\Lambda)$.

8. The condition (6) implies by results of [20] the existence of the semi-invariant $c^{V_u}$ in the coordinate ring $SI(\Lambda, \beta)$. We require that $c^{V_u}(V_w) = -u + w$.

**Definition 3.1.5.** Let $\{V_t, E_{[i,k]}^{(j)}\}$ be the simple modules in $\mathcal{R}(\Lambda)$ of tubular type $(m_1, \ldots, m_s)$. For each regular dimension vector $\beta$ we have the ring of **regular semi-invariants**

$$SI_{\text{reg}}(\Lambda, \beta) := SI(GL(Q, \beta), \mathcal{R}(\Lambda, \beta)).$$

To each admissible path $[i, k]$ in $\Gamma(j)$ we associate the semi-invariant $c^{E_{[i+1,k]}^{(j)}}$. In particular, each path connecting two consecutive vertices labeled by zero is admissible. The product of all semi-invariants of index zero has weight $\langle h, - \rangle$. This includes a
degenerate case of the case when only one vertex of $\Gamma(j)$ is labeled by zero. In such case the whole circle gives a semi-invariant of index zero of weight $\langle h, - \rangle$.

### 3.2 Semi-invariants of Tubes

**Theorem 3.2.1.** Let $\{V_t, E_i^{(j)}\}$ be the simple modules in $R(\Lambda)$ of tubular type $(m_1, \ldots, m_s)$. Let $\beta$ be a regular dimension vector. Then the ring of regular semi-invariants $SI_{\text{reg}}(\Lambda, \beta)$ has the generators and relations described as follows.

(a) The non-homogeneous generators are $c^{E_i^{(j)}_{[i+1,k]}}$ where $[i,k]$ is a $\beta$-admissible path on the $j$-th circle,

(b) The homogeneous generators $c^{V_t}$ which span the space of dimension $p + 1$. We fix a basis $\{c_0, \ldots, c_p\}$ of the weight space $SI_{\text{reg}}(\Lambda, \beta)_{\langle h, - \rangle}$.

(c) There are $s$ relations in $SI_{\text{reg}}(\Lambda, \beta)$ such that each relation expresses the product of non-homogeneous semi-invariants of index zero on the $j$-th circle as a linear combination of the semi-invariants $c_0, \ldots, c_p$.

**Proof.** We use the result of Derksen and Weyman [20] (and Domokos [22]) describing the generators of the rings of semi-invariants for quiver with relations. The component $R(\Lambda, \beta)$ for $p > 0$ is faithful (otherwise change $\Lambda$ to some factor algebra $\Lambda'$, and all modules $V_t$ and $E_i^{(j)}$ will be also modules over $\Lambda'$) so we know that the ring of regular semi-invariants $SI_{\text{reg}}(\Lambda, \beta)$ is generated by semi-invariants $c^W$ where $W$ is a module of projective dimension 1 orthogonal to a general module $V$ in $R(\Lambda)_{\gamma}$. A general module in that component can be written

$$V = V_{t_1} \oplus \ldots \oplus V_{t_p} \oplus \bigoplus_{j=1}^{8} \oplus_{[i,k]} \text{admissible } E_i^{(j)}_{[i,k]}.$$ 

The condition (6) of 3.1.4 implies that the module $W$ has to be regular. This allows us to repeat the reasoning from [40] and describe the generators of the rings.
We have the existence of semi-invariants $c_0, \ldots, c_p$ as defined by Ringel in \cite[(32), 4.1]{ringel}. We know that these $c_i$'s are linear combinations of $c^V_t$ for $V_t$ homogeneous and orthogonal to $V$. Furthermore, $c^V_t = \sum_{i=0}^{p} t^i c_i$.

Following \cite{ringel}, we check to see that for $c^W, W$ is in the orthogonal category of $V$. We know based on 3.1.4.(4) and (6) that $W$ has projective dimension 1. Clearly, for any of the $V_t$ because there are no maps between tubes, we have that

\[
\langle \langle h, \beta \rangle \rangle = \langle \langle h, ph + \sum_{j=1}^{s} \sum_{i=0}^{m_j-1} p_{i}^{(j)} e_{i}^{(j)} \rangle \rangle
\]

\[
= \langle \langle h, ph \rangle \rangle + \langle \langle h, \sum_{j=1}^{s} \sum_{i=0}^{m_j-1} p_{i}^{(j)} e_{i}^{(j)} \rangle \rangle = 0
\]

Similarly, for $c^{E^{(l)}_{[n+1,k]}}$, we have that:

\[
\langle \langle e^{(l)}_{[n+1,k]}, \beta \rangle \rangle = \langle \langle e^{(l)}_{n+1} + e^{(l)}_{n+2} + \cdots + e^{(l)}_{k}, \beta \rangle \rangle
\]

\[
= \langle \langle e^{(l)}_{n+1} + e^{(l)}_{n+2} + \cdots + e^{(l)}_{k}, ph + \sum_{j=1}^{s} \sum_{i=0}^{m_j-1} p_{i}^{(j)} e_{i}^{(j)} \rangle \rangle
\]

\[
= \langle \langle e^{(l)}_{n+1} + e^{(l)}_{n+2} + \cdots + e^{(l)}_{k}, ph \rangle \rangle + \sum_{j=1}^{s} \langle \langle e^{(l)}_{n+1} + e^{(l)}_{n+2} + \cdots + e^{(l)}_{k}, \sum_{i=0}^{m_j-1} p_{i}^{(j)} e_{i}^{(j)} \rangle \rangle
\]

Again, since there are no maps between tubes this sum simplifies to:

\[
= \langle \langle e^{(l)}_{n+1} + e^{(l)}_{n+2} + \cdots + e^{(l)}_{k}, \sum_{i=0}^{m_{l}-1} p_{i}^{(l)} e_{i}^{(l)} \rangle \rangle.
\]

Considering 3.1.4.(5) we have:

\[
= \langle \langle e^{(l)}_{n+1} + e^{(l)}_{n+2} + \cdots + e^{(l)}_{k}, \sum_{i=n}^{k} p_{i}^{(l)} e_{i}^{(l)} \rangle \rangle.
\]
and for any $i \in n + 1, \ldots, k - 1$, we have

$$\langle\langle e_{n+1}^{(l)} + e_{n+2}^{(l)} + \cdots + e_k^{(l)} , p_i e_i^{(l)} \rangle\rangle = -p_i^{(l)} + p_i^{(l)} = 0$$

and

$$\langle\langle e_{n+1}^{(l)} + e_{n+2}^{(l)} + \cdots + e_k^{(l)} , p_n e_n^{(l)} \rangle\rangle = -p_n^{(l)}$$

by the definition of admissible path, we have that $p_n^{(l)} = p_k^{(l)}$ therefore,

$$\langle\langle e_{[n+1,k]}^{(l)} , \beta \rangle\rangle = 0.$$

But still we need the relations and weight multiplicities of $SI_{reg}(\Lambda, \beta)$ to have the full structure of this ring. We need a version of reciprocity result from [16].

Lemma 3.2.2. (Reciprocity). Let $\beta, \gamma$ be two regular dimension vectors. Then

$$\dim(SI_{reg}(\Lambda, \beta)_{\langle\langle \gamma, -\rangle\rangle}) = \dim(SI_{reg}(\Lambda, \gamma)_{\langle\langle -, - \beta \rangle\rangle}).$$

Proof. Assume that the space on the left hand side has dimension $m$. It is spanned by semi-invariants $c^W$ where $W$ are regular modules of dimension $\gamma$. Let us choose $m$ regular modules $W_1, \ldots, W_m$ of dimension $\gamma$ such that $c^{W_1}, \ldots, c^{W_m}$ is a basis of $SI_{reg}(\Lambda, \beta)_{\langle\langle \gamma, -\rangle\rangle}$. Then there exist $m$ regular modules $V_1, \ldots, V_m$ of dimension $\beta$ such that the determinant

$$\det(c^{W_i}(V_j))_{1 \leq i, j \leq m}$$

is not zero. But we have by definition $c^{W_i}(V_j) = c_{V_j}(W_i)$. Thus the functions $c_{V_j}$ are
linearly independent in $SI_{\text{reg}}(\Lambda, \gamma)_{(\langle -, - \rangle)}$. This shows that

$$\dim(SI_{\text{reg}}(\Lambda, \beta)_{(\langle \gamma, - \rangle)}) \leq \dim(SI_{\text{reg}}(\Lambda, \gamma)_{(\langle -, - \rangle)}).$$

The other inequality is proved in exactly the same way. 

\[ \square \]

**Lemma 3.2.3.** (*Weight multiplicities*) Let

\[
\beta = p\hbar + \sum_{j=1}^{s} \sum_{i=0}^{m_j-1} p^{(j)}_i e_i^{(j)}
\]

(where we assume that for each $j = 1, \ldots, s$ we have $\min\{p^{(j)}_i; 1 \leq i \leq m_j - 1\} = 0$), and

\[
\gamma = q\hbar + \sum_{j=1}^{s} \sum_{i=0}^{m_j-1} q^{(j)}_i e_i^{(j)}
\]

(where we assume that for each $j = 1, \ldots, s$ we have $\min\{q^{(j)}_i; 1 \leq i \leq m_j - 1\} = 0$), be two regular dimension vectors. Then

$$\dim(SI_{\text{reg}}(\Lambda, \beta)_{(\langle \gamma, - \rangle)}) = \dim(SI_{\text{reg}}(\Lambda, \gamma)_{(\langle -, - \rangle)}) = \binom{p+q}{q}.$$

Moreover, the subring

$$\oplus_{q \geq 0} SI_{\text{reg}}(\Lambda, \beta)_{(\langle q\hbar, - \rangle)}$$

is a polynomial ring in $p+1$ variables, with the generators occurring in degree 1.

**Proof.** First we will reduce to the case where $\gamma = q\hbar$ and $\beta = p\hbar$. Each semi-invariant of weight $\langle \langle \gamma, - \rangle \rangle$ is a product of a semi-invariant of weight $\langle \langle q\hbar, - \rangle \rangle$ and of a semi-invariant of weight $\langle \langle \gamma - q\hbar, - \rangle \rangle$ because of condition (2) of 3.1.4. But the dimension of the weight space $SI_{\text{reg}}(\Lambda, \beta)_{(\langle \gamma - q\hbar, - \rangle)}$ is equal to one, because of condition (3) of 3.1.4. Also, the ring $SI_{\text{reg}}(\Lambda, \beta)$ is a domain, so the non-zero element in the weight space $SI_{\text{reg}}(\Lambda, \beta)_{(\langle \gamma - q\hbar, - \rangle)}$ is a non-zero divisor. Thus the dimensions of weight spaces
The general module in dimension vector $\beta$ is $V = V_{t_1} \oplus \ldots \oplus V_{t_p}$, and in dimension vector $\gamma$ is $W = V_{u_1} \oplus \ldots \oplus V_{u_q}$. Therefore each semi-invariant in the weight space $SI_{\text{reg}}(\Lambda, ph)(qh, -)$ is a linear combination of monomials of degree $q$ in semi-invariants from the weight space $SI_{\text{reg}}(\Lambda, ph)(q, -)$. Applying reciprocity and using the same argument, plus the condition (7) of 3.1.4, which implies that $\dim(SI_{\text{reg}}(\Lambda, h)(h, -)) = 2$, we see that $\dim(SI_{\text{reg}}(\Lambda, ph)(qh, -)) \leq p + 1$. This in turn shows that

$$\dim(SI_{\text{reg}}(\Lambda, ph)(qh, -)) \leq \binom{p + q}{q}.$$ 

This reasoning shows that $SI_{\text{reg}}(\Lambda, ph)$ is naturally a factor of a polynomial ring in $\leq p + 1$ variables.

To prove the equality we observe that if any relation would occur in $SI_{\text{reg}}(\Lambda, ph)$ (or if the number of generators would actually be smaller that $p + 1$), then the dimension of the factor of $\text{Reg}(\Lambda, ph)$ by the group $GL(Q, ph)$ with the stability condition given by $h$ would be smaller than $p$. But this is impossible, because the modules $V_{t_1} \oplus \ldots \oplus V_{t_p}$ are pairwise nonisomorphic (as $\dim(\text{Hom}_\Lambda(V_t, V_u)) = \delta_{t,u}$ by condition (7) of 3.1.4), and the factor in question has to include their orbits.

Now we can conclude the description of the relations in the rings $SI_{\text{reg}}(\Lambda, \beta)$. The relations given in the formulation of the Theorem certainly exist. We need to show that there are no additional relations. But condition (1) of 3.1.4 implies that the additional relations can occur only between the semi-invariants in the weights $qh$. This is impossible by 3.2.2.

We will now use this theorem to make statements about semi-invariants of Euclidean Quiver and Tubular Algebras.
4 Semi-Invariants of Euclidean Quivers

A Euclidean graph is one of the following diagrams:

\[ \tilde{A}_m : \]
\[ \tilde{D}_m : \]
\[ \tilde{E}_6 : \]
\[ \tilde{E}_7 : \]
\[ \tilde{E}_8 : \]

where for \( \tilde{A}_n \) we have to exclude the cyclic orientation.

Let \( Q \) be a Euclidean Quiver then \( A = KQ \) is tame hereditary. The tubular type of \( \tilde{D}_n \) is \( (n - 2, 2, 2) \), of \( \tilde{E}_6 \) is \( (3, 3, 2) \), of \( \tilde{E}_7 \) is \( (4, 3, 2) \) and of \( \tilde{E}_8 \) is \( (5, 3, 2) \).

The representations of \( Q \) without indecomposable preprojective or preinjective direct summands are called regular. Denote by \( D_r \) the set of dimension vectors of all the
regular representations of $Q$. In each of these cases, for some $V \in \text{Rep}_K(Q)$ there is a one-dimensional subspace of $\mathcal{D}$ on which $\langle \langle \dim V, \dim V \rangle \rangle$ vanishes. For a minimal $\dim V$ we let $h = \dim V$.

If $\beta = ph$, then the general module in dimension vector $\beta$ is $V = V_{t_1} \oplus \cdots \oplus V_{t_p}$ where the $V_{t_i}$ are pairwise non-isomorphic, that is $\dim \text{Hom}(V_{t_i}, V_{t_j}) = \delta_{i,j}$.

**Theorem 4.0.4.** ([32]) Let $Q$ be a tame quiver of dimension $d \in \mathcal{D}$ with canonical decomposition $d = ph + \sum_{i \in I} e_i$. Then there exists semi-invariants $c_0, \ldots, c_p \in SI(Q,d)(h,-)$. These $c_0, \ldots, c_p$ are a basis of a vector space of $c^V$ where $\dim V = h$.

Furthermore, $c^{V_\lambda} = \sum_{i=0}^p c_i \lambda^i$

To construct these $c_i$'s, let $V, W \in \text{Rep}_K(Q,d)$. Notice that $\langle \langle d, d \rangle \rangle = 0$. Therefore, we can use a method similar to the construction of Schofield’s $c^V$'s. Consider a certain matrix with coefficients in the polynomial ring $K[\text{Rep}(Q,d)][\lambda]$ obtained from the same method as [2.4.6]. Then taking the determinant of this matrix we obtain a polynomial homogeneous in degree $p$ of the form $\sum_{i=0}^p c_i \lambda^i$ where the $c_i$'s are in $K[\text{Rep}(Q,d)][h,-]$

**Example 4.0.5.** This example illustrates the how to find the $c_i$'s and $c^V$'s for a Euclidean quiver. Consider $Q$ of type $\tilde{D}_4$ of the subspace orientation. Let $d = h = (1,1,1,1,2)$. So we have $V$:

![Diagram of the quiver $\tilde{D}_4$]

Let $\alpha_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$ with coordinate functions $X_i, Y_i$ for $1 \leq i \leq 4$. Let $W \in \text{Rep}_K(Q,d)$.
$\text{Rep}_K(Q, d)$ be the following:

\[ \begin{array}{ccccccccc}
& & & & K^2 & & & & \\
& & & & \searrow & & \swarrow & & \\
& & & K & & K & & K & \\
& & & b_1 & & b_2 & & b_3 & b_4 \\
& & K & & K & & K & & K \\
& & 1 & & 0 & & 1 & & 0 & \\
& & 0 & & 1 & & 1 & & 1 & \\
& & \lambda & & 0 & & 1 & & 1 & \\
\end{array} \]

with $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $b_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $b_4 = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$. Then the matrix of coefficients of the equations $V(a)X^ha - X^taW(a) = 0$ for $a \in Q_1$ is:

\[ \begin{bmatrix}
-X_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-Y_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -X_2 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -Y_2 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -X_3 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & -Y_3 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & -X_4 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & -Y_4 & 0 & 0 & 1 & 1 \\
\end{bmatrix} \]

The determinant of this matrix is

\[ (X_1X_4Y_2Y_3 - X_1X_2Y_3Y_4 - X_3X_4Y_1Y_2 + X_2X_3Y_1Y_4) \]

\[ +(X_1X_2Y_3Y_4 - X_1X_3Y_2Y_4 - X_2X_4Y_1Y_3 + X_3X_4Y_1Y_2)\lambda \]

So we have two semi-invariants:

\[ c_0 = (X_1X_4Y_2Y_3 - X_1X_2Y_3Y_4 - X_3X_4Y_1Y_2 + X_2X_3Y_1Y_4) \]
and
\[ c_1 = (X_1X_2Y_3Y_4 - X_1X_3Y_2Y_4 - X_2X_4Y_1Y_3 + X_3X_4Y_1Y_2). \]

Notice for \( \lambda = 0, 1, \infty \) we have corresponding regular representations, let them be \( V_0, V_1, V_\infty \) respectively. We can then associate \( c^{V_0} = c_0, c^{V_1} = c_0 + c_1 \) and \( c^{V_\infty} = c_1 \).

**Theorem 4.0.6.** ([13]) Let \( Q \) be a Euclidean quiver. Then there are at most three \( \tau \)-orbits, \( \Delta = \{e_i, i \in I\}, \Delta' = \{e'_i|i \in I'\}, \Delta'' = \{e''_i|i \in I''\} \) of the dimension vectors of nonhomogeneous simple regular representations of \( Q \). We may assume that \( I = \{0, 1, \ldots, u-1\}, I' = \{0, 1, \ldots, v-1\}, I'' = \{0, 1, \ldots, w-1\} \) and \( \tau E_i = E_{i+1} \)

These orbits can be represented graphically as polygons:

Any dimension vector \( d \in \mathcal{D}_r \) can be written uniquely in the form
\[
d = ph + \sum_{i \in I} p_i e_i + \sum_{i \in I'} p'_i e'_i + \sum_{i \in I''} p''_i e''_i
\]

for some non-negative integers \( p, p_i, p'_i, p''_i \) with at least one coefficient in each family \( \{p_i|i \in I\}, \{p'_i|i \in I'\}, \{p''_i|i \in I''\} \) being zero. Label the vertices of the polygons \( \Delta, \Delta', \Delta'' \) with the coefficients \( p_i, p'_i, p''_i \). We say that the labelled arc

\[ p_i \cdots p_j \]
of the labelled polygon $\Delta$ is admissible if $p_i = p_j$ and $p_i < p_k$ for all interior labels $p_k$. We denote such an admissible arc by $[i, j]$ and define $p_i = p_j$ to be the index, $\text{ind}[i, j]$ of $[i, j]$. Let the set of admissible arcs be $\mathcal{A}$.

For each admissible arc, $[i, j] \in \mathcal{A}(d)$ (resp. $\mathcal{A}'(d), \mathcal{A}''(d)$), there exists a semi-invariant $c^V$ (resp. $c^{V'}, c^{V''}$) with $V \cong E_{i+1,j} \cong E_{i+1} \oplus \cdots \oplus E_j$ (resp. $V' \cong E'_{i+1,j}, V'' \cong E''_{i+1,j}$).

**Example 4.0.7.** This example illustrates admissible arcs. Consider a tube of rank 6, with $p_0 = p_3 = 0, p_1 = p_2 = 1$ and $p_4 = 2$. Then we have the following polygon $\Delta$ with admissible arcs denoted by the dotted lines.

![Polygon Diagram]

Therefore the admissible paths are $[0, 3], [3, 0]$ of index 0 and $[1, 2]$ of index 1. So we would get the associated semi-invariants $c^{E_{[1,3]}}, c^{E_{[4,0]}}$ and $c^{E_{[2,2]}}$.

**Theorem 4.0.8.** Let $Q$ be a Euclidean quiver and $d = ph + d'$ the canonical decomposition of $d \in \mathcal{D}$, with $p \geq 1$. Then the algebra $SI(Q, d)$ has the presentation $SI(Q, d) \cong$

$$K[c_0, \ldots, c_p, c^{E_{[i+1,j]}}, c^{E'_{[r+1,s]}}, c^{E''_{[t+1,m]}}] / (c_0 - \prod_{\text{ind}[i,j]=0} c^{E_{[i+1,j]}}, c_p - \prod_{\text{ind}[r,s]=0} c^{E'_{[r+1,s]}}, \sum_{i=0}^p c_i - \prod_{\text{ind}[t,m]=0} c^{E''_{[t+1,m]}})$$

with

$$[i, j] \in \mathcal{A}(d), [r, s] \in \mathcal{A}'(d), [t, m] \in \mathcal{A}''(d)$$

**Example 4.0.9.** Continuing the example from 4.0.5, we have that $\widetilde{D}_4$ is of tubular
type $(2, 2, 2)$. Again, we have the quiver of $V$:

![Quiver Diagram]

The simple regular modules at the mouth of the tube have the following dimensions:

$$
e_0 = (1, 1, 0, 0, 1)$$

$$e_1 = (0, 0, 1, 1, 1)$$

$$e'_0 = (1, 0, 1, 0, 1)$$

$$e'_1 = (0, 1, 0, 1, 1)$$

$$e''_0 = (1, 0, 0, 1, 1)$$

$$e''_1 = (0, 1, 1, 0, 1)$$

Since $d = h$ we have that all $p_i, p_i', p_i'' = 0$ so each edge in $\Delta, \Delta', \Delta''$ is an admissible arc. Therefore we have the semi-invariants that come from the simple regular modules, namely: $c^{E_0}, c^{E_1}, c^{E'_0}, c^{E'_1}, c^{E''_0}, c^{E''_1}$. For example, $c^{E_0}(V)$ is the determinant of the matrix

$$
\begin{bmatrix}
-X_1 & 0 & 1 & 0 \\
-Y_1 & 0 & 0 & 1 \\
0 & -X_2 & 1 & 0 \\
0 & -Y_2 & 0 & 1
\end{bmatrix}
$$

which is $X_2Y_1 - X_1Y_2$ and by a similar calculation $c^{E_1}(V) = X_3Y_4 - X_4Y_3$ then $c^{E_0}c^{E_1} = c_0 + c_1$. 

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Similarly, we can see all the relations: \( c^{V_1} - c^{E_0} c^{E_1}, c^{V_0} - c^{E_0} c^{E_1}, c^{V_1} - c^{V_0} - c^{E'_0} c^{E_1} \)

So the ring of semi-invariants is

\[
K[c^{V_0}, c^{V_1}, c^{E_0}, c^{E_1}, c^{E'_0}, c^{E'_1}, c^{E''_0}, c^{E''_1}] / (c^{V_1} - c^{E_0} c^{E_1}, c^{V_0} - c^{E_0} c^{E_1}, c^{V_1} - c^{V_0} - c^{E'_0} c^{E_1})
\]
5 Tame Concealed Algebras

We will first discuss some properties of tilting modules in order that we can discuss a specific family of algebras obtained from tilting modules from tame hereditary algebras.

**Definition 5.0.10.** For an $A$ module $M$, we denote $\text{add } M$ the smallest additive full subcategory of $\text{mod } A$ containing $M$. An $A$ module $T$ is called a tilting module if the following conditions are satisfied:

1. $\text{proj dim } T \leq 1$  
2. $\text{Ext}^1_A(T, T) = 0$  
3. For any indecomposable projective $A$ module $P$ there exists a short exact sequence  
   \[ 0 \to P \to T' \to T' \to 0 \]

with $T', T'$ in $\text{add } T$.

**Definition 5.0.11.** Algebras of the form $\text{End}(T)$ where $T$ is a preprojective tilting module over a tame hereditary algebra $A$ are called tame concealed algebras.

**Example 5.0.12.** This example illustrates how tame concealed algebras are formed from hereditary algebras. Let $A = \tilde{E}_6$ with the following orientation:
We can find $T_i$ in the preprojective component of $\Gamma_A$ such that $B = \text{End}(\oplus T_i)$ is a tilting module and therefore $B$ is tame concealed:

\begin{center}
\begin{tikzpicture}
\node (T1) at (0,0) {$T_1$};
\node (T2) at (2,0) {$T_2$};
\node (T3) at (4,0) {$T_3$};
\node (T4) at (6,0) {$T_4$};
\node (T5) at (8,0) {$T_5$};
\node (T6) at (10,0) {$T_6$};
\node (T7) at (12,0) {$T_7$};
\end{tikzpicture}
\end{center}

Therefore $Q_B$ is the quiver with relations

\begin{center}
\begin{tikzpicture}
\node (A1) at (0,0) {$\rightarrow$};
\node (A2) at (1,0) {$\rightarrow$};
\node (A3) at (2,0) {$\rightarrow$};
\node (A4) at (3,0) {$\rightarrow$};
\node (A5) at (4,0) {$\rightarrow$};
\node (A6) at (5,0) {$\rightarrow$};
\end{tikzpicture}
\end{center}

### 5.1 Semi-Invariants of Tame Concealed

**Definition 5.1.1.** A pair $(\Sigma, \mathcal{F})$ of full subcategories of $A$-modules is called a **torsion pair** if the following conditions are satisfied.

(a) $\text{Hom}_A(M, N) = 0$ for all $M \in \Sigma, N \in \mathcal{F}$

(b) $\text{Hom}_A(M, -)|_\mathcal{F} = 0$ implies $M \in \Sigma$

(c) $\text{Hom}_A(-, N)|_\mathcal{F} = 0$ implies $N \in \mathcal{F}$

the subcategory $\Sigma$ is called the **torsion class** and its objects are called **torsion objects**. The subcategory $\mathcal{F}$ is called the **torsion-free class** and its objects are called **torsion free objects**.
Consider the full subcategory $\mathfrak{X}(T)$ of mod-$A$ defined by $\mathfrak{X}(T) = \{ M | \operatorname{Ext}^1_A(T, M) = 0 \}$

**Proposition 5.1.2.** ([1], IV.3.2) Let $M, N \in \mathfrak{X}(T)$ then we have functorial isomorphisms:

(a) $\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_B(\operatorname{Hom}_A(T, M), \operatorname{Hom}_A(T, N))$

(b) $\operatorname{Ext}^1_A(M, N) \cong \operatorname{Ext}^1_B(\operatorname{Hom}_A(T, M), \operatorname{Hom}_A(T, N))$

For any $R \in \mathcal{T}$, we know that $\operatorname{Hom}_A(\tau^{-}R, T) = 0$, since we have a separating tubular family and $T \in \mathcal{P}$ so therefore $\operatorname{Ext}^1_A(T, R) = 0$. Therefore we know that all the regular modules of $A$ are in $\mathfrak{X}(T)$

**Theorem 5.1.3.** ([1], VI.4.3) Let $A$ be an algebra $T_A$ a tilting module and $B = \operatorname{End}T_A$. Then the correspondence

$$\dim M \mapsto \dim \operatorname{Hom}_A(T, M) - \dim \operatorname{Ext}^1_A(T, M)$$

where $M$ is an $A$ module induces an isomorphism $f : K_0(A) \rightarrow K_0(B)$ of the Groethendieck groups of $A$ and $B$.

**Proposition 5.1.4.** ([1], VI.4.5) Let $A$ be an algebra of finite global dimension, $T_A$ be a tilting module, $B = \operatorname{End}T_A$ and $f : K_0(A) \rightarrow K_0(B)$ be the isomorphism as in 5.1.3. Then for any $A$-modules $M$ and $N$ we have:

$$\langle \dim M, \dim N \rangle_A = \langle f(\dim M), f(\dim N) \rangle_B.$$
(a) gl.dim $B \leq 2$ and pd $Z \leq 1$ and id $Z \leq 1$, for all but finitely many non-isomorphic indecomposable $B$-modules $Z$ that are preprojective or preinjective.

(b) The Euler form $\langle -, - \rangle_B : K_0(B) \times K_0(B) \to \mathbb{Z}$ of $B$ is $\mathbb{Z}$ congruent to the Euler form $\langle -, - \rangle_A : K_0(A) \times K_0(A) \to \mathbb{Z}$ of $A$, that is there exists $f$ as in 5.1.3 making the following diagram commute

$$
\begin{array}{ccc}
K_0(B) \times K_0(B) & \xrightarrow{\langle -, - \rangle_B} & \mathbb{Z} \\
\downarrow{f \times f} \cong & & \\
K_0(A) \times K_0(A) & \xrightarrow{\langle -, - \rangle_A} & \mathbb{Z}
\end{array}
$$

(c) The Auslander-Reiten quiver $\Lambda_B$ of $B$ has the disjoint union form

$$
\Lambda_B = \mathcal{P}(B) \cup \mathcal{R}(B) \cup \mathcal{Q}(B)
$$

with $\mathcal{P}(B)$ a preprojective component, $\mathcal{Q}(B)$ a preinjective component and $\mathcal{R}(B)$ is a family of regular components.

(d) $\mathcal{R}(B)$ is non-empty and furthermore, pd $Z \leq 1$ and id $Z \leq 1$ for any regular module $Z \in \mathcal{R}(B)$.

**Theorem 5.1.5.** ([33], XI.3.4, XI.3.7) Let $B = \text{End}_A T$ be tame concealed algebra. Let $\text{add} \mathcal{R}(B)$ be the full subcategory of $\text{mod} B$ whose objects are all the regular $B$ modules.

(a) The subcategory $\text{add} \mathcal{R}(B)$ of $\text{mod} B$ is serial, abelian and closed under extensions.

(b) The components of $\mathcal{R}(B)$ form a family $\mathcal{T}^B = \{ \mathcal{T}_\lambda^B \}_{\lambda \in \Lambda}^B$ of pairwise orthogonal stable tubes. Every tube $\mathcal{T}_\lambda^A$ in $\text{mod} B$ is the image of a tube in $\text{mod} A$ under the functor $\text{Hom}_A(T, -)$. Moreover, if $r_\lambda^B$ denotes the rank of $\mathcal{T}_\lambda^B$ and $n = |Q_0|$, 

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then
\[
\sum_{\lambda \in \Lambda} (r^B_\lambda - 1) \leq n - 2.
\]

(c) All but finitely many of the tubes \( T^B_\lambda \) in \( \mathcal{R}(A) \) are homogeneous and there are at most \( n - 2 \) non-homogeneous tubes in \( T^B_\lambda \).

(e) The Euler quadratic form \( q_B : K_0(B) \to \mathbb{Z} \) is positive semidefinite of corank one and there exists a unique positive vector \( h_B \in K_0(B) \) such that the radical \( \text{rad } q_B = \{ x \in K_0(B), q_B(x) = 0 \} \) of \( q_B \) is an infinite cyclic subgroup of \( K_0(B) \) of the form \( \text{rad } q_B = \mathbb{Z} \cdot h_B \).

(f) If \( f : K_0(A) \to K_0(B) \) is a group isomorphism as in 5.1.3 then the following diagram is commutative:

\[
\begin{array}{ccc}
K_0(A) & \xrightarrow{\Phi_A} & K_0(A) \\
\downarrow f & \cong & \downarrow f \\
K_0(B) & \xrightarrow{\Phi_B} & K_0(B)
\end{array}
\]

(g) Let \( \text{add } \mathcal{R}(B) \) be the full subcategory of \( \text{mod } B \) whose objects are all the regular \( B \)-modules. The functor \( \text{Hom}_A(T, -) : \text{mod } A \to \text{mod } B \) restricts to the equivalences \( \text{Hom}_A(T, -) : \text{add } \mathcal{R}(A) \to \text{add } \mathcal{R}(B) \) of categories such that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{add } \mathcal{R}(A) & \xrightarrow{\tau} & \text{add } \mathcal{R}(A) \\
\text{Hom}_A(T, -) \cong & & \text{Hom}_A(T, -) \cong \\
\downarrow & & \downarrow \\
\text{add } \mathcal{R}(B) & \xleftarrow{\tau} & \text{add } \mathcal{R}(B)
\end{array}
\]

As stated in 2.6, in this case where \( A \) is tame concealed, if \( d \) is a preprojective or preinjective dimension vector then \( SI(A, d) \) is a polynomial algebra. Therefore we
will focus our attention on the case where \( d \) is a regular dimension vector.

**Example 5.1.6.** This example illustrates how the isomorphism \( f : K_0(A) \to K_0(B) \) is constructed and what it does to dimension vectors of the simple regular modules. Again as in the example [5.0.12](#), we have \( A = \hat{E}_6 \) with the following orientation:

\[
\begin{array}{c}
1 \rightarrow 4 \\
2 \rightarrow 5 \rightarrow 7 \\
3 \rightarrow 6
\end{array}
\]

Specifically, let \( T_1 = P_2, T_2 = \tau^{-2}P_3, T_3 = \tau^{-2}P_4, T_4 = \tau^{-3}P_2, T_5 = \tau^{-4}P_3, T_6 = \tau^{-4}P_1, T_7 = \tau^{-6}P_2 \)

therefore:

\[
\begin{align*}
\dim T_1 &= (0, 1, 0, 0, 1, 0, 1) \\
\dim T_2 &= (1, 1, 0, 1, 1, 1, 2) \\
\dim T_3 &= (0, 1, 1, 1, 1, 1, 2) \\
\dim T_4 &= (0, 1, 0, 1, 2, 1, 2) \\
\dim T_5 &= (1, 1, 1, 2, 2, 1, 3) \\
\dim T_6 &= (1, 1, 1, 2, 2, 3) \\
\dim T_7 &= (1, 2, 1, 2, 3, 2, 4)
\end{align*}
\]
where the quiver with relations of $B$ is

```
1 → 2 → 5
1 → 4 → 7
3 → 6
```

Then we have $f$ as described in 5.1.3 as the matrix:

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & -1 & -1 & -2 & -2 & -2 & -3
\end{bmatrix}
$$

If we let $S$ be the matrix with column $i = \text{dim} T_i$ then $f = E \cdot S$ where $E$ is the Euler matrix of $A$.

Using this we can easily find the dimension vectors of the simple regular modules and the homogeneous module as in 5.1.5.d.
A key observation of tilting theory is that the tilting module $T_A$ induces a tilting $B$-module, which is the left $B$-module $B T$. Moreover, the algebra $A$ can be recovered from $B$ and $B T$. This is seen from the tilting theorem of Brenner and Butler:

**Theorem 5.1.7.** ([5]) Let $T_A$ be a tilting module and $B = \text{End} T_A$, and $(\mathfrak{T}(T_A), \mathfrak{F}(T_A))$, $(\mathcal{D}\mathfrak{F}(B T), \mathcal{D}\mathfrak{T}(B T))$ be the induced torsion pairs in $\text{mod} A$ and $\text{mod} B$ respectively. Then $T$ has the following properties.

(a) $B T$ is a tilting module and the canonical $K$-algebra homomorphism $A \to \text{End}(B T)^{\text{op}}$, given by

$$a \mapsto (t \mapsto ta)$$

is an isomorphism.

(b) The functors $\text{Hom}_A(T, -)$ and $- \otimes_{B T}$ induce quasi-inverse equivalences between $\mathfrak{T}(T_A)$ and $\mathcal{D}\mathfrak{T}(B T)$.

(c) The functors $\text{Ext}^1_A(T, -)$ and $\text{Tor}^B_1(-, T)$ induce quasi-inverse equivalences between $\mathfrak{F}(T_A)$ and $\mathcal{D}\mathfrak{F}(T_A)$.

---

<table>
<thead>
<tr>
<th>regular module</th>
<th>dimension in A</th>
<th>dimension in B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>$(1, 1, 1, 2, 2, 3)$</td>
<td>$(1, 1, 1, 1, 1, 1)$</td>
</tr>
<tr>
<td>$E_1$</td>
<td>$(1, 0, 0, 1, 0, 1)$</td>
<td>$(0, 0, 0, 1, 0, 0)$</td>
</tr>
<tr>
<td>$E_2$</td>
<td>$(0, 1, 0, 0, 1, 1)$</td>
<td>$(1, 1, 0, 1, 1, 1)$</td>
</tr>
<tr>
<td>$E_3$</td>
<td>$(0, 0, 1, 0, 1, 1)$</td>
<td>$(0, 0, 1, 0, 0, 0)$</td>
</tr>
<tr>
<td>$E'_1$</td>
<td>$(1, 0, 0, 1, 0, 1)$</td>
<td>$(0, 1, 0, 0, 0, 0)$</td>
</tr>
<tr>
<td>$E'_2$</td>
<td>$(0, 1, 0, 0, 1, 1)$</td>
<td>$(0, 0, 0, 0, 1, 0)$</td>
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<tr>
<td>$E'_3$</td>
<td>$(0, 0, 1, 1, 0, 1)$</td>
<td>$(1, 0, 1, 1, 0, 0)$</td>
</tr>
<tr>
<td>$E''_1$</td>
<td>$(1, 1, 1, 1, 1, 2)$</td>
<td>$(1, 1, 1, 0, 1, 1)$</td>
</tr>
<tr>
<td>$E''_2$</td>
<td>$(0, 0, 0, 1, 1, 1)$</td>
<td>$(0, 0, 0, 1, 0, 0)$</td>
</tr>
</tbody>
</table>
From part (b) of this theorem more specifically, we can notice that the functors $\text{Hom}_A(T, -)$ and $- \otimes_B T$ induce quasi-inverse equivalences between $\mathcal{R}(A)$ and $\mathcal{R}(B)$ since $\mathcal{R}(B) \in D\Sigma(BT)$. Similarly, we can find a tilting module $T_B$ in the preprojective component of $B$ such that $A = \text{End}(T_B)$. Therefore we can define a functor

$$\varphi' = \text{Hom}_B(T_B, -) : \mathcal{R}(B) \to \mathcal{R}(A)$$

which induces isomorphism of the regular modules.

**Lemma 5.1.8.** Let $A$ be a tame hereditary algebra, $T$ a tilting module from $A$ and $B = \text{End} T$. Let $\varphi = \text{Hom}_A(T, -) : \text{mod } A \to \text{mod } B$ and $f : K_0(A) \to K_0(B)$ which is an isomorphism of dimension vectors such that $f(\dim M) = \dim \varphi(M)$.

(a) If $E_1, \ldots, E_r$ are the simple regular modules at the mouth of a tube in $A$, then $\varphi(E_1), \ldots, \varphi(E_r)$ are the modules at the mouth of a tube in $B$. Then it follows that rank of $T^A_\lambda$ is $r$ so is the rank of $\varphi(T^A_\lambda) = T^B_\lambda$ is also $r$. We can conclude that the tubular type of $B$ is the same as the tubular type of $A$.

(b) For any module $M \in \mathcal{R}(A)$, $\dim M = p h_A + \sum_{j=1}^s \sum_{i=0}^{m_j-1} p_i^{(j)} e_{A,i}^{(j)}$. Then $\dim \varphi(M) = p h_B + \sum_{j=1}^s \sum_{i=0}^{m_j-1} p_i^{(j)} e_{B,i}^{(j)}$.

(c) Let $M \in A$ be a module of dimension $p h_A + \sum_{j=1}^s \sum_{i=0}^{m_j-1} p_i^{(j)} e_{A,i}^{(j)}$. Let $V \in A$ be orthogonal to $M$ in the sense that $\text{Hom}_A(M, V) = \text{Ext}_A(M, V) = 0$ then $\varphi(V)$ is orthogonal to $\varphi(M)$.

**Proof.** (a) Since we have the equivalence of categories of regular modules from the tilting theorem of Brenner and Butler, this is clear.

(b) By 5.1.5.d we know that $h_B$ exists. We know $f(h_A) = h_B$ since this just follows from (a) $h_A = e_{A,1} + \cdots + e_{A,r}$ and $h_B = f(e_{A,1}) + \cdots + f(e_{A,r})$.

(c) This follows from 5.1.2 since $M, V \in \Sigma(T)$. We have that $\text{Hom}_A(M, V) = \text{Hom}_A(\varphi(M), \varphi(V)) = 0$ and also that $\text{Ext}^1_A(M, V) = \text{Ext}^1_A(\varphi(M), \varphi(V)) = 0$.  

\[49\]
Theorem 5.1.9. Let $A$ be a tame hereditary algebra, $T$ a tilting module from $A$ and $B = \text{End } T$. Let $\varphi = \text{Hom}_A(T, -) : \text{mod } A \to \text{mod } B$ and $f : K_0(A) \to K_0(B)$ which is an isomorphism of dimension vectors such that $f(\dim M) = \dim \varphi(M)$ 5.1.3. Let $d$ be a $Q_A$ is the quiver associated to $A$ and $Q_B$ the quiver associated to $B$. Then we have a natural isomorphism $SI(Q_A, d) \cong SI(Q_B, f(d))$.

Proof. The previous lemma 5.1.8 shows that the conditions (1)-(6) of 3.1.4 hold. By 5.1.7 we have an equivalence of $\mathcal{R}(A)$ and $\mathcal{R}(B)$. From this it follows that the orthogonal categories are the same 5.1.8 (c) and thus condition (7) holds. We know that every module in $\mathcal{R}(B)$ has projective dimension 1 from 5.1.5. In general, we have that if $c^V$ is a generating semi-invariant in $SI(Q_A, d)$ then $c^\varphi(V)$ is a generating semi-invariant in $SI(Q_B, f(d))$. Therefore, the main theorem 3.2.1 is demonstrated in this case. \qed
6 Tubular Algebras

First, we will discuss the construction of the tubular algebras and then the module structure. Given a finite dimensional algebra $A_0$ and and an $A_0$ module $R$ we denote by $A_0[R]$ the one-point extension of $A_0$ by $R$, namely the algebra

$$A_0[R] = \begin{bmatrix} A_0 & R \\ 0 & K \end{bmatrix} = \left\{ \begin{bmatrix} a & r \\ 0 & b \end{bmatrix} \middle| a \in A_0, r \in R, b \in k \right\}$$

The quiver of $A_0[R]$ contains the quiver of $A_0$ as a full subquiver and there is an additional vertex $\omega$ called the extension vertex of $A_0[R]$.

The category of $A_0[R]$ modules can be described as the triples $V = (V_0, V_\omega, \gamma_V)$ corresponding to the $A_0[R]$ module $\begin{bmatrix} V_0 \\ V_\omega \end{bmatrix}$ with $\begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix}$ operating on it via the map $\overline{\gamma}_V : R \otimes_k V_\omega \rightarrow V_0$ adjoint to $\gamma_V$.

The one-point coextension $[V]A$ of $A$ by $V$ is defined by:

$$[V]A = (A^{\text{op}}[DV])^{\text{op}} \approx \begin{bmatrix} K & DV \\ 0 & A \end{bmatrix}$$

The vertex of the quiver of $[V]A$ belonging to the simple projective module $\begin{bmatrix} K \\ 0 \end{bmatrix}$ will be denoted by $-\omega$ and called the coextension vertex. The $[V]A$ modules can be written as triples $W = (W_{-\omega}, W_0, \delta_W)$ where $W_{-\omega}$ is a $K$-vectorspace, $W_0$ an $A$ module and $\delta_W : DV \otimes W_0 \rightarrow W_{-\omega}$ a $K$-linear map.

Definition 6.0.10. A branch in $b$ of length $n$ is any finite full connected subquiver of the quiver.
whose vertices are of the form $b_{i_1...i_n}$ indexed by any possible sequence $i_1, \ldots, i_n \in \{+, -\}$ and arrows

$$\beta_{i_1...i_n^-} : b_{i_1...i_n^-} \rightarrow b_{i_1...i_n}$$

and

$$\beta_{i_1...i_n^+} : b_{i_1...i_n} \rightarrow b_{i_1...i_n^+}$$

with relations $\beta_{i_1...i_n^-} - \beta_{i_1...i_n^+}$.

Definition 6.0.11. Given an algebra $C$ with quiver $Q_C$ and relations $\sigma_i$ with a vertex $b$ of $Q_C$ then the restriction $B$ of $C$ to some full subquiver $Q'$ will be said to be a branch of $C$ in $b$ of length $n$ provided that:

a) $B$ is a branch in $b$ of length $n$.

b) There is a full subquiver $\Lambda$ such that $Q = Q' \cup \Lambda$.

c) $Q' \cap \Lambda = \{b\}$.

d) Any relation $\sigma_i$ has its support in either $Q'$ or $\Lambda$.

If we denote by $\{\rho_i\}$ the set of all relations having suport in $Q$ and let $A = (\Lambda, \{\rho_i\})$, then $C$ is said to be obtained from $A$ by adding the branch $B$ in $b$.

Let $A_0$ be an algebra $E_1, \ldots E_t$ be $A_0$ modules and $B_1, \ldots B_t$ branches. Let $A_0[E_i, B_i]_{i=1}^t$ be inductively defined. Let $A_0[E_1, B_1]$ be obtained from the one-point
extension $A_0[E_1]$ with extension vertex $\omega_1$ by adding the branch $B_1$ in $\omega_1$, and let

\[ A_0[E_i, B_i]_{i=1}^t = (A_0[E_i, B_i]_{i=1}^{t-1})[E_t, B_t]. \]

**Definition 6.0.12.** The algebra $A = A_0[E_i, B_i]_{i=1}^t$ is called a tubular extension of $A_0$ using modules from $T$ provided that the modules $E_1, \ldots E_t$ are pairwise orthogonal ray modules from $T$.

**Definition 6.0.13.** A tubular extension $A$ of a tame concealed algebra of extension type $(2, 2, 2, 2), (3, 3, 3), (4, 4, 2)$ or $(6, 3, 2)$ is called a **tubular algebra**.

If the tame concealed algebra was of type $(m_1, \ldots, m_s)$, an extension on a module $E_i$ in the tube of rank $m_j$ by a branch of length $|B_i|$ is an algebra of extension type $(m_1, \ldots, m_{j-1}, m_j + |B_i|, m_{j+1}, \ldots, m_s)$. An extension on a homogeneous tube by a branch of length $|B_i|$ is an algebra of extension type $(m_1, \ldots, m_s, 1 + |B_i|)$. See Appendix A for a list of possible $(m_1, \ldots, m_s)$ on which a tubular extension can be a tubular algebra.

**Example 6.0.14.** Starting with the tame concealed algebra of 5.0.12 which is of tubular type (3,3,2), we can form tubular algebras of type (3,3,3) using a one point extension on either $E_1''$ or $E_2''$ with extension vertex labeled as 1 as follows

One point extension on $E_1''$, $e_1'' = (1, 1, 1, 0, 1, 1, 1)$:

```
1 ← 2 ← 5 ← 8
3 → 6 → 7 → 4
```

One point extension on $E_2''$, $e_2'' = (0, 0, 0, 1, 0, 0, 0)$:

```
1 ← 3 ← 6
2 → 5 → 8
4 → 7
```
Or we can form a tubular algebra of type (4,4,2) using a one point extension on each of the tubes of rank 3. From this we get a variety of tubular algebras where the asterisk indicates an extension vertex.

We can also form a tubular algebra of type (6,3,2) using a branch extension on one of the tubes of rank 3. Let $B$ be the branch, then $|B| = 3$. We obtain one of the following quivers where the asterisks are the extension vertices and the lines connecting vertices indicate that the arrows could be any orientation:
Theorem 6.0.15. Let $A_0$ be an algebra with a separating tubular family $T$, separating $P$ from $Q$. Let $A = A_0[E_i; B_i]_{i=1}^t$ be a tubular extension using modules from $T$. We define module classes $P_0, T_0, Q_0$ in $\text{mod } A$ as follows: Let $P_0 = P$, let $T_0$ be the module class given by all indecomposable $A$ modules $M$ with either $M|A_0$ (the restriction of $M$ to $A_0$) non-zero and in $T$ or else the support of $M$ being contained in some $B_i$ and $\langle l_{B_i}, \dim M \rangle < 0$. Let $Q_0$ be the module class given by all indecomposable $A$ modules $M$ with either $M|A_0$ non-zero and in $Q$ or else the support of $M$ being contained in some $B_i$ and $\langle l_{B_i}, \dim M \rangle > 0$. Then $\text{mod } A = P_0 \vee T_0 \vee Q_0$ with $T_0$ being a separating tubular family separating $P_0$ from $Q_0$.

The elements $z \in \mathbb{Z}^n$ such that $\chi(z) = \langle z, z \rangle = 0$ are called the radical vectors and they form a subgroup of $\mathbb{Z}^n$ called $\text{rad } \chi$.

An algebra $A$ will be said to be cotubular provided that the opposite algebra $A^{op}$ is tubular. Let $A$ be an algebra which is both tubular and cotubular. Assume that $A$ is an extension of $A_0$ and a coextension of $A_\infty$ where both $A_0$ and $A_\infty$ are tame concealed algebras. Let $\alpha_0$ be the positive radical generator of $K_0(A_0)$ then $\langle \alpha_0, \alpha_0 \rangle = 0$ and $\alpha_\infty$ be the positive radical generator of $K_0(A_\infty)$ then $\langle \alpha_\infty, \alpha_\infty \rangle = 0$. $\alpha_0 + \alpha_\infty$ is sincere and the non-sincere positive radical vectors are the positive multiples of $\alpha_0$. 

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and $\alpha_\infty$.

Consider dimension vector $\beta = p\alpha_0 + q\alpha_\infty$, with $\text{GCD}(p, q) = d$, $p = dp', q = dq'$, we define the component $C(p, q)$ to be the closure of all modules which are direct sums of $d$ copies of regular indecomposable modules of dimension $\beta' = p'\alpha_0 + q'\alpha_\infty$.

Given any $x \in K_0(A)$ we can define the index of $x$ provided that the values of $\langle \alpha_0, x \rangle$ and $\langle \alpha_\infty, x \rangle$ are not both zero. In this case, we call index $(x) = \frac{\langle \alpha_0, x \rangle}{\langle \alpha_\infty, x \rangle}$. For a module $M$ we will also call the index of its dimension vector index $(M)$.

**Theorem 6.0.16.** For any non-zero element in the radical of $\chi$, its index is defined namely index $(p\alpha_0 + q\alpha_\infty) = \frac{q}{p} = \gamma$.

We want to define module classes $P_\gamma$, $T_\gamma$, $Q_\gamma$, $P_0$, $T_0$, $Q_0$ have been defined and dually $P_\infty$, $Q_\infty$, $T_\infty$ can be defined. $Q_\infty$ is the module class of all preinjective $A_\infty$ modules. $T_\infty$ is the module class given by all indecomposable $A$ modules $M$ with $M|A_\infty$ a non-zero regular $A_\infty$ module or else $M|A_\infty$ being zero and $\langle \alpha_0, M \rangle > 0$. $P_\infty$ is the module class given by $M|A_\infty$ a preprojective $A_\infty$ module or else $M|A_\infty$ being zero and $\langle \alpha_0, M \rangle < 0$.

Note that $Q_0 \cap P_\infty$ is the module class given by all indecomposable $A$ modules $M$ with both $\langle \alpha_0, \text{dim}M \rangle > 0$ and $\langle \alpha_\infty, \text{dim}M \rangle < 0$.

**Theorem 6.0.17.** The following module classes are pairwise disjoint:

$$P_0, T_0, Q_0 \cap P_\infty, T_\infty, Q_\infty$$

and they give the complete category of $A$ modules.
For $\gamma = \frac{q}{p}$, we can define $P_\gamma$, $T_\gamma$, $Q_\gamma$ as the module classes of all modules $M$ such that $\langle pa_0 + qa_\infty, \dim M \rangle < 0$, $= 0$ or $> 0$ respectively. We call $T_\gamma$ $\gamma$-separating provided $T_\gamma$ separates $P_\gamma$ from $Q_\gamma$.

The next theorem determines completely the structure of all components of $A$. We are particularly interested in this structure because the structure of the rings of semi-invariants of tubular algebras is related to this module structure.

**Theorem 6.0.18.** (33) The structure of a tubular algebra $A$ is as follows: First, there is a preprojective component $P_0$ which is the module class of all preprojective $A_0$ modules. Then for any $\gamma = q/p$, there is a separating tubular $\mathbb{P}^1_k$ family $T_\gamma$, and all $T_\gamma$ are stable of type $(2, 2, 2, 2)$, $(3, 3, 3)$, $(4, 4, 2)$ or $(6, 3, 2)$ except for $T_0$ and $T_\infty$. Finally, there is the preinjective component $Q_\infty$ which is the module class of all preinjective $A_\infty$ modules.

**Example 6.0.19.** Starting with the tame concealed algebra of 5.0.12 which is of tubular type (3,3,2), we can form tubular algebras of type (3,3,3) using a one point extension on $E_1''$, $e_1'' = (1,1,1,0,1,1,1)$, with extension vertex labeled as 1 as follows.
Let $A = KQ/I$, we have that $A_0$ is tame concealed of type $\tilde{E}_6$ with $h_0 = (1, 1, 1, 1, 1, 1)$ as in example 5.0.12 with the following simple regular non-homogeneous dimension vectors:

<table>
<thead>
<tr>
<th>regular module</th>
<th>dimension in $A_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>$(1, 1, 1, 1, 1, 1)$</td>
</tr>
<tr>
<td>$E_{0,1}$</td>
<td>$(0, 0, 0, 1, 0, 0)$</td>
</tr>
<tr>
<td>$E_{0,2}$</td>
<td>$(1, 1, 0, 1, 0, 1, 1)$</td>
</tr>
<tr>
<td>$E_{0,3}$</td>
<td>$(0, 0, 1, 0, 0, 0, 0)$</td>
</tr>
<tr>
<td>$E_{0,1}'$</td>
<td>$(0, 1, 0, 0, 0, 0, 0)$</td>
</tr>
<tr>
<td>$E_{0,2}'$</td>
<td>$(0, 0, 0, 0, 0, 1, 0)$</td>
</tr>
<tr>
<td>$E_{0,3}'$</td>
<td>$(1, 0, 1, 1, 1, 0, 1)$</td>
</tr>
<tr>
<td>$E_{0,1}''$</td>
<td>$(1, 1, 1, 1, 1, 1, 1)$</td>
</tr>
<tr>
<td>$E_{0,2}''$</td>
<td>$(0, 0, 0, 1, 0, 0, 0)$</td>
</tr>
</tbody>
</table>

$A_\infty$ is the following:

```
3 --6
1 --2 --5
4 --7
```

which is also tame concealed of type $\tilde{E}_6$ with the following simple regular dimension vectors:
Then we have that \( \alpha_0 = (0, 1, 1, 1, 1, 1, 1, 1) \) \( \alpha_\infty = (1, 3, 2, 2, 1, 1, 1, 0) \). In the case where \( \gamma = 1 \) we have \( h_1 = \alpha_0 + \alpha_\infty = (1, 4, 3, 3, 2, 2, 1) \).

**Definition 6.0.20.** Let \( T \) be a tilting \( A \) module, \( T = T_0 \oplus T_p \) with \( T_0 \) a preprojective tilting \( A_0 \) module and \( T_p \) a projective \( A \) module, then \( T \) is called a shrinking module. The corresponding functor \( \Sigma_T = \operatorname{Hom}(T, -) : \text{mod} A \to \text{End}(T) \) will be called a left shrinking functor.

\( T_p \) is uniquely determined by \( A_0 \) and \( A \), it is just the direct sum of indecomposable projective modules \( P_a \) with \( a \) outside \( A_0 \). When \( A \) is a tubular extension of a tame concealed algebra \( A_0 \) then \( \Sigma_T A = B \) where \( B \) is a tubular extension of \( B_0 = \text{End}(T_0) \).

**Theorem 6.0.21.** (33) Let \( A \) be a tubular algebra. Then there exists a left shrinking functor \( \Sigma : \text{mod} A \to \text{mod} B \), where \( B \) is a tubular extension of a tame concealed canonical algebra.

**Theorem 6.0.22.** (33 5.7.1) Let \( A \) be a canonical tubular algebra. Then there exists a proper left shrinking functor from \( \text{mod} A \) to \( \text{mod} C \) where \( C \) is one of the algebras \( C(4, \lambda), C(6), C(7), \) or \( C(8) \).
with relations $(\alpha_{12} - \beta_{12})\gamma = 0, (\alpha_{12} - \lambda\beta_{12})\gamma' = 0$. $C(4, \lambda)$ is of type $(2, 2, 2, 2)$ with $\alpha_0 = (0, 1, 1, 2, 1, 1)$ and $\alpha_\infty = (1, 1, 1, 1, 0, 0)$.

$C(6)$

with relation $(\alpha_{123} - \beta_{123})\gamma = 0$. $C(6)$ is of type $(3, 3, 3)$ with $\alpha_0 = (0, 1, 1, 2, 2, 3, 2, 1)$ and $\alpha_\infty = (1, 1, 1, 1, 1, 0, 0)$.

$C(7)$

with relation $(\alpha_{1234} - \beta_{1234})\gamma = 0$. $C(7)$ is of type $(4, 4, 2)$ with $\alpha_0 = (0, 1, 1, 2, 2, 3, 3, 4, 2)$ and $\alpha_\infty = (1, 1, 1, 1, 1, 1, 1, 0)$. 

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with relation $(\alpha_{123} - \beta_{123456})\gamma = 0$. $C(8)$ is of type $(6,3,2)$ with $\alpha_0 = (0, 2, 3, 1, 2, 3, 4, 5, 6, 3)$ and $\alpha_\infty = (1, 1, 1, 1, 1, 1, 1, 1, 1, 0)$.

In order to find the modules in $T_\gamma$, we will first find the tubular family $T_1$, then we will use shrinking functors to shift $T_1$ to any $T_\gamma, \gamma \in \mathbb{Q}^+$. These shrinking functors are just specifically defined tilting functors.

**Example 6.0.23.** Continuing with example 6.0.19 we can illustrate 6.0.22. We have $A$

First we need a shrinking functor which takes $A$ to a canonical tubular algebra. In this case we take $T_p = P_1$ and the following $T_i$:
That is, $T_2 = \tau^{-2}P_3$, $T_3 = P_2$, $T_4 = \tau^{-2}P_4$, $T_5 = \tau^{-2}P_6$, $T_6 = \tau^{-2}P_5$, $T_7 = \tau^{-2}P_7$ and $T_8 = \tau^{-2}P_8$.

Then $B = \text{End}(T_p \oplus_{i=2}^8 T_i)$ is the following canonical algebra:

\[
\begin{array}{c}
\text{2} & \text{5} \\
\text{1} & \text{3} & \text{6} & \text{8} \\
\text{4} & \text{7}
\end{array}
\]

This algebra is canonical and tubular. In the preprojective component of $B$, we can find $T_i'$ such that $\text{End}(\oplus T_i')$ is

Specifically, we have that:
and take $T'_1 = P_1$ Then we have that $T = \text{End}(\oplus T'_i)$ is of the form

\[
\begin{array}{ccc}
\alpha_3 & 4 & \alpha_1 \\
\beta_3 & 5 & \beta_1 \\
\end{array}
\]

with relation $(\alpha_{123} - \beta_{123})\gamma = 0.$ with $T_i$ corresponding to vertex $i$. This is $C(6)$.

6.1 Semi-Invariants of Tubular Algebras

In this section we will verify the conditions of 3.1.4 for Tubular Algebras to show the main theorem holds in this case.

Let $A = KQ/I$ be a tubular algebra as previously defined of tubular type $\mathbb{T} = (m_1, \ldots, m_s) = (2, 2, 2, 2), (3, 3, 3), (4, 4, 2) \text{ or } (6, 3, 2)$.

We know that the representation space $\text{Rep}(A, d)$ for some $d \in \mathcal{R}(A)$ is irreducible by the following theorem of Bobinski and Skowronski [7].

**Theorem 6.1.1.** [7] Let $A$ be a tame concealed or tubular algebra and $d \in \mathcal{R}(A)$. 
Then the affine variety \( \text{mod}_A(d) \) of all \( A \) modules of dimension vector \( d \) is irreducible, normal, a complete intersection, 

\[
\dim \text{mod}_A(d) = \dim GL(d) - \langle \langle d, d \rangle \rangle
\]

and the maximal \( GL(d) \)-orbits in \( \text{mod}_A(d) \) form the open sheet consisting of nonsingular points. Moreover, a module \( M \) in \( \text{mod}_A(d) \) is a nonsingular point if and only if \( \text{Ext}^2_A(M, M) = 0 \).

We now need to show that a tubular family \( T_\gamma \in A \), satisfies the conditions of the main theorem \( \text{[3.2.1]} \). Primarily, we use the fact that for any \( \gamma \in \mathbb{Q}_0^\infty \), we have that \( T_\gamma \) is a separating tubular \( \mathbb{P}^1_K \) family with all but \( T_0 \) and \( T_\infty \) being stable of tubular type \( T \) \( \text{[32], 5.2.4} \).

From the properties of stable tubular families we get the following conditions are true:

1. For each \( T_\gamma \), we have a family of homogeneous indecomposable modules \( \{V_t\} \), \( t \in \mathbb{P}^1_K \) of dimension \( h_\gamma \), which are the homogeneous modules in \( T_\gamma \). We also have a family of nonhomogeneous modules \( E^{(j)}_i \) of dimension vectors \( e^{(j)}_i \), \( (1 \leq j \leq s, 0 \leq i \leq m_j - 1) \). In particular, for tubular algebras, we have that \( h = p\alpha_0 + q\alpha_\infty \) as defined in section 6.

2. Let \( X \) be an indecomposable module in \( T_\gamma \). If \( X \) belongs to a homogeneous tube then \( \dim X = ph \), \( p \in \mathbb{Z}^+ \) where \( p \) is the regular length of \( X \). If \( X \) is in a non-homogeneous tube of rank \( m_j \), then \( \dim X = ph + e^{(j)}_i + \cdots + e^{(j)}_i + k \) where the regular length is \( m_j p + (k + 1) \). Therefore in general, if \( X \) is any module in add \( T \), with \( \dim X = \beta \) we can write the dimension vector as:

\[
\beta = ph_\gamma + \sum_{j=1}^{s} \sum_{i=0}^{m_j-1} p^{(j)}_i e^{(j)}_i.
\]
Therefore tubular algebras satisfy condition 3.1.4(1).

3. Conditions 3.1.4(2) and (5) hold as a result of the properties of tubular families ([32], 3.1.3')

4. In order to show condition 3.1.4(3), we will need to use the idea of generic decomposition. The following theorems of Crawley-Boevey and Schroer, aid in our proof of this decomposition:

**Theorem 6.1.2.** If \( C \) is an irreducible component in \( \text{mod}^d_A(k) \), then \( C = C_1 \oplus \cdots \oplus C_t \) for some irreducible components \( C_i \) of module varieties \( \text{mod}^d_A(k) \) with \( d = d_1 + \cdots + d_t \) and with the property that the general module in each \( C_i \) is indecomposable. Moreover, \( C_1, \ldots, C_t \) are uniquely determined by this, up to reordering.

**Theorem 6.1.3.** If \( C_i \subseteq \text{mod}^d_A(k) \) are irreducible components \((1 \leq i \leq t)\), and \( d = d_1 + \cdots + d_t \), then \( C \) is a irreducible component of \( \text{mod}^d_A(k) \) if and only if \( \text{Ext}_A^1(C_i, C_j) = 0 \) for all \( i \neq j \).

We will use the notion of admissible paths as defined in 3.1.2. Consider a vector of the form

\[
\beta = p \mathbf{h} + \sum_{j=1}^{s} \sum_{i=0}^{m_j-1} p_i^{(j)} e_i^{(j)}.
\]

Clearly \( p \mathbf{h} \) decomposes into \( p \) homogeneous modules of dimension \( \mathbf{h} \), namely \( V_{t_1} \oplus \cdots \oplus V_{t_p} \). For each \( j, 1 \leq j \leq s, \) we can decompose \( d = \sum_{i=0}^{m_j-1} p_i^{(j)} e_i^{(j)} \) using induction on the index of the admissible path. Beginning with \( k = \max\{p_i^{(j)}\} \), consider the admissible paths \( k_1, \ldots, k_n \) of index \( k \). Let \( E_k^{(j)} \) be the module associated to the admissible path \( k_i \) with dimension \( e_{k_i}^{(j)} \). Then we can write the dimension vector as \( d = d' = \sum_{i=0}^{m_j-1} q_i^{(j)} e_i^{(j)} + e_{k_1}^{(j)} + \cdots + e_{k_n}^{(j)} \) where \( q_i^{(j)} = p_i^{(j)} \) if \( p_i^{(j)} \neq k \) and \( q_i^{(j)} = k - 1 \) if \( p_i^{(j)} = k \). Next consider the admissible paths of
index $k' = k - 1$ and then continue in this manner until the paths of index 1 are considered. Therefore we have the general decomposition:

$$V = V_{t_1} \oplus \ldots \oplus V_{t_p} \oplus \oplus_{j=1}^{s} \oplus [i,k] \text{admissible, length} \geq 1 E_{[i,k]}^{(j)}.$$ 

It remains to be shown that each summand has no extensions with any other. Because there are no maps between tubes of tubular algebras, we just need to check that the modules in the same tube have no extensions. Consider, $E_{[i,k]}^{(j)}$ and $E_{[m,n]}^{(j)}$. Without loss of generality and by the definition of admissible sequences, we have either $0 < m < i < k < n < m_j$ that is $[i,k] \subset [m,n]$ or $0 < i < k < m < n < m_j$ with $m - k > 1$, we cannot have $k = m$ or $k = m - 1$ by the construction of our decomposition. So in the case $[i,k] \subset [m,n]$, we have that

$$\langle \langle e_i + \ldots + e_k, e_m + \ldots + e_n \rangle \rangle = \langle \langle e_i + \ldots + e_k, e_{i-1} + \ldots + e_k \rangle \rangle$$

$$= \sum_{l=i}^{k} \langle \langle e_l, e_{i-1} + \ldots + e_k \rangle \rangle = \sum_{l=i}^{k} \langle \langle e_l, e_{l-1} + e_l \rangle \rangle$$

$$\sum_{l=i}^{k} \langle \langle e_l, e_{l-1} \rangle \rangle + \sum_{l=i}^{k} \langle \langle e_l, e_l \rangle \rangle = -(k-i) + (k-i) = 0.$$ 

Therefore, $\text{Ext}_A^1(E_{[i,k]}^{(j)}, E_{[m,n]}^{(j)}) = 0$. Similarly, $\text{Ext}_A^1(E_{[m,n]}^{(j)}, E_{[i,k]}^{(j)}) = 0$. In the case that $m - k > 1$, it is clear that

$$\langle \langle e_{[i,k]}, e_{[m,n]} \rangle \rangle = \langle \langle e_{[m,n]}, e_{[i,k]} \rangle \rangle = 0$$

based on the conditions of 3.1.4(5). So again, $\text{Ext}_A^1(E_{[i,k]}^{(j)}, E_{[m,n]}^{(j)}) = 0$ and $\text{Ext}_A^1(E_{[m,n]}^{(j)}, E_{[i,k]}^{(j)}) = 0$. 

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5. Condition 3.1.4(4): All modules $X \in T$ have projective and injective dimension 1. $\tau X$ is also in $T$, therefore $\text{Hom}(D(A), \tau X) = 0$ since $T$ is a separating tubular family. Similarly, $\text{Hom}(\tau^- X, A) = 0$ therefore $X$ has injective dimension 1.

6. We can see that tubular algebras easily satisfy condition 3.1.4(6) since we have the condition on tubular algebras that all sincere homogeneous modules are of dimension $h = p\alpha_0 + q\alpha_\infty$ with $p, q \in \mathbb{Z}^+$ and furthermore, $\gamma = q/p$.

7. Condition 3.1.4(7) holds based on the fact that $T_\gamma$ is generated by modules such that $\langle \dim X, h \rangle = 0$. Also, in the tubular case, $\gamma = -\langle \dim X, h \rangle / \langle \dim X, h \rangle$.

**Example 6.1.4.** In the following example, we will continue example 6.0.23 applying the main theorem to the algebra $C(6)$. Recall that $C(6)$ is the following algebra:

```
  8 4 2 1 3 5 6
 / \ / \ / \ / \ / \
\gamma 7 6 \gamma 1 \gamma
 / \ / \ / \ / \ / \
\beta 5 3 \beta 3 2 \beta
```

with relation $(\alpha_{123} - \beta_{123})\gamma = 0$. For $A = C(6)$, the tubular type of $A$ is $(3,3,3)$. We have that $A_0 = \overline{E}_6$ and $A_\infty = \overline{A}_5$. Therefore, $\alpha_0 = (0,1,1,2,2,3,2,1)$ and $\alpha_\infty = (1,1,1,1,1,1,0,0)$. For the purpose of this example as well as the exploration of shrinking functors in the next section, we will choose $h = \alpha_0 + \alpha_\infty = (1,2,2,3,3,4,2,1)$. Thus $\gamma = 1$ and we are dealing with the separating tubular family $T_1$.

We know the dimension vectors $e_i^{(j)}$ for $T_1$:
Clearly we can see the illustration of 3.1.4.(2), that is that for each $j$, we have that $\sum_{i=0}^{2} e_i^{(j)} = h$. Let $\beta = h + e_1^{(1)} + e_0^{(2)} + 2e_1^{(2)}$, so we have $p_0^{(1)} = p_1^{(2)} = 1$ and $p_1^{(2)} = 2$. So we have the polygons:

<table>
<thead>
<tr>
<th></th>
<th>dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_0^{(1)}$</td>
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</tr>
<tr>
<td>$E_1^{(1)}$</td>
<td>$(1, 1, 1, 1, 1, 2, 1, 0)$</td>
</tr>
<tr>
<td>$E_2^{(1)}$</td>
<td>$(0, 0, 0, 1, 1, 1, 0, 0)$</td>
</tr>
<tr>
<td>$E_0^{(2)}$</td>
<td>$(1, 1, 1, 2, 1, 2, 1, 1)$</td>
</tr>
<tr>
<td>$E_1^{(2)}$</td>
<td>$(0, 1, 0, 1, 1, 1, 1, 0)$</td>
</tr>
<tr>
<td>$E_2^{(2)}$</td>
<td>$(0, 0, 1, 0, 1, 1, 0, 0)$</td>
</tr>
<tr>
<td>$E_0^{(3)}$</td>
<td>$(1, 1, 1, 1, 2, 1, 1)$</td>
</tr>
<tr>
<td>$E_1^{(3)}$</td>
<td>$(0, 0, 1, 1, 1, 1, 1, 0)$</td>
</tr>
<tr>
<td>$E_2^{(3)}$</td>
<td>$(0, 1, 0, 1, 0, 1, 0, 0)$</td>
</tr>
</tbody>
</table>

If we indicate an admissible path in $\Delta_j$ by $[i, k]^{(j)}$, we have the admissible paths: $[0, 0]^{(1)}, [1, 1]^{(2)}, [0, 1]^{(3)}, [1, 2]^{(3)}, [2, 0]^{(3)}$ of index 0, $[1, 2]^{(1)}, [0, 0]^{(2)}$ of index 1 and $[2]^{(2)}$ of index 2.

We can then find the generic decomposition of $V$ of dimension $\beta$: $V = V_t \oplus E_{[1,2]}^{(1)} \oplus E_2^{(2)} \oplus E_{[0,2]}^{(2)}$. 

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From the main theorem 3.2.1 we have the semi-invariants from modules in the tubes:

\[ c^{E^{(1)}}_{[0,2]}, c^{E^{(2)}}_{[1,0]}, c^{E^{(3)}}_{0}, c^{E^{(3)}}_{1}, c^{E^{(3)}}_{2} \].

We also have \( c_0, c_1 \) which are linear combinations of \( c^{V_i} \)’s. By direct calculation the relations are \( c_0 = c^{E^{(1)}}_{[0,2]}, c_1 = c^{E^{(2)}}_{[1,0]} \) and \( c_0 + c_1 = c^{E^{(3)}}_0 c^{E^{(3)}}_1 c^{E^{(3)}}_2 \).
7 Shrinking Functors and Semi-Invariants

The main theorem in this paper gives a set of conditions under which the structure of the ring of semi-invariants for a tubular algebra can be seen in terms of generators and relations. Generally speaking, shrinking functors are one of the tools used with Tubular Algebras and they can be used in the explicit calculation of the generating \( c^V \)'s for semi-invariants. Unfortunately, although the existence of shrinking functors is shown in [32], they are not easy to explicitly find for all cases. In the \( C_4, C_6, C_7, C_8 \) examples from theorem 6.0.22, we do have the exact calculation of the shrinking functors. We also have theory relating the canonical tubular algebras and these \( C \) algebras through the use of shrinking functors.

Let \( A \) be a tubular algebra. As stated earlier, if \( T \) be a tilting \( A \) module, \( T = T_0 \oplus T_p \) with \( T_0 \) a preprojective tilting \( A_0 \) module and \( T_p \) a projective \( A \) module, then \( T \) is called a shrinking module. The corresponding functor \( \Sigma_l = \text{Hom}(T, -) : \mod A \rightarrow \text{End}(T) \) will be called a left shrinking functor.

\( T_p \) is uniquely determined by \( A_0 \) and \( A \), it is just the direct sum of indecomposable projective modules \( P_a \) with \( a \) outside \( A_0 \). When \( A \) is a tubular extension of a tame concealed algebra \( A_0 \) then \( \Sigma_T A = B \) where \( B \) is a tubular extension of \( B_0 = \text{End}(T_0) \)\n
Let \( \sigma_l : K_0(A) \rightarrow K_0(B) \) be the linear transformation induced by \( \Sigma_l \).

We need the dual notion of a right shrinking functor on \( A \) which is uniquely determined by \( A \) and \( A_\infty \). Let \( T \) be a cotilting \( A \) module with \( T = T_q \oplus T_\infty \) where \( T_\infty \) is a preinjective cotilting \( A_\infty \) module and \( T_q \) is an injective \( A \) module. \( T_q \) is the direct sum of all indecomposable injective \( A \) modules \( Q_a \) with \( a \) outside \( A_\infty \). If \( B = \text{End}(T) \). The functor \( \Sigma_r : A - \mod \rightarrow B - \mod \) is given by:

\[
\Sigma_r = \text{Hom}_A(-, T) \cdot D = (TD) \otimes -.
\]
\( \Sigma_r \) is called the **right shrinking functor**.

Since the right shrinking functor is the dual notion of the left shrinking functor any result concerning left shrinking functors, there is a corresponding result concerning right shrinking functors. In particular, we have a linear transformation corresponding to \( \Sigma_r \) namely \( \sigma_r : K_0(A) \to K_0(B) \).

Note that \( \sigma_l(\alpha_\infty) = \alpha_0 + \alpha_\infty = \sigma_r(\alpha_0) \) therefore \( \sigma_l(p\alpha_0 + q\alpha_\infty) = (p + q)\alpha_0 + q\alpha_\infty \) and \( \sigma_r(p\alpha_0 + q\alpha_\infty) = p\alpha_0 + (p + q)\alpha_\infty \). Define \( \sigma_l \) (resp. \( \sigma_r \)) to be the map of indices taking \( \gamma \) to \( \frac{\gamma}{\gamma + 1} \) (resp. \( \gamma + 1 \)).

We use shrinking functors to shift \( T_1 \) to \( T_\gamma \), \( \gamma \in \mathbb{Q}^+ \). So first we must find the simple modules in \( T_1 \). So once we know the modules in \( T_1 \), we can explicitly determine the modules in other \( T_\gamma \).

**Proposition 7.0.5.** For a tubular algebra \( A \), \( \Sigma_r \) and \( \Sigma_l \) preserve the structure of the ring of semi-invariants for any \( \gamma \in \mathbb{Q}^+ \)

**Proof.** Following directly from 5.1, we know that for a tubular algebra \( A \), \( \Sigma_r \) and \( \Sigma_l \) preserve generic decomposition. For \( V \in R(A) \), \( \Sigma_r(V^\perp) = (\Sigma_r(V))^\perp \) and \( \Sigma_l(V^\perp) = (\Sigma_l(V))^\perp \). If \( \langle \langle V, W \rangle \rangle = 0 \), then then by 5.1 we have that

\[
\langle \langle \text{Hom}(T, V), \text{Hom}(T, W) \rangle \rangle = 0.
\]

It follows easily from the main theorem 3.2.1 that shrinking functors preserve the structure of the ring of semi-invariants of tubular algebras. In particular, Lemma 3.2.3 holds. If we have that for \( T_1 \) the dimension of the weight space \( \langle \langle h, - \rangle \rangle \) is 2, that is the weight space is generated by a linear combination of semi-invariants \( c^{V_1,0} \) and \( c^{V_1,1} \). We can see that given a specific \( \gamma \in \mathbb{Q}^+ \), we have that the homogeneous modules in \( T_{\gamma_1} \) are of dimension \( h_1 \). If we consider the weight space, \( \langle \langle h_1, - \rangle \rangle \) then we have that each semi-invariant is a linear combination of monomials
of degree 1, $c_{V,0}^\gamma$ and $c_{V,1}^\gamma$ where $\text{Hom}(T, V_{1,i}) = V_{\gamma,i}$. Since $\dim \text{Hom}(V_{\gamma,j}, V_{\gamma,k}) = \dim \text{Hom}(\text{Hom}(T, V_{1,j}), \text{Hom}(T, V_{1,k})) = \dim \text{Hom}(V_{1,j}, V_{1,k})$.

Example 7.0.6. Continuing 6.1.4, we can observe the effect of the shrinking functors on the simple modules of $T_1$ for $A = C(6)$.

So we need to define the left shrinking functor and the right shrinking functor for $C(6)$. Let the left shrinking functor be $\Sigma_l = \text{Hom}_C(T_l, -)$ with $T_l = T_0 \oplus P_1$ where $T_0 := \oplus_{i=2}^8 P_i \tau^{-2}$. We can identify $\text{End}(T_l)$ with $C$. Therefore we also have a linear transformation $\sigma_l$ corresponding to $\Sigma_l$ which is $\sigma_l = E_C^T t$ where $t$ is the matrix whose first column is $\dim P_1$ and whose $i$th column is $\dim(P_i \tau^{-2})$ for $2 \leq i \leq 8$.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 2 & 2 & 1 & 0 \\
1 & 1 & 1 & 2 & 1 & 2 & 1 & 0 \\
2 & 2 & 2 & 3 & 3 & 4 & 2 & 1 \\
1 & 1 & 1 & 2 & 2 & 3 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 \\
\end{bmatrix}
$$

So then $\alpha_\infty \sigma_l = \alpha_0 + \alpha_\infty$.

Similarly, we can define the right shrinking functor for $C(6)$. Let

$$T_r = (Q_1 \oplus Q_2 \oplus Q_4) \tau \oplus Q_3 \oplus Q_5 \oplus Q_6 \oplus Q_7 \oplus Q_8$$

Notice that $(Q_1 \oplus Q_2 \oplus Q_4) \tau \oplus Q_3 \oplus Q_5 \oplus Q_6$ is in the preinjective component of $C(6)$. We
can identify \( \text{End}(T_r) \) with \( C(6) \) and the right shrinking functor is \( \Sigma_r = \text{Hom}_C(-, T_r) \).

We can again define \( \sigma_r = -E_r^T t \) where \( t \) is the matrix whose \( i \)th column is the dimension of the summand of \( T_r \) corresponding to the vertex \( i \) of \( C(6) \).

\[
t = \begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Now again, take \( \beta = h_1 + e^{(1)}_1 + e^{(2)}_0 + 2e^{(2)}_1 \) as in 6.1.4 and consider a shift to \( T_2 \) that is \( h_2 = \alpha_0 + 2\alpha_\infty \). Let \( \Sigma_r((E_{i}^{(j)}) = F^{(j)}_{i} \) and \( \sigma_r(e^{(j)}_i) = f^{(j)}_{i} \).

\[
\Sigma_r(V_{1,t} \oplus E^{(1)}_{[1,2]} \oplus E^{(2)}_2 \oplus E^{(2)}_{[0,2]}) = V_{2,t} \oplus F^{(1)}_{[1,2]} \oplus F^{(2)}_2 \oplus F^{(2)}_{[0,2]}
\]

\[
\sigma_r(\beta) = (4, 7, 7, 9, 10, 11, 6, 2) = h_2 + f^{(1)}_1 + f^{(2)}_0 + 2f^{(2)}_1
\]

Illustrating that shrinking functors preserve generic decomposition.

Furthermore, we can illustrate the isomorphism of rings of semi-invariants. Notice since generic decomposition is preserved, for both \( \beta \) and \( \sigma_r(\beta) \) the admissible sequences of index 0 are the same. In particular, we have the following labelled polygons:
\[
\begin{align*}
\Delta_1 &= \begin{array}{c} 0 \\ 1 \end{array} & \Delta_2 &= \begin{array}{c} 1 \\ 2 \end{array} & \Delta_3 &= \begin{array}{c} 0 \\ 0 \end{array}.
\end{align*}
\]

If we indicate an admissible path in \(\Delta_j\) by \([i, k]^{(j)}\), we have the admissible paths:

\([0, 0]^{(1)}, [1, 1]^{(2)}, [0, 1]^{(3)}, [1, 2]^{(3)}, [2, 0]^{(3)}\) of index 0, \([1, 2]^{(1)}, [0, 0]^{(2)}\) of index 1 and \([2]^{(2)}\) of index 2.

In the space \(SI_{\text{reg}}(\Lambda, \beta)\) we have the semi-invariants which come from the admissible paths of index zero. Therefore, we have the following generators:

\[
\begin{align*}
&c^{E_{(1)}}_{[0, 2]}, c^{E_{(2)}}_{[1, 0]}, c^{E_{(3)}}_0, c^{E_{(3)}}_1, c^{E_{(3)}}_2.
\end{align*}
\]

We also have \(c_0, c_1\) which are linear combinations of \(c^{V_i}\)’s. By direct calculation the relations are \(c_0 = c^{E_{(1)}}_{[0, 2]}, c_1 = c^{E_{(2)}}_{[1, 0]}\) and \(c_0 + c_1 = c^{E_{(3)}}_0 c^{E_{(3)}}_1 c^{E_{(3)}}_2\).

Consider now, the action of the functor, \(\Sigma_{r}\) under which we have that \(\Sigma_{r}((E_i^{(j)}) = F_i^{(j)}\) and \(\Sigma_{r}(V_{1,t}) = V_{2,t}\). Then clearly we have the following generators:

\[
\begin{align*}
&c^{F_{(1)}}_{[0, 2]}, c^{F_{(2)}}_{[1, 0]}, c^{F_{(3)}}_0, c^{F_{(3)}}_1, c^{F_{(3)}}_2.
\end{align*}
\]

We also have \(c'_0, c'_1\) which are linear combinations of \(c^{V_{2,1}}\)’s. Because all the relations come from extensions, and extensions are preserved by \(\Sigma_{r}\), the relations are \(c_0 = c^{F_{(1)}}_{[0, 2]}, c_1 = c^{F_{(2)}}_{[1, 0]}\) and \(c_0 + c_1 = c^{F_{(3)}}_0 c^{F_{(3)}}_1 c^{F_{(3)}}_2\).
A  Tame concealed to tubular algebras

Given that a tubular algebra is a tubular extension of a tame concealed algebra, then there are only the following possibilities:

<table>
<thead>
<tr>
<th>$A_0$ (where $T = \text{End}(T_{A_0})$)</th>
<th>Tubular type of $T$</th>
<th>Possible tubular type of $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{A}_3$</td>
<td>(2, 2)</td>
<td>any</td>
</tr>
<tr>
<td>$\tilde{A}_4$</td>
<td>(3, 2)</td>
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References


