Internal and External Invariance of Abstract Polytopes

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ABSTRACT OF DISSERTATION

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Abstract

In addition to the usual symmetries by reflections and rotations, abstract polytopes can have external symmetries such as self-duality and self-Petriality. In this dissertation, we describe and study a way of measuring the invariance of abstract polytopes under such external operations. We then present methods for constructing abstract polytopes with specified external symmetries. In particular, we describe how to construct polyhedra that are self-dual and self-Petrie, and how to construct polytopes that are self-dual and chiral.
I would like to thank the members of my committee for their time and attention.

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Chapter 1

Background

1.1 Introduction

The study of abstract polytopes is a growing field, uniting combinatorics with geometry and group theory. Abstract polytopes are combinatorial structures with a distinctly geometric flavor, generalizing the face-lattices of convex polytopes. The result is a rich theory that encompasses not only polytopes, but also face-to-face tilings of the plane, tessellations of space-forms, and other fascinating structures.

As in the classical theory, the abstract polytopes that have been most extensively studied are the regular polytopes: those whose automorphism group acts transitively on the set of their flags. These polytopes possess maximal symmetry by abstract reflections. Closely related are the chiral polytopes, which possess two flag-orbits under the action of the automorphism group. Intuitively, a chiral polytope has full rotational symmetry, but it occurs in two mirror-image forms. Compared to regular polytopes, relatively little is known about chiral polytopes. In particular, we know very few examples of chiral polytopes of high rank (dimension).
In addition to the study of polytopes with a high degree of internal symmetry – that is, polytopes with a large number of polytope automorphisms – the study of external symmetries of polytopes and maps on surfaces has been gaining attention; for example, see [15, 32]. The idea is to consider operations that transform one polytope into another, and to then look for (or construct) polytopes that are invariant under this transformation. The classic example is self-duality, but there are many others.

In this dissertation, we focus on external symmetries arising from certain outer automorphisms of groups. In particular, we consider outer automorphisms of the automorphism group of the universal polytope of rank $n$, a polytope that covers all others of the same rank. There are three principal external symmetries this gives rise to: duality, Petrie duality, and mirror symmetry. Our goal is then to measure to what extent a polytope is invariant under these symmetries.

Our main tool for constructing new polytopes is the mixing construction, which creates the minimal common cover of two polytopes [24]. The mix of a polytope with its dual is self-dual, and in general, the mix of a polytope with all of its images under a group of transformations is invariant under that group of transformations. The result, however, is not always a polytope; we must check that separately.

We start by giving background on (abstract) polytopes, including regularity, chirality, duality, and Petrie duality, in Chapter 1. In Chapter 2, we describe the mixing construction for groups and for polytopes, and we examine in detail the structure of the mix of two polytopes. In particular, we describe several circumstances that guarantee that the mix of two polytopes will again be a polytope. Next, we describe how to use mixing to measure the invariance of a polytope under external symmetries in Chapter 3. In Chapter 4, we focus on duality and Petrie duality, and we explicitly describe many self-dual, self-Petrie polyhedra. Finally, in Chapter 5, we construct chiral polytopes that are self-dual – in other
words, self-dual polytopes that lack mirror symmetry.

1.2 Abstract polytopes

Abstract polytopes are essentially generalizations of the face-lattice of a convex polytope, and they retain much of the geometric flavor of convex polytopes. As we build up the definition, given in [23], we will give some geometric justification for the properties we require. Some of our later constructions produce objects that are not quite polytopes, and so we start with these more general definitions.

Definition 1.1. A flagged poset of rank $n$ is a partially ordered set $\mathcal{P}$ such that there is a unique maximal element, a unique minimal element, and such that each maximal chain of $\mathcal{P}$ has $n + 2$ elements. We use $F_n$ to denote the maximal element, and $F_{-1}$ to denote the minimal element.

We call the elements of a flagged poset faces, and we call the maximal chains flags. Since each flag of a flagged poset contains the same number of faces, every flagged poset has a natural rank function. In particular, we define the rank of $F_{-1}$ to be $-1$ and the rank of $F_n$ to be $n$, with the rank of any other face obtained in the natural way from its position in any flag. A face of rank $j$ is called a $j$-face. The faces $F_{-1}$ and $F_n$ are called the improper faces of $\mathcal{P}$; the remaining faces are the proper faces. If $F$ and $G$ are faces of a flagged poset $\mathcal{P}$ such that $F \leq G$, then we define the section $G/F$ to be the poset consisting of those faces $H$ such that $F \leq H \leq G$. In fact, if $F$ is a $j$-face and $G$ is a $k$-face, then the section $G/F$ is a flagged poset of rank $k - j - 1$, where $G$ plays the role of the maximal face, and $F$ plays the role of the minimal face.

We think of the rank of a face as its dimension, informally speaking. Thus, in analogy with convex polytopes, we refer to 0-faces as vertices and 1-faces as edges. Faces of rank
$n - 1$ are called facets. The requirement that a flagged poset of rank $n$ must contain a unique $n$-face and a unique $(-1)$-face is analogous to the fact that a convex $n$-polytope is viewed to have a single face of dimension $n$ (namely, the whole polytope) and a single face of dimension $-1$ (namely, the empty set).

There is a natural identification between each $j$-face $F$ and its section $F/F_{-1}$, and we occasionally blur the distinction between these two concepts. If $F$ is a $k$-face, then the co-

$k$-face at $F$ (or sometimes simply the co-face at $F$) is the section $F_n/F$. In particular, if $F$ is a vertex, then we call the co-face at $F$ a vertex-figure. If $G$ is a facet and $F$ is a vertex, then the section $G/F$ is called a medial section. Note that a medial section is a facet of the vertex-figure $F_n/F$ and also a vertex-figure of the facet $G/F_{-1}$.

The requirement that all flags of a flagged poset have the same length ensures that we get no “holes” and that all maximal proper faces have the same rank. Essentially, we are forcing the set of proper faces to form a polytopal complex [13] where the maximal polytopes have the same dimension (except, of course, our polytopes will be abstract instead of convex).

We note that there is one flagged poset of rank $-1$ (where the minimal and maximal element coincide), and one flagged poset of rank 0 (where the only two elements are the minimal and maximal elements). In rank 1, there are already infinitely many flagged posets.

Flagged posets are really too general to have much geometric flavor. The notion of a pre-polytope is much closer to what we want.

**Definition 1.2.** A flagged poset of rank $n$ is a pre-polytope of rank $n$ (also called an $n$-pre-polytope) if it satisfies the following “diamond condition”: whenever $F < G$ where $F$ is a $(j - 1)$-face and $G$ is a $(j + 1)$-face (for any $j = 0, 1, \ldots, n - 1$), then there are exactly two $j$-faces $H$ such that $F < H < G$.

The diamond condition is a strong restriction on the types of structures we investigate. In


particular, we see that each 1-face (edge) of an \( n \)-pre-polytope contains two 0-faces (vertices), and that each \((n-2)\)-face is contained in two \((n-1)\)-faces. This again parallels the case with convex polytopes. As with flagged posets, we note that every section of a pre-polytope is again a pre-polytope.

There is a way to formulate the diamond condition in terms of flags instead of faces. Given two flags of a flagged poset, we say that they are \textit{adjacent} if they differ in exactly one face. If that face is an \( i \)-face, we say that the flags are \textit{i-adjacent}.

\textbf{Proposition 1.3.} Let \( \mathcal{P} \) be a flagged poset of rank \( n \). Then \( \mathcal{P} \) has the diamond property (and is therefore a pre-polytope) if and only if each flag of \( \mathcal{P} \) has a unique \( i \)-adjacent flag for each \( i = 0, \ldots, n - 1 \).

\textit{Proof.} The claim is trivially true for \( n \leq 0 \). Suppose \( n \geq 1 \), and fix an \( i \) between 0 and \( n - 1 \). Let \( F_{i+1} \) be a face of rank \( i + 1 \) and \( F_{i-1} \) a face of rank \( i - 1 \) such that \( F_{i-1} < F_{i+1} \).

Consider a flag \( \Phi = \{F_{-1}, \ldots, F_{i-1}, F_{i}, F_{i+1}, \ldots, F_{n}\} \). Then any flag that is \( i \)-adjacent to \( \Phi \) yields a proper face in the section \( F_{i+1}/F_{i-1} \), and any proper face other than \( F_i \) in \( F_{i+1}/F_{i-1} \) yields a flag that is \( i \)-adjacent to \( \Phi \). Therefore, the section \( F_{i+1}/F_{i-1} \) has two proper faces if and only if \( \Phi \) has a unique \( i \)-adjacent flag. \hfill \Box

For a flag \( \Phi \) in a pre-polytope, we denote the unique \( i \)-adjacent flag by \( \Phi^i \). We then inductively define \( \Phi^{i_1, \ldots, i_k} \) to be \( (\Phi^{i_1, \ldots, i_{k-1}})^i \). Note that for any \( i \), \( \Phi^{i,i} = \Phi \), and if \( |i - j| \geq 2 \), then \( \Phi^{i,j} = \Phi^{j,i} \).

If \( \mathcal{P} \) and \( \mathcal{Q} \) are \( n \)-pre-polytopes, then we can identify their respective maximal elements and minimal elements to obtain another \( n \)-pre-polytope. This essentially amounts to taking the disjoint union of \( \mathcal{P} \) and \( \mathcal{Q} \). Working with disconnected pre-polytopes does not add much to the theory, so we will restrict ourselves to connected pre-polytopes.
Definition 1.4. Let $\mathcal{P}$ be a flagged poset of rank $n$. We say that $\mathcal{P}$ is connected if $n \leq 1$ or if the Hasse diagram of $\mathcal{P} \setminus \{F_{-1}, F_n\}$ is a connected graph. In other words, $\mathcal{P}$ is connected if for every pair of proper faces $G$ and $H$, there is a chain of proper faces

$$G = G_0 \leq G_1 \geq G_2 \leq G_3 \geq \cdots \geq G_{k-1} \leq G_k = H.$$  

Every section of a pre-polytope is again a pre-polytope, but the sections of a connected pre-polytope need not be connected. For example, consider the “dihedral bowtie”: two triangular dihedra joined at a vertex. It has 5 vertices, 6 edges, and 4 triangular faces: see Figure 1.1. As we see, this is a connected pre-polytope. However, if we take the section $F_3/c$, we get the disjoint union of two digons; see Figure 1.2. Thus, we see that working with connected pre-polytopes is a little unsatisfying. In order to build them from smaller pieces, we may have to consider pre-polytopes that are not connected.

We dodge this difficulty in the simplest possible way: by deciding not to deal with such structures. We say that a flagged poset is strongly connected if all of its sections are connected.
Figure 1.2: Hasse diagram of a section of the dihedral bowtie

(including the whole flagged poset itself). Finally, we come to our desired definition:

Definition 1.5. An (abstract) polytope of rank $n$ (also called an (abstract) $n$-polytope) is a strongly connected $n$-pre-polytope. That is, an $n$-polytope is a partially ordered set $\mathcal{P}$ satisfying the following four properties:

1. $\mathcal{P}$ has a unique maximal element $F_n$ and a unique minimal element $F_{-1}$.
2. Every flag of $\mathcal{P}$ has $n + 2$ elements.
3. (Diamond condition): whenever $F < G$ where $F$ is a $(j - 1)$-face and $G$ is a $(j + 1)$-face, then there are exactly two $j$-faces $H$ such that $F < H < G$.
4. $\mathcal{P}$ is strongly connected.

There is only a single polytope in each of the ranks $-1, 0, \text{ and } 1$. In rank 2, the polytopes are the face-lattices of $k$-gons, for $2 \leq k \leq \infty$. In general, the face-lattice of a convex polytope is indeed an abstract polytope, though there are also many fascinating new structures.

A slightly stronger notion of connectivity proves quite useful. The adjacency relation between flags gives rise to the flag graph of a polytope in a natural way; namely, the vertices of the flag graph are the flags of the flagged poset, and two flags are connected by an edge
if they are adjacent (that is, if they differ in just a single element). We say that a flagged poset is *flag-connected* if its flag graph is connected. That is, if for every pair of flags $\Phi$ and $\Psi$, there is a sequence

$$\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi,$$

such that for each $i$, $\Phi_i$ is adjacent to $\Phi_{i+1}$. Every flag-connected flagged poset is also connected.

Note that the dihedral bowtie, though connected, is not flag-connected. For example, there is no sequence of adjacencies that will take a flag containing $T_1$ to a flag containing $T'_1$.

Like connected pre-polytopes, flag-connected pre-polytopes have the unfortunate property that their sections may not be flag-connected. Nevertheless, we shall often encounter flag-connected pre-polytopes with flag-connected facets and vertex-figures. As such, we make the following definition:

**Definition 1.6.** A semi-polytope of rank $n$ (also called an $n$-semi-polytope) is a flag-connected $n$-pre-polytope such that the facets and vertex-figures are both flag-connected.

Note that for $n \leq 3$, all semi-polytopes are in fact polytopes.

If every section of a flagged poset is flag-connected (including the whole poset itself), then we say that the poset is *strongly flag-connected*. Even though a flagged poset can be connected without being flag connected, Proposition 2A1 in [23] shows that a flagged poset is strongly connected if and only if it is strongly flag-connected.

Let $\mathcal{P}$ be an $n$-pre-polytope such that its sections of rank 2 are all connected. Then for any $(i - 2)$-face $F$ and $(i + 1)$-face $G$ such that $F \leq G$, the section $G/F$ is a $k$-gon for some $k$ ($2 \leq k \leq \infty$). Having fixed $\mathcal{P}$, we define $p_i(F, G)$ to be the number of $i$-faces in the section $G/F$; in other words, if $G/F$ is a $k$-gon, then $p_i(F, G) = k$. If every $p_i(F, G)$ depends only on $i$ (and not on the particular choice of $F$ and $G$), then we define $p_i := p_i(F, G)$ and we say
that \( \mathcal{P} \) has Schl{"a}fli symbol \( \{p_1, \ldots, p_{n-1}\} \) or that \( \mathcal{P} \) is of type \( \{p_1, \ldots, p_{n-1}\} \). A pre-polytope that has a Schl{"a}fli symbol is said to be \textit{equivelar}.

Any section of an equivelar pre-polytope is itself equivelar. In particular, if \( \mathcal{P} \) is equivelar of type \( \{p_1, \ldots, p_{n-1}\} \), and if \( F \) is an \( i \)-face and \( G \) is a \( j \)-face such that \( F \leq G \), then \( G/F \) is equivelar of type \( \{p_{i+2}, \ldots, p_{j-1}\} \). If \( \mathcal{P} \) is an equivelar pre-polytope such that the facets are all isomorphic to \( \mathcal{K} \) and the vertex-figures are all isomorphic to \( \mathcal{L} \), then we say that \( \mathcal{P} \) is of type \( \{\mathcal{K}, \mathcal{L}\} \). (In [23], \( \mathcal{P} \) is said to be in the class \( \langle \mathcal{K}, \mathcal{L}\rangle \).

We now move on to functions between pre-polytopes. Let \( \mathcal{P} \) and \( \mathcal{Q} \) be flagged posets and \( \varphi: \mathcal{P} \to \mathcal{Q} \) a function. We call \( \varphi \) a \textit{(polytope) homomorphism} if whenever \( F \leq G \) in \( \mathcal{P} \) we also have \( F\varphi \leq G\varphi \) in \( \mathcal{Q} \). (Following the convention of [23], polytope homomorphisms always act on the right.) Naturally, if \( \varphi \) is bijective and its inverse is a homomorphism, then we call \( \varphi \) an \textit{isomorphism}, and if \( \varphi \) is an isomorphism from \( \mathcal{P} \) to itself, we call it an \textit{automorphism (of \( \mathcal{P} \))}. The group of automorphisms of a flagged poset \( \mathcal{P} \) is denoted \( \Gamma(\mathcal{P}) \).

The automorphism group of a flagged poset has a natural action on the flags. For any automorphism \( \varphi \) of a flagged poset \( \mathcal{P} \), if the flags \( \Phi \) and \( \Phi' \) are \( i \)-adjacent, then so are \( \Phi\varphi \) and \( \Phi'\varphi \). Thus, if \( \mathcal{P} \) is a pre-polytope, \( \Phi^i\varphi = (\Phi\varphi)^i \) for each flag \( \Phi \). We can extend this result as follows:

\textbf{Proposition 1.7.} Let \( \mathcal{P} \) be a pre-polytope. Let \( \varphi \in \Gamma(\mathcal{P}) \), and let \( \Phi \) be a flag of \( \mathcal{P} \). Then

\[
\Phi^{i_1 \ldots \cdot i_k} \varphi = (\Phi\varphi)^{i_1 \ldots \cdot i_k}.
\]

In particular, if \( \mathcal{P} \) is flag-connected, then \( \varphi \) is determined by the action on \( \Phi \).
Proof. We see that

\[
\Phi^{i_1 \ldots i_k} \varphi = \left( \Phi^{i_1 \ldots i_{k-1}} \right)^{i_k} \varphi = \left( \Phi^{i_1 \ldots i_{k-1}} \right)^{i_k}.
\]

By induction, the first claim is proved. Now, suppose \( P \) is flag-connected and that \( \Phi \varphi \) is known. If \( \Psi \) is any other flag of \( P \), then \( \Psi = \Phi^{i_1 \ldots i_k} \) for some sequence \( i_1, \ldots, i_k \), and thus the action of \( \varphi \) on \( \Psi \) is known. Since we know how \( \varphi \) acts on every flag, we know where it takes every face, and thus \( \varphi \) is completely determined.

\[\square\]

**Corollary 1.8.** Let \( P \) be a flag-connected pre-polytope, and let \( \varphi \in \Gamma(P) \). If \( \varphi \) fixes any flag \( \Phi \) of \( P \), then \( \varphi = \epsilon \), the identity automorphism.

**Proof.** By Proposition 1.7, \( \varphi \) is uniquely determined by where it sends \( \Phi \). Since the automorphism \( \epsilon \) also fixes \( \Phi \), we must have that \( \varphi = \epsilon \).

\[\square\]

In practice, we rarely work with homomorphisms in full generality. We will usually be interested in isomorphisms and automorphisms. However, there is one other type of homomorphism that we will often use.

**Definition 1.9.** Let \( P \) and \( Q \) be flagged posets, and let \( \varphi : P \to Q \) be a homomorphism. Then \( \varphi \) is a covering if it satisfies the following properties:

1. The function \( \varphi \) is surjective.
2. If \( F \) is a face of \( P \), then the rank of \( F \varphi \) is equal to the rank of \( F \).
3. Let \( \Phi \) and \( \Phi' \) be flags of \( P \) that are \( i \)-adjacent. Let \( F_i \) be the \( i \)-face of \( \Phi \) and let \( F'_i \) be the \( i \)-face of \( \Phi' \). Then \( F_i \varphi \neq F'_i \varphi \). That is, \( \Phi \varphi \) and \( \Phi' \varphi \) are also \( i \)-adjacent flags; in particular, they do not coincide.
If there is a covering from $\mathcal{P}$ to $\mathcal{Q}$, we say that $\mathcal{P}$ covers $\mathcal{Q}$. Note that if $\mathcal{P}$ covers $\mathcal{Q}$, then $\mathcal{P}$ and $\mathcal{Q}$ must have the same rank.

**Proposition 1.10.** Let $\mathcal{P}$ be a pre-polytope, let $\mathcal{Q}$ be a flag-connected pre-polytope, and let $\varphi : \mathcal{P} \to \mathcal{Q}$ be a homomorphism. If $\varphi$ satisfies properties 2 and 3 in Definition 1.9, then $\varphi$ is a covering.

**Proof.** It suffices to show that $\varphi$ is surjective. Let $\Phi$ be a flag of $\mathcal{P}$, and let $\Psi = \Phi \varphi$. Since $\mathcal{Q}$ is flag-connected, we can write any flag of $\mathcal{Q}$ as $\Psi^{i_1,\ldots,i_k}$ for some sequence $(i_1,\ldots,i_k)$. Since $\varphi$ preserves flag-adjacency, we get that $\Phi^{i_1,\ldots,i_k} \varphi = \Psi^{i_1,\ldots,i_k}$, and thus $\varphi$ is surjective. \qed

### 1.3 Regularity

Classically, the polytopes that have received the most attention are the regular ones: those having the highest degree of symmetry. There are several equivalent definitions of regular convex polytopes, and it takes some care to determine which of these definitions generalizes nicely for abstract polytopes. Furthermore, whatever definition of regularity we choose, we would like for the face lattice of a regular convex polytope to be a regular abstract polytope.

For example, the face lattice of a regular $k$-gon should be a regular abstract polygon. Of course, the face lattice of any other convex $k$-gon is the same. Thus we see that every abstract polyhedron has regular faces and regular vertex-figures. If we were to define regularity inductively – say, by declaring an $n$-polytope regular if it has regular facets and vertex-figures – then every polytope would be regular! Even if we restrict the definition to additionally require the facets to all be isomorphic and the vertex-figures to all be isomorphic as well, we get a fairly permissive definition. Instead, we use the following definition:

**Definition 1.11.** A flagged poset $\mathcal{P}$ is regular if the action of its automorphism group $\Gamma(\mathcal{P})$
on its flags is transitive.

Though we define regularity for flagged posets, we will usually only work with regular polytopes or regular semi-polytopes, for reasons that will become clear in a moment. Given a flagged poset \( \mathcal{P} \), we usually fix a flag \( \Phi \) that we call the base flag. If \( \mathcal{P} \) is regular, then any two choices of base flag are equivalent, since there is an automorphism of \( \mathcal{P} \) that carries the first choice to the second. If \( \mathcal{P} \) is not regular, however, then choosing a base flag amounts to choosing an orbit of the action of \( \Gamma(\mathcal{P}) \) on the flags.

Now, let \( \mathcal{P} \) be a regular semi-polytope. Since \( \mathcal{P} \) is a pre-polytope, every flag has a unique \( i \)-adjacent flag for each \( i = 0, \ldots, n - 1 \). By the regularity of \( \mathcal{P} \), there are automorphisms \( \rho_0, \ldots, \rho_{n-1} \) (called abstract reflections) such that \( \Phi \rho_i = \Phi^i \). Recalling that \( (\Phi^i)\varphi = (\Phi \varphi)^i \) for any \( \varphi \in \Gamma(\mathcal{P}) \), it follows that \( \Phi \rho_i \cdots \rho_k = \Phi^{i_k \cdots i_1} \). Now, since \( \mathcal{P} \) is flag-connected, any automorphism is determined by where it sends \( \Phi \), so that if \( \Phi \varphi = \Phi^{i_k \cdots i_1} \) we must have \( \varphi = \rho_{i_1} \cdots \rho_{i_k} \). Thus we see that the abstract reflections \( \rho_0, \ldots, \rho_{n-1} \) generate \( \Gamma(\mathcal{P}) \). Furthermore, since \( \Phi^{i \cdot i} = \Phi \) we must have \( \rho_i^2 = \epsilon \), and since \( \Phi^{i \cdot j} = \Phi^{j \cdot i} \) whenever \( |i - j| \geq 2 \), we must have \( (\rho_i \rho_j)^2 = \epsilon \) whenever \( |i - j| \geq 2 \).

If \( \mathcal{P} \) is a polytope (and not just a semi-polytope), then the group \( \Gamma(\mathcal{P}) \) satisfies the following “intersection property” [23]:

\[
\langle \rho_i \mid i \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle.
\]

Semi-polytopes do not necessarily satisfy the intersection property, but they do satisfy some special cases of it.

**Proposition 1.12.** Let \( \mathcal{P} \) be a regular \( n \)-semi-polytope, with automorphism group \( \Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle \). For each \( i = 0, \ldots, n - 1 \), let \( \Gamma_i = \langle \rho_j \mid j \neq i \rangle \). Then \( \Gamma \) satisfies the
following “empty intersection property”:

\[ \Gamma_0 \cap \cdots \cap \Gamma_{n-1} = \langle \epsilon \rangle, \]

and the following “weak intersection property”:

\[ \bigcap_{j \neq i} \Gamma_j = \langle \rho_i \rangle, \text{ for each } i \in \{0, \ldots, n-1\}. \]

Furthermore, the action of \( \Gamma \) on the flags of \( \mathcal{P} \) is simply transitive.

Proof. Let \( \Phi = \{F_0, \ldots, F_{n-1}\} \) be the base flag of \( \mathcal{P} \). (We frequently omit the improper faces when listing the elements of a flag.) For each \( i \), the subgroup \( \Gamma_i \) stabilizes \( F_i \), and thus the group \( \Gamma_0 \cap \cdots \cap \Gamma_{n-1} \) stabilizes \( \Phi \). Suppose that \( \varphi \in \Gamma_0 \cap \cdots \cap \Gamma_{n-1} \), so that it stabilizes \( \Phi \). Then by Corollary 1.8, \( \varphi = \epsilon \), and this proves the empty intersection property. Furthermore, this shows that the action of \( \Gamma \) on the flags of \( \mathcal{P} \) is simply transitive, since if \( \Phi \varphi = \Phi \psi \), then \( \varphi \psi^{-1} \) stabilizes \( \Phi \), and so \( \varphi = \psi \).

Now, the group \( \bigcap_{j \neq i} \Gamma_j \) stabilizes every face of \( \Phi \) except for \( F_i \). It is clear that \( \langle \rho_i \rangle \) is contained in this stabilizer. Since \( \mathcal{P} \) is a pre-polytope, there is exactly one flag that is \( i \)-adjacent to \( \Phi \). Then by simple transitivity, \( \rho_i \) is the only automorphism that sends \( \Phi \) to \( \Phi^i \), and we see that the stabilizer is equal to \( \langle \rho_i \rangle \).

We thus make the following definitions:

**Definition 1.13.** Let \( \Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle \) be a finitely generated group such that the order of each \( \rho_i \) is 2 and such that whenever \( |i - j| \geq 2 \), the order of \( \rho_i \rho_j \) is 2. Then we say that \( \Gamma \) is a string group generated by involutions (or sgg for short.) If \( \Gamma \) also satisfies the empty intersection property and the weak intersection property, then we call \( \Gamma \) a string pre-C-group.
If in fact $\Gamma$ satisfies the full intersection property

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle,$$

then we call $\Gamma$ a string C-group.

The ‘C’ in “string C-group” stands for Coxeter, since string C-groups share many of the properties of string Coxeter groups. In particular, every string Coxeter group is a string C-group. However, the class of string C-groups is much more general. Note that by our preceding discussion, the automorphism group of a regular semi-polytope is a string pre-C-group, and the automorphism group of a regular polytope is a string C-group.

We will soon see how to build regular semi-polytopes and regular polytopes out of string pre-C-groups and string C-groups, respectively. In order to use these constructions effectively, we need to know when an ssgi satisfies the various intersection properties. Section 2E of [23] gives some general results for when an ssgi satisfies the (full) intersection property. Here we present two similar results for the weak intersection property.

**Proposition 1.14.** Let $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ be an ssgi that satisfies the empty intersection property. Then for each $i$ in $\{0, \ldots, n-1\}$,

$$\Gamma_0 \cap \cdots \cap \Gamma_{i-1} = \langle \rho_i, \ldots, \rho_{n-1} \rangle$$

and

$$\Gamma_{i+1} \cap \cdots \cap \Gamma_{n-1} = \langle \rho_0, \ldots, \rho_i \rangle.$$

**Proof.** We will prove the first claim by induction on $i$; the proof of the second is analogous and follows by a ‘dual’ argument, replacing each $\rho_i$ by $\rho_{n-1-i}$. The claim is obvious for $i = 1$. Suppose that $\Gamma_0 \cap \cdots \cap \Gamma_{i-1} = \langle \rho_i, \ldots, \rho_{n-1} \rangle$ and consider $\Gamma_0 \cap \cdots \cap \Gamma_i$. This is equal to
Let \( \langle \rho_0, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_{n-1} \rangle \cap \Gamma_i \). Now, \( \Gamma_i = \langle \rho_0, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_{n-1} \rangle \). The subgroups \( \langle \rho_0, \ldots, \rho_{i-1} \rangle \) and \( \langle \rho_{i+1}, \ldots, \rho_{n-1} \rangle \) commute, so

\[
\Gamma_i = \langle \rho_0, \ldots, \rho_{i-1} \rangle \langle \rho_{i+1}, \ldots, \rho_{n-1} \rangle.
\]

Therefore,

\[
\Gamma_0 \cap \cdots \cap \Gamma_i = \langle \rho_1, \ldots, \rho_{n-1} \rangle \cap \langle \rho_0, \ldots, \rho_{i-1} \rangle \langle \rho_{i+1}, \ldots, \rho_{n-1} \rangle.
\]

Now, let \( w \) be in this intersection. Then \( w \in \langle \rho_0, \ldots, \rho_{i-1} \rangle \langle \rho_{i+1}, \ldots, \rho_{n-1} \rangle \), and so we can write \( w = uv \) with \( u \in \langle \rho_0, \ldots, \rho_{i-1} \rangle \) and \( v \in \langle \rho_{i+1}, \ldots, \rho_{n-1} \rangle \). We also have that \( w \in \langle \rho_1, \ldots, \rho_{n-1} \rangle \). Thus we see that \( u = wv^{-1} \in \langle \rho_1, \ldots, \rho_{n-1} \rangle \), so that in fact \( u \in \langle \rho_0, \ldots, \rho_{i-1} \rangle \langle \rho_1, \ldots, \rho_{n-1} \rangle \). Therefore, \( u \in \Gamma_0 \cap \cdots \cap \Gamma_{n-1} \), and since \( \Gamma \) satisfies the empty intersection property, \( u = \epsilon \). So \( w = v \in \langle \rho_{i+1}, \ldots, \rho_{n-1} \rangle \), which shows that

\[
\Gamma_0 \cap \cdots \cap \Gamma_i \leq \langle \rho_{i+1}, \ldots, \rho_{n-1} \rangle.
\]

The other containment is obvious, and so the claim is proved.

**Corollary 1.15.** Let \( \Gamma \) be an ssgi that satisfies the empty intersection property. Then \( \Gamma \) is a string pre-C-group if and only if for each \( i \) we have \( \langle \rho_0, \ldots, \rho_i \rangle \cap \langle \rho_i, \ldots, \rho_{n-1} \rangle = \langle \rho_i \rangle \).

**Proof.** From Proposition 1.14, we see that if \( \Gamma \) satisfies the empty intersection property, then \( \bigcap_{j \neq i} \Gamma_j = \langle \rho_0, \ldots, \rho_i \rangle \cap \langle \rho_i, \ldots, \rho_{n-1} \rangle \). Since \( \Gamma \) is a string pre-C-group if and only if \( \bigcap_{j \neq i} \Gamma_j = \langle \rho_i \rangle \), the claim follows. \( \square \)

Let \( \mathcal{P} \) be a regular \( n \)-semi-polytope, with base flag \( \Phi = \{ F_{-1}, \ldots, F_n \} \). Then for \( -1 \leq i < j \leq n \), the automorphism group of \( F_j/F_i \) can be identified with the stabilizer of the
chain \( \{F_{-1}, \ldots, F_i, F_j, \ldots, F_n\} \) in \( \Gamma(P) \). If \( P \) is a polytope, then the stabilizer of \( F_k \) is

\[
\Gamma_k := \langle \rho_i \mid i \neq k \rangle.
\]

Therefore,

\[
\Gamma(F_j/F_i) = (\Gamma_0 \cap \cdots \cap \Gamma_i) \cap (\Gamma_j \cap \cdots \cap \Gamma_{n-1}) = \langle \rho_{i+1}, \ldots, \rho_{j-1} \rangle.
\]

If \( P \) is just a semi-polytope, however, the situation is somewhat more complicated. In order for \( \Gamma_k \) to be the stabilizer of \( F_k \), the sections \( F_n/F_k \) and \( F_k/F_{-1} \) must both be flag-connected:

**Proposition 1.16.** Let \( P \) be a regular \( n \)-semi-polytope, with base flag \( \Phi = \{F_{-1}, \ldots, F_n\} \). Then the stabilizer of \( F_k \) in \( \Gamma := \Gamma(P) \) is \( \Gamma_k \) if and only if the sections \( F_n/F_k \) and \( F_k/F_{-1} \) are regular and flag-connected.

**Proof.** First, suppose that the stabilizer of \( F_k \) is \( \Gamma_k \). Let \( \Phi_1 = \{F_{-1}, \ldots, F_k\} \) and let \( \Phi_2 = \{F_k, \ldots, F_n\} \). That is, \( \Phi_1 \) is the part of \( \Phi \) that lies in \( F_k/F_{-1} \), and \( \Phi_2 \) is the part of \( \Phi \) that lies in \( F_n/F_k \). Now, to prove that \( F_k/F_{-1} \) is regular and flag-connected, it suffices to show that for any other flag \( \Psi_1 \) of \( F_k/F_{-1} \), there is an automorphism of \( F_k/F_{-1} \) that sends \( \Phi_1 \) to \( \Psi_1 \). Fix a particular choice of \( \Psi_1 \). We can extend it to \( \Psi \), a flag of \( P \). Since \( P \) is regular, there is an automorphism \( \alpha \) sending \( \Phi \) to \( \Psi \). Now, \( \Phi \) and \( \Psi \) both contain the \( k \)-face \( F_k \), so \( \alpha \) is in the stabilizer of \( F_k \). Therefore, \( \alpha \in \Gamma_k = \langle \rho_i \mid i \neq k \rangle \). Now, we can decompose \( \Gamma_k \) as

\[
\Gamma_k = \langle \rho_0, \ldots, \rho_{k-1} \rangle \langle \rho_{k+1}, \ldots, \rho_{n-1} \rangle.
\]

Therefore, we can write \( \alpha = \alpha_1 \alpha_2 \), where \( \alpha_1 \in \langle \rho_0, \ldots, \rho_{k-1} \rangle \) and \( \alpha_2 \in \langle \rho_{k+1}, \ldots, \rho_{n-1} \rangle \). We can identify \( \alpha_1 \) with an automorphism of \( F_k/F_{-1} \), and this automorphism sends \( \Phi_1 \) to \( \Psi_1 \),
as desired. So $F_k/F_{-1}$ is regular and flag-connected. The proof for $F_n/F_k$ is analogous.

Now suppose that the sections $F_n/F_k$ and $F_k/F_{-1}$ are regular and flag-connected. It is clear that $\Gamma_k$ stabilizes $F_k$, so we only need to show that $\Gamma_k$ is the entire stabilizer. Since $\Gamma$ acts simply transitively on the flags of $\mathcal{P}$, it suffices to show that $\Gamma_k$ acts transitively on all flags containing $F_k$. In fact, it suffices to show that for any flag $\Psi$ containing $F_k$, there is an automorphism $\alpha \in \Gamma_k$ such that $\Phi \alpha = \Psi$. Let $\Phi_1$ be the part of $\Phi$ in $F_k/F_{-1}$. Since $\Psi$ contains $F_k$ as its single $k$-face, we can also take $\Psi_1$ to be the part of $\Psi$ in $F_k/F_{-1}$. Now, $\Phi_1$ and $\Psi_1$ are flags of $F_k/F_{-1}$, so by supposition, there is an automorphism $\alpha_1$ of $F_k/F_{-1}$ that sends $\Phi_1$ to $\Psi_1$. Furthermore, we can identify the generators of $\Gamma(F_k/F_{-1})$ with the generators $\{\rho_1, \ldots, \rho_{k-1}\}$ of $\Gamma$. Now, in a similar fashion, we can take $\Phi_2$ and $\Psi_2$ to be the parts of the flags $\Phi$ and $\Psi$ in $F_n/F_k$, and we obtain an automorphism $\alpha_2$ of $F_n/F_k$ that sends $\Phi_2$ to $\Psi_2$. This automorphism can be written in terms of the generators $\{\rho_{k+1}, \ldots, \rho_{n-1}\}$ of $\Gamma$. Now, set $\alpha = \alpha_1 \alpha_2$. Then $\alpha \in \Gamma_k$ and $\alpha$ sends $\Phi$ to $\Psi$, which is what we wanted to prove.

**Corollary 1.17.** Let $\mathcal{P}$ be a regular $n$-semi-polytope. Then the stabilizer of $F_0$ is $\langle \rho_1, \ldots, \rho_{n-1} \rangle$, and the stabilizer of $F_{n-1}$ is $\langle \rho_0, \ldots, \rho_{n-2} \rangle$.

Let $\mathcal{P}$ be a regular $n$-semi-polytope. Clearly, any regular semi-polytope is equivelar; that is, $\mathcal{P}$ has a Schl"afli symbol. In fact, $\mathcal{P}$ is of type $\{p_1, \ldots, p_{n-1}\}$, where $p_i$ is the order of $\rho_{i-1}\rho_i$. It is also clear that the facets of $\mathcal{P}$ must all be isomorphic, and that the vertex-figures must all be isomorphic as well, so that $\mathcal{P}$ is of type $\{K, L\}$ for some regular $(n-1)$-pre-polytopes $K$ and $L$. (The facets and vertex-figures might not be semi-polytopes; i.e., might not have flag-connected facets and vertex-figures themselves.)

For a given Schl"afli symbol $\{p_1, \ldots, p_{n-1}\}$, there is a regular, universal polytope of that type that covers all semi-polytopes of that type. We use $\{p_1, \ldots, p_{n-1}\}$ to refer to this universal polytope. Whenever this universal polytope is the face-lattice of a regular convex
polytope, the name used here is the same as the usual Schlafli symbol for the polytope (see [7]). The automorphism group of \( \{p_1, \ldots, p_{n-1}\} \) is the string Coxeter group \([p_1, \ldots, p_{n-1}]\).

All polytopes of rank \(n\) are quotients of the universal polytope of rank \(n\)

\[ \mathcal{U}_n := \{\infty, \ldots, \infty\}. \]

This polytope has automorphism group

\[ W_n := [\infty, \ldots, \infty] = \langle \rho_0, \ldots, \rho_{n-1} \mid \rho_0^2 = \cdots = \rho_{n-1}^2 = \epsilon, (\rho_i \rho_j)^2 = \epsilon \text{ when } |i - j| \geq 2 \rangle. \]  

(1.1)

There is a natural action of \(W_n\) on the flags of any regular \(n\)-semi-polytope \(\mathcal{P}\) with base flag \(\Phi\), given by \(\Phi^{i_{j_1} \ldots j_k} \rho_i = \Phi^{i_{j_1} \ldots j_k} \). That is, \(\rho_i\) from \(W_n\) acts in the same way as \(\rho_i\) from \(\Gamma(\mathcal{P})\), justifying our choice of notation. Now, if \(w \in W_n\) stabilizes \(\Phi\), then for any \(i \in \{0, \ldots, n-1\}\),

\[ \Phi \rho_i w \rho_i = \Phi^{i_w} \rho_i \]

\[ = (\Phi w)^i \rho_i \]

\[ = \Phi^i \rho_i \]

\[ = \Phi. \]

Therefore, \(\text{Stab}_{W_n}(\Phi)\) is normal in \(W_n\). Furthermore, if \(w\) stabilizes \(\Phi\), then it must in fact stabilize all flags of \(\mathcal{P}\). Therefore, \(W_n/\text{Stab}_{W_n}(\Phi)\) acts simply transitively on the flags of \(\mathcal{P}\).

In fact, \(\Gamma(\mathcal{P}) \simeq W_n/\text{Stab}_{W_n}(\Phi)\).

In [23], the authors show how to build a regular polytope from a string C-group. In fact, the construction can be applied to any sggi to obtain a flagged poset. The construction is as follows. Let \(\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle\) be an sggi. Let \(\Gamma_j = \langle \rho_i \mid i \neq j \rangle\), as before. We define a
poset \( \mathcal{P}(\Gamma) \) by setting the \( j \)-faces of \( \mathcal{P}(\Gamma) \) to be the cosets \( \Gamma_j \varphi \), where \( \varphi \in \Gamma \). Then we say that \( \Gamma_j \varphi \leq \Gamma_k \psi \) if \( j \leq k \) and \( \Gamma_j \varphi \cap \Gamma_k \psi \neq \emptyset \). Lemma 2E6 in [23] shows that this defines a partial order.

We note that for any sggi \( \Gamma \), there is a natural group homomorphism \( f : \Gamma \to \Gamma(\mathcal{P}(\Gamma)) \) sending \( \psi \) to \( f_{\psi} \), where \( f_{\psi} \) acts on \( \mathcal{P}(\Gamma) \) by

\[(\Gamma_j \varphi) f_{\psi} = \Gamma_j (\varphi \psi) .\]

Several of the results in [23, Ch. 2E] apply not just to string C-groups, but to sggi’s or string pre-C-groups. We reproduce the statements here (with minor modification in some cases).

**Lemma 1.18 (2E8).** Let \( \Gamma \) be an sggi, and let \( K \subseteq \{0, 1, \ldots, n - 1\} \). Then \( \Gamma \) is transitive on all chains of the form \( \{\Gamma_i \varphi_i \mid i \in K\} \).

**Lemma 1.19 (2E9).** Let \( \Gamma \) be a string pre-C-group. Then the stabilizer of the base flag is trivial, and for any \( j \in \{0, \ldots, n - 1\} \), the stabilizer of the chain \( \{\Gamma_0, \ldots, \Gamma_{j-1}, \Gamma_{j+1}, \ldots, \Gamma_{n-1}\} \) is \( \langle \rho_j \rangle \).

**Lemma 1.20 (2E10).** The natural homomorphism \( f : \Gamma \to \Gamma(\mathcal{P}(\Gamma)) \) is injective.

Now we are able to expand [23, Thm. 2E11] to sggi’s and string pre-C-groups.

**Theorem 1.21.** Let \( \Gamma \) be an sggi. Then \( \mathcal{P}(\Gamma) \) is a flag-connected flagged poset. If \( \Gamma \) is also a string pre-C-group, then \( \mathcal{P}(\Gamma) \) is a regular semi-polytope with \( \Gamma(\mathcal{P}(\Gamma)) \simeq \Gamma \).

**Proof.** First, suppose that \( \Gamma \) is an sggi. We have \( \Gamma_{-1} = \Gamma_n = \Gamma \), and these play the role of the minimal and maximal face, respectively. Now, let \( \Omega \) be a chain in \( \mathcal{P}(\Gamma) \); we want to show that it extends to a chain with \( n + 2 \) elements. By Lemma 1.18, we can write \( \Omega \) as \( \Phi_K \varphi \) for
some $\varphi \in \Gamma$, where $\Phi_K = \{\Gamma_i \mid i \in K\}$. Then $\Omega$ extends to the flag $\Phi_\varphi$. Thus, we see that $\mathcal{P}(\Gamma)$ is a flagged poset. In fact, since the action of $\Gamma$ on $\mathcal{P}(\Gamma)$ is transitive on flags (again by Lemma 1.18), we see that $\mathcal{P}(\Gamma)$ is flag-connected.

Now, suppose $\Gamma$ is also a string pre-$C$-group. Then Lemma 1.19 shows that the base flag has a unique $j$-adjacent flag for each $j$ in $\{0, \ldots, n-1\}$. Since the action of $\Gamma$ is transitive on flags, this implies that every flag has a unique $j$-adjacent flag for each $j$ in $\{0, \ldots, n-1\}$. Therefore, $\mathcal{P}(\Gamma)$ is a flag-connected pre-polytope.

We now need to show that the facets and vertex-figures of $\mathcal{P}(\Gamma)$ are flag-connected. Fix a facet $F_{n-1}$ of $\mathcal{P}(\Gamma)$. By Corollary 1.17, the stabilizer of the facet is $\Gamma_{n-1} = \langle \rho_0, \ldots, \rho_{n-2} \rangle$, and we can naturally identify this stabilizer with the automorphism group of the facet. Now, given two flags in the facet, they naturally extend to flags $\Phi$ and $\Psi$ of $\mathcal{P}(\Gamma)$, both containing $F_{n-1}$. Since $\Gamma$ acts transitively on the flags, there are elements $\varphi$ and $\psi$ of $\Gamma$ such that $\Phi = (\Gamma_0\varphi, \ldots, \Gamma_{n-1}\varphi)$ and $\Psi = (\Gamma_0\psi, \ldots, \Gamma_{n-1}\psi)$. Then $\varphi^{-1}\psi$ sends $\Phi$ to $\Psi$. Since $\Gamma_{n-1}\varphi = \Gamma_{n-1}\psi = F_{n-1}$, it follows that $\varphi^{-1}\psi \in \Gamma_{n-1}$. Therefore, there is an automorphism of the facet sending $\Phi$ to $\Psi$, which shows that the facets are flag-connected. The same argument works to show that the vertex-figures are flag-connected as well.

It remains to show that $\mathcal{P}(\Gamma)$ is regular and that $\Gamma(\mathcal{P}(\Gamma)) \simeq \Gamma$. By Lemma 1.20, the homomorphism $f : \Gamma \rightarrow \Gamma(\mathcal{P}(\Gamma))$ is injective. I claim that it is surjective as well. Let $\alpha \in \Gamma(\mathcal{P}(\Gamma))$; we want to find some $\psi \in \Gamma$ such that $f_\psi = \alpha$. By Lemma 1.18, there is an element $\psi \in \Gamma$ such that $f_\psi$ sends the base flag $\Gamma_0 < \cdots < \Gamma_{n-1}$ to the same place that $\alpha$ sends it. Therefore, $f_\psi\alpha^{-1}$ fixes the base flag. Then by Corollary 1.8, this means that $f_\psi\alpha^{-1}$ is the identity, and thus $f_\psi = \alpha$. So $f$ is an isomorphism. Finally, this means that $\Gamma(\mathcal{P}(\Gamma))$ is transitive on the flags of $\mathcal{P}(\Gamma)$, and thus $\mathcal{P}(\Gamma)$ is regular. $\square$

Thus, just as we have a one-to-one correspondence between string $C$-groups of rank $n$
on a distinguished set of generators and regular $n$-polytopes, so too we have a one-to-one correspondence between string pre-C-groups of rank $n$ on a distinguished set of generators and regular $n$-semi-polytopes. In particular, for any regular $n$-semi-polytope $Q$, the semi-polytope $P(\Gamma(Q))$ is naturally isomorphic to $Q$.

This correspondence also respects the natural covering relations:

**Proposition 1.22.** Let $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ and $\Gamma' = \langle \rho'_0, \ldots, \rho'_{n-1} \rangle$ be string pre-C-groups. If $\Gamma$ covers $\Gamma'$ via the epimorphism sending each $\rho_i$ to $\rho'_i$, then $P(\Gamma)$ covers $P(\Gamma')$.

**Proof.** Write $\pi$ for the natural epimorphism from $\Gamma$ to $\Gamma'$. We define a function $\overline{\pi} : P(\Gamma) \to P(\Gamma')$ by $(\Gamma_j \varphi) \overline{\pi} = (\Gamma'_j (\varphi \pi))$. To check that this is a well-defined function, we note that if $\Gamma_j = \Gamma_j \varphi$, then $\varphi \in \Gamma_j = \langle \rho_i \mid i \neq j \rangle$. Therefore, $\varphi \pi \in \Gamma'_j$ and thus $\Gamma'_j = \Gamma'_j (\varphi \pi)$. Next, we need to show that $\overline{\pi}$ preserves the order relation. If $\Gamma_j \varphi \leq \Gamma_k \psi$, then by definition there is some $\alpha \in \Gamma$ such that $\alpha \in \Gamma_j \varphi \cap \Gamma_k \psi$. It follows that $\alpha \pi \in \Gamma'_j (\varphi \pi) \cap \Gamma'_k (\psi \pi)$, and therefore $\Gamma'_j (\varphi \pi) \leq \Gamma'_k (\psi \pi)$. So $\overline{\pi}$ is a polytope homomorphism.

Now we need to show that $\overline{\pi}$ is a covering. It is clear that $\overline{\pi}$ preserves rank, so by Proposition 1.10, it suffices to show that it preserves flag adjacency. Let $\Phi$ be a flag of $P(\Gamma)$. We can write it as $\Phi = (\Gamma_0 \varphi, \ldots, \Gamma_{n-1} \varphi)$ for some $\varphi \in \Gamma$. The $j$-adjacent flag $\Phi^j$ can be written $\Phi^j = (\Gamma_0 \varphi, \ldots, \Gamma_{n-1} \rho_j \varphi)$. Then $\overline{\pi}$ sends $\Phi$ to $(\Gamma_0^j (\varphi \pi), \ldots, \Gamma_{n-1}^j (\varphi \pi))$, and since $(\rho_j \varphi) \pi = (\rho_j \pi) (\varphi \pi) = \rho'_j (\varphi \pi)$, it follows that $\overline{\pi}$ sends $\Phi^j$ to $(\Gamma_0^j \rho'_j (\varphi \pi), \ldots, \Gamma_{n-1}^j \rho'_j (\varphi \pi))$, which is $j$-adjacent to the image of $\Phi$. Therefore, $\overline{\pi}$ preserves adjacency, and thus it is a covering.

In light of the correspondence between string pre-C-groups and regular semi-polytopes, we view regular semi-polytopes $P$ and $Q$ (with chosen base flags) as being equal if and only if the stabilizer in $W_n$ of the base flag is the same in each case. That is, if $\Gamma(P) = W_n/M$ and $\Gamma(Q) = W_n/K$ for some normal subgroups $M$ and $K$ of $W_n$, then $P = Q$ if and only if
We also use the correspondence to characterize covering relations between regular semi-polytopes:

**Proposition 1.23.** Let $\mathcal{P}$ and $\mathcal{Q}$ be regular $n$-semi-polytopes with $\Gamma(\mathcal{P}) \cong W_n/M$ and $\Gamma(\mathcal{Q}) \cong W_n/K$. Then $\mathcal{Q}$ covers $\mathcal{P}$ if and only if $K \leq M$.

**Proof.** Suppose $\varphi : \mathcal{Q} \to \mathcal{P}$ is a covering. Since $\mathcal{P}$ and $\mathcal{Q}$ are regular, we can compose any covering with automorphisms of $\mathcal{P}$ and $\mathcal{Q}$ to obtain a covering that sends the base flag $\Psi$ of $\mathcal{Q}$ to the base flag $\Phi$ of $\mathcal{P}$. Thus, without loss of generality, suppose that $\varphi$ sends $\Psi$ to $\Phi$. Let $\psi = \rho_{i_1} \cdots \rho_{i_k} \in W_n$ stabilize $\Psi$. Then $\Psi^{i_k \cdots i_1} = \Psi$. Now, since $\varphi$ preserves adjacency, it must send $\Psi = \Psi^{i_k \cdots i_1}$ to $\Phi^{i_k \cdots i_1}$. Thus we see that $\Phi^{i_k \cdots i_1} = \Phi$, so that $\psi$ stabilizes $\Phi$ as well. Since $K$ is the stabilizer of $\Psi$ and $M$ is the stabilizer of $\Phi$, we see that $K \leq M$.

Conversely, if $K \leq M$, then $\Gamma(\mathcal{Q})$ naturally covers $\Gamma(\mathcal{P})$, and by Proposition 1.22, $\mathcal{P}(\Gamma(\mathcal{Q}))$ covers $\mathcal{P}(\Gamma(\mathcal{P}))$. Since the former is naturally isomorphic to $\mathcal{Q}$ and the latter is naturally isomorphic to $\mathcal{P}$, the claim is proved.

### 1.4 Direct regularity

The directly regular polytopes form an important subclass of the regular polytopes. Direct regularity is essentially an orientability condition; intuitively, a regular polytope is directly regular if there is no rotation that carries the polytope to its mirror image. In order to make this intuitive idea precise, we need to define rotations.

Let $\mathcal{P}$ be a regular semi-polytope with automorphism group $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$. Then the abstract rotations $\sigma_i := \rho_{i-1} \rho_i$ (for $i = 1, \ldots, n-1$) generate the rotation subgroup $\Gamma^+(\mathcal{P})$ of $\Gamma(\mathcal{P})$, which has index at most 2. If the index is 2, then it is not possible to write any of
the abstract reflections \( \rho_i \) as a product of abstract rotations. This is exactly what we want for direct regularity.

**Definition 1.24.** Let \( \mathcal{P} \) be a regular flag-connected \( n \)-pre-polytope. Then \( \mathcal{P} \) is directly regular if \( n \leq 1 \) or if \( \Gamma^+(\mathcal{P}) \) has index 2 in \( \Gamma(\mathcal{P}) \).

Note that all universal polytopes \( \{p_1, \ldots, p_{n-1}\} \), including the regular convex polytopes, are directly regular.

The flag-connected sections of a directly regular semi-polytope are themselves directly regular. In particular, the rotation subgroup of \( \langle \rho_i, \ldots, \rho_j \rangle \) is \( \langle \sigma_{i+1}, \ldots, \sigma_j \rangle \), and this must have index 2 since \( \langle \sigma_1, \ldots, \sigma_{n-1} \rangle \) has index 2 in \( \langle \rho_0, \ldots, \rho_{n-1} \rangle \).

The rotation subgroups of sgg’s inherit many nice properties, including analogues of the intersection properties. Let \( \mathcal{P} \) be a directly regular semi-polytope, and let \( \Gamma^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle = \Gamma^+(\mathcal{P}) \). We define \( \Gamma^+_0 = \langle \sigma_2, \ldots, \sigma_{n-1} \rangle \), \( \Gamma^+_{n-1} = \langle \sigma_1, \ldots, \sigma_{n-2} \rangle \), and for \( 1 \leq i \leq n-2 \), we define \( \Gamma^+_i = \langle \sigma_1, \ldots, \sigma_i, \sigma_i \sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{n-1} \rangle \). Then \( \Gamma^+_i \) is the stabilizer of the \( i \)-face in the base flag. From this we get the following analogue of the empty intersection property:

\[
\Gamma^+_0 \cap \cdots \cap \Gamma^+_{n-1} = \langle \epsilon \rangle,
\]

and the following analogue of the weak intersection property:

\[
\bigcap_{j \neq i} \Gamma^+_j = \langle \epsilon \rangle.
\]

Note that the latter condition simply says that no element of \( \Gamma^+ \) sends the base flag to an \( i \)-adjacent flag; this is required in order for \( \mathcal{P} \) to be directly regular. In any case, we see that the empty intersection property is redundant when dealing with the rotation subgroups of string pre-C-groups, and we need only consider the weak intersection property.
Now suppose that $\mathcal{P}$ is a directly regular polytope (not just a semi-polytope), and let $\Gamma^+$ be as before. For $1 \leq i \leq j \leq n - 1$ we define

$$
\tau_{i,j} := \sigma_i \sigma_{i+1} \cdots \sigma_j,
$$

and for $0 \leq i \leq n$ let $\tau_{0,i} := \tau_{i,n} := 1$; then $\tau_{i,i} = \sigma_i$ for $i \neq 0, n$. For $I \subseteq \{0, \ldots, n - 1\}$ we define

$$
\Gamma^+_I := \langle \tau_{i,j} \mid i \leq j \text{ and } i - 1, j \in I \rangle.
$$

Then we get the following analogue of the intersection property [30]:

$$
\Gamma^+_I \cap \Gamma^+_J = \Gamma^+_{I \cap J} \text{ for } I, J \subseteq \{0, \ldots, n - 1\}.
$$

In light of these properties, we classify rotation subgroups in a way similar to sgg’s (see Definition 1.13):

**Definition 1.25.** Let $\Gamma^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ be a finitely generated group such that $(\sigma_i \cdots \sigma_j)^2 = \epsilon$ for all $i < j$. Then we call $\Gamma^+$ a string rotation group. If $\Gamma^+$ also satisfies the weak intersection property (for rotation groups), we call $\Gamma^+$ a string pre-chi-group, and if $\Gamma^+$ satisfies the full intersection property (for rotation groups), we call $\Gamma^+$ a string chi-group. (The use of “chi” here emphasizes the connection these groups have with chiral polytopes, which will be defined in the next section.)

### 1.5 Chirality

Chirality is a phenomenon that has been studied in many subjects, including chemistry, biology, and physics. Intuitively, an object is said to be chiral if it is impossible to bring
the object to coincide with its mirror image using only rotations and translations. The
prototypical example is the human hand; indeed, we often refer to a chiral object as being
“left-handed” or “right-handed”.

In the study of abstract polytopes, we take a somewhat more specific definition of chiral-
ity. We will get to the technical definition in a moment, but the idea is that a chiral polytope
has full rotational symmetry, but that there is no automorphism that brings the left-handed
form to the right-handed form. Compare this to the definition of a directly regular polytope,
where there is an automorphism that brings the left-handed form to the right-handed form,
but no such rotation.

If \( \mathcal{P} \) is a directly regular semi-polytope, then the abstract rotation \( \sigma_i := \rho_{i-1}\rho_i \) is an
automorphism of \( \mathcal{P} \) that sends the base flag \( \Phi \) to \( \Phi^{i,i-1} \). On the other hand, if \( \mathcal{P} \) is chiral,
we should have an automorphism that acts the same way as the rotation \( \sigma_i \), whereas no
automorphism acts as the reflection \( \rho_i \). We make this precise below.

**Definition 1.26.** Let \( \mathcal{P} \) be a flag-connected \( n \)-pre-polytope with base flag \( \Phi \). Then \( \mathcal{P} \) is
chiral if there is no automorphism that sends \( \Phi \) to any of its adjacent flags \( \Phi^i \), and if there
are automorphisms \( \sigma_1, \ldots, \sigma_{n-1} \) such that for each \( i \), \( \Phi\sigma_i = \Phi^{i,i-1} \).

As an immediate consequence of the definition, a chiral semi-polytope has two flag-orbits
under the action of \( \Gamma(\mathcal{P}) \), and adjacent flags are always in different orbits.

In analogy with directly regular polytopes, we call the elements \( \sigma_i \) abstract rotations. In
fact, these elements generate the entire group \( \Gamma(\mathcal{P}) \), so that everything that was said in the
previous section about rotation subgroups of directly regular semi-polytopes also applies to
the automorphism groups of chiral semi-polytopes. In particular, the automorphism group of
a chiral polytope is a string chi-group, and the automorphism group of a chiral semi-polytope
is a string pre-chi-group. For convenience, we define \( \Gamma^+(\mathcal{P}) := \Gamma(\mathcal{P}) \) whenever \( \mathcal{P} \) is chiral.
If $\Gamma^+$ is the rotation subgroup of a directly regular semi-polytope $\mathcal{P}$, then conjugation by $\rho_0$ (in $\Gamma$) is a group automorphism of $\Gamma^+$ that sends $\sigma_1$ to $\sigma_1^{-1}$, $\sigma_2$ to $\sigma_1^2\sigma_2$, and fixes every other generator. Similar group automorphisms are induced by conjugation with any $\rho_i$. On the other hand, if $\Gamma^+$ is the automorphism group of a chiral semi-polytope, then $\Gamma^+$ admits no automorphism that moves the generators this way.

If $\sigma_1, \ldots, \sigma_{n-1}$ is the set of generators of $\Gamma^+$ corresponding to the base flag $\Phi$ of a chiral semi-polytope $\mathcal{P}$, then $\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \ldots, \sigma_{n-1}$ is the set of generators of $\Gamma^+$ corresponding to the base flag $\Phi^0$ that is 0-adjacent to $\Phi$. The choice of base flag is essentially a choice of orientation; we say that a chiral semi-polytope occurs in two enantiomorphic (mirror image) forms. We write $\overline{\mathcal{P}}$ for the enantiomorphic form of $\mathcal{P}$, with base flag $\Phi^0$ where $\Phi$ is the base flag of $\mathcal{P}$.

Let $\Gamma^+$ be generated by $\sigma_1, \ldots, \sigma_{n-1}$, and let $w$ be a reduced word in the free group on these generators. We define the enantiomorphic (or mirror image) word $\overline{w}$ of $w$ to be the word obtained from $w$ by replacing every occurrence of $\sigma_1$ by $\sigma_1^{-1}$ and $\sigma_2$ by $\sigma_1^2\sigma_2$, while keeping all $\sigma_j$ with $j \geq 3$ unchanged. Then if $\Gamma^+ = \Gamma^+(\mathcal{P})$ for a directly regular semi-polytope $\mathcal{P}$, the elements of $\Gamma^+(\mathcal{P})$ corresponding to $w$ and $\overline{w}$ are conjugate in $\Gamma(\mathcal{P})$. Note that $\overline{\overline{w}} = w$ for all words $w$.

The flag-connected sections of a chiral $n$-semi-polytope are either chiral or directly regular. Furthermore, the $(n-2)$-faces and the co-faces at edges must be directly regular [30]. This feature makes it difficult to find new examples of chiral semi-polytopes, which is part of what makes them so interesting. Given a regular $n$-polytope $\mathcal{P}$, there are typically many ways to build a regular $(n+1)$-polytope with facets isomorphic to $\mathcal{P}$ (for example, see [27]). On the other hand, if $\mathcal{P}$ is a chiral $n$-polytope with chiral facets, then it is impossible to build a chiral $(n+1)$-polytope with facets isomorphic to $\mathcal{P}$.

The simplest examples of chiral polytopes are the torus maps $\{4,4\}_{(b,c)}$, $\{3,6\}_{(b,c)}$ and
\{6,3\}_{(b,c)}$, with $b, c \neq 0$ and $b \neq c$ (see [8]). Many other examples of chiral 3-polytopes are known, and they mostly come to us from the study of regular irreflexible maps. A fair amount of progress has been made in finding chiral 4-polytopes as well, primarily by extending the known chiral 3-polytopes. In ranks 5 and higher, however, only a handful of examples of chiral polytopes are known [5]. Indeed, it was only recently that we knew for certain that there are chiral polytopes in every rank [28].

Chiral semi-polytopes, like regular semi-polytopes, are always equivelar. A chiral $n$-semi-polytope $P$ is of type \{\(p_1, \ldots, p_{n-1}\}\}, where $p_i$ is the order of the generator $\sigma_i$. Chiral semi-polytopes also have the property that their faces are all isomorphic, and so are there vertex-figures, so a chiral semi-polytope $P$ is of type \{\(K, L\)\} for some directly regular or chiral semi-polytopes $K$ and $L$.

The rotation subgroup $W_n^+$ of $W_n$ (see Equation 1.1) has presentation

\[
W_n^+ := [\infty, \ldots, \infty]^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \mid (\sigma_i \cdots \sigma_j)^2 = \epsilon \text{ for } i < j \rangle. \tag{1.2}
\]

Analogous to the case with regular semi-polytopes, there is a natural action of $W_n^+$ on the flags of any chiral or directly regular $n$-semi-polytope $P$ with base flag $\Phi$. In fact, $\Gamma^+(P) \simeq W_n^+/\text{Stab}_{W_n^+}(\Phi)$.

There is also a way to build a semi-polytope from a string rotation group that satisfies a suitable intersection property. In particular, let $\Gamma^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$. Recall that we defined $\Gamma_0^+ = \langle \sigma_2, \ldots, \sigma_{n-1} \rangle$, $\Gamma_{n-1}^+ = \langle \sigma_1, \ldots, \sigma_{n-2} \rangle$, and for $1 \leq i \leq n-2$, $\Gamma_i^+ = \langle \sigma_1, \ldots, \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{n-1} \rangle$. Then we build a poset $P(\Gamma^+)$ by setting the $j$-faces to be the cosets of $\Gamma_j^+$, and where again we say that $\Gamma_j^+ \varphi \leq \Gamma_k^+ \psi$ if $j \leq k$ and if $\Gamma_j^+ \varphi \cap \Gamma_k^+ \psi \neq \emptyset$.

In [30], the authors prove that when we apply the preceding construction to a string chi-group, we get a polytope that is either directly regular or chiral. As before, the construction
works more generally, and the arguments used with sggi’s and string pre-C-groups can be (somewhat tediously) adapted for string rotation groups and string pre-chi-groups, with the help of some results from [30]. We are able to conclude the following.

**Proposition 1.27.** Let $\Gamma^+$ be a string rotation group. Then $\mathcal{P}(\Gamma^+)$ is a flagged poset.

**Proposition 1.28.** Let $\Gamma^+$ be a string pre-chi-group. Then $\mathcal{P}(\Gamma^+)$ is a semi-polytope with $\Gamma^+(\mathcal{P}(\Gamma^+)) = \Gamma^+$. Furthermore, $\mathcal{P}(\Gamma^+)$ is directly regular if and only if there is an involutory group automorphism that sends $\sigma_1$ to $\sigma_1^{-1}$, $\sigma_2$ to $\sigma_1^2\sigma_2$, and $\sigma_i$ to $\sigma_i$ for $i \geq 3$. Otherwise, $\mathcal{P}(\Gamma^+)$ is chiral.

Thus, there is a natural correspondence between string pre-chi-groups and semi-polytopes that are chiral or directly regular. Using this correspondence, we can characterize covers of chiral and directly regular polytopes in terms of their rotation groups. The proof of the following proposition is essentially the same as for Proposition 1.23.

**Proposition 1.29.** Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular $n$-semi-polytopes with $\Gamma^+(\mathcal{P}) \simeq W_n^+ / M$ and $\Gamma^+(\mathcal{Q}) \simeq W_n^+ / K$. Then $\mathcal{Q}$ covers $\mathcal{P}$ if and only if $K \leq M$.

### 1.6 Duality

The usual geometric operation of dualizing a convex polytope has a natural counterpart for abstract polytopes. For any polytope $\mathcal{P}$, we obtain the **dual of** $\mathcal{P}$ (denoted $\mathcal{P}^\delta$) by simply reversing the partial order. A **duality** from $\mathcal{P}$ to $\mathcal{Q}$ is an anti-isomorphism; that is, a bijection $\delta$ between the face sets such that $F < G$ in $\mathcal{P}$ if and only if $\delta(F) > \delta(G)$ in $\mathcal{Q}$. If a polytope is isomorphic to its dual, then it is called **self-dual**.

If $\mathcal{P}$ is of type $\{\mathcal{K}, \mathcal{L}\}$, then $\mathcal{P}^\delta$ is of type $\{\mathcal{L}^\delta, \mathcal{K}^\delta\}$. Therefore, in order for $\mathcal{P}$ to be self-dual, it is necessary (but not sufficient) that $\mathcal{K}$ is isomorphic to $\mathcal{L}^\delta$ (in which case it is
also true that $K^\delta$ is isomorphic to $L$).

A self-dual regular polytope always possesses a duality that fixes the base flag. For chiral polytopes, this may not be the case. If a self-dual chiral polytope $P$ possesses a duality that sends the base flag to another flag in the same orbit (but reversing its direction), then there is a duality that fixes the base flag, and we say that $P$ is \textit{properly self-dual} \cite{16}. In this case, the groups $\Gamma^+(P)$ and $\Gamma^+(P^\delta)$ have identical presentations. If a self-dual chiral polytope has no duality that fixes the base flag, then every duality sends the base flag to a flag in the other orbit, and $P$ is said to be \textit{improperly self-dual}. In this case, the groups $\Gamma^+(P)$ and $\Gamma^+(P^\delta)$ have identical presentations instead.

If $P$ is a regular polytope with $\Gamma(P) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, then the group of $P^\delta$ is $\Gamma(P^\delta) = \langle \rho'_0, \ldots, \rho'_{n-1} \rangle$, where $\rho'_i = \rho_{n-1-i}$. If $P$ is a directly regular or chiral polytope with $\Gamma^+(P) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$, then the rotation group of $P^\delta$ is $\Gamma^+(P^\delta) = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle$, where $\sigma'_i = \sigma_{n-1}^{-1}$. Equivalently, if $\Gamma^+(P)$ has presentation

$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid w_1, \ldots, w_k \rangle$$

then $\Gamma^+(P^\delta)$ has presentation

$$\langle \sigma'_1, \ldots, \sigma'_{n-1} \mid \delta(w_1), \ldots, \delta(w_k) \rangle,$$

where if $w = \sigma_{i_1} \cdots \sigma_{i_j}$, then $\delta(w) := (\sigma'_{n-i_1})^{-1} \cdots (\sigma'_{n-i_j})^{-1}$.

Suppose $P$ is a chiral or directly regular polytope with $\Gamma^+(P) = W_n^+/M$. Then $\Gamma^+(P^\delta) = W_n^+/\delta(M)$, where $\delta(M) = \{ \delta(w) \mid w \in M \}$. If $\delta(M) = M$, then $\Gamma^+(P) = \Gamma^+(P^\delta)$, so $P$ is properly self-dual.

If $P$ is a chiral polytope, then $P^\delta$ is naturally isomorphic to $\overline{P^\delta}$. Indeed, if $w$ is a word
in the generators $\sigma_1, \ldots, \sigma_{n-1}$ of $\Gamma^+(\mathcal{P})$, then

$$\delta(w) = (\sigma_1 \sigma_2 \cdots \sigma_{n-1}) \overline{\delta(w)} (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^{-1},$$

so we see that the presentation for $\overline{\mathcal{P}^\delta}$ is equivalent to that of $\overline{\mathcal{P}^\delta}$. In particular, if $\Gamma^+(\mathcal{P}) = W_n^+/M$, then

$$\delta(M) = \overline{\delta(M)} := \{ \overline{\delta(w)} \mid w \in W \}$$

since $M$ is a normal subgroup of $W_n^+$, and thus $\delta(\delta(M)) = M$.

### 1.7 Petrie duality

There is a second duality operation that is defined on abstract polyhedra. We start with the idea of the Petrie polygons of a polyhedron. The definition usually used is that a Petrie polygon is a maximal edge-path (closed if the polyhedron is finite) such that every two successive edges lie on a common face, but no three successive edges do. For most polyhedra, this definition works well, but it is slightly problematic for polyhedra having at least one vertex of degree 2. In particular, it is not always possible for three successive edges to avoid lying on a common face. We thus make the following more precise definition:

**Definition 1.30.** Let $\mathcal{P}$ be a flag-connected pre-polyhedron, and let $E = (\ldots, E_1, E_2, \ldots)$ be a (possibly finite) sequence of edges of $\mathcal{P}$. Then $E$ is a Petrie polygon of $\mathcal{P}$ if there is a sequence of flags $(\ldots, \Phi_1, \Phi_2, \ldots)$ of $\mathcal{P}$ such that:

1. For each $i$, $E_i \in \Phi_i$.
2. For $i \neq j$, $\Phi_i \neq \Phi_j$.
3. For each $i$, $\Phi_{i+1} = \Phi_i^{0,1,2}$. 

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If $E$ is finite, say $E = (E_1, \ldots, E_k)$, then $\Phi_1 = \Phi_k^{0,1,2}$.

A Petrie polygon in this sense does have the property that every two successive edges share a common face, and no three successive edges share a common face unless it is unavoidable.

We can now describe the second duality operation. Given a polyhedron $\mathcal{P}$, its Petrie dual $\mathcal{P}^\pi$ consists of the same vertices and edges as $\mathcal{P}$, but its faces are the Petrie polygons of $\mathcal{P}$. Taking the Petrie dual of a polyhedron also forces the old faces to be the new Petrie polygons, so that $\mathcal{P}^\pi\pi \simeq \mathcal{P}$. If $\mathcal{P}$ is isomorphic to $\mathcal{P}^\pi$, then we say that $\mathcal{P}$ is self-Petrie.

In some cases, the Petrie dual of a polyhedron is not a polyhedron, or even a pre-polyhedron. For example, the Petrie dual of the toroidal polyhedron $\{3,6\}_{(1,1)}$ is not a pre-polyhedron [23]. Nevertheless, our later results are able to steer around such complications.

When talking about polyhedra, we will frequently consider their Petrie polygons, so we expand some of our earlier terminology. If $\mathcal{P}$ is a regular polyhedron, then its Petrie polygons all have the same length, and that length is the order of $\rho_0\rho_1\rho_2$ in $\Gamma(\mathcal{P})$. A regular polyhedron of type $\{p,q\}$ and with Petrie polygons of length $r$ is also said to be of type $\{p,q\}_r$. If $\mathcal{P}$ is of type $\{p,q\}_r$ and it covers every other polyhedron of type $\{p,q\}_r$, then we call it the universal polyhedron of type $\{p,q\}_r$ and we denote it by $\{p,q\}_r$. The automorphism group of $\{p,q\}_r$ is denoted by $[p,q]_r$, and this group is the quotient of $[p,q]$ by the single extra relation $(\rho_0\rho_1\rho_2)^r = \epsilon$. We will also extend our notation and use $[p,q]_r$ for the group with presentation

\[
\langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^p = (\rho_0\rho_2)^q = (\rho_1\rho_2)^r = (\rho_0\rho_1\rho_2)^r = \epsilon \rangle,
\]

even when there is no universal polyhedron of type $\{p,q\}_r$.

The operations $\delta$ and $\pi$ form a group of order 6, isomorphic to $S_3$. A regular polyhedron that is self-dual and self-Petrie is invariant under all 6 operations; such a polyhedron must
be of type \( \{n,n\}_n \) for some \( n \geq 2 \). In general, if \( \mathcal{P} \) is of type \( \{p,q\}_r \), then \( \mathcal{P}^\delta \) is of type \( \{q,p\}_r \), and \( \mathcal{P}^\pi \) is of type \( \{r,q\}_p \). Furthermore, the dual and the Petrie dual of a universal polyhedron is again universal.
Chapter 2

The mixing operation

The mixing operation on polytopes ([23], [24]) is analogous to the parallel product of groups [33], the tensor product of graphs, and the join of maps and hypermaps [4]. It gives us a natural way to find the minimal common cover of two regular or chiral polytopes. The basic method is to find the parallel product of the automorphism groups (or rotation groups) of two polytopes, and then to build a poset from the resulting group. There are two main challenges. First, we want to determine how the structure of the mix depends on the two component polytopes. Second, we want to know when the mix of two polytopes is a polytope, and not just a poset. Neither issue is solved in full generality here, and it seems unlikely that a fully general and satisfactory solution exists to either problem. Nevertheless, in a wide variety of cases, it is possible to easily determine the structure and polytopality of the mix.

2.1 Mixing finitely generated groups

Let \( \Gamma = \langle x_1, \ldots, x_n \rangle \) and \( \Gamma' = \langle x'_1, \ldots, x'_n \rangle \) be finitely generated groups on \( n \) generators. Then the elements \( z_i := (x_i, x'_i) \in \Gamma \times \Gamma' \) (for \( i = 1, \ldots, n \)) generate a subgroup of \( \Gamma \times \Gamma' \) that we
call the mix of $\Gamma$ and $\Gamma'$, denoted $\Gamma \diamond \Gamma'$ (see [23, Ch.7A]). In other words, $\Gamma \diamond \Gamma'$ is the diagonal subgroup of $\Gamma \times \Gamma'$, consisting of elements that are equivalent to $(x_{i_1} \cdots x_{i_k}, x'_{i_1} \cdots x'_{i_k})$ for some sequence $i_1, \ldots, i_k$.

The following proposition follows immediately.

**Proposition 2.1.** Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$, and let $z_i = (x_i, x'_i)$ for each $i \in \{1, \ldots, n\}$. Then for $1 \leq i \leq j \leq n$,

$$\langle z_i, \ldots, z_j \rangle = \langle x_i, \ldots, x_j \rangle \diamond \langle x'_i, \ldots, x'_j \rangle.$$

Let us consider some simple examples of the mix of two groups.

**Example 2.2.** Let $\Gamma = \langle x_1 \mid x_1^r = \epsilon \rangle$ and let $\Gamma' = \langle x'_1 \mid (x'_1)^s = \epsilon \rangle$. Then $\Gamma \diamond \Gamma'$ has a single generator $z_1 = (x_1, x'_1)$, and the order of $z_1$ is the least common multiple of $r$ and $s$.

**Example 2.3.** Let

$$\Gamma = \langle x_1, x_2 \mid x_1^2 = x_2^2 = (x_1x_2)^r = \epsilon \rangle,$$

and

$$\Gamma' = \langle x'_1, x'_2 \mid (x'_1)^2 = (x'_2)^2 = (x'_1x'_2)^s = \epsilon \rangle.$$

Then the generators $z_1 = (x_1, x'_1)$ and $z_2 = (x_2, x'_2)$ also have order 2, and thus $\Gamma \diamond \Gamma'$ is a dihedral group. The order of $z_1z_2$ is the least common multiple of $r$ and $s$.

**Example 2.4.** For any $n$-generator group $\Gamma$, the group $\Gamma \diamond \Gamma$ is naturally isomorphic to $\Gamma$.

There is another way to characterize the mix of two groups which is in many ways more useful:

**Proposition 2.5.** Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$. Let $\Lambda = \langle y_1, \ldots, y_n \rangle$, and suppose that there is an epimorphism $f : \Lambda \to \Gamma$ with $f(y_i) = x_i$ and an epimorphism
$f' : \Lambda \to \Gamma'$ with $f'(y_i) = x'_i$. Let $K = \ker f$ and $K' = \ker f'$. Then $\Gamma \circ \Gamma' \simeq \Lambda / (K \cap K')$.

**Proof.** Consider the homomorphism $g : \Lambda \to \Gamma \times \Gamma'$ defined by $g(w) = (f(w), f'(w))$. The image of $y_i$ is $(x_i, x'_i)$, and since $\Lambda$ is generated by the elements $y_i$, the image of $g$ is generated by the diagonal elements $(x_i, x'_i)$, and so $\text{Im } g = \Gamma \circ \Gamma'$. Now, $w \in \ker g$ if and only if $w \in \ker f \cap \ker f' = K \cap K'$. Therefore,

$$\Gamma \circ \Gamma' = \text{Im } g \simeq \Lambda / \ker g = \Lambda / (K \cap K').$$

Note that we can always find such a group $\Lambda$; for instance, take $\Lambda$ to be the free group on $n$ generators. In our applications, we will usually use $\Lambda = W_n$ or $\Lambda = W_n^+$.

Our next result, and many later results, use the idea of a natural cover of a group. We say that a group $\Gamma = \langle x_1, \ldots, x_n \rangle$ naturally covers $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$ if there is a well-defined epimorphism $f : \Gamma \to \Gamma'$ defined by $f(x_i) = x'_i$. If $\Lambda$ is a group on $n$ specified generators and $\Gamma = \Lambda / M$ and $\Gamma' = \Lambda / M'$, then $\Gamma$ naturally covers $\Gamma'$ if and only if $M \leq M'$.

**Corollary 2.6.** Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$. Then $\Gamma \circ \Gamma'$ is the minimal group that naturally covers $\Gamma$ and $\Gamma'$. That is, if $\Lambda$ naturally covers both $\Gamma$ and $\Gamma'$, then it also naturally covers $\Gamma \circ \Gamma'$.

Dual to the mix is the *comix* of two groups. We define the comix $\Gamma \square \Gamma'$ to be the amalgamated free product that identifies the generators of $\Gamma$ with the corresponding generators of $\Gamma'$. That is, if $\Gamma$ has presentation $\langle x_1, \ldots, x_n \mid R \rangle$ and $\Gamma'$ has presentation $\langle x'_1, \ldots, x'_n \mid S \rangle$, then $\Gamma \square \Gamma'$ has presentation

$$\langle x_1, x'_1, \ldots, x_n, x'_n \mid R, S, x_1^{-1}x'_1, \ldots, x_n^{-1}x'_n \rangle.$$
Example 2.7. Let $\Gamma = \langle x_1 \mid x_1^r = \epsilon \rangle$ and let $\Gamma' = \langle x_1' \mid (x_1')^s = \epsilon \rangle$. Then

$$\Gamma \boxempty \Gamma' = \langle x_1 \mid x_1^r = x_1^s = \epsilon \rangle.$$ 

Therefore, the group of $\Gamma \boxempty \Gamma'$ is cyclic, and its order is the greatest common divisor of $r$ and $s$.

Example 2.8. Let

$$\Gamma = \langle \sigma_1, \sigma_2 \mid \sigma_1^4 = \sigma_2^3 = (\sigma_1 \sigma_2)^2 = \epsilon \rangle,$$

the rotation subgroup of the cube, and let

$$\Gamma' = \langle \sigma_1', \sigma_2' \mid (\sigma_1')^3 = (\sigma_2')^3 = (\sigma_1' \sigma_2')^2 = \epsilon \rangle,$$

the rotation subgroup of the tetrahedron. Then a presentation for $\Gamma \boxempty \Gamma'$ is given by

$$\Gamma \boxempty \Gamma' = \langle \sigma_1, \sigma_2 \mid \sigma_1^4 = \sigma_2^3 = (\sigma_1 \sigma_2)^2 = \epsilon \rangle.$$ 

Since $\sigma_1^4 = \sigma_2^3$, the generator $\sigma_1$ must be trivial. Therefore, $\sigma_2^3 = \sigma_2^2 = \epsilon$, and $\sigma_2$ must be trivial as well. So $\Gamma \boxempty \Gamma'$ is the trivial group.

Example 2.9. For any group $\Gamma$ on $n$ generators, the group $\Gamma \boxempty \Gamma$ is naturally isomorphic to $\Gamma$.

As with the mix, there is a simple way to characterize the comix using quotients.

Proposition 2.10. Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x_1', \ldots, x_n' \rangle$ be groups on $n$ generators. Suppose $\Lambda = \langle y_1, \ldots, y_n \rangle$ such that there is an epimorphism $f : \Lambda \to \Gamma$ with $f(y_i) = x_i$ and an epimorphism $f' : \Lambda \to \Gamma'$ with $f(y_i) = x_i'$. Let $K = \ker f$ and $K' = \ker f'$. Then $\Gamma \boxempty \Gamma' \simeq \Lambda / KK'$.
Proof. Since $\Gamma \sqcap \Gamma'$ satisfies all the relations of $\Gamma$ and of $\Gamma'$, it is clear that it is a quotient of both. Thus we get an epimorphism $g$ from $\Lambda$ to $\Gamma \sqcap \Gamma'$. Set $N = \ker g$, so that $\Gamma \sqcap \Gamma' \simeq \Lambda/N$. Since there are natural epimorphisms from $\Gamma$ to $\Gamma \sqcap \Gamma'$ and from $\Gamma'$ to $\Gamma \sqcap \Gamma'$, this means that $K \leq N$ and $K' \leq N$, so that $KK' \leq N$. Now, $K$ is the normal closure in $\Lambda$ of the relators in $R$ and $K'$ is the normal closure in $\Lambda$ of the relators in $S$, while $N$ is the normal closure of the relators in both. Since $K$ and $K'$ are both normal in $\Lambda$, so is $KK'$, and thus the normal closure of the relators in $R$ and $S$ must be contained in $KK'$; i.e., $N \leq KK'$. Thus $N = KK'$ and the claim is proved. 

Corollary 2.11. Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$. Then $\Gamma \sqcap \Gamma'$ is the maximal group that is naturally covered by $\Gamma$ and $\Gamma'$. That is, if $\Lambda$ is naturally covered by both $\Gamma$ and $\Gamma'$, then it is also naturally covered by $\Gamma \sqcap \Gamma'$.

The mixing and comixing operations on groups are commutative, associative, and idempotent in a natural way:

Proposition 2.12. Let $\Gamma = \langle x_1, \ldots, x_n \rangle$, $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$, and $\Gamma'' = \langle x''_1, \ldots, x''_n \rangle$ be groups on $n$ specified generators. Then

1. $\Gamma \circ \Gamma' \simeq \Gamma' \circ \Gamma$
2. $\Gamma \sqcap \Gamma' \simeq \Gamma' \sqcap \Gamma$
3. $(\Gamma \circ \Gamma') \circ \Gamma'' \simeq \Gamma \circ (\Gamma' \circ \Gamma'')$
4. $(\Gamma \sqcap \Gamma') \sqcap \Gamma'' \simeq \Gamma \sqcap (\Gamma' \sqcap \Gamma'')$
5. $\Gamma \circ \Gamma \simeq \Gamma \sqcap \Gamma \simeq \Gamma$,

where the isomorphisms are all given by sending each generator of the first group to the corresponding generator of the second group.
Our next result is obvious and we have already used it implicitly in our earlier examples.

**Proposition 2.13.** Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$. Let $z_i = (x_i, x'_i)$ so that $\Gamma \circ \Gamma' = \langle z_1, \ldots, z_n \rangle$. Then for each word $w = z_{i_1} \cdots z_{i_k}$, the order of $w$ is the least common multiple of the orders of $x_{i_1} \cdots x_{i_k}$ and $x'_{i_1} \cdots x'_{i_k}$.

**Corollary 2.14.** The mix of two sggi’s is an sggi, and the mix of two string rotation groups is a string rotation group.

**Proof.** Suppose $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$ are sggi’s; that is, the order of each $x_i$ and $x'_i$ is 2, and the orders of $x_i x_j$ and $x'_i x'_j$ are also both 2 whenever $|i - j| \geq 2$. Let $z_i = (x_i, x'_i)$ so that $\Gamma \circ \Gamma' = \langle z_1, \ldots, z_n \rangle$. Then by Proposition 2.13, each $z_i$ has order 2, and $z_i z_j$ has order 2 when $|i - j| \geq 2$; therefore, $\Gamma \circ \Gamma'$ is an sggi. The proof for string rotation groups is analogous.  

**Corollary 2.15.** The mix of two string pre-chi-groups is a string pre-chi-group.

**Proof.** Let $\Gamma = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and $\Gamma' = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle$ be string pre-chi-groups. In particular, they are string rotation groups, and Corollary 2.14 tells us that $\Gamma \circ \Gamma'$ is a string rotation group. Now we just have to show that $\Gamma \circ \Gamma'$ satisfies the weak intersection property. For each $i$, let $\beta_i = (\sigma_i, \sigma'_i)$, and write $\Lambda$ for $\Gamma \circ \Gamma' = \langle \beta_1, \ldots, \beta_{n-1} \rangle$. Now, if $\beta_{i_1} \cdots \beta_{i_k} \in \bigcap_{j \neq i} \Gamma_j$, then that means that $\sigma_{i_1} \cdots \sigma_{i_k} \in \bigcap_{j \neq i} \Gamma_j$ and $\sigma'_{i_1} \cdots \sigma'_{i_k} \in \bigcap_{j \neq i} \Gamma'_j$. Since $\Gamma$ and $\Gamma'$ both satisfy the weak intersection property, both of these intersections are trivial, and thus $\beta_{i_1} \cdots \beta_{i_k} = (\epsilon, \epsilon)$, and we see that $\Lambda = \Gamma \circ \Gamma'$ satisfies the weak intersection property as well.  

We note that analogous statements about string pre-C-groups, string C-groups, and string chi-groups are not true. For example, $[4, 3, 3] \circ [3, 3, 4]$ is not a string C-group, and $[4, 3, 3]^+ \circ [3, 3, 4]^+$ is not a string chi-group. In fact, we will see later that the mix of two string C-groups need not even be a string pre-C-group.  

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2.2 Variance groups and the structure of the mix

We observed earlier that the canonical projections from $\Gamma \times \Gamma'$ to its factors restrict to epimorphisms from $\Gamma \vartriangleright \Gamma'$ to its factors. It is natural, then, to study the kernel of these epimorphisms as a way of determining the structure of $\Gamma \vartriangleright \Gamma'$.

**Definition 2.16.** Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$. Let $f : \Gamma \vartriangleright \Gamma' \rightarrow \Gamma$ and $f' : \Gamma \vartriangleright \Gamma' \rightarrow \Gamma'$ be the natural epimorphisms. We denote $\ker f$ by $X(\Gamma'|\Gamma)$ and call it the variance group of $\Gamma'$ with respect to $\Gamma$. Similarly, we denote $\ker f'$ by $X(\Gamma|\Gamma')$ and call it the variance group of $\Gamma$ with respect to $\Gamma'$. In other words, $X(\Gamma'|\Gamma)$ consists of the elements of $\Gamma \vartriangleright \Gamma'$ of the form $(1, w)$ with $w \in \Gamma'$, and $X(\Gamma|\Gamma')$ consists of the elements of $\Gamma \vartriangleright \Gamma'$ of the form $(w, 1)$ with $w \in \Gamma$.

**Example 2.17.** For any group $\Gamma$ on $n$ generators, the variance group $X(\Gamma|\Gamma)$ is trivial.

**Example 2.18.** Suppose $\Gamma$ is cyclic of order $r$ and $\Gamma'$ is cyclic of order $s$, with $r$ and $s$ coprime. Then $X(\Gamma|\Gamma')$ is isomorphic to $\Gamma$ and $X(\Gamma'|\Gamma)$ is isomorphic to $\Gamma'$.

As with the mix of two groups, representing $\Gamma$ and $\Gamma'$ as quotients of a group $\Lambda$ gives us a natural way to determine the structure of $X(\Gamma|\Gamma')$.

**Proposition 2.19.** Let $\Gamma = \Lambda/K$ and $\Gamma' = \Lambda/K'$ be finitely generated groups, where $\Lambda$ is a group with $n$ specified generators. Then:

1. $X(\Gamma|\Gamma') \cong K'/(K \cap K') \cong KK'/K$ and $X(\Gamma'|\Gamma) \cong K/(K \cap K') \cong KK'/K'$.

2. Let $g : \Gamma \rightarrow \Gamma \vartriangleright \Gamma'$ and $g' : \Gamma' \rightarrow \Gamma \vartriangleright \Gamma'$ be the natural epimorphisms. Then $\ker g \cong X(\Gamma|\Gamma')$ and $\ker g' \cong X(\Gamma'|\Gamma)$. In particular, $X(\Gamma|\Gamma')$ and $X(\Gamma'|\Gamma)$ can be viewed as normal subgroups of $\Gamma$ and $\Gamma'$, respectively.
Proof. (1) We prove the claim for $X(\Gamma|\Gamma')$; the other case is analogous. Consider

$$f' : \Gamma \otimes \Gamma' \to \Gamma'.$$

We can rewrite this using quotients of $\Lambda$ to

$$f' : \Lambda / (K \cap K') \to \Lambda / K',$$

and $X(\Gamma|\Gamma')$ is the kernel of $f'$. Since $f'$ sends a coset $w(K \cap K')$ to the coset $wK'$, the kernel consists of those cosets $w(K \cap K')$ with $w \in K'$. Thus the variance group $X(\Gamma|\Gamma')$ is $K'/(K \cap K')$. Furthermore, since $K$ and $K'$ are both normal in $\Lambda$, this is isomorphic to $KK'/K$.

(2) Again, we rewrite the domain and codomain of $g$ using quotients of $\Lambda$, so that

$$g : \Lambda / K \to \Lambda / KK'.$$

This homomorphism sends a coset $wK$ to a coset $wKK'$, so the kernel consists of those cosets $wK$ with $w \in KK'$. Thus we see that the kernel of $g$ is $KK'/K$, and by the previous part, this is isomorphic to $X(\Gamma|\Gamma')$.

\[\square\]

**Corollary 2.20.** Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$. Then $X(\Gamma'|\Gamma)$ is trivial if and only if there is a well-defined homomorphism $\pi : \Gamma \to \Gamma'$ sending $x_i$ to $x'_i$.

**Proof.** Write $\Gamma = F/K$ and $\Gamma' = F/K'$, where $F$ is the free group on $n$ generators and $K$ and $K'$ are the necessary normal subgroups. Then by Proposition 2.19, the group $X(\Gamma'|\Gamma)$ is isomorphic to $K/(K \cap K')$. Now, the required homomorphism from $\Gamma$ to $\Gamma'$ exists if and
only if \( K \leq K' \), and in this case, \( K/(K \cap K') \) is trivial.

The fact that \( f' : \Gamma \odot \Gamma' \to \Gamma' \) and \( g : \Gamma \to \Gamma \square \Gamma' \) have isomorphic kernels is what allows us to use the comix of two polytopes to derive information about the mix. The following properties are immediate:

**Proposition 2.21.** Let \( \Gamma = \langle x_1, \ldots, x_n \rangle \) and \( \Gamma' = \langle x'_1, \ldots, x'_n \rangle \) be finite. Then:

1. \( \Gamma \odot \Gamma' \) is finite, and
   \[
   |\Gamma \odot \Gamma'| = |X(\Gamma|\Gamma')| \cdot |\Gamma'| = |X(\Gamma'|\Gamma)| \cdot |\Gamma|.
   \]

2. \( \Gamma \square \Gamma' \) is finite, and
   \[
   |\Gamma \square \Gamma'| = \frac{|\Gamma|}{|X(\Gamma'|\Gamma')|} = \frac{|\Gamma'|}{|X(\Gamma|\Gamma)|}.
   \]

3. \[
|\Gamma \odot \Gamma'| \cdot |\Gamma \square \Gamma'| = |\Gamma| \cdot |\Gamma'|.
\]

4. \[
\frac{|X(\Gamma'|\Gamma)|}{|X(\Gamma|\Gamma')|} = \frac{|\Gamma|}{|\Gamma'|}.
\]

Intuitively speaking, the group \( X(\Gamma'|\Gamma) \) tells us something about how much \( \Gamma' \) and \( \Gamma \) overlap. If \( \Gamma \) covers \( \Gamma' \), then as we saw above, \( X(\Gamma'|\Gamma) \) is trivial. At the other extreme, if \( \Gamma \odot \Gamma' = \Gamma \times \Gamma' \), then \( \Gamma \) and \( \Gamma' \) have no overlap, and \( X(\Gamma'|\Gamma) = \Gamma' \).

In general, we will be much more interested in the mix of two groups than the comix. However, as we see in Proposition 2.21, knowing some information about the comix gives us information about the mix. This is important because it is often difficult to calculate information about \( \Gamma \odot \Gamma' \) directly. Even if \( \Gamma \) and \( \Gamma' \) have only a few hundred elements – a
fairly modest size for the automorphism groups of regular polytopes – their mix can have tens or hundreds of thousands of elements, and calculating the exact number using Todd-Coxeter enumeration can be practically impossible. On the other hand, for the groups we deal with, $\Gamma \square \Gamma'$ tends to be quite small and easily calculated by a computer or even by hand. Proposition 2.21 is thus invaluable for determining the size of $\Gamma \diamond \Gamma'$.

Even when $\Gamma$ and $\Gamma'$ are both infinite, it often happens that $\Gamma \square \Gamma'$ is trivial. Just as with finite groups, the mix $\Gamma \diamond \Gamma'$ is equal to $\Gamma \times \Gamma'$ whenever $\Gamma \square \Gamma'$ is trivial, even if $\Gamma$ and $\Gamma'$ are infinite. In order to prove this, though, it is helpful to write $\Gamma \diamond \Gamma'$ as an explicit subgroup of $\Gamma \times \Gamma'$.

**Proposition 2.22.** Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x_1', \ldots, x_n' \rangle$. Let $N = X(\Gamma|\Gamma')$ and $N' = X(\Gamma'|\Gamma)$, and let $g : \Gamma/N \to \Gamma'/N'$ be the isomorphism sending $x_iN$ to $x_i'N'$. Then

$$\Gamma \diamond \Gamma' = \{(u, v) \in \Gamma \times \Gamma' | g(uN) = vN'\}.$$  

In particular, $X(\Gamma|\Gamma') \times X(\Gamma'|\Gamma)$ is a normal subgroup of $\Gamma \diamond \Gamma'$.

**Proof.** First of all, by Proposition 2.19,

$$\Gamma/N \simeq \Gamma \square \Gamma' \simeq \Gamma'/N',$$

so that $g$ really is an isomorphism. Let $f : \Gamma \diamond \Gamma' \to \Gamma$ and $f' : \Gamma \diamond \Gamma' \to \Gamma'$ be the natural epimorphisms, so that $N' = \ker f$ and $N = \ker f'$ (see Definition 2.16). Then the first part follows directly by Lemma A.1 (Goursat’s Lemma; see the Appendix). For the last part, note that if $u \in N$ and $v \in N'$, then $uN = N$ and $vN' = N'$. Therefore, $g(uN) = g(N) = N' = vN'$, so that $(u, v) \in \Gamma \diamond \Gamma'$. Then we see that $N \times N'$ is a subgroup of $\Gamma \diamond \Gamma'$, and since $N$ is a normal subgroup of $\Gamma$ and $N'$ is a normal subgroup of $\Gamma'$, it
immediately follows that $N \times N' = X(\Gamma|\Gamma') \times X(\Gamma'|\Gamma)$ is normal in $\Gamma \diamond \Gamma'$.

Now we are able to refine Proposition 2.21 to include infinite groups.

**Theorem 2.23.** Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$. If $\Gamma \square \Gamma'$ is finite of order $k$, then the index of $\Gamma \diamond \Gamma'$ in $\Gamma \times \Gamma'$ is $k$.

**Proof.** If $\Gamma$ and $\Gamma'$ are finite, this follows immediately from the third equation of Proposition 2.21. Now let $\Gamma$ and $\Gamma'$ be of arbitrary size, and set $N = X(\Gamma|\Gamma')$ and $N' = X(\Gamma'|\Gamma)$, as in Proposition 2.22. If $\Gamma \square \Gamma'$ is finite of order $k$, then $N'$ has index $k$ in $\Gamma'$. For each fixed $u \in \Gamma$, Proposition 2.22 says that the element $(u, v)$ is in $\Gamma \diamond \Gamma'$ if and only if $g(uN) = vN'$ in $\Gamma \square \Gamma'$. In other words, having fixed $u$ we can pick any $v$ that lies in the same (corresponding) coset. Then since $N'$ has index $k$ in $\Gamma'$, the set of $(u, v) \in \Gamma \diamond \Gamma'$ for a fixed $u$ has “index” $k$ in $\{u\} \times \Gamma'$. Therefore, letting $u$ range over all elements of $\Gamma$, we see that $\Gamma \diamond \Gamma'$ has index $k$ in $\Gamma \times \Gamma'$.

**Corollary 2.24.** If $\Gamma \square \Gamma'$ is trivial, then $\Gamma \diamond \Gamma' = \Gamma \times \Gamma'$.

We have now examined the structure of $\Gamma \diamond \Gamma'$ in some detail. The next thing we will examine is how the variance group of the mix or comix depends on the component groups.

**Proposition 2.25.** Let $\Gamma$, $\Delta$, and $\Delta'$ be groups with $n$ specified generators. Then $\Gamma \diamond \Delta'$ naturally covers $\Gamma \diamond \Delta$ if and only if $X(\Delta'|\Gamma)$ (viewed as a subgroup of $\Gamma \diamond \Delta'$) naturally covers $X(\Delta|\Gamma)$ (viewed as a subgroup of $\Gamma \diamond \Delta$).

**Proof.** Let $F$ be the free group on $n$ generators. Let $M$, $K$, and $K'$ be the normal subgroups of $F$ such that $\Gamma = F/M$, $\Delta = F/K$, and $\Delta' = F/K'$. Then $X(\Delta'|\Gamma) = MK'/K' \cong M/(M \cap K')$ and $X(\Delta|\Gamma) = MK/K \cong M/(M \cap K)$, by Proposition 2.19. Furthermore, $\Gamma \diamond \Delta' = F/(M \cap K')$, and $\Gamma \diamond \Delta = F/(M \cap K)$, by Proposition 2.5. Thus we see that $\Gamma \diamond \Delta'$
naturally covers $\Gamma \diamond \Delta$ if and only if $M \cap K' \leq M \cap K$, and this is true if and only if $X(\Delta'|\Gamma)$ naturally covers $X(\Delta|\Gamma)$.

\[\text{Corollary 2.26. Let } \Gamma, \Gamma', \text{ and } \Delta \text{ be groups with } n \text{ specified generators. Then}\]

\[X(\Delta \diamond \Gamma' | \Gamma \diamond \Gamma') \simeq X(\Delta | \Gamma \diamond \Gamma').\]

\[\text{Proof. Since } (\Delta \diamond \Gamma') \diamond (\Gamma \diamond \Gamma') = \Delta \diamond (\Gamma \diamond \Gamma'), \text{ each of these mixes naturally covers the other, and Proposition 2.25 tells us that the given variance groups naturally cover each other. Therefore they must be isomorphic.} \]

\[\text{Proposition 2.27. Let } \Gamma, \Delta, \text{ and } \Delta' \text{ be groups with } n \text{ specified generators. Then } X(\Delta \diamond \Delta'|\Gamma) \text{ is isomorphic to a subgroup of } X(\Delta|\Gamma) \times X(\Delta'|\Gamma).\]

\[\text{Proof. Let } F \text{ be the free group on } n \text{ generators, and let } M, K, \text{ and } K' \text{ be the normal subgroups of } F \text{ such that } \Gamma = F/M, \Delta = F/K, \text{ and } \Delta' = F/K'. \text{ Then by Proposition 2.19,}\]

\[X(\Delta'|\Gamma) = MK'/K' \simeq M/M \cap K',\]

\[X(\Delta|\Gamma) = MK/K \simeq M/M \cap K,\]

and

\[X(\Delta \diamond \Delta'|\Gamma) = M(K \cap K')/(K \cap K') \simeq M/(M \cap K \cap K').\]

Now, consider the group homomorphism $\varphi : M(K \cap K') \to MK/K \times MK'/K'$ sending each element $w$ to $(wK, wK')$. This is clearly well-defined, and the kernel is $K \cap K'$. Therefore, $M(K \cap K')/(K \cap K')$ is isomorphic to the image of $\varphi$, which is what we wanted to show. \]

\[\text{Proposition 2.28. Let } \Gamma, \Gamma', \text{ and } \Delta \text{ be groups with } n \text{ specified generators. Then } X(\Delta|\Gamma \diamond \Gamma') \text{ is a normal subgroup of } X(\Delta|\Gamma) \cap X(\Delta|\Gamma').\]
Proof. Let $F$ be the free group on $n$ generators, and let $M$, $M'$, and $K$ be the normal subgroups of $F$ such that $\Gamma = F/M$, $\Gamma' = F/M'$, and $\Delta = F/K$. Then we have $X(\Delta|\Gamma \odot \Gamma') = (M \cap M')K/K$, $X(\Delta|\Gamma) = MK/K$, and $X(\Delta|\Gamma') = M'K/K$. It is clear that $(M \cap M')K$ is a subgroup of $MK \cap M'K$, and since $(M \cap M')K$ is a normal subgroup of $F$, it is also a normal subgroup of $MK \cap M'K$. Therefore, $(M \cap M')K/K$ is a normal subgroup of $MK/K \cap M'K/K = (MK \cap M'K)/K$. \hfill \Box

Proposition 2.29. Let $\Gamma$, $\Gamma'$, $\Delta$, and $\Delta'$ be groups with $n$ specified generators. Then $X(\Delta \odot \Delta'|\Gamma \odot \Gamma')$ is isomorphic to a subgroup of $X(\Delta|\Gamma) \times X(\Delta'|\Gamma')$.

Proof. First, by Proposition 2.27, the group $X(\Delta \odot \Delta'|\Gamma \odot \Gamma')$ is isomorphic to a subgroup of $X(\Delta|\Gamma \odot \Gamma') \times X(\Delta'|\Gamma \odot \Gamma')$. Then Proposition 2.28 shows that $X(\Delta|\Gamma \odot \Gamma') \leq X(\Delta|\Gamma)$ and that $X(\Delta'|\Gamma \odot \Gamma') \leq X(\Delta'|\Gamma')$. \hfill \Box

Proposition 2.30. Let $\Gamma$, $\Gamma'$, and $\Delta$ be groups with $n$ specified generators. Then $X(\Delta \odot \Gamma'|\Gamma \odot \Gamma')$ is isomorphic to a normal subgroup of $X(\Delta|\Gamma)$.

Proof. First, Corollary 2.26 says that $X(\Delta \odot \Gamma'|\Gamma \odot \Gamma') \simeq X(\Delta|\Gamma \odot \Gamma')$; then Proposition 2.28 tells us that this is a normal subgroup of $X(\Delta|\Gamma)$. \hfill \Box

2.3 Mixing polytopes

There are two ways we can apply our definition of mixing to semi-polytopes. First, suppose $\mathcal{P}$ and $\mathcal{Q}$ are regular $n$-semi-polytopes. Then by Corollary 2.14, the mix of $\Gamma(\mathcal{P})$ with $\Gamma(\mathcal{Q})$ is an ssgi. Thus, we can build a regular flagged poset from this group (as detailed in Section 1.3), and we call this poset the mix of $\mathcal{P}$ and $\mathcal{Q}$ and denote it $\mathcal{P} \odot \mathcal{Q}$. Similarly, if $\mathcal{P}$ and $\mathcal{Q}$ are chiral or directly regular $n$-semi-polytopes, then $\Gamma^+(\mathcal{P}) \odot \Gamma^+(\mathcal{Q})$ is a string pre-chi-group by
Corollary 2.15. Therefore, we can build a chiral or directly regular semi-polytope from this group, and we again call this semi-polytope the mix of $\mathcal{P}$ and $\mathcal{Q}$ and denote it $\mathcal{P} \diamond \mathcal{Q}$.

We similarly define the comix of $\mathcal{P}$ and $\mathcal{Q}$ to be the poset built from $\Gamma(\mathcal{P}) \Box \Gamma(\mathcal{Q})$ or $\Gamma^+(\mathcal{P}) \Box \Gamma^+(\mathcal{Q})$. However, these groups often have trivial generators, which makes it problematic to build a semi-polytope out of them.

Whenever $\mathcal{P}$ and $\mathcal{Q}$ are directly regular semi-polytopes, there are two ways we can define their mix. The following proposition says that these two definitions are compatible. As usual, we let $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, $\Gamma(\mathcal{Q}) = \langle \rho'_0, \ldots, \rho'_{n-1} \rangle$, $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$, and $\Gamma^+(\mathcal{Q}) = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle$.

**Proposition 2.31.** Let $\mathcal{P}$ and $\mathcal{Q}$ be directly regular $n$-semi-polytopes. Then $(\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q}))^+ = \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$.

**Proof.** Taking $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q}) = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ and $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = \langle \beta_1, \ldots, \beta_{n-1} \rangle$ as above (where $\alpha_i = (\rho_i, \rho'_i)$ and $\beta_i = (\sigma_i, \sigma'_i)$), we see that $(\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q}))^+$ is generated by the elements $\alpha_i \alpha_i = (\rho_{i-1}, \rho'_{i-1})(\rho_i, \rho'_i) = (\rho_{i-1} \rho_i, \rho'_{i-1} \rho'_i) = (\sigma_i, \sigma'_i) = \beta_i$, where $i = 1, \ldots, n-1$. \qed

Thus we see that whether we build $\mathcal{P} \diamond \mathcal{Q}$ from $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$ or $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$, we get the same poset.

For our purposes, it is generally more useful to work with mixing via rotation groups, but we will occasionally need to mix via full automorphism groups. As we will see, the results governing the mixing of both types of groups are very similar.

We now consider a few examples.

**Example 2.32.** Let $\mathcal{P} = \{p\}$ and $\mathcal{Q} = \{q\}$. Then $\mathcal{P} \diamond \mathcal{Q} = \{r\}$, where $r$ is the least common multiple of $p$ and $q$. 

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Example 2.33. Let $P = \{4, 4\}_{(1,2)}$, the chiral torus map with 5 square faces meeting 4 at each vertex [8]. Let $Q = \{4, 4\}_{(2,1)}$, the mirror image of $P$. Then $P \circ Q = \{4, 4\}_{(5,0)}$, which is a regular torus map with 25 square faces meeting 4 at each vertex [1].

To apply the results of the previous section, we want to look at $X(\Gamma(P)|\Gamma(Q))$ if $P$ and $Q$ are chiral or directly regular, and at $X(\Gamma(P)|\Gamma(Q))$ if $P$ and $Q$ are regular. In the case where $P$ and $Q$ are both directly regular, we can use either one:

**Proposition 2.34.** Let $P$ and $Q$ be directly regular $n$-semi-polytopes. Then

$$X(\Gamma^+(P)|\Gamma^+(Q)) = X(\Gamma(P)|\Gamma(Q)).$$

**Proof.** Recall that for any chiral or directly regular $n$-semi-polytope $P$, we can write $\Gamma^+(P) = W_n^+/M$ for some normal subgroup $M$ of $W_n^+$ (see Equation 1.2). Similarly, for any regular $n$-semi-polytope $P$, we can write $\Gamma(P) = W_n/M$ (Equation 1.1). Let $M$ and $K$ be the normal subgroups of $W_n$ such that $\Gamma(P) = W_n/M$ and $\Gamma(Q) = W_n/K$. Then we must have $\Gamma^+(P) = W_n^+/M$ and $\Gamma^+(Q) = W_n^+/K$. Then by Proposition 2.19, we see that $X(\Gamma^+(P)|\Gamma^+(Q)) \simeq MK/M \simeq X(\Gamma(P)|\Gamma(Q))$. \qed

Therefore, we make the following definition:

**Definition 2.35.** Let $P$ and $Q$ be $n$-semi-polytopes. If $P$ and $Q$ are both regular, we define $X(P|Q) := X(\Gamma(P)|\Gamma(Q))$. If instead $P$ and $Q$ are both chiral or directly regular, we define $X(P|Q) := X(\Gamma^+(P)|\Gamma^+(Q))$.

The following two propositions follow directly from Propositions 2.5 and 2.10.

**Proposition 2.36.** Let $P$ and $Q$ be chiral or directly regular $n$-semi-polytopes, with $\Gamma^+(P) = W_n^+/M$ and $\Gamma^+(Q) = W_n^+/K$. Then $\Gamma^+(P \circ Q) \simeq W_n^+/(M \cap K)$ and $\Gamma^+(P \square Q) \simeq W_n^+/MK$. 53
Proposition 2.37. Let $\mathcal{P}$ and $\mathcal{Q}$ be regular $n$-semi-polytopes, with $\Gamma(\mathcal{P}) = W_n/M$ and $\Gamma(\mathcal{Q}) = W_n/K$. Then $\Gamma(\mathcal{P}) \circ \Gamma(\mathcal{Q}) \simeq W_n/(M \cap K)$ and $\Gamma(\mathcal{P}) \sqcap \Gamma(\mathcal{Q}) \simeq W_n/MK$.

The mixing and comixing operations on polytopes are commutative, associative, and idempotent, just as they are on groups:

Proposition 2.38. Let $\mathcal{P}$ and $\mathcal{Q}$ be $n$-semi-polytopes such that they are both regular or they are both chiral or directly regular. Then:

1. $\mathcal{P} \circ \mathcal{Q} = \mathcal{Q} \circ \mathcal{P}$
2. $\mathcal{P} \sqcap \mathcal{Q} = \mathcal{Q} \sqcap \mathcal{P}$
3. $(\mathcal{P} \circ \mathcal{Q}) \circ \mathcal{R} = \mathcal{P} \circ (\mathcal{Q} \circ \mathcal{R})$
4. $(\mathcal{P} \sqcap \mathcal{Q}) \sqcap \mathcal{R} = \mathcal{P} \sqcap (\mathcal{Q} \sqcap \mathcal{R})$
5. $\mathcal{P} \circ \mathcal{P} = \mathcal{P} \sqcap \mathcal{P} = \mathcal{P}$.

Of course, even if $\mathcal{P} \simeq \mathcal{Q}$, it may be the case that $\mathcal{P} \circ \mathcal{Q} \not\simeq \mathcal{P}$. For example, if $\mathcal{P}$ is a chiral polytope, then $\mathcal{P} \simeq \overline{\mathcal{P}}$, but $\mathcal{P} \circ \overline{\mathcal{P}}$ is not isomorphic to $\mathcal{P}$.

Proposition 2.39. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ both be regular $n$-polytopes, or both be chiral or directly regular $n$-polytopes. If $\mathcal{P}_1$ is of type $\{K_1, L_1\}$ and $\mathcal{P}_2$ is of type $\{K_2, L_2\}$, then $\mathcal{P}_1 \circ \mathcal{P}_2$ is of type $\{K_1 \circ K_2, L_1 \circ L_2\}$. Furthermore, if $\mathcal{P}_1$ is of type $\{p_1, \ldots, p_{n-1}\}$ and $\mathcal{P}_2$ is of type $\{q_1, \ldots, q_{n-1}\}$, then $\mathcal{P}_1 \circ \mathcal{P}_2$ is of type $\{\ell_1, \ldots, \ell_{n-1}\}$, where $\ell_i = \text{lcm}(p_i, q_i)$ for $i \in \{1, \ldots, n-1\}$.

Proof. The first part follows from Proposition 2.1, and the second part follows from Proposition 2.13.
2.4 Polytopality of the mix

Our main goal is to use the mixing operation to construct new polytopes. We have already seen that the mix of chiral or directly regular semi-polytopes is a semi-polytope. What about the mix of two regular semi-polytopes? In other words, suppose $\Gamma$ and $\Gamma'$ are string pre-C-groups, and let $\Lambda = \Gamma \circ \Gamma'$. Under what conditions can $\Lambda$ fail to itself be a string pre-C-group?

We start by noting that Corollary 2.14 says that $\Lambda$ is an sgg. Now, for each $i$, the group $\Lambda_i$ is a subgroup of $\Gamma_i \times \Gamma'_i$, and therefore

$$\Lambda_0 \cap \cdots \cap \Lambda_{n-1} \leq (\Gamma_0 \times \Gamma'_0) \cap \cdots \cap (\Gamma_{n-1} \times \Gamma'_{n-1})$$

$$= (\Gamma_0 \cap \cdots \cap \Gamma_{n-1}) \times (\Gamma'_0 \cap \cdots \cap \Gamma'_{n-1})$$

$$= \langle \varepsilon \rangle \times \langle \varepsilon' \rangle.$$

Thus we see that $\Lambda$ has the empty intersection property. We can apply a similar argument to conclude that

$$\bigcap_{j \neq i} \Lambda_j \leq \langle \rho_i \rangle \times \langle \rho'_i \rangle.$$ 

Therefore, the only way that $\Lambda$ can fail to be a string pre-C-group is if for some $i$, the intersection $\bigcap_{j \neq i} \Lambda_j$ is equal to $\langle \rho_i \rangle \times \langle \rho'_i \rangle$.

Let $\alpha_i = (\rho_i, \rho'_i)$ so that $\Lambda = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$. By Proposition 1.14,

$$\bigcap_{j \neq i} \Lambda_j = \langle \alpha_0, \ldots, \alpha_i \rangle \cap \langle \alpha_i, \ldots, \alpha_{n-1} \rangle.$$ 

Suppose that $\Lambda$ does not have the weak intersection property. Then we can fix a value for $i$
such that
\[(\rho_i, \epsilon) \in \langle \alpha_0, \ldots, \alpha_i \rangle \cap \langle \alpha_i, \ldots, \alpha_{n-1} \rangle.\]

Now, since \((\rho_i, \epsilon) \in \langle \alpha_0, \ldots, \alpha_i \rangle\), there is a word \((\rho_{j_1} \cdots \rho_{j_m}, \rho'_{j_1} \cdots \rho'_{j_m})\) that is equal to \((\rho_i, \epsilon)\), with \(0 \leq j_k \leq i\) for each \(j_k\). If \(m\) is even, then \(\Gamma\) has a relator of odd length. Similarly, if \(m\) is odd, then \(\Gamma'\) has a relator of odd length. So either \(\mathcal{P}(\Gamma)\) or \(\mathcal{P}(\Gamma')\) (or possibly both) must not be directly regular.

Of course, this is not new information; we already knew that the mix of two directly regular semi-polytopes is again a directly regular semi-polytope. But we can conclude something even stronger. Assuming again that \(m\) is even, then we in fact get a relator of odd length in \(\langle \rho_0, \ldots, \rho_i \rangle\). So not only is \(\mathcal{P}(\Gamma)\) not directly regular, but in fact, its \((i + 1)\)-faces are not directly regular. In general, in order for \(\Lambda\) to not have the weak intersection property, there must be some \(i\) such that either \(\langle \rho_0, \ldots, \rho_i \rangle\) or \(\langle \rho'_0, \ldots, \rho'_i \rangle\) has a relator of odd length, and such that \(\langle \rho_i, \ldots, \rho_{n-1} \rangle\) or \(\langle \rho'_i, \ldots, \rho'_{n-1} \rangle\) has a relator of odd length. In particular, since a string pre-C-group on 2 generators is dihedral, having only relators of even length, we must have \(2 \leq i \leq n - 3\), so that in particular, \(n \geq 5\).

**Example 2.40.** Let \(\Gamma' = [2, 3, 3, 2]\), and let \(\Gamma\) be the quotient of \([3, 4, 4, 3]\) with the extra relations \((\rho_0 \rho_1 \rho_2)^3 = (\rho_4 \rho_3 \rho_2)^3 = \epsilon\). A calculation with GAP [12] confirms that \(\Gamma\) is a string C-group of order 576; in fact, \(\Gamma\) is the automorphism group of a regular polytope that is not directly regular. Another calculation shows that the intersection \(\langle \alpha_0, \ldots, \alpha_2 \rangle \cap \langle \alpha_2, \ldots, \alpha_4 \rangle\) has 4 elements, and thus \(\Gamma \odot \Gamma'\) does not have the weak intersection property.

We now move on to determining when the mix of two polytopes is a polytope. In fact, in some cases, we can mix a polytope with a semi-polytope and still get a polytope:

**Proposition 2.41.** Let \(\mathcal{P}\) be a chiral or directly regular \(n\)-polytope with facets isomorphic to \(\mathcal{K}\). Let \(\mathcal{Q}\) be a chiral or directly regular \(n\)-semi-polytope with facets isomorphic to \(\mathcal{K}'\). If


\( K \) covers \( K' \), then \( P \diamond Q \) is polytopal. The same is true if both \( P \) and \( Q \) are regular instead.

**Proof.** Since \( K \) covers \( K' \), the facets of \( P \diamond Q \) are isomorphic to \( K \). Therefore, the canonical projection from \( \Gamma^+(P) \diamond \Gamma^+(Q) \rightarrow \Gamma^+(P) \) is one-to-one on the subgroup of the facets, and by [3, Lemma 3.2], the group \( \Gamma^+(P) \diamond \Gamma^+(Q) \) has the intersection property. Therefore, \( P \diamond Q \) is a polytope.

In general, when we mix \( P \) and \( Q \), we have to verify the full intersection property. But as we shall see, some parts of the intersection property are automatic. Recall that for a subset \( I \) of \( \{0, \ldots, n-1\} \) and a string rotation group \( \Gamma = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle \), we define

\[ \Gamma_I = \langle \tau_{i,j} \mid i \leq j \text{ and } i - 1, j \in I \rangle, \]

where \( \tau_{i,j} = \sigma_i \cdots \sigma_j \).

**Proposition 2.42.** Let \( I, J \subset \{0, \ldots, n-1\} \). Suppose \( \Lambda = \Gamma \diamond \Gamma' \), where \( \Gamma \) and \( \Gamma' \) are string chi-groups. Then \( \Lambda_I \cap \Lambda_J \subseteq \Gamma_I \cap \Gamma_J \times \Gamma'_I \cap \Gamma'_J \).

**Proof.** Let \( \gamma \in \Lambda_I \cap \Lambda_J \). Write \( \gamma = (\gamma_1, \gamma_2) \). Then \( \gamma_1 \in \Gamma_I \cap \Gamma_J \) and \( \gamma_2 \in \Gamma'_I \cap \Gamma'_J \). Now, since \( \Gamma \) and \( \Gamma' \) are string chi-groups, \( \Gamma_I \cap \Gamma_J = \Gamma_{I} \cap \Gamma_{J} \) and \( \Gamma'_I \cap \Gamma'_J = \Gamma'_{I} \cap \Gamma'_{J} \). Therefore, \( \gamma \in \Gamma_{I} \cap \Gamma_{J} \times \Gamma'_{I} \cap \Gamma'_{J} \).

**Corollary 2.43.** Let \( \Lambda = \langle \beta_1, \ldots, \beta_{n-1} \rangle = \Gamma \diamond \Gamma' \), where \( \Gamma = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle \) and \( \Gamma' = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle \) are string chi-groups. Let \( i \leq j < k \leq \ell \). Then \( \langle \beta_i, \ldots, \beta_j \rangle \cap \langle \beta_k, \ldots, \beta_\ell \rangle = \langle \epsilon \rangle \).

**Proof.** Noting that \( \Lambda_{\{i-1, \ldots, j\}} = \langle \beta_i, \ldots, \beta_j \rangle \), we get the desired result by applying Proposition 2.42 with \( I = \{i - 1, \ldots, j\} \) and \( J = \{k - 1, \ldots, \ell\} \).

We can apply these same methods to string C-groups to obtain the following:
Proposition 2.44. Let $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ and $\Gamma' = \langle \rho'_0, \ldots, \rho'_{n-1} \rangle$ be string C-groups. Let $\alpha_i = (\rho_i, \rho'_i)$, so that $\Gamma \diamond \Gamma' = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$. Then

$$\langle \alpha_i \mid i \in I \rangle \cap \langle \alpha_j \mid j \in J \rangle \leq \langle \rho_i \mid i \in I \cap J \rangle \times \langle \rho'_i \mid i \in I \cap J \rangle.$$ 

In particular, if $I$ and $J$ are disjoint, then $\langle \alpha_i \mid i \in I \rangle \cap \langle \alpha_j \mid j \in J \rangle = \langle \epsilon \rangle$.

Corollary 2.45. Let $P$ and $Q$ be chiral or directly regular polyhedra. Then $P \diamond Q$ is a chiral or directly regular polyhedron.

Proof. In order for $P \diamond Q$ to be a polyhedron (and not just a pre-polyhedron), it must satisfy the intersection property. For polyhedra, the only requirement is that $\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \langle \epsilon \rangle$, which holds by Corollary 2.43.

Corollary 2.46. Let $P$ and $Q$ be regular polyhedra. Then $P \diamond Q$ is a regular polyhedron.

Proof. We will show just one of the conditions for the intersection property; the others are verified similarly. Proposition 2.44 says that $\langle \alpha_0, \alpha_1 \rangle \cap \langle \alpha_1, \alpha_2 \rangle \leq \langle \rho_1 \rangle \times \langle \rho'_1 \rangle$. Now, $\langle \alpha_0, \alpha_1 \rangle$ is the group of a regular polygon, and in particular, it is directly regular. So neither $(\rho_1, \epsilon)$ nor $(\epsilon, \rho'_1)$ is in $\langle \alpha_0, \alpha_1 \rangle$. Thus we see that $\langle \alpha_0, \alpha_1 \rangle \cap \langle \alpha_1, \alpha_2 \rangle = \langle \alpha_1 \rangle$, as desired.

Corollary 2.45 and Corollary 2.46 are extremely useful. In addition to telling us that the mix of any two polyhedra is a polyhedron, it makes it simpler to verify the polytopality of the mix of 4-polytopes, since the facets and vertex-figures of the mix are guaranteed to be polytopal.

There is another simple way of seeing that the mix of two polyhedra is polyhedron. A semi-polytope of rank 3 is always a polyhedron, so as long as $\Gamma \diamond \Gamma'$ is a string pre-C-group or a string pre-chi-group, the poset $P(\Gamma \diamond \Gamma')$ will be a polyhedron. Corollary 2.15 tells us that
the mix of two string pre-chi-groups is a string pre-chi-group, so that case is easily settled. The mix of string C-groups need not be a string pre-C-group in general, as we saw earlier, but in order to fail, we must have \( n \geq 5 \). Thus the mix of two string pre-C-groups of rank 3 is again a string pre-C-group of rank 3.

We now prove some general results that work for polytopes in any rank. We start by presenting part of [3, Theorem 9.1], noting that the requirement that \( P \) and \( Q \) are finite are not needed for this part of the theorem.

**Proposition 2.47.** Let \( P \) be a chiral or directly regular \( n \)-polytope of type \( \{p_1, \ldots, p_{n-1}\} \), and let \( Q \) be a chiral or directly regular \( n \)-polytope of type \( \{q_1, \ldots, q_{n-1}\} \). If \( p_i \) and \( q_i \) are relatively prime for each \( i = 1, \ldots, n-1 \), then \( P \odot Q \) is a chiral or directly regular \( n \)-polytope of type \( \{p_1 q_1, \ldots, p_{n-1} q_{n-1}\} \), and \( \Gamma^+(P \odot Q) = \Gamma^+(P) \times \Gamma^+(Q) \).

Note that Example 2.40, in which we mixed a directly regular polytope with a regular polytope that is not directly regular, shows that direct regularity is essential to Proposition 2.47.

**Example 2.48.** Let \( P = \{3, 4\} \), the octahedron, and let \( Q = \{4, 3\} \), the cube. Then \( P \odot Q \) is a polyhedron of type \( \{12, 12\} \) with \( \Gamma^+(P) \odot \Gamma^+(Q) = \Gamma^+(P) \times \Gamma^+(Q) \).

We now work on a generalization of Proposition 2.47.

**Lemma 2.49.** Let \( \Gamma = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle \) and \( \Gamma' = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle \) be string chi-groups. Let \( \beta_i = (\sigma_i, \sigma'_i) \), so that \( \Gamma \odot \Gamma' = \langle \beta_1, \ldots, \beta_{n-1} \rangle \). Then

\[
\langle \beta_1, \ldots, \beta_{n-2} \rangle \cap \langle \beta_2, \ldots, \beta_{n-1} \rangle \leq \langle \sigma_2, \ldots, \sigma_{n-2} \rangle \times \langle \sigma'_2, \ldots, \sigma'_{n-2} \rangle,
\]

with equality if

\[
\langle \beta_1, \ldots, \beta_{n-2} \rangle = \langle \sigma_1, \ldots, \sigma_{n-2} \rangle \times \langle \sigma'_1, \ldots, \sigma'_{n-2} \rangle
\]

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and
\[ \langle \beta_2, \ldots, \beta_{n-1} \rangle = \langle \sigma_2, \ldots, \sigma_{n-1} \rangle \times \langle \sigma'_2, \ldots, \sigma'_{n-1} \rangle. \]

Proof. We have that
\[
\langle \beta_1, \ldots, \beta_{n-2} \rangle \cap \langle \beta_2, \ldots, \beta_{n-1} \rangle \\
= \langle (\sigma_1, \sigma'_1, \ldots, (\sigma_{n-2}, \sigma_{n-2}')) \cap ((\sigma_2, \sigma'_2), \ldots, (\sigma_{n-1}, \sigma'_{n-1})) \rangle \\
\leq ((\sigma_1, \ldots, \sigma_{n-2}) \times (\sigma'_1, \ldots, \sigma'_n) \cap ((\sigma_2, \ldots, \sigma_{n-1}) \times (\sigma'_2, \ldots, \sigma'_{n-1}) \rangle \\
= ((\sigma_1, \ldots, \sigma_{n-2}) \cap (\sigma_2, \ldots, \sigma_{n-1})) \times ((\sigma'_1, \ldots, \sigma'_n) \cap (\sigma'_2, \ldots, \sigma'_{n-1}))) \\
= \langle \sigma_2, \ldots, \sigma_{n-2} \rangle \times \langle \sigma'_2, \ldots, \sigma'_{n-2} \rangle,
\]
where the last line follows from the fact that \( \Gamma \) and \( \Gamma' \) are string chi-groups.

Applying the same technique to string C-groups yields the following result.

**Lemma 2.50.** Let \( \Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle \) and \( \Gamma' = \langle \rho'_0, \ldots, \rho'_{n-1} \rangle \) be string C-groups. Let \( \alpha_i = (\rho_i, \rho'_i) \), so that \( \Gamma \circ \Gamma' = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \). Then
\[ \langle \alpha_0, \ldots, \alpha_{n-2} \rangle \cap \langle \alpha_1, \ldots, \alpha_{n-1} \rangle \leq \langle \rho_1, \ldots, \rho_{n-2} \rangle \times \langle \rho'_1, \ldots, \rho'_{n-2} \rangle, \]
with equality if
\[ \langle \alpha_0, \ldots, \alpha_{n-2} \rangle = \langle \rho_0, \ldots, \rho_{n-2} \rangle \times \langle \rho'_0, \ldots, \rho'_{n-2} \rangle \]
and
\[ \langle \alpha_1, \ldots, \alpha_{n-1} \rangle = \langle \rho_1, \ldots, \rho_{n-1} \rangle \times \langle \rho'_1, \ldots, \rho'_{n-1} \rangle. \]

**Proposition 2.51.** Let \( P \) be a chiral or directly regular \( n \)-polytope of type \( \{K, L\} \), and let \( Q \) be a chiral or directly regular \( n \)-polytope of type \( \{K', L'\} \). Suppose \( K \circ K' \) and \( L \circ L' \) are polytopes. Further, suppose that \( K \) has vertex-figures \( M \) and that \( K' \) has vertex figures \( M' \),
and suppose that $\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$. Then $\mathcal{P} \diamond \mathcal{Q}$ is polytopal. The same is true if we suppose that $\mathcal{P}$ and $\mathcal{Q}$ are both regular and that $\Gamma(\mathcal{M} \diamond \mathcal{M}') = \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}')$.

Proof. We start with the case where $\mathcal{P}$ and $\mathcal{Q}$ are chiral or directly regular. Let $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and let $\Gamma^+(\mathcal{Q}) = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle$. Let $\beta_i = (\sigma_i, \sigma'_i)$, so that $\Gamma^+(\mathcal{P} \diamond \mathcal{Q}) = \langle \beta_1, \ldots, \beta_{n-1} \rangle$. The facets of $\mathcal{P} \diamond \mathcal{Q}$ are $\mathcal{K} \diamond \mathcal{K}'$, and the vertex figures are $\mathcal{L} \diamond \mathcal{L}'$, both of which are polytopal. Thus, to verify the polytopality of $\mathcal{P} \diamond \mathcal{Q}$, it suffices to check that

$$\langle \beta_1, \ldots, \beta_{n-2} \rangle \cap \langle \beta_2, \ldots, \beta_{n-1} \rangle = \langle \beta_2, \ldots, \beta_{n-2} \rangle \quad ([30, Lemma 10]).$$

From Lemma 2.49 we get that

$$\langle \beta_1, \ldots, \beta_{n-2} \rangle \cap \langle \beta_2, \ldots, \beta_{n-1} \rangle \leq \langle \sigma_2, \ldots, \sigma_{n-2} \rangle \times \langle \sigma'_2, \ldots, \sigma'_{n-2} \rangle.$$

The right hand side is just $\Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$, and since this is equal to $\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \langle \beta_2, \ldots, \beta_{n-2} \rangle$ by assumption, the result follows.

When $\mathcal{P}$ and $\mathcal{Q}$ are both regular, then to verify polytopality it suffices to check that

$$\langle \alpha_0, \ldots, \alpha_{n-2} \rangle \cap \langle \alpha_1, \ldots, \alpha_{n-1} \rangle = \langle \alpha_1, \ldots, \alpha_{n-2} \rangle \quad ([23, 2E16(a)]).$$

Then the argument used above, along with Lemma 2.50, proves that the mix is polytopal. □

Theorem 2.52. Let $\mathcal{P}$ be a chiral or directly regular $n$-polytope of type $\{\mathcal{K}, \mathcal{L}\}$. Let $\mathcal{Q}$ be a chiral or directly regular $n$-polytope of type $\{\mathcal{K}', \mathcal{L}'\}$. Let $\mathcal{M}$ be the vertex-figures of $\mathcal{K}$, and let $\mathcal{M}'$ be the vertex-figures of $\mathcal{K}'$. Suppose that $\mathcal{K} \diamond \mathcal{K}'$ and $\mathcal{L} \diamond \mathcal{L}'$ are polytopal, and suppose that $\Gamma^+(\mathcal{K} \diamond \mathcal{K}') = \Gamma^+(\mathcal{K}) \times \Gamma^+(\mathcal{K}')$ and $\Gamma^+(\mathcal{L} \diamond \mathcal{L}') = \Gamma^+(\mathcal{L}) \times \Gamma^+(\mathcal{L}')$. Then $\mathcal{P} \diamond \mathcal{Q}$ is polytopal if and only if $\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$. The same is true if $\mathcal{P}$ and $\mathcal{Q}$ are both regular, and if all rotation groups are replaced with full automorphism groups.

Proof. Let $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and let $\Gamma^+(\mathcal{Q}) = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle$. Let $\beta_i = (\sigma_i, \sigma'_i)$, so that $\Gamma^+(\mathcal{P} \diamond \mathcal{Q}) = \langle \beta_1, \ldots, \beta_{n-1} \rangle$. Proposition 2.51 proves that if $\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$, then $\mathcal{P} \diamond \mathcal{Q}$ is polytopal. □
then $\mathcal{P} \diamond \mathcal{Q}$ is polytopal. Conversely, suppose $\mathcal{P} \diamond \mathcal{Q}$ is polytopal. Since $\mathcal{P} \diamond \mathcal{Q}$ is polytopal,

$$\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \langle \beta_2, \ldots, \beta_{n-2} \rangle = \langle \beta_1, \ldots, \beta_{n-2} \rangle \cap \langle \beta_2, \ldots, \beta_{n-1} \rangle.$$

Now, by Lemma 2.49, the right hand side is equal to $\langle \sigma_2, \ldots, \sigma_{n-2} \rangle \times \langle \sigma'_2, \ldots, \sigma'_{n-2} \rangle$, which is $\Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$. The proof for regular polytopes is essentially the same.

These propositions show us how $\mathcal{P} \diamond \mathcal{Q}$ can fail to be polytopal, even if it has polytopal facets and vertex-figures. In particular, if $\Gamma^+(\mathcal{M} \diamond \mathcal{M}')$ is “small” compared to $\Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$, then we need the inequality in the second line to be “loose”. We will make these notions more precise later.

We are now able to generalize Proposition 2.47.

**Theorem 2.53.** Let $\mathcal{P}$ be a chiral or directly regular $n$-polytope of type $\{p_1, \ldots, p_{n-1}\}$, and let $\mathcal{Q}$ be a chiral or directly regular $n$-polytope of type $\{q_1, \ldots, q_{n-1}\}$. If $p_i$ and $q_i$ are relatively prime for each $i = 2, \ldots, n-2$ (but not necessarily for $i = 1$ or $i = n-1$), then $\mathcal{P} \diamond \mathcal{Q}$ is a chiral or directly regular $n$-polytope. Furthermore, if $n \geq 4$, then $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ is a subgroup of index 4 or less in $\Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$.

**Proof.** We prove the claim by induction. The claim is trivially true for $n \leq 2$, and Corollary 2.45 establishes the case $n = 3$. Now, suppose the claim is true for $(n-1)$-polytopes, and let $\mathcal{P}$ and $\mathcal{Q}$ be $n$-polytopes satisfying the given conditions. Suppose that $\mathcal{P}$ is of type $\{\mathcal{K}, \mathcal{L}\}$ with the vertex-figures of $\mathcal{K}$ being $\mathcal{M}$. Similarly, suppose that $\mathcal{Q}$ is of type $\{\mathcal{K}', \mathcal{L}'\}$ with the vertex-figures of $\mathcal{K}'$ being $\mathcal{M}'$. Now, the facets of $\mathcal{P} \diamond \mathcal{Q}$ are isomorphic to $\mathcal{K} \diamond \mathcal{K}'$. The facet $\mathcal{K}$ has type $\{p_1, \ldots, p_{n-2}\}$, and $\mathcal{K}'$ has type $\{q_1, \ldots, q_{n-2}\}$. Now, $p_i$ and $q_i$ are relatively prime for each $i = 2, \ldots, n-2$, so by the inductive hypothesis, $\mathcal{K} \diamond \mathcal{K}'$ is polytopal. Similarly, $\mathcal{L} \diamond \mathcal{L}'$ is polytopal. Then by Proposition 2.51, to prove polytopality of $\mathcal{P} \diamond \mathcal{Q}$,
it suffices to show that $\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$. We have that $\mathcal{M}$ is of type \{p_2, \ldots, p_{n-2}\}, and $\mathcal{M}'$ is of type \{q_2, \ldots, q_{n-2}\}. Then Proposition 2.47 tells us that since each $p_i$ is relatively prime to $q_i$, we do in fact have that $\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$. Therefore $\mathcal{P} \diamond \mathcal{Q}$ is polytopal.

For the last part, note that if $n \geq 4$, then the generators $\sigma_2, \ldots, \sigma_{n-2}$ are all trivial in $\Gamma^+(\mathcal{P}) \vartriangleleft \Gamma^+(\mathcal{Q})$; in fact, $\sigma_i = \sigma'_i$ in $\Gamma^+(\mathcal{P}) \vartriangleleft \Gamma^+(\mathcal{Q})$, and thus each $\sigma_i$ has order dividing $p_i$ and $q_i$. Combined with the relations $(\sigma_1\sigma_2)^2 = \epsilon$ and $(\sigma_{n-2}\sigma_{n-1})^2 = \epsilon$, this forces $\sigma_1$ and $\sigma_{n-1}$ to have order dividing 2. Therefore, $\Gamma^+(\mathcal{P}) \vartriangleleft \Gamma^+(\mathcal{Q})$ has order at most 4, and thus $\Gamma^+(\mathcal{P}) \circ \Gamma^+(\mathcal{Q})$ has index at most 4 in $\Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$.

Example 2.54. Let $\mathcal{P} = \{3, 3, 3\}$, the 4-simplex, and let $\mathcal{Q} = \{3, 4, 3\}$, the 24-cell. Then $\mathcal{P} \diamond \mathcal{Q}$ is a directly regular polytope. Furthermore, $\Gamma^+(\mathcal{P}) \vartriangleleft \Gamma^+(\mathcal{Q})$ is trivial, and thus $\Gamma^+(\mathcal{P}) \circ \Gamma^+(\mathcal{Q}) = \Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$.

We conclude this section with two more general results.

Theorem 2.55. Let $\mathcal{P}$ be a chiral or directly regular $n$-polytope of type $\{\mathcal{K}, \mathcal{L}\}$. Let $\mathcal{Q}$ be a chiral or directly regular $n$-polytope of type $\{\mathcal{K}', \mathcal{L}'\}$. Let $\mathcal{M}$ be the vertex-figures of $\mathcal{K}$, and let $\mathcal{M}'$ be the vertex-figures of $\mathcal{K}'$. If

$$|\Gamma^+(\mathcal{M} \vartriangleleft \mathcal{M}')| > |\Gamma^+(\mathcal{K} \vartriangleleft \mathcal{K}')| \cdot |\Gamma^+(\mathcal{L} \vartriangleleft \mathcal{L}')|,$$

then $\mathcal{P} \circ \mathcal{Q}$ is not polytopal.

Proof. First, note that

$$\Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}') = (\Gamma^+(\mathcal{K}) \times \Gamma^+(\mathcal{K}')) \cap (\Gamma^+(\mathcal{L}) \times \Gamma^+(\mathcal{L}')).$$
Suppose \( P \circ Q \) is polytopal. Then we know from Theorem 2.52 that
\[
\Gamma^+(K \circ K') \cap \Gamma^+(L \circ L') = \Gamma^+(M \circ M').
\]

Now, by Theorem 2.23 and Proposition A.3 in the Appendix,
\[
|\Gamma^+(M \boxempty M')| = |\Gamma^+(M) \times \Gamma^+(M') : \Gamma^+(M \circ M')|
\]
\[
= [(\Gamma^+(K) \times \Gamma^+(K')) \cap (\Gamma^+(L) \times \Gamma^+(L')) : \Gamma^+(K \circ K') \cap \Gamma^+(L \circ L')]
\]
\[
\leq [\Gamma^+(K) \times \Gamma^+(K') : \Gamma^+(K \circ K')] \cdot [\Gamma^+(L) \times \Gamma^+(L') : \Gamma^+(L \circ L')]
\]
\[
= |\Gamma^+(K \square K')| \cdot |\Gamma^+(L \square L')|.
\]

Proposition 2.56. Let \( P \) and \( Q \) be chiral or directly regular \( n \)-polytopes. Suppose \( P \) is of type \( \{p_1, \ldots, p_{n-1}\} \), and that \( Q \) is of type \( \{q_1, \ldots, q_{n-1}\} \). Let \( r_i = \gcd(p_i, q_i) \) for \( i \in \{1, \ldots, n-1\} \). If there is an integer \( m \in \{2, \ldots, n-2\} \) such that \( r_{m-1} = r_{m+1} = 1 \) and \( r_m \geq 3 \), then \( P \circ Q \) is not a polytope.

Proof. Let \( \Gamma^+(P) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle \), \( \Gamma^+(Q) = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle \), and \( \beta_i = (\sigma_i, \sigma'_i) \) for each \( i \in \{1, \ldots, n-1\} \). To show that \( P \circ Q \) is not polytopal, it suffices to show that
\[
\langle \beta_{m-1}, \beta_m \rangle \cap \langle \beta_m, \beta_{m+1} \rangle \neq \langle \beta_m \rangle.
\]

Now, since \( p_{m-1} \) and \( q_{m-1} \) are relatively prime, there is an integer \( k \) such that \( kp_{m-1} \equiv 1 \pmod{q_{m-1}} \). Then since the order of \( \sigma_{m-1} \) is \( p_{m-1} \) and the order of \( \sigma'_{m-1} \) is \( q_{m-1} \), we see that
\[
\beta_{m-1}^{kp_{m-1}} = (\sigma_{m-1}^{kp_{m-1}}, (\sigma'_{m-1})^{kp_{m-1}}) = (\epsilon, \sigma'_{m-1}),
\]

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and therefore

\[(\beta^{-1}_{m-1} \beta_m)^2 = (\sigma_m^2; (\sigma'_{m-1} \sigma'_m)^2) = (\sigma_m^2, \epsilon),\]

since we have \((\sigma' \sigma'_{i+1})^2 = \epsilon\) for any \(i \in \{1, \ldots, n-2\}\). Thus, \((\sigma_m^2, \epsilon) \in \langle \beta_{m-1}, \beta_m \rangle\). Similarly, there is an integer \(k'\) such that \(k'p_{m+1} \equiv 1 \pmod{q_{m+1}}\), and thus

\[(\beta_m \beta_{m+1}^{k'p_{m+1}})^2 = (\sigma_m^2; (\sigma_m' \sigma'_m)^2) = (\sigma_m^2, \epsilon).\]

Therefore, \((\sigma_m^2, \epsilon) \in \langle \beta_m, \beta_{m+1} \rangle\) as well. So we see that

\[(\sigma_m^2, \epsilon) \in \langle \beta_{m-1}, \beta_m \rangle \cap \langle \beta_m, \beta_{m+1} \rangle.\]

Now, since \(r_m \geq 3\), there is no integer \(k\) such that \(\sigma_m^k = \sigma_m^2\) and \((\sigma'_m)^k = \epsilon\). Therefore, \((\sigma_m^2, \epsilon) \not\in \langle \beta_m \rangle\), and that proves the claim. \(\square\)
Chapter 3

Measuring invariance

In this chapter, we develop the theory of internal and external invariance of polytopes; the distinction is similar to that between inner and outer automorphisms of a group. Our framework provides a unified way to measure the extent to which a polytope is chiral, self-dual, or self-Petrie. Our goal is then to understand how the variance of $P \o Q$ depends on $P$ and $Q$, and to use this knowledge to build polytopes with specified symmetries.

3.1 External and internal invariance

Our study of invariance starts with the symmetries of $W_n = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, the automorphism group of the universal $n$-polytope. Let $\mathcal{P}$ be a regular $n$-semi-polytope with base flag $\Phi$. Recall that $W_n$ acts on the flags of $\mathcal{P}$ by $\Phi^{j_1, \ldots, j_k} \rho_i = \Phi^{i, j_1, \ldots, j_k}$. If $M$ is the stabilizer of the base flag $\Phi$ under this action, then $M$ is normal in $W_n$ and $\Gamma(\mathcal{P}) = W_n/M$.

Suppose $\varphi$ is in $\text{Aut}(W_n)$, the group of group automorphisms of $W_n$, and define $\mathcal{P}^\varphi$ to be the flagged poset built from $W_n/\varphi(M)$. If $\varphi$ fixes $M$, then $\mathcal{P}$ and $\mathcal{P}^\varphi$ are naturally isomorphic, and as before we shall consider them equal. On the other hand, if $\varphi(M) \neq M$,
then the polytopes $\mathcal{P}$ and $\mathcal{P}^\varphi$ are distinct, and they need not be isomorphic, even though $\varphi$ induces an isomorphism of their automorphism groups.

Similarly, if $\mathcal{P}$ is a chiral or directly regular $n$-semi-polytope with base flag $\Phi$, then $W^+_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ acts on that flags of $\mathcal{P}$ by $\Phi^j_1 \cdots \Phi^j_k \sigma_i = \Phi^{i,j-1,j_1} \cdots \Phi^j_k$. If $M$ is the stabilizer of the base flag under this action, then $M$ is normal in $W^+_n$ and $\Gamma^+(\mathcal{P}) = W^+_n / M$. Now, taking $\varphi \in \text{Aut}(W^+_n)$, we similarly define $\mathcal{P}^\varphi$ to be the flagged poset built from $W^+_n / \varphi(M)$.

**Definition 3.1.** Let $\mathcal{P}$ be a regular or chiral $n$-semi-polytope. Let $\varphi$ be a group automorphism of $W_n$ or $W^+_n$ (whichever is appropriate), and let $\mathcal{P}^\varphi$ be defined as above.

1. If $\mathcal{P} = \mathcal{P}^\varphi$, we say that $\mathcal{P}$ is internally $\varphi$-invariant.
2. If $\mathcal{P} \neq \mathcal{P}^\varphi$, we say that $\mathcal{P}$ is internally $\varphi$-variant.
3. If $\mathcal{P} \simeq \mathcal{P}^\varphi$, we say that $\mathcal{P}$ is externally $\varphi$-invariant.
4. If $\mathcal{P} \not\simeq \mathcal{P}^\varphi$, we say that $\mathcal{P}$ is externally $\varphi$-variant.

Of course, if a semi-polytope is internally $\varphi$-invariant, it must also be externally $\varphi$-invariant. Similarly, if a semi-polytope is externally $\varphi$-variant, it must also be internally $\varphi$-variant.

As usual, there are two ways to work with a directly regular polytope: via its automorphism group or via its rotation group. These are compatible in a natural way:

**Proposition 3.2.** Let $\varphi \in \text{Aut}(W_n)$ such that $\varphi(W^+_n) = W^+_n$. Let $\varphi^+ \in \text{Aut}(W^+_n)$ be the automorphism induced by $\varphi$. If $\mathcal{P}$ is a directly regular semi-polytope, then $\mathcal{P}^\varphi = \mathcal{P}^{\varphi^+}$.

**Proof.** Let $\Gamma^+(\mathcal{P}) = W^+_n / M$. Since $\mathcal{P}$ is directly regular, $\Gamma(\mathcal{P}) = W_n / M$. Now, $\mathcal{P}^\varphi$ is the directly regular semi-polytope with group $W_n / \varphi(M)$. This polytope has rotation group $W^+_n / \varphi(M) = W^+_n / \varphi^+(M)$, which is the rotation group of $\mathcal{P}^{\varphi^+}$.

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We now consider several applications.

Example 3.3. Let $P$ be a regular semi-polytope with $\Gamma(P) = W_n/M$, and let $\chi_w \in \text{Aut}(W_n)$ be conjugation by $w \in W_n$. Then since $M$ is normal in $W_n$, $\chi_w$ fixes $M$. Therefore, every regular polytope is internally $\chi_w$-invariant.

Example 3.4. Let $\chi$ be the automorphism of $W_n^+$ that sends $\sigma_1$ to $\sigma_1^{-1}$, $\sigma_2$ to $\sigma_2^2$, and fixes all other generators $\sigma_i$. This is the automorphism induced by conjugation by $\rho_1$ in $W_n$. If $P$ is directly regular, then it is internally $\chi$-invariant. If $P$ is chiral instead, then $P^\chi$ is the enantiomorphic form of $P$ (that is, $P^\chi = \overline{P}$). Since $P^\chi \simeq P$ but $P^\chi \neq P$, we see that chiral semi-polytopes are externally $\chi$-invariant, but internally $\chi$-variant.

Example 3.5. Let $\delta$ be the automorphism of $W_n$ that sends each $\rho_i$ to $\rho_{n-i-1}$, and let $P$ be a regular $n$-semi-polytope. Then $P^\delta$ is the dual of $P$ (and indeed, our notation for the dual was chosen in anticipation of this fact). The semi-polytope $P$ is externally $\delta$-invariant if and only if it is self-dual. Every regular self-dual semi-polytope has a duality that fixes the base flag while reversing the order [23], and therefore if $P$ is regular and self-dual, the polytopes $P$ and $P^\delta$ have the same flag-stabilizer in $W_n$. Thus we see that a regular self-dual semi-polytope is always internally $\delta$-invariant.

Example 3.6. Let $\delta^+$ be the automorphism of $W_n^+$ that sends each $\sigma_i$ to $\sigma_{n-i}$. This is the automorphism induced by $\delta$ in the previous example (and by an abuse of notation, we frequently use $\delta$ to denote this automorphism of $W_n^+$ as well). Then a directly regular or chiral semi-polytope $P$ is externally $\delta^+$-invariant if and only if it is self-dual. If $P$ is properly self-dual (i.e., if there is a duality that fixes the base flag), then it is internally $\delta^+$-invariant; otherwise it is internally $\delta^+$-variant.

Example 3.7. Let $\pi$ be the automorphism of $W_3$ that sends $\rho_0$ to $\rho_0\rho_2$ and fixes every other $\rho_i$. Let $P$ be a regular semi-polytope. Then $P^\pi$ is the Petrie dual of $P$. 

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We note here that, in general, $W_n$ has an automorphism $\psi$ that sends $\rho_2$ to $\rho_0 \rho_2$ while fixing every other $\rho_i$ [18]. In rank 3, that automorphism is $\delta \pi \delta$, and when applied to a polyhedron it produces a polyhedron except in rare cases. On the other hand, in ranks 4 and greater, $P^\psi$ frequently fails to be a polytope. To see why, consider the relation $(\rho_2 \rho_3)^p = \epsilon$ in $\Gamma(P)$, which induces the relation $(\rho_0 \rho_2 \rho_3)^p = \epsilon$ in $\Gamma(P^\psi)$. Since $\rho_0$ commutes with $\rho_2 \rho_3$, we get that $\rho_0^{-p} = (\rho_2 \rho_3)^p$. If $p$ is odd, then we get that $\rho_0 = (\rho_2 \rho_3)^p$, a clear violation of the intersection property. In fact, even the empty intersection property is violated, so we do not even get a semi-polytope. Due to this and other difficulties, we shall not treat the automorphism $\psi$ here.

We will now explore the connection between invariance and polytope covers.

**Proposition 3.8.** Let $P$ be a chiral or regular $n$-semi-polytope, and let $\varphi$ be an automorphism of $W_n$ or $W_n^+$, as appropriate. Suppose $Q$ is a internally $\varphi$-invariant $n$-semi-polytope that covers $P$. Then $Q$ covers $P^\varphi$.

**Proof.** Suppose $P$ and $Q$ are both regular; the proof is essentially the same in the other cases. We have $\Gamma(P) = W_n/M$ and $\Gamma(Q) = W_n/K$ for some normal subgroups $M$ and $K$ of $W_n$. Since $Q$ covers $P$, we get that $K \leq M$, by Proposition 1.23. Then $\varphi(K) \leq \varphi(M)$ as well, and since $Q$ is internally $\varphi$-invariant, $\varphi(K) = K$. Therefore $K \leq \varphi(M)$, and so $Q$ covers $P^\varphi$.

**Corollary 3.9.** Let $P$ be a chiral or regular $n$-semi-polytope, and let $\varphi$ be an automorphism of $W_n$ or $W_n^+$, as appropriate. Suppose that $\varphi$ has finite order $k$, and that $Q$ is an internally $\varphi$-invariant $n$-semi-polytope that covers $P$. Then $Q$ covers $P \circ P^\varphi \circ \cdots \circ P^{\varphi^{k-1}}$.

**Proof.** Repeated application of Proposition 3.8 shows that $Q$ covers $P^\varphi$, $P^{\varphi^2}$, ..., and $P^{\varphi^{k-1}}$. Therefore, it covers their mix.
As we shall see shortly, the mix \( \mathcal{P} \circ \mathcal{P}^\varphi \circ \cdots \circ \mathcal{P}^\varphi^{k-1} \) is actually the minimal internally \( \varphi \)-invariant cover of \( \mathcal{P} \). As such, we make the following definition.

**Definition 3.10.** Let \( \mathcal{P} \) be a chiral or regular \( n \)-semi-polytope, and let \( \varphi \) be an automorphism of \( W_n \) or \( W_n^+ \) (as appropriate) of finite order \( k \). Then we define \( \mathcal{P}^\varphi \) to be \( \mathcal{P} \circ \mathcal{P}^\varphi \circ \cdots \circ \mathcal{P}^\varphi^{k-1} \).

**Proposition 3.11.** Let \( \mathcal{P} \) be a chiral or regular \( n \)-semi-polytope, and let \( \varphi \) be an automorphism of \( W_n \) or \( W_n^+ \) (as appropriate) of order finite \( k \). Then \( \mathcal{P}^\varphi \) is the minimal internally \( \varphi \)-invariant cover of \( \mathcal{P} \).

**Proof.** Since \( \mathcal{P}^\varphi^k = \mathcal{P} \), it is clear that \( (\mathcal{P}^\varphi)^\varphi = \mathcal{P}^\varphi \). So \( \mathcal{P}^\varphi \) is internally \( \varphi \)-invariant. By Corollary 3.9, every internally \( \varphi \)-invariant cover of \( \mathcal{P} \) must cover \( \mathcal{P}^\varphi \) as well. Thus it follows that \( \mathcal{P}^\varphi \) is minimal. \( \square \)

In the rest of this chapter, in order to avoid duplication, we will usually assume that \( \varphi \) is an automorphism of \( W_n^+ \), and that any polytopes we deal with are chiral or directly regular. Note, however, that the definitions below all still make sense if we work with automorphisms of \( W_n \) instead and assume that our polytopes are regular.

Given an automorphism \( \varphi \) of \( W_n^+ \), we can consider the variance groups \( X(\mathcal{P}|\mathcal{P}^\varphi) \) and \( X(\mathcal{P}^\varphi|\mathcal{P}) \). By Proposition 2.19, if \( \Gamma^+(\mathcal{P}) = W_n^+/M \), then the former is isomorphic to \( M\varphi(M)/M \), and the latter is isomorphic to \( M\varphi(M)/\varphi(M) \). Since \( M \simeq \varphi(M) \), the groups \( X(\mathcal{P}|\mathcal{P}^\varphi) \) and \( X(\mathcal{P}^\varphi|\mathcal{P}) \) are isomorphic. We make the following definition:

**Definition 3.12.** Let \( \mathcal{P} \) be a chiral or directly regular semi-polytope of rank \( n \). Let \( \varphi \in \text{Aut}(W_n^+) \). We define

\[
X_{\varphi}(\mathcal{P}) := X(\mathcal{P}|\mathcal{P}^\varphi) := X(\Gamma^+(\mathcal{P})|\Gamma^+(\mathcal{P}^\varphi)),
\]

and we call this the (internal) \( \varphi \)-variance group of \( \mathcal{P} \); we usually omit the word ‘internal’.
In other words, $X_\varphi(P)$ is the kernel of the natural epimorphism from $\Gamma^+(P) \circ \Gamma^+(P^\varphi)$ to $\Gamma^+(P^\varphi)$ (and also the kernel of the natural epimorphism from $\Gamma^+(P)$ to $\Gamma^+(P) \square \Gamma^+(P^\varphi)$).

**Example 3.13.** If $P$ is internally $\varphi$-invariant, then $\Gamma^+(P) \circ \Gamma^+(P^\varphi) \simeq \Gamma^+(P^\varphi)$, and $X_\varphi(P)$ is trivial.

**Example 3.14.** If $P$ is a chiral polytope, then its chirality group $X(P)$ is defined to be the kernel of the natural epimorphism from $\Gamma^+(P) \circ \Gamma^+(P)$ to $\Gamma^+(P)$ [9]. Taking $\varphi = \chi$ (as defined in Example 3.4), we see that the chirality group is the same thing as the $\chi$-invariance group $X_\chi(P)$.

The group $X_\varphi(P)$ gives us a measure of how different $P$ is from $P^\varphi$. At one extreme, $X_\varphi(P)$ might be trivial, in which case $P$ is internally $\varphi$-invariant. At the other extreme, $X_\varphi(P)$ might coincide with the whole group $\Gamma^+(P)$; in that case, we say that $P$ is totally (internally) $\varphi$-variant. (Again, we usually drop the word ‘internally’ for brevity.)

### 3.2 Variance of the mix

Using the tools developed in the previous section, our goal now is to determine how $X_\varphi(P \circ Q)$ depends on $X_\varphi(P)$ and $X_\varphi(Q)$. One of the primary motivations is to determine when $P \circ Q$ is chiral, and indeed, many of the results presented here generalize results from [9] and [3]. We start with a simple result.

**Proposition 3.15.** Let $P$ and $Q$ be chiral or directly regular $n$-semi-polytopes, and let $\varphi \in \text{Aut}(W_n^+)$. Then $(P \circ Q)^\varphi = P^\varphi \circ Q^\varphi$.

**Proof.** Write $\Gamma^+(P) = W_n^+/M$ and $\Gamma^+(Q) = W_n^+/K$. Then $\Gamma^+((P \circ Q)^\varphi) = W_n^+/\varphi(M \cap K)$, while $\Gamma^+(P^\varphi \circ Q^\varphi) = W_n^+/(\varphi(M) \cap \varphi(K))$. Since $\varphi \in \text{Aut}(W_n^+)$, the subgroups $\varphi(M \cap K)$ and $\varphi(M) \cap \varphi(K)$ are equal, which proves the result. \qed
Lemma 3.16. Let $P$ and $Q$ be chiral or directly regular $n$-semi-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Then $P \odot Q$ is internally $\varphi$-invariant if and only if it covers $P^{\varphi}$ and $Q^{\varphi}$.

Proof. If $P \odot Q$ is internally $\varphi$-invariant, then by Corollary 3.9, it covers $(P \odot Q)^{\varphi}$. Furthermore, by Proposition 3.15, the latter polytope is equal to $P^{\varphi} \odot Q^{\varphi}$, which covers both $P^{\varphi}$ and $Q^{\varphi}$. Conversely, if $P \odot Q$ covers $P^{\varphi}$ and $Q^{\varphi}$, then it covers $(P \odot Q)^{\varphi}$, which itself covers $P \odot Q$. Then we must have that $(P \odot Q)^{\varphi} = P \odot Q$; that is, $P \odot Q$ must be internally $\varphi$-invariant. \(\blacksquare\)

We now give the main theorem of this section, from which we will be able to derive many results on the $\varphi$-invariance of $P \odot Q$.

Theorem 3.17. Let $P$ and $Q$ be chiral or directly regular $n$-semi-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Suppose $P \odot Q$ is internally $\varphi$-invariant. Then there is a natural epimorphism from $\Gamma^+(P) \odot \Gamma^+(Q)$ to $\Gamma^+(P) \odot \Gamma^+(P^\varphi)$, and it restricts to an epimorphism from $X(Q|P)$ to $X_\varphi(P^\varphi) = X(P^\varphi|P)$. Similarly, the natural epimorphism from $\Gamma^+(P) \odot \Gamma^+(Q)$ to $\Gamma^+(Q) \odot \Gamma^+(Q^\varphi)$ restricts to an epimorphism from $X(P|Q)$ to $X_\varphi(Q^\varphi) = X(Q^\varphi|Q)$.

Proof. Let $\Gamma^+(P) = W_n^+/M$ and $\Gamma^+(Q) = W_n^+/K$. By Lemma 3.16, since $P \odot Q$ is internally $\varphi$-invariant, it covers $P^{\varphi}$, which covers $P \odot P^\varphi$. Therefore, $\Gamma^+(P) \odot \Gamma^+(Q)$ naturally covers $\Gamma^+(P) \odot \Gamma^+(P^\varphi)$. Since

$$\Gamma^+(P) \odot \Gamma^+(Q) = W_n^+/(M \cap K)$$

and

$$\Gamma^+(P) \odot \Gamma^+(P^\varphi) = W_n^+/(M \cap \varphi(M)),$$

this means that $M \cap K \leq M \cap \varphi(M)$. Thus, the group $M/(M \cap K)$ naturally covers $M/(M \cap \varphi(M))$. By Proposition 2.19, the former is the subgroup $X(Q|P)$ of $\Gamma^+(P) \odot \Gamma^+(Q)$,
and the latter is the subgroup $X(\mathcal{P}^\varphi|\mathcal{P})$ of $\Gamma^+(\mathcal{P}) \circ \Gamma^+(\mathcal{P}^\varphi)$. The result then follows by symmetry.

When $\varphi$ is an involution, the converse of Theorem 3.17 also holds:

**Theorem 3.18.** Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular $n$-semi-polytopes. Let $\varphi \in Aut(W_n^+)$ have order 2. If $X(\mathcal{Q}|\mathcal{P})$ naturally covers $X_\varphi(\mathcal{P}^\varphi)$ and $X(\mathcal{P}|\mathcal{Q})$ naturally covers $X_\varphi(\mathcal{Q}^\varphi)$ (as quotients of $W_n^+$), then $\mathcal{P} \circ \mathcal{Q}$ is internally $\varphi$-invariant.

**Proof.** Suppose $X(\mathcal{Q}|\mathcal{P})$ naturally covers $X_\varphi(\mathcal{P}^\varphi)$. Writing $\Gamma^+(\mathcal{P}) = W_n^+/M$ and $\Gamma^+(\mathcal{Q}) = W_n^+/K$, we again get that $M \cap K \leq M \cap \varphi(M)$. Therefore, $\Gamma^+(\mathcal{P}) \circ \Gamma^+(\mathcal{Q})$ naturally covers $\Gamma^+(\mathcal{P}) \circ \Gamma^+(\mathcal{Q}^\varphi)$. So $\mathcal{P} \circ \mathcal{Q}$ covers $\mathcal{P} \circ \mathcal{P}^\varphi$, which is equal to $\mathcal{P}^\varphi \circ \mathcal{Q}$ since $\varphi$ has order 2. Similarly, $\mathcal{P} \circ \mathcal{Q}$ naturally covers $\mathcal{Q} \circ \mathcal{Q}^\varphi = \mathcal{Q}^\varphi \circ \mathcal{Q}$. Then by Lemma 3.16, $\mathcal{P} \circ \mathcal{Q}$ is internally $\varphi$-invariant.

**Example 3.19.** The automorphism $\chi$ of $W_n^+$ (Example 3.4) has order 2, and a chiral or directly regular semi-polytope $\mathcal{P}$ is directly regular if and only if it is internally $\chi$-invariant. Therefore, Theorem 3.17 and Theorem 3.18 tell us that $\mathcal{P} \circ \mathcal{Q}$ is directly regular if and only if it covers $\mathcal{P} \circ \mathcal{P}^\chi$ and $\mathcal{Q} \circ \mathcal{Q}^\chi$ (i.e., $\mathcal{P} \circ \overline{\mathcal{P}}$ and $\mathcal{Q} \circ \overline{\mathcal{Q}}$).

Now we will see several ways to apply Theorem 3.17.

**Theorem 3.20.** Let $\mathcal{P}$ and $\mathcal{Q}$ be finite chiral or directly regular $n$-semi-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. If $\mathcal{P} \circ \mathcal{Q}$ is internally $\varphi$-invariant, then $|X_\varphi(\mathcal{P})|$ divides $|\Gamma^+(\mathcal{Q})|/|\Gamma^+(\mathcal{P}) \circ \Gamma^+(\mathcal{Q})|$ and $|X_\varphi(\mathcal{Q})|$ divides $|\Gamma^+(\mathcal{P})|/|\Gamma^+(\mathcal{P}) \circ \Gamma^+(\mathcal{Q})|$.

**Proof.** If $\mathcal{P} \circ \mathcal{Q}$ is internally $\varphi$-invariant, then by Theorem 3.17, $X(\mathcal{Q}|\mathcal{P})$ covers $X_\varphi(\mathcal{P}^\varphi)$. If $\mathcal{P}$ and $\mathcal{Q}$ are both finite, then so are $X(\mathcal{Q}|\mathcal{P})$ and $X_\varphi(\mathcal{P}^\varphi)$, and thus $|X_\varphi(\mathcal{P}^\varphi)|$ divides $|X(\mathcal{Q}|\mathcal{P})|$. Furthermore, $|X_\varphi(\mathcal{P}^\varphi)| = |X_\varphi(\mathcal{P})|$ and, by Proposition 2.21, $|X(\mathcal{Q}|\mathcal{P})| = |\Gamma^+(\mathcal{Q})|/|\Gamma^+(\mathcal{P}) \circ \Gamma^+(\mathcal{Q})|$. The result then follows by symmetry.
In practice, we often neglect the term $|\Gamma^+(P) \square \Gamma^+(Q)|$ and just check whether $|X_\varphi(P)|$ divides $|\Gamma^+(Q)|$. For many of the examples we will see, $|\Gamma^+(P) \square \Gamma^+(Q)|$ is 1 or 2, so that discarding this term does not affect our results much.

**Corollary 3.21.** Let $P$ and $Q$ be chiral or directly regular $n$-semi-polytopes, and let $\varphi \in Aut(W^+_n)$ have finite order. Suppose that $P$ has infinite $\varphi$-variance group $X_\varphi(P)$ and that $Q$ is finite. Then $P \circ Q$ is internally $\varphi$-variant (that is, not internally $\varphi$-invariant).

**Proof.** Since $Q$ is finite, so is $X(Q|P)$, which is isomorphic to a subgroup of $Q$. Then there is no epimorphism from the finite group $X(Q|P)$ to the infinite group $X_\varphi(P^\varphi) \simeq X_\varphi(P)$, and thus by Theorem 3.17, $P \circ Q$ must be internally $\varphi$-variant. $\Box$

**Corollary 3.22.** Let $P$ and $Q$ be chiral or directly regular $n$-semi-polytopes, and let $\varphi \in Aut(W^+_n)$ have finite order. Suppose $P$ is internally $\varphi$-variant and that $Q$ has a rotation group $\Gamma^+(Q)$ that is simple. If $X_\varphi(P)$ is not isomorphic to $\Gamma^+(Q)$, then $P \circ Q$ is internally $\varphi$-variant.

**Proof.** Theorem 3.17 says that if $P \circ Q$ is internally $\varphi$-invariant, then $X(Q|P)$ covers $X_\varphi(P^\varphi)$. Now, since $P$ has a nontrivial $\varphi$-variance group, $X_\varphi(P^\varphi)$ is nontrivial, and since $\Gamma^+(Q)$ is simple, the normal subgroup $X(Q|P)$ of $\Gamma^+(Q)$ is either trivial or the whole group $\Gamma^+(Q)$. The only way for $X(Q|P)$ to cover $X_\varphi(P^\varphi)$ is for $X(Q|P)$ to be $\Gamma^+(Q)$, and then the only nontrivial group it covers is itself. Therefore, if $X_\varphi(P)$ (and thus $X_\varphi(P^\varphi)$) is not isomorphic to $\Gamma^+(Q)$, then $X(Q|P)$ cannot cover $X_\varphi(P^\varphi)$, and the mix $P \circ Q$ is internally $\varphi$-variant by Theorem 3.17. $\Box$

Corollary 3.22 has a natural analogue using regular semi-polytopes, where $Q$ has a full automorphism group that is simple. Both forms are particularly useful since there are a number of papers that address which simple groups arise as the automorphism groups or rotation subgroups of abstract polytopes. For a small sampling, see [10, 11, 14, 22].
We see that for finite polytopes, there are several simple tests that we can apply to determine whether \( P \odot Q \) is internally \( \varphi \)-invariant. We would like to extend the results to the mix of three or more polytopes. In order to do that, however, we need to know more about the size of \( X_\varphi(P \odot Q) = X(P \odot Q|(P \odot Q)_\varphi) \). We start with some simple observations.

**Proposition 3.23.** Let \( P \) and \( Q \) be chiral or directly regular \( n \)-polytopes, and let \( \varphi \in \text{Aut}(W_n^+) \) have finite order. Then \( X_\varphi(P \odot Q) \) is isomorphic to a subgroup of \( X_\varphi(P) \times X_\varphi(Q) \).

*Proof.* Apply Proposition 2.29, noting that \( X_\varphi(P \odot Q) = X(P \odot Q|P^\varphi \odot Q^\varphi) \), \( X_\varphi(P) = X(P|P^\varphi) \), and \( X_\varphi(Q) = X(Q|Q^\varphi) \).

**Proposition 3.24.** Let \( P \) and \( Q \) be chiral or directly regular \( n \)-polytopes, and let \( \varphi \in \text{Aut}(W_n^+) \) have finite order. Suppose that \( Q \) is internally \( \varphi \)-invariant. Then \( X_\varphi(P \odot Q) \) is a normal subgroup of \( X_\varphi(P) \).

*Proof.* Since \( Q \) is internally \( \varphi \)-invariant, \( \Gamma^+(Q^\varphi) = \Gamma^+(Q) \). Therefore,

\[
X_\varphi(P \odot Q) = X(\Gamma^+(P) \odot \Gamma^+(Q)|\Gamma^+(P^\varphi) \odot \Gamma^+(Q)),
\]

and we can apply Proposition 2.30 to get the desired result.

Next we generalize Theorem 3.20 to find a lower bound for \( |X_\varphi(P \odot Q)| \).

**Theorem 3.25.** Let \( P \) and \( Q \) be finite chiral or directly regular \( n \)-semi-polytopes, and let \( \varphi \in \text{Aut}(W_n^+) \) have finite order. Then \( |X_\varphi(P \odot Q)| \) is an integer multiple of \( |X_\varphi(P)|/|X(Q|P)| \).

*Proof.* Since

\[
P \odot Q \odot P^\varphi \odot Q^\varphi = (P \odot Q) \odot (P \odot Q)^\varphi,
\]

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The latter covers $P \circ P^\varphi$. Therefore, $|\Gamma^+(P) \circ \Gamma^+(P^\varphi)|$ divides $|\Gamma^+(P \circ Q) \circ \Gamma^+((P \circ Q)^\varphi)|$. The former has size $|\Gamma^+(P)| \cdot |X_\varphi(P)|$, by Proposition 2.21, while the latter has size

$$|\Gamma^+(P \circ Q)| \cdot |X_\varphi(P \circ Q)| = |\Gamma^+(P)| \cdot |X(Q|P)| \cdot |X_\varphi(P \circ Q)|.$$ 

Thus we see that $|X_\varphi(P)|$ divides $|X(Q|P)| \cdot |X_\varphi(P \circ Q)|$, and thus $|X_\varphi(P \circ Q)|$ is an integer multiple of $|X_\varphi(P)|/|X(Q|P)|$.

Thus we see that, for instance, if $P$ has a large $\varphi$-variance group $X_\varphi(P)$ and if $Q$ is fairly small (which forces $X(Q|P)$ to be small), then $X_\varphi(P \circ Q)$ is still large.

A careful refinement lets us make a similar statement about infinite $\varphi$-variance groups:

**Theorem 3.26.** Let $P$ and $Q$ be chiral or directly regular $n$-semi-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. If $X(Q|P)$ is finite and $X_\varphi(P)$ is infinite, then $|X_\varphi(P \circ Q)|$ is infinite.

**Proof.** Consider the commutative diagram below, where the maps are all the natural epimorphisms:

$$
\begin{array}{ccc}
\Gamma^+(P \circ P^\varphi \circ Q \circ Q^\varphi) & f_1 & \Gamma^+(P \circ P^\varphi) \\
\downarrow f_2 & & \downarrow g_1 \\
\Gamma^+(P \circ Q) & g_2 & \Gamma^+(P)
\end{array}
$$

Then $\ker(g_1 \circ f_1) = \ker(g_2 \circ f_2)$. Now, since $X_\varphi(P) = \ker g_1$ is infinite by assumption, it follows that $\ker(g_1 \circ f_1)$ is infinite. We have that $\ker(g_2 \circ f_2) = f_2^{-1}(\ker g_2)$. Now, $f_2 = X_\varphi(P \circ Q)$, and $\ker g_2 = X(Q|P)$, which is finite by assumption. If $X_\varphi(P \circ Q)$ is finite, then so is the preimage of any finite subgroup of $\Gamma^+(P \circ Q)$ under $f_2$. Therefore, $f_2^{-1}(\ker g_2)$ is finite, and since $f_2^{-1}(\ker g_2) = \ker(g_2 \circ f_2) = \ker(g_1 \circ f_1)$, the group $X_\varphi(P \circ Q)$ must be infinite.

Let us apply this result to extend Theorem 3.20 in a different way.
Theorem 3.27. Let $P$, $Q$, and $R$ be finite chiral or directly regular $n$-semi-polytopes, and let $\varphi \in \text{Aut}(W_n^+)$ have finite order. If $P \circ Q \circ R$ is internally $\varphi$-invariant, then $|\Gamma^+(R)|$ is an integer multiple of $|X_\varphi(P)|/|\Gamma^+(Q)|$.

Proof. Theorem 3.20 tells us that if $(P \circ Q) \circ R$ is internally $\varphi$-invariant, then $|X_\varphi(P \circ Q)|$ divides $|\Gamma^+(R)|$. Then Theorem 3.25 says that $|X_\varphi(P \circ Q)|$ is an integer multiple of $|X_\varphi(P)|/|X_\varphi(Q)|$, and since $|X_\varphi(Q)|$ divides $|\Gamma^+(Q)|$, this is an integer multiple of $|X_\varphi(P)|/|\Gamma^+(Q)|$. The result follows. □

Thus we see that it is possible (though a bit unwieldy) to extend Theorem 3.20 to the mix of three and indeed any number of polytopes. In practice, however, we usually mix polytopes two at a time and try to find the exact $\varphi$-variance group of the mix. The following results illustrate a few cases where we can easily do so.

Theorem 3.28. Let $P$ and $Q$ be finite chiral or directly regular $n$-semi-polytopes, and let $\varphi \in \text{Aut}(W_n^+)$ have finite order. Suppose $P$ is internally $\varphi$-variant, with $X_\varphi(P)$ simple, and suppose that $Q$ is internally $\varphi$-invariant. If $|X_\varphi(P)|$ does not divide $|\Gamma^+(Q)|$, then $X_\varphi(P \circ Q) = X_\varphi(P)$.

Proof. By Theorem 3.20, the mix $P \circ Q$ is internally $\varphi$-variant. Proposition 3.24 says that $X_\varphi(P \circ Q)$ is a normal subgroup of $X_\varphi(P)$, which is simple. Since $P \circ Q$ is internally $\varphi$-variant, $X_\varphi(P \circ Q)$ must be nontrivial, and therefore $X_\varphi(P \circ Q) = X_\varphi(P)$. □

Theorem 3.29. Let $P$ and $Q$ be finite chiral or directly regular $n$-semi-polytopes, and let $\varphi \in \text{Aut}(W_n^+)$ have finite order. Suppose that $P$ and $Q$ are internally $\varphi$-variant and that $\Gamma^+(Q)$ is simple. If $X_\varphi(P)$ is not isomorphic to $\Gamma^+(Q)$, then $P \circ Q$ is internally $\varphi$-variant.

Proof. Suppose $P \circ Q$ is internally $\varphi$-invariant. Then by Lemma 3.16, $\Gamma^+(P) \circ \Gamma^+(Q)$ naturally covers $\Gamma^+(P \circ Q)$. Thus, we get the following commutative diagram, where the
maps are all the natural epimorphisms:

\[
\begin{array}{c}
\Gamma^+(P) \circ \Gamma^+(Q) \\
\downarrow f_1 \\
\Gamma^+(P) \circ \Gamma^+(P^\varphi) \\
\downarrow f_2 \\
\Gamma^+(P) \\
\end{array}
\]

Therefore, \( \ker f_2 = \ker(g \circ f_1) \), and in particular, \( \ker f_1 \) is normal in \( \ker f_2 \). Now, we can view \( \ker f_2(= X(Q|P)) \) as a normal subgroup of \( \Gamma^+(Q) \), which is simple. Then \( \ker f_2 \) is simple (possibly trivial), and \( \ker f_1 \) is likewise simple (possibly trivial). Suppose \( \ker f_1 \) is trivial. Then \( \ker f_2 \simeq \ker g = X_\varphi(P) \). Since \( P \) is internally \( \varphi \)-variant, \( \ker g \) is nontrivial. Therefore, \( \ker f_2 \) is nontrivial, so we must have \( \ker f_2 \simeq \Gamma^+(Q) \). But then \( X_\varphi(P) \simeq \Gamma^+(Q) \), violating our assumptions. Now, suppose instead that \( \ker f_1 \simeq \Gamma^+(Q) \). Then we must have \( \ker f_2 \simeq \Gamma^+(Q) \) as well. Then since \( \ker f_2 = \ker(g \circ f_1) \), we must have that \( \ker g \) is trivial, contradicting that \( P \) is internally \( \varphi \)-variant. Therefore, \( P \circ Q \) must be internally \( \varphi \)-variant.

\[\square\]

**Theorem 3.30.** Let \( P \) and \( Q \) be finite chiral or directly regular \( n \)-semi-polytopes and let \( \varphi \in \text{Aut}(W_n^+) \) have finite order. Suppose that \( \Gamma^+(P) \circ \Gamma^+(P^\varphi) \) and \( \Gamma^+(Q) \circ \Gamma^+(Q^\varphi) \) have no nontrivial common quotients. Then \( \Gamma^+(P \circ Q) = \Gamma^+(P) \times \Gamma^+(Q) \) and \( X_\varphi(P \circ Q) = X_\varphi(P) \times X_\varphi(Q) \).

**Proof.** Since \( \Gamma^+(P) \) is a quotient of \( \Gamma^+(P) \circ \Gamma^+(P^\varphi) \) and \( \Gamma^+(Q) \) is a quotient of \( \Gamma^+(Q) \circ \Gamma^+(Q^\varphi) \), the groups \( \Gamma^+(P) \) and \( \Gamma^+(Q) \) also have no nontrivial common quotient. Therefore, \( \Gamma^+(P) \square \Gamma^+(Q) \) is trivial, and \( \Gamma^+(P \circ Q) = \Gamma^+(P) \times \Gamma^+(Q) \), by Proposition 2.21. Now, on
the one hand,

\[
|\Gamma^+(P) \circ \Gamma^+(P^\varphi) \circ \Gamma^+(Q) \circ \Gamma^+(Q^\varphi)| = |(\Gamma^+(P) \circ \Gamma^+(Q)) \circ (\Gamma^+(P) \circ \Gamma^+(Q)^\varphi)|
\]
\[
= |\Gamma^+(P) \circ \Gamma^+(Q)| \cdot |X_\varphi(P \circ Q)|
\]
\[
= |\Gamma^+(P)| \cdot |\Gamma^+(Q)| \cdot |X_\varphi(P \circ Q)|.
\]

On the other hand,

\[
|\Gamma^+(P) \circ \Gamma^+(P^\varphi) \circ \Gamma^+(Q) \circ \Gamma^+(Q^\varphi)| = |(\Gamma^+(P) \circ \Gamma^+(P^\varphi)) \circ (\Gamma^+(Q) \circ \Gamma^+(Q^\varphi))|
\]
\[
= |\Gamma^+(P)| \cdot |X_\varphi(P)| \cdot |\Gamma^+(Q)| \cdot |X_\varphi(Q)|,
\]

again by Proposition 2.21. Therefore,

\[
|X_\varphi(P \circ Q)| = |X_\varphi(P)| \cdot |X_\varphi(Q)|.
\]

By Proposition 3.23, \(X_\varphi(P \circ Q)\) is isomorphic to a subgroup of \(X_\varphi(P) \times X_\varphi(Q)\), and thus it follows that \(X_\varphi(P \circ Q) = X_\varphi(P) \times X_\varphi(Q)\). \(\square\)
Chapter 4

Constructing self-dual and self-Petrie polytopes

We will now start applying the results of Chapter 3 to the operations of duality and Petrie duality. Recall that the dual of \( P \) is denoted \( P^\delta \), and it is obtained from \( P \) by reversing the partial order. The Petrie dual of a polyhedron \( P \) is denoted \( P^\pi \), and it is obtained from \( P \) by keeping the vertices and edges, but replacing the faces with the set of Petrie polygons. Each of these operations also has a corresponding group automorphism; the automorphism \( \delta \) of \( W_n^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle \) sends each \( \sigma_i \) to \( \sigma_{n-i}^{-1} \), and the automorphism \( \pi \) of \( W_3 = \langle \rho_0, \rho_1, \rho_2 \rangle \) sends \( \rho_0 \) to \( \rho_0 \rho_2 \) while fixing \( \rho_1 \) and \( \rho_2 \).

Our first goal is to use the results from Chapter 3 to construct self-dual covers of a given polytope and to determine when such covers are polytopal. Then we will see how to construct polyhedra that are simultaneously self-dual and self-Petrie, and we will give several examples of such polyhedra.
4.1 Self-dual polytopes

By mixing a polytope \( P \) with its dual, we get a properly self-dual cover \( Q \) of \( P \); that is, \( Q \) is internally \( \delta \)-invariant, so \( Q = Q^\delta \). Our first question is: under what conditions is this cover a polytope? Since Corollary 2.45 tells us that the mix of two polyhedra is a polyhedron, we focus on ranks 4 and higher. We start by applying Proposition 2.41 and Theorem 2.53 to the mix of \( P \) with \( P^\delta \).

**Proposition 4.1.** Let \( P \) be a chiral or regular polytope of type \( \{K, L\} \), and suppose \( L^\delta \) covers \( K \). Then \( P \circ P^\delta \) is a self-dual polytope.

**Proof.** The facets of \( P^\delta \) are \( L^\delta \), and the facets of \( P \) are \( K \). Since \( L^\delta \) covers \( K \), Proposition 2.41 says that \( P \circ P^\delta \) is a polytope, and Proposition 3.11 tells us that \( P \circ P^\delta \) is internally \( \delta \)-invariant, i.e., properly self-dual. \( \square \)

**Proposition 4.2.** Let \( P \) be a chiral or directly regular \( n \)-polytope, with \( n = 2m + 1 \). Suppose \( P \) is of type \( \{p_1, \ldots, p_{n-1}\} \), and that \( p_i \) is coprime to \( p_{n-i} \) for every \( i \in \{2, \ldots, m\} \). Then \( P \circ P^\delta \) is a self-dual polytope. Furthermore, if \( p_1 \) is coprime to \( p_{n-1} \), then \( \Gamma(P \circ P^\delta) = \Gamma(P) \times \Gamma(P^\delta) \).

**Proof.** The result follows directly from Theorem 2.53 and Proposition 2.47. \( \square \)

Clearly, if \( n \) is even, then it is impossible to apply Theorem 2.53 to the mix of \( P \) with \( P^\delta \), since in that case, \( P \) and \( P^\delta \) will agree in the middle entry of their Schl"afli symbol. In fact, when \( n \) is even, having certain numbers \( p_i \) relatively prime to \( p_{n-i} \) is actually an impediment to polytopality.

**Proposition 4.3.** Let \( P \) be a chiral or directly regular \( n \)-polytope, with \( n = 2m \). Suppose \( P \) is of type \( \{p_1, \ldots, p_{2m-1}\} \), that \( p_{m-1} \) and \( p_{m+1} \) are relatively prime, and that \( p_m \geq 3 \). Then \( P \circ P^\delta \) is not a polytope.
Proof. Apply Proposition 2.56 to $P$ and $P^\delta$. 

**Example 4.4.** If $P$ is the 4-cube, which is the universal polytope of type $\{4, 3, 3\}$, then $P \circ P^\delta$ is not a polytope.

We will briefly explore a way to build self-dual regular polytopes of high rank. Given any polytope $P$ of type $\{p_1, \ldots, p_{n-1}\}$, there is a polytope $\{P, 2\}$ of type $\{p_1, \ldots, p_{n-1}, 2\}$ and with facets isomorphic to $P$. We build this polytope by inserting two new faces between the maximal element and the facets of $P$, and connecting both new faces to every facet of $P$. It is then easy to show that if $P$ is regular, so is $\{P, 2\}$, and $\Gamma(\{P, 2\}) = \Gamma(P) \times \langle \rho_n \rangle$. This extension also works nicely with direct regularity:

**Proposition 4.5.** Let $P$ be a directly regular $n$-polytope and let $Q = \{P, 2\}$. Then $Q$ is directly regular, and $\Gamma^+(Q) \simeq \Gamma(P)$.

**Proof.** Since $P$ is directly regular, all of the relators in $\Gamma(P)$ have even length. The only relators that we add in extending $P$ to $Q$ are $\rho_n^2$ and relators of the form $(\rho_i \rho_n)^2$, all of which are even. Therefore, $Q$ is still directly regular.

Now, let $w \in \Gamma^+(Q) \leq \Gamma(Q) = \langle \rho_0, \ldots, \rho_n \rangle$. Since $\Gamma(Q) = \Gamma(P) \times \langle \rho_n \rangle$, we can write $w$ uniquely as $w = v \rho_n^i$, where $i = 0$ or 1, and where $v \in \Gamma(P)$.

Now, consider the canonical projection from $\Gamma(Q) = \Gamma(P) \times \langle \rho_n \rangle$ to $\Gamma(P)$, and let $f$ be its restriction to $\Gamma^+(Q)$. That is, $f(v \rho_n^i) = v$. We want to show that $f$ is an isomorphism from $\Gamma^+(Q)$ to $\Gamma(P)$. First, let $w \in \Gamma^+(Q)$, and suppose that $f(w) = \epsilon$. Then writing $w = v \rho_n^i$, we see that $v = \epsilon$. Therefore, $w = \rho_n^i$. Then since $w \in \Gamma^+(Q)$, we must have $w = \epsilon$, and thus $f$ is injective. To prove surjectivity, note that if $v \in \Gamma^+(P)$, then $v \in \Gamma^+(Q)$, and $f(v) = v$, while if $v \in \rho_0 \Gamma^+(P)$, then $\rho_n v \in \Gamma^+(Q)$, and $f(\rho_n v) = v$. Thus we see that $f : \Gamma^+(Q) \rightarrow \Gamma(P)$ is an isomorphism.
Theorem 4.6. Let $\mathcal{P}$ be a directly regular $n$-polytope of type $\{p_1, \ldots, p_{n-1}\}$ with every $p_i$ odd. Then there is a self-dual directly regular $(2n-1)$-polytope of type $\{2p_1, \ldots, 2p_{n-1}, 2n-1\}$, with $n$-faces that cover $\mathcal{P}$ and with rotation group $\Gamma(\mathcal{P}) \times \Gamma(\mathcal{P}^\delta) \times C_2^{2n-4} \simeq \Gamma(\mathcal{P}) \times \Gamma(\mathcal{P} \delta) \times C_2^{2n-4}$.

Proof. We start by extending $\mathcal{P}$ $n-1$ times as above to obtain a $(2n-1)$-polytope $\mathcal{Q}$ of type $\{q_1, \ldots, q_{2n-2}\} := \{p_1, \ldots, p_{n-1}, 2, \ldots, 2\}$, with group $\Gamma(\mathcal{Q}) = \Gamma(\mathcal{P}) \times C_2^{n-1}$. Then by Proposition 4.5, $\Gamma^+(\mathcal{Q}) \simeq \Gamma(\mathcal{P}) \times C_2^{n-2}$. Now, let $\mathcal{Q}^\delta$ be the dual of $\mathcal{Q}$. If $i < n$, then $q_i$ is odd and $q_{2n-1-i} = 2$, so by Proposition 4.2, we get that $\mathcal{Q} \diamond \mathcal{Q}^\delta$ is a self-dual directly regular polytope of type $\{2p_1, \ldots, 2p_{n-1}, 2p_{n-1}, \ldots, 2p_1\}$ and with rotation group $\Gamma^+(\mathcal{Q} \diamond \mathcal{Q}^\delta) = \Gamma^+(\mathcal{Q}) \times \Gamma^+(\mathcal{Q}^\delta) = \Gamma(\mathcal{P}) \times \Gamma(\mathcal{P}^\delta) \times C_2^{2n-4}$. $\square$

For example, if we apply Theorem 4.6 to the $n$-simplex, then we obtain a $(2n-1)$-polytope of type $\{6, \ldots, 6\}$ with rotation group $S_{n+1} \times S_{n+1} \times C_2^{2n-4}$.

4.2 Self-dual, self-Petrie polyhedra

Regular polyhedra that are self-dual and self-Petrie have a high degree of “external” symmetry in addition to the maximal “internal” symmetry that regularity measures. Closely related are the self-dual and self-Petrie maps on surfaces, which have been studied in [19, 20, 29]. Many of these results can be adapted to abstract polyhedra. We do so and present several new results here.

Suppose we want to construct the minimal self-dual, self-Petrie cover of a regular polyhedron $\mathcal{P}$. Our first instinct might be to mix $\mathcal{P}$ with its dual to construct its self-dual cover $\mathcal{Q}$, and then to mix $\mathcal{Q}$ with its Petrial to construct its self-Petrie cover $\mathcal{R}$. However, there is no guarantee that $\mathcal{R}$ is self-dual. Constructing a polyhedron that is simultaneously self-dual
and self-Petrie requires a slightly subtler approach.

In [18], the author proves that the automorphisms $\delta$ and $\pi$ (of $W_3$) generate a symmetric group of order 6. Therefore, a regular polyhedron $\mathcal{P}$ has at most 6 distinct images under these operations. Define $\mathcal{P}^*$ to be the mix of these 6 polyhedra; i.e.,

$$\mathcal{P}^* := \mathcal{P} \circ \mathcal{P}^\delta \circ \mathcal{P}^\pi \circ \mathcal{P}^{\delta\pi} \circ \mathcal{P}^{\pi\delta} \circ \mathcal{P}^{\delta\pi\delta}.$$  

Then $\mathcal{P}^*$ is a regular polyhedron. Now, if $\Gamma(\mathcal{P}) = W_3/M$, then $\Gamma(\mathcal{P}^*)$ is the quotient of $W_3$ by

$$M \cap \delta(M) \cap \pi(M) \cap \delta\pi(M) \cap \pi\delta(M) \cap \delta\pi\delta(M),$$

and since this subgroup is fixed by both $\delta$ and $\pi$, it follows that $\mathcal{P}^*$ is self-dual and self-Petrie. In fact, it is the minimal self-dual, self-Petrie cover of $\mathcal{P}$.

If $\mathcal{P}$ is of type $\{p, q\}_r$, then $\mathcal{P}^\delta$ is of type $\{q, p\}_r$, and $\mathcal{P}^\pi$ is of type $\{r, q\}_p$. Therefore, a polyhedron that is self-dual and self-Petrie must be of type $\{k, k\}_k$ for some $k \geq 2$. In particular, if $\mathcal{P}$ is of type $\{p, q\}_r$, then $\mathcal{P}^*$ is of type $\{k, k\}_k$, where $k$ is the least common multiple of $p$, $q$, and $r$.

Determining the global structure of $\Gamma(\mathcal{P}^*)$, such as its size or isomorphism type, is quite challenging. In general, the problem is completely intractable; the mix of 6 groups is simply too large for the usual algorithms in combinatorial group theory to work with. Therefore, we turn our attention to finding ways to avoid such direct calculation. Usually, such results will hinge on calculating the comix of certain polyhedra, building on Theorem 2.23.

**Theorem 4.7.** Let $\mathcal{P}$ be a regular self-Petrie polyhedron of type $\{p, q\}_p$. Suppose $p$ is odd and that $p$ and $q$ are coprime. Then

$$\Gamma(\mathcal{P}^*) = \Gamma(\mathcal{P}) \times \Gamma(\mathcal{P}^\delta) \times \Gamma(\mathcal{P}^{\delta\pi}).$$

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Proof. First, we note that \( P^\delta \) is of type \( \{q, p\}_p \). In \( \Gamma(P) \square \Gamma(P^\delta) \), the order of \( \rho_0 \rho_1 \) divides \( p \) and \( q \), and since \( p \) and \( q \) are coprime, we get \( \rho_0 \rho_1 = \epsilon \); that is, \( \rho_0 = \rho_1 \). Similarly, \( \rho_1 \rho_2 = \epsilon \), and so \( \rho_1 = \rho_2 \). Now, the order of \( \rho_0 \rho_1 \rho_2 \) divides \( p \). On the other hand, \( \rho_0 \rho_1 \rho_2 = \rho_0 \), and so the order divides 2 as well. Therefore, we must have that \( \rho_0 \rho_1 \rho_2 = \epsilon \). This forces all of the generators to be trivial, and thus \( \Gamma(P) \square \Gamma(P^\delta) \) is trivial and \( \Gamma(P) \diamond \Gamma(P^\delta) = \Gamma(P) \times \Gamma(P^\delta) \), by Corollary 2.24.

Now, \( P \diamond P^\delta \) is of type \( \{pq, pq\}_p \), and \( P^\delta \pi \) is of type \( \{p, p\}_q \). Then in their comix, we get that \( \rho_0 \rho_1 \rho_2 \) has order dividing \( p \) and \( q \), and thus \( \rho_0 \rho_1 \rho_2 \) is trivial. Thus \( \rho_0 = \rho_1 \rho_2 \), and so \((\rho_1 \rho_2)^2 = \epsilon \). On the other hand, we also have that \((\rho_1 \rho_2)^p = \epsilon \) in the comix, and since \( p \) is odd, this means that \( \rho_1 \rho_2 \) is trivial. So \( \rho_0 \) is trivial, and \( \rho_1 = \rho_2 \). Similarly, \( \rho_2 = \rho_0 \rho_1 \), and thus \((\rho_0 \rho_1)^2 = \epsilon \). But again, we also have that \((\rho_0 \rho_1)^p = \epsilon \), and thus \( \rho_0 \rho_1 = \epsilon \). So \( \rho_0 = \rho_1 = \rho_2 = \epsilon \) and we see that the comix is trivial. Therefore, the mix is the direct product \( \Gamma(P) \times \Gamma(P^\delta) \times \Gamma(P^\delta \pi) \), again by Corollary 2.24.

Finally, we note that since \( P \) is self-Petrie, it follows that \( P^\pi = P \), \( P^\pi \delta = P^\delta \), and \( P^\pi \delta \pi = P^\delta \pi \). Therefore, the self-dual, self-Petrie cover of \( P \) consists of just the three distinct polyhedra we have mixed. \( \Box \)

**Corollary 4.8.** Let \( P \) be a regular polyhedron of type \( \{p, q\}_r \). Suppose \( p \) and \( r \) are odd and both coprime to \( q \). Let \( Q = P \diamond P^\pi \). Then

\[
\Gamma(P^*) = \Gamma(Q) \times \Gamma(Q^\delta) \times \Gamma(Q^\delta \pi).
\]

Proof. We start by noting that \( P^* = Q^* \). The polyhedron \( Q \) is a self-Petrie polyhedron of type \( \{\ell, q\}_\ell \), where \( \ell \) is the least common multiple of \( p \) and \( r \). Since \( p \) and \( r \) are both odd and coprime to \( q \), \( \ell \) is also odd and coprime to \( q \). Then we can apply Theorem 4.7 to \( Q \) and the result follows. \( \Box \)
The condition in Theorem 4.7 that $p$ is odd is essential. When $p$ is even, we cannot tell from the type alone whether $\Gamma(P) \sqcap \Gamma(P^\delta)$ has order 1 or 2. However, if $P$ is the universal polyhedron of type $\{p, q\}$, then we can still determine $|\Gamma(P^\star)|$.

**Theorem 4.9.** Let $P = \{p, q\}$, the universal polyhedron of type $\{p, q\}$. Suppose $p$ is even and that $p$ and $q$ are coprime. Then $\Gamma(P^\star)$ is a group of index 8 in $\Gamma(P) \times \Gamma(P^\delta) \times \Gamma(P^\delta\pi)$. In particular, if $P$ is finite, then

$$|\Gamma(P^\star)| = |\Gamma(P)|^3/8.$$ 

**Proof.** Since $p$ and $q$ are coprime, a presentation for $\Gamma(P) \sqcap \Gamma(P^\delta)$ is given by

$$\langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_0 \rho_1 = (\rho_0 \rho_2)^2 = \rho_1 \rho_2 = (\rho_0 \rho_1 \rho_2)^p = \epsilon \rangle,$$

and direct calculation shows that this is a group of order 2. Therefore, $\Gamma(P) \bowtie \Gamma(P^\delta)$ has index 2 in $\Gamma(P) \times \Gamma(P^\delta)$.

Now we consider $(\Gamma(P) \bowtie \Gamma(P^\delta)) \sqcap \Gamma(P^\delta\pi)$. In this group, the order of $\rho_0 \rho_1 \rho_2$ divides both $p$ and $q$, and thus $\rho_0 \rho_1 \rho_2 = \epsilon$. Therefore, $\rho_0 \rho_1 = \rho_2$, which forces $\rho_0 \rho_1$ to have order dividing 2, and similarly $\rho_0 = \rho_1 \rho_2$, which forces $\rho_1 \rho_2$ to have order dividing 2. Therefore, the comix is a (not necessarily proper) quotient of the group $[2, 2]_1$, a group of order 4. Now, since $\Gamma(P^\delta\pi) = [p, p]_q$ and $p$ is even, we see that $\Gamma(P^\delta\pi)$ covers $[2, 2]_1$. We similarly see that $\Gamma(P)$ covers $[2, 1]_1$ and that $\Gamma(P^\delta)$ covers $[1, 2]_1$, and thus $\Gamma(P) \bowtie \Gamma(P^\delta)$ covers $[2, 1]_1 \bowtie [1, 2]_1$, which is equal to $[2, 2]_1$. Thus we see that the group $[2, 2]_1$ is the maximal group covered by both $\Gamma(P) \bowtie \Gamma(P^\delta)$ and $\Gamma(P^\delta\pi)$, and therefore it is their comix. Therefore, $(\Gamma(P) \bowtie \Gamma(P^\delta)) \bowtie \Gamma(P^\delta\pi)$ has index 4 in $(\Gamma(P) \bowtie \Gamma(P^\delta)) \times \Gamma(P^\delta\pi)$, which itself has index 2 in $(\Gamma(P) \times \Gamma(P^\delta)) \times \Gamma(P^\delta\pi)$, and the claim is proved. \qed
We note here that the arguments used can be generalized to give bounds on $|\Gamma(P^*)|$ even when we cannot calculate the exact value. In the next section, however, we will only consider cases where we can calculate $|\Gamma(P^*)|$ exactly.

### 4.3 Self-dual, self-Petrie covers of universal polyhedra

In this section, we will consider several of the finite universal polyhedra $P = \{p,q\}$, and calculate $|\Gamma(P^*)|$, the group order for the minimal self-dual, self-Petrie, regular cover of $P$. The results are summarized in Table 4.1. In most cases, the sizes are easily calculated by applying Corollary 4.8 to $P$ or one of its images under the duality operations. A few others can be found by applying Theorem 4.9 or by calculating the size directly using GAP [12]. We cover the remaining cases here.

We start with $P = \{2,2s\}$. The order of $\rho_0 \rho_1 \rho_2$ in $\Gamma(P)$ is $2s$, and therefore $P = \{2,2s\}_{2s}$, which has an automorphism group of order $8s$. If $s = 1$, then $P$ is already self-dual and self-Petrie. For any $s$, there are only 3 distinct images of $P$ under the duality operations, and thus

$$P^* = P \circ P^\delta \circ P^\pi.$$

Now, $P \circ P^\delta$ is a self-dual regular polyhedron of type $\{2s,2s\}_{2s}$. We note that for any $s$, the group $\Gamma(P) \boxtimes \Gamma(P^\delta)$ is equal to $[2,2]_{2s}$. In fact, this group is equal to $[2,2]_2$, the group of order 8 generated by 3 commuting involutions (i.e., the direct product of three cyclic groups of order 2). Then by Proposition 2.21, we see that $|\Gamma(P) \circ \Gamma(P^\delta)| = 8s^2$. In order to determine whether $P \circ P^\delta$ is self-Petrie, we would like to determine the group $\Gamma(P) \circ \Gamma(P^\delta)$ explicitly. A computation with GAP [12] suggests that the group is always the quotient of $[2s,2s]_{2s}$ by the extra relation $(\rho_1 \rho_0 \rho_1 \rho_2)^2 = \epsilon$. Since this extra relation also holds in $\Gamma(P)$ and $\Gamma(P^\delta)$, this quotient must cover $\Gamma(P) \circ \Gamma(P^\delta)$. Therefore, to prove that this is in fact
the mix, it suffices to show that this quotient has order $8s^2$.

Start by considering the Cayley graph $G$ of

$$\langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_2)^2 = (\rho_1\rho_0\rho_1\rho_2)^2 = \epsilon \rangle,$$

determined by the generators $\rho_0, \rho_1$, and $\rho_2$. Starting at a vertex and building out from it, we see that the Cayley graph of this group is the graph of the uniform tiling 4.8.8 of the plane by squares and octagons. Figure 4.1 gives a local picture of $G$, and shows how this tiling can be thought of as being built from square tiles fitting together edge-to-edge.

![Cayley graph](image)

Figure 4.1: Local picture of the Cayley graph $G$

Now, let us see what happens when we introduce the remaining relations $(\rho_0\rho_1)^{2s} = \epsilon$ and $(\rho_1\rho_2)^{2s} = \epsilon$. We note that the components of $G$ induced by edges labeled 0 and 1 are vertical
zig-zags, while the components of $G$ induced by edges labeled 1 and 2 are horizontal zig-zags. Therefore, adding the relation $(\rho_0 \rho_1)^{2s} = \epsilon$ forces an identification between points that are $s$ square tiles away vertically, and adding the relation $(\rho_1 \rho_2)^{2s} = \epsilon$ forces an identification between points that are $s$ square tiles away horizontally. Therefore, the Cayley graph of $\Gamma(\mathcal{P} \circ \mathcal{P}^\delta)$ consists of an $s \times s$ grid of square tiles, with opposite sides identified. Each square tile has 8 vertices of the Cayley graph, so there are a total of $8s^2$ vertices, which shows that the given group has $8s^2$ elements. Therefore the given group is indeed the mix.

Resuming our discussion of whether $\mathcal{P} \circ \mathcal{P}^\delta$ is self-Petrie, we consider $(\Gamma(\mathcal{P}) \circ \Gamma(\mathcal{P}^\delta)) \square \Gamma(\mathcal{P}^\pi)$, which is the quotient of $[2s, 2s]_2$ by the extra relation $(\rho_1 \rho_0 \rho_1 \rho_2)^2 = \epsilon$. Put another way, we get this group as the quotient of $\Gamma(\mathcal{P}) \circ \Gamma(\mathcal{P}^\delta)$ by the extra relation $(\rho_0 \rho_1 \rho_2)^2 = \epsilon$. Using the Cayley graph from before, we see that the relation $(\rho_0 \rho_1 \rho_2)^2 = \epsilon$ forces us to identify each square tile with the square tiles that touch it at a corner. When $s$ is odd, this forces us to identify every pair of square tiles, leaving us with a Cayley graph with 8 vertices. When $s$ is even, we instead get 2 distinct square tiles, and the Cayley graph has 16 vertices. Therefore, $|\Gamma(\mathcal{P}) \circ \Gamma(\mathcal{P}^\pi)| = 8s^3$ if $s$ is odd, and $4s^3$ if $s$ is even. In other words, if $\mathcal{P} = \{2, 2s\}$ and $s = 2k + 1$, then $|\Gamma(\mathcal{P}^*)| = 8(2k + 1)^3$, while if $s = 2k$, then $|\Gamma(\mathcal{P}^*)| = 32k^3$.

Now suppose that $\mathcal{P} = \{2, 2k + 1\}$. This polyhedron is covered by $\mathcal{Q} = \{2, 4k + 2\}$ (which is equal to $\{2, 4k + 2\}_{4k+2}$, as we observed earlier); therefore, $\mathcal{Q} \circ \mathcal{Q}^\delta$ covers $\mathcal{P} \circ \mathcal{P}^\delta$. Calculating the size of the mix, we find that $|\Gamma(\mathcal{Q}) \circ \Gamma(\mathcal{Q}^\delta)| = |\Gamma(\mathcal{P}) \circ \Gamma(\mathcal{P}^\delta)|$, and thus these two groups must be equal. From this it easily follows that $\mathcal{Q}^* = \mathcal{P}^*$, and thus $|\Gamma(\mathcal{P}^*)| = 8(2k + 1)^3$.

Next we consider the polyhedra $\{4, 4\}_{2s}$, which are the torus maps $\{4, 4\}_{(s,s)}$ with $16s^2$ flags [8]. First, suppose that $s = 2k$, so that $\mathcal{P} = \{4, 4\}_{4k}$. Then $\Gamma(\mathcal{P}) \square \Gamma(\mathcal{P}^\pi) = [4, 4]_4$, which has order 64. Therefore, by Proposition 2.21,

$$|\Gamma(\mathcal{P}) \circ \Gamma(\mathcal{P}^\pi)| = |\Gamma(\mathcal{P})| \cdot |\Gamma(\mathcal{P}^\pi)|/64 = 64k^4.$$
Now, $\mathcal{P} \odot \mathcal{P}^\pi$ is of type $\{4k,4\}_{4k}$, and $\mathcal{P}^\pi \delta$ is the universal polyhedron of type $\{4,4k\}_4$. Thus $(\Gamma(\mathcal{P}) \odot \Gamma(\mathcal{P}^\pi) \Box \Gamma(\mathcal{P}^\pi \delta)$ is $[4,4]_4$ or a proper quotient. Since $\Gamma(\mathcal{P})$ covers $[4,4]_4$, so does $\Gamma(\mathcal{P}) \odot \Gamma(\mathcal{P}^\pi)$. Then since $\Gamma(\mathcal{P}^\pi \delta)$ also covers $[4,4]_4$, so does $(\Gamma(\mathcal{P}) \odot \Gamma(\mathcal{P}^\pi) \Box \Gamma(\mathcal{P}^\pi \delta)$, and thus the comix is the whole group $[4,4]_4$. Thus we see that

$$|\Gamma(\mathcal{P}) \odot \Gamma(\mathcal{P}^\pi) \odot \Gamma(\mathcal{P}^\pi \delta)| = |\Gamma(\mathcal{P}) \odot \Gamma(\mathcal{P}^\pi)| \cdot |\Gamma(\mathcal{P}^\pi \delta)| / 64 = 64k^6.$$ 

Now suppose that $s = 2k + 1$, so that $\mathcal{P} = \{4,4\}_{4k+2}$. Then $\Gamma(\mathcal{P}) \Box \Gamma(\mathcal{P}^\pi) = [2,4]_2$, which has size 8. Therefore,

$$|\Gamma(\mathcal{P}) \odot \Gamma(\mathcal{P}^\pi)| = |\Gamma(\mathcal{P})| \cdot |\Gamma(\mathcal{P}^\pi)| / 8 = 32(2k + 1)^4.$$ 

Now, $\mathcal{P} \odot \mathcal{P}^\pi$ is of type $\{8k+4,4\}_{8k+4}$ and $\mathcal{P}^\pi \delta$ is the universal polyhedron $\{4,4k+2\}_4$. Thus $(\Gamma(\mathcal{P}) \odot \Gamma(\mathcal{P}^\pi) \Box \Gamma(\mathcal{P}^\pi \delta)$ is $[4,2]_4$ or a proper quotient. Clearly, $\Gamma(\mathcal{P}^\pi \delta)$ covers $[4,2]_4$. Furthermore, $\Gamma(\mathcal{P})$ covers $[4,4]_2$ and $\Gamma(\mathcal{P}^\pi)$ covers $[2,4]_4$; therefore, their mix covers $[4,4]_2 \odot [2,4]_4$, which covers $[4,2]_4$. Therefore, the comix is the whole group $[4,2]_4$ of order 16, and we see that

$$|\Gamma(\mathcal{P}) \odot \Gamma(\mathcal{P}^\pi) \odot \Gamma(\mathcal{P}^\pi \delta)| = |\Gamma(\mathcal{P}) \odot \Gamma(\mathcal{P}^\pi)| \cdot |\Gamma(\mathcal{P}^\pi \delta)| / 16 = 32(2k + 1)^6.$$ 

It would be natural here to consider the torus maps $\{3,6\}_{2s} = \{3,6\}_{(s,0)}$. However, in this case there are 6 distinct polyhedra under the duality operations, and the problem seems to be intractible.

Finally, we note that the self-dual, self-Petrie covers of $\{3,3\}_4$ and $\{3,4\}_6$ have groups of the same order. In fact, since $\{3,4\}_6$ covers $\{3,4\}_3$, these two polyhedra have the same self-dual, self-Petrie cover, of type $\{12,12\}$. 

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| $\mathcal{P}$          | $|\Gamma(\mathcal{P})|$        | $|\Gamma(\mathcal{P}^*)|$        | Method  |
|----------------------|--------------------------------|----------------------------------|---------|
| $\{2, 2k+1\}_{4k+2}$ | $4(2k+1)$                      | $8(2k+1)^3$                      | Hand    |
| $\{2, 4k\}_{4k}$    | $16k$                          | $32k^3$                          | Hand    |
| $\{2, 4k+2\}_{4k+2}$| $8(2k+1)$                      | $8(2k+1)^3$                      | Hand    |
| $\{3, 3\}_{4}$      | $24$                           | $(24)^3$                         | Thm. 4.7|
| $\{3, 4\}_{6}$      | $48$                           | $(48)^3/8$                       | GAP     |
| $\{3, 5\}_{5}$      | $60$                           | $(60)^4$                         | Thm. 4.7|
| $\{3, 5\}_{10}$     | $120$                          | $(120)^3$                        | GAP     |
| $\{3, 6\}_{6}$      | $108$                          | $(108)^3/216$                    | GAP     |
| $\{3, 7\}_{8}$      | $336$                          | $(336)^6/8$                      | Cor. 4.8|
| $\{3, 7\}_{9}$      | $504$                          | $(504)^6$                        | Cor. 4.8|
| $\{3, 7\}_{13}$     | $1092$                         | $(1092)^6$                       | Cor. 4.8|
| $\{3, 7\}_{15}$     | $12180$                        | $(12180)^6$                      | Cor. 4.8|
| $\{3, 7\}_{16}$     | $21504$                        | $(21504)^6/8$                    | Cor. 4.8|
| $\{3, 8\}_{8}$      | $672$                          | $(672)^3/8$                      | Thm. 4.9|
| $\{3, 8\}_{11}$     | $12144$                        | $(12144)^6/8$                    | Cor. 4.8|
| $\{3, 9\}_{9}$      | $3420$                         | $(3420)^4$                       | GAP     |
| $\{3, 9\}_{10}$     | $20520$                        | $(20520)^6/216$                  | Cor. 4.8|
| $\{4, 4\}_{4k}$     | $64k^2$                        | $64k^6$                          | Hand    |
| $\{4, 4\}_{4k+2}$   | $16(2k+1)^2$                   | $32(2k+1)^6$                     | Hand    |
| $\{4, 5\}_{5}$      | $160$                          | $(160)^3$                        | Thm. 4.7|
| $\{4, 5\}_{9}$      | $6840$                         | $(6840)^6/8$                     | Cor. 4.8|

Table 4.1: Self-dual, self-Petrie covers of several finite polyhedra $\{p, q\}_r$
Chapter 5

Constructing chiral polytopes

Our final goal is the construction of chiral polytopes. Though we have many examples of chiral polyhedra and maps (see [6, 34]) and chiral 4-polytopes (see [25, 26]), only a handful of examples are known in ranks 5 and higher (see [5, 28]). Mixing cannot, by itself, construct a chiral $n$-polytope from a chiral $(n - 1)$-polytope. However, once we have a chiral $n$-polytope, we can use mixing to create many more chiral $n$-polytopes. Results like Theorem 3.20, Corollary 3.21, and Corollary 3.22 give us simple ways to determine when the mix of a chiral polytope with another polytope is chiral.

5.1 Building chiral polytopes of high rank

Recall that, in Example 3.4, we described an automorphism $\chi$ of $W_n^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ that sends $\sigma_1$ to $\sigma_1^{-1}$, sends $\sigma_2$ to $\sigma_1^2 \sigma_2$, and fixes $\sigma_3, \ldots, \sigma_{n-1}$. As noted earlier, if $\mathcal{P}$ is a chiral or directly regular polytope, then it is always externally $\chi$-invariant. Furthermore, $\mathcal{P}$ is internally $\chi$-invariant if and only if it is directly regular (and therefore it is internally $\chi$-variant if and only if it is chiral).
For the rest of the section, we will change our terminology slightly to fit with earlier accounts on the measurement of chirality (see [2, 3, 9]). We will continue to say that a polytope is directly regular or chiral instead of using the terms “internally $\chi$-invariant” and “internally $\chi$-variant”. For $\mathcal{P}^\chi$ we will use $\overline{\mathcal{P}}$, and we call this the enantiomorphom form of $\mathcal{P}$. Finally, the group $X_\chi(\mathcal{P})$, which is equal to $X(\mathcal{P}|\mathcal{P}^\chi) = X(\mathcal{P}|\overline{\mathcal{P}})$, is denoted by $X(\mathcal{P})$ and called the chirality group of $\mathcal{P}$.

In [23], it is shown that for each $n \geq 4$, there are infinitely many $n$-toroids of type $\{4, 3^{n-3}, 4\}$. In particular, for each $s \geq 2$ and $k = 1, 2$, or $n - 1$, there is a regular toroid $\{4, 3^{n-3}, 4\}(s^k, 0^{n-k-1})$ that has a group of order $2^{n+k-2}(n-1)!s^{n-1}$. These toroids provide a rich source of polytopes to mix with chiral polytopes, and in many cases, the mixing will yield another chiral polytope.

**Theorem 5.1.** Let $n \geq 4$, and let $\mathcal{P}$ be a chiral $n$-polytope of type $\{p_1, \ldots, p_{n-1}\}$ such that $p_1$ and $p_{n-1}$ are odd and such that $p_2, \ldots, p_{n-2}$ are not divisible by 3. If $X(\mathcal{P})$ is infinite, or if the largest prime factor of $|X(\mathcal{P})|$ is greater than $n - 1$, then there are infinitely many chiral $n$-polytopes of type $\{4p_1, 3p_2, \ldots, 3p_{n-2}, 4p_{n-1}\}$, with groups $\Gamma^+(\mathcal{P}) \times [4, 3^{n-3}, 4]^+_{(s^k, 0^{n-k-1})}$.

**Proof.** Let $Q(s, k) = \{4, 3^{n-3}, 4\}(s^k, 0^{n-k-1})$. Proposition 2.47 establishes that $\mathcal{P} \odot Q(s, k)$ is polytopal and that its type is $\{4p_1, 3p_2, \ldots, 3p_{n-2}, 4p_{n-1}\}$. When $X(\mathcal{P})$ is infinite, we can appeal to Corollary 3.21 to see that the mix is chiral for any choice of $s$ and $k$. Otherwise, let $m$ be the largest prime factor of $|X(\mathcal{P})|$. In order for $\mathcal{P} \odot Q(s, k)$ to be directly regular, we must have that $|X(\mathcal{P})|$ divides $|[4, 3^{n-3}, 4]^+_{(s^k, 0^{n-k-1})}|$, by Theorem 3.20. In particular, $m$ must divide $2^{n+k-2}(n-1)!s^{n-1}$. Since $m$ is prime, if $m > n - 1$, the only way this can happen is when $m$ divides $s$. Therefore, for any $k$ and any $s$ such that $m \nmid s$, the mix $\mathcal{P} \odot Q(s, k)$ is chiral.

Another general way of producing new chiral polytopes is by mixing chiral polytopes
with chiral or directly regular polytopes that have a simple rotation group. Here is one particularly nice result of this type.

**Theorem 5.2.** Let \( n \geq 4 \), and let \( \mathcal{P} \) be a chiral \( n \)-polytope of type \( \{p_1, \ldots, p_{n-1}\} \) such that no \( p_i \) is divisible by \( 3 \). Suppose that \( X(\mathcal{P}) \) is not isomorphic to \( A_{n+1} \). Let \( \mathcal{Q} \) be the \( n \)-simplex. Then \( \mathcal{P} \circ \mathcal{Q} \) is a chiral polytope of type \( \{3p_1, \ldots, 3p_{n-1}\} \) with group \( \Gamma^+(\mathcal{P}) \times A_{n+1} \).

**Proof.** The \( n \)-simplex \( \mathcal{Q} = \{3, 3, \ldots, 3\} \) (with \( n-1 \) threes) has rotation group \( A_{n+1} \), which is simple for \( n \geq 4 \). Now, Proposition 2.47 establishes that \( \mathcal{P} \circ \mathcal{Q} \) is polytopal, that its type is \( \{3p_1, \ldots, 3p_{n-1}\} \), and that its group is \( \Gamma^+(\mathcal{P}) \times A_{n+1} \). Furthermore, Corollary 3.22 tells us that \( \mathcal{P} \circ \mathcal{Q} \) is chiral. \( \square \)

What we would really like is a way to build a chiral \((n+1)\)-polytope from a given chiral \( n \)-polytope in some way. In [31], the authors proved that for any chiral \( n \)-polytope \( \mathcal{K} \) with directly regular facets, there is a universal chiral \((n+1)\)-polytope \( U(\mathcal{K}) \) with facets isomorphic to \( \mathcal{K} \). This polytope covers all other chiral \((n+1)\)-polytopes with facets isomorphic to \( \mathcal{K} \). In many cases, this extension has an infinite chirality group:

**Lemma 5.3.** Let \( \mathcal{K} \) be a chiral \( n \)-polytope with directly regular facets such that \( \Gamma^+(\mathcal{K}) \square \Gamma^+(\overline{\mathcal{K}}) \) is finite. (For example, the latter holds if \( \Gamma^+(\mathcal{K}) \) is itself finite.) Let \( \mathcal{P} = U(\mathcal{K}) \). Let \( \Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_n \rangle \) so that \( \Gamma^+(\mathcal{K}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle \). If \( \sigma_{n-1} = \epsilon \) in \( \Gamma^+(\mathcal{K}) \square \Gamma^+(\overline{\mathcal{K}}) \), or if \( \sigma_{n-1} = \sigma_{n-2} \) in \( \Gamma^+(\mathcal{K}) \square \Gamma^+(\overline{\mathcal{K}}) \), then \( X(\mathcal{P}) \) is infinite.

**Proof.** Since \( X(\mathcal{P}) \) is the kernel of the natural epimorphism from \( \Gamma^+(\mathcal{P}) \) to \( \Gamma^+(\mathcal{P}) \square \Gamma^+(\overline{\mathcal{P}}) \) (see Proposition 2.19) and \( \Gamma^+(\mathcal{P}) \) is infinite, it suffices to show that \( \Gamma^+(\mathcal{P}) \square \Gamma^+(\overline{\mathcal{P}}) \) is finite. Now, suppose that \( \sigma_{n-1} = \epsilon \) in \( \Gamma^+(\mathcal{K}) \square \Gamma^+(\overline{\mathcal{K}}) \). Then \( \sigma_{n-1} = \epsilon \) in \( \Gamma^+(\mathcal{P}) \square \Gamma^+(\overline{\mathcal{P}}) \) as well. Furthermore, the relation \( (\sigma_{n-1} \sigma_n)^2 = \epsilon \) also holds in \( \Gamma^+(\mathcal{P}) \square \Gamma^+(\overline{\mathcal{P}}) \), so \( \sigma_n^2 = \epsilon \) in \( \Gamma^+(\mathcal{P}) \square \Gamma^+(\overline{\mathcal{P}}) \). Suppose instead that \( \sigma_{n-1} = \sigma_{n-2} \) in \( \Gamma^+(\mathcal{K}) \square \Gamma^+(\overline{\mathcal{K}}) \) (and thus in
\( \Gamma^+(\mathcal{P}) \boxtimes \Gamma^+(\overline{\mathcal{P}}) \). Then \( (\sigma_{n-2} \sigma_{n-1} \sigma_n)^2 = \epsilon \), and thus \( (\sigma_{n-1}^2 \sigma_n)^2 = \epsilon = (\sigma_{n-1} \sigma_n)^2 \). Therefore, \( \sigma_{n-1} \sigma_n \sigma_{n-1} = \sigma_n \), and so \( \epsilon = (\sigma_{n-1} \sigma_n)^2 = \sigma_n^2 \). In any case, we see that \( \sigma_n \) has order dividing 2 in \( \Gamma^+(\mathcal{P}) \circ \Gamma^+(\overline{\mathcal{P}}) \).

Now, \( \sigma_n \) commutes with \( \sigma_i \) if \( i < n - 2 \). Since \( (\sigma_{n-1} \sigma_n)^2 = \epsilon \) and \( \sigma_n^2 = \epsilon \), it follows that \( \sigma_n \sigma_{n-1} = \sigma_{n-1} \sigma_n \). Finally, since \( (\sigma_{n-2} \sigma_{n-1} \sigma_n)^2 = \epsilon \), we see that

\[
(\sigma_{n-2} \sigma_{n-1} \sigma_n)^2(\sigma_{n-1} \sigma_n) = (\sigma_{n-2} \sigma_{n-1})(\sigma_{n-1} \sigma_n) \\
\sigma_n \sigma_{n-2} = \sigma_{n-2} \sigma_{n-1} \sigma_n.
\]

Therefore, given any word \( w \) in \( \Gamma^+(\mathcal{P}) \boxtimes \Gamma^+(\overline{\mathcal{P}}) \), we can bring every factor of \( \sigma_n \) in \( w \) to the right and write \( w = u \sigma_n^k \), where \( k \) is 0 or 1 and where \( u \) contains only the generators \( \sigma_1, \ldots, \sigma_{n-1} \), and is therefore a word in \( \Gamma^+(\mathcal{K}) \boxtimes \Gamma^+(\overline{\mathcal{K}}) \). Therefore, \( \Gamma^+(\mathcal{P}) \boxtimes \Gamma^+(\overline{\mathcal{P}}) \) is finite; in particular, it is at most twice as large as \( \Gamma^+(\mathcal{K}) \boxtimes \Gamma^+(\overline{\mathcal{K}}) \).

**Theorem 5.4.** Let \( \mathcal{K} \) be a finite chiral \( n \)-polytope with directly regular facets and group \( \Gamma^+(\mathcal{K}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle \). Let \( \mathcal{P} = U(\mathcal{K}) \), and let \( \mathcal{Q} \) be a finite directly regular \((n+1)\)-polytope. If \( \sigma_{n-1} = \epsilon \) or \( \sigma_{n-1} = \sigma_{n-2} \) in \( \Gamma^+(\mathcal{K}) \boxtimes \Gamma^+(\overline{\mathcal{K}}) \), then \( \mathcal{P} \circ \mathcal{Q} \) is a chiral \( n \)-semi-polytope with infinite chirality group \( X(\mathcal{P} \circ \mathcal{Q}) \).

**Proof.** By Lemma 5.3, the conditions given on \( \mathcal{K} \) suffice to ensure that \( \mathcal{P} \) has an infinite chirality group. Then Corollary 3.21 applies to show that \( \mathcal{P} \circ \mathcal{Q} \) also has an infinite chirality group. \( \square \)

For example, let \( \mathcal{K} \) be the chiral polyhedron \( \{4, 4\}_{(b,c)} \) with \( p = b^2 + c^2 \) an odd prime. Then \( \Gamma^+(\mathcal{K}) \boxtimes \Gamma^+(\overline{\mathcal{K}}) = [4, 4]_{(1,0)} \), the group of the torus map \( \{4, 4\}_{(1,0)} \), in which \( \sigma_1 = \sigma_2 \) \([1]\). Therefore, Theorem 5.4 says that we can mix \( U(\mathcal{K}) \) with any finite directly regular polytope to obtain a polytope with infinite chirality group.
Note that in Theorem 5.4, we had to choose $K$ to have directly regular facets. Since the $(n - 2)$-faces of a chiral polytope must be directly regular, there is no universal polytope $U(P)$ if $P$ has chiral facets. In particular, there is no universal polytope $U(U(K))$. However, using mixing, we can build a chiral polytope with directly regular facets $K \circ \bar{K}$:

**Theorem 5.5.** Let $K$ be a finite chiral polytope with directly regular facets. Let $P = U(K)$, and let $Q = \{K \circ \bar{K}, 2\}$. If $X(K) = \Gamma^+(K)$, then $P \circ Q$ is a chiral polytope with directly regular facets.

**Proof.** Since $X(K) = \Gamma^+(K)$, $\Gamma^+(K) \Box \Gamma^+(\bar{K})$ is trivial, by Proposition 2.19. Furthermore, the polytope $Q$ is finite, since $K$ is finite. Then by Theorem 5.4, the mix $P \circ Q$ is chiral. Moreover, the facets of $Q$ cover the facets of $P$, so by Proposition 2.41, the mix is polytopal. \qed

Unfortunately, Theorem 5.5 cannot be repeatedly applied, because it requires that we start with a finite chiral polytope, and it produces an infinite chiral polytope.

Finally, we note that Lemma 5.3, Theorem 5.4, and Theorem 5.5 all still apply when $P$ is any infinite chiral polytope with facets isomorphic to $K$ – the universality of $U(K)$ is not necessary.

### 5.2 Self-dual chiral polytopes

We now see how to construct chiral polytopes that are self-dual. This problem ties together many of the different features we have developed so far. The construction itself is simple; we start with a chiral polytope and mix it with its dual. This procedure always yields something self-dual, but it may not be chiral or polytopal. We dealt with the issue of polytopality in Chapter 4, so all that remains is to determine when $P \circ P^\delta$ is chiral.
The results in this section all mix the chiral polytope $P$ with its dual $P^\delta$, yielding something that is properly self-dual (that is, internally $\delta$-invariant). If we replace $P^\delta$ with $\overline{P^\delta}$, then the results all still hold, but they yield polytopes that are improperly self-dual instead (that is, externally $\delta$-invariant but internally $\delta$-variant).

**Proposition 5.6.** Let $P$ be a finite chiral polytope. If $|\Gamma^+(P) \boxempty \Gamma^+(\overline{P})| < |\Gamma^+(P) \boxempty \Gamma^+(P^\delta)|$, then $P \diamond P^\delta$ is chiral.

**Proof.** If $|\Gamma^+(P) \boxempty \Gamma^+(P^\delta)| > |\Gamma^+(P) \boxempty \Gamma^+(\overline{P})|$, then $|\Gamma^+(P) \diamond \Gamma^+(P^\delta)| < |\Gamma^+(P) \diamond \Gamma^+(\overline{P})|$, by Proposition 2.21. In particular, $P \diamond P^\delta$ cannot cover $P \diamond \overline{P} = P^{\sigma_X}$, and so $P \diamond P^\delta$ is chiral by Lemma 3.16. $\square$

By taking into account the Schl"afli symbol of $P$, we obtain a slightly stronger result.

**Theorem 5.7.** Let $P$ be a finite chiral $n$-polytope of type $\{p_1, \ldots, p_{n-1}\}$. Define $\ell_i = \text{lcm}(p_i, p_{n-i})$ for $i = 1, \ldots, n-1$, and let $\ell = \text{lcm}(\ell_1/p_1, \ldots, \ell_{n-1}/p_{n-1})$. If

$$|\Gamma^+(P) \boxempty \Gamma^+(\overline{P})| < \ell |\Gamma^+(P) \boxempty \Gamma^+(P^\delta)|,$$

then $P \diamond P^\delta$ is chiral.

**Proof.** Suppose $P \diamond P^\delta$ is directly regular. Then $\Gamma^+(P) \diamond \Gamma^+(P^\delta)$ covers $\Gamma^+(P) \diamond \Gamma^+(\overline{P})$, by Lemma 3.16. Let $\pi$ be the corresponding natural epimorphism. Now, $P \diamond P^\delta$ is of type $\{\ell_1, \ldots, \ell_{n-1}\}$, while $P \diamond \overline{P}$ is of type $\{p_1, \ldots, p_{n-1}\}$. Let $\Gamma^+(P \diamond P^\delta) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$. Then $\sigma_i^{p_i} \in \ker \pi$ for each $i = 1, \ldots, n-1$. So $\langle \sigma_1^{p_1}, \ldots, \sigma_{n-1}^{p_{n-1}} \rangle \leq \ker \pi$. Now, the order of $\sigma_i^{p_i}$ in $\Gamma^+(P \diamond P^\delta)$ is $\ell_i/p_i$ since the order of $\sigma_i$ is $\ell_i$ and $p_i$ divides $\ell_i$. Then $\ker \pi$ contains elements of order $\ell_i/p_i$ for $i = 1, \ldots, n-1$, and thus it has size at least $\ell = \text{lcm}(\ell_1/p_1, \ldots, \ell_{n-1}/p_{n-1})$. 97
Now, $|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta)| = |\ker \pi| |\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P})|$, and therefore

$$|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P})| = |\ker \pi| |\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta)| \geq \ell |\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta)|,$$

proving the desired result.

We now illustrate this theorem by looking at a few broad classes of examples where polytopality and chirality of the mix are guaranteed.

**Theorem 5.8.** Let $\mathcal{P}$ be a finite chiral polyhedron of type $\{p, q\}$. Let $\ell_1 = \text{lcm}(p, q)$, and suppose that $|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P})| < \ell_1^2/pq |\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta)|$. Then $\mathcal{P} \diamond \mathcal{P}^\delta$ is a properly self-dual chiral polyhedron of type $\{\ell_1, \ell_1\}$.

**Proof.** From Corollary 2.45, we know that $\mathcal{P} \diamond \mathcal{P}^\delta$ is a chiral or directly regular polyhedron. Now, we apply Theorem 5.7. We have that $\ell = \text{lcm}(\ell_1/p, \ell_1/q) = \ell_1^2/pq$, and therefore, $\mathcal{P} \diamond \mathcal{P}^\delta$ is chiral.

**Theorem 5.9.** Let $\mathcal{P}$ be a finite chiral polytope of odd rank of type $\{p_1, \ldots, p_{n-1}\}$. Suppose that $p_i$ and $p_{n-i}$ are coprime for $i \in \{1, \ldots, n-1\}$, and suppose that

$$|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P})| < \text{lcm}(p_1, \ldots, p_{n-1}).$$

Then $\mathcal{P} \diamond \mathcal{P}^\delta$ is a properly self-dual chiral polytope of type $\{p_1 p_{n-1}, p_2 p_{n-2}, \ldots, p_{n-1} p_1\}$, and with group $\Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{P}^\delta)$.

**Proof.** With the given conditions, Proposition 4.2 applies to show us that $\mathcal{P} \diamond \mathcal{P}^\delta$ is a polytope with group $\Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{P}^\delta)$. To prove chirality, we apply Theorem 5.7, noting that $\ell_i = p_i p_{n-i}$, $\ell = \text{lcm}(p_1, \ldots, p_{n-1})$, and $|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta)| = 1$. 

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Theorem 5.10. Let $\mathcal{P}$ be a chiral polyhedron of type $\{p, q\}$ with simple automorphism group, and suppose that $p \neq q$. Then $\mathcal{P} \circ \mathcal{P}^\delta$ is a properly self-dual chiral polyhedron.

Proof. Since the chirality group $X(\mathcal{P})$ is a normal subgroup of the simple group $\Gamma^+(\mathcal{P})$, and $\mathcal{P}$ is chiral, it follows that $X(\mathcal{P}) = \Gamma^+(\mathcal{P})$. Therefore, $\Gamma^+(\mathcal{P}) \square \Gamma^+(\overline{\mathcal{P}})$ is trivial. Since $p \neq q$, Theorem 5.8 then tells us that $\mathcal{P} \circ \mathcal{P}^\delta$ is a properly self-dual chiral polyhedron. \qed

We note that there are many examples of chiral polyhedra with simple automorphism groups. For example, in [4], the authors give several examples of chiral polyhedra whose automorphism group is the Mathieu group $M_{11}$.

Next we consider the simplest chiral polyhedra: the torus maps. Since the torus map $\{4, 4\}_{(b,c)}$ is already (improperly) self-dual, we work only with $\{3, 6\}_{(b,c)}$ and its dual. Let $\mathcal{P} = \{3, 6\}_{(b,c)}$, where $m := b^2 + bc + c^2$ is a prime and $m \geq 5$. (The primality of $m$ is not essential, but it makes some of our calculations easier.) We have that $|\Gamma^+(\mathcal{P})| = 6m$, $|X(\mathcal{P})| = m$, and $|\Gamma^+(\mathcal{P}) \square \Gamma^+(\overline{\mathcal{P}})| = 6$ [1]. Unfortunately, Theorem 5.8 isn’t enough to establish the chirality of $\mathcal{P} \circ \mathcal{P}^\delta$. We need the following result, which relies on the fact that $\mathcal{P}^\delta = \overline{\mathcal{P}}^\delta$.

**Theorem 5.11.** Let $\mathcal{P}$ be a finite chiral polytope, and suppose that

$$|X(\mathcal{P})|^2 > \left| \left( \Gamma^+(\mathcal{P}) \circ \Gamma^+(\overline{\mathcal{P}}) \right) \square \left( \Gamma^+(\mathcal{P}^\delta) \circ \Gamma^+(\overline{\mathcal{P}}^\delta) \right) \right|.$$

Then $\mathcal{P} \circ \mathcal{P}^\delta$ is chiral.
Proof. Suppose that $\mathcal{P} \diamond \mathcal{P}^\delta$ is directly regular. Then $(\mathcal{P} \diamond \mathcal{P}^\delta) \diamond (\mathcal{P} \diamond \mathcal{P}) = \mathcal{P} \diamond \mathcal{P}^\delta$. Therefore,

$$|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta)| = |\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta) \diamond \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta)|$$

$$= |(\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P})) \diamond (\Gamma^+(\mathcal{P}^\delta) \diamond \Gamma^+(\mathcal{P}^\delta))|$$

$$= \frac{|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P})| \cdot |\Gamma^+(\mathcal{P}^\delta) \diamond \Gamma^+(\mathcal{P}^\delta)|}{|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta)|}$$

$$= \frac{|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P})|^2}{|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta)|},$$

where the third line follows from Proposition 2.21. Rearranging, we get that

$$\frac{|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta)|}{|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\delta)|} \geq \frac{|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P})|^2}{|\Gamma^+(\mathcal{P})|^2} \geq |X(\mathcal{P})|^2,$$

where the last line again follows from Proposition 2.21. The result then follows immediately. \hfill \square

**Corollary 5.12.** Let $\mathcal{P}$ be a chiral polytope of type $\{p_1, \ldots, p_{n-1}\}$, and suppose that

$$|X(\mathcal{P})|^2 > |[p_1, \ldots, p_{n-1}] \square [p_{n-1}, \ldots, p_1]|.$$

Then $\mathcal{P} \diamond \mathcal{P}^\delta$ is chiral.

Proof. Since $\mathcal{P}$ is of type $\{p_1, \ldots, p_{n-1}\}$, so are $\overline{\mathcal{P}}$ and $\mathcal{P} \diamond \overline{\mathcal{P}}$. Similarly, $\mathcal{P}^\delta \diamond \overline{\mathcal{P}}^\delta$ is of type $\{p_{n-1}, \ldots, p_1\}$. Therefore, $(\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\overline{\mathcal{P}})) \square (\Gamma^+(\mathcal{P}^\delta) \diamond \Gamma^+(\overline{\mathcal{P}}^\delta))$ is a quotient of $[p_1, \ldots, p_{n-1}] \square [p_{n-1}, \ldots, p_1]$, and the result follows from Theorem 5.11. \hfill \square

Now, returning to $\mathcal{P} = \{3, 6\}_{(b,c)}$, we see that $|X(\mathcal{P})| = m$ and that $|[3, 6]_{(b,c)} \square [6, 3]_{(b,c)}|$
is at most 12 (the order of $[3, 3]^+$). Since $m \geq 5$, we can conclude that $\mathcal{P} \circ \mathcal{P}^\delta$ is chiral using Corollary 5.12.

It is instructive to calculate the full structure of $\mathcal{P} \circ \mathcal{P}^\delta$ in this case. To do so requires that we first calculate the size of $\Gamma^+(\mathcal{P}) \Join \Gamma^+(\mathcal{P}^\delta)$ directly. Since $m$ is prime, $b$ and $c$ must be coprime, and in particular, at least one of them must be odd. We can assume that $b$ is odd by changing from $\mathcal{P} = \{3, 6\}_{(b,c)}$ to $\overline{\mathcal{P}} = \{3, 6\}_{(c,b)}$ if necessary. Now, in $\Gamma^+(\mathcal{P}) \Join \Gamma^+(\mathcal{P}^\delta)$, we have the relation

\[(\sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2)b(\sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1})^c = \epsilon.\]

Now, using the facts that $(\sigma_1 \sigma_2)^2 = \sigma_1^3 = \sigma_2^3 = \epsilon$ and that $b$ is odd, we see that

\[
(\sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2)^b = (\sigma_1 \sigma_1 \sigma_2 \sigma_2)^b \\
= (\sigma_1^{-1} \sigma_2^{-1})^b \\
= (\sigma_2 \sigma_1)^b \\
= \sigma_2 \sigma_1.
\]

Therefore,

\[
\sigma_2 \sigma_1 (\sigma_2 \sigma_1^{-1} \sigma_2)^c = \epsilon \\
\sigma_2 (\sigma_2 \sigma_1 (\sigma_2 \sigma_1^{-1} \sigma_2)^c) \sigma_2^{-1} = \epsilon \\
(\sigma_2 (\sigma_2 \sigma_1) \sigma_2^{-1})(\sigma_2 (\sigma_2 \sigma_1^{-1} \sigma_2)^c \sigma_2^{-1}) = \epsilon \\
\sigma_2^{-1} \sigma_1 \sigma_2^{-1} (\sigma_2^{-1} \sigma_1^{-1})^c = \epsilon,
\]
and thus

\[ \sigma_2^{-1} \sigma_1 \sigma_2 \sigma_1^{-1} = \epsilon \text{ if } c \text{ is odd,} \]

\[ \sigma_2^{-1} \sigma_1 \sigma_2^{-1} = \epsilon \text{ if } c \text{ is even.} \]

In the first case, we see that \( \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \), and since we also have \((\sigma_1 \sigma_2)^2 = \epsilon\), we see that \( \sigma_1 = \sigma_2^{-1} \). In the second case, we also directly get that \( \sigma_1 = \sigma_2^{-1} \), and therefore \( \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \). In any case, the extra relation from \( \{6, 3\}_{(b,c)} \) is rendered redundant, and we see that \( \Gamma^+(\mathcal{P}) \square \Gamma^+(\mathcal{P}^\delta) \) is generated by \( \sigma_1 \) and has order 3.

We can now determine the full structure of \( \mathcal{P} \circ \mathcal{P}^\delta \). Since \( |\Gamma^+(\mathcal{P}) \circ \Gamma^+(\mathcal{P}^\delta)| = |\Gamma^+(\mathcal{P})|^2 / 3 = 12m^2 \), the polyhedron \( \mathcal{P} \circ \mathcal{P}^\delta \) has \( 24m^2 \) flags. Furthermore, \( \mathcal{P} \circ \mathcal{P}^\delta \) is of type \( \{6, 6\} \), and thus must have \( 2m^2 \) vertices, \( 6m^2 \) edges, and \( 2m^2 \) 2-faces. This polyhedron corresponds to a map on an orientable surface of genus \( m^2 + 1 \).
Goursat’s Lemma characterizes the subgroups of a direct product; it appears as an exercise on page 75 of [21], and a proof is given below.

**Lemma A.1** (Goursat’s Lemma). Let $G_1$ and $G_2$ be groups, and let $H$ be a subgroup of $G_1 \times G_2$ such that the restrictions of the canonical projections $\pi_1 : H \to G_1$ and $\pi_2 : H \to G_2$ are surjective. Let $N_1 = \ker \pi_1$ and $N_2 = \ker \pi_2$. Then we can identify $N_2$ as a normal subgroup of $G_1$ and $N_1$ as a normal subgroup of $G_2$. Furthermore, there is an isomorphism $\varphi : G_1/N_2 \to G_2/N_1$ such that

$$H \simeq \{ (g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2, \varphi(g_1N_2) = g_2N_1 \}.$$ 

**Proof.** We first show that $N_2$ is normal in $G_1 \times 1$. It is clear that $N_2$ is a subgroup of $G_1 \times 1$. Let $(g_1, 1) \in G_1 \times 1$. Since $\pi_1$ is surjective, there exists an element $h = (g_1, g_2) \in H$ for some $g_2 \in G_2$. Now, $N_2$ is a normal subgroup of $H$, so $hN_2h^{-1} = N_2$. In other words, given an element $(n, 1) \in N_2$, it follows that $h(n, 1)h^{-1} \in N_2$, which is to say that $(g_1ng_1^{-1}, 1) \in N_2$. Thus we see that $N_2$ is normal in $G_1 \times 1$, and it can be identified with a normal subgroup of
Similarly, $N_1$ can be identified with a normal subgroup of $G_2$.

Now, consider the homomorphism $\pi : H \to G_1/N_2 \times G_2/N_1$ defined as $\pi(g_1, g_2) = (g_1N_2, g_2N_1)$. Let $h = (g_1, g_2) \in H$ and $h' = (g_1', g_2') \in H$ and suppose that $g_1N_2 = g_1'N_2$. Then $g_1^{-1}g_1' \in N_2$ which means that $(g_1^{-1}g_1', 1) \in H$. Then $h(g_1^{-1}g_1', 1)h'^{-1}$, which is equal to $(1, g_2g_2'^{-1})$, is also an element of $H$, so that $g_2g_2'^{-1} \in N_1$ and thus $g_2N_1 = g_2'N_1$. Conversely, if $g_2N_1 = g_2'N_1$, then $g_1N_2 = g_1'N_2$ by similar reasoning. Therefore, given any coset $g_1N_2$, there is exactly one coset $g_2N_1$ such that $(g_1N_2, g_2N_1) \in \pi(H)$. So there is a well-defined homomorphism $\varphi : G_1/N_2 \to G_2/N_1$ given by $\varphi(g_1N_2) = g_2N_1$, where $(g_1N_2, g_2N_1)$ is the unique element of $\pi(H)$ that has $g_1N_2$ as its first coset. By the previous reasoning, this is clearly one-to-one, and it follows immediately that it is a group homomorphism. Furthermore, it is clear by the definition of $\varphi$ that $H$ is as desired.

\begin{proposition}
Let $H$ and $H'$ be subgroups of a group $G$. Let $K$ be a subgroup of $H$. Then $[H \cap H' : K \cap H'] \leq [H : K]$.
\end{proposition}

\begin{proof}
Proposition 4.8 in \cite{17} tells us that if $A$ and $B$ are subgroups of $C$, then $[B : B \cap A] \leq [C : A]$. Now, taking $A = K$, $B = H \cap H'$, and $C = H$ yields the desired result.
\end{proof}

\begin{proposition}
Let $H$ and $H'$ be subgroups of a group $G$. Let $K$ be a subgroup of $H$, and $K'$ a subgroup of $H'$. Then $[H \cap H' : K \cap K'] \leq [H : K][H' : K']$.
\end{proposition}

\begin{proof}
We have that $[H \cap H' : K \cap K'] = [H \cap H' : K \cap H'][K \cap H' : K \cap K']$. Applying Proposition A.2 to each term on the right-hand side yields the desired result.
\end{proof}
Bibliography


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