Quivers with potentials associated with triangulations of Riemann surfaces

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ABSTRACT OF DISSERTATION

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Abstract

We study the behavior of quivers with potentials and their mutations introduced by Derksen-Weyman-Zelevinsky in the combinatorial framework developed by Fomin-Shapiro-Thurston for cluster algebras that arise from bordered Riemann surfaces with marked points.

In Part I we associate to each ideal triangulation of a bordered surface with marked points a quiver with potential, in such a way that whenever two ideal triangulations are related by a flip of an arc, the respective quivers with potentials are related by a mutation with respect to the flipped arc. We prove that if the surface has non-empty boundary, then the quivers with potentials associated to its triangulations are rigid and hence non-degenerate, and have finite-dimensional Jacobian algebra.

In Part II we define, given an arc and an ideal triangulation of a bordered marked surface, a representation of the quiver with potential constructed in Part I, so that whenever two ideal triangulations are related by a flip, the associated representations are related by the corresponding mutation of representations.
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Chapter 1

Introduction

The main motivation for this work comes from two beautiful papers: On the one hand, in [12], S. Fomin, M. Shapiro and D. Thurston associate to each bordered surface with marked points a cluster algebra, each of whose exchange matrices is defined in terms of the (signed) adjacencies between the arcs of a triangulation of the surface. They prove that the seeds of this cluster algebra are related by a mutation if and only if the triangulations to which the seeds are associated are related by a flip. In particular, if two triangulations are related by a flip, then the (skew-symmetric) matrices associated to them are related by a mutation.

On the other hand, in [8], H. Derksen, J. Weyman and A. Zelevinsky introduce the notion of quivers with potentials (QPs for short), that is, pairs consisting of a quiver and a special element of its (complete) path algebra, and define the mutations of such objects, ultimately leading to the notion of mutation of representations, thus providing a new representation-theoretic interpretation for quiver mutations originated in the theory of cluster algebras, interpretation that generalizes the classical
Bernstein-Gelfand-Ponomarev reflection functors.

In this thesis we make an attempt to relate the two above mentioned papers in two different levels. The first level is the quiver-theoretic one: We associate to each triangulation $\tau$ of a bordered surface with marked points a potential $S(\tau)$ on the quiver $Q(\tau)$ defined by its signed adjacency matrix $B(\tau)$. The idea is quite simple: each interior triangle of $\tau$ gives rise to an oriented triangle in $Q(\tau)$, and each puncture has an oriented cycle of $Q(\tau)$ around it; what we do is to add such oriented triangles and cycles to get the potential $S(\tau)$. Then we show that not only the quivers $Q(\tau)$ and $Q(\sigma)$ are related by a mutation if the ideal triangulations $\tau$ and $\sigma$ are related by a flip, but also the QPs $(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$. When the surface has non-empty boundary, we show that these QPs are non-degenerate. By Derksen-Weyman-Zelevinsky’s results in [9], this means that their representation theory encodes the relevant data to find the expansions of the cluster variables in the corresponding cluster algebra.

The second level is the representation-theoretic extension of the first one: Given an arc $i$ on a surface and a triangulation $\tau$, we define a representation $M(\tau, i)$ of $(Q(\tau), S(\tau))$, in such a way that, if the arc $i$ is kept fixed, ideal triangulations related by a flip give rise to representations related by the corresponding QP-mutation. Because of this property and the fact that every arc belongs to a triangulation, the representations $M(\tau, i)$ turn out to be those representations whose geometric data gives Laurent expansions of cluster variables. Let us be more precise: On the one hand, in [9] it is shown that given a quiver $Q$ and a non-degenerate potential $S$ on it, the $(Q, S)$-representations mutation equivalent to negative simple ones are naturally associated to the cluster variables of any cluster algebra having the quiver $Q$
at one of its seeds. Furthermore, it is proved that the Euler characteristics of the quiver Grassmannians of these \((Q,S)\)-representations are the coefficients of the \(F\)-polynomials associated in \([16]\) to the corresponding cluster variables. On the other hand, in \([12]\), \([13]\), a geometric-combinatorial model has been given for the cluster algebras having a quiver of the form \(Q(\tau)\) at one of its seeds, where \(\tau\) is an ideal (or tagged) triangulation of a bordered surface with marked point. In this model, cluster variables are (tagged) arcs on the surface, and mutation of seeds corresponds to flips of (tagged) arcs. Since \(S(\tau)\) is a non-degenerate potential on \(Q(\tau)\) whenever the underlying surface has non-empty boundary or exactly one puncture, by Theorem 9.2.1 below, the representations \(M(\tau, i)\) are mutation-equivalent to negative simple ones. Therefore, these representations \(M(\tau, i)\) are the representations that can be used to calculate the \(F\)-polynomials of the cluster variables (in the positive stratum) of (any) cluster algebra associated to \(Q(\tau)\). (By results of \([16]\), cluster dynamics is governed to a great extent by \(g\)-vectors and \(F\)-polynomials.)

We now describe the contents of the present work in more detail. The thesis is divided into two parts, reflecting the two levels we have described above. So, Part I deals with the definition and mutation properties of \(S(\tau)\), whereas Part II deals with the construction and mutation properties of the representations \(M(\tau, i)\).

Part I consists of Chapters 2, 3 and 4. Chapter 2 is meant to provide the necessary combinatorial and algebraic background on quiver mutations (Section 2.1), triangulations of surfaces and their flips and (unreduced) signed adjacency quivers (Section 2.2), and the mutation theory of quivers with potentials (Section 2.3).

In Chapter 3 we associate a QP \((Q(\tau), S(\tau))\) to each ideal triangulation \(\tau\) of a surface \(\Sigma\) with marked points \(M\). This QP is defined in Section 3.1 as the reduced part
of a QP on the arrow span of the unreduced signed adjacency quiver \( \tilde{Q}(\tau) \). In Section 3.2 we show that \((Q(\tau), S(\tau))\) can always be obtained via restriction from a QP on the signed adjacency quiver of a triangulation of a surface with empty boundary. As a side result we prove a fact that takes place in the general theory of QP-mutations (and not only in the surface-related setup), namely, that the operation of restriction preserves non-degeneracy of QPs (a similar result was established in [8] for rigid QPs). In Section 3.3 we prove our first main result, Theorem 3.3.1, which states that ideal triangulations related by a flip give rise to QPs related by QP-mutation. Unfortunately, the proof of this fact, despite being simplified by the results of Section 3.2, is done with an analysis case-by-case, the reason being that slight changes in the configuration of the arcs surrounding the arc to be flipped can dramatically affect the associated QP.

The process of QP-mutation deserves some words. It has a delicate point: The underlying quiver of the mutated QP depends on the potential of the original QP. This implies that the underlying quiver of the mutated QP may not coincide with the quiver obtained under "ordinary" quiver mutation from the underlying quiver of the original QP. The difficulty relies on the fact that it is the potential that allows us (or not) to delete the 2-cycles from the quiver. A QP is called non-degenerate if, after any sequence of QP-mutations, the potential always allows us to delete all the 2-cycles of the corresponding quiver. An immediate consequence of Theorem 3.3.1 is the fact that, for unpunctured surfaces and for surfaces without boundary and exactly one puncture, the associated QPs are non-degenerate, for the simple reason that such surfaces do not admit ideal triangulations with self-folded triangles. Non-degeneracy is not so immediate for general surfaces since we have not proved Theorem 3.3.1 for
tagged triangulations. Indeed, we have not even defined $S(\tau)$ in case $\tau$ is a tagged non-ideal triangulation. Thus we take a different approach: A family of QPs for which non-degeneracy is guaranteed without the necessity of checking all possible sequences of QP-mutations is the family of rigid QPs. A QP is called rigid if every cycle on the underlying quiver is cyclically equivalent to an element of the ideal generated by the cyclic derivatives of the underlying potential. In Chapter 4 we prove our second main result, which says that the QPs associated to surfaces with non-empty boundary are rigid and hence non-degenerate. We conjecture that the QPs we associate to surfaces with empty boundary are non-degenerate as well, but non-rigid. We close Chapter 4 with our third main result, showing that if the surface has non-empty boundary, then the Jacobian algebras of the QPs we assign to its triangulations are finite-dimensional.

Part II consists of Chapters 5, 6, 7, 8, 9, 10 and 11. Chapter 5 is devoted to recall some notation and terminology about QPs and their representations.

In Chapter 6 we present the main constructions of Part II: to start, in Section 6.1 we formally define what it means for a curve to surround a puncture or to be parallel to an arrow or a path on $Q(\tau)$. In Section 6.2 we fix an ideal triangulation without self-folded triangles and an arc $i$ on $(\Sigma, M)$ not cutting out a once-punctured monogon, and, looking at how the punctures of $(\Sigma, M)$ are surrounded by certain segments of $i$, we draw some short oriented curves on the surface, which we call the detours of $(\tau, i)$, and arrange some information extracted from these curves into a family of detour matrices $D_{ij}^{\Delta}$; then we define the string representations $m(\tau, i)$ following ideas of Assem-Brüstle-Charbonneau-Plamondon/Caldero-Chapoton-Schiffler, and modify them using detour matrices, thus obtaining the arc representations $M(\tau, i)$, which are the main objects of study of Part II. In Section 6.3 we give the definition of $M(\tau, i)$
for the case where \( i \) is a loop cutting out a once-punctured monogon.

From Chapter 7 to the end of the thesis we will suppose that \( i \) is not a loop cutting out a once-punctured monogon. This is due only for space reasons; actually, all our results remain true for these loops under suitable modifications. In Chapter 7 we provide some tools whose aim is to simplify the proofs of the main results of Chapters 8 and 9. Sections 7.1 and 7.2 are representation-theoretic extensions of Section 3.2. In the first of these sections we define the operation of restriction of a general QP-representation to a subset of the vertex set of the underlying quiver in the obvious way, and prove that it commutes with the operations of reduction, premutation and mutation of representations as long as the vertex subset \( I \subseteq Q_0 \) satisfies certain vanishing condition. In Section 7.2 we show that all the representations \( M(\tau, i) \) can be obtained via restriction from representations associated to arcs on surfaces with empty boundary.

In Section 7.3 we show that the operation of mutation of representations preserves not only direct sums, but also local direct sums. Let us be more precise about this. The mutation of a representation \( M \) with respect to a vertex \( j \) depends on linear maps involving the spaces attached to \( j \) and the vertices connected to \( j \) by some arrow of \( Q \). That is, the mutation \( \mu_j(M) \) depends on a representation \( M(\partial, j) \) that is usually much smaller than the whole \( M \). What we show is that mutation is additive on these smaller representations.

Section 7.4 is devoted to provide a combinatorial way of cutting the general configuration that \( i \) can present around the arc \( j \) to be flipped, so that the segments of the cutting give rise to a decomposition of \( M(\tau, i)(\partial, j) \) as the direct sum of representations associated to them.
Since the segments arising in Section 7.4 are still quite hard to handle in general, in Sections 7.5 and 7.6 we describe a way of cutting them further. Section 7.5 deals with the purely representation-theoretical side; we show that in certain circumstances even the mutation of representations that are locally indecomposable can be calculated in terms of the direct sum of mutations of smaller representations, which are then said to constitute a \textit{dicing} of $M(\partial, j)$. Section 7.6 deals with the combinatorial side; we \textit{dice} each segment $\iota$ produced in Section 7.4 into a family of subsegments that gives rise to a dicing of the direct summand of $M(\tau, i)(\partial, j)$ corresponding to $\iota$. Unfortunately, although the \textit{dicing pieces} of $\iota$ (the segments that result from dicing $i$) are easier to handle than $\iota$ itself, their number is not small (there are 63 possible configurations for each dicing piece). This translates into a lengthy proof of Theorem 9.2.1, the main result of Part II.

In Chapter 8 we prove that our arc representations satisfy the relations imposed on $Q(\tau)$ by the Jacobian ideal $J(S(\tau))$. Chapter 9 presents Theorem 9.2.1, the main result of part II: If the arc $i$ is fixed and we have two ideal triangulations (without self-folded triangles) $\tau$ and $\sigma$ related by a flip, then the arc representations $M(\tau, i)$ and $M(\sigma, i)$ are related by the corresponding mutation of representations. The chapter starts with Section 9.1, where we verify that the linear maps attached by an arc representation $M(\tau, i)$ to an arrow of $Q(\tau)$ not incident at the arc $j$ to be flipped do not change when we perform the flip $f_j$ (we know that they should not change by definition of the mutation $\mu_j$). In Section 9.2, we analyze the behavior of the linear maps attached to the arrows that are incident to the arc $j$ in the configurations obtained in Section 7.6.

An application of arc representations is given in Chapter 10, where we give a very
simple formula to calculate the g-vector of an (ordinary) arc with respect to an ideal triangulation \((\Sigma, M)\).

In Chapter 11 we mention some problems that remain open and whose solution the author believes would help to have a complete explicit model of Derksen-Weyman-Zelevinsky’s QP-mutation theory in the context of surface cluster algebras.

The context in which this paper takes place deserves some comments. Triangulations and flips have been present in models of cluster algebras since the beginning of the theory (see, e.g., Subsection 12.1 of [15]), and also in the effort of categorifying these algebras (cf. [6], [25]). This had been done in a somewhat restricted set-up, until signed adjacency quivers for ideal triangulations of arbitrary bordered surfaces with marked points, and the compatibility between the operations of flip on ideal triangulations and ordinary quiver mutation, appeared in works by V. Fock and A Goncharov [11]; S. Fomin, M. Shapiro and D. Thurston [12]; and M. Gekhtman, M. Shapiro and A. Vainshtein [17]. This yielded a general realization of ideal triangulations as clusters in the cluster algebras whose exchange matrices are determined by the signed adjacency quivers of ideal triangulations. However, not all clusters in such a cluster algebra could be interpreted as ideal triangulations. In [12], S. Fomin, M. Shapiro and D. Thurston introduced the notions of tagged triangulations and their signed adjacency quivers, proving that all arcs in a tagged triangulation are flippable and the corresponding compatibility between flips and ordinary mutations, thus realizing all clusters as tagged triangulations, and seed mutations as flips on tagged triangulations. In [13], S. Fomin and D. Thurston have deepened this realization further to interpret cluster variables in terms of R. Penner’s lambda-lengths and coefficients in terms of W. Thurston’s unbounded measured integral laminations.
A similar story can be said about the mutation properties of cluster-tilted algebras which were present from the beginning of cluster-tilting theory (cf. [4]). As in the case of surfaces, these mutation properties had been studied in quite restrictive set-ups, until mutations of quivers with potentials and their representations were systematized by H. Derksen, J. Weyman and A. Zelevinsky in [8]. The depth and importance of Derksen-Weyman-Zelevinsky’s QP-mutation theory in both Representation theory and Cluster algebra theory has manifested in several recent works, one of which is [9], where the same group of authors makes a heavy use of non-degenerate potentials to prove several conjectures from [16].

After the foundational papers [11], [12] and [17] on the one hand, and [8] and [9] on the other, both surface cluster algebras and quivers with potentials have attracted the attention of several authors (e.g., [1], [2], [5], [10], [13], [18], [19], [22], [23], [24], [26], [27]) for many different reasons. Some of their works are directly related to [18] and the present thesis. For instance, in [2], Assem-Brüstle-Charbonneau-Plamondon define, for each triangulation $\tau$ of an unpunctured surface, a gentle quotient of $Q(\tau)$ which is precisely the Jacobian algebra of the potential $S(\tau)$ defined independently in [18]; they also construct string modules parameterized by the arcs on the surface. These string modules are actually the inspiration for our construction of arc representations, which generalize them to the punctured set-up. In [5], Brüstle and Zhang continue the study initiated in [2] by giving a complete combinatorial description of the Amiot cluster category associated to the QPs coming from unpunctured surfaces. It is worth mentioning that Assem-Brüstle-Charbonneau-Plamondon’s string modules are already a generalization of a construction given by P. Caldero, F. Chapoton and R. Schiffler in [6] for unpunctured polygons. When the bordered marked surface has no
punctures, ABCP/CCS’s construction could be (very roughly) described as “traversing curves with the identity map of the field $K$”. In the presence of punctures the situation becomes more complicated, but strings still function as strong combinatorial parameters for representations that are mutation-equivalent to negative simples.

Another related paper is [23], where, working in the full generality of *tagged triangulations* of punctured surfaces, G. Musiker, R. Schiffler and L. Williams give explicit formulas for the expansion of an arbitrary cluster variable (that is, *tagged arc* on the surface) in terms of an arbitrary initial cluster (that is, tagged triangulation), thus establishing, for example, the positivity conjecture of S. Fomin and A. Zelevinsky [16], independently of the much more general approach of [9].

Finally, let us mention that in C. Amiot’s categorification context [1], precisely because $(Q(\tau), S(\tau))$ is non-degenerate and has finite-dimensional Jacobian algebra, each arc on $(\Sigma, M)$ represents an object of the *Amiot cluster category* $\mathcal{C}$, and each triangulation $\tau$ represents a *cluster-tilting object* whose endomorphism algebra is precisely the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau))$; moreover, for each fixed triangulation there is a functor from $\mathcal{C}$ to the module category of the Jacobian algebra of the triangulation. As a consequence of Theorem 9.2.1 below, the arc representation $M(\tau, i)$ gives an explicit calculation of the image of $i$ under the functor $\mathcal{C} \to \text{mod } \mathcal{P}(Q(\tau), S(\tau))$. 
Part I

Quivers with potentials associated with triangulations of surfaces
Chapter 2

Background on triangulations of surfaces and quivers with potentials

2.1 Quiver mutations

In [14], Fomin and Zelevinsky introduced the notion of cluster algebra, one of whose main ingredients is that of quiver mutation (or, more generally, matrix mutations). Here we briefly describe this operation as a three-step procedure on quivers. Recall that a quiver is a finite directed graph, that is, a quadruple $Q = (Q_0, Q_1, h, t)$, where $Q_0$ is the (finite) set of vertices of $Q$, $Q_1$ is the (finite) set of arrows, and $h : Q_1 \rightarrow Q_0$ and $t : Q_1 \rightarrow Q_0$ are the head and tail functions. Recall also the common notation $a : i \rightarrow j$ to indicate that $a$ is an arrow of $Q$ with $t(a) = i$, $h(a) = j$. We will always deal only with loop-free quivers, that is, with quivers that have no arrow $a$ with $t(a) = h(a)$.

A path of length $d > 0$ in $Q$ is a sequence $a_1 a_2 \ldots a_d$ of arrows with $t(a_j) = h(a_{j+1})$
2.1. Quiver mutations

for \( j = 1, \ldots, d - 1 \). A path \( a_1 a_2 \ldots a_d \) of length \( d > 0 \) is a \( d \)-cycle if \( h(a_1) = t(a_d) \). A quiver is \( 2 \)-acyclic if it has no 2-cycles.

Paths are composed as functions, that is, if \( a = a_1 \ldots a_d \) and \( b = b_1 \ldots b_{d'} \) are paths with \( h(b) = t(a) \), then the product \( ab \) is defined as the path \( a_1 \ldots a_d b_1 \ldots b_{d'} \), which starts at \( t(b_{d'}) \) and ends at \( h(a_1) \). See Figure 2.1.

Figure 2.1: Paths are composed as functions

\[
\bullet \xrightarrow{b_{d'}} \ldots \xrightarrow{b_1} \bullet \xrightarrow{a_d} \ldots \xrightarrow{a_1} \bullet
\]

For \( i \in Q_0 \), an \( i \)-hook in \( Q \) is any path \( ab \) of length 2 such that \( a, b \in Q_1 \) are arrows with \( t(a) = i = h(b) \).

**Definition 2.1.1.** Given a quiver \( Q \) and a vertex \( i \in Q_0 \) such that \( Q \) has no 2-cycles incident at \( i \), we define the mutation of \( Q \) in direction \( i \) as the quiver \( \mu_j(Q) \) with vertex set \( Q_0 \) that results after applying the following three-step procedure:

(Step 1) For each \( i \)-hook \( ab \) introduce an arrow \([ab] : t(b) \to h(a)\).

(Step 2) Replace each arrow \( a : i \to h(a) \) of \( Q \) with an arrow \( a^* : h(a) \to i \), and each arrow \( b : t(b) \to i \) of \( Q \) with an arrow \( b^* : i \to t(b) \).

(Step 3) Choose a maximal collection of disjoint 2-cycles and remove them.

We call the quiver obtained after the 1\( \text{st} \) and 2\( \text{nd} \) steps the premutation \( \tilde{\mu}_j(Q) \).

**Remark 2.1.2.** 1. Note that the mutation \( \mu_j \) is defined for non-necessarily 2-acyclic quivers, but in order to be able to perform mutation at any vertex of a quiver, we need it to be 2-acyclic.
2. The choice of the maximal collection in the 3rd step is not given by a canonical procedure. However, up to this choice, \( \mu_j \) is an involution on the class of 2-acyclic quivers, that is, \( \mu_j^2(Q) \cong Q \) for every 2-acyclic quiver \( Q \).

2.2 Triangulations of surfaces and their flips

In this section we briefly review the material on triangulations of surfaces and their signed adjacency matrices and flips. The reader will find much deeper discussions on the subject in [12].

**Definition 2.2.1** ([12], Definition 2.1). A bordered surface with marked points is a pair \((\Sigma, M)\), where \( \Sigma \) is a compact connected oriented Riemann surface with (possibly empty) boundary, and \( M \) is a finite set of points on \( \Sigma \), called marked points, such that \( M \) is non-empty and has at least one point from each connected component of the boundary of \( \Sigma \). The marked points that lie in the interior of \( \Sigma \) will be called punctures, and the set of punctures of \((\Sigma, M)\) will be denoted \( P \). We will always assume that \((\Sigma, M)\) is none of the following:

- a sphere with less than five punctures;
- an unpunctured monogon, digon or triangle;
- a once-punctured monogon.

Here, by a monogon (resp. digon, triangle) we mean a disk with exactly one (resp. two, three) marked point(s) on the boundary.
2.2. Triangulations of surfaces and their flips

Remark 2.2.2. The reason for excluding the surfaces in the second and third items of the above definition is the fact that their triangulations (in the sense of Definition 2.2.5 below) are empty or there is only one such. The reason for excluding the spheres with less than five punctures is a bit more subtle (self-folded triangles present some "unpleasant" properties in these surfaces).

Definition 2.2.3 ([12], Definition 2.2). Let \((\Sigma, M)\) be a bordered surface with marked points. An (ordinary) arc in \((\Sigma, M)\) is a curve \(i\) in \(\Sigma\) such that:

- the endpoints of \(i\) are marked points in \(M\);
- \(i\) does not intersect itself, except that its endpoints may coincide;
- the relative interior of \(i\) is disjoint from \(M\) and from the boundary of \(\Sigma\);
- \(i\) does not cut out an unpunctured monogon or an unpunctured digon.

We consider two arcs \(i_1\) and \(i_2\) to be the same whenever they are isotopic in \(\Sigma\) rel \(M\), that is whenever there exists an isotopy \(H : I \times \Sigma \to \Sigma\) such that \(H(0, x) = x\) for all \(x \in \Sigma\), \(H(1, i_1) = i_2\), and \(H(t, m) = m\) for all \(t \in I\) and all \(m \in M\). An arc whose endpoints coincide will be called a loop. (Do not confuse the notion of loop in \((\Sigma, M)\) with a loop in a quiver). We denote the set of (isotopy classes of) arcs in \((\Sigma, M)\) by \(A^\circ(\Sigma, M)\).

Two arcs are compatible if there are arcs in their respective isotopy classes whose relative interiors do not intersect (cf. [12], Definition 2.4).

Proposition 2.2.4. Given any collection of pairwise compatible arcs, it is always possible to find representatives in their isotopy classes whose relative interiors do not intersect each other.
2.2. Triangulations of surfaces and their flips

Definition 2.2.5. An \emph{ideal triangulation} of \((\Sigma, M)\) is any maximal collection of pairwise compatible arcs whose relative interiors do not intersect each other (cf. [12], Definition 2.6).

If \(\tau\) is an ideal triangulation of \((\Sigma, M)\) and we take a connected component of the complement in \(\Sigma\) of the union of the arcs in \(\tau\), the closure \(\triangle\) of this component will be called an \emph{ideal triangle} of \(\tau\). An ideal triangle \(\triangle\) is called \emph{interior} if its intersection with the boundary of \(\Sigma\) consists only of (possibly none) marked points. Otherwise it will be called \emph{non-interior}. An interior ideal triangle \(\triangle\) is \emph{self-folded} if it contains exactly two arcs of \(\tau\) (note that every interior ideal triangle contains at least two and at most three arcs of \(\tau\), while each non-interior ideal triangle contains at least one and at most two arcs).

Figure 2.2: Self-folded triangle

The number \(n\) of arcs in an ideal triangulation of \((\Sigma, M)\) is determined by the genus \(g\) of \(\Sigma\), the number \(b\) of boundary components of \(\Sigma\), the number \(p\) of punctures and the number \(c\) of marked points on the boundary of \(\Sigma\), according to the formula
\[ n = 6g + 3b + 3p + c - 6, \]
which can be proved using the definition and basic properties of the Euler characteristic. Hence \(n\) is an invariant of \((\Sigma, M)\), called the \emph{rank} of \((\Sigma, M)\) (because it coincides with the rank of the cluster algebra associated to \((\Sigma, M)\), see [12]).

Let \(\tau\) be an ideal triangulation of \((\Sigma, M)\) and let \(i \in \tau\) be an arc. If \(i\) is not the
2.2. Triangulations of surfaces and their flips

folded side of a self-folded triangle, then there exists exactly one arc \(i'\), different from \(i\), such that \(\sigma = (\tau \setminus \{i\}) \cup \{i'\}\) is an ideal triangulation of \((\Sigma, M)\). We say that \(\sigma\) is obtained by applying a flip to \(\tau\), or by flipping the arc \(i\) (cf. [12], Definition 3.5), and write \(\sigma = f_i(\tau)\). In order to be able to flip the folded sides of self-folded triangles, one has to enlarge the set of arcs with which triangulations are formed. This is done by introducing the notion of tagged arc. Since we will deal only with ordinary arcs in this paper, we refer the reader to [12] and [13] for the definition and properties of tagged arcs and tagged triangulations.

Proposition 2.2.6 ([12], Propositions 3.8 and 7.10.). Any two ideal triangulations are related by a sequence of flips. If \((\Sigma, M)\) is not a surface with empty boundary and exactly one puncture, then any two tagged triangulations are related by a sequence of flips.

To each ideal triangulation \(\tau\) we associate a skew-symmetric \(n \times n\) integer matrix \(B(\tau)\) whose rows and columns correspond to the arcs of \(\tau\) (cf. [12], Definition 4.1). Let \(\pi_\tau : \tau \rightarrow \tau\) be the function that is the identity on the set of arcs that are not folded sides of self-folded triangles of \(\tau\), and sends the folded side of a self-folded triangle to the unique loop of \(\tau\) enclosing it. For each non-self-folded ideal triangle \(\triangle\) of \(\tau\), let \(B_\triangle = b_\triangle^{i,j}\) be the \(n \times n\) integer matrix defined by

\[
B_\triangle^{i,j} = \begin{cases} 
1 & \text{if } \triangle \text{ has sides } \pi_\tau(i) \text{ and } \pi_\tau(j), \text{ with } \pi_\tau(j) \text{ following } \pi_\tau(i) \text{ in the clockwise order defined by the orientation of } \Sigma; \\
-1 & \text{if the same holds, but in the counter-clockwise order;} \\
0 & \text{otherwise.}
\end{cases}
\]
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The signed adjacency matrix $B(\tau)$ is then defined as

$$B(\tau) = \sum_{\Delta} B_{\Delta},$$

(2.2)

where the sum runs over all non-self-folded triangles of $\tau$. Note that $B(\tau)$ is skew-symmetric, and all its entries have absolute value less than 3.

The matrix $B(\tau)$ gives rise to the signed adjacency quiver $Q(\tau)$, whose vertices are the arcs in $\tau$, with $b_{ij}$ arrows from $i$ to $j$ whenever $b_{ij} > 0$. Since $B(\tau)$ is skew-symmetric, $Q(\tau)$ is a 2-acyclic quiver.

**Theorem 2.2.1** ([12], Proposition 4.8). Let $\tau$ and $\sigma$ be ideal triangulations. If $\sigma$ is obtained from $\tau$ by flipping the arc $i$ of $\tau$, then $Q(\sigma) = \mu_j(Q(\tau))$.

**Remark 2.2.7.**

1. This Theorem holds in the more general situation where $\tau$ and $\sigma$ are tagged triangulations related by a flip, see [12], Lemma 9.7.

2. The assignment \{$skew\text{-}symmetric$ $n \times n$ integer matrix $B$\} $\mapsto$ \{$2$-acyclic quiver $Q$\} is general and defines a bijection between skew-symmetric $n \times n$ integer matrices and 2-acyclic quivers on $n$ vertices. See [15] for far-reaching discussions on this matter.


**Definition 2.2.8.** If a puncture is incident to exactly two arcs $i_1$ and $i_2$ of the ideal triangulation $\tau$, then $Q(\tau)$ has no arrows between $i_1$ and $i_2$ (see Figure 2.3). For each such pair of arcs we add to $Q(\tau)$ an arrow from $i_1$ to $i_2$ and an arrow from $i_2$ to $i_1$, and call the resulting quiver the unreduced signed adjacency quiver $\hat{Q}(\tau)$. 
2.2. Triangulations of surfaces and their flips

It is clear that $Q(\tau)$ can be obtained from $\tilde{Q}(\tau)$ by deleting all 2-cycles.

**Example 2.2.9.** In Figure 2.4 we can see some ideal triangulations of the once-punctured square and their signed adjacency quivers drawn on them. In Figure 2.5 we have the same triangulations, but their unreduced signed adjacency quivers instead. Notice that among these four triangulations, the only one for which the signed adjacency quiver is different from the unreduced one is the triangulation appearing farthest right in both Figures 2.4 and 2.5.
2.3 Quivers with potentials and their mutations

In this section we give the background on quivers with potentials and their mutations we shall use in the remaining sections. For a more detailed and elegant treatment of the subject, we refer the reader to [8], all of whose notation we will adopt here. In particular, $K$ will always denote a field. A survey of the topics treated in [8] can be found in [28].

Given a quiver $Q$, we denote by $R$ the $K$-vector space with basis $\{e_i \mid i \in Q_0\}$. If we define $e_i e_j = \delta_{ij} e_i$, then $R$ becomes naturally a commutative semisimple $K$-algebra, which we call the vertex span of $Q$; each $e_i$ is called the path of length zero at $i$. We define the arrow span of $Q$ as the $K$-vector space $A$ with basis the set of arrows $Q_1$. Note that $A$ is an $R$-bimodule if we define $e_i a = \delta_{i,h(a)} a$ and $ae_j = a \delta_{l(a),j}$ for $i \in Q_0$ and $a \in Q_1$. For $d \geq 0$ we denote by $A^d$ the $K$-vector space with basis all the paths of length $d$ in $Q$; this space has a natural $R$-bimodule structure as well. Notice that $A^0 = R$ and $A^1 = A$.

The complete path algebra of $Q$ is the $K$-vector space consisting of all possibly
2.3. Quivers with potentials and their mutations

infinite linear combinations of paths in $Q$, that is,

$$R\langle\langle Q\rangle\rangle = \prod_{d=0}^{\infty} A^d;$$

with multiplication induced by concatenation of paths (cf. [8], Definition 2.2). Note that $R\langle\langle Q\rangle\rangle$ is a $K$-algebra and an $R$-bimodule, and has the usual path algebra

$$R\langle Q\rangle = \bigoplus_{d=0}^{\infty} A^d$$

as $K$-subalgebra and sub-$R$-bimodule. Moreover, $R\langle Q\rangle$ is dense in $R\langle\langle Q\rangle\rangle$ under the $m$-adic topology, whose fundamental system of open neighborhoods around 0 is given by the powers of $m = m(A) = \prod_{d \geq 1} A^d$, the ideal of $R\langle\langle Q\rangle\rangle$ generated by the arrows. A crucial property of this topology is the following:

a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $R\langle\langle Q\rangle\rangle$ converges if and only if

$$\lim_{n \to \infty} x_n = \sum_{d \geq 0} \lim_{n \to \infty} x_n^{(d)},$$

where $x_n^{(d)}$ denotes the degree-$d$ component of $x_n$.

Even though the action of $R$ on $R\langle\langle Q\rangle\rangle$ (and $R\langle Q\rangle$) is not central, it is compatible with the multiplication of $R\langle\langle Q\rangle\rangle$ in the sense that if $a$ and $b$ are paths in $Q$, then $e_{h(a)} ab = ae_{t(a)} b = ab e_{t(b)}$. Therefore we will say that $R\langle\langle Q\rangle\rangle$ (and $R\langle Q\rangle$) are $R$-algebras. Accordingly, any $K$-algebra homomorphism $\varphi$ between (complete) path algebras will be called an $R$-algebra homomorphism if the underlying quivers have the same set of vertices and $\varphi(r) = r$ for every $r \in R$. It is easy to see that every $R$-
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algebra homomorphism between complete path algebras is continuous. The following
is an extremely useful criterion to decide if a given linear map $\varphi : R\langle\langle Q\rangle\rangle \rightarrow R\langle\langle Q'\rangle\rangle$
between complete path algebras (on the same set of vertices) is an $R$-algebra homomorphism or an $R$-algebra isomorphism:

Every pair $(\varphi^{(1)}, \varphi^{(2)})$ of $R$-bimodule homomorphisms $\varphi^{(1)} : A \rightarrow A'$, (2.6)

$\varphi^{(2)} : A \rightarrow m(A')^2$, extends uniquely to a continuous $R$-algebra homomorphism
$\varphi : R\langle\langle Q\rangle\rangle \rightarrow R\langle\langle Q'\rangle\rangle$ such that $\varphi|_A = (\varphi^{(1)}, \varphi^{(2)})$. Furthermore, $\varphi$ is $R$-algebra
isomorphism if and only if $\varphi^{(1)}$ is an $R$-bimodule isomorphism.

A potential on $A$ (or $Q$) is any element of $R\langle\langle Q\rangle\rangle$ all of whose terms are cyclic paths
of positive length (cf. [8], Definition 3.1). The set of all potentials on $A$ is denoted by
$R\langle\langle Q\rangle\rangle_{cyc}$, it is a closed vector subspace of $R\langle\langle Q\rangle\rangle$. Two potentials $S, S' \in R\langle\langle Q\rangle\rangle_{cyc}$
are cyclically equivalent if $S - S'$ lies in the closure of the vector subspace of $R\langle\langle Q\rangle\rangle$
spanned by all the elements of the form $a_1 \ldots a_d - a_2 \ldots a_d a_1$ with $a_1 \ldots a_d$ a cyclic
path of positive length (cf. [8], Definition 3.2).

A quiver with potential is a pair $(A, S)$ (or $(Q, S)$), where $S$ is a potential on $A$ such
that no two different cyclic paths appearing in the expression of $S$ are cyclically equivalent (cf. [8], Definition 4.1). (For instance, the pair $(A, xa_1 \ldots a_d - ya_{i+1} \ldots a_d a_1 \ldots a_i)$
is not a quiver with potential for any choice of different non-zero scalars $x, y \in K$).
We will use the shorthand QP to abbreviate “quiver with potential". The direct sum
of two QPs $(A, S)$ and $(A', S')$ on the same set of vertices is the QP $(A, S) \oplus (A', S') =
(A \oplus A', S + S')$.

If $(A, S)$ and $(A', S')$ are QPs on the same set of vertices, we say that $(A, S)$ is
right-equivalent to \((A', S')\) if there exists a right-equivalence between them, that is, an \(R\)-algebra isomorphism \(\varphi : R(\langle\langle Q\rangle\rangle) \rightarrow R(\langle\langle Q'\rangle\rangle)\) such that \(\varphi(S)\) is cyclically equivalent to \(S'\) (cf. [8], Definition 4.2).

For each arrow \(a \in Q_1\) and each cyclic path \(a_1 \ldots a_d\) in \(Q\) we define the cyclic derivative

\[
\partial_a(a_1 \ldots a_d) = \sum_{i=1}^d \delta_{a,a_i}a_{i+1} \ldots a_da_1 \ldots a_{i-1},
\]

(2.7)

(where \(\delta_{a,a_i}\) is the Kronecker delta) and extend \(\partial_a\) by linearity and continuity to obtain a map \(\partial_a : R(\langle\langle Q\rangle\rangle)_{\text{cyc}} \rightarrow R(\langle\langle Q\rangle\rangle)\) (cf. [8], Definition 3.1). Note that we have \(\partial_a(S) = \partial_a(S')\) whenever the potentials \(S\) and \(S'\) are cyclically equivalent.

The Jacobian ideal \(J(S)\) is the closure of the two-sided ideal of \(R(\langle\langle Q\rangle\rangle)\) generated by \(\{\partial_a(S) \mid a \in Q_1\}\), and the Jacobian algebra \(\mathcal{P}(Q, S)\) is the quotient algebra \(R(\langle\langle Q\rangle\rangle)/J(S)\) (cf. [8], Definition 3.1). Jacobian ideals and Jacobian algebras are invariant under right-equivalences, in the sense that if \(\varphi : R(\langle\langle Q\rangle\rangle) \rightarrow R(\langle\langle Q'\rangle\rangle)\) is a right-equivalence between \((A, S)\) and \((A', S')\), then \(\varphi\) sends \(J(S)\) onto \(J(S')\) and therefore induces an isomorphism \(\mathcal{P}(Q, S) \rightarrow \mathcal{P}(Q', S')\) (cf. [8], Proposition 3.7).

A QP \((Q, S)\) is trivial if \(S \in A^2\) and \(\{\partial_a(S) \mid a \in Q_1\}\) spans \(A\) (cf. [8], Definition 4.3, see also Proposition 4.4 therein). We say that a QP \((A, S)\) is reduced if the degree-2 component of \(S\) is 0, that is, if the expression of \(S\) involves no 2-cycles. Note that the underlying quiver of a reduced QP may have 2-cycles. We say that a quiver \(Q\) (or its arrow span, or any QP on it) is 2-acyclic if it has no 2-cycles.

**Theorem 2.3.1** (Splitting Theorem, [8], Theorem 4.6). For every QP \((A, S)\) there exist a trivial QP \((A_{\text{triv}}, S_{\text{triv}})\) and a reduced QP \((A_{\text{red}}, S_{\text{red}})\) such that \((A, S)\) is right-equivalent to the direct sum \((A_{\text{triv}}, S_{\text{triv}}) \oplus (A_{\text{red}}, S_{\text{red}})\). Furthermore, the right-
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 equivalence class of each of the QPs \((A_{\text{triv}}, S_{\text{triv}})\) and \((A_{\text{red}}, S_{\text{red}})\) is determined by the right-equivalence class of \((A, S)\).

In the situation of Theorem 2.3.1, the QP \((A_{\text{red}}, S_{\text{red}})\) (resp. \((A_{\text{triv}}, S_{\text{triv}})\)) is called the reduced part (resp. trivial part) of \((A, S)\) (cf. [8], Definition 4.13); this terminology is well defined up to right-equivalence.

We now turn to the definition of mutation of a QP. Let \((A, S)\) be a QP on the vertex set \(Q_0\) and let \(i \in Q_0\). Assume that \(Q\) has no 2-cycles incident to \(i\). Thus, if necessary, we replace \(S\) with a cyclically equivalent potential so that we can assume that every cyclic path appearing in the expression of \(S\) does not begin at \(i\). This allows us to define \([S]\) as the potential on \(\tilde{\mu}_j(Q)\) obtained from \(S\) by replacing each \(i\)-hook \(ab\) with the arrow \([ab]\) (see Definition 2.1.1). Also, we define \(\Delta_i(Q) = \sum b^*a^*[ab]\), where the sum runs over all \(i\)-hooks \(ab\) of \(Q\).

**Definition 2.3.1** ([8], equations (5.3) and (5.8) and Definition 5.5). Under the assumptions and notation just stated, we define the premutation of \((A, S)\) in direction \(i\) as the QP \(\tilde{\mu}_j(A, S) = (\tilde{Q}, \tilde{S})\), where \(\tilde{A}\) is the arrow span of \(\tilde{\mu}_j(Q)\) (see Definition 2.1.1) and \(\tilde{S} = [S] + \Delta_i(Q)\). The mutation \(\mu_j(A, S)\) of \((A, S)\) in direction \(i\) is then defined as the reduced part of \(\tilde{\mu}_j(A, S)\).

**Theorem 2.3.2** ([8], Theorem 5.2 and Corollary 5.4). Premutations and mutations are well defined up to right-equivalence. That is, if \((A, S)\) and \((A', S')\) are right-equivalent QPs with no 2-cycles incident to the vertex \(i\), then the QP \(\tilde{\mu}_j(A, S)\) is right-equivalent to \(\tilde{\mu}_j(A', S')\) and the QP \(\mu_j(A, S)\) is right-equivalent to \(\mu_j(A', S')\).

**Theorem 2.3.3** ([8], Theorem 5.7). Mutations are involutive up to right-equivalence. More specifically, if \((A, S)\) is a reduced QP, then \(\mu_j^2(A, S)\) is right-equivalent to \((A, S)\).
“Unfortunately”, as the following easy example shows, 2-acyclicity is not a QP-mutation invariant (in contrast to the ordinary quiver mutation, where 2-acyclicity is ensured by definition).

**Example 2.3.2.** Consider the potentials \( S_1 = 0 \) and \( S_2 = abc \) on the quiver

\[
\begin{array}{ccc}
1 & \rightarrow & 3 \\
| & \downarrow & | \\
2 & \leftarrow & b \\
& \downarrow & \\
c & \leftarrow & 1
\end{array}
\]

If we perform the pre-mutation \( \tilde{\mu}_2 \) on \( (A, S_1) \) and \( (A, S_2) \), we get \( (\tilde{A}, \tilde{S}_1) \) and \( (\tilde{A}, \tilde{S}_2) \), where \( \tilde{A} \) is the arrow span of the quiver

\[
\begin{array}{ccc}
1 & \rightarrow & 3 \\
| & \downarrow & | \\
2 & \leftarrow & b^* \\
& \downarrow & \\
c^* & \leftarrow & 1
\end{array}
\]

and \( \tilde{S}_1 = c^*b^*[bc] \), \( \tilde{S}_2 = a[bc] + c^*b^*[bc] \). Note that \( (\tilde{A}, \tilde{S}_1) \) is already reduced (hence equals \( \mu_2(A, S_1) \)), while the reduced part of \( (\tilde{A}, \tilde{S}_2) \) is \( (\tilde{A}, 0) \) (hence \( \mu_2(A, S_2) = (\tilde{A}, 0) \)), where \( \tilde{A} \) is the arrow span of the quiver

\[
\begin{array}{ccc}
1 & \rightarrow & 3 \\
\end{array}
\]

In particular, we cannot apply the mutations in direction 1 or 3 to \( (\tilde{A}, \tilde{S}_1) \), but we can apply them to \( (\tilde{A}, 0) \).

**Definition 2.3.3 ([8], Definition 7.2).** A QP \( (A, S) \) is non-degenerate if it is 2-acyclic.
and the quiver of the QP obtained after any possible sequence of QP-mutations is 2-acyclic.

**Theorem 2.3.4** ([8], Proposition 7.3 and Corollary 7.4). *If the base field $K$ is uncountable, then every 2-acyclic quiver admits a non-degenerate QP.*

A QP $(A, S)$ is rigid if every cycle in $Q$ is cyclically equivalent to an element of the Jacobian ideal $J(S)$ (cf. [8], Definition 6.10 and equation 8.1). Rigidity is invariant under QP-mutation:

**Theorem 2.3.5** ([8], Corollary 6.11, Proposition 8.1 and Corollary 8.2). *Every reduced rigid QP is 2-acyclic. The class of reduced rigid QPs is closed under QP-mutation. Consequently, every rigid reduced QP is non-degenerate.*

**Proposition 2.3.4** ([8], Corollary 6.6). *Let $(A, S)$ be a non-degenerate QP and $i \in Q_0$ any vertex, then the Jacobian algebra $\mathcal{P}(A, S)$ is finite-dimensional if and only if so is $\mathcal{P}(\mu_j(A, S))$. In other words, finite-dimensionality of Jacobian algebras is invariant under QP-mutations.*

We finish this section and the current chapter describing the operation of restriction of a QP to a subset of the set of vertices.

**Definition 2.3.5** ([8], Definition 8.8). *Let $(A, S)$ be a QP and $I$ be a subset of the vertex set $Q_0$. The restriction of $(A, S)$ to $I$ is the QP $(A|_I, S|_I)$ on the vertex set $Q_0$, with $A|_I = \bigoplus_{i,j \in I} A_{ij}$ and $S|_I = \psi_I(S)$, where $\psi_I : R(\langle A \rangle) \to R(\langle A|_I \rangle)$ is the $R$-algebra homomorphism such that $\psi_I(a) = a$ for $a \in A|_I$ and $\psi_I(b) = 0$ for each arrow $b \notin A|_I$.*

**Remark 2.3.6.** Notice that if $I$ is a proper subset of $Q_0$, then the elements of $Q_0 \setminus I$ are isolated vertices of the restriction to $I$, that is, there are no arrows of $A|_I$ whose head or tail belongs to $Q_0 \setminus I$. 
2.3. Quivers with potentials and their mutations

In Proposition 8.9 of [8], Derksen-Weyman-Zelevinsky prove that restriction preserves rigidity and finite-dimensionality of Jacobian algebras. In Section 3.2 we will prove that it preserves non-degeneracy as well.
Chapter 3

The QP of a triangulation

3.1 Definition of $S(\tau)$

Let $(\Sigma, M)$ be a bordered surface with marked points, with $P \subseteq M$ the set of punctures of $(\Sigma, M)$. For each $p \in P$ choose a non-zero scalar $x_p \in K$; this choice is going to be kept fixed for every triangulation of $(\Sigma, M)$.

**Definition 3.1.1.** Let $\tau$ be an ideal triangulation of $(\Sigma, M)$. Based on our choice $(x_p)_{p \in P}$ we associate to $\tau$ a potential $S(\tau) \in R\langle\langle Q(\tau)\rangle\rangle$ as follows. Let $\hat{A}(\tau)$ denote the arrow span of $\hat{Q}(\tau)$.

- For each interior non-self-folded ideal triangle $\triangle$ of $\tau$ that gives rise to an oriented triangle of $\hat{Q}(\tau)$, let $\hat{S}^{\triangle}$ be such oriented triangle up to cyclical equivalence.

- If the interior non-self-folded ideal triangle $\triangle$ with sides $j$, $k$ and $l$, is adjacent to two self-folded triangles as in the configuration of Figure 3.1,
3.1. Definition of $S(\tau)$

Figure 3.1:

[Diagram of a graph with vertices labeled $i$, $j$, $k$, $l$, $m$, $p$, and $q$.]

Define $\hat{T}^\triangle = \frac{b_i b_j b_k}{x_p x_q}$ (up to cyclical equivalence), where $p$ and $q$ are the punctures enclosed in the self-folded triangles adjacent to $\triangle$. Otherwise, if it is adjacent to less than two self-folded triangles, define $\hat{T}^\triangle = 0$.

- If a puncture $p$ is adjacent to exactly one arc $i$ of $\tau$, then $i$ is the folded side of a self-folded triangle of $\tau$ and around $i$ we have the configuration shown in Figure 3.2.

Figure 3.2:

[Diagram showing a puncture $p$ adjacent to one arc $i$.]

In case both $k$ and $l$ are indeed arcs of $\tau$ (and not part of the boundary of $\Sigma$), then we define $\hat{S}^p = -\frac{a_i b_j b_k}{x_p}$ (up to cyclical equivalence).

- If a puncture $p$ is adjacent to more than one arc, delete all the loops adjacent
3.1. Definition of $S(\tau)$

to $p$ that enclose a self-folded triangle. The arrows between the remaining arcs adjacent to $p$ form a unique cycle $a^p_1 \ldots a^p_d$ that exhausts all such remaining arcs and gives a complete round around $p$ in the counter-clockwise orientation defined by the orientation of $\Sigma$. We define $\hat{S}^p = x_p a^p_1 \ldots a^p_d$ (up to cyclical equivalence).

The un reduced potential $\hat{S}(\tau) \in R\langle\langle \hat{Q}(\tau) \rangle\rangle$ of $\tau$ is then defined by

$$\hat{S}(\tau) = \sum_\Delta (\hat{S}^\Delta + \hat{T}^\Delta) + \sum_{p \in P} \hat{S}^p,$$  \hspace{1cm} (3.1)

where the first sum runs over all interior non-self-folded triangles.

Finally, we define $(Q(\tau), S(\tau))$ to be the (right-equivalence class of the) reduced part of $(\hat{Q}(\tau), \hat{S}(\tau))$.

**Remark 3.1.2.** Note that, since $(\Sigma, M)$ is not a sphere with less than five punctures, each non-self-folded ideal triangle is adjacent to at most two self-folded triangles.

To illustrate Definition 3.1.1, we give some examples.

**Example 3.1.3.** If $(\Sigma, M)$ has no punctures (so that the boundary of $\Sigma$ is non-empty and all marked points lie on the boundary), then for every triangulation $\tau$ of $(\Sigma, M)$ we have $(Q(\tau), S(\tau)) = (\hat{Q}(\tau), \hat{S}(\tau))$ and all the terms of $S(\tau)$ are oriented triangles of $Q(\tau)$ arising from interior triangles of $\tau$. The representation theory of the corresponding Jacobian algebra is studied by Assem-Brüstle-Charbonneau-Plamondon in [2] and by Brüstle-Zhang in [5].

**Example 3.1.4.** If the ideal triangulation $\tau$ of $(\Sigma, M)$ does not have self-folded triangles and is such that each puncture $p$ is incident to at least three arcs of $\tau$, then
3.1. Definition of $S(\tau)$

$(\hat{Q}(\tau), \hat{S}(\tau)) = (Q(\tau), S(\tau))$ and $S(\tau) = \sum_{\Delta} \hat{S}^\Delta + \sum_{p \in P} x_p a_1^p \ldots a_d^p$ is the sum of the oriented triangles of $Q(\tau)$ arising from interior ideal triangles of $\tau$ and non-zero scalar multiples of cycles around the punctures.

**Example 3.1.5.** Consider the ideal triangulation $\tau$ of the twice-punctured hexagon shown in Figure 3.3. Its (unreduced) signed adjacency quiver is shown on the right.

![Figure 3.3:](image)

We have $\hat{S}(\tau) = \alpha \beta \gamma + \delta \eta \varepsilon + \frac{\rho \omega \kappa}{x_p x_q} - \frac{\delta \nu \varepsilon}{x_p} - \frac{\varepsilon \mu \lambda}{x_q}$. Since $(\hat{Q}(\tau), \hat{S}(\tau))$ is clearly 2-acyclic (and hence reduced), we have $(Q(\tau), S(\tau)) = (\hat{Q}(\tau), \hat{S}(\tau))$.

**Example 3.1.6.** The ideal triangulation $\sigma$ shown in Figure 3.4 can be obtained from the triangulation $\tau$ of the previous example by a flip. Its unreduced signed adjacency quiver $\hat{Q}(\sigma)$ is shown on the right, and $\hat{S}(\sigma) = \alpha \beta \gamma + \delta \rho b + a \lambda \eta^* - \frac{\rho \omega \kappa^*}{x_p} + x_q a b$. In this
example \((\widehat{Q}(\sigma), \widehat{S}(\sigma))\) is not reduced. The \(R\)-algebra isomorphism \(\varphi : R\langle \langle \widehat{A}(\sigma) \rangle \rangle \rightarrow R\langle \langle \widehat{A}(\sigma) \rangle \rangle\) whose action on the arrows is given by \(a \mapsto a - \frac{\delta^* \rho}{x_q}, \ b \mapsto b - \frac{\lambda \eta^*}{x_q} + \frac{\omega^*}{x_p x_q}\), and the identity on the rest of the arrows, is a right-equivalence between \((\widehat{Q}(\sigma), \widehat{S}(\sigma))\) and \((\widehat{Q}(\sigma), x a b + \alpha \beta \gamma - \frac{\delta^* \rho \lambda \eta^*}{x_q} + \frac{\delta^* \rho \omega^*}{x_p x_q})\). Therefore, the QP associated to \(\sigma\) is, up to right-equivalence, \((Q(\sigma), S(\sigma))\), where \(S(\sigma) = \alpha \beta \gamma - \frac{\delta^* \rho \lambda \eta^*}{x_q} - \frac{\delta \rho \omega^*}{x_p x_q}\).

**Example 3.1.7.** Consider the ideal triangulation of the once-punctured square shown in Figure 3.5. The unreduced signed adjacency quiver \(\widehat{Q}(\tau)\) has been drawn on the triangulation. We have \(\widehat{S}(\tau) = a a \beta + \gamma \delta b + x a b\), where \(x \in K\) is the non-zero scalar attached to the puncture. The \(R\)-algebra isomorphism \(\varphi : R\langle \langle \widehat{Q}(\tau) \rangle \rangle \rightarrow R\langle \langle \widehat{Q}(\tau) \rangle \rangle\) whose action on the arrows is given by \(a \mapsto a - \frac{\gamma \delta}{x}, \ b \mapsto b - \frac{\alpha \beta}{x}\), and the identity on the rest of the arrows, is a right-equivalence between \((\widehat{Q}(\tau), \widehat{S}(\tau))\) and \((\widehat{Q}(\tau), x a b - \frac{\delta \alpha \beta}{x})\).
3.1. Definition of $S(\tau)$

Therefore, $(Q(\tau), S(\tau)) = (Q(\tau), -\frac{\gamma\delta\alpha\beta}{x})$.

The last two examples have something in common, namely, in both of them there is a puncture that is incident to exactly two arcs of the corresponding ideal triangulation. The reduction procedure applied in these examples is general, as we will see in a moment. Now, of course there could be many 2-cycles in $\hat{Q}(\tau)$, but there are only finitely many of them and each of them has a non-zero scalar multiple appearing as a term of $\hat{S}(\tau)$, which means that the underlying quiver of the reduced part of $(\hat{Q}(\tau), \hat{S}(\tau))$ coincides with the signed adjacency quiver $Q(\tau)$. Moreover, the reduction process leading to $S(\tau)$ can be split into steps, so that one takes care of the 2-cycles one by one.

Take an ideal triangulation $\tau$ of $(\Sigma, M)$, and assume that the puncture $p$ is incident to exactly two arcs of $\tau$ as in Figure 3.6, assume also that $j$ and $k$ are indeed arcs of $\tau$ (and not part of the boundary of $\Sigma$). First, consider the case where none of $j$ or $k$ encloses a self-folded triangle. Then the unreduced potential of $\tau$ is $\hat{S}(\tau) = \gamma\delta b + a\alpha\beta + x_pab + x_{m_1}\beta\gamma c_1 \ldots c_l + x_{m_2}\delta\alpha d_1 \ldots d_t + S'(\tau)$, where $S'(\tau) \in R(\langle \hat{Q}(\tau) \rangle)$ is a potential involving none of the arrows $a, b, \alpha, \beta, \gamma, \delta$. Just as in Example 3.1.7, the $R$-algebra isomorphism $\varphi : R(\langle \hat{Q}(\tau) \rangle) \to R(\langle \hat{Q}(\tau) \rangle)$ whose action on the arrows is given by
a \mapsto a - \frac{\gamma \delta}{x_p}, \ b \mapsto b - \frac{\alpha \beta}{x_p}, \ and \ the \ identity \ on \ the \ rest \ of \ the \ arrows, \ is \ a \ right-equivalence \ between \ \( \widehat{Q}(\tau), \widehat{S}(\tau) \) \ and \ \( \widehat{Q}(\tau), x_pab - \frac{\alpha \beta \gamma \delta}{x_p} + x_{m_1} \beta \gamma c_1 \ldots c_t + x_{m_2} \delta \alpha d_1 \ldots d_t + S'(\tau) \). 
Since \( S'(\tau) \) \ does \ not \ involve \ any \ of \ the \ arrows \ \( a, b, \alpha, \beta, \gamma, \delta, \) \ this \ implies \ that \ the \ term \ -\frac{\alpha \beta \gamma \delta}{x_p} \ will \ appear, \ up \ to \ right-equivalence, \ as \ a \ term \ of \ S(\tau). \ In \ other \ words, \ x_pab + a\alpha\beta + \gamma\delta b \ is \ replaced \ by \ -\frac{\alpha \beta \gamma \delta}{x_p} \ in \ the \ reduction \ process.

Now, assume that the puncture \( p \) \ is incident to exactly two arcs of \( \tau \) \ and that \( k \) \ encloses a self-folded triangle, see Figure 3.7. \ Here \ the \ unreduced \ potential \ is \n\[ \widehat{S}(\tau) = a \lambda \eta + \delta \rho b - \frac{\omega \nu}{x_q} + x_p ab + x_m \rho \omega \nu c_1 \ldots c_t + S'(\tau), \] \ where \( S'(\tau) \in R\langle\langle \widehat{Q}(\tau) \rangle\rangle \) \ is \ a \ potential \ that \ does \ not \ involve \ any \ of \ the \ arrows \ \( a, b, \delta, \eta, \lambda, \rho, \nu, \omega, \). \ Similarly \ to \ Example 3.1.6, \ the \ \( R \)-algebra \ isomorphism \( \varphi: R\langle\langle \widehat{Q}(\tau) \rangle\rangle \to R\langle\langle \widehat{Q}(\tau) \rangle\rangle \) \ whose \ action \ on \ the \ arrows \ is \ given \ by \( a \mapsto a - \frac{\delta \rho}{x_p}, \ b \mapsto b + \frac{\lambda \eta}{x_p} + \frac{\omega \nu}{x_p x_q}, \) \ and \ the \ identity \ on \ the \ rest \ of \ the \ arrows, \ is \ a \ right-equivalence \ between \ \( \langle\langle \widehat{Q}(\tau), \widehat{S}(\tau) \rangle\rangle \) \ and \ \( \langle\langle \widehat{Q}(\sigma), x_p ab - \frac{\delta \rho \lambda \eta}{x_p} + \frac{\delta \rho \omega \nu}{x_p x_q} + x_m \rho \omega \nu c_1 \ldots c_t + S'(\sigma) \rangle\rangle \). \ This, \ together \ with \ the \ fact \ that \( S'(\tau) \) \ involves \ none \ of \ the \ arrows \ \( a, b, \delta, \eta, \lambda, \rho, \nu, \omega, \) \ implies \ that \ the \ potential \ -\frac{\delta \rho \lambda \eta}{x_p} + \frac{\delta \rho \omega \nu}{x_p x_q} \ will \ appear,
up to right-equivalence, as a summand of $S(\tau)$. That is, $x_pab + a\lambda\eta - \frac{a\omega\nu}{x_q} + \delta p\nu$ is replaced by $-\frac{\delta p\lambda\eta}{x_p} + \frac{\delta p\omega\nu}{x_p x_q}$ in the reduction process.

3.2 Towards the first main result: restriction, a simplifying tool

We start with a result valid for general QPs, namely, that the operation of restriction preserves non-degeneracy. As a preparation for the proof we recall the construction, given in [8] to prove the Splitting Theorem, of a reduced part and a trivial part of a
3.2. Towards the first main result: restriction, a simplifying tool

QP. Let \((A, S)\) be any QP and denote by \(S^{(2)}\) the degree-2 component of \(S\). Assume \(S^{(2)} \neq 0\) (otherwise \((A, S)\) is already reduced and there is no construction to be done).

Up to a right-equivalence that acts as the identity on the arrows of \(Q\) not appearing in \(S^{(2)}\), we can assume that

\[
S = \sum_{j=1}^{N} (a_j b_j + a_j u_j + v_j b_j) + S',
\]

where each \(a_j b_j\) is a 2-cycle, the \(2N\) distinct arrows \(a_1, b_1, \ldots, a_N, b_N\) form a basis of \(A_{\text{triv}} = \partial S^{(2)}\), each of \(u_j\) and \(v_j\) belongs to \(m^2\), and \(S' \in m^3\) is a potential neither of whose terms involves any of the arrows \(a_j\) or \(b_j\). If \(u_j = v_j = 0\) for all \(j\), then we already have the decomposition of Theorem 2.3.1. Otherwise, one defines a unitriangular automorphism \(\varphi_1 : R\langle\langle Q\rangle\rangle \rightarrow R\langle\langle Q\rangle\rangle\) by setting

\[
\varphi_1(a_j) = a_j - v_j, \quad \varphi_1(b_j) = b_j - u_j, \quad \varphi_1(c) = c \text{ for } c \in Q_1 \setminus \{a_1, b_1, \ldots, a_N, b_N\}. \tag{3.3}
\]

This automorphism of \(R\langle\langle Q\rangle\rangle\) is a right-equivalence between \((Q, S)\) and \((Q, S_1)\), where \(S_1\) is a potential with \(S_1^{(2)} = S^{(2)}\) and such that, when written in the form (3.2), all the \(u_j\)-factors and \(v_j\)-factors belong to \(m^3\). If all these factors are 0, we already have reached the decomposition of Theorem 2.3.1. Otherwise, we define a unitriangular automorphism \(\varphi_2\) of \(R\langle\langle Q\rangle\rangle\) by the rules (3.3) defined in terms of the \(u_j\)-factors and \(v_j\)-factors of \(S_1\). Then \(\varphi_2 \varphi_1\) is a right-equivalence between \((Q, S)\) and \((Q, S_2)\), where \(S_2\) is a potential with \(S_2^{(2)} = S_1^{(2)} = S^{(2)}\) and such that, when written in the form (3.2), the \(u_j\)-factors and \(v_j\)-factors belong to \(m^5\).

If we keep repeating the above procedure ad infinitum, we get a sequence \((S_n)_{n \geq 1}\), with the corresponding \(u_j\)-factors and \(v_j\)-factors belonging to higher and higher powers.
of \( m \). In the limit, \( \varphi = \lim_{n \to \infty} \varphi_n \ldots \varphi_1 \) will be a right-equivalence between \((Q, S)\) and \((Q, \lim_{n \to \infty} S_n)\), where \( \lim_{n \to \infty} S_n \) is a potential whose degree-2 component is \( S^{(2)} \) and such that, when written in the form (3.2), all its \( u_j \)-factors and \( v_j \)-factors are 0. This provides the required right-equivalence of Theorem 2.3.1.

**Remark 3.2.1.** In many concrete examples (like the ones given in the present work), there is no need of considering the above limit process.

**Lemma 3.2.2.** Let \((A, S)\) be a QP, and let \( I \) be any subset of the vertex set \( Q_0 \). There exist a reduced and a trivial QP, \((A_{\text{red}}, S_{\text{red}})\) and \((A_{\text{triv}}, S_{\text{triv}})\), respectively, such that \((A, S)\) is right-equivalent to \((A_{\text{red}}, S_{\text{red}}) \oplus (A_{\text{triv}}, S_{\text{triv}})\), and with the property that the restriction \((A|_I, S|_I)\) is right-equivalent to \((A_{\text{red}}|_I, S_{\text{red}}|_I) \oplus (A_{\text{triv}}|_I, S_{\text{triv}}|_I)\).

**Proof.** The vector subspace of \( A \) generated by the cyclic derivatives of the degree-2 component of \( S \) is \( A_{\text{triv}} \), and we clearly have \((A|_I)_{\text{triv}} = A_{\text{triv}}|_I\) (see [8], equations 4.3 and 4.4). Therefore we will also have \((A|_I)_{\text{red}} = A_{\text{red}}|_I\). Denote these spaces by \( B_{\text{triv}} = A_{\text{triv}}|_I \) and \( B_{\text{red}} = A_{\text{red}}|_I \), respectively, and let \( T_{\text{red}} \in R\langle B_{\text{red}} \rangle \) and \( T_{\text{triv}} \in R\langle B_{\text{triv}} \rangle \) be potentials such that \((B_{\text{red}}, T_{\text{red}})\) is a reduced QP, \((B_{\text{triv}}, T_{\text{triv}})\) is a trivial QP, and there exists an \( R \)-algebra isomorphism \( \varphi : R\langle A|_I \rangle \to R\langle (B_{\text{red}} \oplus B_{\text{triv}}) \rangle \) such that \( \varphi(S|_I) \) is cyclically equivalent to \( T_{\text{red}} + T_{\text{triv}} \). Also, let us write \( S = S|_I + S' \), where \( S' \in R\langle A \rangle \) is a potential each of whose terms involves at least one arrow that does not belong to \( A|_I \).

We can extend \( \varphi \) to an \( R \)-algebra isomorphism \( \widehat{\varphi} : R\langle A \rangle \to R\langle A_{\text{red}} \oplus A_{\text{triv}} \rangle \) by defining \( \widehat{\varphi}(b) = b \) for every arrow \( b \notin A|_I \). The potential \( \widehat{\varphi}(S) \) is cyclically equivalent to \( T_{\text{red}} + T_{\text{triv}} + \widehat{\varphi}(S') \). Let us denote the degree-2 component of \( \widehat{\varphi}(S') \) by \( T'_{\text{triv}} \). Note that every term of the potential \( \widehat{\varphi}(S') \) involves at least one arrow that
3.2. Towards the first main result: restriction, a simplifying tool

does not belong to $A|_I$; in particular, every arrow appearing in $T'_\text{triv}$ is incident to a vertex outside $I$. Note also that the arrows that appear in $T_{\text{triv}}$ can appear in $\tilde{\varphi}(S')$, but do not appear in $T_{\text{red}}$. These remarks make it clear that if we decompose the QP $(A_{\text{red}} \oplus A_{\text{triv}}, T_{\text{red}} + T_{\text{triv}} + \tilde{\varphi}(S'))$ as the direct sum of a reduced and a trivial QP according to the procedure described before this lemma, we will have $S_n|_I = T_{\text{red}} + T_{\text{triv}}$ for all $n \geq 1$. Therefore, the restriction of the reduced part of $(A, S)$ to $I$ is (right-equivalent to) the reduced part of the restriction of $(A, S)$ to $I$.

**Proposition 3.2.3.** Let $(A, S)$ be a QP and $I$ a subset of $Q_0$. For $i \in I$, the mutation $\mu_j(A|_I, S|_I)$ is right-equivalent to the restriction of $\mu_j(A, S)$ to $I$.

**Proof.** An easy check shows that $([S] + \triangle_i(Q))|_I = [S|_I] + \triangle_i(Q|_I)$. Therefore, the premutation $\tilde{\mu}_j(A|_I, S|_I)$ is equal to the restriction of $\tilde{\mu}_j(A, S)$ to $I$. The proposition then follows from Lemma 3.2.2. \qed

**Corollary 3.2.4.** If $(A, S)$ is a non-degenerate QP, then for every subset $I$ of $Q_0$ the restriction $(A|_I, S|_I)$ is non-degenerate as well. In other words, restriction preserves non-degeneracy.

The following lemma says that, using the operation of restriction of QPs, all the QPs we have associated to triangulations can be obtained from QPs associated to triangulations of surfaces without boundary. It will relatively simplify the proof of our first main result.

**Lemma 3.2.5.** For every QP of the form $(Q(\tau), S(\tau))$ there exists an ideal triangulation $\sigma$ of a surface with empty boundary with the following properties:

- $\sigma$ contains all the arcs of $\tau$;
3.3 First main result: Flip $\leftrightarrow$ mutation compatibility of QPs

- the restriction of $(Q(\sigma), S(\sigma))$ to $\tau$ is $(Q(\tau), S(\tau))$ (except for the fact that the arcs in $\sigma \setminus \tau$ belong to the vertex set of the restriction $(A(\sigma)|_{\tau}, S(\sigma)|_{\tau})$ as isolated vertices, but do not belong to the vertex set of $(Q(\tau), S(\tau))$; see Remark 2.3.6).

Proof. Let $\tau$ be an ideal triangulation of a surface $(\Sigma, M)$ with non-empty boundary. Each boundary component $b$ of $\Sigma$ is homeomorphic to a circle. Let $m_b$ be the number of marked points lying on $b$. If $m_b = 1$ then we can glue $\Sigma$ and a triangulated twice-punctured monogon along $b$; whereas if $m_b > 1$, we can glue $\Sigma$ and a triangulated $m_b$-gon along $b$. After doing this for each boundary component of $\Sigma$, we will end up with an ideal triangulation $\sigma$ of a surface with empty boundary and possessing the desired properties. □

3.3 First main result: Flip $\leftrightarrow$ mutation compatibility of QPs

We are ready to state and prove our first main result. Recall from Section 2.2 that we write $\sigma = f_i(\tau)$ to indicate that the ideal triangulation $\sigma$ can be obtained from the ideal triangulation $\tau$ by flipping the arc $i$.

**Theorem 3.3.1.** Let $\tau$ and $\sigma$ be ideal triangulations of $(\Sigma, M)$. If $\sigma = f_i(\tau)$, then $\mu_i(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$ are right-equivalent QPs.

Proof. By Proposition 3.2.3 and Lemma 3.2.5, we can assume, without loss of generality, that the boundary of $\Sigma$ is empty. We are going to consider several cases, taking account the configurations that $\tau$ and $\sigma$ can present around the arc $i$ to be flipped.
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Before proceeding to the case-by-case check, we describe these cases, and to do this, we recall (a slight modification of) the "puzzle-piece decomposition" mentioned in Remark 4.2 of [12].

Any ideal triangulation of $(\Sigma, M)$ can be obtained by means of the following procedure: Consider the four "puzzle pieces" shown in Figure 3.8 (a triangle, a once-punctured digon enclosing a self-folded triangle, a twice-punctured monogon enclosing two self-folded triangles, and a once-punctured digon with arcs connecting the puncture to both vertices of the digon). Take several copies of these pieces, assign an orientation to each of the outer sides of these copies and fix a partial matching of these outer sides, never matching two sides of the same copy. (In order to obtain a connected surface, any two puzzle pieces in the collection must be connected via matched pairs). Furthermore, if two sides of a triangle are oriented and matched with two oriented sides of another triangle as shown in Figure 3.9, then replace the pair of triangles with a digon as indicated in Figure 3.9 (thus obtaining a new partial matching). Then glue the puzzle pieces along the matched sides, making sure the orientations match. Though some partial matchings may not lead to an (ideal triangulation of an) oriented surface, any ideal triangulation $\tau$ of an oriented surface can be obtained from a suitable partial matching. Any partial matching giving raise to $\tau$ is called a puzzle-piece decomposition of $\tau$. 

![Figure 3.8: Puzzle pieces](image-url)
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Figure 3.9:

Now, if we have such a puzzle-piece decomposition of $\tau$, then each flip occurs either inside a puzzle piece, or involves an arc shared by two puzzle pieces. So the proof of the theorem is done by analyzing how these flips affect the corresponding QPs. Figure 3.10 shows a non-oriented list of possible matchings. Therefore, the proof of the theorem should be carried on by checking the flips that can occur inside a puzzle piece on the one hand, and on the other hand those indicated in Figure 3.10, where the sides of the puzzle pieces have to be given an orientation and glued along the bold arc, which represents the arc $i$ to be flipped.

Figure 3.10:
3.3. First main result: Flip $\leftrightarrow$ mutation compatibility of QPs

**Remark 3.3.1.**

1. Since QP-mutations are involutive up to right-equivalence, there can be some redundancies when doing this checking: for instance, if this theorem is true in case we flip the loop inside the second puzzle piece of Figure 3.8, then it automatically holds also for the flips inside the fourth puzzle piece. Another instance of this redundancy is exemplified by the matchings 3 and 5 of Figure 3.10.

2. Consider the matchings 1, 2, 3, 4, 5, 7 and 9 of Figure 3.10. Depending on the surface $(\Sigma, M)$ and the ideal triangulation $\tau$ decomposed into puzzle pieces, after gluing the corresponding pair of puzzle pieces one needs to further consider how the vertices on the boundary of the resulting small surface are matched (some of these vertices may represent the same marked point of $\tau$) because different identifications of these vertices may lead to potentials that require slight variations of the right-equivalences we show in the cases analyzed below. The cases we analyze here will have the implicit assumption that different vertices on the boundary of the small surface obtained after gluing the pair of puzzle pieces represent different marked points of $\tau$. We leave to the reader the verification of the cases where different vertices represent the same marked point.

3. If none of $\tau$ and $\sigma = f_i(\tau)$ possesses self-folded triangles, then the right-equivalence between $\mu_i(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$ can be given by choosing arrows $t(\eta) \xrightarrow{\eta} i \xrightarrow{\beta} h(\beta)$ contained in different ideal triangles of $\tau$ and multiplying $\beta^*$ and $\eta^*$ by $-1$. This fact will be used in Part II to prove Theorem 9.2.1.

Each of the cases below is divided into three stages: in the first stage, we sketch in
3.3. First main result: Flip $\leftrightarrow$ mutation compatibility of QPs

a Figure the local configuration of $\tau$ around the arc $i$ to be flipped. This configuration will be located at the left of the respective Figure. On the right side of the Figure, the reader will find the corresponding local configuration of the signed adjacency quiver $Q(\tau)$ (this is done also in order to fix some notation for the several arrows we will have to keep track of). In the second stage, we apply to $(Q(\tau), S(\tau))$ the QP-mutation with respect to the arc $i$. In the third stage we flip the arc $i$ to obtain $\sigma$ and compare the corresponding QP with the QP obtained in the second stage. The local configuration that $\sigma$ presents around the flip of $i$ is also sketched at the left of the respective Figure, and on the right side of the Figure, the local configuration of the signed adjacency quiver $Q(\sigma)$.

Having said all this, let us proceed to the case-by-case verification. For the rest of the proof, we shall label each puncture $p$ with the scalar $x_p$; we will also make use, without mentioning it, of the reduction process described after Example 3.1.7.

Case 1. (First matching of Figure 3.10, with only one side of each triangle matched to a side of the other one) Assume that, around the arc $i$, $\tau$ looks like the configuration in Figure 3.11, with $l, m, n, t > 1$ and $\alpha, \beta, \gamma, \delta, \varepsilon, \eta \notin \{a_1, \ldots, a_l, b_1, \ldots, b_m, c_1, \ldots, c_n, d_1, \ldots, d_t\}$. Let us abbreviate $a = a_1 \ldots a_l$, $b = b_1 \ldots b_m$, $c = c_1 \ldots c_n$, $d = d_1 \ldots d_t$. Then

$$S(\tau) = \alpha \beta \gamma + \delta \varepsilon \eta + w \alpha a + x \beta \eta b + y \varepsilon \gamma c + z \delta d + S'(\tau),$$

with $S'(\tau) \in R\langle\langle Q(\tau)\rangle\rangle$ involving none of the arrows $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$. If we perform the premutation $\tilde{\mu}_j$ on $(Q(\tau), S(\tau))$, we get $(\tilde{Q}(\tau), \tilde{S}(\tau))$, where $\tilde{Q}(\tau)$ is the arrow span of the quiver shown in Figure 3.12 and $\tilde{S}(\tau) = \alpha[\beta \gamma] + \delta[\varepsilon \eta] + w \alpha a + x[\beta \eta] b + y[\varepsilon \gamma] c + z \delta d + S'(\tau) + \gamma^* \beta^*[\beta \gamma] + \eta^* \varepsilon^*[\varepsilon \eta] + \eta^* \beta^*[\beta \eta] + \gamma^* \varepsilon^*[\varepsilon \gamma] \in R\langle\langle \tilde{Q}(\tau)\rangle\rangle$. The $R$-algebra
automorphism $\varphi$ of $R(\langle Q(\tau) \rangle)$ whose action on the arrows is given by

$$\alpha \mapsto \alpha - \gamma^* \beta^*, \ [\beta \gamma] \mapsto [\beta \gamma] - wa, \ \delta \mapsto \delta - \eta^* \varepsilon^*, \ [\varepsilon \eta] \mapsto [\varepsilon \eta] - zd,$$
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and the identity in the rest of the arrows, sends $\tilde{S}(\tau)$ to

$$
\varphi(\tilde{S}(\tau)) = \alpha[\beta\gamma] + \delta[\varepsilon\eta] - w\gamma^*\beta^*a + x[\beta\eta]b + y[\varepsilon\gamma]c - z\eta^*\varepsilon^*d + S'(\tau) + \eta^*\beta^*\beta\eta + \gamma^*\varepsilon^*\varepsilon\gamma.
$$

Therefore, the reduced part $\mu_j(Q(\tau), S(\tau))$ of $(\tilde{Q}(\tau), \varphi(\tilde{S}(\tau)))$ is (up to right-equivalence)

the QP on the arrow span $\tilde{Q}(\tau)$ obtained from $\tilde{Q}(\tau)$ by deleting the arrows $\alpha, [\beta\gamma], \delta$

and $[\varepsilon\eta]$, with $\varphi(\tilde{S}(\tau)) - \alpha[\beta\gamma] - \delta[\varepsilon\eta]$ as its potential.

On the other hand, $\sigma = f_i(\tau)$ and its quiver $Q(\sigma)$ look as Figure 3.13, and $S(\sigma) =$

Figure 3.13: Case 1 (1st matching), flip $\sigma = f_i(\tau)$

$$
w\gamma^*\beta^*a + x[\beta\eta]b + y[\varepsilon\gamma]c + z\eta^*\varepsilon^*d + \eta^*\beta^*\beta\eta + \gamma^*\varepsilon^*\varepsilon\gamma + S'(\sigma), \text{ with } S'(\sigma) = S'(\tau).
$$

Thus the $R$-algebra isomorphism $\psi : R\langle\langle Q(\tau)\rangle\rangle \to R\langle\langle Q(\sigma)\rangle\rangle$ whose action on the arrows is given by

$$
\beta^* \mapsto -\beta^*, \eta^* \mapsto -\eta^*,
$$

and the identity in the rest of the arrows, is a right-equivalence between $\mu_j(Q(\tau), S(\tau))$
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and \((Q(\sigma), S(\sigma))\).

Case 2. (First matching of Figure 3.10, with exactly two sides of each triangle matched) Assume that, around the arc \(i\), \(\tau\) looks like the configuration shown in Figure 3.14. Then

**Figure 3.14: Case 2 (1st matching), configuration of \(\tau\) around \(i\)**

\[
S(\tau) = \beta\gamma\alpha + \epsilon\eta\delta + ba_1a_1 + cd_1d_1 + w\beta a\gamma\delta + y\varepsilon d\eta\alpha + S'(\tau),
\]

with \(S'(\tau) \in R(\langle Q \rangle)\) involving none of the arrows \(\alpha, \beta, \gamma, \delta, \varepsilon, \eta\). If we perform the premutation \(\tilde{\mu}_j\) on \((Q(\tau), S(\tau))\), we get \((\tilde{Q}(\tau), \tilde{S}(\tau))\), where \(\tilde{Q}(\tau)\) is the arrow span of the quiver shown in Figure 3.15, and \(S(\tau) = \beta[\gamma\alpha] + \varepsilon[\eta\delta] + ba_1a_1 + cd_1d_1 + w\beta a[\gamma\delta] + y\varepsilon d[\eta\alpha] + S'(\tau) + \alpha^*\gamma^*[\gamma\alpha] + \delta^*\eta^*[\eta\delta] + \delta^*\gamma^*[\gamma\delta] + \alpha^*\eta^*[\eta\alpha] \in R(\langle \tilde{Q}(\tau) \rangle).\)
The $R$-algebra automorphism $\varphi$ of $R(\langle \tilde{Q}(\tau) \rangle)$ whose action on the arrows is given by

$$\beta \mapsto \beta - \alpha^* \gamma^*, \, [\gamma \alpha] \mapsto [\gamma \alpha] - wa[\gamma \delta], \, \varepsilon \mapsto \varepsilon - \delta^* \eta^*, \, [\eta \delta] \mapsto [\eta \delta] - yd[\eta \alpha],$$

and the identity in the rest of the arrows, sends $\tilde{S}(\tau)$ to

$$\varphi(\tilde{S}(\tau)) = \beta[\gamma \alpha] + \varepsilon[\eta \delta] + ba_i a_1 + cd_i d_1 - w\alpha^* \gamma^* a[\gamma \delta] - yd^* \eta^* d[\eta \alpha] + S'(\tau) + \delta^* \gamma^* [\gamma \delta] + \alpha^* \eta^* [\eta \alpha].$$

Therefore, the reduced part $\mu_3(Q(\tau), S(\tau))$ of $(\tilde{Q}(\tau), \tilde{S}(\tau))$ is (up to right-equivalence) the QP on the arrow span $\overrightarrow{Q(\tau)}$ obtained from $\overrightarrow{Q(\tau)}$ by deleting the arrows $\beta$, $[\gamma \alpha]$, $\varepsilon$ and $[\eta \delta]$, with $\varphi(\tilde{S}(\tau)) - \beta[\gamma \alpha] + \varepsilon[\eta \delta]$ as its potential.
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On the other hand $\sigma = f_i(\tau)$ and its quiver $Q(\sigma)$ look as Figure 3.16, and $S(\sigma) =$

Figure 3.16: Case 2 (1st matching), flip $\sigma = f_i(\tau)$

$$ba_1a_1 + cd_t d_1 + \delta^* \gamma^* [\gamma \delta] + \alpha^* \eta^* [\eta \alpha] + w \alpha^* \gamma^* a [\gamma \delta] + y \delta^* \eta^* d [\eta \alpha] + S'(\sigma),$$

with $S'(\sigma) = S'(\tau)$. Thus the $R$-algebra isomorphism $\psi : R(\langle\langle Q(\tau)\rangle\rangle) \to R(\langle\langle Q(\sigma)\rangle\rangle)$ whose action on the arrows is given by

$$\gamma^* \mapsto -\gamma^*, \; \delta^* \mapsto -\delta^*,$$

and the identity in the rest of the arrows, is a right-equivalence between $\mu_j(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$.

Case 3. (Second matching of Figure 3.10) Assume that, around the arc $i$, $\tau$ looks like the configuration in Figure 3.17, with the arc $k$ not enclosing a self-folded triangle. Let us abbreviate $a = a_1 \ldots a_t, \; b = b_1 \ldots b_m, \; d = d_1 \ldots d_t$. Then
3.3. First main result: Flip ↔ mutation compatibility of QPs

Figure 3.17: Case 3 (2\textsuperscript{nd} matching), configuration of $\tau$ around $i$

$$S(\tau) = \alpha\beta\gamma + \eta\nu\xi - \frac{\alpha\delta\varepsilon}{z} + x\alpha\eta a + w\nu b + y\xi\delta\varepsilon dS'(\tau),$$

with $S'(\tau) \in R\langle\langle Q(\tau)\rangle\rangle$ involving none of the arrows $\alpha, \beta, \gamma, \delta, \varepsilon, \eta, \nu, \xi$. If we perform the premutation $\tilde{\mu}_j$ on $(Q(\tau), S(\tau))$, we get $(\tilde{Q}(\tau), \tilde{S}(\tau))$, where $\tilde{Q}(\tau)$ is the arrow span of the quiver shown in Figure 3.18 and $\tilde{S}(\tau) = [\alpha\beta]\gamma + [\xi\eta]\nu - \frac{\alpha\delta}{z} + x[\alpha\eta]a + w\nu b +$

Figure 3.18: Case 3, QP-mutation process $\mu_j(Q(\tau), S(\tau))$

$$yd[\xi\delta]\varepsilon + S'(\tau) + [\alpha\beta]\beta^*\alpha^* + [\alpha\eta]\eta^*\alpha^* + [\alpha\delta]\delta^*\alpha^* + [\xi\beta]\beta^*\xi^* + [\xi\delta]\delta^* + [\xi\eta]\eta^*\xi \in R\langle\langle \tilde{Q}(\tau)\rangle\rangle.$$
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The $R$-algebra automorphism $\varphi$ of $R\langle\langle Q(\tau)\rangle\rangle$ whose action on the arrows is given by

\[ \gamma \mapsto \gamma - \beta^*\alpha^*, \quad [\xi\eta] \mapsto [\xi\eta] - \omega b, \quad \nu \mapsto \nu - \eta^*\xi^*, \quad [\alpha\delta] \mapsto [\alpha\delta] - yzd[\xi\delta], \quad \varepsilon \mapsto \varepsilon - z\delta^*\alpha^*, \]

and the identity in the rest of the arrows, sends $S(\tau)$ to

\[ \varphi(S(\tau)) = [\alpha\beta]\gamma + [\xi\eta]\nu - \frac{[\alpha\delta][\varepsilon]}{z} + x[\alpha\eta]a + S'(\tau) + [\alpha\eta]\eta^*\alpha^* + yzd[\xi\delta]\delta^*\alpha^* + [\xi\beta]\beta^*\xi^* + [\xi\delta]\delta^*\xi^* - w\eta^*\xi^*. \]

Therefore, the reduced part $\mu_j(Q(\tau), S(\tau))$ of $(\widetilde{Q(\tau)}, \varphi(S(\tau)))$ is (up to right-equivalence) the QP on the arrow span $\overline{Q(\tau)}$ obtained from $\widetilde{Q(\tau)}$ by deleting the arrows $\gamma, \varepsilon, \nu, [\alpha\beta], [\alpha\delta]$ and $[\xi\eta]$, with $\varphi(S(\tau)) - [\alpha\beta]\gamma - [\xi\eta]\nu + \frac{[\alpha\delta][\varepsilon]}{z}$ as its potential.

On the other hand, $\sigma = f_i(\tau)$ and its quiver $Q(\sigma)$ look as Figure 3.19, and

Figure 3.19: Case 3 (2nd matching), flip $\sigma = f_i(\tau)$

$S(\sigma) = [\alpha\eta]\eta^*\alpha^* + [\xi\beta]\beta^*\xi^* - \frac{[\xi\delta]\delta^*\xi^*}{z} + x[\alpha\eta]a + \omega\eta^*\xi^*b + y[\xi\delta]\delta^*\alpha^*d + S'(\sigma),$ with $S'(\sigma) = S'(\tau)$. Thus the $R$-algebra isomorphism $\psi : R\langle\langle Q(\tau)\rangle\rangle \rightarrow R\langle\langle Q(\sigma)\rangle\rangle$ whose
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action on the arrows is given by

$$\alpha^* \mapsto -\alpha^*, \delta^* \mapsto -\delta^*, \eta^* \mapsto -\eta^*, [\xi \delta] \mapsto \frac{[\xi \delta]}{z},$$

and the identity in the rest of the arrows, is a right-equivalence between $\mu_j(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$.

Case 4. (Third and fifth matchings of Figure 3.10) Assume that, around the arc $i$, $\tau$ looks like the configuration in Figure 3.20, with $l > 1$ and none of the arcs $j$ and $k$ enclosing a self-folded triangle. Let us abbreviate $c = c_1 \ldots c_l$, $d = d_1 \ldots d_t$. Then

$$S(\tau) = \alpha \beta \gamma + \delta \eta \varepsilon + \frac{\nu \psi \xi}{yz} - \frac{\nu \varepsilon b}{y} - \frac{\delta \psi a}{z} + \frac{x \nu \beta \alpha \psi \xi + wc \gamma}{S'(\tau)},$$

with $S'(\tau) \in R(\langle Q(\tau) \rangle)$ involving none of the arrows $\alpha, \beta, \gamma, \delta, \varepsilon, \eta, \nu, \psi, \xi, a, b$. If we
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perform the premutation \( \tilde{\mu}_j \) on \((Q(\tau), S(\tau))\), we get \((\tilde{Q}(\tau), \tilde{S}(\tau))\), where \( \tilde{Q}(\tau) \) is the arrow span of the quiver shown in Figure 3.21 and \( \tilde{S}(\tau) = [\alpha \beta] \gamma + [\delta \varepsilon] \eta + \frac{[\nu \psi] \xi}{yz} - \)

\[ \frac{[\nu \psi] b}{y} - \frac{[\delta \psi] a}{z} + x[\nu \beta] d[\alpha \psi] \xi + wc \gamma + S'(\tau) + [\alpha \beta] \beta^* \alpha^* + [\alpha \varepsilon] \varepsilon^* \alpha^* + [\alpha \psi] \psi^* \alpha^* + [\delta \beta] \beta^* \delta^* + [\delta \varepsilon] \varepsilon^* \delta^* + [\delta \psi] \psi^* \delta^* + [\nu \beta] \beta^* \nu^* + [\nu \varepsilon] \varepsilon^* \nu^* + [\nu \psi] \psi^* \nu^* \in R(\langle \tilde{Q}(\tau) \rangle). \] The \( R \)-algebra automorphism \( \varphi \) of \( R(\langle \tilde{Q}(\tau) \rangle) \) whose action on the arrows is given by

\[ [\alpha \beta] \mapsto [\alpha \beta] - wc, \ \gamma \mapsto \gamma - \beta^* \alpha^*, \ \eta \mapsto \eta - \varepsilon^* \delta^*, \ [\nu \psi] \mapsto [\nu \psi] - xyz[\nu \beta] d[\alpha \psi], \]

\[ \xi \mapsto \xi - yz \psi^* \nu^*, \ b \mapsto b + y \varepsilon^* \nu^*, \ a \mapsto a + z \psi^* \delta^*, \]
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and the identity in the rest of the arrows, sends \( S(\tau) \) to

\[
\varphi(\widehat{S}(\tau)) = [\alpha \beta] \gamma + [\delta \varepsilon] \eta + \frac{[\nu \psi] \xi}{yz} - \frac{[\nu \varepsilon] b}{y} - \frac{[\delta \psi] a}{z} + S'(\tau) - wc \beta^* \alpha^* + [\alpha \varepsilon] \varepsilon^* \alpha^* + \]

\[
[\alpha \psi] \psi^* \alpha^* + [\delta \beta] \beta^* \delta^* + [+[\nu \beta] \beta^* \nu^* - xyz [\nu \beta] d[\alpha \psi] \psi^* \nu^*].
\]

Therefore, the reduced part \( \mu_j(Q(\tau), S(\tau)) \) of \( (\widehat{Q}(\tau), \varphi(\widehat{S}(\tau))) \) is (up to right-equivalence) the QP on the arrow span \( \widehat{Q}(\tau) \) obtained from \( \widehat{Q}(\tau) \) by deleting the arrows \( [\alpha \beta], \gamma, [\delta \varepsilon], \eta, [\nu \psi], \xi, [\nu \varepsilon], b, [\delta \psi] \) and \( a \), with \( \varphi(\widehat{S}(\tau)) - [\alpha \beta] \gamma - [\delta \varepsilon] \eta - \frac{[\nu \psi] \xi}{yz} + \frac{[\nu \varepsilon]}{y} + \frac{[\delta \psi] a}{z} \) as its potential.

On the other hand, \( \sigma = f_i(\tau) \) and its quiver \( Q(\sigma) \) look as Figure 3.22, and

\[
S(\sigma) = [\alpha \varepsilon] \varepsilon^* \alpha^* + [\delta \beta] \beta^* \delta^* - \frac{[\nu \beta] \beta^* \nu^*}{y} - \frac{[\alpha \psi] \psi^* \alpha^*}{z} + x [\nu \beta] d[\alpha \psi] \psi^* \nu^* + w \beta^* \alpha^* c S'(\sigma),
\]

with \( S'(\sigma) = S'(\tau) \). Thus the \( R \)-algebra isomorphism \( \psi : R(\langle (Q(\tau)) \rangle) \to R(\langle Q(\sigma) \rangle) \)
3.3. First main result: Flip ↔ mutation compatibility of QPs

whose action on the arrows is given by

\[ \beta^* \mapsto -\beta^* \quad [\alpha \psi] \mapsto -\frac{[\alpha \psi]}{z}, \quad [\nu \beta] \mapsto \frac{\nu \beta}{y}, \]

and the identity in the rest of the arrows, is a right-equivalence between \( \mu_j(Q(\tau), S(\tau)) \) and \( (Q(\sigma), S(\sigma)) \).

Case 5. (Fourth matching of Figure 3.10) Assume that, around the arc \( i, \tau \) looks like the configuration shown in Figure 3.23, with \( m, n > 0, \ t > 1 \). Then (again

Figure 3.23: Case 5 (4 matching), configuration of \( \tau \) around \( i \)

abbreviating \( b = b_1 \ldots b_m, c = c_1 \ldots c_n, d = d_1 \ldots d_t \)

\[ S(\tau) = \delta \varepsilon \eta - \frac{\xi \alpha \beta \gamma}{w} + x \alpha \beta \eta b + y \varepsilon \gamma \xi c + z \delta d + S'(\tau), \]

with \( S'(\tau) \in R(\langle Q(\tau) \rangle) \) involving none of the arrows \( \alpha, \beta, \gamma, \delta, \varepsilon, \eta, \xi \). If we perform
3.3. First main result: Flip ↔ mutation compatibility of QPs

the premutation $\tilde{\mu}_j$ on $(Q(\tau), S(\tau))$, we get $(\tilde{Q}(\tau), \tilde{S}(\tau))$, where $\tilde{Q}(\tau)$ is the arrow span of the quiver shown in Figure 3.24, and $\tilde{S}(\tau) = \delta[\varepsilon \eta] - \frac{\xi \alpha[\beta \gamma]}{w} + x \alpha[\beta \eta]b + y[\varepsilon \gamma] \xi c + \mu_j(Q(\tau), S(\tau))$

Figure 3.24: Case 5, QP-mutation process $\mu_j(Q(\tau), S(\tau))$

$$z \delta d + S'(\tau) + \eta^* \varepsilon^*[\varepsilon \eta] + \gamma^* \varepsilon^*[\varepsilon \gamma] + \gamma^* \beta^*[\beta \gamma] + \eta^* \beta^*[\beta \eta] \in R(\langle \tilde{Q}(\tau) \rangle).$$

The $R$-algebra automorphism $\varphi$ of $R(\langle \tilde{Q}(\tau) \rangle)$ whose action on the arrows is given by

$$\delta \mapsto \delta - \eta^* \varepsilon^*, \ [\varepsilon \eta] \mapsto [\varepsilon \eta] - zd,$$

and the identity in the rest of the arrows, sends $\tilde{S}(\tau)$ to

$$\varphi(\tilde{S}(\tau)) = \delta[\varepsilon \eta] - \frac{\xi \alpha[\beta \gamma]}{w} + x \alpha[\beta \eta]b + y[\varepsilon \gamma] \xi c - \eta^* \varepsilon^* d + S'(\tau) + \gamma^* \varepsilon^*[\varepsilon \gamma] + \gamma^* \beta^*[\beta \gamma] + \eta^* \beta^*[\beta \eta].$$

Therefore, the reduced part $\mu_j(Q(\tau), S(\tau))$ of $(\tilde{Q}(\tau), \tilde{S}(\tau))$ is (up to right-equivalence) the QP on the arrows span $\overline{Q(\tau)}$ obtained from $\tilde{Q}(\tau)$ by deleting the arrows $\delta$ and $[\varepsilon \eta]$, with $\varphi(\tilde{S}(\tau)) - \delta[\varepsilon \eta]$ as its potential.
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On the other hand, $\sigma = f_i(\tau)$ and its quiver $Q(\sigma)$ look as Figure 3.25, and

\[ S(\sigma) = \xi \alpha [\beta \gamma] + \eta^* \beta^* [\beta \eta] + \gamma^* \varepsilon^* [\varepsilon \gamma] + w \gamma^* \beta^* [\beta \gamma] + x \alpha [\beta \eta] b + y [\varepsilon \gamma] \xi c + z \eta^* \varepsilon^* d + S'(\sigma), \]

with $S'(\sigma) = S'(\tau)$. Thus, the $R$-algebra isomorphism $\psi : R\langle\langle Q(\tau) \rangle\rangle \to R\langle\langle Q(\sigma) \rangle\rangle$ whose action on the arrows is given by

\[ \alpha \mapsto -\alpha, \ \beta^* \mapsto -\beta^*, \ \gamma \mapsto -\gamma^*, \ \varepsilon^* \mapsto -\varepsilon^*, \ [\beta \gamma] \mapsto w [\beta \gamma], \ [\beta \eta] \mapsto -[\beta \eta], \]

and the identity in the rest of the arrows, is a right-equivalence between $\mu_j (Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$.

Case 6. (Fifth and eighth matchings of Figure 3.10) Assume that, around the arc $i$, $\tau$ looks like the configuration in Figure 3.26, with the arc $k$ not enclosing a self-folded triangle. Let us abbreviate $c = c_1 \ldots c_n$. Then
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Figure 3.26: Case 6 (5th and 8th matchings), configuration of \( \tau \) around \( i \)

\[
S(\tau) = \alpha \beta \gamma + \frac{\eta \delta \varepsilon}{xy} - \frac{\eta \beta b}{y} - \frac{\alpha \delta a}{x} - \frac{\xi \psi \nu \rho}{z} + w \eta \psi \nu \varepsilon \xi \delta \varepsilon + S'(\tau),
\]

with \( S'(\tau) \in R\langle\langle Q(\tau)\rangle\rangle \) involving none of the arrows \( \alpha, \beta, \gamma, \delta, \varepsilon, \eta, \nu, \rho, \psi, \xi, a, b \). If we perform the premutation \( \tilde{\mu}_j \) on \( (Q(\tau), S(\tau)) \), we get \( (\tilde{Q}(\tau), \tilde{S}(\tau)) \), where \( \tilde{Q}(\tau) \)
is the arrow span of the quiver shown in Figure 3.33 and \( \tilde{S}(\tau) = [\alpha \beta] \gamma + \frac{[\eta \delta] \varepsilon}{xy} - \frac{[\eta \beta b]}{x} - \frac{[\alpha \delta a]}{y} - \frac{[\xi \psi] \mu \rho}{z} + w \eta \psi \nu \varepsilon \xi \delta \varepsilon + S'(\tau) + [\alpha \beta] \beta^* \alpha^* + [\alpha \delta] \delta^* \alpha^* + [\alpha \psi] \psi^* \alpha^* [\eta \beta] \beta^* \eta^* + [\eta \delta] \delta^* \eta^* + \eta \psi \xi \eta^* \xi \beta^* \xi^* + \xi \delta \xi^* \varepsilon + [\xi \psi] \psi^* \xi^* + \in R\langle\langle \tilde{Q}(\tau) \rangle\rangle \). The \( R \)-algebra automorphism \( \varphi \) of \( R\langle\langle \tilde{Q}(\tau) \rangle\rangle \) whose action on the arrows is given by

\[
\gamma \mapsto \gamma - \beta^* \alpha^*, \ [\eta \delta] \mapsto [\eta \delta] - wxy[\eta \psi] \nu \varepsilon \rho [\xi \delta], \ \varepsilon \mapsto \varepsilon - xy \delta^* \eta^*,
\]

\[
a \mapsto a + y \delta^* \alpha, \ b \mapsto b + x \beta^* \eta^*,
\]
and the identity in the rest of the arrows, sends $\widehat{S(\tau)}$ to
\[
\varphi(\widehat{S(\tau)}) = [\alpha\beta]\gamma + \frac{[\eta\delta]c}{xy} - \frac{[\eta\beta]b}{x} - \frac{[\alpha\delta]a}{y} - \frac{[\xi\psi]\nu\rho}{z} + S'(\tau) + [\alpha\psi]^*\alpha^* - wxy[\eta\psi]\nu\rho[\xi\delta]\delta^*\eta^* + [\eta\psi]^*\eta^* + [\xi\beta]^*\xi^* + [\xi\delta]^*\xi^* + [\xi\psi]^*\xi^*.
\]

Therefore, the reduced part $\mu_j(Q(\tau), S(\tau))$ of $(\widehat{Q(\tau)}, \varphi(\widehat{S(\tau)}))$ is (up to right-equivalence) the QP on the arrow span $\overline{Q(\tau)}$ obtained from $\overline{Q(\tau)}$ by deleting the arrows $[\alpha\beta], \gamma, [\eta\delta], \varepsilon, [\eta\beta], b, [\alpha\delta]$ and $a$, with $\varphi(\widehat{S(\tau)}) - [\alpha\beta]\gamma - \frac{[\eta\delta]c}{xy} + \frac{[\eta\beta]b}{x} + \frac{[\alpha\delta]a}{y}$ as its potential.

On the other hand, $\sigma = f_i(\tau)$ and its quiver $Q(\sigma)$ look as Figure 3.28, and $S(\sigma) = [\alpha\psi]^*\alpha^* + [\xi\beta]^*\xi^* + [\xi\psi]^*\nu\rho - \frac{|\eta\psi|\nu\rho}{x} - \frac{|\xi\delta|^*\xi^*}{y} + z[\xi\psi]^*\xi^* + w[\eta\psi]\nu\rho[\xi\delta]\delta^*\eta^* + S'(\sigma)$, with $S'(\sigma) = S'(\tau)$. Thus the $R$-algebra isomorphism $\psi : R\langle\langle Q(\tau)\rangle\rangle \to R\langle\langle Q(\sigma)\rangle\rangle$ whose action on the arrows is given by
\[
\alpha^* \mapsto -\alpha^*, \quad \psi^* \mapsto -\psi^*, \quad [\eta\psi] \mapsto \frac{[\eta\psi]}{x}, \quad [\xi\delta] \mapsto -\frac{[\xi\delta]}{y}, \quad [\xi\psi] \mapsto -z[\xi\psi]
\]
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Figure 3.28: Case 6 (5th and 8th matchings), flip σ = f_i(τ)

and the identity in the rest of the arrows, is a right-equivalence between μ_j(Q(τ), S(τ)) and (Q(σ), S(σ)).

Case 7. (Sixth matching of Figure 3.10) Assume that, around i, the triangulation τ looks like the configuration shown in Figure 3.29. Then

Figure 3.29: Case 7 (6 matching), configuration of τ around i
3.3. First main result: Flip $\leftrightarrow$ mutation compatibility of QPs

$$S(\tau) = \alpha \beta \gamma + \frac{\delta \varepsilon \eta}{xy} + \psi \omega \nu - \frac{c \varepsilon \gamma}{x} - \frac{d \beta \eta}{y} - \frac{\xi \varphi \nu}{z} + w \delta \varepsilon \nu b \ldots a \xi \varphi \eta + S'(\tau),$$

with $S'(\tau) \in R\langle\langle Q(\tau) \rangle\rangle$ involving none of the arrows $\alpha, \beta, \gamma, \delta, \varepsilon, \eta, \nu, \varphi, \psi, \omega, \xi, c$ and $d$. If we perform the premutation $\tilde{\mu}_j$ on $(Q(\tau), S(\tau))$ we get $(\tilde{Q}(\tau), \tilde{S}(\tau))$, where $\tilde{Q}(\tau)$ is the arrow span of the quiver shown in Figure 3.30, and

$$\tilde{S}(\tau) = \alpha [\beta \gamma] + \frac{\delta [\varepsilon \eta]}{xy} + \psi [\omega \nu] -$$

Figure 3.30: Case 7, QP-mutation process $\mu_j(Q(\tau), S(\tau))$

$$\frac{c [\varepsilon \gamma]}{x} - \frac{d [\beta \eta]}{y} - \frac{\xi [\varphi \nu]}{z} + w \delta [\varepsilon \nu] b \ldots a \xi [\varphi \eta] + S'(\tau) + \gamma^* \beta^* [\beta \gamma] + \gamma^* \varepsilon^* [\varepsilon \gamma] + \gamma^* \varphi^* [\varphi \gamma] + \gamma^* \omega^* [\omega \gamma] + \eta^* \beta^* [\beta \eta] + \eta^* \varepsilon^* [\varepsilon \eta] + \eta^* \varphi^* [\varphi \eta] + \eta^* \omega^* [\omega \eta] + \nu^* \beta^* [\beta \nu] + \nu^* \varepsilon^* [\varepsilon \nu] + \nu^* \varphi^* [\varphi \nu] + \nu^* \omega^* [\omega \nu].$$

The $R$-algebra automorphism $\varphi_1$ of $R\langle\langle \tilde{Q}(\tau) \rangle\rangle$ whose action on the arrows is given by

$$\alpha \mapsto \alpha - \gamma^* \beta^*, \ \delta \mapsto \delta - x y \eta^* \varepsilon^*, \ [\varepsilon \eta] \mapsto [\varepsilon \eta] - w x y [\varepsilon \nu] b \ldots a \xi [\varphi \eta],$$

$$\psi \mapsto \psi - \nu^* \omega^*, \ c \mapsto c + x \gamma^* \varepsilon^*, \ d \mapsto d + y \eta^* \beta^*, \ \xi \mapsto \xi + z \nu^* \varphi^*,$$
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and the identity in the rest of the arrows, sends \( \overline{S(\tau)} \) to

\[
\varphi_1(\overline{S(\tau)}) = \alpha[\beta\gamma] + \frac{\delta[\varepsilon\eta]}{xy} + \psi[\omega\nu] - \frac{c[\varepsilon\gamma]}{x} - \frac{d[\beta\eta]}{y} - \frac{\xi[\varphi\nu]}{z} + wz\delta[\varepsilon\nu]b \ldots av^*\varphi^*[\varphi\eta] \\
-wxyz\eta^*\varepsilon^*\varepsilon\nu b \ldots av^*\varphi^*[\varphi\eta] + S'(\tau) + \gamma^*\varphi^*[\varphi\gamma] + \gamma^*\omega^*\omega\gamma - wxyz\eta^*\varepsilon^*\varepsilon\nu b \ldots a\xi[\varphi\eta] \\
+ \eta^*\varphi^*[\varphi\eta] + \eta^*\omega^*\omega\eta + \nu^*\beta^*[\beta\nu] + \nu^*\varepsilon^*\varepsilon\nu,
\]

which is cyclically equivalent to

\[
T = \alpha[\beta\gamma] + \frac{\delta[\varepsilon\eta]}{xy} + \psi[\omega\nu] - \frac{c[\varepsilon\gamma]}{x} - \frac{d[\beta\eta]}{y} - \frac{\xi[\varphi\nu]}{z} + wz\delta[\varepsilon\nu]b \ldots av^*\varphi^*[\varphi\eta] \\
-wxyz\eta^*\varepsilon^*\varepsilon\nu b \ldots av^*\varphi^*[\varphi\eta] + S'(\tau) + \gamma^*\varphi^*[\varphi\gamma] + \gamma^*\omega^*\omega\gamma - wxyz\xi[\varphi\eta]\eta^*\varepsilon^*\varepsilon\nu b \ldots a \\
+ \eta^*\varphi^*[\varphi\eta] + \eta^*\omega^*\omega\eta + \nu^*\beta^*[\beta\nu] + \nu^*\varepsilon^*\varepsilon\nu,
\]

which in turn is sent to

\[
\varphi_2(T) = \alpha[\beta\gamma] + \frac{\delta[\varepsilon\eta]}{xy} + \psi[\omega\nu] - \frac{c[\varepsilon\gamma]}{x} - \frac{d[\beta\eta]}{y} - \frac{\xi[\varphi\nu]}{z} - wxyz\eta^*\varepsilon^*\varepsilon\nu b \ldots av^*\varphi^*[\varphi\eta] \\
+S'(\tau) + \gamma^*\varphi^*[\varphi\gamma] + \gamma^*\omega^*\omega\gamma + \eta^*\varphi^*[\varphi\eta] + \eta^*\omega^*\omega\eta + \nu^*\beta^*[\beta\nu] + \nu^*\varepsilon^*\varepsilon\nu
\]

by the \( R \)-algebra automorphism \( \varphi_2 \) of \( R(\overline{Q(\tau)}) \) whose action on the arrows is given by

\[
[\varepsilon\eta] \mapsto [\varepsilon\eta] - wxyz[\varepsilon\nu]b \ldots av^*\varphi^*[\varphi\eta], [\varphi\eta] \mapsto [\varphi\eta] - wxyz[\varphi\eta]\eta^*\varepsilon^*\varepsilon\nu b \ldots a.
\]

Therefore, the reduced part \( \mu_2(Q(\tau), S(\tau)) \) of \( (\overline{Q(\tau)}, \overline{S(\tau)}) \) is (up to right-equivalence) the QP on the arrows span \( \overline{Q(\tau)} \) obtained from \( \overline{Q(\tau)} \) by deleting the arrows \( \alpha, [\beta\gamma], \delta, [\varepsilon\eta], \psi, [\omega\nu], c, [\varepsilon\gamma], d, [\beta\eta], \xi \) and \( [\varphi\nu] \), with \( \varphi_2(T) = \alpha[\beta\gamma] - \frac{\delta[\varepsilon\eta]}{xy} - \psi[\omega\nu] + \frac{c[\varepsilon\gamma]}{x} + \)
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\[ \frac{d[\beta \eta]}{y} + \frac{\xi[\varphi \eta]}{z} \]
as its potential.

On the other hand, $\sigma = f_i(\tau)$ and its quiver $Q(\sigma)$ look as Figure 3.31, and $S(\sigma) =

\text{Figure 3.31: Case 7 (6 matching), flip } \sigma = f_i(\tau)

\[ \gamma^*\omega^*[\omega \gamma] + \frac{\eta^*\varphi^*[\varphi \eta]}{yz} + \nu^*\beta^*[\beta \nu] - \frac{\nu^*\varphi^*[\varphi \eta]}{x} - \frac{\eta^*\omega^*[\omega \eta]}{y} - \frac{\gamma^*\varphi^*[\varphi \gamma]}{z} + w\eta^*\varepsilon^*[\varepsilon \nu]b \ldots a
\]
\[ + \varphi^*[\varphi \eta] + S'(\sigma), \text{ with } S'(\sigma) = S'(\tau). \]

Thus, the $R$-algebra isomorphism $\psi : R\langle\langle Q(\tau)\rangle\rangle \rightarrow R\langle\langle Q(\sigma)\rangle\rangle$ whose action on the arrows is given by

\[ \beta^* \mapsto -x\beta^*, \eta^* \mapsto -\frac{\eta^*}{y}, \nu^* \mapsto -\frac{\nu^*}{x}, \varphi^* \mapsto -\frac{\varphi^*}{z} \]

and the identity in the rest of the arrows, is a right-equivalence between $\mu_j(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$.

Case 8. (Seventh matching of Figure 3.10) Assume that, around the arc $i$, $\tau$ looks like the configuration in Figure 3.32, with none of the arcs $j$ and $k$ enclosing a self-folded triangle. Let us abbreviate $a = a_1 \ldots a_l$, $c = c_1 \ldots c_n$. Then

\[ S(\tau) = \alpha\beta\gamma - \frac{\alpha \delta \varepsilon}{x} - \frac{\eta \nu \psi \xi}{y} + w\eta \varepsilon \alpha \xi + z\alpha \nu \psi c S'(\tau), \]
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Figure 3.32: Case 8 (7th matching), configuration of \( \tau \) around \( i \)

with \( S'(\tau) \in R(\langle Q(\tau) \rangle) \) involving none of the arrows \( \alpha, \beta, \gamma, \delta, \varepsilon, \eta, \nu, \psi, \xi \). If we perform the premutation \( \tilde{\mu}_j \) on \( (Q(\tau), S(\tau)) \), we get \( (\tilde{Q}(\tau), \tilde{S}(\tau)) \), where \( \tilde{Q}(\tau) \) is the arrow span of the quiver shown in Figure 3.33 and \( \tilde{S}(\tau) = [\alpha\beta]\gamma - \frac{[\alpha\delta]\varepsilon}{x} - \frac{[\eta\nu]\psi}{y} + \)

Figure 3.33: Case 8, QP-mutation process \( \mu_j(Q(\tau), S(\tau)) \)
3.3. First main result: Flip $\leftrightarrow$ mutation compatibility of QPs

$$w[\eta\delta]a\xi + z[\alpha\nu]\psi c + S'(\tau) + [\alpha\beta]\beta^*\alpha^* + [\alpha\delta]\delta^*\alpha^* + [\alpha\nu][\nu^*\alpha^* + [\eta\beta]\beta^*\eta^* + [\eta\delta]\delta^*\eta^* + [\eta\nu][\nu^*\eta^* \in R\langle\langle Q(\tau)\rangle\rangle].$$ The $R$-algebra automorphism $\varphi$ of $R\langle\langle Q(\tau)\rangle\rangle$ whose action on the arrows is given by

$$\gamma \mapsto \gamma - \beta^*\alpha^*, \quad [\alpha\delta] \mapsto [\alpha\delta] + wxa\xi[\delta\eta], \quad \varepsilon \mapsto \varepsilon + x\delta^*\alpha^*,$$

and the identity in the rest of the arrows, sends $\widehat{S(\tau)}$ to

$$\varphi(\widehat{S(\tau)}) = [\alpha\beta]\gamma - \frac{[\alpha\delta]\varepsilon}{x} - \frac{[\eta\nu]\psi\xi}{y} + z[\alpha\nu]\psi c + S'(\tau) + wxa\xi[\eta\delta]\delta^*\alpha^* + [\alpha\nu][\nu^*\alpha^* + [\eta\beta]\beta^*\eta^* + [\eta\delta]\delta^*\eta^* + [\eta\nu][\nu^*\eta^*].$$

Therefore, the reduced part $\mu_j(Q(\tau), S(\tau))$ of $(\widehat{Q(\tau)}, \varphi(\widehat{S(\tau)}))$ is (up to right-equivalence) the QP on the arrow span $\overline{Q(\tau)}$ obtained from $\widehat{Q(\tau)}$ by deleting the arrows $[\alpha\beta], \gamma, [\alpha\delta]$ and $\varepsilon$, with $\varphi(\widehat{S(\tau)}) - [\alpha\beta]\gamma + \frac{[\alpha\delta]\varepsilon}{x}$ as its potential.

On the other hand, $\sigma = f_i(\tau)$ and its quiver $Q(\sigma)$ look as Figure 3.34, and $S(\sigma) =$

Figure 3.34: Case 8 (7th matching), flip $\sigma = f_i(\tau)$

$$[\alpha\nu][\nu^*\alpha^* + [\eta\nu]\psi\xi + [\eta\beta]\beta^*\eta^* - \frac{[\eta\delta]\delta^*\eta^*}{x} + y[\eta\nu][\nu^*\eta^* + wa\xi[\eta\delta]\delta^*\alpha^* + z[\alpha\nu]\psi c + S'(\sigma),$$
with $S'(\sigma) = S'(\tau)$. Thus the $R$-algebra isomorphism $\psi : R\langle\langle Q(\tau) \rangle\rangle \to R\langle\langle Q(\sigma) \rangle\rangle$ whose action on the arrows is given by

$$\beta^* \mapsto -\beta^*, \eta \mapsto -\eta^*, [\eta\delta] \mapsto \frac{[\eta\delta]}{x}, [\eta\nu] \mapsto -y[\eta\nu],$$

and the identity in the rest of the arrows, is a right-equivalence between $\mu_j(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$.

**Case 9.** (Ninth matching of Figure 3.10) Assume that, around the arc $i$, $\tau$ looks like the configuration in Figure 3.35. Let us abbreviate $a = a_1 \ldots a_l, d = d_1 \ldots d_t$. Then

$$S(\tau) = -\frac{\alpha\beta\gamma\delta}{x} - \frac{\varepsilon\eta\nu\xi}{y} + wa\xi\beta\gamma + z\delta\alpha\eta\nu + S'(\tau),$$

with $S'(\tau) \in R\langle\langle Q(\tau) \rangle\rangle$ involving none of the arrows $\alpha, \beta, \gamma, \delta, \varepsilon, \eta, \nu, \psi, \xi$. If we perform the premutation $\tilde{\mu}_j$ on $(Q(\tau), S(\tau))$, we get $(\tilde{Q}(\tau), \tilde{S}(\tau))$, where $\tilde{Q}(\tau)$ is the
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arrow span of the quiver shown in Figure 3.36 and $\widehat{S}(\tau) = -\frac{[\alpha\beta]\gamma\delta}{x} - \frac{[\alpha\eta]\nu\xi}{y} + wa\xi[\varepsilon\beta]\gamma +$

Figure 3.36: Case 9, QP-mutation process $\mu_j(Q(\tau), S(\tau))$

\[
z\delta[\alpha\eta]\nu + S'(\tau) + [\alpha\beta]\beta^*\alpha^* + [\alpha\eta]\eta^*\alpha^* + [\varepsilon\beta]\beta^*\varepsilon^* + [\varepsilon\eta]\eta^*\varepsilon^* \in R\langle\langle \widehat{Q}(\tau) \rangle\rangle.~~~\text{Since} \quad (\widehat{Q}(\tau), \widehat{S}(\tau)) \text{ is already reduced, we have} \quad \mu_j(Q(\tau), S(\tau))?(\widehat{Q}(\tau), \widehat{S}(\tau)).
\]

On the other hand, $\sigma = f_i(\tau)$ and its quiver $Q(\sigma)$ look as Figure 3.37, and $S(\sigma) = [\alpha\eta]\eta^*\alpha^* + [\varepsilon\beta]\beta^*\varepsilon^* + [\alpha\beta]\gamma\delta + [\varepsilon\eta]\nu\xi + x[\alpha\beta]\beta^*\alpha^* + y[\varepsilon\eta]\eta^*\varepsilon^* + wa\xi[\varepsilon\beta]\gamma + z\delta[\alpha\eta]\nu + S'(\sigma)$, with $S'(\sigma) = S'(\tau)$. Thus the $R$-algebra isomorphism $\psi : R\langle\langle \widehat{(Q(\tau))} \rangle\rangle \to R\langle\langle Q(\sigma) \rangle\rangle$ whose action on the arrows is given by

\[
\alpha^* \mapsto -\alpha^*, \quad \eta^* \mapsto -\eta^*, \quad [\alpha\beta] \mapsto -x[\alpha\beta], \quad [\varepsilon\eta] \mapsto -y[\varepsilon\eta],
\]

and the identity in the rest of the arrows, is a right-equivalence between $\mu_j(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$.

Case 10. (Flip inside the fourth puzzle piece of Figure 3.8) Assume that, around the
arc $i$, $\tau$ looks like the configuration shown in Figure 3.38, with $l > 1$, and none of $j$ and $k$ enclosing a self-folded triangle. Then

$$S(\tau) = -\frac{\alpha \beta \gamma \delta}{y} + x \alpha \beta a + z \gamma \delta d + S'(\tau),$$
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with $S'(\tau)$ involving none of the arrows $\alpha, \beta, \gamma, \delta$. If we perform the premutation $\tilde{\mu}_j$ on $(Q(\tau), S(\tau))$, we get $(\tilde{Q}(\tau), \tilde{S}(\tau))$, where $\tilde{Q}(\tau)$ is the arrow span of the quiver shown in Figure 3.39, and $\tilde{S}(\tau) = -\frac{[\alpha \beta \gamma \delta]}{y} + x[\alpha \beta]a + z\gamma \delta d + S'(\tau) + \beta^* \alpha^*[\alpha \beta]$. Since

$$(\tilde{Q}(\tau), \tilde{S}(\tau))$$
is already reduced, it coincides with $\mu_j(Q(\tau), S(\tau))$.

On the other hand, $\sigma = f_i(\tau)$ and its quiver $Q(\sigma)$ look as Figure 3.40, and

$$S(\sigma) = \beta^* \alpha^*[\alpha \beta] - \frac{[\alpha \beta \gamma \delta]}{y} + x[\alpha \beta]a + z\gamma \delta d + S'(\sigma),$$
with $S'(\sigma) = S'(\tau)$. Therefore, $\mu_j(Q(\tau), S(\tau)) = (Q(\sigma), S(\sigma))$.  

![Figure 3.39: Case 10, QP-mutation process $\mu_j(Q(\tau), S(\tau))$](image)

![Figure 3.40: Case 10 (inside 4th puzzle piece), flip $\sigma = f_i(\tau)$](image)
3.3. First main result: Flip ↔ mutation compatibility of QPs

Case 11. (Flip inside the fourth puzzle piece of Figure 3.8) Assume that, around the arc \(i\), \(\tau\) looks like the configuration in Figure 3.41. Let us abbreviate \(c = c_1 \ldots c_l\).

![Figure 3.41: Case 11 (inside 4th puzzle piece), configuration of \(\tau\) around \(i\)](image)

Then

\[
S(\tau) = y\delta\alpha d + dc_1 c_l - \frac{\delta\alpha\beta\gamma}{x} + z\beta\gamma + S'(\tau)
\]

with \(S'(\tau) \in R\langle\langle Q(\tau)\rangle\rangle\) involving none of the arrows \(\alpha, \beta, \gamma, \delta, c_1, c_l, d\). If we perform the premutation \(\tilde{\mu}_j\) on \((Q(\tau), S(\tau))\), we get \((\tilde{Q}(\tau), \tilde{S}(\tau))\), where \(\tilde{Q}(\tau)\) is the arrow span of the quiver shown in Figure 3.42 and \(\tilde{S}(\tau) = y[\delta\alpha]d + dc_1 c_l - \frac{[\delta\alpha]\beta\gamma}{x} + z\beta\gamma +

![Figure 3.42: Case 11, QP-mutation process \(\mu_j(Q(\tau), S(\tau))\)](image)

\(S'(\tau) + [\delta\alpha]\alpha^*\delta^* \in R\langle\langle \tilde{Q}(\tau)\rangle\rangle\). The \(R\)-algebra automorphism \(\varphi\) of \(R\langle\langle \tilde{Q}(\tau)\rangle\rangle\) whose
3.3. First main result: Flip $\leftrightarrow$ mutation compatibility of QPs

action on the arrows is given by

$$[\delta \alpha] \mapsto [\delta \alpha] - \frac{c_1 c_l}{y}, \quad d \mapsto d + \frac{\beta \gamma}{xy} - \frac{\alpha \delta^*}{y},$$

and the identity in the rest of the arrows, sends $\tilde{S}(\tau)$ to

$$\varphi(\tilde{S}(\tau)) = y[\delta \alpha]d + \frac{c_1 c_l \beta \gamma}{xy} - \frac{c_1 c_l \alpha \delta^*}{y} + z c \beta \gamma + S'(\tau).$$

Therefore, the reduced part $\mu_j(Q(\tau), S(\tau))$ of $(\tilde{Q}(\tau), \varphi(\tilde{S}(\tau)))$ is (up to right-equivalence) the QP on the arrow span $\tilde{Q}(\tau)$ obtained from $\tilde{Q}(\tau)$ by deleting the arrows $[\delta \alpha]$ and $d$, with $\varphi(\tilde{S}(\tau)) - y[\delta \alpha]d$ as its potential.

On the other hand, $\sigma = f_i(\tau)$ and its quiver $Q(\sigma)$ look as Figure 3.43, and

Figure 3.43: Case 11 (inside 4$^{th}$ puzzle piece), flip $\sigma = f_i(\tau)$

$$S(\sigma) = \frac{c_1 c_l \beta \gamma}{xy} - \frac{c_1 c_l \alpha \delta^*}{y} + z c \beta \gamma + S'(\sigma), \text{ with } S'(\sigma) = S'(\tau).$$

Thus the identity is a right-equivalence between $\mu_j(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$.

Case 12. (Flip inside the third puzzle piece of Figure 3.8) Assume that, around the arc $i$, $\tau$ looks like the configuration in Figure 3.44. Let us abbreviate $c = c_1 \ldots c_n$. 
Then

\[ S(\tau) = \alpha \beta \gamma - \frac{\alpha \delta \varepsilon}{x} - \frac{\eta \psi \gamma}{y} + \frac{\eta \nu \varepsilon}{xy} + w c \eta \nu \varepsilon + S'(\tau), \]

with \( S'(\tau) \in R(\langle Q(\tau) \rangle) \) involving none of the arrows \( \alpha, \beta, \gamma, \delta, \varepsilon, \nu, \psi \). If we perform the premutation \( \tilde{\mu}_j \) on \( (Q(\tau), S(\tau)) \), we get \( (\widetilde{Q(\tau)}, \widetilde{S(\tau)}) \), where \( \widetilde{Q(\tau)} \) is the arrow span of the quiver shown in Figure 3.36 and \( \widetilde{S(\tau)} = [\alpha \beta] \gamma - \frac{[\alpha \delta] \varepsilon}{x} - \frac{\eta \psi \gamma}{y} + \frac{\eta \nu \varepsilon}{xy} + \]

\[ w c \eta \nu \varepsilon + S'(\tau) + [\alpha \beta] \beta^* \alpha + [\alpha \delta] \delta^* \alpha^* \in R(\langle \widetilde{Q(\tau)} \rangle). \]

The \( R \)-algebra automorphism \( \varphi \) of \( R(\langle \widetilde{Q(\tau)} \rangle) \) whose action on the arrows is given by

\[ [\alpha \beta] \mapsto [\alpha \beta] + \frac{\eta \psi}{y}, \quad \gamma \mapsto \gamma - \beta^* \alpha^*, \quad [\alpha \delta] \mapsto [\alpha \delta] + \frac{\eta \nu}{y} + w x c \eta \nu, \quad \varepsilon \mapsto \varepsilon + x \delta^* \alpha^*, \]
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and the identity in the rest of the arrows, sends \( \hat{S}(\tau) \) to

\[
\varphi(\hat{S}(\tau)) = [\alpha \beta] \gamma - \frac{[\alpha \delta] \varepsilon}{x} + \frac{\eta \psi \beta^* \alpha^*}{y} + \frac{\eta \nu \delta^* \alpha^*}{y} + w x c \eta \nu \delta^* \alpha^* + S'(\tau).
\]

Therefore, the reduced part \( \mu_j(Q(\tau), S(\tau)) \) of \( (Q(\tau), \varphi(\hat{S}(\tau))) \) is (up to right-equivalence) the QP on the arrow span \( \overline{Q(\tau)} \) obtained from \( \hat{Q}(\tau) \) by deleting the arrows \([\alpha \beta], \gamma, [\alpha \delta] \text{ and } \varepsilon\) with \( \varphi(\hat{S}(\tau)) - [\alpha \beta] \gamma + \frac{[\alpha \delta] \varepsilon}{x} \) as its potential.

On the other hand, \( \sigma = f_i(\tau) \) and its quiver \( Q(\sigma) \) look as Figure 3.46, and

Figure 3.46: Case 12 (inside 3rd puzzle piece), flip \( \sigma = f_i(\tau) \)

\[
S(\sigma) = -\frac{\alpha^* \eta \psi \beta^*}{y} + \frac{\alpha^* \eta \nu \delta^*}{xy} + w \eta \nu \delta^* \alpha^* c + S'(\sigma), \text{ with } S'(\sigma) = S'(\tau). \text{ Thus the R-algebra isomorphism } \psi : R\langle\langle Q(\tau) \rangle\rangle \to R\langle\langle Q(\sigma) \rangle\rangle \text{ whose action on the arrows is given by }
\]

\[
\alpha^* \mapsto -\alpha^*, \ \delta^* \mapsto -\frac{\delta^*}{x},
\]

and the identity in the rest of the arrows, is a right-equivalence between \( \mu_j(Q(\tau), S(\tau)) \) and \( (Q(\sigma), S(\sigma)) \).
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A similar analysis in the rest of the cases (see Remark 3.3.1) finishes the proof of Theorem 3.3.1.

□
Chapter 4

Non-empty boundary: Rigidity and finite dimension

Our second main result ensures non-degeneracy for the QPs constructed in Definition 3.1.1 provided the boundary of the underlying surface is non-empty.

**Theorem 4.0.2.** Let \((\Sigma, M)\) be a bordered surface with marked points. If \(\Sigma\) has non-empty boundary, then the QP associated in Definition 3.1.1 to any ideal triangulation of \((\Sigma, M)\) (under any choice \((x_p)_{p \in P}\)) is rigid, hence non-degenerate.

**Proof.** Changing the notation a little bit, throughout the proof we will assume that \((\Sigma, M)\) has no punctures, in other words, all the marked points in \(M\) belong to the boundary. We begin by inductively constructing a sequence of triangulations \(\sigma_1, \sigma_2, \ldots\), such that each \(\sigma_n\) is an ideal triangulation of \((\Sigma, M \cup P_n)\) for some set \(P_n = \{p_1, \ldots, p_n\}\) of \(n\) distinct punctures on \(\Sigma\) and

\[
\tau \subseteq \sigma_1 \subseteq \sigma_2 \subseteq \ldots \quad \text{and} \quad Q(\tau) \subseteq Q(\sigma_1) \subseteq Q(\sigma_2) \subseteq \ldots \quad (4.1)
\]
Let $\tau$ be any ideal triangulation of $(\Sigma, M)$ (which we assume to have no punctures). To construct $\sigma_1$ we need to choose a puncture $p_1$ in $\Sigma$. We put $p_1$ inside any non-interior triangle $\Delta_0$ of $\tau$. Then we draw the three arcs emanating from $p_1$ and going to the three vertices of $\Delta_0$. The result is an ideal triangulation $\sigma_1$ of $(\Sigma, M \cup P_1)$. For $n > 1$, once $\sigma_{n-1}$ has been constructed, we put $p_n$ inside a non-interior triangle $\Delta_{n-1}$ of $\sigma_{n-1}$ having $p_{n-1}$ as a vertex, then we draw the three arcs emanating from $p_n$ and going to the three vertices of $\Delta_{n-1}$. The result is an ideal triangulation of $(\Sigma, M \cup P_n)$.

Now, for $n > 0$, let $\tau_n = f_{j_1^n}(\sigma_n)$ be the triangulation obtained by the flip of the arc $j_1^n$ of the triangulation $\sigma_n$, see Figure 4.1. Note that $\sigma_{n-1} \subseteq \tau_n$ and $Q(\sigma_{n-1}) \subseteq Q(\tau_n)$ for $n \geq 1$ if we denote $\sigma_0 = \tau_0 = \tau$.

**Lemma 4.0.2.** With the above notation, every potential on $Q(\sigma_{n-1})$ belonging to $J(S(\sigma_{n-1}))$ is cyclically equivalent to an element of $J(S(\tau_n))$.

**Proof.** For $n = 1$ we actually have $J(S(\tau)) \subseteq J(S(\tau_1))$. So let us treat the case $n > 1$. 

![Figure 4.1: σ_n](image-url)
With the notation of Figure 4.2, we have \( S(\tau_n) = S(\sigma_{n-1}) + \alpha \delta \varepsilon \), and hence, for

\[
\begin{align*}
  a &\in Q_1(\sigma_{n-1}), \quad a \neq \alpha \text{ (note that } \delta, \varepsilon \notin Q_1(\sigma_{n-1})\text{)}, \quad \text{we have } \partial_a(S(\tau_n)) = \partial_a(S(\sigma_{n-1})), \\
  \text{whereas } \partial_a(S(\tau_n)) &= \partial_a(S(\sigma_{n-1}))+\delta \varepsilon = x_{p_{n-1}}\beta \gamma + \delta \varepsilon, \quad \partial_\delta(S(\tau_n)) = \varepsilon \alpha \text{ and } \partial_\varepsilon(S(\tau_n)) = \alpha \delta.
\end{align*}
\]

Let \( c \) be a cycle of \( Q(\sigma_{n-1}) \) that has \( \beta \gamma \) as factor. Then one of the following three cases holds:

- \( c \) is cyclically equivalent to a cycle in \( Q(\sigma_{n-1}) \) that has \( \beta \gamma \alpha \) as factor.

- \( c \) is cyclically equivalent to a cycle in \( Q(\sigma_{n-1}) \) that has \( \alpha \beta \gamma \) as factor.

- The segment labeled \( j_2 \) in Figure 4.2 is an arc of \( \sigma_{n-1} \) (hence the dotted arrows are indeed arrows of \( \sigma_{n-1} \)) and \( c \) is cyclically equivalent to a cycle that has \( \beta_2 \beta \gamma \gamma_1 \) as factor.
Since \( \beta \gamma \alpha = x_{p_{n-1}}^{-1} \partial_\alpha(S(\tau_n)) \alpha - x_{p_{n-1}}^{-1} \delta \varepsilon \alpha \in J(S(\tau_n)) \) and \( \alpha \beta \gamma = x_{p_{n-1}}^{-1} \alpha \partial_\alpha(S(\tau_n)) - x_{p_{n-1}}^{-1} \alpha \delta \varepsilon \in J(S(\tau_n)) \), in the first two cases we see that \( c \) is cyclically equivalent to an element of \( J(S(\tau_n)) \).

In the third case, we have two possibilities: The marked point \( m \) either lies on the boundary of \( \Sigma \) or is a puncture of \( (\Sigma, M \cup P_{n-1}) \). If it lies on the boundary, then \( \beta_2 \beta \gamma_1 = \beta_2 \beta \partial_{\gamma_2}(S(\sigma_{n-1})) = \beta_2 \beta \partial_{\gamma_2}(S(\tau_n)) \in J(S(\tau_n)) \). Otherwise, if \( m \) is a puncture, then \( \beta_2 \beta \gamma_1 = \beta_2 \beta \partial_{\gamma_2}(S(\sigma_{n-1})) - x_m \beta_2 \beta \beta \beta_1 b = \beta_2 \beta \partial_{\gamma_2}(S(\tau_n)) - x_m \beta_2 \beta \beta_2 b \in J(S(\tau_n)) \) for a unique path \( b \) in \( Q(\sigma_{n-1}) \) from the arc \( t(\gamma_1) \) to the arc \( t(\beta_1) \).

This and the fact that \( \partial_\alpha(S(\tau_n)) = \partial_\alpha(S(\sigma_{n-1})) \) for \( a \in Q_1(\sigma_{n-1}) \) imply Lemma 4.0.2.

We now return to the proof of Theorem 4.0.2. Since rigidity is preserved by QP
mutation, by Proposition 2.2.6 and Theorem 3.3.1 it suffices to show that \( (Q(\tau_n), S(\tau_n)) \)
is rigid. We prove this by induction on \( n \geq 0 \).

For \( n = 0 \) we have \( \tau_0 = \tau \). Each arrow of \( Q(\tau) \) appears in at most one term of
\( S(\tau) \), and all the terms of \( S(\tau) \) are oriented triangles coming from interior triangles
of \( \tau \). Also, since there are no punctures, every cycle in \( Q(\tau) \) is cyclically equivalent
to a cycle that has \( ab = \partial_c(S) \) as factor for some oriented cycle \( abc \) that appears (up
to cyclical equivalence) as a term of \( S(\tau) \). Therefore, \( (Q(\tau), S(\tau)) \) is rigid.

For the inductive step, let \( n > 0 \) and assume that the QP \( (Q(\tau_{n-1}), S(\tau_{n-1})) \) associated
to the triangulation \( \tau_{n-1} \) of \( (\Sigma, M \cup P_{n-1}) \) is rigid; then the QP \( (Q(\sigma_{n-1}), S(\sigma_{n-1})) \)
is rigid as well. Take any cycle \( c \) in \( Q(\tau_n) \). If \( c \) is contained in \( Q(\sigma_{n-1}) \), then \( c \) is
cyclically equivalent to an element of \( J(S(\sigma_{n-1})) \), which in turn is cyclically equiva-
lent to an element of \( J(S(\tau_n)) \) by Lemma 4.0.2. If \( c \) is not contained in \( Q(\sigma_{n-1}) \), then \( c \) is cyclically equivalent to a cycle \( c' \) that has \( \delta \varepsilon \) as factor, say \( c' = d\delta \varepsilon \), with \( d \) a path in \( Q(\tau_n) \) from \( h(\delta) \) to \( t(\varepsilon) \). Then \( c' = d\partial_n(S(\tau_n)) - x_{p_n-1}d\beta \gamma \), and we can keep substituting each factor \( \delta \varepsilon \) of \( d \) by \( \partial_c(S(\tau_n)) - x_{p_n-1}\beta \gamma \). After doing this, we see that \( c' \) is the sum of an element of \( J(S(\tau_n)) \) with a scalar multiple of a cycle \( c'' \) in \( Q(\sigma_{n-1}) \). This cycle \( c'' \) is cyclically equivalent to an element of \( J(S(\tau_n)) \) as we have seen above. We conclude that \( (Q(\tau_n), S(\tau_n)) \) is rigid. This finishes the proof of Theorem 4.0.2. \( \square \)

**Remark 4.0.3.** 1. In terms of trace spaces (see [8], Definition 3.4), Lemma 4.0.2 says that the inclusion of quivers \( Q(\sigma_{n-1}) \hookrightarrow Q(\tau_n) \) induces a well defined map between the trace spaces \( \text{Tr}(\mathcal{P}(Q(\sigma_{n-1}), S(\sigma_{n-1}))) \rightarrow \text{Tr}(\mathcal{P}(Q(\tau_n), S(\tau_n))) \).

2. Since every quiver mutation equivalent to a quiver of the form \( Q(\tau) \), with \( \tau \) an ideal triangulation of \( (\Sigma, M) \), is of the same form, Theorem 4.0.2 says in particular that in Definition 3.1.1 we have given an explicit construction of a non-degenerate potential for each of the members of the mutation equivalence class of the quivers that arise as signed adjacency quivers of triangulations of surfaces with non-empty boundary.

**Conjecture 4.0.4.** The \( \mathcal{P} \) associated in Definition 3.1.1 to any triangulation of \( (\Sigma, M) \) is always non-degenerate, regardless of the emptiness of the boundary of \( \Sigma \).

**Conjecture 4.0.5.** If \( (\Sigma, M) \) has empty boundary, then the \( \mathcal{P} \) associated to any triangulation of \( (\Sigma, M) \) is never rigid. In other words, the converse of Theorem 4.0.2 is true as well.

To illustrate this two conjectures, let us look at an example.
Example 4.0.6. Consider the triangulation $\tau$ of the once-punctured torus shown in Figure 4.3. We have $S(\tau) = a_1b_1c_1 + a_2b_2c_2 + xa_1b_2c_1a_2b_1c_2$. If we mutate $(Q(\tau), S(\tau))$ in direction $i = t(b_1)$, we obtain $(Q(\tau), S(\tau))$, where $Q(\tau)$ is the quiver shown in Figure 4.4 and $S(\tau) = c_1^*b_2^*[b_2c_1] + c_2^*b_1^*[b_1c_2] + xc_1^*b_1^*[b_2c_1]c_2^*b_2^*[b_1c_2]$, and we therefore have $(Q(\sigma), S(\sigma)) = \mu_j(Q(\tau), S(\tau))$, where $\sigma = f_i(\tau)$. This example shows in particular that Theorem 3.3.1 holds for the once-punctured torus and that Definition 3.1.1 gives an explicit non-degenerate potential for the “double cyclic triangle" $Q(\tau)$. However, it is known (see [8], example 8.6) that the double cyclic triangle does not admit a rigid potential. In short words, Conjectures 4.0.4 and 4.0.5 hold for the once-punctured torus.

Now we show that non-empty boundary implies also finite dimension of the Jacobian algebra.
**Theorem 4.0.3.** If the surface $\Sigma, M$ has non-empty boundary, then for any triangulation $\tau$ of $\Sigma, M$ the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau))$ is finite dimensional, the ideal $I(\tau)$ of the path algebra $R\langle Q(\tau) \rangle$ generated by $\{\partial_a(S(\tau)) \mid a \in Q_1(\tau)\}$ is admissible (that is, it is contained in the square of the ideal generated by the arrows and contains all paths of sufficiently large length), and $\mathcal{P}(Q(\tau), S(\tau))$ is isomorphic to the quotient $R\langle Q(\tau) \rangle/I(\tau)$.

**Proof.** Assume that $\Sigma, M$ has non-empty boundary and no punctures, and let $\tau$ be any triangulation of $\Sigma, M$. For each marked point $m$ we have the following configuration in a sufficiently small neighborhood of $m$:

![Figure 4.5](image)

where the arrows $a_1^m, \ldots, a_{d_m}^m$ are uniquely determined by $m$. Let $N = \max\{d_m \mid m \in M\}$. Then any path in $Q(\tau)$ having length greater than $N$ must have $ab = \partial_c(S(\tau))$ as a factor for some oriented triangle $abc$ appearing as a term of $S(\tau)$.

Now let $\tau_1$ be the triangulation of $\Sigma, M \cup P_1$ constructed in the proof of Theorem 4.0.2 (see Figure 4.6). Then we have $J(\tau) \subseteq J(\tau_1)$ and $\alpha \delta, \delta \varepsilon, \varepsilon \alpha \in J(\tau_1)$. Therefore any path in $Q(\tau_1)$ of length greater than $N + 2$ belongs to $J(\tau_1)$. Since finite-dimensionality of the Jacobian algebra is preserved under QP-mutation, we have proved the Theorem for non-empty boundary and at most one puncture.
The proof for the case of more than one puncture is similar to the proof of Lemma 4.0.2. So let \( n > 1 \) and, with the notation of Theorem 4.0.2, assume inductively that the Jacobian algebra \( \mathcal{P}(Q(\tau_{n-1}), S(\tau_{n-1})) \) is finite-dimensional. Then \( \mathcal{P}(Q(\sigma_{n-1}), S(\sigma_{n-1})) \) is finite-dimensional as well.

With the notation of Figure 4.2, let \( \mathcal{P} \) denote the set of all paths in \( Q(\sigma_{n-1}) \) that do not start at \( t(\gamma) \) and do not end at \( h(\beta) \). If \( u \in \mathcal{P} \) has \( \beta \gamma \) as a factor, that is, if \( u = u_1 \beta \gamma u_2 \) for some paths \( u_1, u_2 \) in \( Q(\sigma_{n-1}) \), then one of the following three conditions holds:

- \( u \) has \( \beta \gamma \alpha \) as factor.
- \( u \) has \( \alpha \beta \gamma \) as factor.
- The segment labeled \( j_2^n \) in Figure 4.2 is an arc of \( \sigma_{n-1} \) (hence the dotted arrows are indeed arrows of \( Q(\sigma_{n-1}) \)) and \( u \) has \( \beta_2 \beta \gamma \gamma_1 \) as factor.

In any of these three cases, we have \( u \in J(S(\tau_n)) \) just as in the proof of Lemma 4.0.2.

Since \( \mathcal{P}(Q(\sigma_{n-1}), S(\sigma_{n-1})) \) is finite-dimensional, there exists a positive integer \( N' \) such that every element of \( \mathcal{P} \) whose length is greater than \( N' \) belongs to the
Jacobian ideal $J(S(\sigma_{n-1}))$. Let $P_{>N'}$ be the subset of $P$ consisting of paths of length greater than $N'$. By the previous paragraph and because $\partial_a(S(\tau_n)) = \partial_a(S(\sigma_{n-1}))$ for $a \in Q_1(\sigma_{n-1})$, $a \neq \alpha$ (note that $\delta, \varepsilon \notin Q_1(\sigma_{n-1})$), we have $P_{>N'} \subseteq J(S(\tau_n))$.

Now take any path $u$ in $Q(\sigma_{n-1})$ of length greater than $N' + 2$. If $u$ does not start at $t(\gamma)$ and does not end at $h(\beta)$, then $u \in J(S(\tau_n))$. Otherwise, we can write $u = \beta u'$ or $u = u' \gamma$ or $u = \beta u' \gamma$ for some path $u'$ in $Q(\sigma_{n-1})$ (remember that $\delta, \varepsilon \notin Q(\sigma_{n-1})$) that has length greater than $N'$, does not start at $t(\gamma)$ and does not end at $h(\beta)$, and hence belongs to $J(S(\tau_n))$.

Finally, let $v$ be any path in $Q(\tau_n)$ of length greater than $N' + 6$, we claim that $v \in J(S(\tau_n))$. Since $\partial_a(S(\tau_n)) = x_{p_{n-1}} \beta \gamma + \delta \varepsilon$, we can assume, without loss of generality, that $v$ does not have $\delta \varepsilon$ as a factor. Then one of the following cases holds:

- $v$ is contained in $Q(\sigma_{n-1})$;
- $\varepsilon$ is a factor of $v$;
- $\delta$ is a factor of $v$.

In the first case, we have $v \in J(S(\tau_n))$. In the second case, we must have $v = \varepsilon v'$ or $v = \omega \varepsilon v'$ for some path $v'$ in $Q(\tau_n)$ because we are assuming that $v$ does not contain $\delta \varepsilon$ as factor. In any of these two situations, the path $v'$ is either contained in $Q(\sigma_{n-1})$ (and has length greater than $N' + 4$) or can be written as $v' = v'' \delta$ or $v' = v'' \delta \nu$ for some path $v''$ (of length greater than $N' + 2$) contained in $Q(\sigma_{n-1})$. This yields $v \in J(\tau_n)$.

Similarly, in the third case, we must have $v = v' \delta$ or $v = v' \delta \nu$ for some path $v'$ in $Q(\tau_n)$, and in any of these situations, the path is either contained in $Q(\sigma_{n-1})$ (and has
length greater than \(N' + 4\) or can be written as \(v' = \varepsilon v''\) or \(v' = \omega \varepsilon v''\) for some path \(v''\) (of length greater than \(N' + 2\)) contained in \(Q(\sigma_{n-1})\). This yields \(v \in J(S(\tau_n))\).

Therefore, the Jacobian algebra \(\mathcal{P}(Q(\tau_n), S(\tau_n))\) has finite dimension. The theorem follows from Proposition 2.2.6, Theorem 3.3.1 and Proposition 2.3.4. \(\square\)

To finish this section and close the chapter and Part I of this thesis, we include a curious example that came up in discussions with J. Weyman.

**Example 4.0.7** (cf. [18], Example 35). Consider the triangulation \(\tau\) of the once-punctured torus shown in Figure 4.3. We have \(S(\tau) = a_1 b_1 c_1 + a_2 b_2 c_2 + x a_1 b_2 c_1 a_2 b_1 c_2\). By Theorem 3.3.1 (or as we directly checked in Example 4.0.6), flips of (ideal) triangulations of this surface are compatible with QP-mutations and hence the potential is non-degenerate (it is not possible to obtain a tagged non-ideal triangulation from an ideal one by flips). Also, it is easy to see that the quotient \(R\langle Q(\tau) \rangle/I\) is infinite-dimensional, where \(I\) is the ideal generated by the cyclic derivatives of \(S(\tau)\). However, in contrast to Example 11.3 of [8], where the non-degenerate potential \(a_1 b_1 c_1 + a_2 b_2 c_2\) has infinite-dimensional Jacobian algebra, the Jacobian algebra \(\mathcal{P}(Q(\tau), S(\tau))\) is finite-dimensional, as the following elementary calculation shows.

First, notice that the paths \(a_1 b_2 c_1 a_2 b_1 c_2\) and \(a_2 b_1 c_2 a_1 b_2 c_1\) represent the same element in \(\mathcal{P}(Q(\tau), S(\tau))\):

\[
(a_1 b_2 c_1 a_2 b_1) c_2 \equiv -x^{-1} a_2 b_2 c_2 \equiv a_2 (b_1 c_2 a_1 b_2 c_1) \quad \text{mod } J,
\]

where \(J\) is the two-sided ideal of \(R\langle Q(\tau) \rangle\) generated by the cyclic derivatives of \(S(\tau)\) (hence the Jacobian ideal \(J(S(\tau))\) is the topological closure of \(J\)). Notice also that any path in \(Q(\tau)\) can be represented in \(\mathcal{P}(Q(\tau), S(\tau))\) by (a scalar multiple of)
an alternating path, that is, a path whose conforming arrows have subindices that alternate between the numbers 1 and 2. Here is the calculation for paths of length 2:

\[ a_1 b_1 \equiv -x a_2 b_1 c_2 a_1 b_2 \text{ mod } J, \text{ and similarly for the rest of the paths } b_1 c_1, c_1 a_1, a_2 b_2, b_2 c_2, c_2 a_2. \]

Now the calculation for paths of length 3. For the paths \( a_1 b_1 c_1, a_2 c_2 b_2 \) and their rotations it is essentially shown above, whereas for the paths of the form \( A_1 B_2 C_2 \) we have

\[
A_1 B_2 C_2 \equiv -x A_1 B_1 C_2 A_1 B_2 C_1 \\
\equiv x^2 A_2 B_1 C_2 A_1 (B_2 C_2) A_1 B_2 C_1 \\
\equiv -x^3 A_2 B_1 C_2 (A_1 B_1) C_2 A_1 B_2 C_1 A_1 B_2 C_1 \equiv \ldots \text{ mod } J,
\]

from what we see that \( A_1 B_2 C_2 \in J(S(\tau)) \). Similarly, \( A_1 B_1 C_2, A_2 B_1 C_1, A_2 B_2 C_1 \in J(S(\tau)) \).

In length 4 it already happens that the only paths that are not zero in \( P(Q(\tau), S(\tau)) \) are the alternating ones: \( A_1 B_1 C_1 A_1 \equiv x(A_2 B_1 C_2 A_1 B_2) C_1 A_1 \in J(S(\tau)) \) and the rest is an easy check.

Now we claim that all the paths of length 7 are zero in \( P(Q(\tau), S(\tau)) \). After all the above calculations it is clear that we only need to check that the alternating paths of length 7 belong to \( J(S(\tau)) \). But

\[
(A_1 B_2 C_1 A_2 B_1 C_2) A_1 \equiv A_2 B_1 C_2 A_1 B_2 (C_1 A_1) \\
\equiv -x A_2 B_1 C_2 A_1 (B_2 C_2) A_1 B_2 C_1 A_2
\]
\[ \equiv x^2 A_2 B_1 C_2 (A_1 B_1) C_2 A_1 B_2 (C_1 A_1) B_2 C_1 A_2 \equiv \ldots \mod J, \]

and hence \( A_1 B_2 C_1 A_2 B_1 C_2 A_1 \in J(S(\tau)) \). Therefore, all paths of length greater than 6 belong to \( J(S(\tau)) \), which implies the finite-dimensionality of \( \mathcal{P}(Q(\tau), S(\tau)) \).
Part II

Representations of quivers with potentials associated to arcs on surfaces
Chapter 5

Background on QP-representations and their mutations

In this subsection we describe how the notions of right-equivalence and QP-mutation extend to the level of representations. As in the Section 2.3, our main reference is [8].

Recall that the vertex span of a quiver $Q$ is the $K$-vector space $R$ with basis \( \{ e_i \mid i \in Q_0 \} \). This vector space is actually a commutative ring if we define $e_i e_j = \delta_{ij} e_i$.

**Definition 5.0.8** ([8], Definition 10.1). Let $(Q, S)$ be any QP. A decorated $(Q, S)$-representation, or QP-representation, is a quadruple $\mathcal{M} = (Q, S, M, V)$, where $M$ is a finite-dimensional left $\mathcal{P}(Q, S)$-module and $V$ is a finite-dimensional left $R$-module.

By setting $M_i = e_i M$ for each $i \in Q_0$, and $a_M : M_{t(a)} \to M_{h(a)}$ as the multiplication by $a \in Q_1$ given by the $R(\langle Q \rangle)$-module structure of $M$, we easily see that each $\mathcal{P}(Q, S)$-module induces a representation of the quiver $Q$. The following lemma, whose proof can be found in [8], allows us to deduce the relations this representation
Lemma 5.0.9. Every finite-dimensional $R\langle\langle Q\rangle\rangle$-module is nilpotent. That is, if $M$ is a finite-dimensional $R\langle\langle Q\rangle\rangle$-module, then there exists a positive integer $r$ such that $m^r M = 0$. (Remember that $m$ is the ideal of $R\langle\langle Q\rangle\rangle$ generated by the arrows.)

Because of this lemma, we see that any QP-representation is prescribed by the following data:

1. A tuple $(M_i)_{i\in Q_0}$ of finite-dimensional $K$-vector spaces;

2. a family $(a_M : M_{t(a)} \to M_{h(a)})_{a\in Q_0}$ of $K$-linear transformations annihilated by $\{\partial_a(S) \mid a \in Q_1\}$, for which there exists an integer $r \geq 1$ with the property that the composition $a_{1,M} \ldots a_{r,M}$ is identically zero for every $r$-path $a_1 \ldots a_r$ in $Q$.

3. a tuple $(V_i)_{i\in Q_0}$ of finite-dimensional $K$-vector spaces (without any specification of linear maps between them).

Remark 5.0.10. 1. In the literature, the linear map $a_M : M_{t(a)} \to M_{h(a)}$ induced by left multiplication by $a$ is more commonly denoted by $M_a$. We will use both of these notations indistinctly.

2. More generally, we will use the notation $u_M$ (or $M_u$) for the linear map $M \to M$ induced by left multiplication by an arbitrary element $u \in R\langle\langle Q\rangle\rangle$.

3. It is not true that any representation annihilated by the cyclic derivatives of $S(\tau)$ is nilpotent. That is, it is possible to construct $R\langle\langle Q(\tau)\rangle\rangle$-modules that satisfy the cyclic derivatives of $S(\tau)$ but cannot be given the structure of $\mathcal{P}(Q(\tau), S(\tau))$-
module. An example of this is given by the representation

![Diagram](attachment://quiver.png)

of the quiver

![Diagram](attachment://quiver2.png)

which obviously satisfies the cyclic derivatives of the potential $S = a_1 b_1 c_1 + a_2 b_2 c_2 - a_1 b_2 c_1 a_2 b_1 c_2$, but is not nilpotent.

**Definition 5.0.11** ([8], Definition 10.2). Let $(Q, S)$ and $(Q', S')$ be QPs on the same set of vertices, and let $\mathcal{M} = (Q, S, M, V)$ and $\mathcal{M}' = (Q', S', M', V')$ be decorated representations. A triple $\Phi = (\varphi, \psi, \eta)$ is called a *right-equivalence* between $\mathcal{M}$ and $\mathcal{M}'$ if the following three conditions are satisfied:

- $\varphi : R\langle\langle Q \rangle\rangle \to R\langle\langle Q' \rangle\rangle$ is a right-equivalence of QPs between $(Q, S)$ and $(Q', S')$;

- $\psi : M \to M'$ is a vector space isomorphism such that $\psi \circ u_M = \varphi(u)_M' \circ \psi$ for all $u \in R\langle\langle Q \rangle\rangle$;

- $\eta : V \to V'$ is an $R$-module isomorphism.
Example 5.0.12. Consider the QP $(Q, 0)$, where $Q$ is the quiver

$$
\begin{array}{c}
\text{1} \\
\text{3} \\
\end{array}
\xrightarrow{\text{a}}
\begin{array}{c}
\text{2} \\
\end{array}
\xrightarrow{\text{c}}
\begin{array}{c}
\text{b} \\
\end{array}
\xrightarrow{\text{b}}
$$

For any $\lambda \in K$, the QP-representation

$$
\begin{array}{ccc}
K & \xrightarrow{\lambda 1_K} & K \\
\downarrow^{1_K} & & \downarrow^{1_K} \\
V_1 & & V_2 \\
\end{array}
$$

is right-equivalent to the QP-representation

$$
\begin{array}{ccc}
K & \xrightarrow{0} & K \\
\downarrow^{1_K} & & \downarrow^{1_K} \\
V_1 & & V_2 \\
\end{array}
$$

by means of the triple $\Phi = (\varphi, \psi, 1_V)$, where $\varphi : R\langle\langle Q\rangle\rangle \rightarrow R\langle\langle Q\rangle\rangle$ is the $R$-algebra isomorphism whose action on the arrows is given by $a \mapsto \lambda bc - a$, $b \mapsto b$, $c \mapsto c$, and $\psi$ is the identity on each copy of $K$. This example shows in particular that there are right-equivalent representations that are not isomorphic.

Recall that every QP is right-equivalent to the direct sum of its reduced and trivial parts, which are determined up to right-equivalence (Theorem 2.3.1). These facts have representation-theoretic extensions, which we now describe. Let $(Q, S)$ be any QP, and let $\varphi : R\langle\langle Q_{\text{red}} \oplus C\rangle\rangle \rightarrow R\langle\langle Q\rangle\rangle$ be a right equivalence between $(Q_{\text{red}}, S_{\text{red}}) \oplus (C, T)$ and $(Q, S)$, where $(Q_{\text{red}}, S_{\text{red}})$ is a reduced QP and $(C, T)$ is a
trivial QP. Let \( \mathcal{M} = (Q, S, M, V) \) be a decorated representation, and set \( M^\varphi = M \)
as \( K \)-vector space. Define an action of \( R\langle\langle Q_{\text{red}}\rangle\rangle \) on \( M^\varphi \) by setting \( u_{M^\varphi} = \varphi(u)_M \) for \( u \in R\langle\langle Q_{\text{red}}\rangle\rangle \).

**Proposition 5.0.13** ([8], Propositions 4.5 and 10.5). With the action of \( R\langle\langle Q_{\text{red}}\rangle\rangle \) on \( M^\varphi \) just defined, the quadruple \( (Q_{\text{red}}, S_{\text{red}}, M^\varphi, V) \) becomes a QP-representation. Moreover, the right-equivalence class of \( (Q_{\text{red}}, S_{\text{red}}, M^\varphi, V) \) is determined by the right-equivalence class of \( \mathcal{M} \).

**Definition 5.0.14** ([8], Definition 10.4). The (right-equivalence class of the) QP-representation \( \mathcal{M}_{\text{red}} = (Q_{\text{red}}, S_{\text{red}}, M^\varphi, V) \) is the reduced part of \( \mathcal{M} \).

**Remark 5.0.15.** The construction of a right-equivalence between a QP \((Q, S)\) and the direct sum of a reduced QP with a trivial one is not given by a canonical procedure in any obvious way; that is, there is no canonical way to construct \((Q_{\text{red}}, S_{\text{red}})\) nor a right-equivalence \((Q, S) \to (Q_{\text{red}}, S_{\text{red}}) \oplus (Q_{\text{triv}}, S_{\text{triv}})\) (even when the QP is \((Q_{\text{red}}, S_{\text{red}})\) is already known). In [8] a specific right-equivalence \( \varphi \) is defined, satisfying the condition of acting as the identity on all the arrows of \( Q \) that do not appear in the degree-2 component \( S^{(2)} \) of \( S \) as long as no arrow appearing in \( S^{(2)} \) appears in different summands of \( S^{(2)} \). Using this property of \( \varphi \), it is easy to see that given a decorated representation \( \mathcal{M} = (Q, S, M, V) \), the action of \( R\langle\langle Q_{\text{red}}\rangle\rangle \) on \( M^\varphi \) coincides with its action on \( M \) induced by the inclusion of quivers \( Q_{\text{red}} \hookrightarrow Q \). That is, restricting the action of \( R\langle\langle Q\rangle\rangle \) on \( M \) to its subalgebra \( R\langle\langle Q_{\text{red}}\rangle\rangle \) gives us the reduced part \( \mathcal{M}_{\text{red}} \).

We now turn to the representation-theoretic analogue of the notion of QP-mutation (cf. [8], Section 10). Let \((Q, S)\) be a QP. Fix a vertex \( j \in Q_0 \), and suppose that \( Q \) has no 2-cycles incident to \( j \). Denote by \( a_1, \ldots, a_s \) (resp. \( b_1, \ldots, b_t \)) the arrows ending at
\( j \) (resp. starting at \( j \)). Take a QP-representation \( \mathcal{M} = (Q, S, M, V) \) and set

\[
M_{\text{in}} = \bigoplus_{k=1}^{s} M_{t(a_k)}, \quad M_{\text{out}} = \bigoplus_{l=1}^{t} M_{h(b_l)}.
\]

Multiplication by the arrows \( a_1, \ldots, a_s \) and \( b_1, \ldots, b_t \) induces \( K \)-linear maps

\[
a = a_M = [a_1 \ldots a_s] : M_{\text{in}} \to M_j, \quad b = b_M = \begin{bmatrix} b_1 \\ \vdots \\ b_t \end{bmatrix} : M_j \to M_{\text{out}}.
\]

For each \( k \) and each \( l \) let \( c_{k,l} : M_{h(b_l)} \to M_{t(a_k)} \) be the linear map given by multiplication by the the element \( \partial_{[b_t a_k]}([S]) \), and arrange these maps into a matrix to obtain a linear map \( c = c_M : M_{\text{out}} \to M_{\text{in}} \) (remember that \([S]\) is obtained from \( S \) by replacing each \( j \)-hook \( ab \) with the arrow \([ab]\)). Since \( M \) is a \( \mathcal{P}(Q, S) \)-module, we have \( ac = 0 \) and \( cb = 0 \) (cf. [8], Lemma 10.6).

Define vector spaces \( \overline{M}_i = M_i \) and \( \overline{V}_i = V_i \) for \( i \in Q_0, i \neq j \), and

\[
\overline{M}_j = \frac{\ker c}{\im b} \oplus \frac{\ker a}{\im c}, \quad \overline{V}_j = \frac{\ker b}{\ker b \cap \im a}.
\]

We define an action of the arrows of \( \overline{\mu}_j(Q) \) on \( \overline{M} = \bigoplus_{i \in Q_0} \overline{M}_i \) as follows. If \( c \) is an arrow of \( Q \) not incident to \( j \), we define \( c_{\overline{M}} = c_M \), and for each \( k \) and each \( l \) we set \([b_t a_k]_{\overline{M}} = (b_t a_k)_M = b_{tM} a_{kM} \). To define the action of the remaining arrows, choose a linear map \( \tau : M_{\text{out}} \to \ker c \) such that the composition \( \ker c \hookrightarrow M_{\text{out}} \xrightarrow{\tau} \ker c \) is the identity (where \( \ker c \hookrightarrow M_{\text{out}} \) is the inclusion) and a linear map \( s : \frac{\ker a}{\im c} \to \ker a \) such that the composition \( \frac{\ker a}{\im c} \xrightarrow{s} \ker a \to \frac{\ker a}{\im c} \) is the identity (where \( \ker a \to \frac{\ker a}{\im c} \) is the
canonical projection). Then set
\[
[\tilde{b}_1^* \ldots \tilde{b}_r^*] = \tilde{a} = \begin{bmatrix} -p r & -c & 0 & 0 \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ a_1^* & \vdots & \vdots & a_s^* \end{bmatrix} : M_{out} \to \overline{M}_j, \quad \begin{bmatrix} a_1^* \\ \vdots \\ a_s^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \overline{b} = [0 \text{ i is 0}] : \overline{M}_j \to M_{in},
\]

where \( p : \ker c \to \frac{\ker c}{\ker b} \) is the canonical projection and \( i : \ker a \to M_{in} \) is the inclusion.

This action of the arrows of \( \tilde{\mu}_j(Q) \) on \( \overline{M} \) extends uniquely to an action of \( R\langle \tilde{\mu}_j(Q) \rangle \) under which \( \overline{M} \) is an \( R\langle \tilde{\mu}_j(Q) \rangle \)-module. And since \( m^r M = 0 \) for some sufficiently large \( r \), this action of \( R\langle \tilde{\mu}_j(Q) \rangle \) on \( \overline{M} \) extends uniquely to an action of \( R\langle \langle \tilde{\mu}_j(Q) \rangle \rangle \) under which \( \overline{M} \) is an \( R\langle \langle \tilde{\mu}_j(Q) \rangle \rangle \)-module.

**Remark 5.0.16.** Note that the choice of the linear maps \( r \) and \( s \) is not canonical. However, different choices lead to isomorphic \( R\langle \langle \tilde{\mu}_j(Q) \rangle \rangle \)-module structures on \( \overline{M} \), see [8] Proposition 10.9.

**Definition 5.0.17** ([8], Section 10). With the above action of \( R\langle \langle \tilde{\mu} \rangle \rangle \) on \( \overline{M} \) and the obvious action of \( R \) on \( \overline{V} = \bigoplus_i \overline{V}_i \), the quadruple \((\tilde{\mu}_j(Q), \overline{S}, \overline{M}, \overline{V})\) is called the \textit{premutation} of \( \mathcal{M} = (Q, S, M, V) \) in direction \( j \), and denoted \( \tilde{\mu}_j(M) \). The \textit{mutation} of \( \mathcal{M} \) in direction \( j \), denoted by \( \mu_j(M) \), is the reduced part of \( \tilde{\mu}_j(M) \).

The following are a couple of important properties of mutations of QP-representations. The reader is referred to [8] for more precise versions of the statements.

**Theorem 5.0.18.** 1. Premutations and mutations are well defined up to right-equivalence.
2. Mutations of $QP$-representations are involutive up to right-equivalence.
Chapter 6

Definition of arc representations

Let \((\Sigma, M)\) be a bordered surface with marked points and \(P \subseteq M\) be the set of punctures of \((\Sigma, M)\). Fix a choice \((x_p)_{p \in P}\) of non-zero elements of the field \(K\). Throughout the rest of the thesis, \(\tau\) will be an ideal triangulation of \((\Sigma, M)\) without self-folded triangles.

6.1 Curves that *surround* a puncture or are *parallel* to paths on \(Q(\tau)\)

It will be useful for our purposes to formally define when a curve *surrounds* a puncture with respect to \(\tau\) or when it is *parallel* to an arrow of \(Q(\tau)\). Such a curve \(\gamma\) need not be an arc on \((\Sigma, M)\) or even a segment of an arc.

**Definition 6.1.1.** Let \(\gamma\) be a non-selfintersecting oriented curve on \(\Sigma\), not passing through any marked point, whose extremes lie on an arc \(j\) that belongs to \(\tau\). Given a
puncture $p \in M$ we will say that $\gamma$ surrounds $p$ with respect to $\tau$ if $\gamma$ can be divided into segments $[r_0, r_1], \ldots, [r_l, r_{l+1}]$, with the following properties:

- $r_0, r_{l+1} \in j$ are the extremes of $\gamma$ and $\gamma$ is oriented from $r_0$ to $r_{l+1}$;
- the triangulation $\tau$ has an ideal triangle $\triangle$ that contains $j$ and such that $p$ is the vertex of $\triangle$ opposite to $j$;
- $l$ is the number of arcs of $\tau$ incident to the puncture $p$ (counted with multiplicity);
- the only points of the segment $[r_k, r_{k+1}]_\gamma$ that lie on an arc of $\tau$ are $r_k$ and $r_{k+1}$;
- for $r = 1, \ldots, l-1$, the segment $[r_k, r_{k+1}]_\gamma$ is contractible to the puncture $p$ with a homotopy that avoids $M$ and each of whose intermediate maps are segments with endpoints in the arcs of $\tau$ to which $r_k$ and $r_{k+1}$ belong (this is sketched leftmost in Figure 6.1);

- the segments $[r_0, r_1]_\gamma$ and $[r_l, r_{l+1}]_\gamma$ are contained in $\triangle$ (this is sketched at the center of Figure 6.1);
6.1. Curves that surround a puncture or are parallel to paths on $Q(\tau)$

- the union of the oriented curve $\gamma$ and the oriented segment $[r_{l+1}, r_0]_j$ is a closed simple curve contractible to the puncture $p$, and whose complement in $\Sigma$ consists of two connected components, one of which contains exactly one puncture - namely $p$ (see the rightmost part of Figure 6.1);

- the oriented closed curve of the previous item is oriented in the counterclockwise direction.

Note that, since our triangulation $\tau$ does not have self-folded triangles, the definition of the signed adjacency quiver $Q(\tau)$ allows us to think of each arrow $a : j \to k$ as an oriented simple curve going from a point on the arc $j$ to a point on the arc $k$, and contained in the ideal triangle in which the adjacency between $j$ and $k$ witnessed by $a$ takes place. (see Figure 6.2). This way, there is a marked point $p(a)$ canonically associated to $a$, and $a$ is contractible to it within the ideal triangle at which $j$ and $k$ are adjacent.

![Figure 6.2:](image)

Remark 6.1.2. It is possible for two arcs $j$ and $k$ to be adjacent “twice”, that is, with respect to two different ideal triangles of $\tau$, in such a way that there are two arrows from $j$ to $k$. In this case, assuming that the arrows are actually “drawn” on the surface allows to distinguish which one of the two adjacencies each arrow witnesses.
6.1. Curves that *surround* a puncture or are *parallel* to paths on $Q(\tau)$

We can go a bit further, and assume that the arrows are embedded in such a way that, given any pair $a, b \in Q_1(\tau)$ such that $t(a) = h(b)$, the oriented curve $b$ ends at the same point on $\Sigma$ at which the oriented curve $a$ starts. (That is, we think of $Q(\tau)$ as an oriented graph embedded in the surface $\Sigma$). Thus, paths on $Q(\tau)$ are curves on the surface $\Sigma$.

**Definition 6.1.3.** Let $\gamma$ be a non-selfintersecting oriented curve on $\Sigma$, not passing through any marked point, whose extremes lie on arcs $j$ that belongs to $\tau$.

- We will say that $\gamma$ is *parallel* to an arrow $a$ of $Q(\tau)$ (or $\hat{Q}(\tau)$) if the starting (resp. ending) point of $\gamma$ lies on $t(a)$ (resp. $h(a)$), $\gamma$ is entirely contained in the ideal triangle of $\tau$ containing $a$, and $\gamma$ is homotopic to $a$ through a homotopy that avoids $M$ and each of whose intermediate maps are segments with endpoints in $t(a)$ and $h(a)$ (this is sketched in Figure 6.3 on the left).

- We will say that $\gamma$ is *parallel* to a path $a_n \ldots a_1$ of $Q(\tau)$ (or $\hat{Q}(\tau)$) if $\gamma$ can be written as the concatenation of $n$ oriented segments $\gamma_1, \ldots, \gamma_n$, such that $\gamma_l$ is parallel to the arrow $a_l$ for $l = 1, \ldots, n$ (this is sketched in Figure 6.3 on the right).

![Figure 6.3:](image)
6.2 First case: $i$ does not cut out a once-punctured monogon

Throughout this section we will assume that

the arc $i$ is not a loop that cuts out a once-punctured monogon. \hfill (6.1)

We are going to define a representation $M(\tau, i)$ in several stages. First we define the \textit{detours} of $i$ with respect to $\tau$ and encode them into \textit{detour matrices}. Then we define the \textit{string representation} of $Q(\tau)$ with respect to $i$, and use the detour matrices to modify it and obtain the \textit{arc representation} $M(\tau, i)$ of the unreduced QP $(\tilde{Q}(\tau), \tilde{S}(\tau))$. By Remark 5.0.15, $M(\tau, i)$ will actually be a representation of $(Q(\tau), S(\tau))$, where the action of $Q(\tau)$ on $M(\tau, i)$ is given by simply forgetting the action of the arrows appearing in the 2-cycles of $\tilde{Q}(\tau)$.

Let us begin assuming that $i \notin \tau$. Then, replacing $i$ by an isotopic arc if necessary, we can assume that

$$i \text{ intersects transversally each of the arcs of } \tau \text{ (if at all), and} \hfill (6.2)$$

the number of intersection points of $i$ with each of the arcs of $\tau$ is minimal. \hfill (6.3)

In (6.3) we mean that, if $i'$ is isotopic to $i$, then $i'$ does not have a smaller number of intersection points with any of the arcs of $\tau$. The reader can find in [21], Section 5, the "efficient intersection lemma for ideal arcs", which gives a quite useful combinatorial criterion to know when (6.3) is satisfied.
6.2. First case: \( i \) does not cut out a once-punctured monogon

Fix an arc \( j \in \tau \); it is contained in two ideal triangles. Fix one such triangle \( \Delta \), and let \( \mathfrak{M}_{\tau,i,j}^{\Delta,1} \) be the set whose elements are the ordered quadruples \( (q_1, q_2, r_1, p) \) for which we have the situation sketched in Figure 6.4 on the left, where the segment \( \gamma \) of \( i \) that goes from \( q_1 \) to \( q_2 \) surrounds \( p \) with respect to \( \tau \) (see Definition 6.1.1 above).

**Definition 6.2.1.** For each such quadruple \( (q_1, q_2, r_1, p) \in \mathfrak{M}_{\tau,i,j}^{\Delta,1} \) we draw an oriented simple curve \( d_{(q_1,q_2)}^{\Delta,1} \) contained in \( \Delta \) and going from \( r_1 \) to \( q_2 \), and say that \( d_{(q_1,q_2)}^{\Delta,1} \) is a 1-detour of \( (\tau, i) \) (this is sketched in Figure 6.4 on the right). We will write \( b(d_{(q_1,q_2)}^{\Delta,1}) = r_1 \) for the beginning point of \( d_{(q_1,q_2)}^{\Delta,1} \) and \( e(d_{(q_1,q_2)}^{\Delta,1}) = q_2 \) for its ending point. We shall also say that \( p \) is the puncture detoured by \( d_{(q_1,q_2)}^{\Delta,1} \).

For \( n \geq 1 \), after having drawn all \( n \)-detours of \( (\tau, i) \), take an arc \( j \in \tau \) and fix an ideal triangle \( \Delta \) containing \( j \). Let \( \mathfrak{M}_{\tau,i,j}^{\Delta,n+1} \) be the set whose elements are the ordered quadruples \( (q_1, q_2, b(d^n), p) \) for which we have the situation shown in Figure 6.5 on the left, where the curve \( \gamma \) obtained as the union of the segment \([q_1, r_1]_i\), the detour \( d^n \) and the segment \([r_2, q_2]_i\), oriented from \( q_1 \) to \( q_2 \), surrounds the puncture \( p \).
6.2. First case: \( i \) does not cut out a once-punctured monogon

**Figure 6.5:** Drawing \( n + 1 \)-detours after drawing all \( n \)-detours

![Diagram of \( n + 1 \)-detours](image)

**Definition 6.2.2.** For each quadruple \( (q_1, q_2, b(d^n), p) \in B_{ij}^{\Delta,n+1} \) we draw an oriented simple curve \( d_{(q_1, q_2)}^{\Delta,n+1} \) contained in \( \Delta \) and going from \( b(d^n) \) to \( q_2 \), and say that \( d_{(q_1, q_2)}^{\Delta,n+1} \) is an \((n + 1)\)-detour of \((\tau, i)\) (this is sketched in Figure 6.5 on the right). We will write \( b(d_{(q_1, q_2)}^{\Delta,n+1}) \) and \( c(d_{(q_1, q_2)}^{\Delta,n+1}) \) for the beginning and ending points of \( d_{(q_1, q_2)}^{\Delta,n+1} \). We shall also say that \( p \) is the puncture detoured by \( d_{(q_1, q_2)}^{\Delta,n+1} \).

**Example 6.2.3.** In Figure 6.6 we can see how the process of “detour drawing” is performed. Here, the process stops at 3-detours.

**Remark 6.2.4.**

1. Since each detour connects points of intersection of \( i \) with (arcs of) \( \tau \), and since for a triangle \( \Delta \) and intersection points \( q_1 \) and \( q_2 \) there is at most one detour contained in \( \Delta \) and connecting \( q_1 \) with \( q_2 \), the arc \( i \) has only finitely many detours with respect to \( \tau \). In other words, the process of drawing detours stops after finitely many steps;

2. Given \( n \), a triangle \( \Delta \) may contain more than one \( n \)-detour;

3. Given an \((n + 1)\)-detour \( d^{n+1} \) there exists exactly one \( n \)-detour used to define
6.2. First case: $i$ does not cut out a once-punctured monogon

Figure 6.6: Detours are drawn recursively

$d^{n+1}$. This $n$-detour $d^n$ satisfies $b(d^n) = b(d^{n+1})$, point that lies on an arc of $\tau$ that connects the punctures detoured by $d^n$ and $d^{n+1}$. This means that each $n$-detour $d^n$ determines a sequence $(d^1, \ldots, d^m)$ where $d^m$ is an $m$-detour with $1 \leq m \leq n$ and $b(d^m) = b(d^{m+1})$; the sequence of punctures detoured by the members of the sequence alternates between two punctures of $(\Sigma, M)$.

4. If we think of the arrows of $Q(\tau)$ as oriented curves on the surface, then each detour is parallel to exactly one arrow of $Q(\tau)$. Notice that if an arrow $a$ is parallel to a detour, then $a$ is parallel to a 1-detour.
6.2. First case: \( i \) does not cut out a once-punctured monogon

**Definition 6.2.5.** Using the detours of \((\tau, i)\) we define two *detour matrices* for each arc \( j \) as follows. Take an ideal triangle \( \triangle \) containing \( j \). The rows and columns of the *detour matrix* \( D_{i,j}^{\triangle} \) are indexed by the intersection points of \( i \) with the relative interior of \( j \). For each such point \( q \), the corresponding column of \( D_{i,j}^{\triangle} \) is defined according to the following rules:

- the \( q^{\text{th}} \) entry is 1;
- if \( \tilde{q} \in i \cap j \) is the ending point of an \( n \)-detour \( d_{q,\tilde{q}}^{\triangle,n} \) and there is a quadruple \((q, \tilde{q}, b(d_{q,\tilde{q}}^{\triangle,n}), p) \in \mathfrak{B}_{i,j}^{\triangle,n} \), then the \( \tilde{q}^{\text{th}} \) entry is

\[
(-1)^n x_p^{\frac{n+1}{2}} x_p^{l_4} x_p^{l_4},
\]

where \( \{p, p'\} \) is the set of punctures incident to the arc that contains the point \( b(d_{q,\tilde{q}}^{\triangle,n}) \);
- all the remaining entries of the \( q^{\text{th}} \) column are zero.

We now turn to the definition of the *string* and *arc representations* for \( i \). For each \( j \in \tau \), let \( q_{j,1}, \ldots, q_{j,\Delta(i,j)} \) be an enumeration of the intersection points of \( i \) with the relative interior of \( j \). Define

\[
M(\tau, i)_j = m(\tau, i)_j = K^{\Delta(i,j)},
\]

(6.5)

Next we define the linear maps \((m(\tau, i)_a)_{a \in Q_1(\tau)}\). In the paragraphs to follow, for \( t = 1, \ldots, \Delta(i, j) \), we will write \( K_{j,t} \) to denote the copy of the field \( K \) that corresponds to \( q_{j,t} \) in 6.5). For \( 1 \leq s \leq \Delta(i, j) \) and \( 1 \leq r \leq \Delta(i, k) \), set \((m(\tau, i)_a)_{r,s} : K_{j,s} \to K_{k,r}\)
6.2. First case: $i$ does not cut out a once-punctured monogon
to be the identity if and only if the segment $[q_{j,s}, q_{k,r}]_i$ is parallel to the arrow $a$ (see Definition 6.1.3). Otherwise, define $(m(\tau, i))_{r,s} : K_{j,s} \to K_{k,r}$ to be the zero map.

**Definition 6.2.6.** The representation $m(\tau, i)$ just constructed will be called the *string representation* of $\tilde{Q}(\tau)$ induced by $i$.

It is easy to see that, in the presence of punctures, the string representation $m(\tau, i)$ does not necessarily satisfy the cyclic derivatives of $\tilde{S}(\tau)$. Let us illustrate with an example.

**Example 6.2.7.** Consider the arc $i$ and the ideal triangulation $\tau$ of the once-punctured hexagon shown in Figure 6.7, where the representation $m(\tau, i)$ is shown as well. This representation obviously satisfies the cyclic derivatives of the potential $S(\tau) = abc + xa\delta\epsilon\eta g$. Furthermore, after applying the sequence of mutations $\mu_{j_1}, \mu_{j_2}, \mu_{j_3}, \mu_{j_4}, \mu_{j_5}, \mu_{j_6}$, we get the $j_6$th negative simple representation $S^-_6(\mathbb{D}_6, 0) = \mu_{j_6}\mu_{j_5}\mu_{j_4}\mu_{j_3}\mu_{j_1}(m(\tau, i))$ of the QP $(\mathbb{D}_6, 0) = \mu_{j_6}\mu_{j_5}\mu_{j_4}\mu_{j_3}\mu_{j_1}(Q(\tau), S(\tau))$, where $\mathbb{D}_6$ has the following orientation.
6.2. First case: $i$ does not cut out a once-punctured monogon

and labeling of vertices:

![Diagram](image)

If we flip the arc $j_2$ of $\tau$, we obtain the ideal triangulation $\sigma = f_{j_2}(\tau)$ shown in Figure 6.8, where also the representation $m(\sigma, i)$ is shown. This representation does not satisfy all the cyclic derivatives of the potential $S(\sigma) = \gamma \beta \alpha + x \alpha \delta \varepsilon \eta$, namely, it is not annihilated by $\partial_\alpha (S(\sigma)) = \gamma \beta + x \delta \varepsilon \eta$. Therefore, $m(\sigma, i)$ cannot be obtained from $m(\tau, i)$ by applying the $j_2$th mutation. Consequently, the sequence $\mu_2, \mu_1, \mu_3, \mu_4, \mu_5, \mu_6$, does not bring $m(\sigma, i)$ to the negative simple representation $S^-_6(\mathbb{D}_6, 0) = \mu_{j_6} \mu_{j_5} \mu_{j_4} \mu_{j_3} \mu_{j_1} (m(\tau, i))$ of $\mathbb{D}_6$.

We modify $m(\tau, i)$ using the detour matrices as follows. For each arrow $a : j \to k$ of $Q(\tau)$, let $\triangle^a$ be the unique ideal triangle that contains $a$. Define the linear map
6.2. First case: $i$ does not cut out a once-punctured monogon

$M(\tau, i)_a : K^A(i,j) \to K^A(i,k)$ to be given by the matrix product

$$(D_{i,k}^a)(m(\tau, i)_a).$$

Definition 6.2.8. With the action of $R\langle\langle Q(\tau)\rangle\rangle$ induced by the inclusion of quivers $Q(\tau) \hookrightarrow \tilde{Q}(\tau)$, the representation $M(\tau, i)$ will be called the arc representation of $Q(\tau)$ induced by $i$.

Remark 6.2.9. 1. In many cases (even for punctured surfaces), the arc representation $M(\tau, i)$ coincides with the string representation $m(\tau, i)$. When the surface has no punctures, we always have $M(\tau, i) = m(\tau, i)$.

2. For $(\Sigma, M) =$ unpunctured polygon, the arc representations were defined by P. Caldero, F. Chapoton an R. Schiffler (cf. [6]), and have been recently generalized to the situation $(\Sigma, M) =$ unpunctured surface by I. Assem, T. Brüstle, G. Charbonneau-Jodoin and P-G. Plamondon (cf. [2]). (None of these authors, however, has used the term “arc representation”)

3. We use the terms “string representation" and “arc representation" for lack of better terms. Actually, in the presence of punctures, the Jacobian algebras $\mathcal{P}(Q(\tau), S(\tau))$ are not necessarily gentle algebras, hence our string representations may not satisfy the definition of string module most commonly accepted in representation theory of algebras.

To illustrate these constructions, let us have a look at an example.

Example 6.2.10. Consider the triangulation $\tau$ and the arc $i$ on the twice-punctured hexagon shown in Figure 6.9.
It is straightforward to see that

\[ \mathcal{B}_{i,j_1}^{\Delta_1,1} = \{ (p_1, p_2, r, q) \}, \quad \mathcal{B}_{i,j_1}^{\Delta_2,1} = \{ (q_1, q_2, r, p) \}, \quad \mathcal{B}_{i,j_1}^{\Delta_1,2} = \{ (p_1, p_3, r, q) \}, \quad \mathcal{B}_{i,j_1}^{\Delta_2,2} = \emptyset. \]

and for \( n \geq 3 \) and \( j \in Q_1(\tau) \), and

\[ M(\tau, i)_{j_1} = K, \quad M(\tau, i)_{j_2} = K^3, \quad M(\tau, i)_{j_3} = K^2, \quad M(\tau, i)_{j_4} = M(\tau, i)_{j_5} = K, \]

\[ M(\tau, i)_{j_6} = M(\tau, i)_{j_7} = M(\tau, i)_{j_8} = M(\tau, i)_{j_9} = K^2. \]
Hence all detour matrices are the corresponding identities, except

\[
D_{i,j_2}^{\Delta_1} = \begin{bmatrix}
1 & 0 & 0 \\
-y & 1 & 0 \\
xy & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
D_{i,j_6}^{\Delta_2} = \begin{bmatrix}
1 & 0 \\
-x & 1
\end{bmatrix},
\]

where \( x = x_p \) and \( y = x_q \). The linear maps \( m(\tau, i)_a \) of the string representation are given as follows

\[
m(\tau, i)_\alpha = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\quad
m(\tau, i)_\beta = \begin{bmatrix}
1 \\
0
\end{bmatrix},
\quad
m(\tau, i)_\gamma = 0,
\quad
m(\tau, i)_\delta = \begin{bmatrix}
0 & 1
\end{bmatrix},
\]

\[
m(\tau, i)_\varepsilon = \begin{bmatrix}
1 \\
0
\end{bmatrix},
\quad
m(\tau, i)_\eta = 0,
\quad
m(\tau, i)_{d_1} = 1,
\quad
m(\tau, i)_{d_2} = \begin{bmatrix}
1 & 0
\end{bmatrix},
\]

\[
m(\tau, i)_{d_3} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix},
\quad
m(\tau, i)_{b_1} = 1,
\quad
m(\tau, i)_{b_2} = 1,
\quad
m(\tau, i)_{b_3} = 1.
\]

Therefore, the representation \( M(\tau, i) \) is
6.2. First case: \( i \) does not cut out a once-punctured monogon

Note that this representation is actually a \( \mathcal{P}(Q(\tau), S(\tau)) \)-module, that is, it is nilpotent and satisfies the relations imposed by the potential \( S(\tau) = \alpha \beta \gamma + \delta \varepsilon \eta + x \beta \eta d_1 d_2 d_3 + y \varepsilon \gamma b_1 b_2 b_3 \). Therefore, \( \mathcal{M}(\tau, i) = (Q(\tau), S(\tau), M(\tau, i), V(\tau, i)) \) is a QP-representation, where \( V(\tau, i) = 0 \).

If we flip the arc \( j_1 \), we get ideal triangulation \( \sigma = f_{j_1}(\tau) \) shown in Figure 6.10 (abusing notation, we use the same symbol \( j_1 \) in both \( \tau \) and \( \sigma \)) and the following

Figure 6.10:
representation of its signed adjacency quiver

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0
-\epsilon & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
-\gamma & 0 \\
0 & 1 & 0 & -\epsilon
\end{pmatrix}
\]

which obviously satisfies the relations imposed by the potential \(S(\sigma) = [\beta \eta] \gamma \eta^* \beta^* + [\epsilon \gamma] \gamma^* \epsilon^* + [\beta \eta] d_1 d_2 d_3 + [\epsilon \gamma] b_1 b_2 b_3.\)

A straightforward calculation shows that, up to right-equivalence, this representation (with the zero decoration) can be obtained also by performing the mutation \(\mu_{j_1}\) to \(\mathcal{M}(\tau, i)\). That is, the flip of \(j_1\) has the same effect on \(\mathcal{M}(\tau, i)\) as the \(j_1\)th QP-mutation.

6.3 Second case: \(i\) cuts out a once-punctured monogon

In this section we deal with the case where the arc \(i\) is a loop that cuts out a once-punctured monogon from \((\Sigma, M)\). Specifically, throughout this section we will keep assuming that \(\tau\) is an ideal triangulation of \((\Sigma, M)\) without self-folded triangles, that
6.3. Second case: $i$ cuts out a once-punctured monogon

$i$ is an arc on $(\Sigma, M)$, $i \not\in \tau$, satisfying (6.2) and (6.3), and that

the arc $i$ is a loop that cuts out a once-punctured monogon from $(\Sigma, M)$. \hfill (6.7)

Let $\odot$ be the monogon cut out by $i$ and $p$ be the puncture inside $\odot$. Consider
the (unique) arc $k$ that connects $p$ with the marked point $m$ at which $i$ is based and
is contained in $\odot$. If the arc $k$ belongs to $\tau$, then we define the arc representation
$M(\tau, i)$ following the exact same rules of Section 6.2.

So, assume $k$ does not belong to $\tau$. Then all arcs of $\tau$ that have $p$ as an extreme
point intersect the relative interior of $i$. Consider the collection of segments (of arcs
of $\tau$) that are contained in $\odot$ and go from $p$ to a point on $i$. Let $\mathfrak{F}$ be the set of
terminal points of these segments (thus all elements of $\mathfrak{F}$ lie on the relative interior
of $i$, see Figure 6.11). If we traverse $i$ in the clockwise direction around $p$, at some

Figure 6.11: $\mathfrak{F}$ consists of the circled intersection points

moment we will begin passing through the elements of $\mathfrak{F}$. Right after exhausting
these elements, before having finished traversing $i$, we must pass through a point of
$i \cap \tau$ that does not belong to $\mathfrak{F}$ (see Figure 6.11). Let $t$ be the first such point, and
delete the segment of $i$ we have not traversed yet (see Figure 6.12).

The segment of $i$ we have not deleted is a curve $\iota = \iota_{\tau, i}$ on $(\Sigma, M)$ one of whose
endpoints is $m$, a marked point, $t$ being the other endpoint. For $n \geq 1$ we define the $n$-detours of $(\tau, i)$ in the exact same way we did in the previous subsection, but with respect to $i$ instead of $i$. The detour matrices of $(\tau, i)$ are also defined in the exact same way.

Now we turn to the definition of the string representation $m(\tau, i)$. For each arc $j \in \tau$, assume that the points in which the oriented segment $[m, t]_s$ intersects $j$ are $q_{j,1}, \ldots, q_{j,\mathbb{A}(i,j)}$. The vector spaces attached to the vertices of $Q(\tau)$ by the string representation will be given by

$$m(\tau, i)_j = K^{\mathbb{A}(i,j)}.$$  

(6.8)

The linear maps $m(\tau, i)_a$ are defined as follows. Let $a : j \to k$ be an arrow of $\tilde{Q}(\tau)$, and assume that the segment $[m, t]_s$ intersects the relative interior of $j$ (resp. $k$) in the $\mathbb{A}(i, j)$ (resp. $\mathbb{A}(i, k)$) different points $q_{j,1}, \ldots, q_{j,\mathbb{A}(i,j)}$ (resp. $q_{k,1}, \ldots, q_{k,\mathbb{A}(i,k)}$). Let $(m(\tau, i)_a)_r,s : K_{q_{j,s}} \to K_{q_{k,r}}$ be the identity if and only if the segment $[q_{k, r}, q_{j, s}]_i$ is parallel to the arrow $a$. Otherwise, define $(m(\tau, i)_a)_r,s : K \to K$ to be the zero map.

**Definition 6.3.1.** The representation $m(\tau, i)$ just constructed will be called the *string representation* of $Q(\tau)$ induced by $i$.

Just as in Section 6.2, it is easy to see that the string representation $m(\tau, i)$ does
6.3. Second case: \(i\) cuts out a once-punctured monogon

not satisfy the cyclic derivatives of \(S(\tau)\). So we modify it using the detour matrices to produce the arc representation \(M(\tau, i)\). The dimension of this representation will be one less than that of \(m(\tau, i)\). Let \(i'\) be the arc of \(\tau\) containing \(t\); for \(j \neq i'\), we set

\[
M(\tau, i)_j = m(\tau, i)_j.
\]

As for \(i'\), the space \(M(\tau, i)_{i'}\) is defined to be the quotient of \(m(\tau, i)_{i'}\) by the copy of \(K\) that corresponds to the intersection point \(t\). That is, \(M(\tau, i)_{i'}\) takes into account only the intersection points of \([m, f]\) with \(i'\). Note that there is a canonical inclusion \(M(\tau, i)_{i'} \hookrightarrow m(\tau, i)_{i'}\) (because the subspace being mod out is a coordinate subspace).

Now let us define the linear maps of the arc representation. Let \(a : j \rightarrow k\) be an arrow of \(Q(\tau)\), and \(\triangle^a\) be the unique ideal triangle of \(\tau\) that contains \(a\). If \(j \neq i' \neq k\), then \(M(\tau, i)_a : K : M(\tau, i)_j \rightarrow M(\tau, i)_k\) is defined to be \((D_{i,k}^a)(m(\tau, i)_a)\).

If \(i' = j\), then \(M(\tau, i)_a = (D_{i,k}^a)(m(\tau, i)_a)\ell\), where \(\ell : M(\tau, i)_{i'} \rightarrow m(\tau, i)_{i'}\) is the canonical vector space inclusion. And if \(i' = k\), then \(M(\tau, i)_a = \pi(D_{i,k}^a)(m(\tau, i)_a)\), where \(\pi : m(\tau, i)_{i'} \rightarrow M(\tau, i)\) is the canonical vector space projection.

**Definition 6.3.2.** With the action of \(R(\langle Q(\tau) \rangle)\) induced by the inclusion of quivers \(Q(\tau) \hookrightarrow \tilde{Q}(\tau)\), the representation \(M(\tau, i)\) will be called the **arc representation** of \(Q(\tau)\) induced by \(i\).

**Remark 6.3.3.** The arc representation \(M(\tau, i)\) never coincides with the string representation \(m(\tau, i)\).

To illustrate this definition, let us give an example.

**Example 6.3.4.** Consider the triangulation \(\tau\) and the arc \(i\) on the twice-punctured hexagon shown in Figure 6.13. The point \(t\) is indicated there, and farthest right we
can see the segment $i = i_{i, \tau}$ and its detours with respect to $\tau$. It is straightforward to see that

$$\mathcal{B}^{\triangle_{1,1}}_{i,j_1} = \{(q_1, q_3, b(d_{(1,3)}^{\triangle_{1,1}}), p)\}, \quad \mathcal{B}^{\triangle_{2,1}}_{i,j_1} = \{(q_2, q_1, b(d_{(2,1)}^{\triangle_{2,1}}), q)\}$$

and $\mathcal{B}^{\triangle,n}_{i,j} = \varnothing$ for $n \geq 2$ and $j \in Q_1(\tau)$

$$m(\tau,i)_{j_1} = K^3, \ m(\tau,i)_{j_6} = K^2, \ \text{and} \ m(\tau,i)_{j_l} = K \ \text{for} \ l = 2, 3, 4, 5, 7, 8, 9;$$

Hence $D^{\triangle}_{i,j_l} = 1$ for $l = 2, \ldots, 9$ (and each ideal triangle $\triangle$ containing $j_l$), whereas

$$D^{\triangle_{1}}_{i,j_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x & 0 & 1 \end{bmatrix}, \quad \text{and} \quad D^{\triangle_{2}}_{i,j_1} = \begin{bmatrix} 1 & -y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $x = x_p$ and $y = x_q$. The linear maps $m(\tau,i)_a$ of the string representation are
6.3. Second case: \( i \) cuts out a once-punctured monogon

given as follows

\[
m(\tau, i)_\alpha = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad m(\tau, i)_\beta = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad m(\tau, i)_\gamma = 0, \quad m(\tau, i)_\delta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\]

\[
m(\tau, i)_\varepsilon = 0, \quad m(\tau, i)_\eta = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad m(\tau, i)_{d_1} = 1, \quad m(\tau, i)_{d_2} = 1,
\]

\[
m(\tau, i)_{d_3} = 1, \quad m(\tau, i)_{b_1} = 1, \quad m(\tau, i)_{b_2} = 1 \quad m(\tau, i)_{b_3} = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\]

Therefore, the representation \( M(\tau, i) \) is

\[
\begin{array}{c}
\begin{array}{cccc}
K & \xrightarrow{1} & K & \xrightarrow{1} & K \\
1 & \rightarrow & 0 & \rightarrow & 1 \\
K & \xrightarrow{1} & K & \xrightarrow{1} & K
\end{array}
\end{array}
\]

\[
\begin{array}{cccc}
K^2 & \xrightarrow{1} & K^2 & \xrightarrow{1} & K \\
0 & \rightarrow & 0 & \rightarrow & 1 \\
K^2 & \xrightarrow{1} & K^2 & \xrightarrow{1} & K
\end{array}
\]

Note that this representation is actually a \( P(Q(\tau), S(\tau)) \)-module, that it, it satisfies the relations imposed by the potential \( S(\tau) = \alpha \beta \gamma + y b_1 b_2 b_3 \gamma + \delta \varepsilon \eta + x d_1 d_2 d_3 \varepsilon \).

Therefore, \( \mathcal{M}(\tau, i) = (Q(\tau), S(\tau), M(\tau, i), V(\tau, i)) \) is a QP-representation, where \( V(\tau, i) = 0 \).

If we flip the arc \( j_1 \), we get ideal triangulation \( \sigma = f_{j_1}(\tau) \) shown in Figure 6.14 (abusing notation, we use the same symbol \( j_1 \) in both \( \tau \) and \( \sigma \)) and the following
representation of its signed adjacency quiver

![Diagonal Quiver](image)

\[
\begin{pmatrix}
K & K & K & K \\
1 & K & 1 & K \\
\end{pmatrix}
\]

which obviously satisfies the relations imposed by the potential \( S(\sigma) = [\alpha \delta] \delta^* \alpha^* + [\eta \beta] \beta^* \eta^* + x \delta^* \eta^* d_1 d_2 d_3 + y \beta^* \alpha^* b_1 b_2 b_3. \)

A straightforward calculation shows that, up to right-equivalence, this representation can be obtained also by performing the mutation \( \mu_{j_1} \) to \( M(\tau, i) \). That is, the flip of \( j_1 \) has the same effect as the \( j_1^{th} \) QP-mutation.
Chapter 7

Towards the main result of Part II: simplifying tools

7.1 Restriction

The operation of restriction of QPs is a simple, yet useful, operation: In Part I it helped to relatively simplify the proof of Theorem 3.3.1 by allowing to focus on surfaces with empty boundary. In this section we study its representation-theoretic analogue with the same aim in mind: reducing the proof of Theorem 9.2.1, the main result of Part II, to the situation of surfaces with empty-boundary.

Let \((Q, S)\) be a QP and \(I\) be a subset of the vertex set \(Q_0\). Notice that \(Q|_I\) is the quiver whose vertex set is \(Q_0\) and whose arrow set \((Q|_I)_1 = Q_1|_I\) consists of the arrows of \(Q\) having both head and tail in \(I\). We will use the notation \(u|_I = \rho_I(u)\) for the restriction to \(I\) of an element \(u\) of a complete path algebra. It is straightforward to see that \(\partial_a(S|_I) = \partial_a(S)|_I\) for all \(a \in (Q|_I)_1\) and \(\partial_b(S)|_I = 0\) for every arrow \(b\) of
7.1. Restriction

Q not belonging to \((Q|I)_1\).

For a representation \(M\), one can define the restriction \(M|_I\) in the obvious way, namely, by setting \((M|_I)_j = 0\) for \(j \notin I\), \((M|_I)_i = M_i\) for \(i \in I\), and \(a_{M|_I} = a_M : M_{t(a)} \rightarrow M_{h(a)}\) for \(a \in Q_1|_I\). However, if the representation \(M\) satisfies the cyclic derivatives of a potential \(S\), the restriction \(M|_I\) does not necessarily satisfy the relations obtained by restricting the cyclic derivatives of \(S\).

**Definition 7.1.1.** Let \(I\) be a subset of the vertex set \(Q_0\), and \(M\) be a representation of \(Q\). We will say that \(M\) is \(I\)-path-restrictable if for each pair \(i, j \in I\), every path from \(i\) to \(j\) in \(Q\) passing through a vertex \(k \notin I\) acts as zero on \(M_i\). Accordingly, a QP-representation \(M = (Q, S, M, V)\) will be called \(I\)-path-restrictable if \(M\) is \(I\)-path-restrictable.

Clearly, if the representation \(M\) of \(Q\) is \(I\)-path-restrictable and satisfies the cyclic derivatives of the potential \(S\), then \(M|_I\) satisfies the cyclic derivatives of \(S|_I\) (which are also the restrictions of the cyclic derivatives of \(S\) as we noted above). This implies the following:

**Lemma 7.1.2.** Let \(I\) be a subset of the vertex set \(Q_0\) and \(M = (Q, S, M, V)\) be a QP-representation such that \(M\) is \(I\)-path-restrictable. Set \((M|_I)_i = M_i\) for \(i \in I\) and \((M|_I)_j = 0\) for \(j \notin I\). Then \((Q|_I, S|_I, ((M|_I)_i)_{i \in Q_0}, (a_M)_{a \in Q_1|_I}, (V_i)_{i \in Q_0})\) is a QP-representation.

**Definition 7.1.3.** In the situation of Lemma 7.1.2, the QP-representation \((Q|_I, S|_I, ((M|_I)_i)_{i \in Q_0}, (a_M)_{a \in Q_1|_I}, (V_i)_{i \in Q_0})\) will be called the restriction of \(M = (Q, S, M, V)\) to \(I\) and denoted \(M|_I\).
7.1. Restriction

The following lemma and its corollary are a representation-theoretic analogue of Lemma 3.2.2 above.

**Lemma 7.1.4.** If the QP-representations \( \mathcal{M} = (Q, S, M, V) \) and \( \mathcal{M}' = (Q', S', M', V') \) are right-equivalent and \( \mathcal{M} \) is \( I \)-path-restrictable, then \( \mathcal{M}' \) is \( I \)-path-restrictable as well, and the restrictions \( \mathcal{M}|_I \) and \( \mathcal{M}'|_I \) are right-equivalent.

**Proof.** Let \( \Phi = (\varphi, \psi, \eta) \) be a right-equivalence between \( \mathcal{M} = (Q, S, M, V) \) and \( \mathcal{M}' = (Q', S', M', V') \). Consider the \( R \)-algebra homomorphism \( \varphi|_I : R\langle\langle Q|_I\rangle\rangle \to R\langle\langle Q'|_I\rangle\rangle \), defined by the rule \( u \mapsto \varphi(u)|_I \). We claim that \( \varphi|_I(S|_I) = \varphi(S)|_I \). To see this, write \( S = S|_I + S' \), where \( S' \in R\langle\langle Q\rangle\rangle \) is a potential each of whose terms has at least one arrow \( b \notin Q|_I \). Then each term of \( \varphi(S') \) has at least one arrow not from \( Q'|_I \), which means that \( \varphi(S')|_I = 0 \), and hence \( \varphi(S)|_I = \varphi(S|_I)|_I = \varphi|_I(S|_I) \). Now, the \( R \)-algebra homomorphism \( \rho_I : R\langle\langle Q'\rangle\rangle \to R\langle\langle Q'|_I\rangle\rangle \), being continuous, sends cyclically equivalent potentials to cyclically equivalent ones, from which it follows that \( \varphi(S)|_I \) is cyclically equivalent to \( S'|_I \). Therefore, \( \varphi|_I \) is a right-equivalence between \( (Q|_I, S|_I) \) and \( (Q'|_I, S'|_I) \).

To see that \( \mathcal{M}' \) is \( I \)-path-restrictable if \( \mathcal{M} \) is, take any pair of vertices \( i, j \in I \), and let \( u \) be any path from \( i \) to \( j \) in \( Q' \) that passes through some vertex \( k \notin I \). Then for \( m \in M'_i \) we have \( u_{M'M'}m = \varphi(\varphi^{-1}(u))_{M'M'}(\psi^{-1}(m)) = \psi \circ (\varphi^{-1}(u))_{M'}(\psi^{-1}(m)) = 0 \) since \( \varphi^{-1}(u) \) is a (possibly infinite) linear combination of paths that pass through \( k \notin I \). This shows that \( \mathcal{M}' \) is \( I \)-path-restrictable.

Note that \( M|_I \) (resp. \( M'|_I \)) is an \( R \)-submodule of \( M \) (resp. \( M' \)), and \( \psi(M|_I) = M'|_I \). This allows us to define \( \psi|_I : M|_I \to M'|_I \) as the restriction of the map \( \psi \) to \( M|_I \). Clearly \( \psi|_I : M|_I \to M'|_I \) is a \( K \)-vector space isomorphism, and for \( a \in Q|_I \) we
have \( \psi|_I \circ a_{M|_I} = (\psi \circ a_M)|_I = (\varphi(a)_{M'} \circ \psi)|_I = (\varphi|_I(a)_{M'|_I}) \circ \psi|_I \) (the last equality follows from the fact that \( M' \) is \( I \)-path-restrictable).

It follows that the triple \( \Phi|_I = (\varphi|_I, \psi|_I, \eta) \) is a right-equivalence between \( M|_I \) and \( M'|_I \). \( \square \)

**Remark 7.1.5.** The first paragraph of the above proof is an alternative proof of Lemma 3.2.2 above.

**Corollary 7.1.6.** Let \( I \) be a subset of the vertex set \( Q_0 \) and \( M = (Q, S, M, V) \) be an \( I \)-path-restrictable QP-representation. Then \( M_{\text{red}} \) is \( I \)-path-restrictable and \( M_{\text{red}}|_I \) is right-equivalent to the reduced part of \( M|_I \).

**Proof.** Let \( \varphi : R\langle Q_{\text{red}} \oplus C \rangle \rightarrow R\langle Q \rangle \) be a right-equivalence between \( (Q_{\text{red}}, S_{\text{red}}) \oplus (C, T) \) and \( (Q, S) \), where \( (Q_{\text{red}}, S_{\text{red}}) \) is a reduced QP and \( (C, T) \) is a trivial QP. Then the triple \( \Phi = (\varphi, 1_M, 1_V) \) is a right-equivalence between \( (Q_{\text{red}} \oplus C, S_{\text{red}} + T, M^\varphi, V) \) and \( (Q, S, M, V) \) (where \( M^\varphi = M \) as \( K \)-vector spaces, see Proposition 5.0.13 and Definition 5.0.14). By Lemma 7.1.4 \( M^\varphi \) is \( I \)-path-restrictable. By the proof of Lemma 7.1.4, the triple \( \Phi|_I = (\varphi|_I, 1_{M|_I}, 1_V) \) is a right-equivalence between \( (Q_{\text{red}}|_I \oplus C|_I, S_{\text{red}}|_I + T|_I, M'|_I, V) \) and \( (Q|_I, S|_I, M|_I, V) \). By Proposition 5.0.13, this implies that the reduced part of \( M|_I \) is right-equivalent to \( (Q_{\text{red}}|_I, S_{\text{red}}|_I, M'|_I, V) \). \( \square \)

**Theorem 7.1.7.** Let \( (Q, S) \) be a QP, \( I \) a subset of the vertex set \( Q_0 \), and \( j \in I \). If \( M = (Q, S, M, V) \) is an \( I \)-path-restrictable QP-representation, then the mutation \( \mu_j(M) \) is \( I \)-path-restrictable as well, and the restriction \( \mu_j(M)|_I \) is right-equivalent to the mutation \( \mu_j(M|_I) \).
7.2. Gluing disks along boundary components

Proof. An easy check shows that \((\tilde{Q}_I, \tilde{S}_I, \overline{M}_I, \overline{\nu}) = (\tilde{Q}_I, \tilde{S}_I, \overline{M}_I, \overline{\nu})\), where \((\tilde{Q}_I, \tilde{S}_I) = \tilde{\mu}_j(Q_I|_I, S_I|_I)\) and \((\tilde{Q}_I, \tilde{S}_I) = \tilde{\mu}_j(Q, S)|_I\). The theorem follows then from Corollary 7.1.6. □

Remark 7.1.8. It is very easy to give examples where \(I\)-path-restorentability fails and the conclusion of Theorem 7.1.7 does not hold. Consider, for instance the QP-representation \(\mathcal{M}(\tau, i)\) of Example 6.2.10, with \(I = \{j_1\}\). On the other hand, there are conditions weaker than path-restorentability that still ensure the conclusion of Theorem 7.1.7; we do not state these conditions here since we will not need them.

7.2 Gluing disks along boundary components

The following lemma is the representation-theoretic extension of Lemma 3.2.5.

Lemma 7.2.1. For every QP-representation of the form \(\mathcal{M}(\tau, i)\) there exists an ideal triangulation \(\tilde{\tau}\) of a surface \((\tilde{\Sigma}, \tilde{M})\) with empty boundary with the following properties:

1. \(\Sigma \subseteq \tilde{\Sigma}\) and \(M \subseteq \tilde{M}\);

2. \(\tilde{\tau}\) contains all the arcs of \(\tau\);

3. \(\mathcal{M}(\tilde{\tau}, i)\) is \(\tau\)-path-restorentable, and the restriction of \(\mathcal{M}(\tilde{\tau}, i)\) to \(\tau\) is \(\mathcal{M}(\tau, i)\).

Proof. Let \(\tau\) be an ideal triangulation of a surface \((\Sigma, M)\) with non-empty boundary. Each boundary component \(b\) of \(\Sigma\) is homeomorphic to a circle. Let \(m_b\) be the number of marked points lying on \(b\). If \(m_b = 1\) then we can glue \(\Sigma\) and a triangulated twice-punctured monogon (without self-folded triangles) along \(b\); whereas if \(m_b > 1\), we can glue \(\Sigma\) and a triangulated (unpunctured) \(m_b\)-gon along \(b\). After doing this for
each boundary component of $\Sigma$, we end up with an ideal triangulation $\tilde{\tau}$ of a surface $(\tilde{\Sigma}, \tilde{M})$ with empty boundary. By construction, $\tilde{\tau}$ does not have self-folded triangles, and parts (1) and (2) of the lemma are obviously satisfied.

Now, if $i$ is an arc on $(\Sigma, M)$, then it is also an arc on $(\tilde{\Sigma}, \tilde{M})$. Furthermore, the fact that $i$ minimizes the number of intersection points with each of the arcs in $\tilde{\tau}$ follows from the "efficient intersection lemma for ideal arcs" stated in Section 5 of [21]. Thus the representation $\mathcal{M}(\tilde{\tau}, i)$ is well-defined. And since $i$ does not intersect the topological closure of the union of the triangulated disks glued to $(\Sigma, M)$ along the boundary components, it is clear that every path in $Q(\tau)$ that passes through a vertex $k \notin \tau$ acts as zero on $\mathcal{M}(\tilde{\tau}, i)$. Hence $\mathcal{M}(\tilde{\tau}, i)$ is $\tau$-path-restrictable.

Induction on $n \geq 1$ shows that the $n$-detours of $(\tilde{\tau}, i)$ are precisely the $n$-detours of $(\tau, i)$. Therefore, the restriction of $\mathcal{M}(\tilde{\tau}, i)$ to $\tau$ is precisely $\mathcal{M}(\tau, i)$. The lemma follows. \hfill $\Box$

**Remark 7.2.2.** In general, given a quiver $Q$, and a proper subset $I$ of $Q_0$, the elements of $Q_0 \setminus I$ are isolated vertices of the restriction to $I$, that is, there are no arrows of $A|_I$ whose head or tail belongs to $Q_0 \setminus I$. So, strictly speaking, for part (3) of Lemma 7.2.1 to be true, the arcs in $\tilde{\tau}$ have to be added to $Q(\tau)$ as isolated vertices.

### 7.3 Local decompositions

Derksen-Weyman-Zelevinsky’s mutation of QP-representations takes direct sums to direct sums. If one traces the proof of this *additivity*, one realizes that mutation actually takes *local* direct sums to *local* direct sums. This section is devoted to give a precise statement of this fact.
7.3. Local decompositions

Let \((Q, S')\) be an arbitrary QP, and let \(j \in Q_0\) be any vertex. Define a quiver \(Q(\partial) = Q(\partial, j)\) as follows: the set of vertices of \(Q(\partial)\) consists of all heads and all tails of the arrows of \(Q\) that are incident to \(j\). For each \(j\)-hook \(ba\) of \(Q\), introduce one arrow \(\gamma_{ba} : h(b) \rightarrow t(a)\); the set of arrows of \(Q(\partial)\) consists of all the arrows of \(Q\) that are incident to \(j\) and all the arrows of the form \(\gamma_{ba} : h(b) \rightarrow t(a)\). Define also a set of relations on \(Q(\partial)\) by setting \(R(\partial) = R(\partial, j) = \{\sum_{b : t(b) = j} \gamma_{ba} b \mid a \in Q_1, h(a) = j\} \cup \{\sum_{a : h(a) = j} a\alpha_{ba} \mid b \in Q_1, t(b) = j\}\).

Given a representation \(M\) of \(Q\) (not necessarily nilpotent or annihilated by the cyclic derivatives of \(S\)), let \(M(\partial) = M(\partial, j)\) be the representation of \(Q(\partial)\) that attaches to each vertex of \(Q(\partial)\) the same vector space \(M\) attaches to it. As for the linear maps, for each arrow \(a\) of \(Q\) incident to \(j\) let \(M(\partial)_a = M_a\), and for each \(j\)-hook \(ba\) of \(Q\), let \(M(\partial)_{\gamma_{ba}} = \partial_{[ba]}([S])_M : M_{h(b)} \rightarrow M_{t(a)}\). As remarked in [8] and [9],

\[
M \text{ is a } \mathcal{P}(Q, S)\text{-module if and only if } M \text{ is nilpotent and } \quad (7.1)
\]

\(M(\partial, j)\) satisfies all the relations in \(R(\partial, j)\) for all \(j \in Q_0\).

Now let \(\mathcal{M} = (Q, S, S(\tau), M, V)\) be a QP-representation. A quick look at Chapter 5 makes us see that in order to calculate the \(j\)th mutation of \(\mathcal{M}\), it is enough to apply the mutation process with respect to the data defining \(M(\partial)\). The next Proposition, whose proof can be given using only basic linear algebra, tells us that if we decompose \(M(\partial)\) as the direct sum of subrepresentations (which may be possible even when \(M\) is indecomposable), then in order to calculate \(\mu_j(\mathcal{M})\) it is enough to apply the mutation process to each of the summands of \(M(\partial)\) separately.

**Proposition 7.3.1.** Let \((Q, S)\) be any QP and \(\mathcal{M} = (Q, S, M, V)\) be a decorated
(Q, S)-representation. Fix a vertex \( j \in Q_0 \) and, with respect to this vertex, define the quiver \( Q(\partial) \) and the representation \( M(\partial) \) as above. Suppose that \( M(\partial) \) decomposes as

\[
M(\partial) = N^1 \oplus \ldots \oplus N^t,
\]

where the representations \( N^1, \ldots, N^t \), need not be indecomposable. For \( 1 \leq l \leq t \) let \( N^l \) denote the representation \( N^l \) with the zero decoration and \( \mu_j(N^l) = (\overline{N^l}, \overline{V^l}) \) denote the decorated representation obtained from \( N^l \) by applying the mutation process with respect to the data \( a_{N^l}, b_{N^l}, c_{N^l} \) of \( N^l \). Then the mutation \( \mu_j(M) \) is isomorphic, as a representation of \( \mu_j(Q, S) \), to the direct sum of \( \mu_j(Q, S, 0, V) \) and \( (\mu_j(Q, S), M_N, V_N) \), where \( M_N \) is the representation obtained from \( \overline{N^1} \oplus \ldots \oplus \overline{N^t} \) by remembering the spaces and maps attached by \( M \) to the arrows of \( Q \) not incident to \( j \), and \( V_N \) is the decoration obtained from \( \overline{V^1} \oplus \ldots \oplus \overline{V^t} \) by attaching the zero vector space to each of the vertices of \( Q \) that are not head or tail of an arrow incident to \( j \).

### 7.4 Local decompositions of arc representations

Let \((Q, S)\) be a QP. The problem of finding local decompositions of representations of \( Q \) makes sense even for those representations that are not known to be nilpotent or annihilated by the cyclic derivatives of \( S \). In this section we tackle the referred problem for arc representations (whose nilpotency and Jacobian annihilation have not been yet proved). Our underlying assumption will be that the surface \( \Sigma \) we work on has empty boundary (by the results of Sections 7.1 and 7.2 this will not mean any loss of generality). Thus, given an ideal triangulation \( \tau \) of \((\Sigma, M)\), an arc \( j \in \tau \) such that neither \( \tau \) nor \( \sigma = f_j(\tau) \) have self-folded triangles is the diagonal of a quadrilateral
7.4. Local decompositions of arc representations

\(\Diamond\) of \(\tau\) each of whose "sides" are (not necessarily different) arcs of \(\tau\), and \(j\) is tail of exactly two arrows of \(Q(\tau)\) and head of exactly two arrows of \(Q(\tau)\). We adopt the notation of Figure 7.1, so that \(j = j_1\) and \(j_2, j_3, j_4\) and \(j_5\) are the arcs directly connected to \(j_1\) by means of an arrow of \(\hat{Q}(\tau)\).

![Figure 7.1:](image)

Throughout the rest of the thesis we will assume that

\[ i \text{ is not an arc cutting out a once-punctured monogon.} \quad (7.2) \]

This is due to space reasons; actually, for arcs cutting out once-punctured monogons, the forthcoming results about arc representations will remain true under suitable
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Let $I = I_{j_i}$ be the set of points of intersection of $i$ with the arcs $j_1, j_2, j_3, j_4, j_5$ (excluding the endpoints of $i$ if these occur as intersection points with the mentioned arcs). Let $R = R_{j_i}$ be the equivalence relation on $I$ generated by declaring two points $q, q' \in i \cap (j_1 \cup j_2 \cup j_3 \cup j_4 \cup j_5)$ to be equivalent whenever there is an oriented curve $\xi$ on $\Sigma \setminus M$ with $q$ and $q'$ as its extreme points and satisfying any of the following conditions (cf. Definition 6.1.3):

(a) $\xi$ is a segment of $i$ (that is, $\xi = [q, q']$, or $\xi = [q', q]$) and is parallel to any of the following paths on $\bar{Q}(\tau)$: $\alpha, \beta, \gamma, \delta, \epsilon, \eta, a_1 \ldots a_t, b_1 \ldots b_m, c_1 \ldots c_n, d_1 \ldots d_t$;

(b) $\xi$ is the concatenation of a detour parallel to the arrow $a_i$ (resp. $b_m, c_n, d_t$) followed by a segment of $i$ that is parallel to the path $a_1 \ldots a_{t-1}$, (resp. $b_1 \ldots b_{m-1}$, $c_1 \ldots c_{n-1}, d_1 \ldots d_{t-1}$).

**Proposition 7.4.1.** Let $H^1, \ldots, H^t$ be the equivalence classes induced by $R$. For each such class $H^s$ and each arc $j_r$, $1 \leq s \leq t$, $1 \leq r \leq 5$, let $N^s_{j_r}$ be the vector subspace of $M(\tau, i)_{j_r}$ spanned by (basis vectors corresponding to) the intersection points of $i$ with $j_r$ that belong to $H^s$ and are different from the point $t$ of Figure 6.12 (in case $i$ is a loop cutting out a once-punctured monogon). Then each $N^s$ is a subrepresentation of $M(\partial) = M(\tau, i)(\partial)$ and $M(\partial) = N^1 \oplus \ldots \oplus N^t$.

**Remark 7.4.2.** 1. The representations $N^s$ need not be indecomposable.

2. Note that throughout this section we have not supposed that the vertices of $\diamond$ (labeled with the scalars $w, x, y, z$) are different. Thus, for instance, $\alpha$ may
coincide with one of the arrows $a_1, \ldots, a_l, b_1, \ldots, b_m, c_1, \ldots, c_n, d_1, \ldots, d_t$, but this does not affect at all the definition of $\mathcal{R}$ nor the validity of Proposition 7.4.1. This is due to the fact that, although the potential $S(\tau)$ can change drastically if we identify different vertices of $\diamond$, the form of the cyclic derivatives of $S(\tau)$ remains the same (a commutativity relation).

**Example 7.4.3.** In Figure 7.2 we can see a triangulation $\tau$ of a hexagon with five punctures, and an arc $i$ not belonging to $\tau$. The five bold arcs on $\tau$ are $j_1$ and the sides of the quadrilateral $\diamond$ of which $j_1$ is a diagonal. The equivalence relation $\mathcal{R}$ defined above has two equivalence classes, given by the intersection points of the blue and red segments of $i$ with each of the arcs $j_1, j_2, j_3, j_4$ and $j_5$. The segments of $i$ not
relevant to the definition of $\mathcal{R}$ have been drawn as dotted segments. It is fairly easy to check that the red and blue segments give rise to a decomposition of $M(\partial, j_1)$ as the internal direct sum of subrepresentations canonically attached to them.

### 7.5 Dicing direct summands of $M(\partial, j)$

Let $(Q, S)$ be an arbitrary QP, and let $j \in Q_0$ be any vertex. Consider the quiver $Q(\partial) = Q(\partial, j)$ defined in Section 7.3.

**Proposition 7.5.1.** Let $M$ be a representation of $Q(\partial)$ annihilated by all the relations in $R(\partial)$. Suppose we are given subrepresentations $M'$ and $M''$ of $M$, and vector subspaces $H'_{h(b)}$, $N'_{h(b)} \subseteq M'_h(b)$ and $H''_{h(b)}$, $N''_{h(b)} \subseteq M''_{h(b)}$, for all $b \in Q_1$ with $t(b) = j$, and $L_j \subseteq M_j$, that satisfy the following conditions:

1. $M_{t(a)} = M'_{t(a)} \oplus M''_{t(a)}$ for all $a \in Q_1$ with $h(a) = j$, $M_h(b) = M'_h(b) \oplus M''_h(b)$ for all $b \in Q_1$ with $t(b) = j$, and $M'_j \cap M''_j = 0$;

2. $M'_h(b) = H'_{h(b)} \oplus N'_{h(b)}$ and $M''_h(b) = H''_{h(b)} \oplus N''_{h(b)}$ for all $b \in Q_1$ with $t(b) = j$;

3. $M_j = M'_j \oplus L_j \oplus M''_j$;

4. the linear map $\pi' b : M_j \rightarrow M'_\text{out}$ (resp. $\pi'' b : M_j \rightarrow M''_\text{out}$) maps $L_j$ bijectively into the subspace $N'_\text{out} = \bigoplus_{b,t(b)=j} N'_{h(b)}$ (resp. $N''_\text{out} = \bigoplus_{b,t(b)=j} N''_{h(b)}$) of $M'_\text{out}$ (resp. $M''_\text{out}$), where $\pi' : M_\text{out} = M'_\text{out} \oplus M''_\text{out} \rightarrow M'_\text{out}$ and $\pi'' : M_\text{out} = M'_\text{out} \oplus M''_\text{out} \rightarrow M''_\text{out}$ are the canonical projections;

5. $\text{im } b' \subseteq H'_\text{out} = \bigoplus_{b,t(b)=j} H'_{h(b)}$ and $\text{im } b'' \subseteq H''_\text{out} = \bigoplus_{b,t(b)=j} H''_{h(b)}$. 


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Let $P' = \frac{M'}{P'}$ and (resp. $P'' = \frac{M''}{P''}$). Then the representation

\[
\begin{array}{c}
\xymatrix{
M_j \ar[rr]^\pi \ar[rd]_c & & \ar[ll]_{\bar{b}} M_{\text{out}} \\
M_{\text{in}} \ar[rr]_{\bar{b}a} & & M_{\text{out}} \\
}
\end{array}
\]

(7.3)

is isomorphic to the representation

\[
\begin{array}{c}
\xymatrix{
P'_j \oplus P''_j \oplus L_j \ar[rr]^\mathcal{A} \ar[rd]_c & & \ar[ll]_{\mathcal{B}} P'_j \oplus P''_j \\
P'_j \oplus P''_j \ar[rr]_{\bar{b}a} & & P'_j \oplus P''_j \\
}
\end{array}
\]

(7.4)

where

\[
\mathcal{A} = \begin{bmatrix}
\bar{a}_{P'} & 0 \\
0 & \bar{a}_{P''} \\
-(\pi'|_{L_j})^{-1} p' & (\pi''|_{L_j})^{-1} p''
\end{bmatrix}
\quad \text{and} \quad
\mathcal{B} = \begin{bmatrix}
\bar{b}_{P'} & 0 & 0 \\
0 & \bar{b}_{P''} & 0
\end{bmatrix},
\]

$p'$ (resp. $p''$) being the canonical projection $H'_j \oplus N'_j \to N'_j$ (resp. $H''_j \oplus N''_j \to N''_j$). The isomorphism going from (7.4) to (7.3) can be chosen so that its restrictions to $P'_j$, $P''_j$, $P'_j$, $P''_j$, are the corresponding inclusions of these spaces into $M_{\text{in}}$ and $M_{\text{out}}$. Furthermore, \[
\frac{\ker b}{\ker b \cap \ker a} = \frac{\ker b_{P'}}{\ker b_{P'} \cap \ker a_{P'}} \oplus \frac{\ker b_{P''}}{\ker b_{P''} \cap \ker a_{P''}}.
\]

Proof. Let us start by giving an alternative description of $P'$ and $P''$, namely:

\[
P'_j = M'_j \oplus L_j, \quad P'_{t(a)} = M'_{t(a)}, \quad P'_{h(b)} = M'_{h(b)}.
\]
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\[ P''_j = L_j \oplus M''_j, \quad P''_{t(a)} = M''_{t(a)}, \quad P''_{h(b)} = M''_{h(b)} \]

for $a, b \in Q_1$ such that $h(a) = j$, $t(b) = j$, and

\[
\begin{bmatrix}
    a' \\
    0
\end{bmatrix}
\xrightarrow{P'_j}
\begin{bmatrix}
    b' \\
    \pi' b|_{L_j}
\end{bmatrix},
\begin{bmatrix}
    0 \\
    a''
\end{bmatrix}
\xrightarrow{P''_j}
\begin{bmatrix}
    b'' \\
    \pi'' b|_{L_j}
\end{bmatrix}.
\]

Write $b : M_j \rightarrow M_{\text{out}}$ as a matrix of linear maps

\[
b = 
\begin{bmatrix}
    b' & 0 & 0 \\
    0 & \pi' b|_{L_j} & 0 \\
    0 & 0 & b'' \\
    0 & \pi'' b|_{L_j} & 0
\end{bmatrix} : M'_j \oplus L_j \oplus M''_j \rightarrow H'_\text{out} \oplus N'_\text{out} \oplus H''_\text{out} \oplus N''_\text{out}.
\]

Thus,

\[
b_{P'} = 
\begin{bmatrix}
    b' & 0 \\
    0 & \pi' b|_{L_j}
\end{bmatrix} : M'_j \oplus L_j \rightarrow H'_\text{out} \oplus N'_\text{out}
\]

and $b_{P''} = 
\begin{bmatrix}
    0 & b'' \\
    \pi'' b|_{L_j} & 0
\end{bmatrix} : L_j \oplus M''_j \rightarrow H''_\text{out} \oplus N''_\text{out}$.

Since $M'$ and $M''$ are subrepresentations of $M$, and $M_{\text{in}} = M'_{\text{in}} \oplus M''_{\text{in}}$ and $M_{\text{out}} =$
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\[ M'_{\text{out}} \oplus M''_{\text{out}}, \text{ we have} \]

\[ a = \begin{bmatrix} a' & 0 \\ 0 & 0 \\ 0 & a'' \end{bmatrix} : M'_{\text{in}} \oplus M''_{\text{in}} \to M'_j \oplus L_j \oplus M''_j \]

and \( c = \begin{bmatrix} c' & 0 \\ 0 & c'' \end{bmatrix} = \begin{bmatrix} c_{P'} & 0 \\ 0 & c_{P''} \end{bmatrix} : M'_{\text{out}} \oplus M''_{\text{out}} \to M'_{\text{in}} \oplus M''_{\text{in}}, \]

which means that there are direct sum decompositions

\[ \ker a = \ker a' \oplus \ker a'' = \ker a_{P'} \oplus \ker a_{P''} \subseteq M'_{\text{in}} \oplus M''_{\text{in}}, \quad \text{im} \, c = \text{im} \, c_{P'} \oplus \text{im} \, c_{P''} \subseteq M'_{\text{in}} \oplus M''_{\text{in}}, \]

\[ \ker c = \ker c_{P'} \oplus \ker c_{P''} \subseteq M'_{\text{out}} \oplus M''_{\text{out}}, \]

from which we in particular have \( \frac{\ker a}{\text{im} \, c} = \frac{\ker a_{P'}}{\text{im} \, c_{P'}} \oplus \frac{\ker a_{P''}}{\text{im} \, c_{P''}}. \) On the other hand, the function \( g : \frac{\ker c}{\text{im} \, b} \to \frac{\ker c_{P'}}{\text{im} \, b_{P'}} \oplus \frac{\ker c_{P''}}{\text{im} \, b_{P''}} \oplus L_j \) given by \((u', v', u'', v'') + \text{im} \, b \mapsto ((u', 0) + \text{im} \, b_{P'}, (u'', 0) + \text{im} \, b_{P''}, (\pi' b|_{L_j})^{-1}(v') - (\pi'' b|_{L_j})^{-1}(v''))\) is easily seen to be well-defined, linear and bijective.

Whence we obtain an isomorphism

\[ \frac{\ker c}{\text{im} \, b} \oplus \frac{\ker a}{\text{im} \, c} \quad \to \quad \frac{\ker c_{P'}}{\text{im} \, b_{P'}} \oplus \frac{\ker a_{P'}}{\text{im} \, c_{P'}} \oplus \frac{\ker c_{P''}}{\text{im} \, b_{P''}} \oplus \frac{\ker a_{P''}}{\text{im} \, c_{P''}} \oplus L_j, \quad (7.6) \]

which we shall denote by \( f. \)
Choose sections \( s' : \frac{\ker a_{p'}}{\im c_{p'}} \to \ker a_{p'} \), \( s'' : \frac{\ker a_{p''}}{\im c_{p''}} \to \ker a_{p''} \). Then the map

\[
\begin{bmatrix}
  s' & 0 \\
  0 & s''
\end{bmatrix}
\]

is a section. Since \( f^{-1} \) sends the space \( L_j \) into \( \frac{\ker c}{\im b} \), this implies that the linear map

\[
\overline{b} \circ f^{-1} : P'_j \oplus P''_j \oplus L_j \to M'_m \oplus M''_m
\]

has matrix form

\[
\begin{bmatrix}
  b_{p'} & 0 & 0 \\
  0 & b_{p''} & 0
\end{bmatrix}
\]

It is easy to see that \( N'_m \subseteq \ker c_{p'} \), \( N''_m \subseteq \ker c_{p''} \), and hence \( \ker c_{p'} = (H'_m \cap \ker c_{p'}) \oplus N'_m \), \( \ker c_{p''} = (H''_m \cap \ker c_{p''}) \oplus N''_m \). Consequently, if we choose rejections

\[
r' : H'_m \to H'_m \cap \ker c_{p'} , \quad r'' : H''_m \to H''_m \cap \ker c_{p''},
\]

then the maps

\[
\begin{bmatrix}
  r' & 0 \\
  0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
  r'' & 0 \\
  0 & 1
\end{bmatrix}
\]

are rejections. Therefore, if we write the linear map \( f \circ \overline{a} : M'_m \to P'_j \oplus P''_j \oplus L_j \) in terms of the decompositions \( M'_m = P'_m \oplus P''_m \) and \( P'_j \oplus P''_j \oplus L_j = \frac{\ker c_{p'}}{\im b_{p'}} \oplus \frac{\ker c_{p''}}{\im b_{p''}} \oplus \frac{\ker c_{p''}}{\im b_{p''}} \oplus L_j \),
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\[ \text{im } \epsilon^{\prime} \oplus \frac{\ker \epsilon^{\prime}}{\text{im } \epsilon^{\prime}} \oplus \frac{\ker \epsilon^{\prime\prime}}{\text{im } \epsilon^{\prime\prime}} \oplus \text{im } \epsilon^{\prime} \oplus \frac{\ker \epsilon^{\prime\prime}}{\text{im } \epsilon^{\prime\prime}} \oplus L_j, \quad \text{we get} \]

\[
\bar{\alpha} = \begin{bmatrix}
-\mathbf{p}' \mathbf{c}' & 0 \\
-\mathbf{c}_P' & 0 \\
0 & 0 \\
0 & -\mathbf{p}' \mathbf{c}'^{\prime\prime} \\
0 & -\mathbf{c}_P^{\prime\prime} \\
-(\pi' b|_L_j)^{-1} \mathbf{p}' & (\pi' b|_L_j)^{-1} \mathbf{p}'^{\prime\prime}
\end{bmatrix},
\]

where the projection $\mathbf{p} : \ker \mathbf{c} \rightarrow \frac{\ker \mathbf{c}}{\text{im } \mathbf{b}}$ is assumed to have been written in matrix form as

\[
\mathbf{p} = \begin{bmatrix}
\mathbf{p}' & 0 \\
0 & \mathbf{p}^{\prime\prime}
\end{bmatrix} : \ker \mathbf{c}' \oplus \ker \mathbf{c}^{\prime\prime} \rightarrow \frac{\ker \mathbf{c}'}{\text{im } \mathbf{b}'} \oplus \frac{\ker \mathbf{c}^{\prime\prime}}{\text{im } \mathbf{b}^{\prime\prime}},
\]

the diagonal entries being the canonical projections to the corresponding quotients.

We have thus shown that the identity functions $\mathbf{1}_{\text{in}}$ and $\mathbf{1}_{\text{out}}$, together with the map $f : \overline{M}_j \rightarrow \overline{\mathcal{P}}_j \oplus \overline{\mathcal{P}}^{\prime}_j \oplus L_j$ from (7.6), constitute an isomorphism between the representation in (7.3) and the representation in (7.4).

That $\frac{\ker b}{\ker b \cap \text{im } \alpha} = \frac{\ker b_{\mathcal{P}}}{\ker b_{\mathcal{P}} \cap \text{im } a_{\mathcal{P}}} \oplus \frac{\ker b_{\mathcal{P}^{\prime\prime}}}{\ker b_{\mathcal{P}^{\prime\prime}} \cap \text{im } a_{\mathcal{P}^{\prime\prime}}}$ is obvious. \(\Box\)

**Definition 7.5.2.** Let $M$ be a representation of $Q(\partial)$ (not necessarily nilpotent nor annihilated by the relations in $R(\partial)$). If we are given subrepresentations $M'$ and $M''$ of $M$, and vector subspaces $H'_h(b), N'_h(b) \subseteq M'_h(b)$ and $H''_h(b), N''_h(b) \subseteq M''_h(b)$, for all $b \in Q_1$ with $t(b) = j$, and $L_j \subseteq M_j$, that satisfy conditions (1)-(5) of Proposition 7.5.1, we will say that the triple $(M', L_j, M'')$ is a source-dicing of $M$. 

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**Example 7.5.3.** Consider the QP $(Q, S)$ shown in Figure 7.3 and the representation $N$ of $(Q, S)$ shown in Figure 7.4 on the left. The representation $M = N(\partial, j_1)$ is shown in Figure 7.4 on the right. Let $M'$ and $M''$ be the subrepresentations of $M$ whose spaces are given by

\[
M'_{j_1} = \langle (0, 1, 0, 0, 0), (-y, 0, 1, 0, 0) \rangle, \quad M'_{j_2} = K, \quad M'_{j_3} = K \oplus K \oplus 0 \oplus 0,
\]

\[
M'_{j_4} = 0 \quad \text{and} \quad M'_{j_5} = \langle (0, 1, 0, 0), (-y, 0, 1, 0) \rangle;
\]

\[
M''_{j_1} = \langle (0, 0, 0, 1, 0), (0, 0, -x, 0, 1) \rangle, \quad M''_{j_2} = 0, \quad M''_{j_3} = \langle (0, 0, 1, 0), (0, -x, 0, 1) \rangle,
\]
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Figure 7.4:

\[ M''_{j_4} = K, \quad M''_{j_5} = 0 \oplus 0 \oplus K \oplus K. \]

Let also \( L_{j_1} = 0 \oplus 0 \oplus K \oplus 0 \oplus 0 \subseteq M_{j_1} \). Then the triple \( (M', L_{j_1}, M'') \) is a source-dicing of \( M \). To see this, we set

\[ N'_{j_3} = 0 \oplus K \oplus 0 \oplus 0, \quad H'_{j_3} = \langle (-y, 1, 0, 0) \rangle, \quad N''_{j_3} = 0 \quad \text{and} \quad H''_{j_3} = M'_{j_3}. \]

\[ N''_{j_3} = 0, \quad H''_{j_3} = M''_{j_3}, \quad N''_{j_5} = 0 \oplus 0 \oplus K \oplus 0, \quad H''_{j_5} = \langle (0, 0, -x, 1) \rangle \]

By Proposition 7.5.1, the mutation \( \mu_{j_1}(M) \) can be computed in terms of \( \mu_{j_1}(P') \)
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and $\mu_{j_1}(p'')$, where $P' = \frac{M}{M'}$ and $P'' = \frac{M}{M''}$. These quotients are shown in Figure 7.5 (the matrices shown are calculated in terms of the bases chosen above).

Using the bases

\[
\{(0, 1, 0, 0, 0), (-y, 0, 1, 0, 0), (0, 0, 0, 1, 0, 0)\} \text{ of } M'_{j_1} \oplus L_{j_1},
\]

\[
\{(1, 0, 0, 0), (0, 1, 0, 0)\} \text{ of } M'_{j_3}, \quad \{(0, 1, 0, 0), (-y, 0, 1, 0)\} \text{ of } M'_{j_5},
\]

\[
\{(0, 0, 1, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, -x, 0, 1)\} \text{ of } L_{j_3} \oplus M''_{j_3},
\]

\[
\{(0, 0, 0, 1, 0), (0, -x, 0, 1)\} \text{ of } M''_{j_3} \text{ and } \{(0, 0, 1, 0), (0, 0, 0, 1)\} \text{ of } M''_{j_5},
\]
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we get

$P'_{\text{in}} \oplus P''_{\text{in}} \cong P'_{\text{out}} \oplus P''_{\text{out}}$ 

On the other hand, using the bases

$\{(0, 1, 0, 0, 0), (-y, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 1, 0), (0, 0, 0, 1, 0), (0, 0, 0, 1, 0)\}$ of $M_{j_1}$,

$\{(0, 1, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1, 0)\}$ of $M_{j_3}$,

and $\{(0, 1, 0, 0), (-y, 0, 1, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ of $M_{j_5}$,

we get

$M_{\text{in}} \cong M_{\text{out}}$ 

$K^3 \begin{bmatrix} -1 & 0 & 0 & -y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ -\frac{1}{x} & -1 & 0 & 0 & 0 & 0 & 1 & \therefore \end{bmatrix}$
7.5. Dicing direct summands of $M(\partial, j)$

The isomorphism from (7.4) to (7.3) is then given by the matrices $1 : P'_i \oplus P''_i \to M_i$,

$$
\begin{bmatrix}
-\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
: P'_j \oplus P''_j \oplus L_j \to M_j \quad \text{and} \quad
\begin{bmatrix}
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
: P'_\text{out} \oplus P''_\text{out} \to M_\text{out}
$$

For considering it of independent interest, we state and prove the “sink” version of Proposition 7.5.1.

**Proposition 7.5.4.** Let $M$ be a representation of $Q(\partial)$ annihilated by all the relations in $R(\partial)$. Suppose we are given subrepresentations $M'$ and $M''$ of $M$, and vector subspaces $H'_t(a), N'_t(a) \subseteq M'_t(a)$ and $H''_t(a), N''_t(a) \subseteq M''_t(a)$, for all $a \in Q_1$ with $h(a) = j$, and $L'_j \subseteq M'_j, L''_j \subseteq M''_j$, that satisfy the following conditions:

1. $M_t(a) = M'_t(a) \oplus M''_t(a)$ for all $a \in Q_1$ with $h(a) = j$, $M_{h(b)} = M'_h(b) \oplus M''_h(b)$ for all $b \in Q_1$ with $t(b) = j$, and $M_j = M'_j + M''_j$;
2. $M'_t(a) = H'_t(a) \oplus N'_t(a)$ and $M''_t(a) = H''_t(a) \oplus N''_t(a)$ for all $a \in Q_1$ with $h(a) = j$;
3. $M'_j = L'_j \oplus (M'_j \cap M''_j)$ and $M''_j = (M'_j \cap M''_j) \oplus L''_j$;
4. the subspace $N'_i = \bigoplus_{a:h(a)=j} N'_t(a)$ (resp. $N''_i = \bigoplus_{a:h(a)=j} N''_t(a)$) of $M'_i$ (resp. $M''_i$) is mapped bijectively to $M'_j \cap M''_j$ by the linear map $a' : M'_i \to M'_j$ (resp. $a'' : M''_i \to M''_j$);
5. the subspace $H'_i = \bigoplus_{a:h(a)=j} H'_t(a)$ (resp. $H''_i = \bigoplus_{a:h(a)=j} H''_t(a)$) of $M'_i$ (resp. $M''_i$) is mapped into $L'_j$ (resp. $L''_j$) by the linear map $a' : M'_i \to M'_j$ (resp. $a'' : M''_i \to M''_j$).
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\[ a'' : M_{in}'' \rightarrow M_j'' \].

Then the representation

\[ \overline{M_j} \]

is isomorphic to the representation

\[ \overline{M_j'} \oplus \overline{M_j''} \oplus (M_j' \cap M_j'') \]

where

\[ \mathcal{A} = \begin{bmatrix} \overline{a'} & 0 \\ 0 & \overline{a''} \\ 0 & 0 \end{bmatrix}, \text{ and } \mathcal{B} = \begin{bmatrix} \overline{b'} & 0 & (\overline{a'}|_{N_j})^{-1} \\ 0 & \overline{b''} & - (\overline{a''}|_{N_{j''}})^{-1} \end{bmatrix} \].

Furthermore, the isomorphism going from (7.8) to (7.7) can be chosen so that its restrictions to $P_{in}'$, $P_{in}''$, $P_{out}'$, $P_{out}''$, are the corresponding inclusions of these spaces into $M_{in}$ and $M_{out}$.

Proof. Write $a : M_{in} \rightarrow M_j$ as a matrix of linear maps

\[ a = \begin{bmatrix} A_1' & 0 & 0 & 0 \\ 0 & A_2' & 0 & A_2'' \\ 0 & 0 & A_1'' & 0 \end{bmatrix} : H_{in}' \oplus N_{in}' \oplus H_{in}'' \oplus N_{in}'' \rightarrow L_j' \oplus (M_j' \cap M_j'') \oplus L_j''. \]
Then we clearly have the identities

\[
\alpha' = \begin{bmatrix}
 A'_1 & 0 \\
 0 & A'_2 \\
\end{bmatrix} : M'_{in} \to M'_j \quad \text{and} \quad \alpha'' = \begin{bmatrix}
 0 & A''_2 \\
 A''_1 & 0 \\
\end{bmatrix} : M''_{in} \to M''_j.
\]

Since \( N'_{in} \) (resp. \( N''_{in} \)) is mapped bijectively to \( M'_j \cap M''_j \) by \( \alpha' \) (resp. \( \alpha'' \)), the map \( A'_2 \) (resp. \( A''_2 \)) is bijective. Hence, setting \( W = \{(0, (\alpha'|_{N'_{in}})^{-1}(w), 0, - (\alpha''|_{N''_{in}})^{-1}(w)) \mid w \in M'_j \cap M''_j \} \subseteq H'_i \oplus N'_{in} \oplus H''_i \oplus N''_{in} \), we have an internal direct sum decomposition \( \ker \alpha = (\ker \alpha' + \ker \alpha'') \oplus W \), and \( \ker \alpha \) is therefore isomorphic to \( \ker \alpha' \oplus \ker \alpha'' \oplus M'_j \cap M''_j \) by means of the map \( \ker \alpha \to \ker \alpha' \oplus \ker \alpha'' \oplus M'_j \cap M''_j \) given by \( (u_1, u_2, v_1, v_2) \mapsto ((u_1, 0), (v_1, 0)), \alpha'(u_2)) \) (here and below, we are writing \( \ker \alpha' = \ker A'_1 \oplus 0 \subseteq H'_i \oplus N'_{in} \) and \( \ker \alpha'' = \ker A''_1 \oplus 0 \subseteq H''_i \oplus N''_{in} \)).

Since \( M' \) and \( M'' \) are subrepresentations of \( M \), and \( M_{in} = M'_{in} \oplus M''_{in} \) and \( M_{out} = M'_{out} \oplus M''_{out} \), we have \( b'(M'_j \cap M''_j) = 0 = b''(M'_j \cap M''_j) \) and

\[
\begin{bmatrix}
 c' & 0 \\
 0 & c'' \\
\end{bmatrix} : M'_{out} \oplus M''_{out} \to M'_{in} \oplus M''_{in}.
\]

Thus we have direct sum decompositions

\[
\ker c = \ker c' \oplus \ker c'' \subseteq M'_{out} \oplus M''_{out}, \quad \text{im } c = \text{im } c' \oplus \text{im } c'' \subseteq M'_{in} \oplus M''_{in};
\]

\[
\text{im } b = \text{im } b' \oplus \text{im } b'' \subseteq \ker c' \oplus \ker c''.
\]

Hence, \( \ker c \varepsilon_{\text{im } b} = \ker c' \varepsilon_{\text{im } b'} \oplus \ker c'' \varepsilon_{\text{im } b''} \)

From the previous two paragraphs we also see that the isomorphism \( \ker a \to \)
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\( \ker \alpha' \oplus \ker \alpha'' \oplus M'_j \cap M''_j \) induces an isomorphism \( g : \ker \frac{A}{\text{im} c} \oplus \ker \frac{A'}{\text{im} c'} \oplus \ker \frac{A''}{\text{im} c''} \oplus (M'_j \cap M''_j) \)
given by \((u_1, u_2, v_1, v_2) + \text{im} c) \mapsto ((u_1, 0) + \text{im} c', (v_1, 0) + \text{im} c'', \alpha(u_2))\). Therefore we obtain an isomorphism

\[
\ker \frac{c}{\text{im} b} \oplus \text{im} c \oplus \ker \frac{a}{\text{im} c} \rightarrow \ker \frac{c'}{\text{im} b'} \oplus \text{im} c' \oplus \ker \frac{a'}{\text{im} c'} \oplus \ker \frac{c''}{\text{im} b''} \oplus \text{im} c'' \oplus \ker \frac{a''}{\text{im} c''} \oplus (M'_j \cap M''_j), \quad (7.9)
\]

which we will denote by \( f \).

Choose sections \( s' : \ker \frac{a'}{\text{im} c'} \rightarrow \ker \alpha' \), \( s'' : \ker \frac{a''}{\text{im} c''} \rightarrow \ker \alpha'' \). Then the map

\[
s = \begin{bmatrix} s' & 0 & 0 \\ 0 & s'' & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

is a section. Therefore, if we write the linear map \( \overline{b} = [0 \ i \ is] : \overline{M}_j = \ker \frac{c}{\text{im} b} \oplus \text{im} \ c \oplus \ker \frac{a}{\text{im} b} \rightarrow M_{\text{in}} \) in terms of the decompositions \( \overline{M}_j = \ker \frac{c'}{\text{im} b'} \oplus \ker \frac{c''}{\text{im} b''} \oplus \text{im} c' \oplus \text{im} c'' \oplus \ker \frac{a'}{\text{im} c'} \oplus \ker \frac{a''}{\text{im} c''} \oplus W \) and \( M_{\text{in}} = H'_j \oplus N'_j \oplus H''_j \oplus N''_j \), we get

\[
\overline{b} = \begin{bmatrix} 0 & 0 & \pi' t' & 0 & \pi' s' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pi' | W' \\ 0 & 0 & 0 & \pi'' t'' & 0 & \pi'' s'' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pi'' | W'' \end{bmatrix},
\]

where \( t' : \text{im} c' \hookrightarrow H'_j \oplus N'_j = M'_j \) (resp. \( t'' : \text{im} c'' \hookrightarrow H''_j \oplus N''_j = M''_j \)) is the inclusion and \( \pi' : M'_j = H'_j \oplus N'_j \rightarrow H'_j \) (resp. \( \pi'' : M''_j = H''_j \oplus N''_j \rightarrow H''_j \)) is the projection onto the first component (recall that we are writing \( \ker \alpha' = \ker A'_1 \oplus 0 \subseteq H'_j \oplus N'_j \) and \( \ker \alpha'' = \ker A''_1 \oplus 0 \subseteq H''_j \oplus N''_j \), so \( \text{im} c' \subseteq H'_j \oplus 0 \) and \( \text{im} c'' \subseteq H''_j \oplus 0 \), which
7.5. Dicing direct summands of $M(\partial, j)$ explains why we have followed $i'$ and $i''$, respectively, by $\pi'$ and $\pi''$.

Now choose retractions $r': M'_\text{out} \to \ker c'$, $r'': M''_\text{out} \to \ker c''$. Then the map

$$
r = \begin{bmatrix} r' & 0 \\
0 & r'' \end{bmatrix} : M_{\text{out}} = M'_\text{out} \oplus M''_\text{out} \to \ker c' \oplus \ker c'' = \ker c
$$

is a retraction. Therefore, if we write the linear map $\overline{\alpha} = [-pr' - c' 0]^t : M_{\text{out}} \longrightarrow \ker c \oplus \text{im} c \oplus \ker a' \oplus \text{im} a' \oplus \ker a'' \oplus \text{im} a'' \oplus W$, we get

$$
\overline{\alpha} = \begin{bmatrix}
-p'r' & 0 \\
0 & -p''r' \\
-c' & 0 \\
0 & -c'' \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix},
$$

where the projection $p : \ker c \to \ker c \oplus \text{im} b'$ is assumed to have been written in matrix form as

$$
p = \begin{bmatrix} p' & 0 \\
0 & p'' \end{bmatrix} : \ker c' \oplus \ker c'' \to \text{im} b' \oplus \text{im} b'',
$$

the diagonal entries being the canonical projections to the corresponding quotients.

Having computed $\overline{\alpha}$ and $\overline{b}$, a straightforward check shows that the identity functions $1_{M_{\text{in}}}$ and $1_{M_{\text{out}}}$, together with the map $f : \overline{M}_j \to \overline{M}'_j \oplus \overline{M}''_j \oplus (M'_j \cap M''_j)$
from (7.9), constitute an isomorphism between the representation in (7.7) and the representation in (7.8).

\[\square\]

### 7.6 Dicing equivalence classes of \( R_j \)

In this section we dice the equivalence classes induced by the equivalence relation \( R_j \) introduced in Section 7.4, and show that this dicing corresponds to dicing direct summands of \( M(\tau, i)(\partial, j) \). The notation and assumptions from the referred Section will be adopted here.

Let \( C \subseteq \mathcal{I} = \mathcal{I}_{j_1} \) be an equivalence class of \( R = R_{j_1} \), and let \( \iota = \iota_C \) be the union of all segments \( \xi \) of \( i \) whose extremes belong to \( C \). Then \( \iota \) is a connected segment of \( i \).

For each \( v \in C \) such that there is at least one curve \( \xi \) starting at \( v \) and satisfying condition (b) of Section 7.4, let \( s_v \) be the union of all segments of \( i \) whose extreme points are elements of \( C \) that lie on some such curve \( \xi \). Then \( s_v \) is a segment of \( \iota \). Define \( \mathcal{T} = \mathcal{T}_C = \{ [q, q']_\iota \mid q \text{ and } q' \text{ lie on opposite sides of } \diamond, [q, q']_\iota \text{ crosses } j_1 \text{ exactly once, and } [q, q']_\iota \text{ is not contained in any segment of the form } s_v \} \) and \( \mathcal{S} = \mathcal{S}_C = \{ s_v \mid v \in C \text{ is the starting point of a curve } \xi \text{ satisfying condition (b) of Section 7.4} \} \cup \mathcal{T} \).

If \( \mathcal{S} = \emptyset \), then \( \iota \) coincides with one of the configurations shown in Figures 7.6, 7.7, and 7.8. (In Figure 7.6, \( j_2, j_3, j_4, j_5 \) are assumed to be four different arcs of \( \tau \). In Figure 7.7 they are assumed to be exactly three arcs of \( \tau \), with \( j_2 = j_4 \) or \( j_3 = j_5 \); we have indicated this by drawing coinciding arcs as bolder straight-lines. In Figure 7.8, we have \( j_2 = j_4 \) and \( j_3 = j_5 \), so that the underlying surface is a once-punctured
torus. For space reasons, the arc $j_1$, which can be either diagonal of the quadrilateral shown, has not been drawn—drawing it would double the number of configurations shown.)

If $\mathcal{S}$ has exactly one element, then $\iota$ coincides with one of the configurations shown in Figures 7.9, 7.10, 7.11, and 7.12. (Just as before, in Figure 7.10, $j_2, j_3, j_4, j_5$ are assumed to be four different arcs of $\tau$. In Figure 7.11 they are assumed to be exactly three arcs of $\tau$, with $j_2 = j_4$ or $j_3 = j_5$. In Figure 7.12, the underlying surface is a once-punctured torus. Again, the arc $j_1$ has not been drawn.)
7.6. Dicing equivalence classes of $\mathcal{R}_j$

Figure 7.7: $\mathcal{S} = \emptyset$. Bold arcs are to be identified

So, let us suppose that $|\mathcal{S}| > 1$. Then $\mathcal{T} \neq \emptyset$. Give an orientation to $\iota$. We can assume that the elements of $\mathcal{S}$ and the set consisting of the extreme points of the
elements of \( S \) have been ordered along the orientation chosen for \( \ell \). We define two segments \( \mathcal{t}_1 \) and \( \mathcal{t}_1' \) of \( \mathcal{t} \) as follows. Let \( \ell_1 \) be the first element of \( \mathcal{T} \). Then the segment \( \mathcal{t}_1 \) goes from the initial point of \( \mathcal{t} \) to the initial point of \( \ell_1 \), and \( \mathcal{t}_1' \) goes from the final point of \( \ell_1 \) to the final point of \( \mathcal{t} \). Notice that the triple \( (\mathcal{t}_1, \ell_1, \mathcal{t}_1') \) is invariant under the flip of \( j_1 \).

**Definition 7.6.1.** The triple \( (\mathcal{t}_1, \ell_1, \mathcal{t}_1') \) will be called a dicing of \( \mathcal{t} \).

**Remark 7.6.2.** Having diced \( \mathcal{t} \) into the pieces \( \mathcal{t}_1, \ell_1, \mathcal{t}_1' \), we can repeat the process to produce a dicing of \( \mathcal{t}_1' \), say \( (\mathcal{t}_2, \ell_2, \mathcal{t}_2') \), then produce a dicing of \( \mathcal{t}_2' \), say \( (\mathcal{t}_3, \ell_3, \mathcal{t}_3') \) and so on. That is, we can define a recursive procedure to dice \( \mathcal{t} \) into a sequence \( (\mathcal{t}_1, \ell_1, \mathcal{t}_2, \ell_2, \ldots, \ell_{n-1}, \ell_n) \) of pieces that can no longer be diced. For space reasons, we omit the explicit description of this procedure, whose steps, on the other hand, are clear once we know how to perform the first step.
7.6. Dicing equivalence classes of $\mathcal{R}_j$

Figure 7.10: $|S| = 1$
Figure 7.11: \(|S| = 1\). Bold arcs are to be identified

Now we turn to the problem of producing a dicing of the direct summand of \(M(\tau, i)(\partial)\) that corresponds to \(i\). In case \(|S| > 1\) we flip \(j_1\) if necessary so that we have the situation sketched in Figure 7.13 on the left (and not the situation sketched in that same Figure on the right).

**Proposition 7.6.3.** Suppose that \(|S| > 1\). Then the triple \((\iota_1, \ell_1, \ell_2)\) induces a source-
7.6. Dicing equivalence classes of $\mathcal{R}_j$

Figure 7.12: $|S| = 1. (\Sigma, M) =$ once-punctured torus

Figure 7.13: When $|S| > 1$, flip $j_1$ if necessary to obtain the configuration on the left

dicing of the direct summand of $M(\tau, i)(\partial)$ corresponding to $C$.

Proof. Define $L_{j_1}$ to be the one-dimensional subspace of $M(\tau, i)_{j_1}$ spanned by the unique element of $j_1 \cap \ell_1$.

If the unique intersection point of $\ell_1$ with $j_1$ is the ending point of a detour $a^n$ whose beginning point $v$ lies on $\ell_1$ (resp. $\ell'_2$), that is, if we are in the situation shown in Figure 7.14, where the blue segment is either a detour or a segment of $\ell_1$ (resp. $\ell_2$), let $\omega$ be the first (resp. last) point on $\ell_1 \cup \ell_1$ (resp. $\ell_1 \cup \ell'_2$) that happens to be the ending point of some detour whose beginning point is $v$. If $\omega \in \ell_1$, we define $M'_{j_1}$ (resp. $M''_{j_1}$) to be the subspace of $M(\tau, i)_{j_1}$ spanned by the set obtained from $\ell_1 \cap j_1$ (resp. $\ell_2 \cap j_1$) by replacing the point $u$ with the image of $v$ under the map $a = a_{M(\tau, i)}$. If $\omega \notin \ell_1$, let $\xi_\omega$ and $\xi_{\ell_1 \cap j_1}$ be the coefficients of $a(v)$ at $\omega$ and (the unique element of) $\ell_1 \cap j_1$, and define $M'_{j_1}$ (resp. $M''_{j_1}$) to be the subspace of $M(\tau, i)_{j_1}$ spanned by the set
obtained from $\ell_1 \cap j_1$ (resp. $\ell_2 \cap j_1$) by replacing $\omega$ with the vector $\xi_\omega \omega + \xi_{\ell_1 \cap j_1} \ell_1 \cap j_1$.

If, on the other hand, the unique element of $j_1 \cap \ell_1$ is not the ending point of any detour having its beginning point on $\ell_1$ (resp. $\ell_2'$), we set $M'_{j_1}$ (resp. $M''_{j_1}$) to be the space spanned by all intersection points of $\ell_1$ with $j_1$ (resp. $\ell_2'$ with $j_1$).

Having defined $M'_{j_1}$ (resp. $M''_{j_1}$), we define the spaces at the rest of the vertices of $Q(\partial, j)$. For each arrow $b \in \{\beta, \varepsilon\}$ of $Q(\tau)$ (whose tail is $j_1$), we set $M'_{h(b)}$ (resp. $M''_{h(b)}$) to be the space spanned by $\ell_1 \cap \ell_1$ (resp. $\ell_1 \cap \ell_2'$) and the image of $M'_{j_1}$ (resp. $M''_{j_1}$) under the linear map $M(\tau, i)_b$. And for each arrow $a \in \{\gamma, \eta\}$ of $Q(\tau)$ (whose head is $j_1$), we set $M'_{t(a)}$ (resp. $M''_{t(a)}$) to be the space spanned by $t(a) \cap \ell_1$ (resp. $t(a) \cap \ell_2'$).

We claim that the triple $(M', L_{j_1}, M'')$ is a source-dicing of the direct summand of $M(\tau, i)(\partial)$ corresponding to $C$.

Exactly one of the extreme points of $\ell_1$ belongs to $\ell_1$ (resp. $\ell_2'$) and lies on the head on an arrow whose tail is $j_1$. The one-dimensional subspace of $M(\tau, i)$ it spans is $N'_{\text{out}}$ (resp. $N''_{\text{out}}$). For each arrow $b$ with $t(b) = j_1$, $H'_{h(b)}$ (resp. $H''_{h(b)}$) is the image of $M'_{j_1}$ under $M(\tau, i)_b$.
7.6. Dicing equivalence classes of $\mathcal{R}_j$

It is left to the reader to check that $M'$ and $M''$ are subrepresentations of the
direct summand of $M(\tau, i)(\partial)$ corresponding to $C$, and that $H'_\text{out}, N'_\text{out}, H''_\text{out}, N''_\text{out}$ and
$L_j$, satisfy conditions (1)-(5) of Proposition 7.5.1, so that the triple $(M', L_j, M'')$
is a source-dicing of the mentioned direct summand. This proves the first part of
Proposition 7.6.3. $\square$

The representations $M'$ and $M''$ from the proof of Proposition 7.6.4 will be useful
for us since their quotients $P''$ and $P'$ can be directly calculated from segments of
$i$. In general, given a segment of $i$, we can define its detour matrices and string
representations just as we did for entire arcs in Sections 6.2 and 6.3, and combine
them to produce a representation, which will be referred to as segment representation
associated to the given segment of $i$. (Such segment representation need not be a
quotient nor a subrepresentation of $M(\tau, i)$).

**Proposition 7.6.4.** Suppose that $|S| > 1$. With the notation used in the proof of
Proposition 7.6.3, the quotient $P' = \frac{M(\tau, s_1)(\partial)}{M''}$ is isomorphic to $M(\tau, s_1)(\partial)$, where
$M(\tau, s_1)$ is the segment representation associated to the segment $s_1$ that goes from the
first point of $i$ to the unique element of $\ell_1 \cap j_1$. Similarly, the quotient $P'' = \frac{M(\tau, s_2)(\partial)}{M'}$
is isomorphic to $M(\tau, s_2')(\partial)$, where $M(\tau, s_2')$ is the segment representation associated
to the segment $s_2''$ that goes from the unique element of $\ell_1 \cap j_1$ to the last point of $i$.

**Remark 7.6.5.** An inductive argument shows that if $i$ is diced into $(\ell_{1}, \ell_{2}, \ell_{3}, \ldots, \ell_{n-1}, \ell_{n})$ (see Remark 7.6.2), then it is possible to dice $M(\tau, i)(\partial)$ into a collection
of $n$ representations, the $k^{th}$ one of which is isomorphic to a segment representation
associated to a segment $s_k$ obtained from $\ell_k$ by adding small subsegments of $\ell_{k-1}$ and
$\ell_{k}$. Details of the induction are left to the reader.
7.6. Dicing equivalence classes of $\mathcal{R}_j$

Next, we list all possibilities for each of the segments $s_k$. When $|S| \leq 1$, we have $\iota = \iota_1$ and the list of all possibilities has been given above. A segment of the form $s_1$ or $s_n$ arising in the process of dicing $\iota$ when $|S| > 1$ has to coincide with one of the segments depicted in Figures 7.15, 7.16 and 7.17, whereas a segment of the form $s_k$ with $1 < k < n$ has to coincide with one of the segments depicted in Figures 7.18, 7.19 and 7.20.

We close this section with an example of a dicing of an arc and the induced source-dicing of the corresponding representation $M(\tau, i)(\partial)$.

**Example 7.6.6.** Consider the ideal triangulation $\tau$ and the arc $i$ on the five-times-punctured hexagon shown in Figure 7.21. The arcs of the quadrilateral $\diamondsuit$ of which $j_1$ is a diagonal have been drawn bolder. Detours have not been drawn on the triangulation, but rather on the “local” picture appearing on the right of the Figure (in this “local” picture we have not drawn the entire $j$ but only a segment $i$). The equivalence relation $\mathcal{R}$ defined in Section 7.4 has only one equivalence class. Under the orientation of $i$ that starts at the extreme of $i$ that is a puncture, the recursive dicing of $\iota$ consists of only one step, the resulting segments $\iota_1$, $\ell_1$ and $\iota_2$ having been signaled in the Figure with the use of different colors. The QP $(Q(\tau), S(\tau))$ and the representation $M(\tau, i)$ are the ones in Example 7.5.3, and the corresponding source-dicing of $M(\tau, i)(\partial, j_1)$ we have described in this section is precisely the one given in that Example.
7.6. Dicing equivalence classes of $\mathcal{R}_j$

Figure 7.15: Possibilities for $s_1$ and $s_n$ when $|\mathcal{S}| > 1$
Figure 7.16: Possibilities for $s_1$ and $s_n$ when $|S| > 1$. Bold arcs are to be identified

Figure 7.17: Possibilities for $s_1$ and $s_n$ when $|S| > 1$, $(\Sigma, M) =$ once-punctured torus
Figure 7.18: Possibilities for an intermediate segment $s_k$ when $|S| > 1$
7.6. Dicing equivalence classes of $\mathcal{R}_j$

Figure 7.19: Possibilities for an intermediate segment $s_k$ when $|S| > 1$. Bold arcs are to be identified

Figure 7.20: Possibilities for an intermediate segment $s_k$ when $|S| > 1$, $(\Sigma, M) =$ once-punctured torus
Figure 7.21:
Chapter 8

Checking Jacobian relations

Let us prove that our arc representations satisfy the Jacobian ideal.

**Proposition 8.0.7.** Let $\tau$ be an ideal triangulation without self-folded triangles and $i$ be an arc. If $i$ satisfies (6.2) and (6.3), then $M(\tau, i)$ is a nilpotent representation of $Q(\tau)$ that satisfies the cyclic derivatives of $S(\tau)$ and is therefore annihilated by the Jacobian ideal $J(S(\tau))$.

*Proof.* We give the proof in case $i$ is not a loop cutting out a once-punctured monogon; the other case is similar.

By Lemmas 7.1.2 and 7.2.1, we can assume that $\Sigma$ has empty boundary. Furthermore, since $(Q(\tau), S(\tau))$ is the reduced part of $(\hat{Q}(\tau), \hat{S}(\tau))$, by Remark 5.0.15 it suffices to show that $M(\tau, i)$ is a nilpotent representation of $\hat{Q}(\tau)$ and satisfies the cyclic derivatives of $\hat{S}(\tau)$.

For each $j \in \tau$, let $n_j$ be the total number of 1-detours of $(\tau, i)$ whose beginning point lies on $j$. Let $(j_1, \ldots, j_r)$ be any ordering of the arcs of $\tau$. Let $J$ be the two-
sided ideal of $R(\langle \hat{Q} (\tau) \rangle)$ generated by the cyclic derivatives of $\hat{S}(\tau)$ (thus $J(\hat{S}(\tau))$ is the topological closure of $J$ in $R(\langle \hat{Q} (\tau) \rangle)$). We are going to recursively define representations $M_0, \ldots, M_r$ of $Q(\tau)$, with the following properties:

$$M_0 = m(\tau, i), \ M_r = M(\tau, i), \quad (8.1)$$

$$\dim(W_l) \geq \dim(W_{l-1}) + n_{ji} \text{ for } 1 \leq l \leq r, \quad (8.2)$$

where $W_l = \{ w \in M_l \mid Jw = 0 \}$ is the maximal vector subspace of $M_l$ satisfying the cyclic derivatives of $\hat{S}(\tau)$.

In all the representations $M_l$, $0 \leq l \leq r$, the vector space attached to each $j \in \tau$ will be $M(\tau, i)_j$. We define $M_0 = m(\tau, i)$. For $0 \leq l \leq r - 1$, once $M_l$ has been constructed, let $\alpha_1^{l+1}$ and $\alpha_2^{l+1}$ be the arrows of $\hat{Q}(\tau)$ that have $j_{l+1}$ as tail. Define $M_{l+1}$ as follows:

$$(M_{l+1})_{\alpha_1^{l+1}} = (D^{\Delta_{l+1}^{i,h}}_{i,h(\alpha_{l+1}^{1})})(m(\tau, i)_{\alpha_1^{l+1}}), \quad (M_{l+1})_{\alpha_2^{l+1}} = (D^{\Delta_{l+1}^{i,j}}_{i,j(\alpha_{l+1}^{2})})(m(\tau, i)_{\alpha_2^{l+1}}), \quad (8.3)$$

and $(M_{l+1})_a = (M_l)_a$ for $a \notin \{\alpha_1^{l+1}, \alpha_2^{l+1}\}$.

We obviously have $M_r = M(\tau, i)$. Notice that $M_{l+1} \neq M_l$ only if at least one of $\alpha_1^{l+1}$ or $\alpha_2^{l+1}$ is parallel to a detour. As a first step towards proving that $\dim(W_l) \geq \dim(W_{l-1}) + n_{ji}$ for $1 \leq l \leq r$, he have the following.

**Lemma 8.0.8.** For $0 \leq l \leq r - 1$, $W_l \subseteq W_{l+1}$.

**Proof.** With the notation of Figure 8.1, the arrow $\alpha_1^{l+1}$ (resp. $\alpha_2^{l+1}$) appears as a factor of two terms of the potential $\hat{S}(\tau)$, namely $\alpha_1^{l+1} \beta_1 \gamma_1$ and $x_\rho \alpha_1^{l+1} \beta_2 \gamma_2$ (resp. $\alpha_2^{l+1} \beta_2 \gamma_2$...
and $x_q\alpha_2^{l+1}\beta_1 d)$, where we are writing $c = c_1 \ldots c_{s_c}$ and $d = d_1 \ldots d_{s_d}$. If we were to have $W_I \not\subseteq W_{I+1}$, then there would exist an arc $k$, a basis element $v$ of $(M_{I+1})_k$ corresponding to an intersection point of $i$ with $k$, and $\xi \in \rho(k) = \{\partial_a(\bar{S}(\tau)) \mid a \in Q_1(\tau), h(a) = k\}$ such that $\xi_{M_I}v = 0$ but $\xi_{M_{I+1}}v \neq 0$. Since $M_{I+1}$ may differ from $M_I$ only by the action of $\alpha_1^{l+1}$ and $\alpha_2^{l+1}$, this would force $\xi$ to have the form $\xi = \partial_a(\bar{S}(\tau))$ for some $a \in \{c_1, \ldots, c_{s_c}, d_1, \ldots, d_{s_d}, \gamma_1, \gamma_2, \beta_1, \beta_2\}$, and $\alpha_1^{l+1}$ or $\alpha_2^{l+1}$ (or both) would then be parallel to some detour of $(\tau, i)$. Therefore, Lemma 8.0.8 will follow if we establish $\ker(\xi_{M_I}) \subseteq \ker(\xi_{M_{I+1}})$ when $\xi$ has the form $\xi = \partial_a(\bar{S}(\tau))$ for some $a \in \{c_1, \ldots, c_{s_c}, d_1, \ldots, d_{s_d}, \gamma_1, \gamma_2\}$, and $\xi_{M_{I+1}} = 0$ when $\xi = \partial_a(\bar{S}(\tau))$ for $a \in \{\beta_1, \beta_2\}$.

**Lemma 8.0.9.** $\xi_{M_{I+1}} = 0$ when $\xi = \partial_a(\bar{S}(\tau))$ for $a \in \{\beta_1, \beta_2\}$. 
Proof. We will unravel the definition of the maps \((M_{l+1})_{\alpha_{1,l+1}} = (D_{i,j_{l+1}}^{\alpha_{1,l+1}})(m(\tau,i)_{\alpha_{1,l+1}})\) and \((M_{l+1})_{\alpha_{2,l+1}} = (D_{i,j_{l+1}}^{\alpha_{2,l+1}})(m(\tau,i)_{\alpha_{2,l+1}})\). It is enough to check that \(\partial_{\alpha_{1}}(\widehat{S}(\tau))\) and \(\partial_{\alpha_{2}}(\widehat{S}(\tau))\) act as zero on each intersection point of \(i\) with the relative interior of \(j_{l+1}\).

If such an intersection point \(v\) is not the beginning point of a detour parallel to \(\alpha_{1,l+1}\) or \(\alpha_{2,l+1}\), there is nothing to prove. Otherwise, collecting all the detours whose beginning point is \(v\), we have either of the situations sketched in Figure 8.2 (the difference between these two being the parity of the number of detours starting at \(v\)). Let us analyze the upper configuration, the lower one is completely analogous. We have \(\beta_{1} : k_{3} \rightarrow j_{l+1}\) and \(\beta_{2} : k_{4} \rightarrow j_{l+1}\). Whence, with the numbering of intersection points shown in Figure 8.2, the action of \(\partial_{\beta_{1}}(\widehat{S}(\tau))_{M_{l+1}}\) on \(v\) is

\[
\partial_{\beta_{1}}(\widehat{S}(\tau))_{M_{l+1}}v = (\gamma_{1}^{l+1})_{M_{l+1}}v + (x_{q}d_{1} \ldots d_{s_{d}}^{l+1})_{M_{l+1}}v =
\]

\[
= \left[ \begin{array}{cccc}
1 & 0 & \ldots & 0 \\
-x_{q} & x_{p}x_{q} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{n-1}x_{p} & \frac{n-1}{x_{q}} & \frac{n-1}{x_{q}} & \ldots
\end{array} \right]
+ \left[ \begin{array}{c}
1 \\
-x_{p} \\
\vdots \\
(-1)^{n-1}x_{p} \frac{n-1}{x_{q}}
\end{array} \right]
\]

Similarly, the action of \(\partial_{\beta_{2}}(\widehat{S}(\tau))_{M_{l+1}}\) on \(v\) is zero. \(\square\)

Now we prove that \(\ker(\xi_{M_{l}}) \subseteq \ker(\xi_{M_{l+1}})\) when \(\xi = \partial_{a}(\widehat{S}(\tau))\) for \(a \in \{c_{1}, \ldots, c_{s_{c}}, d_{1}, \ldots, d_{s_{d}}, \gamma_{1}, \gamma_{2}\}\).

Case 1. Both \(\alpha_{1,l+1}\) and \(\alpha_{2,l+1}\) parallel to a detour.
Figure 8.2: $\partial_{\beta_1}(\hat{S}(\tau))$ and $\partial_{\beta_2}(\hat{S}(\tau))$ act as zero on $M_{l+1}$
We already noted that if an arrow is parallel to a detour, then it is parallel to a 1-detour (See Remark 6.2.4). So in this case both $\alpha_{1}^{l+1}$ and $\alpha_{2}^{l+1}$ are parallel to 1-detours. There are two ways this can happen, depending on whether $\alpha_{1}^{l+1}$ and $\alpha_{2}^{l+1}$ are parallel to detours with the same beginning point or not. See Figure 8.3. In both situations it is easy to see that $m(\tau, i)_{\beta_{1}}$ and $m(\tau, i)_{\beta_{2}}$ are zero, which implies $(\beta_{1})_{M+1} = 0$ and $(\beta_{2})_{M+1} = 0$.

**Subcase 1.** $\xi = \partial_a(\hat{S}(\tau))$ for some $a \in \{c_1, \ldots, c_s, d_1, \ldots, d_d\}$.

If $a = c_t$ then, using the notation of Figure 8.4, we have $\xi = \partial_a(\hat{S}(\tau)) = x_p c_{t+1} \ldots c_{s} \alpha_{1}^{l+1} \beta_2 c_1 \ldots c_{t-1} + \delta \varepsilon$. Notice that $\varepsilon$ cannot be parallel to a detour of $(\tau, i)$, and this implies that the composition $(\delta \varepsilon)_{M+1}$ is zero. Therefore, $\xi_{M+1} = 0$. The situation $a \in \{d_1, \ldots, d_d\}$ leads to a similar conclusion.

**Subcase 2.** $\xi = \partial_a(\hat{S}(\tau))$ for $a \in \{\gamma_1, \gamma_2\}$. 

Figure 8.3:
Since $m(\tau, i)_{\beta_1} = 0$, every intersection of $i$ with $t(\beta_1)$ is part of a segment of $i$ with endpoints in $t(\beta_1)$ and $t(\gamma_1)$, and this implies that the composition $(g_1 \ldots g_s)_{M_{l+1}}$ is zero (even if some $g_t$ is parallel to some detour). Thus $\partial_{\gamma_1}(\hat{S}(\tau))_{M_{l+1}} = 0$. A similar argument shows that $\partial_{\gamma_2}(\hat{S}(\tau))_{M_{l+1}} = 0$

**Case 2.** $\alpha_{l+1}^1$ parallel to a detour, $\alpha_{l+1}^2$ not parallel to any detour.

**Subcase 1.** $\xi = \partial_a(\hat{S}(\tau))$ for some $a \in \{c_1, \ldots, c_{sc}\}$.

The image of $(\beta_2)_{M_{l+1}}$ has zero intersection with the vector subspace of $(M_{l+1})_{l+1}$ spanned by (the basis vectors corresponding to) intersection points of $i$ with $j_{l+1}$ that are beginning points of detours. And the linear maps $(\alpha_{l+1}^1)_{M_l}$ and $(\alpha_{l+1}^2)_{M_{l+1}}$ agree on the rest of basis vectors of $(M_{l+1})_{l+1}$ that correspond to intersection points of $i$ with $j_{l+1}$. Therefore $\ker(\xi_{M_l}) \subseteq \ker(\xi_{M_{l+1}})$.

**Subcase 2.** $\xi = \partial_a(\hat{S}(\tau))$ for some $a \in \{d_1, \ldots, d_{sd}\}$. 
Since $\alpha_2^{l+1}$ is not parallel to any detour of $(\tau, i)$, we have $\partial_a(\hat{S}(\tau))_{M_l} = \partial_a(\hat{S}(\tau))_{M_{l+1}}$.

Subcase 3. $\xi = \partial_a(\hat{S}(\tau))$ for $a \in \{\gamma_1, \gamma_2\}$.

Again, since $\alpha_2^{l+1}$ is not parallel to any detour of $(\tau, i)$, we have $\partial_{\gamma_2}(\hat{S}(\tau))_{M_l} = \partial_{\gamma_2}(\hat{S}(\tau))_{M_{l+1}}$.

This finishes the proof of Lemma 8.0.8. □

Returning to the proof of Proposition 8.0.7, for each 1-detour $d^1$ whose beginning point lies on $j_{l+1}$, the basis element of $m(\tau, i)$ corresponding to the beginning point $b(d^1)$ belongs to $W_{l+1}$ (by Lemma 8.0.9) but not to $W_l$. This fact and Lemma 8.0.8 prove that property (8.2) is satisfied. Furthermore, we also have

$$\dim(W_0) \geq \dim(M(\tau, i)) - \sum_{j \in \tau} n_j, \quad (8.4)$$

which follows from the observation that if $v \in m(\tau, i)$ is a basis element corresponding to an intersection point that is not a beginning point of a 1-detour of $(\tau, i)$, then $v \in W_0$.

The nilpotency of $M(\tau, i)$ follows by induction on $l = 0, \ldots, r$. Proposition 8.0.7 is proved. □

To close the chapter, let us decorate the representations defined in Sections 6.2 and 6.3.

**Definition 8.0.10.** Let $\tau$ be an ideal triangulation of $(\Sigma, M)$ and $i$ be any arc on $(\Sigma, M)$. We define the decorated arc representation $M(\tau, i)$ to be $(Q(\tau), S(\tau), M(\tau, i), V(\tau, i))$, where $V(\tau, i)_j = \delta_{i,j} K$ ($\delta_{i,j}$ being the Kronecker delta). In other words, $M(\tau, i)$ is the arc representation $M(\tau, i)$ with the zero decoration if $i \notin \tau$, and is the
$i^{th}$ negative simple representation if $i \in \tau$. 
Chapter 9

Flip ↔ mutation compatibility

We now turn to investigate the compatibility between flips of triangulations and mutations of representations. Throughout this section we will be interested in flipping the arc \( j \) of \( \tau \). We will work under the assumption that none of the ideal triangulations \( \tau \) and \( \sigma = f_j(\tau) \) has self-folded triangles.

9.1 Effect of flips on detour matrices

From their very definition, detour matrices depend on the triangles of \( \tau \). Let us be more specific; take an arc \( j \in \tau \) and let \( \triangle_1 \) and \( \triangle_2 \) be the triangles of \( \tau \) that contain \( j \), let also \( \diamondsuit = \triangle_1 \cup \triangle_2 \) be the quadrilateral in \( \tau \) of which \( j \) is a diagonal. Given an arc \( k \in \tau \), \( k \neq j \), \( k \subseteq \triangle_1 \), the detour matrix \( D_{i,k}^{\triangle_1} \) has been defined with respect to \( \triangle_1 \) but it does not even make sense to talk of such the matrix \( D_{i,k}^{\triangle_1} \) with respect to \( \sigma = f_j(\tau) \) because \( \triangle_1 \) is not a triangle of \( \sigma \). This of course does not mean that the arc \( k \) does not have two detour matrices attached according to \( \sigma \), but rather that,
strictly speaking, we should use some notation like $D_{i,k}^{\triangle,\tau}$ to indicate the dependence on the triangulation (we will do so only when it is really necessary).

But the above mentioned change of detour matrices is not the only expectable one when we flip $j$: the existence of many detours contained in the triangles of $\tau$ adjacent to $\triangle$ (which in most cases will remain triangles of $\sigma$) depends on the existence of detours contained in $\triangle_1$ or $\triangle_2$. So, in principle, the existence could be possible of a triangle $\triangle$ present in both $\tau$ and $\sigma$ and an arc $k \in \tau \cap \sigma$, $k \subseteq \triangle$, such that the detour matrices $D_{i,k}^{\triangle,\tau}$ and $D_{i,k}^{\triangle,\sigma}$ were different. This would ultimately lead to the existence of an arrow $a$ not incident to $j$ (hence belonging to both $Q(\tau)$ and $Q(\sigma)$) such that the linear maps $M(\tau, i)_a$ and $M(\sigma, i)_a$ do not coincide. This subsection is devoted to show that this does not happen, that is, that the detour matrices that should not change actually do not. For time and space reasons, we show this only when $i$ is not a loop cutting out a once-punctured monogon, and leave to the reader the task of doing the necessary checks when $i$ is such a loop.

**Lemma 9.1.1.** If the arc $k$ is not contained in any of the ideal triangles that contain $j$, then the flip of $j$ does not affect any of the two detour matrices attached to $k$.

**Proof.** Let $\sigma = f_j(\tau)$ be the ideal triangulation obtained from $\tau$ by flipping $j$. Since $k$ is not contained in any of the ideal triangles of $\tau$ that contain $j$, both of the ideal triangles of $\tau$ that contain $k$ are ideal triangles of $\sigma$ as well. Fix one such triangle $\triangle$, and denote by $D_{i,k}^{\triangle,\tau}$ (resp. $D_{i,k}^{\triangle,\sigma}$) the detour matrix attached to $k$ using the detours of $(\tau, i)$ (resp. $(\sigma, i)$) that are contained in $\triangle$. The assertion of the lemma is that $D_{i,k}^{\triangle,\tau} = D_{i,k}^{\triangle,\sigma}$.

Let $k_1$ be the (unique) arc contained in $\triangle$ such that there is an arrow $\alpha : k_1 \to k$
9.1. Effect of flips on detour matrices

in \( \hat{Q}(\tau) \). Let \( \triangle_1 \) be the (unique) triangle of \( \tau \) that contains \( k_1 \) and is different from \( \triangle \) (see Figure 9.1). Since \( k \) is not contained in any of the ideal triangles of \( \tau \) that contain \( j \), we have \( j \neq k_1 \). If \( j \notin \{k_2, k_3\} \), then all the \( n \)-detours of \((\tau, i)\) whose beginning point lies on \( k_1 \) are \( n \)-detours of \((\sigma, i)\), and clearly \( D_{i,k}^{\triangle,\tau} = D_{i,k}^{\triangle,\sigma} \). To see what happens when \( j \in \{k_2, k_3\} \), let us begin assuming that \( j = k_2 \). Since \( j \neq k_1 \), any detour of \((\tau, i)\) or \((\sigma, i)\) whose ending point lies in \( k \) has its beginning point lying on \( k_1 \). Thus, by Definition 6.2.5, the equality \( D_{i,k}^{\triangle,\tau} = D_{i,k}^{\triangle,\sigma} \) will be proved if we show that for all \( n \geq 1 \), the set \( \mathcal{B}_{i,k}^{\triangle,n} = \mathcal{B}_{i,k}^{\triangle,\tau,n} \) defined in terms of \( \tau \) and \( i \) coincides with the set \( \mathcal{B}_{i,k}^{\triangle,n} = \mathcal{B}_{i,k}^{\triangle,\sigma,n} \) defined in terms of \( \sigma \) and \( i \) (see Sections 6.2 and 6.3). To show that these sets coincide, we take one by one the intersection points of \( i \) with the relative interior of \( k \), and show that for each such point \( v \), a quadruple \((v, w, b(d_{(v,w)}^{\triangle,n}), p)\) belongs to \( \mathcal{B}_{i,k}^{\triangle,\tau,n} \) if and only if it belongs to \( \mathcal{B}_{i,k}^{\triangle,\sigma,n} \). So, let \( v \) be an intersection point of \( i \) with the relative interior of \( k \). Thus \( v \) is an extreme point of a segment \([v, b], \) of \( i \) contained in \( \triangle \). If the other extreme point \( b \) of this segment does not lie in the relative interior of \( k_1 \), then none of the sets \( \mathcal{B}_{i,k}^{\triangle,\tau,n} \) and \( \mathcal{B}_{i,k}^{\triangle,\sigma,n} \) contains a quadruple of the form \((v, w, b(d_{(v,w)}^{\triangle,n}), p)\). So, suppose that the extreme point \( b \) belongs to the relative interior of \( k_1 \). Then for any quadruple \((v, w, b(d_{(v,w)}^{\triangle,n}), p)\) belonging to either \( \mathcal{B}_{i,k}^{\triangle,\tau,n} \) or \( \mathcal{B}_{i,k}^{\triangle,\sigma,n} \) we must have \( b(d_{(v,w)}^{\triangle,n}) = b \). Let us draw all possible detours of \((\tau, i)\)
9.1. Effect of flips on detour matrices

(resp. \((\sigma, i)\)) whose beginning point is \(b\). Then we get the configurations shown in Figures 9.2 and 9.3.

For each \(n \geq 1\) there is at most one \(n\)-detour of \((\tau, i)\) (resp. \((\sigma, i)\)) contained in \(\triangle\) and having \(b\) as its beginning point. Any such detour is clearly an \(n\)-detour of \((\sigma, i)\) (resp. \((\tau, i)\)) as well, and detours the marked point opposite to \(k\) in the triangle \(\triangle\). Hence a quadruple \((v, w, b(d_{(\tau, i)}^{\triangle, n}), p)\) belongs to \(\mathcal{B}_{i,k}^{\triangle, \tau, n}\) if and only if it belongs to \(\mathcal{B}_{i,k}^{\triangle, \sigma, n}\). This finishes the proof of the equality \(D_{i,k}^{\triangle, \tau} = D_{i,k}^{\triangle, \sigma}\) in case \(j = k_2\).

A similar argument also proves that \(D_{i,k}^{\triangle, \tau} = D_{i,k}^{\triangle, \sigma}\) when \(j = k_3\). \(\square\)
Lemma 9.1.2. Let \( \triangle \) be one of the ideal triangles of \( \tau \) that contain \( j_1 \) and \( k \in \tau \), \( k \neq j \), be an arc contained in \( \triangle \). If \( \triangle' \) denotes the unique triangle of \( \tau \) that contains \( k \) and is different from \( \triangle \), then \( \triangle' \) is a triangle of \( \sigma = f_{j_1}(\tau) \) and the detour matrices \( D_{i,k}^{\triangle',\tau} \) and \( D_{i,k}^{\triangle',\sigma} \) coincide.

Proof. Similar to the proof of Lemma 9.1.1. \( \square \)

Corollary 9.1.3. Let \( k_1, k_2 \in \tau \). If \( a : k_1 \to k_2 \) is an arrow of \( \hat{Q}(\tau) \) contained in an ideal triangle of \( \tau \) that does not contain \( j_1 \), then \( a \) is also an arrow of \( \hat{Q}(\sigma) \), where \( \sigma = f_{j_1}(\tau) \), and the linear maps \( M(\tau, i)_a : M(\tau, i)_{k_1} \to M(\tau, i)_{k_2} \) and \( M(\sigma, i)_a : \)
9.2 Main result of Part II: Statement and proof

\[ M(\sigma, i)_{k_1} \rightarrow M(\sigma, i)_{k_2} \text{ coincide.} \]

9.2 Main result of Part II: Statement and proof

Throughout this section we assume that the arc \( i \) satisfies (6.2), (6.3) and that

\[ i \text{ intersects transversally each of the arcs of } \sigma = f_j(\tau) \text{ (if at all), and} \quad (9.1) \]

the number of intersection points of \( i \) with each of the arcs of \( \sigma \) is minimal. \quad (9.2)

The following theorem is the main result of Part II. Its proof is long, due to the separation into several cases, ending in page 232.

**Theorem 9.2.1.** Let \( \tau \) and \( \sigma \) be ideal triangulations without self-folded triangles and \( i \) be an arc satisfying (6.2), (6.3), (9.1), and (9.2). If \( \sigma = f_j(\tau) \) for an arc \( j \in \tau \), then the decorated arc representations \( \mu_j(\mathcal{M}(\tau, i)) \) and \( \mathcal{M}(\sigma, i) \) are right-equivalent. The right-equivalence between these representations can be chosen so that it acts as the identity on the spaces \( M(\tau, i)_k = M(\sigma, i)_k \) for \( k \neq j \).

**Proof.** Our assumptions mean that none of \( \tau \) and \( \sigma \) has self-folded triangles, and that either \( i \in \tau \cup \sigma \), or else intersects transversally each of the arcs in \( \tau \) and each of the arcs in \( \sigma \), and moreover, \( i \) is a representative of its isotopy class that minimizes intersection numbers with each of the arcs in \( \tau \) and each of the arcs in \( \sigma \). We will further assume that \( \Sigma \) has empty boundary (by Theorem 7.1.7 and Lemma 7.2.1 this means no loss of generality). Since the theorem is trivially true in case \( i \in \tau \cup \sigma \), throughout the proof we will suppose that \( i \notin \tau \cup \sigma \). This in particular implies that
all decorated representations will have the zero space as decoration.

Since $\tau$ does not have self-folded triangles, $j$ is contained in two different ideal triangles of $\tau$, say $\Delta_1$ and $\Delta_2$. Furthermore, since $\sigma$ does not self-folded triangles, $j$ is the head (resp. tail) of exactly two arrows of $Q(\tau)$. To establish some notation for these arrows and their heads and tails, suppose $j_2 \xrightarrow{\gamma} j \xleftarrow{\beta} j_3$ are contained in $\Delta_1$ and $j_4 \xrightarrow{\eta} j \xleftarrow{\epsilon} j_5$ are contained in $\Delta_2$. Then $\hat{Q}(\tau)$ has arrows $j_3 \xrightarrow{\alpha} j_2$ and $j_5 \xrightarrow{\delta} j_4$. These arrows may not belong to the arrow set of $Q(\tau)$ (indeed, they may be contained in 2-cycles of $\hat{Q}(\tau)$), so we let $\hat{Q}(\tau)$ be the subquiver of $\hat{Q}(\tau)$ obtained from $Q(\tau)$ by adding all 2-cycles containing $\alpha$ and $\delta$. We also let $\hat{S}(\tau)$ be the potential on $\hat{Q}(\tau)$ that results from adding $S(\tau)$ and all 2-cycles containing $\alpha$ and $\delta$. Thus the $(Q(\tau), S(\tau))$ is the reduced part of the QP $(\hat{Q}(\tau), \hat{S}(\tau))$, and $\mu_j(Q(\tau), S(\tau)) = \mu_j(\hat{Q}(\tau), \hat{S}(\tau))$. Moreover,

\[
\text{the } R\text{-algebra isomorphism } \varphi : R\langle\langle \mu_j(Q(\tau))\rangle\rangle \to R\langle\langle Q(\sigma)\rangle\rangle \quad (9.3)
\]
defined by $\beta^* \mapsto -\beta^*$, $\eta^* \mapsto -\eta^*$,

and the identity on the rest of the arrows of $\mu_j(Q(\tau)) = Q(\sigma)$, is a right equivalence between $\mu_j(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$.

Now, as noted before, the process of reducing a QP can be done in steps, taking care of 2-cycles one by one. From this and Remark 5.0.15 we see that, with the action of $\hat{Q}(\tau)$ on $M(\tau, i)$ obtained from the inclusion of quivers $\hat{Q}(\tau) \hookrightarrow \hat{Q}(\tau)$, and with $V(\tau, i)$ as decoration, $M(\tau, i)$, is a well-defined QP-representation of $(\hat{Q}(\tau), \hat{S}(\tau))$ that has $M(\tau, i)$ as its reduced part and whose $j^{th}$ mutation is $\mu_j(M(\tau, i))$. We therefore see that in order to calculate $\mu_j(M(\tau, i))$ we do not need to delete $\alpha$ or $\delta$ even when
9.2. Main result of Part II: Statement and proof

one of them is contained in a 2-cycles of $\hat{Q}(\tau)$. This justifies the abuse consisting in thinking of the arrows $\alpha$ and $\delta$ as always belonging to $Q(\tau)$.

Consider the set $\mathcal{I} = \mathcal{I}_j$ and the equivalence relation $\mathcal{R} = \mathcal{R}_j$ defined in Section 7.4. For each of the equivalence classes induced by $\mathcal{R}$ on $\mathcal{I}$ take the union of the segments of $i$ whose extremes are elements of the class. This way we obtain segments $i_1, \ldots, i_t$, of $i$, whose intersections with $j_1 \cup j_2 \cup j_3 \cup j_4 \cup j_5$ are precisely the equivalence classes of $\mathcal{R}$. These segments have associated segment representations $M(\tau, i_1), \ldots, M(\tau, i_t)$, that induce a direct sum decomposition of the representation $M(\tau, i)(\partial) = M(\tau, i)(\partial, j)$ of the quiver with relations $(Q(\tau)(\partial), R(\partial)) = (Q(\tau)(\partial, j), R(\partial, j))$ (this is Proposition 7.4.1). Using Proposition 7.3.1 and applying Proposition 7.4.1 to $M(\sigma, i)(\partial)$, we see that Theorem 9.2.1 will be proved if we can prove it separately for each of the segments $i_1, \ldots, i_t$. So, we take one such segment, denote it by $\ell$, and dice it, thus obtaining $(\ell_1, \ell_1, \ell_2, \ell_2, \ldots, \ell_{n-1}, \ell_{n-1}, \ell_n, \ell_n)$ for some $n \geq 1$. Just as in Section 7.6, if $n > 1$, we let $s_1, \ldots, s_n$ be the segments of $\ell$ defined as follows:

- $s_1$ goes from the initial point of $\ell_1$ to the unique element of $\ell_1 \cap j_1$;
- $s_n$ goes from the unique element of $\ell_{n-1} \cap j_1$ to the final point of $\ell_n$;
- for $k = 2, \ldots, n - 1$, the segment $s_k$ goes from the unique point of $\ell_{k-1} \cap j_1$ to the unique element of $\ell_k \cap j_1$.

These segments have associated segment representations $M(\tau, s_1), \ldots, M(\tau, s_n)$. All possible configurations of these segments are shown in Figures 7.6, 7.7, 7.8, 7.9, 7.10, 7.11, 7.12, 7.15 and 7.18.
Lemma 9.2.2. If Theorem 9.2.1 holds for each of the segments \( s_1, \ldots, s_n \), then it holds for \( i \).

Proof. This follows by repeated applications of Proposition 7.5.1. We show the first such application, the remaining ones are totally analogous. Consider the situation \((\ell_1, \ell_1', \ell_2')\) (that is, the first step in the recursive dicing of \( i \) in case \(|S| > 1\)). By Proposition 7.6.3, this triple induces a source-dicing triple \((M', L_j, M'')\) of \( M(\tau, \iota)(\partial, j)\). By Proposition 7.5.1, the mutation \( \mu_j(M(\tau, \iota)(\partial, j)) \) is isomorphic to

\[
\begin{align*}
\mu_j' & \oplus \mu_j'' \oplus L_j \\
\mu_j' & \oplus L_j' \oplus P_{in} \\
P_{in} & \oplus P_{in}'' \oplus P_{out} \oplus P_{out}'' \\
\end{align*}
\]

where \( P' = \frac{M(\tau, \iota)(\partial, j)}{M'} \) and \( P'' = \frac{M(\tau, \iota)(\partial, j)}{M''} \) (abusing, we think of this isomorphism as an identification). Furthermore, by Proposition 7.6.4, \( P' \) and \( P'' \) are respectively isomorphic to \( M(\tau, s_1)(\partial, j) \) and \( M(\tau, s_2')(\partial, j) \). Suppose there are vector space isomorphisms \( \psi_1 : \mu_j(M(\tau, s_1)) \to M(\sigma, s_1) \) and \( \psi_2 : \mu_j(M(\tau, s_2')) \to M(\sigma, s_2') \), with the property of being the identity at the spaces attached to the vertices different from \( j \) and such that \( \psi_1(um_1) = \varphi(u)\psi_1(m_1) \) and \( \psi_2(um_2) = \varphi(u)\psi_2(m_2) \) for \( u \in R(\langle Q(\sigma) \rangle) \), \( m_1 \in \mu_j(M(\tau, s_1)) \) and \( m_2 \in \mu_j(M(\tau, s_2')) \) where \( \varphi \) is the right-equivalence defined by (9.3). To extend this linear isomorphisms to an isomorphism \( \psi : \mu_j(M(\tau, \iota)) \to M(\sigma, \iota) \), we just need to define \( \psi \) on \( L_j \) (which is a one-dimensional space spanned by the unique element of \( \ell_1 \cap j \)). If \( (\pi'b)_{L_j} = \beta_{L_j} \) we define \( \psi \) to send the unique element of \( \ell_1 \cap j \) to the unique intersection point between \( \ell_1 \) and the unique arc in \( \sigma \setminus \tau \). If \( (\pi'b)_{L_j} \neq \beta_{L_j} \) we define \( \psi \) to send the unique element of
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\( \ell_1 \cap j \) to minus the unique intersection point between \( \ell_1 \) and the unique arc in \( \sigma \setminus \tau \).

Because of the specific form of the linear maps \( \mathcal{A} \) and \( \mathcal{B} \), the resulting linear map 
\( \psi : \mu_j(M(\tau, \iota)) \to M(\sigma, \iota) \) satisfies \( \psi(um) = \varphi(u)\psi(m) \) and for \( u \in R(\langle Q(\sigma) \rangle) \) and 
\( m \in \mu_j(M(\tau, \iota)) \).

Returning to the proof of Theorem 9.2.1, by Lemma 9.2.2 it is enough to prove it for each of the dicing pieces \( s_1, \ldots, s_n \). So we take one such and denote it simply by \( s \).

As said above, all possibilities for \( s \) are shown in Figures 7.6, 7.7, 7.8, 7.9, 7.10, 7.11, 7.12, 7.15, 7.16, 7.17, 7.18, 7.19 and 7.20. These figures comprehend three different situations for the arcs \( j_2, j_3, j_4 \) and \( j_5 \), namely,

- when they are four distinct arcs of \( \tau \) (Figures 7.6, 7.9, 7.10, 7.15 and 7.18);
- when they are three distinct arcs of \( \tau \), with \( j_2 = j_4 \) or \( j_3 = j_5 \) (Figures 7.7, 7.11, 7.16 and 7.19);
- when they are two distinct arcs of \( \tau \), with \( j_2 = j_4 \) and \( j_3 = j_5 \) (Figures 7.8, 7.12, 7.17 and 7.20).

**Lemma 9.2.3.** If Theorem 9.2.1 holds for any possible configuration of \( s \) when all four arcs \( j_2, j_3, j_4 \) and \( j_5 \) are different, then it also holds for all possible configurations of \( s \) when the four arcs \( j_2, j_3, j_4 \) and \( j_5 \) are not necessarily different.

**Proof.** Consider any of the configurations in Figures 7.7, 7.11, 7.16, 7.19, 7.8, 7.12, 7.17 and 7.20. Then, whether we identify opposite sides of \( \Diamond \) as instructed in these
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figures or not, the triangle

\[ M(\tau, s)_j \]

\[ \xymatrix{ & M(\tau, s)_j \ar[dl]_a \ar[dr]^b \ar[dl]_c \ar[dr] \cr M(\tau, s)_{\text{in}} & & M(\tau, s)_{\text{out}} } \]

will not change. \qquad \square

We finish the proof of Theorem 9.2.1 by a case-by-case verification, the cases being
given by the configurations shown in Figures 7.6, 7.10, 7.15 and 7.18, all under the
assumption that \( j_2, j_3, j_4 \) and \( j_5 \) are four different arcs of \( \tau \). For time and space
reasons, we are not going to include the \( f\eta p \leftrightarrow \text{mutation} \) analysis of each one of these
configurations; we will analyze some of them and leave the rest as an exercise for the
reader. We will use the notation \( j_1 = j \) indistinctly.

\textit{Case 1} (Configuration 1, Figure 7.6). Assume that, around the arc \( j_1 \) to be flipped,
\( \tau \) and \( i \) look as shown in Figure 9.4. The relevant vector spaces assigned in \( M(\tau, i) \)
to the vertices of $Q(\tau)$ are

\[ M_{j_1} = M(\tau, i)_{j_1} = K, \quad M_{j_2} = M(\tau, i)_{j_2} = 0, \]
\[ M_{j_3} = M(\tau, i)_{j_3} = K, \quad M_{j_4} = M(\tau, i)_{j_4} = 0, \]
\[ \text{and} \quad M_{j_5} = M(\tau, i)_{j_5} = 0. \]

Since none of $\alpha, \beta, \gamma, \delta, \varepsilon$ and $\eta$ is parallel to any detour of $(\tau, i)$, the detour matrices $D_{i,j_1}^{\Delta_1}, D_{i,j_2}^{\Delta_1}, D_{i,j_3}^{\Delta_1}, D_{i,j_4}^{\Delta_2}$ and $D_{i,j_5}^{\Delta_2}$ are identities (of the corresponding sizes). Hence the arrows $\alpha, \beta, \gamma, \delta, \varepsilon$ and $\eta$ act on $M(\tau, i)$ according to the following linear maps:

\[ M(\tau, i)_{\alpha} = 0, \quad M(\tau, i)_{\beta} = 1 : K \to K, \]
\[ M(\tau, i)_{\gamma} = 0, \quad M(\tau, i)_{\delta} = 0, \]
\[ M(\tau, i)_{\varepsilon} = 0, \quad M(\tau, i)_{\eta} = 0. \]

Let us investigate the effect of the $j_1^{th}$ QP-mutation on $\mathcal{M}(\tau, i)$. An easy check using the information about $\mathcal{M}(\tau, i)$ we have collected thus far yields

\[ M_{\text{in}} = M_{j_2} \oplus M_{j_4} = 0, \]
\[ M_{\text{out}} = M_{j_3} \oplus M_{j_5} = K \oplus 0. \]
\[ a = 0 : M_{\text{in}} = 0 \to K = M_{j_1}, \]
\[ b = 1 : M_{j_1} = K \to K \oplus 0 = M_{\text{out}}. \]
\[ c = 0 : M_{\text{out}} = K \oplus 0 \rightarrow 0 = M_{\text{in}}. \]

We see that \( \text{im } b = \ker c = M_{\text{out}}, \text{im } c = 0 \) and \( \ker a = 0 \). Hence \( \overline{M}_j = 0 \)

Let us compute the action of the arrows of \( \bar{\mu}_{j_1}(Q(\tau)) \) on \( \overline{M(\tau, i)} \). A trivial check shows that \( \beta^*, \gamma^*, \epsilon^*, \eta^* \) and \( [\beta \eta] \) act as zero on \( \overline{M(\tau, i)} \). We have thus computed the permutation \( \bar{\mu}_{j_1}(M(\tau, i)) = (\bar{\mu}_{j_1}(Q(\tau)), \bar{S(\tau)}), \overline{M(\tau, i)}, 0) \).

On the other hand, if we flip the arc \( j_1 \) of \( \tau \) we obtain the ideal triangulation \( \sigma = f_{j_1}(\tau) \) sketched at the left of Figure 9.5 (in a clear abuse of notation, we are using the same symbol \( j_1 \) in both \( \tau \) and \( \sigma \)).

Clearly, the triple \( \Phi = (\varphi, \psi, \eta) \) is a right-equivalence between these QP-representations, where

- \( \varphi \) is the right-equivalence whose action on the arrows is given by (9.3);
- \( \psi : \overline{M(\tau, i)} \rightarrow M(\sigma, i) \) is the vector space isomorphism given by the identity \( 1 : \overline{M(\tau, i)}_k \rightarrow M(\sigma, i)_k \) for all \( k \in \sigma \);
- \( \eta \) is the zero map (of the zero space).
This finishes the proof of Theorem 9.2.1 for configuration 1 of Figure 7.6.

Case 2 (Configuration 2, Figure 7.6). Assume that, around the arc $j_1$ to be flipped, $\tau$ and $i$ look as shown in Figure 9.6. The relevant vector spaces assigned in $M(\tau, i)$ to the vertices of $Q(\tau)$ are

\[
M_{j_1} = M(\tau, i)_{j_1} = K, \quad M_{j_2} = M(\tau, i)_{j_2} = K,
\]

\[
M_{j_3} = M(\tau, i)_{j_3} = 0, \quad M_{j_4} = M(\tau, i)_{j_4} = 0,
\]

and $M_{j_5} = M(\tau, i)_{j_5} = 0$.

Since none of $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ and $\eta$ is parallel to any detour of $(\tau, i)$, the detour matrices $D_{i,j_1}^{\Delta_1}$, $D_{i,j_2}^{\Delta_1}$, $D_{i,j_3}^{\Delta_2}$, $D_{i,j_4}^{\Delta_2}$ and $D_{i,j_5}^{\Delta_2}$ are identities (of the corresponding sizes). Hence the arrows $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ and $\eta$ act on $M(\tau, i)$ according to the following linear maps:

\[
M(\tau, i)_\alpha = 0, \quad M(\tau, i)_\beta = 0,
\]

\[
M(\tau, i)_\gamma = 1 : K \to K, \quad M(\tau, i)_\delta = 0,
\]
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\[ M(\tau, i)_{\varepsilon} = 0 \quad M(\tau, i)_{\eta} = 0. \]

Let us investigate the effect of the \( j_1^{th} \) QP-mutation on \( M(\tau, i) \). An easy check using the information about \( M(\tau, i) \) we have collected thus far yields

\[
M_{\text{in}} = M_{j_2} \oplus M_{j_4} = K \oplus 0,
\]

\[
M_{\text{out}} = M_{j_3} \oplus M_{j_5} = 0.
\]

\[ a = 1 \colon M_{\text{in}} = K \oplus 0 \to K = M_{j_1}, \]

\[ b = 0 \colon M_{j_4} = K \to 0 = M_{\text{out}}, \]

\[ c = 0 \colon M_{\text{out}} = 0 \to K \oplus 0 = M_{\text{in}}. \]

We see that \( \text{im } b = \ker c = 0, \text{im } c = 0 \) and \( \ker a = 0 \). Hence \( \overline{M}_{j_1} = 0 \)

Let us compute the action of the arrows of \( \overline{\mu}_{j_1}(Q(\tau)) \) on \( \overline{M}(\tau, i) \). A trivial check shows that \( \beta^*, \gamma^*, \varepsilon^*, \eta^* \), [\( \beta \eta \)] and [\( \varepsilon \gamma \)] act as zero on \( \overline{M}(\tau, i) \). We have thus computed the premutation \( \overline{\mu}_{j_1}(M(\tau, i)) = (\overline{\mu}_{j_1}(Q(\tau)), \overline{S(\tau)}, \overline{M(\tau, i)}, 0). \)

On the other hand, if we flip the arc \( j_1 \) of \( \tau \) we obtain the ideal triangulation \( \sigma = f_{j_1}(\tau) \) sketched in Figure 9.7 (in a clear abuse of notation, we are using the same symbol \( j_1 \) in both \( \tau \) and \( \sigma \)).

Clearly, the triple \( \Phi = (\varphi, \psi, \eta) \) is a right-equivalence between these QP-representations, where

- \( \varphi \) is the right-equivalence whose action on the arrows is given by (9.3);

- \( \psi : \overline{M(\tau, i)} \to M(\sigma, i) \) is the vector space isomorphism given by the identity
1 : $M(\tau,i)_k \rightarrow M(\sigma,i)_k$ for all $k \in \sigma$.

- $\eta$ is the zero map (of the zero space).

This finishes the proof of Theorem 9.2.1 for configurations 2 of Figure 7.6.

Case 3 (Configuration 3, Figure 7.6). Assume that, around the arc $j_1$ to be flipped, $\tau$ and $i$ look as shown in Figure 9.8. Then $M(\tau,i)$ is the positive simple representation at vertex $j_1$. Both flip and mutation take it to the negative simple representation at $j_1$. Thus this case is trivial.
9.2. Main result of Part II: Statement and proof

Case 4 (Configuration 4, Figure 7.6). Assume that, around the arc $j_1$ to be flipped, $\tau$ and $i$ look as shown in Figure 9.9. The relevant vector spaces assigned in $M(\tau, i)$ to the vertices of $Q(\tau)$ are

$$M_{j_1} = M(\tau, i)_{j_1} = K, \quad M_{j_2} = M(\tau, i)_{j_2} = 0,$$

$$M_{j_3} = M(\tau, i)_{j_3} = K, \quad M_{j_4} = M(\tau, i)_{j_4} = 0,$$

and $M_{j_5} = M(\tau, i)_{j_5} = 0$.

Since none of $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ and $\eta$ is parallel to any detour of $(\tau, i)$, the detour matrices $D_{i,j_1}^{\Delta_1}$, $D_{i,j_2}^{\Delta_1}$, $D_{i,j_3}^{\Delta_1}$, $D_{i,j_4}^{\Delta_2}$ and $D_{i,j_5}^{\Delta_2}$ are identities (of the corresponding sizes). Hence the arrows $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ and $\eta$ act on $M(\tau, i)$ according to the following linear maps:

$$M(\tau, i)_\alpha = 0, \quad M(\tau, i)_\beta = 1 : K \to K,$$

$$M(\tau, i)_\gamma = 0, \quad M(\tau, i)_\delta = 0,$$

$$M(\tau, i)_\varepsilon = 0, \quad M(\tau, i)_\eta = 1 : K \to K.$$
9.2. Main result of Part II: Statement and proof

Let us investigate the effect of the \( j_1 \)th QP-mutation on \( \mathcal{M}(\tau, i) \). An easy check using the information about \( \mathcal{M}(\tau, i) \) we have collected thus far yields

\[
M_{\text{in}} = M_{j_2} \oplus M_{j_4} = 0 \oplus K,
\]

\[
M_{\text{out}} = M_{j_3} \oplus M_{j_5} = K \oplus 0.
\]

\( a = 1 : M_{\text{in}} = 0 \oplus K \rightarrow K = M_{j_1}, \)

\( b = 1 : M_{j_1} = K \rightarrow K \oplus 0 = M_{\text{out}}, \)

\( c = 0 : M_{\text{out}} = K \oplus 0 \rightarrow 0 \oplus K = M_{\text{in}}. \)

We see that \( \text{im } b = \ker c, \text{ im } c = 0 \) and \( \ker a = 0 \). Hence \( \overline{M}_{j_1} = 0. \)

Let us compute the action of the arrows of \( \tilde{\mu}_{j_1}(Q(\tau)) \) on \( \overline{M}(\tau, i) \). A trivial check shows that \( \beta^*, \gamma^*, \varepsilon^*, \eta^* \), and \( [\varepsilon \gamma] \) act as zero on \( \overline{M}(\tau, i) \), whereas \( [\beta \eta] = 1 : M_{j_4} \rightarrow M_{j_3}. \) We have thus computed the premutation \( \tilde{\mu}_{j_1}(\mathcal{M}(\tau, i)) = (\tilde{\mu}_{j_1}(Q(\tau)), \tilde{S}(\tau), \overline{M}(\tau, i), 0). \)

On the other hand, if we flip the arc \( j_1 \) of \( \tau \) we obtain the ideal triangulation \( \sigma = f_{j_1}(\tau) \) sketched in Figure 9.10 (in a clear abuse of notation, we are using the same symbol \( j_1 \) in both \( \tau \) and \( \sigma \)). Clearly, the triple \( \Phi = (\varphi, \psi, \eta) \) is a right-equivalence between these QP-representations, where

- \( \varphi \) is the right-equivalence whose action on the arrows is given by (9.3);

- \( \psi : \overline{M}(\tau, i) \rightarrow M(\sigma, i) \) is the vector space isomorphism given by the identity

\[
1 : \overline{M}(\tau, i)_k \rightarrow M(\sigma, i)_k \text{ for all } k \in \sigma,
\]
• \( \eta \) is the zero map (of the zero space).

This finishes the proof of Theorem 9.2.1 for configuration 4 of Figure 7.6.

Case 5 (Configuration 5, Figure 7.6). Assume that, around the arc \( j_1 \) to be flipped, \( \tau \) and \( i \) look as shown in Figure 9.11. The relevant vector spaces assigned in \( M(\tau, i) \) to the vertices of \( Q(\tau) \) are

\[
M_{j_1} = M(\tau, i)_{j_1} = K^2, \quad M_{j_2} = M(\tau, i)_{j_2} = K,
\]
\[ M_{j_3} = M(\tau, i)_{j_3} = K, \quad M_{j_4} = M(\tau, i)_{j_4} = K, \]
\[ \text{and} \quad M_{j_5} = M(\tau, i)_{j_5} = K. \]

Only \( \gamma \) is parallel to a detour. Hence the arrows \( \alpha, \beta, \gamma, \delta, \varepsilon \) and \( \eta \) act on \( M(\tau, i) \) according to the following linear maps:

\[
M(\tau, i)_\alpha = 0, \quad M(\tau, i)_\beta = \begin{bmatrix} 0 & 1 \end{bmatrix} : K^2 \to K,
\]
\[
M(\tau, i)_\gamma = \begin{bmatrix} 1 \\ -w \end{bmatrix} : K \to K^2, \quad M(\tau, i)_\delta = 0,
\]
\[
M(\tau, i)_\varepsilon = \begin{bmatrix} 1 & 0 \end{bmatrix} : K^2 \to K, \quad M(\tau, i)_\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix} : K \to K^2.
\]

Let us investigate the effect of the \( j_1^{\text{th}} \) QP-mutation on \( \mathcal{M}(\tau, i) \). An easy check using the information about \( \mathcal{M}(\tau, i) \) we have collected thus far yields

\[
M_{\text{in}} = M_{j_2} \oplus M_{j_4} = K \oplus K,
\]
\[
M_{\text{out}} = M_{j_3} \oplus M_{j_5} = K \oplus K.
\]
\[
a = \begin{bmatrix} 1 & 0 \\ -w & 1 \end{bmatrix} : M_{\text{in}} = K \oplus K \to K^2 = M_{j_1},
\]
\[
b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : M_{j_1} = K^2 \to K \oplus K = M_{\text{out}},
\]
\[ c = 0 : M_{\text{out}} = K \oplus K \to K \oplus K = M_{\text{in}}. \]

We see that \( \text{im } b = \ker c, \text{im } c = 0 \) and \( \ker a = 0 \). Hence \( \overline{M_{j_1}} = 0 \).

Let us compute the action of the arrows of \( \overline{\mu_{j_1}(Q(\tau))} \) on \( \overline{M(\tau, i)} \). A trivial check shows that \( \beta^*, \gamma^*, \varepsilon^* \) and \( \eta^* \), act as zero on \( \overline{M(\tau, i)} \), whereas \([\beta \eta] = 1 : M_{j_4} \to M_{j_3}\) and \([\varepsilon \gamma] = 1 : M_{j_2} \to M_{j_5}\). We have thus computed the premutation \( \overline{\mu_{j_1}(M(\tau, i))} = (\overline{\mu_{j_1}(Q(\tau))}, \overline{S(\tau)}, \overline{M(\tau, i)}, 0) \).

On the other hand, if we flip the arc \( j_1 \) of \( \tau \) we obtain the ideal triangulation \( \sigma = f_{j_1}(\tau) \) sketched in Figure 9.12 (in a clear abuse of notation, we are using the same symbol \( j_1 \) in both \( \tau \) and \( \sigma \)).

Clearly, the triple \( \Phi = (\varphi, \psi, \eta) \) is a right-equivalence between these QP-representations, where

- \( \varphi \) is the right-equivalence whose action on the arrows is given by (9.3);
- \( \psi : \overline{M(\tau, i)} \to M(\sigma, i) \) is the vector space isomorphism given by the identity 
  \[ 1 : \overline{M(\tau, i)}_k \to M(\sigma, i)_k \text{ for all } k \in \sigma; \]
9.2. Main result of Part II: Statement and proof

- $\eta$ is the zero map (of the zero space).

This finishes the proof of Theorem 9.2.1 for configuration 5 of Figure 7.6.

*Case 6 (Configuration 6, Figure 7.6).* Now we are going to deal with configuration 6 from Figure 7.6. Assume that, around the arc $j_1$ to be flipped, $\tau$ and $i$ look as shown in Figure 9.13.

The relevant vector spaces assigned in $M(\tau, i)$ to the vertices of $Q(\tau)$ are

$$M_{j_1} = M(\tau, i)_{j_1} = K^{2n-1} \oplus W, \quad M_{j_2} = M(\tau, i)_{j_2} = K^{n-1},$$

$$M_{j_3} = M(\tau, i)_{j_3} = K^{n-1} \oplus W, \quad M_{j_4} = M(\tau, i)_{j_4} = K^{n-1} \oplus W,$$
and \( M_{j_5} = M(\tau, i)_{j_5} = K^{n-1} \),

where \( W = K \) if the number of intersection points of \( i \) with \( j_1 \) is \( 2_n \), and \( W = 0 \) if the number of intersection points of \( i \) with \( j_1 \) is \( 2n - 1 \). (That is, \( W \) account for the dotted segment in Figure 9.13, which may be or may be not part of \( i \)).

The arrows \( \alpha, \beta \) act as zero, whereas \( \beta, \gamma, \varepsilon \) and \( \eta \) act on \( M(\tau, i) \) either according to the following linear maps:

\[
M(\tau, i)_\beta = \begin{bmatrix} 0_{(n-1)\times n} & 1_{(n-1)\times (n-1)} \end{bmatrix} : K^{2n-1} \rightarrow K^{n-1},
\]

\[
M(\tau, i)_\gamma = \begin{bmatrix} 1_{n\times n} \\ -w C \end{bmatrix} : K^n \rightarrow K^{2n-1} \text{ where } C = [1_{(n-1)\times (n-1)} \ 0_{(n-1)\times 1}],
\]

\[
M(\tau, i)_\varepsilon = \begin{bmatrix} 0_{(n-1)\times 1} & 1_{(n-1)\times (n-1)} & 0_{(n-1)\times (n-1)} \end{bmatrix} : K^{2n-1} \rightarrow K^{n-1},
\]

and \( M(\tau, i)_\eta = \begin{bmatrix} 0_{1\times (n-1)} \\ -\varepsilon 1_{(n-1)\times (n-1)} \\ 1_{(n-1)\times (n-1)} \end{bmatrix} : K^{n-1} \rightarrow K^{2n-1}, \)

or according to the following maps

\[
M(\tau, i)_\beta = \begin{bmatrix} 0_{n\times n} & 1_{n\times n} \end{bmatrix} : K^{2n} \rightarrow K^n,
\]

\[
M(\tau, i)_\gamma = \begin{bmatrix} 1_{n\times n} \\ -w 1_{n\times n} \end{bmatrix} : K^n \rightarrow K^{2n},
\]

\[
M(\tau, i)_\varepsilon = \begin{bmatrix} 0_{(n-1)\times 1} & 1_{(n-1)\times (n-1)} & 0_{(n-1)\times n} \end{bmatrix} : K^{2n} \rightarrow K^{n-1},
\]
9.2. Main result of Part II: Statement and proof

and \( M(\tau, i)_{\eta} = \begin{bmatrix} \mathbf{0}_{1 \times n} \\ -z \mathbf{E} \\ \mathbf{1}_{n \times n} \end{bmatrix} : K^n \rightarrow K^{2n} \) where \( E = [\mathbf{1}_{(n-1) \times (n-1)} \ 0_{(n-1) \times 1}] \).

In any case, the map \( a : M_{in} \rightarrow M_{j_1} \) is bijective, the map \( b : M_{j_1} \rightarrow M_{out} \) is surjective and the map \( c : M_{out} \rightarrow M_{in} \) is identically zero. This implies

\[
\overline{M}_{j_1} = 0 \text{ and } \nabla_{j_1} = 0. \quad (9.4)
\]

If we flip the arc \( j_1 \) of \( \tau \) we obtain the ideal triangulation \( \sigma = f_{j_1}(\tau) \) sketched in Figure 9.14 (in a clear abuse of notation, we are using the same symbol \( j_1 \) in both \( \tau \) and \( \sigma \)).

Thus the triple \( \Phi = (\varphi, \psi, \eta) \) is a right-equivalence between these QP-representations, where

- \( \varphi \) is the right-equivalence whose action on the arrows is given by (9.3);
- \( \psi : \overline{M}(\tau, i) \rightarrow M(\sigma, i) \) is the vector space isomorphism given by the identity \( 1 : \overline{M}(\tau, i) \rightarrow M(\sigma, i) \) for all \( k \in \sigma \);
- \( \eta \) is the zero map (of the zero space).

This finishes the proof of Theorem 9.2.1 for configuration 6 from Figure 7.6.

Case 7 (Configuration 7, Figure 7.6). Configuration 7 from Figure 7.6 is completely analogous to Configuration 6, so it is left to the reader.

Case 8 (Configuration 27, Figure 7.9). Now we are going to deal with configuration 27 from Figure 7.9. Assume that, around the arc \( j_1 \) to be flipped, \( \tau \) and \( i \) look as
shown in Figure 9.15.

The relevant vector spaces assigned in $M(\tau, i)$ to the vertices of $Q(\tau)$ are

$$M_{j_1} = M(\tau, i)_{j_1} = K, \ M_{j_2} = M(\tau, i)_{j_2} = K,$$

$$M_{j_3} = M(\tau, i)_{j_3} = 0, \ M_{j_4} = M(\tau, i)_{j_4} = K,$$

and $M_{j_5} = M(\tau, i)_{j_5} = 0$.

The arrows $\alpha, \beta, \delta$ and $\epsilon$ act as zero, whereas $M(\tau, i)_{\gamma} = 1 : K \to K$, and $M(\tau, i)_{\eta} = 1 : K \to K$.

The maps $b : M_{j_1} \to M_{\text{out}}$ and $c : M_{\text{out}} \to M_{\text{in}}$ are identically zero, $a : M_{\text{in}} \to M_{j_1}$.
is surjective, and \( \ker a = \{ (v, -v) \mid v \in K \} \cong K \). Therefore,

\[
\overline{M}_{j_1} = K \quad \text{and} \quad \overline{v}_{j_1} = 0. \tag{9.5}
\]

Furthermore, \( \beta^*, \varepsilon^* \), \( [\beta \eta] \) and \( [\varepsilon \gamma] \) obviously act as zero on \( \overline{M(\tau, i)} \), and \( \gamma^* = 1 : \overline{M}_{j_1} = K \to \overline{M}_{j_2} = K \) and \( \eta^* = -1 : \overline{M}_{j_1} = K \to \overline{M}_{j_1} = K \).

If we flip the arc \( j_1 \) of \( \tau \) we obtain the ideal triangulation \( \sigma = f_{j_1}(\tau) \) sketched in Figure 9.16 (in a clear abuse of notation, we are using the same symbol \( j_1 \) in both \( \tau \) and \( \sigma \)).

Thus the triple \( \Phi = (\varphi, \psi, \eta) \) is a right-equivalence between these QP-representations, where

- \( \varphi \) is the right-equivalence whose action on the arrows is given by (9.3);
\[ \psi : M(\tau, i) \to M(\sigma, i) \] is the vector space isomorphism given by the identity
\[ 1 : M(\tau, i)_k \to M(\sigma, i)_k \] for all \( k \in \sigma; \]

• \( \eta \) is the zero map (of the zero space).

This finishes the proof of Theorem 9.2.1 for configuration 27 from Figure 7.9.

Case 9 (Configuration 28, Figure 7.10). Now we are going to deal with configuration 28 from Figure 7.10. Assume that, around the arc \( j_1 \) to be flipped, \( \tau \) and \( i \) look as shown in Figure 9.17.

The relevant vector spaces assigned in \( M(\tau, i) \) to the vertices of \( Q(\tau) \) are

\[ M_{j_1} = M(\tau, i)_{j_1} = K, \ M_{j_2} = M(\tau, i)_{j_2} = K^n, \]

\[ M_{j_3} = M(\tau, i)_{j_3} = K^{n+1}, \ M_{j_4} = M(\tau, i)_{j_4} = K^n, \]
and \( M_{j_5} = M(\tau, i)_{j_5} = K^{n+1} \).

We also have (with some notational abuse regarding the intersection points of \( i \) with the arcs of \( \tau \) )

\[
\mathfrak{B}_{i,j_1}^{\triangle_1,l} = \mathfrak{B}_{i,j_1}^{\triangle_2,l} = \mathfrak{B}_{i,j_2}^{\triangle_1,l} = \mathfrak{B}_{i,j_4}^{\triangle_2,l} = \emptyset \text{ for } l \geq 1; \\
\mathfrak{B}_{i,j_3}^{\triangle_1,l} = \{(t_1, t_{l+1}, b(d_{(t_1,t_{l+1})}^{\triangle_1,l}, y))\} \text{ and } \mathfrak{B}_{i,j_5}^{\triangle_2,l} = \{(t_1, t_{l+1}, b(d_{(t_1,t_{l+1})}^{\triangle_1,l}, x))\} \text{ for } 1 \leq l \leq n.
\]

The relevant detour matrices are therefore defined as follows. The matrices \( D_{i,j_1}^{\triangle_1} \),
9.2. Main result of Part II: Statement and proof

$D_{i,j_1}^\triangle_2$, $D_{i,j_2}^\triangle_1$ and $D_{i,j_4}^\triangle_2$ are identities (of the corresponding sizes), whereas

\[
D_{i,j_3}^\triangle_1 = \begin{bmatrix}
1 \\ -y \\ x \\
\vdots \\
(-1)^{n-1}x_{i,j_3} \frac{1}{y} \frac{1}{2} \\
(-1)^{n}x_{i,j_3} \frac{1}{y} \frac{1}{2}
\end{bmatrix} \begin{bmatrix}
o_{1 \times n} \\ n \times n
\end{bmatrix},
\]

\[
D_{i,j_5}^\triangle_2 = \begin{bmatrix}
1 \\ -x \\ x \\
\vdots \\
(-1)^{n-1}x_{i,j_5} \frac{1}{y} \frac{1}{2} \\
(-1)^{n}x_{i,j_5} \frac{1}{y} \frac{1}{2}
\end{bmatrix} \begin{bmatrix}
o_{1 \times n} \\ n \times n
\end{bmatrix}.
\]

Hence the arrows $\alpha, \beta, \gamma, \delta, \varepsilon$ and $\eta$ act on $M(\tau, i)$ according to the following linear maps:

\[
M(\tau, i)_\alpha = (D_{i,j_2}^\triangle_1)(m(\tau, i)_\alpha) = \begin{bmatrix}
0_{n \times 1} \\ 1_{n \times n}
\end{bmatrix} : K^{n+1} \rightarrow K^n,
\]

\[
M(\tau, i)_\beta = (D_{i,j_3}^\triangle_1)(m(\tau, i)_\beta) = \begin{bmatrix}
1 \\ -y \\ x \\
\vdots \\
(-1)^{n-1}x_{i,j_3} \frac{1}{y} \frac{1}{2} \\
(-1)^{n}x_{i,j_3} \frac{1}{y} \frac{1}{2}
\end{bmatrix} : K \rightarrow K^{n+1},
\]

\[
M(\tau, i)_\gamma = (D_{i,j_1}^\triangle_1)(m(\tau, i)_\gamma) = 0 : K^n \rightarrow K,
\]

\[
M(\tau, i)_\delta = (D_{i,j_4}^\triangle_2)(m(\tau, i)_\delta) = \begin{bmatrix}
0_{n \times 1} \\ 1_{n \times n}
\end{bmatrix} : K^{n+1} \rightarrow K^n,
\]

\[
M(\tau, i)_\varepsilon = (D_{i,j_5}^\triangle_2)(m(\tau, i)_\varepsilon) = \begin{bmatrix}
1 \\ -x \\ x \\
\vdots \\
(-1)^{n-1}x_{i,j_5} \frac{1}{y} \frac{1}{2} \\
(-1)^{n}x_{i,j_5} \frac{1}{y} \frac{1}{2}
\end{bmatrix} : K \rightarrow K^{n+1},
\]

and $M(\tau, i)_\eta = (D_{i,j_1}^\triangle_2)(m(\tau, i)_\eta) = 0 : K^n \rightarrow K$. 
Let us investigate the effect of the \( f_1 \) QP-mutation on \( \mathcal{M}(\tau, i) \). An easy check using the information about \( \mathcal{M}(\tau, i) \) we have collected thus far yields

\[
M_{in} = M_{j2} \oplus M_{j4} = K^n \oplus K^n,
\]

\[
M_{out} = M_{j3} \oplus M_{j5} = K^{n+1} \oplus K^{n+1}.
\]

\( a = 0 : M_{in} = K^n \oplus K^n \to K = M_{j1}, \)

\[
b = \begin{bmatrix}
1 \\
- y \\
x y \\
\vdots \\
(-1)^{n+1} \\
1 \\
- x \\
x y \\
\vdots \\
(-1)^{n+1} \\
\end{bmatrix} : M_{j1} = K \to K^{n+1} \oplus K^{n+1} = M_{out},
\]

\[
c = \begin{bmatrix}
0_{n \times 1} & 1_{n \times n} & y 1_{n \times n} & 0_{n \times 1} \\
x 1_{n \times n} & 0_{n \times 1} & 0_{n \times 1} & 1_{n \times n}
\end{bmatrix}:
\]

\[
M_{out} = K^{n+1} \oplus K^{n+1} \to K^n \oplus K^n = M_{in}.
\]

It is easily seen that \( b \) is injective and that \( \ker c = \{(u, v) \in K^{n+1} \oplus K^{n+1} \mid u_{i+1} + yv_i = xu_i + v_{i+1} = 0 \forall l \in \{1, \ldots, n\}\} = \{(u_1, -yv_1, xyu_1, \ldots, v_1, -xu_1, xyv_1, \ldots) \mid u_1, v_1 \in K\}, \) which is isomorphic to \( K^2 \) under the assignment \( \ell : (u_1, -yv_1, xyu_1, \ldots, v_1, -xu_1, xyv_1, \ldots) \mapsto (u_1, v_1 - u_1). \) Together with a standard dimension counting, this yields surjectivity of \( c. \)

Now \( \text{im } b = \{u, -yu, xyu, \ldots, (-1)^n x^{ \frac{n+1}{2} } y^{ \frac{n+1}{2} } u, -xu, \ldots (-1)^n x^{ \frac{n+1}{2} } y^{ \frac{n+1}{2} } u \} u \in \)
$K$ is mapped by $\ell$ onto $\ell(\text{im} \, b) = \{(u, 0) \mid u \in K\}$. Hence $\frac{\text{ker} \, \xi}{\text{im} \, b} \cong K$ and we can describe the canonical projection $\ker \, c \rightarrow \frac{\text{ker} \, \xi}{\text{im} \, b}$ by means of the matrix

$$p = \begin{bmatrix} 0 & -1 \end{bmatrix} : K^2 \rightarrow K.$$

From the previous two paragraphs we deduce that

$$\overline{M}_j = K \oplus (K^n \oplus K^n) \oplus 0 \oplus 0 \quad \text{and} \quad \nabla_j = 0. \quad (9.6)$$

And from the fact that $\gamma$ and $\eta$ act as zero on $M(\tau, i)$, we conclude that the arrows $[\beta \gamma], [\varepsilon \gamma], [\beta \eta]$ and $[\varepsilon \eta]$ of $\tilde{\mu}_j(Q(\tau))$ act as zero on $\overline{M(\tau, i)}$. Since the arrows of $\tilde{\mu}_j(Q(\tau))$ not incident to $j_1$ act on $\overline{M(\tau, i)}$ in the exact same way they act on $M(\tau, i)$, we just have to find out how the arrows $\beta^*, \gamma^*, \varepsilon^*$ and $\eta^*$ of $\tilde{\mu}_j(Q(\tau))$ act on $\overline{M(\tau, i)}$. To this end, we choose the zero section $s = 0 : \frac{\text{ker} \, a}{\text{im} \, c} = 0 \rightarrow \ker \, a$ and the retraction $r : M_{\text{out}} \rightarrow \ker \, c$ given by the matrix

$$r = \begin{bmatrix} 1 & 0_{2 \times n} & 0 \end{bmatrix} : K^{n+1} \oplus K^{n+1} \rightarrow K^2$$

(here we are thinking of $\ell : \ker \, c \xrightarrow{\cong} K^2$ as an identification). A straightforward check yields

$$-pr = \begin{bmatrix} -1 & 0_{1 \times n} & 1 & 0_{1 \times n} \end{bmatrix}.$$
The action of $\beta^*$ and $\varepsilon^*$ is therefore encoded by the matrix $[\beta^* \varepsilon^*] =$

\[
\begin{pmatrix}
-1^{(n+1) \times (n+1)} & 1 \\
-x 1_{n \times n} 0_{n \times 1} & -y \\
0_{n \times 1} & 0_{(n-1) \times (n-1)} \\
-1_{n \times n} & -y 1_{(n-1) \times (n-1)} \\
0_{n \times 1} & 0_{n \times 1} \\
-1_{n \times n} & -1_{n \times n} \\
\end{pmatrix}
\]

: $K^{n+1} \oplus K^{n+1} \rightarrow K \oplus (K^n \oplus K^n) \oplus 0 \oplus 0$,

whereas the arrows $\gamma^*$ and $\eta^*$ act according to the matrix

\[
\begin{pmatrix}
\gamma^* \\
\eta^* \\
\end{pmatrix} =
\begin{pmatrix}
0_{n \times 1} & 1_{n \times n} & 0_{n \times n} & - & - \\
0_{n \times 1} & 0_{n \times n} & 1_{n \times n} & - & - \\
\end{pmatrix}
: K^{(K^{n+1} \oplus K^{n+1}) \oplus 0 \oplus 0} \rightarrow K^{n+1} \oplus K^{n+1}.
\]

This completes the computation of the action of the arrows of $\tilde{\mu}_{j_1}(Q(\tau))$ on $\overline{M(\tau, i)}$.

We have thus computed the permutation $\tilde{\mu}_{j_1}(M(\tau, i)) = (\tilde{\mu}_{j_1}(Q(\tau)), S(\tau), \overline{M(\tau, i)}, 0)$.

On the other hand, if we flip the arc $j_1$ of $\tau$ we obtain the ideal triangulation $\sigma = f_{j_1}(\tau)$ sketched in Figure 9.18 (in a clear abuse of notation, we are using the same symbol $j_1$ in both $\tau$ and $\sigma$).

The relevant vector spaces attached to the vertices of $Q(\sigma)$ are

\[
N_{j_1} = M(\sigma, i)_{j_1} = K^{n+1} ,\ N_{j_2} = M(\sigma, i)_{j_2} = K^{n},
\]

\[
N_{j_3} = M(\sigma, i)_{j_3} = K^{n+1} ,\ N_{j_4} = M(\sigma, i)_{j_4} = K^{n},
\]
and $N_{j_5} = M(\sigma, i)_{j_5} = K^{n+1}$.

We also have (again with some notational abuse regarding intersection points)

$$\mathcal{B}_{i,j_1}^{\triangle', 1} = \{(t_1, t_2, b(d_{(t_1, t_2)}, x)) \cup \{(s_1, t_{l+1}, b(d_{(s_{l+1}, s_l)}, x)) \mid 2 \leq l \leq n\},$$

$$\mathcal{B}_{i,j_1}^{\triangle'', 1} = \{(t_1, s_2, b(d_{(t_1, s_2)}), y)) \cup \{(t_1, s_{l+1}, b(d_{(t_1, s_l)}, y)) \mid 2 \leq l \leq n\};$$

$$\mathcal{B}_{i,j_1}^{\triangle', r} = \mathcal{B}_{i,j_1}^{\triangle'', r} = \emptyset \text{ for } r \geq 2;$$

and $\mathcal{B}_{i,j_2}^{\triangle', r} = \mathcal{B}_{i,j_3}^{\triangle', r} = \mathcal{B}_{i,j_4}^{\triangle', r} = \mathcal{B}_{i,j_5}^{\triangle', r} = \emptyset$

$$\mathcal{B}_{i,j_2}^{\triangle'', r} = \mathcal{B}_{i,j_3}^{\triangle'', r} = \mathcal{B}_{i,j_4}^{\triangle'', r} = \mathcal{B}_{i,j_5}^{\triangle'', r} = \emptyset \text{ for } r \geq 1.$$

The relevant detour matrices are therefore defined as follows. The matrices $D_{i,j_2}^{\triangle''}$,
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\(D^\Delta_{i,j_1}, D^\Delta_{i,j_2}\) and \(D^\Delta_{i,j_5}\) are identities (of the corresponding sizes), whereas

\[
D^\Delta_{i,j_1} = \begin{bmatrix}
1_{(n+1)\times(n+1)} & 0_{(n+1)\times n} \\
-x1_{n\times n} & 0_{n\times 1} & 1_{n\times n}
\end{bmatrix}
\]

and

\[
D^\Delta_{i,j_5} = \begin{bmatrix}
0_{(2n-1)\times 1} & 1_{n\times n} & 0_{2\times n} \\
0_{n\times n} & -y1_{(n-1)\times(n-1)} & 0_{(n-1)\times 1}
\end{bmatrix},
\]

where the order in which the basis vectors of \(N_{j_1}\) are taken is \((t_1, s_2, \ldots, s_{n+1}, t_2, \ldots, t_{n+1})\).

Hence the arrows \(\beta^*, [\beta\eta], \eta^*, \varepsilon^*\) and \([\varepsilon\gamma]\) act on \(M(\sigma, i)\) according to the following linear maps:

\[
M(\sigma, i)_{\beta^*} = (D^\Delta_{i,j_1})(m(\sigma, i)_{\beta^*}) = \begin{bmatrix}
1_{(n+1)\times(n+1)} \\
-x1_{n\times n} & 0_{n\times 1} & 1_{n\times n}
\end{bmatrix} : K^{n+1} \rightarrow K^{2n+1},
\]

\[
M(\sigma, i)_{[\beta\eta]} = (D^\Delta_{i,j_3})(m(\sigma, i)_{[\beta\eta]}) = 0 : K^n \rightarrow K^{n+1},
\]

\[
M(\sigma, i)_{\eta^*} = (D^\Delta_{i,j_4})(m(\sigma, i)_{\eta^*}) = \begin{bmatrix}
0_{n\times(n+1)} & 1_{n\times n}
\end{bmatrix} : K^{2n+1} \rightarrow K^n,
\]

\[
M(\sigma, i)_{\gamma^*} = (D^\Delta_{i,j_2})(m(\sigma, i)_{\gamma^*}) = \begin{bmatrix}
0_{n\times 1} & 1_{n\times n} & 0_{n\times n}
\end{bmatrix} : K^{2n+1} \rightarrow K^n,
\]

\[
M(\sigma, i)_{\varepsilon^*} = (D^\Delta_{i,j_1})(m(\sigma, i)_{\varepsilon^*}) = \begin{bmatrix}
1_{(n+1)\times(n+1)} & 0_{2\times n} \\
-x1_{n\times n} & -y1_{(n-1)\times(n-1)} & 0_{(n-1)\times 1}
\end{bmatrix} : K^{n+1} \rightarrow K^{2n+1},
\]

\[
M(\sigma, i)_{[\varepsilon\gamma]} = (D^\Delta_{i,j_5})(m(\sigma, i)_{[\varepsilon\gamma]}) = 0 : K^n \rightarrow K^{n+1}.
\]
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We have thus computed the spaces and linear maps of $\mathcal{M}(\sigma, i)$ relevant to the flip of the arc $j_1$. Now we have to compare it to $\mu_{j_1}(\mathcal{M}(\tau, i))$. The triple $\Phi = (\varphi, \psi, \eta)$ is a right-equivalence between these QP-representations, where

- $\varphi$ is the right-equivalence whose action on the arrows is given by (9.3);

- $\psi : \overline{M(\tau, i)} \to \overline{M(\sigma, i)}$ is the vector space isomorphism given by the identity $1 : \overline{M(\tau, i)_{j_1}} \to \overline{M(\sigma, i)_{j_1}}$ for $k \neq j_1$, and the matrix

$$
\psi_{j_1} = \begin{bmatrix}
1_{(n+1)\times(n+1)} & 0_{(n+1)\times n} \\
0_{n\times(n+1)} & -1_{n\times n}
\end{bmatrix} : \overline{M(\tau, i)_{j_1}} \to \overline{M(\sigma, i)_{j_1}};
$$

- $\eta$ is the zero map (of the zero space).

This finishes the proof of Theorem 9.2.1 for configuration 28 from Figure 7.10.

Case 10 (Configuration 29, Figure 7.10). Assume that, around the arc $j_1$ to be flipped, $\tau$ and $i$ look as shown in Figure 9.19.

The relevant vector spaces assigned in $\overline{M(\tau, i)}$ to the vertices of $Q(\tau)$ are

$$
M_{j_1} = \overline{M(\tau, i)_{j_1}} = K^{2n-1} \oplus W, \quad M_{j_2} = \overline{M(\tau, i)_{j_2}} = K^{n-1} \oplus W,
$$

$$
M_{j_3} = \overline{M(\tau, i)_{j_3}} = K^n, \quad M_{j_4} = \overline{M(\tau, i)_{j_4}} = K^{n-1},
$$

and $M_{j_5} = \overline{M(\tau, i)_{j_5}} = K^n \oplus W$,

where $W = 0$ if the number of intersection points of $i$ with $j_1$ is $2n - 1$ and $W = K$ if the number of intersection points of $i$ with $j_1$ is $2n$. (That is, $W$ takes into account the dotted segment of Figure 9.19, which may or not be a part of $i$.)
The arrows $\alpha$ and $\delta$ act as zero on $M(\tau, i)$, whereas $\beta$, $\gamma$, $\varepsilon$ and $\eta$ act either according to the following linear maps:

$$M(\tau, i)_\beta = \begin{bmatrix} 1_{n \times n} & 0_{n \times (n-1)} \end{bmatrix} : K^{2n-1} \to K^n,$$

$$M(\tau, i)_\gamma = \begin{bmatrix} -w1_{(n-1) \times (n-1)} & 0_{1 \times (n-1)} & 1_{(n-1) \times (n-1)} \end{bmatrix} : K^{n-1} \to K^{2n-1},$$

$$M(\tau, i)_\varepsilon = \begin{bmatrix} e_1 & 0_{n \times (n-1)} & e_2 & e_3 & \ldots & e_n \end{bmatrix} : K^{2n-1} \to K^n \text{ (standard basis vectors),}$$
and $M(\tau, i)_{\eta} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ -z & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & -z & 0 & \ldots & 0 & 0 \\ 0 & 0 & -z & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -z & 0 \\ 0 & 0 & 0 & \ldots & 0 & -z \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$ : $K^{n-1} \to K^{2n-1}$,

or according to the maps

$$M(\tau, i)_{\beta} = \begin{bmatrix} 1_{n \times n} & 0_{n \times n} \end{bmatrix} : K^{2n} \to K^n,$$

$$M(\tau, i)_{\gamma} = \begin{bmatrix} -w 1_{n \times n} \\ 1_{n \times n} \end{bmatrix} : K^n \to K^{2n};$$

$$M(\tau, i)_{\varepsilon} = \begin{bmatrix} e_1 & 0_{n \times (n-1)} & e_2 & \ldots & e_n & e_{n+1} \end{bmatrix} : K^{2n-1} \to K^n \text{ (standard basis vectors)},$$

and $M(\tau, i)_{\eta} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ -z & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & -z & 0 & \ldots & 0 & 0 \\ 0 & 0 & -z & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -z & 0 \\ 0 & 0 & 0 & \ldots & 0 & -z \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$ : $K^{n-1} \to K^{2n}$,

Let us investigate the effect of the $j_1^{th}$ QP-mutation on $M(\tau, i)$. An easy check
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using the information about $\mathcal{M}(\tau, i)$ we have collected thus far yields

$$M_{in} = M_{j_2} \oplus M_{j_4} = K^{n-1} \oplus W \oplus K^{n-1},$$

$$M_{out} = M_{j_3} \oplus M_{j_5} = K^n \oplus K^n \oplus W.$$  

Furthermore, the maps $a$ and $b$ are injective, $c$ is identically zero, and the linear function $p : M_{out} = K^n \oplus K^n \oplus W \rightarrow K,$ $(v_1, v_2, \ldots) \mapsto v_1 - v_{n+1}$ induces an isomorphism $\ker_{im} \cong K.$ We deduce that

$$\overline{M}_{j_1} = K \oplus 0 \oplus 0 \oplus 0 \text{ and } \overline{V}_{j_1} = 0. \quad (9.7)$$

The action of $\beta^*$ and $\varepsilon^*$ is encoded by the matrix

$$[\beta^* \varepsilon^*] = \begin{bmatrix} -1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix} : K^n \oplus K^n \oplus W \rightarrow K,$$

whereas the arrows $\gamma^*$ and $\eta^*$ act as zero.

If we flip the arc $j_1$ of $\tau$ we obtain the ideal triangulation $\sigma = f_{j_1}(\tau)$ sketched in Figure 9.20 (in a clear abuse of notation, we are using the same symbol $j_1$ in both $\tau$ and $\sigma$).

The triple $\Phi = (\varphi, \psi, \eta)$ is a right-equivalence between these QP-representations, where

- $\varphi$ is the right-equivalence whose action on the arrows is given by (9.3);
- $\psi : \overline{M}(\tau, i) \rightarrow M(\sigma, i)$ is the vector space isomorphism given by the identity
  $$1 : \overline{M}(\tau, i)_k \rightarrow M(\sigma, i)_k \text{ for all } k \in \sigma;$$
\( \eta \) is the zero map (of the zero space).

This finishes the proof of Theorem 9.2.1 for configuration 29 from Figure 7.10.

**Case 11** (Configuration 30, Figure 7.10). Now we are going to deal with configuration 30 from Figure 7.10. Assume that, around the arc \( j_1 \) to be flipped, \( \tau \) and \( i \) look as shown in Figure 9.21.

The relevant vector spaces assigned in \( M(\tau, i) \) to the vertices of \( Q(\tau) \) are

\[
M_{j_1} = M(\tau, i)_{j_1} = K^m, \quad M_{j_2} = M(\tau, i)_{j_2} = K^{m+1},
\]

\[
M_{j_3} = M(\tau, i)_{j_3} = K, \quad M_{j_4} = M(\tau, i)_{j_4} = K^{m+1},
\]

and \( M_{j_5} = M(\tau, i)_{j_5} = 0. \)
We also have

\[ \mathcal{B}_{i,j_1}^{\triangle_1,l} = \mathcal{B}_{i,j_1}^{\triangle_2,l} = \mathcal{B}_{i,j_3}^{\triangle_1,l} = \mathcal{B}_{i,j_5}^{\triangle_2,l} = \mathcal{B}_{i,j_4}^{\triangle_3,l} = \emptyset \text{ for } l \geq 1, \text{ and} \]

\[ \mathcal{B}_{i,j_2}^{\triangle_1,l} = \{ (s_1, s_{i+1}, b(d_{i,j_2}^{\triangle_1,l}), x) \} \text{ for } 1 \leq l \leq m \text{ and} \]

The relevant detour matrices are therefore defined as follows. The matrices \( D_{i,j_1}^{\triangle_1,l} \),
$D_{i,j_1}^{\Delta_2}, D_{i,j_2}^{\Delta_1}, D_{i,j_3}^{\Delta_2}$ and $D_{i,j_4}^{\Delta_2}$ are identities (of the corresponding sizes), whereas

$$D_{i,j_2}^{\Delta_1} = \begin{bmatrix}
1 \\
-x \\
xw \\
\vdots \\
(-1)^{l-1}x^{\lfloor \frac{m+1}{w} \rfloor}y^{\lfloor \frac{m+1}{w} \rfloor} \\
\vdots \\
(-1)^{m}x^{\lfloor \frac{m+1}{w} \rfloor}y^{\lfloor \frac{m+1}{w} \rfloor} \\
\end{bmatrix}_{0_{1\times m}} \begin{bmatrix} \mathbf{1}_{m \times m} \\
\end{bmatrix}.$$ 

Hence the arrows $\alpha, \beta, \gamma, \delta, \varepsilon$ and $\eta$ act on $M(\tau, i)$ according to the following linear maps:

$$M(\tau, i)_{\alpha} = (D_{i,j_2}^{\Delta_1})(m(\tau, i)_{\alpha}) = \begin{bmatrix} 1 \\
-x \\
xw \\
\vdots \\
(-1)^{l-1}x^{\lfloor \frac{m+1}{w} \rfloor}y^{\lfloor \frac{m+1}{w} \rfloor} \\
\vdots \\
(-1)^{m}x^{\lfloor \frac{m+1}{w} \rfloor}y^{\lfloor \frac{m+1}{w} \rfloor} \\
\end{bmatrix} : K \rightarrow K^{m+1},$$

$$M(\tau, i)_{\beta} = (D_{i,j_3}^{\Delta_1})(m(\tau, i)_{\beta}) = \mathbf{0} : K^m \rightarrow K,$$

$$M(\tau, i)_{\gamma} = (D_{i,j_1}^{\Delta_1})(m(\tau, i)_{\gamma}) = \begin{bmatrix} 0_{m \times 1} & \mathbf{1}_{m \times m} \\
\end{bmatrix} : K^{m+1} \rightarrow K^m,$$

$$M(\tau, i)_{\delta} = (D_{i,j_4}^{\Delta_2})(m(\tau, i)_{\delta}) = \mathbf{0} : K^m \rightarrow 0,$$

$$M(\tau, i)_{\varepsilon} = (D_{i,j_5}^{\Delta_2})(m(\tau, i)_{\varepsilon}) = \mathbf{0} : K^m \rightarrow K^{m+1},$$

and $M(\tau, i)_{\eta} = (D_{i,j_1}^{\Delta_2})(m(\tau, i)_{\eta}) = \mathbf{0} : K^m \rightarrow 0$,

$$M(\tau, i)_{\eta} = (D_{i,j_1}^{\Delta_2})(m(\tau, i)_{\eta}) = \begin{bmatrix} \mathbf{1}_{m \times m} & 0_{n \times 1} \\
\end{bmatrix} : K^{m+1} \rightarrow K^m.$$ 

Let us investigate the effect of the $j_1$ th QP-mutation on $\mathcal{M}(\tau, i)$. An easy check
using the information about $\mathcal{M}(\tau, i)$ we have collected thus far yields

$$M_{\text{in}} = M_{j_2} \oplus M_{j_4} = K^{m+1} \oplus K^{m+1},$$

$$M_{\text{out}} = M_{j_3} \oplus M_{j_5} = K \oplus 0.$$ 

$$a = \begin{bmatrix} 0_{m \times 1} & 1_{m \times m} & 1_{m \times m} & 0_{m \times 1} \end{bmatrix} : M_{\text{in}} = K^{m+1} \oplus K^{m+1} \to K^m = M_{j_1},$$

$$b = 0 : M_{j_1} = K^m \to K \oplus 0 = M_{\text{out}},$$

$$c = \begin{bmatrix} 1 \\ -x \\ wx \\ \vdots \\ (-1)^{m-1} \frac{x}{w \cdot \left\lceil \frac{m-1}{2} \right\rceil} \\ (-1)^{m} \frac{x}{w \cdot \left\lceil \frac{m}{2} \right\rceil} \\ \vdots \\ (-1)^{m} \frac{w}{x \cdot \left\lfloor \frac{m-1}{2} \right\rfloor} \\ (-1)^{m} \frac{w}{x \cdot \left\lfloor \frac{m}{2} \right\rfloor} \end{bmatrix} : M_{\text{out}} = K \oplus 0 \to K^{m+1} \oplus K^{m+1} = M_{\text{in}}.$$ 

It is easily seen that $a$ is surjective and $c$ is injective. Moreover, a straightforward computation shows that $\ker a = \{(u_1, u_2, \ldots, u_{m+1}, -u_2, \ldots, -u_{m+1}, v) \mid u_1, \ldots, u_{m+1}, v \in K\}$, which is isomorphic to $K^{m+2}$ under the linear map $\ell : (u_1, u_2, \ldots, u_{m+1}, -u_2, \ldots, -u_{m+1}, v) \mapsto (u_1, u_2, \ldots, u_{m+1}, v)$. The image of $\text{im } c$ under $\ell$ is $\ell(\text{im } c) = \{(u, -xu, xwu, \ldots, (-1)^m x^{\left\lfloor \frac{m+1}{2} \right\rfloor} w^{\left\lfloor \frac{m}{2} \right\rfloor} u, (-1)^m x^{\left\lfloor \frac{m+2}{2} \right\rfloor} w^{\left\lfloor \frac{m+1}{2} \right\rfloor} u) \mid u \in K\}$. Therefore, $\frac{\ker a}{\text{im } c} \cong K^{m+1}$, and we can describe the canonical projection $\ker a \twoheadrightarrow \frac{\ker a}{\text{im } c}$.
by means of the matrix

$$
\begin{bmatrix}
 x \\
 -xw \\
 \vdots \\
 (-1)^{m+1} x \frac{m+1}{w} \\
 (-1)^{m+1} x \frac{m+1}{w} \\
\end{bmatrix}
\begin{pmatrix}
 1 & (m+1) \times (m+1)
\end{pmatrix}
: \mathbb{K}^{m+2} \to \mathbb{K}^{m+1}
$$

and the inclusion \( \ker a \hookrightarrow M_{in} \) by means of the matrix

$$
i = \begin{bmatrix}
 1 & (m+1) \times (m+1) & 0_{(m+1) \times 1} \\
 0_{m \times 1} & -1_{m \times m} & 0_{m \times 1} \\
 0_{1 \times (m+1)} & & 1
\end{bmatrix}
: \mathbb{K}^{m+2} \to \mathbb{K}^{m+1} \oplus \mathbb{K}^{m+1}.
$$

We deduce that

$$
\overline{M}_{j_1} = 0 \oplus \mathbb{K} \oplus \mathbb{K}^{m+1} \oplus 0 \quad \text{and} \quad \overline{V}_{j_1} = 0.
$$

Now, from the fact that \( \beta, \) and \( \varepsilon \) act as zero on \( M(\tau, i) \), we conclude that the arrows \([\beta \gamma], [\varepsilon \gamma], [\beta \eta]\) and \([\varepsilon \eta]\) of \( \widehat{Q}(\tau) \) act as zero on \( \overline{M(\tau, i)} \). Since the arrows of \( \tilde{\mu}_{j_1}(Q(\tau)) \) not incident to \( j_1 \) act on \( \overline{M(\tau, i)} \) in the exact same way they act on \( M(\tau, i) \), we just have to find out how the arrows \( \beta^*, \gamma^*, \varepsilon^* \) and \( \eta^* \) of \( \widehat{Q}(\tau) \) act on \( \overline{M(\tau, i)} \).

To this end, we choose the zero retraction \( r : M_{out} \to \ker r = 0 \) and the section \( s : \ker a \to \ker a \) given by the matrix

$$
s = \begin{bmatrix}
 0_{1 \times (m+1)} \\
 1_{(m+1) \times (m+1)}
\end{bmatrix}
: \mathbb{K}^{m+1} \to \mathbb{K}^{m+2}.
$$
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A straightforward check yields

\[
is = \begin{bmatrix}
0_{1 \times m} & 0 \\
1_{m \times m} & 0_{m \times 1} \\
-1_{m \times m} & 0_{m \times 1} \\
0_{1 \times m} & 1 \\
\end{bmatrix}.
\]

From all these pieces of information we deduce that the action of \( \beta^* \) and \( \varepsilon^* \) is encoded by the matrix

\[
[\beta^* \, \varepsilon^*] = \begin{bmatrix}
-1_{(m+1) \times 1} & - \\
\end{bmatrix} : K \oplus 0 \rightarrow 0 \oplus K \oplus K^{m+1} \oplus 0,
\]
whereas the arrows $\gamma^*$ and $\eta^*$ act according to the matrix

$$
\begin{pmatrix}
\gamma^*
\eta^*
\end{pmatrix} =
\begin{bmatrix}
-1 & 0_{1 \times m} & 0 \\
-x & 1_{m \times m} & 0_{m \times 1} \\
w x & -1_{m \times m} & 0_{m \times 1} \\
\vdots & \vdots & \vdots \\
(-1)^{m-1} x^{\left\lfloor \frac{m}{2} \right\rfloor} w^{\left\lfloor \frac{m-1}{2} \right\rfloor} & -1_{m \times m} & 0_{m \times 1} \\
\end{bmatrix}.
$$

This completes the computation of the action of the arrows of $\tilde{\mu}_{j_1}(Q(\tau))$ on $\overline{M(\tau, i)}$. We have thus computed the permutation $\tilde{\mu}_{j_1}(M(\tau, i)) = (\tilde{\mu}_{j_1}(Q(\tau)), \overline{S(\tau)}, \overline{M(\tau, i)}, 0)$.

On the other hand, if we flip the arc $j_1$ of $\tau$ we obtain the ideal triangulation $\sigma = f_{j_1}(\tau)$ sketched in Figure 9.22 (in a clear abuse of notation, we are using the same symbol $j_1$ in both $\tau$ and $\sigma$).

The relevant vector spaces attached to the vertices of $Q(\sigma)$ are

$$
N_{j_1} = M(\sigma, i)_{j_1} = K^{m+2}, \quad N_{j_2} = M(\sigma, i)_{j_2} = K^{m+1},
$$
$N_{j_3} = M(\sigma, i)_{j_3} = K, \ N_{j_4} = M(\sigma, i)_{j_4} = K^{m+1},$

and $N_{j_5} = M(\sigma, i)_{j_5} = 0.$

We also have

$$\mathcal{B}_{i,j_1}^{\Delta',l} = \{(s_1, s_{l+1}, b(d_{i,j_1}^{\Delta',l}), x)\} \text{ for } 1 \leq l \leq m + 1,$$

and

$$\mathcal{B}_{i,j_1}^{\Delta',m+1+r} = \mathcal{B}_{i,j_2}^{\Delta',r} = \mathcal{B}_{i,j_3}^{\Delta',r} = \mathcal{B}_{i,j_4}^{\Delta',r} = \mathcal{B}_{i,j_5}^{\Delta',r} =$$

$$= \mathcal{B}_{i,j_1}^{\Delta'',r} = \mathcal{B}_{i,j_2}^{\Delta'',r} = \mathcal{B}_{i,j_3}^{\Delta'',r} = \mathcal{B}_{i,j_4}^{\Delta'',r} = \mathcal{B}_{i,j_5}^{\Delta'',r} = \emptyset \text{ for } r \geq 1.$$

The relevant detour matrices are therefore defined as follows. The matrices $D_{i,j_1}^{\Delta''},$
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\( D_{i,j_2}^\Delta, D_{i,j_3}^\Delta, D_{i,j_4}^\Delta \) and \( D_{i,j_5}^\Delta \) are identities (of the corresponding sizes), whereas

\[
D_{i,j_1}^\Delta' = \begin{pmatrix}
1 \\
-x \\
xw \\
\vdots \\
(-1)^{m+1} & \frac{w+2}{w} & \frac{m+1}{w}
\end{pmatrix}^{(m+1)\times(m+1)}
\]

\[
\mathbf{1}^{(m+1)\times(m+1)}
\]

Hence the arrows \( \beta^*, [\beta\eta], \eta^*, \gamma^*, \varepsilon^* \) and \([\varepsilon\gamma]\) act on \( M(\sigma, i) \) according to the following linear maps:

\[
M(\sigma, i)_{\beta^*} = (D_{i,j_1}^\Delta')(m(\sigma, i)_{\beta^*}) = \begin{pmatrix}
1 \\
-x \\
xw \\
\vdots \\
(-1)^{m+1} & \frac{w+2}{w} & \frac{m+1}{w}
\end{pmatrix}^{(m+1)\times(m+1)} : K \to K^{m+2},
\]

\[
M(\sigma, i)_{[\beta\eta]} = (D_{i,j_3}^\Delta)(m(\sigma, i)_{[\beta\eta]}) = 0 : K^{m+1} \to K,
\]

\[
M(\sigma, i)_{\eta^*} = (D_{i,j_4}^\Delta)(m(\sigma, i)_{\eta^*}) = \begin{pmatrix}
0_{(m+1)\times1} & \mathbf{1}_{(m+1)\times(m+1)}
\end{pmatrix} : K^{m+2} \to K^{m+1},
\]

\[
M(\sigma, i)_{\gamma^*} = (D_{i,j_2}^\Delta)(m(\sigma, i)_{\gamma^*}) = \begin{pmatrix}
\mathbf{1}_{(m+1)\times(m+1)} & 0_{(m+1)\times1}
\end{pmatrix} : K^{m+2} \to K^{m+1},
\]

\[
M(\sigma, i)_{\varepsilon^*} = (D_{i,j_1}^\Delta')(m(\sigma, i)_{\varepsilon^*}) = 0 : 0 \to K^{m+2},
\]

\[
M(\sigma, i)_{[\varepsilon\gamma]} = (D_{i,j_5}^\Delta)(m(\sigma, i)_{[\varepsilon\gamma]}) = 0 : K^{m+1} \to 0.
\]

We have thus computed the spaces and linear maps of \( M(\sigma, i) \) relevant to the flip of the arc \( j_1 \). Now we have to compare it to \( \mu_{j_1}(M(\tau, i)) \). The triple \( \Phi = (\varphi, \psi, \eta) \) is a right-equivalence between these QP-representations, where

- \( \varphi \) is the right-equivalence whose action on the arrows is given by (9.3);
• \( \psi : M(\tau, i) \to M(\sigma, i) \) is the vector space isomorphism given by the identity

\[
1 : M(\tau, i)_k \to M(\sigma, i)_k \text{ for } k \neq j_1, \text{ and the matrix }
\]

\[
\psi_{j_1} = \begin{bmatrix}
1 & 0_{1 \times m} & 0 \\
-2 & 1_{m \times m} & 0_{m \times 1} \\
\vdots & \vdots & \vdots \\
(-1)^{m+1} & 0_{1 \times m} & -1
\end{bmatrix}
\]

: \( M(\tau, i)_{j_1} \to M(\sigma, i)_{j_1} \);

• \( \eta \) is the zero map (of the zero space).

This finishes the proof of Theorem 9.2.1 for configuration 30 from Figure 7.10.

Case 12 (Configuration 31, Figure 7.10). Now we are going to deal with configuration 31 from Figure 7.10. In order to avoid making the notation too cumbersome, we include only the verification of the theorem for the configuration 31.4, the general case is completely similar. Assume that, around the arc \( j_1 \) to be flipped, \( \tau \) and \( i \) look as shown in Figure 9.23.

The relevant vector spaces assigned in \( M(\tau, i) \) to the vertices of \( Q(\tau) \) are

\[
M_{j_1} = M(\tau, i)_{j_1} = K^7, \quad M_{j_2} = M(\tau, i)_{j_2} = K^2, \\
M_{j_3} = M(\tau, i)_{j_3} = K^3, \quad M_{j_4} = M(\tau, i)_{j_4} = K^3, \\
\text{and } M_{j_5} = M(\tau, i)_{j_5} = 5.
\]

The arrows \( \alpha \) and \( \delta \) act as zero on \( M(\tau, i) \), whereas \( \beta, \gamma, \varepsilon \) and \( \eta \) act according to
the following linear maps:

\[ M(\tau, i)_{\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} : K^7 \to K^3, \]

\[ M(\tau, i)_{\gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -w \\ -w & 0 \\ wy & 0 \end{bmatrix} : K^2 \to K^7, \]

\[ M(\tau, i)_{\epsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} : K^7 \to K^4, \]
\[ \begin{bmatrix} 0 & 0 & 0 \\ -z & 0 & 0 \\ 0 & -z & 0 \\ 0 & 0 & -z \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : K^3 \to K^7, \]

An easy check shows that \(a\) is injective, \(b\) is bijective, and \(c\) is identically zero. Hence,

\[ \overline{M}_{j_1} = 0 \quad \text{and} \quad \nabla_{j_1} = 0. \]

If we flip the arc \(j_1\) of \(\tau\) we obtain the ideal triangulation \(\sigma = f_{j_1}(\tau)\) sketched in Figure 9.24 (in a clear abuse of notation, we are using the same symbol \(j_1\) in both \(\tau\) and \(\sigma\)).

\[ \Phi = (\phi, \psi, \eta) \]

Thus, the triple \(\Phi = (\phi, \psi, \eta)\) is a right-equivalence between these QP-representations, where
• $\varphi$ is the right-equivalence whose action on the arrows is given by (9.3);

• $\psi : M(\tau,i) \rightarrow M(\sigma,i)$ is the vector space isomorphism given by the identity
  $1 : M(\tau,i)_k \rightarrow M(\sigma,i)_k$ for all $k \in \sigma$;

• $\eta$ is the zero map (of the zero space).

This finishes the proof of Theorem 9.2.1 for configuration 31.4 from Figure 7.10.

Case 13 (Configuration 32, Figure 7.10). Similar to the proof in configuration 31 and is left to the reader.

Configurations 45, 46, 47, 48, 49 (Figure 7.15), 54, 55, 56 and 57 (Figure 7.18) are very similar to each other and to some of the cases we have dealt with so far, so we include the proof of Theorem 9.2.1 for a couple of them and leave the rest to the reader.

Case 14 (Configuration 46, Figure 7.15). Now we are going to deal with configuration 46 from Figure 7.15. Assume that, around the arc $j_1$ to be flipped, $\tau$ and $i$ look as shown in Figure 9.25.

The relevant vector spaces assigned in $M(\tau,i)$ to the vertices of $Q(\tau)$ are

$$M_{j_1} = M(\tau,i)_{j_1} = K^{n+2}, \quad M_{j_2} = M(\tau,i)_{j_2} = K,$$

$$M_{j_3} = M(\tau,i)_{j_3} = K^n, \quad M_{j_4} = M(\tau,i)_{j_4} = K^2,$$

and $M_{j_5} = M(\tau,i)_{j_5} = K^n$. 

The arrows $\alpha$ and $\delta$ act as zero on $M(\tau, i)$, whereas:

\[
M(\tau, i)_\beta = \begin{bmatrix}
0_{n \times 1} & 1_{n \times n} & 0_{n \times 1}
\end{bmatrix} : K^{n+2} \to K^n,
\]

\[
M(\tau, i)_\gamma = \begin{bmatrix}
1 \\
-w \\
-wy \\
\vdots \\
(-1)^{n-i}w \cdot \frac{n-1}{2} y^{\frac{n-1}{2}} \\
(-1)^{n-i}w \cdot \frac{n+1}{2} y^{\frac{n+1}{2}} \\
0
\end{bmatrix} : K \to K^{n+2},
\]

\[
M(\tau, i)_\varepsilon = \begin{bmatrix}
e_1 & \ldots & e_{n-1} & 0_{n \times 2} & e_n
\end{bmatrix} : K^{n+2} \to K^n,
\]
\[
M(\tau, i)_\eta = \begin{bmatrix}
0_{(n-1) \times 1} & 0_{(n-1) \times 1} \\
1 & 0 \\
0 & 1 \\
-z & 0 \\
\end{bmatrix} : K^2 \to K^{n+2},
\]

Thus the maps \( \alpha \) and \( \beta \) are injective, \( \varepsilon \) is identically zero, and the function \( p : \)
\[
M_{\text{out}} = M_{j_3} \oplus j_5 = K^n \oplus K^n \to K^{n-2} \text{ given by } (u_1, \ldots, u_n, v_1, \ldots, v_n) \mapsto (u_1 - v_2, u_2 - v_3, \ldots, u_{n-2} - v_{n-1}) \text{ induces an isomorphism } \frac{\ker \varepsilon}{\text{im } \beta} \cong K^{n-2}. \]
We deduce that
\[
\overline{M}_{j_1} = K^{n-2} \oplus 0 \oplus 0 \oplus 0 \text{ and } \overline{V}_{j_1} = 0.
\]
Furthermore, \( \gamma^* \) and \( \eta^* \) act as zero on \( \overline{M(\tau, i)} \), and \( \beta^* \) and \( \varepsilon^* \) act according to the matrices
\[
\beta^* = \begin{bmatrix}
-1_{(n-2) \times (n-2)} & 0_{(n-2) \times 2} \\
0_{(n-2) \times 1} & 1_{(n-2) \times (n-2)} & 0_{(n-2) \times 1} \\
\end{bmatrix} : M_{j_3} = K^n \to K^{n-2} = \overline{M}_{j_1}
\]
and \( \varepsilon^* = \begin{bmatrix}
0_{(n-2) \times 1} & 1_{(n-2) \times (n-2)} & 0_{(n-2) \times 1} \\
\end{bmatrix} : M_{j_5} = K^n \to K^{n-2} = \overline{M}_{j_1}
\]
If we flip the arc \( j_1 \) of \( \tau \) we obtain the ideal triangulation \( \sigma = f_{j_1}(\tau) \) sketched in Figure 9.26 (in a clear abuse of notation, we are using the same symbol \( j_1 \) in both \( \tau \) and \( \sigma \); note also the shift in the numbering of intersection points).

The triple \( \Phi = (\varphi, \psi, \eta) \) is a right-equivalence between these QP-representations, where

- \( \varphi \) is the right-equivalence whose action on the arrows is given by \( (9.3) \);
- \( \psi : \overline{M(\tau, i)} \to M(\sigma, i) \) is the vector space isomorphism given by the identity
Figure 9.26:

1: \( M(\tau, i)_k \rightarrow M(\sigma, i)_k \) for all \( k \in \sigma \).

- \( \eta \) is the zero map (of the zero space).

This finishes the proof of Theorem 9.2.1 for configuration 46 from Figure 7.15.

Case 15 (Configuration 56, Figure 7.18). Now we are going to deal with configuration 56 from Figure 7.18. Assume that, around the arc \( j_1 \) to be flipped, \( \tau \) and \( i \) look as shown in Figure 9.27.

The relevant vector spaces assigned in \( M(\tau, i) \) to the vertices of \( Q(\tau) \) are

\[
M_{j_1} = M(\tau, i)_{j_1} = K^{n+3}, \quad M_{j_2} = M(\tau, i)_{j_2} = K, \\
M_{j_3} = M(\tau, i)_{j_3} = K^n, \quad M_{j_4} = M(\tau, i)_{j_4} = K^2,
\]
and $M_{j_5} = M(\tau, i)_{j_5} = K^{n+1}$.

The arrows $\alpha$ and $\delta$ act as zero on $M(\tau, i)$, whereas:

$$M(\tau, i)_\beta = \begin{bmatrix} 0_{n \times 1} & 1_{n \times n} & 0_{n \times 2} \end{bmatrix} : K^{n+3} \rightarrow K^n,$$

$$M(\tau, i)_\gamma = \begin{bmatrix} 1 \\ -w \\ wy \\ \vdots \\ (-1)^{n-1} w^{\frac{n-1}{2}} y^{\frac{n+1}{2}} \\ (-1)^{n} w^{\frac{n+1}{2}} y^{\frac{n-1}{2}} \\ 0 \\ 0 \end{bmatrix} : K \rightarrow K^{n+3},$$

$$M(\tau, i)_\epsilon = \begin{bmatrix} e_1 & \ldots & e_{n-1} & 0_{n \times 2} & e_n & e_{n+1} \end{bmatrix} : K^{n+3} \rightarrow K^{n+1},$$
\[ M(\tau, i)_\eta = \begin{bmatrix} 0_{(n-1)\times 1} & 0_{(n-1)\times 1} \\ 1 & 0 \\ 0 & 1 \\ -z & 0 \\ 0 & -z \end{bmatrix} : K^2 \to K^{n+3}, \]

Thus the maps \( a \) and \( b \) are injective, \( c \) is identically zero, and the function \( p : M_{out} = M_{j_1} \oplus j_5 = K^n \oplus K^{n+1} \to K^{n-2} \) given by \((u_1, \ldots, u_n, v_1, \ldots, v_n) \mapsto (u_1 - v_2, u_2 - v_3, \ldots, u_{n-2} - v_{n-1}) \) induces an isomorphism \( \frac{\ker c}{\im b} \cong K^{n-2} \). We deduce that

\[
\overline{M}_{j_1} = K^{n-2} \oplus 0 \oplus 0 \oplus 0 \text{ and } \overline{V}_{j_1} = 0.
\]

Furthermore, \( \gamma^* \) and \( \eta^* \) act as zero on \( \overline{M(\tau, i)} \), and \( \beta^* \) and \( \varepsilon^* \) act according to the matrices

\[
\beta^* = \begin{bmatrix} -1_{(n-2)\times (n-2)} & 0_{(n-2)\times 2} \end{bmatrix} : M_{j_3} = K^n \to K^{n-2} = \overline{M}_{j_1}
\]

and \( \varepsilon^* = \begin{bmatrix} 0_{(n-2)\times 1} & 1_{(n-2)\times (n-2)} & 0_{(n-2)\times 2} \end{bmatrix} : M_{j_5} = K^{n+1} \to K^{n-2} = \overline{M}_{j_1} \)

If we flip the arc \( j_1 \) of \( \tau \) we obtain the ideal triangulation \( \sigma = f_{j_1}(\tau) \) sketched in Figure 9.26 (in a clear abuse of notation, we are using the same symbol \( j_1 \) in both \( \tau \) and \( \sigma \); note also the shift in the numbering of intersection points).

The triple \( \Phi = (\varphi, \psi, \eta) \) is a right-equivalence between these QP-representations, where

- \( \varphi \) is the right-equivalence whose action on the arrows is given by (9.3);
\begin{itemize}
  \item $\psi : M(\tau, i) \rightarrow M(\sigma, i)$ is the vector space isomorphism given by the identity $1 : M(\tau, i)_k \rightarrow M(\sigma, i)_k$ for all $k \in \sigma$;
  \item $\eta$ is the zero map (of the zero space).
\end{itemize}

This finishes the proof of Theorem 9.2.1 for configuration 56 from Figure 7.15.

The rest of the cases are quite similar, and are left to the reader. \hfill \Box

\textbf{Theorem 9.2.4.} The decorated arc representations $\mathcal{M}(\tau, i)$ are mutation-equivalent to negative simples. More precisely, given any ideal triangulation (without self-folded triangles) $\sigma$ such that $i \in \sigma$, then $\mathcal{M}(\tau, i)$ is mutation-equivalent to the negative simple representation $S^-_i (Q(\sigma), S(\sigma))$. Consequently, the Euler-Poincaré characteristics of the quiver Grassmannians of $\mathcal{M}(\tau, i)$ are the coefficients of the $F$-polynomial that calculates the Laurent expansion of the cluster variable $i$ with respect to the cluster $\tau$. 
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Proof. The assertion about Euler-Poincaré characteristics follows from mutation-equivalence to negative simplices by results of [9]. Since every arc belongs to an ideal triangulation, one way of proving mutation-equivalence to negative simplices is by exhibiting the representation $M(\omega, i)$ for each triangulation $\omega$ with self-folded triangles, and showing that Theorem 9.2.1 remains valid when at least one of the triangulations involved has self-folded triangles. Though this is possible, we will not prove it this way for space reasons. Instead, we will show that

\[
\text{any two ideal triangulations without self-folded triangles are related} \quad (9.8)
\]

\[
\text{by a sequence of flips in such a way that every triangulation arising}
\]

\[
\text{in the sequence is an ideal triangulation without self-folded triangles.}
\]

The fact that any two ideal triangulations are related by a sequence of flips is well-known and has many different proofs (see for example [21], where an elementary proof is given). Now, it is possible to define a function $f$ from the set of all ideal triangulations to the set of ideal triangulations without self-folded triangles, with the following properties:

- $f(\tau) = \tau$ whenever the ideal triangulation $\tau$ does not have self-folded triangles;

- for any two ideal triangulations $\tau$ and $\sigma$, if they are related by a flip, then $f(\tau)$ and $f(\sigma)$ are related by a sequence of flips none of whose intermediate triangulations has self-folded triangles.

(Namely, for an arbitrary ideal triangulation, let $f(\tau)$ be obtained from $\tau$ by simultaneously flipping all loops enclosing self-folded triangles.) In this way, if we start
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with any sequence of flips starting and ending at triangulations without self-folded triangles, by applying $f$ to all the intermediate triangulations of the sequence we get a sequence of flips not involving self-folded triangles at all. □
Chapter 10

An application: g-vectors for the positive stratum

This chapter is devoted to a small application of our arc representations in the cluster algebra context. We assume that the reader is familiar with the tagged arc ↔ cluster variable identification established in [12]. Let \((\Sigma, M)\) be a bordered surface with marked points. Then the positive stratum of any cluster algebra \(\mathcal{A}\) whose exchange matrices are the signed adjacency matrices of the (tagged) triangulations of \((\Sigma, M)\) is, by definition, the subgraph of the exchange graph of \(\mathcal{A}\) whose vertices are the ideal triangulations that do not have self-folded triangles (cf. [12]).

Let \(n\) be the rank of \((\Sigma, M)\), that is, the number of arcs in any ideal triangulation of \((\Sigma, M)\) (cf. [12]). Fix an "initial" ideal triangulation \(\tau = \tau^0 = \{j_1, \ldots, j_n\}\) of a surface \((\Sigma, M)\). In [16], S. Fomin and A. Zelevinsky introduce a \(\mathbb{Z}^n\)-grading for
\[ \mathbb{Z}[j_1^{\pm 1}, \ldots, j_n^{\pm 1}, y_1, \ldots, y_n] \] defined by the formulas

\[ \text{deg}(j_i) = e_i, \quad \text{and} \quad \text{deg}(y_i) = -b_i, \]

where \( e_1, \ldots, e_n \) are the standard basis (column) vectors in \( \mathbb{Z}^n \), and \( b_i = \sum_k b_{ik} e_k \) is the \( l \)th column of \( B(\tau) = B(\tau^0) \). Under this \( \mathbb{Z}^n \)-grading, the principal coefficient cluster algebra \( A_\bullet(B(\tau)) \) is a \( \mathbb{Z}^n \)-graded subalgebra of \( \mathbb{Z}[j_1^{\pm 1}, \ldots, j_n^{\pm 1}, y_1, \ldots, y_n] \) and all cluster variables in \( A_\bullet(B(\tau)) \) are homogeneous elements (cf. [16], Proposition 6.1 and Corollary 6.2). By definition, the \( g \)-vector \( g^\tau_i \) of a cluster variable \( i \in A_\bullet(B(\tau)) \) with respect to the “initial” triangulation \( \tau = \tau^0 \) is its multi-degree with respect to the \( \mathbb{Z}^n \)-grading just defined. Fomin-Zelevinsky have shown in [16] that the mutation dynamics inside cluster algebras are controlled to an amazing extent by \( g \)-vectors and \( F \)-polynomials.

In [9], H. Derksen, J. Weyman and A. Zelevinsky have given a representation-theoretic interpretation of \( g \)-vectors using the mutation theory of quivers with potentials as follows. Let \( i \) be an (ordinary) arc on \( (\Sigma, M) \). As seen in Section 5, for each arc \( j \in \tau = \tau^0 \) the decorated representation \( M(\tau, i) = (Q(\tau), S(\tau), M(\tau, i), V(\tau, i)) \) induces a linear map \( \epsilon_j : M(\tau, i)_{j,\text{out}} \to M(\tau, i)_{j,\text{in}} \) (in Chapter 5 we did not use the subscript \( j \)). In Theorem 31 of [18] it is proved that if \( \partial \Sigma \neq \emptyset \), then the QP \( (Q(\tau), S(\tau)) \) is non-degenerate. Therefore, combining Theorem 9.2.1 above with Equations (1.13), (5.2), and Theorem 5.1 of [9], we see that the \( j \)th entry of the \( g \)-vector \( g^\tau_i \) is

\[ g^\tau_{i,j} = \text{dim ker} \epsilon_j - \text{dim} M(\tau, i)_j + \text{dim} V(\tau, i)_j \quad (10.1) \]

provided the underlying surface \( \Sigma \) has non-empty boundary. (According to Conjecture
33 of [18], the assumption $\partial \Sigma \neq \emptyset$ is superfluous.)

Assume that $i$ is not a loop cutting out a once-punctured monogon. Let $\Diamond$ be the quadrilateral of $\tau$ whose diagonal is $j$. The connected components of the intersection $i \cap \Diamond$ are segments of $i$, each of which falls within one of the types described in Figure 10.1. Let $s(i,j)$ (resp. $r(i,j)$, $t(i,j)$, $v(i,j)$, $z(i,j)$) be the number of components of $i \cap \Diamond$ that fall within type I (resp. II, III, V, VI) in Figure 10.1.

**Theorem 10.0.5.** Under the assumptions and notation just stated, the $j^{th}$ entry of the $g$-vector $g_i^\tau$ is $g_{i,j}^\tau = s(i,j) + t(i,j) - v(i,j) - z(i,j) + \delta_{ij}$ (the Kronecker delta).

**Proof.** If $i \in \tau$, the result is obvious. Otherwise, the theorem follows from the equalities

$$\dim \ker c_j = 2s(i,j) + r(i,j) + t(i,j), \quad \dim M(\tau,i)_j = s(i,j) + r(i,j) + v(i,j) + z(i,j),$$
which are easy to check.

Example 10.0.6. With respect to the triangulations of Figures 6.7 and 6.8, the arc $i$ shown there has the following g-vectors:

$$g_i^\tau = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ -1 \end{bmatrix}, \quad g_i^\sigma = \begin{bmatrix} 1 \\ 0 \\ -1 & 1 \\ -1 \end{bmatrix}$$

Remark 10.0.7. Using a totally different approach, G. Musiker, R. Schiffler and L. Williams have given in [23] combinatorial formulas that calculate g-vectors in full generality, that is, without the restrictions of working with ideal triangulations or arcs not cutting out once-punctured monogons. In that same generality, they have given formulas that calculate the $F$-polynomials of all cluster variables for any given initial cluster, thus establishing, for example the positivity conjecture of Fomin-Zelevinsky [16].
Chapter 11

Some problems

There are some problems whose solution the author thinks would help to have a full model of Derksen-Weyman-Zelevinsky’s representation-theoretic approach to cluster algebras.

**Problem 11.0.8.** In the case of surfaces with no boundary, determine whether the Jacobian algebras of its triangulations are finite-dimensional or not.

**Problem 11.0.9.** Prove or disprove that, in the case of surfaces with empty boundary, the potentials defined in [18] for ideal triangulations are non-degenerate.

**Problem 11.0.10.** Extend the combinatorial recipe for $S(\tau)$ given in [18] when $\tau$ is an ideal triangulation to the general situation of tagged triangulations.

Notice that Problem 11.0.10 is still open even when the underlying surface has non-empty boundary (and at least one puncture), despite the fact that in such situation the non-degeneracy of the potentials is already proved.
**Problem 11.0.11.** Extend the definition of $M(\tau, i)$ to the general situation where $\tau$ is a tagged triangulation and $i$ is a tagged arc.

Related to this problem is the following: If $\Sigma$ has non-empty boundary, then all the QPs $(Q(\tau), S(\tau))$ are non-degenerate and have finite-dimensional Jacobian algebras, so these QPs admit C. Amiot’s categorification [1]. In this context, each arc on $(\Sigma, M)$ represents an object of the Amiot cluster category $\mathcal{C}$, and each triangulation $\tau$ a *cluster-tilting object* whose endomorphism algebra is precisely the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau))$; moreover, for each fixed triangulation there is a functor from $\mathcal{C}$ to the module category of the Jacobian algebra of the triangulation. As a consequence of Theorem 9.2.4 above, the arc representation $M(\tau, i)$ gives an explicit calculation of the image of $i$ under the functor $\mathcal{C} \to \text{mod} \mathcal{P}(Q(\tau), S(\tau))$. For type $\mathbb{D}_n$, a complete geometric model of the cluster category was given by R. Schiffler in [25], and the representations $M(\tau, i)$ can also be seen as an explicit calculation of the image of the objects under the corresponding functor.

**Problem 11.0.12.** Give a complete combinatorial/geometrical description of the Amiot cluster category of the QPs associated to triangulations of surfaces.

**Remark 11.0.13.** Problem 11.0.12 has been successfully tackled by Brüstle-Zhang in the recent paper [5].

**Problem 11.0.14.** Give a cell or CW decomposition of the quiver Grassmannians of the arc representations defined above.

Finally, let us state a couple of challenging problems that have motivated [18] and the present work:
Problem 11.0.15. Give a combinatorial recipe to calculate non-degenerate potentials for arbitrary quivers. If possible, in such a way that given two mutation-equivalent quivers, the potentials calculated by the recipe are QP-mutation-equivalent.

Problem 11.0.16. Give a combinatorial recipe that calculates the decorated representations of arbitrary non-degenerate QPs that are mutation-equivalent to negative simples, without performing any mutation.

Remark 11.0.17. In [9], Derksen-Weyman-Zelevinsky have proved several conjectures from [16] without needing to give such combinatorial recipes. In that same paper, a conjectural characterization of representations mutation-equivalent to negative simples is given in terms of the vanishing of the $E$-invariant.
Bibliography


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