A FUNCTORIAL APPROACH TO LINKAGE AND THE ASYMPTOTIC
STABILIZATION OF THE TENSOR PRODUCT

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ABSTRACT OF DISSERTATION

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Abstract

This thesis consists of three projects.

The first project deals with generalizing the definition of zeroth derived functors to work for any abelian category. The classical definitions of zeroth derived functors require existence of injectives or projectives. In this paper, we give definitions of the zeroth derived functors that do not require the existence of injectives or projectives. The new definitions result in generalized definitions of projective and injective stabilization of functors. The category of coherent functors is shown to admit a zeroth right derived functor. An interesting result of this fact is a counterpart to the Yoneda lemma for coherent functors. Moreover, zeroth derived functors are seen more appropriately as approximations of functors by left exact or right exact functors. Under certain reasonable conditions, the category of coherent functors is shown to have enough injectives. This result was first shown by Ron Gentle. We give an alternate proof of this fact.

The second project deals with extending the definition of horizontally linked modules over semiperfect Noetherian rings to the category of finitely presented functors over arbitrary Noetherian rings. Linkage of modules can be defined using the syzygy and transpose operation. Auslander established the merit of studying modules by studying the functors from the module category into the category of abelian groups. It turns out that the module theoretic notion of linkage can be extended to a functorial notion of linkage and the satellite endofunctors are crucial to this extension.

The third project deals with finding alternate ways of recovering Vogel homology. There are three known ways of generalizing Tate cohomology to a cohomology theory that works over arbitrary rings, that given by Vogel, that given by Mislin, and that given by Buchweitz. Vogel also provided a homological counterpart to his generalization of Tate cohomology. Yoshino attempted to recover this homology theory using an approach similar to Mislin’s approach to recovering Tate cohomology; however, he only produced Vogel homology in positive degrees. This is fixed by returning to Mislin’s construction and observing that it can be dualized. Completely missing from the picture was an approach similar to Buchweitz’s approach to generalizing Tate cohomology. The asymptotic stabilization of the tensor product is introduced to fill this gap.
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Introduction

The functorial approach to representation theory was developed by Maurice Auslander. The main idea is that one may study an abelian category $\mathcal{C}$ by studying the category $(\mathcal{C}, \text{Ab})$ consisting of additive covariant functors from the $\mathcal{C}$ into the category of abelian groups together with the natural transformations between such functors. We now recall some basic properties. The category $(\mathcal{C}, \text{Ab})$ is abelian, the sequence

$$0 \to F \to G \to H \to 0$$

being exact if and only if for every $X \in \mathcal{C}$ the sequence

$$0 \to F(X) \to G(X) \to H(X) \to 0$$

is exact in $\text{Ab}$. One of the fundamental properties used in studying $(\mathcal{C}, \text{Ab})$ is the well known result of Yoneda Lemma

**Lemma 1 (Yoneda).** For any $X \in \mathcal{C}$ and $F : \mathcal{C} \to \text{Ab}$, there is an isomorphism of abelian groups

$$\text{Nat}((X, -), F) \cong F(X)$$

which is natural in $X$ and $F$.

As a result, one sees quite easily that representable functors are projective objects in $(\mathcal{C}, \text{Ab})$. Moreover, any natural transformation $(Y, -) \to (X, -)$ comes from a map $X \to Y$. The Yoneda embedding $\mathcal{Y} : \mathcal{C} \to (\mathcal{C}, \text{Ab})$ given by $\mathcal{Y}(X) = (X, -)$ is a contravariant left exact embedding. By applying $\mathcal{Y}$ to the exact sequence $X \to Y \to Z \to 0$, we get an exact sequence of functors $0 \to (Z, -) \to (Y, -) \to (X, -)$. Hence, the kernel of a natural transformation between representable functors is again representable. This naturally leads to the study of the category of coherent functors $\text{fp}(\mathcal{C}, \text{Ab})$, which is the full subcategory of $(\mathcal{C}, \text{Ab})$ consisting of all $F$ such that there exists $X, Y \in \mathcal{C}$ and exact sequence

$$(Y, -) \to (X, -) \to F \to 0.$$

In [1], Auslander establishes that the category of coherent functors $\text{fp}(\mathcal{C}, \text{Ab})$ has some very nice properties, namely that it is abelian, has enough projectives, and the inclusion $\text{fp}(\mathcal{C}, \text{Ab}) \to (\mathcal{C}, \text{Ab})$ is both exact and
reflects exact sequences which means a sequence

\[ 0 \to F \to G \to H \to 0 \]

is exact in \( \text{fp}(C, \text{Ab}) \) if and only if it is exact when viewed as a sequence in \( (C, \text{Ab}) \).

Every exact sequence

\[ X \to Y \to Z \to 0 \]

in \( C \) gives rise to a coherent functor \( F \) determined by the exact sequence

\[ 0 \to (Z, \_ ) \to (Y, \_ ) \to (X, \_ ) \to F \to 0. \]

Each coherent functor \( F \) may arise from different sequences of this form. Of particular interest are those \( F \) which arise from short exact sequences, that is those \( F \) such that there exists short exact sequence \( 0 \to X \to Y \to Z \to 0 \) such that \( F \) has presentation

\[ 0 \to (Z, \_ ) \to (Y, \_ ) \to (X, \_ ) \to F \to 0. \]

In [1], Auslander constructs a contravariant exact functor \( w: \text{fp}(C, \text{Ab}) \to C \) that measures for each coherent functor \( F \) the deviation from arising through a short exact sequence. We refer to the functor \( w \) as the \textbf{defect}. Moreover, he constructs for each coherent functor \( F \) an exact sequence of coherent functors called the \textbf{defect sequence}

\[ 0 \to F_0 \to F \to (w(F), \_ ) \to F_1 \to 0 \]

with the property that \( F \) agrees with \((w(F), \_ )\) on injectives of \( C \). If \( C \) has enough injectives, then one sees easily that \((w(F), \_ )\) is the zeroth right derived functor of \( F \).

The \textbf{injective stabilization sequence} studied by Auslander and Bridger in [3] is constructed as follows: Let \( C \) be an abelian category with enough injectives and \( D \) be an abelian category. For every functor \( F: C \to D \) recall the definition of the zeroth derived functor \( R^0F \). Given \( X \in C \), take exact sequence \( 0 \to X \to I^0 \to I^1 \) with \( I^0, I^1 \) injective. Then \( R^0F(X) \) is given by the exact sequence

\[ 0 \to R^0F(X) \to F(I^0) \to F(I^1). \]

This completely determines \( R^0F \) as a functor and the assignment is functorial in \( F \) and does not depend on the choices of injectives. One sees easily that there is a natural morphism \( F \to R^0F \). Hence, for any functor \( F: C \to D \), there exists an exact sequence of functors called the \textbf{injective stabilization sequence} of \( F \):

\[ 0 \to F \to R^0F \]
where \( \overline{F} \) is called the **injective stabilization of** \( F \). It is easily seen that \( R^0F \) is left exact and agrees with \( F \) on injectives.

The similarities between the defect sequence and the injective stabilization sequence together with the fact that the defect sequence exists regardless of whether or not \( C \) has enough injectives motivates extending the definition of the zeroth right derived functor to work in the general abelian setting. The main point of Chapter 1 is to give generalized definitions of zeroth derived functors for general abelian categories, even those that do not have enough injectives and those that do not have enough projectives. The generalized definition allows one to view zeroth derived functors more appropriately as approximations. This is how it works: For any abelian categories \( C, D \) let \( S \) denote any full subcategory of \( (C, D) \). Denote by \( \text{Lex}(S) \) the full subcategory of \( S \) consisting of the left exact functors. We say that \( S \) admits a **zeroth right derived functor** if the inclusion \( s: \text{Lex}(S) \rightarrow S \) admits a left adjoint \( r^0: S \rightarrow \text{Lex}(S) \) satisfying the following two properties:

1. The unit of adjunction \( u: 1_S \rightarrow sr^0 \) is an isomorphism on the injectives of \( C \). More precisely, if \( F \in S \) and \( I \in C \) is injective, then the morphism \( (u_F)_I \) is an isomorphism.

2. The composition \( r^0s \) is isomorphic to the identity. That is

\[
r^0s \cong 1_S.
\]

If \( S \) admits a zeroth derived functor, then we define \( R^0 := sr^0 \) to be the zeroth right derived functor on \( S \). One similarly defines the zeroth left derived functor. Suppose the inclusion \( s: \text{Rex}(S) \rightarrow S \) admits a right adjoint \( l_0: S \rightarrow \text{Rex}(S) \) satisfying

1. The counit of adjunction \( c: sl_0 \rightarrow 1_S \) is an isomorphism on the projectives of \( C \). More precisely, if \( F \in S \) and \( P \in C \) is projective, then the morphism \( (c_F)_P \) is an isomorphism.

2. The composition \( l_0s \) is isomorphic to the identity. That is

\[
l_0s \cong 1_S.
\]

If \( S \) admits a zeroth left derived functor, define \( L_0 := sl_0 \) to be the zeroth derived functor on \( S \).

Suppose that \( G \) is left exact and \( \alpha: F \rightarrow G \). Then, since \( u_F \) is the unit of adjunction, the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{u_F} & R^0F \\
\downarrow & & \downarrow \\
G & & 
\end{array}
\]
The zeroth right derived functor of $F$ is thus more appropriately described as a covariant approximation of $F$ by a left exact functor and the zeroth left derived functor is more appropriately described as a contravariant approximation of $F$ by a right exact functor. In the case that $\mathcal{C}$ has enough injectives, the entire functor category $(\mathcal{C}, \mathcal{D})$ admits a zeroth right derived functor as defined above and its construction is as defined using the exact sequence $0 \to X \to I^0 \to I^1$. If $\mathcal{C}$ has enough projectives, then $(\mathcal{C}, \mathcal{D})$ admits a zeroth left derived functor as defined above and its construction is also the usual one. It is also shown that the more general definitions of the zeroth derived functors allow one to generalize the definitions of the injective and projective stabilization of functors given by Auslander and Bridger in [3].

After the category of coherent functors is introduced, we recall Auslander’s construction of both the defect and the defect sequence

\[ 0 \to F_0 \to F \to (w(F), -) \to F_1 \to 0 \]

for any coherent functor $F$. The similarity between the defect sequence and the injective stabilization sequence is then explained using the more general definition of a zeroth right derived functor. We may summarize our results in the following theorem and lemma:

**Theorem 2.** The category of coherent functors admits a zeroth right derived functor. For any coherent functor $F \in \text{fp}(\mathcal{C}, \text{Ab})$, $R^0 F \cong (w(F), -)$ and the map $F \to (w(F), -)$ in the defect sequence is precisely the unit of adjunction evaluated at $F$. Moreover, the injective stabilization sequence of a coherent functor embeds into the defect sequence.

**Lemma 3 (The CoYoneda Lemma).** For any $F \in \text{fp}(\mathcal{C}, \text{Ab})$ and for any $X \in \mathcal{C}$,

\[ \text{Nat}(F, (X, -)) \cong (X, w(F)) \]

The category $\text{fp}(\mathcal{C}, \text{Ab})$ is abelian and has enough projectives for any abelian category $\mathcal{C}$. This is of interest in its own right. In the final section of chapter 1, we investigate under what conditions the category $\text{fp}(\mathcal{C}, \text{Ab})$ has enough injectives. It is shown that if $\mathcal{C}$ has enough projectives, then $\text{fp}(\mathcal{C}, \text{Ab})$ has enough
injectives. Moreover, for every $F \in \text{fp}(\mathcal{C}, \text{Ab})$, there exists injective resolution

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow 0$$

This was first established by Ron Gentle in [7], though the proof he gives is quite different.

As an immediate application, assume that $R$ is a Noetherian ring. In this case the category $\text{mod}(R)$ consisting of the finitely generated $R$-modules is abelian and has enough projectives. Therefore $\text{fp}(\text{mod}(R), \text{Ab})$ has enough injectives. Auslander established that the injectives of $\text{fp}(\text{mod}(R), \text{Ab})$ are precisely the right exact functors. Applying a theorem of Watts, which states that all right exact functors from $\text{mod}(R) \text{Ab}$ are tensor functors, it follows that the injectives of $\text{fp}(\text{mod}(R), \text{Ab})$ are precisely the tensor functors $X \otimes -$ where $X \in \text{mod}(R^{\text{op}})$. We end chapter 1 with the following result first established by Auslander:

**Proposition 4.** Let $R$ be a Noetherian ring. Every functor $F \in \text{fp}(\text{mod}(R), \text{Ab})$ has an injective resolution

$$0 \rightarrow F \rightarrow X \otimes - \rightarrow Y \otimes - \rightarrow Z \otimes - \rightarrow 0$$

We begin chapter 2 by recalling the notion of a connected sequence of functors. For any abelian categories $\mathcal{C}, \mathcal{D}$, a **connected sequence of functors** (cohomological functor) from $\mathcal{C}$ to $\mathcal{D}$ is a sequence $T = (T^n)_{n \in \mathbb{Z}}$ of additive covariant functors $T^n: \mathcal{C} \rightarrow \mathcal{D}$ satisfying the following property: For any commutative diagram with exact rows:

$$
\begin{array}{c}
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \\
0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \\
\end{array}
$$

there is a commutative diagram with underexact (exact) rows:

$$
\begin{array}{c}
\begin{array}{c}
\cdots \rightarrow T^{n-1}(C) \rightarrow T^n(A) \rightarrow T^n(B) \rightarrow T^n(C) \rightarrow T^{n+1}(A) \rightarrow \cdots \\
\cdots \rightarrow T^{n-1}(C) \rightarrow T^n(X) \rightarrow T^n(Y) \rightarrow T^n(Z) \rightarrow T^{n+1}(X) \rightarrow \cdots \\
\end{array}
\end{array}
$$

For an abelian category $\mathcal{C}$, the cohomology functors $H^n$ from the category for complexes in $\mathcal{C}$ to abelian groups form a cohomological functor.

The category of connected triples $\Delta(\mathcal{C}, \mathcal{D})$ is also introduced. For abelian categories $\mathcal{C}, \mathcal{D}$, a triple $(F, \delta, G)$ consists of two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ and the connecting homomorphism $\delta$ which assigns to every short exact sequence $\mathcal{E}$ in $\mathcal{C}$

$$
\begin{array}{c}
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\end{array}
$$
a morphism $\delta_E : F(C) \to F(A)$, this assignment being natural in $\mathcal{E}$. The triple $(F, \delta, G)$ is called connected if it satisfies the following property: For any short exact sequence $\mathcal{E}$ in $\mathcal{C}$:

\[
\begin{array}{c}
0 \\
\downarrow \ne \downarrow \ne \downarrow \ne \downarrow \\
A & \to & B & \to & C & \to & 0
\end{array}
\]

the sequence

\[
\begin{array}{c}
0 \\
\downarrow \ne \downarrow \ne \downarrow \ne \downarrow \\
F(B) & \to & F(C) & \to & G(A) & \to & G(B)
\end{array}
\]

is a complex.

In section 2, the definitions of the satellite endofunctors as given by Fisher-Palmquist and Newell are recalled. Classically the satellites are defined by using syzygy and cosyzygy sequences. For abelian categories $\mathcal{C}, \mathcal{D}$, there are two obvious functors $\pi_1, \pi_2 : \Delta(\mathcal{C}, \mathcal{D}) \to (\mathcal{C}, \mathcal{D})$ given by $\pi_1(F, \delta, G) = F$ and $\pi_2(F, \delta, G) = G$.

The category $(\mathcal{C}, \mathcal{D})$ admits a **right satellite** if the functor $\pi_1$ admits a left adjoint $\Delta S^1 : (\mathcal{C}, \mathcal{D}) \to \Delta(\mathcal{C}, \mathcal{D})$. The category $(\mathcal{C}, \mathcal{D})$ admits a **left satellite** if the functor $\pi_2$ admits a right adjoint $\Delta S^1 : (\mathcal{C}, \mathcal{D}) \to \Delta(\mathcal{C}, \mathcal{D})$.

**Definition 1.** Suppose that $(\mathcal{C}, \mathcal{D})$ admits both a left and right satellite. The **left satellite endofunctor** $S_1 : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}, \mathcal{D})$ is defined as the composition $S_1 := \pi_1 \circ \Delta S^1$. The **right satellite endofunctor** $S^1 : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}, \mathcal{D})$ is defined as the composition $S^1 := \pi_2 \circ \Delta S^1$. By iteration $S_n := (S_1)^n$ and $S^n := (S^1)^n$.

We also use the convention that $S_0 = S^0 = 1_{(\mathcal{C}, \mathcal{D})}$.

**Proposition 5** (Fisher-Palmquist, Newell [6]). Suppose that $(\mathcal{C}, \mathcal{D})$ admits left and right satellites. For every $n \geq 0$, $(S^n, S_n)$ form an adjoint pair of endofunctors.

The definition of the satellites given by Fisher-Palmquist and Newell naturally leads to a simple proof of the well known result of Hilton-Rees:

\[
\text{Nat}(\text{Ext}^1(Y, \_), \text{Ext}^1(X, \_)) \cong \text{Hom}(X, Y)
\]

In section 3, we turn to linkage of algebraic varieties. Horizontal linkage of algebraic varieties is classically an ideal theoretic notion. In [10], Martsinkovsky and Strooker extend the notion of linkage to finitely generated modules over semiperfect Noetherian rings. In order to state the definition, we recall the transpose of a finitely generated module. Given $M$, take a presentation of $M$ be finitely generated projectives

\[
P_1 \to P_0 \to M \to 0
\]
The transpose of $M$, $\text{Tr}(M)$, is given by the exact sequence

$$0 \to M^* \to P_0^* \to P_1^* \to \text{Tr}(M) \to 0$$

The module $M$ is said to be horizontally linked if $M \cong \Omega \text{Tr} \Omega \text{Tr} M$. The main question we would like to answer in chapter 2 is whether or not one can extend the notion of linkage of modules by to linkage of coherent functors. There are two important obstacles:

1. The operations $\Omega$ and $\text{Tr}$ are not functorial operations.
2. It is not clear what “extend” should mean in this case.

To get a clearer sense of what we are asking, we begin by recalling that while the operations $\Omega$ and $\text{Tr}$ are not functorial on $\text{mod}(R)$, they are functorial on the stable category $\text{mod}(R)$, that is the category whose objects are finitely generated modules and whose morphisms are the morphisms of $\text{mod}(R)$ modulo projectives. Moreover, for a Noetherian ring $R$, $(\text{Tr} \Omega \text{Tr}, \Omega)$ form an adjoint pair of endofunctors on $\text{mod}(R)$.

We have the following diagram of functors:

\[
\begin{array}{c}
\text{mod}(R) \\
\downarrow \text{Tr} \\
\text{mod}(R^{\text{op}}) \\
\downarrow \Omega \\
\text{fp}\left(\text{mod}(R), \text{Ab}\right) \\
\end{array}
\]

\[
\begin{array}{c}
\text{mod}(R^{\text{op}}) \\
\downarrow \text{Tr} \\
\text{mod}(R) \\
\downarrow \Omega \\
\text{fp}\left(\text{mod}(R^{\text{op}}), \text{Ab}\right) \\
\end{array}
\]

The idea of extending linkage will require a diagram of functors:

\[
\begin{array}{c}
\text{mod}(R) \\
\downarrow v \\
\text{fp}\left(\text{mod}(R), \text{Ab}\right) \\
\end{array}
\]

\[
\begin{array}{c}
\text{mod}(R^{\text{op}}) \\
\downarrow v \\
\text{fp}\left(\text{mod}(R^{\text{op}}), \text{Ab}\right) \\
\end{array}
\]

such that

1. The vertical functors $v$ are contravariant embeddings.
(2) The following commutativity relations are satisfied:

\[(a) \quad hv = v\text{Tr} \]

\[(b) \quad l^k v = v \Omega^k \text{Tr} \]

(3) There is a counit of adjunction

\[ lhlh \longrightarrow 1 \]

The reason for suspecting that the functor \( v \) is a contravariant embedding comes from two observations. The first is how the right satellite behaves on the extension functors, namely that for \( M \in \text{mod}(R) \),

\[ S^1 \text{Ext}^1(M, \_ \_) \cong \text{Ext}^1(\Omega M, \_ \_) \].

This suggests that \( S^1 \) might be some type of analog of \( \Omega \). The second is the fact that the functor \( HR: \text{mod}(R) \rightarrow \text{fp}(\text{mod}(R), \text{Ab}) \)

given by \( HR(M) := \text{Ext}^1(M, \_ \_) \) is an embedding as shown by Hilton-Rees. Because linkage of modules can be described using the operations \( \Omega \) and \( \text{Tr} \), one would expect analogs of these operations exist on \( \text{fp}(\text{mod}(R), \text{Ab}) \).

In section 4, we calculate the left derived functors of the functor \( Y_R: \text{fp}(\text{Mod}(R), \text{Ab}) \rightarrow \text{fp}(\text{Mod}(R^{op}), \text{Ab}) \)

given by \( Y_R(F) := (F(R), \_ \_) \) using injective resolutions of \( \text{fp}(\text{Mod}(R), \text{Ab}) \). The only nonzero left derived functor is the zeroth left derived functor \( L^0 Y_R: \text{fp}(\text{Mod}(R), \text{Ab}) \rightarrow \text{fp}(\text{Mod}(R), \text{Ab}) \). It is shown that for a Noetherian ring \( R \), \( L^0 Y_R \) restricts to a functor \( D: \text{fp}(\text{mod}(R), \text{Ab}) \rightarrow \text{fp}(\text{mod}(R^{op}), \text{Ab}) \).

**Theorem 6.** The functor \( D: \text{fp}(\text{mod}(R), \text{Ab})) \rightarrow \text{fp}(\text{mod}(R^{op}), \text{Ab}) \) is a duality satisfying the following properties:

1. \( D(X, \_ \_) \cong \_ \_ \otimes X \)
2. \( D(\text{Ext}^1(X, \_ \_)) \cong \text{Tor}_1(\_ \_, X) \)
3. \( D(\_ \_ \otimes X) \cong (X, \_ \_) \)
4. \( D(\text{Tor}_1(\_ \_, X)) \cong \text{Ext}^1(X, \_ \_) \)
5. \( DS^1 \cong S_1 D \)
6. \( DS_1 \cong S^1 D \)

**Theorem 7.** The duality \( D: \text{fp}(\text{mod}(R), \text{Ab})) \rightarrow \text{fp}(\text{mod}(R^{op}), \text{Ab}) \) satisfies the following properties:

1. \( DF(A) \cong \text{Nat}(F, A \otimes \_ \_) \)
2. Given \( F \in \text{fp}(\text{mod}(R), \text{Ab}) \), take presentation \( (Y, \_ \_) \rightarrow (X, \_ \_) \rightarrow F \rightarrow 0 \). Then \( DF \) is completely determined by the exact sequence \( 0 \rightarrow DF \rightarrow \_ \_ \otimes X \rightarrow \_ \_ \otimes Y \).

As a result,

1. \( D \) is the duality first defined by Auslander in [2].
The duality $D$ also governs the defect $w(F)$ of every coherent functor. For any $F \in \text{fp} (\text{mod}(R), \text{Ab})$, $w(F) \cong DF(R)$. This provides use with the following property already established by Auslander: For all $n \geq 0$, $w(\text{Tor}_1(-, X)) \cong \text{Ext}^n(X, R)$.

In section 5, we begin by observing that for a Noetherian ring, the adjoint pair of endofunctors $(S^1, S_1): \text{fp} (\text{Mod}(R), \text{Ab}) \to \text{fp} (\text{Mod}(R^{\text{op}}), \text{Ab})$ restricts to an adjoint pair of endofunctors $(S^1, S_1): \text{fp} (\text{mod}(R), \text{Ab}) \to \text{fp} (\text{mod}(R^{\text{op}}), \text{Ab})$. Moreover, we show that $S^1 D(\text{Ext}^1(X, -)) \cong \text{Ext}^1(\text{Tr}(X), -)$, meaning that $S^1 D$ is the analog of $\text{Tr}$ we have been searching for. Now that we have $S^1$ as the analog of $\Omega$ and $S^1 D$ as the analog of $\text{Tr}$. This allows us to define a linkage for coherent functors. A functor $F \in \text{fp} (\text{mod}(R), \text{Ab})$ is linked, if the counit of adjunction

\[
S^2 S_2 F \xrightarrow{\epsilon} F
\]

is an isomorphism. We have a diagram:

\[
\begin{array}{c}
\text{mod}(R) \\
\text{fp}(\text{mod}(R), \text{Ab})
\end{array}
\xrightarrow{HR}
\begin{array}{c}
\text{mod}(R^{\text{op}}) \\
\text{fp}(\text{mod}(R^{\text{op}}), \text{Ab})
\end{array}
\]

satisfying the following:

1. The vertical arrows are contravariant embeddings.
2. The following commutativity relations are satisfied:
   a) $S^1 DHR = HR \text{Tr}$
   b) $S^k S^1 D \text{Tr} = HR \Omega^k \text{Tr}$
3. There is a counit of adjunction $S^2 S_2 \to 1$.

Marsinkovsky and Strooker extended linkage from algebraic varieties to finitely generated modules over semiperfect Noetherian rings. Our new definition of linkage for coherent functors $F: \text{mod}(R) \to \text{Ab}$ works
over arbitrary Noetherian rings. Martsinkovsky made the observation that one could extend linkage from $\text{mod}(R)$ to the stable category $\text{mod}(R)$.

**Theorem 8.** A module $M$ is linked in $\text{mod}(R)$ if and only if $\text{Ext}^1(M, \_)$ is linked as a coherent functor.

A finitely generated module $M$ over a Noetherian ring is said to have $G$-dimension zero if the following three conditions hold:

1. $\text{Ext}^k(M, R) = 0$ for all $k \geq 1$.
2. $\text{Ext}^k(\text{Tr}M, R) = 0$ for all $k \geq 1$.
3. $M$ is reflexive.

One of the first things that Martsinkovsky and Strooker establish is that all modules of $G$-dimension zero are linked as finitely generated modules over semiperfect Noetherian rings. We have a similar result for functors.

A functor $F : \text{mod}(R) \rightarrow \text{Ab}$ is said to have $G$-dimension zero if all of its satellites are both projectively and injectively stable.

**Theorem 9.** All half exact coherent functors of $G$-dimension zero are linked.

We end the section by using the new definition of linkage of functors to prove a criterion for linkage of modules given by Martsinkovsky and Strooker.

**Theorem 10.** A module $M \in \text{mod}(R)$ is horizontally linked if and only if $\text{Ext}^1(\text{Tr}M, \_)$ is projectively stable.

The final chapter deals with different constructions of certain cohomological functors given by Tate, Vogel, Buchweitz, and Mislin. In the 1950’s, John Tate invented a cohomology theory for finite groups using complete resolutions. This theory was later generalized to include a class of infinite groups by Farrell. There are three known ways to generalize Tate cohomology to a cohomology theory for modules over arbitrary rings. These three approaches are covered in the first three sections. Vogel generalized Tate Cohomology by taking cohomology of certain quotient complexes of graded maps between projective resolutions. Buchweitz generalized Tate cohomology by inverting the syzygy endofunctor on the stable module category. This same generalization was independently discovered by Benson and Carlson and appears in [4]. In [11], Mislin generalized Tate cohomology by using the satellites. All three approaches work over arbitrary rings and produce isomorphic cohomological functors which are commonly referred to as Vogel cohomology.

Vogel’s approach also included a homological counterpart referred to as Vogel homology. Given that Vogel cohomology can be obtained via three different constructions, it seems plausible that Vogel homology may share the same property. This is the topic of section 4. In [13], Yoshino investigated a way of producing
Vogel homology by using the satellites similar to Mislin’s approach to Vogel cohomology. Yoshino was able to recover Vogel homology in positive degrees under certain restrictions placed on the ring and modules involved. This was a good first step; however, the negative degrees of Vogel homology were missing and Yoshino’s approach did not produce a connected sequence of functors. In section 5, we explain how to use Mislin’s approach to produce Vogel homology in all degrees under the same restrictions imposed by Yoshino.

Completely missing from the picture was an approach similar to Buchweitz’s generalization. In section 6, we introduce an object called the asymptotic stabilization of the tensor product. The asymptotic stabilization of the tensor product is obtained by constructing via simple homological methods, a directed system of morphisms between injective stabilizations of tensor products and taking the limit of these systems. This process results in a connected sequence of functors isomorphic to the right satellite completion of the tensor product.

**Theorem 11** (Martsinkovsky, Russell). There is an epimorphism from Vogel homology to the asymptotic stabilization of the tensor product. Under the same restrictions imposed by Yoshino, this epimorphism is an isomorphism.

**Proposition 12** (Martsinkovsky, Russell). The directed systems appearing in Yoshino’s and Mislin’s work are isomorphic to the directed systems appearing in the satellite grid.

Moreover, by passing to the colimit and limit of each directed system, we obtain two connected sequences of functors which we call the left satellite completion and right satellite completion respectively. The satellite completions unify the two approaches of Mislin and Yoshino.

**Theorem 13** (Martsinkovsky, Russell). The left satellite completion of the hom functor is isomorphic to Vogel cohomology. Under the same restrictions imposed by Yoshino, the right satellite completion of the tensor functor is isomorphic to Vogel homology.
CHAPTER 1

Zeroth Derived Functors as Adjoint

1. The Functor Category and The Yoneda Lemma

Let \( \mathcal{C} \) and \( \mathcal{D} \) denote two abelian categories. Recall the definition of the functor category \((\mathcal{C}, \mathcal{D})\). The objects of \((\mathcal{C}, \mathcal{D})\) are additive covariant functors \( F: \mathcal{C} \to \mathcal{D} \) and the morphisms between two such objects are the natural transformations between them. The category \((\mathcal{C}, \mathcal{D})\) is abelian. The addition of natural transformations \( \alpha, \beta \) is defined componentwise

\[
(\alpha + \beta)_X := \alpha_X + \beta_X
\]

A sequence of functors

\[
F \to G \to H
\]

is exact in \((\mathcal{C}, \mathcal{D})\) if and only if for every \( X \in \mathcal{C} \) the sequence

\[
F(X) \to G(X) \to H(X)
\]

is exact in \( \mathcal{D} \). For every \( X \in \mathcal{C} \) the evaluation functor \( \text{ev}_X : (\mathcal{C}, \mathcal{D}) \to \mathcal{D} \) is exact.

The category of abelian groups will be denoted \( \text{Ab} \). A functor \( F \in (\mathcal{C}, \text{Ab}) \) is called **representable** if it is isomorphic to \( \text{Hom}_\mathcal{C}(X, -) \) for some \( X \in \mathcal{C} \). The full subcategory of \((\mathcal{C}, \text{Ab})\) consisting of representable functors will be denoted \( \text{Rep}(\mathcal{C}, \text{Ab}) \). We will abbreviate the representable functors by \((X, -)\). The most important property of representable functors is the following well known lemma of Yoneda:

**Lemma 14 (Yoneda).** For any covariant functor \( F: \mathcal{C} \to \text{Ab} \) and any \( X \in \mathcal{C} \)

\[
\text{Nat}((X, -), F) \cong F(X)
\]

the isomorphism given by \( \alpha \mapsto \alpha_X(1_X) \). This isomorphism is natural in \( F \) and \( X \).

An immediate consequence of the Yoneda lemma is that for any \( X, Y \in \mathcal{C} \), \( \text{Nat}((X, -), (Y, -)) \cong (Y, X) \). Hence all natural transformations between representable functors come from maps between objects in \( \mathcal{C} \). The **Yoneda embedding** is the contravariant functor \( \mathcal{Y}: \mathcal{C} \to (\mathcal{C}, \text{Ab}) \) defined by \( \mathcal{Y}(X) = (X, -) \) and for any \( f: X \to Y \), \( \mathcal{Y}(f) = (f, -) \). Notice that as a result of Yoneda’s lemma, the functor \( \mathcal{Y} \) is fully faithful. One
easily shows that this embedding is also left exact. Given \( \alpha \in (Y, \_)_\to (X, \_)_\), there exists \( f: X \to Y \) such that \((f, \_)_\) = \( \alpha \). The exact sequence

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{Z} \\
0 & \end{array}
\]

embeds into the exact sequence

\[
\begin{array}{ccc}
0 & \to & (Z, \_) \\
\downarrow{g, \_} & & \downarrow{(f, \_)} \\
(Y, \_) & \to & (X, \_) \\
\end{array}
\]

Hence, the kernel of any natural transformation between representable functors is itself representable.

It is worth noting that the Yoneda embedding \( \hat{Y}: C \to \text{Rep}(C, \text{Ab}) \) which is also dense. Therefore the category \( C \) and the category \( \text{Rep}(C, \text{Ab}) \) are contravariantly equivalent via \( \hat{Y} \). As a result there exists a functor \( \hat{w}: \text{Rep}(C, \text{Ab}) \to C \) such that \( \hat{w} \hat{Y} \cong 1 \) and \( \hat{Y} \hat{w} \cong 1 \). Clearly \( \hat{w} \) is defined as follows: If \( F \cong (X, \_) \), then \( \hat{w}(F) = X \). Given \((f, \_): (Y, \_) \to (X, \_)\), \( \hat{w}(f, \_) = f \). Since \( C \) is abelian, so is \( \text{Rep}(C, \text{Ab}) \); however, the inclusion \( \text{Rep}(C, \text{Ab}) \to (C, \text{Ab}) \) is not in general exact. This means that the abelian structures of \( \text{Rep}(C, \text{Ab}) \) and \( (C, \text{Ab}) \) are in general different.

Suppose that

\[
\begin{array}{ccc}
0 & \to & F \\
\to & G & \to H \\
\| & \| & \| \\
0 & \to & F(X) \\
\to & G(X) & \to H(X) \\
\end{array}
\]

is a short exact sequence in \((C, \text{Ab})\). Consider the commutative diagram whose rows are complexes

\[
\begin{array}{ccc}
0 & \xrightarrow{\cong} & \text{Nat}(\{(X, \_), F\}) \xrightarrow{\cong} \text{Nat}(\{(X, \_), G\}) \xrightarrow{\cong} \text{Nat}(\{(X, \_), H\}) \to 0 \\
0 & \to & F(X) \to G(X) \to H(X) \to 0 \\
\end{array}
\]

The vertical maps are the Yoneda isomorphisms and the bottom row is exact. This makes the top row exact. Therefore \( \text{Nat}(\{(X, \_), \_\}): (C, \text{Ab}) \to \text{Ab} \) is exact and a a result, \( (X, \_) \) is projective. Thus, the representable functors are projective objects in \((C, \text{Ab})\). Throughout this paper, we will focus solely on results for covariant functors; however, in all cases, the dual results hold for contravariant functors. The easiest way to see this is to use the equivalence of the category of contravariant functors from \( C \) into \( \text{Ab} \) with the category of covariant functors \((C^{\text{op}}, \text{Ab})\). The exact details are left to the reader.

### 2. Zeroth Derived Functors

Throughout this section we fix abelian categories \( C, D \). If \( C \) has enough injectives, then given \( F \in (C, D) \), the classical way of defining the zeroth right derived functor \( R^0 F \) is to define \( R^0 F \) on its components as follows: For any \( X \in C \), take exact sequence \( 0 \to X \to I^0 \to I^1 \) with \( I^0, I^1 \) injective. The component \( R^0 F(X) \) is defined by the exact sequence
It is easily seen that this assignment is functorial in both $F$ and $X$. Moreover, up to isomorphism $R^0 F$ is independent of the choices of $I^0$ and $I^1$. The following properties may also be established:

1. $R^0 F$ and $F$ agree on injectives.
2. $R^0 F$ is left exact.
3. If $F$ is left exact, then $R^0 F \cong F$.
4. If $G$ is left exact and $G$ agrees on injectives with $F$, then $G \cong R^0 F$.
5. There is a morphism $F \to R^0 F$.
6. For any left exact functor $G$, there exists an isomorphism $(F, G) \cong (R^0 F, G)$

which is natural in $F$ and $G$.

Now drop the assumption that $C$ has enough injectives. It is still possible to give a definition of zeroth right derived functors. Let $S$ denote any subcategory of $(C, D)$. Define $\text{Lex}(S)$ to be the full subcategory of $S$ consisting of the left exact functors, that is, all functors $F \in S$ such that if $0 \to A \to B \to C \to 0$ is exact, then $0 \to F(A) \to F(B) \to F(C)$ is exact. There is an inclusion functor

$$\text{Lex}(S) \xrightarrow{s} S$$

We say that $S$ admits a **zeroth right derived functor** if $s$ has a left adjoint $r^0 : S \to \text{Lex}(S)$ such that

1. The unit of adjunction $u : 1_S \to sr^0$ is an isomorphism on the injectives of $C$. More precisely, if $F \in S$ and $I \in C$ is injective, then the morphism $(u_F)_I$ is an isomorphism.
2. The composition $r^0 s$ is isomorphic to the identity. That is

$$r^0 s \cong 1_S$$

If $S$ admits a zeroth right derived functor, then we define the composition $R^0 = sr^0$ to be the **zeroth right derived functor of $S$**. Clearly if $S \subseteq (C, D)$ admits a zeroth right derived functor $R^0$, then for any functor $F \in S$ and any injective $I \in C$,

$$R^0 F(I) \cong F(I)$$

Moreover, the functor $r^0 : S \to \text{Lex}(S)$ produces for each functor $F$ a left exact functor $r^0 F$ by altering $F$ in the smallest amount possible. To clarify this comment, note that $r^0$ does not change the functors which are already left exact. It also does not change values of $F$ on objects $X$ such that $0 \to X \to Y \to Z \to 0$ splits.
for all \( Y, Z \) as this condition implies that \( 0 \to F(X) \to F(Y) \to F(Z) \) is exact whether or not \( F \) is left exact. These objects are precisely the injectives of \( \mathcal{C} \). If \( \mathcal{C} \) has enough injectives, it is easily seen that the classical definition of \( R^0 \) satisfies the more general definition, that is \((\mathcal{C}, \mathcal{D})\) admits a zeroth right derived functor.

Now suppose that \( \mathcal{S} \subseteq (\mathcal{C}, \mathcal{D}) \) is an abelian subcategory of \((\mathcal{C}, \mathcal{D})\) that admits a zeroth right derived functor \( R^0: \mathcal{S} \to \mathcal{S} \). In this case, there is an exact sequence of functors

\[
0 \to \ker(u) \to 1_{\mathcal{S}} \to u \to R^0
\]

where \( u \) is the unit of adjunction. The functor \( \ker(u) \) is called the injective stabilization functor. Given \( F \in \mathcal{S} \), the functor \( \ker(u)(F) \) will be denoted \( \overline{F} \). The functor \( \overline{F} \) is called the injective stabilization of \( F \)

Evaluating the exact sequence at \( F \) yields an exact sequence of functors

\[
0 \to \overline{F} \to F \to \overline{R^0F}
\]

A functor \( F \in \mathcal{S} \) is called injectively stable if \( R^0F = 0 \). This generalizes the definition of the injective stabilization given by Auslander and Bridger in [3]. The definition given there is essentially the same except that it uses the zeroth right derived functor as defined classically, which requires the existence of injectives in \( \mathcal{C} \).

If \( \mathcal{C} \) has enough projectives, then given \( F \in (\mathcal{C}, \mathcal{D}) \), the classical way of defining the zeroth left derived functor \( L_0F \) is to define \( L_0F \) on its components as follows: For any \( X \in \mathcal{C} \) take exact sequence \( P_1 \to P_0 \to X \to 0 \) with \( P_0, P_1 \) projective. The component \( L_0F(X) \) is defined by the exact sequence

\[
F(P_1) \to F(P_0) \to L_0F(X) \to 0
\]

It is easily seen that this assignment is functorial in both \( F \) and \( X \). Moreover, up to isomorphism \( L_0F \) is independent of the choices of \( P_0 \) and \( P_1 \). The following properties may also be established:

1. \( L_0F \) and \( F \) agree on projectives.
2. \( L_0F \) is right exact.
3. If \( F \) is right exact, then \( L_0F \cong F \).
4. If \( G \) is right exact and \( G \) agrees on projectives with \( F \), then \( G \cong L_0F \).
5. There is a morphism \( L_0F \to F \).
6. For any right exact functor \( G \), there exists an isomorphism

\[
(G, F) \cong (G, L_0F)
\]

which is natural in \( F \) and \( G \).

Now drop the assumption that \( \mathcal{C} \) has enough projectives. It is still possible to give a definition of zeroth left derived functors. Let \( \mathcal{S} \) denote any subcategory of \((\mathcal{C}, \mathcal{D})\). Define \( \text{Rex}(\mathcal{S}) \) to be the full subcategory of \( \mathcal{S} \)
consisting of the right exact functors, that is, all functors \( F \in S \) such that if \( 0 \to A \to B \to C \to 0 \) is exact, then \( F(A) \to F(B) \to F(C) \to 0 \) is exact. There is an inclusion functor
\[
\text{Rex}(S) \xrightarrow{s} S
\]

We say that \( S \) admits a **zeroth left derived functor** if \( s \) has a right adjoint \( l_0 : S \to \text{Rex}(S) \) such that

1. The counit of adjunction \( c : sl_0 \to 1_S \) is an isomorphism on the projectives of \( C \). More precisely, if \( F \in S \) and \( P \in C \) is projective, then the morphism \((c_F)_P\) is an isomorphism.
2. The composition \( l_0s \) is isomorphic to the identity. That is \( l_0s \cong 1_S \).

If \( S \) admits a zeroth left derived functor, then we define the composition \( L_0 = sl_0 \) to be the **zeroth left derived functor** of \( S \).

Clearly if \( S \subseteq (C, D) \) admits a zeroth left derived functor \( R^0 \), then for any functor \( F \in S \) and any projective \( P \in C \),
\[
L_0 F(P) \cong F(P)
\]

Moreover, the functor \( l_0 : S \to \text{Rex}(S) \) produces for each functor \( F \) a right exact functor \( l_0 F \) by altering \( F \) in the smallest amount possible. To clarify this comment, note that \( l_0 \) does not change the functors which are already right exact. It also does not change values of \( F \) on objects \( Z \) such that \( 0 \to X \to Y \to Z \to 0 \) splits for all \( X, Y \) as this condition implies that \( F(X) \to F(Y) \to F(Z) \to 0 \) is exact whether or not \( F \) is right exact. These objects are precisely the projectives of \( C \). If \( C \) has enough projectives, it is easily seen that the classical definition of \( L_0 \) satisfies the more general definition, that is \((C, D)\) admits a zeroth left derived functor.

Now suppose that \( S \subseteq (C, D) \) is an abelian subcategory of \((C, D)\) that admits a zeroth left derived functor \( R^0 : S \to S \). In this case, there is an exact sequence of functors
\[
\begin{array}{ccc}
L_0 & \xrightarrow{c} & 1_S \\
\downarrow & & \downarrow \\
\text{Coker}(c) & \xrightarrow{c} & 0
\end{array}
\]

where \( c \) is the counit of adjunction. The functor \( \text{Coker}(c) \) is called the **projective stabilization functor**. Given \( F \in S \), the functor \( \text{Coker}(c)(F) \) will be denoted \( F' \). The functor \( F' \) is called the **projective stabilization of \( F \)**.

Evaluating the exact sequence at \( F \) yields an exact sequence of functors
\[
\begin{array}{ccc}
L_0 F & \xrightarrow{c_F} & F \\
\downarrow & & \downarrow \\
F' & \xrightarrow{c_F} & 0
\end{array}
\]

A functor \( F \in S \) is called **projectively stable** if \( L_0 F = 0 \). This generalizes the definition of the projective stabilization given by Auslander and Bridger in [3]. The definition given there is essentially the same except that it uses the zeroth left derived functor as defined classically, which requires the existence of projectives in \( C \).
3. Coherent Functors

All of the results in this section are due to Auslander and can be found in [1]. Auslander defined for any abelian category $C$ the category of coherent functors $\text{fp}(C, \text{Ab})$ and studied its formal properties. In his study of $\text{fp}(C, \text{Ab})$, he constructs an exact contravariant functor $w: \text{fp}(C, \text{Ab}) \to C$ that contains certain information about $\text{fp}(C, \text{Ab})$. Auslander’s study of the functor $w$ leads to a certain four term exact sequence of interest. We recall some important properties of $\text{fp}(C, \text{Ab})$ that will be needed later. In particular, we focus on Auslander’s construction of the functor $w: \text{fp}(C, \text{Ab}) \to C$ and the four term exact sequence that $w$ induces.

A functor $F \in (C, \text{Ab})$ is coherent if there exists $X, Y \in C$ and exact sequence

$$
(Y, -) \longrightarrow (X, -) \longrightarrow F \longrightarrow 0
$$

The full subcategory of $(C, \text{Ab})$ consisting of coherent functors is denoted $\text{fp}(C, \text{Ab})$. Auslander showed that $\text{fp}(C, \text{Ab})$ is abelian and the inclusion $\text{fp}(C, \text{Ab}) \to (C, \text{Ab})$ is exact. The representable functors are projective objects of $\text{fp}(C, \text{Ab})$ and therefore $\text{fp}(C, \text{Ab})$ has enough projectives. In fact, the representable objects are the only projective objects of $\text{fp}(C, \text{Ab})$.

**Proposition 15.** The only projectives of $\text{fp}(C, \text{Ab})$ are the representable functors.

**Proof.** Suppose that $F$ is a projective coherent functor. Take presentation

$$
0 \longrightarrow (Z, -) \longrightarrow (Y, -) \longrightarrow (X, -) \longrightarrow F \longrightarrow 0
$$

This embeds into the following commutative diagram with exact rows and columns:

Since $F$ is projective, the column $0 \to G \to (X, -) \to F \to 0$ splits and hence

$$(X, -) \cong G + F$$
As a direct summand of a projective, \( G \) must be projective and hence the row \( 0 \to (Z, _) \to (Y, _) \to G \to 0 \) splits. This results in exact sequence \( 0 \to G \to (Y, _) \to (Z, _) \to 0 \). As a result, \( G \) is a kernel of a natural transformation between representable functors. But this makes \( G \) itself representable. Now returning to the split exact sequence \( 0 \to G \to (X, _) \to F \to 0 \), there exists split exact sequence \( 0 \to F \to (X, _) \to G \to 0 \) making \( F \) a kernel of natural transformation between representable functors. Consequently, \( F \) is representable. ■

At this point we will explicitly construct the functor \( w : \text{fp}(C, \text{Ab}) \to C \). Given \( F \in \text{fp}(C, \text{Ab}) \) we take presentanion

\[
(Y, _) \xrightarrow{(f, -)} (X, _) \xrightarrow{\alpha} F \xrightarrow{k} 0
\]

The value of \( w \) at \( F \) is defined by the exact sequence

\[
0 \xrightarrow{k} w(F) \xrightarrow{k} X \xrightarrow{f} Y
\]

This assigment gives rise to a contravariant additive functor \( w : \text{fp}(C, \text{Ab}) \to C \). Auslander established the following properties concerning the functor \( w \):

1. For any coherent functor \( F \), \( w(F) \) is independent of the chosen projective presentation.
2. For any coherent functor \( F \), \( w(F) = 0 \) if and only if there exists an exact sequence \( 0 \to X \to Y \to Z \to 0 \) such that \( 0 \to (Z, _) \to (Y, _) \to (X, _) \to F \to 0 \) is exact.
3. The functor \( w \) is exact.

We now construct for each coherent functor \( F \) a four term exact sequence

\[
0 \to F_0 \to F \xrightarrow{\varphi} (w(F), _) \xrightarrow{w(F)} F_1 \to 0
\]

that will play a major role in showing that \( \text{fp}(C, \text{Ab}) \) admits a zeroth right derived functor. We start with the exact sequence

\[
0 \to (Z, _) \xrightarrow{(g, -)} (Y, _) \xrightarrow{(f, -)} (X, _) \xrightarrow{\alpha} F \to 0
\]

to which we apply the exact functor \( w \) yielding the exact sequence

\[
0 \to w(F) \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{g} Z \to 0
\]
This exact sequence embeds in the following commutative diagram with exact rows and columns:

```
0 → w(F) → w(F) → 0
k

0 → w(F) → X → Y → Z → 0
k

0 → V → Y → Z → 0
0
```

Applying the Yoneda embedding to this diagram and extending to include cokernels where necessary yields the following commutative diagram with exact rows and columns:

```
0 → (Z, _) → (Y, _) → (V, _) → F_0 → 0
\alpha

0 → (Z, _) → (Y, _) → (X, _) → F → 0
(k, _)

(k, _) → (w(F), _) → (w(F), _)
\varphi

F_1 → F_1

0 → 0
```

This yields the following exact sequence:

```
0 → F_0 → F \xrightarrow{\varphi} (w(F), _) → F_1 → 0
```

where \( w(F_0) = 0 \) and \( w(F_1) = 0 \). This sequence is functorial in \( F \). The map \( \varphi \) is the unique map such that \( \varphi \alpha = (k, _) \).
4. The CoYoneda Lemma and the Zeroth Right Derived Functor of \( \text{fp}(C, \text{Ab}) \)

We begin this section by making the observation that if \( F \in \text{fp}(C, \text{Ab}) \) and \( w(F) = 0 \), then \( F \) vanishes on injectives. To see this note that since \( w(F) = 0 \) there exists exact sequence

\[
0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0
\]

such that the following sequence is exact

\[
0 \rightarrow (Z, _) \xrightarrow{(g, -)} (Y, _) \xrightarrow{(f, -)} (X, _) \xrightarrow{\alpha} F \rightarrow 0
\]

By evaluating this sequence on an injective we have the following exact sequence

\[
0 \rightarrow (Z, I) \xrightarrow{(g, I)} (Y, I) \xrightarrow{(f, I)} (X, I) \xrightarrow{\alpha_I} F(I) \rightarrow 0
\]

Since \( I \) is injective, \( (f, I) \) is an epimorphism. Therefore \( \alpha_I = 0 \). Since \( \alpha_I \) is an epimorphism, \( F(I) = 0 \).

We now turn to one of the major properties of the functor \( w \). It is clear from [1] that Auslander was aware of the following theorem, though he never explicitly stated it. The proof here uses Yoneda’s lemma.

**Theorem 16.** Let \( F \in \text{fp}(C, \text{Ab}) \). Suppose that \( G \in (C, \text{Ab}) \) is left exact. Then there are isomorphisms

\[
G(w(F)) \cong ((w(F), _), G) \cong (F, G)
\]

which are natural in \( F \) and \( G \).

**Proof.** Take exact sequence

\[
(Y, _) \xrightarrow{(f, -)} (X, _) \xrightarrow{\alpha} F \rightarrow 0
\]

Applying \( w \) yields the following exact sequence

\[
0 \rightarrow w(F) \xrightarrow{k} X \xrightarrow{f} Y
\]

Applying the left exact functor \( G \) yields the exact sequence:

\[
0 \rightarrow G(w(F)) \xrightarrow{G(k)} G(X) \xrightarrow{G(f)} G(Y)
\]

By the Yoneda lemma, this embeds into the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \rightarrow & G(w(F)) \\
& & \downarrow \cong \\
0 & \rightarrow & ((w(F), _), G) \\
& & \downarrow \cong \\
& & ((X, _), G) \\
& & \downarrow \cong \\
& & ((Y, _), G)
\end{array}
\]

where the vertical maps are the Yoneda isomorphisms. This commutative diagram embeds into the following commutative diagram with exact rows:
It is easily seen that $\theta_{F,G}$ is an isomorphism which is natural in both $F$ and $G$. Therefore

$$G \langle \omega(F), \_ \rangle \cong \langle (w(F), \_), G \rangle \cong (F,G),$$

as claimed, these isomorphisms being natural in each variable. $\blacksquare$

The remainder of this section is devoted to interpreting the results Auslander in terms of the more general definition of zeroth right derived functors.

**Theorem 17.** The inclusion functor $s: \text{Lex}(\mathfrak{p}(C, \text{Ab})) \to \mathfrak{p}(C, \text{Ab})$ admits a left adjoint $r^0: \mathfrak{p}(C, \text{Ab}) \to \text{Lex}(\mathfrak{p}(C, \text{Ab})).$ Moreover, the unit of adjunction evaluated at coherent functor $F$ is the map $\varphi$ in the exact sequence

$$0 \longrightarrow F_0 \longrightarrow F \xrightarrow{\varphi} \langle w(F), \_ \rangle \longrightarrow F_1 \longrightarrow 0$$

**Proof.** Define $r^0(F) := \langle w(F), \_ \rangle$. From the preceding theorem, for any $F \in \mathfrak{p}(C, \text{Ab})$ and for any $G \in \text{Lex}(\mathfrak{p}(C, \text{Ab}))$, there exists an isomorphism

$$\langle F, s(G) \rangle \cong (r^0F,G)$$

Moreover, this isomorphism is natural in $F$ and $G$. In the commutative diagram with exact rows:

$$0 \longrightarrow G \langle \omega(F) \rangle \xrightarrow{G(k)} G(X) \xrightarrow{G(f)} G(Y) \xrightarrow{\cong} 0$$

choose $G = \langle w(F), \_ \rangle$ and focus on the lower left commutative square:
By definition, the unit of adjunction evaluated at \( F \) is \( u_F = \theta(1) \). From this commutative square, it follows that \( u_F \alpha = (k, _) \). Since \( \varphi \) is the unique map such that \( \varphi \alpha = (k, _) \), it follows that \( \varphi = u_F \).

**Proposition 18.** If \( F \) is representable, then \( r^0 F \cong F \).

**Proof.** Suppose that \( F \cong (X, _) \). Then \( w(F) \cong X \) and therefore

\[
r^0 F = (w(F), _) \cong (X, _) \cong F
\]

these isomorphisms being natural.

It is a well known fact that if \( L : \mathcal{A} \to \mathcal{B} \) and \( R : \mathcal{B} \to \mathcal{A} \) form an adjoint pair, then the unit of adjunction \( u : 1_\mathcal{A} \to RL \) satisfies the following property: Given \( X \in \mathcal{A} \) and \( Y \in \mathcal{B} \), every diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u_X} & RL(X) \\
f \downarrow & & \downarrow g \\
R(Y) & & R(Y)
\end{array}
\]

embeds into a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u_X} & RL(X) \\
f \downarrow & & \downarrow g \\
R(Y) & & R(Y)
\end{array}
\]

Because the morphism \( \varphi \) in the exact sequence

\[
0 \longrightarrow F_0 \longrightarrow F \xrightarrow{\varphi} (w(F), _) \longrightarrow F_1 \longrightarrow 0
\]

is precisely the unit of adjunction for the adjoint pair \((r^0, s)\) evaluated at \( F \), we have the following property:

**Proposition 19.** Let \( F \) be a coherent functor and

\[
0 \longrightarrow F_0 \longrightarrow F \xrightarrow{\varphi} (w(F), _) \longrightarrow F_1 \longrightarrow 0
\]
be the corresponding four term exact sequence. Suppose that there is a natural transformation

\[ 0 \rightarrow F_0 \rightarrow F \xrightarrow{\varphi} (w(F), \_ ) \rightarrow F_1 \rightarrow 0 \]

and \( G \) is left exact coherent. Then there exists \( \psi : (w(F), \_ ) \rightarrow G \) such that the following diagram commutes:

\[ 0 \rightarrow F_0 \rightarrow F \xrightarrow{\varphi} (w(F), \_ ) \rightarrow F_1 \rightarrow 0 \]

We now come to a complete description of the left exact coherent functors.

**Theorem 20.** For any \( F \in \text{fp}(C, \text{Ab}) \), the following are equivalent:

1. \( F \) is representable.
2. \( F \) is projective.
3. \( F \) is left exact.

**Proof.** That (1) and (2) are equivalent follows from Proposition 15 and the Yoneda lemma. (1) clearly implies (3). We will show that (3) implies (2) thus completing the proof. Suppose that \( F \) is left exact. Because \( F \) is left exact, the diagram

\[ 0 \rightarrow F_0 \rightarrow F \xrightarrow{\varphi} (w(F), \_ ) \rightarrow F_1 \rightarrow 0 \]

embeds into commutative diagram

\[ 0 \rightarrow F_0 \rightarrow F \xrightarrow{\varphi} (w(F), \_ ) \rightarrow F_1 \rightarrow 0 \]

Hence \( F \) is a retract of the projective \( (w(F), \_ ) \). Therefore \( F \) is projective. \( \square \)
Theorem 21. The category \( \text{fp}(C, \text{Ab}) \) admits a zeroth right derived functor \( R^0 : \text{fp}(C, \text{Ab}) \rightarrow \text{fp}(C, \text{Ab}) \) and \( R^0 \cong Yw. \) In particular the zeroth right derived functor applied to a coherent functor yields a representable functor.

Proof. Let \( s : \text{Lex}(\text{fp}(C, \text{Ab})) \rightarrow \text{fp}(C, \text{Ab}) \) be the natural inclusion functor. Observe that for any functor \( F, Yw(F) = (w(F), _) \). We have already shown that \( s \) admits a left adjoint \( r^0 : \text{fp}(C, \text{Ab}) \rightarrow \text{Lex}(\text{fp}(C, \text{Ab})) \) sending \( F \) to \((w(F), _)\) and that the unit of this adjunction evaluated at \( F \) is the map \( \varphi \) in the exact sequence

\[
0 \rightarrow F_0 \rightarrow F \xrightarrow{\varphi} (w(F), _) \rightarrow F_1 \rightarrow 0
\]

Since \( w(F_0) = w(F_1) = 0 \), both \( F_0 \) and \( F_1 \) vanish on injectives. Therefore \( \varphi_I \) is an isomorphism whenever \( I \) is injective. Finally, suppose that \( F \) is left exact. Then \( F \) is representable and therefore \( r^0 s(F) \cong F. \) \( \blacksquare \)

Lemma 22 (The CoYoneda Lemma). For any \( F \in \text{fp}(C, \text{Ab}) \) and any \( X \in C \),

\[
\text{Nat}(F, (X, _)) \cong (X, w(F))
\]

this isomorphism being natural in \( F \) and \( X \).

Proof. Since \((X, _)\) is left exact coherent, by Theorem 16,

\[
\text{Nat}(F, (X, _)) \cong (X, _)(w(F)) = (X, w(F))
\]

this isomorphism being natural in \( F \) and \( X \). \( \blacksquare \)

For the functor category \((C, \text{Ab})\), there is a functor \((_)^* : (C, \text{Ab}) \rightarrow (C^{\text{op}}, \text{Ab})\) defined as follows: For each \( F : C \rightarrow \text{Ab}, F^* : C^{\text{op}} \rightarrow \text{Ab} \) is defined by

\[
F^*(X) := \text{Nat}(F, (X, _))
\]

By the CoYoneda lemma, if \( F \in \text{fp}(C, \text{Ab}) \), then \( F^*(X) \cong (X, w(F)). \) Therefore \( F^* \cong (_, w(F)) \). Since \( F^{**} \cong (w(F))^* \cong (w(F), _) \), the endofunctor \( R^0 = Y \circ w : \text{fp}(C, \text{Ab}) \rightarrow \text{fp}(C, \text{Ab}) \) is isomorphic to \((_)^{**}\). Recall the following: For any ring \( R \) and any module \( M \in \text{mod}(R) \), apply the functor \((_)^*\) to any presentation of \( M \):

\[
P_1 \rightarrow P_0 \rightarrow X \rightarrow 0
\]
yielding exact sequence
\[ \begin{array}{cccccc}
0 & \rightarrow & X^* & \rightarrow & (P_0)^* & \rightarrow & (P_1)^* & \rightarrow & \text{Tr}(M) & \rightarrow & 0
\end{array} \]

The module \( \text{Tr}(M) \) is called the transpose of \( M \).

**Proposition 23** (Auslander-Bridger, [3]). For any finitely presented module \( M \), there is an exact sequence
\[ \begin{array}{cccccc}
0 & \rightarrow & \text{Ext}^1(\text{Tr}(M), R) & \rightarrow & M & \rightarrow & M^{**} & \rightarrow & \text{Ext}^2(\text{Tr}(M), R) & \rightarrow & 0
\end{array} \]

This sequence will have its analog in the category \( \text{fp}(\mathcal{C}, \text{Ab}) \). From the defect sequence
\[ \begin{array}{cccccc}
0 & \rightarrow & F_0 & \rightarrow & F & \xrightarrow{\varphi} & (w(F), _) & \rightarrow & F_1 & \rightarrow & 0
\end{array} \]

and the fact that for any \( F \in \text{fp}(\mathcal{C}, \text{Ab}) \) we have \( F^{**} \cong R^0F \cong (w(F), _) \), we have the following exact sequence
\[ \begin{array}{cccccc}
0 & \rightarrow & F_0 & \rightarrow & F & \xrightarrow{\varphi} & F^{**} & \rightarrow & F_1 & \rightarrow & 0
\end{array} \]

This allows one to calculate \( F_0 \) and \( F_1 \) explicitly. Namely \( F_k \cong \text{Ext}^{k+1}(\text{Tr}(F), \mathcal{Y}) \) where
\[ \text{Ext}^{k+1}(\text{Tr}(F), \mathcal{Y})(X) := \text{Ext}^{k+1}(\text{Tr}(F), (_, X)) \]

**Proposition 24.** For any \( F \in (\mathcal{C}, \text{Ab}) \) the defect sequence is equivalent to the following exact sequence:
\[ \begin{array}{cccccc}
0 & \rightarrow & \text{Ext}^1(\text{Tr}(F), \mathcal{Y}) & \rightarrow & F & \xrightarrow{\varphi} & F^{**} & \rightarrow & \text{Ext}^2(\text{Tr}(F), \mathcal{Y}) & \rightarrow & 0
\end{array} \]

### 5. Injective Resolutions of \( \text{fp}(\mathcal{C}, \text{Ab}) \)

In this section it is shown that all coherent functors have injective resolutions under the assumption that \( \mathcal{C} \) has enough projectives. We begin by introducing the coherent closure of a full subcategory of an abelian category. This terminology is not standard.

**Definition 2.** Let \( \mathcal{A} \) denote an abelian category and \( \mathcal{P} \) be a full subcategory of \( \mathcal{A} \). Define the **coherent closure** of \( \mathcal{P} \), denoted by \( \text{pres}(\mathcal{P}) \), to be the full subcategory of \( \mathcal{A} \) consisting of all objects \( X \in \mathcal{A} \) such that there exists exact sequence \( P_1 \rightarrow P_0 \rightarrow X \rightarrow 0 \) with \( P_1, P_0 \in \mathcal{P} \).

Essentially, the category of coherent functors is the coherent closure of the full subcategory of representable functors. The next proposition is stated and proven in slightly more generality by Auslander in [1]; however, for our purposes, this slightly less general statement will suffice.

**Proposition 25** (Auslander). Let \( \mathcal{A} \) denote an abelian category and \( \mathcal{P} \) be a full subcategory of \( \mathcal{A} \) satisfying the following:
(1) $\mathcal{P}$ consists of projectives.
(2) $\mathcal{P}$ is closed under finite sums.
(3) $\mathcal{P}$ is closed under kernels.

Under these conditions, the coherent closure $\text{pres}(\mathcal{P})$ is an abelian category and the inclusion $\text{pres}(\mathcal{P}) \to \mathcal{A}$ is exact and reflects exact sequences.

**Proof.** The goal is to establish that given any exact sequence in the larger category $\mathcal{A}$:

$$
0 \longrightarrow K \longrightarrow H \longrightarrow G \longrightarrow F \longrightarrow 0
$$

if $H, G \in \text{pres}(\mathcal{P})$ then $K, F \in \text{pres}(\mathcal{P})$. The statement of the proposition will follow easily from this fact. First, observe that by the horseshoe lemma, if

$$
0 \longrightarrow U \longrightarrow V \longrightarrow T \longrightarrow 0
$$

is exact in $\mathcal{A}$ and $U, T \in \text{pres}(\mathcal{P})$, then $V \in \text{pres}(\mathcal{P})$. Now assume that $H, G \in \text{pres}(\mathcal{P})$. Then there exists commutative diagram with exact rows and columns

$$
\begin{array}{ccc}
H_1 & \longrightarrow & G_1 \\
\downarrow & & \downarrow \\
H_0 & \longrightarrow & G_0 \\
\downarrow & & \downarrow \\
H & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

where $G_0, G_1, H_0, H_1 \in \mathcal{P}$. This diagram induces the following exact sequence

$$
H_0 + G_1 \longrightarrow G_0 \longrightarrow F \longrightarrow 0
$$

establishing that the $\text{pres}(\mathcal{P})$ is closed under the cokernel operation on $\mathcal{A}$.

The next step is to establish that for any short exact sequence in $\mathcal{A}$:

$$
0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0
$$

if $M, N \in \text{pres}(\mathcal{P})$, then $L \in \text{pres}(\mathcal{P})$. Take a presentation $N_1 \to N_0 \to N \to 0$. Since $N_1, N_0 \in \mathcal{P}$ and $\mathcal{P}$ is closed under the kernel operation on $\mathcal{A}$, there is an exact sequence $0 \to N_2 \to N_1 \to N_0 \to N \to 0$. Let $W$ be the kernel of $N_0 \to N$. There is an exact sequence $0 \to N_2 \to N_1 \to W \to 0$ so in particular $W \in \text{pres}(\mathcal{P})$. Now the diagram:
embeds into the pullback diagram

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
W & W & L & M \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\begin{array}{c}
E \\
\downarrow \\
N_0 \\
\downarrow \\
N_0 \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\end{array}
\]

Since \( W, M \in \text{pres}(P) \), \( E \in \text{pres}(P) \) by the horseshoe lemma. Since \( N_0 \in P \), the exact sequence \( 0 \to L \to E \to N_0 \to 0 \) splits and hence there exists an exact sequence \( 0 \to N_0 \to E \to L \to 0 \). Hence \( L \) is a cokernel of a morphism between two objects from \( \text{pres}(P) \). It follows that \( L \in \text{pres}(P) \) from above.

To finish the proof, suppose that we have the following exact sequence in \( A \):

\[
0 \longrightarrow K \longrightarrow H \longrightarrow G \longrightarrow F \longrightarrow 0
\]

and that \( H, G \in \text{pres}(P) \). This exact sequence embeds into the following commutative diagram with exact rows and columns:
Since $H, G \in \text{pres}(P)$, it follows that $F \in \text{pres}(P)$. But from the exact sequence $0 \to W \to G \to F \to 0$, this means that $W \in \text{pres}(P)$. Finally, from the exact sequence $0 \to K \to H \to W \to 0$, it follows that $K \in \text{pres}(P)$. This completes the proof. 

**Definition 3.** Let $\mathcal{A}$ denote an abelian category and $\mathcal{I}$ be a full subcategory of $\mathcal{A}$. Define the **coherent coclosure** of $\mathcal{I}$, denoted by $\text{copres}(\mathcal{I})$, to be the full subcategory of $\mathcal{A}$ consisting of all objects $X \in \mathcal{A}$ such that there exists exact sequence $0 \to X \to I^0 \to I^1$ with $I^1, I^0 \in \mathcal{I}$.

The dual statement of Proposition 25 holds as well:

**Proposition 26.** Let $\mathcal{A}$ denote an abelian category and $\mathcal{I}$ be a full subcategory of $\mathcal{A}$ satisfying the following:

1. $\mathcal{I}$ consists of injectives.
2. $\mathcal{I}$ is closed under finite sums.
3. $\mathcal{I}$ is closed under cokernels.

Under these conditions, the coherent coclosure $\text{copres}(\mathcal{I})$ is an abelian category and the inclusion $\text{copres}(\mathcal{I}) \to \mathcal{A}$ is exact and reflects exact sequences.

**Theorem 27.** If $\mathcal{C}$ has enough projectives, then for every $F \in \text{fp}(\mathcal{C}, \text{Ab})$, there exists injective resolution

$$0 \to F \to I^0 \to I^1 \to I^2 \to 0$$

**Proof.** Let $\mathcal{I}$ be the full subcategory of $\text{fp}(\mathcal{C}, \text{Ab})$ consisting of all functors $H$ with presentation $$(Q, _) \to (P, _) \to H \to 0$$
where \( Q, P \) are projectives in \( C \). It is easily seen that

1. \( \mathcal{I} \) consists of injectives in \( \text{fp}(C, \text{Ab}) \).
2. \( \mathcal{I} \) is closed under finite sums.
3. \( \mathcal{I} \) is closed under cokernels.

By Proposition 26, the coherent coclosure \( \text{copres}(\mathcal{I}) \) is abelian.

Let \( X \in C \). From presentation \( P_1 \to P_0 \to X \to 0 \), we get exact sequence

\[
0 \to (X, \_\,) \to (P_0, \_\,) \to (P_1, \_\,)
\]

Clearly \( (P_0, \_\,), (P_1, \_\,) \in \mathcal{I} \). Therefore \( (X, \_\,) \in \mathcal{I}^1 \) which means that \( \text{copres}(\mathcal{I}) \) contains the representable functors. Given \( F \in \text{fp}(C, \text{Ab}) \) we have presentation

\[
(Y, \_\,) \to (X, \_\,) \to F \to 0
\]

Since \( (Y, \_\,), (X, \_\,) \in \mathcal{I}^1 \) and \( \mathcal{I}^1 \) is abelian, \( F \in \text{copres}(\mathcal{I}) \). As a result, there exists exact sequence

\[
0 \to F \to I^0 \to I^1
\]

where \( I^0, I^1 \in \mathcal{I} \). Completing this sequence to include cokernels yields the following exact sequence

\[
0 \to F \to I^0 \to I^1 \to C \to 0
\]

Since \( I^0, I^1 \in \mathcal{I} \) so is \( C \) because \( \mathcal{I} \) is closed under cokernels. Therefore there exists exact sequence

\[
0 \to F \to I^0 \to I^1 \to I^2 \to 0
\]

where \( I^0, I^1, I^2 \in \mathcal{I} \). Since \( \mathcal{I} \) consists of injectives, we have provided the desired injective resolution of \( F \). □

The fact that \( \text{fp}(C, \text{Ab}) \) has enough injectives whenever \( C \) has enough projectives was first shown by Ron Gentle in [7], though his approach was different.

**Corollary 28.** If \( C \) has enough projectives, then every coherent functor has injective dimension at most 2.

We end this chapter by characterizing the injectives in \( \text{fp}(C, \text{Ab}) \). Suppose that \( G \in \text{fp}(C, \text{Ab}) \) is injective. Let

\[
0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0
\]
be a short exact sequence in $\mathcal{C}$. Then the sequence

$$0 \rightarrow (Z, \_ \_ \_ \hspace{0.5cm}) \rightarrow (Y, \_ \_ \_ \hspace{0.5cm}) \rightarrow (X, \_ \_ \_ \hspace{0.5cm})$$

is exact in $\text{fp}(\mathcal{C}, \text{Ab})$. Since $G$ is injective, the following sequence is exact

$$\text{Nat}((X, \_ \_ \_ \hspace{0.5cm}), G) \rightarrow \text{Nat}((Y, \_ \_ \_ \hspace{0.5cm}), G) \rightarrow \text{Nat}((Z, \_ \_ \_ \hspace{0.5cm}), G) \rightarrow 0$$

By the Yoneda lemma this sequence is equivalent to the following exact sequence

$$G(X) \rightarrow G(Y) \rightarrow G(Z) \rightarrow 0$$

Therefore $G$ is right exact. One can also easily see that if $G \in \text{fp}(\mathcal{C}, \text{Ab})$ is right exact, then $G$ is injective.

In summary we have

**Proposition 29.** A coherent functor is injective if and only if it is right exact.

For any noetherian ring $R$, the category $\text{mod}(R)$ is abelian and has enough projectives. Therefore $\text{fp}(\text{mod}(R), \text{Ab})$ has enough injectives. In [12], Watts shows that whenever $R$ is noetherian, any right exact functor $F: \text{mod}(R) \rightarrow \text{Ab}$ is isomorphic to the tensor functor $F(R) \otimes \_ \_ \_ \hspace{0.5cm}$. Therefore if $R$ is noetherian ring, then the injectives of $\text{fp}(\text{mod}(R), \text{Ab})$ are tensor functors and for every $F \in \text{fp}(\text{mod}(R), \text{Ab})$ there exists an injective resolution

$$0 \rightarrow F \rightarrow X \otimes \_ \_ \_ \hspace{0.5cm} \rightarrow Y \otimes \_ \_ \_ \hspace{0.5cm} \rightarrow Z \otimes \_ \_ \_ \hspace{0.5cm} \rightarrow 0$$

This result was first shown by Auslander in [2]. There Auslander explicitly constructs the exact sequence

$$0 \rightarrow F \rightarrow X \otimes \_ \_ \_ \hspace{0.5cm} \rightarrow Y \otimes \_ \_ \_ \hspace{0.5cm} \rightarrow Z \otimes \_ \_ \_ \hspace{0.5cm} \rightarrow 0$$

by using a duality which we have not mentioned. We will see later that this duality can be constructed from the injective resolutions. In other words, Auslandes approach was to construct a duality $D: \text{fp}(\text{mod}(R), \text{Ab}) \rightarrow \text{fp}(\text{mod}(R^{op}), \text{Ab})$ and use this duality to construct the exact sequence

$$0 \rightarrow F \rightarrow X \otimes \_ \_ \_ \hspace{0.5cm} \rightarrow Y \otimes \_ \_ \_ \hspace{0.5cm} \rightarrow Z \otimes \_ \_ \_ \hspace{0.5cm} \rightarrow 0$$

However, we shall see that one may turn this approach on its head and use the injective resolutions of $\text{fp}(\text{mod}(R), \text{Ab})$ to recover Auslander’s duality.
A Functorial Approach to Linkage

1. Connected Triples and Connected Sequences of Functors

Let $\mathcal{A}$ be an abelian category. The category $\text{Ses}(\mathcal{A})$ has as objects all short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$ and as morphisms all chain maps

$$
\begin{array}{c}
0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \\
0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0
\end{array}
$$

The term functors $T_1, T_3: \text{Ses}(\mathcal{A}) \rightarrow \mathcal{A}$ are defined as follows: For any short exact sequence $\mathcal{E}$:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$T_1(\mathcal{E}) = A$$

$$T_3(\mathcal{E}) = C.$$ 

Given a morphism $\phi: \mathcal{E}_1 \rightarrow \mathcal{E}_2$ between two short exact sequences:

$$
\begin{array}{c}
0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \\
\downarrow f \quad \downarrow g \\
0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0
\end{array}
$$

$$T_1(\phi) = f$$

$$T_3(\phi) = g.$$ 

A functor $F$ is said to be connected to $G$ via the natural transformation $\delta: FT_3 \rightarrow GT_1$ if for any short exact sequence $\mathcal{E}$:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

the sequence

$$F(B) \longrightarrow F(C) \xrightarrow{\delta_\mathcal{E}} G(A) \longrightarrow G(B)$$
is a complex. For abelian categories \(\mathcal{A}, \mathcal{B}\), a **connected triple** in \((\mathcal{A}, \mathcal{B})\) is a triple \((F, \gamma, G)\) where \(F, G : \mathcal{A} \rightarrow \mathcal{B}\) are additive functors and \(F\) is connected to \(G\) via \(\gamma\). In this case, we refer to \(\gamma\) as the **connecting homomorphism**.

A morphism between two connected triples \((F_1, \beta, F_2)\) and \((G_1, \gamma, G_2)\) in \((\mathcal{A}, \mathcal{B})\) is a pair \((\eta, \mu)\) of natural transformations \(\eta : F_1 \rightarrow G_1\) and \(\mu : F_2 \rightarrow G_2\) such that for every short exact sequence \(\mathcal{E}\):

\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\]

the following diagram commutes:

\[
\begin{array}{ccccccccc}
F_1(A) & \rightarrow & F_1(B) & \rightarrow & F_1(C) & \rightarrow & G_1(A) & \rightarrow & G_1(B) & \rightarrow & G_1(C) \\
| & & | & & | & & | & & | & & | \\
\eta_A & & \eta_B & & \eta_C & & \mu_A & & \mu_B & & \mu_C \\
F_2(A) & \rightarrow & F_2(B) & \rightarrow & F_2(C) & \rightarrow & G_2(A) & \rightarrow & G_2(B) & \rightarrow & G_2(C)
\end{array}
\]

The category of connected triples in \((\mathcal{A}, \mathcal{B})\) is denoted \(\Delta(\mathcal{A}, \mathcal{B})\). This category has as objects the connected triples in \((\mathcal{A}, \mathcal{B})\) and as morphisms the morphisms of connected triples. This category is abelian. A **cohomological triple** is a connected triple \((F, \gamma, G)\) such that for every short exact sequence \(\mathcal{E}\):

\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\]

the sequence

\[
F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow G(A) \rightarrow G(B) \rightarrow G(C)
\]

is exact. We will sometimes simple refer to these triples as being cohomological.

A **connected sequence of functors** \((F, \delta) : \mathcal{A} \rightarrow \mathcal{B}\) is a sequence \((F, \delta) = (F^n, \delta^n)_{n \in \mathbb{Z}}\) of additive covariant functors \(F^n : \mathcal{A} \rightarrow \mathcal{B}\) and natural transformations \(\delta^n : F^n \rightarrow F^{n+1}\) such that for all \(n \in \mathbb{Z}\), \(F^n\) is connected to \(F^{n+1}\) via \(\delta^n\), or if the reader prefers, for all \(n \in \mathbb{Z}\), \((F^n, \delta^n, F^{n+1}) \in \Delta(\mathcal{A}, \mathcal{B})\).

This essentially means that for any chain map between short exact sequences \(\mathcal{E}_1 \rightarrow \mathcal{E}_2\):

\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\]

there is a commutative diagram whose rows are complexes:

\[
\cdots \rightarrow F^{n-1}(C) \xrightarrow{\delta_{\mathcal{E}_1}^{n-1}} F^n(A) \rightarrow F^n(B) \rightarrow F^n(C) \xrightarrow{\delta_{\mathcal{E}_1}^n} F^{n+1}(A) \rightarrow \cdots
\]

\[
\cdots \rightarrow F^{n-1}(Z) \xrightarrow{\delta_{\mathcal{E}_2}^{n-1}} F^n(X) \rightarrow F^n(Y) \rightarrow F^n(Z) \xrightarrow{\delta_{\mathcal{E}_2}^n} F^{n+1}(X) \rightarrow \cdots
\]
A cohomological functor is a connected sequence of functors \((F, \delta)\) such that for all \(n \in \mathbb{Z}\), \((F^n, \delta^n, F^{n+1})\) is a cohomological triple. The category \(\text{Con}_\Delta(A, B)\) has as objects all connected sequences of functors \((T, \delta): A \to B\). Given two connected sequences of functors \((T, \delta)\) and \((U, \gamma)\), a morphism \(\alpha: (T, \delta) \to (U, \gamma)\) is a sequence \((\alpha^n)_{n \in \mathbb{Z}}\) of natural transformations \(\alpha^n: T^n \to U^n\) commuting with the connecting homomorphisms, which means that for every \(n \in \mathbb{Z}\) and for every short exact sequence \(E:\)

\[
\begin{array}{cccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}
\]

The following diagram commutes:

\[
\begin{array}{cccccc}
T^n(A) & \longrightarrow & T^n(B) & \longrightarrow & T^n(C) & \overset{\delta^n}{\longrightarrow} & T^{n+1}(A) & \longrightarrow & T^{n+1}(B) & \longrightarrow & T^{n+1}(C) \\
\alpha^n_A & & \alpha^n_B & & \alpha^n_C & & \alpha^{n+1}_A & & \alpha^{n+1}_B & & \alpha^{n+1}_C \\
U^n(A) & \longrightarrow & U^n(B) & \longrightarrow & U^n(C) & \overset{\gamma^n}{\longrightarrow} & U^{n+1}(A) & \longrightarrow & U^{n+1}(B) & \longrightarrow & U^{n+1}(C)
\end{array}
\]

This is equivalent to the following square commuting for every \(n \in \mathbb{Z}\):

\[
\begin{array}{cccc}
T^n(C) & \overset{\delta^n}{\longrightarrow} & T^{n+1}(A) \\
\alpha^n_C & & \alpha^{n+1}_A \\
U^n(C) & \overset{\gamma^n}{\longrightarrow} & U^{n+1}(A)
\end{array}
\]

We will often simply write \(T\) in place of \((T, \delta)\) for a connected sequence of functors. We will abbreviate \(\text{Hom}_{\text{Con}_\Delta(A, B)}(T, U)\) by \(\text{Hom}_\Delta(T, U)\) when there is no danger of confusion.

2. Satellites

The classical definitions of the satellite endofunctors require the existence of projectives and injectives. In [6], Fisher-Palmquist and Newell give alternate definitions of the satellites that do not require projectives or injectives, which we now recall. For abelian categories \(\mathcal{C}, \mathcal{D}\), there are two functors \(\pi_1, \pi_2: \Delta(\mathcal{C}, \mathcal{D}) \to (\mathcal{C}, \mathcal{D})\) given by \(\pi_1(F, \delta, G) = F\) and \(\pi_2(F, \delta, G) = G\).

\[
\begin{array}{c}
\Delta(\mathcal{C}, \mathcal{D}) \\
\pi_1 \quad \pi_2
\end{array}
\]

The category \((\mathcal{C}, \mathcal{D})\) admits a right satellite if the functor \(\pi_1\) admits a left adjoint \(\Delta S_1: (\mathcal{C}, \mathcal{D}) \to \Delta(\mathcal{C}, \mathcal{D})\).

The category \((\mathcal{C}, \mathcal{D})\) admits a left satellite if the functor \(\pi_2\) admits a right adjoint \(\Delta S_1: (\mathcal{C}, \mathcal{D}) \to \Delta(\mathcal{C}, \mathcal{D})\).

**Definition 4.** Suppose that \((\mathcal{C}, \mathcal{D})\) admits both a left and right satellite. The left satellite endofunctor \(S_1: (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}, \mathcal{D})\) is defined as the composition \(S_1 := \pi_1 \circ \Delta S_1\). The right satellite endofunctor
$S^1 : (C, D) \rightarrow (C, D)$ is defined as the composition $S^1 := \pi_2 \circ \Delta S^1$. By iteration $S_n := (S_1)^n$ and $S^n := (S^1)^n$.

We also use the convention that $S_0 = S^0 = 1_{(C, D)}$.

**Proposition 30** (Fisher-Palmquist, Newell [6]). Suppose that $(C, D)$ admits left and right satellites. For every $n \geq 0$, $(S^n, S_n)$ form an adjoint pair of endofunctors.

We will now recall the classical definitions of the satellites. For details, the reader is referred to [5]. Suppose that $C$ has both enough injectives and projectives. Let $F : C \rightarrow D$ and $X \in C$. Choose syzygy and cosyzygy sequences

$$
0 \longrightarrow \Omega X \longrightarrow P \longrightarrow X \longrightarrow 0
$$

$$
0 \longrightarrow X \longrightarrow I \longrightarrow \Sigma X \longrightarrow 0
$$

Classically $S_1 F$ and $S^1 F$ are defined on the object $X$ by the following exact sequences:

$$
0 \longrightarrow S_1 F(X) \longrightarrow F(\Omega X) \longrightarrow F(P)
$$

$$
F(I) \longrightarrow F(\Sigma X) \longrightarrow S^1 F(X) \longrightarrow 0
$$

This completely determines $S_1$ and $S^1$ as endofunctors on the functor category $(C, D)$.

Let

$$
0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0
$$

be a short exact sequence in $C$. This embeds into the following commutative diagram with exact rows:

$$
0 \longrightarrow \Omega Z \longrightarrow P \longrightarrow Z \longrightarrow 0
$$

$$
0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0
$$

1

This results in the following commutative diagram with exact rows:

$$
0 \longrightarrow S_1 F(Z) \longrightarrow F(\Omega Z) \longrightarrow F(P)
$$

$$
F(X) \longrightarrow F(Y)
$$

The morphism $\nabla_\varepsilon$ is defined by the following commutative diagram:

$$
0 \longrightarrow S_1 F(Z) \longrightarrow F(\Omega Z) \longrightarrow F(P)
$$

$$
\nabla_\varepsilon
$$

$$
F(X) \longrightarrow F(Y)
$$
**Proposition 31** (Cartan-Eilenberg, [5]). For any \( F : \mathcal{C} \to \mathcal{D} \), \((S^1F, \nabla, F)\) is a connected triple. Hence for any short exact sequence

\[
0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0
\]

the sequence

\[
S^1F(X) \longrightarrow S^1F(Y) \longrightarrow S^1F(Z) \xrightarrow{\nabla_F} F(X) \longrightarrow F(Y) \longrightarrow F(Z)
\]

is a complex. Moreover, if \( F : \mathcal{C} \to \mathcal{D} \) is half exact, then the connected triple \((S^1F, \nabla, F)\) is cohomological.

In a dual manner to the construction of \( \nabla \), one may construct for every short exact sequence

\[
0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0
\]

a morphism \( \Delta \varepsilon : F(Z) \to S^1F(X) \).

**Proposition 32** (Cartan-Eilenberg, [5]). For any functor \( F : \mathcal{C} \to \mathcal{D} \), \((F, \Delta, S^1F)\) is a connected triple. Hence for any short exact sequence

\[
0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0
\]

the sequence

\[
F(X) \longrightarrow F(Y) \longrightarrow F(Z) \xrightarrow{\Delta_F} S^1F(X) \longrightarrow S^1F(Y) \longrightarrow S^1F(Z)
\]

is a complex. Moreover, if \( F \) is half exact, then the connected triple \((F, \Delta, S^1F)\) is cohomological.

Define \( \Delta S_1 : (\mathcal{C}, \mathcal{D}) \to \Delta(\mathcal{C}, \mathcal{D}) \) by \( \Delta S_1(F) := (S^1F, \nabla, F) \) and \( \Delta S^1 : (\mathcal{C}, \mathcal{D}) \to \Delta(\mathcal{C}, \mathcal{D}) \) by \( \Delta S^1(F) := (F, \Delta, S^1F) \).

**Proposition 33** (Cartan-Eilenberg [5]). The functor \( \Delta S_1 : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}, \mathcal{D}) \) is the right adjoint of \( \pi_2 \). The functor \( \Delta S^1 \) is the left adjoint to \( \pi_1 \).

**Proof.** We wish to show that \( \Delta S_1 \) is the right adjoint to \( \pi_2 \). We must show that for any connected triple \((G, \delta, H)\) and any functor \( F : \mathcal{C} \to \mathcal{D} \), that there is a natural isomorphism

\[
\text{Nat}(\pi_2(G, \delta, H), F) \cong \text{Hom}_\Delta((G, \delta, H), \Delta S_1(F))
\]

or equivalently a natural isomorphism

\[
\text{Nat}(H, F) \cong \text{Hom}_\Delta((G, \delta, H), (S^1F, \nabla, F))
\]

We will construct the isomorphism and leave the verifications that it is natural and in fact an isomorphism to the reader. Let \( \alpha : H \to F \). Start with any \( X \in \mathcal{C} \). Take a syzygy sequence
Since both \((G, \delta, H)\) and \((S_1F, \nabla, F)\) are connected triples, there exists a commutative diagram of complexes:

\[
\begin{array}{ccccccccc}
G(\Omega X) & \longrightarrow & G(P) & \longrightarrow & G(X) & \overset{\delta}{\longrightarrow} & H(\Omega X) & \longrightarrow & H(P) \\
\downarrow{\alpha_{\Omega X}} & & \downarrow{\alpha_P} & & & & & & \\
0 & \longrightarrow & S_1F(X) & \overset{\nabla}{\longrightarrow} & F(\Omega X) & \longrightarrow & F(P) \\
\end{array}
\]

where the bottom row is exact by definition. This embeds uniquely into the following commutative diagram of complexes:

\[
\begin{array}{ccccccccc}
G(\Omega X) & \longrightarrow & G(P) & \longrightarrow & G(X) & \overset{\delta}{\longrightarrow} & H(\Omega X) & \longrightarrow & H(P) \\
\downarrow{\beta_X} & & \downarrow{\alpha_{\Omega X}} & & \downarrow{\alpha_P} & & & & \\
0 & \longrightarrow & S_1F(X) & \overset{\nabla}{\longrightarrow} & F(\Omega X) & \longrightarrow & F(P) \\
\end{array}
\]

where the bottom row is exact. The morphism \(\beta_X\) determines a natural transformation \(\beta: G \to S_1F\) and it is easily seen that \((\beta, \alpha): (G, \delta, H) \to (S_1F, \nabla, F)\) is a morphism of connected triples. The map \(\alpha \mapsto (\beta, \alpha)\) is the isomorphism. The reader will easily verify that this map satisfies all of the required properties. A similar map may be constructed for the case of \(\Delta S^1\). The details of this proof can be found in [5].  

Essentially, this guarantees that if \(C\) has enough projectives and injectives, then it admits both left and right satellites. In chapter 1, we saw how one can generalize the definition of zeroth derived functors so as to not require the existence of projectives or injectives. Fisher-Palmquist and Newell have provided a definition of the satellites that also does not require the existence of projectives or injectives. In the classical setting, it is well known that \(R^n \cong S^nR^0\) and \(L_n \cong S_nL_0\). Accordingly, one may now give definitions of derived functors that do not require the existence of projectives or injectives.

Suppose that \(C, D\) are abelian categories such that \((C, D)\) admits a zeroth left derived functor and a left satellite. The derived functors \(L_n\) are defined by

\[
L_n := S_nL_0
\]

where \(S_n := (\pi_1 \circ \Delta S_1)^n\). Similarly, if \((C, D)\) admits a zeroth right derived functor and a right satellite, then \(R^n\) is defined by

\[
R^n := S^nR^0
\]

where \(S^n := (\pi_2 \Delta S^1)^n\).
For any abelian category \( C \) with enough projectives, one may construct the stable category \( \underline{C} \). The objects of \( \underline{C} \) are precisely the objects of \( C \). Given \( X, Y \in \underline{C} \), define \( \text{Hom}_\underline{C}(X, Y) = \text{Hom}_C(X, Y)/\mathcal{P}(X, Y) \) where \( \mathcal{P}(X, Y) \) is the subgroup of \( \text{Hom}(X, Y) \) consisting of all morphisms that factor through a projective. One writes \( \text{Hom}(X, Y) \) in place of \( \text{Hom}_\underline{C}(X, Y) \).

Finally, we recall the following from the Auslander-Bridger treatise on stable module theory. If \( C \) has enough projectives, then for any abelian category \( D \), the functor category \( (C, D) \) admits a zeroth derived functor \( L_0 : (C, D) \to (C, D) \). Therefore, there is an exact sequence of endofunctors on \( (C, D) \)

\[
L_0 \xrightarrow{c} 1_{(C, D)} \xrightarrow{} \text{Coker}(c) \xrightarrow{} 0
\]

Similarly, if \( C \) has enough injectives then \( (C, D) \) admits a zeroth right derived functor \( R^0 : (C, D) \to (C, D) \) and we have an exact sequence of endofunctors:

\[
0 \xrightarrow{} \text{Ker}(u) \xrightarrow{} 1_{(C, D)} \xrightarrow{u} R^0
\]

Evaluating this two exact sequences at component \( F : C \to D \) yields exact sequences:

\[
L_0 F \xrightarrow{c F} F \xrightarrow{} F \xrightarrow{} 0
\]

\[
0 \xrightarrow{} \underline{F} \xrightarrow{} F \xrightarrow{u F} R^0 F
\]

where \( \underline{F} \) represents the projective stabilization of \( F \), i.e. \( \text{Coker}(u)(F) \) and \( \underline{F} \) represents the injective stabilization of \( F \), i.e. \( \text{Ker}(u)(F) \). It is easily seen that if \( F = (X, -) \), then \( \underline{F} = \text{Hom}(X, -) \). One of the major results from [3] is the following:

**Proposition 34 (Auslander-Bridger, [3]).** For any half exact functor \( F : C \to D \), \( \underline{F} \cong S_1 S^1(F) \) and \( \underline{F} \cong S^1 S_1(F) \).

In the case that \( F = \text{Hom}(X, -) \), the projective stabilization is the functor \( \underline{\text{Hom}}(X, -) \). Hence \( S_1 S^1 \text{Hom}(X, -) \cong \underline{\text{Hom}}(X, -) \). This result, together with the understanding of the satellites as adjoints allows us to prove the following result of Hilton-Rees:

**Proposition 35 (Hilton-Rees,[9]).** For any \( X, Y \in C \)

\[
\text{Nat}(\text{Ext}^1(Y, -), \text{Ext}^1(X, -)) \cong \text{Hom}(X, Y)
\]

**Proof.** Recall that \( \text{Ext}^1(A, -) \cong S^1(A, -) \). Therefore
2. A FUNCTORIAL APPROACH TO LINKAGE

\[
\text{Nat}(\text{Ext}^1(Y, \_), \text{Ext}^1(X, \_)) \cong \text{Nat}(\text{S}^1(Y, \_), \text{S}^1(X, \_)) \\
\cong \text{Nat}((Y, \_), \text{S} \text{S}^1(X, \_)) \\
\cong \text{Nat}((Y, \_), \text{Hom}(X, \_)) \\
\cong \text{Hom}(X, \_)(Y) \\
\cong \text{Hom}(X, Y) \quad \blacksquare
\]

**Corollary 36.** The functor \( HR : \mathcal{C} \to (\mathcal{C}, \text{Ab}) \) given by \( HR(X) := \text{Ext}^1(X, \_) \) is an embedding.

3. Linkage of Modules

In this section we quickly review the results of Martsinkovsky-Strooker. Horizontal linkage is classically an ideal theoretic notion. In \cite{10}, Martsinkovsky and Strooker showed that this notion can be extended to finitely generated modules over semiperfect Noetherian rings. This will be the first step towards extending linkage to finitely presented functors. In order to state the definition we quickly recall the transpose operation. For any ring \( R \) and any module \( M \in \text{mod}(R) \), apply the functor \( (\_)^* \) to any presentation of \( M \):

\[
P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0
\]

yielding exact sequence

\[
0 \longrightarrow X^* \longrightarrow (P_0)^* \longrightarrow (P_1)^* \longrightarrow \text{Tr}(M) \longrightarrow 0
\]

The module \( \text{Tr}(M) \) is called the transpose of \( M \). We now recall the definition of linkage given by Martsinkovsky and Strooker. Let \( R \) be a semiperfect Noetherian ring. A finitely generated module \( M \) is **horizontally linked** if \( M \cong \Omega \text{Tr} \Omega \text{Tr}(M) \).

One of the main ideas behind Auslander’s functorial approach is that one can study the module category by studying the coherent functors from the module category into abelian groups. The definition of linkage given by Martsinkovsky-Strooker is stated in terms of \( \Omega \) and \( \text{Tr} \) which are module theoretic operations. The main goal of this chapter is to answer the following question:

**Question 1.** Can the notion of linkage on \( \text{mod}(R) \) be extended to \( \text{fp}(\text{mod}(R), \text{Ab}) \)?

There are two important issues that must be addressed:

(1) The operations \( \Omega \) and \( \text{Tr} \) are not functorial.
(2) It is not clear what extension of linkage means.

Martsinkovksy made the fundamental observation that the first issue can be solved by passing to the stable module category. Recall that for any ring $R$, the category $\text{mod}(R)$: The objects of $\text{mod}(R), \text{Ab}$ are finitely presented $R$-modules. Given $X, Y \in \text{mod}(R)$, $\text{Hom}_{\text{mod}(R)}(X, Y) := \text{Hom}_{\text{mod}(R)}(X, Y)/P(X, Y)$ where $P(X, Y)$ is the subgroup of $\text{Hom}(X, Y)$ consisting of those morphisms that factor through projectives... It is well known that the operations $\Omega$ and $\text{Tr}$ on $\text{mod}(R)$ induce functors $\Omega$ and $\text{Tr}$ on the stable category. Moreover, $(\text{Tr}\Omega\text{Tr}, \Omega)$ form an adjoint pair of endofunctors. This means that there is a unit of adjunction

$$1_{\text{mod}(R)} \Rightarrow \Omega\text{Tr}\Omega\text{Tr}$$

This allows one to restate the definition of linkage for stable modules:

**Definition 5.** Let $R$ be a Noetherian ring. A module $M \in \text{mod}(R)$ is linked if $u_M$ is an isomorphism:

$$M \xrightarrow{u_M \cong} \Omega\text{Tr}\Omega\text{Tr}(M)$$

With this in mind we can restate the main goal of this chapter: We have the following diagram of functors:

We would like to produce a diagram of functors:
such that

1. The vertical functors $v$ are contravariant embeddings.
2. The following commutativity relations are satisfied:
   - $hv = vTr$
   - $l^k hv = v\Omega^k Tr$
3. There is a counit of adjunction $\lambda lhlh \rightarrow 1$

If such a diagram exists, we will define a functor $F$ to be linked if $c_F$ is an isomorphism. We will then further require that $M$ is linked in $\text{mod}(R)$ if and only if $v(M)$ is linked in $\text{fp}(\text{mod}(R), \text{Ab})$. We will consider such a situation a satisfactory way of extending the definition of linkage.

As a left adjoint, the functor $S_1$ is right exact. Similarly, as a right adjoint, $S_1$ is left exact. Now suppose that $F \in \text{fp}(\mathcal{C}, \text{Ab})$. Applying the right exact functor $S_1$ to the presentation

$$
(Y, _) \rightarrow (X, _) \rightarrow F \rightarrow 0
$$

produces exact sequence

$$
\text{Ext}^1(Y, _) \rightarrow \text{Ext}^1(X, _) \rightarrow S_1 F \rightarrow 0
$$

Since $\mathcal{C}$ is assumed to have enough projectives, the functors $\text{Ext}^1(Y, _)$ and $\text{Ext}^1(X, _)$ are coherent. Therefore the functor $S_1 F$ is coherent. This means that $S_1: (\mathcal{C}, \text{Ab}) \rightarrow (\mathcal{C}, \text{Ab})$ restricts to a functor $S_1: \text{fp}(\mathcal{C}, \text{Ab}) \rightarrow \text{fp}(\mathcal{C}, \text{Ab})$. Moreover,

$$
S_1\text{Ext}^1(M, _) \cong \text{Ext}^1(\Omega M, _)
$$

Combining this fact with the fact that $HR$, we start to suspect that we should choose $v = HR$ and $l = S_1$. That is, it appears that if we use the Hilton-Rees embedding, we already have an analog for $\Omega$ on the functor category; however, we are still missing the analog of the transpose.

4. Duality

Let $R$ be a ring. The category $\text{Mod}(R)$ is abelian with enough projectives. Therefore the category of coherent functors $\text{fp}(\text{Mod}(R), \text{Ab})$ has enough injectives. There is an obvious contravariant functor $Y_R: \text{fp}(\text{Mod}(R), \text{Ab}) \rightarrow \text{fp}(\text{Mod}(R^{op}), \text{Ab})$ given by $Y_R(F) := (F(R), _)$. Because $\text{fp}(\text{Mod}(R), \text{Ab})$ has enough injectives, one may calculate the left derived functors $L^n(Y_R)$. It is easily seen that for all $n \geq 1$,
L^n(Y_R) = 0. The only surviving derived functor is $L^0$. Define $D: \text{fp}(\text{Mod}(R), \text{Ab}) \to \text{fp}(\text{Mod}(R^{op}), \text{Ab})$ by

$$D := L^0 Y_R$$

**Lemma 37.** Suppose that $\mathcal{A}$ is an abelian category with enough injectives and $\mathcal{B}$ is an abelian category. In addition, assume that every object in $\mathcal{A}$ has injective dimension at most 2. Then for any left exact contravariant functor $F: \mathcal{A} \to \mathcal{B}$, the zeroth derived functor $L^0 F$ is exact.

**Proof.** Take any exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{A}$. There is a commutative diagram with exact rows and columns:

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
& & & \\
0 & 0 & 0 & 0 \\
& & & \\
0 & X & Y & Z & 0 \\
& & & \\
0 & I^0 & J^0 & K^0 & 0 \\
& & & \\
0 & I^1 & J^1 & K^1 & 0 \\
& & & \\
0 & I^2 & J^2 & K^2 & 0 \\
& & & \\
0 & 0 & 0 & 0 \\
\end{array}
$$

where $I^p, J^p, K^p$ are injectives. From this diagram, the fact that $F$ is left exact, and the fact that the rows consisting of injectives split, one easily recovers the following commutative diagram with exact rows and
Applying the snake lemma yields exact sequence

\[ 0 \to L^0F(Z) \to L^0F(Y) \to L^0F(X) \to 0 \]

completing the proof. ■

**Proposition 38.** The functor \( D := L^0Y_R \) is exact.

**Proof.** The functor \( Y_R \) is the composition of evaluation at the ring \( \text{ev}_R \) followed by the Yoneda embedding \( Y \). Since evaluation is exact and the Yoneda embedding is left exact, \( Y_R \) is left exact. The result is now immediate from the preceding lemma. ■

For any noetherian ring \( R \), \( \text{mod}(R) \) is abelian with enough projectives. Hence for any noetherian ring \( R \), the category \( \text{fp}(\text{mod}(R), \text{Ab}) \) has enough injectives. For the remainder of this chapter, we will assume that any ring \( R \) is a noetherian on both sides. Because \( R \) is Noetherian, the category \( \text{mod}(R) \) is abelian. This will result in the following:

**Lemma 39.** Suppose that \( R \) is Noetherian. There is a functor \( E := Yiw: \text{fp}(\text{mod}(R), \text{Ab}) \to \text{fp}(\text{Mod}(R), \text{Ab}) \) where

1. \( Y: \text{Mod}(R) \to \text{fp}(\text{Mod}(R), \text{Ab}) \) is the Yoneda embedding.
(2) \( i : \text{mod}(R) \to \text{Mod}(R) \) is the inclusion.

(3) \( w : \text{fp} \( \text{mod}(R), \text{Ab} \) \to \text{mod}(R) \) is the functor \( w \) as defined by Auslander.

The zeroth derived functor of \( E \) is an exact embedding with \( (L_0E)(X, - ) = (X, - ) \) for all \( X \in \text{mod}(R) \).

The functor \( D : \text{fp} \( \text{Mod}(R), \text{Ab} \) \to \text{fp} \( \text{Mod}(R^{\text{op}}), \text{Ab} \) \) restricts to a functor, also labeled \( D : \text{fp} \( \text{mod}(R), \text{Ab} \) \to \text{fp} \( \text{mod}(R^{\text{op}}), \text{Ab} \) \). In other words, there is a commutative diagram of functors whose horizontal arrows are embeddings:

\[
\begin{array}{ccc}
\text{fp} \( \text{mod}(R), \text{Ab} \) & \xrightarrow{E} & \text{fp} \( \text{Mod}(R), \text{Ab} \) \\
\downarrow{D} & & \downarrow{D} \\
\text{fp} \( \text{mod}(R^{\text{op}}), \text{Ab} \) & \xleftarrow{E} & \text{fp} \( \text{Mod}(R^{\text{op}}), \text{Ab} \)
\end{array}
\]

**Lemma 40.** For any noetherian ring \( R \), the left satellite \( S_1 : (\text{Mod}(R), \text{Ab}) \to (\text{Mod}(R), \text{Ab}) \) restricts to an endofunctor \( S_1 : \text{fp}(\text{mod}(R), \text{Ab}) \to \text{fp}(\text{mod}(R), \text{Ab}) \).

**Proof.** Since \( R \) is noetherian, for any \( F \in \text{fp} \( \text{mod}(R), \text{Ab} \) \) there is an exact sequence

\[
0 \to F \to _{-} \otimes X \to _{-} \otimes Y \to _{-} \otimes Z \to 0
\]

This sequence is also exact in \( (\text{Mod}(R), \text{Ab}) \). Since \( S_1 \) is a right adjoint, it is left exact. Therefore, applying \( S_1 \) yields the following exact sequence

\[
0 \to S_1 F \to S_1(_{-} \otimes X) \to S_1(_{-} \otimes Y).
\]

Because \( S_1(_{-} \otimes A) \cong \text{Tor}_1(_{-}, A) \), this sequence is equivalent to

\[
0 \to S_1 F \to \text{Tor}_1(_{-}, X) \to \text{Tor}_1(_{-}, Y).
\]

Since \( X, Y \in \text{mod}(R) \), \( S_1 F \in \text{fp}(\text{mod}(R), \text{Ab}) \). ■

We will now make some calculations. We begin with \( D(X, - ) \). From the presentation of \( X \):

\[
P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0
\]

there is an exact sequence

\[
0 \longrightarrow (X, - ) \longrightarrow (P_0, - ) \longrightarrow (P_1, - )
\]

Since \( P_i \) are finitely generated projective, it follows that \( (P_i, - ) \cong _{-} \otimes (P_i)^* \). Hence there is an exact sequence.
\[ 0 \longrightarrow (X, _) \longrightarrow P_0^* \otimes _- \longrightarrow P_1^* \otimes _- \longrightarrow \text{Tr}(X) \otimes _- \longrightarrow 0 \]

which is the injective resolution of \((X, _)\). Since \(D := L^0_Y R\) we have that \(D(X, _)\) is determined by the following exact sequence

\[ (P_1^*, _) \longrightarrow (P_0^*, _) \longrightarrow D(X, _) \longrightarrow 0 \]

Again, since \(P_i\) are finitely generated projective, \((P_i^*, _) \cong _- \otimes P_i\) so this sequence is equivalent to the following exact sequence

\[ _- \otimes P_1 \longrightarrow _- \otimes P_0 \longrightarrow D(X, _) \longrightarrow 0 \]

Since

\[ P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0 \]

is exact so is

\[ _- \otimes P_1 \longrightarrow _- \otimes P_0 \longrightarrow _- \otimes X \longrightarrow 0 \]

From this, it follows that

\[ D(X, _) \cong _- \otimes X \]

This calculation has an immediate application. For any \(X \in \text{mod}(R)\), the syzygy sequence

\[ 0 \rightarrow \Omega X \rightarrow P \rightarrow X \rightarrow 0 \]

yields exact sequence

\[ 0 \rightarrow (X, _) \rightarrow (P, _) \rightarrow (\Omega X, _) \rightarrow \text{Ext}^1(X, _) \rightarrow 0. \]

Applying the exact functor \(D\) yields exact sequence

\[ 0 \rightarrow D(\text{Ext}^1(X, _)) \rightarrow D(\Omega X, _) \rightarrow D(P, _) \rightarrow D(X, _) \rightarrow 0. \]

By our previous calculation this sequence is equivalent to

\[ 0 \rightarrow D(\text{Ext}^1(X, _)) \rightarrow _- \otimes \Omega X \rightarrow _- \otimes P \rightarrow _- \otimes X \rightarrow 0 \]

Since we already have the exact sequence

\[ 0 \rightarrow \text{Tor}_1(_, X) \rightarrow _- \otimes \Omega X \rightarrow _- \otimes P \rightarrow _- \otimes X \rightarrow 0 \]

it follows that \(D(\text{Ext}^1(X, _)) \cong \text{Tor}_1(_, X)\).
Given \( X \in \text{mod}(R) \), take presentation \( P_1 \to P_0 \to X \to 0 \). There is an exact sequence
\[
\_ \otimes P_1 \to \_ \otimes P_0 \to \_ \otimes X.
\]
Since \( P_i \) are finitely generated projectives, \( \_ \otimes P_i \cong (P_i^*, \_ ) \) so we have an exact sequence
\[
(P_1^*, \_ ) \to (P_0^*, \_ ) \to \_ \otimes X \to 0
\]
Applying \( D \) yields exact sequence
\[
0 \to D(\_ \otimes X) \to D(P_0^*, \_ ) \to D(P_1^*, \_ )
\]
which by our previous calculation is equivalent to exact sequence
\[
0 \to D(\_ \otimes X) \to P_0^* \otimes \_ \to P_1^* \otimes \_.
\]
Again since \( P_i \) are finitely generated projectives, \( P_i^* \otimes \_ \cong (P_i, \_ ) \). Hence there is an exact sequence
\[
0 \to D(\_ \otimes X) \to (P_0, \_ ) \to (P_1, \_ )
\]
Since \( P_1 \to P_0 \to X \to 0 \) is exact, there exists exact sequence
\[
0 \to (X, \_ ) \to (P_0, \_ ) \to (P_1, \_ )
\]
Therefore
\[
D(\_ \otimes X) \cong (X, \_ ).
\]
If \( X \in \text{mod}(R) \), from the syzygy sequence
\[
0 \to \Omega X \to P \to X \to 0
\]
there is an exact sequence
\[
0 \to \text{Tor}_1(\_, X) \to \_ \otimes \Omega X \to \_ \otimes P
\]
Applying the exact functor \( D \) yields exact sequence
\[
D(\_ \otimes P) \to D(\_ \otimes \Omega X) \to D(\text{Tor}_1(\_, X)) \to 0
\]
which from our previous work is equivalent to
\[
(P, \_ ) \to (\Omega X, \_ ) \to D(\text{Tor}_1(\_, X)) \to 0
\]
From the exact sequence

\[(P, \_ ) \rightarrow (\Omega X, \_ ) \rightarrow \text{Ext}^1(X, \_ ) \rightarrow 0\]

it follows that

\[D(\text{Tor}_1(\_, X)) \cong \text{Ext}^1(X, \_ )\]

For any coherent functor \(F\) we have exact sequence

\[(Y, \_ ) \rightarrow (X, \_ ) \rightarrow F \rightarrow 0\]

from which we get exact sequence

\[0 \rightarrow DF \rightarrow D(X, \_ ) \rightarrow D(Y, \_ )\]

or equivalently

\[0 \rightarrow DF \rightarrow \_ \otimes X \rightarrow \_ \otimes Y.\]

Applying \(D\) to this yields

\[D(\_ \otimes Y) \rightarrow D(\_ \otimes X) \rightarrow D^2F \rightarrow 0\]

which is equivalent to

\[(Y, \_ ) \rightarrow (X, \_ ) \rightarrow D^2F \rightarrow 0.\]

It is now clear that \(D^2F \cong F\). Hence \(D\) is a duality. From the exact sequence

\[(Y, \_ ) \rightarrow (X, \_ ) \rightarrow F \rightarrow 0\]

we apply the right exact functor \(S^1\) to get the exact sequence

\[S^1(Y, \_ ) \rightarrow S^1(X, \_ ) \rightarrow S^1F \rightarrow 0\]

Since \(S^1(A, \_ ) \cong \text{Ext}^1(A, \_ )\), this sequence is equivalent to

\[\text{Ext}^1(Y, \_ ) \rightarrow \text{Ext}^1(X, \_ ) \rightarrow S^1F \rightarrow 0.\]

Applying \(D\) yields exact sequence

\[0 \rightarrow DS^1F \rightarrow \text{Tor}_1(\_, X) \rightarrow \text{Tor}_1(\_, Y).\]

Applying \(D\) to the exact sequence

\[(Y, \_ ) \rightarrow (X, \_ ) \rightarrow F \rightarrow 0\]
yields exact sequence

\[ 0 \to DF \to \_ \otimes X \to \_ \otimes Y. \]

Since \( S_1 \) is a right adjoint, it is left exact. Applying \( S_1 \) to this sequence yields the exact sequence

\[ 0 \to S_1 DF \to S_1 (\_ \otimes X) \to S_1 (\_ \otimes Y). \]

Given that \( S_1 (\_ \otimes A) \cong \text{Tor}_1 (\_, A) \), this sequence is equivalent to

\[ 0 \to S_1 DF \to \text{Tor}_1 (\_, X) \to \text{Tor}_1 (\_, Y). \]

It is now clear that \( S_1 DF \cong DS^1 F \), this isomorphism being natural in \( F \). One may similarly show that \( DS_1 \cong S^1 D \). We summarize these results as follows:

**Theorem 41.** The functor \( D : \text{fp}(\text{mod}(R), \text{Ab}) \to \text{fp}(\text{mod}(R^{op}), \text{Ab}) \) is a duality satisfying the following properties:

1. \( D(X, \_) \cong \_ \otimes X \)
2. \( D(\text{Ext}^1 (X, \_)) \cong \text{Tor}_1 (\_, X) \)
3. \( D(\_ \otimes X) \cong (X, \_) \)
4. \( D(\text{Tor}_1 (\_, X)) \cong \text{Ext}^1 (X, \_ ) \)
5. \( DS^1 \cong S_1 D \)
6. \( DS_1 \cong S^1 D \)

**Theorem 42.** The duality \( D : \text{fp}(\text{mod}(R), \text{Ab}) \to \text{fp}(\text{mod}(R^{op}), \text{Ab}) \) satisfies the following properties:

1. \( DF(A) \cong \text{Nat}(F, A \otimes \_) \)
2. Given \( F \in \text{fp}(\text{mod}(R), \text{Ab}) \), take presentation \( (Y, \_) \to (X, \_) \to F \to 0 \). Then \( DF \) is completely determined by the exact sequence \( 0 \to DF \to \_ \otimes X \to \_ \otimes Y \).

As a result,

1. \( D \) is the duality first defined by Auslander in [2].
2. \( D \) is the duality defined by Hartshorne in [8].

**Proof.** For any \( F \in \text{fp}(\text{mod}(R), \text{Ab}) \) take presentation

\[ (Y, \_) \to (X, \_) \to F \to 0 \]

and apply the functor \( \text{Nat}(\_, A \otimes \_) \) to get exact sequence

\[ 0 \to (F, A \otimes \_) \to \text{Nat}((X, \_), A \otimes \_) \to \text{Nat}((Y, \_), A \otimes \_). \]
which by the Yoneda lemma is equivalent to

$$0 \to (F, A \otimes \_ ) \to A \otimes X \to A \otimes Y$$

Finally, applying the exact functor $D$ to the same presentation of $F$ yields exact sequence

$$0 \to DF \to D(X, \_ ) \to D(Y, \_ )$$
or equivalently

$$0 \to DF \to \_ \otimes X \to \_ \otimes Y.$$

Evaluating at $A$ yields exact sequence

$$0 \to DF(A) \to A \otimes X \to A \otimes Y.$$  

Hence $DF(A) \cong \text{Nat}(F, A \otimes \_ )$ as claimed. □

It should be pointed out that although $D : \text{fp}(\text{mod}(R), \text{Ab}) \to \text{fp}(\text{mod}(R^{op}), \text{Ab})$ is equivalent to Auslander’s duality, the original functor $D : \text{fp}(\text{mod}(R), \text{Ab}) \to \text{fp}(\text{mod}(R^{op}), \text{Ab})$ does not obey the formula $DF(A) = \text{Nat}(F, A \otimes \_ )$. Moreover, we discovered Auslander’s duality by showing that $\text{fp}(\text{Mod}(R), \text{Ab})$ has enough injectives, which is of interest in its own right. We now have an analog of Yoneda’s lemma:

**Lemma 43 (Yoneda Analog for Tensor Product).** If $R$ is a Noetherian ring and $F \in \text{fp}(\text{mod}(R), \text{Ab})$ and $A \in \text{mod}(R^{op})$, then

$$\text{Nat}(F, A \otimes \_ ) \cong DF(A)$$

We end this section by showing how $D$ may be used to calculate the defect $w$ of a coherent functor. Again, recall that we are assuming that $R$ is noetherian. We begin by calculating $w(\_ \otimes X)$. First, take presentation

$$P_1 \to P_0 \to X \to 0.$$  

This yields exact sequence

$$0 \to X^* \to P_0^* \to P_1^* \to \text{Tr}(X) \to 0.$$  

Embedding this into $\text{fp}(\text{mod}(R), \text{Ab})$ yields exact sequence

$$0 \to (\text{Tr}(X), \_ ) \to (P_1^*, \_ ) \to (P_0^*, \_ ) \to F \to 0.$$
Since the $P_i$ are finitely generated projective, this is equivalent to the following exact sequence

$$0 \to (\text{Tr}(X), \_ \_ ) \to \_ \_ \otimes P_1 \to \_ \_ \otimes P_0 \to F \to 0.$$ 

Since we have exact sequence

$$\_ \_ \otimes P_1 \to \_ \_ \otimes P_0 \to \_ \_ \otimes X \to 0$$

it follows that $F = \_ \_ \otimes X$ and we have exact sequence

$$0 \to (\text{Tr}(X), \_ \_ ) \to (P_1^*, \_ \_ ) \to (P_0^*, \_ \_ ) \to \_ \_ \otimes X \to 0.$$ 

By applying $w$ we get exact sequence

$$0 \to w(\_ \_ \otimes X) \to P_0^* \to P_1^* \to \text{Tr}(X) \to 0$$

from which it follows that $w(\_ \_ \otimes X) = X^*$. 

With this information, for any coherent functor $F$ take presentation

$$(Y, \_ \_ ) \to (X, \_ \_ ) \to F \to 0$$

and apply $D$ yielding exact sequence

$$0 \to D(F) \to \_ \_ \otimes X \to \_ \_ \otimes Y.$$ 

Applying $w$ yields the exact sequence

$$Y^* \to X^* \to wD(F) \to 0.$$ 

This forces $wD(F) \cong F(R)$. We may now calculate the defect of a coherent functor using $D$:

**Proposition 44.** For any $F \in \text{fp}(\text{mod}(R), \text{Ab})$, $w(F) \cong DF(R)$

**Proof.** Since $D^2 F \cong F$, we get that $w(F) \cong w(D^2(F)) \cong wD(DF)) \cong DF(R)$ as claimed.  

**Corollary 45.** For all $n \geq 0$, $w(\text{Tor}_n(\_ \_ , X)) = \text{Ext}^n(X, R)$. 

Proof. For \( n = 0 \) the statement is \( w(\_ \otimes X) = X^* \) which we have already shown. Suppose that \( n \geq 1 \). Then

\[
w(\text{Tor}_n(\_, X)) = D(\text{Tor}_n(\_, X))(R)
\]

\[
= D(S_n(\_ \otimes X))(R)
\]

\[
= S^n(D(\_ \otimes X))(R)
\]

\[
= S^n(X^*)(R)
\]

\[
= \text{Ext}^n(X, \_)(R)
\]

\[
= \text{Ext}^n(X, R) \quad \blacksquare
\]

5. Linkage of Coherent Functors

The functor \( D \) will allow us to extend the definition of linkage to the category of coherent functors in a suitable way. We begin by recalling that for any module \( M \in \text{mod}(R) \), \( S^1 \text{Ext}^1(M, \_) \cong \text{Ext}^1(\Omega M, \_) \). Moreover, \( S^1 D(\text{Ext}^1(M, \_)) \cong \text{Ext}^1(\text{Tr} M, \_) \). We have a diagram of functors:

![Diagram of functors](image)

satisfying the following properties:

1. The vertical arrows are contravariant embeddings.
2. The following commutativity relations are satisfied:
   
   (a) \( S^1 DHR = HR \text{Tr} \)
   
   (b) \( S^k S^1 D\text{Tr} = HR \Omega^k \text{Tr} \)

3. There is a counit of adjunction \( S^2 S_2 \to 1 \).
Therefore $S^1D$ is the analog of $\text{Tr}$ and $S^1$ is the analog of $\Omega$. As such, one may define a functor $F \in \text{fp} (\text{mod}(R), \text{Ab})$ to be linked if $F \cong S^1S^1DS^1S^1D(F)$. This is equivalent to saying $F$ is linked if $F \equiv S^2DS^D(F)$. Since $S^nD \cong DS_n$, this is equivalent to saying $F$ is linked if $F \equiv S^2S_2D^2(F) \cong S^2S_2(F)$. Moreover, this is equivalent to requiring the counit of adjunction of the adjoint pair $(S^2, S_2)$ to be an isomorphism at $F$. In other words, one may give the following:

**Definition 6.** A functor $F \in \text{fp} (\text{mod}(R), \text{Ab})$ is linked, to $S^2D(F) \in \text{fp} (\text{mod}(R^{op}, \text{Ab}))$, if the counit of adjunction

$$S^2S_2F \xrightarrow{\epsilon} F$$

evaluated at $F$ is an isomorphism.

As an immediate result, we see that the only functors that have a chance of being linked are the injectively stable functors. One of the first requirements we imposed was that the new definition of linkage be consistent with that given for the stable category $\text{mod}(R)$. We in fact have:

**Theorem 46.** A module $M \in \text{mod}(R)$ is linked if and only if $\text{Ext}^1(M, \_)$ is linked.

**Proof.** If $M$ is linked in $\text{mod}(R)$, then the map $M \to \Omega \text{Tr} \Omega \text{Tr}(M)$ is an isomorphism. As a result applying $\text{HR}$ yields an isomorphism $S^2S_2\text{Ext}^1(M, \_) \to \text{Ext}^1(M, \_)$.

In [10], Martsinkovsky and Strooker establish that all modules of $G$-dimension zero are horizontally linked. The notion of $G$-dimension was introduced by Auslander and Bridger in [3] and the definition is given for half exact functors. Suppose that $\mathcal{C}$ is an abelian category with enough projectives and injectives. A half exact functor $F : \mathcal{C} \to \mathcal{D}$ is said to have $G$-dimension zero if all of its satellites are both projectively and injectively stable. A module is said to have $G$-dimension zero if the functor $\text{Hom}(M, \_)$ has $G$-dimension zero as a functor. That modules of $G$-dimension zero are linked is an immediate result of the more general statement that we can now make:

**Theorem 47.** All half exact coherent functors of $G$-dimension zero are linked.
**Proof.** If $F$ has $G$-dimension zero, then both $S_1F$ and $F$ are injectively stable. Therefore

\[
S^2 S_2 F \cong S^1 S^1 S_1 (S_1 F) \\
\cong S^1 S_1 F \\
\cong S^1 (S_1 F) \\
\cong F \\
\cong F
\]
CHAPTER 3

The Asymptotic Stabilization of the Tensor Product

Throughout this chapter we will assume that \( \mathcal{C} \) is an abelian category with enough projectives.

1. Vogel Cohomology

In this section we recall the cohomological functor \( V^\bullet(A, \_ \_ ) \) defined by Pierre Vogel. Choose \( A, B \in \mathcal{C} \).

Fix projective resolutions \( (P, d) \) of \( A \) and \( (Q, d') \) of \( B \). Define \( \text{Homgr}(P, Q) \) to be the graded object whose degree \( n \) part is given by

\[
\prod_{k \in \mathbb{Z}} \text{Hom}(P_k, Q_{k+n})
\]

One can think of elements of \( \text{Homgr}(P, Q) \) as infinite sequences of slanted maps between the projective resolutions with no commutativity requirements. For example, a degree 0 element of \( \text{Homgr}(P, Q) \) is a sequence of maps

\[
\cdots \to P_4 \to P_3 \to P_2 \to P_1 \to P_0 \to 0
\]

\[
\cdots \to Q_4 \to Q_3 \to Q_2 \to Q_1 \to Q_0 \to 0
\]

with no commutativity requirements, a degree 1 element of \( \text{Homgr}(P, Q) \) is a sequence of maps

\[
\cdots \to P_4 \to P_3 \to P_2 \to P_1 \to P_0 \to 0
\]

\[
\cdots \to Q_4 \to Q_3 \to Q_2 \to Q_1 \to Q_0 \to 0
\]

with no commutativity requirements, and a degree \(-1\) element of \( \text{Homgr}(P, Q) \) is a sequence of maps

\[
\cdots \to P_4 \to P_3 \to P_2 \to P_1 \to P_0 \to 0
\]

\[
\cdots \to Q_4 \to Q_3 \to Q_2 \to Q_1 \to Q_0 \to 0
\]

with no commutativity requirements.
Define $D : \text{Hom}(P, Q) \to \text{Hom}(P, Q)$ by

$$D(f) := d' \circ f - (-1)^{\deg(f)} f \circ d$$

One may easily verify that $D^2 = 0$ which means $\text{Hom}(P, Q)$ together with the differential $D$ forms a complex. Define $\text{Hom}_b(P, Q)$ to be the graded sub-object of $\text{Hom}(P, Q)$ whose degree $n$ part is given by

$$\prod_{k \in \mathbb{Z}} \text{Hom}(P_k, Q_{k+n})$$

One can think of elements of $\text{Hom}_b(P, Q)$ as infinite sequences of slanted maps between the projective resolutions with no commutativity requirements where only finitely many of the slanted maps in the sequence are nonzero. For example, a degree 0 element of $\text{Hom}_b(P, Q)$ is a sequence of maps

$$\cdots \to P_{k+1} \to P_k \to \cdots \to P_4 \to P_3 \to P_2 \to P_1 \to P_0 \to 0$$

with no commutativity requirements and a degree 1 element of $\text{Hom}_b(P, Q)$ is a sequence of maps

$$\cdots \to Q_{k+1} \to Q_k \to \cdots \to Q_4 \to Q_3 \to Q_2 \to Q_1 \to Q_0 \to 0$$

with no commutativity requirements.

The differential $D$ on $\text{Hom}(P, Q)$ restricts to a differential on $\text{Hom}_b(P, Q)$. Therefore there is a short exact sequence of complexes:

$$0 \to \text{Hom}_b(P, Q) \to \text{Hom}(P, Q) \to \hat{\text{Hom}}(P, Q) \to 0$$

The Vogel cohomology in degree $n$ is $n$-th degree cohomology of the complex $\hat{\text{Hom}}(P, Q)$.

$$V^n(A, B) := H^n(\hat{\text{Hom}}(P, Q))$$

**Proposition 48.** Vogel cohomology is a cohomological functor.

**Proof.** Let

$$0 \to X \to Y \to Z \to 0$$
be a short exact sequence in \( \mathcal{C} \). There is an exact sequence of projective resolutions

\[
0 \to Q_X \to Q_Y \to Q_Z \to 0
\]

Applying the additive functor \( \widehat{\text{Hom}}_{\mathcal{C}}(P, \_ ) \) to this sequence yields the short exact sequence of complexes:

\[
0 \to \widehat{\text{Hom}}_{\mathcal{C}}(P, Q_X) \to \widehat{\text{Hom}}_{\mathcal{C}}(P, Q_Y) \to \widehat{\text{Hom}}_{\mathcal{C}}(P, Q_Z) \to 0
\]

Since cohomology \( (H^n)_{n \in \mathbb{Z}} \) is a cohomological functor, we have a long exact sequence

\[
\cdots \to H^n(\widehat{\text{Hom}}_{\mathcal{C}}(P, Q_X)) \to H^n(\widehat{\text{Hom}}_{\mathcal{C}}(P, Q_Y)) \to H^n(\widehat{\text{Hom}}_{\mathcal{C}}(P, Q_Z)) \to H^{n+1}(\widehat{\text{Hom}}_{\mathcal{C}}(P, Q_X)) \to H^{n+1}(\widehat{\text{Hom}}_{\mathcal{C}}(P, Q_Y)) \to H^{n+1}(\widehat{\text{Hom}}_{\mathcal{C}}(P, Q_Z)) \to \cdots
\]

which by definition is the sequence

\[
\cdots \to V^n(A, X) \to V^n(A, Y) \to V^n(A, Z) \to V^{n+1}(A, X) \to V^{n+1}(A, Y) \to V^{n+1}(A, Z) \to \cdots
\]

establishing that \( V^n(A, \_ ) \) is cohomological. ■

2. Vogel Cohomology via Inversion of the Endofunctor \( \Omega \)

Due to Buchweitz’s general construction of inverting an endofunctor, one may invert \( \Omega: \mathcal{C} \to \mathcal{C} \) which yields a new category \( \mathcal{C}[\Omega^{-1}] \). The objects of \( \mathcal{C}[\Omega^{-1}] \) are the same as the objects of \( \mathcal{C} \) which are the same objects as \( \mathcal{C} \). For each \( A, B \in \mathcal{C}[\Omega^{-1}] \),

\[
\text{Hom}_{\mathcal{C}[\Omega^{-1}]}(A, B) = \lim_{\to} \text{Hom}(\Omega^k A, \Omega^k B)
\]

For each \( n \in \mathbb{Z} \) define \( B^n(A, \_ ): \mathcal{C} \to \text{Ab} \):

\[
B^n(A, \_ )(B) := \lim_{\to n+k, k \geq 0} \text{Hom}(\Omega^{k+n} A, \Omega^k B)
\]

Notice that

\[
B^0(A, B) = \text{Hom}_{\mathcal{C}[\Omega^{-1}]}(A, B)
\]

\[
B^n(A, B) = \text{Hom}_{\mathcal{C}[\Omega^{-1}]}(\Omega^n A, B) \quad \text{for all } n \geq 1
\]

\[
B^n(A, B) = \text{Hom}_{\mathcal{C}[\Omega^{-1}]}(A, \Omega^n B) \quad \text{for all } n \leq -1
\]
**Proposition 49.** As defined $\mathbb{B}^\bullet(A, _{-}) \colon \mathcal{C} \to \text{Ab}$ is a cohomological functor.

**Proof.** Let $A \in \mathcal{C}$ and $0 \to X \to Y \to Z \to 0$ be a short exact sequence. We begin by constructing a map from $\mathbb{B}^0(A, Z) \to \mathbb{B}^1(A, X)$. Any element of $\mathbb{B}^0(A, Z)$ is represented by a map $f : \Omega^k A \to \Omega^k Z$. The diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^{k+1} A & \longrightarrow & P_k & \longrightarrow & \Omega^k A & \longrightarrow & 0 \\
& & \downarrow f & & \downarrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^k X & \longrightarrow & \Omega^k Y & \longrightarrow & \Omega^k Z & \longrightarrow & 0 \\
\end{array}
$$

embeds into a commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^{k+1} A & \longrightarrow & P_k & \longrightarrow & \Omega^k A & \longrightarrow & 0 \\
& & \downarrow g & & \downarrow & & \uparrow f & & \\
0 & \longrightarrow & \Omega^k X & \longrightarrow & \Omega^k Y & \longrightarrow & \Omega^k Z & \longrightarrow & 0 \\
\end{array}
$$

which yields a map $g : \Omega^{k+1} A \to \Omega^k X$. This represents an element of $\mathbb{B}^1(A, X)$. ■

**Proposition 50.** The cohomological functor $\mathbb{B}^\bullet(A, _{-})$ is isomorphic to Vogel cohomology.

**Proof.** We construct the isomorphism in degree 0. Fix projective resolutions:

$$
\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0
$$

$$
\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow B \longrightarrow 0
$$

Let $x \in V^0(A, B)$. Then $x$ is an element of $H^0(\widehat{\text{Hom}}(A, B)) = \text{Ker}(\hat{D}_0) / \text{Im}(\hat{D}_{-1})$. This means that $x$ is represented by an infinite sequence $(f_k)_{k \geq 0}$ of maps where $f_k : P_k \to Q_k$ such that there exists some $n$ such that $(f_k)_{k \geq n}$ is a chain map between the complex $(P_k)_{k \geq n}$ and $(Q_k)_{k \geq n}$. Take the smallest such $n$ such that this is true. Then $f_n : P_n \to Q_n$ can be lifted to a map $\tilde{f}_n : \Omega^{n+1} A \to \Omega^{n+1} B$. The map $\tilde{f}_n$ then determines uniquely an element in $\lim \rightarrow \text{Hom}(\Omega^k A, \Omega^k B)$. Define $\phi(x)$ to be this element. So we start with a diagram

$$
\begin{array}{cccccc}
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
& & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \\
\cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & B & \longrightarrow & 0 \\
\end{array}
$$

where only finitely many squares do not commute and for some $n$ we have a commutative diagram.
3. Vogel Cohomology via Satellites

In this section, we review the $P$-completion of a cohomological functor as defined by Mislin. A cohomological functor $T$ from $C$ to $D$ is called $P$-complete if for every $n \in \mathbb{Z}$ the functor $T^n : C \to D$ is projectively stable, that is $T^n$ vanishes on projectives. Denote by $\text{Coh}(C, D)$ the category of cohomological functors from $C$ to $D$ and by $\text{PCoh}(C, D)$ the full subcategory of $\text{Coh}(C, D)$ consisting of $P$-complete functors. In [11], Mislin gives an explicit way of constructing the left adjoint $M : \text{Coh}(C, D) \to \text{PCoh}(C, D)$ to the inclusion functor $i : \text{PCoh}(C, D) \to \text{Coh}(C, D)$. In order to explain this construction, we shall need the following:

**Definition 7.** Suppose that $T$ is a connected sequence of functors from $C$ to $D$. For every $k \in \mathbb{Z}$ we have a connected sequence of functors $\tau(k)T$ from $C$ to $D$ given by

$$(\tau(k)T)^n = \begin{cases} T^n & \text{if } n \geq k \\ 0 & \text{otherwise} \end{cases}$$

We also have a connected sequence of functors $S(k)T$ from $C$ to $D$ given by

$$(S(k)T)^n = \begin{cases} T^n & \text{if } n \geq k \\ S_{k-n}T^k & \text{otherwise} \end{cases}$$

While these definitions are precise, one may visualize these two sequences. Let $T$ be a connected sequence of functors. Then we have the following connected sequences of functors for each $k \in \mathbb{Z}$:
\[ T : \ldots T^{k-2} T^{k-1} T^k T^{k+1} T^{k+2} \ldots \]

\[ \tau_{(k)} T : \ldots 0 0 T^k T^{k+1} T^{k+2} \ldots \]

\[ S_{(k)} T : \ldots S_2 T^k S_1 T^k T^k T^{k+1} T^{k+2} \ldots \]

**Proposition 5.1.** For any \( k \in \mathbb{Z} \), \((\tau_{(k)}, S_{(k)})\) form an adjoint pair of endofunctors on the category of connected sequences of functors. As a result, given any two connected sequences of functors \( T, U \) there is a natural bijection

\[ \text{Hom}_\Delta(\tau_{(k)} T, U) \cong \text{Hom}_\Delta(T, S_{(k)} U) \]

**Proof.** This will follow from repeated application of the adjunction \((\pi_2, \Delta S_1)\). Let us explicitly construct the isomorphism. Let \( T, U \) denote two connected sequences of functors. Suppose that \( \alpha : \tau_{(k)} T \to U \). We have the following picture:

\[ \tau_{(k)} T : \ldots 0 0 T^k T^{k+1} T^{k+2} \ldots \]

\[ U : \ldots U^{k-2} U^{k-1} U^k U^{k+1} U^{k+2} \ldots \]

Since \( T \) is by hypothesis a connected sequence of functors, \((T^{k-1}, \delta, T^k)\) is a connected triple. Then \( \alpha^k \in \text{Nat}(\pi_2(T^{k-1}, \delta, T^k), U^k)\) By the adjunction \( \text{Nat}(\pi_2(T^{k-1}, \delta, T^k), U^k) \cong \text{Hom}_\Delta((T^{k-1}, \delta, T^k), \Delta S_1 U^k) \) there exists \( \beta^1 : T^{k-1} \to S_1 U^k \) such that \((\beta^1, \alpha^k) : (T^{k-1}, \delta, T^k) \to (S_1 U^k, \nabla, U^k)\) is a morphism of connected triples. Hence we have the following picture:

\[ T : \ldots T^{k-2} T^{k-1} T^k T^{k+1} T^{k+2} \ldots \]

\[ S_{(k)} U : \ldots S_2 U^k S_1 U^k U^k U^{k+1} U^{k+2} \ldots \]

By iteration of this process we obtain a morphism \( \beta : T \longrightarrow S_{(k)} U \):

\[ T : \ldots T^{k-2} T^{k-1} T^k T^{k+1} T^{k+2} \ldots \]

\[ S_{(k)} U : \ldots S_2 U^k S_1 U^k U^k U^{k+1} U^{k+2} \ldots \]

It is easily seen that the assignment \( \alpha \mapsto \beta \) is an isomorphism natural in both \( T \) and \( U \). □
For any $k$, it is easily seen that $S_{(k+1)}^{(k+1)} T = S_{(k+1)} T$. Let $T$ be a cohomological functor and let $k \in \mathbb{Z}$. There is a natural map $\eta(k) : S_{(k)} T \to S_{(k+1)} T$ obtained as follows: By the adjunction formula, we have

\[ \text{Hom}_\Delta(\tau_{(k+1)} S_{(k)} T, \tau_{(k+1)} S_{(k)} T) \cong \text{Hom}_\Delta(S_{(k)} T, S_{(k+1)} \tau_{(k+1)} S_{(k)} T) = \text{Hom}_\Delta(S_{(k)} T, S_{(k+1)} T) \]

The image of the identity under this isomorphism is taken as $\eta(k)$. This yields a directed system:

\[ \cdots \xrightarrow{\eta(k-1)} S_{(k)} T \xrightarrow{\eta(k)} S_{(k+1)} T \xrightarrow{\eta(k+1)} S_{(k+2)} T \xrightarrow{\eta(k+2)} \cdots \]

The $P$-completion $M(T)$ is defined to be the colimit of this directed system:

\[ M(T) = \lim_{\rightarrow} S_{(k)} T \]

It should be pointed out that although Mislin defines the $P$-completion for cohomological functors, the same construction may be applied to arbitrary connected sequences of functors. We end this section by recalling the following results of Mislin:

**Proposition 52 (Mislin, [11]).** For a finite group $G$, $M(\text{Ext}(G, \_))$ is isomorphic as a connected sequence of functors to the Tate-cohomology of $G$.

This result is easily generalized to an arbitrary $R$-module $A$:

**Theorem 53 (Mislin, [11]).** Let $R$ be a ring and $A$ an $R$-module. The $P$-completion of $\text{Ext}(A, \_)$ is isomorphic as a connected sequence of functors to the Vogel cohomology of $A$. That is

\[ M(\text{Ext}(A, \_)) \cong \text{V}^\bullet(A, \_) \]

**Proof.** We will construct an isomorphism $\phi : M(\text{Ext}(A, \_)) \to \text{B}(A, \_)$. Since $\text{B}(A, \_) \cong \text{V}(A, \_)$, this will establish the theorem. The key to doing this is to understand the maps in the directed systems of each construction. It will suffice to show that in degree zero, the two directed systems are in fact isomorphic.

Let $B \in \text{Mod}(R)$. By definition $B^0(A, B)$ is the limit of the following directed system:

\[ \text{Hom}(B, A) \xrightarrow{\Omega} \text{Hom}(\Omega A, \Omega B) \xrightarrow{\Omega} \text{Hom}(\Omega^2 A, \Omega^2 B) \xrightarrow{\Omega} \text{Hom}(\Omega^3 A, \Omega^3 B) \xrightarrow{\Omega} \cdots \]

Now we will look at the maps in Mislin’s directed system. To make this easier, we will use the following connected sequence of functors for $\text{Ext}(A, \_)$:

\[ \cdots \xrightarrow{0} (A, \_) \xrightarrow{\text{Ext}^1(A, \_)} \text{Ext}^1(\Omega A, \_) \xrightarrow{\text{Ext}^1(\Omega^2 A, \_)} \cdots \]

We have the following picture to aid us in understanding Mislin’s construction:
If we can show that $\beta$ is the operation $\Omega$, then it will be clear that the two directed systems are identical.

In order to calculate $\beta$, we begin with $\alpha$. In order to do this, recall that every element of $\text{Ext}^1(A, B)$ can be represented by a map $f : \Omega A \to B$. Now take any syzygy sequence of $B$:

$$
\begin{array}{cccc}
0 & \longrightarrow & \Omega B & \longrightarrow & P & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

We have the following commutative diagram with exact rows:

$$
\begin{array}{cccc}
\text{Ext}^1(A, P) & \longrightarrow & \text{Ext}^1(A, B) & \overset{\gamma}{\longrightarrow} & \text{Ext}^1(\Omega A, \Omega B) & \longrightarrow & \text{Ext}^1(\Omega A, P)
\end{array}
$$

which embeds uniquely into the following commutative diagram with exact rows:

$$
\begin{array}{cccc}
\text{Ext}^1(A, P) & \longrightarrow & \text{Ext}^1(A, B) & \overset{\gamma}{\longrightarrow} & \text{Ext}^1(\Omega A, \Omega B) & \longrightarrow & \text{Ext}^1(\Omega A, P)
\end{array}
$$

The connecting homomorphism $\gamma$ works as follows. Given $x \in \text{Ext}^1(A, B)$, $x$ is represented by some map $f : \Omega A \to B$. Therefore the diagram with exact rows:

$$
\begin{array}{cccc}
0 & \longrightarrow & \Omega^2 A & \longrightarrow & P_1 & \longrightarrow & \Omega A & \longrightarrow & 0
\end{array}
$$

embeds into the commutative diagram with exact rows:

$$
\begin{array}{cccc}
0 & \longrightarrow & \Omega^2 A & \longrightarrow & P_1 & \longrightarrow & \Omega A & \longrightarrow & 0
\end{array}
$$

The connecting homomorphism $\gamma$ sends $x$ represented by $f$ to the element $\gamma(x)$ represented by $\Omega f$. Now since $\alpha_B$ is nothing more than the restriction of $\gamma$ to its image, we now understand that $\alpha_B(x)$ sends $x$ which
is represented by \( f: \Omega A \to B \) to the \( \alpha(x) \in \text{Hom}(\Omega A, B) \) represented by \( \Omega f \). Hence we may think of \( \alpha \) as \( \Omega \). Now to calculate \( \beta \). Again from the syzygy sequence

\[
0 \longrightarrow \Omega B \longrightarrow P \longrightarrow B \longrightarrow 0
\]

we have the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}(A, B) & \xrightarrow{\nabla'} & \text{Ext}^1(A, \Omega B) & \longrightarrow & \text{Ext}^1(A, P) \\
& & \downarrow{\alpha_{\Omega B}} & & \downarrow{\alpha_P} & & \\
0 & \longrightarrow & \text{Hom}(\Omega A, \Omega B) & \xrightarrow{\nabla} & \text{Hom}(\Omega A, \Omega B) & \longrightarrow & 0
\end{array}
\]

which embeds uniquely into the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}(A, B) & \xrightarrow{\nabla'} & \text{Ext}^1(A, \Omega B) & \longrightarrow & \text{Ext}^1(A, P) \\
& & \downarrow{\beta_B} & & \downarrow{\alpha_{\Omega B}} & & \downarrow{\alpha_P} \\
0 & \longrightarrow & \text{Hom}(\Omega A, \Omega B) & \xrightarrow{\nabla} & \text{Hom}(\Omega A, \Omega B) & \longrightarrow & 0
\end{array}
\]

Since \( \nabla \) is the identity, \( \nabla' \) is the inclusion, and \( \alpha \) is the operation \( \Omega \), it is clear that \( \beta_B \) is also \( \Omega \). Therefore the two directed systems are identical and hence the cohomological functors are isomorphic. ■

4. Vogel Homology

At the same time that Pierre Vogel defined the cohomological functor \( V^\bullet(A, \_\_) \), he also defined its counterpart \( V_\bullet(A, \_\_) \). Fix a ring \( R \), a right \( R \)-module \( A \), and a left \( R \)-module \( B \). Take a projective resolution \((P, d)\) of \( A \) and injective resolution \((I, d')\) of \( B \). Define \( P \hat{\otimes} I \) to be the graded abelian group whose degree \( n \) part is given by

\[
\prod_{k \in \mathbb{Z}} P_{k+n} \otimes I^k.
\]

We can visualize \( P \hat{\otimes} I \) as follows: A degree 1 element \((s_k)_{k \in \mathbb{Z}}\) of \( P \hat{\otimes} I \) is a “line”: 
where \( s_1 \in P_1 \otimes I^0 \), \( s_2 \in P_2 \otimes I^1 \), and \( s_k \in P_k \otimes I^{k-1} \).

Define \( D : \hat{P} \otimes I \rightarrow \hat{P} \otimes I \) on elementary tensor \( a \otimes b \) as follows:

\[
D(a \otimes b) = d(a) \otimes b + (-1)^{\deg(a)} a \otimes d'(b)
\]

This extends uniquely to a map \( D : \hat{P} \otimes I \rightarrow \hat{P} \otimes I \). One easily verifies that \( D^2 = 0 \). Hence \( \hat{P} \otimes I \) together with \( D \) forms a complex.

Recall that \( P \otimes I \) is the graded abelian subgroup of \( \hat{P} \otimes I \) whose degree \( n \) part is

\[
\bigoplus_{k \in \mathbb{Z}} P_{k+n} \otimes I^k
\]

The differential \( D : \hat{P} \otimes I \rightarrow \hat{P} \otimes I \) restricts to a differential on \( P \otimes I \) and there is a short exact sequence of complexes:

\[
0 \longrightarrow P \otimes I \longrightarrow \hat{P} \otimes I \longrightarrow P \oplus I \longrightarrow 0
\]

The **Vogel homology** in degree \( n \) is \( n + 1 \)-th degree homology of the complex \( P \oplus I \).

\[
V_n(A, B) := H_{n+1}(P \oplus I)
\]

**Proposition 54.** The sequence of functors \( V_\bullet(A, -) \) is a connected sequence of functors. That is, Vogel homology forms a connected sequence of functors.
Proof. Suppose that $0 \to X \to Y \to Z$ is a short exact sequence. Then there exists a short exact sequence of injective resolutions

$$0 \to I^X \to I^Y \to I^Z \to 0$$

Applying the functor $P_A \otimes -$ to this sequence yields the exact sequence of complexes

$$0 \to P_A \otimes I^X \to P_A \otimes I^Y \to P_A \otimes I^Z \to 0$$

This results in a long exact sequence of homology

$$\cdots \to H_n(P_A \otimes I^X) \to H_n(P_A \otimes I^Y) \to H_n(P_A \otimes I^Z) \to H_{n-1}(P_A \otimes I^X) \to \cdots$$

which is by definition the exact sequence

$$\cdots \to V_{n-1}(A, -)(X) \to V_{n-1}(A, -)(Y) \to V_{n-1}(A, -)(Z) \to V_{n-2}(A, -)(X) \cdots$$

The remaining details are left to the reader. ■

5. Vogel Homology via the Satellites

In [3], Auslander and Bridger construct for each half exact functor $F : C \to D$ two directed systems:

$$F \xrightarrow{p_0} S_1 S^1 F \xrightarrow{p_1} S_2 S^2 F \xrightarrow{p_2} \cdots$$

$$\cdots \xrightarrow{p_2} S_2 S^2 F \xrightarrow{q_1} S^1 S_1 F \xrightarrow{q_0} F$$

In [13], Yoshinos defines the Tate-Vogel completions $F^\vee : C \to D$ and $F^\wedge : C \to D$ by looking at the asymptotic behavior of these two directed systems. More precisely,

$$F^\vee := \lim_{\to} \left( F \xrightarrow{p_0} S_1 S^1 F \xrightarrow{p_1} S_2 S^2 F \xrightarrow{p_2} \cdots \right)$$

$$F^\wedge := \lim_{\leftarrow} \left( \cdots \xrightarrow{q_2} S_2 S^2 F \xrightarrow{q_1} S^1 S_1 F \xrightarrow{q_0} F \right)$$

PROPOSITION 55 (Yoshino,[13]). Let $R$ be a ring. Then for any $k \in \mathbb{Z}$ and any right $R$-module $A$, there is a natural map

$$\Theta_k : V_k(A, -) \to \text{Tor}_k(A, -)^\wedge$$

Moreover if the following conditions are satisfied:

1. $R$ is noetherian on the right.
(2) \( A \) is a finitely generated.

(3) There exists \( n \in \mathbb{Z} \) such that for any \( j > n \), \( \text{Ext}^j(A, R) = 0 \).

then for all \( k \geq 0 \), \( \Theta_k \) is an isomorphism:

\[
\text{Tor}_k(A, -)^\wedge \cong V_k(A, -)
\]

Yoshino is able to recover in all positive degrees the Vogel homology under these restrictions. The main issue with Yoshino’s approach is that it only produces Vogel homology in positive degrees. This is easily solved by focusing on a different category than that considered by Yoshino. Yoshino’s method is to assign to each functor \( F: \mathcal{C} \to \mathcal{D} \) two new functors \( F^\vee, F^\wedge: \mathcal{C} \to \mathcal{D} \). By looking at Mislin’s approach to constructing Tate cohomology, it becomes clear that the main point missing is that of a connected sequence of functors.

**Definition 8.** Suppose that \( T \) is a connected sequence of functors from \( \mathcal{C} \) to \( \mathcal{D} \). For every \( k \in \mathbb{Z} \) we have a connected sequence of functors \( \tau^{(k)}T \) from \( \mathcal{C} \) to \( \mathcal{D} \) given by

\[
(\tau^{(k)}T)^n = \begin{cases} 
T^n & \text{if } n \leq k \\
0 & \text{otherwise}
\end{cases}
\]

We also have a connected sequence of functors \( S^{(k)}T \) from \( \mathcal{C} \) to \( \mathcal{D} \) given by

\[
(S^{(k)}T)^n = \begin{cases} 
T^n & \text{if } n \leq k \\
S^{n-k}T^k & \text{otherwise}
\end{cases}
\]

**Proposition 56.** For any \( k \in \mathbb{Z} \), \((S^{(k)}, \tau^{(k)})\) form an adjoint pair of endofunctors on the category of connected sequences of functors. As a result, given any two connected sequences of functors \( T, U \) there is a natural bijection

\[
\text{Hom}_\Delta(S^{(k)}T, U) \cong \text{Hom}_\Delta(T, \tau^{(k)}U)
\]

For any \( k \), it is easily seen that \( S^{(k)}\tau^{(k)}S^{(k+1)} = S^{(k)} \). Let \( T \) be a cohomological functor and let \( k \in \mathbb{Z} \). There is a natural map \( \eta^{(k)}: S^{(k)}T \to S^{(k+1)}T \) obtained as follows: By the adjunction formula, we have

\[
\text{Hom}_\Delta(\tau^{(k)}S^{(k+1)}T, \tau^{(k)}S^{(k+1)}T) \cong \text{Hom}_\Delta(S^{(k)}\tau^{(k)}S^{(k+1)}T, S^{(k+1)}T) \cong \text{Hom}_\Delta(S^{(k)}T, S^{(k+1)}T)
\]

The image of the identity under this isomorphism is taken as \( \eta^{(k)} \). This yields a directed system:

\[
\cdots \xrightarrow{\eta^{(k-1)}} S^{(k)}T \xrightarrow{\eta^{(k)}} S^{(k+1)}T \xrightarrow{\eta^{(k+1)}} S^{(k+2)}T \xrightarrow{\eta^{(k+2)}} \cdots
\]
The $I$-completion $W(T)$ is defined to be the limit of this directed system:

$$W(T) = \lim_{\leftarrow} S^{(k)}T$$

We have informally been referring to this construction as the **Mirror Mislin**. This is because we have taken Mislin’s approach to recovering Tate cohomology and used it as a blueprint to find an approach that produces Vogel homology under certain conditions:

**Proposition 57.** Let $R$ be a ring and $A$ be a right $R$-module satisfying the following conditions:

1. $R$ is noetherian on the right.
2. $A$ is a finitely generated.
3. There exists $n \in \mathbb{Z}$ such that for any $j > n$, $\text{Ext}^j(A, R) = 0$.

Then for any $k \in \mathbb{Z}$ and any right $R$-module $A$,

$$W(\text{Tor}(A, \_)) \cong V_k(A, \_)$$

Notice that under the same conditions imposed by Yoshino we recover all degrees of Vogel homology.

We shall see that unlike Yoshino, we are able to claim the following:

**Proposition 58.** Let $R$ be any ring. For any right $R$-module $A$ there is a surjection $\rho: V(A, \_) \to W(\text{Tor}(A, \_)).$

Denote by $\text{ICon}(A, B)$ the full subcategory of $\text{Con}_\Delta(A, B)$ consisting of all connected sequences of functors $T$ such that every term of $T$ is injectively stable. There is an inclusion $i: \text{ICon}(A, B) \to \text{Con}_\Delta(A, B).$ The functor $W: \text{Con}_\Delta(A, B) \to \text{ICon}(A, B)$ was obtained by mirroring Mislin’s construction of the functor $M: \text{Coh}(A, B) \to \text{PCoh}(A, B).$ In fact, though Mislin constructs for each cohomological functor $T$ a cohomological $M(T)$ such that every term of $M(T)$ is projectively stable, his construction works equally well for any connected sequence of functors $T$. In other words, Mislin’s construction can actually be seen as producing a functor $M: \text{Con}_\Delta(A, B) \to \text{PCoh}(A, B)$ that restricts to a left adjoint of the inclusion $\text{PCoh}(A, B) \to \text{Coh}(A, B).$ We highlight this as a

**Theorem 59 (Mislin, [11]).** There is a functor $M: \text{Con}_\Delta(A, B) \to \text{PCoh}(A, B).$ The functor $M$ takes cohomological functors to cohomological functors and restricts to the left adjoint of the inclusion $\text{PCoh}(A, B) \to \text{Coh}(A, B).$
Theorem 60 (Martsinkovsky-Russell). There is a functor \( W: \text{Con}_\Delta(A, B) \to \text{ICon}(A, B) \) obtained by dualizing Mislin’s approach. The functor \( W \) does not necessarily take cohomological functors to cohomological functors.

6. The Asymptotic Stabilization of the Tensor Product

Completely missing from the picture is a construction of Vogel homology using a method similar to that of Buchweitz’s construction of Vogel cohomology. We will introduce a new object called the asymptotic stabilization of the tensor product that will fill this gap. To begin with we define for every \( A \in \text{Mod}(R^{op}) \) and every \( B \in \text{Mod}(R) \) the abelian group \( A \boxtimes B \). Take an exact sequence

\[
0 \to B \to I
\]

where \( I \) is injective. Then \( A \boxtimes B \) is defined as the kernel of the map \( A \otimes B \to A \otimes I \) and hence determined by the exact sequence

\[
0 \to A \boxtimes B \to A \otimes B \to A \otimes I
\]

Proposition 61. The assignment \( B \mapsto A \boxtimes B \) determines a functor \( A \boxtimes -: \text{Mod}(R) \to \text{Ab} \) which up to isomorphism does not depend on the choice of the monomorphism \( B \to I \) or the injective \( I \).

Proof. We begin by constructing for each \( f: B \to C \) the morphism \( A \boxtimes f: A \boxtimes B \to A \boxtimes C \). The diagram

\[
\begin{array}{ccccccc}
0 & \to & B & \overset{i}{\to} & I \\
& & \uparrow{f} & & \downarrow{g} \\
0 & \to & C & \overset{j}{\to} & J
\end{array}
\]

embeds into the following commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \to & B & \overset{i}{\to} & I \\
& & \uparrow{f} & & \downarrow{g} \\
0 & \to & C & \overset{j}{\to} & J
\end{array}
\]

which yields the following commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \to & A \boxtimes B & \overset{k}{\to} & A \otimes B & \overset{A \otimes i}{\to} & A \otimes I \\
& & \downarrow{A \otimes f} & & \downarrow{A \otimes g} \\
0 & \to & A \boxtimes C & \overset{l}{\to} & A \otimes C & \overset{A \otimes j}{\to} & A \otimes J
\end{array}
\]

which embeds uniquely into the following commutative diagram with exact rows:
Since \( l, k, f \) do not depend on \( g \), neither does \( A \otimes f \) as it is the unique map such that \( lA \otimes f = fk \). One easily verifies that this assignment makes \( A \otimes \_ \) into a functor.

To show that up to isomorphism \( A \otimes \_ \) is independent of the choice of \( I \), we take the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & A \otimes B & \longrightarrow & A \otimes I \\
& & A \otimes f & \downarrow & \downarrow A \otimes g \\
0 & \longrightarrow & A \otimes C & \longrightarrow & A \otimes J
\end{array}
\]

which embeds uniquely into the following diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & A \otimes I & \longrightarrow & A \otimes I \\
& & A \otimes f & \downarrow A \otimes g \\
0 & \longrightarrow & A \otimes J & \longrightarrow & A \otimes J
\end{array}
\]

which embeds uniquely into the following diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & A \otimes I & \longrightarrow & A \otimes I \\
s & \downarrow & 1 & & A \otimes g \\
0 & \longrightarrow & A \otimes J & \longrightarrow & A \otimes J
\end{array}
\]

Therefore \( kr s = k \). Since \( k \) is a monomorphism, \( rs = 1 \) making \( r \) a retraction. Since \( kr = j \) is a monomorphism, \( r \) is also a monomorphism. Therefore \( r \) is an isomorphism and hence \( s \) is an isomorphism. It is easily seen that these isomorphisms are unique and natural. It follows that the functors \( A \otimes I \_ \cong A \otimes J \_ \). This
establishes that $A \otimes -$ up to isomorphism does not depend of the chosen injective $I$ and monomorphism $B \to I$.

Notice that if $P$ is any projective, then the functor $P \otimes -$ is exact. Therefore $P \otimes -$ = 0. Hence the functor Mod$(R^{op}) \to (\text{Mod}(R), \text{Ab})$ given by $A \mapsto A \otimes -$ vanishes on projectives and hence factors through $\text{Mod}(R^{op})$. As a result, one may speak of $\Omega A \otimes B$ without worrying about the choice of the syzygy module $\Omega A$. It is also clear from the definition that $A \otimes -$ is injectively stable. Therefore, one may also speak of $A \otimes \Sigma B$ without worrying about the choice of the cosyzygy module $\Sigma B$.

**Proposition 62.** As a functor $A \otimes - \cong S^1 \text{Tor}_1(A, -)$.

**Proof.** From the short exact sequence

$$0 \longrightarrow B \longrightarrow I \longrightarrow \Sigma B \longrightarrow 0$$

one has the exact sequence

$$\text{Tor}_1(A, \Sigma B) \longrightarrow \text{Tor}_1(A, I) \longrightarrow \text{Tor}_1(A, \Sigma B) \longrightarrow A \otimes B \longrightarrow A \otimes I \longrightarrow A \otimes \Sigma B \longrightarrow 0$$

from which it is clear that $V \cong S^1 \text{Tor}_1(A, B)$ and $V \cong A \otimes B$.

Fix a ring $R$, $A \in \text{Mod}(R^{op})$ and $B \in \text{Mod}(R)$. For every module $X \in \text{Mod}(R)$, fix an injective $I(X)$ such that $0 \to X \to I(X)$ is exact. In this way we have provided for every $X$ an injective resolution $I^X$. In particular we have an injective resolution of $B$:

$$0 \to B \to I^0 \to I^1 \to I^2 \to \cdots$$

Similarly we have a projective resolution of $A$:

$$\cdots \to P_1 \to P_0 \to A \to 0$$
We have the following commutative diagram with exact rows, columns, and diagonals:

![Diagram](image)

Applying the Snake Lemma yields the following diagram whose bottom row is exact:

![Diagram](image)

Because the bottom row is exact, this diagram embeds into the following:

![Diagram](image)

which produces a morphism

\[ \Omega A \boxtimes \Sigma B \overset{\Delta_1}{\longrightarrow} A \boxtimes B \]

Iteration of this processes yields a directed system

\[ \cdots \rightarrow \Omega^2 A \boxtimes \Sigma^2 B \overset{\Delta_2}{\longrightarrow} \Omega A \boxtimes \Sigma B \overset{\Delta_1}{\longrightarrow} A \boxtimes B \]
and more generally, this process may be repeated for any integer \( n \in \mathbb{Z} \) yielding directed systems

\[
M_n(A, B) := \Omega^{k+n} A \boxtimes \Sigma^k B \quad k, k + n \geq 0
\]

**Definition 9.** The asymptotic stabilization of the tensor product in degree \( n \), denoted \( T_n(A, B) \)
is defined as follows:

\[
T_n(A, B) := \lim_{\leftarrow \substack{k, k + n \geq 0 \ \Omega \kern-2.5pt k}} \Omega^{k+n} A \boxtimes \Sigma^k B = \lim_{\leftarrow \substack{k, k + n \geq 0 \ \Omega \kern-2.5pt k}} M_n(A, B)
\]

This yields a sequence of functors \( T_\bullet(A, \_) : \text{Mod}(R) \to \text{Ab} \) where

\[
T_n(A, \_)(B) := T_n(A, B)
\]

For each \( n \in \mathbb{Z} \), it is easily seen that \( T_n(A, \_) \) is injectively stable. We will show that \( T_\bullet(A, \_) \) is actually a connected sequence of functors. We begin by understanding the dimension shift.

**Proposition 63.** For every \( n \in \mathbb{Z} \) and every \( k \in \mathbb{Z}_{\geq 0} \),

\[
T_n(A, \Sigma^k \_) = T_{n-k}(A, \_).
\]

**Proof.** The case \( k = 0 \) is clear. The case \( k = 1 \) follows from the easily verified fact that for any \( B \in \text{Mod}(R) \), the directed systems \( M_n(A, \Sigma^1 B) \) and \( M_{n-1}(A, B) \) are equal.

Suppose that for every \( n \in \mathbb{Z} \),

\[
T_n(A, \Sigma^k \_) = T_{n-k}(A, \_)
\]

Then

\[
\begin{align*}
T_n(A, \Sigma^{k+1} \_) & = T_n(A, \Sigma^k \Sigma \_) \\
& = T_{n-k}(A, \Sigma \_) \\
& = T_{n-k-1}(A, \_) \\
& = T_{n-(k+1)}(A, \_)
\end{align*}
\]

as needed. \( \blacksquare \)

**Proposition 64.** For every \( n \in \mathbb{Z} \) and every \( k \in \mathbb{Z}_{\geq 0} \),

\[
V_n(A, \Sigma^k \_) \cong V_{n-k}(A, \_)
\]
Proposition 65. For every $n \in \mathbb{Z}$,

$$S^1T_n(A, \_ \_ ) \cong T_{n-1}(A, \_ \_ ).$$

Proof. Since $T_n(A, \_ \_ )$ is injectively stable,

$$S^1T_n(A, \_ \_ ) \cong T_n(A, \Sigma \_ \_ ) \cong T_{n-1}(A, \_ \_ )$$

Corollary 66. For every $m \geq 0$

$$S^mT_n(A, \_ \_ ) = T_{n-m}(A, \_ \_ )$$

Corollary 67. The asymptotic stabilization of the tensor product $T_\bullet(A, \_ \_ )$ is a connected sequence of functors.

Proof. For any integer $n \in \mathbb{Z}$, $T_n(A, \_ \_ )$ and $S^1T_n(A, \_ \_ )$ are connected (this is true for any functor $F$ and $S^1F$). Because $T_n(A, \_ \_ )$ is injectively stable, $S^1T_n(A, \_ \_ ) \cong T_n(A, \Sigma \_ \_ ) \cong T_{n-1}(A, \_ \_ )$. Hence the entire sequence $T_\bullet(A, \_ \_ )$ is connected.

Theorem 68. For every $n \in \mathbb{Z}$ there is a natural transformation

$$k_n : V_\bullet(A, \_ \_ ) \rightarrow T(A, \_ \_ )$$

Proof. The map will be constructed in degree 0. All other degrees are similar. An element of $V_0(A, B)$ can be represented by an infinite sequence

$$(s_i)_{i=1}^\infty \in (P_1 \otimes I^0) \times (P_2 \otimes I^1) \times \cdots$$

which vanishes under the differential of $V_\bullet(A, B)$. This means that the element

$$D(s_i) = (d_P(s_1), -d_I(s_1) + d_P(s_2), d_I(s_2) + d_P(s_3), -d_I(s_3) + d_P(s_4), \ldots)$$

is zero in $V^1(A, B)$. Hence only finitely many components of $D(s_i)$ are nonzero. Let $k$ denote the smallest index such that

$$d_I(s_k) = d_P(s_{k+1})$$
\[
d_I(s_{k+1}) = -d_P(s_{k+2}) \\
d_I(s_{k+2}) = d_P(s_{k+3}) \\
d_I(s_{k+3}) = -d_P(s_{k+4}) \\
\vdots
\]

that is for all \( n \geq 0 \)
\[
d_I(s_{k+n}) = (-1)^n d_P(s_{k+n+1})
\]

Observe that since \( s_{k+1} \in P_{k+1} \otimes I^k \), \( d_P(s_{k+1}) \in P_k \otimes I^k \). Denote \( d_P(s_{k+1}) \) by \( \bullet \) in the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
\Omega^{k+1} A \otimes \Sigma^k B & \longrightarrow & \Omega^{k+1} A \otimes I^k \\
\downarrow & & \downarrow \\
0 & \longrightarrow & P_k \otimes \Sigma^k B \\
\downarrow & & \downarrow \\
\Omega^k A \otimes \Sigma^k B & \longrightarrow & \Omega^k A \otimes I^k \\
\downarrow & & \downarrow \\
0 & \longrightarrow & P_k \otimes \Sigma^k B
\end{array}
\]

By a simple diagram chase, pull \( \bullet \) back to \( \Box \) and push \( \Box \) down to \( \Delta \), one produces \( \Delta \in \Omega^k A \otimes \Sigma^k B \). Again by diagram chase the element \( \Delta \) is in the kernel of the horizontal map \( \Omega^k A \otimes \Sigma^k B \to \Omega^k A \otimes I^k \). Since this kernel is by definition \( \Omega^k A \otimes \Sigma^k B \), we have produced an element \( \varphi_k \in \Omega^k A \otimes \Sigma^k B \).

Now perform the same process for \( d_P(s_{k+2}) = -d_I(s_{k+1}) \) letting \( d_P(s_{k+2}) \) be represented by \( \bullet \) in the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
\Omega^{k+2} A \otimes \Sigma^{k+1} B & \longrightarrow & \Omega^{k+2} A \otimes I^{k+1} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & P_{k+1} \otimes \Sigma^{k+1} B \\
\downarrow & & \downarrow \\
\Omega^{k+1} A \otimes \Sigma^{k+1} B & \longrightarrow & \Omega^{k+1} A \otimes I^{k+1} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & P_{k+1} \otimes \Sigma^{k+1} B
\end{array}
\]
By a simple diagram chase, pull $\bullet$ back to $\square$ and push $\square$ down to $\Delta$, one produces $\Delta \in \Omega^{k+1}A \otimes \Sigma^{k+1}B$. Again by diagram chase the element $\Delta$ is in the kernel of the horizontal map $\Omega^{k+1}A \otimes \Sigma^{k+1}B \to \Omega^{k+1}A \otimes I^{k+1}$. Since this kernel is by definition $\Omega^{k+1}A \otimes \Sigma^{k+1}B$, we have produced an element $\varphi_{k+1} \in \Omega^{k+1}A \otimes \Sigma^{k+1}B$. To obtain $\varphi_{k+2}$ perform the same process with $-d_P(s_{k+3})$. To obtain $\varphi_{k+3}$ perform the same process with $-d_P(s_{k+4})$. Now iteration of this process yields $\varphi_n$ for any $n \geq k$. One can easily verify that $(\varphi_k, \varphi_{k+1}, \ldots)$ is coherent. This sequence can be extended to a unique coherent sequence $(\varphi_i)_{i=0}$. ■

**Theorem 69.** For every $n \in \mathbb{Z}$, the natural transformation $k_n$ is an epimorphism.

**Proof.** The proof is primarily diagram chase and only a sketch will be given. Let $(\varphi_0, \varphi_1, \varphi_2, \ldots)$ be a coherent sequence in the asymptotic stabilization of the tensor product. Then $\varphi_k \in \Omega^k A \otimes \Sigma^k B$. We will construct an element in $V_0(A, B)$ which maps onto this coherent sequence. One will benefit from the following diagram:

\[
\begin{array}{ccccccccc}
P_2 \otimes I^0 & \longrightarrow & P_2 \otimes I^1 & \longrightarrow & P_2 \otimes \Sigma^2 B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \ \\\n\Omega^2 A \otimes \Sigma^1 B & \longrightarrow & \Omega^2 A \otimes I^1 & \longrightarrow & \Omega^2 A \otimes \Sigma^2 B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \ \\\n0 & \longrightarrow & P_1 \otimes B & \longrightarrow & P_1 \otimes I^0 & \longrightarrow & P_1 \otimes \Sigma^1 B & \longrightarrow & P_1 \otimes I^1 & \longrightarrow & P_1 \otimes \Sigma^2 B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \ \\\n\Omega^1 A \otimes B & \longrightarrow & \Omega^1 A \otimes I^0 & \longrightarrow & \Omega^1 A \otimes \Sigma^1 B & \longrightarrow & \Omega^1 A \otimes I^1 & \longrightarrow & \Omega^1 A \otimes \Sigma^2 B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \ \\\n0 & \longrightarrow & P_0 \otimes B & \longrightarrow & P_0 \otimes I^0 & \longrightarrow & P_0 \otimes \Sigma^1 B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \ \\\nA \otimes B & \longrightarrow & A \otimes I^0 & \longrightarrow & A \otimes \Sigma B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \ \\\n0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

Start by selecting $s_1 \in P_1 \otimes I^0$ that maps onto $\varphi_1$. Then $d_P(s_1)$ will pullback to $\varphi_0$. Now select $t_2 \in P_2 \otimes I^1$ that maps onto $\varphi_2$. By diagram chase we get that there exists $y_2 \in P_2 \otimes I^0$ such that
3. The Asymptotic Stabilization of the Tensor Product

d_I(s_1) - d_P(t_2) = d_P(d_I(y)) which yields d_I(s_1) = d_P(t_2 - d_I(y)). Define s_2 := t_2 - d_I(y_2). Then s_2 still maps onto \( \varphi_2 \) and \( d_P(s_2) \) pulls back to \( \varphi_1 \).

Now select \( t_3 \in P_3 \otimes I^2 \) that maps onto \( -\varphi_3 \). Then \( -d_P(t_3) \) pulls back to \( \varphi_2 \) as does \( d_I(s_2) \). By diagram chasing, there exists \( y_3 \in P_3 \otimes I^1 \) such that \( d_I(s_2) + d_P(t_3) = d_P(d_I(y_3)) \). Define \( s_3 := t_3 - d_I(y_3) \). Then \( s_3 \) maps onto \( -\varphi_3 \) so \( -d_P(s_3) \) pulls back to \( \varphi_2 \). Moreover \( d_I(s_2) = -d_P(s_3) \).

If we continue this process paying close attention to signs, we can construct an element \( (s_k)_{k=1}^\infty \in V_0(A, B) \) that maps onto the coherent sequence \( (\varphi_k)_{k=1}^\infty \). The details are left to the reader. ■

**Proposition 70.** For all \( n \in \mathbb{Z} \) and any short exact sequence \( 0 \to X \to I(X) \to \Sigma X \to 0 \) the following square is commutative:

\[
\begin{array}{ccc}
V_n(A, \_)(\Sigma X) & \xrightarrow{=} & V_{n-1}(A, \_)(X) \\
\downarrow k_{n, \Sigma X} & & \downarrow k_{n-1, X} \\
T_n(A, \_)(\Sigma X) & \xrightarrow{=} & T_{n-1}(A, \_)(X)
\end{array}
\]

**Lemma 71.** Suppose we have a cube of morphisms in any category:

\[
\begin{array}{c}
A \\
\downarrow \downarrow \\
C \\
\downarrow \downarrow \\
X \\
\downarrow \downarrow \\
Z
\end{array} \quad \begin{array}{c}
B \\
\downarrow \downarrow \\
D \\
\downarrow \downarrow \\
Y \\
\downarrow \downarrow \\
U
\end{array}
\]

If \( m \) is a monomorphism and every face other than the back commutes, then the back face commutes.

**Theorem 72.** The sequence \( k_\bullet = (k_n)_{n \in \mathbb{Z}} : V_n(A, \_ ) \to T_n(A, \_) \) is a map of connected sequences of functors \( k : V_\bullet(A, \_ ) \to T_\bullet(A, \_ ) \).

**Proof.** Suppose that we have an exact sequence \( 0 \to X \to Y \to Z \). Then we have a commutative diagram with exact rows:
7. The \((u,c)\)-Grid

Proposition 73 (Yoshino, [13]). For any \(k \in \mathbb{Z}\), there is a natural map

\[
\Psi_k : \text{Ext}^k(A, -) \rightarrow V^k(A, -)
\]

For all \(k \geq 0\), the map \(\Psi_k\) is an isomorphism

\[
\text{Ext}^k(A, -) \cong V^k(A, -)
\]

Recall that if \(F\) is half exact then the satellite sequence \(SF\) is cohomological. The \(P\)-completion \(M(SF)\) is a connected sequence of functors. The 0th term of \(M(SF)\) is obtained by taking the colimit of the following directed system:

\[
F \xrightarrow{\eta} S_1SF \xrightarrow{\eta} S_2SF \rightarrow \cdots
\]
where the subscripts of $\eta$ have been suppressed. For the same half exact functor $F$, Yoshino’s Tate-Vogel completion $F^\vee$ is obtained by taking the colimit of the following directed system:

$$F \xrightarrow{p_0} S_1 S^1 F \xrightarrow{p_1} S_2 S^2 F \xrightarrow{p_2} \cdots$$

The connection between these two directed systems is not discussed in either Mislin’s or Yoshino’s paper. By passing to the colimit of each directed system, we obtain Vogel cohomology in degree 0 but we would like to understand any connection between the actual directed systems. We will in fact do more. The approaches of Mislin and Yoshino involve creating directed systems from a half exact functor through techniques involving the satellites. We will actually unify the two approaches and in doing so completely explain the connection between the techniques and the resulting directed systems. In order to do this, we will study an object which we call the $(u,c)$-grid.

Fix a category $\mathcal{A}$, endofunctors $L, R: \mathcal{A} \to \mathcal{A}$ and natural transformations $u: 1_{\mathcal{A}} \to RL$, $c: LR \to 1_{\mathcal{A}}$. In order to construct the $(u,c)$-grid associated to the quintuple $(\mathcal{A}, L, R, u, c)$, we will adopt the following convention:

1. $R^0 = L^0 = 1_{\mathcal{A}}$.
2. For any $n \in \mathbb{N}^+$, $L^{-n} := R^n$.
3. $L^n L^m = L^{n+m}$ if and only if $nm \geq 0$

Formally the $(u,c)$-grid is the collection of endofunctors

$$\{L^{n+m} L^{-m} : \mathcal{A} \to \mathcal{A}\}_{(n,m) \in \mathbb{Z} \times \mathbb{Z}}$$

together with the collection of natural transformations

$$\{\varphi_{(n,m)} : L^{n+m} L^{-m} \to L^{n+m-1} L^{-m+1}\}_{(n,m) \in \mathbb{Z} \times \mathbb{Z}}$$

defined by the following rules:

1. $\varphi_{(n,m)}$ is the identity if it is an endomorphism.
2. If $\varphi_{(n,m)}$ is not an endomorphism, then
   
   a) $\varphi_{(n,m)} = L^{n+m} c L^{-m}$ if $m > 0$
   
   b) $\varphi_{(n,m)} = L^{n+m} u L^{-m}$ if $m \leq 0$

The grid may be represented graphically as follows:
Recalling the convention that $L^{-n} = R^n$, this is the same as the following:

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</table>

Notice that the columns of the $(u, c)$-grid form directed systems: Fix a $(u, c)$-grid $D$. For any $n \in \mathbb{Z}$, define the directed system in degree $n$ as follows:

$$D_n := \left\{ \varphi(n,k) : L^{n+k}L^{-k} \rightarrow L^{n+k-1}L^{-k+1} \right\}_{k \in \mathbb{Z}}$$

That is $D_n$ is the directed system obtained by focusing on the $n$-th column in the $(u, c)$-grid. The $(u, c)$-completions are the two sequences of endofunctors $\xrightarrow{n} D$ and $\xleftarrow{n} D$ obtained by considering the asymptotic behavior of the $(u, c)$-grid. They are defined as follows:

$$\xrightarrow{n} D := \lim_{n \to -} D_n$$

$$\xleftarrow{n} D := \lim_{n \to -} D_n$$

The $u$-completion of $X$ is the sequence $\left( \xrightarrow{n} D (X) \right)_{n \in \mathbb{Z}}$ and the $c$-completion of $X$ is the sequence $\left( \xleftarrow{n} D (X) \right)_{n \in \mathbb{Z}}$. 
The \((u, c)\)-completions may not exist. If \(A\) has colimits, then the \(u\)-completion exists. If \(A\) has limits, then the \(c\)-completion exists. There is an obvious situation in which the \((u, c)\)-grid may be constructed: Suppose \((L, R)\) is an adjoint pair of endofunctors on a category \(C\). The \((u, c)\)-grid may be constructed taking \(u\) as the unit of adjunction and \(c\) as the counit of adjunction. If \((L, R)\) form an adjoint pair of endofunctors on a category \(A\), the \((u, c)\)-grid obtained by taking the unit and counit of adjunction is called the \textbf{adjoint grid}. In this case the \(u\)-completion is referred to as the \textbf{unit completion} and the \(c\)-completion is referred to as the \textbf{counit completion}. Again, these may not exist. Suppose that \(C\) is an abelian category with enough projectives and injectives and \(D\) is an abelian category. The satellites \(S_1, S^1 : (C, D) \to (C, D)\) form an adjoint pair \((S_1, S^1)\). The adjoint grid in this case is called the \textbf{satellite grid} and it can be visualized graphically:

\[
\begin{array}{cccccc}
S_2 & S^1 S_2 & S^2 S_2 & S^3 S_2 & S^4 S_2 \\
\downarrow & c S_1 & s^1 c S_1 & s^2 c S_1 & s^3 c S_1 \\
S_2 & S_1 & S^1 S_1 & S^2 S_1 & S^3 S_1 \\
\downarrow & 1 & c & s^1 c & s^2 c \\
S_2 & 1 & 1 & 1 & 1 \\
\downarrow & s^1 u & s^1 u & s^1 u & s^1 u \\
S_3 & S_2 S^1 & S_1 S^1 & S^1 & S^2 \\
\downarrow & s^1 u s^1 & s^1 u s^1 & s^1 u s^1 & 1 \\
S_4 & S_3 S^2 & S_2 S^2 & S_1 S^2 & S^2 \\
\end{array}
\]

If \(D\) is closed under colimits and limits then the unit completion and counit completion exist for the satellite grid. We will call the unit completion in this case the \textbf{left satellite completion} and the counit completion will be called the \textbf{right satellite completion}.

**Proposition 74.** The directed system obtained by evaluating the satellite grid at the functor \(\text{Hom}(A, -)\) is isomorphic to both the directed systems:

\[
F \xrightarrow{\eta} S_1 F \xrightarrow{\eta} S_2 F \xrightarrow{\eta} \cdots
\]

\[
F \xrightarrow{p_1} S_1 F \xrightarrow{p_1} S_2 F \xrightarrow{p_3} \cdots
\]

of Mislin and Yoshino.

**Theorem 75.** Vogel cohomology \(V^\bullet(A, -)\) is isomorphic as a connected sequence of functors to the left satellite completion of \(\text{Hom}(A, -)\).
THEOREM 76. The asymptotic stabilization of the tensor product $T(A, \_)$ is isomorphic as a connected sequence of functors to the right satellite completion of $A \otimes \_$. 
Bibliography


Notation

Categories

$\text{Ab}$ - the category of abelian groups.

$\text{Set}$ - the category of sets.

$(\mathcal{A}, \mathcal{B})$ - category of additive covariant functors $F : \mathcal{A} \to \mathcal{B}$.

$\text{fp}(\mathcal{C}, \text{Ab})$ - category of coherent or finitely presented functor $F : \mathcal{C} \to \text{Ab}$.

$\text{Mod}(R)$ - category of left $R$-modules.

$\text{mod}(R)$ - category of finitely presented left $R$-modules.

$\mathcal{C}$ - category modulo projectives.

$\text{mod}(R)$ - stable module category.

Functors

$S^1$ - right satellite.

$S_1$ - left satellite.

$(X, \_)$ - hom functor $\text{Hom}_A(X, \_)$.

Connected Sequences of Functors

$V^\bullet(X, \_)$ - Vogel cohomology

$V_\bullet(X, \_)$ - Vogel homology

$B^\bullet(X, \_)$ - Buchweitz Inversion of $\Omega$

$T_\bullet(X, \_)$ - Asymptotic Stabilization of the Tensor Product

$M$ - Mislin's left adjoint to the inclusion $\text{PCoh}(\mathcal{A}, \mathcal{B}) \to \text{Coh}(\mathcal{A}, \mathcal{B})$

$W$ - Mirror Mislin