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APOLARITY FOR THE DETERMINANT AND PERMANENT

by

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ABSTRACT OF DISSERTATION

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Abstract

We show that the apolar ideals to the determinant and permanent of a generic matrix, the Pfaffian of a generic skew symmetric matrix and the determinant and the hafnian of a generic symmetric matrix are each generated in degree two. We also show that unlike the previous polynomials, the apolar ideal to the permanent of a generic symmetric matrix is generated in degrees two and three. In each case we specify the generators and give a Gröbner basis of the apolar ideal. As a consequence, using a result of K. Ranestad and F.-O. Schreyer we give lower bounds to the cactus rank and rank of each of these invariants. We compare these bounds with those obtained by J. Landsberg and Z. Teitler.
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Chapter 1

Introduction

The main object of this thesis is to study the apolar algebra of the determinant and permanent of the generic matrix and the generic symmetric matrix. This study was originally motivated by a question from Zach Teitler about the generating degree of the annihilator ideal of the determinant and the permanent of a generic $n \times n$ matrix. Here annihilator is meant in the sense of the apolar pairing, i.e. Macaulay's inverse system. We also determine the generators of the apolar ideal of the Pfaffian of a $2n \times 2n$ generic skew symmetric matrix and the hafnian of a $2n \times 2n$ of a generic symmetric matrix. The reason for Teitler's interest in this problem is the recent paper by Kristian Ranestad and Frank-Olaf Schreyer [RS], which gives a lower bound for cactus rank of the polynomials in terms of the generating degree of the apolar ideal and the dimension of the Artinian apolar algebra defined by the apolar ideal. We apply this and our result to bounding the scheme/cactus length of the determinant and the permanent of the generic matrix and the generic symmetric matrix.

Chapters 2, 3, and 4 of the thesis are essentially the content of the papers [Sh] and [Sh2]. And chapter 5 on invariants contains some results of a work in progress.

Let $k$ be a field of characteristic zero or characteristic $p > 2$, and let $A = (a_{ij})$ be a square matrix of size $n$ with $n^2$ distinct variables. The determinant and permanent of $A$ are homogeneous polynomials of degree $n$. Let $R = k[a_{ij}]$ be a polynomial ring and $S = k[d_{ij}]$ be the ring of inverse polynomials associated to $R$, and let $R_k$ and $S_k$ denote the degree-$k$ homogeneous summands. Then $S$ acts on $R$ by contraction:
(d_{ij})^k \circ (a_{uv})^\ell = \begin{cases} a_{uv}^{\ell-k} & \text{if } (i,j) = (u,v), \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}

If h \in S_k and F \in R_n, then we have h \circ F \in R_{n-k}. This action extends multilinearly to the action of S on R. When the characteristic of the field k is zero or \text{char} k = p \text{ greater than the degree of } F, the contraction action can be replaced by the action of partial differential operators with constant coefficients ([IK], Appendix A, and [Ge]).

**Definition 1.0.1.** To each degree-\(j\) homogeneous element, \(F \in R_j\) we associate \(I = \text{Ann}(F)\) in \(S = k[d_{ij}]\) consisting of polynomials \(\Phi\) such that \(\Phi \circ F = 0\). We call \(I = \text{Ann}(F)\), the \textit{apolar ideal} of \(F\); and the quotient algebra \(S/\text{Ann}(F)\) the \textit{apolar algebra} of \(F\).

Let \(F \in R\), then \(\text{Ann}(F) \subset S\) is an ideal and we have

\[(\text{Ann}(F))_k = \{h \in S_k | h \circ F = 0\}.

**Remark 1.0.2.** Let \(\phi : (S_i, R_i) \rightarrow k\) be the pairing \(\phi(g, f) = g \circ f\), and \(V\) be a vector subspace of \(R_k\), then we have

\[\dim_k(V^\perp) = \dim_k S_k - \dim_k V.\] \tag{1.2}

For \(V \subset R_k\), we denote by \(V^\perp = \text{Ann}(V) \cap S_k\).

Let \(F\) be a form of degree \(j\) in \(R\). We denote by \(<F>_{j-k}\) the vector space \(S_k \circ F \subset R_{j-k}\). ([IK]).

We denote by \(M_k(A)\) the vector subspace of \(R\) spanned by the \(k \times k\) minors of \(A\).

**Lemma 1.0.3.**

\[S_k \circ (\text{det}(A)) = M_{n-k}(A) \subset R_{n-k}.\] \tag{1.3}

**Proof.** It is easy to see that

\[S_k \circ (\text{det}(A)) \subset M_{n-k}(A) \subset R_{n-k}.\]

For the other inclusion, let \(M_{I,J}(A), I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_k), 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n, 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n\) be the \((n-k) \times (n-k)\) minor of \(A\) one obtains by deleting the \(I\) rows and
CHAPTER 1. INTRODUCTION

J columns of $A$. Now it is easy to see that

$$M_{\hat{f}, \hat{j}} = \pm (d_{i_1, j_1} \cdot d_{i_2, j_2} \cdots d_{i_k, j_k}) \circ \det(A).$$

Hence $M_{\hat{f}, \hat{j}} \in S_k \circ \det(A)$. \hfill \qed

Remark 1.0.4. (see [IK], page 69, Lemma 2.15) Let $F \in R$ and $\deg F = j$ and $k \leq j$. Then we have

$$(\text{Ann}(F))_k = \{h \in S_k | h \circ S^{j-k}F = 0\} = (\text{Ann}(S^{j-k}F))_k. \quad (1.4)$$

Remark 1.0.5. By Lemma 1.0.3 and Remark 1.0.4 we have

$$\text{Ann}(\det(A))_k = M_k(A)^\perp.$$

Example 1.0.6. Let $n = 3$,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$ 

Let $P_{ij}$ and $M_{ij}$ be respectively the permanent and the determinant corresponding to the entry $a_{ij}$. Question: Does $P_{11} = d_{22}d_{33} + d_{32}d_{23}$ annihilate $\det(A) = a_{11}M_{11} + a_{12}M_{12} + a_{13}M_{13}$? The following computations allow us to answer this.

$$P_{11} \circ a_{11}M_{11} = (d_{22}d_{33} + d_{32}d_{23}) \circ (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) = a_{11} - a_{11} = 0.$$ 

$$P_{11} \circ a_{12}M_{12} = (d_{22}d_{33} + d_{32}d_{23}) \circ (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) = 0.$$ 

$$P_{11} \circ a_{13}M_{13} = (d_{22}d_{33} + d_{32}d_{23}) \circ (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) = 0.$$ 

Hence $P_{11}$ annihilates the determinant.

It is easy to see that when $n = 3$, $P_{ij} \circ M_{kl} = 0$ for each $1 \leq i, j, k, l \leq 3$. So in the case $n = 3$ the annihilator of the determinant of a generic matrix certainly contains all its $2 \times 2$ permanents.
1.1 Summary of main results

- We show that the following apolar ideals are generated in degree two and we specify the generators:
  - Determinant of a generic $n \times n$ matrix (Theorem 2.2.11). This ideal is generated by $2 \times 2$ permanents and certain “unacceptable” degree two monomials.
  - Permanent of a generic $n \times n$ matrix (Theorem 2.2.12). This ideal is generated by $2 \times 2$ minors and unacceptable degree two monomials.
  - Determinant of a generic symmetric $n \times n$ matrix (Theorem 3.2.11). This ideal is generated by certain $2 \times 2$ permanents, certain degree two trinomials that are hafnians of $4 \times 4$ symmetric submatrices and some monomials.
  - Pfaffian of a generic skew symmetric $2n \times 2n$ matrix (Theorem 4.0.21). This ideal is generated by certain degree two binomials corresponding to each $4 \times 4$ Pfaffian and some monomials.
  - Hafnian of a generic symmetric $2n \times 2n$ matrix (Theorem 4.0.24). This ideal is generated by certain degree two binomials corresponding to $4 \times 4$ hafnians and some monomials.

- We show that exceptionally, the apolar ideal to the permanent of a generic symmetric matrix is generated in degree two and three (Theorem 3.2.23). This ideal is generated by certain $2 \times 2$ minors, certain degree three polynomials corresponding to $6 \times 6$ hafnians and some degree two monomials.

- We apply these results to give a lower bound for:
  - Cactus rank of the determinant and permanent of the generic $n \times n$ matrix (Theorem 2.3.5).
  - Cactus rank of the determinant of a generic symmetric $n \times n$ matrix (Theorem 3.3.1).
  - Rank of the determinant of the generic symmetric $n \times n$ matrix (Proposition 3.3.3).
  - Cactus rank of the permanent of a generic symmetric $n \times n$ matrix (Theorem 3.3.2).

- We give a Gröbner basis for the following ideals:
  - The apolar ideal of the determinant of a generic matrix (Theorem 2.2.16)
– The apolar ideal of the determinant of a generic symmetric matrix (Proposition 3.2.12).
Chapter 2

The apolar algebras associated to the $n \times n$ generic matrix

In this chapter we determine the annihilator ideals of the determinant and the permanent of a generic $n \times n$ matrix. In section 2.1 we review the dimension of the subspace of $k \times k$ minors and permanents of an $n \times n$ generic matrix. In section 2.2 we determine the generators of the apolar ideal to the determinant and permanent of a generic matrix. In section 2.3 we apply our result to find a lower bound for the scheme/cactus rank and rank of the determinant and permanent of the generic matrix (Theorems 2.3.5).

Throughout this chapter $k$ is a field of characteristic zero or characteristic $p > 2$, and $A = (a_{ij})$ be a square matrix of size $n$ with $n^2$ distinct variables. The determinant and the permanent of $A$ are polynomials of degree $n$. Let $R = k[a_{ij}]$ be a polynomial ring and $S = k[d_{ij}]$ be the ring of inverse polynomials associated to $R$, and let $R_k$ and $S_k$ denote the degree-$k$ homogeneous summands. Then $S$ acts on $R$ by contraction.

2.1 Hilbert function and dimension of spaces of minors and permanents

Denote by $\mathfrak{A}_A = S/(\text{Ann}(\det(A)))$ the apolar algebra of the determinant of the matrix $A$. Recall that the Hilbert function of $\mathfrak{A}_A$ is defined by $H(\mathfrak{A}_A)_i = \dim_k(\mathfrak{A}_A)_i$ for all $i = 0, 1, \ldots$.
CHAPTER 2. THE APOLAR ALGEBRAS ASSOCIATED TO THE GENERIC MATRIX

Definition 2.1.1. Let $F$ be a polynomial in $R$, we define the $\deg(\text{Ann}(F))$ to be the length of $S/\text{Ann}(F)$.

The number of the $k \times k$ minors and permanents of a generic $n \times n$ matrix is $\binom{n}{k}^2$. The $k \times k$ minors form a linearly independent set ([BC] Theorem 5.3 and Remark 5.4), and the $k \times k$ permanents form another linearly independent set. To show the linear independence of these two sets we choose a term order, for example the diagonal order (see Definition 2.2.14) where the main diagonal term is a Gröbner initial term. Now the initial terms give a basis for the two spaces ([LS], page 197). So the dimension of the space of $k \times k$ minors of an $n \times n$ matrix and the dimension of the space of $k \times k$ permanents of an $n \times n$ matrix are both $\binom{n}{k}^2$. By Lemma 1.0.3 and Remark 1.0.5 we have

$$H(S/\text{Ann} (\det A))_k = H(S/\text{Ann}(\text{Perm} A))_k = \binom{n}{k}^2.$$ (2.1)

So the length $\dim_k(\mathfrak{A}_A)$ satisfies

$$\dim_k(\mathfrak{A}_A) = \sum_{k=0}^{k=n} \binom{n}{k}^2 = \binom{2n}{n}.$$ (2.2)

A combinatorial proof of the Equation 2.2 can be found in [ST], Example 1.1.17.

2.2 Generators of the apolar ideal

In this section we determine the generators of the apolar ideal of the determinant and permanent of a generic matrix.

Notation. For a generic $n \times n$ matrix $A = (a_{ij})$, the permanent of $A$ is a polynomial of degree $n$ defined as follows:

$$\text{Per}(A) = \sum_{\sigma \in S_n} \Pi a_{i,\sigma(i)}$$

Lemma 2.2.1. Let $A = (a_{ij})$ be a generic $n \times n$ matrix. Then each $2 \times 2$ permanent of $D = (d_{ij})$ annihilates the determinant of $A$.

Proof. Assume we have an arbitrary $2 \times 2$ permanent $d_{ij}d_{kl} + d_{il}d_{kj}$ corresponding to
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\[
P = \begin{pmatrix} d_{ij} & d_{il} \\ d_{kj} & d_{kl} \end{pmatrix}
\]

Recall that \( \det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i,i' \in \sigma} a_{i,i'} \). There are \( n! \) terms in the expansion of the determinant. If a term does not contain the monomial \( a_{ij}a_{kl} \) or the monomial \( a_{il}a_{kj} \) then the result of the action of the permanent \( d_{ij}d_{kl} + d_{il}d_{kj} \) on it will be zero. There are \( (n - 2)! \) terms which contain the monomial \( a_{ij}a_{kl} \) and \( (n - 2)! \) terms which contain the monomial \( a_{il}a_{kj} \). So assume we have a permutation \( \sigma_1 \) of \( n \) objects having \( a_{ij} \) and \( a_{kl} \) respectively in its \( i \)-th and \( k \)-th place. Corresponding to \( \sigma_1 \) we also have a permutation \( \sigma_2 = \tau \sigma_1 \), where \( \tau = (j,l) \) is a transposition and \( \text{sgn}(\sigma_2) = \text{sgn}(\tau \sigma_1) = -\text{sgn}(\sigma_1) \). Thus corresponding to each positive term in the determinant which contains the monomial \( a_{ij}a_{kl} \) or the monomial \( a_{il}a_{kj} \) we have the same term with the negative sign, thus the resulting action of the permanent \( d_{ij}d_{kl} + d_{il}d_{kj} \) on \( \det(A) \) is zero.

**Definition 2.2.2.** Let \( A = (a_{ij}) \) and \( D = (d_{ij}) \) be two generic matrices with entries in the polynomial ring \( R = k[a_{ij}] \), and in the ring of differential operators \( S = k[d_{ij}] \), respectively. Let \( \{\mathcal{P}_A\}, \{\mathcal{M}_A\}, \{\mathcal{P}_D\} \) and \( \{\mathcal{M}_D\} \) denote the set of all \( 2 \times 2 \) permanents and the set of all \( 2 \times 2 \) minors of \( A \) and \( D \), respectively. And let \( \mathcal{P}_A \), \( \mathcal{M}_A = M_2(A) \), \( \mathcal{P}_D \) and \( \mathcal{M}_D = M_2(D) \) denote the spaces they span, respectively.

**Corollary 2.2.3.** Each \( 2 \times 2 \) permanent of \( D \) annihilates \( \mathcal{M}_A \)

**Proof.** By Lemma 2.2.1, \( \mathcal{P}_D \circ \det(A) = 0 \). Let \( F = \det(A) \). We have

\[
(\text{Ann}F)_2 = (\text{Ann}(S_{j=2} \circ F))_2.
\]

Hence

\[
\mathcal{P}_D \circ \det(A) = 0 \iff \mathcal{P}_D \circ S_{j=2}(\det(A)) = 0 \iff \mathcal{P}_D \circ \mathcal{M}_A = 0.
\]

We also know that the square of an element, or any product of two or more elements of the same row or column of \( D \) annihilates \( \det(A) \).

**Definition 2.2.4.** A monomial in the \( n^2 \) variables of the ring \( S = k[d_{ij}] \) is *acceptable*, if it is square free and has no two variables from the same row or column of \( D \). A polynomial is acceptable if it can be written as the sum of acceptable monomials.
We denote by $<X>$ the $k$-vector space span of the set $X$.

**Lemma 2.2.5.** $\mathcal{P}_D \oplus \mathcal{M}_D = <\text{degree 2 acceptable polynomials in } S>$. 

*Proof.* Let $d_{ij}d_{kl}$ be an arbitrary acceptable monomial of degree 2. Since $\text{char}(k) \neq 2$ we have:

$$d_{ij}d_{kl} = 1/2((d_{ij}d_{kl} - d_{ij}d_{kl}) + (d_{ij}d_{kl} + d_{ij}d_{kl})).$$

By Equation 2.1

$$\dim \mathcal{P}_D = \dim \mathcal{M}_D = \binom{n}{2}^2.$$

Let $\Psi = <\text{degree 2 acceptable polynomials in } S>$. Then

$$\dim \Psi = \dim S_2 - \dim \mathcal{U}_D = \left(\frac{n^2 + 1}{2}\right) - \left(\frac{n^2 + \binom{n}{2}(2n)}{2}\right).$$

So we have

$$\dim(\mathcal{P}_D + \mathcal{M}_D) = \left(\frac{n^2 + 1}{2}\right) - \left(\frac{n^2 + \binom{n}{2}(2n)}{2}\right) = \dim \mathcal{P}_D + \dim \mathcal{M}_D.$$ 

Hence $\mathcal{P}_D \cap \mathcal{M}_D = 0.$

**Lemma 2.2.6.** $\text{Ann}(\mathcal{M}_A) \cap S_2 = \mathcal{P}_D + \mathcal{U}_D$. 

*Proof.* By Lemma 2.2.5 we have $\mathcal{P}_D + \mathcal{M}_D$ is complementary to $\mathcal{U}_D$. So we have

$$\dim((\text{Ann}(\mathcal{M}_A))_2) = \dim S_2 - \dim \mathcal{M}_A = \dim \mathcal{P}_D + \mathcal{U}_D.$$
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Notation. We define the homomorphism \( \xi : R \to S \) by setting \( \xi(a_{ij}) = d_{ij} \); for a monomial \( v \in R \) we denote by \( \hat{v} = \xi(v) \) the corresponding monomial of \( S \).

Remark 2.2.7. Let \( f = \sum_{i=1}^{k} \alpha_i v_i \in R_n \) with \( \alpha_i \in k \) and with \( v_i \)'s linearly independent monomials. Then we will have:

\[
\Ann(f) \cap S_n = \langle \alpha_j \hat{v}_1 - \alpha_1 \hat{v}_j, <v_1, ..., v_k>^\perp \rangle,
\]  
(2.3)

where \( <v_1, ..., v_k>^\perp = \Ann(<v_1, ..., v_k>) \cap S_n \).

Lemma 2.2.8.

\[
(P_D + U_D)_k \subset \Ann(M_k(A)) \cap S_k.
\]  
(2.4)

Proof. We have:

(1) \( P_D \circ \det(A) = 0 \iff P_D \circ S_{n-2}(\det(A)) = 0 \iff P_D \circ M_A = 0 \).

(2) \( (\Ann(\det(A))) \cap S_2 = P_D + U_D \Rightarrow S_{k-2}(P_D + U_D) \circ (S_{n-k} \circ \det(A)) = 0 \).

\( \Rightarrow S_{k-2}(P_D + U_D) \circ M_k(A) = 0 \).

\( \Rightarrow (P_D + U_D)_k \circ M_k(A) = 0 \). (By Remark 1.0.4)

So Equation 2.4 holds.

Proposition 2.2.9. For a generic \( n \times n \) matrix \( A \) with \( n \geq 2 \), we have

\[
(P_D + U_D)_n = \Ann(\det(A)) \cap S_n.
\]

Proof. Using Equation 2.4 we only need to show

\[
(P_D + U_D)_n \supset \Ann(\det(A)) \cap S_n.
\]

We use induction on \( n \). For \( n = 2 \) the equality is easy to see. Next we verify that the proposition holds for the case \( n = 3 \). We need to see that the space of \( 2 \times 2 \) permanents of \( D \) generates \( \Ann(\det(A))_3/U_D \), i.e., \( \Ann(M_3(A))_3/U_D \). Corresponding to each term in the determinant, there is a permutation of three objects \( \sigma \) such that we can write the term as \( a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} \). Consider the degree three binomial \( b = a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} - a_{1\tau(1)}a_{2\tau(2)}a_{3\tau(3)} \), where \( \tau \neq \sigma \). Without loss
of generality we can assume that $\sigma$ is the identity, so we consider the binomial $b = a_{11}a_{22}a_{33} - a_{1r(1)}a_{2r(2)}a_{3r(3)}$. If these two monomials have a common variable i.e., $\tau(i) = i$ for some $i = 1, 2, 3$, then the binomial will be of the form $b = a_{ii}(a_{jj}a_{kk} - a_{jk}a_{kj})$, $1 \leq i, j, k, l \leq 3$, so we will have $b = a_{ii}M_{ii}$ and, as we have shown previously, $P_{ii} = d_{jj}d_{kk} - d_{jk}d_{kj}$ annihilates it. Assume that the monomials $a_{11}a_{22}a_{33}$ and $a_{1r(1)}a_{2r(2)}a_{3r(3)}$ do not have any common factor. We can add and subtract another term $a_{ij}(a_{kl}a_{ij} - a_{ij}a_{kl})$, where $ij$ is a permutation, such that it will have one common factor with $a_{11}a_{22}a_{33}$ and one common factor with $a_{1r(1)}a_{2r(2)}a_{3r(3)}$. By reindexing we can take $ij$ as the proposition holds for the binomial $a_{11}a_{22}a_{33}$ and one common factor with $a_{1r(1)}a_{2r(2)}a_{3r(3)}$. By reindexing we can rewrite the binomial $a_{ij}M_{ij} + a_{kl}M_{kl}$, where the first term can be annihilated by the permanent of the matrix $D$ corresponding to $d_{ij}$ and the second term can be annihilated by the permanent of the matrix $D$ corresponding to the element $d_{kl}$. So by Equation 2.3 we are done. For example, if we have the binomial $a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32}$ we can add and subtract the term $a_{11}a_{23}a_{32}$ which has one common factor with $a_{11}a_{22}a_{33}$ and one common factor with $a_{13}a_{21}a_{32}$ so we will get $a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{32}(a_{11}a_{23} - a_{13}a_{21})$ which is $a_{11}M_{11} + a_{32}M_{32}$. And as we have shown before it can be annihilated by the space of $2 \times 2$ permanents. So by Equation 2.3 we are done.

When $n$ is larger than 3 then by the induction assumption we can assume that the proposition holds for all $k \leq n - 1$. By the Remark 2.2.7 it is enough to show that if $b$ is a binomial of the form Equation 2.3, in $\text{Ann}(\det(A)) \cap S_n$, then $b \in (\mathcal{P}_D + \mathcal{U}_D)_n$. Assume $b = b_1 + b_2$ is of degree $n$. If the two terms, $b_1$ and $b_2$ are monomials in $S$ and have a common factor $l$, i.e., $b_1 = la_1$ and $b_2 = la_2$, then $b = l(a_1 + a_2)$ where $a_1$ and $a_2$ are of degree at most $n - 1$. So by the induction assumption the proposition holds for the binomial $a_1 + a_2$, i.e. $a_1 + a_2 \in (\mathcal{P}_D + \mathcal{U}_D)_{n-1}$. Hence we have

$$b = l(a_1 + a_2) \in l(\mathcal{P}_D + \mathcal{U}_D)_{n-1} \subset (\mathcal{P}_D + \mathcal{U}_D)_n.$$ 

If the two terms, $b_1$ and $b_2$ do not have any common factor then with the same method as above we can rewrite the binomial $b$ by adding and subtracting a term of the determinant, $m$ of degree $n$, which has a common factor $m_1$ with $b_1$ and a common factor $m_2$ with $b_2$. Then we will have

$$b_1 + b_2 = b_1 + m + b_2 - m = m_1(c_1 + m') + m_2(c_2 - m''),$$

where $b_1 = m_1c_1$, $m = m_1m' = m_2m''$ and $b_2 = m_2c_2$. Since $c_1 + m'$ and $c_2 - m''$ are of degree at most $n - 1$, the induction assumption yields

$$b_1 + b_2 = m_1(c_1 + m') + m_2(c_2 - m'') \in (\mathcal{P}_D + \mathcal{U}_D)_n.$$
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This completes the induction step and hence the proof of the proposition.

\[ \square \]

Corollary 2.2.10. For a generic \( n \times n \) matrix \( A \) and each integer \( k, 1 \leq k \leq n \), we have

\[
(\mathcal{P}_D + \mathcal{U}_D)_k = \Ann(\det(A)) \cap S_k.
\]

We also have \((\mathcal{U}_D)_{n+1} = S_{n+1}\).

Proof. Using equation 2.4 we only need to show that

\[
\Ann(\det(A)) \cap S_k \subset (\mathcal{P}_D + \mathcal{U}_D)_k.
\]

By Lemma 1.0.3 and Remark 1.0.4 we have

\[
(\Ann(\det(A)))_k = (\Ann(S_{n-k} \circ (\det(A))))_k = (\Ann(M_k(A)))_k
\]

If we label the \( k \times k \) minors of \( A \) by \( f_1, ..., f_s \) we have

\[
(\Ann(M_k(A)))_k = \Ann(<f_1, ..., f_s>)_k = \bigcap_{i=1}^{i=s} (\Ann(f_i))_k
\]

For each \( f_i \) we denote the ring of variables of \( f_i \) by \( R^i \) and the ring of differential operators by \( S^i \), then by Proposition 2.2.9 we have

\[
(\mathcal{P}^i_D + \mathcal{U}^i_D)_k = \Ann(f_i) \cap S^i_k.
\]

Hence

\[
\Ann(\det(A)) \cap S_k \subset (\mathcal{P}_D + \mathcal{U}_D)_k.
\]

Finally, every monomial of degree larger than \( n \) will be unacceptable. So we have \((\mathcal{U}_D)_{n+1} = S_{n+1}\).

\[ \square \]
Theorem 2.2.11. Let $A$ be a generic $n \times n$ matrix. Then the apolar ideal $\text{Ann}(\det(A)) \subset S$ is the ideal $(P_D + U_D)$, and is generated in degree two.

Proof. This follows directly from the Proposition 2.2.9 and Corollary 2.2.10.

Theorem 2.2.12. Let $A$ be a generic $n \times n$ matrix. Then the apolar ideal $\text{Ann}(\text{Per}(A)) \subset S$ to $\text{Per}(A) \in R$ is the ideal $(M_D + U_D)$, generated in degree two.

Proof. The proof follows directly from the proof of the Proposition 2.2.9 and Corollary 2.2.10, by interchanging the determinants and the permanents.

Corollary 2.2.13. Let $A = (a_{ij})$ be an $m \times n$ matrix where $n \geq m$. Let $N$ denote the space generated by all $m \times m$ minors of $A$. Then $\text{Ann}(N)$ is generated in degree two by all $2 \times 2$ permanents of $A$ and the degree two unacceptable monomials.

Proof. Let $s = \binom{n}{m}$, and $f_1, ..., f_s$ denote the $m \times m$ minors of $A$. We have

$$\text{Ann}(N) = \text{Ann}(<f_1, ..., f_s>) = \bigcap_{i=1}^{s} (\text{Ann}(f_i)).$$

Let $R^i$ denote the ring of variables of $f_i$. Hence by Theorem 2.12 we have $\text{Ann}(f_i) \cap S^i$ is generated in degree 2. So we have $\text{Ann}(N)$ is also generated in degree 2.

In [LS], R. Laubenbacher and I. Swanson give a Gröbner bases for the ideal of $2 \times 2$ permanents of a matrix. In this section we first review their result (Theorem 2.2.15) and then state our result for the ideal $\text{Ann}(\det(A))$ and prove it independently (Theorem 2.2.16).

Definition 2.2.14. ([LS], page 197) Let $D = (d_{ij})$ be the matrix of the differential operators as defined in section 1. A monomial order on the $d_{ij}$ is diagonal if for any square submatrix of $D$, the leading term of the permanent (or of the determinant) of that submatrix is the product of the entries on the main diagonal. An example of such an order is the lexicographic order defined by:

$$d_{ij} < d_{kl} \text{ if and only if } l > j \text{ or } l = j \text{ and } k > i.$$
Throughout this section we use a lexicographic diagonal ordering.

**Theorem 2.2.15.** ([LS], page 197) The following collection \( G \) of polynomials is a minimal reduced Gröbner basis for \( \mathcal{P}_D \), with respect to any diagonal ordering:

1. The subpermanents \( d_{ij}d_{kl} + d_{kj}d_{il}, \ i < k, j < l \);
2. \( d_{i_1j_1}d_{i_2j_2}d_{i_3j_3}, i_1 > i_2, j_1 < j_2 < j_3 \);
3. \( d_{i_1j_1}d_{i_2j_2}d_{i_3j_3}, i_1 > i_2, j_1 < j_2 < j_3 \);
4. \( d_{i_1j_1}d_{i_2j_1}d_{i_3j_2}, i_1 < i_2 < i_3, j_1 > j_2 \);
5. \( d_{i_1j_1}d_{i_2j_2}d_{i_3j_2}, i_1 < i_2 < i_3, j_1 > j_2 \);
6. \( d_{i_1j_1}^{e_1}d_{i_2j_2}^{e_2}d_{i_3j_3}^{e_3}, i_1 < i_2 < i_3, j_2 > j_3, e_1e_2e_3 = 2 \).

Monomials of type (2), (3), (4), (5) and (6) in the above theorem are in the ideal generated by all unacceptable monomials.

**Theorem 2.2.16.** The collection of unacceptable degree 2 monomials and \( 2 \times 2 \) subpermanents of \( D \), form a Gröbner basis for \( \text{Ann} (\det (A)) \) with respect to any diagonal ordering.

**Proof.** We will denote \( \mathcal{U}_D \) and \( \mathcal{P}_D \) by \( \mathcal{U}, \mathcal{P} \) respectively in the following, where \( D \) is understood.

The elements of \( (\mathcal{U} + \mathcal{P}) \) generate \( \text{Ann} (\det (A)) \). Since \( \mathcal{U} \) is a set of monomials, it is already Gröbner. We use Buchberger’s algorithm to find a Gröbner basis for \( \mathcal{P} + \mathcal{U} \). We consider several cases:

a) Let \( F \) and \( G \) be distinct permanents of \( D \). Let \( F = a_{ik}a_{jl} + a_{il}a_{jk} \) and \( G = a_{uz}a_{vw} + a_{uw}a_{vz} \) be two permanents in \( \mathcal{P} \).

\[
F = \text{perm} \begin{pmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{pmatrix}.
\]

and

\[
G = \text{perm} \begin{pmatrix} a_{uz} & a_{uv} \\ a_{vz} & a_{vw} \end{pmatrix}.
\]
Let $f_1 = a_{ik}a_{jl}$ be the leading term of $F$, and $g_1 = a_{uw}a_{vw}$ be the leading term of $G$ with respect to the given diagonal ordering. Denote the least common multiple of $f_1$ and $g_1$ by $h_{11}$. Let

$$S(F, G) = \left(\frac{h_{11}}{f_1}\right)F - \left(\frac{h_{11}}{g_1}\right)G = a_{uz}a_{vw}a_{il}a_{jk} - a_{ik}a_{jl}a_{uw}a_{vz}.$$ 

Now using the multivariate division algorithm, reduce all the $S(F, G)$ relative to the set of all permanents. When there is no common factor in the initial terms of $F$ and $G$ the reduction is zero, as one can use $F$ and $G$ again as we show. First we reduce $S(F, G)$ dividing by $F \in \mathcal{P}$, so we will have

$$S(F, G) + a_{uw}a_{vz}(a_{ik}a_{jl} + a_{il}a_{jk}) = a_{uz}a_{vw}a_{il}a_{jk} + a_{uw}a_{vz}a_{il}a_{jk}.$$ 

Then we reduce the result using $G$ this time, so we will have

$$a_{uz}a_{vw}a_{il}a_{jk} + a_{uw}a_{vz}a_{il}a_{jk} - a_{il}a_{jk}(a_{uz}a_{vw} + a_{uw}a_{vz}) = 0.$$ 

So we have shown that for all pairs $F$, $G$ of distinct permanents of $D$, the $S$-polynomials $S(F, G)$ reduces to zero with respect to $\mathcal{P}$.

b) Let $F = a_{ik}a_{jl} + a_{il}a_{jk}$ and $G = a_{ik}a_{jm} + a_{im}a_{jk}$ be two permanents so that their initial terms have a common factor. We have

$$S(F, G) = a_{il}a_{jk}a_{jm} - a_{im}a_{jk}a_{jl} \in \mathcal{U}.$$ 

c) Let $F = a_{in}a_{jn} + a_{in}a_{jm}$ be a permanent and $M = a_{tk}a_{tl}$ be an unacceptable monomial. We have

$$S(F, M) = a_{tk}a_{tl}a_{jm}a_{in} \in \mathcal{U}.$$ 

d) Let $F = a_{il}a_{jm} + a_{im}a_{jl}$ be a permanent and $M = (a_{kn})^2$ be an unacceptable monomial. We have

$$S(F, M) = a_{im}a_{jl}(a_{kn})^2 \in \mathcal{U}.$$
e) Let $F = a_{il}a_{jm} + a_{im}a_{jl}$ be a permanent and $M = (a_{il})^2$ be an unacceptable monomial which has a common factor with the initial term of $F$. We have

$$S(F, M) = a_{il}a_{im}a_{jl} \in \mathcal{U}.$$ 

f) Let $F = a_{il}a_{jm} + a_{im}a_{jl}$ be a permanent and $M = a_{jn}a_{kn}$ be an unacceptable monomial. We have

$$S(F, M) = a_{im}a_{jl}a_{jn}a_{kn} \in \mathcal{U}.$$ 

This exhausts all possibilities, so the generating set $\mathcal{P} + \mathcal{U}$ is itself a Gröbner basis by Buchberger’s algorithm.

$\square$

### 2.3 Application to the ranks of the determinant and the permanent

**Notation.** Let $F \in R = k[a_{ij}]$ be a homogeneous form of degree $d$. A presentation

$$F = l_1^d + \ldots + l_s^d \text{ with } l_i \in R_1.$$ \hspace{1cm} (2.5)

is called a *Waring decomposition* of length $s$ of the polynomial $F$. The minimal number $s$ that satisfies the Equation 2.5 is called the *rank* of $F$.

The apolarity action of $S = k[d_{ij}]$ on $R$, defines $S$ as a natural coordinate ring on the projective space $\mathbf{P}(R_1)$ of 1-dimensional subspaces of $R_1$ and vice versa. A finite subscheme $\Gamma \subset \mathbf{P}(R_1)$ is apolar to $F$ if the homogeneous ideal $I_\Gamma \subset S$ is contained in Ann($F$) ([IK], [RS]).

**Remark 2.3.1.** ([IK] Def. 5.66, [RS]) Let $\Gamma = \{[l_1], \ldots, [l_s]\}$ be a collection of $s$ points in $\mathbf{P}(R_1)$. Then

$$F = c_1l_1^d + \ldots + c_sl_s^d \text{ with } c_i \in k.$$ 

if and only if

$$I_\Gamma \subset \text{Ann}(F) \subset S.$$
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Definition 2.3.2. We have the following ranks ([IK] Def. 5.66, [BR] and [RS]). Here $\Gamma$ is a punctual scheme (possibly not smooth), and the degree of $\Gamma$ is the number of points (counting multiplicities) in $\Gamma$.

a. the rank $r(F)$:

$$r(F) = \min \{ \deg \Gamma | \Gamma \subset \mathbb{P}(R_1) \text{ smooth}, \dim \Gamma = 0, I_{\Gamma} \subset \text{Ann}(F) \}.$$

Note that when $\Gamma$ is smooth, it is the set of points in the Remark 2.3.1 ([IK], page 135).

b. the smoothable rank $sr(F)$:

$$sr(F) = \min \{ \deg \Gamma | \Gamma \subset \mathbb{P}(R_1) \text{ smoothable}, \dim \Gamma = 0, I_{\Gamma} \subset \text{Ann}(F) \}.$$

Note that for the smoothable rank one considers the smoothable schemes, that are the schemes which are the limits of smooth schemes of $s$ simple points ([IK], Definition 5.66).

c. the cactus rank (scheme length in [IK], Definition 5.1 page 135) $cr(F)$:

$$cr(F) = \min \{ \deg \Gamma | \Gamma \subset \mathbb{P}(R_1), \dim \Gamma = 0, I_{\Gamma} \subset \text{Ann}(F) \}.$$

d. the differential rank (Sylvester’s catalecticant or apolarity bound) is the maximal dimension of a homogeneous component of $S/\text{Ann}(F)$:

$$l_{diff}(F) = \max_{i \in \mathbb{N}_0} \{ (H(S/\text{Ann}(F)))_i \}.$$

Note that we give a lower bound for the cactus rank of the determinant and permanent of the generic matrix. We do not have information on the smoothable rank of the generic determinant or permanent. It is still open to find a bound for the smoothable rank. The work of A. Bernardi and K. Ranestad [BR] in the case of generic forms of a given degree and number of variables show that the cactus rank and smoothable rank can be very different.

Proposition 2.3.3. ([IK], Proposition 6.7C) The above ranks satisfy

$$l_{diff}(F) \leq cr(F) \leq sr(F) \leq r(F).$$
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Proposition 2.3.4. (Ranestad-Schreyer) If the ideal of \( \text{Ann}(F) \) is generated in degree \( d \) and \( \Gamma \subset P(T_1) \) is a finite (punctual) apolar subscheme to \( F \), then

\[
\deg \Gamma \geq \frac{1}{d} \deg(\text{Ann}(F)),
\]

where \( \deg(\text{Ann}(F)) = \dim(S/\text{Ann}(F)) \) is the length of the 0-dimensional scheme defined by \( \text{Ann}(F) \).

If in Proposition 2.3.4 we take \( F = \det(A) \) or \( F = \text{Per}(A) \), since we have found that for the determinant and the permanent of a matrix we have \( d = 2 \); we can use the above proposition to find a lower bound for the above ranks of \( F \).

Theorem 2.3.5. Let \( F \) be the determinant or permanent of a generic \( n \times n \) matrix \( A \). We have

\[
\frac{1}{2} \binom{2n}{n} \leq \text{cr}(F) \leq \text{sr}(F) \leq r(F).
\]

Proof. By Theorems 2.2.11 and 2.2.12, Propositions 2.3.4 and 2.3.3, and Equations 2.1 and 2.2 we have for an apolar punctual scheme \( \Gamma \),

\[
\deg \Gamma \geq \frac{1}{d} \deg(\text{Ann}(F)) = \frac{1}{2} \sum_{k=0}^{k=n} \binom{n}{k}^2 = \frac{1}{2} \binom{2n}{n}.
\]

\[\square\]

Notation. [LT] Let \( \Phi \in S^d \mathbb{C}^n \) be a polynomial, we can polarize \( \Phi \) and consider it as a multilinear form \( \tilde{\Phi} \) where \( \Phi(x) = \tilde{\Phi}(x, ..., x) \) and consider the linear map \( \Phi_{s,d-s} : S^s \mathbb{C}^{n^*} \rightarrow S^{d-s} \mathbb{C}^n \), where \( \Phi_{s,d-s}(x_1, ..., x_s)(y_1, ..., y_{d-s}) = \tilde{\Phi}(x_1, ..., x_s, y_1, ..., y_{d-s}) \). Define

\[
\text{Zeros}(\Phi) = \{ [x] \in \mathbb{P} \mathbb{C}^{n^*} | \Phi(x) = 0 \} \subset \mathbb{P} \mathbb{C}^{n^*}.
\]

Let \( x_1, ..., x_n \) be linear coordinates on \( \mathbb{C}^{n^*} \) and define

\[
\Sigma_s(\Phi) := \{ [x] \in \text{Zeros}(\Phi) | \frac{\partial^I \Phi}{\partial x^I}(x) = 0, \forall I, \text{ such that } |I| \leq s \}.
\]

In this notation \( \Phi_{s,d-s} \) is the map from \( S_s \rightarrow R_{n-s} \) taking \( h \) to \( h \circ \Phi \), hence its rank is \( H(\mathfrak{A}_A)_s \).
In the following theorem we use the convention that \( \dim \emptyset = -1 \).

**Theorem 2.3.6. (Landsberg-Teitler)** ([LT]) Let \( \Phi \in S^d \mathbb{C}^n \), Let \( 1 \leq s \leq d \). Then

\[
\text{rank}(\Phi) \geq \text{rank}\Phi_{s,d-s} + \dim \Sigma_s(\Phi) + 1.
\]

**Remark. (Z. Teitler)** If we define \( \Sigma_s(\Phi) \) to be a subset of affine rather than projective space, then the above theorem does not need +1 at the end, and does not need the statement that the dimension of the empty set is \(-1\).

Applying this theorem for the determinant yields

**Corollary 2.3.7. (Landsberg-Teitler)** ([LT])

\[
r(\det_n) \geq \left( \frac{n}{\lfloor n/2 \rfloor} \right)^2 + n^2 - (\lfloor n/2 \rfloor + 1)^2.
\]

**Proposition 2.3.8. (Bernardi-Ranestad)** ([BR], Theorem 1) Let \( F \in R^s \) be a homogeneous form of degree \( d \), and let \( l \) be any linear form in \( S^1_1 \). Let \( F_l \) be a dehomogenization of \( F \) with respect to \( l \). Denote by \( \text{Diff}(F) \) the subspace of \( S^s \) generated by the partials of \( F \) of all orders. Then

\[
\text{cr}(F) \leq \dim_k \text{Diff}(F_l)
\]

We thank Pedro Marques for pointing out that it is easy to show that the length of a polynomial is an upper bound for the length of any dehomogenization of that polynomial. So we have

\[
\text{cr}(F) \leq \dim_k \text{Diff}(F) = \deg(\text{Ann}(F)) \quad (2.6)
\]

**Proposition 2.3.9.** For the monomial \( m = x_1^{b_1} \ldots x_n^{b_n} \), where \( 1 \leq b_1 \leq \ldots \leq b_n \) we have

(a) ([CCG])

\[
r(x_1^{b_1} \ldots x_n^{b_n}) = \prod_{i=2}^{n}(b_i + 1)
\]
(b) ([RS])
\[ sr(x_1^{b_1} \ldots x_n^{b_n}) = cr(x_1^{b_1} \ldots x_n^{b_n}) = \Pi_{i=1}^{i=n-1} (b_i + 1) \]

(c) ([BBT2]) Let \( d = b_1 + \ldots + b_n \), and \( m = l_1^d + \ldots + l_s^d \) with \( r(m) = s \). Let \( I \subset S \) be the homogeneous ideal of functions vanishing on \( Q = \{[l_1], \ldots, [l_s]\} \subset \mathbb{P}^{n-1} \). Then \( I \) is a complete intersection of degrees \( b_2 + 1, \ldots, b_n + 1 \) generated by
\[
y_2^{b_2+1} - \Phi_1 y_1^{b_1+1}, \ldots, y_n^{b_n+1} - \Phi_n y_1^{b_1+1},
\]
for some homogeneous polynomials \( \Phi_i \in S \) of degree \( b_i - b_1 \).

**Example 2.3.10.** Let \( n = 2 \), and
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]
\[
det(A) = ad - bc = (a + d)^2 - (a - d)^2 + (b - c)^2 - (b + c)^2 \text{ so } r(det(A)) = 4.
\]
The corresponding Hilbert sequence for \( n = 2 \) is \((1, 4, 1)\). We have \( l_{diff}(det(A)) = 4 \). Using Theorem 2.3.5 we have:
\[
\text{cr}(det(A)) \geq \frac{1}{d}\deg(\text{Ann}(det(A))) = \frac{1}{2}(6) = 3.
\]
So the lower bound we obtain using Theorem 2.3.5 is 3.

Using Corollary 2.3.7 (Landsberg-Teitler) we obtain:
\[
r(det_2) \geq \left( \frac{2}{\lfloor 2/2 \rfloor} \right)^2 = 4 + 4 - 4 = 4.
\]
On the other hand we have
\[
det(A) = ad - bc = 1/4((a + d)^2 - (a - d)^2) - 1/4((b + c)^2 - (b - c)^2).
\]
Hence
\[
r(det(A)) = cr(det(A)) = sr(det(A)) = l_{diff}(det(A)) = 4.
\]

**Example 2.3.11.** Let \( n = 3 \), and
\[ A = \begin{pmatrix} a & b & e \\ c & d & f \\ g & h & i \end{pmatrix}, \]

\[ \det(A) = g(bf - de) - h(af - ce) + i(ad - bc). \]

Using Macaulay2 for the calculations we obtain the Hilbert sequence \((1, 9, 9, 1)\), and by Theorem 2.3.5 we have:

\[ \text{cr}(\det(A)) \geq \frac{1}{d} \deg(\text{Ann}(\det(A))) = \frac{1}{2} (20) = 10. \]

So the lower bound we find using the Theorem 2.3.5 is 10, which is greater than the \(l_{\text{diff}}(\det(A)) = 9\), so it is a better lower bound than the differential length for the cactus and smoothable ranks introduced above.

Using Corollary 2.3.7 we have:

\[ r(\det_3) \geq \left( \frac{3}{\lfloor 3/2 \rfloor} \right)^2 + 3^2 - (\lfloor 3/2 \rfloor + 1)^2 = 9 + 9 - 4 = 14. \]

On the other hand, for every \(x, y\) and \(z\), it is easy to see that \(r(xyz) \leq 4\):

\[ xyz = \frac{1}{24}((x + y + z)^3 + (x - y - z)^3 - (x - y + z)^3 - (x + y - z)^3). \]

Hence \(14 \leq r(\det(A)) \leq 24\).

If \(a = 1\) in \(\det(A)\), we have that the punctual scheme \(\text{Ann}(\det A_{a=1})\) of degree 18 with Hilbert function \((1, 8, 8, 1)\). So by Proposition 2.3.8 we have:

\[ \text{cr}(\det(A)) \leq 18. \]

Example 2.3.12. Let \(n = 4\), and

\[ A = \begin{pmatrix} a & b & e & j \\ c & d & f & k \\ g & h & i & l \\ m & n & o & p \end{pmatrix}, \]

Using Macaulay2 for the calculations we obtain the Hilbert sequence \((1, 16, 36, 16, 1)\). By Theorem
2.3.5,

\[ cr(\det(A)) \geq \frac{1}{d} \deg(\text{Ann}(\det(A))) = \frac{1}{2}(70) = 35. \]

which is less than the \( l_{diff}(\det(A)) = 36 \). So in this case \( l_{diff} \) is a better lower bound for the cactus rank.

Using Corollary 2.3.7 (Landsberg-Teitler) we have:

\[ r(\det(A)) \geq \left( \frac{4}{\lfloor 4/2 \rfloor} \right)^2 + 4^2 - (\lfloor 4/2 \rfloor + 1)^2 = 36 + 16 - 9 = 43. \]

Now using Proposition 2.3.9 we have

\[ r(\det(A)) \leq (4!)(2^3) = 192 \]

**Example 2.3.13.** Let \( n = 5 \), and

\[
A = \begin{pmatrix}
a & b & e & j & q \\
c & d & f & k & r \\
g & h & i & l & s \\
m & n & o & p & t \\
u & v & w & x & y \\
\end{pmatrix},
\]

Using Macaulay2 for the calculations we obtain the Hilbert sequence \((1, 25, 100, 100, 25, 1)\). By Theorem 2.3.5

\[ cr(\det(A)) \geq \frac{1}{d} \deg(\text{Ann}(\det(A))) = \frac{1}{2}(252) = 126, \]

which is greater than the \( l_{diff}(\det(A)) = 100 \). So it is a better lower bound for cactus rank than \( l_{diff} \).

Using Corollary 2.3.7 (Landsberg-Teitler) we have:

\[ r(\det(A)) \geq \left( \frac{5}{\lfloor 5/2 \rfloor} \right)^2 + 5^2 - (\lfloor 5/2 \rfloor + 1)^2 = 116. \]

So for the first time at \( n = 5 \) Theorem 2.3.5 gives us a better lower bound for the rank than Corollary 2.3.7 (Landberg-Teitler).
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Now using Proposition 2.3.9 we have

\[ r(\det(A)) \leq (5!)(2^4) = 1920 \]

**Example 2.3.14.** Let \( n = 6 \). Using Macaulay2 for the calculations we obtain the Hilbert sequence

\[ H(S/\text{Ann}(\det A)) = (1, 36, 225, 400, 225, 36, 1). \]

Now using Theorem 2.3.5 we have:

\[ cr(\det(A)) \geq \frac{1}{d} \deg(\text{Ann}(\det(A))) = \frac{1}{2}(924) = 462. \]

So the lower bound we can find using Theorem 2.3.5 is 462, which is greater than the \( l_{\text{diff}}(\det(A)) = 400 \), and therefore is a better lower bound for cactus rank than \( l_{\text{diff}} \).

Using Corollary 2.3.7 (Landsberg-Teitler) we have:

\[ r(\det(A)) \geq \left( \frac{6}{\lfloor 6/2 \rfloor} \right)^2 + 6^2 - (\lfloor 6/2 \rfloor + 1)^2 = 420. \]

So again at \( n = 6 \) Theorem 2.3.5 gives us a better lower bound than Corollary 2.3.7 (Landsberg-Teitler).

Now using Proposition 2.3.9 we have

\[ r(\det(A)) \leq (6!)(2^5) = 23040 \]

**Remark 2.3.15.** (a) Using Stirling’s formula, \( n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \), we can approximate \( \binom{2n}{n} \) for large \( n \) by \( 4^n/\sqrt{n\pi} \). Hence for large \( n \) Theorem 2.3.5 gives us a lower bound asymptotic to \( 4^n/2\sqrt{n\pi} \leq cr(\det(A)) \), and the Landsberg-Teitler formula gives us the lower bound \( 2 \cdot 4^n/(n\pi) \leq r(\det(A)) \). The Landsberg-Teitler lower bound for \( r(\det(A)) \) is also asymptotic to \( l_{\text{diff}}(\det(A)) = \left( \frac{n}{\lfloor n/2 \rfloor} \right)^2 \), which is a lower bound for \( cr(\det(A)) \). These are also lower bounds for the corresponding ranks of the permanent of a generic \( n \times n \) matrix.

(b) Using Proposition 2.3.9 the upper bound for the rank of the determinant and permanent of a generic \( n \times n \) matrix is given by \( (n!)2^{n-1} \). This can be approximated for large \( n \), using Stirling’s
formula, by $\sqrt{2\pi n}(\frac{n}{e})^n(2^{n-1})$.

(c) By Equation 2.6 an upper bound for the cactus rank of both the determinant and permanent of a generic $n \times n$ matrix is $2n\choose n$, which is asymptotic to $4^n/\sqrt{n\pi}$.

In the following table we give lower bounds for the ranks of the determinant and permanent of an $n \times n$ generic matrix.

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>$n \gg 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower bound for $cr(det(A))$ by Theorem 2.3.5</td>
<td>3</td>
<td>10</td>
<td>35</td>
<td>126</td>
<td>462</td>
<td>$4^n/2\sqrt{n\pi}$</td>
</tr>
<tr>
<td>lower bound for $r(det(A))$ by Corollary 2.3.7</td>
<td>4</td>
<td>14</td>
<td>43</td>
<td>116</td>
<td>420</td>
<td>$4^n/2n\pi$</td>
</tr>
<tr>
<td>$l_{diff}(det(A))$</td>
<td>4</td>
<td>9</td>
<td>36</td>
<td>100</td>
<td>400</td>
<td>$\left(\frac{n}{\lfloor n/2 \rfloor}\right)^2$</td>
</tr>
</tbody>
</table>
Chapter 3

Generic symmetric matrix

In this chapter we determine the annihilator ideals of the determinant and the permanent of a generic symmetric $n \times n$ matrix. In section 3.1 we review the doset basis of the space of $t \times t$ minors of an $n \times n$ symmetric matrix. In section 3.2 we determine the generators of the apolar ideal to the determinant and permanent of a generic symmetric matrix (Theorems 3.2.11 and 3.2.23). In section 3.3 we apply our result to find a lower bound for the scheme/cactus rank of the determinant and permanent of the generic symmetric matrix (Theorems 3.3.1 and 3.3.2). And finally in section 3.4, we study the unusual case of acting by contraction on the polynomial ring.

Throughout this chapter $k$ is an infinite field of characteristic zero or characteristic $p > 2$, and $X = (x_{ij})$ be a square symmetric matrix of size $n$ with $\frac{n(n+1)}{2}$ distinct variables. The determinant and the permanent of $X$ are polynomials of degree $n$. Let $R^s = k[x_{ij}] (i \leq j)$, be a polynomial ring and $S^s = k[y_{ij}] (i \leq j)$, be the ring of differential operators associated to $R^s$, and let $R^s_k$ and $S^s_k$ denote the degree-$k$ homogeneous summands. We use the superscript $s$ on $R$ and $S$ to indicate that the matrix of variables $X = (x_{ij})$ is symmetric. $S^s$ acts on $R^s$ by differentiation.

Note that the determinant and permanent of the symmetric matrices contain squares of some variables. Hence it makes a difference when one writes the determinant of a symmetric matrix in a divided power ring or in the usual polynomial ring. We can choose to write the determinant in the usual polynomial ring and use the differentiation to find the apolar ideal or write the determinant in the divided powers ring (Definition 3.0.16 below) and use the contraction.

For the main results in the section 3.2 we use the usual polynomial ring and the differentiation unless otherwise stated. In section 3.4 we consider what happens when we write the determinant in
the usual polynomial ring but find the apolar ideals using the contraction instead of differentiation.

Definition 3.0.16. ([IK], Appendix A) Let $k$ be a field of arbitrary characteristic. Let $R = k[x_1, ..., x_r] = \oplus_{j \geq 0} R_j$, Let $\mathcal{D}$ be the graded dual of $R$, i.e.

$$\mathcal{D} = \oplus_{j \geq 0} \text{Hom}_k(R_j, k) = \oplus_{j \geq 0} \mathcal{D}_j.$$ 

We consider the vector space $R_1$ with the basis $x_1, ..., x_r$ and the left action of $GL_r(k)$ on $R_1$ defined by $Ax_i = \sum_{j=1}^r A_{ij}x_j$. Since $R = \oplus_{j \geq 0} \text{Sym}^j R_1$ this action extends to an action of $GL_r(k)$ on $R$. By duality this action determines a left action of $GL_r(k)$ on $\oplus_{j \geq 0} \mathcal{D}_j$. We denote by $x^U = x_1^{u_1}...x_r^{u_r}, |U| = u_1 + ... + u_r = j$ the standard monomial basis of $R_j$. Let $X_1, ..., X_r$ be the basis of $\mathcal{D}_1$ dual to the basis $x_1, ..., x_r$. We denote by

$$X^U = X_1^{[u_1]}...X_r^{[u_r]}$$

the basis of $\mathcal{D}_j$ dual to the basis $\{x^U : |U| = j\}$. We call these elements divided power monomials. We call the elements of $\mathcal{D}_j$ divided power forms, and the elements of $\mathcal{D}$ divided power polynomials. We extend the definition of $X^U$ to multi-degrees $U = (u_1, ..., u_r)$ with negative components by letting $X^U = 0$ if $u_i < 0$ for some $i$.

We define a ring structure on $\mathcal{D}$. One defines multiplication of monomials by the equality

$$X^U \cdot X^V = \binom{U + V}{U} X^{[U+V]},$$

where $\binom{U+V}{U}$ is a product of binomial coefficients. This is extended by linearity and gives $\mathcal{D}$ a structure of a $k$-algebra.

Notation. Throughout this chapter we let $X = (x_{ij})$ with $x_{ij} = x_{ji}$ be an $n \times n$ symmetric matrix of indeterminates in the polynomial ring $R^s = k[x_{ij}]$. Let $\mathcal{D}$ be the corresponding divided power ring. Let $Y = (y_{ij})$ with $y_{ij} = y_{ji}$ be an $n \times n$ symmetric matrix of indeterminates in the ring of inverse polynomials $S^s = k[y_{ij}]$ associated to $R^s$. 
CHAPTER 3. GENERIC SYMMETRIC MATRIX

3.1 Doset basis for the space of $k \times k$ minors

We recall the definition of doset minors and the Gröbner basis for the determinantal ideal of a generic symmetric matrix.

**Definition 3.1.1.** (see [CON]) Let $H$ be the set of all subsequences $(a_1, ..., a_t)$ of $(1, ..., n)$. Let $a, b \in H$. We define on $H$ the partial order

$$a = (a_1, ..., a_t) \leq b = (b_1, ..., b_r) \iff r \leq t,$$

and

$$a_i \leq b_i \text{ for } i = 1, ..., r.$$

We denote by $[a_1, ..., a_t | b_1, ..., b_t]$ the minor $\det(X_{a_i b_j})$, $1 \leq i, j \leq t$ of $X$. Since $X$ is symmetric it is clear that $[a, b] = [b, a]$. A minor $[a_1, ..., a_t | b_1, ..., b_t]$ of $X$ with $a \leq b$ in $H$ is called a doset minor.

Let $\tau$ be a diagonal term order on $R^s = k[x_{ij}]$ such that the initial term of every doset minor $[a_1, ..., a_s | b_1, ..., b_s]$ is $\prod_{i=1}^{s} x_{a_i b_i}$. For instance, we can consider the lexicographic order induced by the variable order

$$x_{11} \geq x_{12} \geq ... \geq x_{1n} \geq x_{22} \geq ... \geq x_{2n} \geq .... \geq x_{n-1n} \geq x_{nn}.$$

**Theorem 3.1.2.** (Conca) [CON, Theorem 2.9] Let $I_t(X)$ be the ideal generated by the $t$-minors of $X$. The set of the doset $t$-minors is a Gröbner basis for $I_t(X)$ with respect to $\tau$.

**Definition 3.1.3.** A Young tableaux of shape $(r_1, ..., r_u)$ is an array of positive integers $A = (a_{ij})$, with $1 \leq i \leq u$, $1 \leq j \leq r_i$, and $r_1 \geq ... \geq r_u$. Such a tableaux is said to be semi-standard if the numbers in each row is strictly increasing from left to right, and the numbers in each column are in non-decreasing order from top to bottom (that is $a_{i,j} < a_{i,j+1}$ for all $i = 1, ..., u, j = 1, ..., r_i - 1$ and $a_{i,j} \leq a_{i+1,j}$ for all $i = 1, ..., u-1, j = 1, ..., r_{i+1}$).

**Example.** An example of a semi-standard Young Tableaux of shape $(4, 3, 2, 2, 1)$ filled with the numbers $\{1, 2, 3, 4\}$ is
Definition 3.1.4. A path composed of horizontal and vertical line segments in the \(x-y\) plane from \((0,0)\) to \((n,n)\) with steps \((0,1)\) and \((1,0)\) is called a lattice path of order \(n\). A lattice path that never rises above the line \(y = x\), is called a Dyck path of order \(n\). The corners of the Dyck path are the points on the path where the direction of the path changes from horizontal to vertical or vice versa.

The total number of Dyck paths of order \(n\) is given by the Catalan number ( [ST], Volume 2, page 221, Exercise 6.16 h)

\[
C_n = \frac{1}{n+1} \binom{2n}{n}.
\]

Lemma 3.1.5. The dimension of the space of \(t \times t\) minors of an \(n \times n\) symmetric matrix is equal to the number of the doset \(t\)-minors of the \(n \times n\) symmetric matrix. This is equal to the number of fillings of a semi-standard Young tableaux of shape \((t,t)\) with the numbers \(\{1, \ldots, n\}\), which is equal to the Narayana number

\[
N(n+1, t) = \binom{n+1}{t} \binom{n+1}{t-1} / (n+1).
\]

Proof. The first statement is true by Conca’s Theorem 3.1.2. To show the second statement one can view the count of the Conca doset \(t\)-minors as giving the coordinates in the \(x-y\) plane of \(t\) points. Then the count is of segmented paths with \(t\) interior corners lying on or below the diagonal, beginning at \((0,0)\) and ending at \((n+1,n+1)\). That is the number of Dyck \(n+1\)-paths with exactly \(t\) vertices, which is given by the Narayana numbers as \(\binom{n+1}{t} \binom{n+1}{t-1} / (n+1)\). See [ST], Volume 2, page 237, Exercise 6.36 a, with a change of +1 to \((1,0)\) and \(-1\) to \((0,1)\).

\[
\square
\]

Corollary 3.1.6. Let \(X\) be a generic symmetric \(n \times n\) matrix, then \(\deg(\text{Ann}(\det(X)))\) is the Catalan number \(C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}\).

Proof. Let \(<I_t>\) be the space of \(t \times t\) minors of a symmetric \(n \times n\) matrix. Note that \(N(n+1, n+1) = \)
1, and $H(S^a/\text{Ann}(\det(A)))_0 = 1$, so

$$\deg(\text{Ann}(\det(X))) = H(S^a/\text{Ann}(\det(A)))_0 + \sum_{t=1}^{t=n} \dim(<I_t>) = \sum_{t=1}^{n+1} N(n+1,t) = C_{n+1}.$$ 

Thus the $\deg(\text{Ann}(\det(A))$ will be the total number of Dyck paths, below or meeting the diagonal through the $(n+1) \times (n+1)$ grid, which is given by the Catalan number $C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}$.

(See [ST], Volume 2, page 237, Exercise 6.36 a)

Table 3.1: The Hilbert sequence of the annihilator of the determinant of the generic symmetric matrix

<table>
<thead>
<tr>
<th>n=2</th>
<th>1</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=3</td>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>n=4</td>
<td>1</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>n=5</td>
<td>1</td>
<td>15</td>
<td>50</td>
</tr>
<tr>
<td>n=6</td>
<td>1</td>
<td>21</td>
<td>105</td>
</tr>
<tr>
<td>n=7</td>
<td>1</td>
<td>28</td>
<td>196</td>
</tr>
<tr>
<td>n=8</td>
<td>1</td>
<td>36</td>
<td>336</td>
</tr>
</tbody>
</table>

### 3.2 Generators of the apolar ideal

In section 3.2.1 we determine the generators of the apolar ideal of the determinant of the $n \times n$ generic symmetric matrix. In section 3.2.2 we determine the generators of the apolar ideal of the permanent of the $n \times n$ generic symmetric matrix.

**Notation.** ([IKO]) Let $F_{2m} \subset \mathfrak{S}_{2m}$ be the set of all permutations $\sigma$ satisfying the following conditions:

1. $\sigma(1) < \sigma(3) < \ldots < \sigma(2m-1)$
2. $\sigma(2i-1) < \sigma(2i)$ for all $1 \leq i \leq m$

We denote by $Hf(X)$ the hafnian of a generic symmetric $2n \times 2n$ matrix $X$, which is defined by

$$Hf(X) = \sum_{\sigma \in F_{2n}} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}\ldots x_{\sigma(2n-1)}x_{\sigma(2n)} \quad (3.1)$$
3.2.1 Apolar ideal of the determinant

In this subsection we determine the apolar ideal of the determinant of the $n \times n$ generic symmetric matrix, and we will show that it is generated by degree two elements. We first determine the ideal in degree two (Proposition 3.2.1). We then show that the generators in degree two generate the annihilator ideal in degree $n$ (Proposition 3.2.6). Then we show that the elements of degree two generate the ideal in each degree $k$ for $2 \leq k \leq n$. A key step is to use triangularity to show that these degree two generators generate all of the apolar ideal (Lemma 3.2.7 and Proposition 3.2.8). This leads to our main result (Theorem 3.2.11).

**Notation.** For the generic symmetric $n \times n$ matrix $X$, the unacceptable monomials of degree $k$ in $S^s_k$ are monomials which do not divide any term of the determinant of $X$. We denote the set of degree $k$ unacceptable monomials by $U_k$. A monomial that divides some term of the determinant is called an acceptable monomial.

**Proposition 3.2.1.** For an $n \times n$ symmetric matrix $X = (x_{ij})$, $\text{Ann}(\det(X)) \subset S^s$, includes the following degree 2 polynomials:

(a) The unacceptable monomials of the form $y_{ii}y_{ij}$ for all $1 \leq i, j \leq n$. The number of these monomials is $n^2$.

(b) All the diagonal $2 \times 2$ binomials of the form $y_{ii}^2 + 2y_{ii}y_{jj}$. The number of these binomials is $\binom{n}{2}$.

(c) All the $2 \times 2$ permanents with one diagonal element, i.e. $y_{jk}y_{il} + y_{jl}y_{ii}$. The number of these binomials is $n \cdot \binom{n-1}{2}$.

(d) The hafnians of all symmetrically chosen $4 \times 4$ submatrices of $X$. The number of these trinomials is $\binom{n}{4}$.

**Proof.** We have $\det(X) = \sum_{\sigma \in S_n} Sgn(\sigma) \Pi x_{i,\sigma(i)}$. First we show that monomials of type (a) are in $\text{Ann}(\det(X))$. By symmetry we have

$$y_{ii}y_{ij} \circ \det(X) = 0 \text{ (where } j \geq i),$$

$$y_{ii}y_{ji} \circ \det(X) = 0 \text{ (where } j \leq i).$$

Next we want to show that binomials of type (b) are in $\text{Ann}(\det(X))$. 


Let $P = 2y_{ii}y_{jj} + y_{ij}^2$. There are $n!$ terms in the expansion of the determinant. If a term doesn’t contain the monomial $x_{ii}x_{jj}$ or the monomial $x_{ij}^2$ then the result of the action of $P$ on it will be zero. Let $\sigma_1$ be a permutation having $x_{ii}$ and $x_{jj}$ respectively in it’s $i$-th and $j$-th place. Corresponding to $\sigma_1$ we also have a permutation $\sigma_2 = \tau\sigma_1$, where $\tau = (i, j)$ is a transposition and $sgn(\sigma_2) = sgn(\tau\sigma_1) = -sgn(\sigma_1)$. Thus, corresponding to each positive term in the determinant which contains the monomial $x_{ii}x_{jj}$ or the monomial $x_{ij}^2$ we have the same term with the negative sign, thus the resulting action of the binomial $P$ on $\text{det}(X)$ is zero.

To show that the binomials of type (c) are in the annihilator ideal we can use the same proof as we used for the binomials of type (b).

Next we want to show that any $4 \times 4$ hafnian of $Y$ annihilates the determinant of an $n \times n$ symmetric matrix $X$. This is easy to check for $n = 4$. So let $n \geq 4$. Let $W$ be a $4 \times 4$ submatrix of $Y$, involving the rows and the columns $i_1, i_2, i_3$ and $i_4$.

$$W = \begin{pmatrix} y_{i_1i_1} & y_{i_1i_2} & y_{i_1i_3} & y_{i_1i_4} \\ y_{i_2i_1} & y_{i_2i_2} & y_{i_2i_3} & y_{i_2i_4} \\ y_{i_3i_1} & y_{i_3i_2} & y_{i_3i_3} & y_{i_3i_4} \\ y_{i_4i_1} & y_{i_4i_2} & y_{i_4i_3} & y_{i_4i_4} \end{pmatrix}$$

Using Equation 3.1 the hafnian of $W$ is

$$H = Hf(W) = y_{i_1i_2}y_{i_3i_4} + y_{i_1i_3}y_{i_2i_4} + y_{i_1i_4}y_{i_2i_3}.$$  

If a term in the determinant does not contain the monomials $x_{i_1i_2}x_{i_3i_4}$ or $x_{i_1i_3}x_{i_2i_4}$ or $x_{i_1i_4}x_{i_2i_3}$, then $H$ annihilates it. If a term in the determinant contains one of the monomials $x_{i_1i_2}x_{i_3i_4}$ or $x_{i_1i_3}x_{i_2i_4}$ or $x_{i_1i_4}x_{i_2i_3}$, then since these monomials do not appear in any other $4 \times 4$ sub matrix of $X$, we can use the Laplace expansion (cofactor expansion) of the determinant and the proof is complete.

\[\Box\]

We denote by $\{V\}$ be set of the degree two elements of type (a), (b), (c) and (d) in Proposition 3.2.1, and by $V$ the vector subspace of $S^a$ spanned by $\{V\}$. We denote by $\{a\}$, $\{b\}$, $\{c\}$ and $\{d\}$ the set of elements in (a), (b), (c) and (d) respectively.
Lemma 3.2.2. The set \( \{V\} \) is linearly independent and we have,

\[
\dim V = n^2 + \binom{n}{2} + n \cdot \binom{n-1}{2} + \binom{n}{4}.
\]

Proof. Each of the four subsets are linearly independent from each other since they involve different variables. So it suffices to show that each subset is linearly independent. The subset \( \{a\} \) is linearly independent since the monomials in \( \{a\} \) form a Gröbner basis for the ideal they generate. The subsets \( \{b\} \) and \( \{c\} \) are linearly independent since by choosing two elements of the matrix, where at least one element is diagonal, we have a unique 2 \( \times \) 2 minor. The subset \( \{d\} \) is linearly independent since the monomials that appear in a hafnian of a 4 \( \times \) 4 symmetric submatrix of \( X \), do not appear in the hafnian of any other 4 \( \times \) 4 symmetric submatrix of \( X \). Hence the set \( \{V\} \) is linearly independent and the dimension of the vector space \( V \) is \( n^2 + \binom{n}{2} + n \cdot \binom{n-1}{2} + \binom{n}{4} \).

\[\square\]

\[\text{Lemma 3.2.3.}\]

\[S_k^a \circ (\det(X)) = M_{n-k}(X) \subset R^s. \tag{3.2} \]

Proof. To show the inclusion

\[S_k^a \circ (\det(X)) \subset M_{n-k}(X) \subset R^s,\]

we use induction on \( k \). For \( k = 1 \), the above inclusion is easy to see. Now assume that the above inclusion holds for \( k-1 \), i.e \( S_{k-1}^a \circ \det(X) \subset M_{n-(k-1)}(X) \), and we want to show that it is true for \( k \). We have

\[S_k^a \circ \det(X) = S_1^a S_{k-1}^a \circ \det(X) \subset S_1^a \circ M_{n-k+1}(X) \subset M_{n-k}(X).\]

Now we want to show the opposite inclusion

\[S_k^a \circ (\det(X)) \supset M_{n-k}(X) \subset R^s,\]

Let \( M_{I,J}(X), I = \{i_1, \ldots, i_k\}, J = \{j_1, \ldots j_k\}, 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n, 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n \).
Let \( \Delta(I,J) = \{(i_r,j_r)|i_r \in I, j_r \in J \text{ and } i_r = j_r \} \).

Let \( \Delta_I = \{i_r|(i_r,j_r) \in \Delta(I,J) \} \) and \( \Delta_J = \{j_r|(i_r,j_r) \in \Delta(I,J) \} \).

Let \( M_{(I,J) - \Delta} \) be the sub matrix of \( Y \) with the rows \( I - \Delta_I \), and the columns \( J - \Delta_J \).

We claim

\[
M_{\tilde{I}, \tilde{J}} = \pm c \prod_{(i_r,j_r) \in \Delta(I,J)} y_{i_r,j_r} \det(M_{(I,J) - \Delta}) \circ \det(X)
\]

where \( c \in k \).

To prove this claim we use induction on \( k = |I| = |J| \), the cardinality of the sets \( I \) and \( J \). First we show the claim is true for \( k = 1 \). Let \( I = \{i_1\} \) and \( J = \{j_1\} \). We have two cases

I. \( i_1 = j_1 \) so \( y_{i_1,j_1} \) is a diagonal element and we have

\[
M_{\tilde{I}, \tilde{J}} = y_{i_1,j_1} \circ (\det(X)).
\]

II. \( i_1 \neq j_1 \) so we have

\[
y_{i_1,j_1} \circ (\det(X)) = 2M_{\tilde{I}, \tilde{J}}.
\]

So for \( k = 1 \) the claim holds. Next assume that the claim holds for all every \( I \) and \( J \) with \( |I| = |J| = k - 1 \) and we want to show that the claim is also true for \( I \) and \( J \) with \( |I| = |J| = k \).

Let \( I = \{i_1, ..., i_k\} \) and \( J = \{j_1, ..., j_k\} \).

Let \( I' = I - \{i_1\} \) and \( J' = J - \{j_1\} \). We have \( |I'| = |J'| = k - 1 \) so by the induction assumption we have

\[
M_{\tilde{I'}, \tilde{J'}} = \pm c \prod_{(i_r,j_r) \in \Delta(I',J')} y_{i_r,j_r} \det(M_{(I',J') - \Delta}) \circ \det(X)
\]

By writing the Laplace expansion of the determinant using row \( i_1 \) or column \( j_1 \) for \( M_{\tilde{I}, \tilde{J}} \), we get
$M_{i,j} = \pm c \prod_{(i_r,j_r) \in \Delta_{(i,j)}} y_{i_r,j_r} \det(M_{i,j} - \Delta) \circ \det(X)$,

where $c \in k$. Hence $M_{i,j} \in S_{n-k}^s \circ (\det(X))$.

**Lemma 3.2.4.** For the generic symmetric $n \times n$ matrix $X$, we have

$$V = \text{Ann}(M_2) \cap S_2^s = (\text{Ann}(\det X))_2$$

**Proof.** By the Lemma 3.2.3 we have

$$\text{Ann}(S_{n-2}^s \circ (\det(X))) \subset \text{Ann}(M_2(X)).$$

By the Proposition 3.2.1 we have

$$(\text{Ann}(\det(X)))_2 \supset V$$

By the Remark 1.0.4 we have

$$(\text{Ann}(\det(X)))_2 = (\text{Ann}(S_{n-2}^s \circ (\det(X))))_2 \subset \text{Ann}(M_2(X))$$

Hence we have

$$V \subset \text{Ann}(M_2).$$

On the other hand, using Lemma 3.2.2 we have

$$\dim(V) = n^2 + \binom{n}{2} + n \cdot \binom{n-1}{2} + \binom{n}{4} = \left(\frac{n^2+n}{2} + 1\right) - \frac{(n+1)(n+1)}{n+1} = \dim S_2^s - \dim M_2(X).$$
So we have
\[ V = \text{Ann}(M_2) \cap S^2_s. \]

By the Proposition 3.2.1, Lemmas 3.2.2 and 3.2.3 we have
\[ (\text{Ann}(\det(X)))_2 \subset \text{Ann}(M_2(X)) \subset V, \]
and therefore
\[ V = (\text{Ann}(\det(X)))_2. \]

Lemma 3.2.5. Let \(2 \leq k \leq n\). We have
\[ (V)_k \subset \text{Ann}(M_k(X)) \cap S^k_s. \quad (3.3) \]

Proof. We have
(A) \( V \circ \det(X) = 0 \iff V \circ S^s_{n-2} \circ (\det(X)) = 0 \iff V \circ M_2(X) = 0. \)

(B) \( (\text{Ann}(\det(X))) \cap S^2_s = V \Rightarrow S^s_{k-2} V \circ (S^s_{n-k} \circ \det(X)) = 0. \)
\[ \Rightarrow S^s_{k-2}(V) \circ M_k(X) = 0. \]
\[ \Rightarrow (V)_k \circ M_k(X) = 0. \quad \text{(By Remark 1.0.4)} \]
which proves the lemma.

Proposition 3.2.6. For \(n \geq 2\) we have
\[ (V)_n = \text{Ann}(\det(X)) \cap S^n_s. \quad (3.4) \]

Proof. One inclusion is given by Lemma 3.2.5. To show the other inclusion holds we use induction on \(n\). For \(n = 2, 3\) the equality is easy to see, and we want to show that equation holds for \(n = 4\).
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\[ X = (x_{ij}) = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix}, \]

\[ Y = (y_{ij}) = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}, \]

We have:

\[
\det(X) = d^2f^2 - 2cdfg + c^2g^2 - d^2eh + 2bdgh - ag^2h + 2cdei - 2bdfi - \\
2bcgi + 2afgi + b^2i^2 - aei^2 - c^2ej + 2bcfj - af^2j - b^2hj + aehj \in \mathbb{R}_4.
\]

If we denote the determinant in the divided power ring by \( \det(X)_{\text{Div}} \) we have:

\[
\det(X)_{\text{Div}} = 4d^2f^2 - 2cdfg + 4c^2g^2 - 2d^2eh + 2bdgh - 2ag^2h + 2cdei - 2bdfi - \\
2bcgi + 2afgi + 4b^2i^2 - 2aei^2 - 2c^2ej + 2bcfj - 2af^2j - 2b^2hj + aehj \in \mathcal{D}_4.
\]

We use the divided powers and the contraction in the following proof. Using the Remark 2.2.7, let \( \psi \) be the binomial in \( \text{Ann}(\det(X)) \cap S_4^4 \).

\[
\psi = \alpha_\sigma(-1)^{\text{sgn}(\eta)}y_{1\eta(1)}y_{2\eta(2)}y_{3\eta(3)}y_{4\eta(4)} - \alpha_\eta(-1)^{\text{sgn}(\sigma)}y_{1\sigma(1)}y_{2\sigma(2)}y_{3\sigma(3)}y_{4\sigma(4)},
\]

where \( \sigma \neq \eta \) are two permutations of the set \( \{1, 2, 3, 4\} \), \( \alpha_\eta \) is the coefficient of the monomial \( y_{1\eta(1)}y_{2\eta(2)}y_{3\eta(3)}y_{4\eta(4)} \) in \( \det(X)_{\text{Div}} \) and \( \alpha_\sigma \) is the coefficient of \( y_{1\sigma(1)}y_{2\sigma(2)}y_{3\sigma(3)}y_{4\sigma(4)} \) in \( \det(X)_{\text{Div}} \). The terms \( y_{1\eta(1)}y_{2\eta(2)}y_{3\eta(3)}y_{4\eta(4)} \) and \( y_{1\sigma(1)}y_{2\sigma(2)}y_{3\sigma(3)}y_{4\sigma(4)} \) cannot have 3 common factors, since if they have 3 variables in common the fourth variable is forced and it contradicts our assumption \( \sigma \neq \eta \). Without loss of generality we can assume \( \eta = \text{id} \). We have three different possibilities.

(i) \( y_{11}y_{22}y_{33}y_{44} \) and \( y_{1\sigma(1)}y_{2\sigma(2)}y_{3\sigma(3)}y_{4\sigma(4)} \) have two common factors. Without loss of generality we can assume that \( \sigma(1) = 1 \) and \( \sigma(2) = 2 \). So we have
\[ \psi = 2AEHJ - (-1)^{sgn(\sigma)} AEy_{3\sigma(3)}y_{4\sigma(4)} = 2AEHJ + AEI^2 = AE(2HJ + I^2) \in (V)_4 \]

(ii) \( y_{11}y_{22}y_{33}y_{44} \) and \( y_{1\sigma(1)}y_{2\sigma(2)}y_{3\sigma(3)}y_{4\sigma(4)} \) have only one common factor. Without loss of generality we can assume that \( \sigma(1) = 1 \). Since the only term in the determinant which has a and does not have \( e, h \) and \( j \) is \( 2afgi \) we have

\[ \psi = 2AEHJ - AFGI = 2AEHJ - AFGI + AF^2I - AF^2J = \]

\[ AJ(2EH + F^2) - AF(GI + FJ) \in (V)_4, \]

since we know that \( 2EH + F^2 \in V \) and \( GI + FJ \in V \).

(iii) \( y_{11}y_{22}y_{33}y_{44} \) and \( y_{1\sigma(1)}y_{2\sigma(2)}y_{3\sigma(3)}y_{4\sigma(4)} \) do not have any common factor. We add and subtract a term which has a common factor with \( y_{11}y_{22}y_{33}y_{44} \) and a common factor with \( y_{1\sigma(1)}y_{2\sigma(2)}y_{3\sigma(3)}y_{4\sigma(4)} \). The reason that such a term exists in the determinant is that if we choose two elements, \( \alpha \) and \( \beta \) not in the same row or column, it is easy to see that we always have a term in the determinant containing \( \alpha \beta \). On the other hand if we choose one variable from \( y_{11}y_{22}y_{33}y_{44} \), say \( y_{11} \), there is always one variable in \( y_{1\sigma(1)}y_{2\sigma(2)}y_{3\sigma(3)}y_{4\sigma(4)} \) which is not in the first row or column, since we only have three elements other than \( y_{11} \) in the first row and column. So we can always choose a term in the determinant with at least one common factor with \( y_{11}y_{22}y_{33}y_{44} \) and at least one common factor with \( y_{1\sigma(1)}y_{2\sigma(2)}y_{3\sigma(3)}y_{4\sigma(4)} \). Then using the cases (i) or (ii) we have

\[ \psi = AEHJ - AFGI = 2AEHJ - AFGI + AF^2J - AF^2J = \]

\[ AJ(2EH + F^2) - AF(GI + FJ) \in (V)_4. \]

This completes the proof of the Equation 3.2.1 for \( n = 4 \). If \( n \) is larger than 4 then by the induction assumption we can assume that the Equation 3.2.1 holds for all integers \( 2 \leq k \leq n - 1 \). Again we use the Remark 2.2.7. Let \( \beta = \beta_1 + \beta_2 \in \text{Ann}(\det(X)) \cap S^*_1 \). If the two terms, \( \beta_1 \) and \( \beta_2 \) have a common factor \( l \), i.e. \( \beta_1 = la_1 \) and \( \beta_2 = la_2 \), then \( \beta = l(a_1 + a_2) \) where \( a_1 \) and \( a_2 \) are of degree at most \( n - 1 \). By the induction assumption the proposition holds for the binomial \( a_1 + a_2 \), i.e.
$a_1 + a_2 \in V_{n-1}$ hence we have

$$\beta = l(a_1 + a_2) \in l(V)_{n-1} \subset (V)_n.$$  

If the two terms, $\beta_1$ and $\beta_2$ do not have any common factor then with the same method as used in (iii), we can rewrite the binomial $\beta$ by adding and subtracting a term $m$ of degree $n$, which has a common factor $m_1$ with $\beta_1$ and a common factor $m_2$ with $\beta_2$, and we will have

$$\beta_1 + \beta_2 = \beta_1 + m + \beta_2 - m = m_1(c_1 + m') + m_2(c_2 - m''),$$

where $\beta_1 = m_1c_1$, $m = m_1m' = m_2m''$ and $\beta_2 = m_2c_2$. Since $c_1 + m'$ and $c_2 - m''$ are of degree at most $n - 1$ so by the induction assumption we have

$$\beta_1 + \beta_2 = m_1(c_1 + m') + m_2(c_2 - m'') \in (V)_n.$$  

This completes the induction step and the proof of the proposition.

Recall that for the generic symmetric $n \times n$ matrix $X$, the unacceptable monomials of degree $k$ in $S^k_s$ are the monomials which do not divide any term of the determinant of $X$, and recall we denote the set of degree $k$ unacceptable monomials by $U_k$.

**Lemma 3.2.7.** We can write each unacceptable monomial of degree $k$ ($2 \leq k \leq n$), as an explicit element of $S^k_{s-2}V$, where $V$ is the space defined in the Proposition 3.2.1.

**Proof.** We use induction on $k$. For $k = 2$ the claim is obviously true. We show that the claim is true for $k = 3$. We need to show that the space $U_3$ of unacceptable monomials in $S^3_s$ are in $S^1_s V$. The unacceptable monomials of degree 3 for the $n \times n$ generic symmetric matrix have one of the following forms:

(a) The form $x^2y$ where $x$ is a diagonal element.

(b) The form $xyz$ where $x$ is a diagonal element, $y \neq x$ is in the same row or column with $x$ and $z \neq x$.

(c) The form $xyz$ where $x, y, z$ are nondiagonal elements from the same row or column (can be
equal to each other).

Unacceptable monomials of type (a) or (b) are multiples of unacceptable monomials of degree 2, so they are in the space $S_1^sU_2$. So we only need to show that the degree 3 unacceptable monomials of type (c) are in $S_1^sV$. The 3 nondiagonal elements in the same row or column of the matrix $X$ are from a symmetric $4 \times 4$ sub-matrix. So without loss of generality we show that a degree 3 monomial of type (c) from the following sub-matrix is in $S_1^sV$. Let $A^s$ be the $4 \times 4$ symmetric sub-matrix of a generic symmetric $n \times n$ matrix,

$$A^s = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix},$$

and $D^s$ be the matrix

$$D^s = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}.$$

Monomials of type (c) in $S_3^s$ annihilating $\det(A^s)$ can be of one of the following forms:

(I) All three non-diagonal variables are distinct. Consider the monomial $\eta_1 = BCD$, a degree three unacceptable monomial of type (c). We have $AF + BC \in V$ (since it is a permanent with one diagonal element), so we have $(AF + BC) \circ \det(A^s) = 0$. Hence,

$$D(AF + BC) \circ \det(A^s) = 0.$$

We also know that $DAF \in S_1^sU_2 \subset U_3$ so $DAF \circ \det(A^s) = 0$. We have $\eta_1 = BCD = D(AF + BC)(\mod S_1^sU_2)$. So we have $\eta_1 \in S_1^sV$.

(II) There are only two distinct non-diagonal variables. Consider the monomial $\eta_2 = B^2C$, also of type (c). We have $AF + BC \in V$ (since it is a permanent with one diagonal element), so we have $(AF + BC) \circ \det(A^s) = 0$. Hence,

$$B(AF + BC) \circ \det(A^s) = 0.$$
We also know that $BAF \in S_{1}^{s}U_{2} \subset U_{3}$ so $B^{2}C \circ \det(A^{*}) = 0$. We have $\eta_{2} = B^{2}C = B(AF + BC)(\text{mod } S_{1}^{s}U_{2})$. So we have $\eta_{2} \in S_{1}^{s}V$.

(III) There is only one non-diagonal variable. Consider the monomial $\eta_{3} = B^{3}$, also of type (c). We have $B^{2} + 2AE \in V$ (since it is a diagonal permanent with the coefficient 2), so we have $(B^{2} + 2AE) \circ \det(A^{*}) = 0$. Hence,

$$B(B^{2} + 2AE) \circ \det(A^{*}) = 0.$$

We also know that $BAE \in S_{1}^{s}U_{2} \subset U_{3}$ so $B^{3} \circ \det(A^{*}) = 0$. We have $\eta_{3} = B^{3} = B(B^{2} + 2AE)(\text{mod } S_{1}^{s}U_{2})$. So we have $\eta_{3} \in S_{1}^{s}V$.

So the lemma is proven for $k = 3$ and we have $U_{3} \subset S_{1}^{s}V$. Let $P$ denote the subspace of $V$ generated by binomials of type (b) and (c) defined in Proposition 3.2.1. We have shown that $U_{3} \subset S_{1}^{s}(U + P)$. Next assume that $k \geq 4$ and the lemma is established for all integers less than $k$. We want to show that the claim is true for $k$. Let $\mu = \mu_{1}\mu_{2}\ldots\mu_{k}$ be an unacceptable monomial of degree $k$. We can write $\mu$ such that $\mu_{2}\ldots\mu_{k}$ is an unacceptable monomial of degree $k - 1$ so we have

$$\mu = \mu_{1}(\mu_{2}\ldots\mu_{k}) \in S_{1}^{s}(S_{k-3}^{s}V) = S_{k-2}^{s}V.$$

So the lemma is true also for $k$.

**Notation.** We use the following definitions and notations in the remaining part of this section.

- By Lexicographic/Conca order we mean the lexicographic term order induced by the variable order,

$$Y_{1,1} > Y_{1,2} > \ldots > Y_{1,n} > Y_{2,2} > \ldots > Y_{2,n} > \ldots > Y_{n-1,n} > Y_{n,n}.$$

By Reverse Lexicographic order we mean the lexicographic term order induced by the variable order,

$$Y_{1,1} < Y_{1,2} < \ldots < Y_{1,n} < Y_{2,2} < \ldots < Y_{2,n} < \ldots < Y_{n-1,n} < Y_{n,n}.$$

- Let $M$ be the $k \times k$ minor of the generic symmetric matrix $X$, with the set of rows $\{a_{1}, \ldots, a_{k}\}$ and the set of the columns $\{b_{1}, \ldots, b_{k}\}$ where $a_{1} < \ldots < a_{k}$ and $b_{1} < \ldots < b_{k}$. Then the initial monomial of $M$ using the lexicographic (Conca) order is $x_{a_{1}b_{1}}\ldots x_{a_{k}b_{k}}$.

- We denote by $[a_{1}, \ldots, a_{k}|b_{1}, \ldots, b_{k}]$ the $k \times k$ doset minor with the sequence of rows $a =$
(a₁, ..., aₖ) and the sequence of columns b = (b₁, ..., bₖ) both subsequences of (1, ..., n) satisfying the following conditions:

\[ a₁ < a₂ < \ldots < aₖ, \]
\[ b₁ < b₂ < \ldots < bₖ, \]
\[ aᵢ ≤ bᵢ, \quad ∀ \quad 1 ≤ i ≤ k. \]

- We denote by \((a₁, ..., aₖ | b₁, ..., bₖ)\) the acceptable monomial \(xᵢ₁b₁...xᵢₖbₖ\). Note that we write the acceptable monomial \(m = (a₁, ..., aₖ | b₁, ..., bₖ)\), with \(a = (aᵢ)\) an increasing sequence. But unlike the doset minors, the sequence \(b = (bᵢ)\) doesn’t need to be increasing.

- For the monomial \(m = xᵢ₁b₁...xᵢₖbₖ\) denoted by \((a₁, ..., aₖ | b₁, ..., bₖ)\), we call a pair \((bᵢ, bⱼ)\), with \(i < j\), a reversal pair if \(bᵢ ≥ bⱼ\).

- Let \(μ = [i₁, ..., iₖ | j₁, ..., jₖ]\) be a \(k \times k\) doset minor. We call the monomial \(mₖ = xᵢ₁j₁...xᵢₖjₖ\) the flag monomial of \(μ\).

- \(Conca\ monomial\) is the initial monomial of a doset minor in lexicographic order. Note that the initial monomial of a doset minor in lexicographic order is the flag monomial of that minor.

- The set of all \(k \times k\) doset minors form a Gröbner basis for the ideal generated by all \(k \times k\) minors (Theorem 3.1.2). Hence the ideal generated by the set of initial monomials of all minors is equal to the ideal generated by the set of the initial monomials of the doset minors.

- Let \(Aₖ\) be the set of acceptable monomials in \(Sₖ^k\).

- Let \(ι : R^s → S^ₖ, ι(xᵢj) = yᵢj,\) and \(Cₖ\) be the subset of \(Aₖ\) defined by

\[ \{ι(μ) | μ \text{ a Conca initial monomial (in lex order) of a } k \times k \text{ doset minor of } X\}. \]

- Let \(C'_k\) be the complementary set to \(C_k\) of acceptable monomials in \(A_k\).

- For each \(μ ∈ A_k\), let \(A>μ\), denote the subset of elements \(ν ∈ A_k\), such that \(ν > μ\) in the lexicographic order of \(S^k\).

**Proposition 3.2.8.** Each acceptable non-Conca monomial of degree \(k\) \((3 ≤ k ≤ n)\), is the initial monomial (in the reverse lex order) of an element of \(Sₖ^k₋2V\).
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Proof. We use the induction on $k \geq 3$. First let $k = 3$. Let $\mu$ be a degree 3 acceptable monomial which is not the initial term of any $3 \times 3$ doset minor in the lexicographic order. The acceptable monomials, $y_{i_1i_2}y_{i_3i_4}y_{i_5i_6}$, of degree 3 for the $n \times n$ generic symmetric matrices can be listed as follows:

(a) All 6 indices are distinct

(b) There is one repeated index

(c) There are 2 repeated indices

(d) There are 3 repeated indices

We discuss each of the above types separately.

(a) all 6 indices are distinct $m = y_{i_1i_2}y_{i_3i_4}y_{i_5i_6}$, $(i_1, i_3, i_5| i_2, i_4, i_6)$. Without loss of generality we can assume these indices are $1,2,3,4,5,6$. In order to have a non-Conca monomial of this kind, it is enough to have at least one reversal pair. Since the monomial $m$ has a reversal pair it is not the initial monomial of any $3 \times 3$ minor of $Y$. So it is not in the ideal generated by all the initial monomials of the $3 \times 3$ minors of $Y$. Hence by Theorem 3.1.2 it is not in the ideal generated by all the initial monomials of all $3 \times 3$ doset minors of $Y$. As an example, in $(1,2,3|6,4,5)$, $6 \geq 4$ so $(6,4)$ is a reversal pair.

In the doset minor $[i_1, i_3, i_5| i_2, i_4, i_6]$, we have

$$i_1 < i_3 < i_5,$$

$$i_2 < i_4 < i_6.$$  

Now assume $i_1 = 1, i_3 = 2, i_5 = 3, i_2 = 4, i_4 = 5$ and $i_6 = 6$ then $y_{14}y_{25}y_{36}$ is the initial term in the corresponding $3 \times 3$ doset minor using the lexicographic order. Consider a non-initial Conca monomial which has at least one reversal pair $(i_j, i_k), (j < k)$ where $j, k \in \{2, 4, 6\}$ such that $i_j \geq i_k$. 
Look at the corresponding $6 \times 6$ symmetric sub-matrix, we have

$$X = \begin{pmatrix}
a & b & c & d & e & f \\
b & g & h & i & j & k \\
c & h & l & m & n & o \\
d & i & m & p & q & r \\
e & j & n & q & s & t \\
f & k & o & r & t & u \\
\end{pmatrix},$$

$$Y = \begin{pmatrix}
A & B & C & D & E & F \\
B & G & H & I & J & K \\
C & H & L & M & N & O \\
D & I & M & P & Q & R \\
E & J & N & Q & S & T \\
F & K & O & R & T & U \\
\end{pmatrix}.$$ 

Consider a non-Conca degree three monomial involving 6 distinct rows and columns. Without loss of generality we consider the monomial $(1, 2, 5|6, 4, 3)$. We have the hafnian of the following $4 \times 4$ symmetric sub-matrix with the rows and columns $1, 2, 4, 6,$

$$\text{Haf} \begin{pmatrix}
A & B & D & F \\
B & G & I & K \\
D & I & P & R \\
F & K & R & U \\
\end{pmatrix} = BR + DK + FI.$$ 

Given $\mu = FIN$, $f_\mu = N(BR + DK + FI) \in S_1^sV$, where $N(BR + DK) \in A_{>\mu}$.

(b) There is one repeated index. Without loss of generality we can assume these indexes are $1, 2, 3, 4, 5$, with one of them repeated. In order to have a non-Conca example of this kind, it is enough to have 1 reversal pair. For example in $(1, 2, 3|4, 1, 5)$, $4 > 1$. We can form a $5 \times 5$ symmetric matrix with these rows and columns
The monomial $\mu = y_{14}y_{21}y_{35} = BDL$ as an acceptable monomial of type (b). We have the hafnian of the following $4 \times 4$ symmetric sub-matrix with the rows and columns 1, 2, 3, 5,

$$\begin{pmatrix} A & B & C & D & E \\ B & F & G & H & I \\ C & G & J & K & L \\ D & H & K & M & N \\ E & I & L & N & O \end{pmatrix} = BL + CI + EG.$$
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\[ Y = \begin{pmatrix}
    A & B & C & D \\
    B & E & F & G \\
    C & F & H & I \\
    D & G & I & J 
\end{pmatrix}. \]

Now we consider the monomial \( \mu = y_{12}y_{21}y_{34} = B^2I \) as an acceptable monomial of type (c). Given \( \mu = B^2I, f_\mu = I(B^2 + 2AE) \in S_1^sV, \) where \( AEI \in A_{>\mu}. \)

(d) There are 3 repeated indices. Without loss of generality we can assume these indexes are 1, 2, 3, all of them repeated. In order to have a non-Conca example of this kind, it is enough to have 1 reversal pair. For example in \( (1, 2, 3|3, 2, 1), \) \( 3 > 1. \) We can form a \( 3 \times 3 \) symmetric matrix with these rows and columns,

\[ X = \begin{pmatrix}
    a & b & c \\
    b & d & e \\
    c & e & f 
\end{pmatrix}, \]

\[ Y = \begin{pmatrix}
    A & B & C \\
    B & D & E \\
    C & E & F 
\end{pmatrix}. \]

Now we consider the monomial \( \mu = y_{13}y_{22}y_{31} = C^2D \) as an acceptable monomial of type (c). Given \( \mu = C^2D, f_\mu = C(BE + CD) \in S_1^sV, \) where \( BEC \in A_{>\mu}. \)

Since all other cases are similar to the above examples, for \( k = 3 \) the claim of Proposition 3.2.8 is true. Now assume that the Proposition 3.2.8 is true for all integers less than \( k. \) We have to show that the Proposition 3.2.8 is also true for \( k. \) Let \( \mu = y_{i_1j_1}...y_{i_kj_k} = (i_1, ..., i_k|j_1, ..., j_k) \) be a degree \( k \) acceptable non-Conca monomial, so it has at least one reversal pair. We can consider \( \mu \) as the product of one variable, \( y_{ab} \) and a degree \( k - 1 \) acceptable non-Conca monomial, \( \mu_1, \) containing at least one reversal pair. Then by the induction assumption \( \mu_1 \) is the initial monomial (in rev. lex.) of an element of \( S_{k-3}^sV. \) So we have \( \mu = y_{ab}t_1 \) as the initial monomial (in rev. lex.) of an element of \( S_{k-2}^sV. \) This completes the proof. \( \Box \)
Example 3.2.9. Consider the case \( n = 3 \). we have

\[
X = \begin{pmatrix}
a & b & c \\
b & d & e \\
c & e & f \\
\end{pmatrix},
\]

\[
Y = \begin{pmatrix}
A & B & C \\
B & D & E \\
C & E & F \\
\end{pmatrix},
\]

There are 5 acceptable degree 3 monomials. Using the lexicographic term order induced by the variable order,

\[
Y_{1,1} > Y_{1,2} > Y_{1,3} > Y_{2,2} > Y_{2,3} > Y_{3,3},
\]

we have the following order on the degree 3 acceptable monomials

\[
ADF > AE^2 > B^2F > BEC > C^2D.
\]

- The set of Conca initial monomials of degree three, \( C_3 \), is the subspace spanned by the set \( \{ADF\} \).
- The set of all acceptable degree three monomials that are not in \( C_3 \) is spanned by

\[
C'_3 = \{AE^2, B^2F, BEC, C^2D\}.
\]

- For \( \mu_1 = C^2D \), \( f_{\mu_1} = C^2D + 2ADF = D(C^2 + 2AF) \in S^*_1V \), where \( ADF \in A_{\mu_1} \).
- For \( \mu_2 = BCE \), \( f_{\mu_2} = BEC + AE^2 = E(BC + AE) \in S^*_1V \), where \( AE^2 \in A_{\mu_2} \).
- For \( \mu_3 = B^2F \), \( f_{\mu_3} = B^2F + 2ADF = F(B^2 + 2AD) \in S^*_1V \), where \( ADF \in A_{\mu_3} \).
- For \( \mu_4 = AE^2 \), \( f_{\mu_4} = A(E^2 + 2DF) \in S^*_1V \), where \( ADF \in A_{\mu_4} \).

Hence each acceptable non-Conca monomial of degree three is the initial monomial (in the reverse Lex. order) of an element of \( S^*_1V \).
Corollary 3.2.10. For $1 \leq k \leq n$ we have

$$(V)_k = \text{Ann}(\text{det}(X)) \cap S_k^s.$$  

We also have $(V)_{n+1} = S_{n+1}^s$.

Proof. By Lemmas 3.2.5 and 3.2.3 we have

$$S_{k-2}^s V = (V)_k \subset \text{Ann}(\text{det}(X)) \cap S_k^s.$$  

By Remark 1.0.4 and Lemma 3.2.3 we have

$$(\text{Ann}(\text{det}(X)))_k = (\text{Ann}(S_{n-k}^s \circ \text{det}(X)))_k = (\text{Ann}(M_k(X)))_k$$  

So we have

$$\dim S_{k-2}^s V \leq \dim(\text{Ann}(\text{det}(X)) \cap S_k^s) = \dim S_k^s - \dim M_k(X).$$  

On the other hand, by definition the sets $U_k$ and $C_k'$ are linearly independent and form a basis for the corresponding subspaces. Hence by Lemma 3.2.7 and Proposition 3.2.8 we have

$$\dim S_k^s - \dim M_k(X) = \dim < C_k'> + \dim < U_k > \leq \dim S_{k-2}^s V.$$  

So we have

$$\dim(V)_k = \dim S_{k-2}^s V = \dim S_k^s - \dim M_k(X) = \dim(\text{Ann}(\text{det}(X)) \cap S_k^s).$$

\[\Box\]

Theorem 3.2.11. Let $X$ be a generic symmetric $n \times n$ matrix. Then the apolar ideal $\text{Ann}(\text{det}(X))$ is the ideal $(V)$ and is generated in degree 2.

Proof. This follows directly from Lemma 3.2.7, Proposition 3.2.8 and Corollary 3.2.10.  

\[\Box\]
Proposition 3.2.12. The set $V$ is a Gröbner basis for the ideal $\text{Ann}(\det(X))$.

Proof. We have shown that $V$ generates $\text{Ann}(\det(X))$, and we use Buchberger’s Algorithm to show that $V$ is a Gröbner basis for the ideal $\text{Ann}(\det(X))$.

(1) Let $F$ and $G$ be two distinct permanents of $Y$ of type (c) in Proposition 3.2.1. Let $F = y_{ii}y_{jk} + y_{ik}y_{ji}$ and $G = y_{uu}y_{zu} + y_{uv}y_{zu}$.

$$F = \text{perm} \begin{pmatrix} y_{ii} & y_{ik} \\ y_{ji} & y_{jk} \end{pmatrix}.$$  

$$G = \text{perm} \begin{pmatrix} y_{uu} & y_{uv} \\ y_{zu} & y_{zu} \end{pmatrix}.$$  

Let $f_1 = y_{ii}y_{jk}$ be the leading term of $F$, and $g_1 = y_{uu}y_{zu}$ be the leading term of $G$ with respect to Conca monomial order. Denote the least common multiple of $f_1$ and $g_1$ by $h$. Then we have:

$$S(F, G) = \left(\frac{h}{f_1} \right) F - \left(\frac{h}{g_1} \right) G = y_{uu}y_{zu}y_{ik} y_{ji} - y_{ii}y_{jk}y_{uv}y_{zu}.$$

Now using the multivariate division algorithm, we reduce $S(F, G)$ relative to the set $V$. When there is no common factor in the initial terms of $F$ and $G$ the reduction is zero. First we reduce $S(F, G)$ dividing by $F$, so we will have

$$S(F, G) + y_{uv}y_{zu}F = y_{uu}y_{zu}y_{ik} y_{ji} + y_{uv}y_{zu}y_{ik} y_{ji}.$$  

Then we reduce the result using $G$ this time, so we will have

$$y_{uu}y_{zu}y_{ik} y_{ji} + y_{uv}y_{zu}y_{ik} y_{ji} - y_{ik} y_{ji} G = 0.$$  

So we have shown that for all pairs $F$ and $G$ of distinct permanents of $Y$ of type (c), the $S$-polynomials $S(F, G)$ reduce to zero with respect to $V$.

(2) Let $F = y_{ii}y_{jk} + y_{ik}y_{ji}$ and $G = y_{ii}y_{im} + y_{im}y_{li}$ be two permanents whose initial terms have a common factor. We have
\[ S(\mathcal{F}, \mathcal{G}) = y_{lm}y_{ik}y_{ji} - y_{jk}y_{im}y_{li}. \]

Without loss of generality we can restrict to a $5 \times 5$ symmetric submatrix. Note that in a $5 \times 5$ symmetric submatrix we can have two hafnians whose initial terms have one common factor, two permanents whose initial terms have a common factor, and a permanent and a hafnian whose initial terms have a common factor. Denote the $5 \times 5$ symmetric submatrix by

\[
\begin{pmatrix}
A & B & C & D & E \\
B & F & G & H & I \\
C & G & J & K & L \\
D & H & K & M & N \\
E & I & L & N & O
\end{pmatrix}
\]

Without loss of generality we consider the two permanents \( F = AG + BC \) and \( G = AN + DE \).

\[ S(\mathcal{F}, \mathcal{G}) = BCN - DEG. \]

We checked using the multivariate division algorithm in Macaulay 2 that the binomial \( BCN - DEG \) reduces to zero mod the initial set of generators \( V \).

Note that any two $4 \times 4$ hafnians with the same initial term are exactly the same. So for the hafnians it is enough to consider the \( S \)-polynomials of hafnians whose initial terms have only one common factor, and of the hafnians whose initial terms do not have a common factor. We should also consider the \( S \)-polynomials in the case that we have a hafnian and a permanent.

(3) Let \( \mathcal{F} \) and \( \mathcal{G} \) be two distinct hafnians of \( Y \) whose initial terms do not have a common factor. Without loss of generality we can restrict to a $5 \times 5$ symmetric matrix as in (2), and consider the two hafnians \( \mathcal{F} = HL + IK + GN \) and \( \mathcal{G} = DG + CH + BK \).

\[ S(\mathcal{F}, \mathcal{G}) = BKHL + BIK^2 - DG^2N - CGHN. \]

The multivariate division algorithm in Macaulay 2 shows that the \( S \)-polynomial \( BKHL + BIK^2 - DG^2N - CGHN \) reduces to zero.

(4) Let \( \mathcal{F} \) and \( \mathcal{G} \) be two distinct hafnians of \( Y \) whose initial terms have a common factor. Without loss of generality we can restrict to a $5 \times 5$ symmetric matrix as in (2),
Without loss of generality we consider the two hafnians $F = CH + DG + BK$ and $G = CI + EG + BL$.

\[ S(F, G) = CHL + DGL - CIK - EGK. \]

Using the multivariate division algorithm in Macaulay 2, it is easy to see that the polynomial, \( CHL + DGL - CIK - EGK \), reduces to zero. We also show the reduction process for this example directly. We want to reduce the polynomial \( S(F, G) \) using the set \( V \). The initial term for this polynomial is \( CHL \). So we should find elements of the set \( V \) other than \( F \) and \( G \) whose initial terms divide \( CHL \). We have the following three possibilities:

(a) The initial term is \( CH \). There is no permanent or hafnian with this initial term in the set \( V \).

(b) The initial term is \( CL \). The only element of the set \( V \) with this initial term is the permanent \( CL + EJ \)

(c) The initial term is \( HL \). There is no permanent or hafnian with this initial term in the set \( V \).

So we reduce \( S(F, G) \) using \( CL + EJ \), and we get

\[ S' = S(F, G) - H(CL + EJ) = -CIK + DGL - EGK - JHE. \]

Now the initial term of \( S' \) is \( CIK \), and we again do the reduction process. Here we have three different possibilities to choose an element from \( V \).

(a') The initial term is \( CI \). There is no permanent or hafnian with this initial term in the set \( V \).

(b') The initial term is \( CK \). The only element of the set \( V \) with this initial term is the permanent \( CK + DJ \)

(c') The initial term is \( IK \). There is no permanent or hafnian with this initial term in the set \( V \).

So we reduce \( S' \) using \( CK + DJ \), and we get

\[ S'' = S' - I(CK + DJ) = DGL - EGK - JHE + DIJ. \]

Again we look at the three different degree 2 monomials which divide the initial term of \( S'' \), we have
(a”) The initial term is $DG$. There is no permanent or hafnian with this initial term in the set $V$.

(b”) The initial term is $GL$. The only element of the set $V$ with this initial term is the permanent $GL + IJ$.

(c”) The initial term is $DL$. There is no permanent or hafnian with this initial term in the set $V$.

So we reduce $S''$ using $GL + IJ$, and we get

$$S''' = S'' - D(GL + IJ) = -E(GK + HJ) \in V.$$ 

So the $S$-polynomial can be reduced to zero using the set $V$.

(5) Let $F$ be a permanent and $G$ be a hafnian of $Y$ whose initial terms do not have a common factor. Without loss of generality we can restrict to a $5 \times 5$ symmetric matrix as in (2),

Without loss of generality we consider two permanents $F = 2FJ + G^2$ and $G = CI + EG + BL$.

$$S(F, G) = BLG^2 - 2FJCI - 2FJEG.$$ 

The multivariate division algorithm in Macaulay 2 shows that the $S$-polynomial, $BLG^2 - 2FJCI - 2FJEG$ can be reduced to zero using the set $V$.

(6) Let $F$ be a permanent and $G$ be a hafnian of $Y$ whose initial terms have a common factor. Without loss of generality we can restrict to a $5 \times 5$ symmetric matrix as in (2), and consider two permanents $F = BG + CF$ and $G = CI + EG + BL$.

$$S(F, G) = CFL - CIG - EG^2.$$ 

The multivariate division algorithm in Macaulay 2 shows that the $S$-polynomial, $CFL - CIG - EG^2$ reduces to zero.
3.2.2 Apolar ideal of the permanent

In this section we determine the apolar ideal of the permanent of the $n \times n$ generic symmetric matrix, and we will show that it is generated by degree two and degree three polynomials. We first determine the generators of degree two (Proposition 3.2.13). We then determine the degree three generators, which occur when $n \geq 6$ (Lemma 3.2.18). A key step is to use triangularity to show that these degree two and degree three generators, generate all of the apolar ideal (Lemma 3.2.19 and Proposition 3.2.21). This leads to our main result (Theorem 3.23).

Analogous to Proposition 3.2.1 we have:

**Proposition 3.2.13.** For an $n \times n$ symmetric matrix $X = (x_{ij})$, $\text{Ann}(\text{Perm}(X)) \subset S^*$, includes the following degree 2 polynomials:

(a) Unacceptable monomials including $y_{ii}y_{ij}$ for all $1 \leq i, j \leq n$. The number of these monomials are $n^2$.

(b) All the diagonal $2 \times 2$ minors with a coefficient 2 on the diagonal term, i.e. $y_{ij}^2 - 2y_{ii}y_{jj}$. The number of these binomials is $\binom{n}{2}$.

(c) All the $2 \times 2$ minors with one diagonal element, i.e. $y_{jk}y_{il} - y_{jl}y_{ii}$. The number of these binomials is $n \cdot \binom{n-1}{2}$.

**Proof.** We have $\text{Perm}(X) = \sum_{\sigma \in S_n^*} \Pi x_{i, \sigma(i)}$. First we show that monomials of type (a) are in $\text{Ann}(\text{Perm}(X))$. By symmetry we have

$$y_{ii}y_{ij} \circ \text{Perm}(X) = 0 (\text{where } j \geq i),$$

$$y_{ii}y_{ji} \circ \text{perm}(X) = 0 (\text{where } j \leq i).$$

Next we want to show that the binomials of type (b) are in $\text{Ann}(\text{perm}(X))$. Let $M = 2y_{ii}y_{jj} - y_{ij}^2$.

There are $n!$ terms in the expansion of the permanent. If a term, $t$, does not contain the monomial $x_{ii}x_{jj}$ or the monomial $x_{ij}^2$ then $M \circ t = 0$. Let $\sigma_1$ be a permutation having $x_{ii}$ and $x_{jj}$ respectively in it’s $i$-th and $j$-th place. Corresponding to $\sigma_1$ we also have a permutation $\sigma_2 = \tau \sigma_1$, where $\tau = (i, j)$ is a transposition. Thus the resulting action of the minor $M$ on $\text{perm}(X)$ is zero.
To show that the binomials of type (c) are in the annihilator ideal we can use a similar proof to that used for the binomials of type (b).

We denote by $\{W\}$ be set of the degree 2 elements of type (a), (b) and (c) in Proposition 3.2.13, and by $W$ the vector subspace of $S^s$ spanned by $\{W\}$. We denote by $\{a\}, \{b\},\text{and} \{c\}$ the set of elements in (a), (b),and (c) respectively.

Analogous to Lemma 3.2.2 we have:

**Lemma 3.2.14.** The set $\{W\}$ is linearly independent and we have,

$$\dim_k W = n^2 + \binom{n}{2} + n \cdot \binom{n-1}{2}.$$  

*Proof.* Each of the three subsets are linearly independent from each other since they involve different variables. So if we show that each subset is linearly independent we are done. The subset $\{a\}$ is linearly independent since the monomials in $\{a\}$ form a Gröbner basis for the ideal they generate. The subsets $\{b\}$ and $\{c\}$ are linearly independent since by choosing two elements of the matrix, where at least one element is diagonal, we have a unique $2 \times 2$ minor. Hence the set $\{W\}$ is linearly independent and the dimension of the vector space $W$ is $n^2 + \binom{n}{2} + n \cdot \binom{n-1}{2}$.

**Notation.** For a generic symmetric $n \times n$ matrix $X$, we denote by $P_k(X)$ the space of the permanents of all $k \times k$ submatrices of $X$.

Analogous to Lemma 3.2.3 we have:

**Lemma 3.2.15.** Let $1 \leq k \leq n$. We have

$$S_k^s \circ (\text{perm}(X)) = P_{n-k}(X) \subset R^s.$$  

*Proof.* To show the inclusion

$$S_k^s \circ (\text{perm}(X)) \subset P_{n-k}(X) \subset R^s,$$
we use induction on \( k \). Let \( P_{ij} \) denote the permanent of the sub matrix obtained by deleting the \( i \)-th row and \( j \)-th column. For \( k = 1 \), we have two different cases:

I) for a diagonal element \( y_{ii} \) we have

\[
y_{ii} \circ (\perm(X)) = P_{ii} \in P_{n-1}(X).
\]

II) Let \( y_{ij} \) be a non-diagonal element. Without loss of generality we can consider \( y_{12} \). We have

\[
y_{12} = y_{21}.
\]

The monomial \( y_{12}^2 \) appears in exactly \((n-2)!\) terms coming from \( y_{12}^2 \cdot (P_{12})_{21} \).

We also have \( 2((n-1)! - (n-2)!))\) terms in the permanent which contain \( y_{12} \) but do not contain \( y_{12}^2 \). These terms come from the sub-permanent obtained by deleting the first or second row.

So we have

\[
y_{ij} \circ (\perm(X)) = 2P_{ij} \in P_{n-1}(X).
\]

Assume that the above inclusion holds for \( k - 1 \), i.e

\[
S_k^s \circ (\perm(X)) \subset P_{n-(k-1)}(X),
\]

and we want to show that it is true for \( k \). We have

\[
S_k^s \circ \perm(X) = S_k^sS_{k-1}^s \circ \perm(X) \subset S_1^s \circ P_{n-k+1}(X) \subset P_{n-k}(X),
\]

as required. Next we want to show the opposite inclusion

\[
S_k^s \circ (\perm(X)) \supset P_{n-k}(X) \subset R_s,
\]

Let \( P_{\bar{I},\bar{J}}(X), I = \{i_1, \ldots, i_k\}, J = \{j_1, \ldots, j_k\}, 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n, 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n \) be the \((n-k) \times (n-k)\) permanent of the submatrix of \( X \) one obtains by deleting the \( I \) rows and \( J \) columns of \( X \). Let

\[
\Delta_{(I,J)} = \{(i_r, j_r) | i_r \in I, j_r \in J \text{ and } i_r = j_r\}.
\]

Let \( \Delta_I = \{i_r | (i_r, j_r) \in \Delta_{(I,J)} \} \) and \( \Delta_J = \{j_r | (i_r, j_r) \in \Delta_{(I,J)} \} \). Let \( P_{(I,J)-\Delta} \) be the sub matrix of \( Y \) with the rows \( I - \Delta_I \), and the columns \( J - \Delta_J \).
Claim:

\[ P_{\hat{I}, \hat{J}} = c \prod_{(i_r, j_r) \in \Delta(I, J)} y_{i_r j_r} \text{perm}(P_{(I, J) - \Delta}) \circ (\text{perm}(X)) \]

where \( c \neq 0 \in k \).

To prove the claim we use induction on \(|I| = |J| = k\), the cardinality of the sets \( I \) and \( J \). First we show the claim is true for \( k = 1 \). Let \( I = \{i_1\} \) and \( J = \{j_1\} \). We have two cases

I. \( i_1 = j_1 \) so \( y_{i_1 j_1} \) is a diagonal element and we have

\[ P_{\hat{I}, \hat{J}} = y_{i_1 j_1} \circ (\text{perm}(X)). \]

II. \( i_1 \neq j_1 \) so we have

\[ y_{i_1 j_1} \circ (\text{perm}(X)) = 2P_{\hat{I}, \hat{J}}. \]

So for \( k = 1 \) the claim holds. Assume that the claim holds for every \( I \) and \( J \) with \(|I| = |J| = k - 1\) and we want to how that the claim is also true for \( I \) and \( J \) with \(|I| = |J| = k\).

Let \( I = \{i_1, ..., i_k\} \) and \( J = \{j_1, ..., j_k\} \).

Let \( I' = I - \{i_1\} \) and \( J' = J - \{j_1\} \). We have \(|I'| - |J'| = k - 1\) so by the induction assumption we have

\[ P_{\hat{I'}, \hat{J'}} = c \prod_{(i_r, j_r) \in \Delta(I', J')} y_{i_r j_r} \text{perm}(P_{(I', J') - \Delta}) \circ (\text{perm}(X)) \]

Writing the Laplace expansion of the permanent using row \( i_1 \) or column \( j_1 \) for \( P_{\hat{I'}, \hat{J'}} \), we get

\[ P_{\hat{I}, \hat{J}} = c \prod_{(i_r, j_r) \in \Delta(I, J)} y_{i_r j_r} \text{perm}(P_{(I, J) - \Delta}) \circ (\text{perm}(X)), \]

where \( c \neq 0 \in k \). Hence \( P_{\hat{I}, \hat{J}} \in S_{n-k}^k \circ (\text{perm}(X)) \).

\[ \square \]
Lemma 3.2.16. Let $X$ be a generic symmetric matrix. We have

$$H(S^s/\text{Ann(Perm}(X)))_k = \frac{{n \choose k} \left( \frac{{n \choose k}}{2} + 1 \right)}{2}.$$ 

So the length $\dim S^s/\text{Ann(Perm}(X))$ satisfies the following equation

$$\dim S^s/\text{Ann(Perm}(X)) = \frac{(2^n) + 2^n}{2}.$$ 

Proof. Let $P_k$ denote the space of $k \times k$ permanents of the $n \times n$ generic symmetric matrix $X$. Using Lemma 3.2.15 we have

$$H(S^s/\text{Ann(Perm}(X)))_k = \dim_k P_k = \frac{{n \choose k} \left( \frac{{n \choose k}}{2} + 1 \right)}{2}.$$ 

Hence A combinatorial proof of the Equation 2.2 can be found in [ST], Example 1.1.17.

$$\dim S^s/\text{Ann(Perm}(X)) = \sum_{k=0}^{k=n} \frac{{n \choose k} \left( \frac{{n \choose k}}{2} + 1 \right)}{2} = \frac{(2^n) + 2^n}{2}. \quad (3.5)$$

A combinatorial proof of the Equation 3.5 can be found in [ST], Example 1.1.17.

Table 3.2: The Hilbert sequence of the Permanent of the generic symmetric matrix

<table>
<thead>
<tr>
<th>n=2</th>
<th>1</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=3</td>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>n=4</td>
<td>1</td>
<td>10</td>
<td>21</td>
</tr>
<tr>
<td>n=5</td>
<td>1</td>
<td>15</td>
<td>55</td>
</tr>
<tr>
<td>n=6</td>
<td>1</td>
<td>21</td>
<td>120</td>
</tr>
<tr>
<td>n=7</td>
<td>1</td>
<td>28</td>
<td>231</td>
</tr>
<tr>
<td>n=8</td>
<td>1</td>
<td>36</td>
<td>406</td>
</tr>
</tbody>
</table>

Analogous to Lemma 3.2.4 we have

Lemma 3.2.17. For the generic symmetric $n \times n$ matrix $X$, we have

$$W = \text{Ann}(P_2) \cap S^s_2 = (\text{Ann(perm}(X)))_2.$$
Proof. By the Lemma 3.2.15 we have

$$\text{Ann}(S_{n-2}^{s} \circ (\text{perm}(X))) = \text{Ann}(P_{2}(X)).$$

Remember $W$ consists of the degree two monomials. By the Proposition 3.2.13 we have

$$(\text{Ann}(\text{perm}(X)))_{2} \supset W$$

By the Remark 1.0.4 we have

$$(\text{Ann}(\text{perm}(X)))_{2} = (\text{Ann}(S_{n-2}^{s} \circ (\text{perm}(X))))_{2} \subset \text{Ann}(P_{2}(X))$$

Hence we have

$$W \subset \text{Ann}(P_{2}).$$

On the other hand, using Lemma 3.2.16 we have

$$\dim(W) = n^2 + \binom{n}{2} + n \cdot \left( \frac{n-1}{2} \right) = \left( \frac{n^2 + n}{2} + 1 \right) - \left( \frac{n}{2} \right) \left( \frac{n}{2} + 1 \right) = \dim S_{2}^{s} - \dim P_{2}(X).$$

So we have the equality

$$W = \text{Ann}(P_{2}) \cap S_{2}^{s}.$$
\[ W = (\text{Ann}(\text{perm}(X)))_2. \]

The apolar ideal of the permanent of the generic symmetric matrix is not generated in degree two in general. For \( n = 2, 3, 4, 5 \) the apolar ideal is generated in degree 2 with the generators \( W \) introduced in Proposition 3.2.13. We will show starting from \( n = 6 \) there are generators of degree 3 in the annihilator ideal (Lemma 3.2.18). Here, for the readers’ convenience we summarize the information/observation we have about these examples:

- For \( n = 2 \) we have

\[
X = \begin{pmatrix} a & b \\ b & c \end{pmatrix},
\]

so \( \text{perm}(X) = b^2 + ac \). The apolar ideal \( I = (C^2, BC, B^2 - 2AC, AB, A^2) \). The corresponding Hilbert sequence is \( H = (1, 3, 1) \)

- For \( n = 2, 3, 4, 5 \) the apolar ideal is generated by \( W \) in degree 2.

- We show that for \( n = 6, 7, 8 \) the apolar ideal has some degree 3 generators:

The terms \( y_{i_1i_2}y_{i_3i_4}y_{i_5i_6} \) that appear in the degree 3 polynomials of the apolar ideal for \( n = 6, 7, 8 \) follow these rules:

\[
i_1 \leq i_3 \leq i_5,
\]

\[
i_1 \leq i_2, i_3 \leq i_4, i_5 \leq i_6.
\]

So the terms that appear in the degree three generators of the apolar ideals are exactly the terms that appear in the hafnians of the \( 6 \times 6 \) symmetric sub-matrices.

- For each \( 6 \times 6 \) symmetric submatrix, we have five linearly independent homogeneous polynomials (forms) of degree three among the generators. Three of them have six terms and two of them have eight terms. So the number of degree three generators of an \( n \times n \) symmetric
matrix is equal to $5 \cdot \binom{n}{6}$. We write these five degree-three forms below for $n = 6$ in Lemma 3.2.18 that follows.

**Lemma 3.2.18.** Let $X$ be a generic symmetric $n \times n$ matrix. For each symmetric $6 \times 6$ submatrix of $X$ we have five minimal generators of degree three in the apolar ideal of the permanent as listed below:

$$
\begin{pmatrix}
  a & b & c & d & e & f \\
  b & g & h & i & j & k \\
  c & h & l & m & n & o \\
  d & i & m & p & q & r \\
  e & j & n & q & s & t \\
  f & k & o & r & t & u
\end{pmatrix},
$$

- $F_1 = EIO - DJO - EHR + CJR + DHT - CIT$
- $F_2 = DKN - DJO - CKQ + BOQ + CJR - BNR$
- $F_3 = FIN - DJO - FHQ + BOQ + CJR - BNR + DHT - CIT$
- $F_4 = EKM - DJO - CKQ + BOQ - EHR + CJR + DHT - BMT$
- $F_5 = FJM - DJO - FHQ + BOQ + DHT - BMT$

These $5 \cdot \binom{n}{6}$ polynomials of $Y$ annihilate the permanent of the matrix $X$.

**Proof.** We use induction on $n$. For $n = 6$ it is easy to check that $F_1, \ldots, F_5$ annihilate the permanent of $X$. So we assume that for all integer values less than $n$ we have that all the five polynomials coming from the symmetric $6 \times 6$ submatrices annihilate the permanent of $X$. We want to show this for $n$. Let $N$ be a $6 \times 6$ symmetric submatrix of $Y$, involving the rows and the columns $i_1, \ldots, i_6$. If a term in the permanent does not contain any of the 15 degree three monomials of the $6 \times 6$ hafnian then $F_1, \ldots, F_5$ annihilate it. Suppose a term in the permanent contains one of the fifteen monomials in the symmetric $6 \times 6$ hafnian. Since these monomials do not appear in any other $6 \times 6$ submatrix of $X$, by the first induction step we have shown that these monomials annihilate the permanent of $X$. These generators are linearly independent mod $(W)_3$, since they involve different variables.
Lemma 3.2.19. We can write each unacceptable monomial of degree \( k \) (\( 2 \leq k \leq n \)), as an explicit element of \( S_{k-2}^sW \), where \( W \) is the space defined in the Proposition 3.2.13.

Proof. We use induction on \( k \). For \( k = 2 \) the claim is obviously true. To show that the claim is true for \( k = 3 \), we need to show that the space \( U_3 \) of unacceptable monomials in \( S_3^s \) are in \( S_1^sW \). The unacceptable monomials of degree 3 for the \( n \times n \) generic symmetric matrix are of one of the following forms:

(a) Unacceptable of the form \( x^2y \) where \( x \) is a diagonal element. The number of these monomials is \( n\left(\frac{n(n+1)}{2}\right) \).

(b) Unacceptable of the form \( xyz \) where \( x \) is a diagonal element, \( y \neq x \) in the same row or column with \( x \) and \( z \neq x \). The number of these monomials is \( n(n-1)\left(\frac{n(n+1)}{2}\right) - 1 \).

(c) Unacceptable of the form \( xyz \) where \( x, y, z \) are non diagonal elements from the same row or column (can be equal to each other). The number of these monomials is \( \binom{n}{3}^{(n-1+3-1)} \).

Unacceptable monomials of type (a) or (b) are multiples of unacceptable monomials of degree 2, so they are in the space \( S_1^sU_2 \). So we only need to show that the degree 3 unacceptable monomials of type (c) are in \( S_1^sW \). The 3 nondiagonal elements in the same row or column of the matrix \( X \) are from a symmetric \( 4 \times 4 \) sub-matrix. So without loss of generality we show that a degree 3 monomial of type (c) from the following sub-matrix is in \( S_1^sW \). Let \( A^s \) be the \( 4 \times 4 \) symmetric sub-matrix of a generic symmetric \( n \times n \) matrix,

\[
A^s = \begin{pmatrix}
a & b & c & d \\
b & e & f & g \\
c & f & h & i \\
d & g & i & j \\
\end{pmatrix},
\]

and \( D^s \) be the matrix
Monomials of type (c) in $S_3^s$ can have one of the following forms. In each case we prove the claim for one monomial in the given form. The proof for any other monomial is similar to what we show below.

(I) All three non-diagonal variables are distinct. The monomials of the form $\eta_1 = BCD$ is a degree three unacceptable monomial of type (c). We have $AF - BC \in W$ (since it is a minor with one diagonal element), so we have $(AF - BC) \circ \text{perm}(A^s) = 0$. Hence,

$$D(AF - BC) \circ \text{perm}(A^s) = 0.$$  

We also know that $DAF \in S_1^sU_2 \subset U_3$ so $DAF \circ \text{perm}(A^s) = 0$. We therefore have $\eta_1 = BCD = -D(AF - BC)(\text{mod } S_1^sU_2)$. So we have $\eta_1 \in S_1^sW$.

(II) There are two distinct non-diagonal variables. Consider the monomial $\eta_2 = B^2C$, also of type (c). We have $AF - BC \in W$ (since it is a minor with one diagonal element), so we have $(AF - BC) \circ \text{perm}(A^s) = 0$. Hence,

$$B(AF - BC) \circ \text{perm}(A^s) = 0.$$  

We also know that $BAF \in S_1^sU_2 \subset U_3$ so $BAF \circ \text{perm}(A^s) = 0$. We therefore have $\eta_2 = B^2C = -B(AF - BC)(\text{mod } S_1^sU_2)$. So we have $\eta_2 \in S_1^sW$.

(III) There is only one non-diagonal variable. Consider the monomial $\eta_3 = B^3$, also of type (c). We have $-B^2 + 2AE \in W$ (since it is a diagonal minor with the coefficient 2), so we have $(-B^2 + 2AE) \circ \text{perm}(A^s) = 0$. Hence,

$$B(-B^2 + 2AE) \circ \text{perm}(A^s) = 0.$$  

We also know that $BAE \in S_1^sU_2 \subset U_3$ so $BAE \circ \text{perm}(A^s) = 0$. We therefore have $\eta_3 = B^3 = -B(-B^2 + 2AE)(\text{mod } S_1^sU_2)$. So we have $\eta_3 \in S_1^sV$.

So the lemma is true for $k = 3$ and we have $U_3 \subset S_1^sW$. Let $M$ denote the subspace of $V$
CHAPTER 3. GENERIC SYMMETRIC MATRIX

generated by binomials of type (b) and (c) as defined in Proposition 3.2.13. We have shown that $U_3 \subset S_1^s(U + M)$.

Finally assume that $k \geq 4$ and the lemma is true for all integers less than $k$. We want to show that the claim is true for $k$. Let $\mu = \mu_1 \mu_2 ... \mu_k$ be an unacceptable monomial of degree $k$. We can write $\mu$ such that $\mu_2 ... \mu_k$ is an unacceptable monomial of degree $k - 1$ so we have

$$\mu = \mu_1 (\mu_2 ... \mu_k) \in S_1^s(S_{k-3}^s W) = S_{k-2}^s W,$$

and the lemma is true also for $k$.

Definition 3.2.20. Let $H$ be the ideal of the degree three polynomials listed in the Lemma 3.2.18. Let $W^+ = W + H$ denote the ideal generated by the degree 2 polynomials defined in Proposition 3.2.13 and the degree 3 polynomials corresponding to the $6 \times 6$ symmetric submatrices discussed in the Lemma 3.2.18.

- The number of $k \times k$ permanents of the $n \times n$ generic symmetric matrix is

$$\frac{1}{2} \binom{n}{k} \cdot \binom{n}{k} + \frac{1}{2} \binom{n}{k},$$

choosing two from a subset of $\binom{n}{k}$ elements.

- Let $\{P_k\}$ be the set of all $k \times k$ permanents of $X$.

- As in the determinant case, let $A_k$ be the set of acceptable monomials in $S_k^s$.

- Let $\iota : R^s \rightarrow S^s, \iota(x_{ij}) = y_{ij}$, and $E_k$ be the subset of $A_k$ defined by

$$\{\iota(\mu)|\mu\text{ an initial monomial (in Lex. order) of some element of }\{P_k\}\}.$$

- Let $E'_k = A_k \setminus E_k$ be the complementary set to $E_k$ in $A_k$.

- For each $\mu \in A_k$, let $A_{\geq \mu}$, denote the subset of elements $\nu \in A_k$, such that $\nu > \mu$ in the lexicographic order of $S^s$.

- Let $[a_1, ..., a_k|b_1, ..., b_k]_\mu$ be the permanent of the $k \times k$ sub matrix with the rows $\{a_1, ..., a_k\}$ and the columns $\{b_1, ..., b_k\}$. Recall that for a monomial

$$m = y_{a_1 b_1} ... y_{a_k b_k} = (a_1, ..., a_k|b_1, ..., b_k),$$
we call a pair \((b_i, b_j)\), with \(i < j\), a reversal pair if \(b_i \geq b_j\). The initial term of the \(k \times k\) permanent \([a_1, ..., a_k | b_1, ..., b_k]_p\) is the term \(y_{i_1,j_1}y_{i_2,j_2}...y_{i_k,j_k}\) such that \(i_1 \leq i_2 \leq ... \leq i_k\) and \(j_1 \leq j_2 \leq ... \leq j_k\) where \(\{i_1, ..., i_k\} = \{a_1, ..., a_k\}\) and \(\{b_1, ..., b_k\} = \{j_1, ..., j_k\}\).

**Proposition 3.2.21.** Each acceptable monomial in \(E'_k\) (\(3 \leq k \leq n\)), is the initial monomial (in the reverse Lex. order) of an element of \(W^+_k\).

**Proof.** We use induction on \(k\), and start with \(k = 3\). Let \(\mu\) be a degree 3 acceptable monomial which is not the initial term of any \(3 \times 3\) permanent in the lexicographic order. The acceptable monomials, \(x_{i_1,i_2}x_{i_3,i_4}x_{i_5,i_6}\), of degree 3 for the \(n \times n\) generic symmetric matrices can be listed as follows:

(a) All 6 indices are distinct

(b) There is one repeated index

(c) There are 2 repeated indices

(d) There are 3 repeated indices

We discuss each of the above types separately in each case for one monomial of the given form. The proof for any other monomial of the given type is similar to what we show.

(a) all 6 indices are distinct \(x_{i_1,i_2}x_{i_3,i_4}x_{i_5,i_6}\), \((i_1, i_3, i_5 | i_2, i_4, i_6)\). Without loss of generality we can assume these indices are \(1,2,3,4,5,6\). In order to have a non-initial monomial of this kind, it is enough to have at least one reversal pair. For example in \((1,2,3|6,4,5)\), \(6 \geq 4\) so \((6,4)\) is a reversal pair.

In the doset minor \((i_1, i_3, i_5 | i_2, i_4, i_6)\), without loss of generality we may arrange that

\[i_1 < i_3 < i_5.\]

Now assume \(i_1 = 1, i_3 = 2, i_5 = 3\), \(i_2 = 4, i_4 = 5\) and \(i_6 = 6\) then \(x_{14}x_{25}x_{36}\) is the initial term in the corresponding \(3 \times 3\) permanent using the lexicographic order. So in order to have a non-initial monomial we need to assign to \(i_2, i_4\) and \(i_6\) the numbers 4, 5 and 6 but not in order. So we have at least one reversal pair \((i_j, i_k)\), \((j < k)\) where \(j, k \in \{2, 4, 6\}\) such that \(i_j \geq i_k\).
Next we look at the corresponding $6 \times 6$ symmetric sub-matrix. we have

$$X = \begin{pmatrix} a & b & c & d & e & f \\ b & g & h & i & j & k \\ c & h & l & m & n & o \\ d & i & m & p & q & r \\ e & j & n & q & s & t \\ f & k & o & r & t & u \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C & D & E & F \\ B & G & H & I & J & K \\ C & H & L & M & N & O \\ D & I & M & P & Q & R \\ E & J & N & Q & S & T \\ F & K & O & R & T & U \end{pmatrix},$$

Consider a degree three non-initial monomial, the terms coming from 6 distinct rows and columns. A general example of this kind is $(1, 2, 5|6, 4, 3)$. But we have:

$$FIN - DJO - FHQ + BOQ + CJR - BNR + DHT - CIT \in H,$$

so given $\mu = FIN$, we have

$$f_\mu = FIN - DJO - FHQ + BOQ + CJR - BNR + DHT - CIT \in W_3^+, $$

where

$$-DJO - FHQ + BOQ + CJR - BNR + DHT - CIT \in A_{\geq \mu}.$$

(b) There is one repeated index. Without loss of generality we can assume the indices are 1, 2, 3, 4, 5, with one of them repeated. In order to have a non-initial example of this kind, it is enough to have 1 reversal pair. For example in $(1, 2, 3|4, 1, 5)$, $4 \geq 1$. We can form a $5 \times 5$ symmetric matrix with these rows and columns,
Now consider the monomial $\mu = y_{14} y_{21} y_{35} = BDL$ as an acceptable monomial of type (b). We have the minor $AH - BD \in W$. Given $\mu = BDL$, $f_\mu = -L(AH - BD) \in S_1^W \subset W_3^+$, where $AHL \in A_{>\mu}$.

(c) There are 2 repeated indices, Without loss of generality we can assume the indices are 1,2,3,4, with two of them repeated. In order to have a non-initial example of this kind, it is enough to have 1 reversal pair. For example in $(1,2,3|2,1,4)$, $2 \geq 1$. We can form a $4 \times 4$ symmetric matrix with these rows and columns,

$$X = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix},$$

and consider the monomial $\mu = y_{12} y_{21} y_{34} = B^2I$ as an acceptable monomial of type (c). Given $\mu = B^2I$, $f_\mu = I(B^2 - 2AE) \in S_1^W \subset W_3^+$, where $AEI \in A_{>\mu}$.

(d) There are 3 repeated indices, Without loss of generality we can assume the indices are 1,2,3,
each of them repeated. In order to have a non-initial example of this kind, it is enough to have
1 reversal pair. For example in (1, 2, 3|3, 2, 1), in the second column of tableau 3 \( \geq 1 \). So we can
form a \( 3 \times 3 \) symmetric matrix with these rows and columns,

\[
X = \begin{pmatrix}
a & b & c \\
b & d & e \\
c & e & f \\
\end{pmatrix},
\]

\[
Y = \begin{pmatrix}
A & B & C \\
B & D & E \\
C & E & F \\
\end{pmatrix},
\]

and we look at the monomial \( \mu = y_{13}y_{22}y_{31} = C^2D \) as an acceptable monomial of type (c). Given
\( \mu = C^2D, f_\mu = -C(BE - CD) \in S^*_1W \subset W^+_3 \), where \( BEC \in A_{> \mu} \).

So for \( k=3 \) the claim is true, and assume that the Proposition 3.2.21 is true for all integers less than
\( k \). We have to show that the Proposition 3.2.21 is also true for \( k \). Let \( \mu = \mu_1\mu_2...\mu_k \) be a degree
\( k \) acceptable non-initial monomial, we can write \( \mu \) such that \( \mu_2...\mu_k \) is a degree \( k-1 \) acceptable
non-initial monomial (it is enough that the monomial \( \mu_2...\mu_k \) includes one reversal). Then by the
induction assumption \( \mu_2...\mu_k \) is the initial monomial (in rev. lex.) of an element of \( W^+_{k-1} \). So we
have \( \mu = \mu_1(\mu_2...\mu_k) \) as the initial monomial (in rev. lex.) of an element of \( W^+_k \). This completes
the proof.

\[ \blacksquare \]

**Corollary 3.2.22.** For \( 2 \leq k \leq n \) we have

\[
(W^+)_k = \Ann(\perm(X)) \cap S^*_k
\]

**Proof.** We have

(A) \( W^+ \circ \perm(X) = 0 \iff W^+ \circ S^*_{n-2} \circ (\perm(X)) = 0 \iff W^+ \circ P_2(X) = 0. \)

(B) \( (\Ann(\perm(X))) \cap S^*_2 = (W^+)_2 \) (By Lemma 3.2.17) \( \Rightarrow S^*_{k-2}(W^+) \circ (S^*_{n-k} \circ \perm(X)) = 0. \)
\( \Rightarrow S^*_{k-2}(W^+) \circ P_k(X) = 0. \)
\( \Rightarrow (W^+)_k \circ P_k(X) = 0. \) (By Remark 1.0.4)

Therefore
(W^+)_k \subset \text{Ann}(P_k(X)) \cap S^a_k.

By Remark 1.0.4 and Lemma 3.2.15 we have

\[(\text{Ann}(\text{perm}(X)))_k = (\text{Ann}(S^a_{n-k} \circ (\text{perm}(X)))_k = (\text{Ann}(P_k(X)))_k\]

So we have

\[\dim W^+_k \leq \dim(\text{Ann}(\text{perm}(X)) \cap S^a_k) = \dim S^a_k - \dim P_k(X).\]

On the other hand by the Definition 3.2.20 the sets \(U_k\) and \(E'_k\) are linearly independent and form a basis for the corresponding subspaces. So by Lemma 3.2.19 and Proposition 3.2.21, we have

\[\dim(W^+)_k \geq \dim S^a_k - \dim P_k(X) = \dim <E'_k> + \dim <U_k>.

So we have

\[\dim(W^+)_k = \dim S^a_k - \dim P_k(X) = \dim(\text{Ann}(\text{perm}(X)) \cap S^a_k).

\[\square\]

**Theorem 3.2.23.** Let \(X\) be a generic symmetric \(n \times n\) matrix. Then the apolar ideal \(\text{Ann}(\text{perm}(X))\) is the ideal \(W^+\) generated in degrees two and three.

**Proof.** This follows directly from Proposition 3.2.21 and Corollary 3.2.22. \[\square\]

### 3.3 Application to the ranks of the determinant and permanent of the generic symmetric matrix

In this section we apply our result from sections 3.1 and 3.2 to find some lower bounds for the the cactus rank and the rank of the determinant and permanent of the generic symmetric matrix.
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Using the Ranestad-Schreyer Proposition 2.3.4 and our results in 3.1 and 3.2, we have

**Theorem 3.3.1.** For the determinant of a generic symmetric $n \times n$ matrix $X$, we have

$$\frac{1}{2(n+2)} \binom{2n+2}{n+1} \leq cr(\det(X)) \leq sr(\det(X)) \leq r(\det(X)).$$

*Proof.* This follows directly from Propositions 2.3.3 and 2.3.4, Theorem 3.2.11 and Corollary 3.1.6.

**Theorem 3.3.2.** For the permanent of a generic symmetric $n \times n$ matrix $X$, we have

$$\frac{(2n)^n + 2^n}{6} \leq cr(\operatorname{perm}(X)) \leq sr(\operatorname{perm}(X)) \leq r(\operatorname{Perm}(X)).$$

*Proof.* This follows directly from Proposition 2.3.3 and 2.3.4, Theorem 3.2.23 and Table 3.2.

Using the Landsberg-Teitler Theorem 2.3.6 we have:

**Proposition 3.3.3.** Let $X$ be a generic symmetric $n \times n$ matrix. For each $t$, $1 \leq t \leq n$, we have

$$r(\det(X)) \geq \frac{(n+1)\binom{n+1}{t+1}}{n+1} + \frac{(n-t-1)(n-t)}{2} + (t+1)(n-t-1) + 1.$$  

The maximum of the right hand side of the above inequality occurs at $t = \lfloor n/2 \rfloor$.

*Proof.* By Lemma 3.1.5 the dimension of the space of $t \times t$ is the Narayana number $\frac{(n+1)\binom{n+1}{t+1}}{n+1}$. The determinant of $X$ vanishes to order $t+1$ if and only if every minor of $X$ of size $n-t$ vanishes. Thus $\Sigma_t(\det(A))$ is the locus of matrices of rank at most $n-t-1$ so the dimension of $\Sigma_t(\det(X))$ is $\frac{(n-t-1)(n-t)}{2} + (t+1)(n-t-1)$ ([BH], page 304). By unimodality of the binomial coefficients the first term of the right hand side is maximum at $t = \lfloor n/2 \rfloor$.

**Example 3.3.4.** Let $X$ be a $4 \times 4$ generic symmetric matrix. The Hilbert sequence will be $H = (1, 10, 20, 10, 1)$. Now using Theorem 3.3.1 we have:

$$cr(\det(X)) \geq \frac{1}{d} \deg(\text{Ann}(\det(A))) = \frac{1}{2}(42) = 21.$$
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Now using the Proposition 3.3.3 we have

\[
r(\det_4) \geq \frac{(\binom{4+1}{2})(\binom{4+1}{2+1})}{4+1} + \frac{(4-2-1)(4-2)}{2} + (2+1)(4-2-1) + 1 = 25.
\]

The following table gives the lower bounds for the rank and cactus rank of the determinant of an \( n \times n \) generic symmetric matrix \( X \), for \( 2 \leq n \leq 6 \), and also for \( n \gg 0 \) using the Stirling formula. Asymptotically the RS lower bound is \( \approx 2^{n+1} \) times that from LT.

Table 3.3: The determinant of the generic symmetric matrix

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
<th>( n \gg 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower bound for ( cr(\det(X)) ) using RS</td>
<td>2.5</td>
<td>7</td>
<td>21</td>
<td>66</td>
<td>209.5</td>
<td>( \frac{2^{n+1}}{(n+1)\sqrt{(n+1)\pi}} )</td>
</tr>
<tr>
<td>lower bound for ( r(\det(X)) ) using LT</td>
<td>4</td>
<td>7</td>
<td>25</td>
<td>56</td>
<td>187</td>
<td>( \frac{2^n}{n\sqrt{n\pi}} )</td>
</tr>
<tr>
<td>( l_{diff}(\det_n) )</td>
<td>3</td>
<td>6</td>
<td>20</td>
<td>50</td>
<td>175</td>
<td>( \left(\frac{n}{\lfloor n/2 \rfloor}\right)^2/\lfloor n/2 \rfloor )</td>
</tr>
</tbody>
</table>

The following table gives the lower bounds for the cactus rank of the permanent of an \( n \times n \) generic symmetric matrix \( X \), for \( 2 \leq n \leq 6 \), and also for \( n \gg 0 \) using the Stirling formula.

Table 3.4: The permanent of the generic symmetric matrix

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
<th>( n \gg 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower bound for ( cr(\perm(X)) ) using RS</td>
<td>1.6</td>
<td>4.6</td>
<td>14.3</td>
<td>47.3</td>
<td>164.6</td>
<td>( \frac{2^n(2^n+\sqrt{n\pi})}{6\sqrt{n\pi}} )</td>
</tr>
<tr>
<td>( l_{diff}(\perm(X)) )</td>
<td>3</td>
<td>6</td>
<td>21</td>
<td>55</td>
<td>210</td>
<td>( \frac{\sqrt{2}}{\sqrt{\pi n}}2^{n-1} )</td>
</tr>
</tbody>
</table>

Note that for \( n \leq 8 \) and \( n = 10 \) the \( l_{diff}(\perm(X)) \) is a larger lower bound. For \( n = 9 \) and \( n \geq 11 \) our result using RS is larger than the \( l_{diff}(\perm(X)) \).

3.4 Rank using contraction on the polynomial ring

In this section we use the contraction on the usual polynomial ring. This is an unusual choice and gives an answer that is less regular compared to section three. We first looked at this case, and realized in comparing our calculations with some kindly sent us by A. Conca for the annihilator of the determinant of a symmetric matrix, that they were different. What is happening here is that taking the contraction yields information about writing \( \det(X) \) as the sum of divided powers, not usual powers. Hence both the Hilbert function and generators of the apolar ideal are different.
for contraction versus differentiation. Throughout this section, $S^s = k[y_{ij}]$ acts on $R^s = k[x_{ij}]$ by contraction as follows:

$$(y_{ij})^k \circ_{\text{co}} (x_{uv})^\ell = \begin{cases} x_{uv}^{\ell-k} & \text{if } (i,j) = (u,v), \\ 0 & \text{otherwise.} \end{cases} \quad (3.6)$$

This action extends multilinearly to the action of $S^s$ on $R^s$.

**Notation.** In order to prevent confusion, we use the following notations in this section.

- We use contraction on usual polynomials (see 3.6).
- We use the notation $\circ_{\text{co}}$ for the contraction on the usual polynomials. We denote by $\text{Ann}_{\text{co}}(F)$ the apolar ideal to the usual polynomial $F$, using the contraction. We denote by $H_{\text{co}}(F)$ the corresponding Hilbert function, and $c_{\text{r,co}}, l_{\text{diff,co}},$ and $r_{\text{co}}$ the corresponding ranks.
- We denote by $F_{\text{div}}$ the divided power form of polynomial $F$ in the divided power ring $D$.

**Definition 3.4.1.** ([IK], page 267) Let $D$ be the divided power ring. Let $L = a_1 x_1 + \ldots + a_r x_r \in D_1$. The divided power $L[j]$ is defined as

$$L[j] = \sum_{j_1 + \ldots + j_r = j} a_1^{j_1} \ldots a_r^{j_r} x_1^{[j_1]} \ldots x_r^{[j_r]}.$$ 

The Hilbert function and generators of the apolar ideal of a polynomial are different when we use contraction versus differentiation. This can be seen with an easy example $(x + y)^2$ which is discussed in Example A. This example explains the difference between the following two cases:

I. The unusual case: We consider the polynomial $F$, in the usual polynomial ring $R^s$, and find the apolar ideal and the corresponding Hilbert function using contraction.

II. The usual case: We consider the polynomial $F_{\text{div}}$, in the divided power ring $D$, and find the apolar ideal and the corresponding Hilbert function using contraction; this gives us the same Hilbert function and apolar ideal as when we write the polynomial $F$ in the polynomial ring and use differentiation, provided the characteristic is at least degree$(F)$ or zero.
Example A. Let $R = k[x, y]$, and $D$ be its corresponding divided power ring. Let $F = (x + y)^2 = x^2 + 2xy + y^2 \in R$

I. The apolar ideal of $F = x^2 + 2xy + y^2 \in R$ using contraction is $I = (xy - 2y^2, x^2 - y^2)$, and the corresponding Hilbert function is $H_{co} = (1, 2, 1)$.

II. The apolar ideal of $F_{div} = 2x^{[2]} + 2xy + 2y^{[2]} \in D$ using contraction is $I = (x - y, y^3)$, and the corresponding Hilbert function is $H = (1, 1, 1)$.

Example 3.4.2. Let $A^s$ be the symmetric $2 \times 2$ matrix,

$$A^s = \begin{pmatrix} x & y \\ y & z \end{pmatrix}.$$

Let

$$D^s = \begin{pmatrix} X & Y \\ Y & Z \end{pmatrix}.$$

Let $S^s = k[X, Y, Z]$ act on $R^s = k[x, y, z]$ by contraction. Let

$$f = \det(A^s) = xz - y^2$$

and let $G = XZ + Y^2$ be the permanent of $D^s$. Then we have

$$G \circ_{co} f = (XZ + Y^2) \circ_{co} (xz - y^2) = 1 - 1 = 0.$$

Thus $G \in \text{Ann}_{co}(f)$.

In this case the number of $1 \times 1$ linearly independent minors are the number of variables which is 3. Hence the Hilbert sequence corresponding to that is $H_{co} = (1, 3, 1)$. Now using the Ranestad-Schreyer Proposition we have:

$$cr_{co}(\det(A^s)) \geq \frac{1}{d} \deg(\text{Ann}_{co}(\det(A^s))) = \frac{1}{2}(5).$$

Here the lower bound given by the Ranestad-Schreyer Proposition for $cr_{co}(\det(A^s))$ is the same as the lower bound for $cr(\det(A^s))$ (see Table 3.3).

In this example $l_{diff}(\det(A^s)) = 3$. 
On the other hand we have
\[ f = \det(A^s) = xz - y^2 = 1/2(x + z)^2 - 1/2(x - z)^2 - 1/2y^2. \]
so we have
\[ l_{diff,co}(\det(A^s)) = 3 \leq r(\det(A^s)) \leq 3. \]

So by Proposition 2.3.3 we have
\[ r_{co}(\det(A^s)) = sr_{co}(\det(A^s)) = cr_{co}(\det(A^s)) = l_{diff,co}(\det(A^s)) = 3. \]

**Example 3.4.3.** Let \( A^s \) be a symmetric \( 3 \times 3 \) matrix,
\[ A^s = \begin{pmatrix} x & y & u \\ y & z & v \\ u & v & w \end{pmatrix}. \]

Let
\[ D^s = \begin{pmatrix} X & Y & U \\ Y & Z & V \\ U & V & W \end{pmatrix}. \]

Here \( S^s = k[X, Y, U, Z, V, W] \) acts on \( R^s = k[x, y, u, z, v, w] \) by contraction. Let \( f = \det(A^s) = -u^2z + 2yuv - xv^2 - y^2w + xzw \). Let \( P_{ii} \) be the permanent corresponding to the entry \( d_{ii} \) of the matrix \( D^s \). We have
\[ P_{11} \circ_{co} f = (ZW + V^2) \circ_{co} (x(zw - v^2) - y(yw - uv) + u(yv - zu)) = x - x = 0. \]

Thus \( P_{11} \in \text{Ann}_{co}(f) \). It is easy to see that when \( n = 3 \), \( P_{ij} \circ_{co} f = 0 \) for each \( 1 \leq i, j \leq 3 \). So in the case \( n = 3 \) the annihilator of the determinant of a symmetric matrix certainly contains all its principal \( 2 \times 2 \) permanents of \( D^s \). Using Macaulay 2 for calculations we have:
\[ \text{Ann}_{co}(\det(A^s)) = (W^2, VW, UW, V^2 + ZW, ZV, UV + 2YW, Z^2, 2UZ + YV, YZ, U^2 + XW, YU + 2XV, Xu, Y^2 + XZ, XY, X^2) \]
In this case the number of $1 \times 1$ linearly independent minors are the number of variables which is 6. The number of $2 \times 2$ linearly independent minors is also 6. So the Hilbert sequence corresponding to that is $H_{co} = (1, 6, 6, 1)$, which is the same as what we had in the usual II case (see Table 3.1).

Using the Ranestad-Schreyer Proposition we have:

$$cr_{co}(\det(A^*)) \geq \frac{1}{d} \deg(\operatorname{Ann}_{co}(\det(A^*))) = \frac{1}{2}(14) = 7.$$ 

Here the lower bound given by the Ranestad-Schreyer Proposition for $cr_{co}(\det(A^*))$ is the same as the lower bound for $cr(\det(A^*))$ (see Table 3.3).

In this example $l_{diff,co}(\det(A^*)) = 6$.

On the other hand, [LT] table 1, describes the ranks of cubic forms as follows:

$$r_{co}(x^2y) = 3, r_{co}(xyz) = 4$$

So by the expansion of the determinant and the above equations we have

$$7 \leq cr_{co}(\det(A^*)) \leq r_{co}(\det(A^*)) \leq 17$$

By Landsberg-Teitler Proposition a lower bound for the rank is 10.

**Example 3.4.4.** When $A^*$ is a $4 \times 4$ symmetric matrix,

$$A^* = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix},$$

and

$$D^* = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}.$$

Using Macaulay 2 for calculations, we find that the corresponding Hilbert sequence will be $H_{co} =$
\( (1, 10, 33, 10, 1), \) and using the Ranestad-Schreyer Proposition we have:

\[
cr_{co}(\det(A^s)) \geq \frac{1}{d} \deg(\Ann_{co}(\det(A^s))) = \frac{1}{3}(55) = 18.33.
\]

This is the first example where the lower bound given by the Ranestad-Schreyer Proposition for \( cr_{co}(\det(A^s)) \) is smaller than the lower bound for \( cr(\det(A^s)) \) (see Table 3.3).

In this example \( l_{\text{diff},co}(\det(A^s)) = 33 \) and using the Landsberg-Teitler Proposition a lower bound for the rank of the determinant as a sum of divided powers is 38.

**Example 3.4.5.** When \( A^s \) is a 5 \( \times \) 5 symmetric matrix, the Hilbert sequence will be \( H_{co} = (1, 15, 85, 85, 15, 1) \) according to our calculations using Macaulay 2 and the maximum degree of the generators of \( \Ann_{co}(\det(A^s)) \) is 3. Here using the Ranestad-Schreyer Proposition we have:

\[
cr_{co}(\det(A^s)) \geq \frac{1}{d} \deg(\Ann_{co}(\det(A^s))) = \frac{1}{3}(202) = 67.33,
\]

and the lower bound given by the Ranestad-Schreyer Proposition for \( cr_{co}(\det(A^s)) \) is larger than the lower bound for \( cr(\det(A^s)) \) that is 66 (see Table 3.3).

In this example \( l_{\text{diff},co}(\det(A^s)) = 85 \) and using the Landsberg-Teitler Proposition a lower bound for the rank as a sum of divided powers is 95.

**Example 3.4.6.** When \( A^s \) is a 6 \( \times \) 6 symmetric matrix, the Hilbert sequence will be \( H_{co} = (1, 21, 180, 485, 180, 21, 1) \) according to our calculations using Macaulay 2 and the maximum degree of the generators of \( \Ann_{co}(\det(A^s)) \) is 4. Using the Ranestad-Schreyer Proposition we have:

\[
cr_{co}(\det(A^s)) \geq \frac{1}{d} \deg(\Ann_{co}(\det(A^s))) = \frac{1}{4}(889) = 222.25.
\]

Here the lower bound given by the Ranestad-Schreyer Proposition for \( cr_{co}(\det(A^s)) \) is larger than the lower bound for \( cr(\det(A^s)) \) that is 209.5 (see Table 3.3).

In this example \( l_{\text{diff},co}(\det(A^s)) = 485 \) and using the Landsberg-Teitler Proposition a lower bound for the rank as a sum of divided powers is 497.

**Example 3.4.7.** When \( A^s \) is a 7 \( \times \) 7 symmetric matrix, the Hilbert sequence will be \( H_{co} = (1, 28, 336, 1505, 1505, 336, 28, 1) \) according to our calculations using Macaulay 2 and the maximum
degree of the generators of \( \text{Ann}_{\text{co}}(\det(A^s)) \) is 4. Here using the Ranestad-Schreyer Proposition we have:

\[
\text{cr}_{\text{co}}(\det(A^s)) \geq \frac{1}{d} \deg(\text{Ann}_{\text{co}}(\det(A))) = \frac{1}{4}(3740) = 935,
\]

and the lower bound given by the Ranestad-Schreyer Proposition for \( \text{cr}_{\text{co}}(\det(A^s)) \) is larger than the lower bound for \( \text{cr}(\det(A^s)) \) that is 715.

In this example \( l_{\text{diff},\text{co}}(\det(A)) = 1505 \), and using the Landsberg-Teitler Proposition a lower bound for the rank as a sum of divided powers is 1524.

The next proposition concerns generators for the apolar ideal in the unusual setting of contraction acting on the usual determinant of \( A^s \).

**Proposition 3.4.8.** Let \( A^s \) be a generic symmetric \( n \times n \) matrix. For \( n > 3 \), the generators of degree 2 of the ideal \( \text{Ann}_{\text{co}}(\det(A^s)) \) are:

(a) \( x^2 \), where \( x \) is a diagonal element.

(b) \( xy \), where \( x \) is a diagonal element and \( y \) is in the same row or column as \( x \).

(c) all the diagonal \( 2 \times 2 \) permanents.

Therefore

\[
\dim_k(\text{Ann}_{\text{co}}(\det(A^s)))_2 = n^2 + \binom{n}{2}.
\]

**Proof.** Let \( A^s = (x_{ij}) \) and \( D^s = (y_{ij}) \) be symmetric matrices in \( R^s \) and \( S^s \) respectively, we have \( \det(A^s) = \sum_{\sigma \in S_n} \text{Sgn}(\sigma) \Pi x,_{\sigma(i)} \). First we show that the monomials of type (a) and (b) are in \( \text{Ann}(\det(A^s)) \). We have

\[
y_{ii}y_{ij} \circ_{\text{co}} \sum_{\sigma \in S_n} \text{Sgn}(\sigma) \Pi x,_{\sigma(i)} = 0 (\text{where } j \geq i)
\]

\[
y_{ii}y_{ji} \circ_{\text{co}} \sum_{\sigma \in S_n} \text{Sgn}(\sigma) \Pi x,_{\sigma(i)} = 0 (\text{where } j \leq i)
\]
Next we want to show that the binomials of type (c) are in Ann(det($A^s$)). Let 
\[ P = y_{ii}y_{jj} + y_{ij}^2, \]
be a $2 \times 2$ diagonal permanent. Assume that we have an arbitrary $2 \times 2$ diagonal permanent 
\[ P = y_{ii}y_{jj} + y_{ij}^2. \]

\[ P = \text{perm} \begin{pmatrix} y_{ii} & y_{ij} \\ y_{ij} & y_{jj} \end{pmatrix}, \]

\[ \det(A) = \sum_{\sigma \in S_n} Sgn(\sigma) \Pi x_{i,\sigma(i)}. \]

There are $n!$ terms in the expansion of the determinant. If a term doesn’t contain the monomial 
$x_{ii}x_{jj}$ or the monomial $x_{ij}^2$ then the result of action of the permanent $P$ on it will be zero. Let 
$\sigma_1$ be a permutation having $x_{ii}$ and $x_{jj}$ respectively, in it’s $i$-th and $j$-th place. Corresponding 
to $\sigma_1$ we also have a permutation $\sigma_2 = \tau \sigma_1$, where $\tau = (i, j)$ is a transposition and $sgn(\sigma_2) = 
sgn(\tau \sigma_1) = -sgn(\sigma_1)$. Thus corresponding to each positive term in the determinant which contains 
the monomial $x_{ii}x_{jj}$ or the monomial $x_{ij}^2$ we have the same term with the negative sign, thus the 
resulting action of the permanent $P$ on $\det(A^s)$ is zero.

Let $V$ be the space generated by forms of type (a), (b) and (c), we have shown that 
\[ V \subset \text{Ann}_{co}(\det(A^s))_2 = (\text{Ann}_{co}(S_{n-2}^{s} \circ_{co} (\det(A^s))))_2. \tag{3.7} \]

Let $W = V^\perp \subset S_{n}^{s}$. The space $W$ is generated by generated by:

- (a’) $y_{ij}y_{kl}$, where $i \neq j, k \neq l$ and $(i, j) \neq (k, l)$.
- (b’) $y_{ii}y_{jk}$, where $i \neq j, k$.
- (c’) $y_{ii}y_{jj} - y_{ij}^2$.

We want to show $W = S_{n-2}^{s} \circ_{co} (\det(A^s))$, we have 
\[ W \supset S_{n-2}^{s} \circ_{co} (\det(A^s)). \]

So we need to show that $W = V^\perp \subset S_{n-2}^{s} \circ_{co} (\det(A^s))$. 
By Laplace’s expansion theorem for determinant, we have
\[ y_{ii}y_{jj} - y_{ij}^2 \in S_{n-2} \circ co \left( \det(A^s) \right). \]

It is enough to act by the corresponding cofactor on the determinant to get \( y_{ii}y_{jj} - y_{ij}^2 \). Next we need to show that monomials of the form \((a')\) are also contained in the ideal \( S_{n-2} \circ co \left( \det(A^s) \right) \).

For \( y_{ij}y_{kl} \) where \( i, j, k \) and \( l \) are all distinct without loss of generality we can consider \( y_{12}y_{34} \). If we act by the permutation \( \rho = (1, i)(2, j)(3, k)(4, l) \in S_n \) on \( y_{ij}y_{kl} \) we have
\[ (1, i)(2, j)(3, k)(4, l) \circ co \; y_{ij}y_{kl} = y_{12}y_{34}. \]

If we consider the 4 × 4 case in the Example 3.4.4 we need to show \( BI \in S_2^s \circ co \left( \det(A^s) \right) \). Which by the expansion of determinant in the Example 3.4.4 is easy to see. We can extend this result to \( n \times n \) symmetric matrix.

If in \((a')\), \( i, j, k \) and \( l \) are not all distinct, then without loss of generality we can consider \( y_{12}y_{23} \), since we have:
\[ (1, i)(2, j)(3, k) \circ co \; y_{ij}y_{jk} = y_{12}y_{23}. \]

If we consider the 4 × 4 matrix in the Example 3.4.4 we need to show \( BF \in S_2^s \circ co \left( \det(A^s) \right) \). We can see this easily by using the expansion of the determinant as in the Example 3.4.4.

It is easy to see that monomials of type \((b')\) are also contained in \( S_{n-2}^s \circ co \left( \det(A^s) \right) \), using the same method as used for \((a')\).

Hence we have
\[ \dim_k W = \dim_k V^\perp = \dim_k (S_{n-2} \circ co \left( \det(A^s) \right)). \]

We also have
\[ \dim_k V + \dim_k V^\perp = \dim_k S_2^s. \]

Hence by Equation 3.7 we have
\[ V = \text{Ann}_{co}(\det(A^s))_2. \]

**Corollary 3.4.9.** Let \( A^s = (x_{ij}) \) be a generic symmetric matrix and \( D^s = (y_{ij}) \) be the corresponding symmetric matrix of differential operators. The only monomials of degree two and three which appear among the minimal generators of \( \text{Ann}_{co}(\det(A^s)) \) using contraction are:

(a) The monomials of degree two that are the product of a diagonal entry of \( D^s \) either with itself or with an entry in same row or column.

(b) The monomials of degree three that are the product of three distinct off diagonal entries of \( D^s \) which are in the same row or column.

**Proof.** From Proposition 3.4.8 we know that all the degree two monomials of type (a) are in \( \text{Ann}_{co}(\det(A^s)) \) and these monomials are the only monomials of degree two among the generators. So we want to show that the monomials of type (b) are contained in \( \text{Ann}_{co}(\det(A^s)) \). Let \( y_{ij}y_{ik}y_{il} \) be an arbitrary monomial of type (b). Since \( D^s \) is symmetric we have \( y_{ij} = y_{ji}, y_{ik} = y_{ki} \) and \( y_{il} = y_{li} \). We know that in the expansion of the determinant of \( A^s \) we cannot have a term which has two elements from the same row or column. So a monomial term \( x_{ij} \) and \( x_{ki} \) it cannot contain \( x_{il} = x_{li} \). Since \( x_{il} = x_{li} \) occurs only for two entries of the matrix, so it should be either in the same row with \( x_{ij} \) or in the same column with \( x_{ki} \), which is not possible. Hence we have

\[ y_{ij}y_{ik}y_{il} \circ_{co} \sum_{\sigma \in S_n} \text{Sgn}(\sigma) \Pi x_{i,\sigma(i)} = 0. \]

Next we show that there is no other monomial among the generators of \( \text{Ann}_{co}(\det(A^s)) \). Let \( \mu \) be an arbitrary monomial of degree three or higher not of type (b) but among the generators of \( \text{Ann}_{co}(\det(A^s)) \), so we should have

\[ \mu \circ_{co} \sum_{\sigma \in S_n} \text{Sgn}(\sigma) \Pi x_{i,\sigma(i)} = 0 \]

If \( \mu \) is a monomial in \( \text{Ann}_{co}(\text{Det}(A^s)) \) of degree three that is not of type (b) nor a multiple of a monomial of type (a) then we claim \( \mu \) must have one of the following forms:
(a') \( y_{ij}y_{kl}y_{mn} \), where \( i, j, k, l, m \) and \( n \) are all distinct.

(b') \( y_{ij}y_{ik}y_{lm} \), where \( i, j, k, l \) and \( m \) are distinct

(c') \( y_{ii}y_{jj}y_{kk} \), where \( i, j \) and \( k \) are distinct.

(d') \( y_{ii}y_{jj}y_{kl} \), where \( i, j, k \) and \( l \) are distinct.

(e') \( y_{ii}y_{jk}y_{jl} \), where \( i, j, k \) and \( l \) are distinct.

(f') \( y_{ii}y_{jk}y_{lm} \), where \( i, j, k, l \) and \( m \) are distinct.

Using the expansion of the determinant for \( n \geq 4 \) it is easy to see that none of the above forms can annihilate the determinant, and we are done.

\[\square\]

**Remark 3.4.10.** In the Examples 3.4.2-3.4.7, using Macaulay 2 for calculations we have, the Hilbert function for \( S^s/\text{Ann}_{co}(\text{Per}(A^s)) \) is the same as the Hilbert function of the apolar algebra \( S^s/\text{Ann}_{co}(\text{det}(A^s)) \). Comparing the degree 2 generators we have:

For \( n > 3 \), the generators of degree 2 of the ideal \( \text{Ann}_{co}(\text{Per}(A^s)) \) are:

(a) \( x^2 \), where \( x \) is a diagonal element,

(b) \( xy \), where \( x \) is a diagonal element and \( y \) is in the same row or column as \( x \),

(c) all the diagonal \( 2 \times 2 \) minors,

and the only degree 3 monomials that appear among the minimal generators of the apolar algebra \( S^s/\text{Ann}_{co}(\text{Per}(A^s)) \) are the product of 3 distinct off diagonal entries of \( D^s \) which are in the same row or column.

In the following tables we also summarize the information about the Hilbert sequence of the apolar algebra of the determinant and permanent of the generic symmetric matrix in the unusual setting, acting by contraction on the polynomial ring. For a comparison to the usual case see Tables 3.1 and 3.2. Note that in Table 3.5, polynomial \( F \) is the determinant or permanent of the generic symmetric matrix.

In the following tables we summarize the information about the lower bound for the length of the determinant and permanent of the generic symmetric matrix as a divided power sum. For a
Table 3.5: The Hilbert sequence corresponding to $F$, unusual contraction

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 2$</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>1</td>
<td>10</td>
<td>33</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>1</td>
<td>15</td>
<td>85</td>
</tr>
<tr>
<td>$n = 6$</td>
<td>1</td>
<td>21</td>
<td>180</td>
</tr>
<tr>
<td>$n = 7$</td>
<td>1</td>
<td>28</td>
<td>336</td>
</tr>
</tbody>
</table>

Comparison to the usual case see Tables 3.3 and 3.4. Note that in Table 3.6, polynomial $F$ is the determinant or permanent of the generic symmetric matrix.

Table 3.6: The ranks of determinant and permanent of $X$, unusual contraction

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower bound for $cr_{co}(F)$ using RS</td>
<td>2.5</td>
<td>7</td>
<td>18.33</td>
<td>67.33</td>
<td>222.25</td>
<td>935</td>
</tr>
<tr>
<td>lower bound for $r_{co}(F)$ using LT</td>
<td>3</td>
<td>10</td>
<td>38</td>
<td>95</td>
<td>497</td>
<td>1524</td>
</tr>
<tr>
<td>$l_{diff,co}(F)$</td>
<td>3</td>
<td>6</td>
<td>33</td>
<td>85</td>
<td>485</td>
<td>1505</td>
</tr>
</tbody>
</table>
Chapter 4

Annihilator of the Pfaffian and hafnian

In this section we discuss the annihilator ideals of the Pfaffians and of the hafnians. We show that the annihilator ideal of the Pfaffian of a generic skew symmetric $2n \times 2n$ matrix and the annihilator ideal of the hafnian of generic symmetric $2n \times 2n$ matrix are both generated in degree 2.

In the following discussion we let $X_{sk}^{m} = (x_{ij})$ with $x_{ij} = -x_{ji}$ be an $m \times m$ skew symmetric matrix of indeterminates in the polynomial ring $R^{sk} = k[x_{ij}]$. Let $Y_{sk}^{m} = (y_{ij})$ with $y_{ij} = -y_{ji}$ be an $m \times m$ skew symmetric matrix of indeterminates in the ring of differential operators $S^{sk} = k[y_{ij}]$. We denote the Pfaffian of the matrix $X_{sk}^{m}$ by $Pf(X_{sk}^{m})$. It is well known that for any odd number $m$ we have $\det(X_{sk}^{m}) = 0$. It is also well known that the square of the Pfaffian is equal to the determinant of a skew symmetric matrix. So in the following we are going to consider the annihilator of the Pfaffian of generic $m \times m$ skew symmetric matrices, where $m = 2n$ is an even number. Recall that

**Notation.** Let $F_{2n} \subset S_{2n}$ be the set of all permutations $\sigma$ satisfying the following conditions:

1. $\sigma(1) < \sigma(3) < ... < \sigma(2n - 1)$
2. $\sigma(2i - 1) < \sigma(2i)$ for all $1 \leq i \leq n$

- For a $2n \times 2n$ generic skew symmetric matrix $X^{sk}$, we denote by $Pf(X^{sk})$ the Pfaffian of $X^{sk}$ defined by
\[ Pf(X^s) = \sum_{\sigma \in F_{2n}} sgn(\sigma)x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}\cdots x_{\sigma(2n-1)}x_{\sigma(2n)} \] (4.1)

- (IKO) We denote by \( Hf(X^s) \) the hafnian of a generic symmetric \( 2n \times 2n \) matrix \( X^s \) defined by

\[ Hf(X^s) = \sum_{\sigma \in F_{2n}} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}\cdots x_{\sigma(2n-1)}x_{\sigma(2n)} \] (4.2)

Let \( J_{2n} = \text{Ann}(Pf(X_{2n}^{sk})) \). We first give some examples and then some partial results concerning \( \text{Ann}(Pf(X_{2n}^{sk})) \). Using Macaulay2 for calculations we have the following results:

(a) Let \( X_2 \) be a generic skew symmetric \( 2 \times 2 \) matrix, then we have \( H(S^{sk}/J_2) = (1, 1) \). And the maximum degree of the generators of the annihilator ideal \( J_2 \) is 2. So using the Ranestad-Schreyer Proposition we have:

\[ cr(Pf(X_2^{sk})) \geq \frac{1}{d} \deg(\text{Ann}(Pf(X_2^{sk}))) = \frac{1}{2}(2) = 1, \]

which is the same as the differential length in this case. Evidently, in this case \( r(Pf(X_2^{sk}) = 1 \), so we have

\[ r(Pf(X_2^{sk}) = cr(Pf(X_2^{sk}) = sr(Pf(X_2^{sk}) = l_{diff}(Pf(X_2^{sk}) = 1. \]

(b) Let \( X_4 \) be a generic skew symmetric \( 4 \times 4 \) matrix. Using Macaulay2 for calculations we have \( H(S^{sk}/J_4) = (1, 6, 1) \), and the maximum degree of the generators of the annihilator ideal \( J_4 \) is 2. Using the Ranestad-Schreyer Proposition we have:

\[ cr(Pf(X_4^{sk}) \geq \frac{1}{d} \deg(\text{Ann}(Pf(X_4^{sk}))) = \frac{1}{2}(8) = 4, \]

which is less than \( l_{diff} = 6. \)

(c) Let \( X_6 \) be a generic skew symmetric \( 6 \times 6 \) matrix. Using Macaulay2 for calculations we have \( H(S^{sk}/J_6) = (1, 15, 15, 1) \), and the maximum degree of the generators of the annihilator ideal \( J_6 \) is 2. Using the Ranestad-Schreyer Proposition we have:

\[ cr(Pf(X_6^{sk}) \geq \frac{1}{d} \deg(\text{Ann}(Pf(X_6^{sk}))) = \frac{1}{2}(32) = 16, \]
which is larger than $l_{\text{diff}} = 15$.

(d) Let $X_8$ be a generic skew symmetric $8 \times 8$ matrix. Using Macaulay2 for calculations we have $H(S^{sk}/J_8) = (1, 28, 70, 28, 1)$, and the maximum degree of the generators of the annihilator ideal $J_8$ is 2. From the Ranestad-Schreyer Proposition we have:

$$cr(Pf(X_8^{sk})) \geq \frac{1}{d} \deg(\text{Ann}(Pf(X_8^{sk}))) = \frac{1}{2}(128) = 64,$$

which is less than $l_{\text{diff}} = 70$.

(e) Let $X_{10}$ be a generic skew symmetric $10 \times 10$ matrix. Using Macaulay2 for calculations we have

$$H(S^{sk}/J_{10}) = (1, 45, 210, 210, 45, 1).$$

The maximum degree of the generators of the annihilator ideal $J_{10}$ is 2. From the Ranestad-Schreyer Proposition we have:

$$cr(Pf(X_{10}^{sk})) \geq \frac{1}{d} \deg(\text{Ann}(Pf(X_{10}^{sk}))) = \frac{1}{2}(512) = 256,$$

which is larger than $l_{\text{diff}} = 210$.

**Remark 4.0.11.** The Hilbert sequence for the apolar algebra of the Pfaffian of a generic $2n \times 2n$ matrix is given by $\binom{2n}{2t}$, and we have $\sum_{t=0}^{t=n} \binom{2n}{2t} = 2^{2n-1}$.

**Definition 4.0.12.** A $2t$-Pfaffian minor of a skew symmetric matrix $X$ is a Pfaffian of a submatrix of $X$ consisting of rows and columns indexed by $i_1, i_2, ..., i_{2t}$ for some $i_1 < i_2 < ... < i_{2t}$.

The number of $2t$-Pfaffian minors of a $2n \times 2n$ skew symmetric matrix is clearly $\binom{2n}{2t}$. We denote by $\{P_{2t}(X^{sk})\}$ the set of the $2t$-Pfaffians of $X^{sk}$. Furthermore, we denote by $P_{2t}(X^{sk})$ the vector space generated by $\{P_{2t}(X^{sk})\}$ in $R^{sk}_t$ and we denote by $(P_{2t}(X^{sk}))$ the ideal generated by $\{P_{2t}(X^{sk})\}$ in $R^{sk}$. Let $\tau$ be the lexicographic term order on $R^{sk} = k[x_{ij}]$ induced by the following order on the indeterminates:

$$x_{1,2n} \geq x_{1,2n-1} \geq ... \geq x_{1,2} \geq x_{2,2n} \geq x_{2,2n-1} \geq ... \geq x_{2n-1,2n}.$$
Theorem 4.0.13. (Herzog-Trung [HT], Theorem 4.1) The set \( \{ P_{2t}(X) \} \) of the \( 2t \)-Pfaffians of the matrix \( X^{sk} \) is a Gröbner basis of the ideal \( (P_{2t}(X)) \) with respect to \( \tau \).

Corollary 4.0.14. The dimension of the space of \( 2t \times 2t \) Pfaffians of a \( 2n \times 2n \) generic skew symmetric matrix \( X^{sk} \) is \( \binom{2n}{2t} \). So we have

\[
\dim(S^{sk}/\text{Ann}(Pf(X^{sk}))) = 2^{2n-1}.
\]

Proof. The proof follows directly from the Theorem 4.0.13 and the combinatorial identity:

\[
\sum_{t=0}^{t=n} \binom{2n}{2t} = 2^{2n-1}.
\]

This identity is easy to show; e.g., it follows immediately by evaluating at \( x = 1 \) and \( x = -1 \) the binomial expansion of \((x + 1)^{2n}\).

The examples strongly suggest that the apolar ideal of the Pfaffian is generated in degree 2. In the remaining part of this section we prove that this is always the case.

Definition 4.0.15. Let \( W \) be the vector subspace of \( S^{sk} \) spanned by degree 2 elements of type (a), (b) and (c) defined as follows

(a) square of each element of \( Y^{sk} \). The number of these monomials is \( 2n^2 - n \).

(b) product of each element of \( Y^{sk} \) with another element in the same row or column of the matrix \( Y^{sk} \). The number these monomials is \((2n^2 - n)(2n - 2)\).

(c) Given any \( 4 \times 4 \) submatrix of \( X^{sk} \) of the rows and columns \( i_1, i_2, i_3 \) and \( i_4 \),

\[
Q = \begin{pmatrix}
0 & x_{i_1i_2} & x_{i_1i_3} & x_{i_1i_4} \\
-x_{i_1i_2} & 0 & x_{i_2i_3} & x_{i_2i_4} \\
-x_{i_1i_3} & -x_{i_2i_3} & 0 & x_{i_3i_4} \\
-x_{i_1i_4} & -x_{i_2i_4} & -x_{i_3i_4} & 0
\end{pmatrix},
\]

we have \( Pf(Q) = x_{i_1i_2}x_{i_3i_4} - x_{i_1i_3}x_{i_2i_4} + x_{i_1i_4}x_{i_2i_3} \). Corresponding to \( Pf(Q) \) we have 3 binomials which annihilate \( Pf(Q) \) hence annihilate \( Pf(X^{sk}) \). These binomials are \( y_{i_1i_2}y_{i_3i_4} + y_{i_1i_3}y_{i_2i_4} \),
\[ y_{i_1i_2}y_{i_3i_4} - y_{i_1i_4}y_{i_2i_3} \text{ and } y_{i_1i_3}y_{i_2i_4} + y_{i_1i_4}y_{i_2i_3}. \] However these three binomials are not linearly independent, and we can write one of them as the sum of the other 2 binomials. So corresponding to each \(4 \times 4\) Pfaffian we have 2 linearly independent binomials in the annihilator ideal, and using Theorem 4.0.13, the number of these binomials is \(2 \cdot \left(\begin{smallmatrix} 2n \\ 4 \end{smallmatrix}\right)\).

**Remark.** For a \(2n \times 2n\) skew symmetric matrix \(X^{sk}\), we have \(W \subset \text{Ann}(Pf(X^{sk}))\).

**Lemma 4.0.16.** For the generic skew symmetric \(2n \times 2n\) matrix \(X^{sk}\), we have

\[ W = \text{Ann}(P_4(X^{sk})) \cap S_{2}^{sk}. \]

**Proof.** The monomials of type (a) and (b) correspond to unacceptable monomials discussed earlier and are linearly independent from any binomial in (c). The binomials in (c) are linearly independent by Theorem 4.0.13. Hence we have

\[ \dim_k(W) = 2\left(\begin{smallmatrix} 2n \\ 4 \end{smallmatrix}\right) + (2n^2 - n)(2n - 2) + 2n^2 - n = \left(\begin{smallmatrix} 2n^2 - n + 1 \\ 2 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 2n \\ 4 \end{smallmatrix}\right). \] (4.3)

According to Remark 1.0.4 we have

\[ \dim_k(\text{Ann}(P_4(X^{sk})) \cap S_{2}^{sk}) = \dim_k S_{2}^{sk} - \dim_k(P_4(X^{sk})). \]

So we have

\[ \dim_k(\text{Ann}(P_4(X^{sk})) \cap S_{2}^{sk}) = \left(\begin{smallmatrix} 2n^2 - n + 1 \\ 2 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 2n \\ 4 \end{smallmatrix}\right). \] (4.4)

Using Equations 4.3 and 4.4 we obtain

\[ \dim_k(W) = \dim_k(\text{Ann}(P_4(X^{sk})) \cap S_{2}^{sk}). \] (4.5)

On the other hand, evidently we have

\[ W \subset \text{Ann}(P_4(X^{sk})) \cap S_{2}^{sk}. \] (4.6)
CHAPTER 4. ANNIHILATOR OF THE PFAFFIAN AND HAFNIAN

Using Equations 4.5 and 4.6 we have

$$W = \text{Ann}(P_4(X^{sk})) \cap S^2_{sk}.$$ 

\[ \square \]

**Lemma 4.0.17.** Let $X^{sk}$ be a $2n \times 2n$ skew symmetric matrix ($n \geq 2$). We have,

$$S_{n-2} \circ Pf(X^{sk}) = P_4(X^{sk}) \subset P_2^{sk}.$$ 

*Proof.* First we show

$$S_{n-2} \circ Pf(X^{sk}) \supset P_4(X^{sk}). \quad (4.7)$$

We use induction on the size of the matrix.

The first step is $2n = 6$. We denote by $f = [i_1, i_2, i_3, i_4] \in P_4(X^{sk})$ the Pfaffian of the sub matrix with the rows and columns $i_1, i_2, i_3$ and $i_4$. We have $\binom{6}{4} = 15$ choices for $f$. For any of these choices we get the Pfaffian of a $2 \times 2$ sub matrix of the form

$$\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix},$$

as the coefficient of $f$ in the Pfaffian of the matrix $X^{sk}$. So if we differentiate the $6 \times 6$ Pfaffian with respect to that variable $x$, we get the $4 \times 4$ Pfaffian $f = [i_1, i_2, i_3, i_4]$.

Assume that Equation 4.7 holds for the generic skew symmetric $(2n - 2) \times (2n - 2)$ matrix. We want to show it holds for the $2n \times 2n$ generic skew symmetric matrix. The Pfaffian of the skew symmetric $2n \times 2n$ matrix $X^{sk}$ can be computed recursively as

$$Pf(X^{sk}) = \sum_{i=2}^{i=2n} (-1)^{i} x_{ii}^{sk} Pf(X_{ii}^{sk}), \quad (4.8)$$

where $X_{ii}^{sk}$ denotes the matrix $X^{sk}$ with both the first and the $i$-th rows and columns removed. So $X_{ii}^{sk}$ is a $(2n - 2) \times (2n - 2)$ matrix and Equation 4.7 holds for it. So for each choice of $[i_1, i_2, i_3, i_4]$ of the matrix $X_{ii}^{sk}$ we can find $n - 3$ variables of $X_{ii}^{sk}$ such that differentiating $Pf(X_{ii}^{sk})$ with respect to those variables gives us $[i_1, i_2, i_3, i_4]$. If we call those variable $a_1, ..., a_{n-3}$, then using Equation 4.8 if we add $x_{ii}^{sk}$ to our set of $n - 3$ variables we will have a set of $n - 2$ variables such that differentiating
\( Pf(X^{sk}) \) with respect to those \( n - 2 \) variables we will get \([i_1, i_2, i_3, i_4]\). Since we could write the recursive formula for the Pfaffian with respect to any other row or column, the result follows.

For the opposite inclusion to Equation 4.7 we have

\[
W \subset (\text{Ann}(Pf(X^{sk})))_2 \subset (\text{Ann}(P_4(X^{sk})))_2.
\]

But we have shown in Lemma 4.0.16 that

\[
W = (\text{Ann}(P_4(X^{sk})))_2.
\]

So we have

\[
(\text{Ann}(Pf(X^{sk})))_2 = (\text{Ann}(P_4(X^{sk})))_2.
\]

By Remark 1.0.4 we have

\[
(\text{Ann}(Pf(X^{sk})))_2 = \text{Ann}(S_{n-2} \circ (Pf(X^{sk}))).
\]

Hence we have

\[
S_{n-2} \circ Pf(X^{sk}) = P_4(X^{sk}).
\]

Recall that we denote by \( P_{2k}(X^{sk}) \) the vector subspace of \( R^{sk} \) spanned by the \( 2k \)–Pfaffian minors of \( X^{sk} \) [Definition 4.0.12].

**Lemma 4.0.18.** For \( 1 \leq k \leq n - 1 \) we have

\[
S_k \circ (Pf(X^{sk})) = P_{2n-2k}(X^{sk}). \tag{4.9}
\]

**Proof.** First we want to show

\[
S_k \circ (Pf(X^{sk})) \subset P_{2n-2k}(X^{sk}).
\]
We use induction on $k$. For $k = 1$, we need to prove

$$S_1 \circ (Pf(X^{sk})) \subset P_{2n-2}(X^{sk}).$$

so we need to show for any monomial $y_{ij} \in S_1$ we have

$$y_{ij} \circ (Pf(X^{sk})) \subset P_{2n-2}(X^{sk}).$$

It is enough to show the above inclusion holds for $y_{12}$. Using equation 4.8 we have

$$y_{12} \circ (Pf(X^{sk})) = y_{12} \circ \sum_{i=2}^{i=2n}(-1)^i x_{1i}^s Pf(X_{i1}^{sk}) = Pf(X_{11}^{sk}) + \sum_{i=3}^{i=2n}(-1)^i x_{1i}^s Pf(X_{i1}^{sk}) \in P_{2n-2}(X^{sk}).$$

So indeed

$$S_1 \circ (Pf(X^{sk})) \subset P_{2n-2}(X^{sk}).$$

Next assume $S_k \circ (Pf(X^{sk})) \subset P_{2n-2k}(X^{sk})$. We want to show

$$S_{k+1} \circ (Pf(X^{sk})) \subset P_{2n-2k-2}(X^{sk}).$$

We have

$$S_{k+1} \circ (Pf(X^{sk})) = S_1 \circ (S_k \circ (Pf(X^{sk})) \subset S_1 \circ (P_{2n-2k}(X^{sk}) \subset P_{2n-2k-2}(X^{sk}).$$

For the other inclusion, we again use induction on $k$. First we show the inclusion holds for $k = 1$. Let $\eta \in P_{2n-2}(X^{sk})$ be a $(2n-2) \times (2n-2)$ Pfaffian minor of $X^{sk}$. Corresponding to $\eta$ there exists a $2 \times 2$ matrix of the form

$$\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix},$$

where $x$ is not in the $2n-2$ rows and columns of $\eta$. If we differentiate the Pfaffian of $X^{sk}$ with respect to $x$ we will get $\eta$. So we have $\eta \in S_1 \circ (Pf(X^{sk}))$.

Next assume $P_{2n-2k}(X^{sk}) \subset S_k \circ (Pf(X^{sk}))$, we have
\[ P_{2n-2k-2}(X^{sk}) \subset S_1 \circ (P_{2n-2k}(X^{sk})) \subset S_1 \circ (S_k \circ (Pf(X^{sk}))) = S_{k+1} \circ (Pf(X^{sk})). \]

Thus by induction the equality holds. 

Recall that \((W)\) is the ideal of \(S^{sk}\) spanned by degree 2 elements of type (a), (b) and (c) as in Definition 4.0.15.

**Proposition 4.0.19.** For the \(2n \times 2n\) generic skew symmetric matrix \(X^{sk}\) we have

\[ (W)_n = \text{Ann}(Pf(X^{sk})) \cap S_n^{sk} \tag{4.10} \]

**Proof.** Let \(2 \leq k \leq n\). By Remark 1.0.4 and Lemma 4.0.18 we have

1. \(W \circ Pf(X^{sk}) = 0 \iff W \circ S_{n-2}^{sk}Pf(X^{sk}) = 0 \iff W \circ P_4(X^{sk}) = 0.\)

2. \((\text{Ann}(Pf(X^{sk}))) \cap S_2 = W \Rightarrow S_{n-k} \circ (Pf(X^{sk})) = 0.\)

\[ \Rightarrow (W)_{k} \circ P_{2k}(X^{sk}) = 0. \]

Therefore for all integers \(k, 2 \leq k \leq n\), we have

\[ (W)_{k} \subset \text{Ann}(P_{2k}(X^{sk})) \cap S_k^{sk}. \tag{4.11} \]

We need to show

\[ (W)_n \supset \text{Ann}(Pf(X^{sk})) \cap S_n^{sk}. \tag{4.12} \]

We use induction on \(n\). For \(n = 1, 2\), we have the \(2 \times 2\) and \(4 \times 4\) skew symmetric matrices and the equality is easy to see. Now we want to show that the proposition holds for \(n = 3\).

We use the Remark 2.2.7. Let \(\eta\) be a binomial in \(\text{Ann}(Pf(X^{sk})) \cap S_3^{sk}\). Without loss of generality we can write

\[ \eta = y_{12}y_{34}y_{56} - y_{\sigma(1)\sigma(2)}y_{\sigma(3)\sigma(4)}y_{\sigma(5)\sigma(6)}. \]
Where \( \sigma \in S_6 \), \( \text{sgn}(\sigma) = 1 \) and we have \( \sigma(1) < \sigma(3) < \sigma(5) \) and \( \sigma(1) < \sigma(2) \), \( \sigma(3) < \sigma(4) \) and \( \sigma(5) < \sigma(6) \).

If the two terms of the binomial \( \eta \) have a common factor then without loss of generality we can assume that the common factor is \( y_{12} \) so we can write \( \eta \) as

\[
\eta = y_{12}(y_{34}y_{56} - y_{\sigma(3)}y_{\sigma(4)}y_{\sigma(5)}y_{\sigma(6)})
\]

But by the definition of \((W)_3\) the monomial \( y_{34}y_{56} - y_{\sigma(3)}y_{\sigma(4)}y_{\sigma(5)}y_{\sigma(6)} \) is included in \( W \) since it is of the form (c). So we have \( \eta \in (W)_3 \).

On the other hand, assume that the two terms of \( \eta \), i.e. \( y_{12}y_{34}y_{56} \) and \( y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}y_{\sigma(4)}y_{\sigma(5)}y_{\sigma(6)} \) do not have any common factor. We can add and subtract another term of the Pfaffian \( \tau = y_{\beta(1)}y_{\beta(2)}y_{\beta(3)}y_{\beta(4)}y_{\beta(5)}y_{\beta(6)} \) such that \( \beta \) is a permutation in \( S_6 \) and we have \( \beta(1) < \beta(3) < \beta(5) \) and \( \beta(1) < \beta(2), \beta(3) < \beta(4) \) and \( \beta(5) < \beta(6) \). and \( \tau \) has one common factor with \( y_{12}y_{34}y_{56} \) and one common factor with \( y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}y_{\sigma(4)}y_{\sigma(5)}y_{\sigma(6)} \). Without loss of generality we can take \( \beta(5) = 5, \beta(6) = 6 \) and \( \beta(1) = \sigma(1), \beta(2) = \sigma(2) \). So we have

\[
\eta - \tau + \tau = \eta - y_{\sigma(1)}y_{\sigma(2)}y_{\beta(3)}y_{\beta(4)}y_{5,6} + y_{\sigma(1)}y_{\sigma(2)}y_{\beta(3)}y_{\beta(4)}y_{5,6}.
\]

Hence we have

\[
\eta = y_{5,6}(y_{12}y_{34} - y_{\sigma(1)}y_{\sigma(2)}y_{\beta(3)}y_{\beta(4)}) + y_{\sigma(1)}y_{\sigma(2)}(y_{\beta(3)}y_{\beta(4)}y_{5,6} - y_{\sigma(3)}y_{\sigma(4)}y_{\sigma(5)}y_{\sigma(6)}).
\]

But by the definition of \( W \) we know that \( y_{12}y_{34} - y_{\sigma(1)}y_{\sigma(2)}y_{\beta(3)}y_{\beta(4)} \) and \( y_{\beta(3)}y_{\beta(4)}y_{5,6} - y_{\sigma(3)}y_{\sigma(4)}y_{\sigma(5)}y_{\sigma(6)} \) are both elements of \( W \) of type (c). So we have \( \eta \in (W)_3 \).

When \( n \) is larger than 3 then by the induction assumption we can assume that the proposition holds for all integers \( k \leq n - 1 \). Again we use the Remark 2.2.7. Assume \( b = b_1 + b_2 \) is of degree \( n \). If the two terms, \( b_1 \) and \( b_2 \) are monomials in \( S^k \) and have a common factor \( l \), i.e. \( b_1 = la_1 \) and \( b_2 = la_2 \), then \( b = l(a_1 + a_2) \) where \( a_1 \) and \( a_2 \) are of degree at most \( n - 1 \). By the induction assumption the proposition holds for the binomial \( a_1 + a_2 \), i.e., \( a_1 + a_2 \in W_{n-1} \), hence we have

\[
b = l(a_1 + a_2) \in l(W)_{n-1} \subset (W)_n.
\]
If the two terms, $b_1$ and $b_2$ do not have any common factor then with the same method as above we can rewrite the binomial $b$ by adding and subtracting a term $m$ of degree $n$, which has a common factor $m_1$ with $b_1$ and a common factor $m_2$ with $b_2$, and we will have

$$b_1 + b_2 = b_1 + m + b_2 - m = m_1(c_1 + m') + m_2(c_2 - m''),$$

where $b_1 = m_1c_1$, $m = m_1m' = m_2m''$ and $b_2 = m_2c_2$. Since $c_1 + m'$ and $c_2 - m''$ are of degree at most $n - 1$, the induction assumption yields

$$b_1 + b_2 = m_1(c_1 + m') + m_2(c_2 - m'') \in (W)_n.$$

This completes the induction step and the proof of the proposition. \qed

**Corollary 4.0.20.** For $1 \leq k \leq n$ we have

$$(W)_k = \text{Ann}(Pf(X^{sk})) \cap S^{sk}_k$$

We also have $(W)_{n+1} = S^{sk}_{n+1}$.

**Proof.** Using Equation 4.11 we only need to show that

$$\text{Ann}(Pf(X^{sk})) \cap S^{sk}_k \subset (W)_k$$

By Remark 1.0.4 and Lemma 4.0.18 we have

$$(\text{Ann}(Pf(X^{sk})))_k = (\text{Ann}(S_{n-k} \circ Pf(X^{sk})))_k = (\text{Ann}(P_{2k}(X^{sk})))_k$$

If we label the $2k \times 2k$ Pfaffians of $X^{sk}$ by $f_1, ..., f_s$ we have

$$\text{Ann}(P_{2k}(X^{sk}))_k = (\text{Ann} < f_1, ..., f_s >)_k = (\bigcap_{i=1}^{i=s}(\text{Ann}(f_i)))_k$$

Let $R^i$ denote the ring in the variables of $f_i$ and $W(i)$ the $f_i$ variables that are involved. By Proposition 4.0.19 we have

$$(W(i))_k = \text{Ann}(f_i) \cap S^{i}_k$$
So we have

$$\text{Ann}(Pf(X^{sk})) \cap S^k_k \subset (W)_k$$

To prove the second part, it is easy to see that every monomial of degree larger than $n$ will be unacceptable, of type (a) or (b), so in $W$, and we have $(W)_{n+1} = S^k_{n+1}$.

**Theorem 4.0.21.** Let $X^{sk}$ be a generic skew symmetric $2n \times 2n$ matrix. Then the apolar ideal $\text{Ann}(Pf(X^{sk}))$ is the ideal $W$ and is generated in degree 2.

**Proof.** This follows directly from Proposition 4.0.19 and Corollary 4.0.20.

**Corollary 4.0.22.** Let $X^{sk}$ be a $2n \times 2n$ generic skew symmetric matrix. We have

$$2^{2n-2} \leq \text{cr}(Pf(X^{sk})) \leq 2^{2n-1}$$

(4.13)

**Proof.** By the Ranestad-Schreyer Proposition, Corollary 4.0.14 and Theorem 4.0.21 we have

$$\text{cr}(Pf(X^{sk})) \geq \frac{1}{2} \dim(S^{sk}/\text{Ann}(Pf(X^{sk}))) = \frac{1}{2}(2^{2n-1}) = 2^{2n-2}.$$

The second inequality is true by Equation 2.6.

**Remark 4.0.23.** For $n \geq 5$ it can be easily seen that the lower bound for the cactus rank given by Corollary 4.12 is larger than $l_{diff} = (\frac{2n}{2t_0})$, where $t_0 = \lfloor n/2 \rfloor$.

**Theorem 4.0.24.** Let $X^s$ be a generic symmetric $2n \times 2n$ matrix. Then the apolar ideal $\text{Ann}(Hf(X^s))$ is generated in degree 2, and the inequality 4.13 also holds for $(Hf(X^s))$.

**Proof.** By the definition of the hafnian, it is easy to see that none of the diagonal elements appear in $Hf(X^s)$, so for $1 \leq i \leq 2n$ we have

$$y_{ii} \circ Hf(X^s) = 0$$

Hence without loss of generality we can restrict our discussion to the case where $X^s$ is a generic zero-diagonal symmetric matrix. By changing the Pfaffians to hafnians and vice versa, the proof
follows directly from the proofs that we have for the Pfaffian of a generic skew symmetric matrix.
Chapter 5

Invariants

In this chapter we discuss some invariants that are related to the apolar ideal of the determinant and permanent of the generic and generic symmetric matrices.

5.1 Hafnian invariants

In this section we discuss some properties of the space of the monomials that are involved in the hafnian of a generic symmetric matrix. The original goal was to understand the degree three generators of the Ann(perm($X$)), for a generic symmetric matrix (see Lemma 3.2.18). Throughout this section we use the same notations that we used in chapter 3. Recall that $X = (x_{ij})$ is a generic symmetric matrix, and the differential operator ring $S^s = k[y_{ij}]$ acts on the polynomial ring $R^s = k[x_{ij}]$ by differentiation.

Notation. Throughout this section, $\mathfrak{S}_n$ is the symmetric group of order $n$, $X = (x_{ij})$ is a generic symmetric matrix, and $\sigma \in \mathfrak{S}_n$ acts on $R = k[x_{ij}]$ as follows:

$$\sigma (x_{ij}) = x_{\sigma^{-1}(i)\sigma^{-1}(j)}.$$ 

(5.1)

Let $\text{MonHaf}_{2k}(X)$ denote the space of the monomials of the hafnian (Equation 4.2) of a $2k \times 2k$ generic symmetric matrix $X$. 

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Remark 5.1.1. [Vi] Let $X$ be a $2k \times 2k$ symmetric matrix. We can write the hafnian of $X$ as

$$Hf(X) = \frac{1}{2^k \cdot k!} \sum_{\sigma \in S_{2k}} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}\ldots x_{\sigma(2k-1)}x_{\sigma(2k)}.$$  \hspace{1cm} (5.2)$$

By Equation 5.2 it is easy to see that $Hf(X)$ is invariant under $S_{2k}$. and we have

$$\dim_k \text{MonHaf}_{2k}(X) = \frac{(2k)!}{2^k \cdot k!}.$$  \hspace{1cm} (5.3)$$

Definition 5.1.2. Let $n = 2k$, and $X$ be a generic symmetric $n \times n$ matrix. Recall that $P_k(X)$ is the space of permanents of $k \times k$ submatrices of $X$, and $M_k(X)$ the space of $k \times k$ minors of $X$. We define the maps $\Phi$ and $\Psi$ as follows

$$\Phi : \text{MonHaf}_{2k}(Y) \rightarrow S_k \circ \text{perm}(X) = P_k(X),$$

$$\Phi(h) = h \circ \text{perm}(X).$$

Let $\Omega$ denote the kernel of the map $\Phi$, and

$$\Psi : \text{MonHaf}_{2k}(Y) \rightarrow S_k \circ \text{det}(X) = M_k(X),$$

$$\Psi(h) = h \circ \text{det}(X).$$

Example 5.1.3. Let $n = 4$,

$$X = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix},$$

$$Y = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}. $$

In this case $Hf(Y) = BI + CG + DF$ so we have $\dim_k \text{MonHaf}_4(Y) = 3$. Consider the map

$$\Phi : \text{MonHaf}_4(Y) \rightarrow S_2 \circ \text{perm}(X) = P_2(X)$$

$$\Phi(h) = h \circ \text{perm}(X).$$
We have

\[ \text{Im}(\Phi) = \langle 2df + 2cg + 4bi, 2df + 4cg + 2bi, 4df + 2cg + 2bi \rangle = \langle bi, cg, df \rangle \]

Hence \( \dim_k \text{Im}(\Phi) = 3 \). So \( \text{Ker}\Phi = 0 \).

Now let

\[ \Psi : \text{MonHaf}_4 \rightarrow S_2 \circ \det(X) = M_2(X) \]

\[ \Psi(h) = h \circ \det(X). \]

The kernel of the map \( \Psi \) is \( \text{MonHaf}_4 \cap \text{Ann}(\det(X)) \).

We have

\[ \text{Im}\Psi = \langle -2df - 2cg + 4bi, -2df + 4cg - 2bi, 4df - 2cg - 2bi \rangle = \langle cg - bi, df - bi \rangle \]

is a two dimensional space. The kernel of \( \Psi \) is \( \langle BI + CG + DF \rangle \) that is a one dimensional space. Note also that

\[ \langle \text{Ann}(\text{Hf}(X)) \rangle = \langle (CG - BI, DF - BI) \rangle \]

The character table of \( S_4 \) acting on the space \( \text{MonHaf}_4 \) is as follows:

<table>
<thead>
<tr>
<th>conjugacy class of ( S_4 )</th>
<th>number of elements</th>
<th>( 1 )</th>
<th>( 6 )</th>
<th>( 8 )</th>
<th>( 6 )</th>
<th>( 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (1^4) )</td>
<td>( (1^2 2) )</td>
<td>1</td>
<td>6</td>
<td>8</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>( (2^2) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \chi_{\text{MonHaf}_4}(g) )</td>
<td></td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Here we have \( 3^2(1) + 6 + 3(3^2) + 6 = 48 \). So \( |\chi(\text{MonHaf}_4)| = 2 \). This representation is the sum of two irreducible representations of \( S_4 \) (see [FH], page 17). Indeed the character table of \( S_4 \), Table 5.2, shows that \( \text{MonHaf}_4 = U \oplus W \) corresponding to the partitions \( [4] \) and \( [2, 2] \). Hence the image of \( \Psi \) is the irreducible representation \( W \) and the Kernel of \( \Psi \) corresponds to the trivial representation \( U \). We also see that the image of \( \Phi \) corresponds to the representation of \( \text{MonHaf}_4 = U \oplus W \).
Table 5.2: character table of $S_4$

<table>
<thead>
<tr>
<th>Conjugacy Class</th>
<th>1</th>
<th>6</th>
<th>8</th>
<th>6</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trivial $U$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Alternating $U'$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Standard $V$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$V' = V \otimes U'$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$W$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Lemma 5.1.4. For $n = 6$, the map $\Phi$ is an $S_6$ equivariant map.

Proof. Let $h \in Haf_6(Y)$. Let

$$X = \begin{pmatrix} a & b & c & d & e & f \\ b & g & h & i & j & k \\ c & h & l & m & n & o \\ d & i & m & p & q & r \\ e & j & n & q & s & t \\ f & k & o & r & t & u \end{pmatrix},$$

and let $Y$ be the corresponding matrix in the ring of differential operators in the variables $A - U$. Without loss of generality we can take $h = BMT$. We want to show for any element $\sigma \in S_6$ we have:

$$\Phi(\sigma \cdot h) = \sigma \cdot \Phi(h) = \sigma \cdot (h \circ \text{perm}(X)).$$

Let $\sigma$ be a transposition in $S_6$. Without loss of generality we can assume $\sigma = (14)$. We have

$$(14) \cdot BMT = CIT$$

Hence we need to show:

$$\Phi(CIT) = (14) \cdot (BMT \circ \text{perm}(X)).$$

We have:

$$\text{CIT} \circ \text{perm}(X) = 2fjm + 2ekm + 4fin + 2dkn + 4eio + 2djo + 2fhq + 4ckq + 2boq + 2ehr + 4cjr + 2bmr + 4dht + 8cit + 4bmt$$
We also have

\[ \text{BMT} \circ \text{perm}(X) = 4fjm + 4ekm + 2fin + 2dkn + 2eio + 2djo + 2fhq + 2ckq + 4boq + 2ehr + 2cjr + 4bmr + 4dht + 4cit + 8bmt \]

Hence

\[ (14) \cdot (\text{BMT} \circ \text{perm}(X)) = \text{CIT} \circ \text{perm}(X). \]

**Example 5.1.5.** Let \( n = 6 \), and \( X \) be the generic symmetric matrix in the Lemma 5.1.4. In this case \( \dim_k \text{MonHaf}_6(Y) = 15 \). The kernel of the map

\[ \Phi : \text{MonHaf}_6 \rightarrow S_3 \circ \text{perm}(X) = P_3(X) \]

\[ \Phi(\mathfrak{h}) = \mathfrak{h} \circ \text{perm}(X). \]

is a five dimensional space \( \Omega = \langle F_1, \ldots, F_5 \rangle \), where \( F_1, \ldots, F_5 \) are the polynomials introduced in the Lemma 3.2.18. Hence \( \dim_k \Omega = 5 \), and \( \dim_k \text{Im} \Phi = 10 \).

Now let

\[ \Psi : \text{MonHaf}_6 \rightarrow S_3 \circ \text{det}(X) = M_3(X) \]

\[ \Psi(\mathfrak{h}) = \mathfrak{h} \circ \text{det}(X). \]

The kernel of \( \Psi \) is \( \text{MonHaf}_6 \cap \text{Ann}(\text{det}(X)) \). Using Macaulay 2 for calculations we have

\[ \text{Im} \Psi = \langle F_1(X), \ldots, F_5(X) \rangle, \]

is a five dimensional space. Hence the kernel of \( \Psi \) is a 10-dimensional space.

The character table of \( S_6 \) acting on the space \( \text{MonHaf}_6 \) is as follows:

<table>
<thead>
<tr>
<th>( \chi_{\text{MonHaf}_6}(g) )</th>
<th>15</th>
<th>3</th>
<th>0</th>
<th>3</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>7</th>
<th>1</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (1^6) )</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>( (1^42) )</td>
<td></td>
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<tr>
<td>( (1^33) )</td>
<td></td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>( (1^22^2) )</td>
<td></td>
<td></td>
<td></td>
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<td></td>
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<td></td>
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<tr>
<td>( (1^24) )</td>
<td></td>
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<tr>
<td>( (123) )</td>
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<tr>
<td>( (12^3) )</td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( (15) )</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>( (2^3) )</td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( (24) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>( (3^2) )</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( (6) )</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
</tr>
</tbody>
</table>
\begin{equation}
| < \chi_{\text{MonHaf}_6}, \chi_{\text{MonHaf}_6} > | /720 = 3 \tag{5.4}
\end{equation}

So using Equation 5.4, Table 5.3, and the character table of $\mathcal{S}_6$, the representation of $\text{MonHaf}_6$ is the sum of three irreducible representations of $\mathcal{S}_6$, namely to those corresponding to the partitions $[6], [4,2]$ and $[2,2,2]$ of dimensions 1, 9 and 5 respectively. It follows that the 10-dimensional image of the map $\Phi$ is the sum of the irreducible representations corresponding to the partitions $[6]$ and $[4,2]$, and the 5-dimensional kernel corresponds to the irreducible representation of partition $[2,2,2]$. Hence we have a duality between $\Phi$ and $\Psi$, since the kernel of one is the image of the other one where we replace the small variables with large variables and vice versa.

In a conversation with Claudiu Raicu, he explained that the representation of $\text{MonHaf}_{2k}$ is the sum of the irreducible representations of $\mathcal{S}_{2k}$ corresponding to the partitions with all even parts. We checked using a program we wrote in JAVA the analogous result for $n = 8$, namely that $\text{MonHaf}_8$ is a direct sum of irreducible representations of $\mathcal{S}_8$ corresponding to the partitions with all even parts.

**Theorem 5.1.6.** ([KE], page 272, Theorem 5.8.3) For each $n \in \mathbb{N}$ we have the following decomposition of plethysm of identity representations of symmetric group

$$[2] \odot [n] = \sum_{\alpha \vdash n} [2\alpha],$$

if $2\alpha = (2\alpha_1, 2\alpha_2, \ldots)$.

$\text{MonHaf}_{2m}$ is a subspace of $S_m(S_2 V)$ and since it does not involve any diagonal element it can be considered as a subspace of $S_m(\wedge^2 V)$. The following Theorem about the $\text{Gl}(V)$ representations of the space $S_m(S_2 V)$ seems relevant.

**Theorem 5.1.7.** ([We], page 65) Let $k$ be a commutative ring of characteristic 0.

$$S_m(S_2 E) = \bigoplus_{|\lambda| = 2m, \lambda_i \text{ even for all } i} LE,$$

$$S_m(\wedge^2 E) = \bigoplus_{|\lambda| = 2m, \lambda_i \text{ even for all } i} LE.$$
5.2 Other Immanents

The determinant of an $n \times n$ matrix is the immanent corresponding to the alternating irreducible representation of $\mathfrak{S}_n$, which comes from the partition $[1, 1, \ldots, 1]$. The permanent of an $n \times n$ matrix is the immanent corresponding to the trivial irreducible representation of $\mathfrak{S}_n$, which comes from the partition $[n]$. The partitions $[1, 1, \ldots, 1]$ and $[n]$ are conjugate partitions, and in the section 5.1, we discussed a duality between the determinant and permanent of a matrix (see Example 5.1.5). In this section we look for some duality between the generators of the apolar ideal of other immanents corresponding to the other irreducible representations which come from dual partitions. The apolar ideal for the other immanents of the generic and generic symmetric matrix are still unknown to us. Here we give some examples for $n = 3$ and $n = 4$.

**Definition 5.2.1.** Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of $n$, and let $\chi_\lambda$ be the corresponding irreducible representation-theoretic character of the symmetric group $\mathfrak{S}_n$. The immanent of an $n \times n$ matrix $A = (a_{ij})$ associated with the character $\chi_\lambda$ is defined as the expression

$$\text{Imm}_\lambda(A) = \sum_{\sigma \in \mathfrak{S}_n} \chi_\lambda(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

**Example 5.2.2.** Consider the standard representation of $\mathfrak{S}_3$, corresponding to the self-conjugate partition $[2,1]$.

Table 5.4: character table of $\mathfrak{S}_3$

<table>
<thead>
<tr>
<th>$\chi(V)$</th>
<th>Standard</th>
<th>2</th>
<th>0</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi(\sigma)$</td>
<td>Standard</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi(\sigma)$</td>
<td>(12)</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi(\sigma)$</td>
<td>(3)</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $n = 3$. Consider the generic $3 \times 3$ matrix:

$$A = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}.$$

We have

$$\text{Imm}_V(A) = 2aei - dhc - bfg.$$
Ann(Imm\(V\)(A)) = (I^2, HI, GI, FI, DI, CI, BI, H^2, GH, FH, EH, BH, AH, G^2, EG, DG, CG, AG, F^2, EF, DF, CF, AF, E^2, DE, CE, BE, D^2, BD, AD, C^2, BC, AC, B^2, AB, A^2, 2CDH + AEI, 2BFG + AEI)

and the corresponding Hilbert sequence is (1, 9, 9, 1).

Now consider the generic symmetric 3 × 3 matrix

\[
X = \begin{pmatrix}
a & b & c \\
b & d & e \\
c & e & f \\
\end{pmatrix}
\]

We have

\[
\text{Imm}_V(X) = 2adf - 2bce.
\]

Ann(Imm\(V\)(X)) = (f^2, ef, cf, bf, e^2, de, ae, d^2, cd, bd, c^2, ac, b^2, ab, a^2, bce + adf)

and the corresponding Hilbert sequence is (1, 6, 6, 1).

**Example 5.2.3.** Let \(n = 4\). The character table of \(S_4\) is Table 5.2. Let \(X\) be the generic symmetric matrix

\[
X = \begin{pmatrix}
a & b & c & d \\
b & e & f & g \\
c & f & h & i \\
d & g & i & j \\
\end{pmatrix}
\]

We have:

\[
\text{Imm}_V(X) = 3ahj + hjb^2 + ejc^2 + ehd^2 + ajf^2 + ahg^2 + aei^2 - bfid - cgif - dfg - bgec - cbig - dbfi - b^2i^2 - c^2g^2 - d^2f^2
\]

Calculations in Macaulay 2 shows that Ann(Imm\(V\)(X)) is generated in degrees two and three and corresponding Hilbert sequence is (1, 10, 39, 10, 1).

\[
\text{Imm}_W(X) = 2ahj - bfcj - bgd - cei - cfbj - degi - afi - agfi + 2b^2i^2 + 2c^2g^2 + 2d^2f^2
\]

Again for the Immanant corresponding to \(W\) calculations show that the apolar ideal is generated in degree two and three and the corresponding Hilbert sequence is (1, 10, 39, 10, 1).
\[ \text{Imm}_{V'}(X) = 3aehj - hjb^2 - ejc^2 - ehf^2 - ahg^2 - aeij + bfid + cgfd + dfeg + bgci + cbig + dbfi - b^2i^2 - c^2g^2 - d^2f^2 \]

For the Immanent corresponding to \( V' \) calculations show that the apolar ideal is generated in degree two and three and the corresponding Hilbert sequence is \((1, 10, 38, 10, 1)\).

In this case as we expected calculations in Macaulay 2 show some duality between the generators the apolar ideals of \( \text{Imm}_V \) and \( \text{Imm}_{V'} \), which correspond to the dual partitions \([3, 1]\) and \([2, 1, 1]\) of \( 4 \) respectively. For example \( BI^2 + 2BHJ \in \text{Ann}(\text{Imm}_V(X)) \) and dual to this we have \( BI^2 - 2BHJ \in \text{Ann}(\text{Imm}_{V'}(X)) \).
Bibliography


