Constructions of $k$-orbit Abstract Polytopes

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ABSTRACT OF DISSERTATION

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Abstract

The desire to derive new polytopes from old polytopes dates back to the classical study of polytopes, as many of the Archimedean solids can be obtained from Platonic solids through the act of truncation. In this dissertation, we apply these ideas to the setting of abstract polytopes. We present a number of constructions of abstract polytopes, which will generally share some properties with the polytopes from which they were derived. Most notably, we are interested in the circumstances under which the automorphism group of the derived polytope is isomorphic to the automorphism group of the original polytope. We construct polytopes called $k$-bubbles which generalize truncated polytopes, and which will generally have $k$ flag-orbits and retain the automorphism group of the polytopes from which they are derived. Additionally, among the polytopes that we will construct will be examples of two-orbit polytopes, as well as semiregular polytopes, which we can construct given a preassigned automorphism group. We will also construct polytopes with $k$ flag-orbits, for arbitrary $k$, with a preassigned automorphism group. Finally, we focus on $k$-orbit polytopes whose automorphism groups are certain quotients of Coxeter groups called string C-groups. We will prove that there almost always exists a $k$-orbit polytope whose automorphism group is a given string C-group. In particular, we will prove that in every odd rank there is exactly one counterexample.
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Introduction

This thesis will focus on the theory of abstract polytopes. An abstract polytope can be thought of as a combinatorial generalization of the face lattice of a convex polytope, though the category is considerably more general. In addition to granting us more generality, the study of abstract polytopes allows us to focus on the combinatorics of polytopes, specifically the interaction between faces of a polytope, without relying on the ambient geometry. The shift towards a more combinatorial approach to the theory of polytopes is in part inspired by Grünbaum in [14]. Grünbaum’s ideas were more formalized and much of the early groundwork was developed in [13] and [30, 31].

Until recently, much of the research that has been done on abstract polytopes has focused on regular polytopes (those with a single flag-orbit under the automorphism group) (see [23]). As the field grew, the focus began to shift to a specific class of polytopes with two flag-orbits, called chiral polytopes [32]. Chiral polytopes share many properties with regular polytopes. However, while examples of regular polytopes are abundant ([22, 28]), until recently very few examples of chiral polytopes were known, especially in high ranks, where it was not even known for some time if such polytopes existed. We now have examples of chiral polytopes in every rank [29], and many more examples in low ranks [16]. In particular, we have ways of constructing new chiral polytopes from existing chiral polytopes [1, 11] and from interesting classes of groups [4]. The study of chiral polytopes has led to a greater interest in other polytopes with two flag-orbits (see [17, 19, 20]). Expanding on this, there has also been
some recent work done on polytopes with \( k \) flag-orbits for arbitrary \( k \) (see [12, 27]). In this thesis we will construct polytopes with \( k \) flag-orbits for arbitrary \( k \). We will also devote special attention to the case \( k = 2 \), as it is an important field of study right now.

In this thesis, we will study certain polytopes which can be derived from regular polytopes, but are not themselves regular. Historically, there have been many operations which have been applied to regular geometric polytopes in order to obtain a polytope which is not regular, such as truncations, medials, and snubs (see [7]). Some of these operations have also been examined from the combinatorial point of view of abstract polytopes. These operations often result in polytopes which retain the automorphism group of the polytope from which they were derived. However, the derived polytope will often have more faces, and therefore more flag-orbits than the original polytope. In most cases, the number of flag-orbits of the transformed polytope is determined by the construction. In this thesis, we will be looking at constructions of polytopes which allow us more freedom in choosing the number of flag-orbits than any of the previous constructions. In particular, we will be studying constructions which allow us to build polytopes with \( k \) flag-orbits for \( k \geq 1 \), in such a way that the constructed polytopes will have automorphism groups which are isomorphic to the groups of the polytopes from which they are derived. Specifically, if these polytopes are derived from regular polytopes, we will be looking at polytopes which are not themselves regular, but whose automorphism groups satisfy the necessary properties needed to be the automorphism group of a regular polytope.

This thesis will be divided as follows. In the first chapter, we discuss much of the background theory and provide definitions which will be needed in the remainder of this thesis. In the second chapter, we define our first construction, which we call the \( k \)-bubble, and explore many of the properties of \( k \)-bubbles. We begin by introducing \( k \)-bubbles as a generalization of truncations. However, the construction is considerably more general. These \( k \)-bubbles will be vertex-transitive and have at most two types of facets. Thus, the
set of facets will have at most two orbits under the action of the automorphism group. However, these polytopes will, in general, have more than two orbits on the set of flags. We will examine the automorphism groups of $k$-bubbles, specifically focusing on the conditions under which a $k$-bubble retains the automorphism group of the polytope from which it was derived. We will also focus on the number of flag-orbits a $k$-bubble has.

In the third chapter, we narrow our focus to a specific case of $k$-bubbles. The definition of $k$-bubbles given in Chapter 2 is general enough to also encompass a class of polytopes which has come up a number of times in the recent literature, specifically in the study of two-orbit polytopes and semiregular polytopes, which are closely related growing fields. The goal of this chapter is to demonstrate how $k$-bubbles fit into those theories.

There is a class of groups, called string C-groups, which are quotients of Coxeter groups that generally satisfy certain non-Coxeter type relations. One of the fundamental results in the study of abstract polytopes is that given a string C-group $\Gamma$, one can construct an abstract regular polytope which has automorphism group $\Gamma$. Additionally, the automorphism groups of all abstract regular polytopes are string C-groups. We note that taking the $k$-bubble of a regular polytope will, under certain circumstances, allow us to construct other polytopes whose automorphism groups are string C-groups, and that, to a limited extent, we can choose the number of flag-orbits. In Chapters 4 and 5, we try to extend this result. From this point of view, the biggest drawback of a $k$-bubble is that the number of flag-orbits is bounded by the rank of the polytope. In Chapter 4, we give another construction which allows us to build polytopes with arbitrarily many flag-orbits. We will look at the circumstances under which this construction retains the automorphism group of the original polytope, as well as when we have the freedom to construct such a polytope with arbitrarily many flag-orbits. The polytopes which we will construct in Chapter 4 have the added benefit of being semiregular [6, 8].

Finally, in Chapter 5 we formally turn our attention to the following question: Given an
abstract regular polytope \( P \) with automorphism group \( \Gamma \), and some integer \( k \geq 1 \), is there always a \( k \)-orbit polytope whose group of automorphisms is isomorphic to \( \Gamma \)? If not, under what circumstances does such a polytope exist? Alternatively, this can be phrased as: Given a string C-group \( \Gamma \) and an integer \( k \geq 1 \), is there a \( k \)-orbit polytope whose automorphism group is isomorphic to \( \Gamma \)?

We will answer the first question in the negative by providing a string C-group \( \Gamma \) and a choice of \( k \) for which no \( k \)-orbit polytope exists whose automorphism group is isomorphic to \( \Gamma \). However, we will show that there almost always does exist such a polytope. In fact, we will show that if \( P \) is a polytope of rank \( n \geq 3 \), with \( n \) odd, then the provided counterexample is the only possible counterexample.
Chapter 1

Background

1.1 Abstract polytopes

In this section we will define abstract polytopes and give some of their basic properties. Throughout this thesis, we will refer to them simply as polytopes. While the study of polytopes is traditionally a geometric field, abstract polytopes will be defined combinatorially, as posets which satisfy certain conditions. These conditions are required so that abstract polytopes resemble the face lattices of convex polytopes. While we will be working with these objects combinatorially, the required conditions ensure that they make sense geometrically. Often we will think of abstract polytopes as geometric objects via their geometric realizations. However, we are only concerned with the combinatorics of the polytopes. That is to say, we are only concerned with the relationships between faces of the polytopes. For example, in this theory, any two triangles are equivalent because their set of faces with a partial order given by inclusion are isomorphic posets. Therefore, we do not concern ourselves with properties of geometric triangles such as size, angles or location. Before giving the formal definition, we will discuss some of the properties which these posets should have in order to behave like geometric polytopes. Most of the definitions and results in this section can be
found in [23].

Let $\mathcal{P}$ be a poset, with a partial order denoted by $\leq$, or sometimes $\leq_{\mathcal{P}}$. We will call the elements of $\mathcal{P}$ faces. Let $F$ and $G$ be faces of $\mathcal{P}$. If $F \leq G$ or $G \leq F$, then we call $F$ and $G$ incident. Intuitively, we think of two faces being incident if one is contained in the other. In order to call $\mathcal{P}$ a polytope, the first property that it should have is:

(P1) $\mathcal{P}$ has a unique minimal face and a unique maximal face.

Intuitively, we think of the minimal face as the empty set, so that it is contained in every other face of $\mathcal{P}$, and the maximal face as being the polytope itself, so that it contains every other face of $\mathcal{P}$. Rather than defining an abstract polytope in general, we will define an abstract polytope of rank $n$ (or an $n$-polytope). In this case, the minimal face of $\mathcal{P}$ will be denoted $F_{-1}$, and the maximal face of $\mathcal{P}$ will be denoted $F_n$. These faces are called improper faces of $\mathcal{P}$, and all other faces are called proper faces.

Maximal chains (by inclusion) are called flags. The set of flags of $\mathcal{P}$ is denoted $\mathcal{F}(\mathcal{P})$. In order to call $\mathcal{P}$ an abstract polytope of rank $n$, the second property that we will require of $\mathcal{P}$ is:

(P2) Every flag of $\mathcal{P}$ contains exactly $n + 2$ faces.

These two conditions make $\mathcal{P}$ a ranked poset. Faces of rank $i$ are called $i$-faces, for $i = -1, \ldots, n$. We give names to the faces of large and small ranks, since they are referred to commonly. The $(n - 1)$-faces of $\mathcal{P}$ are called facets, the $(n - 2)$-faces of $\mathcal{P}$ are called ridges, the $(n - 3)$-faces of $\mathcal{P}$ are called subridges, the 0-faces of $\mathcal{P}$ are called vertices, and the 1-faces of $\mathcal{P}$ are called edges.

The condition (P2) essentially means that you cannot “skip” a rank in a flag. Each flag must contain exactly one face of each rank. As an example of the geometric implications of this requirement, this would exclude situations such as the object depicted in Figure 1.1 where there is a vertex (point) which lies in a 2-face, however there is no intermediate edge.
which is incident to both the vertex and the 2-face.

Figure 1.1

If $\Phi$ is a flag of a poset $\mathcal{P}$ which satisfies (P1) and (P2), then $\mathcal{P}$ contains a unique $i$-face for $i = -1, \ldots, n$. We will denote this unique $i$-face $(\Phi)_i$.

Next, we require a connectedness condition. There are several equivalent conditions which can be used. Before presenting any of them, we will begin with a few definitions:

**Definition 1.1** If $F \leq G$, then define the section $G/F$ to be $\{H : F \leq H \leq G\}$ (with the inherited partial order). Then $F$ and $G$ are called improper faces of the section, and all other faces are called proper faces. If $G$ is an $i$-face of $\mathcal{P}$, and $F$ is a $j$-face of $\mathcal{P}$, then we say that the section $G/F$ is a section of rank $i - j - 1$, or an $(i - j - 1)$-section. A flag of a section is a maximal chain of the section.

We associate the section $F/F_{-1}$ with the face $F$ itself. This is consistent with our claim that we can think of the maximal face $F_n$ as the polytope itself, since the polytope $\mathcal{P}$ is exactly the section $F_n/F_{-1}$. Additionally, we call the section $F_n/F$ the co-face of $F$. Let $V$ be a vertex of $\mathcal{P}$. Then we call the section $F_n/V$ the vertex figure of $\mathcal{P}$ at $V$. If every vertex figure of $\mathcal{P}$ is isomorphic, then we can simply call this section the vertex figure of $\mathcal{P}$.

**Definition 1.2** Let $\mathcal{P}$ be a ranked poset. A section of $\mathcal{P}$ (including $\mathcal{P}$ itself) is called connected if it is either a section of rank less than $2$, or it is a section of rank greater than or equal to $2$, and given any two proper faces $F$ and $G$ of the section there is a sequence of proper faces

$$F = F_0, F_1, \ldots, F_k = G$$
such that $F_i$ is incident to $F_{i+1}$ for $i = 0, \ldots, k-1$. Moreover, $\mathcal{P}$ is called strongly connected if every section of $\mathcal{P}$ is connected.

To call $\mathcal{P}$ a polytope, we require:

(P3) $\mathcal{P}$ is strongly connected.

We will often use Hasse diagrams to describe abstract polytopes. Because we require $\mathcal{P}$ to be a ranked poset, we will define Hasse diagrams in the form they will take for ranked posets, rather than defining the Hasse diagram of a general poset.

**Definition 1.3** Given a ranked poset $\mathcal{P}$, we define the Hasse diagram of $\mathcal{P}$ to be a graph with a node for every face of $\mathcal{P}$ and an edge connecting two nodes if they are incident and of adjacent ranks.

The following lemma is immediate:

**Lemma 1.4** A ranked poset $\mathcal{Q}$, of rank at least 2, is connected if and only if the improper faces can be removed from the Hasse diagram of $\mathcal{Q}$, and the result is a connected graph.

Notice that this lemma can be applied to the Hasse diagram of $\mathcal{P}$ itself, or to any section of $\mathcal{P}$ of rank at least 2. Therefore, verifying the strong connectedness of $\mathcal{P}$ is equivalent to verifying the strong connectedness of the Hasse diagram of $\mathcal{P}$.

Two flags are called adjacent if they differ in exactly one face. Two flags are called $i$-adjacent if they differ in the face of rank $i$.

**Definition 1.5** Let $\mathcal{P}$ be a ranked poset. A section of $\mathcal{P}$ (including $\mathcal{P}$ itself) is called flag-connected, if it is either a section of rank less than 2, or it is a section of rank greater than or equal to 2, and given any two flags $\Phi$ and $\Psi$ of the section, there is a sequence of flags of the section from $\Phi$ to $\Psi$ such that successive flags in the sequence are adjacent. Moreover, $\mathcal{P}$ is called strongly flag-connected if every section of $\mathcal{P}$ is flag-connected.
The proof of the following proposition can be found on page 24 of [23].

**Proposition 1.6 [23, Prop. 2A1]** Let \( \mathcal{P} \) be a poset with properties (P1) and (P2). Then \( \mathcal{P} \) is strongly connected if and only if it is strongly flag-connected.

Therefore, condition (P3) is equivalent to the following:

\( (P3') \) \( \mathcal{P} \) is strongly flag-connected.

The following lemma is immediate, and gives us another equivalent condition to check.

**Lemma 1.7** Let \( \mathcal{P} \) be a poset with properties (P1) and (P2). Then \( \mathcal{P} \) is strongly flag-connected if and only if given any two flags \( \Phi \) and \( \Psi \) of \( \mathcal{P} \), there is a sequence of flags of \( \mathcal{P} \) from \( \Phi \) to \( \Psi \) such that successive flags in the sequence are adjacent, and \( \Phi \cap \Psi \) is contained in every flag of the sequence.

The condition \( (P3') \) excludes cases such as the shape given in Figure 1.2, which gives a connected poset of rank \( n = 2 \) which is not strongly connected. If the central node in Figure 1.2 is denoted \( a \), then there is no sequence of proper faces of \( F_n/a \) from a proper face of this section in the right triangle to a proper face of this section in the left triangle. This is also an important condition needed to guarantee that every section of a polytope is itself a polytope.

![Figure 1.2](image)

The final condition that we will require in order to call \( \mathcal{P} \) an abstract polytope of rank \( n \) is the following:
(P4) For $i = 0, \ldots, n - 1$, if $F$ is a face of $\mathcal{P}$ of rank $i - 1$ and $G$ is a face of $\mathcal{P}$ of rank $i + 1$, which is incident to $F$, then the section $G/F$ contains exactly two faces of $\mathcal{P}$ of rank $i$.

We call condition (P4) the diamond condition, because it requires that the Hasse diagram of every 1-section have the (diamond) form given in Figure 1.3.

![Figure 1.3](image)

This is an important condition which is certainly true of all convex polytopes. It encompasses many conditions such as the following: In any polytope, each edge contains exactly two vertices; in a polygon (a 2-polytope), each vertex is incident to exactly two edges; in a polyhedron (a 3-polytope), each edge is incident to exactly two 2-faces; in a 4-polytope, each 2-face is incident to exactly two 3-faces. In general, each ridge of a polytope $\mathcal{P}$ is contained in exactly two facets. Since intuitively, we think of the rank of a polytope as its dimension, it would not make sense geometrically to have more than two $i$-faces in any such 1-section.

![Figure 1.4](image)

Additionally, this excludes cases such as the simplicial complex given in Figure 1.4. In it, there are edges which are only contained in a single 2-face. This exclusion is based on the assumption that we are considering the simplicial complex to be a poset of rank three. If we consider it only a poset of rank two, then it violates the condition that there is a single maximal element. This distinction is important to note because it is not the case that the
diamond condition excludes a single triangle from being a polytope, even though the edges of the triangle are each only incident to a single 2-face.

The following lemma follows directly from the diamond condition.

**Lemma 1.8** Let $\Phi$ be a flag of a ranked poset $\mathcal{P}$ of rank $n$, where $\mathcal{P}$ satisfies the diamond condition ($P_4$). Then $\Phi$ has exactly one $i$-adjacent flag for each $i = 0, \ldots, n - 1$.

If $\Phi$ is a flag of $\mathcal{P}$, where $\mathcal{P}$ satisfies the diamond condition, then we call the unique $i$-adjacent flag $\Phi^i$. Additionally, we define $\Phi^{i_1 \cdots i_j}$ to be the flag $(\Phi^{i_1 \cdots i_{j-1}})^{i_j}$. Note that the following also follows directly from the diamond condition and Lemma 1.8:

**Lemma 1.9** Let $\Phi$ be a flag of a ranked poset $\mathcal{P}$ of rank $n$, where $\mathcal{P}$ satisfies the diamond condition ($P_4$). Additionally, let $0 \leq j, k \leq n - 1$ be integers. Then:

1. $\Phi^{jj} = \Phi$;

2. $\Phi^{jk} = \Phi^{kj}$ if $|k - j| \geq 2$.

We are now able to formally define an abstract polytope:

**Definition 1.10** A poset $\mathcal{P}$ is called an abstract polytope of rank $n$ or an abstract $n$-polytope if it satisfies conditions ($P_1$)-(P4).

Throughout the remainder of this dissertation, we will use the term *polytope* to refer to an abstract polytope. When referring to traditional types of polytopes we will distinguish them by calling them convex polytopes or geometric polytopes.

The following lemma is easily verified:

**Lemma 1.11** Every section of a polytope is itself a polytope. In particular, a $k$-section of a polytope is an $k$-polytope.
A formal proof of the following proposition would be somewhat geometric in nature, and would involve a formal definition of convex polytope. We choose to omit this proof because we have been justifying the required conditions by discussing how they apply in the context of convex polytopes, so the result should be clear:

**Proposition 1.12** The face lattice of every convex polytope is an abstract polytope.

While we can think of abstract polytopes as generalizations of the face lattices of convex polytopes, it is not always the case that an abstract polytope is a lattice. We will now give some examples of abstract polytopes in small ranks, focusing on abstract polytopes which are not convex.

The only polytopes of ranks 0 and 1 are a vertex and an edge, respectively, viewed combinatorially. Any finite polytope of rank two, which we call a *polygon*, is a *k*-gon for some $k \geq 2$. That is to say, any finite poset which satisfies the conditions to be an abstract 2-polytope is equivalent to the face lattice of a convex $k$-gon for some $k \geq 3$, or (when $k = 2$) is a polytope which we will call a *digon*, or 2-gon. We will return to the case of the digon and focus on the case $k \geq 3$ for now. While the study of convex polytopes distinguishes between $k$-gons of different sizes and shapes, every convex $k$-gon is combinatorially equivalent to a regular convex $k$-gon, so combinatorially, there is just one $k$-gon for each $k$; the regular convex $k$-gon is a geometric realization of the underlying abstract $k$-gon. However, there are non-convex polygons which are also geometric realizations of abstract polygons. Notice that the Hasse diagram given in 1.5 (c) can represent either a geometric pentagon or pentagram, shown in Figure 1.5 (a) and (b). In the setting of abstract polytopes, we do not distinguish between these two geometric objects because they are combinatorially equivalent. While the pentagram is not a convex polytope, it is combinatorially equivalent to a convex polytope, so this is not the best example to demonstrate the generality of the definition of abstract polytopes.
There are two abstract polygons which are not combinatorially equivalent to any convex polygons. The first is the aforementioned digon. The digon has two vertices and two edges. It is shown in Figure 1.6 (a), and its Hasse diagram is in Figure 1.6 (b). Notice that the digon is not a lattice. For example, the two vertices do not have a unique join. From the point of view of geometric polytopes, this would be considered degenerate, because if the edges were represented with straight lines, the two edges would lie on top of each other since they connect the same two vertices, and there would be no space for a 2-face. However, it is easily verified that the digon does satisfy conditions (P1)-(P4), and is an abstract polytope. Secondly, notice that there is no finiteness requirement (or even a requirement of local finiteness) in the definition of abstract polytope, so this allows for polytopes such as the apeirogon, which can be thought of a tiling of the real line by finite segments. The Hasse diagram for an apeirogon is given in Figure 1.6 (c). This is another abstract polytope which cannot be geometrically realized as a convex polytope.

In rank three, we have examples of polytopes which can be viewed as maps on surfaces
other than the sphere or the real plane. For example, the hemicube is the polytope shown in Figure 1.7. Faces with the same label are identified. It is a map on the projective plane, and it can be obtained from the cube by identifying opposite vertices, edges and faces. Every abstract polyhedron with finite 2-faces and finite vertex figures can be viewed as a map on a surface (see [9]). Therefore, there is a close relationship between studying maps on surfaces and abstract polyhedra (see [27]). However, the converse is not true. A map can fail to be an abstract polytope by being too small in some way. For example, any map with a single 2-face is too small to be an abstract polytope, since it would not satisfy the diamond condition.

![Figure 1.7: Hemicube [33]](image)

Now that we have a basic understanding of abstract polytopes, we will discuss a few more basic notions which we will use in this dissertation. Describing abstract polytopes as posets makes it very easy to define the dual of a polytope by simply reversing the partial order. If we compare this to how the dual of a convex polytope is constructed in [15], we see that our definition is much simpler.

**Definition 1.13** Given an abstract polytope $\mathcal{P}$, we define the dual of $\mathcal{P}$, denoted $\mathcal{P}^*$, to be the polytope with the same face-set as $\mathcal{P}$, and for two faces $F$ and $G$ of $\mathcal{P}$, $F \leq_\mathcal{P} G$ if and only if $G \leq_{\mathcal{P}} F$.

**Definition 1.14** An isomorphism of polytopes is a rank- and incidence-preserving bijection between the polytopes. An isomorphism from a polytope to itself is called an automorphism.
The group of automorphisms of \( P \) is denoted \( \Gamma(P) \) and is sometimes referred to as the group of \( P \).

The proof of the following proposition is straightforward, and can be found in [23, p. 28]. This is a very useful result because it means that for all polytopes, we can choose a base flag and describe elements of \( \Gamma(P) \) by their actions on the base flag.

**Proposition 1.15** [23, Prop. 2A4] For every polytope \( P \), the group \( \Gamma(P) \) acts freely on \( \mathcal{F}(P) \), the set of flags of \( P \).

In this thesis, we will write the action of \( \Gamma(P) \) on flags of \( P \) as a left action, though it is sometimes written as a right action in other sources. While we generally are interested in the action of the automorphism group on flags of \( P \), we will sometimes look at the action of the automorphism group on faces of \( P \). Therefore, the following definition will be useful.

**Definition 1.16** A polytope is called \( k \)-face-transitive if the automorphism group acts transitively on the faces of rank \( k \). A polytope is called fully transitive if it is \( k \)-face transitive for \( k = -1, \ldots, n \).

In this dissertation, we will be particularly interested in the number of orbits that the set of flags has under \( \Gamma(P) \), so the following definition is important.

**Definition 1.17** A polytope \( P \) is called regular if \( \Gamma(P) \) acts transitively on the flags of \( P \). A polytope \( P \) is called a \( k \)-orbit polytope, or is said to have \( k \) orbits if the set of flags of \( P \) has \( k \) orbits under the action of \( \Gamma(P) \).

We will discuss regularity more in the next section, but for now we will point out a property which regular polytopes have. If \( F \) and \( G \) are faces of a regular polytope, such
that $F \leq G$, then the section $G/F$ is isomorphic to any section $G'/F'$ where $F' \leq G'$, $F'$ has the same rank as $F$, and $G'$ has the same rank as $G$. As we have noted, every 2-section of an abstract polytope is an $p$-gon for some $p \geq 2$, where we allow $p$ to be infinity. If $G$ is an $(i + 1)$-face of $\mathcal{P}$ and $F$ is an $(i - 2)$-face of $\mathcal{P}$, then define $p_i$ to be such that $G/F$ is a $p_i$-gon. Then, the 2-sections of a regular polytope can be described simply by listing the values of $p_i$ for $i = 1, \ldots, n - 1$. Polytopes which share this property with regular polytopes are called *equivelar*. The definition is given formally below. It is important to note that there are many equivelar polytopes which are not regular. In fact, there are many equivelar polytopes which are not even fully transitive.

**Definition 1.18** If the number of $i$-faces of $\mathcal{P}$ in the section $G/F$, for $G$ an $(i + 1)$-face of $\mathcal{P}$ and $F$ an incident $(i - 2)$-face of $\mathcal{P}$, is independent of the choice of $F$ and $G$, then we denote this number $p_i$. If this is the case for all $i = 1, \ldots, n - 1$, then we call $\mathcal{P}$ equivelar, and call $\{p_1, \ldots, p_{n-1}\}$ the Schl"afli type, or Schl"afli symbol, of $\mathcal{P}$.

Note that since the section $G/F$ is a $p_i$-gon, there are also $p_i$ faces of $\mathcal{P}$ of rank $(i - 1)$ in the section $G/F$. Occasionally, the Schl"afli symbol of a polytope $\mathcal{P}$ is padded with additional information (such as the length of Petrie polygons (see [7, Ch. 2]), etc.) to provide a "refined Schl"afli symbol." However, in this dissertation we will be primarily concerned with Schl"afli types of polytopes, that is with the basic Schl"afli symbol. Moreover, we will sometimes refer to the universal polytope of a given type. Every polytope of a given Schl"afli type is a quotient of the universal polytope of that type. For example, the cube is the universal polytope of type $\{4,3\}$ because every other abstract polytope of type $\{4,3\}$ (such as the hemicube previously depicted in Figure 1.7) can be obtained from the cube by identifying faces of the cube. We will briefly return to the idea of universal polytopes from the perspective of regular polytopes in the next section.

Two polyhedra which will be of particular concern to us are the universal polytopes of
Schläfli types \( \{p, 2\} \) and \( \{2, p\} \). As it turns out, these universal polytopes are the only polypopes of their respective Schläfli types (see Lemma 5.10). Both of these polyhedra are best thought of as maps on spheres. The polyhedron \( \{p, 2\} \) can be viewed as a sphere with an equator made of \( p \) vertices and \( p \) edges, so that the boundary of \( \{p, 2\} \) is made of two \( p \)-gons, identified along their boundaries. This is an example of a ditope, a polytope with exactly two facets. By the diamond condition, every ridge of a ditope must be incident to both facets, so a ditope has two isomorphic facets which are identified along their boundaries. If \( P \) is a ditope with facets isomorphic to \( Q \), then we say that \( P \) is a ditope over \( Q \).

The polyhedron \( \{2, p\} \) can be viewed as a sphere with a vertex at the north pole and a vertex at the south pole with \( p \) edges connecting them, so that the boundary of \( \{2, p\} \) is made of \( p \) digons, arranged in a cyclic manner.

Finally, we will give some definitions which describe polytopes which have some symmetry, but are not quite regular.

**Definition 1.19** A polytope of rank \( n \geq 3 \) is called uniform if its facets are uniform and it is vertex-transitive. To begin this inductive definition, all polygons (2-polytopes) are called uniform.

A much stronger condition is leads to a more restrictive concept [6, 8].

**Definition 1.20** A polytope is called semiregular if it has regular facets and is vertex-transitive.

Clearly, all regular polytopes are semiregular, and all semiregular polytopes are uniform. Additionally, all of the classical Archimedean solids are semiregular.
1.2 Regularity

While this dissertation will mostly focus on polytopes which are not regular, it is important to have an understanding of regular polytopes. In particular, we are interested in the automorphism groups of regular polytopes because we will be defining operations on regular polytopes which do not change the automorphism group, so we will be working with groups of regular polytopes, and thus will use some of the properties that these groups have.

Let $\mathcal{P}$ be a regular polytope. By Proposition 1.15, the action of a given automorphism on the flags (and hence faces) of $\mathcal{P}$ can be described by the action of that automorphism on any given flag. As such, we will choose a flag, which we will call the base flag, and describe the automorphisms in $\Gamma(\mathcal{P})$ by their actions on the base flag.

Let $\Phi$ be the base flag of a regular polytope $\mathcal{P}$. For each flag $\Psi$ of $\mathcal{P}$, there is a unique automorphism which takes $\Phi$ to $\Psi$. As such, once a base flag has been chosen, there is a one to one correspondence between flags of $\mathcal{P}$ and automorphisms of $\mathcal{P}$. In particular, if $\mathcal{P}$ is finite, then $\#F(\mathcal{P}) = \#\Gamma(\mathcal{P})$, where $\#$ denotes the cardinality.

For $j = 0, \ldots, n - 1$, we define $\rho_j$ to be the automorphism such that $\rho_j \Phi = \Phi^j$. In other words, $\rho_j$ takes $\Phi$ to its $j$-adjacent flag. In light of the following result, we call the $\rho_j$ distinguished generators of $\mathcal{P}$ (with respect to $\Phi$). Notice that $\rho_j$ is defined once we have chosen a base flag. It is not the case that $\rho_j$ takes every flag to its $j$-adjacent flag. We can think of the $\rho_j$ as abstract reflections, as a reflection of a geometric polytope takes a base flag to an adjacent flag. Since $\mathcal{P}$ is regular, there is an automorphism which takes $\Phi$ to any other flag, so any two base flags are equivalent and hence any two sets of distinguished generators are equivalent (in fact, conjugate to each other). The proof of the following proposition can be found in [23, p. 34].

**Proposition 1.21** [23, Prop. 2B8] Let $\mathcal{P}$ be a regular $n$-polytope, and let $\rho_0, \ldots, \rho_{n-1}$ be
the distinguished generators of $\mathcal{P}$ with respect to some flag. Then $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$.

The subgroups which are generated by distinguished generators are especially important in the theory of regular polytopes, as we will see. We therefore give the following definition.

**Definition 1.22** If $\mathcal{P}$ is a regular $n$-polytope with group $\Gamma := \Gamma(\mathcal{P})$, then, for $J \subseteq \{0, \ldots, n-1\}$, let $\Gamma_J := \langle \rho_j \mid j \not\in J \rangle$. We call such subgroups of $\Gamma$ distinguished subgroups. We will also denote $\Gamma_{\{j\}}$ as $\Gamma_j$.

These subgroups are important because $\Gamma_J$ is the stabilizer of the subset of the base flag which contains the faces of ranks which are elements of $J$. In particular, $\Gamma_j$ is the stabilizer of the base $j$-face (the $j$-face of the base flag) [23, Prop. 2B7].

We will now begin to explore some of the properties of the groups of regular polytopes. First, by Lemma 1.9, the $\rho_j$ are involutions, and $\rho_i$ and $\rho_j$ commute if $|i - j| \geq 2$. We call any group whose generators have these two properties a *string group generated by involutions*, or *sggi* for short. In order to understand why they are called string groups, we will first briefly discuss Coxeter diagrams. A *Coxeter diagram* is a graph which can be defined for any group which satisfies Coxeter-style relations, even if it is not actually a Coxeter group. It is a graph whose nodes represent the distinguished generators of the group and in which the branch connecting a pair of nodes $i$ and $j$ is labeled with the order of the element $\rho_i \rho_j$; by convention, if this order is 2 (i.e. $\rho_i$ and $\rho_j$ commute), we omit this branch. Therefore if distinguished generators with non-adjacent indices commute, then the underlying Coxeter diagram is a string diagram.

In addition to knowing that the groups of polytopes are string groups, we know the other Coxeter relations which they satisfy (and hence we know the other labels on the branches of the Coxeter diagram). More precisely, if $\mathcal{P}$ is a regular polytope $\mathcal{P}$ with Schl"afli type $\{p_1, \ldots, p_{n-1}\}$, then $p_k$ is the order of $\rho_{k-1} \rho_k$ [23, Prop. 2B9].
One of the most important things which distinguishes the groups of regular polytopes is that the distinguished generators of a regular polytope satisfy a condition known as the intersection condition. The condition is as follows, and the proof can be found in [23].

**Proposition 1.23** Let \( \mathcal{P} \) be a regular \( n \)-polytope, and let \( \rho_0, \ldots, \rho_{n-1} \) be the distinguished generators of \( \mathcal{P} \) with respect to some flag. Additionally, let \( N := \{0, \ldots, n - 1\} \), and let \( I, J \subseteq N \). Then:

\[
\langle \rho_i | i \in I \rangle \cap \langle \rho_i | i \in J \rangle = \langle \rho_i | i \in I \cap J \rangle.
\] (1.1)

This is natural because if \( \bar{I} := N \setminus I \), then \( \langle \rho_i | i \in I \rangle \) is \( \Gamma_{\bar{I}} \), the stabilizer of the chain comprised of faces of the base flag with ranks not in \( I \). Likewise, \( \langle \rho_i | i \in J \rangle \) is \( \Gamma_{\bar{J}} \), the stabilizer of the chain comprised of faces of the base flag with ranks not in \( J \). Therefore, an element is in their intersection if and only if it stabilizes the chain comprised of faces of the base flag with ranks other than those in \( I \cap J \). In other words, their intersection is \( \Gamma_{\bar{I} \cap \bar{J}} \), which is \( \langle \rho_i | i \in I \cap J \rangle \).

**Definition 1.24** Any group generated by involutions which satisfies equation (1.1) is called a C-group. Any sggi which satisfies equation (1.1) is called a string C-group.

The “C” in C-group stands for Coxeter, as string C-groups are always quotients of Coxeter groups. The groups of abstract polytopes are all string C-groups, by the fact that they are sggis and by Proposition 1.23. The universal polytope of a given Schlӓfli type has an automorphism group which satisfies only the Coxeter relations which are implied from its Schlӓfli symbol, and no additional independent relations. The group of a universal polytope of a given Schlӓfli type is always a Coxeter group, and since Coxeter groups are always C-groups ([21, Theorem 5.5]), the universal polytope of any Schlӓfli type is a regular polytope. We previously stated that all polytopes of a given type can be obtained from the universal polytope by identifying faces. Now we can add that the group of every polytope of a given
type is a quotient of the universal polytope, which can be obtained by adding additional relations which the generators must satisfy. We will denote by \([p_1, \ldots, p_{n-1}]\) the string Coxeter group which is the automorphism group of the universal polytope of type \(\{p_1 \ldots, p_{n-1}\}\). As it turns out, not only are the groups of abstract polytopes always string C-groups, but the converse holds as well.

**Proposition 1.25** [23, Prop. 2E13] The string C-groups are precisely the groups of regular polytopes.

Given a string C-group \(\Gamma\), we can construct a regular polytope \(\mathcal{P} = \mathcal{P}(\Gamma)\), such that \(\Gamma(\mathcal{P}) = \Gamma\). The construction will rely on the distinguished subgroups of \(\Gamma\). If we start with a string C-group \(\Gamma\) from which we want to construct a regular polytope whose automorphism group is \(\Gamma\), we cannot yet think of the distinguished subgroups as stabilizers because we do not yet have a polytope on which this group is acting. However, we will define a polytope \(\mathcal{P} = \mathcal{P}(\Gamma)\) using these distinguished subgroups, keeping in mind that we ultimately want them to be stabilizers.

The polytope \(\mathcal{P}\) is defined as follows: For the improper faces of \(\mathcal{P}\), we take two distinct copies of \(\Gamma\), which we call \(\Gamma_{-1}\) and \(\Gamma_n\). For \(j = 1, \ldots, n - 1\), we take the \(j\)-faces to be all left cosets \(\varphi \Gamma_j\) of \(\Gamma_j\) with \(\varphi \in \Gamma\). For a partial order, we say that \(\varphi \Gamma_j \leq \psi \Gamma_k\) if and only if \(-1 \leq j \leq k \leq n\) and \(\varphi \Gamma_j \cap \psi \Gamma_k \neq \emptyset\). See [23, Pages 52-53] for the proof that this definition results in a polytope \(\mathcal{P}\), and that \(\Gamma(\mathcal{P}) = \Gamma\).

Finally, we will examine the dual of a regular polytope from the point of view of its automorphism group. We know that we simply have to reverse the partial order of a polytope to obtain the dual. If we think of the set of \(i\)-faces of the polytope as the set of cosets of \(\Gamma_i\), then when we reverse the partial order, the new \(i\)-faces are the cosets of \(\Gamma_{n-i-1}\). Thus, the group of the dual of \(\mathcal{P}\) can be obtained by reversing the indices of the distinguished generators. From this string C-group, we can construct \(\mathcal{P}^*\). The automorphism group of
a polytope and its dual are isomorphic as groups, but if a polytope $P$ and its dual are regular polytopes, their automorphism groups are considered to be distinct string C-groups, since the distinguished generators occur in a different order, and hence when applying the construction, they will result in different polytopes.

1.3 Two-orbit polytopes

Recently, there has been some research done on polytopes with precisely two flag-orbits. In her doctoral dissertation [17] and subsequent paper [20], Isabel Hubard described the groups of two-orbit polyhedra. Since then, Hubard and Egon Schulte have been extending this description to polytopes of higher ranks. See [19], which is in preparation.

One important fact that allows two-orbit polytopes to be nicely analyzed is the following result. Note that this result cannot be extended to polytopes with more than flag-two orbits.

**Proposition 1.26** [18, Lemma 15] Let $P$ be a two-orbit polytope and $\Phi$ a flag of $P$. If, for some $i \in \{0, \ldots, n-1\}$, $\Phi$ and $\Phi^i$ are in the same flag-orbit under $\Gamma(P)$, then any flag $\Psi$ is in the same flag-orbit under $\Gamma(P)$ as its $i$-adjacent flag $\Psi^i$.

This allows us to partition the set of two-orbit polytopes into disjoint classes as follows:

**Definition 1.27** We say that a two-orbit polytope $P$ is in class $2_I$, where $I$ is a proper subset of $N := \{0, 1, \ldots, n-1\}$, if a flag of $P$ is in the same orbit under $\Gamma(P)$ as its $i$-adjacent flag if and only if $i \in I$.

We exclude the case where $I = N$, because polytopes in this class would be regular. Often we drop the set brackets, for example writing class $2_{1,2}$ instead of $2_{\{1,2\}}$. Additionally, we denote the class $2_\emptyset$ as $2$, and call polytopes in this class chiral. Thus a polytope $P$ is
chiral if and only if $\Gamma(P)$ has two orbits on the flags such that any two adjacent flags are in distinct orbits. Chiral polytopes are a very interesting class of polytopes which are similar to regular polytopes in many ways. In general, a flag-orbit is determined by the faces of the flag of ranks not in $I$. For example, for a polytope in class $2_0,\ldots,n-2$, the orbit of a flag is determined by its facet. On the other hand, to determine the orbit of a flag of a chiral polytope, we need to consider every face of the flag.

We can now begin to talk about further symmetric properties of two-orbit polytopes. Specifically, we will be interested in determining for which $i$ a two-orbit polytope is $i$-face-transitive. Clearly, if $P$ is a two-orbit polytope in class $2_I$, then $P$ is $i$-face transitive for every $i \in I$ [20, Lemma 3]. This follows from the flag-connectedness of $P$ and the fact that flags in the same orbit under $\Gamma(P)$ have their $j$-faces in the same orbit under $\Gamma(P)$ for all $j$.

We will use this fact in order to give a necessary and sufficient condition for a two-orbit polytope in class $2_I$ to be $i$-face transitive. The following lemma is adapted from the work of Isabel Hubard and Egon Schulte.

**Lemma 1.28** If $P$ is a two-orbit polytope in class $2_I$, and $j \in N$, then $\Gamma(P)$ acts transitively on the set of $j$-faces of $P$ if and only if $I$ is not the set $N \setminus \{j\}$. In particular, if the cardinality of $I$ is less than $n-1$, then $P$ is fully transitive.

**Proof** We will first assume that $I$ is not the set $N \setminus \{j\}$. Then there is some $k \neq j$ such that $k$ is not in $I$. Let $\Phi$ be a flag of $P$. Since $k$ is not in $I$, the flag $\Phi$ and its $k$-adjacent flag $\Phi^k$ are in different orbits. Now $\Phi$ and $\Phi^k$ differ only in their $k$-faces so they must share a $j$-face, namely $(\Phi)_j$. If we can show that for any $j$-face $F$ there is an automorphism of $P$ which takes $F$ to $(\Phi)_j$, then $P$ must be $j$-face transitive. Let $F$ be a $j$-face of $P$. Let $\Psi$ be a flag containing $F$. Then $\Psi$ is in the same flag-orbit as either $\Phi$ or $\Phi^k$. Thus, there is some automorphism $\sigma$ which takes $\Psi$ to either $\Phi$ or $\Phi^k$. Either way, $\sigma$ takes $F$ to $(\Phi)_j$, which completes the first direction of the proof.
Secondly, we will assume that \( I \) is the set \( N \setminus \{j\} \). We must show that \( \Gamma(P) \) is not \( j \)-face transitive. For the sake of contradiction, assume that \( \Gamma(P) \) is \( j \)-face transitive. Let \( \Phi \) be a flag of \( P \). Let \( F \) be \((\Phi)_j\), the \( j \)-face of \( \Phi \), and let \( G \) be \((\Phi^j)_j\), the \( j \)-face of \( \Phi^j \), which is the flag which is \( j \)-adjacent to \( \Phi \). Let \( \gamma \) be an automorphism of \( P \) such that \( G = \gamma(F) \). This automorphism exists because we assumed \( j \)-face transitivity. Let \( \Psi = \gamma \Phi \). Now \( \Psi \) and \( \Phi^j \) each contain \( G \) as their respective \( j \)-face, so by the strong flag-connectivity of \( P \), there is a sequence of adjacent flags from \( \Phi^j \) to \( \Psi \) all containing \( G \). Since \( G \) is fixed throughout the sequence, every adjacency in the sequence of successively adjacent flags is an \( i \)-adjacency for some \( i \neq j \). Therefore, every adjacency in the sequence of successively adjacent flags is an \( i \)-adjacency for some \( i \in I = N \setminus \{j\} \). Since \( P \) is in the class \( 2_I \), each flag in this sequence must be in the same flag-orbit as the next flag, so \( \Psi \) and \( \Phi^j \) must be in the same flag-orbit. But \( \Psi \) was defined to be \( \gamma \Phi \), so \( \Psi \) and \( \Phi \) are in the same flag-orbit, thus \( \Phi \) and \( \Phi^j \) are in the same flag-orbit, contradicting the assumption that \( I \) is the set \( N \setminus \{j\} \). This completes the proof. \( \square \)

In particular, since the two-orbit polytopes which we will be considering will primarily be in the class \( 2_{0,1,\ldots,n-2} \), we will make use of the following corollary:

**Corollary 1.29** A two-orbit polytope is not facet-transitive if and only if it is in class \( 2_{0,1,\ldots,n-2} \).

Just as an equivelar polytope has a Schl"afli type, we will associate with each two-orbit polytope \( P \) a double Schl"afli type, which can be reduced to the usual Schl"afli type of \( P \) in the case that \( P \) is equivelar. Let \( P \) be a two-orbit polytope with flag-orbits \( O_1 \) and \( O_2 \). Let \( \Phi \) be a flag in \( O_1 \) and \( \Psi \) be a flag in \( O_2 \). Since flags in a single orbit are equivalent, the number of \( i \)-faces in \( (\Phi)_{i+1}/(\Phi)_{i-2} \) is independent of choice of \( \Phi \) from \( O_1 \). Let this number be called \( p_i(O_1) \). Likewise, let \( p_i(O_2) \) be the number of \( i \)-faces in \( (\Psi)_{i+1}/(\Psi)_{i-2} \). Then we define the double Schl"afli type of \( P \) to be the array whose top row consists of \( p_i(O_1) \) for
\( i = 1, \ldots, n - 1 \), and whose bottom row consists of \( p_i(\mathcal{O}_2) \) for \( i = 1, \ldots, n - 1 \). If for some \( i \), \( p_i(\mathcal{O}_1) = p_i(\mathcal{O}_2) \), we only need to write this number once. Hence if \( \mathcal{P} \) is equivelar (as is the case in every fully-transitive class), \( p_i(\mathcal{O}_1) = p_i(\mathcal{O}_2) \) for all \( i \), so the double Schl"afli type is reduced to an ordinary Schl"afli type. The double Schl"afli type is unique up to interchanging the top and bottom rows.

We will now discuss the automorphism groups of two-orbit polytopes. A complete characterization of these groups will be given in Hubard and Schulte’s upcoming paper [19], but in this dissertation, we will simply discuss certain automorphisms which must be elements of the group of any two orbit polytope.

Like in the case of regular polytopes, we will start our discussion of the automorphisms of a two-orbit polytope \( \mathcal{P} \) by choosing a base flag. Unlike in the case of regular polytopes, \( \mathcal{P} \) is not flag-transitive, so the automorphisms that we will describe will depend on the choice of base flag.

Now assume that \( \mathcal{P} \) is in class 2I. Let \( \Phi \) be the base flag of \( \mathcal{P} \). Then for each \( i \in I \), the flag \( \Phi \) is in the same orbit as \( \Phi^i \). Therefore for each \( i \in I \), there is an automorphism of \( \mathcal{P} \) which takes \( \Phi \) to \( \Phi^i \). We shall call this automorphism \( \rho_i \). Additionally, for \( j, k \not\in I \), \( \Phi^{jk} \) must not be in the same orbit as \( \Phi^j \), so it must be in the same orbit as \( \Phi \). Therefore, there is an automorphism which takes \( \Phi \) to \( \Phi^{jk} \), and we call this automorphism \( \alpha_{k,j} \). Finally, if \( i \in I \) and \( j \not\in I \), then \( \Phi^{jij} \) must be in the same flag-orbit as \( \Phi \). We will define \( \alpha_{j,i,j} \) to be the automorphism which takes \( \Phi \) to \( \Phi^{jij} \).

As it turns out, these automorphisms generate \( \Gamma(\mathcal{P}) \), so we will call them distinguished generators, though we will not prove this fact. For our purposes, it is sufficient to note that \( \Gamma(\mathcal{P}) \) contains these generators and that the generators satisfy certain relations, which we will give shortly. In the cases that we are interested in, we will show by other means that these automorphisms do in fact generate the group.
The following conditions, which must be satisfied by the distinguished generators follow from Lemma 1.9. For all \( i \in I \) and \( j, k \in N \setminus I \):

\[
\rho_i^2 = \alpha_{j,k}^2 = \alpha_{j,i,j}^2 = \epsilon \quad \text{if } |j - k| \geq 2; \quad (1.2)
\]

\[
\alpha_{k,j} = \alpha_{j,k}^{-1}; \quad (1.3)
\]

\[
\alpha_{j,i,j} = \rho_i, \quad \text{if } |j - i| \geq 2. \quad (1.4)
\]

Finally, the groups of two-orbit polytopes must satisfy an intersection condition similar to the one satisfied by regular polytopes. We will state it here without proof, and will not use it in this dissertation, but it is an important property of the groups \( \Gamma := \Gamma(P) \) of two-orbit polytopes in class \( 2_I \).

For \( J \subset N \), let

\[
\Gamma_J := \langle \rho_i, \alpha_{j,k}, \alpha_{j,i,j} | i \in I \cap \overline{J}; j, k \in \overline{I} \cap \overline{J} \rangle.
\]

Note that the requirements that \( i \in I \) and \( j, k \not\in I \) simply indicate that this set of generators is a subset of our set of distinguished generators of \( \Gamma \). The key restriction here is that \( i, j, \) and \( k \) are not in \( J \). As is the case with regular polytopes, we call these subgroups distinguished subgroups of \( \Gamma(P) \). Then the following must hold:

\[
\Gamma_J \cap \Gamma_K = \Gamma_{J \cup K}, \quad (J, K \subseteq N). \quad (1.5)
\]

If we define \( \Gamma^J \) to be \( \Gamma_{\overline{J}} \), then (1.5) is equivalent to:
This second form of the intersection condition for two-orbit polytopes is reminiscent of the intersection condition for regular polytopes.

As we mentioned previously, the set of distinguished generators with respect to a flag $\Phi$ is not equivalent to a set of distinguished generators chosen with respect to a flag in a distinct flag-orbit from $\Phi$. The following lemma describes the relationship between sets of distinguished generators chosen with respect to adjacent flags in different orbits.

**Lemma 1.30** Let $\mathcal{P}$ be a two-orbit polytope in class $2_I$ with base flag $\Phi$ and distinguished generators (with respect to $\Phi$) $\rho_i, \alpha_{j,k}$ and $\alpha_{j,i,j}$ for $i \in I$, and $j, k \not\in I$. Let $j_0$ be in $N \setminus I$ so that $\Phi$ and $\Phi_{j_0}$ are in distinct flag-orbits. Let $\rho'_i, \alpha'_{j,k}$, and $\alpha'_{j,i,j}$ for $i \in I$, $j, k \not\in I$ be the distinguished generators with respect to $\Phi_{j_0}$. Then:

a) $\rho'_i = \alpha_{j_0,i,j_0}$;

b) $\alpha'_{j,k} = \alpha_{k,j_0} \alpha_{j_0,j}$;

c) $\alpha'_{j,i,j} = \alpha_{j,j_0} \rho_i \alpha_{j_0,j}$.

**Proof** Each of these is easily verified by considering the actions of $\rho'_i, \alpha'_{j,k}$ and $\alpha'_{j,i,j}$ on the flag $\Phi$. 

\[ \Gamma^J \cap \Gamma^K = \Gamma^{J \cap K} . \]  

(1.5')
Chapter 2

Constructing $k$-bubbles

In this section we present a construction of a class of abstract polytopes which can be seen as a generalization of truncation in a combinatorial setting. Under certain circumstances, this construction allows us to build polytopes with preassigned automorphism group and preassigned number of flag-orbits. This construction will start with a fixed polytope $\mathcal{P}$, and the face structure of $\mathcal{P}$ will be used to define a new set of faces as well as a partial order on those faces which satisfies the definition of an abstract polytope. In this section we will discuss several of the properties of these polytopes. Additionally, much of this section will focus on determining conditions under which the new polytope will have the same automorphism group as the original polytope $\mathcal{P}$.

2.1 Truncation

We begin with an informal discussion of the geometric concept of truncation (or more exactly, vertex truncation) and how it relates to our construction in the familiar setting of convex polytopes. Unfortunately, we cannot give a general description of our construction entirely in the setting of convex polytopes because, even when the original polytope is a convex
polytope, the resulting polytope will, in general, be degenerate, in the sense that two faces
would lie on top of each other.

For our purposes, it is sufficient to discuss the combinatorial properties of geometric
truncation. When a single vertex of a geometric polytope is truncated, the vertex is “cut
off” and replaced with a facet. This new facet is a vertex figure of the original polytope. We
can say that the original polytope has been truncated at that vertex. We say that a polytope
is truncated or fully vertex-truncated when it is truncated at every vertex, in such a way that
any two hyperplanes used in the truncation process meet only outside of the polytope.

The concept of truncation is very old, as many of the Archimedean solids can be ob-
tained from Platonic solids by truncation. Generally, this term has been used in reference
to geometric polytopes, where truncation is accomplished by intersecting the polytope with
a hyperplane. However, it is clear that mathematicians as early as Johannes Kepler have
thought about truncation combinatorially, as evidenced by the fact that the object which Ke-
pler calls the truncated cuboctahedron is only combinatorially equivalent to a geometrically
truncated cuboctahedron. Kepler recognized this fact, describing a figure:

... which I call a truncated cuboctahedron not because it can be formed by
truncation, but because it is like a cuboctahedron that has been truncated. [10,
p. 81]

It should be noted that there is more than one way to (combinatorially) “truncate” a poly-
tope. In our use of the term “truncation,” the vertices are cut off and replaced with their
vertex figures, but some of each original edge still remains. This is the more traditional
version of “truncation,” which was used by Kepler, when he first named the Archimedean
solids [10, p. 81]. Additionally, this is the version which has been used in other combinatorial
contexts (see [27]).

It should be noted, however that Coxeter [7, Ch. 8] defines “truncation” to be an operation
which “truncates” each vertex up to the midpoints of the edges emanating from the vertex, so that there is only a single vertex and no edge remaining where the original edge used to be. In other sources, this operation is generally called rectification, or medial operation. Coxeter describes the operation which we call truncation as “intermediate truncation.”

When a polytope is truncated (in our sense) and a vertex is replaced with its vertex figure, every face of the vertex figure is also a face of the truncated polytope, so in addition to new facets, truncation also creates new faces of every proper rank. If we further consider the effects of truncation on a convex polytope, we see that truncating a polytope also truncates every face of the original polytope. Every proper face of the truncated polytope falls into one of these two categories; it is either the truncation of a face of the original polytope or a face of a vertex figure of \( P \).

Let \( tP \) be the (full vertex) truncation of a convex \( n \)-polytope \( P \). As we mentioned, when truncating \( P \), each face of \( P \) of every rank \( i = 1, \ldots, n \) becomes truncated as well. In particular, if \( F \) is an \( i \)-face of \( P \), with \( 1 \leq i \leq n \), then, under the truncation operation, \( F \) is replaced with an \( i \)-face \( F' \) in \( tP \) such that the section \( F'/F_{i-1} \) in \( tP \) is isomorphic to the truncation of the section \( F/F_{i-1} \) of \( P \).

For example, consider \( \{3, 3\} \), the regular tetrahedron. When a vertex of a tetrahedron is “cut off”, what remains is the vertex figure of the “cut off” vertex, which is a triangle. If each vertex of the tetrahedron is “cut off” and replaced with a triangle, the result is \( t\{3, 3\} \), a truncated tetrahedron, shown in Figure 2.1. Now consider the 2-faces of \( t\{3, 3\} \). In addition to the newly formed triangular 2-faces which are the vertex figures of the tetrahedron, there are 2-faces which remain in the same planes as the original triangular 2-faces, though they have different shapes than the original 2-faces. While these are technically different 2-faces which are subsets of the original triangular 2-faces, we can combinatorially associate them with the faces that they are replacing. Since each vertex has been cut off, the triangular faces of the tetrahedron have been replaced with hexagons. Each hexagon has three edges
which are incident to the triangular vertex figures, and three edges which are the truncations of the original edges of that 2-face. In summary, each triangular 2-face has been replaced with a truncation of itself, $t\{3\}$.

Consider the effect of truncation on the edges of a convex polytope $\mathcal{P}$. Each edge is replaced by a shorter edge and two new vertices. Combinatorially, we can associate the shortened edge $tE$ with the longer edge $E$ which it replaced. If an edge $E$ with vertices $V_1$ and $V_2$ is truncated to $tE$, then the two new vertices which fall in the interior of the original edge $E$ can be combinatorially associated with the ordered pairs $(V_1, E)$ and $(V_2, E)$. When truncating $\mathcal{P}$, every edge of $\mathcal{P}$ is truncated as well, so $t\mathcal{P}$ will have an edge corresponding to each edge of the original polytope $\mathcal{P}$, and a vertex corresponding to each ordered pair $(V, E)$ such that $V$ is a vertex of $\mathcal{P}$ and $E$ is an incident edge of $\mathcal{P}$. Note that these edges do not comprise the entire edge-set of $t\mathcal{P}$ (though these vertices do comprise the entire vertex-set of $t\mathcal{P}$).

In general, consider the effect of truncation on the faces of $\mathcal{P}$ of rank $i + 1$ for $i = 1, \ldots, n - 1$. When the convex polytope $\mathcal{P}$ is truncated, each $(i + 1)$-face is itself truncated, so if we restrict to looking at the effect of truncation on a single $(i + 1)$-face, then each vertex of the $(i + 1)$-face is replaced with an $i$-face. For example, consider the case where $V$ is a vertex of $\mathcal{P}$ and $F$ is an $(i + 1)$-face of $\mathcal{P}$ that contains $V$. Then when $\mathcal{P}$ is truncated, $F$ (or rather $F/F_{-1}$) is as well. In the section $F/F_{-1}$, the vertex $V$ is replaced by an $i$-face, which is the vertex figure $F/V$ of $V$ in $F/F_{-1}$.
This is the case for every \((i + 1)\)-face \(F\) to which a fixed vertex \(V\) is incident. Thus, in \(P\), each vertex is replaced with several \(i\)-faces. In particular, each vertex is replaced by one \(i\)-face for each \((i + 1)\)-face to which it was incident in \(P\). These new \(i\)-faces correspond to ordered pairs \((V, F)\) such that \(V\) is a vertex of \(P\) and \(F\) is an \((i + 1)\)-face of \(P\) such that \(V\) and \(F\) are incident in \(P\). These \(i\)-faces are precisely the \(i\)-faces of the facet of \(tP\) that replaces the vertex \(V\) of \(P\) and which is isomorphic to the section \(F_n/V\) of \(P\), the vertex figure of \(V\) in \(P\). Additionally, the \((i + 1)\)-face \(F\) is replaced with a different \((i + 1)\)-face, namely \(tF\) (or rather \(t(F/F_{i+1})\)), so there remains an \((i + 1)\)-face of \(tP\) for every \((i + 1)\)-face of \(P\). Just as we can associate a shortened edge with the longer edge which it replaced, we can associate \(tF\) with the original face \(F\).

In summary, if \(P\) is a convex \(n\)-polytope and \(P_i\) is the set of \(i\)-faces of \(P\), then the proper faces of \(tP\) can be combinatorially described as follows:

The set of vertices of \(tP\) can be identified with \(\{(V, E)|V \in P_0, E \in P_1, V \leq_P E\}\).

For \(i = 1, \ldots, n - 1\), the set of \(i\)-faces of \(tP\) can be identified with:

\[P_i \cup \{(V, F)|V \in P_0, F \in P_{i+1}, V \leq_P F\} \]

This set includes both the newly formed \(i\)-faces, which replace the vertices in each original \((i + 1)\)-face, as well as faces which we combinatorially associate with the original \(i\)-faces.

The improper faces of the truncation can be identified with the improper faces of the original polytope.

Notice that when \(i = n - 1\), there is only a single \((i + 1)\)-face of \(P\), namely \(F_n\), and it is incident to every vertex of \(P\), so the ordered pairs of the form \((V, F_n)\) such that \(V\) is a vertex of \(P\) incident to \(F_n\) simply correspond to the vertices of \(P\). These \((n - 1)\)-faces are the new facets of the truncated polytope which replace the old vertices, and hence there is
a natural bijection between them and the original vertices.

2.2 The Definition of $[\mathcal{P}]_k$

We will give an informal description of our construction before preceding to the formal definition. The primary way in which this construction generalizes truncation is that the role of the vertices in the truncation operation can be fulfilled by faces of any rank between 0 and $n - 1$. In other words, rather than replacing the vertices of a polytope with new facets, a rank $k$ with $0 \leq k \leq n - 1$ is chosen, and each $k$-face is replaced with a new facet. Given a polytope $\mathcal{P}$ of rank $n \geq 3$, we denote this derived polytope $[\mathcal{P}]_k$. Informally, we will refer to it as the rank $k$-bubble of $\mathcal{P}$, or just the $k$-bubble of $\mathcal{P}$. However, because we have the written notation $[\mathcal{P}]_k$, we will not often use this term in writing. We call $[\mathcal{P}]_k$ a $k$-bubble of $\mathcal{P}$ because we can think of each $k$-face of $\mathcal{P}$ as essentially ballooning out in a certain way to create a new facet which replaces the $k$-face of $\mathcal{P}$. When each $k$-face of $\mathcal{P}$ is replaced with a facet in $[\mathcal{P}]_k$, it is done in a way such that no new faces of rank less than $k$ are created, and for any $(k - 1)$-face $F$ of $[\mathcal{P}]_k$, the co-faces of $F$ in $[\mathcal{P}]_k$ (sections in which the vertices of the section are $k$-faces of $[\mathcal{P}]_k$) are (vertex-) truncations of the corresponding section of the original polytope $\mathcal{P}$.

More precisely, we can describe the proper faces of $[\mathcal{P}]_k$ of rank greater than or equal to $k$ as follows:

The $k$-faces of $[\mathcal{P}]_k$ correspond to the ordered pairs $(F, G)$ such that $F$ is a $k$-face of $\mathcal{P}$, and $G$ is a $(k + 1)$-face of $\mathcal{P}$ incident with $F$.

The $i$-faces of $[\mathcal{P}]_k$ for $i = k + 1, \ldots, n - 1$ include all the $i$-faces of $\mathcal{P}$, as well as some new faces. These new faces correspond to the ordered pairs $(F, G)$ such that $F$ is a $k$-face of $\mathcal{P}$, and $G$ an $(i + 1)$-face of $\mathcal{P}$ incident with $F$. These new $i$-faces are the $i$-faces of the new
facets which replace the $k$-faces. For $H$ a $(k-1)$-face, the coface of $H$ in $\mathcal{P}_k$ is the (vertex-) truncation of the coface of $H$ in $\mathcal{P}$. Therefore, the reason that the $i$-faces of the new facets correspond to ordered pairs $(F,G)$ is the same reason that in truncation the $j$-faces (where $j = i - k$) of the new facets (original vertex figures) correspond to ordered pairs $(V,G)$ such that the vertex $V$ is incident to the $(j+1)$-face $G$ in $\mathcal{P}$.

The replacement of a $k$-face $F$ by a new facet $(F,F_n)$ is done in such a way that every proper face of $F$ in $\mathcal{P}$ is a face of $\mathcal{P}_k$ and is incident to every face of $(F,F_n)$ of rank greater than or equal to $k$ in $\mathcal{P}_k$.

Until this point, we have thought of the $k$-bubble of $\mathcal{P}$ as being obtained by replacing each $k$-face of $\mathcal{P}$ with a new facet. This line of thinking considers the faces of intermediate ranks $i = k+1, \ldots, n-2$ to be consequences of that replacement: as faces of the new facets, they must also be faces of the new polytope.

It is possible, however, to view the $k$-bubble of $\mathcal{P}$ as being obtained by replacing each $k$-face of $\mathcal{P}$ with several new $k$-faces. This approach to the same construction considers the faces of ranks greater than $k$ to be consequences of that replacement. More precisely, each $k$-face is replaced with several new $k$-faces, one for each $(k+1)$-face of $\mathcal{P}$ to which it was incident. Each of these new $k$-faces is incident to all of the faces of $\mathcal{P}$ of lower rank to which the original $k$-face was incident in $\mathcal{P}$, so that each $k$-face is replaced with several copies of itself. If we think of the $k$-bubble of $\mathcal{P}$ in this way, then the new faces of ranks greater than $k$ are essentially being created by the new $k$-faces, and are necessary in order to obtain a polytope. These new faces can be viewed geometrically as filling the spaces that are being carved out by the new $k$-faces. This alternative interpretation of the new polytope is responsible for our use of the term “$k$-bubble.”

We will show that $\mathcal{P}_{n-1}$ is isomorphic to the original $n$-polytope $\mathcal{P}$. Thus, after discarding the case $k = n - 1$, for a given $n$-polytope $\mathcal{P}$, we can use this construction to derive
$n - 1$ new $n$-polytopes, one for each $k = 0, \ldots, n - 2$. When $\mathcal{P}$ is a convex polytope, the case $k = 0$ is exactly the geometric vertex-truncation operation we described before. We can consider the case $k = 0$ to be a combinatorial truncation, even when $\mathcal{P}$ is not convex.

We will now describe what the construction looks like in the case that $\mathcal{P}$ is a rank 3 polytope. Now, when $\mathcal{P}$ is finite, the operations can be carried on the surface on which $\mathcal{P}$ is realized as a map. Given a polyhedron $\mathcal{P}$, we can (non-trivially) derive $[\mathcal{P}]_0$ and $[\mathcal{P}]_1$. As previously mentioned, $[\mathcal{P}]_0$ is the (vertex-) truncation of $\mathcal{P}$. The polytope $[\mathcal{P}]_1$ is obtained from $\mathcal{P}$ by replacing each edge with two edges and their spanning in a digon. Each digon has two vertices, and these are the same two vertices which were incident to the edge which was replaced, thus the vertices of $[\mathcal{P}]_1$ are the same as the vertices of $\mathcal{P}$. Because every original edge was replaced, every edge of $[\mathcal{P}]_1$ is an edge of a digon. In addition to being incident to a digon, each edge of $[\mathcal{P}]_1$ is incident to one of the original faces of $\mathcal{P}$ (which may have also been a digon). We claim that each edge of $[\mathcal{P}]_1$ corresponds to a pair of the form $(F, G)$ with $F$ an edge of $\mathcal{P}$ and $G$ an incident 2-face of $\mathcal{P}$. Here, $G$ is the original 2-face of $\mathcal{P}$ to which the new edge is incident, and $F$ is the original edge of $\mathcal{P}$ which was replaced by the digon incident to the new edge. The edge $F$ must be incident to $G$ in $\mathcal{P}$, and furthermore, this correspondence is bijective. The 2-faces of $[\mathcal{P}]_1$ are the original 2-faces of $\mathcal{P}$ as well as the digons, the latter corresponding to the edges of $\mathcal{P}$, or equivalently, pairs of the form $(F, F_3)$ where $F$ is an edge of $\mathcal{P}$, and $F_3$ is the unique maximal face of $\mathcal{P}$. Figure 2.2 (a) shows a local picture of the Hasse diagram of an edge $F$ of $\mathcal{P}$. Figure 2.2 (b) shows a local picture of $[\mathcal{P}]_1$, once the edge $F$ has been replaced with two edges $(F, G_1)$ and $(F, G_2)$ and a digon $(F, F_n)$. The two edges $(F, G_1)$ and $(F, G_2)$ are incident to the same vertices to which $F$ was incident in $\mathcal{P}$.

Next, we will informally describe the construction in the case where $\mathcal{P}$ is a rank 4 equivelar polytope. Given a 4-polytope $\mathcal{P}$, we can (non-trivially) derive $[\mathcal{P}]_0$, $[\mathcal{P}]_1$, and $[\mathcal{P}]_2$. Again, $[\mathcal{P}]_0$ is the truncation of $\mathcal{P}$ in which each vertex is replaced by its vertex figure.
Let \( \mathcal{P} \) be an equivelar 4-polytope of Schl"afli type \( \{p,q,r\} \). To obtain \([\mathcal{P}]_1\), each edge of \( \mathcal{P} \) is ultimately replaced by a (regular) polyhedron of type \( \{2,r\} \). Figure 2.3 shows a local picture of Hasse diagrams which depict the replacement of an edge \( E \) in the case that \( r = 3 \). Figure 2.3 (a) shows the incidences of \( E \) in \( \mathcal{P} \). Figure 2.3 (b) shows the faces which replace \( E \), and their incidences. The facet which replaces \( E \) in this case is \((E,F_n)\). Notice that in this case, since \( r = 3 \), \((E,F_n)\) has three 2-faces and three edges, which are all incident to the same two vertices. An edge \( E \) of \( \mathcal{P} \) is incident to \( r \) 2-faces and \( r \) 3-faces in \( \mathcal{P} \). When \( E \) is replaced by a new facet of type \( \{2,r\} \), the \( r \) edges of the facet correspond to the \( r \) ordered pairs of the form \((E,F)\) where \( F \) is one of the \( r \) 2-faces to which \( E \) was incident in \( \mathcal{P} \). The edge \((E,F)\) which corresponds to the 2-face \( F \) is incident to the face of \([\mathcal{P}]_1\) corresponding to \( F \). The \( r \) 2-faces of the new facet correspond to the \( r \) ordered pairs of the form \((E,G)\)
where $G$ is one of the $r$ 3-faces to which $E$ was incident in $\mathcal{P}$. The 2-face $(E,G)$ which corresponds to the 3-face $G$ is incident to the face which corresponds to $G$ in $[\mathcal{P}]_1$. Thus the vertex figures of these new facets are the edge figures of $\mathcal{P}$. Additionally, when looking at the original facets of $\mathcal{P}$, they have each of their edges replaced by a digon, and thus each facet $G_i/F_{-1}$ of $\mathcal{P}$ is replaced with $[G_i/F_{-1}]_1$. Combinatorially, we associate this new facet which is isomorphic to $[G_i/F_{-1}]_1$ with $G_i$ itself.

If $\mathcal{P}$ is not equivelar, the description above can still be used as a local description of $\mathcal{P}$. Specifically, if an edge of $\mathcal{P}$ is incident to $r$ 2-faces (and thus $r$ 3-faces), it is replaced by the polyhedron $\{2,r\}$. Each facet $Q$ of $\mathcal{P}$ is still replaced by $[Q]_1$. Again, let $\mathcal{P}$ be an equivelar 4-polytope of Schl"afli type $\{p,q,r\}$. Then $[\mathcal{P}]_2$ is obtained from $\mathcal{P}$ by replacing the 2-faces of $\mathcal{P}$ (which are $p$-gons) with polyhedra of type $\{p,2\}$. We will refer to these polyhedra as dihedra $\{p,2\}$ or $p$-gonal ditopes. Just as replacing an edge with a digon essentially doubles the edge and adds in a 2-face to fill the space in between, replacing a $p$-gon with a $p$-gonal ditope essentially doubles the 2-face and adds in a 3-face to fill the space in between. Thus the facets of $[\mathcal{P}]_2$ consist of facets which are isomorphic to the old facets of $\mathcal{P}$ in addition to these ditopes. The edges and vertices of $[\mathcal{P}]_2$ are those of $\mathcal{P}$. Again, if $\mathcal{P}$ is not equivelar, this can be used as a local description of $[\mathcal{P}]_2$ to describe what happens to a $p$-gonal 2-face of $\mathcal{P}$. Figure 2.2 is also a local picture of the Hasse diagram of $[\mathcal{P}]_2$ since $n = 4$. In this case, $(F,G_1)$ and $(F,G_2)$ are both copies of the 2-face $F$ of $\mathcal{P}$, so the section $(F,F_n)/F_{-1}$, which we associate with the facet $(F,F_n)$, is two copies of $F$, identified along their boundaries.

Notice that $[\mathcal{P}]_0$, $[\mathcal{P}]_1$, and $[\mathcal{P}]_2$ all have two types of facets. Facets of the first type are those which replace the $k$-faces of $\mathcal{P}$, and facets of the second type correspond to the original facets of $\mathcal{P}$, though as polytopes they are not, in general, isomorphic to the original facets of $\mathcal{P}$. It is only in $[\mathcal{P}]_2$ that the facets which correspond to the old facets are actually isomorphic to the old facets. This is because if $Q$ is a facet of $\mathcal{P}$, then the facet which
corresponds to \( Q \) in \([P]_2\) is isomorphic to \([Q]_2\) and hence to \( Q \) (since \( Q \) is a polyhedron).

We now give the formal definition for our construction. We will first define the faces of the polytope \([P]_k\), then we will define the partial order. At this point, we are simply defining \([P]_k\) as a ranked poset, and we will subsequently show that \([P]_k\) is a polytope. We first define \([P]_k\) as a face set with a rank function as follows:

**Definition 2.1** Let \( P \) be a polytope of rank \( n \geq 3 \) and let \( P_i \) be the set of \( i \)-faces of \( P \). For \( 0 \leq k \leq n - 1 \), define \([P]_k\) to be the ranked set, of rank \( n \), whose faces are as follows:

1. For \(-1 \leq i \leq k - 1 \) and for \( i = n \), the set of \( i \)-faces of \([P]_k\) is \( P_i \).

2. The set of \( k \)-faces of \([P]_k\) is:

\[
\{(F,G) | F \in P_k, G \in P_{k+1}, F \preceq P G\}.
\]

3. For \( k + 1 \leq i \leq n - 1 \), the set of \( i \)-faces of \([P]_k\) is:

\[
P_i \cup \{(F,G) | F \in P_k, G \in P_{i+1}, F \preceq P G\}.
\]

In particular, note that the set of facets of \([P]_k\) can be identified with \( P_k \cup P_{n-1} \) (since in this case \( F_n \) is the only element of \( P_{i+1} \) and \( F \preceq F_n \) holds for all \( k \)-faces \( F \)). We further call all faces of \([P]_k\) which are of the form \( F \) with \( F \in P \) of type 1, and those faces which are of the form \((F,G)\) of type 2. In writing \((F,G)\) for a face of \([P]_k\) of type 2 we always are implicitly assuming that \( F \preceq P G \). Note that the rank of \( F \) is always \( k \) in this case.

We will now define a partial order on \([P]_k\), which is determined (by transitivity) by the following order relationships between faces of consecutive ranks \( i - 1, i \) for \( i = 0, \ldots, n \).
**Definition 2.2** Let $\mathcal{P}$ be a polytope of rank $n \geq 3$, and let $[\mathcal{P}]_k$ be as in Definition 2.1. Then we define an order relationship as follows:

1. For $0 \leq i \leq k - 1$, $F \in \mathcal{P}_{i-1} = ([\mathcal{P}]_k)_{i-1}$, $G \in \mathcal{P}_i = ([\mathcal{P}]_k)_i$:
   
   $F \leq_{[\mathcal{P}]_k} G$ if and only if $F \leq \mathcal{P} G$;

2. For $F \in \mathcal{P}_{k-1}$, $G \in \mathcal{P}_k$, $H \in \mathcal{P}_{k+1}$:
   
   $F \leq_{[\mathcal{P}]_k} (G, H)$ if and only if $F \leq \mathcal{P} G$ (and hence $F \leq \mathcal{P} G \leq \mathcal{P} H$);

3. For $k + 1 \leq i \leq n - 1$, $F, F' \in \mathcal{P}_k$, $G, G' \in \mathcal{P}_i$, $H \in \mathcal{P}_{i+1}$:
   
   a. $(F, G) \leq (F', H)$ if and only if $F = F'$ and $G \leq \mathcal{P} H$;
   
   b. $(F, G) \leq G'$ if and only if $G = G'$;

4. For $k + 2 \leq i \leq n - 1$, $F \in \mathcal{P}_k$, $K \in \mathcal{P}_{i-1}$, $G \in \mathcal{P}_i$, $H \in \mathcal{P}_{i+1}$:
   
   a. $K \leq_{[\mathcal{P}]_k} G$ if and only if $K \leq \mathcal{P} G$;
   
   b. $K$ is not incident to $(F, H)$;

5. The $n$-face (which is unique) is incident to every $(n - 1)$-face.

**Definition 2.3** We call $[\mathcal{P}]_k$ the $k$-bubble of $\mathcal{P}$. When $k = 0$, we call $[\mathcal{P}]_k = [\mathcal{P}]_0$ the (combinatorial vertex-) truncation of $\mathcal{P}$.

First we make the following observation.

**Lemma 2.4** For $\mathcal{P}$ an $n$-polytope, $[\mathcal{P}]_{n-1} \cong \mathcal{P}$.

**Proof** Recall that at this point, we have merely defined $[\mathcal{P}]_k$ as a ranked poset, so this isomorphism is an isomorphism of posets. For $i \neq n - 1$, the $i$-faces of $[\mathcal{P}]_{n-1}$ are exactly the $i$-faces of $\mathcal{P}$ by Definition 2.1, part 1. The set of $(n - 1)$-faces of $[\mathcal{P}]_{n-1}$ is the set $\{(F, F_n) | F \in \mathcal{P}_{n-1}\}$, which is clearly in one-to-one correspondence with the set of $(n - 1)$-faces of $\mathcal{P}$. The partial order is directly inherited from $\mathcal{P}$ by Definition 2.2, parts 1 and 2. □
Our goal is to show that the $k$-bubble $[\mathcal{P}]_k$ is an abstract $n$-polytope for each $k$. Before proceeding with the formal proof, we describe three examples and establish several lemmas which help clarify the structure of $k$-bubbles.

In the following three examples, let $\mathcal{P}$ be the rank 4 cubical tessellation of $\mathbb{R}^3$, which is the universal polytope with Schl"afli symbol $\{4,3,4\}$. Thus $n = 4$. Since we know that $[\mathcal{P}]_3$ is isomorphic to $\mathcal{P}$, we will describe $[\mathcal{P}]_k$ for $k = 0, 1, 2$.

**Example 1** We will start by describing $[\mathcal{P}]_0$, the truncation of $\mathcal{P}$. Recall from our discussion of truncation in geometry that each face of the truncation is either the truncation of a face of $\mathcal{P}$ or a face of the vertex figure which replaces a vertex. The former are (in one-to-one correspondence with) the faces of type 1, and the latter are the faces of type 2. Figure 2.4 shows a local picture of $[\mathcal{P}]_0$. Notice that the facets of type 1 are truncated cubes and the facets of type 2 are octahedra, the vertex figures of $\mathcal{P}$. Let $\{F_{-1}, V, E, F, G, F_n\}$ be a flag of $\mathcal{P}$. We will describe every proper face of $[\mathcal{P}]_0$ which are derived from faces in this flag, beginning with the facets and moving down in rank.

![Figure 2.4: Truncated Cubic Tessellation [35]](image)

The first facet of $[\mathcal{P}]_0$ is $G$, which is a facet of type 1 corresponding to the facet $G$ of $\mathcal{P}$. In $\mathcal{P}$, $G$ is a cube. The face $G$ of $[\mathcal{P}]_0$ is a truncated cube. Notice that though $G$ does not
retain the same shape as in \( \mathcal{P} \), if we think geometrically, the face \( G \) of \( [\mathcal{P}]_0 \) is in the same place as the face \( G \) of \( \mathcal{P} \) (meaning it is a subset of the same dimension). The second facet of \( [\mathcal{P}]_0 \) which we will examine is the octahedron which replaces the vertex \( V \) of \( \mathcal{P} \). This facet is a facet of type 2 and is denoted by the ordered pair \((V, F_n)\). Recall that we can think of the ordered pair \((V, F_n)\) as corresponding to the vertex \( V \) itself, so that in \( [\mathcal{P}]_0 \) there is one facet for each original facet \( G \) of \( \mathcal{P} \) and one facet for each vertex \( V \) of \( \mathcal{P} \).

There are two kinds of 2-faces in \( [\mathcal{P}]_0 \). Restricting to the 2-faces which correspond to our given flag of \( \mathcal{P} \), there is a triangle which is a face of both the octahedron \((V, F_n)\) and the truncated cube \( G \), and there is an octagon which is a face of only the truncated cube \( G \). The octagon is really just a truncation of the square face \( F \), so this is a face of type 1 corresponding to the square face \( F \) of \( \mathcal{P} \), so we call it \( F \). Note that according to Definition 2.2, part 4 (with \( i = 2 \)), \( F \) is incident to \( G \), but \( F \) is not incident to the octahedron \((V, F_n)\). This is consistent with our picture. The triangular face is a face of the octahedron which replaced the vertex \( V \), so it is a face of type 2. Namely, this face is \((V, G)\). The \( V \) in the ordered pair tells us that this is a face of the vertex figure which replaced the vertex \( V \). The \( G \) in the ordered pair tells us two things. First, since \( G \) is a 3-face, it tells us that \((V, G)\) is a 2-face. Secondly, the \( G \) in the ordered pair tells us that \((V, G)\) is the 2-face of \((V, F_n)\) which is “closest” to the face \( G \). If we think geometrically, the face \((V, G)\) lies within the space occupied by the original face \( G \) of \( \mathcal{P} \). Notice that \((V, G)\) is incident to both \((V, F_n)\) and \( G \), by Definition 2.2, part 3. This tells us that this triangle is incident to both the octahedron \((V, F_n)\) and the cube \( G \).

There are two types of edges. There are those which are edges of both truncated cubes and octahedra and those which are only edges of truncated cubes. The first edge with which we are concerned is an edge of the octahedron \((V, F_n)\) as well as the truncated cube \( G \). This is an edge of type 2; it is the edge \((V, F)\). It is incident to the triangle \((V, G)\) as well as the octagon \( F \), again by Definition 2.2, part 3. Again, the \( V \) in the ordered pair tells us the
vertex figure to which this edge is incident, and the $F$ tells us the rank of the face $(V,F)$ as well as the fact that this is the edge which is incident to the face $F$. The edge which is incident to only truncated cubes is a truncation of the edge $E$ of the polytope $P$. It is therefore a type 1 face of $[P]_0$, and we will also call this edge $E$. Then $E$ is incident to $F$, but not to $(V,G)$. This can be seen in the picture, and is consistent with Definition 2.2, part 4.

In contrast to the proper faces of higher ranks, there is only one type of vertex. Every vertex in $[P]_0$ is incident to four truncated cubes and one octahedron. Since these vertices are faces of the octahedral vertex figures, and are not truncations of any faces of $P$, they are all of type 2. The vertex that we are concerned with is the vertex $(V,E)$. It is incident to the edge $E$, as well as the edge $(V,F)$ by Definition 2.2, part 3.

![Figure 2.5](image1)

**Figure 2.5**

![Figure 2.6](image2)

**Figure 2.6**

**Example 2** Figure 2.5 shows a local picture of $[P]_1$, the 1-bubble of $P$. Notice that the facets of type 1 are 1-bubbles of cubes, which are derived from cubes by replacing each edge with a digon. Since each edge of $P$ is incident to four cubes in $P$, each edge of $P$ is replaced with the 3-polytope $\{2,4\}$. These polytopes are the facets of type 2.
Again, we will focus on the proper faces of $[\mathcal{P}]_1$ which correspond to faces of the flag $\{F_{-1}, V, E, F, G, F_n\}$ of $\mathcal{P}$, and we will move down in rank. The facet of type 1 corresponding to this flag corresponds to the cube $G$ of $\mathcal{P}$. In $[\mathcal{P}]_1$ it has the shape of the 1-bubble of a cube, shown in Figure 2.6 (a), and it is called $G$ as well. The facet of type 2 corresponding to this flag is the universal polytope of type $\{2, 4\}$, shown in Figure 2.6 (b), which replaced the edge $E$. It is the face $(E, F_n)$. Again, notice that we can associate this facet with $E$ itself.

There are two types of 2-faces: the 2-faces of type 2 are 2-faces of the facets of type 2, and are digons in shape. The 2-face of type 2 that we are interested in is the face $(E, G)$. It is incident to both $(E, F_n)$ and $G$ by Definition 2.2, part 3. Notice that the $E$ in the ordered pair tells us in which facet of type 2 this digon is located, and the $G$ in the ordered pair tells us the rank of the face as well as its location in that facet. The 2-face of type 1 that we are interested in is the square $F$. Since $F$ is a 2-face of $\mathcal{P}$, the 1-bubble of $F$ is isomorphic to $F$ itself, by Lemma 2.4. This is why when $F$ is viewed as a face of $[\mathcal{P}]_1$ it retains the same combinatorial shape of the face $F$ of $\mathcal{P}$. In $[\mathcal{P}]_1$, $F$ is incident to the type 1 facet $G$, but not the type 2 facet $(E, F_n)$ by Definition 2.2, part 4.

There is only one type of edge of $[\mathcal{P}]_1$. Every edge of $[\mathcal{P}]_1$ is of type 2. The edge with which we are concerned is the edge $(E, F)$. It is incident to both the digon $(E, G)$ and the square $F$ by Definition 2.2, part 3. Again, notice that the $F$ in the ordered pair tells us that of the two edges of the digon $(E, G)$, we are referring to the edge which is incident to the face $F$.

Finally, the vertices of $[\mathcal{P}]_1$ are exactly the vertices of $\mathcal{P}$. Thus the only vertex that we are concerned with is the vertex $V$. It is incident to the edge $(E, F)$ since $V$ is incident to $E$ by Definition 2.2, part 2.

**Example 3** The 2-bubble $[\mathcal{P}]_2$ has the 1-skeleton of the cubical tessellation $\mathcal{P}$. However,
instead of each cycle of four edges being the boundary of a single square face, each cycle of four edges is the boundary of two square faces, identified along their edges with a 3-face in between them. You can picture this by simply taking the tessellation $\mathcal{P}$ and replacing each square with a “double layered” square, and then separating the interiors of the two squares slightly so that they have a square-shaped bubble between them. The facets of type 1 are cubes and the facets of type 2 are dihedral squares (universal polytopes of type $\{4, 2\}$). Again, we will look at the proper faces of $[\mathcal{P}]_2$ which correspond to faces of the flag $\{F_{-1}, V, E, F, G, F_n\}$ of $\mathcal{P}$, and we will move down in rank.

The facet of type 1 with which we are concerned is the cube $G$. It is isomorphic to a 2-bubble of the 3-face $G$ of $\mathcal{P}$, which is isomorphic to $G$ itself by Lemma 2.4. Hence $G$ retains the combinatorial shape of the cube which it had in $\mathcal{P}$. The facet of type 2 which corresponds to our given flag is the dihedral square $(F, F_n)$. Again, we can think of this face as corresponding to the face $F$ itself, where we are informally considering the square $F$ as having been replaced with a 3-face.

The 2-faces of $[\mathcal{P}]_2$ are all of type 2 and are all squares (copies of 2-faces of facets of $\mathcal{P}$). The face that we are concerned with is the face $(F, G)$. The $F$ tells us that this is a face of the dihedral square $(F, F_n)$ and the $G$ tells us the rank as well as the fact that we are looking at the square on the side of the bubble closer to the cube $G$. By Definition 2.2, part 3, this face is incident to both the cube $G$ and the dihedral square $(F, F_n)$.

The vertices and edges of $[\mathcal{P}]_2$ are exactly the vertices and edges of $\mathcal{P}$. We will look at the vertex $V$ and the edge $E$. The vertex $V$ is still incident to $E$ by Definition 2.2, part 1. The edge $E$ is incident to $(F, G)$ because $E$ is incident to $F$ in $\mathcal{P}$ and by Definition 2.2, part 2.

The following lemmas follow directly from Definition 2.1 and Definition 2.2. In addition to their usefulness in the proofs of later results, these lemmas serve to give a further un-
derstanding of the structure of $[\mathcal{P}]_k$. We have already used many of them implicitly in the previous examples.

**Lemma 2.5** Every face of $[\mathcal{P}]_k$ of rank $i < k$ is of type 1, and every $k$-face of $[\mathcal{P}]_k$ is of type 2.

The next lemma extends the order relation given in Definition 2.2 to describe when any two faces, not necessarily of adjacent ranks, are incident. Note that the conditions of the lemma could have served as an alternative (rank-free) way of defining the partial order in $[\mathcal{P}]_k$.

**Lemma 2.6** Let $H$ and $H'$ be faces of $[\mathcal{P}]_k$ of type 1, and let $(F,G)$ and $(F',G')$ be faces of $[\mathcal{P}]_k$ of type 2. Then:

1. $H \leq_{[\mathcal{P}]_k} H'$ if and only if $H \leq_{\mathcal{P}} H'$;

2. $H \leq_{[\mathcal{P}]_k} (F,G)$ if and only if $H \leq_{\mathcal{P}} F$ (and hence $H \leq_{\mathcal{P}} F \leq_{\mathcal{P}} G$);

3. $(F,G) \leq_{[\mathcal{P}]_k} H$ if and only if $G \leq_{\mathcal{P}} H$ (and hence $F \leq_{\mathcal{P}} G \leq_{\mathcal{P}} H$);

4. $(F,G) \leq_{[\mathcal{P}]_k} (F',G')$ if and only if $F = F'$ and $G \leq_{\mathcal{P}} G'$.

Note that the requirement $F = F'$ in Lemma 2.6, part 4 is equivalent to the requirement $F \leq_{\mathcal{P}} F'$, since the first component of a face of type 2 is always a face of $\mathcal{P}$ of rank $k$.

From now on we will drop the subscript on partial order signs, unless we want to stress that we are referring to incidence in a specific polytope. This is unambiguous by Lemma 2.6, part 1.

The following lemma describes which faces can be incident to faces of type 2. In particular, it states that the only faces of type 1 which are incident to faces of type 2 and are of smaller
rank are those of rank less than \(k\) (which are necessarily of type 1 by Lemma 2.5). It then uses this fact to describe the types of chains which can occur in \([\mathcal{P}]_k\).

**Lemma 2.7** If \((F,G)\) is a face of \([\mathcal{P}]_k\) of type 2, and \(H\) is a face of \([\mathcal{P}]_k\) of type 1, with \(H \leq (F,G)\), then \(H\) is a face of rank \(i < k\). Furthermore, if \((F,G)\) is a face of \([\mathcal{P}]_k\) of type 2, then any maximal chain with greatest element \((F,G)\) in \([\mathcal{P}]_k\) has the form

\[
\{F_{-1}, \ldots, F_{k-1}, (F, F_{k+1}), (F, F_{k+2}), \ldots, (F, G)\},
\]

where \(\{F_{-1}, \ldots, F_{k-1}, F, F_{k+1}, \ldots, G\}\) is a maximal chain with greatest element \(G\) in \(\mathcal{P}\). In particular, all faces in this chain of rank less than \(k\) are of type 1, and all faces of rank greater than or equal to \(k\) are of type 2 and have \(F\) as the first component of the ordered pair.

The first part of Lemma 2.7 is consistent with Lemma 2.6, part 2, since in Lemma 2.6, \(F\) is a \(k\)-face of \(\mathcal{P}\). Thus the requirement \(H \leq F\) in Lemma 2.6, part 2, implies that \(H\) is a face of rank less than \(k\).

The following lemma extends Lemma 2.7 to describe the types of the faces in any flag of \([\mathcal{P}]_k\). Since every flag of \([\mathcal{P}]_k\) contains a face of type 2 (namely, the \(k\)-face), every flag of \([\mathcal{P}]_k\) contains a chain of the form described in Lemma 2.7. In fact, every flag of \([\mathcal{P}]_k\) has a maximal face of type 2, so it must contain a chain of the form described in Lemma 2.7 such that all faces in the flag of ranks larger than those included in that chain are of type 1. Additionally, every flag of \([\mathcal{P}]_k\) has \(F_n\) as a maximal element, which is of type 1.

**Lemma 2.8** Every flag of \([\mathcal{P}]_k\) is of the form

\[
\{F_{-1}, \ldots, F_{k-1}, (F_k, F_{k+1}), (F_k, F_{k+2}), \ldots, (F_k, F_i), F_i, F_{i+1}, \ldots, F_n\}
\]
for some $i$ such that $k + 1 \leq i \leq n$ and some flag $\{F_{-1}, \ldots, F_k, \ldots, F_i, \ldots, F_n\}$ of $\mathcal{P}$.

The following lemma formalizes the notion that when constructing $[\mathcal{P}]_k$ from $\mathcal{P}$, the $k$-faces are replaced by several copies of themselves.

**Lemma 2.9** Let $\mathcal{P}$ be an $n$-polytope, let $-1 \leq i < k \leq n - 1$, let $H$ be an $i$-face of $\mathcal{P}$, $F$ be a $k$-face of $\mathcal{P}$, and $G$ be a $(k + 1)$-face of $\mathcal{P}$ with $H \leq F \leq G$. Then the section $(F, G)/H$ of $[\mathcal{P}]_k$ is isomorphic to the section $F/H$ of $\mathcal{P}$. In particular, (letting $H = F_{-1}$) every $k$-face of $[\mathcal{P}]_k$ is isomorphic to the $k$-face of $\mathcal{P}$ which it replaced.

The following lemma follows from Lemma 2.7. In particular, if we take $(F, G)$ to be a facet of type 2 (so $G = F_n$), then this new lemma tells us that the section $(F, G)/H$ is isomorphic to the $k$-face figure of $\mathcal{P}$. This is consistent with our claim that $[\mathcal{P}]_k$ is a generalization of the truncation operation, in which the new facets are vertex figures of the original polytope.

**Lemma 2.10** Let $\mathcal{P}$ be an $n$-polytope, let $0 \leq k < i \leq n$ and let $F$ be a $k$-face of $\mathcal{P}$, $G$ be an $i$-face of $\mathcal{P}$, $H$ be a $(k - 1)$-face of $\mathcal{P}$ with $H \leq F \leq G$. Then the section $(F, G)/H$ of $[\mathcal{P}]_k$ is isomorphic to the section $G/F$ of $\mathcal{P}$.

**Proof** By Lemma 2.7, every proper face of this section must be of the form $(F, K)$ for some face $K$ of $\mathcal{P}$. The existence of the face $(F, K)$ implies that $F \leq \mathcal{P} K$, and the fact that $(F, K)$ is in the section (and therefore is incident to $(G, K)$) implies that $K \leq \mathcal{P} G$. If $K$ is a face of $\mathcal{P}$ with $F \leq \mathcal{P} K \leq \mathcal{P} G$, then it must be the case that $H \leq \mathcal{P} K$, and thus $(F, K)$ is a face of the section $(F, G)/H$ of $[\mathcal{P}]_k$. Therefore, there is a one-to-one correspondence between proper faces of $(F, G)/H$ and faces $K$ of $\mathcal{P}$ with $F \leq K \leq G$, which are exactly the proper faces of the section $G/F$ of $\mathcal{P}$. By Lemma 2.6, part 4, $(F, K) \preceq [\mathcal{P}]_k (F, K')$ if and only if $K \preceq \mathcal{P} K'$. This completes the proof. □
The following lemma describes the structure of the facets of $[\mathcal{P}]_k$ of type 1. It also points out that the $[\mathcal{P}]_k$-construction is somewhat recursive in nature. Note that we have yet to prove that $[\mathcal{P}]_k$ is a polytope, so we cannot assume that the facets of $[\mathcal{P}]_k$ are polytopal. Therefore for the sake of the following lemma, for $\mathcal{K}$ a ranked poset, we define $[\mathcal{K}]_k$ to be the ranked poset whose face structure is determined by Definition 2.1 and Definition 2.2.

**Lemma 2.11** Let $K$ be a facet of $\mathcal{P}$, let $\mathcal{K}$ be the ranked poset of rank $n - 1$ isomorphic to the section $K/F_{-1}$ of $\mathcal{P}$, and let $0 \leq k \leq n - 2$. When viewed in $[\mathcal{P}]_k$, the faces $K$ and $F_{-1}$ determine a section $K/F_{-1}$ of $[\mathcal{P}]_k$ which is isomorphic to $[\mathcal{K}]_k$.

**Proof** Let $i \neq k$, and let $H$ be an $i$-face of $[\mathcal{P}]_k$ of type 1. Then $H$ is in the section $K/F_{-1}$ of $[\mathcal{P}]_k$ if and only if $H \leq_P K$, by Lemma 2.6, part 1. Now let $i \geq k$, and let $(F, G)$ be an $i$-face of $[\mathcal{P}]_k$ of type 2 (so that $F$ and $G$ are $k$- and $(i + 1)$-faces of $\mathcal{P}$, respectively). Then $(F, G)$ is in the section $K/F_{-1}$ of $[\mathcal{P}]_k$ if and only if $F \leq G \leq K$, by Lemma 2.6, part 3. These conditions describe exactly the faces of $[\mathcal{P}]_k$ which are elements of $[\mathcal{K}]_k$. The partial order is clearly the same. □

**Lemma 2.12** Assume that $k \geq 1$, and that $F, G$ and $H$ are faces of $\mathcal{P}$ of ranks $k$, $k + 2$ and $k - 2$, respectively, such that $H \leq F \leq G$. Then $H$ and $(F, G)$ are faces of ranks $k - 2$ and $k + 1$ of $[\mathcal{P}]_k$, respectively, and the 2-section $(F, G)/H$ of $[\mathcal{P}]_k$ is a digon.

**Proof** There are only two $k$-faces of $[\mathcal{P}]_k$ incident to $(F, G)$, namely $(F, G')$ and $(F, G'')$, for $G'$ and $G''$ the only two $(k + 1)$-faces that are incident to both $F$ and $G$ in $\mathcal{P}$. They are both incident to $H$ because $H \leq F$. In addition, the only $(k - 1)$-faces in the section are the two $(k - 1)$-faces in the section $F/H$ of $\mathcal{P}$. □

It is because of this lemma that we call $[\mathcal{P}]_k$ a $k$-bubble. If $k \neq 0$, $[\mathcal{P}]_k$ is degenerate in the sense that “bubbles” occur at the $k$th level. However, some 2-sections at that level are not digons in general, so these bubbles may not occur everywhere at the $k$th level.
Lemma 2.13 Every \((k+1)\)-face of \([P]_k\) of type 2 is a ditope over a \(k\)-face of \(P\). In particular, the \((k+1)\)-face \((F,G)/F-1\) of \([P]_k\) is a ditope over the \(k\)-face \(F\) of \(P\), meaning two copies of \(F\) identified along their “boundaries” (or more exactly, a \((k+1)\)-polytope with just two facets, each isomorphic to \(F\)).

Proof If \(G'\) and \(G''\) are the two \((k+1)\)-faces which lie between \(F\) and \(G\) in \(P\), as in the proof of Lemma 2.12, the sections \((F,G')/F-1\) and \((F,G'')/F-1\) of \([P]_k\) are both isomorphic to the section \(F/F-1\) of \(P\) by Lemma 2.9. Therefore, this result follows from Lemma 2.12.

We now are ready to establish the main theorem of this section, saying that \([P]_k\) is an abstract polytope of rank \(n\) for each \(k\).

Theorem 2.14 If \(P\) is an \(n\)-polytope, then \([P]_k\) is an \(n\)-polytope, for \(k = 0, \ldots, n - 1\).

Proof Since \([P]_{n-1} \cong P\) by Lemma 2.4, this is clearly true when \(k = n - 1\), so we may assume that \(k \leq n - 2\). The first two conditions which need to be satisfied for \([P]_k\) to be a polytope, namely the existence of a least and greatest element, and the fact that maximal chains have length \(n + 2\), are clear from the definition.

We will now prove a third required condition, the diamond condition. We will prove this by cases. In each case we will show that the diamond condition is satisfied. We will continuously make use of the fact that \(P\) is a polytope, and thus every 1-section of \(P\) has exactly two proper faces. Every 1-section of \([P]_k\) falls into one of the following cases for some flag \(\{F_{-1}, F_0, \ldots, F_n\}\) of \(P\). Unless otherwise specified, the sections referenced in each of the following cases are sections of \([P]_k\).

Any 1-section of \([P]_k\) whose maximal face is a face of rank less than \(k\) is an example of Case 1 for some flag \(\{F_{-1}, F_0, \ldots, F_n\}\) of \(P\).

Case 1: For \(0 \leq i \leq k-2\), the section \(F_{i+1}/F_{i-1}\) of \([P]_k\) is isomorphic to the corresponding section \(F_{i+1}/F_{i-1}\) of \(P\), thus it satisfies the diamond condition.
Any 1-section of $[\mathcal{P}]_k$ whose maximal face is a $k$-face is an example of Case 2 for some flag $\{F_{-1}, F_0, \ldots, F_n\}$ of $\mathcal{P}$.

**Case 2:** The set of $(k-1)$-faces of $[\mathcal{P}]_k$ is exactly the set of $(k-1)$-face of $\mathcal{P}$. Additionally, if $F$ is a $(k-1)$-face of $[\mathcal{P}]_k$, then $F \leq_F (F_k, F_{k+1})$ if and only if $F \leq \mathcal{P} F_k$. Therefore, the set of proper faces of $(F_k, F_{k+1})/F_{k-2}$ is

$$\{F \in \mathcal{P}_{k-1} : F_{k-2} \leq F \leq F_k\}.$$ 

Thus this section is clearly isomorphic to the 1-section $F_k/F_{k-2}$ of $\mathcal{P}$.

Any 1-section of $[\mathcal{P}]_k$ whose maximal face is a $(k+1)$-face is an example of one of the following two subcases of Case 3, for some flag $\{F_{-1}, F_0, \ldots, F_n\}$ of $\mathcal{P}$.

**Case 3a:** The set of proper faces of $(F_k, F_{k+2})/F_{k-1}$ is

$$\{(F_k, G) : F_{k-1} \leq F_k \leq G \leq F_{k+2}\} = \{(F_k, G) : F_k \leq G \leq F_{k+2}\}.$$ 

Thus this section is clearly isomorphic to the 1-section $F_{k+2}/F_k$ of $\mathcal{P}$.

**Case 3b:** The set of proper faces of $F_{k+1}/F_{k-1}$ is $\{(G, F_{k+1}) : F_{k-1} \leq G \leq F_{k+1}\}$. Thus this section is clearly isomorphic to the 1-section $F_{k+1}/F_{k-1}$ of $\mathcal{P}$.

Any 1-section of $[\mathcal{P}]_k$ whose maximal face is an $(i+1)$-face with $k+1 \leq i \leq n-1$ is an example of one of the following three subcases of Case 4, for some flag $\{F_{-1}, F_0, \ldots, F_n\}$ of $\mathcal{P}$.

**Case 4a:** For $k+1 \leq i \leq n-1$, the proper faces of the section $F_{i+1}/(F_k, F_i)$ are $F_i$ and $(F_k, F_{i+1})$. Thus this section clearly satisfies the diamond condition.

**Case 4b:** For $k+1 \leq i \leq n-1$, the set of proper faces of the section $(F_k, F_{i+2})/(F_k, F_i)$ is $\{(F_k, G) : F_i \leq G \leq F_{i+2}\}$. Thus this section is clearly isomorphic to the 1-section $F_{i+2}/F_i$.
of $\mathcal{P}$.

**Case 4c:** For $k + 2 \leq i \leq n - 1$, the set of proper faces of the section $F_{i+1}/F_{i-1}$ is \( \{ G : F_{i-1} \leq G_i \leq F_{i+1} \} \). Thus this section is clearly isomorphic to the 1-section $F_{i+1}/F_{i-1}$ of $\mathcal{P}$.

Thus $[\mathcal{P}]_k$ satisfies the diamond condition.

We will now prove the last required condition, namely that $[\mathcal{P}]_k$ is strongly flag-connected. We need to prove that every section of $[\mathcal{P}]_k$ is flag-connected. The proof will consider four cases of sections of $[\mathcal{P}]_k$. The first two cases are both cases in which no facet of $[\mathcal{P}]_k$ is a proper face of the section under consideration. These two cases are exhaustive of all such sections, though they are not mutually exclusive. The last two cases assume that the section under consideration contains facets of $[\mathcal{P}]_k$ as proper faces. These last two cases are mutually exclusive.

**Case 1:** We will consider a section of $[\mathcal{P}]_k$ which is included in a section $(F,F_n)/F_{n-1}$ of $[\mathcal{P}]_k$ where $(F,F_n)$ is a facet of type 2. In particular, any such section can be constructed as follows. For $-1 \leq j < i \leq n - 1$, let $H_i$ be an $i$-face of $[\mathcal{P}]_k$ (of either type) such that $H_i \leq (F,F_n)$ and let $H_j$ be a $j$-face of $[\mathcal{P}]_k$ (of either type) such that $H_j \leq H_i$. We will consider the section $H_i/H_j$. Every face of $H_i/H_j$ is incident to $(F,F_n)$, which is a face of type 2. Thus, by Lemma 2.7, all $m$-faces of $[\mathcal{P}]_k$ with $m \geq k$ which are contained in the section $H_i/H_j$ are of type 2.

If $j \geq k - 1$ then all proper faces of this section are of type 2. In particular, since the proper faces are all incident to $(F,F_n)$, they are all of the form $(F,G)$ for some $G$, by Lemma 2.6, part 4. If $j > k - 1$, $H_i = (F,H)$ and $H_j = (F,H')$ say, then the section $H_i/H_j$ is isomorphic to the section $H/H'$ of $\mathcal{P}$ (where the isomorphism sends each face $(F,G)$ of $[\mathcal{P}]_k$ to the face $G$ of $\mathcal{P}$) and is thus flag-connected. If $j = k - 1$ and $H_i = (F,H)$, then $H_i/H_j$ is isomorphic to the section $H/F$ of $\mathcal{P}$, by Lemma 2.10. In either case, the section
$H_i/H_j$ is isomorphic to a section of $\mathcal{P}$ and therefore flag-connected.

If $i < k$ then $H_i/H_j$ is just a section of $\mathcal{P}$ and is thus flag-connected. Further, if $i = k$ and again $H_i = (F, H)$, then the section $H_i/H_j$ is isomorphic to the section $F/H_j$ of $\mathcal{P}$ by Lemma 2.9, and thus is flag-connected.

Thus, it remains to consider the case where $j < k - 1$ and $i > k$. In this case, there are $k$-faces and $(k - 1)$-faces among the proper faces of the section. We need to join two flags of the section by a sequence of successively adjacent flags.

Suppose we have two flags of this section. By Lemma 2.7 and the fact that $i > k$, and thus $H_i = (F, F_{i+1})$ for some $F_{i+1}$, they are necessarily of the form:

$$\Phi = \{H_j, F_{j+1}, \ldots, F_{k-1}, (F, F_{k+1}), (F, F_{k+2}), \ldots, (F, F_i), (F, F_{i+1})\}$$

and

$$\Psi = \{H_j, G_{j+1}, \ldots, G_{k-1}, (F, G_{k+1}), (F, G_{k+2}), \ldots, (F, G_i), (F, F_{i+1})\},$$

where $H_i = (F, F_{i+1})$ and $\{H_j, F_{j+1}, \ldots, F_{k-1}, F, F_{k+1}, F_{k+2}, \ldots, F, F_{i+1}\}$ and $\{H_j, G_{j+1}, \ldots, G_{k-1}, F, G_{k+1}, G_{k+2}, \ldots, G_i, F_{i+1}\}$ are chains of faces of $\mathcal{P}$ of successively adjacent ranks. In particular, note that $G_{k-1} \leq F$ in $\mathcal{P}$ so $G_{k-1} \leq (F, F_{k+1})$ in $[\mathcal{P}]_k$.

By the argument that we used in the case that $H_i$ is a $k$-face, we know that every section contained in $(F, F_n)/F_{-1}$ whose maximal face is a $k$-face is flag-connected, and thus we know that $(F, F_{k+1})/H_j$ is flag-connected. Therefore there is a sequence of successively adjacent flags from

$$\Phi = \{H_j, F_{j+1}, \ldots, F_{k-1}, (F, F_{k+1}), (F, F_{k+2}), \ldots, (F, F_i), (F, F_{i+1})\}$$

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\[ \Phi' = \{ H_j, G_{j+1}, \ldots, G_{k-1}, (F, F_{k+1}), (F, F_{k+2}), \ldots, (F, F_i), (F, F_{i+1}) \}, \]

which can be derived from a similar such sequence of flags in \((F, F_{k+1})/H_j\) by appending the chain \(((F, F_{k+2}), \ldots, (F, F_i), (F, F_{i+1}))\) to each flag of \((F, F_{k+1})/H_j\) in the sequence. Similarly, by the argument that we used in the case that \(H_j\) is a \((k-1)\)-face, we know that every section whose minimal face is a \((k-1)\)-face is flag-connected, and thus \(H_i/G_{k-1}\) is flag-connected. Therefore, there is a sequence of successively adjacent flags from

\[ \Phi' = \{ H_j, G_{j+1}, \ldots, G_{k-1}, (F, F_{k+1}), (F, F_{k+2}), \ldots, (F, F_i), (F, F_{i+1}) \} \]

to

\[ \Psi = \{ H_j, G_{j+1}, \ldots, G_{k-1}, (F, G_{k+1}), (F, G_{k+2}), \ldots, (F, G_i), (F, F_{i+1}) \} \]

which can be derived from the corresponding sequence of flags in \(H_i/G_{k-1}\) by appending the chain \(\{H_j, G_{j+1}, \ldots, G_{k-1}\}\) to each flag of \(H_i/G_{k-1}\) in the sequence. Joining these two sequences of flags together yields a sequence of successively adjacent flags from \(\Phi\) to \(\Psi\).

Thus any section of \([P]_k\) which is included in a facet of type 2 is flag-connected.

**Case 2:** We now consider the case where the section is included in a facet of \([P]_k\) of type 1, say \(K = F_{n-1}/F_{-1}\). Then this is a section of \([K]_k\), by Lemma 2.11. By induction on \(n\), the rank of the poset, holding \(k\) fixed, along with the fact that the \(k\)-bubble of a rank \(k+1\) poset is the poset itself, any such section must be flag-connected in \([K]_k\), and hence in \([P]_k\).

**Case 3:** We now consider sections of \([P]_k\) which include facets as proper faces. We will first consider a section of the form \(F_n/H\) where \(H\) is a face of type 1. (That is, the section is the co-face of \(H\) in \([P]_k\)).

For \(\Phi = \{F_{-1}, F_0, F_1, \ldots, F_n\} \in \mathcal{F}(P)\), let \(f(\Phi) \in \mathcal{F}([P]_k)\) denote the flag
\{F_{-1}, F_0, F_1, \ldots, F_{k-1}, (F_k, F_{k+1}), F_{k+1}, \ldots, F_n\}.

We will also refer to any “restriction” of the map $f$ to a section of $\mathcal{P}$ as $f$.

We want to show that for any two flags of this section, there is a sequence of successively adjacent flags from one to the other. We first consider the special case that these two flags are of the form $f(\Phi)$ and $f(\Psi)$ for some flags $\Phi$ and $\Psi$ of the coface of $H$ in $\mathcal{P}$, which is a section of $\mathcal{P}$ of the same rank. We will deal with the general case later.

By the strong flag-connectivity of $\mathcal{P}$, there is a sequence of successively adjacent flags in the section of $\mathcal{P}$ from $\Phi$ to $\Psi$. We claim that this sequence gives rise to a sequence of successively adjacent flags of the section of $[\mathcal{P}]_k$ from $f(\Phi)$ to $f(\Psi)$. Clearly, it is sufficient to explain the case that $\Phi$ and $\Psi$ are adjacent. Now, if two flags $\Phi$ and $\Psi$ are $i$-adjacent in a section of $\mathcal{P}$, then in $[\mathcal{P}]_k$ either $f(\Phi)$ is $i$-adjacent to $f(\Psi)$ (and then $i \neq k + 1$ and we are done), or $i = k + 1$, since only the replacement of the face in $\Phi$ of rank $k + 1$ could result in a change of more than one face in $f(\Phi)$ (and then, in fact, in the change of exactly two faces).

Now suppose $i = k + 1$. Then the $(k + 1)$-faces must be proper faces of the section. Additionally, the $k$-faces must be proper faces of the section because we assumed $H$ to be of type 1, and all $k$-faces are of type 2 by Lemma 2.5. Hence we may assume that $\Phi$ and $\Psi$, respectively, contain the chains $\{F_k, F, F_{k+2}\}$ and $\{F_k, G, F_{k+2}\}$, where $F$ and $G$ are the two $(k + 1)$-faces of $\mathcal{P}$ which are incident to both $F_k$ and $F_{k+2}$. Then we can define a sequence of successively adjacent flags

$$f(\Phi) = \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3 = f(\Psi)$$

as follows:

We noted earlier that $f(\Phi)$ and $f(\Psi)$ only differ in the $k$- and $(k+1)$-faces. Accordingly,
in this sequence of adjacent flags each adjacency will be either a \( k \)- or \((k + 1)\)-adjacency. Thus, for simplicity, we will not list all of the faces of the \( \Lambda_j \) which remain fixed throughout the sequence. More explicitly,

\[
\begin{align*}
\Lambda_0 & \text{ is } f(\Phi) \text{ so it contains the chain } \{(F_k, F), F, F_{k+2}\}; \\
\Lambda_1 & \text{ is } (k + 1)\text{-adjacent to } \Lambda_0 \text{ so it contains the chain } \{(F_k, F), (F_k, F_{k+2}), F_{k+2}\}; \\
\Lambda_2 & \text{ is } (k)\text{-adjacent to } \Lambda_1 \text{ so it contains the chain } \{(F_k, G), (F_k, F_{k+2}), F_{k+2}\}; \\
\Lambda_3 & \text{ is } (k + 1)\text{-adjacent to } \Lambda_2 \text{ so it contains the chain } \{(F_k, G), G, F_{k+2}\}.
\end{align*}
\]

Since all of the other faces of \( f(\Phi) \) and \( f(\Psi) \) have remained fixed, we note that \( \Lambda_3 = f(\Psi) \).

Observe that in essence we have only worked on the section \( F_{k+2}/F_k \) of \([\mathcal{P}]_k\). Therefore the sequence of successively adjacent flags in the section of \( \mathcal{P} \) from \( \Phi \) to \( \Psi \) gives rise to a sequence of successively adjacent flags in \([\mathcal{P}]_k\) from \( f(\Phi) \) to \( f(\Psi) \).

We now deal with the case of arbitrary flags of \([\mathcal{P}]_k\), which are not necessarily of the form \( f(\Phi) \) for any flag \( \Phi \) of \( P \). Say that \( \Lambda \) and \( \Lambda' \) are any flags in the section \( F_n/H \) of \([\mathcal{P}]_k\) (recall here our basic assumption for Case 3). In order to construct a sequence of successively adjacent flags from \( \Lambda \) to \( \Lambda' \), it is sufficient to construct sequences of successively adjacent flags from \( \Lambda \) and \( \Lambda' \) to some flags of the form \( f(\Phi) \) and \( f(\Psi) \), respectively, and then to appeal to the previous case. In fact, it is sufficient to show this for only one of them, say \( \Lambda \).

First consider the case where the facet of the flag \( \Lambda \) is of type 1, say, \( F_{n-1} \). Then \( F_{n-1} \) and \( H \) are both faces of \([\mathcal{P}]_k\) of type 1, so both are also faces of \( \mathcal{P} \), and determine a section \( F_{n-1}/H \) of \( \mathcal{P} \). Let \( \Phi \) be a flag of this section of \( \mathcal{P} \). Then \( f(\Phi) \) is in the section \( F_{n-1}/H \) of \([\mathcal{P}]_k\). Additionally, the section \( F_{n-1}/H \) of \([\mathcal{P}]_k\) has already been shown to be flag-connected (under Case 2). Thus there is a sequence of successively adjacent flags in the section \( F_{n-1}/H \) of \([\mathcal{P}]_k\) from \( \Lambda \setminus \{F_n\} \) to \( f(\Phi) \setminus \{F_n\} \). This sequence gives a sequence from \( \Lambda \) to \( f(\Phi) \) by simply adding \( F_n \) to each flag in that sequence. This settles the case that the facet in \( \Lambda \) is
of type 1.

If the facet of \( \Lambda \) is of type 2, say of the form \((F_k, F_n)\), then \( \Lambda \) must contain an \((n-2)\)-face of type 2, by Lemma 2.7 and the assumption that \( k \leq n - 2 \). Say this \((n-2)\)-face is \((F_k, F_{n-1})\). The \((n-1)\)-adjacent flag \( \Lambda^{n-1} \) of \( \Lambda \) has \( F_{n-1} \) as its facet, and thus determines (by omission of \( F_n \)) a flag of \( F_{n-1}/H \). We have just shown (in the previous case) that \( \Lambda^{n-1} \) can be joined to a flag of type \( f(\Phi) \) by a sequence of successively adjacent flags. Since we can add \( \Lambda \) to the beginning of this sequence, \( \Lambda \) can also be joined to a flag of type \( f(\Phi) \) by a sequence of successively adjacent flags.

**Case 4:** The case where the section of \([\mathcal{P}]_k\) includes facets as proper faces and the minimal face of the section is a face of type 2 is very similar. Consider the section \( F_n/(F_k, F_{i+1}) \), where \((F_k, F_{i+1})\) is an \( i \)-face of type 2 of \([\mathcal{P}]_k\). Notice that \( i \geq k \) by Lemma 2.5.

We will consider the section \( F_n/F_{i+1} \) of \( \mathcal{P} \). Notice that this is a section of rank one less than the rank of the section \( F_n/(F_k, F_{i+1}) \) of \([\mathcal{P}]_k\). For \( \Phi = \{F_{i+1}, F_{i+2}, \ldots, F_n\} \) a flag of the section \( F_n/F_{i+1} \) of \( \mathcal{P} \), let \( g(\Phi) \) be the flag \( \{(F_k, F_{i+1}), F_{i+1}, F_{i+2}, \ldots, F_n\} \) of \( F_n/(F_k, F_{i+1}) \), so that \( g(\Phi) \) contains all of the faces of \( \Phi \) in addition to the face \((F_k, F_{i+1})\). Again, we will also refer to any “restriction” of \( g \) as \( g \).

Here, too, we will first consider the special case that the two flags we wish to join are of the form \( g(\Phi) \) and \( g(\Psi) \) for some flags \( \Phi \) and \( \Psi \) of the section \( F_n/F_{i+1} \) of \( \mathcal{P} \). For \( g(\Phi) \) and \( g(\Psi) \), the existence of a sequence of successively adjacent flags is clear by the strong flag-connectivity of \( \mathcal{P} \). In fact, there is a sequence of successively adjacent flags from \( \Phi \) to \( \Psi \), and this sequence gives rise to a sequence of successively adjacent flags from \( g(\Phi) \) to \( g(\Psi) \) by simply adjoining the face \((F_k, F_{i+1})\) to each flag in this sequence of flags.

Again, we must now show that there is a sequence of successively adjacent flags in \( F_n/(F_k, F_{i+1}) \) from an arbitrary flag \( \Lambda \) to a flag of the form \( g(\Phi) \) for some \( \Phi \). Let \( F_{n-1} \) be an \((n-1)\)-face of \( F_n/(F_k, F_{i+1}) \) of type 1. Let \( \Phi \) be a flag of the section \( F_{n-1}/F_{i+1} \) of \( \mathcal{P} \).
so that \( g(\Phi) \) is in \( F_{n-1}/(F_k, F_{i+1}) \). Just as in Case 3 (where the minimal face of the section was of type 1), we know that each flag of \( F_n/(F_k, F_{i+1}) \) either has a facet of type 1 or is \((n-1)\)-adjacent to such a flag, and we know that sections which are contained in facets of type 1 are strongly flag-connected (by Case 2). Thus, by adjoining \( F_n \) to each flag in a sequence of flags, we can derive a corresponding sequence of flags in \( F_n/(F_k, F_{i+1}) \) from \( \Lambda \) to \( g(\Phi) \). This completes the proof that \([\mathcal{P}]_k\) is strongly flag-connected. Thus \([\mathcal{P}]_k\) is an \( n \)-polytope. \( \square \)

2.3 The automorphism group of \([\mathcal{P}]_k\)

By Theorem 2.14 we can now think of \([\mathcal{P}]_k\) as a polytope, rather than a ranked poset. We can, therefore, begin to concentrate on the symmetries of \([\mathcal{P}]_k\) from a more geometric point of view. In particular, we will attempt to determine its automorphism group. The following lemma says that \([\mathcal{P}]_k\) has all of the symmetries that \( \mathcal{P} \) has. Notice that this does not mean that \([\mathcal{P}]_k\) is as highly symmetric as \( \mathcal{P} \) may be because, in general, \([\mathcal{P}]_k\) has more faces.

**Lemma 2.15** Let \( \mathcal{P} \) be a polytope of rank \( n \geq 3 \), and let \( 0 \leq k \leq n - 1 \). Then \( \Gamma(\mathcal{P}) \) is isomorphic to a subgroup of \( \Gamma([\mathcal{P}]_k) \).

**Proof** We will construct an injective homomorphism \( f \) from \( \Gamma(\mathcal{P}) \) to \( \Gamma([\mathcal{P}]_k) \) as follows. For \( \sigma \) an automorphism of \( \mathcal{P} \), we define \( f(\sigma) \) by \( f(\sigma)(F) = \sigma(F) \) for any face \( F \) of type 1 and \( f(\sigma)((F, G)) = (\sigma(F), \sigma(G)) \) for any face \( (F, G) \) of type 2. The fact that \( f(\sigma) \) is a rank- and incidence-preserving bijection of \([\mathcal{P}]_k\) follows from the corresponding facts about \( \sigma \) in \( \mathcal{P} \). Thus \( f \) is a map into \( \Gamma([\mathcal{P}]_k) \). It is likewise clear that \( f \) is an injective homomorphism from \( \Gamma(\mathcal{P}) \) to \( \Gamma([\mathcal{P}]_k) \). \( \square \)

The following result begins to discuss some of the symmetries of \([\mathcal{P}]_k\) when \( \mathcal{P} \) is a regular \( n \)-polytope (that is, \( \Gamma(\mathcal{P}) \) is flag-transitive). In fact, weaker assumptions on \( \mathcal{P} \) would suffice.
Proposition 2.16  If $\mathcal{P}$ is a regular $n$-polytope and $0 \leq k \leq n - 1$, then $[\mathcal{P}]_k$ is $i$-face-transitive for $i = -1, 0, \ldots, k$.

Proof  For $-1 \leq i \leq k - 1$ this follows immediately from the transitivity of $\Gamma(\mathcal{P})$ on the $i$-faces of $\mathcal{P}$, and the fact that $\Gamma(\mathcal{P}) \leq \Gamma([\mathcal{P}]_k)$. The transitivity on the $k$-faces follows from the transitivity of $\Gamma(\mathcal{P})$ on the chains of $\mathcal{P}$ of type $\{k, k+1\}$ and the fact that $\Gamma(\mathcal{P}) \leq \Gamma([\mathcal{P}]_k)$. □

In determining the automorphism group $\Gamma([\mathcal{P}]_k)$ it will be of particular importance to determine when automorphisms of $[\mathcal{P}]_k$ send a face of type 1 to a face of type 2, or vice-versa. This can create some problems, so in general we want $[\mathcal{P}]_k$ to be such that all automorphisms preserve all face types, 1 or 2. This motivates the following definition.

Definition 2.17  Let $\mathcal{P}$ be a polytope and let $[\mathcal{P}]_k$ be its $k$-bubble. Then we say that an automorphism $\sigma$ of $[\mathcal{P}]_k$ preserves face types if $\sigma(F)$ is of type 1 if and only if $F$ is of type 1. (Clearly, this implies that $\sigma(F)$ is of type 2 if and only if $F$ is of type 2.) Furthermore, we say that $\Gamma([\mathcal{P}]_k)$ preserves face types if $\sigma$ preserves face types for every $\sigma$ in $\Gamma([\mathcal{P}]_k)$.

In the following lemma we will prove that if all face types of faces of any given rank are preserved by a given automorphism, then face types of faces of faces of all ranks are preserved by that automorphism. Because this result makes the distinction unnecessary, we did not formally define what it means for an automorphism to preserve face types for a given rank, though we use it in the lemma. Notice also that the face types of all faces of rank $k$ or lower are determined completely by the rank of the face (the type is 1 if the rank is less than $k$, and 2 if the rank is $k$). Therefore, the following lemma encompasses any possible rank in which face types could potentially not be preserved. Observe that the subgroup $\Gamma(\mathcal{P})$ of $\Gamma([\mathcal{P}]_k)$ preserves face types.

Lemma 2.18  Let $\mathcal{P}$ be an $n$-polytope, let $0 \leq k < i < j \leq n - 1$ and let $\sigma$ be an automorphism of $[\mathcal{P}]_k$. Then $\sigma$ preserves face types of all faces of $[\mathcal{P}]_k$ of rank $i$ if and only if it
preserves face types of all faces of $[P]_k$ of rank $j$.

**Proof** We first assume that $\sigma$ does not preserve face types of faces of rank $i$. In this case there is some $i$-face of $[P]_k$ of type 2, say $(F,G)$ such that $\sigma((F,G)) = H$ for $H$ an $i$-face of $[P]_k$ of type 1. We can make this assumption because $\sigma$ is an automorphism, so if $\sigma$ takes an $i$-face of type 1 to an $i$-face of type 2, it must also take an $i$-face of type 2 to an $i$-face of type 1. Now $(F,G)$ is contained in some $j$-face of $[P]_k$ of type 2, say $(F,G')$ where $F \leq G \leq G'$ in $P$. Because $\sigma$ preserves incidences, $\sigma((F,G))$ is contained in $\sigma((F,G'))$. Therefore, $H$ is contained in $\sigma((F,G'))$. Since $H$ is an $i$-face of type 1 with $k < i < j$, the only $j$-faces which contain $H$ are of type 1, by Lemma 2.7. Thus $\sigma((F,G'))$ is of type 1 and $\sigma$ does not preserve face types of the $j$-faces.

We now assume that $\sigma$ does not preserve face types of faces of rank $j$. In this case there is some $j$-face of $[P]_k$ of type 1, say $H$, such that $\sigma(H) = (F,G)$ for $(F,G)$ a $j$-face of $[P]_k$ of type 2. Now $H$ contains some $i$-face of $[P]_k$ of type 1, say $H'$ for some $i$-face $H'$ of $P$ such that $H' \leq H$. Because $\sigma$ preserves incidences, $\sigma(H)$ must contain $\sigma(H')$. Therefore, $(F,G)$ contains $\sigma(H')$. Since $(F,G)$ is a $j$-face of type 2 with $k < i < j$, the only $i$-faces which are contained in $(F,G)$ are of type 2, by Lemma 2.7. Thus $\sigma(H')$ is of type 2 and $\sigma$ does not preserve face types of the $i$-faces. ∎

We will soon prove that whenever $\Gamma([P]_k)$ preserves face types, $\Gamma([P]_k)$ is isomorphic to $\Gamma(P)$. However, before we do so, it is helpful to discuss certain situations in which $\Gamma([P]_k)$ preserves face types so that we have specific examples in mind. In fact, it turns out that in most cases we are interested in $\Gamma([P]_k)$ preserves face types if $k \geq 1$.

**Proposition 2.19** Let $n \geq 3$ and let $1 \leq k \leq n - 1$. Let $P$ be a polytope of rank $n$ such that none of its 2-sections $F/G$ with $F$ a $(k+1)$-face and $G$ a $(k-2)$-face is a digon. Then $\Gamma([P]_k)$ preserves face types.
**Proof** Suppose that there exists $\tau \in \Gamma([P]_k)$ that does not preserve face types. By Lemma 2.18, then $\tau$ does not preserve the face types of the $(k+1)$-faces. Thus, we have $\tau((F,G)) = G'$ for some pair of $(k+1)$-faces $(F,G)$ and $G'$ of $[P]_k$ such that $G'$ is a face of type 1. Note that we assumed that $k \geq 1$, and thus $k - 2 \geq -1$. By Lemma 2.12, for $H$ a $(k-2)$-face such that $H \subseteq F$, the section $(F,G)/H$ of $[P]_k$ is a digon. Therefore, the section $\tau((F,G))/\tau(H)$, which is the section $G'/\tau(H)$ of $[P]_k$, is also a digon. It follows that there are only two $(k-1)$-faces incident to both $\tau(H)$ and $G'$ in $[P]_k$. But $\tau(H)$ is a $(k-2)$-face and thus is necessarily of type 1, so $\tau(H)$ is also a face of $P$. Because the $(k-1)$-faces of $P$ are exactly the $(k-1)$-faces of $[P]_k$, and the 2-section $G'/\tau(H)$ of $P$ must have at least two $(k-1)$-faces, the section $G'/\tau(H)$ of $P$ is necessarily a digon. But this possibility was excluded by assumption. \hfill \Box

**Proposition 2.20** Let $n \geq 3$ and let $1 \leq k \leq n - 1$. Let $P$ be an equivelar $n$-polytope of Schl"afli type $\{p_1, p_2, \ldots, p_{n-1}\}$ with $p_k \neq 2$. Then $\Gamma([P]_k)$ preserves face types.

**Proof** In this case $P$ satisfies the conditions of Proposition 2.19. \hfill \Box

The converse of this proposition is not true. For example, let $P$ be the universal 4-polytope with Schl"afli symbol $\{2, 4, 2\}$. This polytope is equivelar and when $k = 1$, $p_k = 2$. In $[P]_1$, the facets of type 1 have Schl"afli symbol $\{2, 8\}$, because they are 1-bubbles of the facets of $P$, which are the polyhedra $\{2,4\}$. To take the 1-bubble of $\{2,4\}$, we replace each edge with a digon, leaving the polyhedron $\{2,8\}$. The facets of type 2 have Schl"afli symbol $\{2,2\}$, because the 2-faces of $P$ are digons, and these facets are dihedral digons by Lemma 2.13; alternatively, and more intuitively, in $P$ an edge is surrounded by only two facets, so the facets of $[P]_1$ of type 2 must be of type $\{2,2\}$. Thus as automorphisms of $[P]_1$ preserve isomorphism types of 3-faces, then face types of the 3-faces must be preserved, and hence all face types must be preserved by all automorphisms of this polytope. Notice that even though there are digons of types 1 and 2 (which are isomorphic as polytopes), face types
are still preserved. In the case that \( P \) is regular, we are able to come up with a condition for when face types are preserved which does not require that \( k \neq 0 \) or that \( p_k \neq 2 \). This will be given in Proposition 2.29.

**Theorem 2.21** Let \( P \) be an \( n \)-polytope such that \( \Gamma([P]_k) \) preserves face types, when \( 0 \leq k \leq n - 1 \). Then \( \Gamma([P]_k) \) is isomorphic to \( \Gamma(P) \). In particular, there is a natural group isomorphism \( f : \Gamma(P) \rightarrow \Gamma([P]_k) \) that associates with \( \sigma \in \Gamma(P) \) an \( \sigma \in \Gamma([P]_k) \) given by \( f(\sigma)(F) = \sigma(F) \) for any face \( F \) of \([P]_k \) of type 1, and \( f(\sigma)((F, G)) = (\sigma(F), \sigma(G)) \) for any face \((F, G)\) of \([P]_k \) of type 2.

**Proof** In the proof of Lemma 2.15, we showed that, for \( \sigma \in \Gamma(P) \), the mapping \( f(\sigma) \) described above is a rank- and incidence-preserving bijection of \([P]_k \) and that \( f \) is an injective homomorphism. We must now show that \( f \) is surjective.

Recall that every face of \( P \) with rank not equal to \( k \) is also a face of \([P]_k \). Given \( \tau \in \Gamma([P]_k) \), since \( \tau \) preserves face types, we can define \( \sigma \in \Gamma(P) \) by \( \sigma(F) := \tau(F) \) if \( F \) is a face of \( P \) with rank not equal to \( k \), and \( \sigma(F) := G \) if \( F \) is a \( k \)-face of \( P \) and \( \tau((F, F_n)) = (G, F_n) \). Recall that we can associate the facets of \([P]_k \) of type 2 with \( k \)-faces of \( P \). Under this association, \((F, F_n)\) would be associated with the \( k \)-face \( F \) of \( P \) and \((G, F_n)\) would be associated with the \( k \)-face \( G \) of \( P \). Thus, this is a natural way to define \( \sigma(F) \). The fact that \( \sigma \) is a rank- and incidence-preserving bijection of \( P \) follows from the corresponding facts about \( \tau \) for \([P]_k \).

We claim that \( \tau = f(\sigma) \). Clearly, by the definition of \( \sigma \) we have \( \tau(F) = \sigma(F) = f(\sigma)(F) \) for all faces \( F \) of \([P]_k \) of type 1. Now suppose that \((F, H)\) is a face of \([P]_k \) of type 2. We first show that \( \tau((F, H)) = (\sigma(F), \tau(H)) \). Because \( \tau \) is an automorphism of \([P]_k \) it must preserve ranks and incidences, and we assumed that it preserves face types. Since \((F, H) \leq H \), we have \( \tau((F, H)) \leq \tau(H) \). The only faces of type 2 that are incident to \( \tau(H) \) and of the same rank as \((F, H)\) are of the form \((G, \tau(H))\) for some \( k \)-face \( G \) of \( P \). Therefore, \( \tau((F, H)) \)
must be such a face. Additionally, \( \tau((F, H)) \leq \tau((F, F_n)) \), because \( \tau \) preserves incidence, and \( \tau((F, F_n)) = (\sigma(F), F_n) \), by definition of \( \sigma \). Thus, \( \tau((F, H)) \leq (\sigma(F), F_n) \). Thus it is necessarily the case that \( \tau((F, H)) = (\sigma(F), H') \), for some \( H' \) (which we have already determined to be \( \tau(H) \)), and thus \( \tau((F, H)) = (\sigma(F), \tau(H)) \), as claimed. It follows that \( \tau((F, H)) = (\sigma(F), \tau(H)) = (\sigma(F), \sigma(H)) = f(\sigma)((F, H)) \), as desired.

Hence, every automorphism of \([\mathcal{P}]_k\) which preserves face types has a (unique) preimage under \( f \). Therefore, if \( \Gamma([\mathcal{P}]_k) \) is such that every \( \tau \in \Gamma([\mathcal{P}]_k) \) preserves face types, then \( f \) is surjective, and therefore an isomorphism.

Many examples of polytopes to which Theorem 2.21 applies can be derived from Propositions 2.19 and 2.20. However, notice that these propositions require that \( k \geq 1 \). However, from a geometric standpoint, the case \( k = 0 \) is the most recognizable, as it is exactly the truncation operation. It is easy to picture examples of truncating a polytope where the automorphism group of the original polytope is preserved, as this is often the case. However there are cases when the truncation operation strictly increases the automorphism group. One example is the universal polytope with Schl"afli symbol \( \mathcal{P} = \{2, 4\} \). In this case, \([\mathcal{P}]_0\) is combinatorially a cube, and has a different, larger automorphism group. In this case, the facets of types 1 and 2 are both squares. The facets of type 1 are squares because the truncation of a digon is a square. The facets of type 2 are squares because the vertex figures of \( \mathcal{P} \) are squares. The automorphism group of the cube takes squares of type 1 to squares of type 2, and vice versa. Recall that the 1-bubble of the polytope \( \{2, 4, 2\} \) had digonal 2-faces of both types 1 and 2, yet the automorphism group still preserved face types, and did not take digons of one type to digons of another type. Thus, demonstrating that there are squares of both types 1 and 2 is not sufficient to claim that face types are not preserved, though in the case of \( \{2, 4\} \) they are not.

Another example where face types are not preserved by \( \Gamma([\mathcal{P}]_0) \) is slightly different. Let \( \mathcal{P} \) be the regular tessellation of \( \mathbb{R}^2 \) with Schl"afli symbol \( \{3, 6\} \). Then \([\mathcal{P}]_0\) is the tessellation
\{6,3\} whose automorphism group \(\Gamma([\mathcal{P}]_0)\) contains \(\Gamma(\mathcal{P})\) as a proper subgroup. However, this case is different because in this case \(\Gamma([\mathcal{P}]_0)\) is also isomorphic to \(\Gamma(\mathcal{P})\), but this group isomorphism is not the specific isomorphism \(f\) of Theorem 2.21. In particular, there are polytope automorphisms which do not preserve face types. In \([\mathcal{P}]_0\) the 2-faces of types both 1 and 2 are hexagons. The facets of type 1 are truncated triangles, and the facets of type 2 are the vertex figures of the original tessellation. Because the tessellation \(\{6,3\}\) is regular, the automorphism group acts 2-face transitively, and thus there are automorphisms which take hexagons of type 1 to hexagons of type 2. In other words, there are automorphisms of \([\mathcal{P}]_0\) outside of \(\Gamma(\mathcal{P})\) (when viewed as a subgroup of \(\Gamma([\mathcal{P}]_0)\)) which do not preserve face types. This example demonstrates that there are examples when the automorphism group of \([\mathcal{P}]_k\) is actually isomorphic to the automorphism group of \(\mathcal{P}\), but the isomorphism is not the specific isomorphism identified in Theorem 2.21. These examples can occur when the face types are not preserved.

As mentioned earlier, if \(k \geq 1\) there are nice sufficient conditions that guarantee that face types are preserved. We now investigate similar conditions for the case \(k = 0\).

There are many instances, however, where \(\Gamma([\mathcal{P}]_0)\) does preserve face types, thus allowing the conclusion that the groups \(\Gamma(\mathcal{P})\) and \(\Gamma([\mathcal{P}]_0)\) are isomorphic. We would like to give a condition for the case \(k = 0\), which would guarantee that \(\Gamma([\mathcal{P}]_0)\) preserves face types. Consider the rank 3 case for equivelar polytopes. Wherever there originally was a facet with \(p_1\) edges (and \(p_1\) vertices), there is now (after truncation) a facet with an edge for each former edge and each former vertex (the latter corresponding to the pair of the former vertex and the facet in question), so the new facet has \(2p_1\) edges. Where there originally was a vertex with vertex figure a \(p_2\)-gon, there is now a \(p_2\)-gon. If \(2p_1 \neq p_2\), then it is clear that the face types of the facets will be preserved. By Lemma 2.18, this is sufficient to prove that face types of all ranks are preserved.

In the following proposition, we extend this condition to ranks greater than 3.
Proposition 2.22 If $\mathcal{P}$ is an equivelar polytope of rank $n \geq 3$ of Schl"afli type $\{p_1, p_2, \ldots, p_{n-1}\}$, which is not the type $\{p_1, 2p_1, 3, \ldots, 3\}$ (that is to say that either $p_2 \neq 2p_1$ or $p_i \neq 3$ for some $3 \leq i \leq n-1$), then $\Gamma([\mathcal{P}]_0)$ preserves face types and therefore $\Gamma([\mathcal{P}]_0) \cong \Gamma(\mathcal{P})$.

Proof By Lemma 2.18, it is sufficient to show that the face types of faces of a single (positive) rank are preserved. In each of the two cases case $2p_1 \neq p_2$ and $p_i \neq 3$ for some $i \geq 3$ we will show that faces of $[\mathcal{P}]_0$ of type 1 and rank greater than or equal to 3 are not equivelar.

We know that all faces of $[\mathcal{P}]_0$ of type 2 are equivelar because they are contained in facets of type 2 of rank greater than or equal to 3, which are vertex figures of $\mathcal{P}$ by Lemma 2.10. An automorphism cannot take an equivelar face to a face which is not equivelar, so in these two cases, face types are preserved under automorphisms, and the groups are isomorphic.

When $2p_1 \neq p_2$, then we consider a 3-face $H$ of $[\mathcal{P}]_0$ of type 1. The section $H/F_{-1}$ of $[\mathcal{P}]_0$ is isomorphic to $[H/F_{-1}]_0$ (the truncation of the corresponding section of $\mathcal{P}$) by Lemma 2.11 and induction. By the above discussion of truncation of polyhedra, this polyhedron has 2-faces which are $p_2$-gons, and 2-faces which are $2p_1$-gons and is therefore not equivelar, since $2p_1 \neq 2p_2$.

When $p_i \neq 3$ for some $i \geq 3$, then we consider an $(i+1)$-face $H$ of type 1. We also consider an $(i-3)$-face $(V, G)$ of type 2 where $V$ is a vertex of $\mathcal{P}$, and $G$ is an incident $(i-2)$-face. The 3-section $H/(V, G)$ of $[\mathcal{P}]_0$ contains $i$-faces of both types 1 and 2. In particular, for every $i$-face $F$ of $\mathcal{P}$ such that $G \leq F \leq H$ in $\mathcal{P}$, the corresponding face $F$ of $[\mathcal{P}]_0$ is an $i$-face of type 1 in the section. Additionally, the face $(V, H)$ is an $i$-face of type 2 in the section. (In fact, it is the only such $i$-face.) We will show that the number of $(i-1)$-faces in the section $H/(V, G)$ to which an $i$-face is incident is different for $i$-faces of types 1 and 2. Once we have shown this, we may conclude that this 3-section is not equivelar, so the type 1 face $H/F_{-1}$ of $[\mathcal{P}]_0$ is also not equivelar. Let $F$ be an $i$-face of type 1 contained in $H/(V, G)$ so that in $\mathcal{P}$, we have $V \leq G \leq F \leq H$. We will count the number of $(i-1)$-faces to which $F$ is incident.
in the section \( H/(V,G) \). If \( F \) is incident to an \((i - 1)\)-face \( K \) of type 1 in this section, then \( K \) is also an \((i - 1)\)-face of \( P \), and \((V,G) \leq K \leq F \) in \([P]_0\). Hence, \( G \leq K \leq F \) in \( P \). Recall that \( G \) is an \((i - 2)\)-face of \( P \), so there are exactly two such \( K \), by the diamond condition of \( P \). Now if \( F \) is incident to an \((i - 1)\)-face \((V,K')\) of type 2 in this section, then \( K' \) is an \( i \)-face of \( P \), and hence \( K' \) must be \( F \) by Definition 2.2, part 3b. Hence \((V,F)\) is the only \((i - 1)\)-face to which \( F \) is incident in this section. Thus, an \( i \)-face \( F \) of type 1 is incident to exactly three \((i - 1)\)-faces in this section, so the 2-section \( F/(V,G) \) is a triangle.

Now, the only \( i \)-face of type 2 contained in \( H/(V,G) \) is \((V,H)\). Every \((i - 1)\)-face of \([P]_0\) incident to both \((V,H)\) and \((V,G)\) is of type 2, by Lemma 2.7. An \((i - 1)\)-face \((V,L)\) is incident to both \((V,H)\) and \((V,G)\) if and only if \( G \leq L \leq H \) in \( P \). Recall that \( G \) is an \((i - 2)\)-face, \( H \) is an \((i + 1)\)-face and that \( L \) is necessarily an \( i \)-face. By the ranks of \( G, L \) and \( H \), the number of such \( L \) is \( p_i \). Since \( p_i \neq 3 \), the 2-section \((V,H)/(V,G)\) is not a triangle. Thus \( H \) is not equivelar.

Since in either case we were able to show that all the faces of type 1 of a given rank are not equivelar, and all the faces of type 2 are equivelar, face types are preserved under automorphisms, which completes the proof. \( \square \)

In Proposition 2.29, we will prove a generalization of Proposition 2.22 in the case that \( P \) is regular. Next we investigate the number of flag-orbits of \([P]_k\) under the automorphism group. Recall that a polytope is an \( m \)-\emph{orbit polytope} if the polytopes has exactly \( m \) flag-orbits under its automorphism group.

**Theorem 2.23** Let \( P \) be an \( m \)-orbit \( n \)-polytope and let \( 0 \leq k \leq n - 1 \). Assume that \( \Gamma([P]_k) \) preserves face types so that \( \Gamma([P]_k) = \Gamma(P) \) (see Theorem 2.21). Then \([P]_k\) is an \( m(n - k) \)-orbit polytope. In particular, if \( P \) is a regular polytope such that \( \Gamma([P]_k) \) preserves face types, then \([P]_k\) is an \((n - k)\)-orbit polytope.
Proof Recall that \( \mathcal{F}(\mathcal{Q}) \) denotes the set of all flags of a polytope \( \mathcal{Q} \). Let \( g : \mathcal{F}([\mathcal{P}]_k) \to \mathcal{F}(\mathcal{P}) \) be the surjective map which takes a flag \( \Psi \) of \( [\mathcal{P}]_k \) to the flag of \( \mathcal{P} \) which contains all of the faces of type 1 of \( \Psi \) as well as all the components of the faces of type 2 of \( \Psi \). For example, if \( \Psi \) is the flag \( \{ F_{-1}, \ldots, F_{k-1}, (F_k, F_{k+1}), \ldots, (F_i, F_{i+1}), F_{i+1}, \ldots, F_n \} \), then \( g(\Psi) = \{ F_{-1}, \ldots, F_{k-1}, F_k, F_{k+1}, \ldots, F_i, F_{i+1}, \ldots, F_n \} \). This map exists by Lemma 2.8, which guarantees that the set of specified faces which are elements of \( g(\Psi) \) is a flag of \( \mathcal{P} \) for each \( \Psi \).

Each flag in \( \mathcal{F}(\mathcal{P}) \) has exactly \( n - k \) preimages under \( g \). This follows from Lemma 2.8 and the fact that for each \( i = k, k+1, \ldots, n-1 \), there is exactly one preimage whose maximal face of type 2 is a face of rank \( i \).

Then the mapping \( \Phi \to \{ \Psi : g(\Psi) = \Phi \} \) is a bijection between \( \mathcal{F}(\mathcal{P}) \) and a partition of \( \mathcal{F}([\mathcal{P}]_k) \) consisting of sets of size \( n - k \). But every automorphism of \( [\mathcal{P}]_k \) preserves face types so \( \Gamma(\mathcal{P}) \cong \Gamma([\mathcal{P}]_k) \), so \( [\mathcal{P}]_k \) has at most \( m(n - k) \) flag-orbits, regardless of whether \( \mathcal{P} \) is finite or infinite. On the other hand, no two flags of \( [\mathcal{P}]_k \) with the same image under \( g \) can be equivalent under \( \Gamma([\mathcal{P}]_k) \), once again by the fact that automorphisms of \( [\mathcal{P}]_k \) preserve face types (for each face in a flag), and each preimage of \( \Phi \) under \( g \) is a flag of \( [\mathcal{P}]_k \) with a distinct sequence of face types (since each preimage has a distinct rank occupied by the maximal face of type 2.) Thus there are exactly \( m(n - k) \) flag-orbits under \( \Gamma([\mathcal{P}]_k) \). \( \square \)

Proposition 2.24 Let \( \mathcal{P} \) be a regular polytope of rank \( n \geq 3 \), and let \( 0 \leq k \leq n - 1 \). Then any two flags of \( [\mathcal{P}]_k \) with the same sequence of face types are equivalent under \( \Gamma([\mathcal{P}]_k) \).

Proof Let \( g : \mathcal{F}([\mathcal{P}]_k) \to \mathcal{F}(\mathcal{P}) \) be as in the proof of Theorem 2.23 such that \( g \) takes a flag of \( [\mathcal{P}]_k \) to the flag of \( \mathcal{P} \) which contains all of its faces of type 1 and components of faces of type 2, and let \( f : \Gamma(\mathcal{P}) \to \Gamma([\mathcal{P}]_k) \) be as in the proof of Lemma 2.15 such that for \( \sigma \in \Gamma(\mathcal{P}) \), we define \( f(\sigma) \) by \( f(\sigma)(F) = \sigma(F) \) for any face \( F \) of type 1 and \( f(\sigma)((F, G)) = (\sigma(F), \sigma(G)) \) for any face \((F, G)\) of type 2. Let \( \Psi \) and \( \Lambda \) be two flags of \( [\mathcal{P}]_k \) which have the same sequence of face types. Because \( \mathcal{P} \) is regular, there is some automorphism \( \sigma \) of \( \mathcal{P} \) which takes \( g(\Psi) \)
to $g(\Lambda)$. As an automorphism of $[P]_k$, $f(\sigma)$ acts naturally on the flags of $[P]_k$. Under this natural action, $f(\sigma)(\Psi) = \Lambda$. □

The following proposition and Theorem 2.23 tell us that if $P$ is a regular polytope, then $[P]_k$ either has $n - k$ flag-orbits or a single flag-orbit.

**Proposition 2.25** Let $P$ be a regular polytope of rank $n \geq 3$, and let $0 \leq k \leq n - 1$. If $\Gamma([P]_k)$ does not preserve face types, then $[P]_k$ is regular.

**Proof** First of all, note that by Proposition 2.24, flags with the same sequence of face types are all contained in the same orbit under $\Gamma([P]_k)$. Also note that the sequence of face types of a flag of $[P]_k$ is determined uniquely by the largest rank for which there is a face of type 2, by Lemma 2.8. Let $O_1$ be the flag-orbit which contains all of the flags of $[P]_k$ whose only face of type 2 is the $k$-face. Of course $O_1$ may contain other flags (with different sequences of face types) as well. If $\Gamma([P]_k)$ does not preserve face types, then it does not preserve face types of some of the $(k+1)$-faces, by Lemma 2.18. Therefore, we may assume that there is some $(k+1)$-face of type 2, say $(F,G)$, and some automorphism $\tau$ of $[P]_k$ such that $\tau((F,G))$ is of type 1. Since $\tau((F,G))$ is a $(k+1)$-face of type 1, every flag which contains it must not have any faces of type 2 of ranks higher than $k$ (once again, see Lemma 2.8), and therefore must be in $O_1$. Then $\tau$ must take each flag whose $(k+1)$-face is $(F,G)$ to a flag in $O_1$, and every such flag must itself be in $O_1$. Additionally, if $\Psi$ is a flag of $[P]_k$ which contains $(F,G)$, then every flag with the same sequence of face types as $\Psi$ must be in the same flag-orbit under $\Gamma([P]_k)$ as $\Psi$, and therefore must be in $O_1$. But for every possible sequence of face types, except the sequence of face types of flags whose only face of type 2 is the $k$-face, there is a flag containing $(F,G)$. Thus, every flag must be in $O_1$, so there must only be a single flag-orbit and $P$ must be regular.

**Corollary 2.26** Let $P$ be a regular polytope of rank $n \geq 3$, and let $0 \leq k \leq n - 1$. Then $\Gamma([P]_k)$ is a string C-group.
This follows from Proposition 2.25 in the case that \( \Gamma([\mathcal{P}]_k) \) does not preserve face types, and from Theorem 2.21 in the case that \( \Gamma([\mathcal{P}]_k) \) does preserve face types. \( \square \)

**Remark** Let \( \mathcal{P} \) be a regular polytope and \( \mathcal{Q} \) a finite quotient polytope of \( \mathcal{P} \). Then \([\mathcal{Q}]_k\) is the polytope obtained from \([\mathcal{P}]_k\) by identifying any two faces of type 1 which were identified in \( \mathcal{Q} \) as well as any two faces of type 2 whose components were identified in \( \mathcal{Q} \). The latter means that if \( F \) and \( F' \) are \( k \)-faces of \( \mathcal{P} \) identified in \( \mathcal{Q} \), and \( G \) and \( G' \) are faces of \( \mathcal{P} \) with \( F < G \) and \( F < G' \) identified in \( \mathcal{Q} \), then \( (F,G) \) and \( (F',G') \) are identified in \([\mathcal{Q}]_k\).

**Proposition 2.27** If \( \mathcal{P} \) is a regular \( n \)-polytope of Schl"afli type \( \{p_1,p_2,\ldots,p_{n-1}\} \), then the section \( (F,F_n)/F_{-1} \) of \([\mathcal{P}]_k\) corresponding to a facet \( (F,F_n) \) of type 2 is a regular \((n-1)\)-polytope of Schl"afli type \( \{p_1,p_2,\ldots,p_{k-1},2,p_{k+2},\ldots,p_{n-1}\} \).

**Proof** We first show that \( (F,F_n)/F_{-1} \) is regular. By Lemma 2.7, every flag of \( (F,F_n)/F_{-1} \) is of the form
\[
\{F_{-1}, \ldots, (F,F_{k+1}), (F,F_{k+2}), \ldots, (F,F_n)\}
\]
for some flag \( \{F_{-1}, \ldots, F_{k-1}, F, F_{k+1}, F_{k+2}, \ldots, F_n\} \) of \( \mathcal{P} \). Consider the map \( g_F \) between the set of flags of \( (F,F_n)/F_{-1} \) and the set of flags of \( \mathcal{P} \) which removes the first component \( F \) from each face of the form \( (F,F_i) \) in any flag of \( (F,F_n)/F_{-1} \) and then inserts \( F \) between \( F_{k-1} \) and \( F_{k+1} \). Then \( g_F \) is a restriction of the map \( g \) described in the proof of Theorem 2.23, and is injective. In particular, \( g_F \) is a bijection between the set of flags of \( (F,F_n)/F_{-1} \) and the set of flags of \( \mathcal{P} \) which contain \( F \). Since \( \Gamma(\mathcal{P}) \) acts transitively on \( \mathcal{F}(\mathcal{P}) \), the stabilizer of the \( k \)-face \( F \) in \( \Gamma(\mathcal{P}) \) acts transitively on the set of flags of \( \mathcal{P} \) containing \( F \). But \( \Gamma(\mathcal{P}) \) is a subgroup of \( \Gamma([\mathcal{P}]_k) \) so this stabilizer is a subgroup of the group of the section \( (F,F_n)/F_{-1} \). Because the specific group isomorphism \( f \) given in Theorem 2.21 commutes with \( g_F \), this stabilizer subgroup acts transitively on the flags of \( (F,F_n)/(F_{-1}) \). Hence, this section is a regular \((n-1)\)-polytope.
The Schlafli type follows from the fact that \((F,F_n)/F_{k-1}\) is isomorphic to the section \(F_n/F\) of \(\mathcal{P}\) by Lemma 2.10, the fact that \((F,F_{k+1})/F_{-1}\) is isomorphic to the section \(F/F_{-1}\) of \(\mathcal{P}\) by Lemma 2.9, and the fact that \((F,F_{k+2})/F_{-2}\) is a digon by Lemma 2.12. □

**Remark** In particular, if \(k = 0\) in Proposition 2.27, the components of the Schlafli type begin with \(p_{k+2} = p_2\), in which case the facets of type 2 are regular polytopes of Schlafli type \(\{p_2, \ldots, p_{n-1}\}\), which are vertex figures of \(\mathcal{P}\). This fact was also previously derived from Lemma 2.10, and we have used it several times in examples.

For the following proposition recall the definitions of “uniform” and “semiregular” from Definitions 1.19 and 1.20, respectively.

**Proposition 2.28** If \(\mathcal{P}\) is a regular \(n\)-polytope, then \([\mathcal{P}]_k\) is uniform for all \(k = 1, 2, \ldots, n-1\). Moreover, \([\mathcal{P}]_{n-2}\) is semiregular for all regular \(n\)-polytopes, \(\mathcal{P}\).

**Proof** A uniform polytope is vertex-transitive with uniform facets. The vertex-transitivity of such polytopes follows from Proposition 2.16. Inductively, the uniformity of the facets of type 1 follows from Lemma 2.11, and the regularity, and the uniformity of the facets of type 2 follows from Proposition 2.27. To be semiregular, the facets must additionally be regular. We have shown that the facets of type 2 are regular. The regularity of the facets of type 1 of \([\mathcal{P}]_{n-2}\) follows from Lemma 2.11 and Lemma 2.4. □

Finally, we return to our discussion of when face types are preserved, so that much of the above theory can be applied. As we mentioned previously, the following is a generalization of Proposition 2.22 to higher values of \(k\).

**Proposition 2.29** Let \(\mathcal{P}\) be a regular \(n\)-polytope, and let \(0 \leq k \leq n-3\). Additionally, let \(\mathcal{P}\) be of Schlafli type \(\{p_1, \ldots, p_{n-1}\}\), such that either \(p_{k+2} \neq 2p_{k+1}\) or \(p_i \neq 3\) for some \(i \geq k+3\). Then \(\Gamma([\mathcal{P}]_k)\) preserves face types and thus \(\Gamma([\mathcal{P}]_k) \cong \Gamma(\mathcal{P})\). In particular, this is the case even when \(p_k = 2\).
We will show that the 2-sections whose maximal faces are \((k+2)\)-faces of type 1 are not isomorphic to any 2-sections whose maximal faces are \((k+2)\)-faces of type 2, and therefore no \((k+2)\)-face of type 1 is isomorphic to a \((k+2)\)-face of type 2, so face types of the \((k+2)\)-faces are preserved under automorphisms of \([\mathcal{P}]_k\). By Lemma 2.18 it is sufficient to show that the face types of faces of a single rank larger than \(k\) are preserved.

If \(p_{k+2} \neq 2p_{k+1}\) then consider a \((k+2)\)-face \((F,G)\) of \([\mathcal{P}]_k\) of type 2. Note that \(G\) is a \((k+3)\)-face of \(\mathcal{P}\), and that \(k+3 \leq n\). Let \(\mathcal{F}\) denote the polytope \((F,G)/F_1\). By the fact that \(\mathcal{F}\) is included in a facet of type 2, and Proposition 2.27, \(\mathcal{F}\) is a regular \((k+2)\)-polytope of type \(\{p_1, p_2, \ldots, p_{k-1}, 2, p_{k+2}\}\). In particular, given any \((k-1)\)-face \(H\) incident to \((F,G)\), the 2-section \((F,G)/H\) is a \(p_{k+2}\)-gon.

Now, let \(K\) be a \((k+2)\)-face of \([\mathcal{P}]_k\) of type 1, and hence a \((k+2)\)-face of \(\mathcal{P}\). Now let \(H\) be a \((k-1)\)-face of \([\mathcal{P}]_k\) incident to \(K\), and consider the 2-section \(K/H\) of \([\mathcal{P}]_k\). This section is a \(p\)-gon for \(p\) the number of \(k\)-faces in the section \(K/H\) of \([\mathcal{P}]_k\). Each \(k\)-face \(F\) of \(\mathcal{P}\) which is incident to both \(H\) and \(K\) in \(\mathcal{P}\) is incident to exactly two \((k+1)\)-faces of \(\mathcal{P}\), say \(G\) and \(G'\), which are also incident to \(K\), by the diamond condition on \(\mathcal{P}\). The \(k\)-faces of the section \(K/H\) of \([\mathcal{P}]_k\) are ordered pairs \((F,G)\) with \(F\) a \(k\)-face of \(\mathcal{P}\), and \(G\) a \((k+1)\)-face of \(\mathcal{P}\) with \(H \leq F \leq G \leq K\) in \(\mathcal{P}\). Thus, for every \(k\)-face \(F\) in the section \(K/H\) of \(\mathcal{P}\), there are two \(k\)-faces in the section \(K/H\) of \([\mathcal{P}]_k\), specifically \((F,G)\) and \((F,G')\) for \(F \leq G, G' \leq K\) in \(\mathcal{P}\). By the Schl"afl"i type of \(\mathcal{P}\), there are \(p_{k+1}\) \(k\)-faces in the section \(K/H\) of \(\mathcal{P}\), and hence \(2p_{k+1}\) \(k\)-faces in the section \(K/H\) of \([\mathcal{P}]_k\). Thus this section is a \((2p_{k+1})\)-gon. Since \(p_{k+2} \neq 2p_{k+1}\), the \((k+2)\)-faces of types 1 and 2 are not isomorphic. Thus the face types of the \((k+2)\)-faces of \([\mathcal{P}]_k\) must be preserved.

If \(p_i \neq 3\) for some \(i \geq k+3\), then the proof is very similar to the corresponding section of the proof of Proposition 2.22, if we require that \(i \geq k+3\) and replace each vertex \(V\) with a \(k\)-face \(L\). \(\square\)
Combining Proposition 2.20 and Proposition 2.29 with Theorem 2.21 we see that if $\mathcal{P}$ is a regular $n$-polytope of Schläfli type \{\(p_1, \ldots, p_{n-1}\}\), then $\Gamma([\mathcal{P}]_k)$ is isomorphic to $\Gamma(\mathcal{P})$ unless all of the following are true:

1. $p_k = 2$

2. $p_{k+2} = 2p_{k+1}$

3. $p_i = 3$ for all $i \geq k + 3$.  


Chapter 3

The special case $[\mathcal{P}]_{n-2}$ when $\mathcal{P}$ is regular

In this section we will concentrate on polytopes of the form $[\mathcal{P}]_{n-2}$, where $\mathcal{P}$ is a regular $n$-polytope of Schlafli type $\{p_1, \ldots, p_{n-1}\}$ such that $p_{n-2} \neq 2$. To begin with, these $(n-2)$-bubbles are of interest because they are semiregular by Proposition 2.28. Specifically, they are alternating semiregular polytopes in the sense of the term used by Barry Monson and Egon Schulte in [26]. Essentially, this means that they are semiregular polytopes with two types of facets which occur in an alternating fashion around faces of rank $n - 3$. We refer to [26] for more details on such polytopes. We will come back to this aspect of $[\mathcal{P}]_{n-2}$ at the end of this section after we present another construction which is given in another context. We will first focus on the fact that $[\mathcal{P}]_{n-2}$ is a two-orbit polytope when $\mathcal{P}$ is a regular polytope of type $\{p_1, \ldots, p_{n-1}\}$ and $p_{n-2} \neq 2$. Work has been done by Isabel Hubard in [17] and [20] and there is ongoing work by Hubard and Egon Schulte in [19] on classifying two-orbit polytopes, so it is natural to ask where these polytopes fit into the general theory of two-orbit polytopes. The construction which we will give first comes from Hubard and Schulte’s work on two-orbit polytopes. While this construction is not generally equivalent to Monson
and Schulte’s construction, there is a case in which they overlap and this is exactly the case $[\mathcal{P}]_{n-2}$ when $\mathcal{P}$ is regular. Before we present either of these constructions, we will start with a basic description of what $[\mathcal{P}]_{n-2}$ looks like.

For $\mathcal{P}$ any polytope (not necessarily regular), the $(n-2)$-bubble $[\mathcal{P}]_{n-2}$ can be derived from $\mathcal{P}$ by simply replacing each ridge ($(n-2)$-face) with a ditope over itself. When doing this, the original facets are not affected, so the facets of $[\mathcal{P}]_{n-2}$ of type 1 are isomorphic to the facets of $\mathcal{P}$. This is formalized in the following lemma. The difference between $\mathcal{P}$ and $[\mathcal{P}]_{n-2}$ is that the facets of type 1 are now separated by “bubbles” which are the facets of type 2. If two facets of $\mathcal{P}$ share a ridge in $\mathcal{P}$ (and therefore all faces of that ridge) then in $[\mathcal{P}]_{n-2}$, they are no longer both incident to that ridge itself, but they are both still incident to all the proper faces of that ridge.

**Lemma 3.1** If $\mathcal{P}$ is an $n$-polytope, then each facet of $[\mathcal{P}]_{n-2}$ of type 1 is isomorphic to the facet of $\mathcal{P}$ to which it is associated. In particular, if $\mathcal{P}$ is equivelar (respectively, regular), then the facets of type 1 of $[\mathcal{P}]_{n-2}$ are also equivelar (respectively, regular). In the case that $\mathcal{P}$ is regular, every facet of $[\mathcal{P}]_{n-2}$ of type 1 is isomorphic to every facet of $\mathcal{P}$.

**Proof** This follows directly from Lemma 2.4 and Lemma 2.11. □

### 3.1 $[\mathcal{P}]_{n-2}$ as a two-orbit polytope

For the remainder of this section, we will assume that $\mathcal{P}$ is a regular $n$-polytope of type $\{p_1, \ldots, p_{n-1}\}$ such that $p_{n-2} \neq 2$. Thus, $[\mathcal{P}]_{n-2}$ is a two-orbit polytope, by Theorem 2.23. Recall from Section 1.3 that every two-orbit polytope is in a class $2_I$ for some $I \subset N = \{0, 1, \ldots, n - 1\}$, and that each two-orbit polytope has a double Schlafli type which is determined by the Schlafli types of flags in each of the two orbits. The following proposition gives the class of $[\mathcal{P}]_{n-2}$ as well as its double Schlafli type.
Proposition 3.2 If $\mathcal{P}$ is a regular polytope of rank $n \geq 3$ of Schl"afli type $\{p_1, p_2, \ldots, p_{n-1}\}$ with $p_{n-2} \neq 2$, then $[\mathcal{P}]_{n-2}$ is a two-orbit polytope in class $2_{0,1, \ldots, n-2}$ of double Schl"afli type $\{p_1, p_2, \ldots, p_{n-3}, \frac{p_{n-2}}{2}, 2p_{n-1}\}$. Moreover, $\Gamma([\mathcal{P}]_{n-2}) \cong \Gamma(\mathcal{P})$.

Proof We know that $[\mathcal{P}]_{n-2}$ is a two-orbit polytope from Proposition 2.20 and Theorem 2.23. In this case $\Gamma([\mathcal{P}]_{n-2})$ preserves facet types (since $p_{n-2} \neq 2$), so $[\mathcal{P}]_{n-2}$ cannot be facet transitive. Therefore, by Corollary 1.29, $[\mathcal{P}]_{n-2}$ is in class $2_{0,1, \ldots, n-2}$. Recall that Proposition 2.16 states that $[\mathcal{P}]_{n-2}$ is $i$-face transitive for $i = 0, \ldots, n-2$, so this is consistent with the fact that a two-orbit polytope is $i$-face-transitive for all $i \in I$. It follows that the two flag-orbits are determined by the facets. Meaning, one flag-orbit is the set of all flags which contain a facet of type 1 and the other flag-orbit is the set of all flags which contain a facet of type 2. Note that the statement about the groups now follows from Theorem 2.21.

All but the last entry in each row of the double Schl"afli type can be obtained by looking at the structures of facets of each of the two facet types. Thus, the double Schl"afli type follows from Lemma 3.1 which tells us that we can obtain the Schl"afli type of the facets of type 1 from the Schl"afli type of facets of $\mathcal{P}$, which is $\{p_1, \ldots, p_{n-2}\}$, as well as from Proposition 2.27, which tells us the Schl"afli type of the facets of type 2, which is $\{p_1, \ldots, p_{n-3}, 2\}$.

The final entry in each row of the Schl"afli type for $[\mathcal{P}]_{n-2}$ is the number of $(n-2)$-faces in the section $F_n/H$, where $H$ is an $(n-3)$-face. Since every $(n-3)$-face of $[\mathcal{P}]_{n-2}$ is of type 1, there is only one such type of section, so this entry of the Schl"afli type will be the same on the top and bottom rows, so they can be merged into one. Now since $H$ is a face of type 1, it is a face of $\mathcal{P}$. Therefore, we can consider the section $F_n/H$ of $\mathcal{P}$. For every $(n-2)$-face $F$ of this section of $\mathcal{P}$, there are two $(n-1)$-faces $G$ and $G'$ to which $F$ is incident in $\mathcal{P}$. Both of these faces are in the section $F_n/H$ of $\mathcal{P}$. For each such $(n-2)$-face $F$, there are two $(n-2)$-faces $(F, G)$ and $(F, G')$ in the section $F_n/H$ of $[\mathcal{P}]_{n-2}$. Since the section $F_n/H$ of $\mathcal{P}$ has $p_{n-1}$ $(n-2)$-faces, the section $F_n/H$ of $[\mathcal{P}]_{n-2}$ has $2p_{n-1}$ $(n-2)$-faces. Alternatively,
the facets in the section $F_n/H$ of $[P]_{n-2}$ are of the form $G$, where $G$ is a facet in the section $F_n/H$ of $P$, or $(F, F_n)$, where $F$ is a ridge in the section $F_n/H$ of $P$. Since the section $F_n/H$ of $P$ was a $p_{n-1}$-gon, there are $p_{n-1}$ such $G$ and $p_{n-1}$ such $F$, so the section $F_n/H$ of $[P]_{n-2}$ has $2p_{n-1}$ facets. Note that the two kinds of facets alternate around $H$. \[\square\]

For $P$ a regular $n$-polytope of Schl"afli type $\{p_1, p_2, \ldots, p_{n-1}\}$, with group $\Gamma(P) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ the (weak) condition $p_{n-2} \neq 2$ is sufficient, but not necessary, for $[P]_{n-2}$ to be a two-orbit polytope. In fact, the condition $p_{n-2} \neq 2$ only guarantees that $[P]_{n-2}$ is not equivelar. What turns out to be a necessary and sufficient condition for $[P]_{n-2}$ to be a two-orbit polytope (rather than a regular polytope) is that there is no group automorphism of $\Gamma(P)$ which exchanges $\rho_{n-2}$ and $\rho_{n-1}$, while fixing the other generators $\rho_i$. Clearly if $p_{n-2} \neq 2$, then no such automorphism exists because in that case $\rho_{n-3}$ commutes with $\rho_{n-1}$, but not $\rho_{n-2}$. However, there certainly are string C-groups for which $p_{n-2} = 2$ but there are additional relations which involve one of $\rho_{n-1}$ and $\rho_{n-2}$, but not the other. We will see in this section that in $[P]_{n-2}$ the facets of types 1 and 2 can be thought of as cosets of subgroups which differ by simply exchanging $\rho_{n-1}$ with $\rho_{n-2}$. Thus it is reasonable to claim that the condition that no group automorphism of $\Gamma(P)$ exchanges $\rho_{n-1}$ and $\rho_{n-2}$ is equivalent to the condition that facet types are preserved by every automorphism of $P$, which we know is a necessary and sufficient condition for $[P]_{n-2}$ to be a two-orbit polytope. In the next section, we will be able to cite a result which will formally prove this claim. For now, it is enough to use the sufficient condition that $p_{n-2} \neq 2$.

In [19], Isabel Hubard and Egon Schulte describe a method for constructing a polytope which is either two-orbit or regular from a group which satisfies certain conditions. This is of interest to us because $[P]_{n-2}$ is a specific case of this construction whenever $P$ is regular, regardless if $[P]_{n-2}$ is regular or two-orbit. (Recall that Proposition 2.25 tells us that if $[P]_{n-2}$ is not a two-orbit polytope, then it is regular.) As [19] is still in progress, we will present our results independently, but it should be noted that our interest is motivated by
the work of Hubard and Schulte, and we will continue to reference the relationship between their construction and the restricted version which is presented here.

In particular, given a group $\Gamma$ with certain distinguished generators, we will define a polytope in Definition 3.4, which we will call $Q = Q(\Gamma)$. As we will point out, the polytope $Q$ is a restriction of Hubard and Schulte’s construction to a specific case, but we will define it in our case explicitly, and we will use only the definition given in Definition 3.4. In order to apply Hubard and Schulte’s construction, a group $\Gamma$ needs to satisfy certain conditions. We will be applying this construction to string C-groups, so for the sake of integration with their work, we will show (in Lemma 3.3) that string C-groups do satisfy the necessary relations. We will define $Q(\Gamma)$ to be a ranked poset which can be derived from any group with a certain set of distinguished generators. The relations that we will prove in Lemma 3.3 are satisfied by string C-groups are necessary conditions for $Q$ to be a polytope, and can be used to prove that the more general construction produces polytopes. However, we will not actually need the fact that these relations are satisfied. Instead, we will show that the poset $Q$ is isomorphic to a polytope of the form $[P]_{n-2}$, and is thus a polytope.

We mentioned in Section 1.3 (without proof) that the automorphism group $\Gamma$ of a two-orbit polytope has certain distinguished generators, such that for a polytope in class $2_I$,

$$\Gamma = \langle \rho_i, \alpha_{j,k}, \alpha_{j,i,j} \mid i \in I; j, k \notin I \rangle.$$  

We also gave a number of necessary conditions that these generators must satisfy, namely relations (1.2)-(1.5); in particular, these relations include $\alpha_{j,i,j} = \rho_i$ when $|j - i| \geq 2$.

Applying these latter relations to a general automorphism group of a two-orbit polytope in class $2_{0,1,\ldots,n-2}$, we observe that the automorphism group has a set of distinguished generators of the form $\{\rho_0, \rho_1, \ldots, \rho_{n-2}, \alpha_{n-1,n-2,n-1}\}$. In particular, in the case we are considering, $[P]_{n-2}$ is a two-orbit polytope in class $2_{0,1,\ldots,n-2}$ whose automorphism group must
have those generators. But, in this case, the automorphism group of \([\mathcal{P}]_{n-2}\) is isomorphic to the automorphism group of the original polytope \(\mathcal{P}\). Since \(\mathcal{P}\) is regular, it has an automorphism group which is usually described in terms of distinguished generators \(\rho'_0, \ldots, \rho'_{n-1}\). Because these two groups are isomorphic, it suggests that there is a uniform way to represent the set \(\{\rho_0, \rho_1, \ldots, \rho_{n-2}, \alpha_{n-1,n-2,n-1}\}\) in terms of the \(\rho'_i\). As it turns out this is actually as easy as renaming \(\rho_i\) as \(\rho'_i\) and \(\alpha_{n-1,n-2,n-1}\) as \(\rho'_{n-1}\), as we will describe below.

Thus since the two sets \(\{\rho'_0, \ldots, \rho'_{n-1}\}\) and \(\{\rho_0, \rho_1, \ldots, \rho_{n-2}, \alpha_{n-1,n-2,n-1}\}\) are equivalent, and we know that \(\{\rho'_0, \ldots, \rho'_{n-1}\}\) is a generating set for \(\Gamma(\mathcal{P})\), we can then claim that the set \(\{\rho_0, \rho_1, \ldots, \rho_{n-2}, \alpha_{n-1,n-2,n-1}\}\) (and hence the set \(\{\rho_i, \alpha_{j,k}, \alpha_{i,j}| i \in I, j, k \notin I, I = \{0, \ldots, n-2\}\}\) is a generating set for the automorphism group of \([\mathcal{P}]_{n-2}\), provided that group is isomorphic to the automorphism group of \(\mathcal{P}\). Here we referred to distinguished generators of \(\Gamma(\mathcal{P})\) as \(\rho'_i\), for \(i = 0, \ldots, n-1\), in order to distinguish them from the \(\rho_i\) which were distinguished generators of \(\Gamma([\mathcal{P}]_{n-2})\), but since these generating sets turn out to be equivalent, we no longer have any need to refer to them as \(\rho'_i\), and we will refer to them simply as \(\rho_i\).

From now on, we only consider the case \(I = \{0, \ldots, n-2\}\). As noted above, then distinguished generators \(\rho_0, \ldots, \rho_{n-2}, \alpha_{n-1,n-2,n-1}\) of a two-orbit polytope in class \(2_{0,\ldots,n-2}\) must satisfy conditions (1.2)-(1.5), given in Section 1.3. Additionally, we will show that they satisfy certain commutation relations given in (3.2) below.

In the following definitions, \(i \in I, j, k \notin I\), where again \(I = \{0, \ldots, n-2\}\) (so \(\alpha_{j,k} = \epsilon\) in each case). We define:

\[
\Gamma^-_l := \langle \rho_i, \alpha_{j,k}, \alpha_{i,j}| i, j, k < l \rangle = \Gamma_{\{l, \ldots, n-1\}} \\
\Gamma^+_l := \langle \rho_i, \alpha_{j,k}, \alpha_{i,j}| i, j, k > l \rangle = \Gamma_{\{0, \ldots, l\}}
\]

(3.1)
Then we consider the following condition:

\[ \text{For } l = 0, \ldots, n - 1, \text{ each generator of } \Gamma_l^- \text{ commutes with each generator of } \Gamma_l^+. \quad (3.2) \]

This condition is saying that \( \Gamma_l^- \) and \( \Gamma_l^+ \) commute element-wise for each \( l \).

We will show in Lemma 3.3 that when the group of a regular polytope \( P \) is presented as a string C-group with distinguished generators \( \rho_0, \rho_1, \ldots, \rho_{n-1} \), then renaming the last generator \( \rho_{n-1} \) as \( \alpha_{n-1,n-2,n-1} \) is sufficient to give a set of distinguished generators which satisfy the commutation relations (3.2).

**Lemma 3.3** Let \( P \) be a regular \( n \)-polytope with automorphism group \( \Gamma(P) = \langle \rho_0, \ldots, \rho_{n-1} \rangle \). Then renaming \( \rho_{n-1} \) as \( \alpha_{n-1,n-2,n-1} \) results in a group of the form

\[ \langle \rho_i, \alpha_{j,k}, \alpha_{j,i,j} | i \in I, j, k \notin I \rangle, \]

for \( I = \{0, \ldots, n-2\} \), such that the generators satisfy relations (1.2)-(1.5) and (3.2).

**Proof** We begin by noting that given \( \Gamma(P) = \langle \rho_0, \ldots, \rho_{n-1} \rangle \), renaming \( \rho_{n-1} \) as \( \alpha_{n-1,n-2,n-1} \) results in a group \( \Gamma = \langle \rho_i, \alpha_{j,k}, \alpha_{j,i,j} | i \in I, j, k \notin I \rangle \) for \( I = \{0, \ldots, n-2\} \), once we note that there are no (non-trivial) generators of the form \( \alpha_{j,k} \) in this class, and use relation (1.4), namely that \( \alpha_{j,j} = \rho_i \) when \( |i - j| \geq 2 \), to remove redundant generators. We will continue to refer to \( \Gamma(P) \) as the group generated by \( \rho_i \), for \( i = 0, \ldots, n-1 \), and to the “new” group \( \Gamma \) as \( \langle \rho_i, \alpha_{j,k}, \alpha_{j,i,j} | i \in I, j, k \notin I \rangle \) (with \( I = \{0, \ldots, n-2\} \)), even though these groups are isomorphic. We will now prove that \( \Gamma \) satisfies the conditions (1.2), (1.5), and (3.2), in addition to any specific relations implied by the generators of \( P \). Relation (1.3), namely that \( \alpha_{j,k} = \alpha_{k,j}^{-1} \), is not applicable in this class, since there are no non-trivial generators of the form \( \alpha_{j,k} \).
These conditions will follow directly from the fact that the generators of $\Gamma$ are generators of the group of a regular polytope, and thus satisfy the relations of string C-groups. Relation (1.2), namely that $\rho_i^2 = \alpha_{j,i,j}^2 = \epsilon$ for $i = 0, \ldots, n - 2, j = n - 1$, is satisfied because the $\rho_i$ and $\alpha_{j,i,j}$ are distinguished generators of $\Gamma(\mathcal{P})$, when this group is presented as a string C-group, and the distinguished generators of a string C-group are all involutions. The required intersection condition (1.5) follows from the intersection condition for regular polytopes. First of all, as we mentioned in Section 1.3, if we define $\Gamma^J$ to be $\Gamma_J$, then (1.5) is equivalent to (1.5'), which is that $\Gamma^J \cap \Gamma^K = \Gamma^{J \cap K}$ for $J, K \subseteq \{0, \ldots, n - 1\}$, so the appearance of the intersection condition is equivalent to that for regular polytopes. Additionally, since $I = \{0, \ldots, n - 2\}$, every distinguished subgroup of $\Gamma$ is a distinguished subgroup of $\Gamma(\mathcal{P})$. The distinguished subgroup $\Gamma^J$ of $\Gamma$ is given by $\langle \rho_l \mid l \in J \rangle$, except when $n - 1 \in J, n - 2 \notin J$; if $n - 1 \in J$ and $n - 2 \notin J$, then $\Gamma^J = \langle \rho_l \mid l \in J \setminus \{n - 1\} \rangle$. The intersection condition, $\Gamma^J \cap \Gamma^K = \Gamma^{J \cap K}$ for $\Gamma$ now follows directly from the intersection condition for $\Gamma(\mathcal{P})$. Note that this is true even in the exceptional case when one, or both, of $J$ and $K$ contain $n - 1$, but not $n - 2$. Thus the intersection condition for $\Gamma$ is inherited from that for $\Gamma(\mathcal{P})$.

Finally, we need to show that every generator of $\Gamma^-_l$ commutes with every generator of $\Gamma^+_l$ (which is relation (3.2)). This follows from the fact that $\Gamma(\mathcal{P})$ is a string C-group, so the only generators of $\Gamma(\mathcal{P})$ which do not commute have adjacent indices. Generators of $\Gamma^-_l$ all have indices strictly less than $l$ and generators of $\Gamma^+_l$ all have indices strictly greater than $l$. Clearly, if these definitions were based on indices the way that they are defined in $\Gamma(\mathcal{P})$, this would be satisfied since no generator of $\Gamma^-_l$ has an index which is adjacent to the index of a generator of $\Gamma^+_l$. The change of name from $\rho_{n-1}$ to $\alpha_{n-1,n-2,n-1}$ only precludes this distinguished generator from being a generator of $\Gamma^+_{n-2}$, and there are no other changes which would be made when switching the names of the distinguished generators to the way they are presented in $\Gamma$. Therefore, (3.2) is actually a slightly weaker condition than what is guaranteed by the fact that $\Gamma(\mathcal{P})$ is a string C-group. □
As we mentioned, in [19], Isabel Hubard and Egon Schulte describe a method for constructing a polytope which is either two-orbit or regular from a group that “looks like” the group of a two-orbit polytope. In particular, restricting their results to the class $2_{0,1,...,n-2}$, given a group $\Gamma$ generated by $\{\rho_0, \rho_1, \ldots, \rho_{n-2}, \alpha_{n-1,n-2,n-1}\}$ which satisfies relations (1.2)-(1.5) and (3.2), they construct a polytope which is either a two-orbit polytope in class $2_{0,1,...,n-2}$ or is regular. If the polytope is two-orbit, then $\Gamma$ is its automorphism group. If the polytope is regular, then $\Gamma$ is a subgroup of index 2 of its automorphism group.

Since we showed in Lemma 3.3 that string C-groups do satisfy these relations, Hubard and Schulte’s construction can be applied. The complete details of the construction, in full generality, will be given in their paper. What we will give here are the details of the construction only in the case when the construction is applied to a group generated by $\{\rho_0, \rho_1, \ldots, \rho_{n-2}, \alpha_{n-1,n-2,n-1}\}$. In other words, we will take a construction which is general enough to build two-orbit polytopes in any class, and restrict it to groups for which the construction can result in two-orbit polytopes in the class $2_I$, with $I = N \setminus \{n-1\}$. We will do so because this is the class of $[\mathcal{P}]_{n-2}$, when it is a two-orbit polytope. Again, everything that we will prove will be based on the definition of the polytope $Q$ as given in the following Definition 3.4. While this definition only presents $Q$ as a ranked poset, we are primarily interested in cases where $Q$ is a polytope and we might therefore refer to it as such.

**Definition 3.4 The poset $Q$.**

Let $\Gamma = \langle \rho_0, \rho_1, \ldots, \rho_{n-2}, \alpha_{n-1,n-2,n-1} \rangle$ and for $l = 0, 1, \ldots, n-1$, let $\Gamma_l$ be the subgroup of $\Gamma$ defined by $\langle \rho_i, \alpha_{j,i,j} \mid 1 \leq i \leq n-2; j = n-1; i, j \neq l \rangle$. Define $Q$ to be a ranked poset of rank $n$ whose faces are as follows.

For the faces of $Q$ of ranks $-1$ and $n$, we take two distinct copies $F_{-1}$ and $F_n$, respectively, of $\Gamma$, which we call improper faces. Let these faces be incident to all other faces of other ranks. For $0 \leq l \leq n-2$, take the $l$-faces of $Q$ to be the left cosets of $\Gamma_l$ in $\Gamma$. We define the
(n - 1)-faces of \( Q \) to be the left cosets of either \( \Gamma_{n-1} \) or \( \Gamma'_{n-1} \), where \( \Gamma'_{n-1} = \langle \rho'_i | i \neq n - 1 \rangle \) with \( \rho'_i := \alpha_{n-1,i,n-1} \) for each \( i \neq n - 1 \).

The partial order of the proper faces is given by:

\[
\varphi \Gamma_l \leq Q \psi \Gamma_m \text{ if and only if } l \leq m \text{ and } \varphi \Gamma_l \cap \psi \Gamma_m \neq \emptyset;
\]

\[
\varphi \Gamma_l \leq Q \psi \Gamma'_{n-1} \text{ if and only if } l \leq n - 1 \text{ and } \varphi \Gamma_l \cap \psi \Gamma'_{n-1} \neq \emptyset.
\]

We should add a few comments, in order to make clear that this is, in fact, simply a restriction the construction in [19]. If we think of \( \Gamma \) as being the group of a two orbit polytope with a base flag, then \( \Gamma'_{n-1} \) is the subgroup which would have been the distinguished subgroup \( \Gamma_{n-1} \) had the \((n - 1)\)-adjacent flag been chosen as the base flag. As such, the most general definition of \( \Gamma'_{n-1} \) is \( \langle \rho'_i, \alpha'_{j,k}, \alpha'_{j,i,j} | i, j, k \neq n - 1; i \in I, j, k \notin I \rangle \). However, in Definition 3.4, we gave the definition of \( \Gamma'_{n-1} \) in a way which was specific to the class that we are considering. In particular, since \( N \setminus I = \{n - 1\} \) (where \( N := \{0, \ldots, n - 1\} \) is the set of potential indices), there do not exist indices \( j \) and \( k \) which are at the same time equal to \( n - 1 \) and not equal to \( n - 1 \), so there are no generators of the form \( \alpha'_{j,k} \) or \( \alpha'_{j,i,j} \). So in this case, we have \( \Gamma'_{n-1} = \langle \rho'_i | i \neq n - 1 \rangle \). Also, note that the construction in [19] was given using right cosets, rather than left cosets.

Secondly, the general case of construction in [19] includes a third partial order condition on proper faces, which is:

\[
\Gamma'_{n-1} \varphi \leq Q \Gamma_m \varphi_m \text{ if and only if } n - 1 \leq m \text{ and } \Gamma'_{n-1} \varphi \cap \Gamma_m \varphi_m \neq \emptyset.
\]

However, this condition is not applicable to our situation since there are no proper faces of rank greater than \( n - 1 \).

The reason that this construction is important to us is given in the following theorem. In it, we prove that if we begin with the group \( \Gamma(P) \), for \( P \) a regular polytope, the polytope \( Q \) is exactly \([P]_{n-2}\). In fact, Theorem 3.5 establishes independently that \( Q \) is a polytope.
Theorem 3.5 Let \( P \) be a regular \( n \)-polytope with automorphism group \( \Gamma(P) = \langle \rho_0, \ldots, \rho_{n-1} \rangle \) and let \([P]_{n-2}\) be its \((n-2)\)-bubble. Let \( Q \) be the polytope obtained from \( \Gamma(P) \) by renaming the distinguished generator \( \rho_{n-1} \) as \( \alpha_{n-1, n-2, n-1} \) and applying Definition 3.4. Then \([P]_{n-2}\) is isomorphic to \( Q \).

As we mentioned, this theorem is important because it gives the relationship between \([P]_{n-2}\) and Hubard and Schulte’s construction which allows us to view \([P]_{n-2}\) within the general structure of two-orbit polytopes. However, this theorem is also important because it gives us an algebraic understanding of \([P]_{n-2}\), in addition to the geometric and combinatorial understandings which have been used previously.

Before proving the theorem, we will prove a lemma. This lemma appears to be a merely technical statement to be used in the proof, however this lemma will actually help shine some light onto the relationship between the algebraic definition of \([P]_{n-2}\) (in Definition 3.4) and the combinatorial definition. We note that both geometrically and combinatorially, the defining characteristic of \([P]_{n-2}\) is that each ridge ((\(n-2\))-face) of \( P \) is replaced with two ridges each of which is incident to the same faces of rank \( j < n - 2 \) to which the original ridge was incident. Lemma 3.6 will demonstrate that this structure can also be seen from the algebraic definition. Recall that if \( \Gamma \) is a rank \( n \) string C-group generated by \( \{\rho_0, \ldots, \rho_{n-1}\} \), then for \( J \subset N \) we defined \( \Gamma_J \) to be \( \langle \rho_i | i \notin J \rangle \), and denoted \( \Gamma_{\{j\}} \) as \( \Gamma_j \). Recall also that the left cosets of \( \Gamma_{n-2} \) are ridges of the underlying \( n \)-polytope \( P = P(\Gamma) \). If we set \( J := \{n-2, n-1\} \), then we will show that the cosets of \( \Gamma_J \) are actually the ridges of \([P]_{n-2}\). Given that, the cosets \( \varphi \Gamma_J \), and \( \varphi \rho_{n-1} \Gamma_J \), which are mentioned in Lemma 3.6, then are ridges of \([P]_{n-2}\). What Lemma 3.6 tells us is that these two ridges of \([P]_{n-2}\) are both incident to the same faces, \( \Lambda \), of lower rank as is the ridge \( \varphi \Gamma_{n-2} \) of \( P \). Thus we can view the ridge \( \varphi \Gamma_{n-2} \) of \( P \) as being replaced by two identical faces each of which is incident to all of the same faces of lower rank, or (once a facet has also been added to fill in the space) a ditope over itself. 

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Lemma 3.6 Let $\Gamma$ be a rank $n$ string $C$-group generated by $\{\rho_0, \ldots, \rho_{n-1}\}$, and let $J := \{n-2, n-1\}$. Let $\Lambda = \psi \Gamma_i$ for some $i \notin J$ and $\psi \in \Gamma$, where as usual $\Gamma_i := (\rho_i | l \neq i)$. Then for any $\varphi \in \Gamma$, if $\Lambda$ has a non-empty intersection with any of $\varphi \Gamma_J, \varphi \rho_{n-1} \Gamma_J$, or $\varphi \Gamma_{n-2}$, then it has a non-empty intersection with each of them.

Proof We first prove that $\Lambda$ has a non-empty intersection with $\varphi \Gamma_J$ if and only if it has a non-empty intersection with $\varphi \rho_{n-1} \Gamma_J$. We only need to prove one implication because $\varphi$ is an arbitrary element of $\Gamma$ and $(\rho_{n-1})^2 = \epsilon$ so if $\varphi$ is replaced with $\varphi' = \varphi \rho_{n-1}$ then $\varphi' \rho_{n-1} = \varphi \rho_{n-1} \rho_{n-1} = \varphi$, so the reverse implication is equivalent.

Assume that $\Lambda$ has non-empty intersection with $\varphi \Gamma_J$, say $\gamma \in \varphi \Gamma_J \cap \Lambda$. Then $\gamma \rho_{n-1} \in \Lambda$ because $\Lambda = \psi \Gamma_i$, $\rho_{n-1} \in \Gamma_i$ (since $i \notin J$ and hence $i \neq n-1$) and $\gamma \in \Lambda$. Additionally $\gamma \rho_{n-1} \in \varphi \Gamma_J \rho_{n-1}$ since $\gamma \in \varphi \Gamma_J$. However $\rho_{n-1}$ commutes with every generator of $\Gamma_J$, since $J = \{n-2, n-1\}$, and therefore $\rho_{n-1}$ commutes with every element of $\Gamma_J$. Therefore, $\gamma \rho_{n-1} \in \varphi \rho_{n-1} \Gamma_J$. Thus, $\gamma \rho_{n-1}$ is in the intersection of $\Lambda$ and $\varphi \rho_{n-1} \Gamma_J$, and this intersection is non-empty.

We now prove that $\Lambda$ has a non-empty intersection with $\varphi \Gamma_{n-2}$ if and only if it has a non-empty intersection with $\varphi \Gamma_J$ (or equivalently, $\varphi \rho_{n-1} \Gamma_J$, since we proved that these two statements are equivalent). Assume that $\Lambda$ has non-empty intersection with $\varphi \Gamma_J$, say $\gamma \in \varphi \Gamma_J \cap \Lambda$. Then $\gamma \rho_{n-1} \in \Lambda$ for the reasons described above. Additionally, $\gamma \rho_{n-1}$ is an element of $\varphi \Gamma_{n-2}$ because $J = \{n-2, n-1\}$ so $\Gamma_J \rho_{n-1} \subseteq \Gamma_{n-2}$ and thus since $\gamma \in \varphi \Gamma_J$, then $\gamma \rho_{n-1} \in \varphi \Gamma_J \rho_{n-1}$, and hence $\gamma \rho_{n-1} \in \varphi \Gamma_{n-2}$. Thus, $\gamma \rho_{n-1} \in \varphi \Gamma_{n-2} \cap \Lambda$, and this intersection is also non-empty.

Now, say $\gamma \in \varphi \Gamma_{n-2} \cap \Lambda$. We know that $\Gamma_{n-2} = \Gamma_J \cup \rho_{n-1} \Gamma_J$, since $\rho_{n-1}$ commutes with every other generator of $\Gamma_{n-2}$, the other generators of $\Gamma_{n-2}$ generate $\Gamma_J$, and $\rho_{n-1}$ is an involution. Because $\gamma \in \varphi \Gamma_{n-2}$, we know that $\gamma = \varphi \xi$ for some $\xi \in \Gamma_{n-2}$. Thus $\xi \in \Gamma_J$ or $\xi \in \rho_{n-1} \Gamma_J$. Therefore, $\gamma \in \varphi \Gamma_J$ or $\gamma \in \varphi \rho_{n-1} \Gamma_J$. Since $\gamma \in \Lambda$, we know that $\Lambda$ has a
non-empty intersection with either $\varphi \Gamma_J$ or $\varphi \rho_{n-1} \Gamma_J$. Either way, by what we showed above, then $\Lambda$ has a non-empty intersection with $\varphi \Gamma_J$. \qed

We can now prove Theorem 3.5.

**Proof of Theorem** We begin by discussing the notation that we will use over the course of this proof to avoid the ambiguity associated with renaming generators. As in the statement of the proof, $\Gamma(P)$ will be a string C-group generated by $\{\rho_0, \ldots, \rho_{n-1}\}$. We will also define $\Gamma$ to be the same group as $\Gamma(P)$, but taken with the new generating set $\{\rho_0, \ldots, \rho_{n-2}, \alpha_{n-1,n-2,n-1}\}$. (Note that this notation will only be used in the course of this proof, and that in Lemma 3.6 the group $\Gamma$ was defined to be a string C-group generated by $\{\rho_0, \ldots, \rho_{n-1}\}$.) Given this notation, we define $\Gamma(P)_J$ to be the subgroup of $\Gamma(P)$ defined by $\langle \rho_i \mid i \not\in J \rangle$. If $J = \{l\}$, we will denote this group $\Gamma(P)_l$. We will continue to use the notation $\Gamma_l = \langle \rho_i, \alpha_{j,i,j} \mid i,j \not= l \rangle$ for subgroups of $\Gamma$, so that $\Gamma_l = \Gamma(P)_l$, except in the case that $l = n - 2$. In this exceptional case, we have $\Gamma_{n-2} = \langle \rho_0, \ldots, \rho_{n-3} \rangle$, which is $\Gamma(P)_{(n-2,n-1)}$.

Recall that the subgroup $\Gamma'_{n-1}$ of $\Gamma$ is defined to be $\langle \rho'_i \mid i \not= n - 1 \rangle$, where $\rho'_i := \alpha_{n-1,i,n-1}$ for each $i$, so this group is $\langle \alpha_{n-1,i,n-1} \mid i \not= n - 1 \rangle$. However, $\alpha_{n-1,i,n-1} = \rho_i$ for $i \leq n - 3$, so we have that $\Gamma'_{n-1} = \langle \rho_0, \rho_1, \ldots, \rho_{n-3}, \alpha_{n-1,n-2,n-1} \rangle$, which is the subgroup of $\Gamma$ generated by every distinguished generator except $\rho_{n-2}$. Notice that, since $\alpha_{n-1,n-2,n-1}$ is just $\rho_{n-1}$ renamed, $\Gamma'_{n-1}$ is simply the subgroup of $\Gamma(P)$ generated by $\{\rho_i \mid i \not= n - 2\}$, that is, $\Gamma(P)_{n-2}$.

Given this, we can now begin to define an isomorphism from $[P]_{n-2}$ to $Q$. We begin by defining a function $g$ which is a rank-preserving map from $[P]_{n-2}$ to $Q$, and then we will prove that this map is bijective and incidence preserving. The definition of $g$ makes use of the fact that the faces of $P$ can be described as cosets of distinguished subgroups of $\Gamma(P)$.

The images of the improper faces of $[P]_{n-2}$ under $g$ are trivial, because we required $g$ to be rank-preserving. For $0 \leq j \leq n - 3$, the $j$-faces of $[P]_{n-2}$ are exactly (or rather, identified
with) the \( j \)-faces of \( P \). Therefore, we can consider these faces to be left cosets of \( \Gamma(P)_j \) and therefore of \( \Gamma_j \). This is exactly how the \( j \)-faces of \( Q \) are defined for \( 0 \leq j \leq n - 3 \), so we define the map \( g \) to act trivially on these faces, sending each coset of \( \Gamma_j \) to itself.

Recall that the \( (n - 2) \)-faces of \( [P]_{n-2} \) are defined as ordered pairs \((F, G)\), where \( F \) is an \( (n - 2) \)-face of \( P \), and \( G \) is in incident \( (n - 1) \)-face of \( P \). When we consider these faces of \( P \) to be cosets, \( F \) is a left coset of \( \Gamma(P)_{n-2} \) and \( G \) is a left coset of \( \Gamma(P)_{n-1} \). Since \( F \) is incident to \( G \), these two cosets of \( \Gamma(P) \) have a non-empty intersection. Say \( \varphi \) is in the intersection of \( F \) and \( G \). Then we define \( g((F, G)) \) to be \( \varphi \Gamma(P)_{\{n-2,n-1\}} \). This is well defined, as this coset is the intersection of the cosets \( F \) and \( G \) because if \( H \) and \( H' \) are subgroups of a given group, then if cosets of \( H \) and \( H' \) have a non-empty intersection, then that intersection is a coset of \( H \cap H' \). Additionally, by the intersection condition on \( \Gamma(P) \), the intersection of \( \Gamma(P)_{n-2} \) and \( \Gamma(P)_{n-1} \) is \( \Gamma(P)_{\{n-2,n-1\}} \).

Note that because \( \Gamma(P)_{\{n-2,n-1\}} \) is simply \( \Gamma_{n-2} \), the left cosets of \( \Gamma(P)_{\{n-2,n-1\}} \) are exactly the \( (n - 2) \)-faces of \( Q \), so \( g \) does, in fact, take \( (n - 2) \)-faces of \( [P]_{n-2} \) to \( (n - 2) \)-faces of \( Q \).

The set of facets of \( [P]_{n-2} \) is identified with the set of facets of \( P \) together with the set of \( (n - 2) \)-faces (ridges) of \( P \), which are left cosets of \( \Gamma(P)_{n-1} \) and \( \Gamma(P)_{n-2} \), respectively. Recall that \( \Gamma(P)_{n-1} \) is just \( \Gamma_{n-1} \), and that we saw that \( \Gamma(P)_{n-2} \) is just \( \Gamma'_{n-1} \). The facets of \( Q \) were defined to be left cosets of \( \Gamma_{n-1} \) and left cosets of \( \Gamma'_{n-1} \). Thus, we can define \( g \) to trivially send cosets of \( \Gamma(P)_{n-1} \) to cosets of \( \Gamma_{n-1} \), and to trivially send cosets of \( \Gamma(P)_{n-2} \) to cosets of \( \Gamma'_{n-1} \) so that this mapping sends facets of \( [P]_{n-2} \) to facets of \( Q \).

We now claim that the map \( g \) is a bijection. On faces of all ranks except \( n - 2 \), \( g \) is a trivial map between equal sets so the inverse map is trivial as well. We must only show that \( g \) is bijective on the set of faces of rank \( n - 2 \). Let \( F \) be an \( (n - 2) \)-face of \( Q \), say \( F = \varphi \Gamma(P)_{\{n-2,n-1\}} \). Then \( (\varphi \Gamma(P)_{n-2}, \varphi \Gamma(P)_{n-1}) \) is a preimage of \( F \) under \( g \). It is clear that \( g((\varphi \Gamma(P)_{n-2}, \varphi \Gamma(P)_{n-1})) = \varphi \Gamma(P)_{n-2} \) because \( \varphi \) is in the intersection \( \varphi \Gamma(P)_{n-1} \cap \varphi \Gamma(P)_{n-2} \).
Additionally, every $(n - 2)$-face of $[P]_{n-2}$ is of the form $(\psi \Gamma(P)_{n-2}, \psi \Gamma(P)_{n-1})$ for some $\psi$ since the $(n - 2)$-faces of $[P]_{n-2}$ are ordered pairs of a ridge and an incident facet in $P$, and (when viewed as cosets of $\Gamma(P)_{n-2}$ and $\Gamma(P)_{n-1}$) a ridge and a facet are incident if and only if they can be written as $\psi \Gamma(P)_{n-2}$ and $\psi \Gamma(P)_{n-1}$ for some $\psi$, which can be chosen to be any element of their intersection. In other words, $F$ has just one preimage under $g$.

Finally, to prove that $g$ is an isomorphism, we claim that $g$ preserves incidence. This is where Lemma 3.6 comes in. Given two faces of $[P]_{n-2}$ we want to show that their images under $g$ in $Q$ are incident if and only if the original faces were incident. Because the map is trivial for all but the $(n - 2)$-faces, we may assume without loss of generality that one of the faces is an $(n - 2)$-face.

We first consider the case where the $(n - 2)$-face is the face of greater rank. In $[P]_{n-2}$, every $(n - 2)$-face is of type 2, and every face of lower rank is of type 1. Let $H$ and $(F,G)$ be $j$- and $(n - 2)$- faces of $[P]_{n-2}$, respectively, with $j < n - 2$. Then $F \leq G$ in $P$ so, viewing $F$ and $G$ as cosets of subgroups of $\Gamma(P)$, there is some face, say $\varphi$, in their intersection. Thus, $F = \varphi \Gamma(P)_{n-2}$, $G = \varphi \Gamma(P)_{n-1}$ and $g((F,G)) = \Gamma(P)_{\{n-2,n-1\}}$. If $H = \psi \Gamma(P)_{j}$, then $g(H) = \psi \Gamma(P)_{j}$ as well. By Lemma 3.6 $\psi \Gamma(P)_{j}$ has a non-empty intersection with $\varphi \Gamma(P)_{\{n-2,n-1\}}$ if and only if it has a non-empty intersection with $\varphi \Gamma(P)_{n-2}$. In other words, $g(H)$ is incident to $g((F,G))$ in $Q$ if and only if $H$ is incident to $F$ in $P$. Since $H$ is incident to $(F,G)$ in $[P]_{n-2}$ if and only if $H$ is incident to $F$ in $P$, we have that $g(H)$ is incident to $g((F,G))$ in $Q$ if and only if $H$ is incident $(F,G)$ in $[P]_{n-2}$.

It remains to show that $g$ preserves the incidence of an $(n - 2)$-face and an $(n - 1)$-face. Let $(F,G)$ be an $(n - 2)$ face of $[P]_{n-2}$. Then $(F,G)$ is incident to exactly two facets of $[P]_{n-2}$, namely $G$ and $(F,F_n)$, where we identify $(F,F_n)$ with $F$ itself. Now $g((F,G)) = \varphi \Gamma(P)_{\{n-2,n-1\}}$ for some $\varphi$. Then $g(F) = \varphi \Gamma(P)_{n-2}$, and $g(G) = \varphi \Gamma(P)_{n-1}$, as before. It is clear that $\varphi$ is in the intersection of $g((F,G))$ and $g(F)$ as well as in the intersection of $g((F,G))$ and $g(G)$, so these two intersections are, in particular, non-empty. Thus $g((F,G))$
is incident to \( g(F) \) and \( g(G) \). On the other hand, if there is some facet \( H \) of \( Q \) such that \( H \) is incident to \( g(F, G) \), then \( H \) would have to be a coset of either \( \Gamma(P)_{n-2} \) or \( \Gamma(P)_{n-1} \) which has a non-empty intersection with \( \varphi \Gamma(P)_{(n-2,n-1)} \). However, any element of \( \varphi \Gamma(P)_{(n-2,n-1)} \) is an element of both \( \varphi \Gamma(P)_{n-2} \) and \( \varphi \Gamma(P)_{n-1} \), so \( H \) must have a non-empty intersection with \( \varphi \Gamma(P)_{n-2} \) and \( \varphi \Gamma(P)_{n-1} \). Since distinct cosets of \( \Gamma(P)_{n-2} \) (respectively \( \Gamma(P)_{n-1} \)) are disjoint, \( H \) must be either \( \varphi \Gamma(P)_{n-2} \) or \( \varphi \Gamma(P)_{n-1} \). So \( H \) must be either \( g(F) \) or \( g(G) \). Thus incidences are preserved, and this is an isomorphism of posets and hence of polytopes. □

Notice that Theorem 3.5 holds for all regular \( n \)-polytopes \( P \) regardless of if \( \Gamma(P) \) is actually isomorphic to \( \Gamma([P]_{n-2}) \). As previously mentioned, the construction given in [19] will result in a polytope which is either two-orbit or regular. Therefore, Theorem 3.5 can give us more information about the construction given in [19], since we know that when \([P]_{n-2}\) is a two-orbit polytope, \( Q \) must also be a two-orbit polytope. We know that if \( P \) is of type \( \{p_1, \ldots, p_{n-1}\} \) and \( p_{n-2} \neq 2 \) then \([P]_{n-2}\) is a two-orbit polytope, by Proposition 2.20 and Theorem 2.23. Since \( p_{n-2} \) is the order of \( \rho_{n-3}\rho_{n-2} \) in \( \Gamma(P) \), if \( \rho_{n-3} \) and \( \rho_{n-2} \) do not commute, then \( Q \), the polytope derived from \( \Gamma = \langle \rho_0, \rho_1, \ldots, \rho_{n-2}, \alpha_{n-1,n-2}, n-1 \rangle \), is a two-orbit polytope. It is known that \( Q \) is a two-orbit polytope if \( \Gamma \) has index 1 in \( \Gamma(Q) = \Gamma([P]_{n-2}) \). If \( \Gamma \) has index 2 in \( \Gamma(Q) = \Gamma([P]_{n-2}) \), then \( Q \) is regular. However, one must know \( \Gamma(Q) \) in order to apply this. We now have an alternative sufficient condition to check (in the class \( 2_0, \ldots, n-2 \)), which is based only on the group \( \Gamma \). As we have alluded to, we will actually be able to give a necessary and sufficient condition for \( Q \) to be a two-orbit polytope which can easily be checked from the structure of \( \Gamma \), and that is simply that there is no automorphism which acts on \( \Gamma \) and exchanges \( \rho_{n-2} \) and \( \rho_{n-1} \). We have shown that this is a stronger sufficient condition, and we will show that that is necessary as well. However, in this case, a sufficient condition is more important than a necessary condition since the goal is to construct two-orbit polytopes, since we can easily construct regular polytopes without this construction.

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While we can use our construction to learn more about the number of orbits of Hubard and Schulte’s construction, the knowledge that if Γ has index 1 in Γ(Q) then Q is a two-orbit polytope, and otherwise Γ has index 2 in Γ(Q), and Q is regular, does not give us any new information about our construction. We had shown previously that if Γ was isomorphic to Γ([P]_{n-2}), then Γ([P]_{n-2}) was a two-orbit polytope. We also knew that if these two groups were not isomorphic, then [P]_{n-2} was regular. We know that in this case Γ is a subgroup of index 2 of Γ([P]_{n-2}) because we know it is a subgroup, and that Γ acts transitively on flags with a certain sequence of face types. Since in [P]_{n-2} it is only the facets for which the face types can differ, and there are only two face types of the facets, Γ[P]_{n-2} must be generated by Γ and a single automorphism which takes a facet of type 1 to a facet of type 2.

3.2 [P]_{n-2} from the abstract Wythoff construction

There is a geometric construction called Wythoff’s construction which can be used to construct uniform polytopes. Full details can be found in [5] as well as [7, Ch. 5]. In [26] Barry Monson and Egon Schulte define an abstract version of Wythoff’s construction which can be applied to groups which they call tail-triangle C-groups. This combinatorial version of Wythoff’s construction is very useful, and has been used to construct polytopes such as the Tomotope, which is an important non-regular polytope, because it has infinitely many distinct minimal regular covers [25]. We will begin with a brief discussion of these groups before discussing Monson and Schulte’s construction.

We will start by defining a tail-triangle C-group. Our definition will differ from that given by Monson and Schulte only in that we will replace the rank n with the rank n – 1 because when we ultimately apply their construction, we will want to construct a polytope of rank n, rather than n + 1, and changing the ranks given in the definition will simplify the process for us.
Definition 3.7 Let $\Gamma = \langle \alpha_0, \ldots, \alpha_{n-3}, \alpha_{n-2}, \beta_{n-2} \rangle$, with $n \geq 2$, be a group generated by involutions such that $(\alpha_i \alpha_j)^2 = \epsilon$ for $|i - j| \geq 2$, and $(\alpha_i \beta_{n-2})^2 = \epsilon$ for $i \neq n - 2, n - 3$. Assume also that for $I, J \subseteq \{\alpha_0, \ldots, \alpha_{n-3}, \alpha_{n-2}, \beta_{n-2}\}$, $\langle I \rangle \cap \langle J \rangle = \langle I \cap J \rangle$. Then we call $\Gamma$ a tail-triangle C-group.

We call such groups tail-triangle groups because their generators satisfy the relations given in the Coxeter diagram shown in Figure 3.1 with all, but one, branch label suppressed. (We ignore the label $k$ on the branch connecting $\alpha_{n-3}$ and $\beta_{n-2}$ for now.)

\begin{center}
\begin{tikzpicture}
    \node (a0) at (0,0) {$\alpha_0$};
    \node (a1) at (1,0) {$\alpha_1$};
    \node (am1) at (2,0) {$\ldots$};
    \node (am3) at (3,0) {$\alpha_{n-3}$};
    \node (am4) at (4,0) {$\alpha_{n-4}$};
    \node (bn2) at (5,0) {$\beta_{n-2}$};
    \node (an2) at (6,0) {$\alpha_{n-2}$};
    \path (a0) edge (a1);
    \path (a1) edge (am1);
    \path (am1) edge (am3);
    \path (am3) edge (am4);
    \path (am4) edge (an2);
    \path (an2) edge (bn2);
\end{tikzpicture}
\end{center}

Figure 3.1

Additionally, these groups are C-groups because the generators satisfy the intersection condition. For our purposes, we need the following lemma:

\textbf{Lemma 3.8} Every string C-group is a tail triangle C-group.

\textbf{Proof.} Given a string C-group of rank $n$, which is generated by $\{\rho_0, \ldots, \rho_{n-1}\}$, then we can rename $\rho_i$ as $\alpha_i$ for $i = 0, \ldots, n - 2$ and rename $\rho_{n-1}$ as $\beta_{n-2}$. It is clear that the conditions necessary to be a tail-triangle C-group are satisfied. $\square$

A string C-group is essentially a tail-triangle C-group with the additional condition that $\alpha_{n-3}$ and $\beta_{n-2}$ commute. Notice that if we take the tail-triangle diagram given in Figure 3.1 and let $k = 2$, then this edge of the diagram disappears, and we have a string diagram.

Lemma 3.8 allows us to apply the abstract Wythoff’s construction of [26] to the group of an abstract polytope of rank $n$. When this abstract Wythoff’s construction is applied
to a string C-group $\Gamma(\mathcal{P})$, for some regular polytope $\mathcal{P}$, we will show that the construction results in the polytope $[\mathcal{P}]_{n-2}$.

We will now give the details of the abstract Wythoff’s construction. The major change that we are making here from the way that it is presented in [26] is that we are lowering all of the ranks by one so that we end up with a polytope of rank $n$, rather than a polytope of rank $n + 1$. We had already set a precedent for this in how we defined a tail triangle group in Definition 3.7. Additionally, rather than renaming the groups which Monson and Schulte refer to as $\Gamma^n_\mathcal{P}$ and $\Gamma^n_\mathcal{Q}$ as $\Gamma^{n-1}_\mathcal{P}$ and $\Gamma^{n-1}_\mathcal{Q}$, we instead rename them as $\Gamma'_{n-1}$ and $\Gamma''_{n-1}$. Finally, we will be defining the faces as left cosets, rather than right cosets.

**Definition 3.9 ([26, Def. 4.2])** The polytope $\mathcal{S}$.

Suppose that the group $\Gamma = \langle \alpha_0, \ldots, \alpha_{n-3}, \alpha_{n-2}, \beta_{n-2} \rangle$ is a tail-triangle C-group. Take the improper faces of $\mathcal{S}$ to be two distinct copies $\Gamma_{n-1}$ and $\Gamma_n$ of $\Gamma$. Next, for $0 \leq j \leq n - 3$, define the $j$-faces of $\mathcal{S}$ to be all left cosets in $\Gamma$ of

$$
\Gamma_j := \langle \alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{n-2}, \beta_{n-2} \rangle .
$$

The $(n-2)$-faces, or ridges, of $\mathcal{S}$ are all left cosets of

$$
\Gamma_{n-2} := \langle \alpha_0, \ldots, \alpha_{n-4}, \alpha_{n-3} \rangle .
$$

Finally, the $(n-1)$-faces, or facets, of $\mathcal{S}$ are all left cosets of either

$$
\Gamma'_{n-1} = \langle \alpha_0, \ldots, \alpha_{n-4}, \alpha_{n-3}, \alpha_{n-2} \rangle
$$

or

$$
\Gamma''_{n-1} = \langle \alpha_0, \ldots, \alpha_{n-4}, \alpha_{n-3}, \beta_{n-2} \rangle .
$$
Finally we define what turns out to be a partial order on the set of all such faces by taking

\[ \nu \Gamma_j < \mu \Gamma_k \]

whenever \(-1 \leq j < k \leq n\) and \(\nu \Gamma_j \cap \mu \Gamma_k \neq \emptyset\), for \(\mu, \nu \in \Gamma\), where \(\Gamma_{n-1}\) can refer to either \(\Gamma'_{n-1}\) or \(\Gamma''_{n-1}\).

Notice that after renaming \(\alpha_i\) as \(\rho_i\) for \(i = 1, \ldots, n - 2\) and \(\beta_{n-2}\) as \(\alpha_{n-1,n-2,n-1}\), the polytope \(S\) is defined exactly as the polytope \(Q\) was defined in Section 3.1. Therefore, we have the following result.

**Proposition 3.10** Let \(P\) be a regular polytope of rank \(n\). Then, when the abstract Wythoff construction is applied to \(\Gamma(P)\), the resulting polytope is then \((n - 2)\)-bubble \([P]_{n-2}\).

Given this, we can now apply results which are known about the abstract Wythoff’s construction to \([P]_{n-2}\). In particular, we will apply [26, Proposition 4.8] to the case of \([P]_{n-2}\). We will state this result here, adapting it only to match the terminology that we have been using for \([P]_{n-2}\)

**Proposition 3.11** Let \([P]_{n-2}\) be the \((n - 2)\)-bubble of a regular polytope \(P\) of Schlafli type \(\{p_1, \ldots, p_{n-1}\}\). Let \(\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle\) be the automorphism group of \(P\).

(a) Then \([P]_{n-2}\) is a regular polytope if and only if \(\Gamma\) admits a group automorphism which swaps \(\rho_{n-2}\) and \(\rho_{n-1}\), while fixing the remaining \(\rho_j\)’s. In this case the facets of \([P]_{n-2}\) of types 1 and 2 are isomorphic, with Schlafli type \(\{p_1, \ldots, p_{n-2}\}\), and \([P]_{n-2}\) is regular of type \(\{p_1, \ldots, p_{n-2}, 2p_{n-1}\}\); moreover, \(\Gamma([P]_{n-2}) \simeq \Gamma \times C_2\).

(b) If \([P]_{n-2}\) is not regular, then \([P]_{n-2}\) is a 2-orbit polytope and \(\Gamma([P]_{n-2}) \simeq \Gamma\). In particular, this is so if the facets of types 1 and 2 are non-isomorphic (as is the case, for example, if \(p_{n-2} \neq 2\)).
This result proves our earlier claim that $[\mathcal{P}]_{n-2}$ is a two-orbit polytope if and only if there is no automorphism of $\Gamma(\mathcal{P})$ which exchanges $\rho_{n-1}$ and $\rho_{n-2}$ while fixing the other generators. We already knew that $[\mathcal{P}]_{n-2}$ was either two-orbit or regular, and that if $p_{n-2} \neq 2$, then it was a two-orbit polytope. However, this result does give us more information about the group of $[\mathcal{P}]_{n-2}$ in the case that it is regular.
Chapter 4

Additional Constructions

In this section, given an \( n \)-polytope \( P \), we describe a construction which is a generalization of the \((n - 2)\)-bubble \([P]_{n-2}\). While in some ways this construction is less versatile than the general \( k \)-bubble \([P]_k\), one benefit to this construction will be that it allows us to construct polytopes with an arbitrary finite number of flag-orbits, whereas the number of flag-orbits of \([P]_k\) was limited by the rank and number of flag-orbits of \( P \). When \( P \) is regular, these polytopes will also be vertex-transitive, uniform, and semiregular. They will also be \( i \)-face transitive for all ranks \( i \leq n - 2 \).

4.1 The definition of \([P]^k\)

Like the construction of \([P]_{n-2}\), this new construction takes the ridges ((\(n - 2\))-faces) of \( P \) and replaces them with several copies of themselves, preserving all incidences with faces of lower ranks, and adds in additional facets which we can think of as space filling facets between the new ridges. As is the case in \([P]_{n-2}\), each of these space filling facets will be a ditope over a ridge. We will denote this construction \([P]^k\) for some \( k \geq 1 \).
Informally, the construction of $[P]^k$ will replace each ridge of $P$ with $k$ ridges, so that $[P]^2 = [P]_{n-2}$. Note that when we defined $[P]_k$ (for $k \geq 0$), the subscript $k$ referred to the rank of the faces of $P$ which are replaced by several copies of themselves. In this case, the number of copies is determined by the structure of $P$. In particular, in $[P]_k$, the number of copies of each $k$-face is determined by the number of $(k+1)$-faces to which it was incident in $P$. However, in $[P]^k$, it is always the ridges which are being replaced, and the superscript $k$ refers to the number of copies which replace each ridge. In particular, in $[P]^k$ each ridge is replaced with $k$ ridges and $k - 1$ facets, in an alternating manner, beginning and ending with a ridge. The formal definition is given shortly.

We will consider the differences between $[P]^2$, which is the familiar case $[P]_{n-2}$, and $[P]^k$ in general. For example, if $P$ is a polyhedron, the ridges are edges. While in $[P]^2$ (which is the familiar case $[P]_1$, since $n = 3$), each edge was replaced with two edges and a digon to fill in the space between them, in $[P]^k$ each edge is replaced with $k$ edges, each of which is incident to the same two vertices to which the original edge was incident. Because $[P]^k$ is a polyhedron, it cannot have proper faces of rank 3, so the edges which replace the original edge must be arranged in sequence, with digons between consecutive edges to fill in the space in a way such that the digons are all facets of $[P]^2$. Thus, you can view each edge as being replaced by $k$ edges and $k - 1$ digons in an alternating manner so that we have an edge, which is incident to a digon, which is incident to an edge and so forth. Additionally, the original 2-faces of $P$ remain intact. The combinatorics is illustrated in Figure 4.1. Figure 4.1(a) shows a local picture at an edge $F$ of $P$, and Figure 4.1(b) shows a local picture of the faces which replace $F$ in $[P]^3$, where $G_1$ and $G_2$ are the original 2-faces, the $H_i$ are new digons, and the $F_i$ are the new edges replacing $F$.

We will illustrate an important difference between the way that faces are replaced in $[P]^k$ and $[P]_k$. Let $P$ be a polyhedron, and $Q$ be a 4-polytope such that each edge is incident to three 2-faces. Let $k = 3$. Then in both $[P]^3$ and $[Q]_1$, each edge is replaced with three edges
with digons separating them. However, in the 4-polytope \([Q]_1\), these three new edges are arranged in a cycle so that they, along with the digons between them, create a new facet, \(\{2,3\}\). In particular, there are also three digons which replace the edge. In the polyhedron \([P]^3\), the three edges are arranged in a sequence, and the digons separating them are facets of the whole polyhedron. In particular, there are only two digons which are needed to fill in the spaces, and the outer two edges are incident to original facets as well as a digon.

When \(P\) is a rank 4-polytope, then the ridges of \(P\) are 2-faces. If a 2-face of \(P\) is a \(p\)-gon, then in \([P]^k\), it is replaced with \(k\) \(p\)-gons, identified along their boundaries such that \(k - 1\) facets of type \(\{p,2\}\) (ditopes over the \(p\)-gon) are created. Again, these \(k\) \(p\)-gons and \(k - 1\) ditopes are arranged in an alternating manner between the the original facets of \(P\), which remain intact. Again, note that each of these faces of type \(\{p,2\}\) is a facet of \([P]^k\), as opposed to being a face of a newly created facet, as is the case when faces of type \(\{p,2\}\) are created in \([Q]_2\), for \(Q\) a polytope of rank greater than 4.

In general, if \(P\) is an \(n\)-polytope, \(F\) is a ridge of \(P\), and \(H\) is a ditope over \(F\), then in \([P]^k\), \(F\) is replaced with \(k\) distinct copies of itself which are identified along their boundaries and \(k - 1\) distinct copies of \(H\), which fill in the spaces so that the \(k\) copies of \(F\) and \(k - 1\) copies of \(H\) are arranged in an alternating manner between the original facets of \(P\) meeting at \(F\), so that a copy of \(F\) is incident to a copy of \(H\), which is in turn incident to a copy of \(F\), and so forth. This can also be illustrated by the Hasse diagram in Figure 4.1, where in
this case, we think of $G_1$ and $G_2$ as original $(n - 1)$-faces, the $H_i$ as new facets, and the $F_i$ as the new ridges replacing $F$.

**Remark** In the construction of the $k$-bubble $[\mathcal{P}]_k$ we used the name $(F, G)$ to refer to a face which had two properties. First of all, it was one of the faces which replaced the face $F$. Secondly, it was located “in the direction of $G$.” These two properties, along with the rank of $G$ were enough to uniquely identify the face. However, in the new construction $[\mathcal{P}]^k$, for a ridge $F$ and a facet $G$ of $\mathcal{P}$, there are several new ridges as well as several new facets with both of these properties. Thus, simply referring to the new faces as ordered pairs of faces of $\mathcal{P}$ as was done in the first construction is not sufficient. We will still use the pairing of a ridge $F$ with a facet $G$ to refer to faces which are among the faces which are replacing $F$, and fall “in the direction of $G$.” However, we will need to give more information in order to distinguish a face from other faces with these properties. In order to distinguish the ridges from the facets, $F$ will always represent a ridge of $\mathcal{P}$, and $G$ will always represent a facet of $\mathcal{P}$. The ridges of $[\mathcal{P}]^k$ will be ordered pairs of the form $(F, G)_j$ and the facets of $[\mathcal{P}]^k$ will be ordered pairs of the (reversed) form $(G, F)_j$, where in each case the subscripts distinguish between multiple copies of each pair, with higher subscripts indicating that a face is closer to the “middle” of the sequence of faces. The reason that we cannot simply denote the $k$ ridges which replace $F$ by $F_i$ for $i = 1, \ldots, k$ is that there is no uniform way to choose an orientation for each ridge $F$. That is to say, if $F$ is incident to $G$ and $G'$ in $\mathcal{P}$, then there is no uniform way to choose if $F_1$ is the ridge of $[\mathcal{P}]^k$ which is incident to $G$ or $G'$ (where $F_k$ would be incident to the other). However, we can denote the new ridge closest to $G$ by $(F, G)_1$, and the new ridge closest to $G'$ by $(F, G')_1$. In general, we need to use the two facets $G$ and $G'$ to denote which side of $F$ a new ridge or facet falls on, even for faces other than those which are actually incident to $G$ or $G'$.

When replacing a ridge with an alternating sequence of ridges and facets, beginning and ending with a ridge, the “middle” face of this sequence cannot be associated with the facet
at either end of the sequence in a uniform way. Therefore, while most of the new ridges and facets can be described using an ordered pair of a ridge and a facet of \( P \), the “middle” face of the sequence can only be identified with the ridge \( F \) which it is replacing. When \( k \) is even, this “middle” face is a facet, and when \( k \) is odd, this “middle” face is a ridge. Thus every ridge of \( P \) can be identified uniquely and uniformly with a face of \([P]^k\); but the rank of this face depends on the parity of \( k \). Therefore, the definition of \([P]^k\) distinguishes between the cases \( k \) even and \( k \) odd.

**Definition 4.1** Let \( P \) be an \( n \)-polytope, let \( P_i \) denote the set of \( i \)-faces of \( P \), and let \( k \geq 1 \). Define \([P]^k\) to be the ranked poset of rank \( n \) with the following faces:

For \(-1 \leq i \leq n - 3 \) and \( i = n \), the \( i \)-faces of \([P]^k\) are exactly the \( i \)-faces of \( P \).

For \( k \) even, the set of \((n - 2)\)-faces (ridges) of \([P]^k\) is

\[
\{(F, G)_j \mid F \in P_{n-2}, G \in P_{n-1}, F \leq P G, 1 \leq j \leq \frac{k}{2}\}.
\]

Additionally, for \( k \) even, the set of \((n - 1)\)-faces (facets) of \([P]^k\) is

\[P_{n-2} \cup P_{n-1} \cup \{(G, F)_j \mid F \in P_{n-2}, G \in P_{n-1}, F \leq P G, 1 \leq j \leq \frac{k}{2} - 1\}.
\]

For \( k \) odd, the set of \((n - 2)\)-faces (ridges) of \([P]^k\) is

\[P_{n-2} \cup \{(F, G)_j | F \in P_{n-2}, G \in P_{n-1}, F \leq P G, 1 \leq j \leq \frac{k-1}{2}\}.
\]

Additionally, for \( k \) odd, the set of \((n - 1)\)-faces (facets) of \([P]^k\) is

\[P_{n-1} \cup \{(G, F)_j | F \in P_{n-2}, G \in P_{n-1}, F \leq P G, 1 \leq j \leq \frac{k-1}{2}\}.
\]
Finally, for \( k \) even, we define \((G,F)_{k/2} := F\) for \( F \) a ridge of \( \mathcal{P} \) and \( G \) an incident facet of \( \mathcal{P} \). For \( k \) odd, we define \((F,G)_{k+1} := F\) for \( F \) a ridge of \( \mathcal{P} \) and \( G \) an incident facet of \( \mathcal{P} \).

The partial order on \([\mathcal{P}]^k\) will be defined after the following remark.

**Remark** As remarked previously, when \( k \) is even, the ridges of \( \mathcal{P} \) are facets of \([\mathcal{P}]^k\). That is to say that given a ridge \( F \) of \( \mathcal{P} \), the “middle” of the sequence of faces which replaces \( F \) is a facet of \([\mathcal{P}]^k\), which we call \( F \). Given such a facet \( F \) of \([\mathcal{P}]^k\), it is sometimes convenient to think of \( F \) as both \((G,F)_{k/2}\) and \((G',F)_{k/2}\), for \( G \) and \( G' \) the two facets incident to \( F \) in \( \mathcal{P} \). It is important to remember though, that this still is a single facet, \( F \). Likewise, if \( k \) is odd, it can be useful to think of a “middle” ridge \( F \) of \([\mathcal{P}]^k\) which also was a ridge of \( \mathcal{P} \) as both \((F,G)_{k+1/2}\) and \((F,G')_{k+1/2}\), for \( G \) and \( G' \) the two facets incident to \( F \) in \( \mathcal{P} \). This will simplify how the partial order is defined, as well as some of the following proofs.

**Definition 4.1 (cont.)** The partial order on \([\mathcal{P}]^k\) is defined (via transitivity) by the following order relation between faces of consecutive ranks, independent of the parity of \( k \).

1. For an \( i \)-face \( H \) and an \((i+1)\)-face \( H' \) of \( \mathcal{P} \), with \(-1 \leq i \leq n-4\), define \( H \leq_{[\mathcal{P}]^k} H' \) if and only if \( H \leq_{\mathcal{P}} H' \).

2. For an \((n-3)\)-face \( H \), an \((n-2)\)-face \( F \), and an \((n-1)\)-face \( G \) of \( \mathcal{P} \), with \( F \leq_{\mathcal{P}} G \), define \( H \leq_{[\mathcal{P}]^k} (F,G)_j \) if and only if \( H \leq_{\mathcal{P}} F \) (and hence \( H \leq_{\mathcal{P}} F \leq_{\mathcal{P}} G \)).

3. For an \((n-2)\)-face \( F \) and an \((n-1)\)-face \( G \) of \( \mathcal{P} \), with \( F \leq_{\mathcal{P}} G \), define
   a. \((F,G)_j \leq_{[\mathcal{P}]^k} G\) only when \( j = 1 \);
   b. \((F,G)_j \leq_{[\mathcal{P}]^k} (G,F)_j\) for \( 1 \leq j \leq \frac{k}{2} \);
   c. \((F,G)_j \leq_{[\mathcal{P}]^k} (G,F)_{j-1}\) for \( 2 \leq j \leq \frac{k+1}{2} \).

Furthermore, these are the only situations in which a face of rank \( n-2 \) is incident to a face of rank \( n-1 \).
4. Every \((n - 1)\)-face is incident to \(F_n\).

Note that when the condition \(j \leq \frac{k}{2}\) is given, the highest possible value for \(j\) is either \(\frac{k}{2}\) or \(\frac{k-1}{2}\), depending on the parity of \(k\), and when the condition \(j \leq \frac{k+1}{2}\) is given, the highest possible value for \(j\) is either \(\frac{k+1}{2}\) or \(\frac{k}{2}\), depending on the parity of \(k\). Note also that in giving this partial order, we utilized the preceding remark in order to give the incidences for the middle ridge or facet \(F\).

**Definition 4.2** We say that a face of \([P]^k\) which is of the form \((F,G)_j\) or \((G,F)_j\) is of type \(j\). Furthermore, we call a face of \([P]^k\) which is of the form \(G\), with \(G\) a facet of \(P\), of type 0. (Notice that if \(F\) is a ridge of \(P\), then in \([P]^k\), the face \(F\) is of type \(\frac{k}{2}\) or \(\frac{k+1}{2}\), depending on the parity of \(k\). We do not define face types for faces which are not of ranks \(n - 1\) or \(n - 2\).)

Notice that when \(k = 1\), \(k\) is odd and there do not exist any values of \(j\) such that \(1 \leq j \leq \frac{k-1}{2}\). Therefore, we have the following result.

**Lemma 4.3** For \(P\) an \(n\)-polytope, \([P]^1 \cong P\).

Earlier, we stated that each ridge of \(P\) is replaced by a sequence of alternating ridges and facets. We now can specify that a ridge \(F\) which is incident to facets \(G\) and \(G'\) in \(P\) is replaced by the sequence of successively incident faces

\[
(F,G)_1, (G,F)_1, (F,G)_2, \ldots, (G,F)_{k/2} = F = (G',F)_{k/2}, \ldots, (F,G')_2, (G',F)_1, (F,G')_1
\]

with \((F,G)_1\) incident to \(G\) and \((F,G')_1\) incident to \(G'\), if \(k\) is even, and the sequence

\[
(F,G)_1, (G,F)_1, (F,G)_2, \ldots, (F,G)_{k+1/2} = F = (F,G')_{k+1/2}, \ldots, (F,G')_2, (G',F)_1, (F,G')_1,
\]
with \((F, G)_1\) incident to \(G\) and \((F, G')_1\) incident to \(G'\), if \(k\) is odd. All faces in these sequences are incident to the same faces of ranks less than or equal to \(n - 3\) to which \(F\) was incident. This is illustrated in the Hasse diagram given in Figure 4.2. This is the exact same Hasse diagram which was given in Figure 4.1, which showed the replacement of a ridge with two new facets and three new ridges in a polyhedron \([\mathcal{P}]^3\). Now we have simply renamed the faces in accordance with our definition. Notice that the node in (b) labeled \(F\) could also be labeled \((F, G)_2\) or \((F, G')_2\).

![Hasse diagram](image)

Figure 4.2

Notice that by part 3 of the partial order given in Definition 4.1, a ridge can be incident to a facet in three different ways. These three cases of the partial order give rise to three types of flags. Flags of the first kind have a facet of type 0, i.e., a facet \(G\) which was an original facet of \(P\). In this case, the ridge of the flag must be of the form \((F, G)_1\) since only the “outermost” ridge of the form \((F, G)_j\) is incident to \(G\). Flags of the second or third kind have facets which were not facets of \(P\). The difference between these two kinds of flags is simply that in the second type of flag, the facet is to the “inside” of the ridge, and in the third type of flag, the facet is to the “outside” of the ridge, where “inside” and “outside” refer to proximity to the middle of the sequence of ridges and facets that replace each ridge of \(P\). These three types of flags are defined formally in the following lemma.

**Lemma 4.4** A flag of \([\mathcal{P}]^k\) can take one of the following three forms. In each case, we assume that \(\{F_{-1}, F_0, \ldots, F_{n-3}, F, G, F_n\}\) is a flag of \(\mathcal{P}\).
1. \( \{F_{-1}, F_0, \ldots, F_{n-3}, (F, G)_1, G, F_n\} \);

2. \( \{F_{-1}, F_0, \ldots, F_{n-3}, (F, G)_j, (G, F)_j, F_n\} \) for \( 1 \leq j \leq k \); (If \( j = \frac{k}{2} \), then the facet \((G, F)_j\) is to be identified with \( F \).)

3. \( \{F_{-1}, F_0, \ldots, F_{n-3}, (F, G)_j, (G, F)_{j-1}, F_n\} \) for \( 1 \leq j \leq \frac{k+1}{2} \). (If \( j = \frac{k+1}{2} \), then the ridge \((F, G)_j\) is to be identified with \( F \).)

Now we can prove that \([\mathcal{P}]^k\) is, in fact, a polytope.

**Theorem 4.5** If \( \mathcal{P} \) is an \( n \)-polytope, then \([\mathcal{P}]^k\) is also an \( n \)-polytope.

**Proof** We defined \([\mathcal{P}]^k\) as a ranked poset whose improper faces are exactly the improper faces of \( \mathcal{P} \), so conditions (P1) and (P2) are clear.

Next we verify the diamond condition. Let \( \Phi \) be a flag of \([\mathcal{P}]^k\). Recall that \((\Phi)_l\) denotes the \( l \)-face contained in \( \Phi \) for \( l = -1, \ldots, n \). If \( i \leq n-3 \), then \((\Phi)_i/(\Phi)_{i-2}\) is exactly a section of \( \mathcal{P} \), and therefore satisfies the diamond condition. If \( i = n-2 \), then \((\Phi)_i\) is of the form \((F, G)_j\) for some \( j \). Here, \((F, G)_j\) is incident to the same faces of lower ranks as \( F \) was in \( \mathcal{P} \), so the section \((F, G)_j/(\Phi)_{i-2}\) is isomorphic to the section \( F/(\Phi)_{i-2} \) of \( \mathcal{P} \), and therefore satisfies the diamond condition. Therefore, only the cases \( i = n-1 \) and \( i = n \) require attention.

We first look at the case \( i = n-1 \). The section \((\Phi)_{n-1}/(\Phi)_{n-3}\) can take one of three forms, corresponding to three possible types of facets in \([\mathcal{P}]^k\). The first possible form for a section \((\Phi)_{n-1}/(\Phi)_{n-3}\) of \([\mathcal{P}]^k\) is \( G/H \) where \( H \) is an \((n-3)\)-face of \( \mathcal{P} \) and \( G \) is a face of \([\mathcal{P}]^k\) of type 0, and hence an \((n-1)\)-face of \( \mathcal{P} \) with \( H \leq G \). In this case, the section \( G/H \) of \( \mathcal{P} \) is given by \( \{H, F', F'', G\} \), for some ridges \( F' \) and \( F'' \) of \( \mathcal{P} \), by the diamond condition on \( \mathcal{P} \). In \([\mathcal{P}]^k\), the ridges incident to \( G \) are of the form \((F, G)_1\) with \( F \) a ridge of \( \mathcal{P} \) and \( F \leq G \). Such faces which are also incident to \( H \) must have \( H \leq F \) in \( \mathcal{P} \). Thus, the only such faces
are \((F, G)_1\) and \((F, G')_1\). The diamond condition is satisfied in this case, since there are exactly two proper faces of the section.

The second possible form of such a section is \((G, F)_j/H\), where \(H\) is an \((n - 3)\)-face, \(F\) is an \((n - 2)\)-face, and \(G\) is an \((n - 1)\)-face of \(P\) with \(H \leq F \leq G\), and \(1 \leq j \leq \frac{k - 1}{2}\). The only ridges of \([P]^k\) which are incident to \((G, F)_j\) are \((F, G)_j\) and \((F, G)_{j+1}\). Notice that since \(j \leq \frac{k - 1}{2}\), \((F, G)_{j+1}\) is a face of \([P]^k\). Since these are both also incident to \(H\), the diamond condition is satisfied in this case.

The third possible form is \(F/H\), where \(H\) is an \((n - 3)\)-face of \(P\) and \(F\) is \((n - 2)\) face of \(P\). In this case, \(F\) is a facet of \([P]^k\) which falls in the middle of the sequence of faces which replace the ridge \(F\) of \(P\). Thus, while \(F\) is a ridge of \(P\), it is a facet of \([P]^k\). In this case, \(k\) must be even. In \(P\), there are two facets, \(G\) and \(G'\) (say), incident to \(F\) (as a ridge). Because \(F\) can be thought of as both \((G, F)_{k/2}\) and \((G', F)_{k/2}\), the only ridges to which this facet is incident are \((F, G)_{k/2}\) and \((F, G')_{k/2}\), both of which are also incident to \(H\). So the diamond condition is satisfied in this case as well.

Finally, when \(i = n\), we consider the section \((\Phi)_n/(\Phi)_{n-2}\). In this case, it is only necessary to show that every ridge of \([P]^k\) is incident to exactly two facets, since every face is incident to \((\Phi)_n = F_n\).

A ridge of the form \((F, G)_1\) is incident to only the two facets \(G\) and \((G, F)_1\).

For \(2 \leq j \leq \frac{k}{2}\), a ridge \((F, G)_j\) is incident to the two facets \((G, F)_j\) and \((G, F)_{j-1}\), and only these facets.

Say \(F\) is a ridge of \(P\) and also a ridge of \([P]^k\). Then \(k\) must be odd. In \(P\), there are two facets to which \(F\) is incident, say \(G\) and \(G'\). Then in \([P]^k\), \(F\) can be thought of as either \((F, G)_{\frac{k+1}{2}}\) or \((F, G')_{\frac{k+1}{2}}\). Hence, \(F\) is incident to the two facets \((G, F)_{\frac{k-1}{2}}\) and \((G', F)_{\frac{k-1}{2}}\), and only these facets.

This completes the proof of the diamond condition for \([P]^k\).
It remains to prove that \([\mathcal{P}]^k\) is strongly connected.

Given a section \((\Phi)_i/(\Phi)_l\) of \([\mathcal{P}]^k\), where \((\Phi)_i\) and \((\Phi)_l\) are faces of some flag \(\Phi\), we want to show that the section is connected. This is trivial, unless \(l < n - 2 < i\), because if \(n - 2 \leq l < i \leq n\), then \((\Phi)_i/(\Phi)_l\) is a 0-section or a 1-section, which is trivially connected, and if \(i \leq n - 2\), then \((\Phi)_i/(\Phi)_l\) is isomorphic to a section of \(\mathcal{P}\), which is connected because \(\mathcal{P}\) is a polytope.

Thus we may assume that some ridges of \([\mathcal{P}]^k\) are proper faces of \((\Phi)_i/(\Phi)_l\). First we will look at the case in which no facet is a proper face of the section, that is, the case \(i = n - 1\). Because all 1-sections are trivially connected, we may assume that not only are some ridges proper faces of the section, but that there are also faces of rank less than \(n - 2\) which are proper faces of the section as well. This breaks down into two subcases.

First, suppose that the section \((\Phi)_i/(\Phi)_l\) is contained in a facet \((G, F)_j\) with \(F\) a ridge and \(G\) a facet of \(\mathcal{P}\) with \(F \leq G\). There are only two ridges which are incident to \((G, F)_j\) (namely the ridges \((F, G)_j\) and \((F, G)_{j+1}\), if \(j \neq \frac{k}{2}\), and the ridges \((F, G)_j\) and \((F, G')_j\), if \(j = \frac{k}{2}\) and \((G, F)_j = (G', F)_j = F\) and they are both incident to the same faces of rank less than \(n - 2\), namely all those incident to \(F\) in \(\mathcal{P}\). So every ridge in this section is incident to every face of rank less than \(n - 2\) in this section. Now, to prove connectedness of the section, if \(K\) and \(K'\) are proper faces of \((\Phi)_i/(\Phi)_l\), then either both are ridges, exactly one is a ridge, or neither are ridges. If both are ridges, then choose some face \(L\) in the section \((\Phi)_i/(\Phi)_l\) of rank less than \(n - 2\). Then \(K\) and \(K'\) are both incident to \(L\), so the sequence \(K, L, K'\) is a sequence of successively incident faces from \(K\) to \(K'\). If exactly one of \(K\) and \(K'\) is a ridge, then they are already incident. Finally, if neither is a ridge then choose some ridge \(L\) in the section \((\Phi)_i/(\Phi)_l\). Then \(K\) and \(K'\) are both incident to \(L\), so the sequence \(K, L, K'\) is a sequence of successively incident faces from \(K\) to \(K'\).
Second, suppose \((\Phi)_i/(\Phi)_l\) is contained in a facet \(G\) of \([\mathcal{P}]^k\), where \(G\) is of type 0, and hence is also facet of \(\mathcal{P}\). The only ridges in this section are of the type \((F,G)_1\) for \(F \leq \mathcal{P} G\). There clearly is an isomorphism from \((\Phi)_i/(\Phi)_l\) to a section of \(G\) in \(\mathcal{P}\), which sends \((F,G)_1\) to \(F\), for each face of the form \((F,G)_1\) in the section, and keeps all of the other faces fixed. This section of \(\mathcal{P}\) is connected because \(\mathcal{P}\) is a polytope, so the corresponding section of \([\mathcal{P}]^k\) is connected as well.

Next, we will deal with the case that the section \((\Phi)_i/(\Phi)_l\) contains facets of \([\mathcal{P}]^k\) as proper faces (that is \(i = n\)). Because every 1-section is defined to be connected, we may assume that there are ridges which are proper faces of the section. First we will show that for every proper face \(K\) of this section, there is a sequence of successively incident proper faces in the section joining \(K\) to a facet of type 0. Now clearly \(K\) is incident to (or equal to) some facet of \([\mathcal{P}]^k\). If \(K\) is already incident to (or equal to) some facet of type 0, then we are done. Otherwise, choose any facet to which \(K\) is incident (or equal). It is of the form \((G,F)_j\) for some ridge \(F\) and some facet \(G\) of \(\mathcal{P}\) with \(F \leq \mathcal{P} G\) and some \(j\). Since \((G,F)_j\) is in the section \((\Phi)_i/(\Phi)_l\), \((G,F)_m\) and \((F,G)_m\) must be in the section as well for all \(m\) such that such faces exist in \([\mathcal{P}]^k\). The facet \((G,F)_j\) is incident to the ridge \((F,G)_j\), which is also incident to the facet \((G,F)_{j-1}\), which in turn is incident to the ridge \((F,G)_{j-1}\). Continuing inductively, we can see that there is a sequence of successively incident faces in the section \((\Phi)_i/(\Phi)_l\) from \(K\) to the ridge \((F,G)_1\), and hence also to the face \(G\) of type 0 which is incident to \((F,G)_1\). (We note that if subridges \(((n-3)\text{-faces})\) are proper faces of the section, then this sequence can be replaced with a much shorter facet-subridge-facet sequence. However, in general we cannot make that assumption.)

Now that we have joined proper faces of the given section to facets in that section which are facets of \(\mathcal{P}\), it suffices to show that any two facets \(G\) and \(G'\) of type 0 in the section can be joined by a sequence of successively incident proper faces in the section. Clearly, there is a sequence of successively incident proper faces from \(G\) to \(G'\) in the corresponding section of
Each face of \( \mathcal{P} \) in this sequence is also a face of \([\mathcal{P}]^k\), though ridges of \( \mathcal{P} \) may be facets of \([\mathcal{P}]^k\), and incidence between faces might change. Given two faces of this section of \( \mathcal{P} \) which are incident in \( \mathcal{P} \), then one of the following three possibilities must occur.

The first possibility is that the two faces are also incident in \([\mathcal{P}]^k\). The second possibility is that one face, \( G \), is a facet of \( \mathcal{P} \), while the other, \( F \), is a ridge of \( \mathcal{P} \), such that \( F \) is also a ridge in \([\mathcal{P}]^k\). In this case, there is a sequence of successively incident proper faces from \( G \) to \( F \) in the section of \([\mathcal{P}]^k\), namely

\[
G, (F,G)_1, (G,F)_1, (F,G)_2, \ldots, (F,G)_{(k+1)/2} = F.
\]

The third possibility is that one face, \( G \), is a facet of \( \mathcal{P} \), while the other, \( F \), is a ridge of \( \mathcal{P} \), such that \( F \) is a facet in \([\mathcal{P}]^k\). In this case, there again is a sequence of incident proper faces from \( G \) to \( F \) in the section of \([\mathcal{P}]^k\), namely

\[
G, (F,G)_1, (G,F)_1, (F,G)_2, \ldots, (G,F)_{k/2} = F.
\]

(In the second and third cases, these sequences could again be shortened if subridges were proper faces of the section, but that is not an assumption we make.) Therefore, the sequence of successively incident faces from \( G \) to \( G' \) in the corresponding section of \( \mathcal{P} \) can be extended to a sequence of successively incident faces in \([\mathcal{P}]^k\). This completes the proof that \([\mathcal{P}]^k\) is strongly connected, and therefore that \([\mathcal{P}]^k\) is a polytope.

We have been informally describing \([\mathcal{P}]^k\) as a generalization of \([\mathcal{P}]_{n-2}\). At this point we can formally prove this claim:

**Proposition 4.6** Let \( \mathcal{P} \) be an \( n \)-polytope. Then \([\mathcal{P}]^2 \cong [\mathcal{P}]_{n-2}\).
Proof We will prove this by defining a rank- and incidence-preserving bijection $f$ from $[\mathcal{P}]^2$ to $[\mathcal{P}]_{n-2}$. For $i \leq n - 3$ and $i = n$, the $i$-faces of both $[\mathcal{P}]^2$ and $[\mathcal{P}]_{n-2}$ are defined to be the $i$-faces of $\mathcal{P}$. Additionally, since in $[\mathcal{P}]^k$, if $k$ is even and, in particular, if $k$ is 2, the set of facets of $[\mathcal{P}]^2$ is the union of the set of facets and the set of ridges of $\mathcal{P}$. This is exactly how the set of facets of $[\mathcal{P}]_{n-2}$ is defined. Thus, let $f$ be a rank-preserving function from $[\mathcal{P}]^2$ to $[\mathcal{P}]_{n-2}$ which is trivial on faces of ranks $i \neq n - 2$.

Now the set of ridges of $[\mathcal{P}]^2$ is $\{(F, G) | F \in \mathcal{P}_{n-2}, G \in \mathcal{P}_{n-1}, F \preceq G\}$, and the set of ridges of $[\mathcal{P}]_{n-2}$ is $\{(F, G) | F \in \mathcal{P}_{n-2}, G \in \mathcal{P}_{n-1}, F \preceq G\}$. Define $f((F, G)_1) := (F, G)$. Then $f$ is defined on faces of all ranks and clearly is a rank- and incidence-preserving bijection. □

Finally, we have the following result which has been clear from our discussion of $[\mathcal{P}]^k$, but we will now prove it formally.

**Lemma 4.7** Let $\mathcal{P}$ be an $n$-polytope ($n \geq 3$), and let $k$ be an integer greater than or equal to 2. Let $(G, F)_i$ be a facet of $[\mathcal{P}]^k$ which is not of type 0. Additionally, let $H$ be an incident $(n-4)$-face of $[\mathcal{P}]^k$. Then the section $(G, F)_i/H$ is a digon.

**Proof.** In the proof of Theorem 4.5 we showed that $(G, F)_i$ is incident to exactly two ridges. Thus, these two ridges must be incident to $H$ since $[\mathcal{P}]^k$ is a polytope, so there are exactly two $(n - 2)$-faces in the section $(G, F)_i/H$.

### 4.2 The automorphism group of $[\mathcal{P}]^k$

We will now begin to determine when the automorphism group $\Gamma([\mathcal{P}]^k)$ is isomorphic to $\Gamma(\mathcal{P})$. Our strategy will be similar to the strategy we used with $[\mathcal{P}]_k$. Though we have defined face types differently, we will still show that if face types are preserved by $\Gamma([\mathcal{P}]^k)$, then $\Gamma([\mathcal{P}]^k)$ is isomorphic to $\Gamma(\mathcal{P})$. We will start with some results which will help us to determine when face types are preserved.
Lemma 4.8 (a) The shortest sequence of successively incident ridges and facets in \([\mathcal{P}]^k\) from a ridge of type \(j\) to a face (facet) of type 0 has length \(2j\). The shortest sequence of successively incident ridges and facets in \([\mathcal{P}]^k\) from a facet of type \(j\) to a face of type 0 has length \(2j + 1\). Furthermore, if a sequence of length \(2j\) exists from a ridge \((F,G)_j\) of type \(j\) to a face of type 0 then if \(j \neq \frac{k+1}{2}\), that face of type 0 must be \(G\) and the sequence is unique. If \(j = \frac{k+1}{2}\), and \((F,G)_j = (F,G')_j = F\), then the face of type 0 is \(G\) or \(G'\), and there are exactly two sequences connecting \((F,G)_j\) to a face of type 0. Additionally, if a sequence of length \(2j + 1\) from a facet \((G,F)_j\) to a face of type 0 exists, then, if \(j \neq \frac{k}{2}\) then that face of type 0 must be \(G\) and the sequence is unique; and if \(j = \frac{k}{2}\), and \((G,F)_j = (G',F)_j = F\), then the face of type 0 is \(G\) or \(G'\) and there are exactly two sequences connecting \((F,G)_j\) to a face of type 0.

(b) The shortest sequence of successively incident ridges and facets in \([\mathcal{P}]^k\) from a ridge of type \(j\) to a face of the form \(F\), where \(F\) is a ridge of \(\mathcal{P}\), has length \(k + 2 - 2j\). The shortest sequence of successively incident ridges and facets in \([\mathcal{P}]^k\) from a facet of type \(j\) to a face of the form \(F\), where \(F\) is a ridge of \(\mathcal{P}\), has length \(k + 1 - 2j\). Furthermore, if there exists a sequence of length \(k + 2 - 2j\) from a ridge \((F,G)_j\) or a sequence of length \(k + 1 - 2j\) from a facet \((G,F)_j\) to a face of the form \(F'\), where \(F'\) is a ridge of \(\mathcal{P}\), then \(F'\) must be \(F\).

Proof Part (a) is clear because if you begin with a ridge \((F,G)_j\) of \([\mathcal{P}]^k\) such a sequence necessarily has the form

\[(F,G)_j, (G,F)_{j-1}, (F,G)_{j-1}, \ldots, (F,G)_1, G.\]

In this case the sequence of face types of faces in this sequence is \(j, j-1, j-1, \ldots, 1, 1, 0\), and the sequence has length \(2j\). If you begin with a facet \((G,F)_j\) of \([\mathcal{P}]^k\), the sequence
necessarily has the form

\[(G, F)_j (F, G)_j, (G, F)_{j-1}, (F, G)_{j-1}, \ldots, (F, G)_1, G.\]

In this case the sequence of face types of faces in this sequence is \(j, j, j-1, j-1, \ldots, 1, 1, 0\), and the sequence has length \(2j + 1\). In either case the sequence is unique, unless \(j = \frac{k+2}{2}\) or \(j = \frac{k}{2}\), respectively. In the latter cases we may replace \(G\) by \(G'\), so there are exactly two possible sequences.

The proof of Part (b) follows from Part (a) once we note two things. First of all, if \(F\) is a ridge of \(P\), with \(F \leq_P G\), then regardless of if \(F\) is a ridge or a facet of \([P]^k\), the shortest sequence of successively incident facets and ridges from \(G\) to \(F\) in \([P]^k\) has length \(k + 1\).

Secondly, if we let \((F, G)_j\) be a ridge of \([P]^k\), then the length of a sequence of successively incident ridges and facets from \(G\) to \(F\) in \([P]^k\) is one less than the sum of the length of the sequence of successively incident ridges and facets from \(G\) to \((F, G)_j\) for some \(j\) and the length of the sequence of successively incident ridges and facets from \((F, G)_j\) to \(F\), since \((F, G)_j\) was counted in both sequences. \(\square\)

**Lemma 4.9** If an automorphism of \([P]^k\) preserves the face types of the faces of type \(j\), for some \(j\), then it preserves the face types of faces of all types with a defined face type.

**Proof** Recall that only ridges and facets of \([P]^k\) were assigned face types. We will prove that if an automorphism of \([P]^k\) preserves the face types of the faces of type 0, then it preserves the face types of faces of all types. We will then prove that if an automorphism of \([P]^k\) preserves the face types of the faces of type \(j\), for some \(j\) then it preserves the face types of faces of type \(j - 1\). Together, these two statements complete the proof.

We will first let \(\sigma\) be an automorphism of \([P]^k\), such that \(\sigma\) preserves the face types of the faces of type 0. Then \(\sigma^{-1}\) must also preserve the face types of all faces of type 0. Let \(H\) be
a ridge (respectively, facet) of $[P]^k$ of type $j$. We want to show that $\sigma(H)$ is also of type $j$.

Consider the sequence of successively incident ridges and facets of length $2j$ (respectively $2j + 1$) from $H$ to a face $G$ of type 0. Since $\sigma$ preserves ranks and incidences, applying $\sigma$ to every term of this sequence will result in a sequence of successively incident ridges and facets of length $2j$ (respectively $2j + 1$) from $\sigma(H)$ to $\sigma(G)$. Since $\sigma$ preserves the face types of the faces of type 0, this is a sequence of successively incident ridges and facets of length $2j$ (respectively $2j + 1$) from $\sigma(H)$ to a face of type 0. For the sake of contradiction, assume that there is a shorter sequence of successively incident ridges and facets from $\sigma(H)$ to a face $G'$ of type 0. Then applying $\sigma^{-1}$ to each term of this sequence would result in a shorter sequence of successively incident ridges and facets from $H$ to $\sigma^{-1}(G')$, where $\sigma^{-1}(G')$ is of type 0. This contradicts the assumption that $H$ was of type $j$. Therefore, the shortest sequence of successively incident ridges and facets from $\sigma(H)$ to a face of type 0 has length $2j$ (respectively $2j + 1$). Since this is not true of faces of any other type, $\sigma(H)$ must be of type $j$.

Now assume that $\sigma$ preserves face types of every face of type $j$. Assume that $j \neq \frac{k+1}{2}$, so that there exists a facet $(G,F)_j$ of type $j$. If $j \neq 1$, then let $(G,F)_{j-1}$ be a facet of type $j-1$, and $(F,G)_{j-1}$ be a ridge of type $j - 1$. We will show that $\sigma((G,F)_{j-1})$ is of type $j - 1$, and that $\sigma((F,G)_{j-1})$ is also of type $j - 1$. Since the only faces of type $j - 1$ are ridges and facets, showing that $\sigma$ preserves the face types of these two faces is sufficient. If $j = 1$, then every face of type $j - 1$ is a facet of the form $G$ for some $G$ in $P$. If this facet is incident to a ridge of the form $(F,G)_1$, then, in order to simplify our argument, we will define $(G,F)_0$ to be $G$. In this case, we only need to show that the face type of $G$ is preserved.

Now $\sigma$ preserves the face types of the $j$-faces so $\sigma((G,F)_j)$ is a facet of type $j$, and $\sigma((F,G)_j)$ is a ridge of type $j$. Say $\sigma((G,F)_j)$ is $(G',F')_j$. Then $\sigma((F,G)_j)$ must be $(F',G')_j$, since this is the only ridge of type $j$ to which $(G',F')_j$ is incident, and $\sigma$ preserves incidences. Additionally, $(F,G)_j \leq (G,F)_{j-1}$, so $\sigma((F,G)_j) \leq \sigma((G,F)_{j-1})$, and $(G,F)_{j-1} \neq (G,F)_j$, 

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so \( \sigma((G, F)_{j-1}) \neq \sigma((G, F)_j) \). Now since \( \sigma((F, G)_j) \) is \((F', G')_j\), the face \( \sigma((G, F)_{j-1}) \) is a facet incident to \((F', G')_j\) which is not \((G', F')_j\). The only such face is \((G', F')_{j-1}\) (or \(G'\) if \(j = 1\)). So \( \sigma((G, F)_{j-1}) \) is a facet of type \(j-1\). Now, if \(j \neq 1\), \( \sigma((F, G)_{j-1}) \leq \sigma((G, F)_{j-1}) \) and \( \sigma((F, G)_{j-1}) \neq \sigma((F, G)_j) \). So \( \sigma((F, G)_{j-1}) \) must be a ridge incident to \((F', G')_{j-1}\) which is not \((G', F')_j\). Thus, \( \sigma((F, G)_{j-1}) \) must be \((F', G')_{j-1}\), which is a ridge of type \(j-1\).

Now, if \(j = \frac{k+1}{2}\), then the only faces of type \(j\) are ridges. Let \((G, F)_{j-1}\) be a facet of type \(j-1\), and \((F, G)_{j-1}\) be an incident ridge of type \(j-1\). Let \(F = (F, G)_j = (F, G')_j\) be the incident ridge of type \(j\). Then \(\sigma(F)\) is also a ridge of type \(j\). The only facets to which \(\sigma(F)\) are incident are of type \(j-1\). Thus \(\sigma((G, F)_{j-1})\) must be of type \(j-1\). Also, \(\sigma((F, G)_{j-1})\) must be a ridge which is incident to \(\sigma((G, F)_{j-1})\), but is not of type \(j\), so it must be of type \(j-1\). This completes the proof.

**Proposition 4.10** If \(\mathcal{P}\) is an equivelar (in particular, regular) polytope of rank \(n \geq 3\) of Schl"afli type \(\{p_1, \ldots, p_{n-1}\}\) with \(p_{n-2} \neq 2\), then the automorphism group \(\Gamma([\mathcal{P}]^k)\) preserves face types.

**Proof** By Lemma 4.9, we only need to show that every automorphism in \(\Gamma([\mathcal{P}]^k)\) preserves the face types of the faces of type 0. Let \(\sigma\) be an automorphism of \([\mathcal{P}]^k\). Since \(p_{n-2} \neq 2\), the facets of \(\mathcal{P}\) cannot be ditopes. Since all facets of \([\mathcal{P}]^k\) are combinatorially facets of \(\mathcal{P}\) or ditopes, and isomorphism types are preserved under automorphisms, \(\sigma\) must take a non-ditope facet to a non-ditope facet. That is, \(\sigma\) preserves the face types of the faces of type 0.

**Theorem 4.11** Let \(\mathcal{P}\) be a polytope of rank \(n \geq 3\). Then \(\Gamma(\mathcal{P})\) is embedded as a subgroup in \(\Gamma([\mathcal{P}]^k)\). Moreover, if \(\Gamma([\mathcal{P}]^k)\) preserves face types, then \(\Gamma([\mathcal{P}]^k)\) is isomorphic to \(\Gamma(\mathcal{P})\).

**Proof** First observe that automorphisms of \(\mathcal{P}\) induce automorphisms of \([\mathcal{P}]^k\). In fact, if \(\rho\) is an automorphism of \(\mathcal{P}\), then the map \(\tilde{\rho} : [\mathcal{P}]^k \to [\mathcal{P}]^k\) defined by \(\tilde{\rho}(H) := \rho(H)\) and
\[ \tilde{\rho}((H, H')_j) := (\rho(H), \rho(H'))_j, \text{ for } H, H' \text{ faces of } \mathcal{P}, \text{ is an automorphism of } [\mathcal{P}]^k. \] This follows directly from the fact that \( \rho \) preserves ranks and incidences. In fact, the mapping \( \rho \rightarrow \tilde{\rho} \) is an injective homomorphism from \( \Gamma(\mathcal{P}) \) to \( \Gamma([\mathcal{P}]^k) \).

Now suppose \( \Gamma([\mathcal{P}]^k) \) preserves face types. Given an automorphism \( \sigma \) of \( [\mathcal{P}]^k \), we will show that \( \sigma \) is determined uniquely by its action on faces of ranks less than \( n - 3 \), faces of type 0 and “middle” faces \( F \) for ridges \( F \) of \( \mathcal{P} \). In other words, \( \sigma \) is uniquely determined by its action on faces which are faces of \( \mathcal{P} \). Therefore \( \sigma \) is \( \tilde{\rho} \) for some automorphism \( \rho \) of \( \mathcal{P} \).

For any face \( K \) of \( [\mathcal{P}]^k \) which is not also a face of \( \mathcal{P} \), either \( K = (F, G)_j \) or \( K = (G, F)_j \) for some \( (n - 2) \)-face \( F \) and \( (n - 1) \)-face \( G \) of \( \mathcal{P} \), with \( F \leq G \). Additionally, \( \sigma(K) \) is a face of type \( j \) of the same rank as \( K \). We will assume that \( K \) is a ridge of the form \( (F, G)_j \), but the proof in the case that \( K \) is a facet is the same. By Lemma 4.8(a) there is a sequence of successively incident ridges and facets of length \( 2j \) from \( K \) to \( G \) and a sequence of successively incident ridges and facets of length \( k + 1 - 2j \) from \( K \) to \( F \). Applying \( \sigma \) to each term in these sequences yields a sequence of successively incident ridges and facets of length \( 2j \) from \( \sigma(K) \) to \( \sigma(G) \) and a sequence of successively incident ridges and facets of length \( k + 1 - 2j \) from \( \sigma(K) \) to \( \sigma(F) \), since \( \sigma \) preserves ranks and incidences. Since \( \sigma \) preserves face types of \( K, G, \) and \( F \), it must be the case that \( \sigma(K) = (\sigma(F), \sigma(G))_j \), by Lemma 4.8(b). Therefore, if we let \( \rho \) denote the automorphism of \( \mathcal{P} \) defined as the restriction of \( \sigma \) to the set of faces of \( \mathcal{P} \), then \( \sigma = \tilde{\rho} \).

Just as in the case of \( [\mathcal{P}]_k \), we now turn our attention to the number of flag-orbits of \( [\mathcal{P}]^k \). One main purpose of defining \( [\mathcal{P}]^k \) was to construct polytopes with an arbitrary number of flag-orbits, so the following theorem is a central result of this section.

**Theorem 4.12** If \( \mathcal{P} \) is a regular \( n \)-polytope of Schl"afli type \( \{p_1, \ldots, p_{n-1}\} \) with \( p_{n-2} \neq 2 \), then \( [\mathcal{P}]^k \) is a \( k \)-orbit polytope with group \( \Gamma([\mathcal{P}]_k) = \Gamma(\mathcal{P}) \).
Proof As in the proof that $[\mathcal{P}]_k$ is an $m(n-k)$-orbit polytope, we can count orbits by simply counting the number of flags that $[\mathcal{P}]^k$ has for each flag of $\mathcal{P}$ because the automorphism groups are the same by Theorem 4.11 and Proposition 4.10. This will take the form of a bijection between $\mathcal{F}(\mathcal{P})$, the set of flags of $\mathcal{P}$, and a partition of $\mathcal{F}([\mathcal{P}]^k)$ into sets of cardinality $k$. In particular, we will show that for every flag $\Phi$ of $\mathcal{P}$, we can derive $k$ distinct flags of $\mathcal{F}([\mathcal{P}]^k)$, and that these will comprise all of the flags of $[\mathcal{P}]^k$.

Given a flag $\Phi = \{F_{-1}, F_0, \ldots, F_{n-3}, F_{n-2}, F_{n-1}, F_n\}$ of $\mathcal{P}$, we will derive a set of flags of $\mathcal{F}([\mathcal{P}]^k)$, whose faces and components of faces are faces in $\Phi$. For the sake of counting, we will partition these flags into three distinct sets which we will call $A_\Phi$, $B_\Phi$ and $C_\Phi$. We will first define these sets in the case that $k$ is odd, and then for $k$ even. In the case that $k$ is odd, we define $A_\Phi$, $B_\Phi$ and $C_\Phi$ to be subsets of $\mathcal{F}([\mathcal{P}]^k)$ as follows.

The set $A_\Phi$ consists of all flags $\mathcal{F}([\mathcal{P}]^k)$ whose faces and components of faces are faces of $\Phi$, and whose facets and ridges have types 1 or 0. That is, $A_\Phi$ is the set containing only the flags

$$\{F_{-1}, F_0, \ldots, F_{n-3}, (F_{n-2}, F_{n-1})_1, F_{n-1}, F_n\}$$

and

$$\{F_{-1}, F_0, \ldots, F_{n-3}, (F_{n-2}, F_{n-1})_1, (F_{n-1}, F_{n-2})_1, F_n\}.$$

The set $B_\Phi$ contains any flag in $\mathcal{F}([\mathcal{P}]^k)$ whose faces and components of faces are faces of $\Phi$, and which contains a face of type at least 2, but does not contain any face of type $\frac{k+1}{2}$. That is, a flag in this set does not contain the ridge $F_{n-2}$ of $[\mathcal{P}]^k$, where $F_{n-2}$ is the ridge of $\mathcal{P}$ contained in the flag $\Phi$. In particular, $B_\Phi$ is the set containing precisely the flags

$$\{F_{-1}, F_0, \ldots, F_{n-3}, (F_{n-2}, F_{n-1})_j, (F_{n-1}, F_{n-2})_j, F_n\}$$
and
\[ \{ F_{-1}, F_0, \ldots, F_{n-3}, (F_{n-2}, F_{n-1})_j, (F_{n-1}, F_n)_{j-1}, F_n \} \]
for each \( j \) such that \( 2 \leq j \leq \frac{k-1}{2} \). Notice that this set includes the case where the facet of the flag is of type 1 and the ridge of the flag is of type 2, since we only required that there be at least one face of type at least 2.

Finally, \( C_\Phi \) is the set containing any flag in \( \mathcal{F}(\mathcal{P}^k) \) whose faces and components of faces are faces of \( \Phi \), and which contains a ridge of type \( \frac{k+1}{2} \). That is, this ridge is the face \( F_{n-2} \), where \( F_{n-2} \) is the ridge of \( \mathcal{P} \) contained in the flag \( \Phi \). In this case, \( C_\Phi \) only contains the following flag:
\[ \{ F_{-1}, F_0, \ldots, F_{n-3}, F_{n-2}, (F_{n-1}, F_{n-2})_{(k-1)/2}, F_n \} \]

Notice that \( \{ A_\Phi \cup B_\Phi \cup C_\Phi | \Phi \in \mathcal{F}(\mathcal{P}) \} \) is a partition of \( \mathcal{F}(\mathcal{P}^k) \). Notice also that for each \( \Phi \), the set \( A_\Phi \) has cardinality 2, the set \( B_\Phi \) has cardinality \( 2(\frac{k-1}{2} - 1) \) which is \( k-3 \), and the set \( C_\Phi \) has cardinality 1. These sets are disjoint, so for each \( \Phi \), the cardinality of the union \( A_\Phi \cup B_\Phi \cup C_\Phi \) is \( 2 + k - 3 + 1 \) which is \( k \). Thus, in the case that \( k \) is odd, \( \{ A_\Phi \cup B_\Phi \cup C_\Phi | \Phi \in \mathcal{F}(\mathcal{P}) \} \) is a partition of \( \mathcal{F}(\mathcal{P}^k) \) into sets of cardinality \( k \).

If \( k \) is even, then we also define \( A_\Phi \) to be the set containing any flag in \( \mathcal{F}(\mathcal{P}^k) \) whose faces and components of faces are faces of \( \Phi \), and whose ridges and facets have types 1 or 0. Here too, \( A_\Phi \) is the set containing only
\[ \{ F_{-1}, F_0, \ldots, F_{n-3}, (F_{n-2}, F_{n-1})_1, (F_{n-1}, F_n)_1, F_n \} \]
and
\[ \{ F_{-1}, F_0, \ldots, F_{n-3}, (F_{n-2}, F_{n-1})_1, (F_{n-1}, F_{n-2})_1, F_n \}. \]
The set $B_\Phi$ consists of all flags in $\mathcal{F}([\mathcal{P}]^k)$ whose faces and components of faces are faces of $\Phi$, and which contains a face of type at least 2, but does not contain any face of type $\frac{k}{2}$. In particular, in this case, $B_\Phi$ is the set containing precisely the flags

$$\{F_{-1}, F_0, \ldots, F_{n-3}, (F_{n-2}, F_{n-1})_j, (F_{n-1}, F_{n-2})_j, F_n\}$$

and

$$\{F_{-1}, F_0, \ldots, F_{n-3}, (F_{n-2}, F_{n-1})_j, (F_{n-1}, F_{n-2})_{j-1}, F_n\}$$

for each $j$ such that $2 \leq j \leq \frac{k}{2} - 1$.

Finally, $C_\Phi$ is the set containing any flag in $\mathcal{F}([\mathcal{P}]^k)$ whose faces and components of faces are faces of $\Phi$, and which contains a face of type $\frac{k}{2}$. In this case, $C_\Phi$ contains two flags, namely

$$\{F_{-1}, F_0, \ldots, F_{n-3}, (F_{n-2}, F_{n-1})_{k/2}, F_{n-2}, F_n\}$$

and

$$\{F_{-1}, F_0, \ldots, F_{n-3}, (F_{n-2}, F_{n-1})_{k/2}, (F_{n-1}, F_{n-2})_{k^2 - 1}, F_n\}.$$ 

Notice again that \{A_\Phi \cup B_\Phi \cup C_\Phi | \Phi \in \mathcal{F}(\mathcal{P})\} is a partition of $\mathcal{F}([\mathcal{P}]^k)$. Notice also that in this case for each $\Phi$, the set $A_\Phi$ has cardinality 2, the set $B_\Phi$ has cardinality $2(\frac{k}{2} - 2)$ which is $k - 4$, and the set $C_\Phi$ has cardinality 2. These sets are disjoint, so the cardinality of the union $A_\Phi \cup B_\Phi \cup C_\Phi$ is $2 + k - 4 + 2$ which is $k$. Thus, in the case that $k$ is even, \{A_\Phi \cup B_\Phi \cup C_\Phi | \Phi \in \mathcal{F}(\mathcal{P})\} is a partition of $\mathcal{F}([\mathcal{P}]^k)$ into sets of cardinality $k$.

We now define a bijection $f$ from $\mathcal{F}(\mathcal{P})$ to this partition of $\mathcal{F}([\mathcal{P}]^k)$ by $f(\Phi) = A_\Phi \cup B_\Phi \cup C_\Phi$. Since the cardinality of $A_\Phi \cup B_\Phi \cup C_\Phi$ is $k$ for any $k$ and any $\Phi$, there are at most $k$ orbits. Since every automorphism of $[\mathcal{P}]^k$ preserves face types, there is no automorphism which takes one flag in $A_\Phi \cup B_\Phi \cup C_\Phi$ to another, and this completes the proof. \qed
Informally, the orbits are defined by the face types of the ridge and facet of a flag. Any two flags whose ridges and facets have the same respective face types are in the same orbit. In this section we will present one final result about the combinatorial symmetries of $[P]^k$:

**Proposition 4.13** If $P$ is regular, then $[P]^k$ is semiregular.

**Proof** If $P$ is regular, then all the facets of $[P]^k$ are regular. This is because they are either ditopes of ridges of $P$ or isomorphic to facets of $P$. Moreover, $[P]^k$ is also vertex transitive, so $[P]^k$ is semiregular. □

Notice that while $[P]^k$ is semiregular, it is not, in general, an alternating semiregular polytope in the sense of [26].

### 4.3 The dual of $[P]^k$

Recall that the *dual* of an abstract polytope can be obtained by reversing the partial order, so that the $i$-faces of a polytope are the $(n - i - 1)$-faces of its dual. We denote the dual of $P$ by $P^*$. Clearly, dualizing a polytope is an involutory operation, so that $(P^*)^* = P$. Additionally, we will define a *duality* $\delta$ from an $n$-polytope $P$ to an $n$-polytope $Q$ as an incidence-preserving and rank-reversing bijection from $P$ to $Q$, so that $F \leq G$ in $P$ if and only if $\delta(F) \geq \delta(G)$ in $Q$. In particular, if $F$ is an $i$-face of $P$, then $\delta(F)$ is an $(n - i - 1)$-face of $Q$.

In addition to allowing us to construct polytopes with an arbitrary number of flag-orbits, the polytopes $[P]^k$ have the additional property that their duals are very easy to visualize. We will show that the dual of $[P]^k$ is a subdivision of the dual of $P$.

In graph theory, the term *subdivision* refers, informally, to adding a vertex in the interior of an edge. Formally, an edge with endpoints $\{a, b\}$ is replaced with a vertex $v$ and two new
edges \{a, v\} and \{v, b\}. This definition can be extended inductively to add \(k\) vertices in the interior of an edge \(e\), by replacing \(e\) with \(k\) vertices and \(k + 1\) edges which alternate to form a path.

Applying this idea to polytopes, we can subdivide the edges of a polytope. Informally, this still means placing a vertex (or several vertices) in the interior of an edge. Now, after replacing a given edge with several edges and several vertices, any face of higher rank which was incident to that edge becomes incident to all of the new edges and vertices.

Figure 4.3 depicts two examples of subdivided polytopes. Notice that when the square is subdivided by adding two additional vertices per edge, the result is a (combinatorial) dodecagon (12-gon). In particular, notice that this object is (combinatorially) vertex-transitive and edge-transitive. There are automorphisms which take newly added vertices to vertices which were original vertices of the square. When the tetrahedron is subdivided by adding an extra vertex in the middle of each edge, the result is a new polytope with hexagonal 2-faces. However, in this case, no new symmetries have been added and the automorphism group is the same as that of the tetrahedron. In particular, this object is not vertex-transitive. The new vertices are 2-valent whereas and the original vertices are 3-valent. In keeping with the terminology of this dissertation, we can say that the automorphism group of the subdivided tetrahedron preserves face types, while the automorphism group of the subdivided square does not preserve face types. This motivates the following proposition.
Proposition 4.14 Given an $n$-polytope $\mathcal{P}$, the polytope $([\mathcal{P}^*]^k)^*$, which is the polytope obtained by taking the dual of $\mathcal{P}$, then replacing each ridge with a sequence of $k$ ridges and $k-1$ facets, then taking the dual of that, can be obtained by starting with $\mathcal{P}$ and subdividing the edges of $\mathcal{P}$ by adding $k-1$ vertices to each edge while maintaining all induced incidences with faces of higher ranks. Equivalently, starting with $\mathcal{P}^*$, the dual of $\mathcal{P}$, and subdividing the edges by adding $k$ new vertices per edge, and taking the dual polytope, results in $[\mathcal{P}]^k$.

Proof We will first show that the two claims made in the proposition are, in fact, equivalent. This is because for any operations $f$ and $g$ on polytopes, if $(f(\mathcal{P}^*))^* = g(\mathcal{P})$ for any polytope $\mathcal{P}$, then exchanging the roles of $\mathcal{P}$ and $\mathcal{P}^*$, we get $(f(\mathcal{P}))^* = g(\mathcal{P}^*)$, so $f(\mathcal{P}) = (g(\mathcal{P}^*))^*$.

We will prove that there is a duality between the subdivided $\mathcal{P}$ and $[\mathcal{P}^*]^k$. In particular, we will prove that there is a bijection between $i$-faces of the subdivided $\mathcal{P}$ and $(n-i-1)$-faces of $[\mathcal{P}^*]^k$, which preserves incidences.

For $i \geq 2$, we can define our bijection trivially. This is because in the subdivided $\mathcal{P}$ the $i$-faces are the $i$-faces of $\mathcal{P}$, and all their incidences to faces of ranks other than 0 and 1 are exactly the incidences in $\mathcal{P}$. When $i \geq 2$, then $n-i-1 \leq n-3$, and hence the $(n-i-1)$-faces of $[\mathcal{P}^*]^k$ are the $(n-i-1)$-faces of $\mathcal{P}^*$, which are the $i$-faces of $\mathcal{P}$. Here too, the incidences of these faces to faces of ranks other than 0 and 1 are exactly the incidences in $\mathcal{P}$. Thus we can define a bijection which is a trivial map from the set of $i$-faces of $\mathcal{P}$ to itself for faces of ranks $i \geq 2$, such that the incidence of these faces with all faces except possibly faces of ranks 0 and 1 are preserved.

We now consider the cases when $i$ is 0 or 1, that is to say the edges and vertices of the subdivided $\mathcal{P}$, and the ridges and facets of $[\mathcal{P}^*]^k$. In the subdivided $\mathcal{P}$ each edge of the polytope $\mathcal{P}$ was replaced with a sequence of alternating edges and vertices, with the first and last edges incident to existing vertices. Each of these edges and vertices is incident to all the faces of higher rank that the original edge was incident to. Likewise, in $[\mathcal{P}^*]^k$ each
$(n - 2)$-face of $P^*$ (each of which is an edge of $P$), is replaced with an alternating sequence of ridges and facets, with the first and last ridges incident to existing facets. These new ridges and facets are incident to all of the faces of lower rank to which the original ridge was incident. These ridges and facets are, of course, edges and vertices of $P$, so the bijection is clear, and incidences are preserved.
Chapter 5

Existence of Polytopes with
Preassigned Automorphism Group
and Number of Flag-Orbits

It is well known that given a string C-group $\Gamma$, there is a regular polytope whose automorphism group is isomorphic to $\Gamma$ (Proposition 1.25). In this section, we will use the constructions defined earlier in this thesis to answer a related question: Given a string C-group $\Gamma$ and an integer $j \geq 1$, when is there a $j$-orbit polytope, $P$, with $\Gamma(P) \cong \Gamma$? Note that often a string C-group is thought of as a group together with a set of distinguished generators. We are only asking about the existence of a group isomorphism between $\Gamma(P)$ and $\Gamma$, irrespective of choice of distinguished generators of $\Gamma$. In particular, we do not distinguish between a string C-group and its dual. Additionally, since all polytopes of ranks less than three are regular, we will only be considering string C-groups of rank greater than or equal to three. Clearly, if $j = 1$, there always exists such a polytope, so we will primarily be looking at the case $j > 1$.

Because of the natural correspondence between string C-groups and regular polytopes, we
will be looking at a number of results which begin with a regular polytope \( Q \) and determine if there is a \( j \)-orbit polytope \( P \), for a fixed \( j \), such that \( \Gamma(P) \cong \Gamma(Q) \).

We begin with a limited result which deals with the case for \( j \leq n - 1 \) and follows from the construction given in Chapter 1. We will then extend this result to arbitrary \( j \) using the construction given in Chapter 4.

**Theorem 5.1** Given any regular polytope \( P \) of rank \( n \geq 3 \) whose Schl"afli symbol does not contain a 2, there exists a \( j \)-orbit \( n \)-polytope whose automorphism group is isomorphic to \( \Gamma(P) \) for \( j = 1, 2, \ldots, n - 1 \). In other words, given any string C-group \( \Gamma \) of rank \( n \geq 3 \) in which no two consecutive distinguished generators commute, there exists \( j \)-orbit polytopes for \( j = 1, 2, \ldots, n - 1 \) whose automorphism groups are isomorphic to \( \Gamma \).

**Proof** If the Schl"afli symbol of \( P \) does not contain a 2, then Theorem 2.23 together with Proposition 2.20 tells us that \([P]_k\) is an \((n - k)\)-orbit polytope for \( k = 1, 2, \ldots, n - 1 \). Therefore, we can set \( k = n - j \), so that \([P]_k\) is a \( j \)-orbit polytope for \( j = 1, 2, \ldots, n - 1 \). □

More generally, we have the following theorem:

**Theorem 5.2** Let \( P \) be a regular \( n \)-polytope of Schl"afli type \( \{p_1, \ldots, p_{n-1}\} \) with \( p_2 \) and \( p_{n-2} \) not both 2, and automorphism group \( \Gamma = \Gamma(P) \). Then for every integer \( k \geq 1 \) there is a \( k \)-orbit polytope with automorphism group \( \Gamma \).

**Proof** This is a direct corollary of Theorem 4.12, since if \( P \) is of Schl"afli type \( \{p_1, \ldots, p_{n-1}\} \) with \( p_{n-2} \neq 2 \), then \([P]^k\) is a \( k \)-orbit polytope whose automorphism group is isomorphic to \( \Gamma \). Additionally, if \( P \) is of Schl"afli type \( \{p_1, \ldots, p_{n-1}\} \) such that \( p_2 \neq 2 \), then \( P^* \), the dual of \( P \), is a regular polytope of Schl"afli type \( \{p_{n-1}, p_{n-2}, \ldots, p_2, p_1\} \) whose automorphism group is isomorphic to \( \Gamma \) (though it is not the same string C-group). In this case, \([P^*]^k\) is a \( k \)-orbit polytope whose automorphism group is isomorphic to \( \Gamma \). □
However, the answer to the question of whether or not such a polytope exists for every $\Gamma$ and every $j$ is in general negative. We will prove that for any rank $n$, there is no two-orbit $n$-polytope whose automorphism group is isomorphic to the rank $n$ string Coxeter group $[2, \ldots, 2]$. The latter is the automorphism group of the regular $n$-polytope $\{2, \ldots, 2\}$ and is a string C-group. The following results will be useful in the proof.

**Proposition 5.3** If $\mathcal{P}$ is a two-orbit $n$-polytope which is not facet transitive, then every facet of $\mathcal{P}$ must be isomorphic to a regular polytope.

**Proof** Let $\mathcal{P}$ be a two-orbit polytope which is not facet transitive. Then $\mathcal{P}$ is in class $2_I$ with $I = \{0, \ldots, n-2\}$, by Corollary 1.29.

Let $F$ be a facet of $\mathcal{P}$ and let $\Phi$ and $\Psi$ be two flags of $\mathcal{P}$ containing $F$. Since $\mathcal{P}$ is strongly flag-connected, there is a sequence of successively adjacent flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that each $\Phi_j$ contains $F$. Thus for each $j$, $\Phi_j$ and $\Phi_{j+1}$ are $i_j$-adjacent for some $i_j \in \{0, \ldots, n-2\}$ since both $\Phi_j$ and $\Phi_{j+1}$ contain $F$. Therefore $i_j \in I$ and $\Phi_j$ and $\Phi_{j+1}$ are in the same flag-orbit for each $j = 0, \ldots, k-1$. Hence $\Phi$ and $\Psi$ are in the same orbit.

Thus $\Gamma(\mathcal{P})$ acts transitively on flags of $\mathcal{P}$ containing $F$. Any automorphism which takes a flag containing $F$ to another flag containing $F$ is in the stabilizer of the section $F/F_{-1}$. Therefore this stabilizer acts transitively on flags containing $F$. Because this stabilizer is contained in the automorphism group of the facet $F/F_{-1}$, and every flag of the facet $F/F_{-1}$ can be extended to a flag of $\mathcal{P}$ containing $F$, $F$ is regular. □

This result can clearly be dualized to give an equivalent statement for the vertex figures.

**Remark** In the proof of Proposition 5.3, we used the fact that the stabilizer of the section $F/F_{-1}$ is contained in the automorphism group of the facet $F/F_{-1}$. In fact, these two groups are isomorphic, because two-orbit polytopes in this class are a type of polytope called hereditary polytopes [24, Theorem 2].
Remark Polytopes of the type described in Proposition 5.3 are semiregular of alternating type, since any semiregular polytope which is not of alternating type would not be \((n-2)\)-face transitive.

The following general results about abstract polytopes will also be useful. For results on the smallest regular polytopes once we have excluded the somewhat trivial case where the Schl"afli type contains a 2 (as is the case in Lemma 5.4), see [3].

**Lemma 5.4** An \(n\)-polytope has at least \(2^n\) flags, and if \(P\) is an equivelar \(n\)-polytope with more than \(2^n\) flags, then \(P\) has at least \(3(2^{n-1})\) flags.

**Proof** If a flag of a polytope \(P\) is selected by beginning with the \((-1)\)-face of \(P\) and then choosing incident faces of successively larger ranks, then the diamond condition on \(P\) guarantees that, once the faces of ranks \((-1)\) through \(i\) for \(i = -1, \ldots, n - 2\) have been chosen, there are at least two choices for the \((i + 1)\)-face which can be in that flag. Since the choice of a face is made for \(i = -1, \ldots, n - 2\), it is made \(n\) times, and hence \(P\) has at least \(2^n\) flags.

If \(P\) is equivelar and the same process is used to select a flag, then for each \(i = -1, 0 \ldots, n - 2\) the number of \((i + 1)\)-faces which can be in a given flag is independent of choice of faces of lower rank. In particular, \(\#(\mathcal{F}(P))\) is a product of \(n\) integer factors each of which is at least 2, and if they are not all 2, then the number of flags of \(P\) is at least \(3(2^{n-1})\). \(\square\)

**Lemma 5.5** Let \(P\) be an \(n\)-polytope, and let \(F\) be a facet of \(P\). Then the number of flags of \(P\) containing \(F\) is equal to the number of flags of \(F/F_{-1}\).

**Proof** If \(F_n\) is the \(n\)-face of \(P\), then there is a natural bijection from the set of flags of \(P\) containing \(F\) to the set of flags of \(F/F_{-1}\), which is defined by removing \(F_n\) from each flag. \(\square\)
Proposition 5.6 The Hasse diagram of an abstract polytope, $\mathcal{P}$, cannot properly contain (as a subgraph) the Hasse diagram of another polytope $\mathcal{Q}$ of the same rank.

Proof Let $\Phi$ be a flag of $\mathcal{Q}$. Then $\Phi$ is also a flag of $\mathcal{P}$. Let $\Psi$ be a flag of $\mathcal{P}$. We will show that $\Psi$ is also a flag of $\mathcal{Q}$. By the flag-connectedness of $\mathcal{P}$ there is a sequence of successively adjacent flags from $\Phi$ to $\Psi$. By Lemma 1.8 each flag of the sequence has a unique $i$-adjacent flag (in both $\mathcal{P}$ and $\mathcal{Q}$), and therefore the $i$-adjacent flag in $\mathcal{P}$ must also be the $i$-adjacent flag in $\mathcal{Q}$. Therefore, every flag of the sequence is also a flag of $\mathcal{Q}$, so $\mathcal{P} = \mathcal{Q}$. □

Finally, we are able to provide a counterexample to the question of the existence of a $k$-orbit polytope with automorphism group $\Gamma$ for any string C-group $\Gamma$ and any $k \geq 1$. This counterexample will be given in Corollary 5.8

Theorem 5.7 The only $n$-polytopes with $2^{n+1}$ flags which are either regular or two-orbit are the regular universal polytopes of Schl"afli type $\{p_1, \ldots, p_{n-1}\}$ with $p_i = 4$ for some $i$ and $p_j = 2$ for $j \neq i$.

Proof The proof will be done by induction on $n$, the rank of the polytope. We first prove the base case of the induction, which is the case $n = 2$. All finite rank 2 polytopes are $p$-gons for some $p$. A $p$-gon has $2p$ flags, and its automorphism group is the dihedral group with $2p$ elements. Thus the only rank 2 polytope with 8 flags is the square, $\{4\}$.

We now assume that the theorem holds for polytopes of rank less than $n$, where $n \geq 3$. Let $\mathcal{P}$ be an $n$-polytope with $2^{n+1}$ flags such that $\mathcal{P}$ is either regular or two-orbit. We need to show that $\mathcal{P}$ is a regular universal polytope of Schl"afli type $\{p_1, \ldots, p_{n-1}\}$ with $p_i = 4$ for some $i$ and $p_j = 2$ for $j \neq i$.

If $\mathcal{F}(\mathcal{P})$ is the set of flags of $\mathcal{P}$, then

$$\# \mathcal{F}(\mathcal{P}) = \sum_{F \text{ is a facet of } \mathcal{P}} \# \mathcal{F}(F/F_{-1}),$$

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by Lemma 5.5 and the fact that every flag of $P$ contains exactly one facet.

By Lemma 5.4, and the fact that the section $F/F_{-1}$ is isomorphic to an $(n - 1)$-polytope, $\#F(F/F_{-1}) \geq 2^{n-1}$ for each facet $F$ of $P$. Since $P$ has $2^{n+1}$ flags, $P$ has at most 4 (and at least 2) facets. We will consider the cases that $P$ has 2, 3, and 4 facets individually.

We first assume that $P$ has exactly 2 facets, say $A$ and $B$. Let $G$ be a ridge ($\binom{n}{2}$-face) of $P$. Then $G \leq A$ and $G \leq B$ because $F_n/G$ must have 2 proper faces. Because this is true for every ridge of $P$, the facets $A$ and $B$ must be incident to every ridge of $P$, and therefore every face of $P$. In particular, $A/F_{-1}$ and $B/F_{-1}$ are isomorphic, and thus $\#F(A/F_{-1}) = \#F(B/F_{-1})$. Since the total number of flags of $P$ must sum to $2^{n+1}$, and every flag contains either $A$ or $B$, then $\#F(A/F_{-1}) = \#F(B/F_{-1}) = 2^n$. Now, $A/F_{-1}$ and $B/F_{-1}$ are isomorphic to polytopes of rank $n - 1$. Additionally, since $P$ has no more than two flag-orbits, $A/F_{-1}$ and $B/F_{-1}$ have no more than two flag-obits, so by induction they are universal regular polytopes of Schl"{a}fli type $\{p_1, \ldots, p_{n-2}\}$ with $p_i = 4$ for some $i$ and $p_j = 2$ for $j \neq i$.

Because $A$ and $B$ have the same incidences, $P$ is a ditope over $A/F_{-1}$. Thus $P$ is the universal regular polytope of Schl"{a}fli type $\{p_1, \ldots, p_{n-2}, 2\}$, with $p_1, \ldots, p_{n-2}$ as in the type of $A/F_{-1}$, so $p_i = 4$ for some $i$ and $p_j = 2$ for $j \neq i$.

We now consider the case that $P$ has 3 facets, say $A, B$ and $C$. Let $A, B,$ and $C$ be the polytopes isomorphic to $A/F_{-1}, B/F_{-1},$ and $C/F_{-1}$, respectively. Because 3 does not divide $2^{n+1}$, not all three facets can have the same number of flags, and hence they are not all isomorphic. Because $P$ has at most two orbits (and hence, exactly two orbits), two of the three facets must be isomorphic. We may assume that $B$ is isomorphic to $C$. Since $P$ has two flag-orbits, every flag containing $B$ or $C$ needs to be contained in one orbit and every flag containing $A$ must be contained in the other. By Proposition 5.3, $A, B$ and $C$ must all be regular, and hence equivelar.
Within the case that $\mathcal{P}$ has three facets, we will first consider the subcase where the number of flags of $\mathcal{P}$ containing $A$ is the least possible, that is $2^{n-1}$. In this case there are $2^{n+1} - 2^{n-1} = 3(2^{n-1})$ flags of $\mathcal{P}$ which do not contain $A$. Since $\mathcal{B}$ is isomorphic to $\mathcal{C}$, there must be the same number of flags of $\mathcal{P}$ containing $B$ as there are flags containing $C$. Therefore, there are $3(2^{n-2})$ flags containing $B$ and the same number of flags containing $C$.

If there are $2^{n-1}$ flags of $\mathcal{P}$ containing $A$ then there are $2^{n-1}$ flags of $\mathcal{A}$, by Lemma 5.5. Therefore $\mathcal{A}$ must be the polytope $\{2^{n-2}\}$ since this is the only $(n - 1)$-polytope with $2^{n-1}$ flags. Additionally, given an $(n - 2)$-face $F$ of $\mathcal{P}$ which is incident to $A$, $F/F_{-1}$ is isomorphic to the polytope $\{2^{n-3}\}$. By the diamond condition, $F$ must be incident to two facets of $\mathcal{P}$, so, without loss of generality we may assume that $F$ is incident to $B$. Now since $F$ is an $(n - 2)$-face of $\mathcal{B}$, and $\mathcal{B}$ is equivelar, $\mathcal{B}$ is of type $\{2^{n-3}, p_{n-2}\}$ for some $p_{n-2}$. Since $\mathcal{B}$ has $3(2^{n-2})$ flags, $p_{n-2} = 3$. In particular, there must be at least three $(n - 2)$-faces of $\mathcal{P}$ incident to $B$. However, not all of these $(n - 2)$-faces are in the same $(n - 2)$-face orbit as $F$ because $\mathcal{A}$ only contains two $(n - 2)$-faces, so they are not all incident to $A$, and because every automorphism of $\mathcal{P}$ preserves incidences and fixes $A$ since $\mathcal{A}$ is not isomorphic to either of the other two facets. Therefore, not every flag of $\mathcal{P}$ containing $B$ is contained in a single flag-orbit, which contradicts the definition of $B$. Therefore, the subcase considered for $A$ is not a possible case.

Now if $\mathcal{P}$ has 3 facets, and there are more than $2^{n-1}$ flags of $\mathcal{P}$ containing $A$, then there must be fewer than $3(2^{n-2})$ flags of $\mathcal{P}$ containing $B$ (and fewer than $3(2^{n-2})$ flags of $\mathcal{P}$ containing $C$). The polytopes $\mathcal{B}$ and $\mathcal{C}$ must be equivelar, so by Lemma 5.4, $\mathcal{B}$ and $\mathcal{C}$ must each have $2^{n-1}$ flags, and hence there are $2^{n-1}$ flags of $\mathcal{P}$ containing each of $B$ and $C$. Therefore there are $2^{n+1} - 2(2^{n-1})$, which is $2^n$, flags of $\mathcal{P}$ which contain $A$, and hence $\mathcal{A}$ has $2^n$ flags. But $\mathcal{A}$ must be regular, so by induction, $\mathcal{A}$ is a universal regular $(n - 1)$-polytope of Schl"afli type $\{p_1, \ldots, p_{n-2}\}$ with $p_i = 4$ for some $i$ and $p_j = 2$ for $j \neq i$. Since $\mathcal{B}$ and $\mathcal{C}$ must be $\{2^{n-2}\}$, and $A$ must be incident to a face of rank $n - 2$ which is incident to $B$ or
$C$, the facet $A$ must be $\{2^{n-3}, 4\}$. In particular, there are exactly four $(n - 2)$-faces incident to $A$. Two of the $(n - 2)$-faces incident to $A$ are also incident to $B$ and the other two are incident to $C$. This is because $B$ and $C$ are each incident to two $(n - 2)$-faces of $\mathcal{P}$ and the diamond condition requires that each $(n - 2)$-face of $\mathcal{P}$ be incident to two $(n - 1)$-faces of $\mathcal{P}$. Let the two $(n - 2)$-faces which are incident to both $A$ and $B$ be $F$ and $G$. Let $H$ be an $(n - 4)$-face incident to both $F$ and $G$. (This exists because every $(n - 4)$ face incident to $B$ is incident to every $(n - 2)$ face incident to $B$ by the type of $B$.) By the Schlafli type of $\mathcal{A}$, the 2-section $A/H$ is a square, and in particular, is not a digon, so there is at most one $(n - 3)$ face incident to both $F$ and $G$. However $B/H$ also contains both $F$ and $G$ and (by the Schlafli type of $B$), $B/H$ is a digon, so there must be at least two $(n - 3)$-faces incident to both $F$ and $G$. Therefore, we have a contradiction, and this subcase is impossible as well. Thus $\mathcal{P}$ cannot have three facets.

The final case is that $\mathcal{P}$ has four facets. In this case, each facet must lie in exactly $2^{n-1}$ flags of $\mathcal{P}$. Thus the polytopes corresponding to each facet must be isomorphic to the polytopes $\{2^{n-2}\}$. There is only one polytope with four facets all isomorphic to $\{2^{n-2}\}$, and that polytope must be $\{2^{n-2}, 4\}$. To see that the polytope is uniquely determined by the fact that there are four facets, first consider the possible ridges of the polytope. If the four facets are $A, B, C$ and $D$, then by the diamond condition, the only two options for the top three layers of the Hasse diagram of the polytope are shown in Figure 5.1.

![Figure 5.1](image)

This can be seen by considering the ridge $a$. The ridge $a$ must be incident to exactly two facets by the diamond condition. Without loss of generality, we may assume that they are
A and B. By the Schlafli type of B, the facet B must be incident to exactly one other ridge, which we will call b. The ridge b must be incident to exactly one other facet, by the diamond condition. If this ridge is A, then the top three layers of the Hasse diagram must be as in Figure 5.1 (b), since by the Schlafli type of A, the facet A cannot be incident to any ridges other than a and b. We can argue similarly for C and D. If b is incident to a facet other than A and B, we can assume it is C. Now C must be incident to exactly one other ridge. This ridge cannot be a, since then the section $F_n/a$ would have three proper faces, violating the diamond condition. Thus C must be incident to c. Now c must be incident to a facet other than C. It cannot be B since the Schlafli type of B indicates that B cannot be incident to a, b and c. If c were incident to A, then for any ridge d incident to D, the section $F_n/d$ would only have one proper face, because d could not be incident to A, B or C, since each would already be “saturated” with ridges. Thus, the Schlafli types of the facets, together with the diamond condition tell us that we have to be in one of the situations shown in Figure 5.1.

However, any ranked poset whose Hasse diagram has the diagram in Figure 5.1 (b) as the top three rows would not be strongly connected and hence can be eliminated. In fact, for any strongly connected n-polytope the layer of the Hasse diagram consisting of the $(n - 2)$-faces and $(n - 1)$-faces forms a connected graph (this is the edge graph of the dual polytope). Thus the diagram in Figure 5.1 (a) is the only option.

Next we will consider the faces of lower ranks of this polytope and we will show that the only possible n-polytope has the Hasse diagram shown in Figure 5.2, where the dashed lines indicate that for any rank $m$ less than $n - 3$ there are exactly two faces of rank $m$, both of which are incident to faces of all other ranks. This is the Hasse diagram of the universal regular polytope $\{2^{n-2}, 4\}$. Thus, we will show that any polytope with exactly four facets and $2^{n+1}$ flags must be this polytope.

We know that the Hasse diagram of any such polytope must include the diagram in Figure 5.2 because we have already discussed the top rows, and we know the Schlafli type
of each facet to be \(\{2^{n-2}\}\). To see this, observe that by the Schl"afli type of \(A\), as well as the above arguments, the Hasse diagram must include the entire above diagram, except for possibly the edges connecting \(F_{n-3}\) to \(b\), \(F_{n-3}\) to \(c\), \(G_{n-3}\) to \(b\), and \(G_{n-3}\) to \(c\). However, each of these edges must exist by the diamond condition, as well as the fact that we know there are no more \((n-2)\)-faces. For example, the edge connecting \(F_{n-3}\) to \(b\) must be present because the section \(B/F_{n-3}\) must contain exactly two \((n-2)\)-faces. They must be \(a\) and \(b\). Thus \(F_{n-1}\) is incident to \(b\). The arguments for the remaining three edges are identical. Now by Proposition 5.6, since the Hasse diagram of our polytope contains the Hasse diagram of the universal regular \(n\)-polytope of type \(\{2^{n-2},4\}\), and is also of rank \(n\), this it must be this polytope.

This completes the proof of the inductive hypothesis, and hence establishes the theorem. \(\square\)

**Corollary 5.8** For any rank \(n \geq 2\), there is no two-orbit \(n\)-polytope whose group of automorphisms is isomorphic to the \(n\)-generator string Coxeter group \([2, \ldots, 2] =: [2^{n-1}]\).
**Proof** First of all, note that \([2^{n-1}] = C_2^n\), since \([2^{n-1}]\) is generated by \(n\) involutions all of which commute. Any two-orbit \(n\)-polytope whose automorphism group is isomorphic to \(C_2^n\) would have \(2^{n+1}\) flags (twice the cardinality of the group). This is impossible by Theorem 5.7.

Even though (by Corollary 5.8) there is no two-orbit \(n\)-polytope with automorphism group \([2^{n-1}] = C_2^n\), there are \(n\)-polytopes with \(k\) orbits for any integer \(k > 2\) which have this automorphism group. In particular, let \(\Gamma\) denote the Coxeter group \([2^{n-1}]\), and let \(\mathcal{P}\) be the universal regular \(n\)-polytope \(\{2^{n-1}\}\) with \(\Gamma(\mathcal{P}) \cong \Gamma\). Then, though there is no two-orbit polytope whose automorphism group is isomorphic to \(\Gamma\), Proposition 2.29 guarantees that for \(k = 0, \ldots, n-3\), the group \(\Gamma\) is isomorphic to \(\Gamma([\mathcal{P}]_k)\), and hence by Theorem 2.23 there are polytopes with any number of orbits from 3 to \(n\) whose automorphism groups are isomorphic to \(\Gamma\). Unfortunately, in this case \(p_{n-2} = 2\), so applying the construction given in Chapter 4 does not result in a polytope with automorphism group \(\Gamma\), and \(p_2 = 2\) so we cannot apply this to the dual of \(\mathcal{P}\) either to create a polytope with automorphism group \(\Gamma\). Nonetheless, \(k\)-orbit polytopes with group \(\Gamma\) do exist for each \(k > 2\), as can be seen from the following construction.

Let \(\mathcal{P}\) be the universal regular \(n\)-polytope with Schl"afli type \(\{2, 2, \ldots, 2, 2k - 2, 2\}\), for some \(k \geq 3\). Let \(G\) be an \((n - 1)\)-face and \(H\) be an \((n - 4)\)-face of \(\mathcal{P}\); there are only two faces of these ranks. Now, \(G\) and \(H\) are each incident to every face of rank other than their own, so \(G\) an \(H\) are incident and every \((n - 2)\)-face of \(\mathcal{P}\) is in the section \(G/H\). Choose an \((n - 2)\)-face \(F\). The section \(G/H\) is a \((2k - 2)\)-gon, and \(2k - 2\) is even, so there is another face \((n - 2)\)-face \(F'\), which falls “across” from \(F\). We now replace both \(F\) and \(F'\) (but not the other \((n - 2)\)-faces) with ditopes over themselves to obtain a new polytope \(\mathcal{P}'\).

For example, Figure 5.3 parts (a) and (b) show the case \(k = 3\). In this case, we have taken the polytope \(\mathcal{P}\) (shown in (a)) and we have replaced it with the polytope labeled \(\mathcal{P}'\) (shown in (b)), where new faces are labeled as they would have been had they been faces
in $[P]_{n-2}$. The incidences identified by dotted lines are all incidences that occurred in the original polytope and we are not concerned with them. Notice that while $F$ and $F'$ were both incident to both of the two facets of $P$, they are each only incident to one of the original facets, and one new facet in the polytope $P'$.

![Graphs of $P$ and $P'$](image)

Figure 5.3

Thus, the 3-sections $F_n/H$ in $P'$ where $H$ is an $(n-4)$-face of $P$ are comprised of two $(2k-2)$-gons, identified along all but two opposite edges as in Figure 5.3 (c), and two digons.

**Proposition 5.9** The $n$-polytope described above, for $k \geq 3$, is a $k$-orbit polytope with automorphism group $[2^{n-1}]$.

**Proof** The statement about the automorphism group follows from the fact that every automorphism of $P'$ is an involution, and the group is therefore abelian. The group $[2^{n-1}]$ is the only abelian string C-group of rank $n$. We can look at this construction in terms of the group of $P$. The group of $P$ has automorphisms $\rho$ of order $2k-2$. However, out of the powers of $\rho$, only $\rho^{k-1}$ survives as an automorphism of $P'$, and has order 2.
It is clear that the orbit of a flag is dependent on the ridge, subridge and the facet of the flag. This is because every face of rank less than \( n - 3 \) has the same upperrank incidences, so for any flag \( \Phi \) of \( \mathcal{P}' \) and chain \( \Psi \) of type \( \{ -1, 0, \ldots, n - 4 \} \) there is an automorphism which fixes the faces of ranks \( n, n - 1, n - 2, \) and \( n - 3 \) of \( \Phi \), and sends the faces of lower ranks to those of \( \Psi \). Thus, the number of flag-orbits is equal to the number of orbits of flags of a co-(\( n - 4 \))-face of \( \mathcal{P}' \). Such a coface is pictured in Figure 5.3 (c). Since this is a co-(\( n - 4 \))-face, the subridges of \( \mathcal{P}' \) are represented by vertices in the picture, and the ridges are represented by edges. The “top” and “bottom” are the new facets \( (F, F_n) \) and \( (F', F_n) \), where \( (F, F_n) \) is a ditope over the original ridge (or edge as it is represented in the diagram) \( F \), and \( (F', F_n) \) is a ditope over the original ridge \( F' \). The “front” and “back” \((2k-2)\)-gons are the original facets \( G_1 \) and \( G_2 \). Notice that the flags which contains either \( (F, F_n) \) or \( (F', F_n) \) (i.e., the flags whose facets are ditopes of original ridges) form a single orbit. There are \((k-1)\) flag-orbits which contain facets of the form \( G \), with \( G \) an original facet of \( \mathcal{P} \), and these flag-orbits are determined by the ridges and subbridges. This can be seen by examining Figure 5.3(c) and noticing that for each vertex along the right side, if that vertex is paired with the edge below it (or in the case of the bottom vertex, with one of the edges to the left of it), then that pairing defines a unique flag-orbit, and each flag of \( \mathcal{P}' \) is in one of the orbits defined by those pairings. Since there are \( k - 1 \) vertices along the right side, there are \( k - 1 \) flag-orbits which are comprised of flags which include original faces as their facets, and one flag-orbit consisting of all flags which have a new facet. Thus, there are \( k \) flag-orbits. \(\square\)

It is clear that the universal \( n \)-polytope of type \( \{[2^{n-1}]\} \) is the only polytope of this type for each \( n \), because the polytope is simply too small to identify any faces without collapsing the entire polytope. Additionally, this fact follows from the following lemma, which we will use later as well. Therefore, our discussion of the group \( \Gamma = [2^{n-1}] \) encompasses all polytopes of this type.

**Lemma 5.10** Let \( \Gamma \) be a string \( C \)-group of type \( \{p_1, p_2, \ldots, p_{n-1}\} \) such that either \( p_j = 2 \)
for all even \( j \), or \( p_j = 2 \) for all odd \( j \). Then \( \Gamma \) is the Coxeter group \([p_1, p_2, \ldots, p_{n-1}]\).

**Proof** Suppose \( \Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle \), where \( \rho_0, \ldots, \rho_{n-1} \) are the distinguished generators of \( \Gamma \). We need to show that the only relations which are satisfied by the distinguished generators of \( \Gamma \) are the ones implied by the Schlafli type of \( \Gamma \). Let \( \rho_{i_1}\rho_{i_2}\ldots, \rho_{i_k} = \epsilon \) be a relation satisfied by the distinguished generators of \( \Gamma \). If the set \( \{i_1, \ldots, i_k\} \) has cardinality at most two, then this is a relation which is implied by the Schlafli type of \( \Gamma \). Assume the set \( \{i_1, \ldots, i_k\} \) has cardinality greater than two. Let \( j = i_1 \). By the Schlafli type of \( \Gamma \), the generator \( \rho_j \) commutes with every generator except possibly either \( \rho_{j+1} \) or \( \rho_{j-1} \). Without loss of generality, we may assume that \( p_t = 2 \) for all \( t \equiv j \mod 2 \). Then \( \rho_j \) commutes with \( \rho_{j-1} \), but might not commute with \( \rho_{j+1} \). Then, \( \rho_{j+1} \) also commutes with \( \rho_i \) for \( i \neq j \) (because \( p_{j+2} = 2 \) or \( j + 1 = n - 1 \), so \( \rho_{n+2} \) commutes with \( \rho_{n-1} \) if it exists). Therefore, we can rewrite this relation as \((\rho_j\rho_{j+1})^p\rho_j^{s}(\rho_{i_1}\ldots\rho_{l_m}) = \epsilon\), where \( s = 0, 1 \) for some \( p \), where \( \{l_1, \ldots, l_m\} \cap \{j, j + 1\} = \emptyset \).

Therefore \((\rho_j\rho_{j+1})^p\rho_j^{s}\) and \(\rho_{i_1}\ldots\rho_{l_m}\) are inverse elements in \( \Gamma \), and therefore each is in the intersection \( \langle \rho_j, \rho_{j+1} \rangle \cap \langle \rho_{i_1}, \ldots, \rho_{l_m} \rangle \). However, since \( \{l_1, \ldots, l_m\} \cap \{j, j + 1\} = \emptyset \), the intersection condition requires this intersection to be \( \langle \epsilon \rangle \). Thus \((\rho_j\rho_{j+1})^p\rho_j^{s}\) and \(\rho_{i_1}, \ldots, \rho_{l_m}\) are both \( \epsilon \).

The relation \((\rho_j\rho_{j+1})^p\rho_j^{s} = \epsilon\) implies that \( s = 0 \) and \( p \) is divisible by \( p_{j+1} \). In particular, this relation is implied by the Schlafli type of \( \Gamma \). We can apply the same process to the relation \( \rho_{i_1}, \ldots, \rho_{l_m} = \epsilon \). Since this is a finite product, we can continue inductively to show that this relation can be broken down into relations implied by the Schlafli type of \( \Gamma \). \( \square \)

We will now examine the string C-groups which have Schlafli types \( \{p_1, \ldots, p_{n-1}\} \) with \( p_2 = p_{n-2} = 2 \), yet there is some \( p_i \neq 2 \). In this case, we have not yet proven that polytopes exist whose automorphism groups are these string C-groups which have \( k \) orbits for any \( k \geq 1 \), nor have we proven that this is impossible.
Because we only require that $\Gamma$ is isomorphic to the automorphism group of our polytope, we can reorder and rename the generators, as long as we maintain the structure of a string C-group. In particular, if we reindex the distinguished generators, we must make sure that generators with non-consecutive indices commute. Taking the dual by sending $\rho_i$ to $\rho_{n-i-1}$ for $i = 0, \ldots, n - 1$ can always be done, but there are not always other ways to reindex the distinguished generators and retain the structure of a string C-group. However, if $\Gamma$ is a string C-group which has a Schlafli type such that $p_2 = p_{n-2} = 2$, then $\Gamma$ nontrivially splits into a direct product of groups generated by distinguished generators. Because of this, there are ways (other than dualizing) in which we can reorder the generators and still have a string C-group.

**Proposition 5.11** Let $\Gamma$ be a string C-group of rank $n \geq 3$, with $n$ odd, with Schlafli type $\{p_1, \ldots, p_{n-1}\}$ such that the $p_i$ are not all 2. Then, there is a way to reindex (possibly trivially) the distinguished generators to obtain a group $\Gamma'$ which is isomorphic (as a group) to $\Gamma$, but has the structure of a string C-group with $p_{n-2} \neq 2$.

**Proof** First of all, notice that reindexing the distinguished generators of a string group does not effect the fact that the group is generated by involutions, nor does it effect the intersection condition. Therefore, when reindexing the distinguished generators, we only have to be careful that we retain a string group structure. Let $\Gamma$ be split into a direct product of groups generated by consecutive distinguished generators in a way such that no factor can be further split into a direct product in this way. It may be the case that there is only one factor, but if this is the case then $p_{n-2} \neq 2$, and the reindexing that we will describe is trivial. Notice that because these factors cannot be split further into a direct product, consecutive generators within a single factor do not commute. Because $n$ is odd, $\Gamma$ has an odd number of distinguished generators, so at least one of these factors must be generated by an odd number of distinguished generators. Choose a factor which
is generated by an odd number $j$ of distinguished generators. Let $\Gamma'$ be a string C-group with distinguished generators $\rho'_0, \ldots, \rho'_{n-1}$, obtained by reindexing the generators of $\Gamma$, such that the last $j$ generators $\rho'_{n-j}, \rho'_{n-j+1}, \ldots, \rho'_{n-1}$ are these $j$ generators in the order that they appeared as distinguished generators of $\Gamma$. Since this factor cannot have been split into a direct product, and the order of the generators was preserved, consecutive generators (if any) do not commute. In particular, if $j \geq 3$, then $\rho'_{n-3}$ and $\rho'_{n-2}$ are generators from this factor and do not commute, so in $\Gamma'$ we have $p_{n-2} \neq 2$. If $j < 3$, then $j = 1$ because $j$ is odd, so the chosen factor has a single generator, which we have called $\rho'_{n-1}$. In this case there must be some other factor with more than one generator (because we assumed that $\Gamma$ is not $C_2^n$). Choose another factor with $l$ distinguished generators, where $l > 1$. Call these distinguished generators $\rho'_{n-l-1}, \ldots, \rho'_{n-2}$ in the order that they were indexed in $\Gamma$. Then both $\rho'_{n-3}$ and $\rho'_{n-2}$ are distinguished generators which were consecutive generators from an original factor of $\Gamma$, and therefore do not commute, so $p_{n-2} \neq 2$. \hfill $\square$

**Corollary 5.12** Let $n$ be an odd integer such that $n \geq 3$, and let $k$ be a positive integer. If a rank $n$ string C-group $\Gamma$ is not the automorphism group of a $k$-orbit $n$-polytope, then $\Gamma = C_2^n = [2^{n-1}]$ and $k = 2$.

**Proof** Assume that a rank $n$ string C-group $\Gamma$ is not the automorphism group of a $k$-orbit $n$-polytope. If $\Gamma = C_2^n$, then $k = 2$ by Proposition 5.9. It remains to show that $\Gamma = C_2^n$. By Proposition 5.11, if $\Gamma$ were not $C_2^n$, the distinguished generators could be reindexed in a way such that $p_{n-2} \neq 2$. In this case, there exists a $k$-orbit $n$-polytope with automorphism group $\Gamma$ by Theorem 5.2, which contradicts our choice of $\Gamma$. \hfill $\square$

Notice that this corollary can actually be stated as a biconditional, since we know that no such two-orbit polytope exists for $\Gamma = C_2^n$. Thus when $n$ is odd, we have given a necessary and sufficient condition for the existence of a $k$-orbit polytope with automorphism group isomorphic to the rank $n$ string C-group $\Gamma$. 

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We have a similar result for the case when \( n \) is even.

**Proposition 5.13** Let \( n \) be an even integer such that \( n \geq 4 \), and let \( k \) be a positive integer. If a rank \( n \) string C-group \( \Gamma \) of Schl"afli type \( \{p_1, \ldots, p_{n-1}\} \) is not the automorphism group of a \( k \)-orbit \( n \)-polytope, then it is either the case that \( p_i = 2 \) for all \( i \) (and \( \Gamma = C_2^n \)), or that \( p_i = 2 \) for \( i \) even, and \( p_i \neq 2 \) for \( i \) odd.

**Proof** As in the proof of Proposition 5.11, whenever \( \Gamma \) is such that there must be a minimal factor with an odd number of distinguished generators, as well as a factor with more than one distinguished generator, we may reindex the generators such that \( p_{n-2} \neq 2 \). As long as it is not the case that \( p_i = 2 \) for all \( i \), there is a minimal factor with more than one distinguished generator. If there are two consecutive distinguished generators which are either both 2 or both not 2, then there is a minimal factor with an odd number of distinguished generators. Finally, note that if \( p_i = 2 \) for \( i \) odd, and \( p_i \neq 2 \) for \( i \) even, then \( n - 2 \) is even and therefore \( p_{n-2} \neq 2 \), so no reindexing is necessary. Therefore, we only have to exclude the cases that \( p_i = 2 \) for all \( i \), \( p_i = 2 \) for \( i \) even, and \( p_i \neq 2 \) for \( i \) odd. The rest of the proof is identical to that of Corollary 5.12. \( \square \)

Notice that by Lemma 5.10 there is only one string C-group of every type referenced in Proposition 5.13, and these groups which are not \( C_2^n \) are groups such that \( \Gamma \) is a direct product of dihedral groups, none of which is \( C_2 \times C_2 \).

There are two possible cases in Proposition 5.13. In the first case, namely when \( p_i = 2 \) for all \( i \), we know that a \( k \)-orbit \( n \)-polytope exists with group \( \Gamma \) precisely when \( k \neq 2 \). However, in the second case, namely the case that \( \Gamma \) is a direct product of dihedral groups, none of which is the group \( C_2 \times C_2 \), is still open. We conjecture that this remaining open case behaves the same way as does the group \( C_2^n \). Specifically, we conjecture that there is no two-orbit polytope whose automorphism group is isomorphic to such a group, but there does exist a \( k \)-orbit polytope for each \( k \geq 3 \) whose automorphism group is isomorphic to
any such group.
Bibliography


