Monodromy theorems
in the affine setting

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by

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ABSTRACT OF DISSERTATION

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Abstract

My doctoral work revolves around a principle which first appeared in the work of T. Kohno and V. Drinfeld and states, roughly speaking, that quantum groups are natural receptacles for the monodromy representations of certain flat connections of Fuchsian type. It stems more specifically from the monodromic interpretation of quantum Weyl groups obtained by V. Toledano Laredo. This shows that these operators describe the monodromy of the Casimir connection, a flat connection with logarithmic singularities on the root hyperplanes of a semisimple Lie algebra $\mathfrak{g}$, and values in any finite dimensional $\mathfrak{g}$-modules. The main goal of this thesis is to extend these results to the case of an affine Kac-Moody algebra $\mathfrak{g}$.

The method we follow is close to that developed by Toledano Laredo in the semisimple case, and relies on the notion of a quasi–Coxeter quasitriangular quasibialgebra (qCqtqba), which is informally a bialgebra carrying actions of a given generalized braid group and Artin’s braid groups on the tensor products of its modules. A cohomological rigidity result shows that there is at most one such structure with prescribed local monodromies on the classical enveloping algebra $U\mathfrak{g}[[h]]$. It follows that the generalized braid group actions arising from quantum Weyl groups and the monodromy of the Casimir connection are
equivalent, provided the quasi–Coxeter quasitriangular quasibialgebra structure responsible for the former can be transferred from $U_{\hbar}\mathfrak{g}$ to $U\mathfrak{g}[[\hbar]]$. The proof of this fact relies on a modification of the Etingof–Kazhdan quantization functor, and yields an isomorphism between (appropriate completions of) $U_{\hbar}\mathfrak{g}$ and $U\mathfrak{g}[[\hbar]]$ preserving a given chain of subalgebras.
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CHAPTER 1

Introduction

My doctoral work revolves around a principle which first appeared in the work of T. Kohno and V. Drinfeld and states, roughly speaking, that quantum groups are natural receptacles for the monodromy representations of certain flat connections of Fuchsian type [Ko, D2]. It stems more specifically from the monodromic interpretation of quantum Weyl groups obtained by Toledano Laredo in [TL2, TL4]. This shows that these operators describe the monodromy of the Casimir connection, a flat connection with logarithmic singularities on the root hyperplanes of a semisimple Lie algebra $\mathfrak{g}$, and values in any finite dimensional $\mathfrak{g}$-modules. The main goal of my doctoral work is to extend these results to the case of an affine Kac-Moody algebra $\mathfrak{g}$.

1. Monodromy theorems in the affine setting

1.1. Motivation: the Knizhnik–Zamolodchikov equations. Around 1990, Kohno [Ko] and Drinfeld [D2] proved a rather astonishing result, now known as the Kohno-Drinfeld Theorem. The theorem states that quantum groups can be used to describe the monodromy of certain first order Fuchsian PDE’s known as the Knizhnik-Zamolodchikov (KZ) equations. Given a simple Lie algebra $\mathfrak{g}$, a representation $V$ of $\mathfrak{g}$ and a positive integer $n$, the KZ
equations are the following system of PDEs

\[ \frac{\partial \Phi}{\partial z_i} = \hbar \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \Phi \]

where

(i) \( \Phi \) is a function on the configuration space of \( n \) ordered points in \( \mathbb{C} \)

\[ X_n := \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \} \]

with values in \( V^\otimes n \),

(ii) \( \Omega \in \mathfrak{g} \otimes \mathfrak{g} \) is the Casimir tensor,

(iii) \( \hbar \) is a complex deformation parameter.

This system is integrable and equivariant under the natural \( \mathfrak{S}_n \)-action and hence defines a one-parameter family of monodromy representations of Artin’s braid group \( B_n = \pi_1(X_n/\mathfrak{S}_n) \) on \( V^\otimes n \). The Kohno-Drinfeld theorem asserts that this representation is equivalent to the \( R \)-matrix representation of \( B_n \) on \( V^\otimes n \) arising from the quantum group \( U_{\hbar}\mathfrak{g} \). Here \( V \) is a quantum deformation of \( V \), i.e., a \( U_{\hbar}\mathfrak{g} \)-module such that \( V/\hbar V \cong V \).

1.2. The Casimir connection. In subsequent work, J. Millson and V. Toledano Laredo [MTL, TL2], and independently C. De Concini (unpublished) and G. Felder et al. [FMTV], constructed another flat connection \( \nabla_C \), now known as the Casimir connection, which is described as follows. Let \( \mathfrak{h} \subset \mathfrak{g} \) be a Cartan subalgebra, and \( \mathfrak{h}_{\text{reg}} \) be the complement of the root hyper-planes in \( \mathfrak{h} \), that is

\[ \mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in \Phi} \ker(\alpha) \]
where \( \Phi = \{ \alpha \} \subset \mathfrak{h}^* \) is the set of roots of \( \mathfrak{g} \). For a finite-dimensional \( \mathfrak{g} \)-module \( V \), \( \nabla_C \) is the following connection on the trivial vector bundle \( \mathfrak{h}_{\text{reg}} \times V \to \mathfrak{h}_{\text{reg}} \):

\[
\nabla_C = d - \frac{\hbar}{2} \sum_{\alpha \in \Phi^+} \frac{d\alpha}{\alpha} C_\alpha
\]

where

(i) the summation is over a chosen system of positive roots \( \Phi^+ \subset \Phi \);

(ii) \( C_\alpha \) is the Casimir operator of the \( \mathfrak{sl}_2 \)-subalgebra of \( \mathfrak{g} \) corresponding to the root \( \alpha \).

This connection is equivariant under the action of the Weyl group \( W \) of \( \mathfrak{g} \), and gives rise to a one-parameter family of monodromy representations on \( V \) of the generalized braid group

\[ B_\mathfrak{g} = \pi_1(\mathfrak{h}_{\text{reg}}/W) \]

If \( \mathcal{V} \) is a quantum deformation of \( V \), the quantum Weyl group operators \( T_i \in \mathcal{U}_\hbar \mathfrak{g} \) defined by G. Lusztig, A.N. Kirillov–N. Reshetikhin and Y. Soibelman, \([L, KR, S]\) define a representation of \( B_\mathfrak{g} \) on \( \mathcal{V} \). In this setting a Kohno–Drinfeld theorem was obtained by V. Toledano Laredo, stating the equivalence of the above two representations of \( B_\mathfrak{g} \) \([TL1, TL2, TL3, TL4, TL5]\).

The proof of this theorem is inspired by Drinfeld’s proof of the Kohno–Drinfeld theorem \([D2]\). It proceeds along the following lines.
1. INTRODUCTION

(i) The notion of quasi-Coxeter algebras is introduced. Informally speaking, these are algebras which carry a natural $B_g$-action on their finite-dimensional modules. They replace the role of Drinfeld’s quasitriangular quasibialgebras with respect to Artin’s braid group $B_n$. Just as quasitriangular quasi–Hopf algebras are related to the Deligne–Mumford compactification of the configuration space of $n$ points in $\mathbb{C}$, quasi–Coxeter algebras are related to the compactifications of hyperplane arrangements constructed by De Concini–Procesi [DCP1, DCP2].

(ii) It is shown that the monodromy of the Casimir connection and the quantum Weyl group representations of $B_g$ arise from suitable quasi-Coxeter algebra structures on $U[\hbar]$ and $U_h[\hbar]$ respectively.

(iii) It is shown that the quasi-Coxeter structure on $U_h[\hbar]$ can be cohomologically transferred to one on $U[\hbar]$.

(iv) An appropriate deformation cohomology for quasi-Coxeter algebras (called Dynkin diagram cohomology) is introduced. It shows however that quasi-Coxeter algebras on $U[\hbar]$ are not rigid. To remedy this, it is necessary to also take into account the quasitriangular quasibialgebra structure on $U[[\hbar]]$.

(v) The notion of quasi-Coxeter algebras is extended to that of quasi-Coxeter quasitriangular quasibialgebras, whose deformation theory
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is controlled by a bicomplex, namely the *Dynkin-Hochschild* bicomplex. Finally, one proves the uniqueness of quasi-Coxeter quasitriangular quasibialgebra structures on $U\mathfrak{g}[[h]]$.

1.3. Affine extension of rational Casimir connection. It is natural to ask if it is possible to generalize the previous construction to the affine setting. If $\mathfrak{l}$ is a symmetrizable Kac-Moody algebra, and $U_h\mathfrak{l}$ is its quantized universal enveloping algebra, the corresponding quantum Weyl group operators give an action of the braid group of type $\mathfrak{l}$ on any integrable $U_h\mathfrak{l}$-module. In particular, if $\mathfrak{l} =  \hat{\mathfrak{g}}$ is the affinization of a semi–simple Lie algebra $\mathfrak{g}$, we obtain an action of the affine braid group $\hat{B}_\mathfrak{g}$ of type $\mathfrak{g}$ on integrable $\hat{\mathfrak{g}}$-modules.

The purpose of my thesis is to solve the following

**Problem.** Give a monodromic description of the quantum Weyl group action of the generalized affine braid group $\hat{B}_\mathfrak{g}$ on integrable $\hat{\mathfrak{g}}$-modules in category $\mathcal{O}$.

The strategy to attack this problem consists in adapting the machinery developed by Toledano Laredo in [TL4] to the affine setting.

Two major tools involved in the proof of the main theorem in [TL4] are still applicable to the affine case. The theory of quasi-Coxeter algebras refers to a general connected graph and it is therefore possible to consider affine Dynkin diagrams as well. In addition, the Kohno-Drinfeld theorem was extended to all symmetrizable Kac-Moody algebras by P. Etingof and D. Kazhdan in
[EK1, EK2, EK6].

We proceed along the following steps, according to the principle that the quantum Weyl group action is encoded by an appropriate quasi-Coxeter structure.

(i) Show that the quantum Weyl group representation of $\hat{B}_g$ and the $R$-matrix representation of $B_n$ arise from a quasi-Coxeter quasitriangular quasibialgebras structure on the quantum affine algebra $U_\hbar \hat{g}$.

(ii) Show that this structure on $U_\hbar \hat{g}$ can be transferred to $U_\hat{g}[[\hbar]]$.

(iii) Prove the rigidity of quasi-Coxeter quasitriangular quasibialgebra structures on $U_\hat{g}[[\hbar]]$.

(iv) Define an affine extension of the rational Casimir connection, whose monodromy representations are encoded by a quasi-Coxeter structure on $U_\hat{g}[[\hbar]]$

The main difficulty in carrying out the above program lies in step (ii). This step is carried out in the semisimple case by relying on the vanishing of Hochschild cohomology groups, which does not hold for affine Lie algebras. An alternative solution which relies instead on the Etingof-Kazhdan quantization functor is explained in Sections 2 and 3 of this chapter. The ongoing work on the solution to steps (iii) and (iv) is presented in Sections 4 and 5, respectively. The step (i) is trivial.
2. Results - Part I

In order to correctly state the problem in the infinite–dimensional setting, we first need to rephrase the notion of transfer of the quasi–Coxeter structure in a category theoretic way.

We first outline the definition of quasi–Coxeter algebras, following [TL4].

We then give the description of their categorical counterpart.

2.1. Quasi-Coxeter algebras.

2.1.1. Let $D$ be a connected diagram, that is an undirected graph, without loops or multiple edges, $SD(D)$ the collection of subdiagrams of $D$ and $CSD(D)$ the collection of connected subdiagrams. The following combinatorial definitions are due to De Concini–Procesi [DCP1, DCP2]. Two elements $B, B' \in CSD(D)$ are compatible if either $B \subseteq B'$, $B' \subseteq B$ or $B \perp B'$, where the latter means that $B$ and $B'$ are disjoint and that no two vertices $\alpha \in B, \alpha' \in B'$ are connected by an edge of $D$. A maximal nested set on $D$ is a maximal collection of pairwise compatible elements in $CSD(D)$. If $D$ is the Dynkin diagram of type $A_{n-1}$, this notion is equivalent to that of complete bracketing on the non-associative monomial $x_1 \ldots x_n$, by mapping the connected diagram $[i, j]$ to the bracketing $x_1 \ldots x_{i-1}(x_i \ldots x_{j+1})x_{j+2}\ldots x_n$. We denote by $Mns(D)$ the collection of maximal nested sets on $D$ and by $N_D$ the poset of nested sets on $D$, ordered by reverse inclusion.

2.1.2. A $D$-algebra $A$ is an algebra $A$ endowed with a family of subalgebras $\{A_B\}_{B \in CSD(D)}$ satisfying

\[ A_B \subseteq A_{B'} \text{ if } B \subseteq B' \quad \text{and} \quad [A_B, A_{B'}] = 0 \text{ if } B \perp B' \]
For any \( B \in \text{SD}(D) \) with connected components \( \{B_k\} \), we denote by \( A_B \) the subalgebra generated by \( \{A_{B_k}\} \).

A naïve morphism of \( D \)-algebras \( \Psi : A \rightarrow A' \) is a homomorphism such that \( \Psi(A_B) \subseteq A'_B \) for any \( B \in \text{SD}(D) \). This notion is too restrictive for applications however. The correct notion of morphism of \( D \)-algebras \( \Psi : A \rightarrow A' \) is a collection of homomorphisms \( \Psi_F : A \rightarrow A' \) indexed by \( F \in \text{Mns}(D) \), such that \( \Psi_F(A_B) \subseteq A'_B \) for any \( B \in F \).

2.1.3. A \( D \)-bialgebra \( A \) is a \( D \)-algebra and a bialgebra with coproduct \( \Delta \) such that \( \Delta(A_B) \subseteq A_B \otimes A_B \). A \( D \)-quasitriangular quasibialgebra \( A \) is a \( D \)-bialgebra in which every \( A_B \) is equipped with an additional quasitriangular quasibialgebra structure of the form \( (A_B, \Delta, \Phi_B) \), so that the category \( \text{Rep}(A_B) \) of \( A_B \)-modules is a braided tensor category. Moreover, for each inclusion \( B \subseteq B' \), the restriction functor \( \text{Rep}(A_{B'}) \rightarrow \text{Rep}(A_B) \) is endowed with a tensor structure. The latter depends upon the additional choice of a chain:

\[
B' = B_0 \supset B_1 \supset B_2 \cdots \supset B_n = B
\]  

(2.1)

where \(|B_i| = |B'| - i\). These tensor structures compose in the obvious way under the concatenation of chains. For any \( F \in \text{Mns}(D) \), we denote by \( J_F \in A^\otimes 2 \) the structural twist defining the tensor structure on the forgetful functor \( \text{Rep}(A) \rightarrow \text{Vect} = \text{Rep}(A_\emptyset) \).

2.1.4. A quasi-Coxeter quasitriangular quasibialgebra (qCqtqb) of type \( D \) is a \( D \)-quasitriangular quasibialgebra with two additional data. First, for any \( G, F \in \text{Mns}(D) \), we are given a gauge transformation \( \Phi_{FG} \in A \) relating the
corresponding twists, i.e.,

\[ J_G = \Phi_{G,F}^\otimes 2 \cdot J_F \cdot \Delta(\Phi_{G,F})^{-1} \]

These satisfy additional relations, in particular,

**Orientation:** for any \( F, G \in \text{Mns}(D) \),

\[ \Phi_{FG} = \Phi_{G,F}^{-1} \]

**Transitivity:** for any \( F, G, H \in \text{Mns}(D) \),

\[ \Phi_{FH} = \Phi_{FG} \cdot \Phi_{GH} \]

In the following, we shall refer to the gauge transformation \( \Phi_{FG} \) as *Casimir associators*.

We recall that a quasi–Coxeter algebra admits an equivalent description in which the relations satisfied by the elements \( \Phi_{FG} \) are labeled by the two–dimensional faces of the *De Concini–Procesi associahedron*, i.e., the regular CW–complex whose poset of nonempty faces is \( N_D \) [TL4, 3.8].

Assume now that \( D \) is labeled, i.e., an integer \( m_{ij} \in \{2,3,\ldots,\infty\} \) is given for any pair \( \alpha_i, \alpha_j \) of distinct vertices of \( D \), such that \( m_{ij} = m_{ji} \) and \( m_{ij} = 2 \) if and only if \( \alpha_i \) and \( \alpha_j \) are orthogonal. The second piece of data is given by the *local monodromies* \( \{S_\alpha\}_{\alpha \in D} \), which define the action of the generalized braid group \( B_D \) attached to \( D \) and its labeling. For any vertex \( \alpha \in D \), \( S_\alpha \) is not an element of \( A_\alpha \), but rather an element of the completion of \( A_\alpha \) with respect to a fixed full subcategory \( C_\alpha \subset \text{Rep}(A_\alpha) \). In [TL4], \( C_\alpha \) is the subcategory of
finite-dimensional representations. Moreover, the elements \( \{ S_\alpha \} \) and \( \{ \Phi_{G,F} \} \) satisfy the following

**Braid relations:** for any pair of vertices \( \alpha_i \neq \alpha_j \in D \) such that \( 2 < m_{ij} < \infty \), and for any \( G, F \in \text{Mns}(D) \) such that \( \alpha_i \in F, \alpha_j \in G \),

\[
\text{Ad}(\Phi_{G,F})(S_{\alpha_i}) \cdot S_{\alpha_j} \cdots = S_{\alpha_j} \cdot \text{Ad}(\Phi_{G,F})(S_{\alpha_i}) \cdots
\]

where the number of factors on both sides is equal to \( m_{ij} \).

The local monodromies are not assumed to preserve the structural twists. Rather, they satisfy the coproduct identity

\[
\Delta_{J_F}(S_\alpha) = (R^{21}_\alpha)_{J_F} \cdot (S_\alpha \otimes S_\alpha)
\]

for any \( F \in \text{Mns}(D) \) such that \( \alpha \in F \).

**2.2. Categorical analogue of quasi–Coxeter algebras.** It is clear from the previous section that every ingredient in the definition of a quasi–Coxeter quasitriangular quasibialgebra has a categorical interpretation. Moreover, the definition of the local monodromies seems more natural in the categorical setting. Implicitly, we are using this approach to describe a quasi–Coxeter structure on the completion of a \( D \)-bialgebra with respect to a fixed compatible subcategory of representations. A detailed description of quasi–Coxeter categories is presented in Chapter 2.

A quasi–Coxeter braided monoidal category is to a quasi–Coxeter quasitriangular quasibialgebra what a braided monoidal category is to a quasitriangular quasibialgebra. Following this principle, we define a quasi–Coxeter braided
monoidal category of type $D$ as a tuple

$$\mathcal{C} = (\{(\mathcal{C}_B, \otimes_B, \Phi_B, \beta_B)\}, \{(F_{BB'}, J_{BB'}^F)\}, \{\Phi_{FG}\}, \{S_i\})$$

where

- $\{(\mathcal{C}_B, \otimes_B, \Phi_B, \beta_B)\}$ are braided monoidal categories indexed by subdiagrams of $D$;
- $\{(F_{BB'}, J_{BB'}^F)\} : \mathcal{C}_{B'} \to \mathcal{C}_B$ are monoidal functors, corresponding to the inclusion $B' \subset B$, with a tensor structure depending upon the choice of a maximal nested set in $\text{Mns}(B, B')$;
- A collection of invertible endomorphisms $\{S_i \in \text{End}(F_i)\}_{i \in D}$ and natural transformations

$$\Phi_{FG} \in \text{Nat}_{\otimes}( (F_{BB'}, J_{BB'}^G), (F_{BB'}, J_{BB'}^F) )^\times$$

for any $\mathcal{F}, \mathcal{G} \in \text{Mns}(B, B')$, satisfying analogous relations to those described in the previous section.

We then obtain the description of morphisms of quasi–Coxeter structure as the datum of

- for any $B \subseteq D$ a functor $H_B : \mathcal{C}_B \to \mathcal{C}'_B$;
- for any $B \subseteq B'$ and $\mathcal{F} \in \text{Mns}(B, B')$, a natural transformation

$$\begin{array}{ccc}
\mathcal{C}_B & \xrightarrow{H_B} & \mathcal{C}'_B \\
\downarrow F_{BB'} & & \downarrow F'_{BB'} \\
\mathcal{C}_B & \xrightarrow{\gamma_{BB'}^F} & \mathcal{C}'_B
\end{array}$$
satisfying the obvious compatibility condition with respect to the quasi–Coxeter braided monoidal structure.

Interestingly perhaps, these notions can be concisely rephrased in terms of a 2–functor from a combinatorially defined 2–category $qC(D)$ to the 2–categories $\text{Cat}^\otimes$ of tensor categories. The objects of $qC(D)$ are the subdiagrams of the Dynkin diagram $D$ of $B$ and, for two subdiagrams $D' \subseteq D''$, $\text{Hom}_{qC(D)}(D'', D')$ is the fundamental 1–groupoid of the De Concini–Procesi associahedron for the quotient diagram $D''/D'$.

3. Results - Part II

For a semisimple Lie algebra $\mathfrak{g}$, the transfer of structure ultimately rests on the vanishing of the first and second Hochschild cohomology groups of $U\mathfrak{g}[[h]]$, and in particular on the fact that $U_h\mathfrak{g}$ and $U\mathfrak{g}[[h]]$ are isomorphic as algebras, a fact which does not hold for affine Kac–Moody algebras. Rather than the cohomological methods of [TL4], we use instead the Etingof–Kazhdan (EK) quantization functor [EK1, EK2, EK6], which yields a canonical isomorphism

$$\Psi^{EK} : \widehat{U_h\mathfrak{g}} \xrightarrow{\sim} \widehat{U\mathfrak{g}[[h]]}$$

between the completions of $U_h\mathfrak{g}$ and $U\mathfrak{g}[[h]]$ with respect to category $\mathcal{O}$.

Surprisingly perhaps, and despite its functorial construction, the isomorphism $\Psi^{EK}$ does not preserve the inclusions of subalgebras

$$U_{\mathfrak{g}_D} \subseteq U_{\mathfrak{g}} \quad \text{and} \quad U_{\mathfrak{g}_D}[[h]] \subseteq U_{\mathfrak{g}[[h]]}$$

determined by a subdiagram $D$ of the Dynkin diagram of $\mathfrak{g}$, something which is clearly required by the transfer of structure.
The isomorphism $\Psi^{\text{EK}}$ has a categorical origin. Namely, it comes from the equivalence between the category $\mathcal{O}$ of $U\mathfrak{g}[[\hbar]]$ and $U_h\mathfrak{g}$ established by the Etingof–Kazhdan functor. Similarly, the compatibility conditions with the subalgebras $\mathfrak{g}_D$ can be expressed in terms of natural relations with the corresponding restriction functors. It follows that the desired isomorphism needs to be obtained from an equivalence of quasi–Coxeter categories, as discussed in the previous section. Such equivalence is constructed in Chapter 3.

3.1. To outline our construction, which works more generally for an inclusion $(\mathfrak{g}_D, \mathfrak{g}_{D-}, \mathfrak{g}_{D+}) \subset (\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+)$ of Manin triples over a field $k$ of characteristic zero, recall first that the main steps of the EK construction are as follows (cf. Ch.3, Section 1).

(i) One considers the Drinfeld category $\mathcal{D}_\Phi(\mathfrak{g})$ of (deformation) equicontinuous $\mathfrak{g}$–modules, with associativity constraints given by a fixed Lie associator $\Phi$ over $k$. This category can be thought of as a topological analogue of category $\mathcal{O}$ when $\mathfrak{g}$ is the Manin triple associated to a Kac–Moody algebra. It can equivalently be described as the category of Drinfeld–Yetter modules over the Lie bialgebra $\mathfrak{g}_-$. 

(ii) One constructs a tensor functor $F$ from $\mathcal{D}_\Phi(\mathfrak{g})$ to the category $\text{Vect}_k[[\hbar]]$ of topologically free $k[[\hbar]]$–modules. The algebra of endomorphisms $H = \text{End}(F)$ is then a topological bialgebra, i.e., it is endowed with a coproduct $\Delta$ mapping $H$ to a completion of $H \otimes H$.

(iii) Inside $H$, one constructs a subalgebra $U_h\mathfrak{g}_-$ such that $\Delta(U_h\mathfrak{g}_-) \subset U_h\mathfrak{g}_- \otimes U_h\mathfrak{g}_-$, and which is a quantization of $U\mathfrak{g}_-$. The quantum group $U_h\mathfrak{g}$ is then defined as the quantum double of $U_h\mathfrak{g}_-$.
(iv) By construction, $U\mathfrak{g}$ acts and coacts on any $F(V), V \in \mathcal{D}_\Phi(g)$, so that the functor $F$ lifts to $\tilde{F} : \mathcal{D}_\Phi(g) \to \text{Rep}(U_h\mathfrak{g})$ where, by definition, the latter is the category of Drinfeld–Yetter modules over $U_h(\mathfrak{g})$.

(v) Finally, one proves that $\tilde{F}$ is an equivalence of categories.

Since $F$ is isomorphic to the forgetful functor $f : \mathcal{D}_\Phi(g) \to \text{Vect}_{k[[h]]}$ as abelian functors, we obtain the following diagram

$$
\begin{array}{cccccc}
\mathcal{D}_\Phi(g) & \xrightarrow{\tilde{F}} & \mathcal{D}_\Phi(g) & \xrightarrow{f} & \text{Rep}(U_h\mathfrak{g}) \\
\downarrow{f} & & \downarrow{F} & & \downarrow{f_h} \\
\text{Vect}_{k[[h]]} & = & \text{Vect}_{k[[h]]} & = & \text{Vect}_{k[[h]]}
\end{array}
$$

where $f_h : \text{Rep}(U_h\mathfrak{g}) \to \text{Vect}_{k[[h]]}$ is the forgetful functor. The EK isomorphism $\Psi_{\text{EK}}$ is then given by the identifications

$$
\widehat{U\mathfrak{g}}[[h]] := \text{End}(f) \cong \text{End}(F) = \text{End}(f_h \circ \tilde{F}) \cong \text{End}(f_h) =: \widehat{U_h\mathfrak{g}}
$$

It is easy to show that in the Kac–Moody case the construction is compatible with the weight decomposition. Therefore, by restriction of the above diagram to the category $\mathcal{O}$, we get the desired isomorphism.
3.2. Overlaying the above diagrams for an inclusion \( i_D : \mathfrak{g}_D \hookrightarrow \mathfrak{g} \) of Manin triples shows that constructing an isomorphism \( \hat{U}_h \mathfrak{g} \sim \hat{U}_h \mathfrak{g}[\hbar] \) compatible with \( i_D \) may be achieved by filling in the diagram

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{D}_\Phi(\mathfrak{g}) & \xrightarrow{i_D^*} & \mathcal{D}_\Phi(\mathfrak{g}_D) \\
\mathcal{D}_\Phi(\mathfrak{g}) & \xrightarrow{\Gamma} & \mathcal{D}_\Phi(\mathfrak{g}_D) \\
\mathcal{D}_\Phi(\mathfrak{g}_D) & \xrightarrow{f_D} & \mathcal{D}_\Phi(\mathfrak{g}_D) \\
\mathcal{D}_\Phi(\mathfrak{g}_D) & \xrightarrow{\mathcal{F}_D} & \text{Rep}(U_h \mathfrak{g}_D) \\
\mathcal{D}_\Phi(\mathfrak{g}) & \xrightarrow{\mathcal{F}} & \text{Rep}(U_h \mathfrak{g}) \\
\text{Vect}_{k[\hbar]} & \xrightarrow{f} & \text{Vect}_{k[\hbar]} \\
\text{Vect}_{k[\hbar]} & \xrightarrow{f_{D,h}} & \text{Vect}_{k[\hbar]} \\
\end{array}
\end{array}
\]

where \( f_D, f_{D,h} \) are forgetful functors, \( F_D \) the EK functor for \( \mathfrak{g}_D \), and \( i_{D,h} : U_h \mathfrak{g}_D \rightarrow U_h \mathfrak{g} \) is the inclusion derived from the functoriality of the quantization.

To do so, we first construct a relative fiber functor, \textit{i.e.}, a (monoidal) functor \( \Gamma \) on \( \mathcal{D}_\Phi(\mathfrak{g}) \) whose target category is \( \mathcal{D}_\Phi(\mathfrak{g}_D) \) rather than \( \text{Vect}_{k[\hbar]} \), and which is isomorphic as abelian functor to the restriction \( i_D^* \). We then show the existence of a natural transformation between the composition \( \mathcal{F}_D \circ \Gamma \) and \( i_{D,h}^* \circ \mathcal{F} \). Our constructions do not immediately yield a commutative diagram, \textit{i.e.}, the two factorizations \( F \cong F_D \circ \Gamma \) deduced from \( f = f_D \circ i_D^* \) and \( f_{h} = f_{D,h} \circ i_{D,h}^* \) do not coincide, but this can easily be adjusted by using a different identification \( F \cong f \), which amounts to modifying the original EK isomorphism.

3.3. The construction of the functor \( \Gamma \) (Ch.3 Section 2) is very much inspired by [\text{EK1}]. The principle adopted by Etingof and Kazhdan is the following. In a \( k \)-linear monoidal category \( \mathcal{C} \), a coalgebra structure on an
object $C \in \text{Obj}(\mathcal{C})$ induces a tensor structure on the Yoneda functor

$$h_C = \text{Hom}_\mathcal{C}(C, -) : \mathcal{C} \to \text{Vect}_k$$

If $\mathcal{C}$ is braided and $C_1, C_2$ are coalgebra objects in $\mathcal{C}$, then so is $C_1 \otimes C_2$, and there is therefore a canonical tensor structure on $h_{C_1 \otimes C_2}$.

If $\mathfrak{g}$ is finite-dimensional, the polarization $U\mathfrak{g} \simeq M_- \otimes M_+$, where $M_\pm$ are the Verma modules $\text{Ind}_{\mathfrak{g}_\pm}^{\mathfrak{g}} k$, realizes $U\mathfrak{g}$ as the tensor product of two coalgebra objects in $\mathcal{D}_\Phi(U\mathfrak{g}[[\hbar]])$. This yields a tensor structure on the forgetful functor

$$h_{U\mathfrak{g}} : \mathcal{D}_\Phi(U\mathfrak{g})[[\hbar]] \to \text{Vect}_k[[\hbar]]$$

Our starting point is to apply the same principle to the (abelian) restriction functor $i_D^* : \mathcal{D}_\Phi(U\mathfrak{g}) \to \mathcal{D}_\Phi(U\mathfrak{g}_D[[\hbar]])$. We therefore factorize $U\mathfrak{g}$ as a tensor product of two coalgebra objects $L_-, N_+$ in the braided monoidal category of $(\mathfrak{g}, \mathfrak{g}_D)$–bimodules, with associator $(\Phi \cdot \Phi_D^{-1})$, where $\Phi_D^{-1}$ acts on the right. Just as the modules $M_-, M_+$ are related to the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$, $L_-$ and $N_+$ are related to the asymmetric decomposition

$$\mathfrak{g} = \mathfrak{m}_- \oplus \mathfrak{p}_+$$

where $\mathfrak{m}_- = \mathfrak{g}_- \cap \mathfrak{g}_D^1$ and $\mathfrak{p}_+ = \mathfrak{g}_D \oplus \mathfrak{m}_+$. This factorization induces a tensor structure on the functor $\Gamma = h_{L_- \otimes N_+}$, canonically isomorphic to $i_D^*$ through the right $\mathfrak{g}_D$–action on $N_+$. As in [EK1, Part II], this tensor structure can also be defined in the infinite–dimensional case.
3. RESULTS - PART II

3.4. To construct a natural transformation making the following diagram commute

\[
\begin{array}{ccc}
\mathcal{D}_\Phi(\mathfrak{g}) & \xrightarrow{\tilde{F}} & \text{Rep}(U_h \mathfrak{g}) \\
\Gamma & \downarrow & \downarrow (i_D)_h^* \\
\mathcal{D}_\Phi(\mathfrak{g}_D) & \xrightarrow{\tilde{F}_D} & \text{Rep}(U_h \mathfrak{g}_D)
\end{array}
\]

we remark, as suggested to us by P. Etingof, that a quantum analogue \( \Gamma_h \) of \( \Gamma \) can be similarly defined using a quantum version \( L^h_-, N^h_+ \) of the modules \( L_-, N_+ \). The functor \( \Gamma_h = \text{Hom}_{U_h \mathfrak{g}}(L^h_-, \mathfrak{g}, N^h_+, \mathfrak{g}) \) is naturally isomorphic to \( (i_D)_h^* \) as tensor functor, since there is no associator involved on this side. Moreover, an identification

\[
\tilde{F}_D \circ \Gamma \simeq \Gamma_h \circ \tilde{F}
\]

is readily obtained, provided one establishes isomorphisms of \( (U_h \mathfrak{g}, U_h \mathfrak{g}_D) \)–bimodules

\[
\tilde{F}_D \circ \tilde{F}(L_-) \simeq L^h_- \quad \text{and} \quad \tilde{F}_D \circ \tilde{F}(N_+) \simeq N^h_+
\]

3.5. While for \( M_\pm \) it is easy to construct an isomorphism between \( \tilde{F}(M_\pm) \) and the quantum counterparts of \( M_\pm \), the proof for \( L_-, N_+ \) is more involved and is presented in Ch.3, Section 4. It relies on a description of the quantization functor \( F^{\text{EK}} \) in terms of \text{PROP} categories (cf. [EK2, EG]) and the
realization of $L_-, N_+$ as universal objects in a suitable colored PROP describing the inclusion of bialgebras $\mathfrak{g}_D \subset \mathfrak{g}$. This yields in particular a relative extension of the EK functor with input a pair of Lie bialgebras $\mathfrak{a}, \mathfrak{b}$ which is split, \textit{i.e.}, endowed with maps $\mathfrak{a} \xrightarrow{p} \mathfrak{b}$ such that $p \circ i = \text{id}$.

3.6. The previous construction leads to the following

**Theorem.** Let $\mathfrak{g}, \mathfrak{g}_D$ be Manin triples with a finite $\mathbb{N}$–grading on $\mathfrak{g}_-, \mathfrak{g}_D$ and $i_D : \mathfrak{g}_D \subset \mathfrak{g}$ an inclusion of Manin triples compatible with the grading. Then, there exists an algebra isomorphism

$$\Psi : \widehat{U}^\text{EK}_h \mathfrak{g} \to \widehat{U}_{\mathfrak{g}[[\hbar]]}$$

restricting to $\Psi^\text{EK}_D$ on $\widehat{U}^\text{EK}_h \mathfrak{g}_D$, where the completion is given with respect to Drinfeld–Yetter modules.

This result naturally extends to a chain of Manin triples of arbitrary length (Ch.3, Section 5).

**Theorem.**

(i) For any chain of Manin triples

$$\mathcal{C} : \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n \subset \mathfrak{g}$$

there exists an isomorphism of algebras

$$\Psi_{\mathcal{C}} : \widehat{U}^\text{EK}_h \mathfrak{g} \to \widehat{U}_{\mathfrak{g}[[\hbar]]}$$

such that $\Psi_{\mathcal{C}}(\widehat{U}_h^{\text{EK}} \mathfrak{g}_k) = U\mathfrak{g}_k[[\hbar]]$ for any $\mathfrak{g}_k \in \mathcal{C}$. 
(ii) Given two chains $C, C'$, the natural transformation

$$\Phi_{CC'} := \gamma_C^{-1} \circ \gamma_{C'} \in \text{Aut}(F_{EK})$$

satisfies

$$\text{Ad}(\Phi_{CC'}) \Psi_{C'} = \Psi_C$$

A straightforward computation shows that the natural transformations $\Phi$ satisfy the properties of orientation and transitivity required by the Casimir associators.

3.7. Given a maximal nested set $\mathcal{F}$, by iteration of the previous procedure, we obtain an algebra isomorphism $\Psi_{\mathcal{F}} : \hat{U}_h g \rightarrow \hat{U}_g[[h]]$ such that $\Psi_{\mathcal{F}}(\hat{U}_h g_D) = \hat{U}_g D[[h]]$ for any $D \in \mathcal{F}$. This isomorphism of $D_g$-algebras transfers the quasi-Coxeter quasitriangular quasibialgebra structure of $\hat{U}_h g$ to $\hat{U}_g[[h]]$.

**Theorem.** Let $g$ be a symmetrizable Kac–Moody algebra with a fixed $D_g$–structure and $\hat{U}_h g$ the corresponding Drinfeld–Jimbo quantum group with the analogous $D_g$–structure. For any choice of a Lie associator $\Phi$, there exists an equivalence of quasi–Coxeter categories between

$$\mathcal{O} := \left\{ \left( (O^\text{int}_{hB}, \otimes_B, \Phi_B, \sigma R_B) \right), \left( (\Gamma_{BB'}, J_{\mathcal{F}}^{BB'}) \right), \left( \Phi_{g\mathcal{F}} \right), \left\{ S_i, \sigma \right\} \right\}$$

and

$$\mathcal{O}_h := \left\{ \left( (O^\text{int}_{U_h gB}, \otimes_B, \text{id}, \sigma R_B^h) \right), \left( (\Gamma_{BB'}, \text{id}) \right), \left\{ \text{id}, \left\{ S_i^h \right\} \right\} \right\}$$
where $\otimes_B$ denotes the standard tensor product in $\mathcal{O}_{\tilde{g}}^{int}$ and

\[
S_{i,C} = \tilde{s}_i \exp\left(\frac{\hbar}{2} \cdot C_i\right)
\]
\[
\Phi_B \equiv 1 \mod \hbar^2
\]
\[
R_B = \exp\left(\frac{\hbar}{2} \Omega_D\right)
\]
\[
\text{Alt}_2 J^{BB'}_F \equiv \frac{\hbar}{2} \left(\frac{r_{BB'} - r^{21}_{BB'}}{2} - \frac{r_{BB'} - r^{21}_{BB'}}{2}\right) \mod \hbar^2
\]

and $\Phi_{g,F}$, $J^{BB'}_F$ are weight zero elements.

We note that Kac–Moody algebras of finite, affine and hyperbolic type are easily endowed with a $D$–structure. This is canonical in finite and affine type and depends upon an initial choice in the hyperbolic case. For an arbitrary Kac–Moody is not always possible to define a $D$–structure, but it is still unclear how to characterize this special class of Lie algebras (cf. Ch.3, Section 6).

The results of Chapters 2 and 3 are contained in the preprint $[\text{ATL1}]$.

4. Results - Part III

4.1. The quasi–Coxeter braided monoidal category naturally associated to $U_{\hbar}\mathfrak{g}$ admits an equivalent counterpart for $U\mathfrak{g}[[\hbar]]$, as illustrated in the previous section. The next goal is to show that there is at most one quasi–Coxeter braided monoidal category for the classical enveloping algebra $U\mathfrak{g}[[\hbar]]$ with prescribed local monodromies (Chapter 4). This result is needed to show that the quasi–Coxeter braided monoidal category naturally attached to the quantum affine algebra $U_{\hbar}\mathfrak{g}$ is equivalent to the one constructed for $U\mathfrak{g}[[\hbar]]$, which underlies the monodromy of the KZ and Casimir connections of $\mathfrak{g}$. 
4.2. In the semisimple case, a central role is played by the Dynkin–Hochschild bicomplex of a $D$–bialgebra $A$, which controls the deformations of quasi–Coxeter quasitriangular quasibialgebra structures on $A$. In Chapter 4 Section 2, we extend the notion of the Dynkin–Hochschild bicomplex to the case of monoidal $D$–categories, using the language of endomorphisms algebra of monoidal functors and the corresponding cohomology theory. In analogy with the semisimple case, this bicomplex controls the deformations of the quasi–Coxeter structures on the $D$–category.

4.3. Let $\mathfrak{g}$ be a Kac–Moody algebra of affine type with extended Dynkin diagram $D_\mathfrak{g}$. For any proper connected subdiagram $D \subset D_\mathfrak{g}$, let $\mathfrak{g}_D \subset \mathfrak{l}_D \subset \mathfrak{g}$ be the corresponding simple and Levi subalgebras, with decomposition $\mathfrak{l}_D = \mathfrak{g}_D \oplus \mathfrak{c}_D$. Denote by

$$
\Omega_D = x_a \otimes x^a, \quad C_D = x_a \cdot x^a \quad \text{and} \quad \tilde{r}_{\mathfrak{g}_D} = \sum_{\alpha \succ 0: \supp(\alpha) \subseteq D} \frac{(\alpha, \alpha)}{2} \cdot e_\alpha \wedge f_\alpha
$$

where $\{x_a\}_a, \{x^a\}_a$ are dual basis of $\mathfrak{g}_D$ with respect to $\langle \cdot, \cdot \rangle$, the corresponding invariant tensor, Casimir operator and standard solution of the modified classical Yang–Baxter equation for $\mathfrak{g}_D$ respectively.

We then consider two quasi–Coxeter braided monoidal categories of type $D$ of the form:

$$
\left\{ \mathcal{O}_0^{\text{int}}[[\hbar]], \mathcal{O}^{\text{int}}_D[[\hbar]], \otimes, R_D, \Phi_D^{KZ}, S_{i,c}, \Phi^k_{(D;\alpha_i)}, F_{(D;\alpha_i)}^k, J_{(D;\alpha_i)}^k \right\}
$$
for $k = 1, 2$, where $\otimes$ is the standard tensor product in $O_{\mathfrak{g}_D}$, $(F_{(D;\alpha_i)}, J^k_{(D;\alpha_i)})$ are fiber functors,

$$S_{i,c} = \tilde{s}_i \cdot \exp(\hbar/2 \cdot C_i) \quad (4.1)$$

$$R_D = \exp(\hbar/2 \cdot \Omega_D) \quad (4.2)$$

$$\text{Alt}_2 J^k_{(D;\alpha_i)} \equiv \frac{\hbar}{2} \left( \tilde{r}_{\mathfrak{g}_D} - \tilde{r}_{\mathfrak{g}_D \setminus \{\alpha_i\}} \right) \text{ mod } \hbar^2 \quad (4.3)$$

and $\Phi^k_{(D;\alpha_i, \alpha_j)}, J^k_{(D;\alpha_i)}$ are of weight 0, i.e., they properly extend to the corresponding Levi’s subalgebras.

We are explicitly assuming that the braided monoidal structure on $O_{\mathfrak{g}_D}^{\text{int}}$ is induced in both categories by the KZ equations for $\mathfrak{g}_D$. This assumption is not strictly necessary and can be easily dropped by Drinfeld’s uniqueness theorem [D3, Prop. 3.5]. The proof of the rigidity property amounts to match the relative twists $\{J^k_{(D;\alpha_i)}\}$ and the Casimir associator $\{\Phi^k_{(D;\alpha_i, \alpha_j)}\}$.

4.4. The procedure matching the relative twists relies on the essential uniqueness (up to gauge transformations) of solutions of the equation

$$F(\Phi_{VWZ}) \circ J_{V \otimes W, Z} \circ (J_{V,W} \otimes \text{id}_Z) = J_{V,W \otimes Z} \circ (\text{id}_V \otimes J_{W,Z}) \circ (\Phi_D)_{F(V)F(W)F(Z)}$$

in $\text{Hom}_{U_{\mathfrak{g}_D}}(F(V) \otimes F(W) \otimes F(Z), F(V \otimes W \otimes Z))$, where $F = F_{D_{\mathfrak{g}_D}}$ and $V, W, Z \in O_{\mathfrak{g}_D}^{\text{int}}$. 
In the semisimple case (cf. [TL3, Thm. 6.1]), it is possible to construct elements \( u \in (Ug)^{\theta_D}[[h]] \) and \( \lambda \in \Lambda^2 c_D[[h]] \) such that

\[
\exp \lambda \cdot J_2 = u \star J_1
\]

where \( u \star J = \Delta(u^{-1}) \cdot J \cdot u \otimes u \), i.e., the lack of uniqueness of such solutions (up to gauge transformation) is measured by an element \( \lambda \in \Lambda^2 c_D \). We illustrate two cases in which is possible to eliminate \( \lambda \) and obtain the gauge equivalence.

When \( g \) is a finite type Lie algebra and \( |D_g \setminus D| = 1 \), the space \( c_D \) is one dimensional and \( \lambda = 0 \). Therefore, the twists \( J_{(D, \alpha_i)} \) are uniquely determined up to gauge transformation. If we try to apply a similar result when \( g \) is an affine Kac–Moody algebra, we can assert the uniqueness of the twists \( J_{(D, \alpha_i)} \) only for proper subdiagrams \( D \subset D_g \). Indeed, when \( D = D_g \) and \( D' \) is a subdiagram satisfying \( |D_g \setminus D'| = 1 \), \( \dim c_{D'} = 2 \) and \( \lambda \) is not necessarily zero.

### 4.5.

Alternatively, we can take into the account the behavior of the relative twists with respect to the Chevalley involution \( \theta \) of \( g \). It is easy to show that if the relative twists satisfy the additional condition

\[
J^\theta = J^{21}
\]

then \( \lambda \) can be chosen to be zero and it is possible to determine a unique gauge transformation \( u \) satisfying \( u^\theta = u \).
Once again, this condition cannot be directly applied to our case. First of all, we have to make sense of the identity in the setting of category $\mathcal{O}$, where $\theta$ does not define an involution of the category. This can be easily rephrased in terms of a natural compatibility of the monoidal functor with the duality functor in category $\mathcal{O}$.

More importantly, we have to make sure that the relative twists, described in the previous section, satisfy (up to gauge equivalence) the compatibility condition with the Chevalley involution. The solution to this problem presents several difficulties. The current results are presented in Chapter 4 Section 3.

4.6. An easy computation shows that this is true for the Etingof–Kazhdan twist in its finite–dimensional formulation \textit{i.e.}, as a tensor structure on the functor $h_{M_+} \otimes M_-$. This is proved by identifying $J^\theta$ and $J^{21}$ as tensor structures on $h_{M_-} \otimes M_+$ given by the same formulae. This case is extremely special and it can be shown, working modulo $\hbar^3$, that this condition is not satisfied by the twist $J_{\text{EK}}$, in its universal form, or by the relative twist. We observe that in both cases the symmetry between $M_+$ and $M_-$ is broken.

We then observe that the condition $J^\theta = J^{21}$ can be weakened, assuming there exists a natural transformation $\chi$ satisfying $\chi^\theta = \chi^{-1}$ and $\chi \ast J^\theta = J^{21}$. Since the relative twist is gauge equivalent to a composition of twists of Etingof–Kazhdan type, we reduced the problem to find a suitable $\chi$ for $J_{\text{EK}}$. 

This element is easily constructed in the finite–dimensional case, but its existence is not yet proven in the infinite–dimensional case.

We also have to remark here that the construction of the elements $u$ and $\lambda$ above strongly relies on the assumption that the Hochschild cohomology of the completion of $U\mathfrak{g}$ with respect to category $\mathcal{O}$ can be described in terms of the exterior algebra of $\mathfrak{g}$, possibly completed. The result is well–known for $U\mathfrak{g}$ [D3, 2.2], but it is not present in the literature for completed universal enveloping algebras.

4.7. It remains to construct the natural transformation matching the Casimir associators $\{\hat{\Phi}^k_{(D;\alpha_i,\alpha_j)}\}$ and preserving the local monodromies. This amounts to solving the following problem. Given a Cartan 2–cocycle in the Dynkin complex, i.e., a special set of elements $\varphi = \{\varphi_{(D;\alpha_i,\alpha_j)}\} \subset \mathfrak{h}_D$, there exists a Cartan 1–cochain $a$ such that $d^D a = \varphi$. The construction is recursive and the initial choice $a_{(\alpha_i,\alpha_i)} = 0$ allows to preserve the local monodromies.

The original proof of this fact in [TL4] is actually incomplete, since it is implicitly assumed that the fundamental coweights $\{\lambda^\vee_{i,D}\} \subset \mathfrak{h}_D$ do not depends upon $D$ or, in other words, that $\lambda^\vee_{i,D'} \in \mathbb{C}\lambda^\vee_{i,D}$ for $D' \subset D$. We then present a corrected proof of the semisimple case and we extend it to the affine case. Since the proof strongly relies on the combinatorics of the Dynkin diagram of $\mathfrak{g}$, it is actually necessary to go through the entire computation for an affine Dynkin diagram. It is unclear at present whether such result holds for an arbitrary symmetrizable Kac–Moody algebra. The results of Chapter 4
are available in [ATL2].

5. Towards a complete solution

To complete the solution to Problem 1.3, we still need two steps, namely, an affine extension of the rational Casimir connection, and the construction of a monodromic quasi–Coxeter braided monoidal structure on the category of integrable modules in category $\mathcal{O}$ of $U\mathfrak{g}[[h]]$.

5.1. The affine extension of the Casimir connection. A rational Casimir connection was defined for any symmetrisable Kac–Moody algebra in [FMTV]. It has the form

$$\nabla_C^\text{aff} = d - \hbar \sum_{\alpha>0} \frac{d\alpha}{\alpha} :K_{\alpha}:,$$

where $\alpha$ ranges over all positive (real and imaginary) roots, $K_{\alpha}$ is the truncated Casimir operator corresponding to $\alpha$, and

$$:K_{\alpha}: = \sum_{i=1}^{\text{mult}(\alpha)} 2e_{-\alpha}^{(i)}e_{\alpha}^{(i)}$$

is its Wick ordered form.

This ordering makes the sum over $\alpha$ locally finite on a module in category $\mathcal{O}$, but breaks the equivariance of $\nabla_C^\text{aff}$ with respect to the Weyl group of $\mathfrak{g}$. If $\mathfrak{g}$ is affine, this equivariance can however be restored by adding to $\nabla_C^\text{aff}$ a closed one–form $A$ with values in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ [TL6]. The monodromy of the connection $\nabla_C^\text{aff} + A$ then gives rise to a one–parameter family of representations of the generalized affine braid group on $V$. 
5. TOWARDS A COMPLETE SOLUTION

5.2. The Fusion operator. In the semisimple case, the quasi-Coxeter quasitriangular quasibialgebra structure on \( U_\mathfrak{g}[[\hbar]] \) underlying the monodromy of the KZ and Casimir connections is obtained as follows [TL5]. The \( D_\mathfrak{g} \)-algebra structure is the standard one. For any \( D \subseteq D_\mathfrak{g} \), the quasitriangular quasibialgebra structure on \( U_\mathfrak{g}_D[[\hbar]] \) is given by the associator of the KZ equations for \( \mathfrak{g}_D \) and the corresponding \( R \)-matrix. The gauge transformations \( \Phi_{\mathcal{G}_F} \) and local monodromies are given by the De Concini–Procesi associators of the Casimir connection, and its local monodromies. Finally, the twists \( J_F \) are obtained as suitable limits of a joint solution with prescribed asymptotics of the system (cf. [FMTV])

\[
d_\mu J = \frac{\hbar}{2} \sum_{\alpha > 0} \frac{d\alpha}{\alpha} \left[ \Delta(K_\alpha)J - J(K_\alpha^{(1)} + K_\alpha^{(2)}) \right] + (z_1 \text{ad}(d\mu^{(1)}) + z_2 \text{ad}(d\mu^{(2)}))J
\]

\[
d_z J = \hbar \frac{d(z_1 - z_2)}{z_1 - z_2} \Omega_{12} J + (\text{ad}(\mu^{(1)})dz_1 + \text{ad}(\mu^{(2)})dz_2) J
\]

where \( (z, \mu) \in \mathbb{C}^2 \times \mathfrak{h} \) and \( J \) has values in \( U_\mathfrak{g}[[\hbar]] \otimes^2 \).

Following [Ka], we write

\[
K_\alpha = :K_\alpha: + \text{mult}(\alpha) \cdot \nu^{-1}(\alpha)
\]

where \( \nu : \mathfrak{h} \to \mathfrak{h}^* \) is the isomorphism induced by the non-degenerate bilinear form on \( \mathfrak{h} \). Since \( J \) is of weight zero, the above equations can be written with the Wick ordered \( :K_\alpha: \) instead of \( K_\alpha \), and therefore make sense for \( \hat{\mathfrak{g}} \). Moreover, preliminary computations show that the construction of the joint solution \( J \) carries over to affine Lie algebras and, in fact, arbitrary symmetrizable Kac–Moody algebras.
6. Outline of the dissertation

6.1. Chapter 2 is devoted to rephrasing the definition of quasi–Coxeter quasitriangular quasibialgebra in categorical terms. This yields the notion of a quasi–Coxeter category, which is to a generalized braid group $B$ what a braided tensor category is to Artin’s braid groups, and of a quasi–Coxeter braided monoidal category. Both notions will be concisely rephrased in terms of a 2–functor from a combinatorially defined 2–category $qC(D)$ to the 2–categories $\text{Cat}, \text{Cat}^\otimes$ of categories and monoidal categories respectively.

6.2. In Chapter 3, we establish an equivalence of quasi–Coxeter braided monoidal categories for $U\mathfrak{g}[[\hbar]]$ and $U_h\mathfrak{g}$, where $\mathfrak{g}$ is a Kac–Moody algebra endowed with a $D$–algebra structure.

In Section 1, we start with a short review the construction of the Etingof–Kazhdan quantization functor and the associated equivalence of categories, following [EK1, EK6]. In Section 2, we modify this construction by using generalized Verma modules $L_-, N_+$, and we obtain a relative fiber functor $\Gamma : \mathcal{D}_\Phi(\mathfrak{g}) \to \mathcal{D}_{\Phi_D}(\mathfrak{g}_D)$. In Section 3, we define the quantum generalized Verma modules $L^h_-$ and $N^h_+$. Using suitably defined PROPS we then show, in Section 4 that these are isomorphic to the EK quantization of their classical counterparts. In Section 5, we use these results to show that, for any given chain of Manin triples ending in a given $\mathfrak{g}$, there exists a natural transformation that preserves the given chain. Finally, in Section 6, we apply these results to the case of a Kac–Moody algebra $\mathfrak{g}$ and obtain the desired transport of its quasi–Coxeter quasitriangular quasibialgebra structure to the completion of $U\mathfrak{g}[[\hbar]]$ with respect to category $\mathcal{O}$, integrable modules.
6.3. The goal of Chapter 4 is to show that, under appropriate hypotheses, there is at most one quasi–Coxeter braided monoidal category for the classical enveloping algebra $U\mathfrak{g}[[\hbar]]$ with prescribed local monodromies (Thm 4.2).

In Section 1, we review a number of basic notions from \textit{TL4}, in particular that of the Dynkin–Hochschild bicomplex of a $D$–bialgebra $A$ which controls the deformations of quasi–Coxeter quasitriangular quasibialgebra structures on $A$. In Section 2, we extend the notion of the bicomplex to the case of completion of algebras, adopting the language of endomorphisms algebra of monoidal functors and the corresponding cohomology theory. In Section 3 we study invariant properties of the relative twist under the Chevalley involution of $\mathfrak{g}$ and we prove a uniqueness property up to gauge transformation. Section 4 contains the rigidity result, consisting in the construction of a natural transformation matching the Casimir associators and preserving the local monodromies.

6.4. Finally, in Appendix 1 we discuss the isomorphism between the completion of the universal enveloping algebra described in \textit{D5, EK6} and the algebra of the endomorphism of the forgetful functor in category $\mathcal{O}$. In Appendix 3 we use the relative twist to construct an Hopf algebra object in the Drinfeld category of $\mathfrak{g}_D$. The corresponding Radford algebra realizes a \textit{partial} quantization of $U\mathfrak{g}$. 
CHAPTER 2

Quasi–Coxeter Categories

This chapter is devoted to rephrasing the definition of quasi–Coxeter quasitriangular quasibialgebra given in [TL4, Ch. 6] in categorical terms. This yields the notion of a quasi–Coxeter category, which is to a generalized braid group $B$ what a braided tensor category is to Artin’s braid groups, and of a quasi–Coxeter braided monoidal category. Interestingly, both notions can be concisely rephrased in terms of a 2–functor from a combinatorially defined 2–category $qC(D)$ to the 2–categories $\text{Cat}, \text{Cat}^\otimes$ of categories and monoidal categories respectively. The objects of $qC(D)$ are the subdiagrams of the Dynkin diagram $D$ of $B$ and, for two subdiagrams $D’ \subseteq D’’$, $\text{Hom}_{qC(D)}(D’’, D’)$ is the fundamental 1–groupoid of the De Concini–Procesi associahedron for the quotient diagram $D’’/D’$ [DCP2, TL4].

1. Diagrams and nested sets

We review in this section a number of combinatorial notions associated to a diagram $D$, in particular the definition of nested sets on $D$ and of the De Concini–Procesi associahedron of $D$ following [DCP2] and [TL4, Section 2].

1.1. Nested sets on diagrams. By a diagram we shall mean an undirected graph $D$ with no multiple edges or loops. We denote the set of vertices of $D$ by $V(D)$ and set $|D| = |V(D)|$. A subdiagram $B \subseteq D$ is a full subgraph of $D$, that is, a graph consisting of a subset $V(B)$ of vertices of $D$, together
with all edges of $D$ joining any two elements of $\mathcal{V}(B)$. We will often abusively identify such a $B$ with its set of vertices and write $i \in B$ to mean $i \in \mathcal{V}(B)$. We denote by $\text{SD}(D)$ the set of subdiagrams of $D$ and by $\text{CSD}(D)$ the set of connected subdiagrams.\footnote{The definition of \textit{diagram}, provided in [TL4, 2.1], did not allow empty graphs. In Section 2, we need to include the empty set in $\text{SD}(D)$ and we are therefore removing the non–emptiness condition.}

The union $B_1 \cup B_2$ of two subdiagrams $B_1, B_2 \subset D$ is the subdiagram having $\mathcal{V}(B_1) \cup \mathcal{V}(B_2)$ as its set of vertices. Two subdiagrams $B_1, B_2 \subset D$ are \textit{orthogonal} if $\mathcal{V}(B_1) \cap \mathcal{V}(B_2) = \emptyset$ and no two vertices $i \in B_1, j \in B_2$ are joined by an edge in $D$. $B_1$ and $B_2$ are \textit{compatible} if either one contains the other or they are orthogonal.

\textbf{Definition.} A \textit{nested set} on a diagram $D$ is a collection $\mathcal{H}$ of pairwise compatible, connected subdiagrams of $D$ which contains the empty set and the connected components $D_1, \ldots, D_r$ of $D$.

\textbf{1.2. The De Concini–Procesi associahedron.} Let $\mathcal{N}_D$ be the partially ordered set of nested sets on $D$, ordered by reverse inclusion. $\mathcal{N}_D$ has a unique maximal element, consisting of the connected components $\{D_i\}$ of $D$ and the empty set. Its minimal elements are the \textit{maximal nested sets}. We denote the set of maximal nested sets on $D$ by $\text{Mns}(D)$. Every nested set $\mathcal{H}$ on $D$ is uniquely determined by a collection $\{\mathcal{H}_i\}_{i=1}^r$ of nested sets on the connected components of $D$. We therefore obtain canonical identifications

\[\mathcal{N}_D = \prod_{i=1}^r \mathcal{N}_{D_i} \quad \text{and} \quad \text{Mns}(D) = \prod_{i=1}^r \text{Mns}(D_i)\]
The De Concini–Procesi associahedron $\mathcal{A}_D$ is the regular CW–complex whose poset of (nonempty) faces is $\mathcal{N}_D$. It easily follows from the definition that

$$\mathcal{A}_D = \prod_{i=1}^{r} \mathcal{A}_{D_i}$$

It can be realized as a convex polytope of dimension $|D| - r$. For any $\mathcal{H} \in \mathcal{N}_D$, we denote by $\dim(\mathcal{H})$ the dimension of the corresponding face in $\mathcal{A}_D$.

### 1.3. The rank function of $\mathcal{N}_D$

For any nested set $\mathcal{H}$ on $D$ and $B \in \mathcal{H}$, we set $i_{\mathcal{H}}(B) = \bigcup_{i=1}^{m} B_i$ where the $B_i$’s are the maximal elements of $\mathcal{H}$ properly contained in $B$.

**Definition.** Set $\alpha_{\mathcal{H}}^B = B \setminus i_{\mathcal{H}}(B)$. We denote by

$$n(B; \mathcal{H}) = |\alpha_{\mathcal{H}}^B| \quad \text{and} \quad n(\mathcal{H}) = \sum_{B \in \mathcal{H}} (n(B; \mathcal{H}) - 1)$$

An element $B \in \mathcal{H}$ is called *unsaturated* if $n(B; \mathcal{H}) > 1$.

**Proposition.**

(i) For any nested set $\mathcal{H} \in \mathcal{N}_D$,

$$n(\mathcal{H}) = |D| - |\mathcal{H}| = \dim(\mathcal{H})$$

(ii) $\mathcal{H}$ is a maximal nested set if and only if $n(B; \mathcal{H}) = 1$ for any $B \in \mathcal{H}$.

(iii) Any maximal nested set is of cardinality $|D| + 1$.

For any $\mathcal{F} \in \text{Mns}(D)$, $B \in \mathcal{F}$, $i_{\mathcal{F}}(B)$ denotes the maximal element in $\mathcal{F}$ properly contained in $B$ and $\alpha_{\mathcal{F}}^B = B \setminus i_{\mathcal{F}}(B)$ consists of one vertex, denoted $\alpha_{\mathcal{F}}^B$. 
For any \( F \in \text{Mns}(D) \), \( B \in F \), we denote by \( F_B \in \text{Mns}(B) \) the maximal nested set induced by \( F \) on \( B \).

1.4. Quotient diagrams. Let \( B \subseteq D \) be a subdiagram with connected components \( B_1, \ldots, B_m \).

**Definition.** The set of vertices of the quotient diagram \( D/B \) is \( V(D) \setminus V(B) \). Two vertices \( i \neq j \) of \( D/B \) are linked by an edge if and only if the following holds in \( D \)

\[
i \nmid j \quad \text{or} \quad i, j \nmid B_k \quad \text{for some } k = 1, \ldots, m
\]

For any connected subdiagram \( C \subseteq D \) not contained in \( B \), we denote by \( \overline{C} \subseteq D/B \) the connected subdiagram with vertex set \( V(C) \setminus V(B) \).

1.5. Compatible subdiagrams of \( D/B \).

**Lemma.** Let \( C_1, C_2 \nsubseteq B \) be two connected subdiagrams of \( D \) which are compatible. Then

(i) \( \overline{C}_1, \overline{C}_2 \) are compatible unless \( C_1 \perp C_2 \) and \( C_1, C_2 \nperp B_i \) for some \( i \).

(ii) If \( C_1 \) is compatible with every \( B_i \), then \( \overline{C}_1 \) and \( \overline{C}_2 \) are compatible.

In particular, if \( \mathcal{F} \) is a nested set on \( D \) containing each \( B_i \), then \( \overline{\mathcal{F}} = \{ \overline{C} \} \), where \( C \) runs over the elements of \( \mathcal{F} \) such that \( C \nsubseteq B \), is a nested set on \( D/B \).

Let now \( A \) be a connected subdiagram of \( D/B \) and denote by \( \tilde{A} \subseteq D \) the connected subdiagram with vertex set

\[
V(\tilde{A}) = V(A) \bigcup_{i : B_i \nsubseteq V(A)} V(B_i)
\]
Clearly, $A_1 \subseteq A_2$ or $A_1 \perp A_2$ imply $\tilde{A}_1 \subseteq \tilde{A}_2$ and $\tilde{A}_1 \perp \tilde{A}_2$ respectively, so the lifting map $A \to \tilde{A}$ preserves compatibility.

1.6. Nested sets on quotients. For any connected subdiagrams $A \subseteq D/B$ and $C \subseteq D$, we have

$$\tilde{A} = A \quad \text{and} \quad \tilde{C} = C \cup \bigcup_{i: B_i \not\subseteq C} B_i$$

In particular, $\tilde{C} = C$ if, and only if, $C$ is compatible with $B_1, \ldots, B_m$ and not contained in $B$. The applications $C \to \tilde{C}$ and $A \to \tilde{A}$ therefore yield a bijection between the connected subdiagrams of $D$ which are either orthogonal to or strictly contain each $B_i$ and the connected subdiagrams of $D/B$. This bijection preserves compatibility and therefore induces an embedding $\mathcal{N}_{D/B} \hookrightarrow \mathcal{N}_D$.

This yields an embedding

$$\mathcal{N}_{D/B} \times \mathcal{N}_B = \mathcal{N}_{D/B} \times (\mathcal{N}_{B_1} \times \cdots \times \mathcal{N}_{B_m}) \hookrightarrow \mathcal{N}_D$$

with image the poset of nested sets on $D$ containing each $B_i$. Similarly, for any $B \subseteq B' \subseteq B''$, we obtain a map

$$\cup : \mathcal{N}_{B''/B'} \times \mathcal{N}_{B'/B} \hookrightarrow \mathcal{N}_{B''/B}$$

The map $\cup$ restricts to maximal nested sets. For any $B \subset B'$, we denote by $\text{Mns}(B', B)$ the collection of maximal nested sets on $B'/B$. Therefore, for any $B \subset B' \subset B''$, we obtain an embedding

$$\cup : \text{Mns}(B'', B') \times \text{Mns}(B', B) \to \text{Mns}(B'', B)$$
such that, for any $\mathcal{F} \in \text{Mns}(B'', B'), \mathcal{G} \in \text{Mns}(B', B)$,

$$(\mathcal{F} \cup \mathcal{G})_{B'/B} = \mathcal{G}$$

1.7. Elementary and equivalent pairs.

**Definition.** An ordered pair $(\mathcal{G}, \mathcal{F})$ in $\text{Mns}(D)$ is called *elementary* if $\mathcal{G}$ and $\mathcal{F}$ differ by one element. A sequence $\mathcal{H}_1, \ldots, \mathcal{H}_m$ in $\text{Mns}(D)$ is called *elementary* if $|\mathcal{H}_{i+1} \setminus \mathcal{H}_i| = 1$ for any $i = 1, 2, \ldots, m - 1$.

**Definition.** The *support* $\text{supp}(\mathcal{F}, \mathcal{G})$ of an elementary pair in $\text{Mns}(D)$ is the unique unsaturated element of $\mathcal{F} \cap \mathcal{G}$. The *central support* $\mathfrak{z}\text{supp}(\mathcal{F}, \mathcal{G})$ is the union of the maximal elements of $\mathcal{F} \cap \mathcal{G}$ properly contained in $\text{supp}(\mathcal{F}, \mathcal{G})$. Thus

$$\mathfrak{z}\text{supp}(\mathcal{F}, \mathcal{G}) = \text{supp}(\mathcal{F}, \mathcal{G}) \setminus \alpha_{\text{supp}(\mathcal{F}, \mathcal{G})}^\text{supp}(\mathcal{F}, \mathcal{G})$$

**Definition.** Two elementary pairs $(\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')$ in $\text{Mns}(D)$ are *equivalent* if

$$\text{supp}(\mathcal{F}, \mathcal{G}) = \text{supp}(\mathcal{F}', \mathcal{G}')$$

$$\alpha_{\text{supp}(\mathcal{F}, \mathcal{G})}^\text{supp}(\mathcal{F}, \mathcal{G}) = \alpha_{\text{supp}(\mathcal{F}', \mathcal{G}')}^\text{supp}(\mathcal{F}', \mathcal{G}')$$

2. Quasi–Coxeter categories

The goal of this section is to rephrase the notion of quasi–Coxeter quasi-triangular quasibialgebra defined in $[\text{TL4}]$ in terms of categories of representations.
2.1. Algebras arising from fiber functors. We shall repeatedly need the following elementary

**Lemma.** Consider the following situation

\[
\begin{array}{c}
\mathcal{C} \\ H \downarrow \alpha \\ F \\
\downarrow \quad \\
\mathcal{D} \\ \downarrow G \\ \quad \downarrow A
\end{array}
\]

where \( \mathcal{A}, \mathcal{C}, \mathcal{D} \) are additive \( k \)-linear categories, \( F, G, H \) functors, and \( \alpha \) is an invertible transformation. If \( H \) is an equivalence of categories, the map \( \text{End}(G) \to \text{End}(F) \) given by

\[
\{g_W\} \mapsto \{\text{Ad}(\alpha_V^{-1})(g_{H(V)})\}
\]

is an algebra isomorphism.

2.2. \( D \)-categories. Recall [TL4, Section 3] that, given a diagram \( D \), a \( D \)-algebra is a pair \( (A, \{A_B\}_{B \in \text{SD}(D)}) \), where \( A \) is a unital associative \( k \)-algebra and \( \{A_B\}_{B \in \text{SD}(D)} \) is a collection of subalgebras indexed by \( \text{SD}(D) \) and satisfying

\[
A_B \subseteq A_{B'} \quad \text{if} \ B \subseteq B' \quad \text{and} \quad [A_B, A_{B'}] = 0 \quad \text{if} \ B \perp B'
\]

with \( A_\emptyset = k \). Moreover, if \( B \in \text{SD}(D) \) has connected components \( \{B_k\} \), then \( A_B \) is generated by the subalgebras \( \{A_{B_k}\} \).

The following rephrases the notion of \( D \)-algebras in terms of their category of representations.

**Definition.** A \( D \)-category \( \mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\}) \) is the datum of
2. QUASI-COXETER CATEGORIES

- a collection of \( k \)-linear additive categories \( \{C_B\}_{B \subseteq D} \)
- for any pair of subdiagrams \( B \subseteq B' \), an additive \( k \)-linear functor \( F_{BB'} : C_{B'} \to C_B \)
- for any \( B \subset B', B' \perp B'', B', B'' \subset B''' \), a homomorphism of \( k \)-algebras
  \[
  \eta : \text{End}(F_{BB'}) \to \text{End}(F_{(B \cup B'')B'''})
  \]
satisfying the following properties

- For any \( B \subseteq D \), \( F_{BB} = \text{id}_{C_B} \).
- For any \( B \subseteq B' \subseteq B'' \), \( F_{BB'} \circ F_{B'B''} = F_{BB''} \).
- For any \( B' \perp B'' \) and any \( B \subset B' \) and any \( B'' \supset B', B'' \), the following diagram of algebra homomorphisms commutes:

\[
\begin{array}{ccc}
\text{End}(F_{BB'}) & \overset{1 \otimes \eta}{\longrightarrow} & \text{End}(F_{B(B \cup B'')B'''}) \\
\downarrow & & \downarrow \\
\text{End}(F_{BB'}) \otimes \text{End}(F_{B'B''}) & \overset{id \otimes 1}{\longrightarrow} & \text{End}(F_{BB''}) \otimes \text{End}(F_{(B \cup B'')B'''}) \\
\end{array}
\]

**Remark.** We will usually think of \( C_\emptyset \) as a base category and at the functors \( F \) as forgetful functors. Then the family of algebras \( \text{End}(F_B) \) defines, through the morphisms \( \alpha \), a structure of \( D \)-algebra on \( \text{End}(F_D) \). Conversely, every \( D \)-algebra \( A \) admits such a description setting \( C_B = \text{Rep}_A B \) for \( B \neq \emptyset \) and \( C_\emptyset = \text{Vect}_k \), \( F_{BB'} = i_{B' B}^* \), where \( i_{B' B} : A_B \subseteq A_{B'} \) is the inclusion.

\[\text{We adopt the convention } F_B = F_{\emptyset B}.\]
Remark. The conditions satisfied by the maps $\eta$ imply that, for any $B' \perp B''$ and any $B \subset B'$ and $B''' \supset B', B''$, the images in $\text{End}(F_{BB''})$ of the maps

$$\text{End}(F_{BB'}) \to \text{End}(F_{BB'B'B''}) = \text{End}(F_{BB''})$$

$$\text{End}(F_{B(B\cup B'')}) \to \text{End}(F_{B(B\cup B'')F_{B'(B''B)}}) = \text{End}(F_{BB''})$$

commute. In particular, given $B''' = \bigsqcup_{j=1}^r B_j$, with $B_j \in \text{SD}(D)$ pairwise orthogonal, the images in $\text{End}(F_{B'''}B''')$ of the maps

$$\text{End}(F_{B_j}) \to \text{End}(F_{B_jB_j'B''}) = \text{End}(F_{B_j''''})$$

pairwise commute. This condition replaces the $D$–algebra axiom

$$[A_{B'}, A_{B''}] = 0 \quad \forall \quad B' \perp B''$$

which is equivalent to the condition that $A_{B'} \subset A_{B''}$, for any $B \supset B', B''$.

Remark. It may seem more natural to replace the equality of functors $F_{BB'} \circ F_{B'B''} = F_{BB''}$ by the existence of invertible natural transformations $\alpha_{BB''}^{B'B''} : F_{BB'} \circ F_{B'B''} \Rightarrow F_{BB''}$ for any $B \subseteq B'$ satisfying the associativity constraints $\alpha_{BB''}^{B'B''} \circ F_{BB'}(\alpha_{B'B''}^{B''}) = \alpha_{BB''}^{B''} \circ (\alpha_{B'B''}^{B''})_{F_{B'B''}}$ for any $B \subseteq B' \subseteq B'' \subseteq B'''$. A simple coherence argument shows however that this leads to a notion of $D$–category which is equivalent to the one given above.

Remark. The above definition of $D$–category may be rephrased as follows. Let $I(D)$ be the category whose objects are subdiagrams $B \subseteq D$ and morphisms $B' \to B$ the inclusions $B \subset B'$. Then a $D$–category is a functor $\mathcal{C} : I(D) \to \text{Cat}$ which is compatible with the disjoint union of orthogonal subdiagrams.
2.3. **Strict morphisms of $D$–categories.** The interpretation of $D$–categories in terms of $I(D)$ suggests that a morphism of $D$–categories $\mathcal{C}, \mathcal{C}'$ is one of the corresponding functors

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mathcal{C}'} & \text{Cat} \\
I(D) & \downarrow & \\
\mathcal{C}' & \xrightarrow{\mathcal{C}} & \\
\end{array}
\]

This yields the following definition. For simplicity, we shall assume throughout that $\mathcal{C}_\emptyset = \mathcal{C}'_\emptyset$.

**Definition.** A strict morphism of $D$–categories $\mathcal{C}, \mathcal{C}'$ is the datum of

- for any $B \subseteq D$, a functor $H_B : \mathcal{C}_B \to \mathcal{C}'_B$
- for any $B \subseteq B'$, a natural transformation

\[
\begin{array}{ccc}
\mathcal{C}_B' & \xrightarrow{H_B'} & \mathcal{C}'_B' \\
\mathcal{C}_B & \xrightarrow{H_B} & \mathcal{C}'_B \\
\end{array}
\]

such that

- $H_\emptyset = \text{id}$
- $\gamma_{BB} = \text{id}_{H_B}$
- For any $B \subseteq B' \subseteq B''$,

\[
\gamma_{BB''} = \gamma_{BB'} \circ \gamma_{B'B''}
\]
where $\circ$ is the composition of natural transformations defined by

\[
\begin{array}{ccc}
C_{B''} & \longrightarrow & C'_{B''} \\
\downarrow & & \downarrow \\
C_{B'} & \longrightarrow & C'_{B'} \\
\downarrow & & \downarrow \\
C_B & \longrightarrow & C'_B
\end{array}
\] (2.2)

The diagram (2.1), with $B = \emptyset$, induces an algebra homomorphism $\text{End}(F'_{B'}) \to \text{End}(F_B)$ which, by (2.2) is compatible with the maps $\text{End}(F_B) \to \text{End}(F_{B'})$ and $\text{End}(F'_B) \to \text{End}(F'_{B'})$ for any $B \subset B'$. As pointed out in [TL4, 3.3], this condition is too restrictive and will be weakened in the next paragraph.

### 2.4. Morphisms of $D$–categories.

**Definition.** A morphism of $D$–categories $C, C'$, with $C_\emptyset = C'_\emptyset$, is the datum of

- for any $B \subseteq D$ a functor $H_B : C_B \to C'_B$
- for any $B \subseteq B'$ and $F \in \text{Mns}(B, B')$, a natural transformation

\[
\begin{array}{ccc}
C_{B'} & \xrightarrow{H_{B'}} & C'_{B'} \\
\downarrow^F & \nearrow^{\gamma_F} & \downarrow_{F'_{B'}} \\
C_B & \xrightarrow{H_B} & C'_B
\end{array}
\]

such that

- $H_\emptyset = \text{id}$
- $\gamma_{BB}^F = \text{id}_{H_B}$
for any $B \subseteq B' \subseteq B''$, $\mathcal{F} \in \text{Mns}(B, B')$, $\mathcal{G} \in \text{Mns}(B', B'')$,

$$\gamma^\mathcal{F}_{BB'} \circ \gamma^\mathcal{G}_{B'B''} = \gamma^\mathcal{F} \vee \gamma^\mathcal{G}_{BB''}$$

**Remark.** For any $\mathcal{F} \in \text{Mns}(B')$, the natural transformation $\gamma^\mathcal{F}_{B'}$ induces an algebra homomorphism $\Psi^\mathcal{F}_{B'} : \text{End}(F'_{B'}) \to \text{End}(F_{B'})$ such that the following diagram commutes for any $B \in \mathcal{F}$

$$\begin{array}{ccc}
\text{End}(F'_{B'}) & \xrightarrow{\Psi^\mathcal{F}_{B'}} & \text{End}(F_{B'}) \\
\uparrow & & \uparrow \\
\text{End}(F'_{B}) & \xrightarrow{\Psi^\mathcal{F}_{B}} & \text{End}(F_{B})
\end{array}$$

In particular, the collection of homomorphisms $\{\Psi^\mathcal{F}_{B'}\}$ defines a morphism of $D$–algebras $\text{End}(F'_{D}) \to \text{End}(F_{D})$ in the sense of [TL4, 3.4].

**Remark.** The above definition may be rephrased as follows. Let $M(D)$ be the category with objects the subdiagrams $B \subseteq D$ and morphisms $\text{Hom}(B', B) = \text{Mns}(B', B)$, with composition given by union and lift (cf. 1.6). There is a forgetful functor $M(D) \to I(D)$ which is the identity on objects and maps $\mathcal{F} \in \text{Mns}(B', B)$ to the inclusion $B \subseteq B'$. Given two $D$–categories $\mathcal{C}, \mathcal{C}' : I(D) \to \text{Cat}$ a morphism $\mathcal{C} \to \mathcal{C}'$ as defined above coincides with a morphism of the functors $M(D) \to \text{Cat}$ given by the composition

$$M(D) \xrightarrow{\mathcal{C}} I(D) \xrightarrow{\mathcal{C}'} \text{Cat}$$

2.5. Quasi–Coxeter categories.
2. QUASI-COXETER CATEGORIES

**Definition.** A labeling of the diagram $D$ is the assignment of an integer \( m_{ij} \in \{2, 3, \ldots, \infty\} \) to any pair \( i, j \) of distinct vertices of $D$ such that

\[
 m_{ij} = m_{ji} \quad m_{ij} = 2
\]

if and only if \( i \perp j \).

Let $D$ be a labeled diagram.

**Definition.** The *generalized braid group of type $D$ $B_D$* is the group generated by elements $S_i$ labeled by the vertices $i \in D$ with relations

\[
 S_i S_j \cdots = S_j S_i \cdots \quad \text{for any } i \neq j \text{ such that } m_{ij} < \infty.
\]

We shall also refer to $B_D$ as the braid group corresponding to $D$.

**Definition.** A *quasi-Coxeter category of type $D$*

\[
 C = (\{C_B\}, \{F_{BB'}\}, \{\Phi_{FG}\}, \{S_i\})
\]

is the datum of

- a $D$–category $C = (\{C_B\}, \{F_{BB'}\})$
- for any $\mathcal{F}, \mathcal{G}$ in $\text{Mns}(B, B')$, a natural transformation

\[
 \Phi_{\mathcal{F}\mathcal{G}} \in \text{Aut}(F_{BB'})
\]

- for any vertex $i \in \mathcal{V}(D)$, an element

\[
 S_i \in \text{Aut}(F_i)
\]
satisfying the following conditions

- **Orientation.** For any $\mathcal{F}, \mathcal{G} \in \text{Mns}(B, B')$, 
  \[ \Phi_{\mathcal{G}\mathcal{F}} = \Phi_{\mathcal{G}\mathcal{F}}^{-1} \]

- **Transitivity.** For any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \text{Mns}(B, B')$, 
  \[ \Phi_{\mathcal{F}\mathcal{G}} \circ \Phi_{\mathcal{G}\mathcal{H}} = \Phi_{\mathcal{F}\mathcal{H}} \]

- **Factorization.** The assignment 
  \[ \Phi : \text{Mns}(B, B')^2 \to \text{Aut}(F_{B'B'}) \]
  is compatible with the embedding 
  \[ \cup : \text{Mns}(B, B') \times \text{Mns}(B', B'') \to \text{Mns}(B, B'') \]
  for any $B'' \subset B' \subset B$, i.e., the diagram

\[
\begin{array}{ccc}
\text{Mns}(B, B')^2 \times \text{Mns}(B', B'')^2 & \xrightarrow{\Phi \times \Phi} & \text{Aut}(F_{B''B'}) \times \text{Aut}(F_{B'B'}) \\
\cup \downarrow & & \downarrow \\
\text{Mns}(B, B'')^2 & \xrightarrow{\Phi} & \text{Aut}(F_{B''B'})
\end{array}
\]

is commutative.

- **Braid relations.** For any pairs $i, j$ of distinct vertices of $B$, such that $2 < m_{ij} < \infty$, and $\mathcal{F}, \mathcal{G}$ in $\text{Mns}(B)$ such that $i \in \mathcal{F}, j \in \mathcal{G}$, the following relations hold in $\text{End}(F_B)$
  \[ \text{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S_i) \cdot S_j \cdots = S_j \cdot \text{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S_i) \cdots \]
where, by abuse of notation, we denote by $S_i$ its image in $\text{End}(F_B)$ and the number of factors in each side equals $m_{ij}$.

The elements $S_i$ will be referred to as local monodromies.

**Remark.** The factorization property implies the support and forgetful properties of [TL4, 3.12].

- **Support.** For any elementary pair $(\mathcal{F}, \mathcal{G})$ in $\text{Mns}(B, B')$, let $S = \text{supp}(\mathcal{F}, \mathcal{G}), Z = 3 \text{supp}(\mathcal{F}, \mathcal{G}) \subseteq D$. Then there are uniquely defined maximal nested sets $\tilde{\mathcal{F}}, \tilde{\mathcal{G}} \in \text{Mns}(Z, S)$ such that

$$\Phi_{\mathcal{F}\mathcal{G}} = \text{id}_{\mathcal{B}Z} \circ \Phi_{\tilde{\mathcal{F}}\tilde{\mathcal{G}}} \circ \text{id}_{\mathcal{B}'S}$$

where the expression above denotes the composition of natural transformations

- **Forgetfulness.** For any equivalent elementary pairs $(\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')$ in $\text{Mns}(B, B')$

$$\Phi_{\mathcal{F}\mathcal{G}} = \Phi_{\mathcal{F}'\mathcal{G}'}$$
Remark. To rephrase the above definition, consider the 2–category $qC(D)$ obtained by adding to $M(D)$ a unique 2–isomorphism $\varphi_{B'B'}^{BF} : F \to G$ for any pair of 1–morphisms $F, G \in Mns(B', B)$, with the compositions

$$
\varphi_{HG}^{BB'} \circ \varphi_{G}^{BB'} = \varphi_{H}^{BB'} 
$$

and

$$
\varphi_{FG}^{BB'} \circ \varphi_{F}^{BB'} = \varphi_{F'G'}^{BB'}
$$

where $F, G, H \in Mns(B', B), B \subset B' \subseteq B''$ and $F_1, G_1 \in Mns(B'', B'), F_2, G_2 \in Mns(B', B)$. There is a unique functor $qC(D) \to I(D)$ extending $M(D) \to I(D)$, and a quasi–Coxeter category is the same as a 2–functor $qC(D) \to \text{Cat}$ fitting in a diagram

$$
\begin{array}{ccc}
qC(D) & \longrightarrow & \text{Cat} \\
\downarrow & & \downarrow \\
I(D) & \nearrow & \nearrow
\end{array}
$$

Note that, for any $B \subset B'$, the category $\text{Hom}_{qC(D)}(B', B)$ is the 1–groupoid of the De Concini–Procesi associahedron on $B'/B$ [TL4].

2.6. Morphisms of quasi–Coxeter categories.

Definition. A morphism of quasi–Coxeter categories $C, C'$ of type $D$ is a morphism $(H, \gamma)$ of the underlying $D$–categories such that

- For any $i \in B$, the corresponding morphism $\Psi_i : \text{End}(F'_i) \to \text{End}(F_i)$ satisfies
  $$
  \Psi_i(S'_i) = S_i
  $$

- For any elementary pair $(F, G)$ in $Mns(B, B')$,
  $$
  H_B(\Phi_{FG}) \circ \gamma_{B'B'}^{F} \circ (\Psi'_i)_{H_B} = \gamma_{B'B'}^{G}
  $$
in \( \text{Nat}(F_{BB'}' \circ H_{B'}, H_B \circ F_{BB'}) \), as in the diagram

\[
\begin{array}{ccc}
C_B & \xrightarrow{\psi_{F\gamma}} & C_B' \\
\downarrow \Phi & & \downarrow \Phi' \\
C_B & \xrightarrow{\gamma_F} & C_B'
\end{array}
\]

**Remark.** Note that the above condition can be alternatively stated in terms of morphisms \( \Psi_F \) as the identity

\[\Psi_G \circ \text{Ad}(\Phi_{GF}) = \text{Ad}(\Phi'_{GF}) \circ \Psi_F\]

### 2.7. Strict monoidal \( D \)-categories.

**Definition.** A **strict monoidal \( D \)-category** \( \mathcal{C} = (\{C_B\}, \{F_{BB}\}, \{J_{BB}\}) \) is a \( D \)-category \( \mathcal{C} = (\{C_B\}, \{F_{BB}\}) \) where

- for any \( B \subseteq D \), \( (C_B, \otimes_B) \) is a strict monoidal category
- for any \( B \subseteq B' \), the functor \( F_{BB'} \) is endowed with a tensor structure \( J_{BB'} \)

with the additional condition that, for every \( B \subseteq B' \subseteq B'' \), \( J_{BB'} \circ J_{B'B''} = J_{BB''} \).

**Remark.** The tensor structure \( J^B \) induces on \( \text{End}(F_B) \) a coproduct \( \Delta_B : \text{End}(F_B) \to \text{End}(F^2_B) \), where \( F^2_B := \otimes \circ (F_B \boxtimes F_B) \), given by

\[
\{g_V\}_{V \in C_B} \mapsto \{\Delta_B(g)_{VW} := \text{Ad}(J^B_{VW})(g_{V \otimes W})\}_{V, W \in C_B}
\]
Moreover, for any $B \subseteq B'$, $\text{End}(F_B)$ is a subbialgebra of $\text{End}(F_{B'})$, i.e., the following diagram is commutative

$$
\begin{array}{ccc}
\text{End}(F_B) & \xrightarrow{\Delta_B} & \text{End}(F_B^2) \\
\downarrow & & \downarrow \\
\text{End}(F_{B'}) & \xrightarrow{\Delta_{B'}} & \text{End}(F_{B'}^2)
\end{array}
$$

**Remark.** Note that a strict monoidal $D$–category can be thought of as a functor

$$
\mathcal{C} : I(D) \to \text{Cat}_0^\otimes
$$

where $\text{Cat}_0^\otimes$ denotes the 2–category of strict monoidal category, with monoidal functors and gauge transformations.

**Definition.** A morphism of strict monoidal $D$–categories is a natural transformation of the corresponding 2–functors $M(D) \to \text{Cat}_0^\otimes$, obtained by composition with $M(D) \to I(D)$.

### 2.8. Monoidal $D$–categories.

**Definition.** A *monoidal $D$–category*

$$
\mathcal{C} = (\{(C_B, \otimes_B, \Phi_B)\}, \{F_{BB'}\}, \{J_{BB'}^F\})
$$

is the datum of

- A $D$–category $(\{(C_B)\}, \{F_{BB'}\})$ such that each $(C_B, \otimes_B, \Phi_B)$ is a tensor category, with $C_{\emptyset}$ a strict tensor category, i.e., $\Phi_{\emptyset} = \text{id}$.
- for any pair $B \subseteq B'$ and $F \in \text{Mns}(B, B')$, a tensor structure $J_{BB'}^F$ on the functor $F_{BB'} : C_{B'} \to C_B$
with the additional condition that, for any \( B \subseteq B' \subseteq B'' \), \( \mathcal{F} \in \text{Mns}(B'', B') \), \( \mathcal{G} \in \text{Mns}(B', B) \),

\[
J_{BB''}^\mathcal{G} \circ J_{B'B''}^\mathcal{F} = J_{BB''}^{\mathcal{F} \cup \mathcal{G}}
\]

**Remark.** The usual comparison with the algebra of endomorphisms leads
to a collection of bialgebras \((\text{End}(F_B), \Delta_F, \varepsilon)\) endowed with multiple coproducts, indexed by \( \text{Mns}(B) \).

**Remark.** A monoidal \( D \)-category can be thought of as a functor \( M(D) \rightarrow \text{Cat}^\otimes \) fitting in a diagram

\[
\begin{array}{c}
M(D) \\
\downarrow \\
I(D)
\end{array} \Rightarrow \begin{array}{c}
\text{Cat}^\otimes \\
\text{Cat}
\end{array}
\]

Accordingly, a morphism of monoidal \( D \)-categories is one of the corresponding functors.

\[
\begin{array}{c}
M(D) \\
\downarrow \\
\text{Cat}^\otimes
\end{array} \Rightarrow \begin{array}{c}
\text{Cat}^\otimes \\
\text{Cat}^\otimes
\end{array}
\]

**2.9. Fibered monoidal \( D \)-categories.** We shall often be concerned
with monoidal \( D \)-categories such that the underlying categories \((C_B, \otimes_B)\) are
strict, and the functors \( F_{BB'} : (C_{B'}, \otimes_{B'}) \rightarrow (C_B, \otimes_B) \) are tensor functors. This
may be described in terms of the category \( M(D) \) as follows. Let \( \text{DCat}^\otimes \) be
the 2-category of Drinfeld categories, that is strict tensor categories \((C, \otimes)\)
endowed with an additional associativity constraint \( \Phi \) making \((C, \otimes, \Phi)\) a
monoidal category. There is a canonical forgetful 2-functor \( \text{DCat}^\otimes \rightarrow \text{Cat}^\otimes_0 \).
We shall say that a monoidal $D$–category fibers over a strict monoidal $D$–category if the corresponding functor $M(D) \to \text{Cat}^\otimes$ maps into $\text{DCat}^\otimes$ and fits in a commutative diagram

\[
\begin{array}{ccc}
M(D) & \longrightarrow & \text{DCat}^\otimes \\
\downarrow & & \downarrow \\
I(D) & \longrightarrow & \text{Cat}_0^\otimes
\end{array}
\]

In this case, the coproduct $\Delta_F$ on a bialgebra $\text{End}(F_B)$ is the twist of a reference coassociative coproduct $\Delta_0$ on $\text{End}(F_D)$ such that $\Delta_0 : \text{End}(F_B) \to \text{End}(F_B^2)$.

2.10. Braided monoidal $D$–categories.

Definition. A braided monoidal $D$–category

\[\mathcal{C} = \{(C_B, \otimes_B, \Phi_B, \beta_B), \{(F_{BB'}, J_{BB'}^F)\}\}\]

is the datum of

- a monoidal $D$–category $\{(C_B, \otimes_B, \Phi_B), \{(F_{BB'}, J_{BB'}^F)\}\}$
- for every $B \subseteq D$, a commutativity constraint $\beta_B$ in $\mathcal{C}_B$, defining a braiding in $(\mathcal{C}_B, \otimes_B, \Phi_B)$.

Remark. Note that the tensor functors $(F_{BB'}, J_{BB'}^F) : \mathcal{C}_B \to \mathcal{C}_B$ are not assumed to map the commutativity constraint $\beta_{B'}$ to $\beta_B$.

Definition. A morphism of braided monoidal $D$–categories from $\mathcal{C}$ to $\mathcal{C}'$ is a morphism of the underlying monoidal $D$–categories such that the functors $H_B : \mathcal{C}_B \to \mathcal{C}'_B$ are braided tensor functors.
Remark. The fact that $H_B$ are braided tensor functors automatically implies that

$$
\Psi_{\mathcal{F}}(R_B) \circ J_{\mathcal{F}} = (R'_B) \circ J_{\mathcal{F}}
$$

in analogy with \([TL4]\), where $R_B = (12) \circ \beta_B$. We are assuming that $C_\emptyset = C'_\emptyset$ is a symmetric strict tensor category.

2.11. Quasi–Coxeter braided monoidal categories.

Definition. A quasi–Coxeter braided monoidal category of type $D$

$$
\mathcal{C} = (\{(C_B, \otimes_B, \Phi_B, \beta_B)\}, \{(F_{BB'}, J_{BB'}^\mathcal{F})\}, \{\Phi_{F\mathcal{G}}\}, \{S_i\})
$$

is the datum of

- a quasi–Coxeter category of type $D$,

  $$
  \mathcal{C} = (\{C_B\}, \{F_{BB'}\}, \{\Phi_{F\mathcal{G}}\}, \{S_i\})
  $$

- a braided monoidal $D$–category

  $$
  \mathcal{C} = (\{(C_B, \otimes_B, \Phi_B, \beta_B)\}, \{(F_{BB'}, J_{BB'}^F)\})
  $$

satisfying the following conditions

- for any $B \subseteq B'$, and $\mathcal{G}, \mathcal{F} \in \text{Mns}(B, B')$, the natural transformation $\Phi_{F\mathcal{G}} \in \text{Aut}(F_{BB'})$ determines an isomorphism of tensor functors from $(F_{BB'}, J_{BB'}^\mathcal{G})$ to $(F_{BB'}, J_{BB'}^\mathcal{F})$, that is for any $V, W \in C_{B'}$,

  $$(\Phi_{g\mathcal{F}})_{V \otimes W} \circ (J_{BB'}^\mathcal{G})_{V, W} = (J_{BB'}^\mathcal{F})_{V, W} \circ ((\Phi_{g\mathcal{F}})_{V} \otimes (\Phi_{g\mathcal{F}})_W)$$
for any $i \in D$, the following holds:

$$\Delta_i(S_i) = (R_i)_{\text{i}} \cdot (S_i \otimes S_i)$$

A morphism of quasi–Coxeter braided monoidal categories of type $D$ is a morphism of the underlying quasi–Coxeter categories and braided monoidal $D$–categories.

**Remark.** A quasi–Coxeter braided monoidal category of type $D$ determines a 2–functor $qC(D) \to \text{Cat}^\otimes$ fitting in a diagram

$$
\begin{array}{ccc}
qC(D) & \longrightarrow & \text{Cat}^\otimes \\
\downarrow & & \downarrow \\
I(D) & \longrightarrow & \text{Cat}
\end{array}
$$

Note however that this functor does not entirely capture the quasi–Coxeter braided monoidal category since it does not encode the commutativity constraints $\beta_B$ and automorphisms $S_i$. 
CHAPTER 3

An equivalence of quasi–Coxeter categories

In this chapter, we establish an equivalence of quasi–Coxeter braided monoidal categories for \( U_\mathfrak{g}[[\hbar]] \) and \( U_\hbar \mathfrak{g} \), where \( \mathfrak{g} \) is a Kac–Moody algebra endowed with a \( D \)–algebra structure. The construction is based on the equivalence, defined by Etingof and Kazhdan in [EK6], between the category \( \mathcal{O} \) for \( U_\mathfrak{g}[[\hbar]] \) and \( U_\hbar \mathfrak{g} \).

This amounts essentially to building a suitable tensor structure on the restriction functors and natural transformations as described in 2.6. Our construction works more generally for an inclusion \((\mathfrak{g}_D, \mathfrak{g}_D^-, \mathfrak{g}_D^+) \subset (\mathfrak{g}, \mathfrak{g}^-, \mathfrak{g}^+)\) of Manin triples over a field \( k \) of characteristic zero, so we adopt this setting until Section 6, where we apply the results to the Kac–Moody case.

In Section 1, we start with a short review the construction of the Etingof–Kazhdan quantization functor and the associated equivalence of categories, following [EK1, EK6]. In Section 2, we modify this construction by using generalized Verma modules \( L_-, N_+ \), and we obtain a relative fiber functor \( \Gamma : \mathcal{D}_\Phi(\mathfrak{g}) \to \mathcal{D}_{\Phi_D}(\mathfrak{g}_D) \). In Section 3, we define the quantum generalized Verma modules \( L_-^\hbar \) and \( N_+^\hbar \). Using suitably defined PROPs we then show, in Section 4 that these are isomorphic to the EK quantization of their classical counterparts. In Section 5, we use these results to show that, for any given chain of Manin triples ending in a given \( \mathfrak{g} \), there exists a natural transformation...
that preserves the given chain. Finally, in Section 6, we apply these results to the case of a Kac–Moody algebra $\mathfrak{g}$ and obtain the desired transport of its quasi–Coxeter quasitriangular quasibialgebra structure to the completion of $U\mathfrak{g}[[\hbar]]$ with respect to category $O$, integrable modules.

1. Etingof-Kazhdan quantization

We review in this section the results obtained in [EK1, EK6]. More specifically, we follow the quantization of Lie bialgebras given in [EK1, Part II] and the case of generalized Kac–Moody algebras from [EK6].

1.1. Topological vector spaces. The use of topological vector spaces is needed in order to deal with convergence issues related to duals of infinite dimensional vector spaces and tensor product of such spaces.

Let $k$ be a field of characteristic zero with the discrete topology and $V$ a topological vector space over $k$. The topology on $V$ is linear if open subspaces in $V$ form a basis of neighborhoods of zero. Let $V$ be endowed with a linear topology and $p_V$ the natural map

$$p_V : V \to \lim(V/U)$$

where the limit is taken over the open subspaces $U \subseteq V$. Then $V$ is called separated if $p_V$ is injective and complete if $p_V$ is surjective. Throughout this section, we shall call topological vector space a linear, complete, separated topological space.
If $U$ is an open subspace of a topological vector space $V$, then the quotient $V/U$ is discrete. It is then possible, given two topological vector spaces $V$ and $W$, to define the topological tensor product as

$$V \hat{\otimes} W := \lim V/V' \otimes W/W'$$

where the limit is taken over open subspaces of $V$ and $W$. We then denote by $\text{Hom}_k(V, W)$ the topological vector space of continuous linear operators from $V$ to $W$ equipped with the weak topology. Namely, a basis of neighborhoods of zero in $\text{Hom}_k(V, W)$ is given by the collection of sets

$$Y(v_1, \ldots, v_n, W_1, \ldots, W_n) := \{ f \in \text{Hom}_k(V, W) \mid f(v_i) \in W_i, i = 1, \ldots, n \}$$

for any $n \in \mathbb{N}, v_i \in V$ and $W_i$ open subspace in $W$ for all $i = 1, \ldots, n$. In particular, if $W = k$ with the discrete topology, the space $V^* = \text{Hom}_k(V, k)$ has a basis of neighborhoods of zero given by orthogonal complements of finite-dimensional subspaces in $V$. When $V$ is finite-dimensional, $V^*$ coincides with the linear dual and the weak topology coincides with the discrete topology.

The space of formal power series in $\hbar$ with coefficients in a topological vector space $V$, $V[[\hbar]] = V \hat{\otimes} k[[\hbar]]$, is also a complete topological space with a natural structure of a topological $k[[\hbar]]$-module. A topological $k[[\hbar]]$-module is complete if it is isomorphic to $V[[\hbar]]$ for some complete $V$. The additive category of complete $k[[\hbar]]$-module, denoted $\mathcal{A}$, where morphisms are continuous $k[[\hbar]]$-linear maps, has a natural symmetric monoidal structure. Namely, the tensor product on $\mathcal{A}$ is defined to be the quotient of the tensor product $V \hat{\otimes} W$ by the image of the operator $\hbar \otimes 1 - 1 \otimes \hbar$. This tensor product will be still
denoted by $\hat{\otimes}$. There is an extension of scalar functor from the category of topological spaces to $\mathcal{A}$, mapping $V$ to $V[[h]]$. This functor respects the tensor product, i.e., $(V \hat{\otimes} W)[[h]]$ is naturally isomorphic to $V[[h]] \hat{\otimes} W[[h]]$.

1.2. Equicontinuous modules. Fix a topological Lie algebra $\mathfrak{g}$.

**Definition.** Let $V$ be a topological vector space. We say that $V$ is an *equicontinuous* $\mathfrak{g}$-module if:

- the map $\pi_V : \mathfrak{g} \to \text{End}_k V$ is a continuous homomorphism of topological Lie algebras;
- $\{\pi_V(g)\}_{g \in \mathfrak{g}}$ is an equicontinuous family of linear operators, i.e., for any open subspace $U \subseteq V$, there exists $U'$ such that $\pi_V(g)U' \subset U$ for all $g \in \mathfrak{g}$.

Clearly, a topological vector space with a *trivial* $\mathfrak{g}$-module structure is an equicontinuous $\mathfrak{g}$-module. Moreover, given equicontinuous $\mathfrak{g}$-modules $V,W,U$, the tensor product $V \hat{\otimes} W$ has a natural structure of equicontinuous $\mathfrak{g}$-module and $(V \hat{\otimes} W) \hat{\otimes} U$ is naturally identified with $V \hat{\otimes} (W \hat{\otimes} U)$. The category of equicontinuous $\mathfrak{g}$-modules is then a symmetric monoidal category, with braiding defined by permutation of components. We denote this category by $\text{Rep}^{\text{eq}} \mathfrak{g}$.

1.3. Lie bialgebras and Manin triples. A Manin triple is the data of a Lie algebra $\mathfrak{g}$ with

- a nondegenerate invariant inner product $\langle , \rangle$;
- isotropic Lie subalgebras $\mathfrak{g}_\pm \subset \mathfrak{g}$;

such that

- $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector space;
• the inner product defines an isomorphism \( g_+ \simeq g_-^* \);
• the commutator of \( g \) is continuous with respect to the topology obtained by putting the discrete and the weak topology on \( g_-, g_+ \) respectively.

Under these assumptions, the commutator on \( g_+ \simeq g_-^* \) induces a cobracket on \( g_- \), satisfying the cocycle condition [D1]. Therefore, \( g_- \) is canonically endowed with a Lie bialgebra structure. Notice that, in absolute generality, \( g_+ \) is only a topological Lie bialgebra, \( i.e., \delta(g_+) \subset g_+ \otimes g_+ \). The inner product also gives rise to an isomorphism of vector spaces \( g_- \simeq g_-^{**} \simeq g_+^* \), where the latter is the continuous dual, though this isomorphism does not respect the topology. Conversely, every Lie bialgebra \( a \) defines a Manin triple \( (a \oplus a^*, a, a^*) \).

1.4. Verma modules. In [EK1], Etingof and Kazhdan constructed two main examples of equicontinuous \( g \)-modules in the case when \( g \) belongs to a Manin triple \( (g, g_+, g_-) \). The modules \( M_{\pm} \), defined as

\[
M_+ = \text{Ind}_{g_-}^{g_+} k \quad M_- = \text{Ind}_{g_+}^{g_-} k
\]

are freely generated over \( U(g_{\pm}) \) by a vector \( 1_{\pm} \) such that \( g_{\pm} 1_{\pm} \). Therefore, they are naturally identified, as vector spaces, to \( U(g_{\pm}) \) via \( x 1_{\pm} \rightarrow x \). The modules \( M_- \) and \( M_+^* \), with appropriate topologies, are equicontinuous \( g \)-modules.

The module \( M_- \) is an equicontinuous \( g \)-module with respect to the discrete topology. The topology on \( M_+ \) comes, instead, from the identification of vector spaces

\[
M_+ \simeq U(g_+) = \bigcup_{n \geq 0} U(g_+)_n
\]
where $U(\mathfrak{g}_+)_n$ is the set of elements of degree at most $n$. The topology on $U(\mathfrak{g}_+)_n$ is defined through the linear isomorphism

$$\xi_n : \bigoplus_{j=0}^{n} S^j\mathfrak{g}_+ \to U(\mathfrak{g}_+)_n$$

where $S^j\mathfrak{g}_+$ is considered as a topological subspace of $(\mathfrak{g}_-^\otimes j)^*$, embedded with the weak topology. Finally, $U(\mathfrak{g}_+)$ is equipped with the topology of the colimit. Namely, a set $U \subseteq U(\mathfrak{g}_+)$ is open if and only if $U \cap U(\mathfrak{g}_+)_n$ is open for all $n$. With respect to the topology just described, the action of $\mathfrak{g}$ on $M_+$ is continuous.

Consider now the vector space of continuous linear functionals on $M_+$

$$M_+^* = \text{Hom}_k(M_+, k) \simeq \text{colim} \text{Hom}_k(U(\mathfrak{g}_+)_n, k)$$

It is natural to put the discrete topology on $U(\mathfrak{g}_+)_n^*$, since, as a vector space,

$$U(\mathfrak{g}_+)_n^* \simeq \bigoplus_{j=0}^{n} S^j\mathfrak{g}_+^* \simeq \bigoplus_{j=0}^{n} S^j\mathfrak{g}_- \simeq U(\mathfrak{g}_-)_n$$

We then consider on $M_+^*$ the topology of the limit. This defines, in particular, a filtration by subspaces $(M_+^*)_n$ satisfying

$$0 \to (M_+^*)_n \to M_+^* \to (U(\mathfrak{g}_+)_n)^* \to 0$$

and such that $M_+^* = \text{lim} M_+^*/(M_+^*)_n$. The topology of the limit on $M_+^*$ is, in general, stronger than the weak topology of the dual. Since the action of $\mathfrak{g}$ on $M_+$ is continuous, $M_+^*$ has a natural structure of $\mathfrak{g}$–module. In particular, this is an equicontinuous $\mathfrak{g}$–action.
1.5. Drinfeld category. The natural embedding

\[ g_- \otimes g^*_+ \subset \text{End}_k(g_-) \]

induces a topology on \( g_- \otimes g^*_+ \) by restriction of the weak topology in \( \text{End}_k(g_-) \). With respect to this topology, the image of \( g_- \otimes g^*_+ \) is dense in \( \text{End}_k(g_-) \) and the topological completion \( g_- \otimes g^*_+ \) is identified with \( \text{End}_k(g_-) \). Under this identification, the identity operator defines an element \( r \in g_- \otimes g^*_+ \).

Given two equicontinuous \( g \)-modules \( V, W \), the map

\[ \pi_V \otimes \pi_W : g_- \otimes g^*_+ \to \text{End}_k(V \hat{\otimes} W) \]

naturally extends to a continuous map \( g_- \hat{\otimes} g^*_+ \to \text{End}_k(V \hat{\otimes} W) \). Therefore, the Casimir operator

\[ \Omega = r + r^{op} \in g_- \hat{\otimes} g^*_+ \oplus g^*_+ \hat{\otimes} g_- \]

defines a continuous endomorphism of \( V \hat{\otimes} W \), \( \Omega_{VW} = (\pi_V \otimes \pi_W)(\Omega) \), commuting with the action of \( g \).

Following [D2], it is possible to define a structure of braided monoidal category on the category of deformed equicontinuous \( g \)-module, depending on the choice of a Lie associator \( \Phi \), the bifunctor \( \hat{\otimes} \) and the Casimir operator \( \Omega \). The commutativity constraint is explicitly defined by the formula

\[ \beta_{VW} = (12) \circ e^{\frac{h}{2} \Omega_{VW}} \in \text{Hom}_g(V \hat{\otimes} W, W \hat{\otimes} V)[[h]] \]
We denote this braided tensor category braided tensor category $\mathcal{D}_\Phi(U\mathfrak{g})$. The category of equicontinuous $\mathfrak{g}$–modules is equivalent to the category of Yetter-Drinfeld module over $\mathfrak{g}_-$, $\mathcal{YD}(\mathfrak{g}_-)$. The equivalence holds at the level of tensor structure induced by the choice of an associator $\Phi$,

$$\mathcal{D}_\Phi(U\mathfrak{g}) \simeq \mathcal{YD}_\Phi(U\mathfrak{g}_-[[\hbar]])$$

1.6. Verma modules. The modules $M_\pm$ are identified, as vector spaces, with the enveloping universal algebras $U\mathfrak{g}_\pm$. Their comultiplications induce the $U\mathfrak{g}$–intertwiners $i_\pm : M_\pm \rightarrow M_\pm \hat{\otimes} M_\pm$, mapping the vectors $1_\pm$ to the $\mathfrak{g}_+$-invariant vectors $1_\pm \otimes 1_\pm$.

For any $f, g \in M_+^*$, consider the linear functional $M_+ \rightarrow k$ defined by $v \mapsto (f \otimes g)(i_+(v))$. This functional defines a map $i_+^* : M_+^* \otimes M_+^* \rightarrow M_+^*$, which is continuous and extends to a morphism in $\text{Rep} \, \mathfrak{g}[[\hbar]]$, $i_+^* : M_+^* \hat{\otimes} M_+^* \rightarrow M_+^*$. The pairs $(M_-, i_-)$ and $(M_+^*, i_+^*)$ form, respectively, a coalgebra and an algebra object in $\mathcal{D}_\Phi(U\mathfrak{g})$.

For any $V \in \mathcal{D}_\Phi(U\mathfrak{g})$, the vector space $\text{Hom}_\mathfrak{g}(M_-, M_+^* \hat{\otimes} V)$ is naturally isomorphic to $V$, as topological vector space, through the isomorphism $f \mapsto (1_+ \otimes 1)f(1_-)$.

1.7. The fiber functor and the EK quantization. We will now recall the main results from [EK1, EK2]. Where no confusion is possible, we will abusively denote $\hat{\otimes}$ by $\otimes$. Let then $F$ be the functor

$$F : \mathcal{D}_\Phi(U\mathfrak{g}) \rightarrow \mathcal{A} \quad F(V) = \text{Hom}_{\mathcal{D}_\Phi(U\mathfrak{g})}(M_-, M_+^* \otimes V)$$
There is a natural transformation

\[ J \in \text{Nat}(\otimes \circ (F \boxtimes F), F \circ \otimes) \]

defined, for any \( v \in F(V), w \in F(W) \), by

\[ J_{WW}(v \otimes w) = (i^\lor \otimes 1 \otimes 1)A^{-1}\beta_{23}^{-1}A(v \otimes w)i_\cdot \]

where \( A \) is defined as a morphism

\[
(V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \to V_1 \otimes ((V_2 \otimes V_3) \otimes V_4)
\]

by the action of \((1 \otimes \Phi_{2,3,4})\Phi_{1,2,34}\).

**Theorem.** The natural transformation \( J \) is invertible and defines a tensor structure on the functor \( F \).

The tensor functor \((F, J)\) is called fiber functor. The algebra of endomorphisms of \( F \) is therefore naturally endowed with a topological bialgebra structure, as described in the previous section.\(^1\)

The object \( F(M_-) \in \mathcal{A} \) has a natural structure of Hopf algebra, defined by the multiplication

\[ m : F(M_-) \otimes F(M_-) \to F(M_-) \quad m(x, y) = (i^\lor \otimes 1)\Phi^{-1}(1 \otimes y)x \]

\(^1\)By topological bialgebra we do not mean topological over \( k[[\hbar]] \). We are instead referring to the fact that the algebra \( \text{End}(F) \) has a natural comultiplication \( \Delta : \text{End}(F) \to \text{End}(F^2) \), where \( \text{End}(F^2) \) can be interpreted as an appropriate completion of \( \text{End}(F)^{\otimes 2} \).
and the comultiplication

$$\Delta : F(M_-) \to F(M_-) \otimes F(M_-) \quad \Delta(x) = J^{-1}(F(i_-)(x))$$

The algebra $F(M_-)$ is naturally isomorphic as a vector space with $M_-[[h]] \simeq U_{g_-}[[h]]$ and

**Theorem.** The algebra $U^E_{K} g_- = F(M_-)$ is a quantization of the algebra $U_{g_-}$.

In [EK2], it is shown that this construction defines a functor

$$Q^E_{K} : LBA(k) \to QUE(K)$$

where $LBA(k)$ denotes the category of Lie bialgebras over $k$ and $QUE(K)$ denotes the category of quantum universal enveloping algebras over $K = k[[h]]$. Another important result in [EK2] states the invertibility of the functor $Q^E_{K}$.

The map

$$m_- : U^E_{K} g_- \to \text{End}(F) \quad m_-(x)v = (i_-^\vee \otimes 1)\Phi^{-1}(1 \otimes v)x$$

where $V \in \mathcal{YD}_\Phi(U_{g_-}[[h]])$ and $v \in F(V)$, is, indeed, an inclusion of Hopf algebras. The map $m_-$ defines an action of $U^E_{K} g_-$ on $F(V)$. Moreover, the map

$$F(V) \to F(M_-) \otimes F(V) \quad v \mapsto R_J(1 \otimes v)$$

where $R_J$ denotes the twisted $R$–matrix, defines a coaction of $U^E_{K} g_-$ on $F(V)$ compatible with the action, therefore
Theorem. The fiber functor $F : \mathcal{YD}_\Phi(U g_-[[h]]) \rightarrow A$ lifts to an equivalence of braided tensor categories

$$\tilde{F} : \mathcal{YD}_\Phi(U g_-[[h]]) \rightarrow \mathcal{YD}(U^{EK}_h g_-)$$

1.8. Generalized Kac-Moody algebras. Denote by $k$ a field of characteristic zero. We recall definitions from [Ka] and [EK6]. Let $A = (a_{ij})_{i,j \in I}$ be an $n \times n$ symmetrizable matrix with entries in $k$, i.e. there exists a (fixed) collection of nonzero numbers $\{d_i\}_{i \in I}$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of $A$. It means that $\mathfrak{h}$ is a vector space of dimension $2n - \text{rank}(A)$, $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{h_1, \ldots, h_n\} \subset \mathfrak{h}$ are linearly independent, and $(\alpha_i, h_j) = a_{ji}$.

Definition. The Lie algebra $\tilde{g} = \tilde{g}(A)$ is generated by $\mathfrak{h}, \{e_i, f_i\}_{i \in I}$ with defining relations

$$[h, h'] = 0 \quad h, h' \in \mathfrak{h}; \quad [h, e_i] = (\alpha_i, h)e_i$$

$$[h, f_i] = -(\alpha_i, h)f_i; \quad [e_i, f_j] = \delta_{ij}h_i$$

There exists a unique maximal ideal $\mathfrak{r}$ in $\tilde{g}$ that intersects $\mathfrak{h}$ trivially. Let $g := \tilde{g}/\mathfrak{r}$. The algebra $g$ is called generalized Kac-Moody algebra. The Lie algebra $g$ is graded by principal gradation $\deg(e_i) = 1, \deg(f_i) = -1, \deg(h) = 0$, and the homogenous component are all finite-dimentional.

Let us now choose a non–degenerate bilinear symmetric form on $\mathfrak{h}$ such that $\langle h, h_i \rangle = d_i^{-1}(\alpha_i, h)$. Following [Ka], there exists a unique extension of the form $\langle, \rangle$ to an invariant symmetric bilinear form on $\tilde{g}$. For this extension,
one gets $\langle e_i, f_j \rangle = \delta_{ij}d_i^{-1}$. The kernel of this form on $\tilde{g}$ is $r$, therefore it descends to a non-degenerate bilinear form on $g$.

Let $n_\pm, b_\pm$ be the nilpotent and the Borel subalgebras of $g$, i.e., $n_\pm$ are generated by $\{e_i\}, \{f_i\}$, respectively, and $b_\pm := n_\pm \oplus h$. Since $[n_\pm, h] \subset n_\pm$, we get Lie algebra maps $\bar{\cdot} : b_\pm \to h$ and we can consider the embeddings of Lie subalgebras $\eta_\pm : b_\pm \to g \oplus h$ given by

$$\eta_\pm(x) = (x, \pm \bar{x})$$

Define the inner product on $g \oplus h$ by $\langle \cdot, \cdot \rangle_{g \oplus h} = \langle \cdot \rangle_g - \langle \cdot \rangle_h$.

**Proposition.** The triple $(g \oplus h, b_+, b_-)$ with inner product $\langle \cdot, \cdot \rangle_{g \oplus h}$ and embeddings $\eta_\pm$ is a graded Manin triple.

Under the embeddings $\eta_\pm$, the Lie subalgebras $b_\pm$ are isotropic with respect to $\langle \cdot, \cdot \rangle_{g \oplus h}$. Since $\langle \cdot \rangle_g$ and $\langle \cdot \rangle_h$ are invariant symmetric non-degenerate bilinear form, so is $\langle \cdot, \cdot \rangle_{g \oplus h}$.

The proposition implies that $g \oplus h, b_+, b_-$ are Lie bialgebras. Moreover, $b_+^* \simeq b_-^{\text{cop}}$ as Lie bialgebras (where $b_+^* := \bigoplus (b_+)_n^*$ denotes the restricted dual space and by $\text{cop}$ we mean the opposite cocommutator). The Lie bialgebra structures on $b_\pm$ are then described by the following formulas:

$$\delta(h) = 0, \quad h \in h \subset b_\pm;$$

$$\delta(e_i) = \frac{d_i}{2}(e_i \otimes h_i - h_i \otimes e_i) = \frac{d_i}{2}e_i \wedge h_i; \quad \delta(f_i) = \frac{d_i}{2}f_i \wedge h_i$$
The Lie subalgebra \( \{(0, h) \mid h \in \mathfrak{h} \} \) is therefore an ideal and a coideal in \( \mathfrak{g} \oplus \mathfrak{h} \). Thus, the quotient \( \mathfrak{g} = (\mathfrak{g} \oplus \mathfrak{h})/\mathfrak{h} \) is also a Lie bialgebra with Lie subbialgebras \( \mathfrak{b}_\pm \) and the same cocommutator formulas.

1.9. Quantization of Kac–Moody algebras and category \( \mathcal{O} \). In [EK6], Etingof and Kazhdan proved that, for any symmetrizable irreducible Kac-Moody algebra \( \mathfrak{g} \), the quantization \( U_{\hbar}^{\mathrm{EK}} \mathfrak{g} \) is isomorphic with the Drinfeld–Jimbo quantum group \( U_{\hbar} \mathfrak{g} \).

In particular, they construct an isomorphism of Hopf algebras \( U_{\hbar} \mathfrak{b}_+ \simeq U_{\hbar}^{\mathrm{EK}} \mathfrak{b}_+ \), inducing the identity on \( U_{\hbar}[[[\hbar]]] \), where \( \mathfrak{b}_+ \) is the Borel subalgebra and \( \mathfrak{h} \) is the Cartan subalgebra of \( \mathfrak{g} \). Thanks to the compatibility with the doubling operations

\[
\mathcal{D} U_{\hbar}^{\mathrm{EK}} \mathfrak{b}_+ \simeq U_{\hbar}^{\mathrm{EK}} \mathfrak{b}_+
\]

proved by Enriquez and Geer in [EG], the isomorphism for the Borel subalgebra induces an isomorphism \( U_{\hbar} \mathfrak{g} \simeq U_{\hbar}^{\mathrm{EK}} \mathfrak{g} \).

Recall that the category \( \mathcal{O} \) for \( \mathfrak{g} \), denoted \( \mathcal{O}_\mathfrak{g} \) is defined to be the category of all \( \hbar \)-diagonalizable \( \mathfrak{g} \)-modules \( V \), whose set of weights \( P(V) \) belong to a union of finitely many cones

\[
\mathcal{D}(\lambda_s) = \lambda_s + \sum_i \mathbb{Z}_{\geq 0} \alpha_i \quad \lambda_s \in \mathfrak{h}^*, \; s = 1, \ldots, r
\]
and the weight subspaces are finite-dimensional. We denote by $O_g[[h]]$ the category of deformation $g$-representations, i.e., representations of $g$ on topologically free $k[[h]]$-modules with the above properties (with weights in $h^*[[h]]$).

In a similar way, one defines the category $O_{U_h g}$: it is the category of $U_h g$-modules which are topologically free over $k[[h]]$ and satisfy the same conditions as in the classical case.

The morphism of Lie bialgebras

$$\mathcal{Db}_+ \to g \simeq \mathcal{Db}_+/(h \simeq h^*)$$

gives rise to a pullback functors

$$O_g \to \mathcal{YD}(Ub_+) \quad O_{g,\Phi}[[h]] \to \mathcal{YD}_{\Phi}(Ub_+[[h]])$$

where $O_{g,\Phi}$ denotes the category $O_g$ with the tensor structure of the Drinfeld category. Similarly, the morphism of Hopf algebras

$$DU_h^{EK} b_+ \to U_h^{EK} g \simeq U_h g$$

gives rise to a pullback functor

$$O_{U_h g} \to \mathcal{YD}(U_h^{EK} b_+)$$

**Theorem.** The equivalence $\tilde{F}$ reduces to an equivalence of braided tensor categories

$$\tilde{F}_O : O_{g,\Phi}[[h]] \to O_{U_h g}$$
which is isomorphic to the identity functor at the level of $\mathfrak{h}$-graded $k[[\hbar]]$-modules.

1.10. **The isomorphism** $\Psi^{EK}$. In [EK6], Etingof–Kazhdan showed that the equivalence $\tilde{F}$ induces an isomorphism of algebras

$$\Psi^{EK} : \widehat{U}\mathfrak{g}[[\hbar]] \rightarrow \widehat{U}_h\mathfrak{g}$$

where

$$\widehat{U}\mathfrak{g} = \lim U_\beta \quad U_\beta = U\mathfrak{g}/I_\beta, \beta \in \mathbb{N}^I$$

$I_\beta$ being the left ideal generated by elements of weight less or equal $\beta$ (analogously for $\widehat{U}_h\mathfrak{g}$, cf. [EK6, Sec. 4] and [D5, 8.1]).

**Proposition.** The isomorphism $\Psi^{EK}$ coincides with the isomorphism induced by the equivalence $\tilde{F}_\mathcal{O}$, as explained in Section 2.1.

**Proof.** The identification of the two isomorphism is constructed in the following way:

(a) First, we show that there is a canonical map

$$\text{End}(\mathcal{f}_\mathcal{O}) \rightarrow C_{\text{End}(\widehat{U})}(\text{End}_\mathfrak{g}(\widehat{U}))$$

(b) There is a *canonical* multiplication in $\widehat{U}$, so that

(i) There is a canonical map

$$C_{\text{End}(\widehat{U})}(\text{End}_\mathfrak{g}(\widehat{U})) \rightarrow \widehat{U}$$
(ii) For every $V \in \mathcal{O}$ the action of $U\mathfrak{g}$ lifts to an action of $\hat{U}$

$U\mathfrak{g} \longrightarrow \text{End}(V) \quad \downarrow \quad \hat{U}$

(c) It defines a map $\hat{U} \to \text{End}(\mathcal{F})$ and we have an isomorphism of algebras

$\hat{U} \simeq \text{End}(\mathcal{F})$

A detailed proof of the equivalence is given in Appendix A Section 1. □

If $\mathfrak{g}$ is a semisimple Lie algebra, the equivalence of categories $\tilde{F}$ leads to an isomorphism of algebras

$U(\mathcal{D}\mathfrak{b}_+)[[h]] \simeq \mathcal{D}U^{\text{EK}}_h \mathfrak{b}_+ \longrightarrow U\mathfrak{g}[[h]] \simeq U_h\mathfrak{g}$

which is the identity modulo $h$. Toledano Laredo proved in [TL4, Prop. 3.5] that such an isomorphism cannot be compatible with all the isomorphisms

$U\mathfrak{sl}_2^\alpha[[h]] \simeq U_h\mathfrak{sl}_2^\alpha \quad \forall \alpha$

where $\{\alpha_i\}$ are the simple roots of $\mathfrak{g}$. This amounts to a simple proof that the isomorphism $\Psi^{\text{EK}}$ cannot be, in general, an isomorphism of $D$–algebras.

2. A relative Etingof–Kazhdan functor

2.1. In this section, we consider a split inclusion of Manin triples

$i_D : (\mathfrak{g}_D, \mathfrak{g}_{D,+}, \mathfrak{g}_{D,-}) \hookrightarrow (\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$
We then define a relative version of the Verma modules $M_{\pm}$, and use them to prove the following theorem.

**Theorem.** There is a tensor functor

$$
\Gamma : \mathcal{D}_{\Phi}(Ug) \rightarrow \mathcal{D}_{\Phi_D}(Ug_D)
$$

canonically isomorphic, as abelian functor, to the restriction functor $i^{\ast}_D$.

**2.2. Split inclusions of Manin triples.**

**Definition.** An embedding of Manin triples $i : (g_D, g_D, -, g_D, +) \rightarrow (g, g^-, g^+)$ is a Lie algebra homomorphism $i : g_D \rightarrow g$ preserving inner products, and such that $i(g_D, \pm) \subset g^{\pm}$.

Denote the restriction of $i$ to $g_{D, \pm}$ by $i_{\pm}$. $i_{\pm}$ give rise to maps $p_{\pm} = i^{\ast}_{\mp} : g_{\pm} \rightarrow g_{D, \pm}$, defined via the identifications $g_{\pm} \simeq g^*_{\pm}$ and $g_{D, \pm} \simeq g^*_{D, \mp}$ by

$$
\langle p_{\pm}(x), y \rangle_D = \langle x, i_{\mp}(y) \rangle
$$

for any $x \in g_{\pm}$ and $y \in g_{D, \mp}$. These map satisfy $p_{\pm} \circ i_{\pm} = \text{id}_{g_{D, \pm}}$ since, for any $x \in g_{D, \pm}, y \in g_{D, \mp},$

$$
\langle p_{\pm} \circ i_{\pm}(x) - x, y \rangle_D = \langle i_{\pm}(x), i_{\mp}(y) \rangle - \langle x, y \rangle_D = 0
$$

This yields in particular a a direct sum decomposition $g_{\pm} = i(g_{D, \pm}) \oplus m_{\pm}$, where

$$
m_{\pm} = \text{Ker}(p_{\pm}) = g_{\pm} \cap i(g_D)\perp
$$
Definition. The embedding \( i: (\mathfrak{g}_D, \mathfrak{g}_D, -\), \mathfrak{g}_D, +) \rightarrow (\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+) \) is called \textit{split} if the subspaces \( m_\pm \subset \mathfrak{g}_\pm \) are Lie subalgebras.

2.3. Split pairs of Lie bialgebras. For later use, we reformulate the above notion in terms of bialgebras via the double construction.

Definition. A \textit{split pair} of Lie bialgebras is the data of

- Lie bialgebras \((\mathfrak{a}, [\cdot, \cdot]_\mathfrak{a}, \delta_\mathfrak{a})\) and \((\mathfrak{b}, [\cdot, \cdot]_\mathfrak{b}, \delta_\mathfrak{b})\).
- Lie bialgebra morphisms \( i: \mathfrak{a} \rightarrow \mathfrak{b} \) and \( p: \mathfrak{b} \rightarrow \mathfrak{a} \) such that \( p \circ i = \text{id}_\mathfrak{a} \).

Proposition. There is a one-to-one correspondence between split inclusions of Manin triples and split pairs of Lie bialgebras. Specifically,

(i) If \( i: (\mathfrak{g}_D, \mathfrak{g}_D, -\), \mathfrak{g}_D, +) \rightarrow (\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+) \) is a split inclusion of Manin triples, then \((\mathfrak{g}_D, -\), i_-, i_+^*\) is a split pair of Lie bialgebras.

(ii) Conversely, if \((\mathfrak{a}, \mathfrak{b}, i, p)\) is a split pair of Lie bialgebras, then \( i \oplus p^*: (\mathfrak{D}a, a, a^*) \rightarrow \mathfrak{D}b, b, b^*\) is a split inclusion of Manin triples.

2.4. Proof of (i) of Proposition 2.3. Given a split inclusion \( i = i_- \oplus i_+: (\mathfrak{g}_D, \mathfrak{g}_D, -\), \mathfrak{g}_D, +) \rightarrow (\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+) \)

we need to show that \( i_- \) and \( i_+^* \) are Lie bialgebra morphisms. By assumption, \( i_- \) is a morphism of Lie algebras, and \( i_+^* \) one of coalgebras. Since \( i_- = (i_+^*)^* \), it suffices to show that \( p_\pm = i_+^* \) preserve Lie brackets.

We claim to this end that \( m_\pm \) are ideals in \( \mathfrak{g}_\pm \). Since \([m_\pm, m_\pm] \subseteq m_\pm \) by assumption, this amounts to showing that \([i(\mathfrak{g}_D, \pm), m_\pm] \subseteq m_\pm \). This follows
from the fact that \([i(g_{D,\pm}), m_\pm] \subseteq g_\pm\), and from

\[
\langle [i(g_{D,\pm}), m_\pm], i(g_{D,\mp}) \rangle = \langle m_\pm, [i(g_{D,\pm}), i(g_{D,\mp})] \rangle \subseteq \langle m_\pm, i(g_{D,\pm}) \rangle + \langle m_\pm, i(g_{D,\mp}) \rangle
\]

where the first term is zero since \(g_\pm\) is isotropic, and the second one is zero by definition of \(m_\pm\).

Let now \(X_1, X_2 \in g_\pm\), and write \(X_j = i_\pm(x_j) + y_j\), where \(x_j \in g_{D,\pm}\) and \(y_j \in m_\pm\). Since \(m_\pm = \text{Ker}(p_\pm)\) and \(p_\pm \circ i_\pm = \text{id}\), we have \([p_\pm(X_1), p_\pm(X_2)] = [x_1, x_2]\), while

\[
p_\pm[X_1, X_2] = p_\pm (i_\pm[x_1, x_2] + [i_\pm x_1, y_2] + [y_1, i_\pm x_2] + [y_1, y_2]) = [x_1, x_2]
\]

where the last equality follows from the fact that \(m_\pm\) is an ideal.

2.5. Proof of (ii) of Proposition 2.3. The bracket on \(\mathfrak{D} a\) is defined by

\[
[a, \phi] = \text{ad}^*(a)(\phi) - \text{ad}^*(\phi)(a) = -\langle \phi, [a, -]\rangle_a + \langle \phi \otimes \text{id}, \delta_a(a) \rangle
\]

for any \(a \in \mathfrak{a}, \phi \in \mathfrak{a}^*\). Analogously for \(\mathfrak{D} b\). Therefore, the equalities

\[
\langle p^*(\phi) \otimes \text{id}, \delta_b(i(a)) \rangle = \langle \phi \otimes \text{id}, (p \otimes \text{id})(i \otimes i)\delta_a(a) \rangle
\]

\[
= \langle \phi \otimes \text{id}, (\text{id} \otimes i)\delta_a(a) \rangle = i(\langle \phi \otimes \text{id}, \delta_a(a) \rangle)
\]

and

\[
\langle p^*(\phi), [i(a), b]_b \rangle = \langle \phi, p([i(a), b])_b \rangle = \langle \phi, [a, p(b)]_a \rangle
\]
for all \( a \in \mathfrak{a} \) and \( b \in \mathfrak{b} \), imply that the map \( i \oplus p^* \) is a Lie algebra map. It also respects the inner product, since for any \( a \in \mathfrak{a}, \phi \in \mathfrak{a}^* \),

\[
\langle p^*(\phi), i(a) \rangle = \langle \phi, p \circ i(a) \rangle = \langle \phi, a \rangle
\]

Finally, \( \mathfrak{m}_- = \text{Ker}(p) \) and \( \mathfrak{m}_+ = \text{Ker} i^* \) are clearly subalgebras.

2.6. Parabolic Lie subalgebras. Let

\[
i_D = i_- \oplus i_+: (\mathfrak{g}_D, \mathfrak{g}_{D,-}, \mathfrak{g}_{D,+}) \to (\mathfrak{g}, \mathfrak{g}_{-}, \mathfrak{g}_+)
\]

be a split embedding of Manin triples. We henceforth identify \( \mathfrak{g}_D \) as a Lie subalgebra of \( \mathfrak{g} \) with its induced inner product, and \( \mathfrak{g}_{D,\pm} \) as subalgebras of \( \mathfrak{g}_\pm \) noting that, by Proposition 2.3, \( \mathfrak{g}_{D,-} \) is a sub Lie bialgebra of \( \mathfrak{g}_- \).

The following summarizes the properties of the subspaces \( \mathfrak{m}_\pm = \mathfrak{g}_\pm \cap \mathfrak{g}_D^\perp \) and \( \mathfrak{p}_\pm = \mathfrak{m}_\pm \oplus \mathfrak{g}_D \).

**Proposition.**

(i) \( \mathfrak{m}_\pm \) is an ideal in \( \mathfrak{g}_\pm \), so that \( \mathfrak{g}_\pm = \mathfrak{m}_\pm \rtimes \mathfrak{g}_{D,\pm} \).

(ii) \([\mathfrak{g}_D, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm\), so that \( \mathfrak{p}_\pm = \mathfrak{m}_\pm \times \mathfrak{g}_D \) are Lie subalgebras of \( \mathfrak{g} \).

(iii) \( \delta(\mathfrak{m}_-) \subset \mathfrak{m}_- \otimes \mathfrak{g}_{D,-} + \mathfrak{g}_{D,-} \otimes \mathfrak{m}_- \), so that \( \mathfrak{m}_- \subseteq \mathfrak{g}_- \) is a coideal.

**Proof.** (i) was proved in 2.4. (ii) Since

\[
\langle [\mathfrak{g}_D, \mathfrak{m}_\pm], \mathfrak{g}_D \rangle = \langle \mathfrak{m}_\pm, [\mathfrak{g}_D, \mathfrak{g}_D] \rangle = 0
\]
we have \([g_D, m_\pm] \subset g_D^\perp = m_- \oplus m_+\). Moreover,

\[
\langle [g_D, m_\pm], m_\pm \rangle = \langle g_D, [m_\pm, m_\pm] \rangle = \langle g_D, m_\pm \rangle = 0
\]

since \(m_\pm\) is a subalgebra, and it follows that \([g_D, m_\pm] \subset m_\pm\). (iii) is clear since \(m_-\) is the kernel of a Lie coalgebra map. □

**Remark.** If the inclusion \(i_D\) is compatible with a finite type \(\mathbb{N}\)–grading, then \(m_+ \subset g_+\) is a coideal. Moreover, \(p_\pm\) are Lie subbialgebras of \(g\) such that the projection \(p_\pm \to g_D\) is a morphism of bialgebras. Namely, a finite type \(\mathbb{N}\)–grading allows to define a Lie bialgebra structure on \(g, g_+\). We then get a vector space decomposition \(g_\pm = m_\pm \oplus g_{D, \pm}\) and a Lie bialgebra map \(g_\pm \to g_{D, \pm}\). It is also possible to define the Lie subalgebras

\[
p_\pm = m_\pm \oplus g_D \subset g
\]

If we assume the existence of a compatible grading on \(g\) and \(g_D, i.e.,\) preserved by \(i_D\), then the natural maps

\[
p_\pm \subset g \quad p_\pm \to g_D
\]

are morphisms of Lie bialgebras.

**2.7. The relative Verma Modules.**

**Definition.** Given a split embedding of Manin triples \(g_D \subset g\), and the corresponding decomposition \(g = m_- \oplus p_+\), let \(L_-, N_+\) be the relative Verma modules defined by

\[
L_- = \text{Ind}_{p_+}^{g_+} k \quad \text{and} \quad N_+ = \text{Ind}_{m_-}^g k
\]
Proposition. The $\mathfrak{g}$-modules $L_-$ and $N^*_+$ are equicontinuous.

The description of the appropriate topologies on $L_-$ and $N^*_+$, and the proof of their equicontinuity will be carried out in 2.8–2.11.

2.8. Equicontinuity of $L_-$. As vector spaces,

$$L_- \simeq U m_- \subset U \mathfrak{g}_-$$

so it is natural to equip $L_-$ with the discrete topology. The set of operators $\{\pi_{L_-}(x)\}_{x \in \mathfrak{g}}$ is then an equicontinuous family, and the continuity of $\pi_{L_-}$ reduces to checking that, for every element $v \in L_-$, the set

$$Y_v = \{b \in \mathfrak{g}_+ | b.v = 0\}$$

is a neighborhood of zero in $\mathfrak{g}_+$. Since $U m_- \subset U \mathfrak{g}_-$ the proof is identical to [EK1, Lemma 7.2]. We proceed by induction on the length of $v = a_{i_1} \ldots a_{i_n} 1_-$. If $n = 0$, then $v = 1_-$ and $Y_v = \mathfrak{g}_+$. If $n > 1$, then assume $v = a_jw$, with $w = a_{i_1} \ldots a_{i_{n-1}} 1_-$ and $Y_w$ open in $\mathfrak{g}_+$. For every $x \in \mathfrak{g}_+$

$$x.v = x.(a_jw) = [x, a_j].w + (a_jx).w$$

Call $Z$ the subset of $\mathfrak{g}_+$

$$Z = \{x \in \mathfrak{g}_+ | [x, a_j] \in Y_w\}$$

$Z$ is open in $\mathfrak{g}_+$, by continuity of bracket $[.,.]$, and clearly $Z \cap Y_w \subset Y_v$.

2.9. Topology of $N_+$. As vector spaces,

$$N_+ = \text{Ind}^\mathfrak{g}_{m_-} k \simeq U p_+ \simeq \colim U_n p_+$$
where \( \{ U_n p_+ \} \) denotes the standard filtration of \( U p_+ \), so that

\[
U_n p_+ \simeq \bigoplus_{m=0}^{n} S^m p_+ = \bigoplus_{i+j \leq n} (S^i g_+ \otimes S^j g_{D,-})
\]

We turn this isomorphism into an isomorphism of topological vector spaces, by taking on \( S^i g_+ \) and \( S^j g_{D,-} \) the topologies induced by the embeddings

\[
S^i g_+ \hookrightarrow (g^*)^i \quad \text{and} \quad S^j g_{D,-} \hookrightarrow g_{D,-}^{\otimes j}
\]

With respect to these topologies, \( U_m p_+ \) is closed inside \( U_n p_+ \) for \( m < n \), and we equip \( N_+ \) with the direct limit topology. We shall need the following

**Lemma.** For any \( x \in g \), the map \( \pi_{N_+}(x) : N_+ \to N_+ \) is continuous.

**Proof.** We need to show that for any neighborhood of the origin \( U \subset N_+ \), there exists a neighborhood of zero \( U' \subset N_+ \) such that \( \pi_{N_+}(x)U' \subset U \). The topology on \( N_+ \) comes from the decomposition \( U p_+ \simeq U g_+ \otimes U g_{D,-} \), so that an open neighborhood of zero in \( N_+ \) has the form \( U \otimes U g_{D,-} + U g_+ \otimes V \), with \( U \) open in \( U g_+ \) and \( V \) open in \( U g_{D,-} \). We apply the same procedure used in [EK1, Lemma 7.3] to construct a set \( U' \otimes U g_{D,-} \), with \( U' \) open in \( U g_+ \), such that

\[
\pi_{N_+}(x)(U' \otimes U g_{D,-}) \subset U \otimes U g_{D,-} \subset U \otimes U g_{D,-} + U g_+ \otimes V
\]

Since the topology on \( U g_{D,-} \) is discrete, the set \( U' \otimes U g_{D,-} \) is open in \( N_+ \) and the lemma is proved.

\[ \square \]

**2.10. Topology of \( N_+^* \).** As vector spaces,

\[
N_+^* \simeq (U p_+)^* \simeq \lim(U_n p_+)^*
\]
Define a filtration \( \{(N^*_+)_n\} \) on \( N_+^* \) by
\[
0 \to (N_+^*)_n \to (U_p^+)^* \to (U_n p_+)^* \to 0
\]
so that \( N_+^* \supset (N_+^*)_0 \supset (N_+^*)_1 \supset \cdots \), and we get an isomorphism of vector spaces
\[
N_+^* \simeq \lim N_+^*/(N_+^*)_n
\]
Finally, we use the isomorphism to endow \( N_+^* \) with the inverse limit topology.

**Lemma.** \( \{\pi_{N^*_+}(x)\}_{x \in \mathfrak{g}} \) is an equicontinuous family of operators.

**Proof.** Since \( p_+ \) acts on \( N_+ \) by multiplication,
\[
p_+(N_+^*)_n \subset (N_+^*)_n-1
\]
If \( x \in \mathfrak{m}_- \) and \( x_i \in U p_+ \) for \( i = 1, \ldots, n \), then in \( U \mathfrak{g} \),
\[
xx_1 \cdots x_n = x_1 \cdots x_n x - \sum_{i=0}^{n} x_1 \cdots x_{i-1}[x_i, x] x_{i+1} \cdots x_n
\]
where \([x_i, x] \in \mathfrak{g}\). Iterating shows that \( (x.f)(x_1 \cdots x_n) = 0 \) if \( f \in (N_+^*)_n \), so that \( x(N_+^*)_n \subset (N_+^*)_n \). Then, for any neighborhood of zero of the form \( U = (N_+^*)_n \), it is enough to take \( U' = (N_+^*)_n+1 \) to get \( \mathfrak{g}(N_+^*)_n+1 \subset (N_+^*)_n \). \( \square \)

**2.11. Equicontinuity of \( N_+^* \).**

**Lemma.** The map \( \pi_{N_+^*} : \mathfrak{g} \to \text{End}(N_+^*) \) is a continuous map.

**Proof.** Since \( \mathfrak{g}_- \) is discrete, it is enough to check that, for any \( f \in N_+^* \) and \( n \in \mathbb{N} \), the subset
\[
Y(f, n) = \{ b \in \mathfrak{g}_+ | b.f \in (N_+^*)_n \}
\]
is open in $\mathfrak{g}_+$, i.e.
\[ b^i.f \in (N^*_+)_n \quad \text{for a.a. } i \in I \]

Since $f \in N^*_+ \simeq \lim N^*_+/(N^*_+)_n$, we have $f = \{f_n\}$ where $f_n$ is the class of $f$ modulo $(N^*_+)_n$. Therefore $b^i.f \in (N^*_+)_n$ iff
\[ (b^i.f)_n = b^i.f_{n+1} = 0 \]

Now, for any $x_1 \cdots x_n \in U_n\mathfrak{p}_+$, we have
\[ b^i.f_{n+1}(x_1 \cdots x_n) = -f_{n+1}(b^i x_1 \cdots x_n) = 0 \]

for a.a. $i \in I$ and the lemma is proved (it is enough to exclude the indices corresponding to the generators involved in the expression of $f_{n+1}$).

As a vector spaces, we can identify
\[ \mathfrak{p}^*_+ = \mathfrak{g}^*_+ \oplus \mathfrak{g}^*_D, - \simeq \mathfrak{g}_- \oplus \mathfrak{g}^{D, +} = \mathfrak{p}_- \]

We can give as a basis for $\mathfrak{p}_+$ and $\mathfrak{p}_-$
\[ \mathfrak{p}_+ \supset \{\{b^i\}_{i \in I}, \{a_r\}_{r \in I(D)}\} \quad \mathfrak{p}_- \supset \{\{a_i\}_{i \in I}, \{b^r\}_{r \in I(D)}\} \]

and obvious relations
\[ (b^i, a_j) = \delta_{ij} \quad (b^i, b^r) = 0 \]
\[ (a_r, a_j) = 0 \quad (a_r, b^s) = \delta_{rs} \]
with \( i, j \in I, r, s \in I(D) \). We can then identify \( f_{n+1} \) with an element in \( U_{n+1}p_- \). Call \( T_{n+1}(f) \) the set of indices of all \( a_i \) involved in the expression of \( f_{n+1} \). Excluding these finite set of indices we get the result. \( \square \)

2.12. **Coalgebra structure on** \( L_-, N_+ \). Define \( g \)-module maps

\[
i_- : L_- \rightarrow L_- \hat{\otimes} L_- \quad \text{and} \quad i_+ : N_+ \rightarrow N_+ \hat{\otimes} N_+
\]

by mapping \( 1_\pm \) to \( 1_\pm \otimes 1_\pm \). Note that, under the identification \( L_- \simeq Um_- \) and \( N_+ \simeq Up_+ \), \( i_\pm \) correspond to the coproduct on \( Um_- \) and \( Up_+ \) respectively.

Following [D3, Prop. 1.2], we consider the invertible element \( T \in (Ug \hat{\otimes} Ug)[[h]] \) satisfying relations:

\[
S^{\otimes 3}(\Phi^{321}) \cdot (T \otimes 1) \cdot (\Delta \otimes 1)(T) = (1 \otimes T)(1 \otimes \Delta)(T) \cdot \Phi
\]

\[
T\Delta(S(a)) = (S \otimes S)(\Delta(a))T
\]

Let \( N_+^* \) be as before and \( f, g \in N_+^* \). Consider the linear functional in \( \text{Hom}_k(N_+, k) \) defined by

\[
v \mapsto (f \otimes g)(T \cdot i_+(v))
\]

This functional is continuous, so it belongs to \( N_+^* \) and allow us to define the map

\[
i_+^\vee \in \text{Hom}_k(N_+^* \otimes N_+^*, N_+^*)[[h]] \quad , \quad i_+^\vee(f \otimes g)(v) = (g \otimes f)(T \cdot i_+(v))
\]
This map is continuous and extends to a map from $N_+^* \hat{\otimes} N_+^*$ to $N_+^*$. For any $a \in \mathfrak{g}$, we have

$$i_+^\vee(a(f \otimes g))(v) = (f \otimes g)((S \otimes S)(\Delta(a))T \cdot i_+(v)) =$$

$$= (f \otimes g)(T\Delta(S(a)) \cdot i_+(v)) =$$

$$= i_+^\vee(f \otimes g)(S(a).v) = (a. i_+^\vee(f \otimes g))(v)$$

and then $i_+^\vee \in \text{Hom}_\mathfrak{g}(N_+^* \hat{\otimes} N_+^*, N_+^*)[[h]]$.

The following shows that $L_-$ and $N_+$ are coalgebra objects in the Drinfeld categories of $\mathfrak{g}$–modules and $(\mathfrak{g}, \mathfrak{g}_D)$–bimodules respectively.

**Proposition.** The following relations hold

(i) $\Phi(i_- \otimes 1)i_- = (1 \otimes i_-)i_-.$

(ii) $i_+^\vee(1 \otimes i_+^\vee)\Phi = i_+^\vee(i_+^\vee \otimes 1)S^{S3}(\Phi_D^{-1})^\rho$

where $(-)^\rho$ denotes the right $\mathfrak{g}_D$–action on $N_+^*$.

**Proof.** We begin by showing that

$$\Phi(1^{\otimes 3}) = 1^{\otimes 3} \quad \text{and} \quad \Phi(1_+^{\otimes 3}) = \Phi_D(1_+^{\otimes 3}) \quad (2.1)$$

To prove the first identity, it is enough to notice that, since $\mathfrak{g}_+1_- = 0$ and $\Omega = \sum (a_i \otimes b^j + b^j \otimes a_i)$,

$$\Omega_{ij}(1^{\otimes 3}) = 0$$
Then \( \Phi(1_\otimes^3) = 1_\otimes^3 \). To prove the second one, we notice that \( m_-1_+ = 0 \) and that we can rewrite

\[
\Omega = \sum_{j \in I_D} (a_j \otimes b^j + b^j \otimes a_j) + \sum_{i \in I \setminus I_D} (a_i \otimes b^i + b^i \otimes a_i) = \Omega_D + \sum_{i \in I \setminus I_D} (a_i \otimes b^i + b^i \otimes a_i)
\]

where \( \{a_j\}_{j \in I_D} \) is a basis of \( g_D^- \) and \( \{b^j\}_{j \in I_D} \) is the dual basis of \( g_D^+ \). Then

\[
\Omega_{ij}(1_\otimes^3) = \Omega_{D,ij}(1_\otimes^3)
\]

and, since for any element \( x \in g_D \), the right and the left \( g_D \)-action coincide on \( 1_+ \), i.e. \( x.1_+ = 1_+.x \), we have

\[
\Omega_{ij}(1_\otimes^3) = (1_\otimes^3)\Omega_{D,ij}
\]

and consequently \( \Phi(1_\otimes^3) = \Phi_D(1_\otimes^3) \).

To prove (i), note that since the comultiplication in \( U m_- \) is coassociative, we have \( (i_- \otimes 1)i_- = (1 \otimes i_-)i_- \). We therefore have to show that \( \Phi(i_- \otimes 1)i_- = (1 \otimes i_-)i_- \). This is an obvious consequence of (2.1) and the fact that \( m_- \) is generated by \( 1_- \).
To prove (ii), consider $v \in N_+$,

$$i_+^\vee (1 \otimes i_+^\vee) (\Phi(f \otimes g \otimes h))(v) =$$

$$= (h \otimes g \otimes f)((S^\otimes 3(\Phi^{321}) \cdot (T \otimes 1) \cdot (\Delta \otimes 1)(T)) \cdot (i_+ \otimes 1)i_+(v)) =$$

$$= (h \otimes g \otimes f)((1 \otimes T)(1 \otimes \Delta)(T) \cdot \Phi(i_+ \otimes 1)i_+(v)) =$$

$$= (h \otimes g \otimes f)((1 \otimes T)(1 \otimes \Delta)(1 \otimes i_+)i_+(v)\Phi_D) =$$

$$= (S^\otimes 3(\Phi_D)^\rho (h \otimes g \otimes f))((1 \otimes T)(1 \otimes \Delta)(1 \otimes i_+)i_+(v)) =$$

$$= i_+^\vee(i_+ \otimes 1)(S^\otimes 3(\Phi_D^{321})^\rho (f \otimes g \otimes h))(v) =$$

$$= i_+^\vee(i_+ \otimes 1)S^\otimes 3(\Phi_D^{-1})^\rho (f \otimes g \otimes h)(v)$$

and (ii) is proved. \qed

2.13. The fiber functor over $g_D$. To any representation $V[[h]] \in \text{Rep} \ Ug[[h]]$, we can associate the $k[[h]]$–module

$$\Gamma(V) = \text{Hom}_g(L_-, N_+^* \hat{\otimes} V)[[h]]$$

where $\text{Hom}_g$ is the set of continuous homomorphisms, equipped with the weak topology. The right $g_D$–action on $N_+^*$ endows $\Gamma(V)$ with the structure of a left $g_D$–module.

**Proposition.** The complete vector space $\text{Hom}_g(L_-, N_+^* \hat{\otimes} V)$ is isomorphic to $V$ as equicontinuous $g_D$–module. The isomorphism is given by

$$\alpha_V : f \mapsto (1_+ \otimes 1)f(1_-)$$

for any $f \in \text{Hom}_g(L_-, N_+^* \hat{\otimes} V)$. 
3. AN EQUIVALENCE OF QUASI-COXETER CATEGORIES

PROOF. By Frobenius reciprocity, we get an isomorphism

\[ \text{Hom}_g(L_-, N_-^* \hat{\otimes} V) \simeq \text{Hom}_{\hat{p}_+}(k, N_+^* \hat{\otimes} V) \simeq \text{Hom}_k(k, V) \simeq V \]

given by the map

\[ f \mapsto (1_+ \otimes 1)f(1_-) \]

For \( f \in \Gamma(V) \) and \( x \in Ug_D \), \( x.f \in \Gamma(V) \) is defined by

\[ x.f = (S(x)^{\rho} \otimes \text{id}) \circ f \]

For any \( x \in Ug_D \), we have

\[ \sum_{i,j} x_i^{(1)} f_j \otimes x_i^{(2)} v_j = \varepsilon(x)f(1_-) \]

where \( \Delta(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)} \) and \( f(1_-) = \sum_j f_j \otimes v_j \). Using the identity

\[ 1 \otimes x = \sum_i (S(x_i^{(1)}) \otimes 1) \cdot \Delta(x_i^{(2)}) \]

holding in any Hopf algebra, we obtain

\[ (1 \otimes x)f(1_-) = \sum_i (S(x_i^{(1)} \varepsilon(x_i^{(2)})) \otimes 1) f(1_-) = (S(x) \otimes 1)f(1_-) \]

Finally, we have

\[ x.\alpha_V(f) = \langle 1_+ \otimes \text{id}, (1 \otimes x)f(1_-) \rangle = \]

\[ = \langle 1_+ \otimes \text{id}, (S(x) \otimes 1)f(1_-) \rangle = \]

\[ = \langle 1_+ \otimes \text{id}, (S(x)^{\rho} \otimes 1)f(1_-) \rangle = \alpha_V(x.f) \]

Therefore, \( \Gamma(V) \) is isomorphic to \( V[[\hbar]] \) as equicontinuous \( g_D \)-module. \( \square \)
2.14. For any continuous \( \varphi \in \text{Hom}_g(V, V') \), define a map \( \Gamma(\varphi) : \Gamma(V) \to \Gamma(V') \) by

\[
\Gamma(\varphi) : f \mapsto (\text{id} \otimes \varphi) \circ f
\]

This map is clearly continuous and for all \( x \in \mathfrak{g}_D \)

\[
\Gamma(\varphi)(x.f) = (S(x) \otimes \varphi) \circ f = x.\Gamma(\varphi)(f)
\]

then \( \Gamma(\varphi) \in \text{Hom}_{\mathfrak{g}_D}(\Gamma(V), \Gamma(V')) \).

Since the diagram

\[
\begin{array}{ccc}
\Gamma(V) & \xrightarrow{\Gamma(\varphi)} & \Gamma(V') \\
\alpha_V & & \alpha_{V'} \\
V[[h]] & \xrightarrow{\varphi} & V'[[h]]
\end{array}
\]

is commutative for all \( \varphi \in \text{Hom}_g(V, V') \), we have a well-defined functor

\[
\Gamma : \text{Rep}^{\text{eq}} U\mathfrak{g}[[h]] \to \text{Rep}^{\text{eq}} U\mathfrak{g}_D[[h]]
\]

which is naturally isomorphic to the pullback functor induced by the inclusion

\( i_D : \mathfrak{g}_D \hookrightarrow \mathfrak{g} \) via the natural transformation

\[
\alpha_V : \Gamma(V) \simeq i_D^* V[[h]]
\]

2.15. Tensor structure on \( \Gamma \). Denote the tensor product in the categories \( \mathcal{D}_g(U\mathfrak{g}) \), \( \mathcal{D}_{\mathfrak{g}_D}(U\mathfrak{g}_D) \) by \( \otimes \), and let \( B_{1234} \) and \( B'_{1234} \) be the associativity constraints

\[
B_{1234} : (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \to V_1 \otimes ((V_2 \otimes V_3) \otimes V_4)
\]
and

\[ B'_{1234} : (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \to (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \]

For any \( v \in \Gamma(V), w \in \Gamma(W) \), define \( J_{VW}(v \otimes w) \) to be the composition

\[
\begin{align*}
L_- & \overset{i}{\to} L_- \otimes L_- \overset{\nu \otimes \nu}{\to} (N_+^* \otimes V) \otimes (N_+^* \otimes W) \\
& \overset{\beta_{12}^{-1}}{\to} N_+^* \otimes ((N_+^* \otimes V) \otimes W) \\
& \overset{A'}{\to} (N_+^* \otimes N_+^*) \otimes (V \otimes W) \\
& \overset{\gamma \otimes 1}{\to} N_+^* \otimes (V \otimes W)
\end{align*}
\]

where the pair \((A, A')\) can be chosen to be \((B_{N_+^*,V,N_+^*,W}, B_{N_+^*,V,N_+^*,W}^{-1})\) or \((B'_{N_+^*,V,N_+^*,W}, B'_{N_+^*,V,N_+^*,W}^{-1})\).

The map \( J_{VW}(v \otimes w) \) is clearly a continuous \( g\)-morphism from \( L_- \) to \( N_+^* \otimes (V \otimes W) \), so we have a well-defined map

\[ J_{VW} : \Gamma(V) \otimes \Gamma(W) \to \Gamma(V \otimes W) \]

**Proposition.** The maps \( J_{VW} \) are isomorphisms of \( g_D \)-modules and define a tensor structure on the functor \( \Gamma \).

The proof of Proposition 2.15 is given in 2.16–2.19.

**2.16.** We easily observe that the maps \( J_{VW} \) are compatible with the \( g_D \)-action. Indeed, the map \( i^\gamma_+ \) is compatible with the right \( g_D \)-action on \( N_+^* \) and, for any \( x \in g_D \),

\[
x.J_{VW}(v \otimes w) = (S^\theta(x) \otimes \text{id})(i^\gamma_+ \otimes \text{id} \otimes \text{id}) \tilde{A}(v \otimes w)i_- = \\
= (i^\gamma_+ \otimes \text{id} \otimes \text{id})(\Delta(S(x))^\theta)_{12} \tilde{A}(v \otimes w)i_- = \\
= (i^\gamma_+ \otimes \text{id} \otimes \text{id}) \tilde{A}((S \otimes S)(\Delta(x))^\theta)_{13}(v \otimes w)i_- = J_{VW}(x.(v \otimes w))
\]
where $\tilde{A} = A'\beta_{32}^{-1}A$.

$J_{VW}$ is an isomorphism, since it is an isomorphism modulo $h$. Indeed,

$$J_{VW}(v \otimes w) \equiv (i^*_+ \otimes 1)(1 \otimes s \otimes 1)(v \otimes w)i_- \mod h$$

Finally, to prove that $J_{VW}$ define a tensor structure on $\Gamma$, we need to show that, for any $V_1, V_2, V_3 \in D\Phi(Ug)$ the following diagram is commutative

$$(\Gamma(V_1) \otimes \Gamma(V_2)) \otimes \Gamma(V_3) \xrightarrow{J_{12,1}} \Gamma(V_1 \otimes V_2) \otimes \Gamma(V_3) \xrightarrow{J_{12,3}} \Gamma((V_1 \otimes V_2) \otimes V_3)$$

$$(\Gamma(V_1) \otimes (\Gamma(V_2) \otimes \Gamma(V_3))) \xrightarrow{1 \otimes J_{23}} \Gamma(V_1) \otimes (\Gamma(V_2 \otimes V_3)) \xrightarrow{J_{1,23}} \Gamma((V_1 \otimes (V_2 \otimes V_3)))$$

where $J_{ij}$ denotes the map $J_{V_i,V_j}$ and $J_{ij,k}$ the map $J_{V_i \otimes V_j,V_k}$.

2.17. For any $v_i \in \Gamma(V_i)$, $i = 1, 2, 3$, the map $\Gamma(\Phi)J_{12,3}J_{12}1(v_1 \otimes v_2 \otimes v_3)$ is given by the composition

$$(1 \otimes \Phi)(i^*_+ \otimes 1^{\otimes 3})A_4(1 \otimes \beta_{1 \otimes 2,N^*_+,1} \otimes 1)A_3((i^*_+ \otimes 1) \otimes 1^{\otimes 3})(A_2 \otimes 1 \otimes 1)$$

$$\cdot (1 \otimes \beta_{N^*_+,1} \otimes 1^{\otimes 3})(A_1 \otimes 1 \otimes 1)(v_1 \otimes v_2 \otimes v_3)(i_- \otimes 1)i_-$$

where

$$A_1 = B_{N^*_+,1,N^*_+,2} \quad A_3 = B_{N^*_+,1 \otimes 2,N^*_+,3}$$

$$A_2 = B_{N^*_+,N^*_+,1,2}^{-1} \quad A_4 = B_{N^*_+,N^*_+,1 \otimes 2,3}^{-1}$$
illustrated by the diagram

\[
\begin{array}{c}
L_+ \\
\downarrow^{i_-} \\
L_+ \otimes L_+ \\
\downarrow^{i_- \otimes 1} \\
(L_+ \otimes L_+) \otimes L_-
\end{array}
\]

\[
\begin{array}{c}
v_1 \otimes v_2 \otimes v_3 \\
\rightarrow ((N_+^\ast \otimes V_1) \otimes (N_+^\ast \otimes V_2)) \otimes (N_+^\ast \otimes V_3) \\
A_1 \otimes 1 \otimes 1 \\
\rightarrow (N_+^\ast \otimes ((V_1 \otimes N_+^\ast) \otimes V_2)) \otimes (N_+^\ast \otimes V_3)
\end{array}
\]

\[
\begin{array}{c}
1 \otimes \beta_{1,2, N_+^\ast} \otimes 1 \otimes 1 \\
\rightarrow (N_+^\ast \otimes ((N_+^\ast \otimes (V_1 \otimes V_2)) \otimes V_3) \\
A_2 \otimes 1 \otimes 1 \\
\rightarrow ((N_+^\ast \otimes N_+^\ast) \otimes (V_1 \otimes V_3)) \otimes (N_+^\ast \otimes V_3)
\end{array}
\]

By functoriality of associativity and commutativity isomorphisms, we have

\[
A_3(i_+^\ast \otimes 1 \otimes 4) = (i_+^\ast \otimes 1 \otimes 4)A_5
\]

where \(A_5 = B_{N_+^\ast \otimes N_+^\ast, 12, N_+^\ast, 3} \).

\[
(1 \otimes \beta_{12, N_+^\ast} \otimes 1)i_+^\ast \otimes 1 \otimes 4 = (i_+^\ast \otimes 1 \otimes 4)(1 \otimes 2 \otimes \beta_{12, N_+^\ast} \otimes 1 \otimes 2)
\]

and

\[
A_4(i_+^\ast \otimes 1 \otimes 4) = (i_+^\ast \otimes 1 \otimes 4)A_6
\]

where \(A_6 = B_{N_+^\ast \otimes N_+^\ast, 1 \otimes 2, 3}^{-1} \). Finally, we have

\[
\Gamma(\Phi)J_{12,3}(J_{12} \otimes 1)(v_1 \otimes v_2 \otimes v_3)
\]

\[
= (1 \otimes 3 \otimes \Phi_{123})((i_+^\ast (i_+^\ast \otimes 1)) \otimes 1 \otimes 3)A(v_1 \otimes v_2 \otimes v_3)(i_- \otimes 1)i_-(2.2)
\]
where

\[ A = A_0(1 \otimes 2 \otimes \beta_{1 \otimes 2, N^*_1} \otimes 1 \otimes 2)A_5(A_2 \otimes 1 \otimes 2)(1 \otimes \beta_{N^*_1, 1} \otimes 1 \otimes 3)(A_1 \otimes 1 \otimes 1) \]

2.18. On the other hand, \(J_{1,2\otimes 3}(1 \otimes J_{23})\Phi_D(v_1 \otimes v_2 \otimes v_3)\) corresponds to the composition

\[ (i^*_+ \otimes 1 \otimes 3)A'_4(1 \otimes \beta_{N^*_1, 1} \otimes 1 \otimes 2)A'_3(1 \otimes 2 \otimes i^*_+ \otimes 1 \otimes 2)(1 \otimes 1 \otimes A'_2) \]

\[ (1 \otimes 3 \otimes 2, N^*_1 \otimes 1)(1 \otimes 1 \otimes A'_1)\Phi_D(v_1 \otimes v_2 \otimes v_3)(1 \otimes i_-)i_- \]

where

\[ A'_1 = B_{N^*_1, 2, N^*_1, 3} \quad A'_3 = B_{N^*_1, 1, N^*_1, 2, 3} \]

\[ A'_2 = B_{N^*_1, N^*_1, 2, 3} \quad A'_1 = B_{N^*_1, 1, 2, 3} \]

illustrated by the diagram

\[ \xymatrix{ L_- \ar[r]^{i_-} & L_- \otimes L_- \ar[r]^{1 \otimes i_-} & L_- \otimes (L_- \otimes L_-) } \]

\[ \Phi_D(v_1 \otimes v_2 \otimes v_3) \ar[r] & (N^*_1 \otimes V_1) \otimes ((N^*_1 \otimes V_2) \otimes (N^*_1 \otimes V_3)) \]

\[ 1 \otimes 3 \otimes 2, N^*_1 \otimes 1 \ar[r] & (N^*_1 \otimes V_1) \otimes (N^*_1 \otimes ((N^*_1 \otimes V_2) \otimes V_3)) \]

\[ 1 \otimes 3 \otimes 2, 1 \otimes 3 \otimes 1 \otimes 2 \ar[r] & (N^*_1 \otimes V_1) \otimes (N^*_1 \otimes (V_1 \otimes V_3) \otimes V_3) \]

By functoriality of associativity and commutativity isomorphisms, we have

\[ A'_3(1 \otimes 2 \otimes i^*_+ \otimes 1 \otimes 2) = (1 \otimes 2 \otimes i^*_+ \otimes 1 \otimes 2)A'_5 \]
where $A'_5 = B_{N^*_+1,N^*_+} \otimes N^*_+2 \otimes 3$,
\[
(1 \otimes \beta_{1,N^*_+} \otimes 1^{\otimes 2})(1^{\otimes 2} \otimes i_+^* \otimes 1^{\otimes 2}) = (1 \otimes i_+^* \otimes 1^{\otimes 3})(1 \otimes \beta_{1,N^*_+} \otimes N^*_+ \otimes 1^{\otimes 2})
\]
and
\[
A'_4(1 \otimes i_+^* \otimes 1^{\otimes 3}) = (1 \otimes i_+^* \otimes 1^{\otimes 3})A'_6
\]
where $A'_6 = B_{N^*_+1,N^*_+}^{-1} \otimes N^*_+,1,2 \otimes 3$. Thus,
\[
J_{1,23}(1 \otimes J_{23}) \Phi_D(v_1 \otimes v_2 \otimes v_3)
\]
\[
= (i_+^* \otimes 1^{\otimes 3})((1 \otimes i_+^*) \otimes 1^{\otimes 3}) B \Phi_D(v_1 \otimes v_2 \otimes v_3)(1 \otimes i_-)i_- \quad (2.3)
\]
where
\[
B = A'_6(1 \otimes \beta_{1,N^*_+} \otimes N^*_+ \otimes 1^{\otimes 2})A'_5(1^{\otimes 2} \otimes A'_2)(1^{\otimes 3} \otimes \beta_{2,N^*_+} \otimes 1)(1 \otimes 1 \otimes A'_1)
\]
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2.19. Comparing (2.2) and (2.3), we see that it suffices to show that the outer arrows of the following form a commutative diagram.

Using the pentagon and the hexagon axiom, we can show that

\[(\Phi \otimes \Phi)A = B\Phi\]

We have to show that

\[\Gamma(\Phi)J_{12,3}(J_{12} \otimes 1)(v_1 \otimes v_2 \otimes v_3) = J_{1,23}(1 \otimes J_{23})\Phi_D(v_1 \otimes v_2 \otimes v_3)\]

in \(\text{Hom}_g(L_-, N_+ ^* \otimes (V_1 \otimes (V_2 \otimes V_3)))\):
\[ J_{1,23}(\id \otimes J_{23}) \Phi_D(v_1 \otimes v_2 \otimes v_3) = \]

\[ = (i_+^\gamma (\id \otimes i_+^\gamma) \otimes \id^{\otimes 3}) B \Phi_D(v_1 \otimes v_2 \otimes v_3)(\id \otimes i_-) i_- \]

\[ = (i_+^\gamma (\id \otimes i_+^\gamma) \otimes \id^{\otimes 3}) B \Phi_D(v_1 \otimes v_2 \otimes v_3)(i_- \otimes \id) i_- \]

\[ = (i_+^\gamma (\id \otimes i_+^\gamma) \otimes \id^{\otimes 3}) B \Phi\Phi D(v_1 \otimes v_2 \otimes v_3)(i_- \otimes \id) i_- \]

\[ = (i_+^\gamma (\id \otimes i_+^\gamma) \Phi \otimes \Phi) A \Phi_D(v_1 \otimes v_2 \otimes v_3)(i_- \otimes \id) i_- \]

\[ = (i_+^\gamma (\id \otimes i_+^\gamma) \Phi S^{\otimes 3}(\Phi_D)^{\rho} \otimes \Phi^{\rho}) A (v_1 \otimes v_2 \otimes v_3)(i_- \otimes \id) i_- \]

\[ = (i_+^\gamma (i_+^\gamma \otimes \id) \otimes \Phi) A (v_1 \otimes v_2 \otimes v_3)(i_- \otimes \id) i_- \]

\[ = \Gamma(\Phi) J_{12,3}(J_{12} \otimes \id)(v_1 \otimes v_2 \otimes v_3) \]

where the second and seventh equalities follow from Proposition 2.12, the fifth one from the definition of the \( \mathfrak{g}_D \)-action on the modules \( \Gamma(V_i) \) and the others from functoriality of the associator \( \Phi \). This complete the proof of Theorem 2.1.

**2.20. 1–Jets of relative twists.** The following is a straightforward extension of the computation of the 1–jet of the Etingof–Kazhdan twist given in [EK1].

**Proposition.** Under the natural identification

\[ \alpha_V : \Gamma(V) \to V[[\hbar]] \]
the relative twist $J_\Gamma$ satisfies

$$\alpha_{V \otimes W} \circ J_\Gamma \circ (\alpha^{-1}_V \otimes \alpha^{-1}_W) \equiv 1 + \frac{\hbar}{2} (r + r^{21}_D) \mod \hbar^2$$

in $\text{End}(V \otimes W)[[\hbar]]$.

PROOF. For $v \in V, w \in W$, let

$$\alpha^{-1}_V(v)(1_-) = \sum f_i \otimes v_i \quad \alpha^{-1}_W(w)(1_-) = \sum g_j \otimes w_j$$

in $(N^*_+ \otimes V)^{p+}$ and $(N^*_+ \otimes W)^{p+}$ respectively. Then using

$$\langle (1_+ \otimes 1)^{\otimes 2}, \Omega_{23}, \sum_{i,j} f_i \otimes v_i \otimes g_j \otimes w_j \rangle = -\tau(v \otimes w)$$

and

$$\langle (1_+ \otimes 1)^{\otimes 2}, \Omega_{D,23}, \sum_{i,j} f_i \otimes v_i \otimes g_j \otimes w_j \rangle = -\Omega_D(v \otimes w)$$

where $\Omega = \Omega + \Omega_D$, we get

$$\alpha_{V \otimes W} \circ J_\Gamma \circ (\alpha^{-1}_V \otimes \alpha^{-1}_W)(v \otimes w) \equiv v \otimes w + \frac{\hbar}{2} (r + r^{21}_D)(v \otimes w) \mod \hbar^2$$

because the definition of $J_\Gamma$ involves the braiding $\beta_{XY}^\prime = \beta_{YX}^{-1}$.

COROLLARY. The relative twist $J_\Gamma$ satisfies

$$\text{Alt}_2 J_\Gamma \equiv \frac{\hbar}{2} \left( \frac{r - r^{21}}{2} - \frac{r_D - r^{21}_D}{2} \right) \mod \hbar^2$$

3. Quantization of Verma modules

This section and the next contain results about the quantization of classical Verma modules, which are required to construct the morphism of $D$–categories
between the representation theory of $U\mathfrak{g}[[\hbar]]$ and that of $U_h\mathfrak{g}$. In particular, from now on, we will assume the existence of a finite $\mathbb{N}$–grading on $\mathfrak{g}_-$, which induces on $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$ a Lie bialgebra structure and allows us to consider the quantization of $\mathfrak{g}$ through the Etingof–Kazhdan functor, $U_h^{\mathfrak{E}K} \mathfrak{g}$.

3.1. Quantum Verma Modules. Let $(\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+)$ be a graded Manin triple. This means that we have an isomorphism between $\mathfrak{g}_+$ and the graded dual of $\mathfrak{g}_-$ and that $\mathfrak{g}_+$ is naturally endowed with a Lie bialgebra structure. Because of the functoriality of the quantization defined by Etingof and Kazhdan in [EK2], in the category of Drinfeld-Yetter modules over $U_h^{\mathfrak{E}K} \mathfrak{g}_-$ we can similarly define quantum Verma modules.

The standard inclusions of Lie bialgebras $\mathfrak{g}_\pm \subset \mathfrak{g} \simeq \mathcal{D} \mathfrak{g}_-$ lift to $U_h^{\mathfrak{E}K} \mathfrak{g}_\pm \subset U_h^{\mathfrak{E}K} \mathfrak{g} \simeq D U_h^{\mathfrak{E}K} \mathfrak{g}_-$, and we can define the induced modules of the trivial representation over $U_h^{\mathfrak{E}K} \mathfrak{g}_\pm$

$$M^h_\pm = \text{Ind}_{U_h^{\mathfrak{E}K} \mathfrak{g}_\pm}^{U_h^{\mathfrak{E}K} \mathfrak{g}} k[[\hbar]]$$

Similarly, we have Hopf algebra maps $U_h^{\mathfrak{E}K} \mathfrak{p}_\pm \subset U_h^{\mathfrak{E}K} \mathfrak{g}$ and $U_h^{\mathfrak{E}K} \mathfrak{p}_\pm \rightarrow U_h^{\mathfrak{E}K} \mathfrak{g}_D$, and we can define induced modules

$$L^h_\pm = \text{Ind}_{U_h^{\mathfrak{E}K} \mathfrak{p}_\pm}^{U_h^{\mathfrak{E}K} \mathfrak{g}} k[[\hbar]] \quad N^h_\pm = \text{Ind}_{U_h^{\mathfrak{E}K} \mathfrak{p}_-}^{U_h^{\mathfrak{E}K} \mathfrak{g}} U_h^{\mathfrak{E}K} \mathfrak{g}_D$$

We want to show that the equivalence $\tilde{\mathcal{F}} : \mathcal{YD}_{\mathfrak{g}}(U\mathfrak{g}_-)[[\hbar]] \rightarrow \mathcal{YD}_{U_h^{\mathfrak{E}K} \mathfrak{g}_-}$ matches these modules. We start proving the statement for $M_-, M^*_+$.

3.2. Quantization of $M_\pm$. We denote by $(M^h_+)^*$ the $U_h^{\mathfrak{E}K} \mathfrak{g}$–module

$$\text{Hom}_k(\text{Ind}_{U_h^{\mathfrak{E}K} \mathfrak{g}_-}^{U_h^{\mathfrak{E}K} \mathfrak{g}} k[[\hbar]], k[[\hbar]])$$
THEOREM. In the category of left $U_{\hbar}^{EK} g$-modules,

(a) $\tilde{F}(M_-) \simeq M_{\hbar}^-$

(b) $\tilde{F}(M_+^*) \simeq (M_{\hbar}^+)^*$

PROOF. The Hopf algebra $U_{\hbar}^{EK} g_-$ is constructed on the space $F(M_-)$ with unit element $u \in F(M_-)$ defined by $u(1_-) = \epsilon_+ \otimes 1_-$, where $\epsilon_+ \in M_+^*$ is defined as $\epsilon_+(x1_+) = \epsilon(x)$ for any $x \in Ug_+$. Consequently, the action of $U_{\hbar}^{EK} g_-$ on $u \in F(M_-)$ is free, as multiplication with the unit element. The coaction of $U_{\hbar}^{EK} g_-$ on $F(M_-)$ is defined using the $\mathcal{R}$-matrix associated to the braided tensor functor $F$, i.e.,

$$\pi_{M_-}^*: F(M_-) \to F(M_-) \otimes F(M_-), \quad \pi^*(x) = \mathcal{R}(u \otimes x)$$

where $x \in F(M_-)$ and $\mathcal{R}_{VW} \in \text{End}_{U_{\hbar}^{EK} g}(F(V) \otimes F(W))$ is given by $\mathcal{R}_{VW} = \sigma J^{-1}_{VW} F(\beta_{VW}) J_{VW}$, $\{J_{VW}\}_{V,W \in \mathcal{YD}_{U_{\hbar} g_-}}$ being the tensor structure on $F$. It is easy to show that $J(u \otimes u)|_{1_-} = \epsilon_+ \otimes 1_- \otimes 1_-$, and, since $\Omega(1_- \otimes 1_-) = 0$, we have

$$\mathcal{R}(u \otimes u) = u \otimes u$$

For a generic $V \in \mathcal{YD}_{U_{\hbar} g_-}[[\hbar]]$, the action of $U_{\hbar}^{EK} g_+$ is defined as

$$F(M_-)^* \otimes F(V) \to F(M_-)^* \otimes F(M_-) \otimes F(V) \to F(V)$$

This means, in particular that, for every $\phi \in I \subset U_{\hbar}^{EK} g_+$, where $I$ is the maximal ideal corresponding to $u^+$, we have $\phi.u = 0$. This proves (a).
The module $M^*_+$ satisfies the following universal property: for any $V$ in the Drinfeld category of *equicontinuous* $U\mathfrak{g}$-modules, we have

$$\text{Hom}_{U\mathfrak{g}}(V, M^*_+) \cong \text{Hom}_{U\mathfrak{g}_-}(V, k)$$

Indeed, to any map of $U\mathfrak{g}$-modules $f : V \to M^*_+$, we can associate $\hat{f} : V \to k$, $\hat{f}(v) = \langle f(v), 1_+ \rangle$. It is clear that $\hat{f}$ factors through $V/\mathfrak{g}_-V$. The equicontinuity property is necessary to show the continuity of $\hat{f}$ with respect to the topology on $V$.

Since $F$ defines an equivalence of categories, we have

$$\text{Hom}_{U_{\mathfrak{g}_+}^{\mathbb{E}_K}}(F(V), F(M^*_+)) \cong \text{Hom}_{U\mathfrak{g}_-}(V, M^*_+[\hbar]) \cong \text{Hom}_{U\mathfrak{g}_-}(V, k)[[\hbar]]$$

Using the natural isomorphism $\alpha_V : F(V) \to V[[\hbar]]$, defined by

$$\alpha_V(f) = \langle f(1_-), 1_+ \otimes \text{id} \rangle$$

we obtain a map $\text{Hom}_{U\mathfrak{g}_-}(V, k)[[\hbar]] \to \text{Hom}_k(F(V), k[[\hbar]])$. Consider now the linear isomorphism $\alpha : U_{\hbar}^{\mathbb{E}_K} \mathfrak{g}_- \to U\mathfrak{g}_-[[\hbar]]$ and for any $x \in U\mathfrak{g}_-$ consider the $\mathfrak{g}$-intertwiner $\psi_x : M_+ \to M^*_+ \otimes M_-$ defined by $\psi_x(1_-) = \epsilon_+ \otimes x 1_-$. It is clear that, if $f(1_-) = f(1) \otimes f(2)$ in Swedler's notation,

$$\alpha_V(\psi_x.f) = \langle (i_+^V \otimes \text{id})\Phi^{-1}(\text{id} \otimes f)(\epsilon_+ \otimes x.1_-), 1_+ \otimes \text{id} \rangle$$

$$= \langle \Phi^{-1}(\epsilon_+ \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta(x))(\text{id} \otimes f(1) \otimes f(2)), (T \otimes \text{id})(1_+ \otimes 1_+ \otimes \text{id}) \rangle$$

$$= \langle \Delta(x)(f(1) \otimes f(2)), 1_+ \otimes \text{id} \rangle$$

$$= \langle f(1), 1_+ \rangle x.f(2)$$

$$= x.\alpha_V(f)$$
using the fact that \((\epsilon \otimes 1 \otimes 1)(\Phi) = 1^{\otimes 2}\) and \((\epsilon \otimes 1)(T) = 1\). So, clearly, if \(\phi \in \text{Hom}_{Ug}(V, k)\), then \(\phi \circ \alpha_V \in \text{Hom}_{U_{E^K}g_0}(F(V), k[[\hbar]])\). Then \(F(M^*_+)\) satisfies the universal property of \(\text{Hom}_k(\text{Ind}_{U_{E^K}g_0}^U k[[\hbar]], k[[\hbar]])\) and (b) is proved. \(\Box\)

3.3. Quantization of relative Verma modules. The proof of (b) shows that the linear functional \(F(M^*_+) \to k[[\hbar]]\) is, in fact, the trivial deformation of the functional \(M^*_+ \to k\). These results extend to the relative case and hold for the right \(g_D\)–action on \(L_-, N^*_+\).

**Theorem.** In the category \(\mathcal{YD}_{U_{E^K}g_0}\)

\[
\begin{align*}
(a) \quad \tilde{F}(L_-) &\simeq L^-_h \\
(b) \quad \tilde{F}(N^*_+) &\simeq (N^*_h)^* \\
(c) \quad \tilde{F}_D(L_-) &\simeq L^-_h \\
(d) \quad \tilde{F}_D(N^*_+) &\simeq (N^*_h)^*
\end{align*}
\]

Moreover, as right \(U^E_{E_K}g_D\)–module

\[
\begin{align*}
(c) \quad \tilde{F}_D(L_-) &\simeq L^-_h \\
(d) \quad \tilde{F}_D(N^*_+) &\simeq (N^*_h)^*
\end{align*}
\]

The proof of (a) and (b) amounts to constructing the morphisms

\[
k[[\hbar]] \to \tilde{F}(L_-) \quad \tilde{F}(N^*_+) \to U^E_{E_K}g_D
\]
equivariant under the action of \(U^E_{E_K}p_+\) and \(U^E_{E_K}p_-\) respectively.

A direct construction along the lines of the proof of Theorem 3.2 is however not straightforward. We prove this theorem in the next section by using a description of the modules \(L_-, N^*_+\) and their images through \(\tilde{F}\) via \(\text{Prop}\).
categories. These descriptions show that the classical intertwiners
\[ k \to L_- \quad N_+^* \to U \mathfrak{g}_D^* \]
satisfy the required properties and yield canonical identifications
\[ \tilde{F}(L_-) \simeq L_-^h \quad \tilde{F}(N_+^*) \simeq (N_+^h)^* \]

4. Universal relative Verma modules

In this section, we prove Theorem 3.3 by using suitable PROP \((product-permutation)\) categories compatible with the EK universal quantization functor [EK2, EG].

4.1. PROP description of the EK quantization functor. We will briefly review the construction of Etingof–Kazhdan in the setting of PROP categories [EK2].

A PROP is a symmetric tensor category generated by one object. More precisely, a cyclic category over \(S\) is the datum of

- a symmetric monoidal \(k\)-linear category \((C, \otimes)\) whose objects are non-negative integers, such that \([n] = [1]^{\otimes n}\) and the unit object is \([0]\)
- a bigraded set \(S = \bigcup_{m,n \in \mathbb{Z}_{\geq 0}} S_{nm}\) of morphism of \(C\), with \(S_{nm} \subset \text{Hom}_C([m],[n])\)

such that any morphism of \(C\) can be obtained from the morphisms in \(S\) and permutation maps in \(\text{Hom}_C([m],[m])\) by compositions, tensor products or linear combinations over \(k\). We denote by \(\mathcal{F}_S\) the free cyclic category over \(S\).
Then there exists a unique symmetric tensor functor $\mathcal{F}_S \to \mathcal{C}$, and the following holds (cf. [EK2])

**Proposition.** Let $\mathcal{C}$ be any cyclic category generated by a set $S$ of morphisms. Then $\mathcal{C}$ has the form $\mathcal{F}_S / \mathcal{I}$, where $\mathcal{I}$ is a tensor ideal in $\mathcal{F}_S$.

Let $\mathcal{N}$ be a symmetric monoidal $k$–linear category, and $X$ an object in $\mathcal{N}$. A linear algebraic structure of type $\mathcal{C}$ on $X$ is a symmetric tensor functor $\mathcal{G}_X : \mathcal{C} \to \mathcal{N}$ such that $\mathcal{G}_X([1]) = X$. A linear algebraic structure of type $\mathcal{C}$ on $X$ is a collection of morphisms between tensor powers of $X$ satisfying certain consistency relations.

We mainly consider the case of non–degenerate cyclic categories, i.e., symmetric tensor categories with injective maps $k[\mathcal{G}_n] \to \text{Hom}_\mathcal{C}([n], [n])$. We first consider the Karoubian envelope of $\mathcal{C}$ obtained by formal addition to $\mathcal{C}$ of the kernel of the idempotents in $k[\mathcal{G}_n]$ acting on $[n]$. Furthermore, we consider the closure under inductive limits. In this category, denoted $S(\mathcal{C})$, every object is isomorphic to a direct sum of indecomposables, corresponding to irreducible representations of $\mathcal{G}_n$ (cf. [EK2, EG]). In particular, in $S(\mathcal{C})$, we can consider the symmetric algebra

$$S[1] = \bigoplus_{n \geq 0} S^n[1]$$

If $\mathcal{N}$ is closed under inductive limits, then any linear algebraic structure of type $\mathcal{C}$ extends to an additive symmetric tensor functor

$$\mathcal{G}_X : S(\mathcal{C}) \to \mathcal{N}$$

We introduce the following fundamental Props.
• **Lie bialgebras.** In this case the set $S$ consists of two elements of bidegrees $(2, 1), (1, 2)$, the universal commutator and cocommutator. The category $\mathcal{C} = LBA$ is $\mathcal{F}_S / \mathcal{I}$, where $\mathcal{I}$ is generated by the classical five relations.

• **Hopf algebras.** In this case, the set $S$ consists of six elements of bidegrees $(2, 1), (1, 2), (0, 1), (1, 0), (1, 1), (1, 1)$, the universal product, coproduct, unit, count, antipode, inverse antipode. The category $\mathcal{C} = HA$ is $\mathcal{F}_S / \mathcal{I}$, where $\mathcal{I}$ is generated by the classical four relations.

The quantization functor described in Section 1 can be described in this generality, as stated by the following (cf. [EK2, Thm.1.2])

**Theorem.** There exists a universal quantization functor $Q : HA \to S(LBA)$.

Let $\mathfrak{g}_-$ be the canonical Lie dialgebra $[1]$ in $LBA$ with commutator $\mu$ and cocommutator $\delta$. Let $U\mathfrak{g}_- := S\mathfrak{g}_- \in S(LBA)$ be the universal enveloping algebra of $\mathfrak{g}_-$. The construction of the Etingof–Kazhdan quantization functor amounts to the introduction of a Hopf algebra structure on $U\mathfrak{g}_-$, which coincides with the standard one modulo $\langle \delta \rangle$, and yields the Lie bialgebra structure on $\mathfrak{g}_-$ when considered modulo $\langle \delta^2 \rangle$. This Hopf algebra defines the object $Q[1]$, where $[1]$ is the generating object in $HA$. The formulae used to defined the Hopf structure coincide with those defined in [EK1, Part II] and described in Section 1. In particular, they rely on the construction of the Verma modules

$$M_- := S\mathfrak{g}_- \quad M_+^* = S\hat{\mathfrak{g}}_-$$
realized in the category of Drinfeld–Yetter modules over \( g_- \) as object of \( \text{LBA} \).

**4.2. Props for split pairs of Lie bialgebras.** Let \((g_-, g_{D,-})\) be a split pair of Lie bialgebras, i.e., there are Lie bialgebra maps

\[
  g_{D,-} \xrightarrow{i} g_- \xrightarrow{p} g_{D,-}
\]
such that \( p \circ i = \text{id} \). These maps induce an inclusion \( \mathcal{D}g_{D,-} \subset \mathcal{D}g_- \) and consequently an inclusion of Manin triple \((g_D, g_{D,-}, g_{D,+}) \subset (g, g_-, g_+)\), as described in Section 2.6.

**Definition.** We denote by \( \text{PLBA} \) the Karoubian envelope of the multi-colored \( \text{PROP} \), whose class of objects is generated by the Lie bialgebra objects \([g_-], [g_{D,-}]\), related by the maps \( i : [g_{D,-}] \to [g_-] \), \( p : [g_-] \to [g_{D,-}] \), such that \( p \circ i = \text{id}_{[g_{D,-}]} \).

The Karoubian envelope implies that \([m_-] := \ker(p) \in \text{PLBA}\).

**Proposition.** The multi-colored \( \text{PROP} \) \( \text{PLBA} \) is endowed with a pair of functors \( U, L \)

\[
U, L : \text{LBA} \to \text{PLBA} \quad U[1] := [g_-], \quad L[1] := [g_{D,-}]
\]

and natural transformations \( i, p \), induced by the maps \( i, p \) in \( \text{PLBA} \),

\[
\begin{array}{c}
\text{LBA} \\
\xrightarrow{i} \\
\xrightarrow{U}
\end{array} \quad \begin{array}{c}
\downarrow \quad i \\
\quad p
\end{array} \quad \begin{array}{c}
\text{PLBA} \\
\xleftarrow{L}
\end{array}
\]
such that \( p \circ i = \text{id} \). Moreover, it satisfies the following universal property: for any tensor category \( \mathcal{C} \), closed under kernels of projections, with the same property as \( \mathcal{PLBA} \), there exists a unique tensor functor \( \mathcal{PLBA} \rightarrow \mathcal{C} \) such that the following diagram commutes

4.3. Props for split pairs of Hopf algebras. We can analogously define suitable PROP categories corresponding to split pairs of Hopf algebras. In particular, we consider the PROP \( \mathcal{PHA} \) characterized by functors \( U_h, L_h \) and natural transformations \( p_h, i_h \) satisfying

where \( \mathcal{HA} \) denotes the PROP category of Hopf algebras. These also satisfy

\[
\begin{array}{ccc}
\mathcal{HA} & \xrightarrow{Q_{\mathcal{EK}}} & S(\mathcal{LBA}) \\
\downarrow & & \downarrow \\
\mathcal{PHA} & \xrightarrow{Q_{\mathcal{PLBA}}} & S(\mathcal{PLBA})
\end{array}
\]

where \( Q_{\mathcal{PLBA}} \) is the extension of the Etingof–Kazhdan quantization functor to \( \mathcal{PLBA} \), obtain by the universal property described above with \( \mathcal{C} = S(\mathcal{PLBA}) \).
4.4. Props for parabolic Lie subalgebras. In order to describe the module $N^+_1$ it is necessary to deal with the Lie bialgebra object $\mathfrak{p}_-$ or, in other words to introduce the double of $\mathfrak{g}_{D,-}$ and the Prop $D_{\oplus}(LBA)$ [EG]. We then introduce the multicolored Prop as a cofiber product of $\text{PLBA}$ and $D_{\oplus}(LBA)$ over $LBA$.

**Proposition.** The multicolored Prop $\text{PLBAD}$ is endowed with canonical functors

$$D_{\oplus}(LBA) \to \text{PLBAD} \leftarrow \text{PLBA}$$

and satisfies the following universal property:

\[
\begin{array}{ccc}
LBA & \xrightarrow{\text{double}} & D_{\oplus}(LBA) \\
\downarrow & & \downarrow \\
\text{PLBA} & \xrightarrow{} & \text{PLBAD} \\
\end{array}
\]

where double is the Prop map introduced in [EG].

In $\text{PLBAD}$ we can consider the Lie bialgebra object $[\mathfrak{p}_-]$.

4.5. Props for parabolic Hopf subalgebras. Similarly, we introduce the multicolored Prop $\text{PHAD}$, endowed with canonical functors (cf. [EG])

$$D_{\otimes}(HA) \to \text{PHAD} \leftarrow \text{PHA}$$
and satisfying an analogous universal property:

\[
\begin{array}{c}
HA \xrightarrow{\text{double}} D_{\otimes}(HA) \\
\downarrow \\
PHA \xrightarrow{} PHAD
\end{array}
\]

Moreover, we then have a canonical functor

\[Q_{PLBAD} : PHAD \to S(PLBAD)\]

obtained applying such universal property with \(C = S(PLBAD)\) and satisfying

\[
\begin{array}{c}
HA \xleftarrow{L_{HA}} \xrightarrow{\text{double}} D_{\otimes}(HA) \\
\downarrow \\
PHA \xrightarrow{Q_{PLBA}} PHAD \\
\downarrow \\
S(LBA) \xrightarrow{S(\text{double})} S(D_{\otimes}(LBA)) \\
\downarrow \\
S(PLBA) \xrightarrow{} S(PLBAD)
\end{array}
\]

The commutativity of the square on the back is given by the compatibility of the quantization functor with the doubling operations, proved in [EG].

4.6. Prop description of \(L_-, N^*_+\). The modules \(L_-, N^*_+\) can be realized in \(S(PLBAD)\). The module \(L_-\) is constructed over the object \(Sm_- \in\)
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$S(PLBA)$. The structure of Drinfeld-Yetter module over $g_-$ is determined in the following way:

- the free action of the Lie algebra object $m_-$ is defined by the map

  $$Sm_- \otimes Sm_- \rightarrow Sm_-$$

  given by Campbell-Hausdorff series, describing on $Sm_-$ the multiplication in $Um_-$. 

- we define the action of $g_{D,-}$ to be trivial on $1 \rightarrow Sm_-$. 

- The actions of $m_-, g_{D,-}$, the relation

  $$\pi \circ ([,] \otimes 1) = \pi \circ (1 \otimes \pi) - \pi \circ (1 \otimes \pi) \circ \sigma_{12}$$

  and the map $[,] : g_{D,-} \otimes m_- \rightarrow m_-$ define the action of $g_-$. 

- We then impose the trivial coaction on $1 \rightarrow Sm_-$ and the compatibility condition between action and coaction

  $$\pi^* \circ \pi = (1 \otimes \pi)\sigma_{12}(1 \otimes \pi^*) - (1 \otimes \pi)(\delta \otimes 1) + (\mu \otimes 1)(1 \otimes \pi^*)$$

  determines the coaction for $Sm_-$. The action defined is compatible with $[,] : g_{D,-} \otimes m_- \rightarrow m_-$. 

  Similarly, the module $N^*_+\mathfrak{p}$ can be realized on the object $\widehat{Sp}_-$, formally added to $S(PLBAD)$. 

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We determine the formulae for the action and the coaction of $g_-$ by direct inspection of the action of $g = g_- \oplus g_+$ on $N_+$ in the category Vect. Namely, the identification $N_+^* = \hat{Sp}_-$ is clearly obtained through the invariant bilinear form $\langle -, - \rangle$ and there are topological formulae expressing the action of $g$ on $N_+$. Therefore we determine action and coaction on $N_+^*$ in the following way:

- the $g_+$–action on $N_+ = Sp_+ = Sg_+ \otimes Sg_{D,-}$ is given by the free action on the first factor $Sg_+$ expressed by Campbell–Hausdorff series.

- the action of $g_- = m_+ \oplus g_{D,-}$ on the subspace $Sg_{D,-} \subset Sp_+$ is given by the trivial action of $m_-$ and the usual free action of $g_{D,-}$ by multiplication.

- The action of $g_-$ is then interpreted as a topological coaction of $g_+$ and the aforementioned compatibility condition between action and coaction allows to extend the formula for the topological $g_+$ coaction on the entire space $Sp_+$.

- Through the invariant bilinear form $\langle -, - \rangle$, these formulae are carried over $N_+^* = \hat{Sp}_-$, by switching, in particular, the bracket and the topological cobracket on $g_+$ with the cobracket and the bracket in $g_-$, respectively.

- The obtained formulae, describing the action and the coaction of $g_-$ on $N_+^*$, are well–defined in the category $PLBAD$ and define the requested structure of Drinfeld-Yetter module over $g_-$. 
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4.7. Proof of Theorem 3.3. The relative Verma module

\[ N_+ = \text{Ind}^g_{m-} k \simeq \text{Ind}^g_{p-} U g_D \]

satisfies

\[ \text{Hom}_{U g}(N_+, V) \simeq \text{Hom}_{U p-}(U g_D, V) \]

for every \( U g \)-module \( V \). We have a canonical map of \( p_- \)-modules \( \rho_D : U g_D \to N_+ \) corresponding to the identity in the case \( V = N_+ \). We get a map of \( p_- \)-modules \( \rho_D^* : N_+^* \to U g_D^* \) inducing an isomorphism

\[ \text{Hom}_{U g}(V, N_+^*) \simeq \text{Hom}_{U p-}(V, U g_D^*) \]

The morphism \( \rho_D^* \) can indeed be thought as

\[
\begin{array}{ccc}
U p_- \otimes N_+^* & \longrightarrow & N_+^* \\
\downarrow \downarrow \downarrow \downarrow & & \downarrow \rho_D^* \\
U g_D \otimes U g_D^* & \longrightarrow & U g_D^*
\end{array}
\]

Assuming the existence of a suitable finite \( \mathbb{N} \)-grading, a split pair of Lie bialgebras \((g_-, g_{D,-})\), gives rise to a functor

\[ PLBAD \to \text{Vect} \]

Consider now the trivial split pair given by \((g_{D,-}, g_{D,-})\). We have a natural transformation

\[
\begin{array}{ccc}
PLBAD & \longrightarrow & \text{Vect} \\
\downarrow p & & \downarrow \\
(g_{D,-}, g_{D,-}) & & (g_{D,-}, g_{D,-})
\end{array}
\]
where \( p \) naturally extends to the projection \( p_- \to g_D \).

The module \( U(g_D)^* \) is indeed the module \( N^*_+ \) with respect to the trivial pair \((g_{D,-}, g_{D,-})\). Consequently, the existence of the \( p_- \)-intertwiner \( \rho_D^* \) can be interpreted as a simple consequence of the existence of natural transformation \( p \).

The quantization functor \( Q_{PLBAD} \) extends the natural transformation \( p \) to

\[
\begin{array}{ccc}
PHAD & \longrightarrow & S(PLBAD) \\
\downarrow & & \downarrow S(p) \\
& \Rightarrow & Vect \\
& \Rightarrow & \\
& \Rightarrow & \uparrow S(i) \\
& \Rightarrow & \uparrow \\
& \Rightarrow & Vect
\end{array}
\]

and shows that

\[
F(N^*_+) \simeq (N^h_+)^*
\]

Similarly, we can consider the natural transformation \( S(i) \) and the diagram

\[
\begin{array}{ccc}
PHAD & \longrightarrow & S(PLBAD) \\
\downarrow & & \downarrow S(i) \\
& \Rightarrow & Vect \\
& \Rightarrow & \\
& \Rightarrow & \\
& \Rightarrow & \uparrow \uparrow \uparrow
\end{array}
\]

implying

\[
F(L_-) \simeq L^h_-
\]
We can make analogous considerations for the right $\mathfrak{g}_D$-action on $L_-, N_+^h$. This leads to isomorphisms of right $U_h^{\mathfrak{g}_D}$-modules

$$\widetilde{F}_D(N_+^h) \simeq (N_+^h)^* \quad \widetilde{F}_D(L_-) \simeq L_-^h$$

5. Chains of Manin triples

5.1. Chains of length 2. In Section 3, given an inclusion of Manin triples $i_D : \mathfrak{g}_D \subseteq \mathfrak{g}$, we introduced the relative quantum Verma modules

$$L_+^h = \text{Ind}_{U_h^{\mathfrak{g}_D}[p_+]}^{U_h^{\mathfrak{g}}} k[[h]] \quad N_+^h = \text{Ind}_{U_h^{\mathfrak{g}_D}[p_-]}^{U_h^{\mathfrak{g}} U_h^{\mathfrak{g}_D}}$$

These modules allow to define the functor

$$\Gamma_h : \mathcal{D}(U_h^{\mathfrak{g}_D}) \to \mathcal{D}(U_h^{\mathfrak{g}_D})$$

by

$$\Gamma_h(V) = \text{Hom}_{U_h^{\mathfrak{g}_D}}(L_+^h, (N_+^h)^* \otimes V)$$

**Lemma.** The functor $\Gamma_h$ is naturally tensor isomorphic to the restriction functor $(U_h^{\mathfrak{g}_D}(i_D^h))^*$.

**Proof.** The proof of the existence of the natural isomorphism as $U_h^{\mathfrak{g}_D}$-module is identical to that of Prop. 2.13. The isomorphism respects the tensor structures, because there are only trivial associators involved. \qed

We now prove the following

**Theorem.** Let $\mathfrak{g}, \mathfrak{g}_D$ be Manin triples with a finite $\mathbb{Z}$-grading and $i_D : \mathfrak{g}_D \subseteq \mathfrak{g}$ an inclusion of Manin triples compatible with the grading. Then, there
exists an algebra isomorphism
\[ \Psi : \widehat{U}_h \mathfrak{g} \rightarrow \widehat{U}_g[[h]] \]

restricting to \( \Psi_{EK}^D \) on \( U_h^{EK} \mathfrak{g}_D \), where the completion is given with respect to Drinfeld–Yetter modules.

**Proof.** In the previous section, we showed that the quantization of the \((U_h^{EK} \mathfrak{g}, U_h^{EK} \mathfrak{g}_D)\)-modules \( N_+, L_- \) gives

\[
\tilde{F}(N_+) \xrightarrow{U_h^{EK} \mathfrak{g}} (N_+)^* \xleftarrow{U_h^{EK} \mathfrak{g}_D} F_D^{EK}(N_+)
\]
\[
\tilde{F}(L_-) \xrightarrow{U_h^{EK} \mathfrak{g}} L_- \xleftarrow{U_h^{EK} \mathfrak{g}_D} F_D^{EK}(L_-)
\]

Recall that the standard natural transformations \( \alpha_V : \tilde{F}(V) \simeq V[[h]] \), \( (\alpha_D)_V : \tilde{F}_D(V) \simeq V[[h]] \) give isomorphisms of right \( U \mathfrak{g}_D[[h]] \)-modules

\[
\tilde{F}(N_+) \simeq N_+[[h]] \quad \tilde{F}(L_-) \simeq L_-[[h]]
\]

and isomorphisms of \( U \mathfrak{g}[[h]] \)-modules

\[
F_D^{EK}(N_+) \simeq N_+[[h]] \quad F_D^{EK}(L_-) \simeq L_-[[h]]
\]

In particular, we get isomorphisms of right \( U_h^{EK} \mathfrak{g}_D \)-modules

\[
F_D^{EK} \circ \tilde{F}(N_+) \simeq F_D^{EK}(N_+) \simeq (N_+^h)^* \quad F_D^{EK} \circ \tilde{F}(L_-) \simeq F_D^{EK}(L_-) \simeq L_-^h
\]

and isomorphisms of \( U_h^{EK} \mathfrak{g} \)-modules

\[
F_D^{EK} \circ \tilde{F}(N_+) \simeq \tilde{F}(N_+) \simeq (N_+^h)^* \quad F_D^{EK} \circ \tilde{F}(L_-) \simeq \tilde{F}(L_-) \simeq L_-^h
\]
5. CHAINS OF MANIN TRIPLES

We have a natural isomorphism through $J$:

$$\text{Hom}_{U_{\mathfrak{g}}}(F(L_{-}), F(N^{*}_{+} \otimes F(V))) \simeq \text{Hom}_{\mathfrak{g}}(L_{-}, N^{*}_{+} \otimes V)[[\hbar]]$$

This is indeed an isomorphism of $U_{\mathfrak{g}}[D][[\hbar]]$-modules, since, for $x \in U_{\mathfrak{g}}$, $\phi \in \text{Hom}_{U_{\mathfrak{g}}}(F(L_{-}), F(N^{*}_{+} \otimes F(V))$, we have

$$x.\phi := (F(x) \otimes \text{id}) \circ \phi \quad J \circ (F(x) \otimes \text{id})) = F(x \otimes \text{id}) \circ J$$

Quantizing both sides and using the isomorphism $F^{ek}_{D} \circ F(N^{*}_{+}) \simeq (N^{h}_{+})^{*}$, we obtain a natural transformation

$$\gamma_{D} : \Gamma \circ \tilde{F} \simeq \tilde{F}_{D} \circ \Gamma$$

making the following diagram commutative

$$\xymatrix{ \mathcal{D}_{\Phi}(U_{\mathfrak{g}}) \ar[r]^{\tilde{F}} \ar[d]^{\Gamma} & \mathcal{D}(U_{\mathfrak{g}}) \ar[d]^{\Gamma_{h}} \ar[dl]_{\gamma_{D}} \ar[dl]_{\tilde{F}_{D}} & \mathcal{D}_{\Phi}(U_{\mathfrak{g}})[[\hbar]] \ar[r] & \mathcal{D}(U_{\mathfrak{g}})[[\hbar]] }$$

Applying the construction above to the algebra of endomorphisms of the fiber functor, we get the result. □

5.2. Chains of arbitrary length. For any chain

$$0 = \mathfrak{g}_{0} \subseteq \mathfrak{g}_{1} \subseteq \cdots \subseteq \mathfrak{g}_{n-1} \subseteq \mathfrak{g}_{n} = \mathfrak{g}$$

of inclusions of Manin triples, the natural transformations

$$\gamma_{i,i+1} \in \text{Nat}_{\Phi}(\Gamma_{i,i+1}^{h} \circ \tilde{F}_{i+1}, \tilde{F}_{i} \circ \Gamma_{i,i+1})$$
where \( 0 \leq i \leq n - 1 \), \( \Gamma_{0,1} : D_\Phi(g_1) \to \text{Vect}_{k[[[\hbar]]]} \) is the EK fiber functor, and \( F_{0}^{\text{EK}} = \text{id} \), yield a natural transformation

\[
\gamma_C = \gamma_{0,1} \circ \cdots \circ \gamma_{n-1,n}
\]

\[
\in \text{Nat} \odot (\Gamma_{0,1}^h \circ \cdots \circ \Gamma_{n-1,n}^h \circ \tilde{F}_n, \Gamma_{0,1} \circ \cdots \circ \Gamma_{n-1,n})
\]

\[
\cong \text{Nat} \odot ((i_{0,n}^*)^h \circ \tilde{F}_n, \Gamma_{0,1} \circ \cdots \circ \Gamma_{n-1,n})
\]

\[
= \text{Nat} \odot (\tilde{F}_n, \Gamma_{0,1} \circ \cdots \circ \Gamma_{n-1,n})
\]

where we used \( \Gamma_{i,i+1}^h \cong (i_{i,i+1}^*)^h \), and the fact that the composition \( (i_{0,n}^*)^h \circ \tilde{F}_n \) is the EK fiber functor for \( g_n \), which we denote by the same symbol as \( \tilde{F}_n \).

This proves the following

**Theorem.**

(i) For any chain of Manin triples

\[ C : g_0 \subseteq g_1 \subseteq \cdots \subseteq g_n \subseteq g \]

there exists an isomorphism of algebras

\[ \Psi_C : \hat{U}_h^{\text{EK}} g \to \hat{U}_g[[\hbar]] \]

such that \( \Psi_C(U_h^{\text{EK}} g_i) = \hat{U}_g_i[[\hbar]] \) for any \( g_i \in C \).

(ii) Given two chains \( C, C' \), the natural transformation

\[ \Phi_{CC'} := \gamma_C^{-1} \circ \gamma_{C'} \in \text{Aut}(F_{0}^{\text{EK}}) \]

satisfies

\[ \text{Ad}(\Phi_{CC'}) \Psi_{C'} = \Psi_C \]
The following proposition is clear.

**Proposition.** The natural transformations \( \{ \Phi_{CC'} \}_{C, C'} \) satisfy the following properties

(i) **Orientation.** Given two chains \( C, C' \)

\[ \Phi_{CC'} = \Phi_{C'C}^{-1} \]

(ii) **Transitivity.** Given the chains \( C, C', C'' \)

\[ \Phi_{CC'} \circ \Phi_{C'C''} = \Phi_{CC''} \]

(iii) **Factorization.** Given the chains

\[ C, C' : g_0 \subseteq g_0 \subseteq \cdots \subseteq g_n \]
\[ D, D' : g_0 \subseteq \cdots \subseteq g_{n+n'} \]

\[ \Phi_{(C \cup D)(C' \cup D')} = \Phi_{CC'} \circ \Phi_{DD'} \]

5.3. **Abelian Manin triples and central extensions.** We will now consider the following special case, that generalizes the role of Levi subalgebras for Kac–Moody algebras.

**Proposition.** If \( g \) admits a Manin subtriple \( l_D \), obtained by a central extension of \( g_D \), then the relative twists and the gauge transformations are invariant under \( l_D \). In particular, the Etingof–Kazhdan constructions are invariant under abelian Manin subtriples.

**Proof.** For \( g_D = \{0\} \), the statement reduces to prove that the Etingof–Kazhdan functor preserves the action of an abelian Manin subtriple \( a \subseteq g \)
(cf. [EK6, Thm. 4.3], with $a_- = h$). Under this assumption, the natural map

$$Ua_- \longrightarrow U_{h}^{\text{EK}} a_- := \tilde{F}(M_-)$$

defines an inclusion of bialgebras. For any $V[[h]] \in D_{\Phi}(U\mathfrak{g})$, the natural identification

$$\alpha_{V} : F(V) \rightarrow V[[h]]$$

is then an isomorphism of $Ua$-modules. This gives the following commutative diagram

$$
\begin{array}{ccc}
D_{\Phi}(U\mathfrak{g}) & \xrightarrow{\tilde{F}} & D(U_{h}^{\text{EK}} \mathfrak{g}) \\
\downarrow F & & \downarrow \leftarrow \\
D(Ua[[h]]) & & \leftarrow
\end{array}
$$

We can observe that the tensor restriction functor fits in an analogous diagram. It is easy to show that the object $\Gamma_{D}(L_-)$ is naturally a pointed Hopf algebra in the category $D_{\Phi_{D}}(U\mathfrak{g}_{D})$. We denote by $D_{\mathfrak{g}_{D}}(\Gamma_{D}(L_-))$ the category of Drinfeld–Yetter modules over $\Gamma_{D}(L_-)$ in the category $D_{\Phi_{D}}(U\mathfrak{g}_{D})$. This category is naturally equivalent to the category of Drinfeld–Yetter module over the Radford’s product $U\mathfrak{g}_{D,-}[[h]]\#\Gamma_{D}(L_-)$ and there is a natural identification

$$
\begin{array}{ccc}
D_{\Phi}(U\mathfrak{g}) & \xrightarrow{\#} & D_{\mathfrak{g}_{D}}(\Gamma_{D}(L_-)) \\
\downarrow \Gamma_{D} & & \downarrow \\
D_{\Phi_{D}}(U\mathfrak{g}_{D}) & &
\end{array}
$$
Moreover, there is a natural inclusion of bialgebras

\[ UI_D \subset U g_D[[\hbar]] \# \Gamma_D(L_-) \]

and a natural \( UI_D \)-module identification \( \Gamma_D(V) \to V[[\hbar]] \). This originates natural identifications

\[ \xymatrix{ \mathcal{D}_\Phi(U g) \ar@/_/[r] \ar[r] \ar[d]^\Gamma_D & \mathcal{D}_{g_D}(\Gamma_D(L_-)) \ar[d] \\
\mathcal{D}(UI_D) & \mathcal{D}_{\Phi_D}(U g_D) \ar[l] } \]

This proves that relative twists are invariant under \( I_D \). It is clear that the Casimir operator \( \Omega_D \in (g_D \otimes g_D)^{I_D} \) defines a braided tensor structure on \( \mathcal{D}(UI_D[[\hbar]]) \) which is preserved by the restriction functor induced by the inclusion \( j_D : g_D \subset I_D \). Given the decomposition \( I_D = g_D \rtimes c_D \), the natural map \( Uc_D \to \text{End}_{g_D}(j_D^*V) \) induces an action of \( Uc_D \) on \( \widetilde{F_D}(j_D^*V) \), commuting with the action of \( U_{EK}^h g_D \). Therefore, we obtain a naturally commutative diagram

\[ \xymatrix{ \mathcal{D}_{\Phi_D}(UI_D) \ar[r]^{\widetilde{F_D}} \ar[d]^{j_D} & \mathcal{D}(U_{EK}^h I_D) \ar[d] \\
\mathcal{D}_{\Phi_D}(U g_D) \ar[r]^{\widetilde{F_D}} & \mathcal{D}(U_{EK}^h g_D) } \]

where \( \widetilde{F_D} \) is the tensor functor induced by the composition \( \widetilde{F_D} \circ j_D^* \). The natural transformation \( \gamma \) automatically lifts to the level of \( I_D \), as showed in
the following diagram

\[
\begin{array}{ccc}
\mathcal{D}_g(Ug) & \xrightarrow{\tilde{F}} & \mathcal{D}(U^Eh g) \\
\downarrow{\Gamma} & & \downarrow{\Delta_0} \\
\mathcal{D}_{\Phi_D}(U_{D}) & \xrightarrow{\tilde{F}_D} & \mathcal{D}(U^Eh I_D) \\
\downarrow{j_D} & & \downarrow{\Delta_D} \\
\mathcal{D}_{\Phi_D}(Ug_D) & \xrightarrow{\tilde{F}_D} & \mathcal{D}(U^Eh g_D) \\
\end{array}
\]

\[
\square
\]

6. The equivalence of quasi–Coxeter categories

The following is the main result of this Chapter.

**Theorem.** Let \( g \) be a symmetrizable Kac–Moody algebra with a fixed \( D_g \)-structure. Then the completion \( \hat{U}_h g \) is isomorphic to a quasi-Coxeter quasitriangular quasibialgebra of type \( D_g \) on the quasitriangular \( D_g \)-quasibialgebra

\[
(Ug[[\hbar]], \{Ug_D[[\hbar]]\}, \Delta_0, \{\Phi_D^{KZ}\}, \{R_D^{KZ}\})
\]

where the completion is taken with respect to the integrable modules in category \( O \).

6.1. \( D \)-structures on Kac–Moody algebras. Let \( A = (a_{ij})_{i,j \in I} \) be a complex \( n \times n \) matrix and \( g = g(A) \) the corresponding generalized Kac–Moody algebra defined in Section 1. Let \( J \) be a nonempty subset of \( I \). Consider the submatrix of \( A \) defined by

\[
A_J = (a_{ij})_{i,j \in J}
\]

We recall the following proposition from [Ka, Ex.1.2]
Proposition. Let

\[ \Pi_J := \{ \alpha_j \mid j \in J \} \quad \Pi_J^\vee := \{ h_j \mid j \in J \} \]

Let \( h'_J \) be the subspace of \( h \) generated by \( \Pi_J^\vee \) and

\[ t_J = \bigcap_{j \in J} \ker \alpha_j = \{ h \in h \mid \langle \alpha_j, h \rangle = 0 \ \forall j \in J \} \]

Let \( h''_J \) be a supplementary subspace of \( h'_J + t_J \) in \( h \) and let

\[ h_J = h'_J \oplus h''_J \]

Then,

(i) \((h_J, \Pi_J, \Pi_J^\vee)\) is a realization of the generalized Cartan matrix \( A_J \).

(ii) The subalgebra \( g_J \subset g \), generated by \( \{e_j, f_j\}_{j \in J} \) and \( h_J \), is the Kac–Moody algebra associated to the realization \((h_J, \Pi_J, \Pi_J^\vee)\) of \( A_J \).

Set

\[ Q_J = \sum_{j \in J} \mathbb{Z} \alpha_j \subset Q \quad g = g(A) = \bigoplus_{\alpha \in Q} g_\alpha \]

Then,

(iii) \[ g_J = h_J \oplus \bigoplus_{\alpha \in Q_J \setminus \{0\}} g_\alpha \]

Let \( A \) be a symmetrizable matrix with a fixed decomposition and \( (\cdot | \cdot) \) be the standard normalized non-degenerate bilinear form on \( h \). Then,

(iv) The restriction of \((\cdot | \cdot)\) to \( h_J \) is non-degenerate.
Proof. Since \( \dim(\mathfrak{h}_J' \cap t_J) = \dim(\mathfrak{z}(\mathfrak{g}_J) = n_J - l_J \), where \( n_J = |J| \) and \( l_J = \text{rank}(A_J) \), it follows that

\[
\dim \mathfrak{h}_J'' = n_J - l_J \quad \dim \mathfrak{h}_J = 2n_J - l_J
\]

Moreover, by construction, the restriction of \( \{\alpha_j\}_{j \in J} \) to \( \mathfrak{h}_J \) are linearly independent. Indeed, since

\[
\langle \sum_{j \in J} c_j \alpha_j, t_J \rangle = 0 \quad \forall c_j \in \mathbb{C},
\]

\[
\langle \sum_{j \in J} c_j \alpha_j, \mathfrak{h}_J \rangle = 0 = \langle \sum_{j \in J} c_j \alpha_j, \mathfrak{h} \rangle = 0 \implies c_j = 0
\]

This proves (i). The proof of (ii) and (iii) is clear.

Assume now that \( A \) is irreducible and symmetrizable and there exists \( h \in \mathfrak{h}_J \) such that

\[
(h|h') = 0 \quad \forall h' \in \mathfrak{h}_J
\]

In particular, \( (h|\alpha^\vee_j) = 0 \) and \( h \in \mathfrak{h}_J' \cap t_J \subset \mathfrak{h}_J' \). Therefore, \( h = \sum c_j \alpha^\vee_j \) and

\[
\langle \sum c_j \alpha^\vee_j | h' \rangle = \sum c_j (\alpha^\vee_j | h') = \langle \sum c_j d_j \alpha_j, h' \rangle = 0
\]

Since the operators \( \{\alpha_j\} \) are linearly independent over \( \mathfrak{h}_J \) and \( d_j \neq 0 \), we have \( c_j = 0 \) and \( h = 0 \). We conclude that (i) is non-degenerate on \( \mathfrak{h}_J \) and (iv) is proved. \( \square \)

Remark. The derived algebra \( \mathfrak{g}_J' = [\mathfrak{g}_J, \mathfrak{g}_J] \) is generated by \( \{e_j, f_j, h_j\}_{j \in J} \), where \( h_j = [e_j, f_j] \). Therefore, it does not depend of the choice of the subspace \( \mathfrak{h}_J'' \). The assignment \( J \mapsto \mathfrak{g}_J' \) defines a structure that coincides with the one provided in [TL4, 3.2.2].
Let now $A$ be an irreducible, generalized Cartan matrix. Let $D_g = D(A)$ be the Dynkin diagram of $g$, that is, the connected graph having $I$ as vertex set and an edge between $i$ and $j$ if $a_{ij} \neq 0$. For any $i \in I$, let $sl_2^i \subset g$ be the three-dimensional subalgebra spanned by $e_i, f_i, h_i$.

Any connected subdiagram $D \subseteq D_g$ defines a subset $J_D \subset I$. We would like to use the assignment $J \mapsto g_J$ to define a $D_g$–algebra structure on $g = g(A)$.

**Remark.** For any subset $J$ of finite type, $\dim h''_J = n_J - l_J = 0$ and $h_J = h'_J$. Therefore, if $A$ is a generalized Cartan matrix of finite type, $h'_J = \{0\}$ for any subset $J \subset I$. The $D_g$–algebra structure on $g = g(A)$ is then uniquely defined by the subalgebras $\{sl_2^i\}_{i \in I}$ and the Cartan subalgebra is defined for any subdiagram $D \subset D_g$ by

$$h_D = \{h_i \mid i \in V(D)\}$$

If $A$ is a generalized Cartan matrix of affine type, we obtain diagrammatic Cartan subalgebras $h_D$, where

$$h_D = \begin{cases} \{h_i \mid i \in V(D)\} & \text{if } D \subset D_g \\ h & \text{if } D = D_g \end{cases}$$

If $A$ is an irreducible generalized Cartan matrix of hyperbolic type, i.e., every submatrix is of finite or affine type, it is still possible to define a $D_g$–algebra structure, depending upon the choice of the subspaces $h''_J$ for $|I \setminus J| = 1$.

It is not always possible to define a $D_g$–algebra structure for a generic matrix of order $\geq 3$. In order to obtain a $D_g$–algebra structure on $g = g(A)$,
we have to satisfy the following condition:

\[ h_J \subset t_{J^\perp} \cap \bigcap_{J \subset J'} h_{J'} \]

Since \( t_{J^\perp} + t_J = h \), we can always choose \( h_J \subseteq t_{J^\perp} \).

**Lemma.** Assume given a \( D_6 \)-algebra structure on \( g = g(A) \). Then for any two subsets \( J', J'' \subset I \),

\[ \text{corank}(A_{J' \cap J''}) \leq \text{corank}(A_{J'}) + \text{corank}(A_{J''}) \]

In particular, if \( \text{corank}(A_{J'}) = \text{corank}(A_{J''}) = 0 \), then \( \text{corank}(A_{J' \cap J''}) = 0 \).

**Proof.** The result is an immediate consequence of the estimate, given by the construction,

\[ \dim(h_{J'} \cap h_{J''}) \leq |J' \cap J''| + (\text{corank}(A_{J'}) + \text{corank}(A_{J''})) \]

and the constraint

\[ h_{J' \cap J''} \subseteq h_{J'} \cap h_{J''} \]

\[ \square \]

**Remark.** Indeed, it is easy to show that the symmetric irreducible Cartan matrix

\[ A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \]
does not admit any $D_g$–algebra structure on $\mathfrak{g}(A)$, since $\dim \mathfrak{h}_{23} = 3$ and $\dim \mathfrak{h}_{123} \cap \mathfrak{h}_{234} = 2$.

The previous condition on the corank is not sufficient to obtain a $D_g$–algebra structure on $\mathfrak{g}(A)$. Consider the symmetric Cartan matrix

$$A = \begin{bmatrix}
2 & -2 & 0 & 0 \\
-2 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{bmatrix}$$

A clearly satisfies the above condition. Nonetheless, a suitable $\mathfrak{h}''_{12}$, complement in $\mathfrak{h}$ of $(\mathfrak{h}'_{12} + \mathfrak{t}_{12})$, should satisfies:

$$\mathfrak{h}''_{12} \subset \mathfrak{h}_{123} = \mathfrak{h}'_{123} \quad \text{and} \quad \mathfrak{h}''_{12} \subseteq \mathfrak{t}_4 = \langle \mathfrak{h}'_{12}, -2\alpha_3^\vee + \alpha_4^\vee \rangle$$

that are clearly not compatible conditions. Therefore, there is no suitable structure for $A$.

In the following, we will consider only symmetrizable Kac–Moody algebras $\mathfrak{g}$ that admit such a structure. It automatically defines an analogue structure on $U_{\hbar}^{\textit{ek}} \mathfrak{g}$.

6.2. $\mathfrak{qCqtqba}$ structure on $U_{\hbar} \mathfrak{g}$. Given a fixed $D_g$–structure on the Kac–Moody algebra $\mathfrak{g}$, the quantum enveloping algebra $U_{\hbar} \mathfrak{g}$ is naturally endowed with a quasi–Coxeter quasitriangular quasibialgebra structure of type $D_g$ defined by
3. AN EQUIVALENCE OF QUASI–COXETER CATEGORIES

(i) $D_g$-algebra: for any $D \in SD(D_g)$, let $g_D \subset g$ be the corresponding Kac–Moody subalgebra. The $D_g$-algebra structure is given by the subalgebras $\{U_\hbar g_D\}$.

(ii) Quasitriangular quasibialgebra: the universal $R$-matrices $\{R_{\hbar,D}\}$, with trivial associators $\Phi_D = 1^{\otimes 3}$ and structural twists $F_F = 1^{\otimes 2}$.

(iii) Quasi-Coxeter: the local monodromies are the quantum Weyl group elements $\{S_i^\hbar\}_{i \in I}$. The Casimir associators $\Phi_{G,F}$ are trivial.

We transfer this qCqtqba structure on $U_g[[\hbar]]$. More precisely, we define an equivalence of quasi–Coxeter categories between the representation theories of $U_\hbar g$ and $U_g[[\hbar]]$.

6.3. Gauge transformations for $g(A)$. For any $D \subset D_g$, the inclusion $g_D \subset g$, defined in the previous section, lifts to an inclusion of Manin triples

$$ g_D \oplus \hbar D \subset g \oplus \hbar $$

We denote by $\tilde{g}_D = (g_D \oplus \hbar D, b_{D,+}, b_{D,-})$ the Manin triple attached to $g_D$, for any $D \subseteq D_g$.

**Theorem.** There exists an equivalence of braided $D_g$–monoidal categories from

$$ \left\{ \left\{ D_{\Phi_B}(U_{\hbar g_B}[[\hbar]]), \otimes_B, \Phi_B, \sigma R_B \right\}, \left\{ (\Gamma_{BB'}, J^{BB'}_F) \right\} \right\} $$

to

$$ \left\{ \left\{ D(U_{\hbar g_B}), \otimes_B, \text{id}, \sigma R^B_\hbar \right\}, \left\{ (\Gamma_{BB'}, \text{id}) \right\} \right\} $$
given by $\left\{ \left\{ \tilde{F}_B, \gamma^{\hbar}_{BB'} \right\} \right\}$.
Proof. The natural transformations $\gamma_{BB'}$, $B \subseteq B' \subseteq D_{\mathcal{A}}$ constructed in Section 5, define, by vertical composition, a natural transformation

$$\gamma^C_{BB'} \in \text{Nat}_\otimes(\Gamma^h_{BB'} \circ \widetilde{F}_{B'}, \widetilde{F}_{B} \circ \Gamma_{BB'})$$

for any chain of maximal length

$$C : B = C_0 \subset C_1 \subset \cdots \subset C_r = B'$$

Any chain of maximal length defines uniquely a maximal nested set $\mathcal{F}_C \in \text{Mns}(B, B')$, but this is not a one to one correspondence. For example, for $D = A_3$, the maximal nested set

$$\mathcal{F} = \{\{\alpha_1\}, \{\alpha_3\}, \{\alpha_1, \alpha_2, \alpha_3\}\}$$

corresponds to two different chains of maximal length

$$C_1 : \{\alpha_1\} \subset \{\alpha_1\} \cup \{\alpha_3\} \subset A_3 \quad C_2 : \{\alpha_3\} \subset \{\alpha_1\} \cup \{\alpha_3\} \subset A_3$$

In order to prove that the natural transformations $\gamma$ define a morphism of braided $D_{\mathcal{A}}$–monoidal categories, we need to prove that the transformation $\gamma^C_{BB'}$ depend only on the maximal nested set corresponding to $C$.\]
In particular, we have to prove that, for any $B_1 \perp B_2$ in $I(D)$, the construction of the fiber functor

\[ C_{B_1 \sqcup B_2} \xrightarrow{F_{B_2 \times B_1 \sqcup B_2}} C_{B_1} \xleftarrow{F_{B_1 \times B_1 \sqcup B_2}} C_{B_2} \]

is independent of the choice of the chain. In our case,

\[ C_{B_1 \sqcup B_2} = \mathcal{D}(U\tilde{g}_{B_1}[[h]] \otimes U\tilde{g}_{B_2}[[h]]) \]

and the braided tensor structure is given by product of the braided tensor structures on

\[ C_{B_i} = \mathcal{D}_{B_i} (U\tilde{g}_{B_i}[[h]]) \]

Similarly, the tensor structure on the forgetful functor

\[ C_{B_1 \sqcup B_2} \rightarrow C_{B_i} \quad i = 1, 2 \]

is obtained killing the tensor structure on $C_{B_i}, i = 1, 2$, i.e., applying the tensor structure on $C_{B_i} \rightarrow C_{\emptyset}$. In particular, the tensor structure on $F_{B_1} \circ F_{B_1 \times B_1 \sqcup B_2}$ and $F_{B_2} \circ F_{B_2 \times B_1 \sqcup B_2}$ coincide, since $[\tilde{g}_{B_1}, \tilde{g}_{B_2}] = 0$.

Analogously we have an equality of natural transformation

\[ \gamma_{B_1} \circ \gamma_{B_1 \times B_2} = \gamma_{B_2} \circ \gamma_{B_2 \times B_1 \sqcup B_2} \]
Therefore, for any maximal nested set $\mathcal{F} \in \text{Mns}(B, B')$, it is well defined a natural transformation

$$\gamma^\mathcal{F}_{BB'} \in \text{Nat}_\otimes(\Gamma^h_{BB'} \circ \tilde{F}_{B'}, \tilde{F}_B \circ \Gamma_{BB'})$$

so that the data $(\{\tilde{F}_B\}, \{\gamma^\mathcal{F}_{BB'}\})$ define an isomorphism of $D$–categories from $\{\mathcal{D}_{\Phi_B}(U_{\hbar B}[[[h]]])\}$ to $\{\mathcal{D}(U_{\hbar \widehat{g}})\}$. □

### 6.4. Extension to Levi subalgebras.

In analogy with [TL4, Thm. 9.1], we want to show that the relative twists and the Casimir associators are weight zero elements. This corresponds to show that the corresponding tensor functors $\Gamma$ and the natural transformations $\gamma$ lift to the level of Levi subalgebras:

$$\mathfrak{g}_D \subset \mathfrak{l}_D = \mathfrak{n}_{D,+} \oplus \mathfrak{h} \oplus \mathfrak{n}_{D,-} \subset \mathfrak{g}$$

**Proposition.** The relative twists and the Casimir associators are weight zero elements.

**Proof.** For $D = \emptyset$, the statement reduces to prove that the Etingof–Kazhdan functor preserves the $\mathfrak{h}$–action [EK6, Thm. 4.3]. The result is a consequence of Proposition 5.3 applied to Levi subalgebras. □

### 6.5. Reduction to category $O^{\text{int}}$. 

The Etingof–Kazhdan functor gives rise, by restriction, to an equivalence of categories

$$\tilde{F} : \mathcal{O}_{\mathfrak{g}}[[[h]]] \to \mathcal{O}_{U_{\hbar \mathfrak{g}}}$$
We will show now that this equivalence can be further restricted to integrable modules in category $\mathcal{O}$, i.e., modules in category $\mathcal{O}$ with a locally nilpotent action of the elements $\{e_i, f_i\}_{i \in I}$ (respectively $E_i, F_i$).

**Proposition.** The Etingof–Kazhdan functor restricts to an equivalence of braided tensor categories

$$\tilde{F} : \mathcal{O}^\text{int}_{\mathfrak{g}}[[h]] \rightarrow \mathcal{O}^\text{int}_{U_h \mathfrak{g}}$$

which is isomorphic to the identity functor at the level of $\mathfrak{h}$–graded $k[[h]]$–modules.

**Proof.** Let $V \in \mathcal{O}^\text{int}_{\mathfrak{g}}$. Then, the elements $e_i, f_i$ for $i \in I$ act nilpotently on $V$. Then, by [Ka], for all $\lambda \in \mathcal{P}(V)$, there exist $p, q \in \mathbb{Z}_{\geq 0}$ such that

$$\{t \in \mathbb{Z} \mid \lambda + t\alpha_i \in \mathcal{P}(V)\} = [-p, q]$$

Since the Cartan subalgebra $\mathfrak{h}$ is not deformed by the quantization, the functor $\tilde{F}$ preserves the weight decomposition. In $U_h \mathfrak{g}$, for any $h \in \mathfrak{h}$ and $i \in I$, we have

$$[h, E_i] = \alpha_i(h)E_i$$

Therefore the action of the $E_i$'s on $V$ is locally nilpotent. The action of the $F_i$'s is always locally nilpotent, since

$$\mathcal{P}(V) \subset \bigcup_{s=1}^r \mathcal{D}(\lambda_s)$$

The result follows. $\square$
Corollary.

(i) There exists an equivalence of braided $D_\theta$–monoidal categories between

$$\mathcal{O} := \left\{ \left( \mathcal{O}_{\mathcal{B}}^{\text{int}}, \otimes_B, \Phi_B, \sigma R_B \right), \left\{ (\Gamma_{BB'}, J_{F,BB'}) \right\} \right\}$$

and

$$\mathcal{O}_h := \left\{ \left( \mathcal{O}_{\mathcal{B}}^{\text{int}, \mathcal{U}}, \otimes_B, \sigma^{R_B^h} \right), \left\{ (\Gamma_{BB'}, \text{id}) \right\} \right\}$$

(ii) There exists an isomorphism of $D_\theta$–algebras

$$\Psi_F : \widehat{U}_{\mathcal{B}} \rightarrow \widehat{U}_{\mathcal{B}}[[\hbar]]$$

such that $\Psi_F(\widehat{U}_{\mathcal{B}}(D_i)) = \widehat{U}_{\mathcal{B}}(D_i)[[\hbar]]$ for any $D_i \in \mathcal{F}$, where the completion is taken with respect to the integrable modules in category $\mathcal{O}$.

6.6. Quasi–Coxeter structure. The previous equivalence of braided $D_\theta$–monoidal categories induces on

$$\mathcal{O} = \left\{ \left( \mathcal{O}_{\mathcal{B}}^{[\hbar]}, \otimes_B, \Phi_B, \sigma R_B \right), \left\{ (\Gamma_{BB'}, J_{F,BB'}) \right\} \right\}$$

a structure of quasi–Coxeter category of type $D_\theta$, given by the Casimir associators $\Phi_{\mathcal{G},\mathcal{F}} \in \text{Nat}_{\otimes}(\Gamma_F, \Gamma_E)$ and the local monodromies $S_i \in \text{End}(\Gamma_i)$ defined for any $\mathcal{G}, \mathcal{F} \in \text{Mns}(B, B')$ and $i \in I(D)$ by

$$\bar{F}_B(\Phi_{\mathcal{G},\mathcal{F}}) = (\gamma_{BB'})^{-1} \circ \gamma_{BB'}^G, \quad S_i = \Psi_{E^K}(S^h_i)$$
where $\Psi_{\text{EK}} : \hat{U}_h \mathfrak{sl}_2 \rightarrow \hat{U}_h \mathfrak{sl}_2[[h]]$ is the isomorphism induced at the $\mathfrak{sl}_2$ level by the Etingof–Kazhdan functor.

**Proposition.** The equivalence of braided $D_g$–monoidal categories $O \rightarrow O_h$ induces a structure of quasi–Coxeter category on $O$.

**Proof.** In order to prove the proposition, we have to prove the compatibility relations of the elements $\Phi_{G,F}, S_i$ with the underlying structure of braided $D_g$–monoidal category on $O$.

The element $S_i$’s satisfy the relation

$$\Delta_F(S_i) = (R_i)_F^{21} \cdot (S_i \otimes S_i)$$

since $\Psi_F$ is given by an isomorphism of braided monoidal $D$–categories and therefore

$$\Psi_F((R_i)_F^h) = (R_i)_F$$

Similarly, the braid relations are easily satisfied, since

$$\text{Ad}(\Phi_{G,F})\Psi_F = \Psi_G$$

The elements $\Phi_{F,G}$ defined above satisfy all the required properties:

(i) **Orientation** For any elementary pair $(F, G)$ in $\text{Mns}(B, B')$

$$\bar{F}_B(\Phi_{F,G}) = (\gamma_{BB'}^F)^{-1} \circ \gamma_{BB'}^G = (\bar{F}_B(\Phi_{G,F}))^{-1}$$
(ii) **Coherence** For any $F, G, H \in \text{Mns}(B, B')$

\[
\tilde{F}_B(\Phi_{FG}) = (\gamma_{BB'})^{-1} \gamma_{BB'} \circ (\gamma_{BB'})^{-1} \circ \gamma_{BB'} = 
\]

\[
= \tilde{F}_B(\Phi_{FH}) \circ H_B(\Phi_{HG})
\]

This property implies the coherence.

(iii) **Factorization.** Clear by construction.

Finally, the elements $\Phi_{g,F}$ satisfy

\[
\Delta(\Phi_{g,F}) \circ J_F = J_g \circ \Phi_{g,F}^\otimes \otimes
\]

because they are given by composition of invertible natural tensor transformations. □

### 6.7. Normalized isomorphisms.

In the completion $\widehat{U_{\sl_2}}[[h]]$ with respect to category $O$ integrable modules, there are preferred element $S_{i,C}$

\[
S_{i,C} = \tilde{s}_i \exp(\frac{h}{2} C_i)
\]

where

\[
\tilde{s}_i = \exp(e_i) \exp(-f_i) \exp(e_i) \quad C_i = \frac{(\alpha_i, \alpha_i)}{2}(e_i f_i + f_i e_i + \frac{1}{2} h_i^2)
\]

**Proposition.** There exists an equivalence of quasi-Coxeter categories of type $D_g$ between

\[
O := \{(\text{O}_{BB}^\text{int}, \otimes_B, \Phi_B, \sigma R_B)\}, \{\Gamma_{BB'}, (J_F^{BB'})\}, \{\Phi_{g,F}\}, \{S_{i,C}\}\}
\]
and

\[ \mathcal{O}_h := \left\{ \left( (\mathcal{O}_{U_B}^{\text{int}}, \otimes_B, \text{id}, \sigma R^h_B) \right) \right\} \}

\{ (\Gamma^h_{BB'}, \text{id}) \}, \{ \text{id} \}, \{ S^h_i \}\}

**Proof.** Using the result of Proposition 6.6, it is enough to prove that the natural transformation \( \gamma_i \)

![diagram](diagram.png)

can be modified in such a way that the induced isomorphism at the level of endomorphism algebras \( \widehat{U}_h \mathfrak{sl}_2 \to \widehat{U} \mathfrak{sl}_2[[h]] \) maps \( S^h_i \) to \( S_{i,C} \). The natural transformation used in Corollary 6.5 induces the Etingof–Kazhdan isomorphism

\[ \Psi^{\text{EK}}_i : \widehat{U}_h \mathfrak{sl}_2 \to \widehat{U} \mathfrak{sl}_2[[h]] \]

which is the identity mod \( h \) and the identity on the Cartan subalgebra. As above, we denote by \( S_i \) the element \( \Psi^{\text{EK}}_i(S^h_i) \). Then \( S_i \equiv S_i \mod h \) and, by [TL4, Proposition 8.1, Lemma 8.4], we have

\[ S^2_i = S^2_{i,C} \quad S_i = \text{Ad}(x)(S_{i,C}) \]

on the integrable modules in category \( \mathcal{O} \), for \( x = (S_{i,C} \cdot S_i^{-1})^{\frac{1}{2}} \). Therefore, the modified isomorphism

\[ \Psi_i := \text{Ad}(x) \circ \Psi^{\text{EK}}_i \]
maps $S_i^\hbar$ to $S_i,C$. Moreover, $\Psi_i$ correspond with the natural transformation given by the composition of $\gamma_i$ with $x \in Usf_2[[\hbar]] = \text{End}(f)$

$$
\begin{array}{c}
\mathcal{O}_i^{\text{int}} \xrightarrow{\mathcal{O}_i^{\text{int}}} \mathcal{O}_i^{\text{int}} \xrightarrow{\tilde{F}} \mathcal{O}_{i,h}^{\text{int}} \\
f \overset{x}{\downarrow} \quad \mathcal{A} \quad \gamma_i \quad \mathcal{A} \quad \f_i \\
\end{array}
$$

The result follows substituting $\gamma_i$ with $x \circ \gamma_i$ in Proposition 6.6. \qed

6.8. The main theorem. We now state in more details the main theorem of this Chapter and summarize the proof outlined in the previous results.

**Theorem.** Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra with a fixed $D_\mathfrak{g}$–structure and $U_\hbar\mathfrak{g}$ the corresponding Drinfeld–Jimbo quantum group with the analogous $D_\mathfrak{g}$–structure. For any choice of a Lie associator $\Phi$, there exists an equivalence of quasi–Coxeter categories between

$$
\mathcal{O} := (\{(\mathcal{O}_{BB}^{\text{int}}, \otimes_B, \Phi_B, \sigma R_B)\}, \{(\Gamma_{BB'}, J_{BB'})\}, \{\Phi_{\mathcal{F}}\}, \{S_{i,C}\})
$$

and

$$
\mathcal{O}_h := (\{(\mathcal{O}_{BB}^{\text{int}}, \otimes_B, \text{id}, \sigma R_B^h)\}, \{(\Gamma_{BB'}, \text{id})\}, \{\text{id}\}, \{S_i^h\})
$$

where $\otimes_B$ denotes the standard tensor product in $\mathcal{O}_{BB}^{\text{int}}$ and

$$
S_{i,C} = \tilde{s}_i \exp(\frac{h}{2} \cdot C_i) \\
\Phi_B = 1 \mod h^2
$$
\[ R_B = \exp\left(\frac{\hbar}{2} \Omega_D\right) \]
\[ \text{Alt}_2 J^{BB'}_F = \frac{\hbar}{2} \left( \frac{r_B - r_B'}{2} - \frac{r_B - r_B'}{2} \right) \]

and \( \Phi_{g_F}, J^{BB'}_F \) are weight zero elements.

**Proof.** The existence of an equivalence is a consequence of the constructions of Section 5 and proved in Theorem 6.3 and Proposition 6.6, 6.7, concerning the local monodromies \( S_{i,C} \).

The properties of associators \( \Phi_B \) and \( R \)-matrices \( R_B \) are direct consequences of the construction in Section 1.2. The relation satisfied by the relative twists \( J^{BB'}_F \) is proven by a simple application of Proposition 2.20 and Corollary 2.20. It is easy to check that the 1–jet of the twist \( J^{BB'}_F \) differs from the 1–jet of the twist \( J^{BB'} \) (as defined in Section 2) by a symmetric element that cancels out computing the alternator. Therefore, Corollary 2.20 holds for \( J^{BB'}_F \) as well.

Finally, as previously explained, the weight zero property of the relative twists \( J^{BB'}_F \) and the Casimir associators \( \Phi_{g_F} \) is proved in Proposition 5.3, 6.4. This complete the proof of Theorem 6.8. \( \square \)
CHAPTER 4

A rigidity theorem for quasi–Coxeter categories

The goal of this chapter is to show that, under appropriate hypotheses, there is at most one quasi–Coxeter braided monoidal category on the classical enveloping algebra $U\mathfrak{g}[[\hbar]]$ with prescribed local monodromies (Thm 4.2). This result is needed to show that the quasi–Coxeter braided monoidal category naturally attached to the quantum affine algebra $U_{\hbar}\mathfrak{g}$ is equivalent to the one constructed for $U\mathfrak{g}[[\hbar]]$, which underlies the monodromy of the rational KZ and Casimir connections of $\mathfrak{g}$.

In Section 1, we review a number of basic notions from [TL4], in particular that of the Dynkin–Hochschild bicomplex of a $D$–bialgebra $A$ which controls the deformations of quasi–Coxeter quasitriangular quasibialgebra structures on $A$. In Section 2, we extend the notion of the bicomplex to the case of completion of algebras, using the language of endomorphisms algebra of monoidal functors and the corresponding cohomology theory. In Section 3 we study invariant properties of the relative twist under the Chevalley involution of $\mathfrak{g}$ and we prove a uniqueness property up to gauge transformation. Section 4 contains the rigidity result, consisting in the construction of a natural transformation matching the Casimir associators and preserving the local monodromies.
4. A RIGIDITY THEOREM FOR QUASI-COXETER CATEGORIES

1. The Dynkin-Hochschild bicomplex

In this section, we briefly recall the definition of $D$–bialgebras and the construction of the Dynkin–Hochschild complex introduced in [TL4].

1.1. The Dynkin complex of a $D$–algebra. We begin by defining the category of coefficients of the Dynkin complex of $D$–algebra $A$ (2.2).

DEFINITION. A $D$–bimodule over $A$ is an $A$–bimodule $M$, with left and right actions denoted by $am$ and $ma$ respectively, endowed with a family of subspaces $M_{D_1}$ indexed by the connected subdiagrams $D_1 \subseteq D$ of $D$ such that the following properties hold

- for any $D_1 \subseteq D$, $A_{D_1} M_{D_1} \subseteq M_{D_1}$ and $M_{D_1} A_{D_1} \subseteq M_{D_1}$

- For any pair $D_2 \subseteq D_1 \subseteq D$, $M_{D_2} \subseteq M_{D_1}$

- For any pair of orthogonal subdiagrams $D_1, D_2$ of $D$, $a_{D_1} \in A_{D_1}$ and $m_{D_2} \in M_{D_2}$, $a_{D_1} m_{D_2} = m_{D_2} a_{D_1}$

A morphism of $D$–bimodules $M, N$ over $A$ is an $A$–bimodule map $T : M \to N$ such that $T(M_D) \subseteq N_D$ for all $D \subseteq D$.

Clearly, $A$ is a $D$–bimodule over itself. We denote by $\text{Bimod}_D(A)$ the abelian subcategory of $\text{Bimod}(A)$ consisting of $D$–bimodules over $A$. If $M \in$
Bimod$_D(A)$, and $D_1 \subseteq D$ is a subdiagram, we set
\[ M^{D_1} = \{ m \in M \mid am = ma \text{ for any } a \in A_{D_1} \} \]
where $D_1^i$ are the connected components of $D_1$. In particular, if $D_1, D_2 \subseteq D$ are orthogonal, and $D_1$ is connected, then $M_{D_1} \subseteq M^{D_2}$.

Let $M \in \text{Bimod}_D(A)$. For any integer $0 \leq p \leq n = |D|$, set
\[ C^p(A; M) = \bigoplus_{\alpha \subseteq D_1, |\alpha| = p} M^{D_1 \setminus \alpha} \]
where the sum ranges over all connected subdiagrams $D_1$ of $D$ and ordered subsets $\alpha = \{ \alpha_1, \ldots, \alpha_p \}$ of cardinality $p$ of $D_1$ and $M^{D_1 \setminus \alpha} = (M_{D_1})^{D_1 \setminus \alpha}$. We denote the component of $m \in C^p(A; M)$ along $M^{D_1 \setminus \alpha}$ by $m_{(D_1; \alpha)}$.

**Definition.** The group of Dynkin $p$–cochains on $A$ with coefficients in $M$ is the subspace $CD^p(A; M) \subset C^p(A; M)$ of elements $m$ such that
\[ m_{(D_1; \sigma \alpha)} = (-1)^\sigma m_{(D_1; \alpha)} \]
where, for any $\sigma \in \mathfrak{S}_p$, $\sigma\{\alpha_1, \ldots, \alpha_p\} = \{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(p)}\}$.

Note that
\[ CD^0(A; A) = \bigoplus_{D_1 \subseteq D} Z(A_{D_1}) \quad \text{and} \quad CD^n(A; M) \cong M_D \]

For $1 \leq p \leq n - 1$, define a map $d^p_D : C^p(A; M) \to C^{p+1}(A; M)$ by
\[ d_p m_{(D_1; \alpha)} = \sum_{i=1}^{p+1} (-1)^{i-1} \left( m_{(D_1; \alpha \setminus \alpha_i)} - m_{(D_1 \setminus \alpha_i; \alpha \setminus \alpha_i)} \right) \]
where $\alpha = \{\alpha_1, \ldots, \alpha_{p+1}\}$, $D_1^{\alpha_i}$ is the connected component of $D_1 \setminus \alpha_i$ containing $\alpha \setminus \alpha_i$ if one such exists and the empty set otherwise, and we set $m(\emptyset, -) = 0$. For $p = 0$, define $d^0_D : C^0(A; M) \to C^1(A; M)$ by

$$d^0_D m(D_1; \alpha_i) = m_{D_1} - m_{D_1 \setminus \alpha_i}$$

where $m_{D_1 \setminus \alpha_i}$ is the sum of $m_{D_2}$ with $D_2$ ranging over the connected components of $D_1 \setminus \alpha_i$. Finally, set $d^p_D = 0$. It is easy to see that the Dynkin differential $d_D$ is well-defined and that it leaves $CD^*(A; M)$ invariant.

**Theorem.** $(CD^*(A; M), d_D)$ is a complex. The cohomology groups

$$HD^p(A; M) = \text{Ker}(d^p_D)/\text{Im}(d^{p-1}_D)$$

for $p = 0, \ldots, |D|$ are called the Dynkin diagram cohomology groups of $A$ with coefficients in $M$.

**1.2. The Dynkin–Hochschild bicomplex of a $D$–bialgebra.** Let $D$ be a connected diagram and $A$ a $D$–bialgebra (the algebraic counterpart of strict $D$–monoidal categories 2.7, cf. [TL4, 6.1]). By combining the Dynkin complex of $A$ with the cobar complexes of its subalgebras $A_B$, we review the definition of a bicomplex which controls the deformations of quasi–Coxeter quasibialgebra structures on $A$ introduced in [TL4, 7.1].

Let $A$ be a bialgebra and $C^*(A)$ the cobar complex of $A$, regarded as a coalgebra. If $C \subseteq A$ is a sub–bialgebra, let

$$C^n(A)^C = \{a \in C^n(A) | [a, \Delta^{(n)}(c)] = 0 \text{ for any } c \in C\}$$
where $\Delta^{(n)} : C \to C^\otimes n$ is the $n$th iterated coproduct, be the submodule of $C$–invariants. It is easy to check that $C^*(A)^C$ is a subcomplex of $C^*(A)$.

Assume now that $A$ is a $D$–bialgebra. For $p \in \mathbb{N}$ and $0 \leq q \leq |D|$, let

$$CD^q(A; A^\otimes p) \subset \bigoplus_{\alpha \subseteq B \subseteq D, |B| = q} (A^\otimes_B)^B \backslash \alpha$$

be the group of Dynkin $q$–cochains with values in the $D$–bimodule $A^\otimes p$ over $A$. The Dynkin differential $d_{DD}$ defines a vertical differential $CD^q(A; A^\otimes p) \to CD^{q+1}(A; A^\otimes p)$ while the Hochschild differential $d_H$ defines a horizontal differential $CD^q(A; A^\otimes p) \to CD^q(A; A^\otimes (p+1))$. A straightforward computation yields the following.

**Theorem.** One has $d_{DD} \circ d_H = d_H \circ d_{DD}$. The corresponding cohomology of the bicomplex $CD^q(A; A^\otimes p)$ is called the Dynkin–Hochschild cohomology of the $D$–bialgebra $A$.

2. Dynkin–Hochschild complex of monoidal functors

We use the language of monoidal functor and cohomology of monoidal functor to reformulate the definition of the Dynkin–Hochschild bicomplex in a more general setting. This approach allows to consider the Dynkin–Hochschild cohomology for completions of bialgebras with respect to a fixed tensor subcategory.

2.1. Hochschild complex of monoidal functors. Let $F : \mathcal{C} \to \mathcal{C'}$ be a monoidal functor of associative monoidal $k$–linear categories with associated
tensor structure $J$. The $n$th tensor power of $F$ is the functor

$$F^\otimes n : \mathcal{C}^n \to \mathcal{C}' \quad F^\otimes n(X_1, \ldots, X_n) = \bigotimes_{i=1}^n F(X_i)$$

Through repeated applications of $J$, we obtain a canonical isomorphism

$$F^\otimes n(X_1, \ldots, X_n) \simeq F\left(\bigotimes_{i=1}^n X_i\right)$$

The $k$–algebra of endomorphisms $\text{End}(F^\otimes n)$ form the cosimplicial complex of algebras

$$\text{End}_{\mathcal{C}'}(1) \Rightarrow \text{End}(F) \Rightarrow \text{End}(F^\otimes 2) \Rightarrow \text{End}(F^\otimes 3) \cdots$$

with face homomorphisms $d^i_n : \text{End}(F^\otimes n) \to \text{End}(F^\otimes n+1)$, $i = 0, \ldots, n+1$, defined as follows:

$$(d^0_0 f)_X : F(X) \to F(X) \otimes 1 \to F(X) \otimes 1 \to F(X)$$

$$(d^i_0 f)_X : F(X) \to 1 \otimes F(X) \to 1 \otimes F(X) \to F(X)$$

for $X \in \mathcal{C}$, $f \in \text{End}_{\mathcal{C}'} 1$, and

$$(d^i_n f)_{X_1, \ldots, X_{n+1}} = \begin{cases} 
\text{id} \otimes f_{X_2, \ldots, X_{n+1}} & i = 0 \\
\text{Ad}(J_{X_i, X_{i+1}}) f_{X_1, \ldots, X_i \otimes X_{i+1}, \ldots, X_{n+1}} & 1 \leq i \leq n \\
f_{X_1, \ldots, X_n} \otimes \text{id} & i = n + 1
\end{cases}$$

for $f \in \text{End}(F^\otimes n)$, $X_i \in \mathcal{C}, i = 1, \ldots, n + 1$. The Hochschild differential is defined in the usual way:

$$d_n = \sum_{i=0}^{n+1} (-1)^i d^i_n : \text{End}(F^\otimes n) \to \text{End}(F^\otimes n+1)$$
The degeneration homomorphisms $s^i_n : \text{End}(F^\otimes n) \to \text{End}(F^\otimes n-1)$ are defined, for $i = 1, \ldots, n$, by

\[(s^i_n f)_{X_1, \ldots, X_{n-1}} := f_{X_1, \ldots, X_{i-1}, 1, X_i, \ldots, X_{n-1}}\]

The morphisms $\{s^i_n\}, \{d^i_n\}$ satisfy the standard relations

\[d^j_{n+1} d^i_n = d^j_{n+1} d^{j-1}_n \quad i < j\]
\[s^j_n s^i_{n+1} = s^i_n s^{j+1}_{n+1} \quad i \leq j\]
\[s^j_{n+1} d^i_n = \begin{cases} d^j_{n-1} s^{j-1}_n & i < j \\ \text{id} & i = j, j + 1 \\ d^{i-1}_n s^j_n & i > j + 1 \end{cases}\]

2.2. Composition of monoidal functor and Hochschild subcomplexes. Let $F : C \to C'$ and $G : C' \to C''$ be monoidal functors of associative monoidal categories with associated tensor structures $J$ and $K$, respectively. For any $n \in \mathbb{N}$ there are well–defined homomorphisms

\[\text{End}(F^\otimes n) \otimes \text{End}(G^\otimes n) \to \text{End}((G \circ F)^\otimes n)\]

mapping the tensor $\alpha \otimes \beta$ into $\alpha \circ \beta \in \text{End}((G \circ F)^\otimes n)$ defined to be the composition $\alpha \circ \beta = \text{Ad}(K)(G(\alpha)) \circ \beta$

\[
\begin{array}{ccc}
\bigotimes_{i=1}^{n} GF(X_i) & \xrightarrow{\beta_F} & \bigotimes_{i=1}^{n} GF(X_i) \\
\downarrow & \quad & \downarrow \\
G(\bigotimes_{i=1}^{n} F(X_i)) & \xrightarrow{G(\alpha)} & G(\bigotimes_{i=1}^{n} F(X_i))
\end{array}
\]

The functor $G \circ F$ is naturally endowed with a tensor structure induced by $J$ and $K$. A quick computation proves the following
Proposition.

(i) The composition maps are compatible with the Hochschild differential maps, i.e., the diagram

\[
\begin{array}{ccc}
\text{End}(F^\otimes n) \otimes \text{End}(G^\otimes n) & \xrightarrow{\circ_n} & \text{End}((G \circ F)^n) \\
\downarrow \quad \quad d_n^F \otimes d_n^G & & \downarrow d_{n+1}^{G \circ F} \\
\text{End}(F^\otimes n+1) \otimes \text{End}(G^\otimes n+1) & \xrightarrow{\circ_{n+1}} & \text{End}((G \circ F)^{n+1})
\end{array}
\]

is commutative for any \( n \in \mathbb{N} \).

(ii) The natural maps

\[
\begin{align*}
\text{End}(F^\otimes n) & \rightarrow \text{End}((G \circ F)^\otimes n) \\
\text{End}(G^\otimes n) & \rightarrow \text{End}((G \circ F)^\otimes n)
\end{align*}
\]

are well-defined morphisms of the associated cochain complexes.

Remark. If \( F = G = \text{id} \), then the composition law described above coincides with the multiplication in \( \text{End}(\text{id}^\otimes n) \).

2.3. Cohomology of monoidal functors. Let \( \mathcal{C} \) be a \( k \)-linear abelian monoidal category and \( F : \mathcal{C} \rightarrow \text{Vect} \) a faithful, exact, associative monoidal functor. Then we can consider on \( \text{End}(F) \) the initial topology with basis of neighborhood of zero the kernels of the maps \( \text{End}(F) \rightarrow \text{End}(F(X)) \). We get canonical isomorphisms

\[
\text{End}(F) \simeq \lim_{X \in \mathcal{C}} \text{End}(F(X))
\]
and
\[ \text{End}(F^\otimes n) \simeq \text{End}(F)^\hat{\otimes n} \]
where \( \hat{\otimes} \) denotes the completion of the tensor product with respect to the topology of the inverse limit. This identification allows in particular to look at the canonical map
\[ \Delta : \text{End}(F) \to \text{End}(F^{\otimes 2}) \simeq \text{End}(F)^\hat{\otimes} \text{End}(F) \]
as a topological coproduct on \( \text{End}(F) \).

**Definition.** The cohomology of the cochain complex \( (\text{End}(F^{\otimes \bullet}), d) \) is called *additive cohomology of the monoidal functor* \( F \) and denoted \( H^\bullet(F) \).

The external multiplication
\[ \text{End}(F^{\otimes n}) \otimes \text{End}(F^{\otimes m}) \to \text{End}(F^{\otimes n+m}) \]
defined in the obvious way on the complex \( \text{End}(F^{\otimes \bullet}) \) induces a structure of graded ring on \( H^\bullet(F) \). This follows from the relations
\[
d^n_{i+m} = \begin{cases} 
d^n_i(\alpha) \cdot \beta & i \leq n \\
\alpha \cdot d^m_{n-i}(\beta) & i > n \end{cases} = \alpha \cdot d^n_{m+1}(\beta)
\]
The differential graded algebra \( (\text{End}(F^{\otimes \bullet}), \otimes, d) \) is homotopically commutative [D]. Thus the cohomology \( H^n(F) \) is a skew–commutative graded algebra. Moreover, the homotopy of the commutativity defines on \( H^\bullet(F) \) a structure of Lie superalgebra \( [\cdot, \cdot] : H^n(F) \otimes H^m(F) \to H^{n+m-1}(F) \), which coincides on the
subalgebra $\bigwedge H^1(F)$ with the Schouten bracket

$$[u_1 \wedge \cdots \wedge u_n, v_1 \wedge \cdots \wedge v_m] = \sum_{i,j} (-1)^{i+j} [u_i, v_j] \wedge u_1 \wedge \cdots \wedge u_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_m$$

### 2.4. Cohomology of the forgetful functor in category $\mathcal{O}$. Let $\mathfrak{g}$ be a Lie algebra over a field $k$, with char $k = 0$, and $F : \text{Rep} U\mathfrak{g} \to \text{Vect}$ be the forgetful functor. Tannaka–Krein duality gives a canonical isomorphism $U\mathfrak{g} \simeq \text{End}(F)$. The cochain complex $\text{End}(F^{\otimes \bullet})$ coincides with the co–Hochschild complex of $U\mathfrak{g}$ and $H^1(F) = \text{Prim}(U\mathfrak{g}) = \mathfrak{g}$. Since $d_H(\mathfrak{g}^{\otimes n}) = 0$, we obtain a map from $T^*\mathfrak{g} \to H^*(F)$ and we denote by $\mu$ the restriction to the exterior algebra

$$\mu : \bigwedge H^1(F) \to H^*(F)$$

It is well–known, in this case, that the map $\mu$ is an isomorphism [D3].

As mentioned above, for an arbitrary monoidal functor $F : \mathcal{C} \to \mathcal{C}'$ we have a canonical inclusion

$$\mu : \bigwedge H^1(F) \to H^*(F)$$

observing that $d_H(H^1(F)^{\otimes n}) = 0$. For an arbitrary monoidal functor, the morphism $\mu$ is not surjective. Easy counterexamples are given by the identity functor on $\text{Rep} U\mathfrak{g}$ (cocommutative case) or the forgetful functor $\text{Rep} H \to \text{Vect}$ where $H$ is the Sweedler’s Hopf algebra (non cocommutative). In both cases we have $H^1(F) = 0$ and $H^*(F) \neq 0$.

Our main interest lies in the forgetful functor $F : \mathcal{O}_\mathfrak{g} \to \text{Vect}$, where $\mathfrak{g}$ is a Kac–Moody algebra and $\mathcal{O}_\mathfrak{g}$ is the corresponding category $\mathcal{O}$. In this case,
we assume that there exists an appropriate completion of the exterior algebra that extends $\mu$ to an isomorphism.

**Conjecture.** Let $\mathfrak{g}$ be a Kac–Moody algebra and $F : \mathcal{O}_g \to \text{Vect}$ be the forgetful functor from the category $\mathcal{O}$, then

$$H^*(F) \cong \bigwedge^\bullet \mathfrak{g}, \quad H^1(F)_\mathfrak{h} \cong \mathfrak{h}$$

2.5. Dynkin complex of $D$–categories. Let $\mathcal{C} = (C_B, F_{BB'})_{B, B' \in SD(D)}$ be a $D$–category. We recall that, by definition, for any $B \subset B' \subset B''$, we are given maps

$$\text{End}(F_{BB'}) \to \text{End}(F_{BB''}) \quad \text{End}(F_{B'B''}) \to \text{End}(F_{BB''})$$

and, for any $B' \perp B''$, $B', B'' \subset B'''$, and $B \subset B'$,

$$\text{End}(F_{BB'}) \to \text{End}(F_{(B \cup B'')B'''}$$

For any integer $0 \leq p \leq n = |D|$, we set

$$C^p(\mathcal{C}) = \bigoplus_{\alpha \subseteq B \in \text{Csd}(D), \, |\alpha| = p} \text{End}(F_{(B, \alpha)})$$

where $F_{(B, \alpha)} := F_{(B \setminus \alpha)B}$.

**Definition.** The group of Dynkin $p$–cochains on $\mathcal{C}$ is the subspace $CD^p(\mathcal{C}) \subset C^p(\mathcal{C})$ of elements $m$ such that

$$f_{(B, \sigma \alpha)} = (-1)^\sigma f_{(B, \alpha)}$$

where, for any $\sigma \in S_p$, $\sigma\{\alpha_1, \ldots, \alpha_p\} = \{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(p)}\}$. 
Note that 

\[ CD^0(\mathcal{C}) = \bigoplus_{B \subset D} \text{End}(\text{id}_B) \quad CD^n(\mathcal{C}) = \text{End}(F_D) \]

For \(1 \leq p \leq n - 1\), we define the map \(d_D^p : CD^p(\mathcal{C}) \to CD^{p+1}(\mathcal{C})\) by 

\[
(d_Df)(B;\alpha) = \sum_{i=1}^{p+1} (-1)^{i-1} (f(B;\alpha_{i}) - f(B^{\alpha_{i}};\alpha_{i}))
\]

as in Section 1.1. By abuse of notation, by \(f(B;\alpha_{i})\) we mean its image under the map 

\[ \text{End}(F(B;\alpha_{i})) \to \text{End}(F_B) \]

and by \(f(B^{\alpha_{i}};\alpha_{i})\) its image under the map 

\[ \text{End}(F(B^{\alpha_{i}};\alpha_{i})) \to \text{End}(F(B;\alpha_{i})) \to \text{End}(F_B) \]

For \(p = 0\), define \(d_D^0 : CD^0(\mathcal{C}) \to CD^1(\mathcal{C})\) by 

\[ d_D^0 f(B;\alpha) = f_B - f_{B^{\alpha}} \]

where \(f_{B^{\alpha}}\) is the sum over \(B'\), the connected components of \(B \setminus \alpha\), of \(f_{B'}\). Finally, we set \(d_D^n = 0\). The Dynkin differential is well-defined and preserves \(CD^*(\mathcal{C})\).

**Proposition.** \((CD^*(\mathcal{C});d_D)\) is a cochain complex.

### 2.6. Dynkin-Hochschild bicomplex for associative \(D\)–monoidal categories.

We now combine the Dynkin complex for a \(D\)–category and the
Hochschild complex for monoidal functors to define a suitable bicomplex for associative $D$–monoidal categories, which controls the deformation of the quasi–Coxeter monoidal structure.

For $p \in \mathbb{N}$ and $0 \leq q \leq |D|$, we denote by

$$C^D p(C^q) \subset \bigoplus_{B \in \mathrm{Com}(D), \ |B| = p} \mathrm{End}(F^{\otimes q}_{(B,A)})$$

the group of Dynkin $p$–cochains for $C^{\otimes q}$. The Dynkin differential $d_D$ defines a vertical differential

$$d_D^p : C^D p(C^q) \to C^D p+1(C^q)$$

while the Hochschild differential $d_H$ defines a horizontal differential

$$d_H^q : C^D p(C^q) \to C^D p(C^{q+1})$$

A simple computation yields the following.

**Proposition.** One has $d_D \circ d_H = d_H \circ d_D$.

The corresponding cohomology of the bicomplex $C^D(C^p)$ is called the Dynkin–Hochschild cohomology of the monoidal $D$–category $C$. As in [TL4, 7.2], the Dynkin–Hochschild bicomplex of $C$ controls the formal, one–parameter deformations of the trivial quasi-Coxeter monoidal structure on $C$.

3. Uniqueness of the relative twist

3.1. Chevalley invariance. We denote by $\theta$ the Chevalley involution on $g$ and the induced morphism on the Manin triple attached to $g$. In this section
we will study the following invariant property of a twist under $\theta$:

$$J^\theta = J^{21} \quad (3.1)$$

**Proposition.** The Etingof–Kazhdan twist $J_{fd}^{EK}$, defined in [EK1, Part I], satisfies

$$(J_{fd}^{EK})^\theta = (J_{fd}^{EK})^{21}$$

**Proof.** The twist $J_{fd}^{EK}$ defines a tensor structure on the functor $h_{M_+ \otimes M_-}$ and it can be identified with the action on $V \otimes W$ of the element $J \in U\mathfrak{g}^\otimes[[\hbar]]$

$$J := (\varphi \otimes \varphi)^{-1}(\Phi_{1,2,34}^{-1} \cdot \Phi_{234} \cdot e^{4\Omega_{23}} \cdot \Phi_{324}^{-1} \cdot \Phi_{1,3,24})(1_+ \otimes 1_- \otimes 1_+ \otimes 1_-)$$

where $\varphi : U\mathfrak{g} \xrightarrow{\sim} M_+ \otimes M_-$. The involution $\theta$ defines an autoequivalence of $D_{\Phi}(\mathfrak{g})$ and a natural identification

$$h_{M_+ \otimes M_-} \circ \theta^* = h_{M_+ \otimes M_-}^\theta$$

Using the isomorphism $M_\pm^\theta \simeq M_\pm$ and the relations $\Omega^\theta = \Omega$, $\Phi^\theta = \Phi$, we observe that the twist induced on the functor $h_{M_+ \otimes M_-} \circ \theta^*$, denoted $(J_{fd}^{EK})^\theta$, is given by the same formula defining $J_{fd}^{EK}$ on the functor $h_{M_- \otimes M_+}$. It corresponds to the action on $V \otimes W$ of the element

$$J^\theta := (\psi \otimes \psi)^{-1}(\Phi_{1,2,34}^{-1} \cdot \Phi_{234} \cdot e^{4\Omega_{23}} \cdot \Phi_{324}^{-1} \cdot \Phi_{1,3,24})(1_- \otimes 1_+ \otimes 1_- \otimes 1_+)$$
where \( \psi : U \mathfrak{g} \cong M_\ast M_+ \). It is easy to check that \( \varphi \circ \psi^{-1} : M_\ast M_+ \to M_+ \ast M_- \) corresponds to the permutation \( \sigma_{12} \). We then obtain

\[
J^{21} = (\psi^{-1} \otimes \psi^{-1}) \sigma_{14} \sigma_{23} (\Phi_{1,2,34}^{-1} \Phi_{234} \cdot e^b \Omega_{23} \cdot \Phi_{324}^{-1} \Phi_{1,3,24}) (1_+ \otimes 1_- \otimes 1_+ \otimes 1_-) =
\]

\[
= (\psi^{-1} \otimes \psi^{-1}) (\Phi_{4,3,21}^{-1} \cdot \Phi_{321} \cdot e^b \Omega_{23} \cdot \Phi_{231}^{-1} \Phi_{4,2,31}) (1_- \otimes 1_+ \otimes 1_- \otimes 1_+) =
\]

\[
= (\psi^{-1} \otimes \psi^{-1}) (\Phi_{12,3,4}^{-1} \cdot \Phi_{123}^{-1} \cdot e^b \Omega_{23} \cdot \Phi_{132} \cdot \Phi_{13,2,4}) (1_- \otimes 1_+ \otimes 1_- \otimes 1_+) = J^\theta
\]

where the last equivalence follows by the pentagon axiom. Alternatively, we can also notice that \( J^{21} \) can be thought as the twist of the functor \( F^{\text{EK}} \) with respect to the opposite tensor product, naturally identified with \( h_{M_+ \ast M_-} \). □

The Etingof–Kazhdan functor \( F^{\text{EK}} \) can be naturally identified with \( h_{M_+ \otimes M_-} \), in the finite dimensional setting, via the natural transformation \( \chi : F^{\text{EK}} \to h_{M_+ \otimes M_-} \)

\[
\chi_V(v) := (ev \otimes \text{id}) \circ \Phi^{-1} \circ (\text{id} \otimes v)
\]

The identification preserves the tensor structure, but it does not satisfy \( \chi^\theta = \chi \). We therefore conclude that the twist \( J^{\text{EK}} \) does not satisfy 3.1. The following lemma shows that we are allowed to weaken the requirement.

**Lemma.** Let \( u \in \text{End}(F)[[\hbar]] \) an invertible natural transformation of the tensor functor \((F,J)\) satisfying

\[
u \ast J^\theta = J^{21}
\]

where \( u \ast J := \Delta(u^{-1}) \cdot J \cdot u \otimes u \). If \( u^\theta = u^{-1} \), \( J \) is gauge equivalent to a solution of 3.1.
Proof. We first prove that the condition $u^\theta = u^{-1}$ is equivalent to the existence of an invertible element $v \in \End(F)[[\hbar]]$ such that $u = v^{-1}v^\theta$. One implication is trivial.

We have $u \equiv 1 \mod \hbar$ and $u^\theta = u^{-1}$. Denote $u = 1 + \sum_{n \geq 0} u_n \hbar^n$, so that $u_1^\theta = -u_1$. The element $v_1 := -u_1/2$ satisfies

$$v_1^\theta - v_1 = u_1$$

Consider the natural transformation

$$u^{(2)} := (1 + v_1 \hbar) \cdot u \cdot (1 + v_1^\theta \hbar)^{-1}$$

Then

$$u^{(2)} \equiv 1 \mod \hbar^2 \quad (u^{(2)})^\theta = (u^{(2)})^{-1}$$

Assume we constructed $u^{(n)}$ satisfying $u^{(n)} \equiv 1 \mod \hbar^n$ and $(u^{(n)})^\theta = (u^{(n)})^{-1}$. Similarly, consider the element $v_n = -u_n^{(n)}/2$ and

$$u^{(n+1)} := (1 + v_n \hbar^n) \cdot u^{(n)} \cdot (1 + v_n^\theta \hbar^n)^{-1}$$

Then $u^{(n+1)}$ satisfies $u^{(n+1)} \equiv 1 \mod \hbar^{n+1}$ and $(u^{(n+1)})^\theta = (u^{(n+1)})^{-1}$. By induction, we obtain

$$u = \prod_{n \geq 0} (1 + v_n \hbar^n)^{-1} \cdot \prod_{n \geq 0} (1 + v_n \hbar^n)^\theta$$

and we denote by $v$ the natural transformation $\prod_{n \geq 0} (1 + v_n \hbar^n)$. The factorization is uniquely defined up to a natural transformation $w \in \End(F)[[\hbar]]$. 

4. A RIGIDITY THEOREM FOR QUASI-COXETER CATEGORIES
invariant under $\theta$.

Finally, the relation $u \star J^\theta = J^{21}$ and the factorization $u = v^{-1}v^\theta$ imply

$$(v \star J)^{\theta} = (v \star J)^{21}$$

□

An easy computation on the 2–jet of $J_\Gamma$ (cf. 2.15) shows that the relative twist does not satisfy, in the finite dimensional setting, the condition 3.1. We observe that the natural transformation constructed in 5.1 gives rise to a gauge transformation relating $J_\Gamma$ and the composition of $J_{EK}$ and $(J_{D,EK})^{-1}$. We get the following

COROLLARY. In the finite–dimensional setting, there exists a natural transformation $u \in \text{End}(\Gamma)$ such that

$$(u \star J_{\Gamma})^{\theta} = (u \star J_{\Gamma})^{21}$$

where $(u \star J)_{V \otimes W} := u_{V \otimes W}^{-1}J_{VW}u_{V} \otimes u_{W}$ for any $V, W \in \mathcal{D}_\Phi(g)$.

3.2. Uniqueness property. We now prove a uniqueness property of the relative twist. By 2.4 it is easy to check that

$$H^2_{S}(\Gamma) := H^2((\bigwedge H^1(F))^\circ, \{\tilde{r} - \tilde{r}_D, \}_D) = (\bigwedge^2 I_D)^\circ = \bigwedge^2 c_D$$

THEOREM. Let $J_i, i = 1, 2$ be two tensor structures on the functor $\Gamma : \mathcal{O}^{\text{int}}(g) \to \mathcal{O}^{\text{int}}(I_D)$ such that $J_i \equiv 1 \otimes 2 + h f_i \mod h$ and

$$\text{Alt}_2 J_i \equiv 1 \otimes 2 + \frac{h}{2} \left( \frac{r - r^{21}}{2} - \frac{r_D - r_{D}^{21}}{2} \right) \mod h^2 \quad i = 1, 2$$
Then there exist
\[ u \in \text{End}(\Gamma) \quad \lambda \in H^2_S(\Gamma) \]
such that
\[ \exp \lambda \cdot J_2 = u \star J_1 \]

In the finite dimensional case, under the additional assumption that
\[ J_{\theta_i} = J_{21}^{21} \]
we have \( \lambda = 0 \) and \( u \) is uniquely determined by the property \( u^\theta = u \).

**Proof.** Denote
\[ \tilde{r} = \frac{r - r_{21}^{21}}{2} \quad \tilde{r}_D = \frac{r_D - r_{21}^{21}}{2} \]
and assume
\[ \text{Alt}_2 f_i = \frac{1}{2}(\tilde{r} - \tilde{r}_D) + \nu_i \quad \nu_i \in H^2_S(\Gamma) \]
Then there exist \( g_i \in \text{End}(\Gamma) \) such that
\[ f_i - \text{Alt}_2 f_i = d_H g_i \]
Therefore, we can replace \( J_i \) with
\[ \exp(-h\nu_i)(1 - h g_i) \star J_i \]
and assume that \( f_i = 1/2(\tilde{r} - \tilde{r}_D) \). In the particular case of the relative twist \( J_\Gamma \) we have
\[ f - \text{Alt}_2 f = \frac{1}{4}(\Omega - \Omega_D) = \frac{1}{4}d_H(m(r_D) - m(r)) \]
where \( m(r) \) is the element obtained by multiplication of the components of the \( r \)-matrix.

The additional assumption \( J_i^\theta = J_i^{21} \) implies

\[
d_H g_i^\theta + \nu_i^\theta = d_H g_i + \nu_i^{21} \quad \Rightarrow \quad d_H g_i^\theta = d_H g_i, \quad \nu_i^\theta = \nu_i^{21}
\]

since \( d_H g_i \) is symmetric and \( (\tilde{r} - \tilde{r}_D)^\theta = -(\tilde{r} - \tilde{r}_D) = (\tilde{r} - \tilde{r}_D)^{21} \). Therefore, we have \( \nu_i = 0 \), since \( \theta \) acts as the identity on \( \bigwedge^2 \mathfrak{c}_D \), and we may assume \( g_i^\theta = g_i \), replacing \( g_i \) with \( \frac{1}{2} (g_i + g_i^\theta) \).

We will now construct inductively two sequences

\[
v_n \in \text{End}(\Gamma) \quad \mu_n \in \text{H}^2_S(\Gamma)
\]

such that

\[
J_2 \equiv \exp(\lambda_n)(u_n \ast J_1) \mod \hbar^{n+1}
\]

where

\[
u_n = \prod_{i=1}^{n} (1 + \hbar^n \nu_n) \quad \lambda_n = \sum_{i=1}^{n} \hbar^i \mu_i
\]

Since we reduced both twists to the form

\[
J_i \equiv 1 \otimes 2 + \frac{\hbar}{2} (\tilde{r} - \tilde{r}_D) \mod \hbar^2
\]

we may assume \( v_1 = 0, \mu_1 = 0 \). Therefore, we assume elements \( v_k, \mu_k \) defined for \( k = 1, \ldots, n, \ n \geq 1 \). Denote \( J'_1 \) the element \( \exp(\lambda_n)(u_n \ast J_1) \) so that

\[
J_2 = J'_1 + \hbar^{n+1} \eta \mod \hbar^{n+2} \quad \eta \in \text{End}(\Gamma^{\otimes 2})
\]
We clearly have $\Phi_{J'_1} = \Phi_D = \Phi_{J_2}$. Computing $\mod h^{n+2}$ we obtain

$$1 \otimes \eta - (\Delta \otimes \text{id})(\eta) + (\text{id} \otimes \Delta)(\eta) - \eta \otimes 1 = 0$$

i.e., $d_H \eta = 0$. Moreover, if $J^\theta_i = J^2_1$, we get $\eta^\theta = \eta^2_1$. Then, there exist $v \in \text{End}(\Gamma)$ and $\mu \in H^2(\Gamma) = (\Lambda^2 H^1(F))^\theta \nu$ such that $\eta = d_H v + \mu$ and

$$v^\theta = v \quad \mu^\theta = \mu^2 = -\mu$$

Therefore, we set $v_{n+1} = -v$ and

$$J''_1 = (1 + h^{n+1} v_{n+1}) \star J'_1 = (1 + h^{n+1} v_{n+1}) \star (\exp(\lambda_n \cdot (u_n \star J_1))) = \exp(\lambda_n \cdot ((1 + h^{n+1} v_{n+1}) \cdot u_n) \star J_1$$

since clearly $v_{n+1}$ and $\lambda_n$ commute. We now have

$$J_2 \equiv \exp(-h^{n+1} \mu) \cdot J''_1 \mod h^{n+2}$$

So we set $\mu_{n+1} = -\mu$ and it remains to show that $\mu$ lies in $H^2_3(\Gamma)$.

Consider the truncation of $J_2 \mod h^{n+2}$

$$J_2^{(n+2)} = 1 \otimes 2 + \frac{h}{2} (\vec{r} - \vec{r}_D) + \sum_{i=2}^{n+1} h^i f_i$$

Then we have

$$\Phi_{J_2^{(n+2)}} \equiv \Phi_D + h^{n+2} \xi \mod h^{n+3}$$
for some \( \xi \in \text{End}(\Gamma^{\otimes 3}) \). Since \( J_2^{(n+2)} \) is the truncation of a relative twist modulo \( \mathcal{H}^{n+3} \), we have \( d_H \xi = 0 \) and \( \text{Alt}_3 \xi = 0 \). Similarly, we can consider the truncation of \( J_1'' \) modulo \( \mathcal{H}^{n+2} \), \( J_1''^{(n+2)} \), and denote by \( \xi'' \) the corresponding error. Then it satisfies \( d_H \xi'' = 0 \) and \( \text{Alt}_3 \xi'' = 0 \) (cf. [TL3, 5.4]).

Now, since \( J_1''^{(n+2)} = J_2^{(n+2)} + h^{n+1} \mu \mod \mathcal{H}^{n+2} \), denote by \( f = \frac{1}{2}(\tilde{r} - \tilde{r}_D) \) and we obtain

\[
\xi'' - \xi = f^{23}(\mu^{12} + \mu^{13}) + \mu^{23}(f^{12} + f^{13}) - f^{12}(\mu^{13} + \mu^{23}) - \mu^{12}(f^{13} + f^{23})
\]

then we get \([f, \mu] = 0\). Then we have

\[
\mu = [f, x] + y \quad \text{for some} \quad x \in H^1(F)^{\ell_D} = c_D \subset \mathfrak{h}, y \in H^2_S(\Gamma)
\]

Since \( f \) is a weight zero element, we have \([f, x] = -\text{ad}(x)f = 0\). We conclude that

\[
\mu \in H^2_S(\Gamma) = H^2((\bigwedge H^1(F)^{\ell_D}, [\tilde{r} - \tilde{r}_D, ])) \simeq \bigwedge^2 c_D[[\mathfrak{h}]]
\]

Since we are assuming \( J_i^\theta = J_i^{21} \), we get \( \mu^\theta = -\mu \) and \( \mu = 0 \).

Let \( u \in \text{End}(\Gamma) \) be such that

\[
u * J_1 = J_1
\]

with \( u^\theta = u \). Assume \( u \equiv 1 \mod \mathcal{H}^n \) and write \( u \equiv 1 + h^n u_n \mod \mathcal{H}^{n+1} \), with \( u_n^\theta = u_n \). Taking the coefficient of \( \mathcal{H}^{n+1} \) we get \( d_H u_n = 0 \), implying \( u_n \in \mathfrak{h} \), since it is of weight zero. Then we get \( u_n = 0, \theta \) acting on \( \mathfrak{h} \) by \(-1\). \( \square \)
4. The main theorem

4.1. The $D$–bialgebra structure on $U\mathfrak{g}$. Let $A = (a_{ij})_{i,j \in I}$ be an irreducible, symmetrizable, generalised Cartan matrix of affine type and $\mathfrak{g} = \mathfrak{g}(A)$ the corresponding affine Kac–Moody algebra [Ka]. Let $D_{\mathfrak{g}} = D(A)$ be the Dynkin diagram of $\mathfrak{g}$, that is the connected graph having $I$ as its vertex set and an edge between $i$ and $j$ if $a_{ij} \neq 0$. For any $i \in I$, let $\mathfrak{sl}_2^i \subseteq \mathfrak{g}$ be the three–dimensional subalgebra spanned by $e_i, f_i, h_i$. In this case, any proper connected subdiagram of $D_{\mathfrak{g}}$ is a Dynkin diagram of finite type. Therefore, for any proper subdiagram $D \subset D_{\mathfrak{g}}$, we consider the subalgebra $\mathfrak{g}_D$ spanned by $\{e_i, f_i, h_i\}_{i \in D}$. For $D = D_{\mathfrak{g}}$, we consider the entire algebra $\mathfrak{g}$. Then $(U\mathfrak{g}, \{U\mathfrak{g}_D\})$ is a $D_{\mathfrak{g}}$–algebra over $\mathbb{C}$.

4.2. The rigidity theorem. Extend the bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}$ to a non–degenerate, symmetric, bilinear, ad–invariant form on $\mathfrak{g}$. For any proper connected subdiagram $D \subset D_{\mathfrak{g}}$, let $\mathfrak{g}_D \subset I_D \subset \mathfrak{g}$ be the corresponding simple and Levi subalgebras. Denote by

$$\Omega_D = x_a \otimes x^a, \quad C_D = x_a \cdot x^a \quad \text{and} \quad \tilde{r}_{\mathfrak{g}_D} = \sum_{\alpha > 0: \text{supp}(\alpha) \subseteq D} \frac{(\alpha, \alpha)}{2} \cdot e_{\alpha} \wedge f_{\alpha}$$

where $\{x_a\}_a, \{x^a\}_a$ are dual basis of $\mathfrak{g}_D$ with respect to $(\cdot, \cdot)$, the corresponding invariant tensor, Casimir operator and standard solution of the modified classical Yang–Baxter equation for $\mathfrak{g}_D$ respectively. Abbreviate $\mathfrak{sl}_2^{\alpha_i}, \Omega_{\alpha_i}$ and $C_{\alpha_i}$ to $\mathfrak{sl}_2^i, \Omega_i$ and $C_i$ respectively and let $\tilde{s}_i$ be the triple exponentials

$$\tilde{s}_i = \exp(e_i) \exp(-f_i) \exp(e_i)$$
where $e_i = e_{\alpha_i}$, $f_i = f_{\alpha_i}$ are a fixed choice of simple root vectors.

In the next sections 4.3-4.5, we outline the proof of the following

**Theorem.** Up to equivalence, there exists a unique quasi-Coxeter braided monoidal $D$–category of type $D_g$ of the form

$$ (\mathcal{O}_g^{\text{int}}[[\hbar]], \{\mathcal{O}_{\theta D}^{\text{int}}[[\hbar]], \otimes, R_D, \Phi_D\}, \{S_{i,c}\}, \{\Phi_{(D;\alpha_i,\alpha_j)}\}, \{F_{(D;\alpha_i)}\}, \{J_{(D;\alpha_i)}\}) $$

where $\otimes$ is the standard tensor product in $\mathcal{O}_g$, $(F_{(D;\alpha_i)}, J_{(D;\alpha_i)})$ are fiber functors,

$$ S_{i,c} = \tilde{s}_i \cdot \exp(\hbar/2 \cdot C_i) \quad (4.1) $$
$$ R_D = \exp(\hbar/2 \cdot \Omega_D) \quad (4.2) $$
$$ \text{Alt}_2 J_{(D;\alpha_i)} = \frac{\hbar}{2} \left( \tilde{r}_{\theta D} - \tilde{r}_{\theta D \setminus \{\alpha_i\}} \right) \mod \hbar^2 \quad (4.3) $$

and $\Phi_{(D;\alpha_i,\alpha_j)}, J_{(D;\alpha_i)}$ are of weight $0$, i.e., they properly extend to the corresponding Levi’s subalgebras.

**4.3. Proof of Theorem 4.2.** Let

$$ (\mathcal{O}_g^{\text{int}}[[\hbar]], \{\mathcal{O}_{\theta D}^{\text{int}}[[\hbar]], \otimes, R_D, \Phi_D^k\}, \{S_{i,c}\}, \{\Phi_{(D;\alpha_i,\alpha_j)}^k\}, \{F_{(D;\alpha_i)}\}, \{J_{(D;\alpha_i)}^k\}) $$

$k = 1, 2$ be two quasi–Coxeter braided monoidal categories.

By Drinfeld’s uniqueness, [TL3, TL4] and the uniqueness of the twist, we may assume that

$$ \Phi_D^2 = \Phi_D^1 \quad J_{(D;\alpha_i)}^2 = J_{(D;\alpha_i)}^1 $$
and that
\[ \Phi_1^{(D; \alpha_i, \alpha_j)} = \Phi_2^{(D; \alpha_i, \alpha_j)} + h^n \varphi^{(D; \alpha_i, \alpha_j)} \mod h^{n+1} \]
for some \( \varphi^{(D; \alpha_i, \alpha_j)} \in \text{End}(F_{D \setminus \{\alpha_i, \alpha_j\}}) \).

Let \( \alpha_i \neq \alpha_j \in D \subseteq D_g \) and \( \mathcal{F}, \mathcal{G} \in \text{Mns}(D_g) \) such that
\[ \mathcal{F} \setminus \mathcal{G} = \mathcal{C}^{D \setminus \{\alpha_i\}}_{\alpha_j} \quad \mathcal{G} \setminus \mathcal{F} = \mathcal{C}^{D \setminus \{\alpha_j\}}_{\alpha_i} \]
Subtracting the equations
\[ J_{\mathcal{F}} \cdot \Delta(\Phi_1^{(D; \alpha_i, \alpha_j)}) = \Phi_2^{(D; \alpha_i, \alpha_j)} \otimes 2 \cdot J_{\mathcal{G}} \]
\[ J_{\mathcal{F}} \cdot \Delta(\Phi_2^{(D; \alpha_i, \alpha_j)}) = \Phi_1^{(D; \alpha_i, \alpha_j)} \otimes 2 \cdot J_{\mathcal{G}} \]
and equating the coefficients of \( h^{n+1} \), we find
\[ \Delta(\varphi^{(D; \alpha_i, \alpha_j)}) - \varphi^{(D; \alpha_i, \alpha_j)} \otimes 1 - 1 \otimes \varphi^{(D; \alpha_i, \alpha_j)} = 0 \]
so that \( \varphi^{(D; \alpha_i, \alpha_j)} \) is a primitive element of \( \text{End}(F_D) \). Since \( \varphi^{(D; \alpha_i, \alpha_j)} \) is also of weight 0, we find that
\[ \varphi^{(D; \alpha_i, \alpha_j)} \in \mathfrak{h}_D \]
Moreover, since \( \Phi_s^{(D; \alpha_i, \alpha_j)} \) satisfy the generalized pentagon identities corresponding to the 2-faces of the De Concini–Procesi associahedron, we also find
\[ d_D \varphi^{(D; \alpha_i, \alpha_j)} = 0 \]
We are then reduced to prove the following
Proposition. Let $\varphi = \{\varphi(D;\alpha_1,\alpha_2)\}$ be a 2–cocycle in the Dynkin complex of $\text{End}(F)$ such that

$$\varphi(D;\alpha_1,\alpha_2) \in \mathfrak{h}_D$$

for any $\alpha_i \neq \alpha_j \in D \subseteq D_\mathfrak{g}$. Then, there exists a Dynkin 1–cochain $a = \{a(D;\alpha_1)\}$ such that

$$a(D;\alpha_1) \in \mathfrak{h}_D \quad \text{and} \quad d_Da = \varphi$$

The element $a$ may be chosen such that $a(D;\alpha_1) = 0$ for all $\alpha_1$ and is then unique with this additional property.

The element $a = \{1 - h^na(D;\alpha_1)\}_{\alpha_1 \in D \subseteq D_\mathfrak{g}}$ defines the twist matching the two qCtqba structures. Notice that, since $a(D;\alpha_1) = 0$, then $(S_{i,c})_a = S_{i,c}$ for all $i$. The theorem is proved.

4.4. Proof of Prop. 4.3: simple case. Let us first give a proof of Prop.4.3 for the simple case.¹

Let then $\mathfrak{g}$ be a simple complex Lie algebra of rank $l$, $D_\mathfrak{g}$ be the Dynkin diagram associated to $\mathfrak{g}$ with vertices $\{\alpha_1, \ldots, \alpha_l\}$. For any $i \in [1, l]$, let $\mathfrak{sl}^3_{\alpha_i} \subset \mathfrak{g}$ be the three dimensional subalgebra spanned by $\{e_i, f_i, h_i\}$. Then the collection $\{U\mathfrak{sl}^3_{\alpha_i}\}$ defines a $D_\mathfrak{g}$–algebra structure on $U\mathfrak{g}$.

Let $\mathfrak{h}_D \subset \mathfrak{h}$ be the subalgebra generated by $\{\alpha_i^\vee \ | \ \alpha_i \in D\}$, where $D \subset D_\mathfrak{g}$, and $\{\lambda_i^\vee_{i,D}\}$ be the fundamental coweights associated to $\mathfrak{h}_D$, for any $D \subset D_\mathfrak{g}$.

¹the original proof in [TL4] is incomplete, since it is implicitly assumed that $\lambda_i^\vee_{i,D}$ does not depends upon $D$ or, in other words, that $\lambda_i^\vee_{i,D'} \in \mathbb{C}\lambda_i^\vee_{i,D}$ for $D' \subset D$.\]
We wish to solve the equation \( \varphi = d_D a \). In components, this reads

\[
\varphi(D;\alpha_i,\alpha_j) = a_{(D;\alpha_i,\alpha_j)} - a_{(D;\alpha_i,\alpha_j \setminus \{\alpha_i\})} - a_{(D;\alpha_j,\alpha_i \setminus \{\alpha_j\})} + a_{(D;\alpha_j,\alpha_i)} \quad (4.4)
\]

for any connected subdiagram \( D \subseteq D_g \) and \( \alpha_i \neq \alpha_j, \alpha_i, \alpha_j \in D \). The assumptions \( \varphi(D;\alpha_i,\alpha_j), a_{(D;\alpha_i)} \in \mathfrak{h}_D \) and the fact that \( \varphi, a \) lie in the Dynkin complex of \( g \) imply that

\[
\varphi(D;\alpha_i,\alpha_j) \in \mathbb{C}\lambda_{i,D}^\vee \oplus \mathbb{C}\lambda_{j,D}^\vee \quad \text{and} \quad a_{(D;\alpha_k)} \in \mathbb{C}\lambda_{k,D}^\vee \quad \text{for } k = i, j
\]

respectively, where \( \lambda_{k,D}^\vee \) is the fundamental coweight in \( \mathfrak{h}_D \) dual to \( \alpha_k \mid_{\mathfrak{h}_D} \). Projecting (4.4) on \( \lambda_{i,D}^\vee \) and \( \lambda_{j,D}^\vee \) we therefore find that it is equivalent to

\[
a_{(D;\alpha_i)} = a^i_{(D;\alpha_i \setminus \{\alpha_i\})} - a^i_{(D;\alpha_i \setminus \{\alpha_j\})} - \varphi^i_{(D;\alpha_i,\alpha_j)}
\]

\[
a_{(D;\alpha_j)} = a^j_{(D;\alpha_j \setminus \{\alpha_i\})} - a^j_{(D;\alpha_j \setminus \{\alpha_j\})} + \varphi^j_{(D;\alpha_i,\alpha_j)}
\]

where

- by \((-)^i\) we denote the component along \( \lambda_{i,D}^\vee \) of the projection

\[
p_{i,D} : \mathfrak{h}_D = \bigoplus_{\alpha_j \in D} \mathbb{C}\lambda_{j,D}^\vee \longrightarrow \mathbb{C}\lambda_{i,D}^\vee
\]

- for any subdiagram \( D' \subset D \), we look at

\[
a_{(D';\alpha_i)} \in \mathfrak{h}_{D'} \subset \mathfrak{h}_D
\]

as an element of the bigger subspace, since in general

\[
\lambda_{i,D'}^\vee \not\in \mathbb{C}\lambda_{i,D}^\vee \quad \lambda_{i,D'}^\vee \in \bigoplus_{\alpha_k \in D} \lambda_{k,D}^\vee
\]
• we are identifying $a_{(D;\alpha_i)}$, $a_{(D;\alpha_j)}$ with their components along $\lambda_{i,D}^\vee$, $\lambda_{j,D}^\vee$ respectively.

Thus, since $\varphi_{(D;\alpha_i,\alpha_j)} = -\varphi_{(D;\alpha_j,\alpha_i)}$, $\varphi = d_D a$ iff

$$a_{(D;\alpha_i)} = a^i_{(L_{a_i} \cap \{\alpha_j\}; \alpha_i)} - a^i_{(L_{a_j} \cap (\alpha_i); \alpha_j)} - \varphi^i_{(D;\alpha_i,\alpha_j)}$$

(4.5)

for all $D$ and $1 \leq i \neq j \leq n$ with $\alpha_i, \alpha_j \in D$. Induction on the cardinality of $D$ readily shows that the above equations possess at most one solution once the values of $a_{(\alpha_i;\alpha_i)}$ are fixed. To prove that one solution exists, assume that $a_{(D;\alpha_i)}$ have been constructed for all $D$ with at most $m$ vertices in such a way that the equations (4.5) hold for all such $D$. We claim that (4.5) may be used to define $a_{(D;\alpha_i)}$ for all $D$ with $|D| = m + 1$ in a consistent way, i.e., independently of $j \neq i$ such that $\alpha_j \in D$. This amounts to showing that for all such $D$, and distinct vertices $\alpha_i, \alpha_j, \alpha_k \in D$, one has

$$a^i_{(L_{a_i} \cap \{\alpha_j\}; \alpha_i)} - a^i_{(L_{a_j} \cap (\alpha_i); \alpha_j)} - \varphi^i_{(D;\alpha_i,\alpha_j)} = a^i_{(L_{a_i} \cap \{\alpha_k\}; \alpha_i)} - a^i_{(L_{a_k} \cap (\alpha_i); \alpha_k)} - \varphi^i_{(D;\alpha_i,\alpha_k)}$$

(4.6)

To see this, consider the $(D;\alpha_i,\alpha_j,\alpha_k)$ component of $d_D \varphi$ i.e., the sum

$$\left( \varphi_{(D;\alpha_j,\alpha_k)} - \varphi_{(a_{\alpha_j,\alpha_k} \cap (\alpha_i); \alpha_j,\alpha_k)} \right)$$

$$- \left( \varphi_{(D;\alpha_i,\alpha_k)} - \varphi_{(a_{\alpha_i,\alpha_k} \cap (\alpha_j); \alpha_i,\alpha_k)} \right)$$

$$+ \left( \varphi_{(D;\alpha_i,\alpha_j)} - \varphi_{(a_{\alpha_i,\alpha_j} \cap (\alpha_k); \alpha_i,\alpha_j)} \right)$$
Since \( d_D \varphi = 0 \) we get, by projecting on \( \lambda_D^V \) and using \( \varphi_{(D; \alpha_i, \alpha_k)} = 0, \)

\[
\varphi_{(D; \alpha_i, \alpha_j)} = \varphi_{(D; \alpha_i, \alpha_k)} - \varphi_{(\ell_{\alpha_i, \alpha_k}; \alpha_i, \alpha_k)} + \varphi_{(\ell_{\alpha_j, \alpha_k}; \alpha_j, \alpha_k)} + \varphi_{(\ell_{\alpha_i, \alpha_j}; \alpha_i, \alpha_j)}
\]

so that (4.6) holds iff

\[
a_i^{(\ell_{\alpha_i, \alpha_j}; \alpha_i)} - a_i^{(\ell_{\alpha_i, \alpha_k}; \alpha_i)} - a_i^{(\ell_{\alpha_j, \alpha_k}; \alpha_i)} + a_i^{(\ell_{\alpha_i, \alpha_j}; \alpha_i)} = -\varphi_i^{(\ell_{\alpha_i, \alpha_j}; \alpha_i, \alpha_k)} + \varphi_i^{(\ell_{\alpha_j, \alpha_k}; \alpha_i, \alpha_k)} + \varphi_i^{(\ell_{\alpha_i, \alpha_j}; \alpha_i, \alpha_j)} = (4.7)
\]

\[
a_i^{(\ell_{\alpha_i, \alpha_k}; \alpha_i)} - a_i^{(\ell_{\alpha_j, \alpha_k}; \alpha_i)} - \varphi_i^{(\ell_{\alpha_i, \alpha_k}; \alpha_j, \alpha_k)} = a_i^{(\ell_{\alpha_i, \alpha_k}; \alpha_i)} + \varphi_i^{(\ell_{\alpha_i, \alpha_k}; \alpha_i, \alpha_j)} - a_i^{(\ell_{\alpha_i, \alpha_j}; \alpha_i)} - \varphi_i^{(\ell_{\alpha_i, \alpha_k}; \alpha_i, \alpha_k)} = (4.8)
\]

The inductive assumption yields the following relation for any distinct \( \alpha_i, \alpha_j, \alpha_k \in D \)

\[
\varphi_{(\ell_{\alpha_i, \alpha_k}; \alpha_i, \alpha_k)} = a_i^{(\ell_{\alpha_i, \alpha_k}; \alpha_i, \alpha_k)} - a_i^{(\ell_{\alpha_i, \alpha_k}; \alpha_i, \alpha_k)} - a_i^{(\ell_{\alpha_i, \alpha_k}; \alpha_i, \alpha_k)} + a_i^{(\ell_{\alpha_i, \alpha_k}; \alpha_i, \alpha_k)} = (4.9)
\]

We consider five separate cases.

4.4.1. At least two connected component among \( C_{\alpha_i, \alpha_k}^{(\alpha_j \setminus \alpha_k)}, C_{\alpha_i, \alpha_j}^{(\alpha_k \setminus \alpha_i)}, C_{\alpha_i, \alpha_k}^{(\alpha_j \setminus \alpha_k)} \) are empty. We may assume \( C_{\alpha_i, \alpha_k}^{(\alpha_j \setminus \alpha_k)} = \emptyset, C_{\alpha_i, \alpha_j}^{(\alpha_k \setminus \alpha_i)} = \emptyset \). This case cannot arise since the first condition implies that any path in \( D \) from \( \alpha_i \) to \( \alpha_k \) must pass through \( \alpha_j \) before it reaches \( \alpha_k \) while the second one implies that the portion of this path linking \( \alpha_i \) to \( \alpha_j \) must first pass through \( \alpha_k \).
4.4.2. $C_{\alpha_i,\alpha_k}^{D\{\alpha_j\}} = \emptyset$, $C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}} \neq \emptyset$, $C_{\alpha_j,\alpha_k}^{D\{\alpha_i\}} \neq \emptyset$. In this case,

$$\varphi_{(a_{\alpha_i,\alpha_k};\alpha_i,\alpha_k)} = 0$$

and (4.7) reads, using (4.9),

$$a_i^{\left(C_{\alpha_i,\alpha_k}^{D\{\alpha_j\}\setminus\{\alpha_k\}}\right)} - a_i^{\left(C_{\alpha_j,\alpha_k}^{D\{\alpha_j\}\setminus\{\alpha_j\}}\right)} - a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}\setminus\{\alpha_k\}}\right)}$$

$$+ a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}\setminus\{\alpha_j\} \setminus\{\alpha_k\}}\right)} + a_i^{\left(C_{\alpha_j,\alpha_k}^{D\{\alpha_j\}\setminus\{\alpha_j\} \setminus\{\alpha_k\}}\right)} - a_i^{\left(C_{\alpha_j,\alpha_k}^{D\{\alpha_k\}\setminus\{\alpha_j\}}\right)} =$$

$$= a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}\setminus\{\alpha_j\} \setminus\{\alpha_k\}}\right)} + a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}\setminus\{\alpha_j\}}\right)} - a_i^{\left(C_{\alpha_j,\alpha_k}^{D\{\alpha_k\}\setminus\{\alpha_j\}}\right)}$$

$$- a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}\setminus\{\alpha_j\}}\right)} + a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}\setminus\{\alpha_j\}}\right)} - a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}}\right)} (\text{4.7 reads, using (4.9),})$$

This equation holds since:

- $C_{\alpha_i}^{D\{\alpha_k\}} = C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}}$, so $a_i^{\left(a_{\alpha_i,\alpha_k}^{D\{\alpha_k\}};\alpha_i\right)} = a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}};\alpha_i\right)}$

- $C_{\alpha_k}^{D\{\alpha_i\}} = C_{\alpha_j,\alpha_k}^{D\{\alpha_i\}}$, so $a_i^{\left(a_{\alpha_i,\alpha_k}^{D\{\alpha_k\}};\alpha_k\right)} = a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}};\alpha_k\right)}$

- $C_{\alpha_j}^{D\{\alpha_i\}} = C_{\alpha_i,\alpha_k}^{D\{\alpha_i\}}$, so $a_i^{\left(a_{\alpha_i,\alpha_k}^{D\{\alpha_i\}};\alpha_j\right)} = a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_i\}};\alpha_j\right)}$

- Since $C_{\alpha_i,\alpha_j}^{D\{\alpha_i\}} \neq \emptyset$, then

$$C_{\alpha_i}^{D\{\alpha_j\}} = C_{\alpha_i}^{D\{\alpha_j,\alpha_k\}} = C_{\alpha_i}^{D\{\alpha_j\}\setminus\{\alpha_j\}} \implies a_i^{\left(a_{\alpha_i,\alpha_k}^{D\{\alpha_j\}\setminus\{\alpha_j\}};\alpha_i\right)} = a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_j\}\setminus\{\alpha_j\}};\alpha_i\right)}$$

- Similarly,

$$C_{\alpha_j}^{D\{\alpha_k\}\setminus\{\alpha_j\}} = C_{\alpha_j}^{D\{\alpha_i,\alpha_k\}} = C_{\alpha_j}^{D\{\alpha_k\}\setminus\{\alpha_j\}} \implies a_i^{\left(a_{\alpha_i,\alpha_k}^{D\{\alpha_k\}\setminus\{\alpha_j\}};\alpha_i\right)} = a_i^{\left(C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}\setminus\{\alpha_j\}};\alpha_i\right)}$$
Condition (4.7) reduces to:

\[ a^i_{(α_i, α_j)} = a^i_{(α_i, α_k)} \]

that is

\[ a^i_{D \setminus (α_k)} = a^i_{(α_j, α_k)} \]

We claim that for all distinct \( i, j, k \in I \),

\[ a^i_{(α_j, α_k)} = 0 \]

that is equivalent to:

\[ \lambda^\vee_{j, D \setminus (α_k)} \in \mathbb{C} \lambda^\vee_{j, D} \oplus \mathbb{C} \lambda^\vee_{k, D} \tag{4.10} \]

Indeed, if \( i \in I^k_j := \{ r \in I \mid α_r \in C_{α_j}^{D \setminus (α_k)} \}, i \neq j \), then

\[ \langle α_i, \lambda^\vee_{j, D \setminus (α_k)} \rangle = 0 \]

by definition of \( \lambda^\vee_{j, D \setminus (α_k)} \). On the other hand, if \( i \notin I^k_j \), then, by orthogonality, we have:

\[ \forall s \in I^k_j \quad \langle α_i, α_s^\vee \rangle = 0 \]

Consequently, since \( \lambda^\vee_{j, D \setminus (α_k)} \in h_{α_j}^{D \setminus (α_k)} \), we have

\[ \langle α_i, \lambda^\vee_{j, D \setminus (α_k)} \rangle = 0 \]

and (4.10) is proved.

4.4.3. \( C_{α_i, α_k} \neq \emptyset \), \( C_{α_i, α_j} = \emptyset \), \( C_{α_j, α_k} \neq \emptyset \). This case reduces to the previous one under the interchange \( α_j \leftrightarrow α_k \).
4.4.4. $C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}} \neq \emptyset$, $C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}} \neq \emptyset$, $C_{\alpha_j,\alpha_k}^{D\{\alpha_i\}} = \emptyset$. In this case,

$$\varphi^{a_{\alpha_j^{D\{\alpha_k\}}}^{\alpha_i,\alpha_k}} = 0$$

and (4.7) reads, using (4.9),

$$a^i_{\alpha_j^{D\{\alpha_k\}};\alpha_k} - a^i_{\alpha_j^{D\{\alpha_k\}};\alpha_i} =\
= a^i_{\alpha_j^{D\{\alpha_k\}};\alpha_i} + a^i_{\alpha_j^{D\{\alpha_k\}};\alpha_j} - a^i_{\alpha_j^{D\{\alpha_k\}};\alpha_j} - a^i_{\alpha_j^{D\{\alpha_k\}};\alpha_k} + a^i_{\alpha_j^{D\{\alpha_k\}};\alpha_i} + a^i_{\alpha_j^{D\{\alpha_k\}};\alpha_i} - a^i_{\alpha_j^{D\{\alpha_k\}};\alpha_k} + a^i_{\alpha_j^{D\{\alpha_k\}};\alpha_k}$$

This equation holds, since:

- $C_{\alpha_j}^{D\{\alpha_j\}} = C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}}$, so $a_{\alpha_j}^{a_{\alpha_j}^{D\{\alpha_k\}};\alpha_i} = a_{\alpha_j}^{a_{\alpha_j}^{D\{\alpha_k\}};\alpha_i}$
- $C_{\alpha_i}^{D\{\alpha_j\}} = C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}}$, so $a_{\alpha_i}^{a_{\alpha_j}^{D\{\alpha_k\}};\alpha_i} = a_{\alpha_i}^{a_{\alpha_j}^{D\{\alpha_k\}};\alpha_i}$
- Since $C_{\alpha_i,\alpha_j}^{D\{\alpha_i\}} = \emptyset$, then

$$C_{\alpha_i}^{D\{\alpha_i\}} = C_{\alpha_i,\alpha_j}^{D\{\alpha_k\}} = C_{\alpha_i}^{D\{\alpha_i\}};\alpha_k} = a_{\alpha_i}^{a_{\alpha_j}^{D\{\alpha_k\}};\alpha_i} = a_{\alpha_i}^{a_{\alpha_j}^{D\{\alpha_k\}};\alpha_i}$$

- Similarly,

$$C_{\alpha_i,\alpha_j}^{D\{\alpha_j\}} = C_{\alpha_i,\alpha_k}^{D\{\alpha_k\}} = C_{\alpha_i,\alpha_j}^{D\{\alpha_j\}} = C_{\alpha_i,\alpha_k}^{D\{\alpha_k\}} = C_{\alpha_i,\alpha_j}^{D\{\alpha_j\}} = C_{\alpha_i,\alpha_k}^{D\{\alpha_k\}}$$
implies

\[
a\left(\bigcap_{\alpha_i, \alpha_k} \left\{ \alpha_j \right\}_i \bigcap_{\alpha_i} \left\{ \alpha_j \right\}_k \right) = a\left(\bigcap_{\alpha_i, \alpha_k} \left\{ \alpha_j \right\}_i \bigcap_{\alpha_i} \left\{ \alpha_j \right\}_k \right)
\]

As before, it remains to check that:

\[
a_i \left(\bigcap_{\alpha_i, \alpha_j} \left\{ \alpha_k \right\}_j \bigcap_{\alpha_i} \left\{ \alpha_j \right\}_j \right) = a_i \left(\bigcap_{\alpha_i, \alpha_k} \left\{ \alpha_k \right\}_k \bigcap_{\alpha_i} \left\{ \alpha_j \right\}_k \right) = 0
\]

\[
a_i \left(\bigcap_{\alpha_i, \alpha_k} \left\{ \alpha_j \right\}_k \bigcap_{\alpha_i} \left\{ \alpha_j \right\}_k \right) = a_i \left(\bigcap_{\alpha_i, \alpha_k} \left\{ \alpha_k \right\}_k \bigcap_{\alpha_i} \left\{ \alpha_j \right\}_k \right) = 0
\]

that it is true by (4.10).

4.4.5. \(\mathcal{C}_{\alpha_i, \alpha_k} \neq \emptyset, \mathcal{C}_{\alpha_i, \alpha_j} \neq \emptyset, \mathcal{C}_{\alpha_j, \alpha_k} \neq \emptyset\). It is interesting to observe that this case can occur only if there is a cycle in \(D\) passing through \(\{\alpha_i, \alpha_j, \alpha_k\}\) that is not our case, since \(g\) is simple. Instead, in the affine case \(\hat{g}\), it can happen only if \(g = A_n\) and \(D = D_{\hat{g}}\).

Even in this case, anyway, the consistency relation holds. Indeed, (4.7) reads,
using (4.9),

\[
\begin{align*}
   a^i & \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) - a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) - a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) + a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) \\
   + a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) - a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) - a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) = \\
   = a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) + a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) - a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) - a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) \\
   + a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) - a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) + a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) + a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) \\
   + a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) - a^i \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right)
\end{align*}
\]

This equation holds, since:

- \( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} = \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \), so \( a \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) = a \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) \)
- \( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} = \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \), so \( a \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) = a \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) \)
- \( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} = \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \), so \( a \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) = a \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) \)
- \( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} = \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \), so \( a \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) = a \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) \)

Similarly,

\[
\mathcal{C}^{D \{ \alpha_i \}; \alpha_k} = \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \]

implies

\[
\left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right) = a \left( \mathcal{C}^{D \{ \alpha_i \}; \alpha_k} \right)
\]
then
\[ C^{D\setminus\{\alpha_i\}\setminus\{\alpha_k\}}_{\alpha_j,\alpha_k} = C^{D\setminus\{\alpha_i,\alpha_k\}}_{\alpha_j} = C^{D\setminus\{\alpha_k\}}_{\alpha_j,\alpha_k} \]
implies
\[ a_j \left( C^{D\setminus\{\alpha_i\}\setminus\{\alpha_k\}}_{\alpha_j,\alpha_k}, \right) = a_j \left( C^{D\setminus\{\alpha_i\}\setminus\{\alpha_k\}}_{\alpha_j} \right) \]
and finally
\[ C^{D\setminus\{\alpha_i\}\setminus\{\alpha_k\}}_{\alpha_k,\alpha_j} = C^{D\setminus\{\alpha_i,\alpha_j\}}_{\alpha_k} = C^{D\setminus\{\alpha_j\}\setminus\{\alpha_k\}}_{\alpha_k} \]
implies
\[ a_j \left( C^{D\setminus\{\alpha_j\}\setminus\{\alpha_k\}}_{\alpha_k,\alpha_j}, \right) = a_j \left( C^{D\setminus\{\alpha_j\}\setminus\{\alpha_k\}}_{\alpha_k} \right) \]
As before it remains to check that:
\[ a_j^i \left( C^{D\setminus\{\alpha_i,\alpha_k\}}_{\alpha_j} \right) = a_j^i \left( C^{D\setminus\{\alpha_i\}\setminus\{\alpha_k\}}_{\alpha_j} \right) = 0 \]
\[ a_j^i \left( C^{D\setminus\{\alpha_j\}\setminus\{\alpha_k\}}_{\alpha_k} \right) = a_j^i \left( C^{D\setminus\{\alpha_j\}\setminus\{\alpha_k\}}_{\alpha_k} \right) = 0 \]
that is true, by (4.10).

This concludes the proof of Proposition 4.3

\[ \square \]

4.5. Proof of Prop. 4.3: Affine case. We will now discuss the previous proof using the combinatorics of the Dynkin diagrams of affine type.

Let \( \mathfrak{g} \) be a simple complex Lie algebra of rank \( l \) and \( \hat{\mathfrak{g}} \) be the affine Lie algebra of type \( \mathfrak{g} \). We denote by \( D_\hat{\mathfrak{g}} \) the Dynkin diagram associated to \( \hat{\mathfrak{g}} \) with
vertices \( \{ \alpha_0, \alpha_1, \ldots, \alpha_l \} \) and by \( \hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d \subset \hat{\mathfrak{g}} \) the Cartan subalgebra generated by \( \{ \alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_l^\vee, \alpha_{l+1}^\vee \} \) where

\[
\alpha_0^\vee = \frac{c}{(\theta|\theta)} - \theta^\vee \otimes 1 \quad \alpha_{l+1}^\vee = d
\]

Let \( \mathcal{B} = \{ \alpha_0, \alpha_1, \ldots, \alpha_l, \alpha_{l+1} \} \) be a basis for \( \hat{\mathfrak{h}}^* \), where \( \alpha_{l+1} = c^* \) is defined by

\[
\forall i \in \{0, 1, \ldots, l, l+1\} \quad \langle \alpha_{l+1}, \alpha_i^\vee \rangle = \delta_{0i}
\]

We denote by \( \{ \lambda_0^\vee, \lambda_1^\vee, \ldots, \lambda_l^\vee, \lambda_{l+1}^\vee \} \) the fundamental coweights of \( \hat{\mathfrak{h}} \). for any subdiagram \( D \subseteq D_\hat{\mathfrak{g}} \) is given a subalgebra

\[
\hat{\mathfrak{h}}_D = \begin{cases} 
\{ \langle \alpha_i^\vee | \alpha_i \in D \} & \text{if } D \neq D_\hat{\mathfrak{g}} \\
\hat{\mathfrak{h}} & \text{if } D = D_\hat{\mathfrak{g}} 
\end{cases}
\]

As in the previous section, we aim at proving the following

**Proposition.** Let \( \varphi = \{ \varphi_{(D, \alpha_i, \alpha_j)} \} \) be a 2-cocycle in the Dynkin complex of \( U\hat{\mathfrak{g}} \) such that

\[
\varphi_{(D, \alpha_i, \alpha_j)} \in \hat{\mathfrak{h}}_D
\]

for any \( \alpha_i \neq \alpha_j \in D \subseteq D_\hat{\mathfrak{g}} \). Then, there exists a Dynkin 1-cochain \( a = \{ a_{(D, \alpha_i)} \} \) such that

\[
a_{(D, \alpha_i)} \in \hat{\mathfrak{h}}_D \quad \text{and} \quad d_D a = \varphi
\]

The element \( a \) may be chosen such that \( a_{(\alpha_i, \alpha_i)} = 0 \) for all \( \alpha_i \) and is then unique (up to a coboundary) with this additional property.
Proof. We will follow the same outline of Prop. 4.3. Indeed, as before, the equation $\varphi = d_D a$, in components, reads

$$\varphi(D;\alpha_i,\alpha_j) = a(D;\alpha_j) - a(D^{\alpha_i \setminus \{\alpha_j\}};\alpha_j) - a(D;\alpha_i) + a(D^{\alpha_i \setminus \{\alpha_j\}};\alpha_i)$$

(4.11)

for any connected subdiagram $D \subseteq \hat{\mathfrak{g}}$ and $i < j$ such that $\alpha_i, \alpha_j \in D$. If $D$ is a proper subdiagram, then $\hat{\mathfrak{g}}_D$ is of finite type and the discussion is the same that before.

If $D = \hat{\mathfrak{g}}$ is the Dynkin diagram of $\hat{\mathfrak{g}}$, then

$$\begin{cases} 
\varphi(D;\alpha_i,\alpha_j) & \in \mathbb{C} \lambda^\vee_j \oplus \mathbb{C} \lambda^\vee_l \oplus \mathbb{C} \lambda^\vee_{l+1} \\
 a_i(D;\alpha_i) & \in \mathbb{C} \lambda^\vee_i \oplus \mathbb{C} \lambda^\vee_{l+1} 
\end{cases}$$

since $\lambda_{l+1}^\vee = c$. Componentwise (4.11) reads

$$\begin{align*}
\varphi^i(D;\alpha_i,\alpha_j) &= a^i(D^{\alpha_i \setminus \{\alpha_j\}};\alpha_i) - a^i(D^{\alpha_i \setminus \{\alpha_j\}};\alpha_j) - a^i(D;\alpha_i) \\
\varphi^j(D;\alpha_i,\alpha_j) &= a^j(D^{\alpha_i \setminus \{\alpha_j\}};\alpha_j) - a^i(D^{\alpha_i \setminus \{\alpha_j\}};\alpha_i) + a^j(D;\alpha_j) \\
\varphi^{l+1}(D;\alpha_i,\alpha_j) &= a^{l+1}(D;\alpha_j) - a^{l+1}(D^{\alpha_i \setminus \{\alpha_j\}};\alpha_j) - a^{l+1}(D;\alpha_i) + a^{l+1}(D^{\alpha_i \setminus \{\alpha_j\}};\alpha_i)
\end{align*}$$

(4.12) (4.13) (4.14)

It is useful to notice that, if $D = \hat{\mathfrak{g}}$, (4.10) becomes

$$\lambda^\vee_j(D^{\alpha_i \setminus \{\alpha_k\}}) \in \mathbb{C} \lambda^\vee_j, \mathbb{C} \lambda^\vee_k, \mathbb{C} \lambda^\vee_{l+1,\hat{\mathfrak{g}}_j}$$

(4.15)

then the consistency of (4.11) along $\lambda^\vee_j, \lambda^\vee_k$ is granted by the proof of (4.3). Moreover, it is easy to see that, if $\alpha_0 \not\in D^{\alpha_i \setminus \{\alpha_k\}}$, then

$$\lambda^\vee_j(D^{\alpha_i \setminus \{\alpha_k\}}) \in \mathbb{C} \lambda^\vee_j, \mathbb{C} \lambda^\vee_k$$

(4.16)
or, in other words,
\[ a_{(\mathcal{L}_{\alpha_j}^{\cup \{\alpha_0\}};\alpha_j)}^{t+1} = 0 \]  \hspace{1cm} (4.17)

Indeed, if
\[ \lambda_j^\vee_{\mathcal{L}_{\alpha_j}^{\cup \{\alpha_k\}}} = \sum_{i=0}^{t+1} y_{j,k}^{i} \alpha_i^\vee = \sum_{i=0}^{t+1} x_{j,k}^{i} \lambda_i^\vee \]
we have \( x_{j,k}^{t+1} = y_{j,k}^{0} \).

It remains to prove the consistency of (4.11) along \( \lambda_{l+1}^\vee \). Up to a coboundary, we may assume \( a_{(\mathcal{L};\alpha_0)}^{t+1} = 0 \). It is enough to observe that the element \( a' = \{ a_{(\mathcal{L}';\alpha_i)} \} \) defined by
\[
a'_{(\mathcal{L}';\alpha_i)} = \begin{cases} 
0 & \text{if } D' \neq D_0 \\
\delta_0 c & \text{if } D' = D_0
\end{cases}
\]
is indeed a coboundary. Assume then \( a_{(\mathcal{L};\alpha_0)}^{t+1} = 0 \). By (4.17), for each \( j \neq 0 \),
\[
\varphi_{(\mathcal{L};\alpha_0,\alpha_j)}^{t+1} = a_{(\mathcal{L};\alpha_j)}^{t+1} + a_{(\mathcal{L}_{\alpha_j}^{\cup \{\alpha_0\}};\alpha_j)}^{t+1}
\]  \hspace{1cm} (4.18)

Using this relation in (4.14), we get
\[
\varphi_{(\mathcal{L};\alpha_i,\alpha_j)}^{t+1} = \varphi_{(\mathcal{L};\alpha_0,\alpha_j)}^{t+1} - a_{(\mathcal{L}_{\alpha_j}^{\cup \{\alpha_0\}};\alpha_0)}^{t+1} - a_{(\mathcal{L}_{\alpha_j}^{\cup \{\alpha_0\}};\alpha_j)}^{t+1}
\]  \hspace{1cm} (4.19)

To see this, consider the \((\mathcal{L};\alpha_0,\alpha_i,\alpha_j)\) component of \( d_{\mathcal{L}} \varphi \) i.e., the sum
\[
\left( \varphi_{(\mathcal{L};\alpha_i,\alpha_j)} - \varphi_{(\mathcal{L}_{\alpha_j}^{\cup \{\alpha_0\}};\alpha_i,\alpha_j)} \right)
- \left( \varphi_{(\mathcal{L};\alpha_0,\alpha_j)} - \varphi_{(\mathcal{L}_{\alpha_j}^{\cup \{\alpha_0\}};\alpha_0,\alpha_j)} \right)
+ \left( \varphi_{(\mathcal{L};\alpha_0,\alpha_i)} - \varphi_{(\mathcal{L}_{\alpha_i}^{\cup \{\alpha_0\}};\alpha_0,\alpha_i)} \right)
\]
Since $d_D \varphi = 0$ and $\varphi^{l+1}_{(b_{\alpha_j, \alpha_i}, \alpha_j, \alpha_i)} = 0$ by (4.16), we get, by projecting on $\lambda^\vee_{l+1, D}$,

$$
\varphi^{l+1}_{(b_{\alpha_j, \alpha_i}, \alpha_j, \alpha_i)} - \varphi^{l+1}_{(c_{\alpha_j, \alpha_i}, \alpha_j, \alpha_i)} = a^{l+1}_{(c_{\alpha_j, \alpha_i}, \alpha_j, \alpha_i)} + a^{l+1}_{(c_{\alpha_i, \alpha_j}, \alpha_j, \alpha_i)} - a^{l+1}_{(b_{\alpha_j, \alpha_i}, \alpha_j, \alpha_i)} - a^{l+1}_{(c_{\alpha_j, \alpha_i}, \alpha_j, \alpha_i)} \quad (4.20)
$$

Recall that $a = \{a_{(D^\prime, \alpha_i)}\}$ is determined by induction on $D$ and the relation

$$
da^{l+1}_{(D, \alpha_i)} = \varphi^{l+1}_{(D, \alpha_i, \alpha_i)} - a^{l+1}_{(c_{\alpha_i, \alpha_j}, \alpha_j, \alpha_i)} \quad (4.21)
$$

gives the recursion formula for the component along $\lambda^\vee_{l+1}$. In particular, induction on $D$ gives us the following relation:

$$
\varphi^{l+1}_{(b_{\alpha_j, \alpha_i}, \alpha_j, \alpha_i)} = a^{l+1}_{(c_{\alpha_j, \alpha_i}, \alpha_j, \alpha_i)} - a^{l+1}_{(c_{\alpha_i, \alpha_j}, \alpha_i, \alpha_i)} \quad (4.22)
$$

Projecting along $\lambda^\vee_{l+1, D}$, by (4.16),

$$
\varphi^{l+1}_{(c_{\alpha_j, \alpha_i}, \alpha_j, \alpha_i)} = a^{l+1}_{(c_{\alpha_j, \alpha_i}, \alpha_j, \alpha_i)} - a^{l+1}_{(b_{\alpha_j, \alpha_i}, \alpha_j, \alpha_i)} + a^{l+1}_{(c_{\alpha_j, \alpha_i}, \alpha_i, \alpha_i)} \quad (4.23)
$$

We consider four separate cases.

4.5.1. $C_{\alpha_0, \alpha_i} = \emptyset$ and $C_{\alpha_0, \alpha_j} = \emptyset$. This case cannot arise since the first condition implies that any path in $D$ from $\alpha_0$ to $\alpha_i$ must pass through $\alpha_j$ before it reaches $\alpha_i$ while the second one implies that the portion of this path linking $\alpha_0$ to $\alpha_j$ must first pass through $\alpha_i$. 

4.5.2. \( C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} \neq \emptyset \) and \( C_{\alpha_0, \alpha_j}^{D \setminus \{ \alpha_i \}} = \emptyset \). In this case,

\[ \varphi_{(\alpha_0, \alpha_j), (\alpha_0, \alpha_j)} = 0 \]

and (4.20) reads

\[
a_{l+1}^{D \setminus \{ \alpha_i \}} (C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} : \alpha_0) - a_{l+1}^{D \setminus \{ \alpha_j \}} (C_{\alpha_0, \alpha_j}^{D \setminus \{ \alpha_i \}} : \alpha_0) + a_{l+1}^{D \setminus \{ \alpha_j \}} (C_{\alpha_0, \alpha_j}^{D \setminus \{ \alpha_i \}} \setminus \{ \alpha_i \} : \alpha_0)
\]

\[= a_{l+1}^{D \setminus \{ \alpha_i \}} (C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} : \alpha_0) + a_{l+1}^{D \setminus \{ \alpha_j \}} (C_{\alpha_0, \alpha_j}^{D \setminus \{ \alpha_i \}} : \alpha_0) - a_{l+1}^{D \setminus \{ \alpha_i \}} (C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} : \alpha_0) - a_{l+1}^{D \setminus \{ \alpha_j \}} (C_{\alpha_0, \alpha_j}^{D \setminus \{ \alpha_i \}} : \alpha_0) = 0 \quad \text{(4.24)}
\]

This equation holds since

- \( C_{\alpha_0}^{D \setminus \{ \alpha_j \}} = C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} \), so \( a_{l+1}^{D \setminus \{ \alpha_i \}} (C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} : \alpha_i) = a_{l+1}^{D \setminus \{ \alpha_i \}} (C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} : \alpha_i) \)
- \( C_{\alpha_0}^{D \setminus \{ \alpha_j \}} = C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} \), so \( a_{l+1}^{D \setminus \{ \alpha_i \}} (C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} : \alpha_0) = a_{l+1}^{D \setminus \{ \alpha_i \}} (C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} : \alpha_0) \)
- Since \( C_{\alpha_0, \alpha_j}^{D \setminus \{ \alpha_i \}} = \emptyset \),

\[ C_{\alpha_0}^{D \setminus \{ \alpha_i \}} = C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} = C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} \setminus \{ \alpha_i \} \implies a_{l+1}^{D \setminus \{ \alpha_i \}} (C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} : \alpha_0) = a_{l+1}^{D \setminus \{ \alpha_j \}} (C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_i \}} : \alpha_0) \]

So, we need to verify

\[ a_{l+1}^{D \setminus \{ \alpha_j \}} (C_{\alpha_0, \alpha_j}^{D \setminus \{ \alpha_i \}} : \alpha_j) = 0 \]

Since \( C_{\alpha_0, \alpha_j}^{D \setminus \{ \alpha_i \}} = \emptyset \), then \( \alpha_0 \notin C_{\alpha_j}^{D \setminus \{ \alpha_i \}} \) and, by (4.17), we can conclude.

4.5.3. \( C_{\alpha_0, \alpha_i}^{D \setminus \{ \alpha_j \}} = \emptyset \) and \( C_{\alpha_0, \alpha_j}^{D \setminus \{ \alpha_i \}} \neq \emptyset \). This case reduces to the previous one under the interchange \( \alpha_j \leftrightarrow \alpha_k \).
4.5.4. $\mathcal{C}_{\alpha_0,\alpha_i}^{D\{\alpha_j\}} = \emptyset$ and $\mathcal{C}_{\alpha_0,\alpha_j}^{D\{\alpha_i\}} \neq \emptyset$. This case is possible when $\alpha_0$ is a vertex in $D$ of degree at least 2 \textit{i.e.}, only for $\hat{g} = \hat{s}_{\alpha_i}$. Equation (4.20) reads:

$$a_{(\alpha_0,\alpha_i):\alpha_0}^{l+1} - a_{(\alpha_0,\alpha_j):\alpha_0}^{l+1} + a_{(\alpha_0,\alpha_i):\alpha_0}^{l+1} + a_{(\alpha_0,\alpha_j):\alpha_0}^{l+1} = a_{(\alpha_0,\alpha_i):\alpha_0}^{l+1} - a_{(\alpha_0,\alpha_i):\alpha_0}^{l+1} + a_{(\alpha_0,\alpha_j):\alpha_0}^{l+1} - a_{(\alpha_0,\alpha_j):\alpha_0}^{l+1}$$

This equation holds since

- $\mathcal{C}_{\alpha_0,\alpha_i}^{D\{\alpha_j\}} = \mathcal{C}_{\alpha_0,\alpha_i}^{D\{\alpha_i\}}$, so $a_{(\alpha_0,\alpha_i):\alpha_0}^{l+1} = a_{(\alpha_0,\alpha_i):\alpha_0}^{l+1}$
- $\mathcal{C}_{\alpha_0,\alpha_j}^{D\{\alpha_j\}} = \mathcal{C}_{\alpha_0,\alpha_j}^{D\{\alpha_j\}}$, so $a_{(\alpha_0,\alpha_j):\alpha_0}^{l+1} = a_{(\alpha_0,\alpha_j):\alpha_0}^{l+1}$
- $\mathcal{C}_{\alpha_0,\alpha_i}^{D\{\alpha_i\}} = \mathcal{C}_{\alpha_0,\alpha_i}^{D\{\alpha_i\}}$, so $a_{(\alpha_0,\alpha_i):\alpha_0}^{l+1} = a_{(\alpha_0,\alpha_i):\alpha_0}^{l+1}$
- $\mathcal{C}_{\alpha_0,\alpha_j}^{D\{\alpha_i\}} = \mathcal{C}_{\alpha_0,\alpha_j}^{D\{\alpha_i\}}$, so $a_{(\alpha_0,\alpha_j):\alpha_0}^{l+1} = a_{(\alpha_0,\alpha_j):\alpha_0}^{l+1}$
- we have

$$\mathcal{C}_{\alpha_0,\alpha_i}^{D\{\alpha_j\} \setminus \{\alpha_i\}} = \mathcal{C}_{\alpha_0,\alpha_j}^{D\{\alpha_i\} \setminus \{\alpha_j\}} = \mathcal{C}_{\alpha_0,\alpha_i}^{D\{\alpha_j\} \setminus \{\alpha_i\}} = \mathcal{C}_{\alpha_0,\alpha_j}^{D\{\alpha_i\} \setminus \{\alpha_j\}} = a_{(\alpha_0,\alpha_i):\alpha_0}^{l+1} = a_{(\alpha_0,\alpha_j):\alpha_0}^{l+1}$$

This concludes the proof of Proposition 4.5 and the proof of the main theorem for the affine case.

\[\square\]
In this Appendix, we provide a detailed proof of two results used in Chapter 3. Namely, in Section 1, we prove that the algebra $\hat{U}_g$, described by Drinfeld in [D5, 8.1] and by Etingof–Kazhdan in [EK6], is isomorphic to the algebra of endomorphisms of the forgetful functor with respect to the category $O$ of $g$. In Section 2, we prove that the isomorphism $\Psi^{EK}$ between $\hat{U}_g[[h]]$ and its quantum counterpart given in [EK6] corresponds, under the previous identification, with the natural isomorphism induced by the equivalence of categories $O[[h]] \simeq O_h$.

Finally, in Section 3, we prove, in analogy with the results of [EK1, Part II], that the relative fiber functor $\Gamma : D_\Phi(g) \to D_{\Phi_D}(g_D)$ gives rise naturally to a braided Hopf algebra object in $D_{\Phi_D}(g_D)$. This result is used in Chapter 3 Section 5 to study the behavior of the functor and its tensor structure with respect to abelian extensions of $g_D$.

1. The completion of $Ug$ in category $O$

Let $Q$ be the root lattice of $g$. Consider the modules

$$U_\beta := U g / I_\beta$$
where $I_\beta \subset U\mathfrak{g}$ is the left ideal generated by all elements of weight $\leq \beta$. Define

$$\hat{U}\mathfrak{g} := \lim_{\beta \in \mathbb{Q}} U_\beta$$

It is easy to prove that there is an injection

$$\lim_{\beta} \text{colim}_\gamma \text{Hom}_{U\mathfrak{g}}(U_\beta, U_\gamma) \longrightarrow \text{Hom}_{U\mathfrak{g}}(\hat{U}\mathfrak{g}, \hat{U}\mathfrak{g})$$

and we want to prove that, in this case, this is an isomorphism obtained by the identification

$$\text{colim}_\gamma \text{Hom}_{U\mathfrak{g}}(U_\gamma, U_\beta) \simeq \text{Hom}_{U\mathfrak{g}}(\hat{U}\mathfrak{g}, U_\beta)$$

1.1. The $U\mathfrak{g}$-module structure of $U_\beta$. The module $U_\beta$ is associated to the short exact sequence

$$0 \longrightarrow I_\beta \overset{i_\beta}{\longrightarrow} U_\mathfrak{g} \overset{p_\beta}{\longrightarrow} U_\beta \longrightarrow 0$$

In particular, $U_\beta$ is a cyclic module generated by an element $1_\beta$ such that $I_\beta.1_\beta = 0$.

**Lemma.** For each $v \in U_\beta$, there exists a weight $\beta' = \beta'(v) \in \mathbb{Q}$, $\beta' \leq \beta$, such that $I_{\beta'}.v = 0$

**Proof.** We can assume $v = a_1...a_n.1_\beta$ for some $a_i \in U\mathfrak{g}$. Proceed by induction. If $n = 0$, then $v = 1_\beta$ and $\beta' = \beta$. Now assume $v = a.w$, where $w = a_1...a_n.1_\beta$ and there exists $\beta' = \beta'(w)$ such that $I_{\beta'}.w = 0$. 
1. THE COMPLETION OF $U_{\mathfrak{g}}$ IN CATEGORY $\mathcal{O}$

(i) if $a \in (U_{\mathfrak{g}})_{\mu}, \mu \geq 0$, then for each element $x$ of weight $\lambda \leq \beta'$ (i.e. for each generator of $I_{\beta'}$) we have

$$x.v = [x,a].w + a.x.w = [x,a].w$$

and $[x,a] \in (U_{\mathfrak{g}})_{\lambda+\mu}$. Now, if we choose $\lambda \leq \beta' - \mu$, we have $[x,a] \in I_{\beta'}$, so $[x,a].w = 0$. We conclude that, for $\beta'' := \beta' - \mu \leq \beta'$, $I_{\beta''}.v = 0$.

(ii) if $a \in (U_{\mathfrak{g}})_{\mu}, \mu \leq 0$, for each generator $x$ of $I_{\beta'}$ of weight $\lambda$, the commutator $[x,a]$ has weight $\lambda + \mu \leq \beta'$, since $\lambda \leq \beta'$ and $\mu \leq 0$. Then $I_{\beta'}.v = 0$.

For general $a \in U_{\mathfrak{g}}$, use the weight decomposition

$$a \in U_{\mathfrak{g}} = \bigoplus_{\beta \in Q} (U_{\mathfrak{g}})_{\beta} \Rightarrow v = a.w = \sum_{i=1}^{m} a_{\mu_i}.w = \sum_{i=1}^{m} v_i$$

where $a_{\mu_i} \in (U_{\mathfrak{g}})_{\mu_i}$. Then there exist $\beta_1...\beta_m$ such that $I_{\beta_i}.v_i = 0$. Choosing $\beta'' := \min\{\beta_i, i = 1..m\}$ (i.e. $\beta'' = \sum_{\alpha \in \Delta} \min\{((\beta_i, \alpha)\} \alpha)$, we have

$$I_{\beta''}.v = \left( \bigcap_{i=1}^{m} I_{\beta_i} \right).v = 0$$

□

As a consequence we can say that the action of $U_{\mathfrak{g}}$ on $U_{\beta}$, for every element $v$, factors through $U_{\beta'(v)}$, $\beta'(v) \leq \beta$, i.e. for every element $u \in U_{\mathfrak{g}}$ $u.v = u_{\beta'}.v$, in the sense that the action of $u$ on $v$ is the same that the action of any representative of the class of $u$ in $U_{\beta'}$, i.e. $u_{\beta'} = p_{\beta'}(u)$.

1.2. The $U_{\mathfrak{g}}$-module structure of $\widehat{U_{\mathfrak{g}}}$. Define an action $a : U_{\mathfrak{g}} \otimes \widehat{U_{\mathfrak{g}}} \to \widehat{U_{\mathfrak{g}}}$ by $u.(x_{\beta}) = \{u.x_{\beta}\}$. Since the maps $\pi_{\beta'\beta}$ are maps of $U_{\mathfrak{g}}$-modules, the
map $Ug \otimes \widehat{Ug} \to \widehat{Ug}$ is well-defined, because for every $\beta' \leq \beta$,

$$u \cdot x_\beta = u \cdot \pi_{\beta' \beta}(x_{\beta'}) = \pi_{\beta' \beta}(u \cdot x_{\beta'})$$

and it’s obviously an action of $Ug$-modules.

1.3. The multiplication map in $\widehat{Ug}$. Using the action $Ug \otimes U_\beta \to U_\beta$

for every $\beta \in Q$, we can define a bilinear map:

$$\begin{array}{ccc}
\widehat{U} \otimes U_\beta & \to & U_\beta \\
\otimes & \downarrow & \\
x \otimes v & \mapsto & x_{\omega(v)} \cdot v
\end{array}$$

This map lifts up to a multiplication in $\widehat{Ug}$

$$\begin{array}{ccc}
\widehat{Ug} \otimes \widehat{Ug} & \overset{\tilde{m}}{\longrightarrow} & \widehat{Ug} \\
\otimes & & \\
x \otimes y & \longmapsto & \{x_{\omega(y_\beta)} \cdot y_\beta\}
\end{array}$$

where $\omega(y_\beta)$ is a weight such that $I_{\omega(y_\beta)} \cdot y_\beta = 0$. In order to show that this map is well-defined, we need to prove that, for every $\beta' \leq \beta$,

$$x_{\omega(y_\beta)} \cdot y_\beta = \pi_{\beta' \beta}(x_{\omega(y_{\beta'})} \cdot y_{\beta'})$$

Indeed, we have $y_{\beta'} \in U_{\beta'}$ and $I_{\omega(y_{\beta'})} \cdot y_{\beta'} = 0$. Then

$$0 = \pi_{\beta' \beta}(I_{\omega(y_{\beta'})} \cdot y_{\beta'}) = I_{\omega(y_{\beta'})} \cdot \pi_{\beta' \beta}(y_{\beta'}) = I_{\omega(y_{\beta'})} \cdot y_\beta$$
So, \( \omega(y_{\beta'}) \leq \omega(y_{\beta}) \). In particular,

\[
x_{\omega(y_{\beta})} = \pi_{\omega(y_{\beta})\omega(\pi_{\beta})}(x_{\omega(y_{\beta})})
\]

Consequently, their action on \( y_{\beta} \) is the same, i.e.

\[
x_{\omega(y_{\beta})} \cdot y_{\beta} = x_{\omega(y_{\beta'})} \cdot y_{\beta} = x_{\omega(y_{\beta'})} \cdot \pi_{\beta' \beta}(y_{\beta'}) = \pi_{\beta' \beta}(x_{\omega(y_{\beta'})} \cdot y_{\beta'})
\]

So, we have an algebra structure on \( \hat{U_g} \).

1.4. Compatibility with the \( U_g \)-action. We have a map \( i : U_g \rightarrow \hat{U_g} \), sending each element \( u \) in the collection of its image in \( U_{\beta} \) for all \( \beta \in Q \),

\[
i(u) := \{ u_{\beta} \}_{\beta \in Q}.
\]

Since

\[
\text{Ker}(i) = \bigcap_{\beta \in Q} I_{\beta} = 0
\]

\( i \) is an injection. It’s very easy to show that the multiplication map is compatible with the \( U_g \)-action in the following way:

(a) By definition, the action of \( U_g \) on \( \hat{U_g} \) is given by the formula

\[
a(u, x) = u \cdot x = \{ u_{\beta} \} \cdot x = \hat{m}(i(u), x)
\]

so that the following diagram commutes:

\[
\begin{array}{ccc}
U_g \otimes \hat{U_g} & \xrightarrow{a} & \hat{U_g} \\
\downarrow_{i \otimes 1} & & \downarrow \hat{m} \\
\hat{U_g} \otimes \hat{U_g} & & \\
\end{array}
\]
(b) Since the injection map is a map of $U\mathfrak{g}$-modules, we have a commutative diagram

\[
\begin{array}{c}
U\mathfrak{g} \otimes U\mathfrak{g} \\ 1 \otimes i \\ U\mathfrak{g} \otimes \widehat{U}\mathfrak{g} \end{array} \xrightarrow{m} \begin{array}{c} U\mathfrak{g} \\ i \\ \widehat{U}\mathfrak{g} \end{array}
\]

Then

\[
\begin{array}{c}
\widehat{U}\mathfrak{g} \otimes \widehat{U}\mathfrak{g} \\ \hat{m} \\ \widehat{U}\mathfrak{g}
\end{array} \xleftarrow{i \otimes 1} \begin{array}{c} U\mathfrak{g} \otimes \widehat{U}\mathfrak{g} \\ a \\ \widehat{U}\mathfrak{g}
\end{array} \xleftarrow{1 \otimes i} \begin{array}{c} U\mathfrak{g} \otimes U\mathfrak{g} \\ i \\ \widehat{U}\mathfrak{g}
\end{array} \xleftarrow{m} \begin{array}{c} U\mathfrak{g} \\ \widehat{U}\mathfrak{g} \\ \widehat{U}\mathfrak{g} \otimes \widehat{U}\mathfrak{g}
\end{array}
\]

is commutative and $U\mathfrak{g} \subset \widehat{U}\mathfrak{g}$ is a subalgebra.

1.5. Topology on $\widehat{U}\mathfrak{g}$. We consider the modules $U\beta$ with the discrete topology and we equip $\widehat{U}\mathfrak{g} = \lim U\beta$ with the topology induced by limit. In particular, a basis for the open sets is given by the sets $U(\beta_1...\beta_n, u_1...u_n)$ for $n \geq 0$, $\beta_i \in Q$, $u_i \in U\beta_i$, where

\[U(\beta_1...\beta_n, u_1...u_n) := \{x \in \widehat{U}\mathfrak{g} \mid x_{\beta_i} = u_i \ \forall i\}\]
Let $\beta := \min\{\beta_i, i = 1...n\}$. Then, by surjectivity of $p_\beta$, there exists an element $u \in U\mathfrak{g}$ such that $u_{\beta_i} = u_i$.

\[ U\mathfrak{g} \quad \xrightarrow{p_\beta} \quad U_\beta \]

\[ U_{\beta_1} \quad \cdots \quad U_{\beta_n} \]

Then we have proved that $U\mathfrak{g}$ is dense in $\hat{U}\mathfrak{g}$.

For any element $u \in \hat{U}\mathfrak{g}$, we can find $u_i \in U\mathfrak{g}$ such that

\[ u = \lim_{i \to \infty} u_i \]

In particular, if $\phi \in \text{End}_{U\mathfrak{g}}(\hat{U}\mathfrak{g})$, then, for any $u, x \in \hat{U}\mathfrak{g}$, we have

\[ \phi(u.x) = \phi((\lim_i u_i).x) = \lim_i \phi(u_i.x) = (\lim_i u_i).\phi(x) = u.\phi(x) \]

and

\[ \text{End}_{U\mathfrak{g}}(\hat{U}\mathfrak{g}) \simeq \text{End}_{\hat{U}\mathfrak{g}}(\hat{U}\mathfrak{g}) \simeq \hat{U}\mathfrak{g} \]

**1.6.** $\text{Hom}_{U\mathfrak{g}}(\hat{U}\mathfrak{g}, U_\beta)$. The map

\[ \hat{U}\mathfrak{g} \otimes U_\beta \longrightarrow U_\beta \]

is in fact an action of the algebra $\hat{U}\mathfrak{g}$ on the module $U_\beta$. Using the same density argument as before, we have

\[ \text{Hom}_{U\mathfrak{g}}(\hat{U}\mathfrak{g}, U_\beta) \simeq \text{Hom}_{\hat{U}\mathfrak{g}}(\hat{U}\mathfrak{g}, U_\beta) \]
Consequently, for any map $\phi \in \text{Hom}_{U_\beta}(\widehat{U}_g, U_\beta)$, we have

$$\phi(u) = u.\phi(1)$$

for any $u \in \widehat{U}_g$. In particular, $\phi$ is completely determined by $\phi(1) \in U_\beta$. By the previous discussion, we know that there exists a weight $\omega = \omega(\phi(1))$ such that $I_\omega.\phi(1) = 0$. Then the action of $u$ on $\phi(1)$ factors through $U_\omega$, i.e.

$$\phi(u) = u.\phi(1) = u_\omega.\phi(1) =: \tilde{\phi}(u_\omega)$$

and

$$\widehat{U}_g \xrightarrow{\phi} U_\beta$$

$$\pi_\omega \downarrow \quad \phi$$

$$U_\omega \xrightarrow{\tilde{\phi}}$$

where $\tilde{\phi} \in \text{Hom}_{U_\beta}(U_\omega, U_\beta)$. It’s clear that we can associate to $\phi$ a uniquely determined class $[\tilde{\phi}] \in \text{colim}_\gamma \text{Hom}_{U_\beta}(U_\gamma, U_\beta)$ and consequently

$$\text{colim}_\gamma \text{Hom}_{U_\beta}(U_\gamma, U_\beta) \simeq \text{Hom}_{U_\beta}(\widehat{U}_g, U_\beta)$$

and

$$\lim_{\beta} \text{colim}_\gamma \text{Hom}_{U_\beta}(U_\gamma, U_\beta) \simeq \lim_{\beta} \text{Hom}_{U_\beta}(\widehat{U}_g, U_\beta) \simeq \text{Hom}_{U_\beta}(\widehat{U}_g, \widehat{U}_g)$$

Notice that the same argument can be used to show that, for any module $V \in \mathcal{O}$, we have

$$\text{Hom}_{U_\beta}(\widehat{U}_g, V) \simeq \text{Hom}_{U_\beta}(\widehat{U}_g, V)$$
and consequently for any map \( f \in \text{Hom}_{U \hat{g}}(\hat{U}_g, V) \) there exists \( \beta \in Q \) and \( f_\beta \in \text{Hom}_{U \hat{g}}(U_\beta, V) \) such that

\[
\begin{array}{ccc}
\hat{U}_g & \xrightarrow{f} & V \\
\downarrow \pi_\beta & & \downarrow f_\beta \\
U_\beta & \xleftarrow{\pi_\beta} & \end{array}
\]

is commutative.

### 2. The isomorphism between \( \hat{U}_g[[h]] \) and \( \hat{U}_h g \)

The equivalence of categories between \( O[[h]] \simeq O_h \), established by the Etingof–Kazhdan functor, carries an isomorphism

\[
\text{Hom}_{U \hat{g}}(U_\gamma, U_\beta)[[h]] \simeq \text{Hom}_{U \hat{h} g}(U_h^\gamma, U_h^\beta)
\]

and then

\[
\begin{array}{ccc}
\lim_\beta \text{colim}_\gamma \text{Hom}_{U \hat{g}}(U_\gamma, U_\beta)[[h]] & \xrightarrow{\simeq} & \lim_\beta \text{colim}_\gamma \text{Hom}_{U \hat{h} g}(U_h^\gamma, U_h^\beta) \\
\downarrow \iota & & \downarrow \iota \\
\text{End}_{U \hat{g}}(\hat{U}_g)[[h]] & \xrightarrow{\simeq} & \text{End}_{U \hat{h} g}(\hat{U}_h g) \\
\downarrow \iota & & \downarrow \iota \\
\hat{U}_g[[h]] & \xrightarrow{\simeq} & \hat{U}_h g
\end{array}
\]

On the other hand, we can consider the forgetful functors

\[
F : O[[h]] \to A \quad F_h : O_h \to A
\]
and the associated completions \( \text{End}(F), \text{End}(F_h) \). Using the equivalence between \( |\text{Co}[[h]]| \) and \( \mathcal{O}_h \) we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}[[h]] & \xrightarrow{\phi} & \mathcal{O}_h \\
\downarrow{F} & & \downarrow{F_h} \\
\mathcal{A} & & 
\end{array}
\]

that induces a natural isomorphism

\[
\text{End}(F) \simeq \text{End}(F_h)
\]

We want to show that this construction is equivalent to the previous one, in particular we want to show that there exists an algebra isomorphism

\[
\widehat{U}_g[[h]] \xrightarrow{\sim} \text{End}(F)
\]

2.1. The map \( \phi \). First of all, let us construct a map \( \phi : \widehat{U}_g[[h]] \rightarrow \text{End}(F) \).

**Lemma.** For any \( v \in V \) there exists \( \beta \in Q \) such that \( I_\beta.v = 0 \)

**Proof.** Assume first \( v \in V_{\lambda} \). Set \( \mu := \min \{ \lambda_i | \lambda \in D(\lambda_i) \} \). For any \( x \in (U_{\mathfrak{g}})_\lambda \) where \( \lambda' \leq \mu(v) := \mu - \lambda \), we have \( x.v = 0 \). Then \( I_{\mu(v)}.v = 0 \).

In general, \( v = \sum v_\lambda \), where \( v_\lambda \in V_\lambda \). Set \( \mu(v) := \min_{\lambda \in P(V)} \{ \mu(v_\lambda) \} \). Then \( I_{\mu(v)}.v = 0 \) \( \square \)

For each \( V \in \mathcal{O} \) we have a \( \widehat{U}_g \)-action given by, for any \( u \in \widehat{U}_g \),

\[
u.v = u_{\mu(v)}.v
\]
where by $u_{\mu(v)}$ we mean any representative of the class $u_{\mu(v)} \in U_{\mu(v)}$ (to be precise, any representative of the classes $u_\beta \in U_\beta$ with $\beta \leq \mu(v)$).

In particular, for any $V \in \mathcal{O}$ we have a map

$$\phi_V : \hat{U}_g \rightarrow \text{End}(V)$$

Define $\phi(u) := \{\phi_V(u)\}$.

**2.2. $\phi$ is well-defined.** We need to show that for any $V, W \in \mathcal{O}$ and $f \in \text{Hom}_{U_g}(V, W)$ we have

$$
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\phi_V(u) \downarrow & & \downarrow \phi_W(u) \\
V & \xrightarrow{f} & W
\end{array}
$$

i.e. for any $v \in V$

$$f(u_{\mu(v)}.v) = u_{\mu(f(v))}.f(v)$$

Since $f \in \text{Hom}_{U_g}(V, W)$, then

$$0 = f(I_{\mu(v)}.v) = I_{\mu(v)} \cdot f(v)$$

so $\mu(v) \leq \mu(f(v))$ and the action of $u_{\mu(v)}$ is equivalent to that one of $u_{\mu(f(v))}$.

So,

$$u_{\mu(f(v))}.f(v) = u_{\mu(v)}.f(v) = f(u_{\mu(v)}.v)$$

Then we have a well-defined map

$$\phi : \hat{U}_g[[h]] \rightarrow \text{End}(F)$$
2.3. \( \phi \) is injective. Recall that for any \( \beta \in Q \) \( U_\beta \in \mathcal{O} \). In particular, for any \( v \in U_\beta \)

\[
\phi_\beta(u)(v) = u_\beta.v
\]

Then, if \( \phi(u) = 0 \), we have \( u_\beta = 0 \) for all \( \beta \in Q \), and \( u = 0 \).

2.4. \( \phi \) is surjective. For any \( f \in \text{End}(F) \), \( \beta' \leq \beta \), we have a commutative diagram

\[
\begin{array}{ccc}
U_{\beta'} & \xrightarrow{\pi_{\beta'\beta}} & U_{\beta} \\
\downarrow{f_{\beta'}} & & \downarrow{f_{\beta}} \\
U_{\beta'} & \xrightarrow{\pi_{\beta'\beta}} & U_{\beta}
\end{array}
\]

By universal property of limits in the category Vec\( (k) \), there exists a map \( \tilde{f} \in \text{Hom}_{k}(\widehat{U}_g, \widehat{U}_g) \) such that the diagram

\[
\begin{array}{ccc}
\widehat{U}_g & \xrightarrow{\pi_{\beta}} & U_{\beta} \\
\downarrow{\tilde{f}} & \xrightarrow{\pi_{\beta'}} & \downarrow{f_{\beta}} \\
\widehat{U}_g & \xrightarrow{\pi_{\beta}} & U_{\beta}
\end{array}
\]

is commutative.

2.5. Commutativity in \( \mathcal{O} \). Recall that \( \text{Hom}_{U_g}(\widehat{U}_g, V) \simeq \text{Hom}_{\widehat{U}_g}(\widehat{U}_g, V) \).

Then for any \( g \in \text{Hom}_{U_g}(\widehat{U}_g, V) \), there exists \( \mu := \mu(g, V) \in Q \) and \( g_\mu \in \)
Hom\textsubscript{\(U_\mathfrak{g}\)}(\(U_\mu, V\)) such that

\[
\begin{array}{ccc}
\widehat{U}_\mathfrak{g} & \xrightarrow{g} & V \\
\downarrow{\pi_\mu} & & \downarrow{g_\mu} \\
U_\mu & & 
\end{array}
\]

Consequently, the following diagram

\[
\begin{array}{ccc}
\widehat{U}_\mathfrak{g} & \xrightarrow{g} & V \\
\downarrow{\pi_\mu} & & \downarrow{g_\mu} \\
\tilde{f} & \longrightarrow & \widetilde{U}_\mu \\
\downarrow{f_\mu} & & \downarrow{f_V} \\
\widehat{U}_\mathfrak{g} & \xrightarrow{g} & V \\
\downarrow{\pi_\mu} & & \downarrow{g_\mu} \\
U_\mu & & 
\end{array}
\]

is commutative and then, for any \(V \in \mathcal{O}\) and \(g \in \text{Hom}_{U_\mathfrak{g}}(\widehat{U}_\mathfrak{g}, V)\), the diagram

\[
\begin{array}{ccc}
\widehat{U}_\mathfrak{g} & \xrightarrow{g} & V \\
\downarrow{\tilde{f}} & & \downarrow{f_V} \\
\widehat{U}_\mathfrak{g} & \xrightarrow{g} & V \\
\downarrow{\pi_\mu} & & \downarrow{g_\mu} \\
U_\mu & & 
\end{array}
\]

is commutative. This fact, in particular, is true for \(U_\beta, \beta \in \mathcal{Q}\).

### 2.6. Commutativity for \textbf{End}_{U_\mathfrak{g}}(\widehat{U}_\mathfrak{g})

Suppose now \(g \in \text{Hom}_{U_\mathfrak{g}}(\widehat{U}, \widehat{U}) \simeq \lim_\beta \text{Hom}_{U_\mathfrak{g}}(\widehat{U}, U_\beta)\). It means that for any \(\beta\), there exists \(g_\beta \in \text{Hom}_{U_\mathfrak{g}}(\widehat{U}_\mathfrak{g}, U_\beta)\)
such that the diagram
\[
\begin{array}{ccc}
\hat{U}_g & \xrightarrow{g} & \hat{U}_g \\
\downarrow{g_\beta} & & \downarrow{\pi_\beta} \\
U_\beta & & 
\end{array}
\]

By density, there exists $\beta' \leq \beta$ and $g_{\beta'\beta} \in \text{Hom}_{Ug}(U_{\beta'}, U_\beta)$ such that the diagram
\[
\begin{array}{ccc}
\hat{U}_g & \xrightarrow{g} & \hat{U}_g \\
\downarrow{\pi_{\beta'}} & & \downarrow{\pi_\beta} \\
U_{\beta'} & \xrightarrow{g_{\beta'\beta}} & U_\beta \\
\end{array}
\]
is commutative. Then we have a box

\[
\begin{array}{ccccccccc}
\hat{U}_g & \xrightarrow{g} & \hat{U}_g & \xrightarrow{\pi_{\beta'}} & \hat{U}_g \\
\downarrow{\pi_\beta} & & \downarrow{g_{\beta'\beta}} & & \downarrow{\pi_\beta} \\
U_{\beta'} & \xrightarrow{f_{\beta'}} & \hat{U}_g & \xrightarrow{f} & U_\beta \\
\downarrow{\pi_{\beta'}} & & \downarrow{g} & & \downarrow{f_\beta} \\
\hat{U}_g & \xrightarrow{f_{\beta'}} & \hat{U}_g & \xrightarrow{\pi_\beta} & \hat{U}_g \\
\downarrow{g_{\beta'\beta}} & & \downarrow{\pi_\beta} & & \downarrow{g_{\beta'\beta}} \\
U_{\beta'} & \xrightarrow{g_{\beta'\beta}} & U_\beta & & U_\beta \\
\end{array}
\]

where every face is commutative except one behind, i.e. we have

\[
\pi_\beta \circ g \circ \hat{f} = \pi_\beta \circ \hat{f} \circ g
\]

It means that the maps $\hat{f} \circ g$ and $g \circ \hat{f}$ define the same endomorphism of $\hat{U}_g$ as vector space. Then, for any $g \in \text{Hom}_{Ug}(\hat{U}_g, \hat{U}_g)$ we have a commutative
2. THE ISOMORPHISM BETWEEN $\hat{\mathcal{U}}_g[[h]]$ AND $\hat{\mathcal{U}}_{h\mathcal{G}}$

Diagram

\[
\begin{array}{ccc}
\hat{\mathcal{U}}_g & \overset{g}{\longrightarrow} & \hat{\mathcal{U}}_g \\
\downarrow \hat{f} & & \downarrow \hat{f} \\
\hat{\mathcal{U}}_g & \overset{g}{\longrightarrow} & \hat{\mathcal{U}}_g
\end{array}
\]

2.7. The inverse map $\phi^{-1}$. Notice that, for each element $y \in \hat{\mathcal{U}}_g$, the map

\[m^y : \hat{\mathcal{U}}_g \rightarrow \hat{\mathcal{U}}_g | m^y(x) = x \cdot y\]

is in $\text{Hom}_{U_g}(\hat{\mathcal{U}}_g, \hat{\mathcal{U}}_g)$. Indeed, for any $u \in U_g$,

\[m^y(u \cdot x) = m^y(i(u) \cdot x) = i(u) \cdot x \cdot y = u \cdot m^y(x)\]

and, for any $u \in \hat{\mathcal{U}}_g$,

\[\hat{f}(u) = \hat{f}(m^u(1)) = m^u(\hat{f}(1)) = u \cdot \hat{f}(1)\]

Then we have defined a map

\[\phi^{-1} : \text{End}(F) \rightarrow \hat{\mathcal{U}}_g | \phi^{-1}(f) = \hat{f}(1)\]

and we have also proved the surjectivity of $\phi$. Indeed, for any $v \in V$, take the map $g \in \text{Hom}_{U_g}(\hat{\mathcal{U}}_g, V)$ defined by $g(1) = v$. We have a commutative diagram

\[
\begin{array}{ccc}
\hat{\mathcal{U}}_g & \overset{g}{\longrightarrow} & V \\
\downarrow \hat{f} & & \downarrow f_V \\
\hat{\mathcal{U}}_g & \overset{g}{\longrightarrow} & V
\end{array}
\]

i.e.

\[f_V(v) = f_V(g(1)) = g(\hat{f}(1)) = \hat{f}(1).g(1) = \hat{f}(1).v\]
It is equivalent to say that

\[ f = \phi(\tilde{f}(1)) \]

We can conclude that the diagram

\[
\begin{array}{ccc}
\hat{U}g[[h]] & \xrightarrow{z} & \hat{U}h\mathfrak{g} \\
\downarrow{t} & & \downarrow{t} \\
\text{End}(F) & \xrightarrow{z} & \text{End}(F_h)
\end{array}
\]

is commutative.
3. A braided Hopf algebra in $\mathcal{D}_{\Phi_D}(g_D)$

3.1. This section is entirely dedicated to the proof of the following

**Theorem.**

(i) The object $\Gamma(L_-)$ has a structure of braided Hopf algebra in $\mathcal{D}_{\Phi_D}(g_D)$.

(ii) On every object $\Gamma(V)$ there are well-defined actions and coalitions of $\Gamma(L_-)$, so that the functor $\Gamma$ naturally factors as

$$
\begin{array}{c}
\mathcal{D}_{\Phi}(g) \\
\downarrow_{\Gamma} \\
\mathcal{D}_{\Phi_D, Ug_D}(\Gamma(L_-)) \\
\downarrow_{\Gamma} \\
\mathcal{D}_{\Phi_D}(g_D) \\
\end{array}
\Rightarrow
\begin{array}{c}
\mathcal{D}_{\Phi_D}(\Gamma(L_-))^\flat Ug_{D,-} \\
\downarrow_{f_0} \\
\mathcal{D}_{\Phi_D}(g_D) \\
\end{array}
$$

where $\Gamma(L_-)^\flat Ug_{D,-}$ denotes the Radford algebra attached to $\Gamma(L_-)$.

Theorem 3.1 is used in Chapter 3 Section 5 to study the behavior of the relative twists with respect to abelian extensions of $g_D$.

3.2. Let $(\Gamma, J_{\Gamma}) : \mathcal{D}_{\Phi}(g) \to \mathcal{D}_{\Phi_D}(g_D)$ be the monoidal functor introduced in Chapter 3. Following [EK1], for every $V \in \mathcal{D}_{\Phi}(g)$ we can define a map

$$
\mu_V : \Gamma(L_-) \otimes \Gamma(V) \to \Gamma(V) \\
\mu_V(x \otimes v) = (i'_+ \otimes \text{id})\Phi^{-1}(\text{id} \otimes v)x
$$

**Proposition.** For every $V \in \mathcal{D}_{\Phi}(g)$, the map $\mu_V$ is a morphism of $g_D$-modules. Moreover, it satisfies

$$
\mu_V(\text{id} \otimes \mu_V)\Phi_D = \mu_V(\mu_{L_-} \otimes \text{id})
$$
Therefore, \((\Gamma(L_-), \mu)\) is an associative algebra object in \(\mathcal{D}_{\Phi_D}(g_D)\) and acts on the image of the functor \(\Gamma\).

**Proof.** Let \(\Delta(a) = a_1 \otimes a_2\) for \(a \in Ug_D\). Then

\[
\mu_V(a_1.x \otimes a_2.v) = (i^\vee_+ \otimes \text{id})\Phi^{-1}(\text{id} \otimes S(a_2)^\rho \otimes \text{id})(\text{id} \otimes v)(S(a_1)^\rho \otimes \text{id})x
\]

\[
= (i^\vee_+ \otimes \text{id})\Phi^{-1}(S(a_1)^\rho \otimes S(a_2)^\rho \otimes \text{id})(\text{id} \otimes v)x
\]

\[
= (i^\vee_+ \otimes \text{id})(S(a_1)^\rho \otimes S(a_2)^\rho \otimes \text{id})\Phi^{-1}(\text{id} \otimes v)x
\]

\[
= (S(a)^\rho \otimes \text{id})(i^\vee_+ \otimes \text{id})\Phi^{-1}(\text{id} \otimes v)x
\]

\[
= a.\mu_V(x \otimes v)
\]

The second statement is also easily proved. Let \(x, y \in \Gamma(L_-), v \in \Gamma(V)\) and \(\Phi_D = \sum P_k \otimes Q_k \otimes R_k\). Then

\[
\mu_V(\text{id} \otimes \mu_V)\Phi_D(x \otimes y \otimes v) =
\]

\[
= (i^\vee_+ \otimes \text{id})\Phi^{-1}(\text{id} \otimes i^\vee_+ \otimes \text{id})(\text{id} \otimes \Phi^{-1})(\text{id} \otimes \text{id} \otimes R_k.v)(\text{id} \otimes Q_k.y)(P_k.x)
\]

\[
= (i^\vee_+ \otimes \text{id})(\text{id} \otimes i^\vee_+ \otimes \text{id})\Phi^{-1}_{1,23,4} \Phi^{-1}_{2,3,4} \Phi^{-1}_{1,2,3,4}(\text{id} \otimes \text{id} \otimes v)(\text{id} \otimes y)x
\]

\[
= (i^\vee_+ \otimes \text{id})(i^\vee_+ \otimes \text{id} \otimes \text{id})\Phi_{12,3,4} \Phi_{D,123} \Phi^{-1}_{D,123} \Phi^{-1}_{1,2,3,4}(\text{id} \otimes \text{id} \otimes v)(\text{id} \otimes y)x
\]

\[
= (i^\vee_+ \otimes \text{id})(i^\vee_+ \otimes \text{id} \otimes \text{id})\Phi^{-1}_{12,3,4} \Phi^{-1}_{1,2,3,4}(\text{id} \otimes \text{id} \otimes v)(\text{id} \otimes y)x
\]

\[
= (i^\vee_+ \otimes \text{id})\Phi^{-1}(\text{id} \otimes \text{id} \otimes v)(i^\vee_+ \otimes \text{id})\Phi^{-1}(\text{id} \otimes y)x
\]

\[
= \mu_V(\mu_{L_-} \otimes \text{id})(x \otimes y \otimes v)
\]

where the equalities are obtained by applications of the pentagon axiom and the associativity of \(i^\vee_+\). \(\square\)
3.3. The tensor structure on $\Gamma$ and the canonical coalgebra structure on $L_-$ define the map

$$\delta : \Gamma(L_-) \to \Gamma(L_-) \otimes \Gamma(L_-) \quad \delta = J^{-1}_\Gamma \circ \Gamma(i_-)$$

**Proposition.** The map $\delta$ is a morphism of $g_D$–modules and it satisfies

$$\Phi_D(\delta \otimes \text{id})\delta = (\text{id} \otimes \delta)\delta$$

Therefore, $(\Gamma(L_-), \delta)$ is a coassociative algebra object in $D_{\Phi_D}(g_D)$.

**Proof.** Let $a \in g_D$. The maps $J_\Gamma$ and $\Gamma(i_-)$ are both morphisms of $g_D$–modules, therefore we have

$$\delta(a.x) = \Delta(a).\delta(x)$$

Moreover,

$$\Phi_D(\delta \otimes \text{id}) = \Phi_D J^{-1}_{\Gamma,1,2}(\Gamma(i_-) \otimes \text{id}) J^{-1}_\Gamma \Gamma(i_-)$$

$$= \Phi_D J^{-1}_{\Gamma,1,2} J^{-1}_{\Gamma,12,3} \Gamma(i_- \otimes \text{id}) \Gamma(i_-)$$

$$= J^{-1}_{\Gamma,12,3} J^{-1}_{\Gamma,123} \Gamma(\Phi) \Gamma(i_- \otimes \text{id}) \Gamma(i_-)$$

$$= J^{-1}_{\Gamma,12,3} (\text{id} \otimes \Gamma(i_-)) J^{-1}_\Gamma \Gamma(i_-)$$

$$= (\text{id} \otimes \delta)\delta$$

$\square$

3.4. We now want to study the action of $\Gamma(L_-)$ on the tensor product.
PROPOSITION. The object $(\Gamma(L_-), \mu, \delta)$ is a bialgebra object in $D_{\Phi_D}(g_D)$.

The action on the tensor product is compatible with the coproduct and the tensor structure $J_\Gamma$, i.e.,

$$\mu_{V \otimes W}(\text{id} \otimes J_\Gamma) = J_\Gamma(\mu_V \otimes \mu_W)\beta_{D,23}(\delta \otimes \text{id} \otimes \text{id})$$

as morphisms from $\Gamma(L_-) \otimes \Gamma(V) \otimes \Gamma(W)$ to $\Gamma(V \otimes W)$

PROOF. We ignored the action of the associator $\Phi_D$ since we proved that it is compatible with $\mu_V$ and $J_\Gamma$. The equality above corresponds to the identity in $\Gamma(V \otimes W)$

$$(i_+^V \otimes \text{id} \otimes \text{id})(\text{id} \otimes i_+^W \otimes \text{id} \otimes \text{id})\beta_{34}(\text{id} \otimes v \otimes w)(\text{id} \otimes i_-)x = (i_+^V \otimes \text{id} \otimes \text{id})\beta_{23}(i_+^V \otimes \text{id} \otimes i_+^V \otimes \text{id})(\text{id} \otimes b.v \otimes \text{id} \otimes w)(x_1 \otimes a_i.x_2)i_-
$$

where $v \in \Gamma(V), w \in \Gamma(W), x \in \Gamma(L_-), \delta(x) = x_1 \otimes x_2$ and $\exp(-\hbar\Omega_D/2) = a_i \otimes b_i$. It is more convenient, in this case, to follow a pictorial proof. We have to show that the following diagrams are equivalent:
The relation satisfied by \( \delta \) is represented by

\[
\delta(x) = \delta(x)
\]

Since the action of \( g_D \) on the objects \( \Gamma(V) \) is given by right action on \( N_+^* \), we represent the braiding \( \beta_D^{-1} \) as

\[
a_1 x_2 = x_2 \quad \text{and} \quad b_1 v = v
\]

We notice that we are allowed to move the black and the white bullet along the lines, since the commute with the left action of \( g \). Clearly, the RHS corresponds to

\[
= =
\]
We now use the fact that the map $i_+^\vee$ satisfies

$$i_+^\vee \circ \beta \circ (\beta_D^{-1})^p = i_+^\vee$$

\[= \]

\[= \]

Finally we get

\[= \]

3.5. The tensor functor $(\Gamma, J_\Gamma)$ induces a natural braiding on the subcategory generated in $\mathcal{D}_{\Phi}(\mathfrak{g}_D)$ by the objects $\Gamma(V)$, for any $V \in \mathcal{D}_{\Phi}(\mathfrak{g})$. The braiding is given by the usual formula

$$\beta_{J_\Gamma} := J_{\Gamma,21}^{-1} \circ \Gamma(\beta) \circ J_\Gamma$$
where $\beta = \sigma \cdot R$ is the braiding in $D_{\Phi}(\mathfrak{g})$. Clearly, $\beta_{J_\Gamma}$ is a morphism of $\mathfrak{g}_D$–module and so is the relative $R$–matrix $\mathcal{R}_D := \sigma \beta_{J_\Gamma}$.

Let now $u \in \Gamma(L_-)$ be the unit element. For every $V \in D_{\Phi}(\mathfrak{g})$ we define the trivial comodule structure on $\Gamma(V)$ by

$$\eta_V : \Gamma(V) \rightarrow \Gamma(L_-) \otimes \Gamma(V) \quad \eta_V(v) = u \otimes v$$

The maps $\eta$ are clearly morphisms of $\mathfrak{g}_D$–modules and they satisfy

$$\Phi_D(\eta \otimes \text{id})\eta_V = (\text{id} \otimes \eta_V)\eta_V$$

where $\eta := \eta_{L_-}$ and $\eta(u) = u \otimes u = \delta(u)$. Consider

$$\mathcal{R}_{D,V} \in \text{End}_{U_{\Phi}}(\Gamma(L_-), \Gamma(V))$$

and define the following map:

$$\delta_V : \Gamma(V) \rightarrow \Gamma(L_-) \otimes \Gamma(V) \quad \delta_V = \mathcal{R}_{D,V} \circ \eta_V$$

**Proposition.** The maps $\delta_V$ define on any $\Gamma(V), V \in D_{\Phi}(\Phi)$, a structure of comodule over $\Gamma(L_-)$.

**Proof.** Clearly we have that the relative $R$–matrix satisfies

$$\mathcal{R}_{D,1,23} = \Phi_{D,231}^{-1}\mathcal{R}_{D,13}\Phi_{D,213}\mathcal{R}_{D,12}\Phi_{D,123}^{-1}$$

$$\mathcal{R}_{D,12,3} = \Phi_{D,312}\mathcal{R}_{D,13}\Phi_{D,132}^{-1}\mathcal{R}_{D,23}\Phi_{D,123}$$
and therefore

\[ \mathcal{R}_{D,12} \Phi_{D,312} \mathcal{R}_{D,13} \Phi_{D,132}^{-1} \mathcal{R}_{D,23} \Phi_{D,123} = \Phi_{D,321} \mathcal{R}_{D,23} \Phi_{D,231}^{-1} \mathcal{R}_{D,13} \Phi_{D,213} \mathcal{R}_{D,12} \]

Moreover, we have the obvious relations

\[ \mathcal{R}_{D,1,23} (\text{id} \otimes \eta_{V}) = (\text{id} \otimes \eta_{V}) \mathcal{R}_{D} \quad \mathcal{R}_{D,1,2,3} (\eta_{V} \otimes \text{id}) = (\eta_{V} \otimes \text{id}) \mathcal{R}_{D} \]

for \( V \in \mathcal{D}_{\Phi}(g) \). Then

\[
(id \otimes \delta_{V}) \delta_{V} = \mathcal{R}_{D,23} (id \otimes \eta_{V}) \mathcal{R}_{D} \eta_{V}
\]

\[
= \mathcal{R}_{D,23} \mathcal{R}_{D,1,23} (id \otimes \eta_{V}) \eta_{V}
\]

\[
= \Phi_{D} \mathcal{R}_{D,12} \mathcal{R}_{D,12,3} \Phi_{D}^{-1} (id \otimes \eta_{V}) \eta_{V}
\]

\[
= \Phi_{D} \mathcal{R}_{D,12} (\eta \otimes \text{id}) \mathcal{R}_{D} \eta_{V}
\]

\[
= \Phi_{D} (\delta \otimes \text{id}) \delta
\]

This concludes the proof of the Theorem 3.1.
Bibliography


[TL5] ____., Quasi-Coxeter quasitriangular quasibialgebras and the Casimir connection. (forthcoming)