QUANTUM $F$-POLYNOMIALS IN THE THEORY OF CLUSTER ALGEBRAS

by

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Abstract of Dissertation

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Abstract: $F$-polynomials and $g$-vectors were defined by Fomin and Zelevinsky to give a formula which expresses cluster variables in a cluster algebra in terms of the initial cluster data. A quantum cluster algebra is a certain noncommutative deformation of a cluster algebra. In this thesis, we define and prove the existence of analogous quantum $F$-polynomials for quantum cluster algebras. We compute quantum $F$-polynomials and $g$-vectors for a certain class of cluster variables, which includes all cluster variables in type $A_n$ quantum cluster algebras. Finally, we give formulas for $F$-polynomials and quantum $F$-polynomials in classical types when the initial exchange matrix is acyclic.
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DEDICATION

I dedicate this thesis to my parents and to my boyfriend David. This thesis would not have been possible without their unconditional love and support.
Abstract of Dissertation
Acknowledgements
Dedication
1. Introduction
2. Cluster Algebras of Geometric Type
3. $F$-polynomials and $g$-vectors
4. $F$-polynomials in Classical Types
   4.1. $F$-polynomials and $g$-vectors corresponding to trees
   4.2. $F$-polynomials in type $A_n$
   4.3. $F$-polynomials in Classical Types for Acyclic Initial Exchange Matrix
   4.4. Type $D_n$
   4.5. Projections of $F$-polynomials and $g$-vectors
   4.6. Type $C_n$
   4.7. Type $B_n$
5. Quantum Cluster Algebras
6. $F$-polynomials in Quantum Cluster Algebras
7. Properties of Quantum $F$-polynomials
8. Quantum $F$-polynomials corresponding to trees
9. Quantum $F$-polynomials in Classical Types for Acyclic Initial Matrix
   9.1. Combinatorial realizations of cluster algebras of types $B_n, C_n, D_n$
   9.2. Type $D_n$
   9.3. Type $B_n$
   9.4. Type $C_n$
References
1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in [8] in order to study total positivity and canonical bases in semisimple groups. Let \( m \geq n \) be positive integers. Roughly speaking, a cluster algebra is a subalgebra generated by a distinguished collection of generators called cluster variables inside of an ambient field \( \mathcal{F} \) which is isomorphic to the field of rational functions in \( m \) independent variables. To obtain these cluster variables, one begins with an initial seed. A seed is a pair \((\tilde{x}, \tilde{B})\) such that \( \tilde{x} \) is an \( m \)-tuple of elements from \( \mathcal{F} \) with the first \( n \) terms being cluster variables and the remaining \( m - n \) terms being coefficient variables, and such that \( \tilde{B} \) is an \( m \times n \) integer matrix whose top \( n \times n \) submatrix is skew-symmetrizable; the \( m \)-tuple \( \tilde{x} \) is called the extended cluster of the seed. Seed mutations are certain operations which transform a seed into another seed. In mutating from one seed \((\tilde{x}, \tilde{B})\) to another seed \((\tilde{x}', \tilde{B}')\), one cluster variable \( x \) is exchanged for another cluster variable \( x' \); the elements \( x, x' \) satisfy a certain exchange relation which is of the form \( xx' = M^+ + M^- \), where \( M^+, M^- \) are monomials on disjoint subsets of variables from \( \tilde{x} - \{x\} \). The role of the matrix \( \tilde{B} \) is to dictate exactly what these monomials \( M^+, M^- \) are. The set of cluster variables which generate a given cluster algebra is obtained by mutating the initial seed with all possible sequences of mutations and taking all cluster variables from the seeds that result.

Quantum cluster algebras were defined by Berenstein and Zelevinsky in [1]. A quantum cluster algebra is a certain noncommutative deformation of a cluster algebra with an additional generator \( q \) lying in the center. Under this deformation, each extended cluster \((x_1, \ldots, x_m)\) in the cluster algebra is replaced by the \( m \)-tuple \((X_1, \ldots, X_m)\), where the elements \( X_1, \ldots, X_m \) now quasi-commute, i.e., for each \( 1 \leq i, j \leq m \), there exists \( \lambda_{ij} \in \mathbb{Z} \) such that \( X_iX_j = q^{\lambda_{ij}}X_jX_i \). The exchange relations are also altered to allow for the fact that extended cluster elements must quasi-commute.

In [10], Fomin and Zelevinsky defined \( F \)-polynomials and \( g \)-vectors corresponding to an initial \( n \times n \) skew-symmetrizable integer matrix \( B^0 \). They gave recurrence relations for the \( F \)-polynomials and \( g \)-vectors which essentially only depend on \( B^0 \). One of the main results of [10] was a formula expressing any cluster variable in terms of \( F \)-polynomials and \( g \)-vectors using only the initial cluster of \( A \).

In this thesis, we demonstrate the existence of quantum \( F \)-polynomials, which is an analogue of \( F \)-polynomials in the quantum cluster algebra setting. Quantum \( F \)-polynomials satisfy the property that any cluster variable in a quantum cluster algebra may be computed (up to a multiple
of a power of $q$) in a formula with the appropriate quantum $F$-polynomial and $g$-vector using only the initial extended cluster. It is conjectured (and proven in some cases) that this formula can be sharpened so that the multiple of $q$ does not appear (see Theorem 7.1). By setting $q = 1$ in a quantum $F$-polynomial, we obtain the appropriate $F$-polynomial for (nonquantum) cluster algebras.

A (quantum) cluster algebra is of finite type if the total set of cluster variables is finite. In [9], it was proven that finite type cluster algebras are classified by the same Cartan-Killing types as semisimple Lie algebras or finite root systems. It was proven in [1] that an identical classification also holds for quantum cluster algebras of finite type. In particular, finite type cluster algebras include those of classical types $A_n$, $B_n$, $C_n$, and $D_n$.

In this thesis, we recall a formula for $F$-polynomials from [4] given for cluster variables corresponding to induced trees of the quiver which can be defined using the initial matrix $B^0$, and state and prove a formula for the corresponding quantum $F$-polynomials. In addition, we show that these formulas may be applied to find all $F$-polynomials and quantum $F$-polynomials in type $A_n$. Furthermore, formulas for $F$-polynomials and for quantum $F$-polynomials are given in classical types for the case where the initial matrix is acyclic (Theorem 4.16 and Theorem 9.1). For $F$-polynomials, different formulas were given in the cases where the initial matrix is bipartite [10], acyclic [18], and in general in [14]. Also, a formula was given in [15] for $F$-polynomials and $g$-vectors for cluster algebras arising from surfaces with marked points as defined in [7], and from this formula, one may compute the $F$-polynomials corresponding to type $A_n$ and $D_n$. Theorem 4.16 differs from these other formulas in the statement of the formula, the methods for proving it, or both. The formula for the finite type $F$-polynomials in this thesis gives a combinatorial recipe for determining which monomials occur with nonzero coefficient and what the coefficient of the monomial is in that case. The formula for type $A_n$ was already proven in [4] using a formula for $F$-polynomials given in terms of quiver representations. To prove the result in type $D_n$ here, the same formula is used. To finish the proof in the other types, we show that the $F$-polynomials can be obtained as certain “projections” of $F$-polynomials from type A and D.

This thesis is based on two papers by the author, one of which will appear in the journal *Algebras and Representation Theory* [16] and the other of which has been submitted [17]. The organization of the thesis is as follows. In Section 2, we recall the definition of cluster algebras. In Section 3, we recall the definition of $F$-polynomials and $g$-vectors as well as the aforementioned formula for cluster
variables (Theorem 3.6). In Section 4, we recall the result about $F$-polynomials corresponding to induced trees as described above, and the results about $F$-polynomials in classical types are stated and proven. Section 5 is devoted to recalling the definition of quantum cluster algebras. In Section 6, (left) quantum $F$-polynomials are defined. Theorem 6.3 is devoted to proving their existence. Properties of quantum $F$-polynomials are given in Section 7. Proposition 7.4 shows how to easily compute “right” quantum $F$-polynomials once the “left” ones are known. A recurrence relation for quantum $F$-polynomials is given in Theorem 7.10. Section 8 gives quantum $F$-polynomials corresponding to induced trees. In Section 9, the formulas for quantum $F$-polynomials in classical types are stated and proven.

2. Cluster Algebras of Geometric Type

Following [10, Section 2], we give the definition of a cluster algebra of geometric type as well as some properties of these cluster algebras. The proofs of any statements given in this section can be found in [10, Section 2].

Definition 2.1. Let $J$ be a finite set of labels, and let $\text{Trop}(u_j : j \in J)$ be an abelian group (written multiplicatively) freely generated by the elements $u_j (j \in J).$ We define the addition $\oplus$ in $\text{Trop}(u_j : j \in J)$ by

\[
\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)},
\]

and call $(\text{Trop}(u_j : j \in J), \oplus, \cdot)$ a tropical semifield. If $J$ is empty, we obtain the trivial semifield consisting of a single element 1. The group ring of $\text{Trop}(u_j : j \in J)$ (as a multiplicative group) is the ring of Laurent polynomials in the variables $u_j$.

Fix two positive integers $m, n$ with $m \geq n$. Let $\mathbb{P} = \text{Trop}(x_{n+1}, \ldots, x_m)$, and let $\mathcal{F}$ be the field of rational functions in $n$ independent variables with coefficients in $\mathbb{Q}\mathbb{P}$, the field of fractions of the integral group ring $\mathbb{Z}\mathbb{P}$. (Note that the definition of $\mathcal{F}$ does not depend on the auxiliary addition $\oplus$ in $\mathbb{P}$.) The group ring $\mathbb{Z}\mathbb{P}$ will be the ground ring for the cluster algebra $\mathcal{A}$ to be defined, and $\mathcal{F}$ will be the ambient field, with $n$ being the rank of $\mathcal{A}$.

Definition 2.2. A labeled seed in $\mathcal{F}$ is a pair $(\tilde{x}, \tilde{B})$ where

- $\tilde{x} = (x_1, \ldots, x_m)$, where $x_1, \ldots, x_n$ are algebraically independent over $\mathbb{Q}\mathbb{P}$, and $\mathcal{F} = \mathbb{Q}\mathbb{P}(x_1, \ldots, x_n)$, and
• $\tilde{B}$ is an $m \times n$ integer matrix such that the submatrix $B$ consisting of the top $n$ rows and columns of $\tilde{B}$ is skew-symmetrizable (i.e., $DB$ is skew-symmetric for some $n \times n$ diagonal matrix $D$ with positive integer diagonal entries).

We call $\tilde{x}$ the extended cluster of the labeled seed, $(x_1, \ldots, x_n)$ the cluster, $\tilde{B}$ the exchange matrix, and the matrix $B$ the principal part of $\tilde{B}$.

We fix some notation to be used throughout the thesis. For $x \in \mathbb{Q}$,

$$[x]_+ = \max(x, 0);$$

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x > 0; \end{cases}$$

Also, for $i, j \in \mathbb{Z}$, write $[i, j]$ for the set $\{k \in \mathbb{Z} : i \leq k \leq j\}$. In particular, $[i, j] = \emptyset$ if $i > j$.

**Definition 2.3.** Let $k \in [1, n]$. We say that an $m \times n$ matrix $\tilde{B}'$ is obtained from an $m \times n$ matrix $\tilde{B} = (b_{ij})$ by matrix mutation in direction $k$ if the entries of $\tilde{B}'$ are given by

$$(2.2) \quad b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \text{sgn}(b_{ik}) \prod_{i=1}^{m} x_i^{[b_{ik}]_+} + \prod_{i=1}^{m} x_i^{[-b_{ik}]_+} & \text{otherwise}. \end{cases}$$

**Definition 2.4.** Let $(\tilde{x}, \tilde{B})$ be a labeled seed in $\mathcal{F}$ as in Definition 2.2, and write $\tilde{B} = (b_{ij})$. The seed mutation $\mu_k$ in direction $k$ transforms $(\tilde{x}, \tilde{B})$ into the labeled seed $\mu_k(\tilde{x}, \tilde{B}) = (\tilde{x}', \tilde{B}')$, where

• $\tilde{x}' = (x'_1, \ldots, x'_m)$, where $x'_j = x_j$ for $j \neq k$, and

$$(2.3) \quad x'_k = x_k^{-1} \left( \prod_{i=1}^{m} x_i^{[b_{ik}]_+} + \prod_{i=1}^{m} x_i^{[-b_{ik}]_+} \right),$$

• $\tilde{B}'$ is obtained from $\tilde{B}$ by matrix mutation in direction $k$.

One may check that the pair $(\tilde{x}', \tilde{B}')$ obtained is again a labeled seed. Furthermore, the seed mutation $\mu_k$ is involutive, i.e., applying $\mu_k$ to $(\tilde{x}, \tilde{B})$ yields the original labeled seed $(\tilde{x}, \tilde{B})$. 
Definition 2.5. Let $T_n$ be the $n$-regular tree whose edges are labeled with $1, \ldots, n$ in such a way that for each vertex, the $n$ edges emanating from that vertex each receive different labels. Write $t \xrightarrow{k} t'$ to indicate $t, t' \in T_n$ are joined by an edge with label $k$.

Definition 2.6. A cluster pattern is an assignment of a labeled seed $(\tilde{x}_t, \tilde{B}_t)$ to every vertex $t \in T_n$ such that if $t \xrightarrow{k} t'$, then the labeled seeds assigned to $t, t'$ may be obtained from one another by seed mutation in direction $k$. Write $\tilde{x}_t = (x_{1,t}, \ldots, x_{m,t})$, $\tilde{B}_t = (b_{ij}^t)$, and denote by $B^t$ the principal part of $\tilde{B}_t$.

Definition 2.7. For a given cluster pattern, write

$$\mathcal{X} = \{x_{j,t} : j \in [1, n], t \in T_n\}.$$ 

The elements of $\mathcal{X}$ are the cluster variables. The cluster algebra $\mathcal{A}$ associated to this cluster pattern is the $\mathbb{Z}_P$-subalgebra of $\mathcal{F}$ generated by all cluster variables. That is, $\mathcal{A} = \mathbb{Z}_P[\mathcal{X}]$.

3. F-polynomials and g-vectors

The reference for all of the information given in this section (except the definition and properties of extended g-vectors) is [10]. For this section, fix an $n \times n$ skew-symmetrizable integer matrix $B_0$ and an initial vertex $t_0 \in T_n$. Assume that any cluster algebra $\mathcal{A}$ in this section has initial exchange matrix $\tilde{B}_0$ with principal part $B_0$.

Definition 3.1. We say that a cluster pattern $t \mapsto (\tilde{x}_t, \tilde{B}_t)$ (or its corresponding cluster algebra) has principal coefficients at $t_0$ if $\tilde{x}_{t_0} = (x_1, \ldots, x_{2n})$ (i.e., $m = 2n$) and the exchange matrix at $t_0$ is the principal matrix corresponding to $B_0$ given by

$$\tilde{B}_0 = \begin{pmatrix} B_0 \\ I \end{pmatrix}$$

where $I$ is the $n \times n$ identity matrix. Denote the corresponding cluster algebra by $\mathcal{A}_* = \mathcal{A}_*(B_0, t_0)$.

Definition 3.2. Let $\mathcal{A}_* = \mathcal{A}_*(B_0, t_0)$ be the cluster algebra with principal coefficients, with labeled seed at $t_0$ written as

$$\tilde{x}_{t_0} = (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

$$\tilde{B}_0 = (b_{ij}^0)$$
Let $\mathbb{Q}_{sf}(z_1, \ldots, z_\ell)$ denote the set of all rational functions in $\ell$ independent variables $z_1, \ldots, z_\ell$ which can be expressed as subtraction-free rational expression in $z_1, \ldots, z_\ell$. Observe that by iterating the exchange relations in (2.3), any cluster variable $x_{\ell,t} \in \mathcal{A}_\bullet$ can be expressed as a unique rational function in $x_1, \ldots, x_n, y_1, \ldots, y_n$ given as a subtraction-free rational expression. Denote this rational function by 

\begin{equation}
X_{\ell,t} = X_{\ell,t}^{B_0,t_0} \in \mathbb{Q}_{sf}(x_1, \ldots, x_n, y_1, \ldots, y_n).
\end{equation}

Let $F_{\ell,t} = F_{\ell,t}^{B_0,t_0} \in \mathbb{Q}_{sf}(u_1, \ldots, u_n)$ denote the rational expression obtained by setting $x_i = 1$ and $y_i = u_i$ for all $i$ in $X_{\ell,t}$:

\begin{equation}
F_{\ell,t} = X_{\ell,t}(1, \ldots, 1, u_1, \ldots, u_n).
\end{equation}

Using [9, Proposition 11.2], which is a sharpened version of the “Laurent phenomenon” (see [8, Theorem 3.1]) for cluster algebras with principal coefficients, the next theorem shows that the $F_{\ell,t}$ functions are indeed polynomials:

**Theorem 3.3.** Let $\mathcal{A}_\bullet = \mathcal{A}_\bullet(B_0, t_0)$ be a cluster algebra with principal coefficients at $t_0$. Suppose that the initial seed in $\mathcal{A}_\bullet$ is given as in (3.2). Then

\begin{align}
X_{\ell,t} &\in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n], \\
F_{\ell,t} &\in \mathbb{Z}[u_1, \ldots, u_n].
\end{align}

The $F$-polynomials may be computed using the following recurrence.

**Proposition 3.4.** Let $t \mapsto \tilde{B}^t = (b_{ij}^t) \in \mathbb{T}_n$ be the family of $2n \times n$ matrices associated with the cluster algebra $\mathcal{A}_\bullet(B_0, t_0)$. Then the polynomials $F_{\ell,t} = F_{\ell,t}^{B_0,t_0}(u_1, \ldots, u_n)$ are uniquely determined by the initial conditions

\begin{equation}
F_{\ell,t_0} = 1 \quad (\ell = 1, \ldots, n),
\end{equation}

together with the recurrence relations

\begin{align}
F_{\ell,t'} &= F_{\ell,t} \quad \text{for } \ell \neq k; \\
F_{k,t'} &= \frac{\prod_{i=1}^n F_{i,t}^{b_{ii}^t} \prod_{j=1}^n u_j^{b_{i+j,j}^t}}{F_{k,t}} + \frac{\prod_{i=1}^n F_{i,t}^{-b_{ii}^t} \prod_{j=1}^n u_j^{-b_{i+j,j}^t}}{F_{k,t}},
\end{align}
for every edge \( t \rightarrow t' \) such that \( t \) lies on the (unique) path from \( t_0 \) to \( t' \) in \( T_n \).

In [10, Section 6], the following \( \mathbb{Z}^n \)-grading of \( \mathcal{A} = \mathcal{A}(B^0, t_0) \) was introduced:

**Proposition 3.5.** Define a \( \mathbb{Z}^n \)-grading of \( \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}] \) by

\[
\text{deg}(x_i) = e_i, \quad \text{deg}(y_j) = -b^0_j,
\]

where \( e_1, \ldots, e_n \) are the standard basis vectors for \( \mathbb{Z}^n \) and \( b_j^0 \) is the \( j \)th column of \( B^0 \). Under this \( \mathbb{Z}^n \)-grading, the cluster algebra \( \mathcal{A}(B^0, t_0) \) is a \( \mathbb{Z}^n \)-graded subalgebra of \( \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}] \), and every cluster variable in \( \mathcal{A}(B^0, t_0) \) is a homogeneous element.

The \( \mathbb{Z}^n \)-degree of the cluster variable \( x_{\ell,t} \in \mathcal{A}(B^0, t_0) \) will be denoted by \( g_{\ell,t} = \mathbf{g}_{\ell,t} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \in \mathbb{Z}^n \); we refer to \( \mathbf{g}_{\ell,t} \) as the \( \mathbf{g} \)-vector of \( x_{\ell,t} \).

For any cluster algebra \( \mathcal{A} \) such that the initial extended cluster at \( t_0 \) is given by \( \tilde{x} = (x_1, \ldots, x_m) \) and the exchange matrix at \( t_0 \) is \( \tilde{B}^0 = (b^0_{ij}) \), let

\[
y_j = \prod_{i=n+1}^{m} b^0_{ij} x_i^{y_j}, \quad \hat{y}_j = \prod_{i=1}^{m} b^0_{ij} x_i^{\hat{y}_j}
\]

for \( j \in [1, n] \).

The next theorem shows that any cluster variable in \( \mathcal{A} \) may be computed in terms of the initial extended cluster if the corresponding \( F \)-polynomial and \( \mathbf{g} \)-vector are known.

**Theorem 3.6.** Let \( \mathcal{A} \) be a cluster algebra such that the principal part of the exchange matrix at \( t_0 \) is \( B^0 \). Using the notation just given for the initial seed at \( \mathcal{A} \), any cluster variable \( x_{\ell,t} \) in \( \mathcal{A} \) may be expressed as

\[
x_{\ell,t} = \frac{F_{\ell,t}^{B^0,t_0}(\hat{y}_1, \ldots, \hat{y}_n)}{F_{\ell,t}^{B^0,t_0}|\tilde{\varphi}(y_1, \ldots, y_n)} x_1^{g_1} \cdots x_n^{g_n},
\]
where \( \mathbf{g}_{\ell,t}^{B^0,t_0} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \) and \( \mathbb{P} = \text{Trop}(x_{n+1}, \ldots, x_m) \).

Furthermore, for \( j \in [1, n] \), \( u_j \) does not divide \( F_{\ell,t} \). Thus, in the cluster algebra \( A_\bullet = A_\bullet(B^0, t_0) \),

\[
x_{\ell,t} = F_{\ell,t}^{B^0,t_0}(\hat{y}_1, \ldots, \hat{y}_n)x_1^{q_1} \cdots x_n^{q_n}.
\]

(3.14)

Next, we introduce extended \( g \)-vectors which generalize the \( g \)-vectors to any cluster algebra \( A \).

**Definition 3.7.** Let \( A \) be any cluster algebra with initial extended cluster \( \hat{x} = (x_1, \ldots, x_m) \), and let \( \ell \in [1, n] \), \( t \in T_n \). Define \( \mathbf{g}_{\ell,t}^{B^0,t_0} = (g_1, \ldots, g_m) \) to be the unique vector in \( \mathbb{Z}^m \) such that the cluster variable \( x_{\ell,t} \in A \) satisfies

\[
x_{\ell,t} = F_{\ell,t}^{B^0,t_0}(\hat{y}_1, \ldots, \hat{y}_n)x_1^{q_1} \cdots x_m^{q_m}.
\]

(3.15)

For \( \ell \in [n+1, m] \), define \( \mathbf{g}_{\ell,t}^{B^0,t_0} = e_\ell \in \mathbb{Z}^m \). We call \( \mathbf{g}_{\ell,t}^{B^0,t_0} \) the extended \( g \)-vector.

Observe that the existence of \( \mathbf{g}_{\ell,t} \) follows immediately from Theorem 3.6. Explicitly, the first \( n \) coordinates of \( \mathbf{g}_{\ell,t} \) are given by \( \mathbf{g}_{\ell,t}^{B^0,t_0} \), and the remaining \( m - n \) coordinates are given by the exponents of \( x_{n+1}, \ldots, x_m \) in the expression

\[
1
\text{\[F_{\ell,t}^{B^0,t_0}\]_{\mathbb{P}(y_1, \ldots, y_n)},}
\]

where \( \mathbb{P} = \text{Trop}(x_{n+1}, \ldots, x_m) \). In particular, \( \mathbf{g}_{\ell,t}^{B^0,t_0} = \mathbf{g}_{\ell,t}^{B^0,t_0} \) when \( \hat{B}^0 \) is principal. (Here, we extend \( \mathbf{g}_{\ell,t}^{B^0,t_0} \) to \( \mathbb{Z}^n \) to an element of \( \mathbb{Z}^{2n} \) by appending \( n \) 0's.)

The next proposition gives a recurrence relation for the extended \( g \)-vectors. Let \( \tilde{b}^j \) denote the \( j \)th column of the matrix \( \tilde{B}^0 \) (\( j \in [1, n] \)). Let \( \tilde{B}^t = (b^t_{ij}) \) denote the matrix obtained from \( \tilde{B}^0 \) by mutating the matrix from \( t_0 \) to vertex \( t \in T_n \), and let \( (b^t_{ij}) \) be the \( 2n \times n \) matrix obtained by mutating the principal matrix corresponding to \( B^0 \) from \( t_0 \) to \( t \).

**Proposition 3.8.** For \( \ell \in [1, m] \), \( t \in T_n \), \( \mathbf{g}_{\ell,t} \) is given by the initial conditions

\[
\mathbf{g}_{\ell,t_0} = e_\ell \quad (\ell = 1, \ldots, m)
\]

(3.17)
together with the recurrence relations

\begin{align}
\tilde{g}_{\ell,t'} &= \tilde{g}_{\ell,t} \quad \text{for } \ell \neq k; \\
\tilde{g}_{k,t'} &= -\tilde{g}_{k,t} + \sum_{i=1}^{m} [-b^i_{ik}] + \tilde{g}_{i,t} - \sum_{j=1}^{n} [-b^*_{n+j,k}] + \tilde{B}^j,
\end{align}

where the (unique) path from \( t_0 \) to \( t' \) in \( T_n \) ends with the edge \( t \rightarrow t' \).

**Proof.** Equation (3.17) is clear from the fact that \( F_{\ell,t_0} = 1 \) for all \( \ell \in [1,n] \). Let \( t \rightarrow t' \) in \( T_n \). If \( \ell \neq k \), then \( F_{\ell,t} = F_{\ell,t'} \), and (3.18) follows. Now consider \( \tilde{g}_{k,t'} \).

In section 3 of [10], an element \( Y_{k,t} \in \mathbb{Q}_{sf}(u_1,\ldots,u_n) \) was defined satisfying certain properties that we recall below. By [10, Proposition 3.12],

\begin{equation}
F_{k,t'} = \frac{Y_{k,t} + 1}{(Y_{k,t} + 1)|Trop(u_1,\ldots,u_n)} \cdot F_{k,t}^{-1} \prod_{i=1}^{n} F_{i,t}^{-[b^i_{ik}]} + ,
\end{equation}

where the right hand side is computed in the field \( \mathbb{Q}(u_1,\ldots,u_n) \) of rational functions. By [10, (3.16)], the cluster variable \( x_{k,t'} \in A \) is given by

\begin{equation}
x_{k,t'} = \frac{(Y_{k,t} + 1)|Trop(\hat{y}_1,\ldots,\hat{y}_n)}{(Y_{k,t} + 1)|Trop(y_1,\ldots,y_n)} \cdot x_{k,t}^{-1} \prod_{i=1}^{n} x_{i,t}^{-[b^i_{ik}]} + ,
\end{equation}

where \( \mathbb{P} = \mathrm{Trop}(x_{n+1},\ldots,x_m) \), and \( \hat{y}_i, y_i \) are elements of the ambient field \( \mathcal{F} \) as defined at (3.12).

By [10, (3.14)],

\begin{equation}
Y_{k,t}|\mathbb{P}(y_1,\ldots,y_n) = \prod_{i=n+1}^{m} x_i^{b^i_{ik}} \in \mathcal{F}.
\end{equation}

By (3.22),

\begin{equation}
(Y_{k,t} + 1)|\mathbb{P}(y_1,\ldots,y_n) = \prod_{i=n+1}^{m} x_i^{\min(0,b^i_{ik})}.
\end{equation}

Since \( \min(0,b^i_{ik}) = -[-b^i_{ik}]_+ \), equation (3.21) may be rewritten as

\begin{equation}
x_{k,t'} = (Y_{k,t} + 1)(\hat{y}_1,\ldots,\hat{y}_n)x_{k,t}^{-1} \prod_{i=1}^{m} x_{i,t}^{-[b^i_{ik}]} + ,
\end{equation}
By Definition 3.7, it follows from equation (3.24) that

\[(3.25)\]

\[x_{k;t'} = P_{k;t}(\hat{y}_1, \ldots, \hat{y}_n)x_1^{g_1'} \ldots x_m^{g_m'},\]

where

\[(3.26)\]

\[P_{k;t} = (Y_{k;t} + 1)F_{k;t}^{-1} \prod_{i=1}^{n} F_{i,t}^{-|b_{i,k}|} + \prod_{i=1}^{n} F_{i,t}^{-|b_{i,k}'|},\]

\[(3.27)\]

\[(g_1', \ldots, g_m') = -\tilde{g}_{k;t} + \sum_{i=1}^{m} [-b_{i,k}'] + \tilde{g}_{i;t}.\]

In the particular case, where \(A\) has principal coefficients, (3.22) implies that

\[(3.28)\]

\[(Y_{k;t} + 1)|_{Trop(y_1, \ldots, y_n)}(y_1, \ldots, y_n) = \prod_{j=1}^{n} y_j^{\min(0, b_{j+n,k})} \in Q_{sf}(u_1, \ldots, u_n).\]

Equation (3.20) may be rewritten as

\[(3.30)\]

\[F_{k;t'} = (Y_{k;t} + 1)F_{k;t}^{-1} \prod_{i=1}^{n} F_{i,t}^{-|b_{i,k}'|} + \prod_{j=1}^{n} u_j^{\min(0, b_{j+n,k}')}.\]

It follows that

\[(3.31)\]

\[x_{k;t'} = F_{k;t'}(\hat{y}_1, \ldots, \hat{y}_n)x_1^{g_1'} \ldots x_m^{g_m'} \prod_{j=1}^{n} \hat{y}_j^{-b_{j+n,k}'+},\]

and equation (3.19) follows from Definition 3.7.

\[\square\]

**Proposition 3.9** ([10, Proposition 6.6]). In the particular case when \(\tilde{B}^0\) is principal and \(g_{i;t}^{B^0_{i;t}} = \tilde{g}_{i;t}\) for all \(i \in [1, n], t \in T_n\), the recurrence (3.19) may be replaced by the following recurrence:

\[(3.32)\]

\[g_{k;t'} = -g_{k;t} + \sum_{i=1}^{2n} [\epsilon b_{i,k}'] + g_{i;t} - \sum_{j=1}^{n} [\epsilon b_{n+j,k}] + \tilde{b}^j,\]

for \(\epsilon \in \{+, -\} \}.\]
4. F-polynomials in Classical Types

Formulas for F-polynomials and quantum F-polynomials will be given in terms of denominator vectors. We recall from [8] the definition of these vectors and the recurrence relations from which they may be computed. Let $A$ be any cluster algebra whose initial exchange matrix has principal part $B^0$. Suppose $A$ has initial extended cluster $(x_1, \ldots, x_m)$. By the Laurent phenomenon ([8, Theorem 3.1]), any cluster variable $x_{j:t} \in A$ may be expressed as

\[ x_{j:t} = \frac{N(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}, \tag{4.1} \]

where $N(x_1, \ldots, x_n)$ is a polynomial with coefficients in $\mathbb{Z}[x_{n+1}^{\pm1}, \ldots, x_m^{\pm1}]$ which is not divisible by any $x_i$. Let $d_{j:t}^{B_0:t_0} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$, and call this the denominator vector of the cluster variable $x_{j:t}$.

The vectors $d_{j:t} = d_{j:t}^{B_0:t_0}$ are uniquely determined by the initial conditions

\[ d_{j:t}^{B_0:t_0} = -e_j \tag{4.2} \]

(where $e_1, \ldots, e_n$ are the standard basis vectors in $\mathbb{Z}^n$) together with the recurrence relations implied by the exchange relation (2.3): for $t \rightarrow t'$ in $T_n$, we have

\[ d_{j:t'} = \begin{cases} d_{j:t} & \text{if } j \neq k; \\ -d_{k:t} + \max \left( \sum_{i=1}^n [b_{ik}^t] + d_{i:t}, \sum_{i=1}^n [-b_{ik}^t] + d_{i:t} \right) & \text{if } j = k \end{cases} \tag{4.3} \]

Here, the max operation on vectors is performed component-wise. Observe that the denominator vector depends on $B^0, t_0, t, j$, not on the choice of coefficients.

4.1. F-polynomials and g-vectors corresponding to trees. For any $n \times n$ integer skew-symmetrizable matrix $B = (b_{ij})$, we may define a quiver $Q(B)$ on the set of vertices $[1, n]$, with $|b_{ij}|$ arrows from $i$ to $j$ if and only if $b_{ij} < 0$. Let $B^0$ be any $n \times n$ skew-symmetric integer matrix, and let $Q^0 = Q(B^0)$. Fix $T \subset [1, n]$ such that the subgraph of $Q^0$ induced by $T$ is a tree. In particular, there is at most one edge in $Q^0$ connecting any given pair of vertices in $T$. Without loss of generality, assume that $T = [1, \ell] \subset [1, n]$. Furthermore, we may assume that the vertices of $T$
are labeled so that for each \( i \in [1, \ell] \), the subgraph of \( T \) induced by the set \([1, i]\) is also a tree, and in that tree, the vertex \( i \) is a leaf.

In this section, we associate to \( T \) a cluster variable in \( \mathcal{A} \_\bullet (B^0, t_0) \) and give a formula for the corresponding \( F \)-polynomials which is known from [4].

For \( S \subset [1, n] \), write

\[
e_S = \sum_{i \in S} e_i \in \mathbb{Z}^n.
\]

The next two propositions are an immediate consequence of Proposition 5.7 and Corollary 5.8 of [4]:

**Proposition 4.1.** There exists a cluster variable \( x_T \) in \( \mathcal{A} \_\bullet (B^0, t_0) \) such that the denominator vector of \( x_T \) is \( e_T \). Furthermore, this cluster variable may be obtained from the initial cluster by mutating in directions \( k = 1, \ldots, \ell \) and taking the \( \ell \)th cluster variable in the resulting cluster.

Let \( t_1, \ldots, t_\ell \) in \( T_n \) such that \( t_{i-1} \rightarrow i \rightarrow t_i \) in \( T_n \) for \( i = 1, \ldots, \ell \). Then the proposition implies that the cluster variable \( x_{t_1, \ldots, t_\ell} \in \mathcal{A} \_\bullet (B^0, t_0) \) is equal to \( x_{[1, i]} \) for each \( i \). Write \( F_{\ell, t_1, \ldots, t_\ell}^c = F_{\ell, t_1, \ldots, t_\ell}^{B^0, t_0} \) and \( g_T = g_{\ell, t_1, \ldots, t_\ell}^{B^0, t_0} \) for the nonquantum (or “classical”) \( F \)-polynomial and \( g \)-vector corresponding to \( T \), respectively.

**Proposition 4.2.** [4, Corollary 5.8] The \( F \)-polynomial corresponding to \( T \) is given by

\[
F_{T}^c(u_1, \ldots, u_n) = \sum_S \prod_{i \in S} u_i
\]

where the summation ranges over all subsets \( S \subset T \) such that the following condition holds:

\[
(4.6) \quad \text{if } j \in S \text{ and } i \in T \text{ are such that } j \rightarrow i \text{ in } Q^0, \text{ then } i \in S.
\]

For \( k \in [1, n] \), define

\[
I_{in}(k) = \{ i \in T : i \rightarrow k \text{ in } Q^0 \}, \quad I_{out}(k) = \{ j \in T : k \rightarrow j \text{ in } Q^0 \}.
\]

**Proposition 4.3.** [4, Remark 5.9] For \( k \in T \), the \( k \)th component of \( g_T \) is equal to

\[
g_k = |I_{out}(k)| - 1.
\]
Next, a formula for the remaining components of the $g$-vector will be given in Proposition 4.5 in a particular situation. This formula was given in [4] in terms of representations of quivers with potentials. Let $B^0$ be an $n \times n$ skew-symmetric matrix with entries from $\{0, 1, -1\}$. We recall some notation and definitions from [4], omitting certain technical details which are not needed here. A representation $M$ (over $\mathbb{C}$) of a quiver $Q$ is given by assigning $\mathbb{C}$-vector space $M_i$ to each vertex $i \in [1, n]$ and assigning a $b$-tuple $(\varphi_{ji}^{(1)}, \ldots, \varphi_{ji}^{(b)})$ of linear maps $\varphi_{ji}^{(k)} : M_i \to M_j$ to every arrow $i \to j$ of multiplicity $b = b_{ji}$. Let $M_0$ be the quiver representation of $Q^0$ such that $M_i = \mathbb{C}$ if $i \in T$, and $M_i = 0$ otherwise. Also, for the arrow $a : i \to j$, let $a_M : M_i \to M_j$ be the identity map if $i, j \in T$, and let $a_M$ be the 0-map otherwise.

For $k \in [1, n]$, let

$$M_{in}(k) = \bigoplus_{i \in I_{in}(k)} M_i, \quad M_{out}(k) = \bigoplus_{j \in I_{out}(k)} M_j,$$

where $I_{in}(k), I_{out}(k)$ were defined at (4.7).

A certain linear map $\gamma_k : M_{out}(k) \to M_{in}(k)$ was defined in [4]. This map is given by the matrix $\gamma_k = (\gamma_k(i, j))$, where the rows of the matrix are indexed by $i \in I_{in}(k)$, the columns are indexed by $j \in I_{out}(k)$, and $\gamma_k(i, j)$ is either a nonzero element of $\mathbb{C}$ if there exists a directed path from $j$ to $i$ in $Q^0$ with all vertices contained in $T$, and $\gamma_k(i, j) = 0$ otherwise.

**Lemma 4.4.** Let $k \in [1, n]$. The rank of $\gamma_k$ is equal to the maximum number $r \in \mathbb{Z}_{\geq 0}$ for which there exist distinct $j_1, \ldots, j_r \in I_{out}(k)$ and distinct $i_1, \ldots, i_r \in I_{in}(k)$ such that there is a directed path from $j_s$ to $i_s$ in $T$ for all $s \in [1, r]$. (In particular, $\text{rank}(\gamma_k)$ does not depend on the specific nonzero values in the matrix $\gamma_k$.)

**Proof.** If $r = 0$, then $\gamma_k = 0$, and the lemma holds. Assume for the remainder of the proof that $r \geq 1$. Let $j_1, \ldots, j_r, i_1, \ldots, i_r$ be vertices as in the statement of the lemma.

We claim that there does not exist $\sigma$ in the symmetric group $S_r$, $\sigma \neq \text{id}$, such that $\gamma_k(i_{\sigma(s)}, j_s) \neq 0$ for all $s \in [1, r]$. For the sake of contradiction, assume that such a $\sigma$ exists. Let $p$ be the order of $\sigma$ in $S_r$. Then there exists a directed path from $j_s$ to $i_{\sigma(s)}$ in $T$ for all $s \in [1, r]$, so it follows that there is a cycle in $T$ containing the vertices $i_1, j_1, i_{\sigma(1)}, j_{\sigma(1)}, \ldots, j_{\sigma^{p-1}(1)}, i_{\sigma^{p}(1)} = i_1$, which contradicts that fact that the subgraph of $Q^0$ induced by $T$ is a tree.
Let $\Gamma$ be the $r \times r$ submatrix of $\gamma_k$ with rows indexed by $i_1, \ldots, i_r$, and columns indexed by $j_1, \ldots, j_r$. Then the determinant of $\Gamma$ is $\prod_{s=1}^{r} \gamma_k(i_s, j_s)$, which is nonzero since $\gamma_k(i_s, j_s) \neq 0$ for all $s \in [1, r]$. This proves that $\text{rank}(\gamma_k) \geq r$.

Now let $r' = \text{rank}(\gamma_k)$, and suppose that $\Gamma'$ is an $r' \times r'$ invertible submatrix of $\gamma_k$ with rows indexed by $i'_1, \ldots, i'_{r'} \in I_{\text{in}}(k)$ and columns indexed by $j'_1, \ldots, j'_{r'} \in I_{\text{out}}(k)$. In order for the determinant of $\Gamma'$ to be nonzero, there must exist $\sigma \in S_{r'}$ such that $\gamma_{i'_{\sigma(s)}, j_s} \neq 0$ for all $s \in [1, r']$. Thus, there is a directed path from $j_s$ to $i_{\sigma(s)}$ in $T$ for each $s \in [1, r']$, which means that $\text{rank}(\gamma_k) \leq r$.

□

**Proposition 4.5.** The $g$-vector $g_T$ corresponding to $T$ is given by $(g_1, \ldots, g_n)$, where

\begin{align}
\text{(4.10)} & \quad g_k = \dim(\ker(\gamma_k)) - \dim(M_k) \\
\text{(4.11)} & \quad = |I_{\text{out}}(k)| - \text{rank}(\gamma_k) - \dim(M_k).
\end{align}

for all $k \in [1, n]$.

**Proof.** The proposition follows immediately from Theorem 5.1 and Proposition 5.7 of [4]. □

4.2. $F$-polynomials in type $A_n$. Let $B^0$ be an $n \times n$ exchange matrix of type $A_n$. Recall that such a matrix $B^0$ may be obtained via a sequence of matrix mutations from the $n \times n$ matrix $B = (b_{ij})$, where

\begin{equation}
\text{(4.12)} \quad b_{ij} = \begin{cases} 
1 & \text{if } j = i + 1 \\
-1 & \text{if } j = i - 1 \\
0 & \text{otherwise}
\end{cases}
\end{equation}

We will show that each cluster variable in the cluster algebra $A_\bullet(B^0, t_0)$ corresponds to an induced chain $C$, which means that Theorem 4.2 may be used to compute all $F$-polynomials for type $A_n$. Also, we compute $g$-vectors for type $A_n$ using Proposition 4.5.

Denote by $\Phi_+(B^0)$ the collection of subsets $C$ of $[1, n]$ such that the subgraph of $Q^0$ induced by $C$ is a chain. (A *chain* is an alternating sequence $v_1, e_1, v_2, \ldots, e_p, v_p$ of distinct vertices and edges such that the edge $e_i$ has vertices $v_i, v_{i+1}$ for $i = 1, \ldots, p - 1$, and there are no other edges connecting the vertices $v_1, \ldots, v_p$.)
Proposition 4.6. The cluster variables in $\mathcal{A}$ which are not in the initial cluster are in bijective correspondence with the elements of $\Phi_+(B^0)$. To be more specific, if a cluster variable corresponds to the set $C \in \Phi_+(B^0)$, then its denominator vector is $e_C = \sum_{i \in C} e_i$.

Remark 4.7. For the acyclic case in finite type, it was proven in [9] that the cluster variables not in the initial cluster are in bijective correspondence with the positive roots of the given type (see Section 4.3). In the acyclic case, Proposition 4.6 is a consequence of this result. More recently, Proposition 4.6 was independently stated and proven in [14].

To prove the proposition, we will use the following combinatorial description of cluster algebras of type $A_n$ given in [9]. In this realization, the cluster variables are in bijective correspondence with the diagonals of the regular $(n+3)$-gon $\mathbb{P}_{n+3}$, and the clusters correspond to maximal sets of noncrossing diagonals, i.e., to triangulations of $\mathbb{P}_{n+3}$. (Note that two diagonals cross if they intersect in an interior point.) Let $T = (\beta_1, \ldots, \beta_n)$ be a list of pairwise distinct noncrossing diagonals of $\mathbb{P}_{n+3}$. Write $\Delta(T)$ for the corresponding set of triangles. In this setting, cluster mutations are encoded as follows. Let $k \in [1,n]$. To mutate $T$ in direction $k$, let $\Delta_1, \Delta_2$ be the two triangles in $\Delta(T)$ which have $\beta_k$ as a side, and let $a, b$ be the vertices opposite the side $\beta_k$ in each of these triangles. Then $\mu_k(T)$ is the list of diagonals obtained from $T$ by replacing $\beta_k$ by the diagonal $ab$.

One may associate to $T$ a quiver $Q(T)$ on the set of vertices $[1,n]$. For the edges, let $i, j \in [1,n], i \neq j$. If there is no triangle in $T$ such that $\beta_i, \beta_j$ are sides of the triangle, then there is no edge between $i$ and $j$. Otherwise, suppose that the endpoints of $\beta_i$ are $a,b$, and the endpoints of $\beta_j$ are $a,c$. Then $i \to j$ if $a,b,c$ are in clockwise order, and $i \leftarrow j$ if $a,b,c$ are in counterclockwise order. The principal part of the exchange matrix $B(T) = (b_{ij})$ corresponding to $T$ is given by $b_{ij} = 0$ if there is no edge between $i$ and $j$, $b_{ij} = -1$ if $i \to j$, and $b_{ij} = 1$ if $i \leftarrow j$. Note that $Q(B(T)) = Q(T)$.

In [3], Buan and Vatne characterize all type $A_n$ quivers (i.e., all quivers $Q$ of the form $Q = Q(T)$ where $T$ is a triangulation of $\mathbb{P}_{n+3}$).

Lemma 4.8. [3, Proposition 2.4] Let $Q$ be a quiver with vertex set $[1,n]$. Then $Q$ is of type $A_n$ if and only (1)-(4) hold:

(1) Any induced cycle in $Q$ is an oriented 3-cycle. (In particular, there are no multiple edges.)

(2) The degree of any vertex is at most 4.
(3) If a vertex $i$ has degree 4, then two of the edges containing $i$ are in a 3-cycle, and the other two edges containing $i$ are in another 3-cycle.

(4) If a vertex $i$ has degree 3, then two of the edges containing $i$ are in a 3-cycle, and the other edge does not belong to any 3-cycle.

In particular, it follows that any induced tree in $Q$ must be a chain.

Let $T^0 = \{\alpha_1, \ldots, \alpha_n\}$ be the triangulation of $\mathbb{P}_{n+3}$ corresponding to the cluster at $t_0$. Proposition 4.6 is an immediate consequence of the following lemma:

**Lemma 4.9.** The diagonals of $\mathbb{P}_{n+3}$ which are not in $T^0$ are in bijective correspondence with the elements of $\Phi_+(B^0)$. Under this correspondence, if $\beta$ is a diagonal not in $T^0$, then the corresponding subset of $[1, n]$ consists precisely of those $i \in [1, n]$ for which $\beta$ crosses $\alpha_i$. Furthermore, if $C = \{p_1, \ldots, p_j\} \in \Phi_+(B^0)$ such that there is some edge between $p_i$ and $p_{i+1}$ for $i = 1, \ldots, j - 1$, then the cluster variable corresponding to $C$ can be obtained from the initial cluster by mutating in directions $p_1, \ldots, p_j$. The denominator vector of this cluster variable is $e_C$.

**Proof.** First, consider a diagonal $\beta$ of $\mathbb{P}_{n+3}$ that is not in $T^0$. Let $T'$ be the set of diagonals in $T^0$ which $\beta$ intersects. This set $T'$ may be constructed as follows: Start with one endpoint of $\beta$. This endpoint is the vertex of a triangle $\Delta_1$ in the initial triangulation such that $\beta$ passes through the interior of the triangle. The diagonal $\beta$ crosses another diagonal $\alpha_{p_1}$ which is a side of the triangle $\Delta_1$. The diagonal $\alpha_{p_1}$ is a side of another triangle $\Delta_2$ in the initial triangulation. Either $\beta$ intersects a vertex of $\Delta_2$, in which case $T' = \{\alpha_{p_1}\}$, or $\beta$ crosses another side $\alpha_{p_2}$ of $\Delta_2$. In the latter case, $\alpha_{p_2}$ is the side of another triangle $\Delta_3$ in the initial triangulation, and it follows that either $T' = \{\alpha_{p_1}, \alpha_{p_2}\}$, or that $\alpha$ crosses another side $\alpha_{p_3} \neq \alpha_{p_2}$ of $\Delta_2$. Continuing this process, we get $T' = \{\alpha_{p_1}, \ldots, \alpha_{p_j}\}$, and triangles $\Delta_1, \ldots, \Delta_j$ such that for each $i = 1, \ldots, j - 1$, the diagonal $\alpha_{p_i}$ is a side of the triangles $\Delta_i$ and $\Delta_{i+1}$. It is clear that the subgraph of $Q^0$ induced by the vertices $p_1, \ldots, p_j$ does not contain a cycle, since Lemma 4.8 implies that the vertices in any induced cycle in $Q^0$ correspond to the diagonals in a triangle in $T^0$, and $\beta$ cannot cross all of the sides of a triangle. By Lemma 4.8, the subgraph of $Q^0$ induced by $p_1, \ldots, p_j$ must be a chain.

Next, consider a sequence $p_1, \ldots, p_j$ of vertices from $[1, n]$ such that the subgraph of $Q^0$ induced by these vertices is a chain (with an edge between any two consecutive vertices in the list). The goal is to find a diagonal $\beta$ of $\mathbb{P}_{n+3}$ which crosses $\alpha_{p_1}, \ldots, \alpha_{p_j}$ and no other diagonals in $T^0$. If $j = 1$, then let $\Delta_0, \Delta_1$ be the triangles in $\Delta(T^0)$ which have $\alpha_{p_1}$ as a side. For $j \geq 2$, any two
consecutive diagonals $\alpha_{p_i}, \alpha_{p_{i+1}}$ are the sides of a triangle $\Delta_i$ in the initial triangulation; also, there exist triangles $\Delta_0, \Delta_j$ with $\Delta_0 \neq \Delta_1, \Delta_j \neq \Delta_{j-1}$, such that $\alpha_{p_1}$ is a side of $\Delta_0$, and $\alpha_{p_j}$ is a side of $\Delta_j$. Let $a_0$ be the vertex of $\Delta_0$ which is opposite the side $\alpha_{p_1}$. For $i = 1, \ldots, j$, let $a_i$ be the vertex of $\Delta_i$ which is opposite the side $\alpha_{p_i}$.

We claim that $\beta = \overline{a_0a_j}$ is the desired diagonal. Observe that all of the triangles $\Delta_0, \ldots, \Delta_j$ are distinct; otherwise, there would be three diagonals from the list $\alpha_{p_1}, \ldots, \alpha_{p_j}$ as sides of a triangle, which would mean that the vertices $p_1, \ldots, p_j$ induce a cycle in $Q^0$. By construction, the total set of vertices from the triangles $\Delta_0, \ldots, \Delta_j$ contains at most $j + 3$ vertices. Any triangulation of $P = \text{Conv}(\Delta_0 \cup \ldots \cup \Delta_j)$ has at most $j + 1$ triangles, so it follows that $\Delta_0, \ldots, \Delta_j$ is a triangulation of $\text{Conv}(\Delta_0 \cup \ldots \cup \Delta_j)$. Thus, $P = \Delta_0 \cup \ldots \cup \Delta_j$ is convex. This means that the only diagonals from $T^0$ that $\beta$ can intersect are in the list $\alpha_{p_1}, \ldots, \alpha_{p_j}$. To show that these are exactly the diagonals from $T^0$ which $\beta$ intersects, consider the diagonals of $T^0$ which $\beta$ passes through as one moves from the endpoint $a_0$ to the other endpoint $a_j$. If $j = 1$, then $\beta$ intersects only the diagonal $\alpha_{p_1}$. Otherwise, $\beta$ intersects the interior of $\Delta_1$, and passes through the interior of another side of $\Delta_1$ different from $\alpha_{p_1}$. This side must be $\alpha_{p_2}$, since the side of $\Delta_1$ different from $\alpha_{p_1}$ and $\alpha_{p_2}$ is on the boundary of $P$. Continuing this argument, it is easy to show that $\beta$ intersects $\alpha_{p_1}, \ldots, \alpha_{p_j}$.

Suppose that the triangulation $T^0$ is mutated in directions $p_1, \ldots, p_j$, and call the resulting sequence of triangulations $T^1, \ldots, T^j$. One verifies by induction that when $T^{i-1}$ is mutated to $T^i$, the diagonal $\alpha_{p_i}$ is flipped to $\overline{a_0a_i}$. The assertion about the denominator vector follows from Proposition 4.1. 

\[\square\]

Remark 4.10. In [7], Fomin, Shapiro, and Thurston associate cluster algebras to punctured Riemann surfaces. The final assertion in Lemma 4.9 about the denominator vector is proven in greater generality for all triangulated surfaces in [7, Theorem 8.6].

Example 4.11. Let $T^0$ be the triangulation of $\mathbb{P}_8$ given in Figure 1. The quiver $Q^0 = Q(T^0)$ of type $A_5$ is given below.

```
3 ← 5
  ↘
  ↗
1 → 2 ← 4
```
Then the diagonals $af$ and $dg$ correspond to cluster variables in $A_\bullet(B^0, t_0)$ with denominator vectors $e_1 + e_2 + e_3 + e_5$ and $e_3 + e_4$, respectively.

**Proposition 4.12.** The $g$-vector of the cluster variable corresponding to the set $C \in \Phi_+(B^0)$ is $(g_1, \ldots, g_n)$, where

- if $k \in C$, then $g_k = |I_{\text{out}}(k)| - 1$;
- if $k \notin C$, the subgraph of $Q^0$ induced by $C \cup \{k\}$ is a chain, and $k \to j$ in $Q^0$ for some $j \in C$, then $g_k = 1$;
- otherwise, $g_k = 0$.

** Remark 4.13.** Proposition 4.12 was independently stated and proven in [14], and generalized to other classical types. For finite type, formulas for (nonquantum) $F$-polynomials and $g$-vectors were given in [10] in the bipartite case and in [18] for the acyclic case (see Proposition 4.18 below).

**Proof.** The first part follows from Proposition 4.3. Using Proposition 4.5 and the notation preceding it (with $T$ replaced by $C$), it suffices to compute $g_k = \dim(\ker(\gamma_k))$ for $k \in [1, n] - C$. Fix such an index $k$.

First, assume that there exists $j \in C$ such that $k \to j$ in $Q^0$, and the subgraph of $Q^0$ induced by $C \cup \{k\}$ is a chain. Then $M_{\text{in}}(k) = 0$, so $\ker(\gamma_k) = M_{\text{out}}(k) = M_j$, which means that $\dim(\ker(\gamma_k)) = 1$. 

```
Next, suppose that there exists \( j \in C \) such that \( k \to j \) in \( Q^0 \), and the subgraph of \( Q^0 \) induced by \( C \cup \{k\} \) is not a chain (which means that it contains a cycle by Lemma 4.8). By the same lemma, the only induced cycles in \( Q^0 \) are directed 3-cycles, so it follows that there exists \( p \in C \) such that \( k \to j \to p \to k \) in \( Q^0 \). It may also be deduced from the same lemma that there is no edge between \( k \) and another vertex \( j' \in C - \{j,p\} \). Thus, \( M_{\text{out}}(k) = M_j \) and \( M_{\text{in}}(k) = M_p \). Since \( \gamma_k \) is an isomorphism between \( M_j \) and \( M_p \), it follows that \( \dim(\ker(\gamma_k)) = 0 \).

Finally, suppose that there is no \( j \in C \) such that \( k \to j \) in \( Q^0 \). Then \( M_{\text{out}}(k) = 0 \), so \( \ker(\gamma_k) = 0 \). □

### 4.3. \( F \)-polynomials in Classical Types for Acyclic Initial Exchange Matrix.

Let \( B^0 = (b^0_{ij}) \) be an acyclic \( n \times n \) exchange matrix of type \( A_n, B_n, C_n, \) or \( D_n \). Write \( Q^0 = Q(B^0) \). (Recall that the matrix \( B^0 \) is acyclic if the quiver \( Q^0 \) is acyclic, i.e., it does not contain any directed cycles.) In particular, \( b_{ij} = \pm a_{ij} \) for \( i \neq j \), where \( (a_{ij}) \) is the Cartan matrix of the corresponding type. Note that we will use the convention for Cartan matrices given in [13] which is different from the one given in [2].

Denote by \( \Phi_+ = \Phi_+(B^0) \) the set of all denominator vectors of cluster variables in \( A_*(B^0, t_0) \) which do not occur in the initial cluster. It was proven in [18] that these cluster variables are in bijective correspondence with the positive roots corresponding to the type of \( B^0 \), and this correspondence is via denominator vectors. To be more precise, if a cluster variable corresponds to \( \alpha = \sum d_i \alpha_i \), where the simple roots are given by \( \alpha_1, \ldots, \alpha_n \), then the denominator vector of the cluster variable is \( (d_1, \ldots, d_n) \in \mathbb{Z}^n \).

The sets \( \Phi_+ \) are given in each type by the following:

**Type** \( A_n \): \( \Phi_+ = \{e_i + \ldots + e_j : 1 \leq i \leq j \leq n\} \)

**Type** \( B_n \): \( \Phi_+ = \{e_i + \ldots + e_j : 1 \leq i \leq j \leq n\} \cup \{e_i + \ldots + e_{j-1} + 2e_j + \ldots + 2e_n : 1 \leq i < j \leq n\} \)

**Type** \( C_n \): \( \Phi_+ = \{e_i + \ldots + e_j : 1 \leq i \leq j \leq n\} \cup \{e_i + \ldots + 2e_j + \ldots + 2e_{n-1} + e_n : 1 \leq i < j < n\} \cup \{2e_i + \ldots + 2e_{n-1} + e_n : 1 \leq i \leq n-1\} \)

**Type** \( D_n \): \( \Phi_+ = \{e_i + \ldots + e_j : 1 \leq i \leq j \leq n\} \cup \{e_i + \ldots + e_{n-2} + e_n : 1 \leq i \leq n-2\} \cup \{e_i + \ldots + e_{j-1} + 2e_j + \ldots + 2e_{n-2} + e_{n-1} + e_n : 1 \leq i < j \leq n-2\} \)
Remark 4.14. In the case that $B^0$ is of type $A_n$, the underlying graph of the quiver $Q^0$ is simply the type $A_n$ Dynkin diagram, i.e., a path on the vertices $1, \ldots, n$. In this case, $\Phi_+(B^0)$ as defined here can be obtained from the $\Phi_+(B^0)$ defined in Section 4.2 by replacing each $S \subset [1, n]$ by $e_S$.

Let $F_d = F_{d_{B^0}}$ be the $F$-polynomial corresponding to the cluster variable with denominator vector $d$.

Define a partial order $\geq$ on $\mathbb{Z}^n$ by

\[(4.13) \quad a \geq a' \text{ if } a - a' \in \mathbb{Z}^n_{\geq 0}.\]

Let $d = (d_1, \ldots, d_n) \in \Phi_+(B^0)$ and $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ such that $0 \leq e \leq d$.

Definition 4.15. We call an arrow $i \to j$ in $Q^0$ acceptable (with respect to the pair $(d, e)$) if $e_i - e_j \leq [d_i - d_j]_+$. Also, an arrow $i \to j$ is called critical (with respect to $(d, e)$) if either

\[(d_i, e_i) = (2, 1), \quad (d_j, e_j) = (1, 0),\]

or

\[(d_j, e_j) = (2, 1), \quad (d_i, e_i) = (1, 1).\]

Note that a critical arrow is always acceptable. Let $S$ be the induced subgraph of $Q^0$ on the set of vertices $\{i : (d_i, e_i) = (2, 1)\}$. For a connected component $C$ of $S$, define $\nu(C)$ as the number of critical arrows having a vertex in $C$.

Theorem 4.16. Fix $d = (d_1, \ldots, d_n) \in \Phi_+(B^0)$, $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$, and define $S$ as above. The coefficient of the monomial $u_1^{e_1} \cdots u_n^{e_n}$ in $F_d$ is nonzero if and only if

1. $0 \leq e \leq d$;
2. all arrows in $Q^0$ are acceptable;
3. $\nu(C) \leq 1$ for all components $C$ of $S$;
4. if $B^0$ is of type $C_n$, then
   - $e_n = 1$, $d_{n-1} = 2$, and $n \to n - 1$ in $Q^0$ imply that $e_{n-1} = 2$;
   - $e_{n-1} \geq 1$, $d_n = 1$, and $n - 1 \to n$ in $Q^0$ imply that $e_n = 1$.
5. If $B^0$ is of type $B_n$, $S$ consists of a single component which contains the vertex $n - 1$, and there is a critical arrow in $S$, then $S$ does not contain the vertex $n$. 

If the conditions above are all satisfied, then the coefficient of $u_1^{e_1} \cdots u_n^{e_n}$ is $2^c$, where $c$ is the number of components $C$ such that $\nu(C) = 0$.

**Remark 4.17.** If $0 \leq d_i \leq 1$ for all $i \in [1, n]$, then $S$ is empty, so the third condition is automatically satisfied.

The formula for $g$-vectors in the next theorem will be useful in computing quantum $F$-polynomials for the classical types.

**Theorem 4.18.** [18] The $g$-vector $g_d$ corresponding to $d = (d_1, \ldots, d_n) \in \Phi_+(B^0)$ is given by

\[
- \sum_{i \in [1,n]} d_i e_i + \sum_{i,j \in [1,n]} d_i [-b_{ji}^0]_+ e_j.
\]

(4.14)

**Proof.** This follows from Theorems 1.8 and 1.10 of [18].

**Example 4.19.** Let

\[
B^0 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 \\
0 & 0 & 2 & 0
\end{pmatrix}
\]

(4.15)

Then $B^0$ is of type $B_4$, and $Q^0$ is the quiver below:

\[
1 \rightarrow 2 \leftarrow 3 \rightarrow 4
\]

Suppose that $d = e_1 + 2e_2 + 2e_3 + 2e_4$. Note that the only possible critical arrow in $Q^0$ is $1 \rightarrow 2$, and this arrow is critical if $e_1 = e_2 = 1$. Let $e = (e_1, e_2, e_3, e_4) \in \mathbb{Z}^4$. Then $u_1^{e_1}u_2^{e_2}u_3^{e_3}u_4^{e_4}$ occurs with nonzero coefficient in $F_d$ if and only if $0 \leq e \leq d$, the three inequalities $e_1 \leq e_2$, $e_2 \geq e_3$, and $e_4 \geq e_3$ are satisfied, and $e \neq (1, 1, 1, 1)$. Using these conditions, it is easy to check that the
\( F \)-polynomial corresponding to \( d \) is

\[
F_{e_1+2e_2+2e_3+2e_4} = u_1^2 u_2^2 u_3^2 u_4^2 + 2u_1 u_2 u_3 u_4^2 + 2u_2^2 u_3^2 u_4^2 + 2u_1 u_2 u_3 u_4 + u_1 u_2 u_3 u_4 + u_1 u_2^2 u_4 + u_1 u_2 u_4^2 + u_2^2 u_3^2 u_4 + u_2 u_3 u_4^2 + 2u_2 u_4^2 + u_2^2 u_4^2 + u_1 u_2 u_3 u_4^2 + 2u_1 u_2 u_3 u_4 + u_1 u_2 + 4u_2 u_4 + 2u_4 + 1.
\]

Some other \( F \)-polynomials corresponding to \( B^0 \) are given below.

\[
F_{e_2} = u_2 + 1
\]

\[
F_{e_2+e_3} = u_2 u_3 + u_2 + 1
\]

\[
F_{e_2+e_3+e_4} = u_2 u_3 u_4 + u_2 u_4 + u_4 + u_2 + 1
\]

\[
F_{e_2+2e_3+2e_4} = u_2 u_3^2 u_4 + 2u_2 u_3 u_4^2 + 2u_2 u_3 u_4 + u_3 u_4^2 + u_2 u_4^2 + 2u_2 u_4 + u_2 + u_4^2 + 2u_4 + 1
\]

The corresponding \( g \)-vectors are given below.

\[
ge_{e_2} = e_1 - e_2 + e_3
\]

\[
ge_{e_2+e_3} = e_1 - e_2
\]

\[
ge_{e_2+e_3+e_4} = e_1 - e_2 + e_3 - e_4
\]

\[
ge_{e_2+2e_3+2e_4} = e_1 - e_2 + e_3 - 2e_4
\]

\[
ge_{e_1+2e_2+2e_3+2e_4} = e_1 - 2e_2 + 2e_3 - 2e_4
\]

For type \( A_n \), Theorem 4.16 follows from Proposition 4.2 and Proposition 4.6. The remainder of this section is devoted to proving Theorem 4.16 in the remaining types. The following easily verified proposition will be useful in checking if an arrow \( i \to j \) in \( Q^0 \) is acceptable.

**Proposition 4.20.** Let \( d = (d_1, \ldots, d_n) \in \Phi_+(B^0) \), \( e = (e_1, \ldots, e_n) \in \mathbb{Z}^n \) such that \( 0 \leq e_i \leq d_i \) for all \( i \in [1, n] \), and let \( i \to j \) be an arrow in \( Q^0 \). If at least one of the equalities \( d_i = 0 \), \( d_j = 0 \), \( e_i = 0 \), or \( e_j = d_j \) holds, then \( i \to j \) is an acceptable arrow.
4.4. Type $D_n$. In this subsection, let $B^0$ be an acyclic $n \times n$ exchange matrix of type $D_n$. The underlying undirected graph of the quiver $Q^0 = Q(B^0)$ is simply the Dynkin diagram of the corresponding type. We will use the formula for $F$-polynomials given in [4]. First, recall some terminology and notation for representations of quivers.

The dimension of a representation $M$ is the vector $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ given by $d_i = \dim(M_i)$. A subrepresentation $N$ of a representation $M$ is given by a collection of subspaces $N_i \subset M_i$ for each $i \in [1, n]$ such that $\varphi_j^i(N_i) \subset N_j$ for all $i, j, k$. For a representation $M$ of dimension $d$ and for an integer vector $e = (e_1, \ldots, e_n)$ such that $0 \leq e \leq d$ (i.e., $0 \leq e_i \leq d_i$ for all $i$), let $\text{Gr}_e(M)$ denote the variety of all subrepresentations of $M$ of dimension $e$. Thus, $\text{Gr}_e(M)$ is a closed subvariety of the product of Grassmanians $\prod \text{Gr}_{e_i}(M_i)$. Denote by $\chi_e(M)$ the Euler-Poincare characteristic of $\text{Gr}_e(M)$ (see [12, Section 4.5]).

To compute $F_d$, we use the following result adapted from [4]:

\textbf{Theorem 4.21.} Assume $B^0$ is as above. Let $d \in \Phi_+(B^0)$, and let $M$ be an indecomposable representation of $Q^0$ of dimension $d$. Then

\begin{equation}
F_d = \sum \chi_e(M)u_1^{e_1} \cdots u_n^{e_n},
\end{equation}

where the summation ranges over $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ such that $0 \leq e \leq d$.

The following lemma will be useful in determining when there exist subrepresentations of $M$ of dimension $e$:

\textbf{Lemma 4.22.} Let $M'$ and $M''$ be vector spaces of dimensions $d'$ and $d''$, respectively, and $\varphi : M' \to M''$ be a linear map of maximal possible rank $\min(d', d'')$. Let $e'$ and $e''$ be two integers such that $0 \leq e' \leq d'$ and $0 \leq e'' \leq d''$. Then the following conditions are equivalent:

1. There exist subspaces $N' \subseteq M'$ and $N'' \subseteq M''$ such that $\dim N' = e'$, $\dim N'' = e''$, and $\varphi(N') \subseteq N''$.
2. $e' - e'' \leq [d' - d'']_+$.

\textbf{Proof.} First, observe that

\begin{equation}
\dim(\text{Ker}\ \varphi) = d' - \min(d', d'') = [d' - d'']_+.
\end{equation}
For the proof that (1) implies (2),

\[(4.28) \quad [d' - d'']_+ \geq \dim(\ker \varphi|_{N'}) = e' - \dim(\im \varphi|_{N'}) \geq e' - e''.\]

Now assume that (2) holds. Choose a subspace \(N'\) of \(M'\) of dimension \(e'\) so that \(N' \subseteq \ker \varphi\) if \(e' \geq [d' - d'']_+\), or \(N' \subset \ker \varphi\) if \(e' \leq [d' - d'']_+\). Then

\[(4.29) \quad \dim(\im \varphi|_{N'}) = e' - \dim(\ker \varphi|_{N'})
\]

\[(4.30) \quad = \begin{cases} e' - [d' - d'']_+ & \text{if } e' \geq [d' - d'']_+ \\ 0 & \text{otherwise.} \end{cases}\]

Since \(\dim(\im \varphi|_{N'}) \leq e''\) in either case, any \(e''\)-dimensional subspace \(N''\) of \(M''\) which contains \(\im \varphi|_{N'}\) will work. \(\square\)

We may restrict attention to \(e \in \mathbb{Z}^n\) which satisfy the first two conditions given in Theorem 4.16. (If these conditions do not hold, Theorem 4.21 together with Lemma 4.22 imply that the coefficient of \(u_1^{e_1} \ldots u_n^{e_n}\) in \(F_d\) equals 0.) If \(d_i = 0\) or 1 for all \(i\), then Theorem 4.16 follows from Proposition 4.2. If \(d_i > 1\) for some \(i\), then \(d = e_p + \ldots + e_{r-1} + 2e_r + \ldots + 2e_{n-2} + e_{n-1} + e_n\) for some \(p, r \in \mathbb{Z}\) satisfying \(1 \leq p < r \leq n - 2\). An indecomposable representation \(M = (M_i)\) of dimension \(d\) can be selected so that for \(p \leq i \leq r - 2\) and for \(r - 1 \leq i \leq n - 3\), the map between \(M_i\) and \(M_{i+1}\) (which is either \(\varphi_{i,i+1}\) or \(\varphi_{i+1,i}\)) is the identity map. If \(r - 1 \rightarrow r\) in \(Q^0\), then let \(V_r = \im(\varphi_{r,r-1})\); otherwise, let \(V_r = \ker(\varphi_{r-1,r})\). For each value \(j = n - 1\) and \(j = n\), if \(n - 2 \rightarrow j\) in \(Q^0\), then let \(V_j = \ker(\varphi_{j,n-2})\); otherwise, let \(V_j = \im(\varphi_{n-2,j})\). Then \(V_r, V_{r-1}, \) and \(V_n\) are three distinct one-dimensional subspaces of the two-dimensional space \(M_{n-2}\).

To compute \(\chi_e(M)\), we will demonstrate how to construct all possible subrepresentations which have dimension vector \(e\) (if there are any), from which it will be easy to compute \(\chi_e(M)\). In order to verify that a given \(N = (N_i)\) (with dimension \(e\)) is a subrepresentation of \(M\), we need to check that for each \(i, j \in [1, n]\) which are connected by an edge in \(Q^0\), the following property is satisfied:

\[(4.31) \quad \varphi_{ji}(N_i) \subseteq N_j \text{ if } i \rightarrow j \text{ in } Q^0, \text{ or } \varphi_{ij}(N_j) \subseteq N_i \text{ if } j \rightarrow i \text{ in } Q^0.\]

Observe that for any \(i \in [1, n] - S\), there is only one possible subspace of \(M_i\) of dimension \(e_i\). Consider a component \(C\) of \(S\). If \(i, j\) are vertices in \(C\) and \(i \rightarrow j\), then (4.31) implies that \(N_i = N_j\). Thus, once a subspace of dimension 1 is chosen for one vertex of the component, all of the vertices
in that component must be assigned that same subspace. If \(i, j \in [p, n]\) such that \(i \rightarrow j\) in \(Q^0\) and so that at least one of \(i, j\) is in \([r, n - 2] - S\), then it is easy to check that \(\varphi_{ji}(N_i) \subset N_j\). For example, if \(i \in [r, n - 2] - S\), then \(\epsilon_i = 0\) or \(\epsilon_i = 2\). If \(\epsilon_i = 0\), then \(\varphi_{ji}(N_i) = 0\). If \(\epsilon_i = 2\), then the fact that \(i \rightarrow j\) is acceptable implies \(e_j = d_j\), so \(N_j = M_j \supset \varphi_{ji}(N_i)\). One can also easily check that for if \(i \rightarrow j\) with \(i, j \in [1, r - 1]\), then \(\varphi_{ji}(N_i) \subset N_j\). Thus, it only remains to verify for each proposed subrepresentation \((N_i)\) that the property (4.31) holds for \((i, j) = (r - 1, r)\) if \(r \in S\), and that the property holds for \((i, j) = (n - 2, n - 1), (n - 2, n)\) if \(n - 2 \in S\).

Next, consider which 1-dimensional subspaces may be assigned to each component. Observe that when \((i, j) = (r - 1, r)\) is a critical pair, property (4.31) is satisfied for \((i, j)\) by \(N\) if and only if \(N_r = V_r\). Also, when \((i, j) = (n - 2, n)\) (resp. \((i, j) = (n - 2, n - 1)\)) is a critical pair, property (4.31) is satisfied for \((i, j)\) if and only if \(N_{n - 2} = V_n\) (resp. \(N_{n - 2} = V_{n - 1}\)).

Let \(C\) be a component of \(S\). If \(\nu(C) = 0\), then any 1-dimensional subspace of \(C^2\) may be assigned to the vertices of \(C\). Suppose \(\nu(C) = 1\). If \((r - 1, r)\) is a critical pair and \(r \in C\), then the subspace on the vertices of \(C\) must by \(V_r\). If \((n - 2, n)\) (resp. \((n - 2, n - 1)\)) is a critical pair and \(n - 2 \in C\), then the subspace on the vertices of \(C\) must \(V_n\) (resp. \(V_{n - 1}\)). Finally, if \(\nu(C) \geq 2\), then there is no 1-dimensional space which can be assigned to the vertices of \(C\) so that (4.31) is satisfied. Since the Euler-Poincare characteristic of the projective line \(\mathbb{P}^1\) is 2, it follows that the contribution to \(\chi_e(M)\) is a multiplicative factor of 0, 1, or 2 when \(\nu(C) = 0\), \(\nu(C) = 1\), or \(\nu(C) \geq 2\), respectively. This concludes the proof of Theorem 4.16 in type \(D_n\).

4.5. **Projections of \(F\)-polynomials and \(g\)-vectors.** To prove Theorem 4.16 in the remaining types, we will show that \(F\)-polynomials and \(g\)-vectors of type \(B_n\) and \(C_n\) can be obtained as certain “projections” of \(F\)-polynomials and \(g\)-vectors of type \(D_{n+1}\) and \(A_{2n-1}\), respectively. In this subsection, we prove a more general result (Theorem 4.24) concerning such projections. For this subsection, let \(B^0 = (b_{ij}^0)\) be an arbitrary \(n \times n\) skew-symmetric integer matrix such that \(B^0\) is acyclic. The methods are similar to those used by Dupont in [5] for coefficient-free cluster algebras, but to give the desired results, we will need to work with cluster algebras with principal coefficients.

We begin by recalling some terminology and notation from [5]. For \(\sigma \in S_n\), let \(\bar{\sigma} \in S_{2n}\) be given by \(\bar{\sigma}(i) = \sigma(i)\) and \(\bar{\sigma}(i + n) = \sigma(i) + n\) for \(i \in [1, n]\). (Thus, \(\sigma\) acts on the set \([1, 2n]\) via \(\bar{\sigma}\).) If \(\tilde{B} = (b_{ij})\) is in \(M_{2n,n}(\mathbb{Z})\), then define an action of \(\sigma\) on \(\tilde{B}\) by \(\sigma \tilde{B} = (b_{\bar{\sigma}^{-1}(i), \bar{\sigma}^{-1}(j)})\). We may also define an action of \(\sigma\) on matrices \(B\) in \(M_{n,n}(\mathbb{Z})\) in a similar way. Say that a subgroup \(G\) of \(S_n\) is an automorphism group for \(\tilde{B}\) (resp. \(B\)) if \(\sigma \tilde{B} = \tilde{B}\) (resp. \(\sigma B = B\)) for all \(\sigma \in G\).
Fix a group $G$ of automorphisms of the matrix $B^0$ for the remainder of this subsection. (Note that $G$ is also a group of automorphisms for $\tilde{B}^0$.)

Let $I = [1,n]$. Write $\mathcal{I}$ for the set of orbits under the action of $G$ on $I$, and $\mathcal{I}i$ for the orbit $Gi$. Fix some ordering $\mathcal{I} = \{\mathcal{I}_1, \ldots, \mathcal{I}_r\}$, where $r = |\mathcal{I}|$. For a matrix $B \in M_{n,n}(\mathbb{Z})$, define the quotient matrix $\overline{B} = (\overline{b}_{i,j})$ to be the $|\mathcal{I}| \times |\mathcal{I}|$ matrix whose entries are given by

$$\overline{b}_{i,j} = \sum_{\ell \in \mathcal{I}_i} b_{\ell,j},$$

where $j$ is fixed representative of $\mathcal{I}_j$. It is easy to show that the definition of quotient matrix does not depend on the choice of representative from the orbit $\mathcal{I}_j$. Furthermore, one may verify that if $B$ is skew-symmetric and $D$ is the $|\mathcal{I}| \times |\mathcal{I}|$ diagonal matrix with entries $d_{\mathcal{I}_i} = |\text{stab}_G(i)|$, where $\text{stab}_G(i)$ is the stabilizer of $i$ under the action of $G$, then $D\overline{B}$ is skew-symmetric. That is, $\overline{B}$ is skew-symmetrizable.

**Definition 4.23.** Let $\tilde{B} = (b_{ij})$ be a $2n \times n$ matrix with automorphism group $G$. We say that $\tilde{B}$ is **admissible** if $b_{ij} = 0$ whenever $i,j$ are in the same $G$-orbit of $I$. (Equivalently, if $B$ is the principal part of $\tilde{B}$, then admissibility means that there is no arrow between $i$ and $j$ in $Q(B)$ whenever $i,j$ are in the same $G$-orbit.)

For a $G$-orbit $\Omega = \{i_1, \ldots, i_s\}$ of $I$, let $\mu_\Omega = \mu_{i_1} \ldots \mu_{i_s}$ be a composition of mutations; we call $\mu_\Omega$ an **orbit mutation**. By using the same argument as in [5], it is not difficult to check that orbit mutations are well-defined, i.e. independent of the order of the elements in $\Omega$.

Let the variables for the $F$-polynomials corresponding to $B^0$ be given by $u_{\mathcal{I}_i}$ for $\mathcal{I}_i \in \mathcal{I}$. Define the **projection** $\pi$ to be the homomorphism of algebras given by

$$\pi : \mathbb{Z}[u_i : i \in I] \rightarrow \mathbb{Z}[u_{\mathcal{I}_i} : \mathcal{I}_i \in \mathcal{I}]$$

$$(4.33) \quad u_i \mapsto u_{\mathcal{I}_i}.$$

Also, for $g = (g_1, \ldots, g_n) \in \mathbb{Z}^n$, define the **quotient** of $g$ to be

$$\overline{g} = \left( \sum_{\ell \in \mathcal{I}_i} g_{\ell}, \ldots, \sum_{\ell \in \mathcal{I}_i} g_{\ell} \right) \in \mathbb{Z}^{|\mathcal{I}|}$$

$$(4.35) \quad g \mapsto \overline{g}.$$

Let $\overline{T}_n$ be the $|\mathcal{I}|$-regular tree such that the $|\mathcal{I}|$ edges emanating from a vertex are labeled by distinct elements of $\mathcal{I}$; write $\mathcal{I}_0$ for the initial vertex in $\overline{T}_n$. 
Theorem 4.24 says that certain $F$-polynomials and $g$-vectors corresponding to the initial matrix $\overline{B}^0$ can be obtained as “projections” or “quotients” of $F$-polynomials or $g$-vectors corresponding to the initial matrix $B^0$.

**Theorem 4.24.** Suppose $\Omega_1, \ldots, \Omega_s$ are $G$-orbits with $\Omega_i = \overline{k}_i$. For $p \in [1, s]$, let

$$B^p = (b^p_{ij}) = \mu_{\Omega_p} \ldots \mu_{\Omega_1}(B^0).$$  \hspace{1cm} (4.36)

Suppose that $k_p$ is a sink or source in $Q(B^p - 1)$ for each $p = 1, \ldots, s$. Let $t \in \mathbb{T}_n$ be the vertex obtained by a path from $t_0$ whose edges are labeled by all of the elements of $\Omega_1$, followed by all the edges in $\Omega_2$, and so on, to finally the edges in $\Omega_s$. Let $\overline{t} \in \overline{\mathbb{T}}_n$ be the path obtained from the initial vertex $\overline{t}_0$ via the edges $\overline{k}_1, \ldots, \overline{k}_s$. Let $j \in [1, n]$.

Then

$$F_{j; \overline{t}} = \pi(F_{j; t_0}),$$  \hspace{1cm} (4.37)

$$g_{j; \overline{t}} = \overline{g}_{j; t_0}.$$  \hspace{1cm} (4.38)

**Proof.** To prove the theorem, we need to first consider the cluster algebras $A_\bullet(B^0; t_0)$ and $A_\bullet(\overline{B}^0; \overline{t}_0)$, and show that the seed at $\overline{t}$ in $A_\bullet(\overline{B}^0; \overline{t}_0)$ is the “projection” of the seed at $t$ in $A_\bullet(B^0; t_0)$. In [5], the seeds of the coefficient-free cluster algebras with initial exchange matrices $B^0$ and $\overline{B}^0$ were related. Some of the results proven in [5] easily carry over to the current situation with principal coefficients (namely, Lemma 4.26 and Lemma 4.28), so the proofs are omitted here. However, for finishing the proof of the result about the projection of the seeds, some stronger conditions are needed (see Proposition 4.29).

Let the initial seed of $A_\bullet(B^0, t_0)$ be given by $(\bar{x}, \bar{B}^0)$, where

$$\bar{B}^0 = \begin{pmatrix} B^0 \\ I_n \end{pmatrix}$$  \hspace{1cm} (4.39)

$$\bar{x} = (x_1, \ldots, x_n, y_1, \ldots, y_n).$$  \hspace{1cm} (4.40)
By abuse of notation, let the projection $\pi$ be the homomorphism of algebras given by

$$
\pi : \mathbb{Z}[x_i^{\pm 1}, y_i^{\pm 1} : i \in I] \rightarrow \mathbb{Z}[x_i^{\pm 1}, y_i^{\pm 1} : \bar{i} \in \mathcal{T}]
$$

(4.41)

$$
\begin{align*}
&x_i \mapsto x_i \\
y_i \mapsto y_i
\end{align*}
$$

(4.42)

(4.43)

We also apply $\pi$ to ordered $2n$-tuples $\tilde{v} = (v_1, \ldots, v_n, y_1, \ldots, y_n)$ (with each $v_j$ in $\mathbb{Z}[x_i^{\pm 1}, y_i^{\pm 1} : i \in I]$):

$$
\pi(\tilde{v}) = (\pi(v_{j_1}), \ldots, \pi(v_{j_r}), y_{j_1}, \ldots, y_{j_r}).
$$

(4.44)

Note that $\pi(\tilde{v})$ is a $2|\mathcal{T}|$-tuple of elements from $\mathbb{Z}[x_i^{\pm 1}, y_i^{\pm 1} : \bar{i} \in \mathcal{T}]$.

Let $\mathcal{A}_{\bullet}(\mathcal{B}^0, \bar{t}_0)$ be the cluster algebra with principal coefficients such that the initial extended cluster is given by

$$
\pi(\tilde{x}) = (x_{\bar{j_1}}, \ldots, x_{\bar{j_r}}, y_{\bar{j_1}}, \ldots, y_{\bar{j_r}}).
$$

(4.45)

By the Laurent phenomenon ([10, Theorem 3.1]),

$$
\begin{align*}
&\mathcal{A}_{\bullet}(\mathcal{B}^0, \bar{t}_0) \subset \mathbb{Z}[x_i^{\pm 1}, y_i^{\pm 1} : i \in I] \\
&\mathcal{A}_{\bullet}(\mathcal{B}^{\bar{t}}, \bar{t}_0) \subset \mathbb{Z}[x_i^{\pm 1}, y_i^{\pm 1} : \bar{i} \in \mathcal{T}]
\end{align*}
$$

(4.46)

(4.47)

If $\bar{B} = (b_{ij}) \in M_{2n,n}(\mathbb{Z})$, then define the quotient matrix $\overline{B} = (\overline{b}_{ij})$ using (4.32). (Equivalently, if $\bar{B} = \begin{pmatrix} B & \end{pmatrix}$, where $B, C$ are $n \times n$ matrices, then $\overline{B} = \begin{pmatrix} B \\ C \end{pmatrix}$.) The first $r$ rows of this matrix are indexed by $\mathcal{T}$, the next $r$ rows are indexed by $\mathcal{T}' = \{\bar{j}_1 + n, \ldots, \bar{j}_r + n\}$ (i.e., the $G$-orbits of $[n + 1, 2n]$), and the columns are indexed by $\mathcal{T}$.

Define an action of $G$ on $\mathbb{Z}[x_i^{\pm 1}, y_i^{\pm 1} : i \in I]$ by

$$
\begin{align*}
&\sigma x_i = x_{\sigma i} \\
&\sigma y_i = y_{\sigma i}
\end{align*}
$$

(4.48)

(4.49)

**Definition 4.25.** A seed $(\tilde{v} = (v_1, \ldots, v_n, y_1, \ldots, y_n), \bar{B})$ in $\mathcal{A}_{\bullet}(\mathcal{B}^0, \bar{t}_0)$ is $G$-invariant if for every $\sigma \in G$, we have $\sigma \bar{B} = \bar{B}$ and $\sigma v_i = v_{\sigma i}$.
Lemma 4.26. Let \( \Omega \) be a \( G \)-orbit of \( I \), and let \((\tilde{v}, \tilde{B})\) a seed in \( A_\bullet(B^0; t_0) \) such that \( \tilde{B} \) is admissible. If \((\tilde{v}, \tilde{B})\) is a \( G \)-invariant seed, then so is \((\tilde{v}', \tilde{B}') = \mu_\Omega(\tilde{v}, \tilde{B})\).

Definition 4.27. Let \( \tilde{B} = (b_{ij}) \) be a \( 2n \times n \) matrix with automorphism group \( G \). We say that \( \tilde{B} \) is strongly admissible if it is admissible and satisfies the following conditions:

1. For \( i, j \in [1, 2n] \) in the same orbit and \( \ell \in [1, n] \), \( b_{i\ell}, b_{j\ell} \) are both nonnegative or both nonpositive.
2. For each \( i, j \in [1, n] \) in the same orbit and \( \ell \in [1, 2n] \), \( b_{\ell i}, b_{\ell j} \) are both nonnegative or both nonpositive.

Lemma 4.28. Let \( \Omega = k \) be a \( G \)-orbit of \( I \), and let \((\tilde{v}, \tilde{B})\) be a \( G \)-invariant seed of \( A_\bullet(B^0, t_0) \) such that \( \tilde{B} \) is strongly admissible. Put \((\tilde{v}', \tilde{B}') = \mu_\Omega(\tilde{v}, \tilde{B})\). Then

\[ (\pi(\tilde{v}'), \overline{\tilde{B}}') = \mu_{\overline{k}}(\pi(\tilde{v}), \overline{\tilde{B}}), \]

i.e., projection commutes with orbit mutation when applied to \((\tilde{v}, \tilde{B})\).

Proposition 4.29. Suppose \( \Omega_1, \ldots, \Omega_s \) are \( G \)-orbits with \( \Omega_i = \overline{k}_i \). Let

\[ \tilde{B}^p = (b^p_{ij}) = \mu_{\Omega_p} \cdots \mu_{\Omega_1} (\tilde{B}^0), \]

and let \( B^p \) be the principal part of \( \tilde{B}^p \). Suppose that \( k_p \) is a sink or source in \( Q(B^{p-1}) \) for each \( p = 1, \ldots, s \). Put

\[ (\tilde{x}', \tilde{B}') = \mu_{\Omega_s} \cdots \mu_{\Omega_1} (\tilde{x}, \tilde{B}^0). \]

Then each \( \tilde{B}^p \) is strongly admissible, and

\[ (\pi(\tilde{x}'), \overline{\tilde{B}}') = \mu_{\overline{k_s}} \cdots \mu_{\overline{k_1}} (\pi(\tilde{x}), \overline{\tilde{B}}^0). \]

Thus, with the notation of Theorem 4.24, the seed at \( \overline{t} \) in \( A_\bullet(\overline{B}^0, \overline{t}_0) \) can be obtained by applying the projection to the seed \((\tilde{x}', \tilde{B}')\) in \( A_\bullet(B^0; t_0) \).

Proof. We only need to prove that each \( \tilde{B}^p \) is strongly admissible. Once this is done, the proposition follows from Lemmas 4.26 and 4.28 by induction on \( s \), the number of orbit mutations.

First, we check that \( Q(B^p) \) is acyclic. Write \( \tilde{B}^p = (b^p_{ij}) \). If \( k_p \) is a sink in \( Q(B^{p-1}) \), then \( b^p_{i,k_p} \leq 0 \) and \( b^p_{k_p,j} \geq 0 \) for all \( i, j \in [1, n] \). If \( k_p \) is a source in \( Q(B^{p-1}) \), then \( b^p_{i,k_p} \geq 0 \) and \( b^p_{k_p,j} \leq 0 \) for all...
\(i, j \in [1, n]\). It follows that

\[
b_{ij}^p = \begin{cases} 
  b_{ij}^{p-1} & \text{if } i = k_p \text{ or } j = k_p \\
  b_{ij}^{p-1} & \text{otherwise.}
\end{cases}
\]

(4.54)

That is, \(Q(B^p)\) is obtained from \(Q(B^{p-1})\) by reversing the arrows which contain the vertex \(k_p\). By an inductive argument, it is easy to show that \(Q(B^p)\) is acyclic.

Let \(i, j \in I\) such that \(i, j\) are in the same \(G\)-orbit. Then \(j = \sigma i\) for some \(\sigma \in G\). Since \(G\) is a finite group, there exists some \(r > 0\) such that \(\sigma^r = 1\).

If \(i \rightarrow j\) in \(Q(B^p)\), then the \(G\)-invariance of \(\tilde{B}^p\) implies that the following cycle occurs in \(Q(B^p)\):

\[
i \rightarrow \sigma i \rightarrow \sigma^2 i \rightarrow \cdots \rightarrow \sigma^r i = i.
\]

(4.55)

This cycle cannot occur, so \(i \not\rightarrow j\) in \(Q(B^p)\), which proves that \(B^p\) is admissible.

Let \(\ell \in [1, n]\). If \(i \rightarrow \ell \rightarrow j\) in \(Q(B^p)\), then the \(G\)-invariance of \(\tilde{B}^p\) implies that the following cycle occurs in \(Q(B^p)\):

\[
i \rightarrow \ell \rightarrow \sigma i \rightarrow \sigma \ell \rightarrow \sigma^2 i \rightarrow \sigma^2 \ell \rightarrow \cdots \rightarrow \sigma^{r-1} \ell \rightarrow \sigma^r i = i.
\]

(4.56)

Thus, the path \(i \rightarrow \ell \rightarrow j\) cannot occur in \(Q(B^p)\), and by similar reasoning, \(j \rightarrow \ell \rightarrow i\) cannot occur in \(Q(B^p)\). This forces \(b_{\ell i}^p, b_{\ell j}^p\) to be both nonnegative or both nonpositive, and the same thing is true of \(b_{\ell i}^p, b_{\ell j}^p\).

From [4], it is known that every \(F\)-polynomial \(F_{j;\ell}^{B^0; t_0}\) has constant term 1 (under the assumption that \(B^0\) is skew-symmetric). From [10, Proposition 5.6], it follows that the entries in the bottom \(n\) rows of the \(\ell\)th column of \(\tilde{B}^p\) are all nonnegative or nonpositive. This shows condition (1) for the strong admissibility of \(\tilde{B}^p\) is true.

Finally, suppose that \(\ell \in [n + 1, 2n]\). Then \(b_{\ell i}^p = b_{\ell,\sigma i}^p = b_{\sigma^{-1} \ell, i}^p\). Since \(b_{\sigma^{-1} \ell, i}^p, b_{\ell i}^p\) are both nonnegative or both nonpositive, this proves condition (2) for strong admissibility. \(\square\)

**Proposition 4.30.** Let \(\Omega_i\) be \(G\)-orbits, and let \(t \in T_n, \ell \in \overline{T_n}\), with the same assumptions as in Theorem 4.24. Let \(j \in [1, n]\), and write \(x_{j;\ell}, x_{\overline{j};\overline{\ell}}\) for the cluster variables in \(A\bullet(B^0; t_0)\) and \(A\bullet(\overline{B^0}; t_0)\), respectively. Then

\[
x_{\overline{j};\overline{\ell}} = \pi(x_{j;\ell}).
\]

(4.57)
Proof. The proposition follows immediately from Proposition 4.29. □

Finally, the proof of Theorem 4.24 may be concluded. By (3.14),

\[ x_{j,t} = F_{j,t}^{B_0,t_0}(\hat{y}_1, \ldots, \hat{y}_n)x_1^{g_1} \cdots x_n^{g_n}, \]

where \( g_{j,t}^{B_0,t_0} = (g_1, \ldots, g_n) \). Applying \( \pi \) to both sides, we get

\[ x_{j,t} = \pi(F_{j,t}^{B_0,t_0}(\hat{y}_1, \ldots, \hat{y}_n))x_1^{g_1} \cdots x_n^{g_n}. \]

Let

\[ \hat{y}_j = y_j \prod_{\ell \in \mathcal{I}} x_{j,\ell}^{\mathcal{I} \setminus \ell}. \]

These are the \( \hat{y}_j \) elements corresponding to \( \mathcal{A}_\bullet(B_0, \vec{t}_0) \) (see (3.12)). It is straightforward to check that \( \pi(\hat{y}_j) = \hat{y}_j \) for \( j \in [1, n] \). Thus,

\[ x_{j,t} = \pi(F_{j,t}^{B_0,t_0})(\hat{y}_1, \ldots, \hat{y}_n) \prod_{\ell \in \mathcal{I}} x_{j,\ell}^{\sum_{\ell \in \mathcal{I}} g_{\ell}}. \]

From [4], it is known that \( F_{j,t}^{B_0,t_0} \) has constant term 1, so the same is true of \( \pi(F_{j,t}^{B_0,t_0}) \). The theorem follows from Proposition 3.6. □

Given a seed \((\vec{x}, \vec{B})\) in a cluster algebra \( \mathcal{A} \), where \( \vec{B} \) has principal part \( B \), we say that \( \mu_k \) is a source mutation (resp. sink mutation) if the vertex \( k \) is a source (resp. sink) in \( Q(B) \). The following result is proven in [18]:

**Theorem 4.31.** [18] If \( B^0 \) is of finite type, then every cluster variable in \( \mathcal{A}_\bullet(B^0, t_0) \) may be obtained from the initial cluster by a sequence of mutations, each of which is either a sink or source mutation.

4.6. **Type C\(_n\).** In this subsection, we use the results of the previous subsection to finish the proof of Theorem 4.16. Let \( \vec{B}_0 = (b_{ij}) \) be an \( n \times n \) acyclic exchange matrix of type \( C_n \). Let \( G \) be the subgroup of \( S_{2n-1} \) generated by the involution \( \sigma \), where \( \sigma(i) = 2n - i \) for all \( i \in [1, 2n-1] \). Then \( G \) is a group of automorphisms for the matrix \( B^0 = (b_{ij}) \) of type \( A_{2n-1} \) whose entries are given
Suppose the cluster variable \(d\) below:

\[
\begin{align*}
4.62 & \quad b_{ij} = b_{2n-i,2n-j} = \overline{b}_{ij} \text{ if } 1 \leq i, j \leq n, (i, j) \neq (n-1, n) \\
4.63 & \quad b_{n-1,n} = b_{n+1,n} = \text{sgn}(\overline{b}_{n-1,n}) \in \{+1, -1\} \\
4.64 & \quad b_{n,n-1} = b_{n,n+1} = -b_{n-1,n} \\
4.65 & \quad b_{ij} = b_{2n-i,2n-j} = 0 \text{ if } n+1 \leq i \leq 2n-1 \text{ and } 1 \leq j \leq n-1
\end{align*}
\]

The matrix \(\overline{B}\) is indeed the quotient matrix of \(B^0\), which justifies the choice of notation.

**Lemma 4.32.** Suppose the cluster variable \(x \in A_\bullet(B^0)\) has denominator vector \(d'\). Then \(\pi(x)\) is a cluster variable in \(A_\bullet(\overline{B})\) with denominator vector \(\overline{d}'\).

**Proof.** The fact that \(\pi(x)\) is a cluster variable in \(A_\bullet(\overline{B})\) follows from Proposition 4.30 and Theorem 4.31. Define the function \(\psi: \Phi_+(B^0) \to \Phi_+(\overline{B})\) in the following way: if \(x \in A_\bullet(B^0)\) has denominator vector \(d' \in \Phi_+(B^0)\), then let \(\psi(d')\) be the denominator vector of the cluster variable \(\pi(x)\). Thus, we must show that

\[
\psi(d') = \overline{d}'.
\]

Clearly, \(\psi(d') \leq \overline{d}'\) and \(\psi(d') = \psi(\sigma(d'))\). It is easy to verify that the number of \(\sigma\)-orbits of \(\Phi_+(B^0)\) is equal to \(|\Phi_+(\overline{B})|\). It follows that \(\psi(d') = \psi(d'')\) if and only if \(d' = d''\) or \(d' = \sigma(d'')\).

We use the natural ordering given by \(\leq\) on \(\Phi_+(B^0)\) to prove (4.66). The least elements of \(\Phi_+(B^0)\) are \(e_i\) \((i \in [1, n])\). Since \(\psi(e_i) \leq e_i\), equality must hold. Now suppose that \(d'\) is an element of \(\Phi_+(B^0)\) such that \(\psi(d'') = \overline{d}'\) for all \(d'' \in \Phi_+(B^0)\) satisfying \(d'' < d'\). If \(d' = e_i + \cdots + e_j\) for some \(1 \leq i < j \leq n\), then any denominator vector in \(\Phi_+(\overline{B})\) that is less than \(\overline{d}' = d'\) is of the form \(e_{i'} + \cdots + e_{j'}\), where \(i \leq i' \leq j' \leq j\), and this is equal to \(\psi(e_{i'} + \cdots + e_{j'})\) by assumption. Thus, (4.66) must hold in this case. If \(d' = e_i + \cdots + e_{2n-j}\) for some \(1 \leq i < j \leq n-1\), then any denominator vector in \(\Phi_+(\overline{B})\) that is less than \(\overline{d}' = e_i + \cdots + 2e_j + \cdots + 2e_{n-1} + e_n\) is either of the form \(e_{i'} + \cdots + e_{j'}\) \((i \leq i' \leq j' \leq n)\) or of the form \(e_{i'} + \cdots + 2e_{j'} + \cdots + 2e_{n-1} + e_n\), where \(i \leq i'\) and \(j' \geq j\). The former expression is equal to \(\psi(e_{i'} + \cdots + e_{j'})\), while the latter is equal to \(\psi(e_{i'} + \cdots + e_{2n-j})\). Again, (4.66) holds. Similar reasoning applies to denominator vectors of the form \(d' = e_i + \cdots + e_{2n-i}\). The remaining denominator vectors in \(\Phi_+(B^0)\) are obtained.
from the ones already mentioned by applying \( \sigma \), so the result again follows by using the fact that 
\[
\psi(\sigma(d')) = \psi(d') \quad \text{for} \quad d' \in \Phi_+(B^0).
\]

To calculate \( F^{B^0, t_0}_d \) for \( d \in \Phi_+(\overline{B^0}) \), we need to use the formula for \( F^{B^0, t_0}_{d'} \) for \( d' \in \Phi_+(B^0) \) such that \( \overline{d'} = d \), and then apply the projection \( \pi \). In the current setting, \( \pi(u_i) = \pi(u_{2n-i}) = u_i \) for \( i \in [1, n] \).

Let \( d = (d_1, \ldots, d_n) \in \Phi_+(\overline{B^0}) \). If \( d = \sum_{i=p}^r e_i \) for some \( 1 \leq p \leq r \leq n \), then we can let \( d' = d \). In this case,

\[
(4.67) \quad F^{B^0, t_0}_d = \pi(F^{B^0, t_0}_{d'}).
\]

It is easy to verify the theorem works in this case.

Now suppose that \( d = \sum_{i=p}^n e_i + \sum_{i=r}^{n-1} e_i \) for some \( 1 \leq p \leq r \leq n - 1 \). Then we can take 
\[
d' = \sum_{i=p}^{2n-r} e_i = (d'_1, \ldots, d'_{2n-1}).
\]
Let \( e = (e_1, \ldots, e_n) \) such that \( u_1^{e_1} \cdots u_n^{e_n} \) occurs with nonzero coefficient in \( \pi(F^{B^0, t_0}_d) \). Then there exists \( (e'_1, \ldots, e'_{2n-1}) \in \mathbb{Z}^{2n-1} \) such that \( u_1^{e_1'} \cdots u_n^{e_{2n-1}'} \) occurs with nonzero coefficient in \( F^{B^0, t_0}_{d'} \) and \( \pi(u_1^{e_1'} \cdots u_n^{e_{2n-1}'}) = u_1^{e_1} \cdots u_n^{e_n} \) (i.e., \( e_i = e'_i + e'_{2n-i} \) for \( i \in [1, n-1] \) and \( e_n = e_n' \)).

First, we will check that properties (1)-(4) in Theorem 4.16 hold for \( u_1^{e_1} \cdots u_n^{e_n} \). (1) is immediate from the fact that (1) holds for \( F^{B^0, t_0}_{d'} \).

Let \( Q^0 = Q(B^0) \) and \( \overline{Q^0} = Q(\overline{B^0}) \). Observe that \( i \rightarrow j \) in \( \overline{Q^0} \) if and only if \( i \rightarrow j \) and \( 2n - i \rightarrow 2n - j \) in \( Q^0 \). Also, if \( i \rightarrow j \) in \( Q^0 \) and \( d'_i \leq d'_j \), then the fact that arrows in \( Q^0 \) are acceptable implies that \( e'_i \leq e'_j \).

For (4), part (a), if \( n \rightarrow n-1 \) in \( \overline{Q^0} \) and \( e_n = 1 \), then \( e'_n = 1 \) implies that \( e'_{n-1} = 1 \), \( e'_{n+1} = 1 \). Thus, \( e_{n-1} = e'_{n-1} + e'_{n+1} = 2 \). For part (b), suppose that \( n-1 \rightarrow n \) in \( \overline{Q^0} \) and \( e_{n-1} \geq 1 \). Then \( e'_{n-1} = 1 \) or \( e'_{n+1} = 1 \). Since \( e'_n \geq e'_{n-1} \), \( e'_n \geq e'_{n+1} \), part (b) follows.

For (3), observe that (4) implies that the arrow connecting \( n-1 \) and \( n \) in \( \overline{Q^0} \) cannot be critical, so there is only one possible critical arrow which may occur between \( r-1 \) and \( r \).

For (2), let \( i \rightarrow j \) in \( \overline{Q^0} \). If \( i,j \) are both in \( [1, p-1] \), both in \( [p, r-1] \), or both in \( [r, n-1] \), then \( d'_i = d'_j, d'_{2n-i} = d'_{2n-j} \), and \( d_i = d_j \). Since \( i \rightarrow j \) and \( 2n - i \rightarrow 2n - j \) in \( Q^0 \), it follows that \( e'_i - e'_j \leq 0 \) and \( e'_{2n-i} - e'_{2n-j} \leq 0 \), which means that \( e_i - e_j \leq 0 = [d_i - d_j] \). If \( i = p-1 \) or \( j = p-1 \), then use the fact that if \( d_i = 0 \) or \( d_j = 0 \), then condition (2) is automatically satisfied. Now suppose that \( r > p \) and \( \{i,j\} = \{r-1,r\} \). Note that \( d'_{r-1} = 1, d'_{2n-r+1} = 0, d'_r = 1, d'_{2n-r} = 1, d_{r-1} = 1, \)
\( d_r = 2 \). If \( r - 1 \rightarrow r \) in \( \overline{Q}^0 \), then \( e'_{r-1} - e'_r \leq 0, e'_{2n-r+1} - e'_{2n-r} \leq 0 \), so \( e_{r-1} - e_r \leq 0 \). If \( r \rightarrow r - 1 \) in \( \overline{Q}^0 \), then \( e'_i - e'_{r-1} \leq 0, e'_{2n-r} - e'_{2n-r+1} \leq 1 \), so \( e_r - e_{r-1} \leq 1 \). Finally, if \( \{i, j\} = \{n - 1, n\} \), then (4) implies that \( i \rightarrow j \) is acceptable.

Now assume that \( e = (e_1, \ldots, e_n) \in \mathbb{Z}^n \) satisfies conditions (1)-(4). Since all the coefficients in \( F_{d'}^{R_0, t_0} \) are either 0 or 1, we need to construct all \( e' = (e'_1, \ldots, e'_{2n-1}) \in \mathbb{Z}^n_{\geq 0} \) such that \( u_1 e'_1 \cdots u_n e'_n \) occurs with nonzero coefficient in \( F_{d'}^{R_0, t_0} \), \( e_i = e'_i + e'_{2n-i} \) for \( i \in [1, n - 1] \), and \( e_n = e'_n \).

If \( e_i = 0 \), then this forces \( e'_i = e'_{2n-i} = 0 \). If \( e_i = d_i \), then we must have \( e'_i = d'_i \) and \( e'_{2n-i} = d'_{2n-i} \). Thus, if \( (d_i, e_i) \neq (2, 1) \), then there is only one possible choice for \( e'_i \) and \( e'_{2n-i} \).

Let \( S \) be the subgraph of \( \overline{Q}^0 \) induced by the vertex set \( \{i \in [1, n] : (d_i, e_i) = (2, 1)\} \). If \( i, j \) are vertices in \( S \) such that \( i \rightarrow j \) in \( \overline{Q}^0 \), then \( e'_i \leq e'_j \) and \( e'_{2n-i} \leq e'_{2n-j} \). It follows that for each component \( C \) of the graph \( S \), exactly one of the following possibilities holds: either \( e'_i = 1 \) and \( e'_{2n-i} = 0 \) for all vertices \( i \) in \( C \), or \( e'_i = 0 \) and \( e'_{2n-i} = 1 \) for all vertices \( i \) in \( C \). Thus, there are two possible choices for each such \( C \), except in the case where \( r \) is a vertex of \( C \) and \( (r - 1, r) \) is a critical pair. In this case, we must have \( e'_r = 1 \) and \( e'_{2n-r} = 0 \) if \( r - 1 \rightarrow r \) in \( \overline{Q}^0 \), or \( e'_r = 0 \) and \( e'_{2n-r} = 1 \) if \( r \rightarrow r - 1 \) in \( \overline{Q}^0 \).

Now suppose that \( e' = (e'_1, \ldots, e'_{2n-1}) \) is chosen as above. It remains to verify that all arrows \( i \rightarrow j \) in \( Q^0 \) are acceptable with respect to \( e' \) and \( d' \). If \( i, j \in [r, n - 1] \) and both \( i, j \) are vertices in \( S \), note that \( i \rightarrow j \) and \( 2n - i \rightarrow 2n - j \) are both acceptable by construction.

Let \( i \rightarrow j \) in \( \overline{Q}^0 \). We will prove that \( i \rightarrow j \) and \( 2n - i \rightarrow 2n - j \) are acceptable arrows in \( Q^0 \). If \( i, j \in [1, p] \), then Proposition 4.20 implies that these arrows are acceptable. If \( i, j \in [p, r - 1] \), then \( e'_{2n-i} = e'_{2n-j} = 0 \), so \( 2n - i \rightarrow 2n - j \) is acceptable. Also, \( e_i = e'_i, e_j = e'_j \), so the fact that \( i \rightarrow j \) is acceptable in \( \overline{Q}^0 \) implies that the arrow is acceptable in \( Q^0 \).

Suppose that \( i \) is in \([r, n - 1]\) but is not a vertex of \( S \). If \( e_i = 0 \), then \( e'_i = e'_{2n-i} = 0 \). If \( e_i = 2 \), then \( e_j = d_j \), so \( e'_j = d'_j \) and \( e'_{2n-j} = d'_{2n-j} \). In either case, Proposition 4.20 implies that \( i \rightarrow j \) and \( 2n - i \rightarrow 2n - j \) are acceptable arrows in \( Q^0 \). The argument in the case that \( j \) is in \([r, n - 1]\) but not in \( S \) is similar.

By Proposition 4.20, the arrow between \( 2n - r + 1 \) and \( 2n - r \) is acceptable since \( d'_{2n-r+1} = 0 \). Next, we need to consider the arrow in \( Q^0 \) between the vertices \( r - 1 \) and \( r \). We may assume that \( r \) is in \( S \). If \( (r - 1, r) \) is a critical pair in \( \overline{Q}^0 \), note that the construction implies that \( r - 1 \rightarrow r \) or \( r \rightarrow r - 1 \) is acceptable in \( Q^0 \). If \( r - 1 \rightarrow r \) in \( Q^0 \), \( e_r = 1 \), and \( (r - 1, r) \) is not a critical pair, then
$e_{r-1} = 0$, so $e'_{r-1} = 0$. This means that $r - 1 \rightarrow r$ in $Q^0$ is acceptable. If $r \rightarrow r - 1$ in $Q^0$, $e_r = 1$ and $(r - 1, r)$ is not critical, then $e_{r-1} = 1 = e'_{r-1} = d'_{r-1}$, so $r \rightarrow r - 1$ is acceptable in $Q^0$.

Finally, we consider the arrow in $Q^0$ between $n - 1$ and $n$. We may assume that $n - 1 \in S$, so $e_{n-1} = 1$. If $n \rightarrow n - 1$ in $Q^0$, then (4) implies that $e'_n = 0$. If $n - 1 \rightarrow n$ in $Q^0$, then (4) implies that $e'_n = 1 = d'_n$. Thus, in either case, the arrow between $n - 1$ and $n$ and the arrow between $n + 1$ and $n$ are both acceptable in $Q^0$.

This concludes the proof of Theorem 4.16 in type C$_n$.

4.7. Type B$_n$. In this subsection, we finish the proof of Theorem 4.16 using the same strategy as in the previous subsection. Let $B_0 = (b_{ij})$ be an $n \times n$ acyclic exchange matrix of type B$_n$. Let $G$ be the subgroup of $S_{n+1}$ generated by the involution $\sigma$, where $\sigma(i) = i$ for all $i \in [1, n - 1]$, $\sigma(n + 1) = n$, $\sigma(n) = n + 1$. Then $G$ is a group of automorphisms for the matrix $B_0 = (b_{ij})$ of type D$_{n+1}$ whose entries are given below:

(4.68) \[ b_{ij} = \bar{b}_{ij} \text{ if } 1 \leq i, j \leq n \text{ and } (i, j) \neq (n, n - 1) \]

(4.69) \[ b_{n,n-1} = -b_{n-1,n} = \text{sgn}(\bar{b}_{n,n-1}) \in \{+1, -1\} \]

(4.70) \[ b_{n+1,n-1} = -b_{n-1,n+1} = \text{sgn}(\bar{b}_{n,n-1}) \in \{+1, -1\} \]

(4.71) \[ b_{n+1,j} = b_{j,n+1} = 0 \text{ if } 1 \leq j \leq n + 1, j \neq n - 1. \]

The matrix $\bar{B}_0$ is indeed the quotient matrix of $B_0$, which justifies the choice of notation. Write $Q^0 = Q(B_0)$ and $\bar{Q}^0 = Q(\bar{B}_0)$.

In this case, it is easy to verify Lemma 4.32 holds using a similar argument. Let $d \in \Phi_+$. If $d = \sum_{i=1}^{n} d_i e_i$ with each $d_i = 0$ or 1, then we can take $d' = d$ and the same reasoning as in type C$_n$ holds here.

Assume for the remainder of this subsection that $d = \sum_{i=p}^{r-1} e_i + \sum_{i=r}^{n} 2e_i$ for some $1 \leq p < r \leq n$. Then we can take $d' = \sum_{i=p}^{r-1} e_i + \sum_{i=r}^{n-2} 2e_i + e_{n-1} + e_n$. Write $d = (d_1, \ldots, d_n)$ and $d' = (d'_1, \ldots, d'_{n+1})$. Then $d_i = d'_i$ for all $i \in [1, n - 1]$ and $d_n = d'_n + d'_{n+1}$. Let $(e'_1, \ldots, e'_{n+1}) \in \mathbb{Z}^{n+1}$ such that $u_1^{e'_1} \cdots u_{n+1}^{e'_{n+1}}$ occurs with nonzero coefficient in $F_{d'}^{B_0, t_0}$. Let $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ be given by $e_i = e'_i$ for $i \in [1, n - 1]$, $e_n = e'_n + e'_{n+1}$. Then $u_1^{e_1} \cdots u_{n}^{e_{n}}$ occurs with nonzero coefficient in $F_{d}^{B_0, t_0} = \pi(F_{d'}^{B_0, t_0})$, so we need to verify that $e$ satisfies conditions (1)-(3) and (5) in Theorem 4.16 with respect to $\bar{Q}^0$ and $d$. (1) is immediate since $0 \leq e'_i \leq d'_i$ for all $i \in [1, n + 1]$. The only possible critical pair is $(r - 1, r)$, so (3) also follows easily.
For (2), let \( i \to j \) be an arrow in \( Q^0 \). If \( i, j \in [1, n-1] \), then the fact that \( i \to j \) is an acceptable arrow in \( Q^0 \) implies the same thing is true of \( i \to j \) in \( Q^0 \).

If \( (i, j) = (n-1, n) \), then the fact that \( n-1 \to n \) and \( n-1 \to n+1 \) are acceptable arrows in \( Q^0 \) implies that \( e'_{n-1} - e'_n \leq 1 \) and \( e'_{n-1} - e'_{n+1} \leq 1 \). If \( e'_{n-1} = 2 \), the inequalities imply that \( e'_n = e'_{n+1} = 1 \). If \( e'_{n-1} = 1 \), then \( e'_n \geq 1 \) or \( e'_{n+1} \geq 1 \), since otherwise \( n-1 \to n \) and \( n-1 \to n+1 \) are both critical arrows in \( Q^0 \). In any case, it is clear that \( e'_{n-1} - e'_n - e'_{n+1} \leq 0 \), which means that \( n-1 \to n \) is an acceptable arrow in \( Q^0 \).

If \( (i, j) = (n, n-1) \), then the fact that \( n \to n-1 \) and \( n+1 \to n-1 \) are acceptable arrows in \( Q^0 \) means that \( e'_n - e'_{n-1} \leq 0 \) and \( e'_{n+1} - e'_{n-1} \leq 0 \). If \( e'_{n-1} = 0 \), then the inequalities imply that \( e'_{n+1} = e'_n = 0 \). If \( e'_{n-1} = 1 \), then \( e'_n + e'_{n+1} \leq 1 \), since \( (n-1, n) \) and \( (n-1, n+1) \) cannot both be critical pairs in \( Q^0 \). If \( e'_{n-1} = 2 \), then \( e'_n \leq 1 \) and \( e'_{n+1} \leq 1 \), so \( e'_n + e'_{n+1} - e'_{n-1} \leq 0 \). In any case, it holds that \( i \to j \) is an acceptable arrow in \( Q^0 \).

Let \( S \) be the subgraph of \( Q^0 \) induced by the set of vertices \( \{i \in [r, n] : e_i = 1\} \), and let \( S' \) be the subgraph of \( Q^0 \) induced by the set of vertices \( \{i \in [r, n-1] : e'_i = 1\} \). Note that \( S' \) is obtained from \( S \) by deleting the vertex \( n \) (if \( n \) is in \( S \)) and any edges that contain \( n \).

For condition (5), suppose that \( S \) consists of a single component containing \( n-1 \), and the arrow between \( r-1 \) and \( r \) is critical in \( Q^0 \). Then the arrow between \( r-1 \) and \( r \) is also critical in \( Q^0 \), and the equalities \( e_r = \cdots = e_{n-1} = 1 \) and \( e'_r = \cdots = e'_{n-1} = 1 \) hold. Assume for the sake of contradiction that \( e_n = 1 \). Then \( e'_n = 1 \) and \( e'_{n+1} = 0 \), or \( e'_n = 0 \) and \( e'_{n+1} = 1 \). In either case, it is easy to check that one of the arrows between \( n-1 \) and \( n \) or between \( n-1 \) and \( n+1 \) in \( Q^0 \) is critical with respect to \( (d', e') \), which contradicts (3) for \( F_{d'} \).

Now suppose that \( e = (e_1, \ldots, e_n) \in \mathbb{Z}^n \) satisfies conditions (1)-(3) and (5) in Theorem 4.16. We compute the coefficient of \( u_1^{e_1} \ldots u_n^{e_n} \) in \( \pi(F_{d'}^0) \).

If \( e_n = 0 \) or \( e_n = 0 \), then there is exactly one \( (e'_1, \ldots, e'_{n+1}) \in \mathbb{Z}^{n+1} \) such that \( u_1^{e'_1} \ldots u_{n+1}^{e'_{n+1}} \) occurs with nonzero coefficient in \( F_{d'}^0 \) and \( \pi(u_1^{e'_1} \ldots u_{n+1}^{e'_{n+1}}) = u_1^{e_1} \ldots u_n^{e_n} \). Note that \( (n-1, n) \) is not critical in \( Q^0 \), and \( (n-1, n), (n, n+1) \) are not critical in \( Q^0 \). Since \( (r-1, r) \) is critical in \( Q^0 \) if and only if it is critical in \( Q^0 \), and the components of \( S \) are the same as the components of \( S' \), the theorem follows in this case from the type \( D_n \) case.

Assume for the remainder of this proof that \( e_n = 1 \). Then there are two possible \( e' = (e'_1, \ldots, e'_{n+1}) \) such that \( u_1^{e'_1} \ldots u_{n+1}^{e'_{n+1}} \) occurs with nonzero coefficient in \( F_{d'}^0 \) and \( \pi(u_1^{e'_1} \ldots u_{n+1}^{e'_{n+1}}) = u_1^{e_1} \ldots u_n^{e_n} \). To be more precise, \( e'_i = e_i \) for \( i \in [1, n-1] \), and either \( e'_n = 1 \) and \( e'_{n+1} = 0 \), or \( e'_n = 0 \)
and $e_{n+1}' = 1$. Note that the coefficient of $u_{e_1} \cdots u_{e_{n+1}}'$ in $F_{d}^{e_0,t_0}$ for either $e'$ is equal to $2^c$, where $c$ is the number of components of $S'$, so we need to show that the number of components of $S'$ is $c + 1$.

Suppose that $n - 1 \to n$ in $Q^0$. Then the fact that this arrow is acceptable implies that $e_{n-1} = 0$ or $1$. If $e_{n-1} = 0$, then the arrows $n - 1 \to n$ and $n - 1 \to n + 1$ in $Q^0$ are not critical. If $e_{n-1} = 1$, then one of the arrows $n - 1 \to n$ or $n - 1 \to n + 1$ is critical in $Q^0$. In either case, the components $C$ of $S$ for which $\nu(C) = 0$ are the components of $S'$ for which $\nu(C) = 0$, plus an additional component containing the vertex $n$. By similar reasoning, the same statement may be proven when $n \to n - 1$ in $Q^0$. This finishes the proof of Theorem 4.16.

5. Quantum Cluster Algebras

In this section, we define quantum cluster algebras and state properties to be used later. See [1, Sections 3 and 4] for additional information.

Let $L$ be a lattice of rank $m$, with associated skew-symmetric bilinear form $\Lambda : L \times L \to \mathbb{Z}$. We also introduce a formal variable $q$ which commutes with all other elements, and consider the ring $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ of integer Laurent polynomials in the variable $q^{\frac{1}{2}}$.

**Definition 5.1.** [1, Definition 4.1] The based quantum torus associated with $L$ is the $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-algebra $\mathcal{T}$ with distinguished $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-basis $\{X^e : e \in L\}$ and multiplication given by

\[
X^e X^f = q^{\Lambda(e,f)/2} X^{e+f} \quad (e, f \in L).
\]

It is easy to check that $\mathcal{T}$ is associative. The basis elements satisfy the following commutation relations:

\[
X^e X^f = q^{\Lambda(e,f)} X^f X^e.
\]

In addition, we have

\[
X^0 = 1, \quad (X^e)^{-1} = X^{-e} \quad (e \in L).
\]

The based quantum torus $\mathcal{T}$ is an Ore domain, which means that it is contained in $\mathcal{F}$, its skew-field of fractions. The quantum cluster algebra to be defined below will be a $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-subalgebra of $\mathcal{F}$. 
Our first goal is to define the quantum analogue of a labeled seed. First, we must make some definitions.

**Definition 5.2.** [1, Definition 4.3] A *toric frame* in $\mathcal{F}$ is a mapping $M: \mathbb{Z}^m \to \mathcal{F} - \{0\}$ of the form
\begin{equation}
M(c) = \phi(X^\eta(c)),
\end{equation}
where $\phi$ is an automorphism of $\mathcal{F}$, and $\eta: \mathbb{Z}^m \to L$ is an isomorphism of lattices.

By definition, the elements $M(c)$ form a basis of $\phi(T)$, which is an isomorphic copy of $T$ in $\mathcal{F}$. The commutation relations of these elements are given by
\begin{align}
M(c)M(d) &= q^{\Lambda_M(c,d)/2}M(c + d), \\
M(c)M(d) &= q^{\Lambda_M(c,d)}M(d)M(c),
\end{align}
where the bilinear form $\Lambda_M$ on $\mathbb{Z}^m$ is obtained by transferring the form $\Lambda$ from $L$ via the lattice isomorphism $\eta$. We also have
\begin{equation}
M(0) = 1, \quad M(c)^{-1} = M(-c) \quad (c \in \mathbb{Z}^m).
\end{equation}

We use the same symbol $\Lambda_M$ to denote the skew-symmetric matrix whose entries are given by
\begin{equation}
\lambda_{ij} = \Lambda_M(e_i, e_j)
\end{equation}
where $e_1, \ldots, e_m$ are the standard basis vectors of $\mathbb{Z}^m$. For a given toric frame, write $X_i = M(e_i)$ for $i \in [1, m]$. The elements $X_i$ satisfy the quasi-commutation relations:
\begin{equation}
X_iX_j = q^{\lambda_{ij}}X_jX_i.
\end{equation}

Using (5.5) and (5.8), it follows that for any $c = (c_1, \ldots, c_m) \in \mathbb{Z}^m,$
\begin{equation}
M(c) = q^{\frac{1}{2} \sum_{k<l} c_k c_l \lambda_{kl}} X_1^{c_1} \ldots X_m^{c_m},
\end{equation}
so that the toric frame $M$ is uniquely determined by the elements $X_1, \ldots, X_m$.

**Definition 5.3.** [1, Definition 3.1] Let $\bar{B} = (b_{ij})$ be an $m \times n$ integer matrix, and let $\Lambda = (\lambda_{ij})$ be a skew-symmetric $m \times m$ integer matrix. We say that the pair $(\Lambda, \bar{B})$ is *compatible* if for every
\( j \in [1, n] \) and \( i \in [1, m] \), we have

\[
\sum_{k=1}^{m} b_{kj} \lambda_{ki} = \delta_{ij} d_j
\]

for some positive integers \( d_j \) \( (j \in [1, n]) \). In other words, the product \( \tilde{B}^T \Lambda \) is equal to the \( n \times m \) matrix \((D|0)\), where \( D \) is a \( n \times n \) diagonal matrix with positive integer diagonal entries \( d_1, \ldots, d_n \).

**Proposition 5.4.** [1, Proposition 3.3] If a pair \((\Lambda, \tilde{B})\) is compatible, then the principal part of \( \tilde{B} \) is skew-symmetrizable.

**Definition 5.5.** [1, Definition 4.5] A quantum seed is a pair \((M, \tilde{B})\), where

- \( M \) is a toric frame in \( \mathcal{F} \), and
- \( \tilde{B} \) is an \( m \times n \) integer matrix such that \((\Lambda_M, \tilde{B})\) is a compatible pair in the sense of Definition 5.3.

Suppose that in the classical limit (i.e., taking \( q = 1 \)), the elements \( X_1, \ldots, X_m \) specialize to \( x_1, \ldots, x_m \). Then these elements \( x_1, \ldots, x_m \) form a free generating set of the ambient field, so by Proposition 5.4, we have that \((\tilde{x} = (x_1, \ldots, x_m), \tilde{B})\) is a labeled seed.

The next goal is to define mutation for quantum seeds. First, we extend the definition of matrix mutations to compatible pairs. Let \( k \in [1, n] \), and pick a sign \( \epsilon \in \{\pm 1\} \). Denote by \( E_\epsilon \) the \( m \times m \) matrix with entries given by

\[
e_{ij} = \begin{cases} 
\delta_{ij} & \text{if } j \neq k \\
-1 & \text{if } i = j = k \\
\max(0, -\epsilon b_{ik}) & \text{if } i \neq j = k.
\end{cases}
\]

Set

\[
\Lambda' = E_\epsilon^T \Lambda E_\epsilon.
\]

Then \( \Lambda' \) is skew-symmetric. Furthermore, one may verify that \( \Lambda' \) is independent of sign \( \epsilon \) and that \((\Lambda', \mu_k(\tilde{B}))\) is a compatible pair (see [1, Proposition 3.4]).

**Definition 5.6.** [1, Definition 3.5] Let \((\Lambda, \tilde{B})\) be a compatible pair, and let \( k \in [1, n] \). We say that \((\Lambda', \mu_k(\tilde{B}))\) (with \( \Lambda' \) as above) is obtained by mutation in direction \( k \) from \((\Lambda, \tilde{B})\), and write \( \mu_k(\Lambda, \tilde{B}) = (\Lambda', \mu_k(\tilde{B})) \).
Write
\[
\binom{r}{p}_t = \frac{(t^r - t^{-r}) \cdots (t^{r-p+1} - t^{-r+p-1})}{(t^p - t^{-p}) \cdots (t - t^{-1})}
\]
for the \(t\)-binomial coefficient. The \(t\)-binomial coefficients satisfy the equation
\[
(5.15) \quad \prod_{p=0}^{r-1} (1 + t^r - 1 - 2p x) = \sum_{p=0}^{r} \binom{r}{p}_t x^p
\]

For mutations of toric frames, let \((M, \tilde{B})\) be a quantum seed, with \(\tilde{B} = (b_{ij})\). Fix an index \(k \in [1, n]\) and a sign \(\epsilon \in \{\pm 1\}\). Define a mapping \(M' : \mathbb{Z}^m \to \mathcal{F} - \{0\}\) by setting, for \(c = (c_1, \ldots, c_n) \in \mathbb{Z}^m\) such that \(c_k \geq 0\),
\[
(5.16) \quad M'(c) = \sum_{p=0}^{c_k} \binom{c_k}{p} M(E_\epsilon c + \epsilon p b^k), \quad M'(-c) = M'(c)^{-1},
\]
where \(b^k\) denotes the \(k\)th column of \(\tilde{B}\), and the matrix \(E_\epsilon\) is given at (5.12).

**Proposition 5.7.** [1, Proposition 4.7]

1. The mapping \(M'\) is a toric frame which does not depend on the choice of sign \(\epsilon\).
2. The pair \((\Lambda_{M'}, \mu_k(\tilde{B}))\) is compatible and is obtained from \((\Lambda_M, \tilde{B})\) by mutation in direction \(k\).
3. \((M', \mu_k(\tilde{B}))\) is a quantum seed.

This proposition justifies the next definition:

**Definition 5.8.** [1, Definition 4.8] Let \((M, \tilde{B})\) be a quantum seed, and let \(k \in [1, n]\). Suppose that \(M'\) is given at (5.16) and \(\tilde{B}' = \mu_k(\tilde{B})\). Then we say that the quantum seed \((M', \tilde{B}')\) is obtained from \((M, \tilde{B})\) by mutation in direction \(k\), and write \((M', \tilde{B}') = \mu_k(M, \tilde{B})\).

*Quantum cluster patterns* may be defined in a completely analogous way to cluster patterns by simply replacing the labeled seeds \((x_t, \tilde{B}_t)\) by the quantum seeds \((M_t, \tilde{B}_t')\) in Definition 2.6. In this case, we write \(X_{j:t} = M_t(x_j)\) for \(j \in [1, m]\), \(t \in \mathbb{T}_n\). The *cluster variables* are the elements \(X_{j:t}\) for \(j \in [1, n]\), \(t \in \mathbb{T}_n\). Observe that \(X_{j:t} = X_{j:t_0}\) for \(i \in [n + 1, m], \ j \in \mathbb{T}_n\); these are the \(m - n\) *coefficient variables* which do not depend on the seed \(t\). The *cluster* at the seed \(t\) is \((X_{1:t}, \ldots, X_{n:t})\), and the *extended cluster* is \((X_{1:t}, \ldots, X_{m:t})\).
The next proposition provides the analogue in quantum cluster algebras of the exchange relation given at (2.3).

**Proposition 5.9.** [1, Proposition 4.9] Let \((M, \tilde{B})\) be a quantum seed, and suppose the quantum seed \((M', \tilde{B}')\) is obtained from \((M, \tilde{B})\) by mutation in direction \(k \in [1, n]\). For \(i \in [1, n]\), set \(X_i = M(e_i), X'_i = M'(e_i)\). Then \(X'_i = X_i\) for \(i \neq k\), and

\[
X'_k = M(-e_k + \sum_{i \in [1,m]} [b_{ik}]_+ e_i) + M(-e_k + \sum_{i \in [1,m]} [-b_{ik}]_+ e_i).
\]

Finally, we are ready for the definition of quantum cluster algebra. Observe that one consequence of Proposition 5.9 is that in a given quantum cluster pattern, \(X_{j; t} = X_{j; t'}\) for any \(j \in [n+1, m]\), \(t, t' \in T_n\). Put \(X_j = X_{j; t}\) for \(j \in [n+1, m]\), and write \(X = \{X_{j; t} : j \in [1, n], t \in T_n\}\) for the collection of all cluster variables.

**Definition 5.10.** [1, Definition 4.12] For a given quantum cluster pattern \(t \mapsto (M_t, \tilde{B}_t)\), the associated quantum cluster algebra \(A\) is the \(\mathbb{Z}[-\frac{1}{2}, X_{n+1}^{\pm 1}, \ldots, X_m^{\pm 1}]\)-subalgebra of the ambient skew-field \(F\), generated by the elements of \(X\).

The next result will be referred to as the quantum Laurent phenomenon:

**Theorem 5.11.** [1, Corollary 5.2] The cluster algebra \(A\) as above is contained in \(\mathbb{Z}[-\frac{1}{2}, X_{n+1}^{\pm 1}, \ldots, X_m^{\pm 1}]\).

6. F-polynomials in Quantum Cluster Algebras

For this section, fix the following:

- a vertex \(t_0 \in T_n\);
- an \(n \times n\) skew-symmetrizable integer matrix \(B^0 = (b^0_{ij})\);
- an \(n \times n\) diagonal matrix \(D\) with positive integer entries \(d_1, \ldots, d_n\) on the diagonal satisfying the property that \(DB^0\) is skew-symmetric.

In this section and the next, we will assume all quantum cluster algebras \(A\) under consideration have the following properties. The extended exchange matrix at \(t_0\), which will be denoted by \(\tilde{B}^0\), has principal part \(B^0\). The \(m \times m\) skew-symmetric integer matrix which gives the quasi-commutation relations for the cluster variables and coefficients at \(t_0\) will be denoted by \(\Lambda_0\), and this matrix will be assumed to satisfy the following compatibility condition with \(\tilde{B}^0\):

\[
(\tilde{B}^0)^T \Lambda_0 = (D|0).
\]
For any such quantum cluster algebra $\mathcal{A}$, denote the ambient skew-field of the quantum cluster algebra $\mathcal{A}$ by $\mathcal{F}$. Let $\tilde{B}^t = (b_{ij}^t)$ denote the extended exchange matrix at the vertex $t$. Write $\tilde{b}^{jt} \in \mathbb{Z}^m$ for the $j$th column of $\tilde{B}^t$, and set $\tilde{b}^j = \tilde{b}^{j;0}$. Let $M_t$ denote the toric frame at the vertex $t \in T_n$, and put $M_0 := M_{t_0}$. Let $\Lambda_t$ denote the skew-symmetric bilinear form on $\mathbb{Z}^m$ corresponding to $M_t$. (We will also use $\Lambda_t$ for the $m \times m$ skew-symmetric integer matrix which gives the quasi-commutation relations in the cluster at $t$). Denote the cluster variables in the corresponding cluster by $X_j^t := M_t(e_j)$ for $1 \leq j \leq n$, and write $X_j = X_j^{t_0}$ for the cluster variables in the initial cluster. For $j \in [1,n]$, $t \in T_n$, write

$$\hat{Y}^t_{jt} = M_t(\tilde{b}^{jt}), \quad \hat{Y}^t_j = \hat{Y}^{t_0}_{jt}$$

From Remark 4.6 of [1], it is known that

$$\Lambda_t(\tilde{b}^i, \tilde{b}^j) = d_i b_t^{ij}.$$  \hspace{1cm} (6.3)

Equivalently, the elements $\hat{Y}^t_{jt}$ satisfy the quasi-commutation relation

$$\hat{Y}^{t'}_{i't'} \hat{Y}^t_{jt} = q^{d_i b_t^{ij}} \hat{Y}^{t'}_{j't'} \hat{Y}^{t'}_{i't'},$$  \hspace{1cm} (6.4)

Note that these quasi-commutation relations depend only on the entries of the matrices $D$ and $B^0$.

Let $R = R_{B^0,D}$ be the $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-algebra with generating set $\{Z_j : 1 \leq j \leq n\}$ such that the $Z_i$ satisfy the same quasi-commutation relations as the $\hat{Y}_i$:

$$Z_i Z_j = q^{d_i b^0_{ij}} Z_j Z_i.$$  \hspace{1cm} (6.5)

Then $R$ is a based quantum torus in the sense of Definition 5.1. Thus, we can consider the skew-field $\mathcal{F}(R)$ of right fractions of $R$:

$$\mathcal{F}(R) = \{FG^{-1} : F, G \in R\}.$$  \hspace{1cm} (6.6)

For $F \in \mathcal{F}(R)$, denote by $F(\hat{Y})$ the element of $\mathcal{F}$ obtained from $F$ by setting each $Z_i$ to $\hat{Y}_i$ for all $1 \leq i \leq n$. In analogy to the notation in (5.10), for $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$, define

$$Z^c = q^{\frac{1}{2} \sum_{1 \leq i < j \leq n} d_i b^0_{ij} c_i c_j} Z_1^{c_1} \cdots Z_n^{c_n}.$$  \hspace{1cm} (6.7)
**Proposition 6.1** (Example 0.5, [19]). Suppose \( \Lambda \) is an \( n \times n \) skew-symmetric integer matrix. The cluster algebra \( \mathcal{A}_\bullet = \mathcal{A}_\bullet(B^0, t_0) \) has a unique quantization such that the quasi-commutation relations of the cluster variables in the cluster at \( t_0 \) are given by \( \Lambda \). The unique \( 2n \times 2n \) matrix \( \Lambda_0 \) satisfying (6.1) is given by

\[
(6.8) \quad \Lambda_0 = \begin{pmatrix}
\Lambda & -\Lambda B^0 - D \\
-(B^0)^T \Lambda + D & (B^0)^T \Lambda B^0 + (B^0)^T D
\end{pmatrix}.
\]

**Definition 6.2.** We say that quantum cluster algebra defined in Proposition 6.1 has *principal coefficients*, and denote it by \( \mathcal{A}_\bullet = \mathcal{A}_\bullet(B^0, D, \Lambda, t_0) \).

**Theorem 6.3.** Let \( j \in [1, n] \), \( t \in \mathbb{T}_n \), \( \Lambda \) an \( n \times n \) skew-symmetric integer matrix, and put \( \mathcal{A}_\bullet = \mathcal{A}_\bullet(B^0, D, \Lambda, t_0) \). Let \( g_{j;t} = g_{j;t}^{B^0,t_0} \) be the \( g \)-vector (as in Proposition 3.5).

1. There exists a unique polynomial \( F_{j;t} = F_{j;t}^{B^0,D,t_0} \) in \( Z_1, \ldots, Z_n \) with coefficients in \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \)

such that the cluster variable \( X_{j;t} \in \mathcal{A}_\bullet \) is given by

\[
(6.9) \quad X_{j;t} = F_{j;t}(\hat{Y}) M_0(g_{j;t}),
\]

where \( M_0 \) is the toric frame at \( t_0 \) for \( \mathcal{A}_\bullet \), and \( \hat{Y}_1, \ldots, \hat{Y}_n \) are defined as at (6.2) using the columns of the principal matrix with respect to \( B^0 \). (Here, we consider \( g_{j;t} \) as an element in \( \mathbb{Z}^{2n} \) by appending \( n \) 0’s to the end of the vector.) The polynomial \( F_{j;t} \) does not depend on the choice of the matrix \( \Lambda \). We call this polynomial a (left) quantum \( F \)-polynomial.

2. Let \( \hat{B}^0 \) be an \( m \times n \) integer matrix with principal part \( B^0 \). Then there exists \( \lambda_{j;t}^{\hat{B}^0,t_0} = \lambda_{j;t} \in \frac{1}{2} \mathbb{Z} \) such that in any quantum cluster algebra \( \mathcal{A} \) whose initial exchange matrix is \( \hat{B}^0 \), we have that the cluster variable \( X_{j;t} \in \mathcal{A} \) is given by

\[
(6.10) \quad X_{j;t} = q^{\lambda_{j;t}} F_{j;t}(\hat{Y}) M_0(\hat{g}_{j;t}^{\hat{B}^0,t_0}),
\]

where \( M_0 \) is the toric frame at \( t_0 \) for \( \mathcal{A} \), and \( \hat{Y}_1, \ldots, \hat{Y}_n \) are defined using the columns of \( \hat{B}^0 \). (Note that \( \lambda_{j;t} = \lambda_{j;t}^{\hat{B}^0,t_0} \) depends on \( \hat{B}^0 \) but not on \( \mathcal{A} \) or the choice of initial quasi-commutation relations.)

3. Setting \( q = 1 \) and \( Z_i = u_i \) (\( i \in [1, n] \)) in \( F_{j;t} \) yields the (nonquantum) \( F \)-polynomial \( F_{j;t}^{B^0,t_0} \).

**Proof.** We start by proving the existence of \( F_{j;t} \in \mathcal{F}(R) \) which satisfy (6.10) so that \( \lambda_{j;t} = 0 \) if \( \hat{B}^0 \) is principal. Proceed by induction on the distance of \( t \) from the initial vertex \( t_0 \) in the tree \( \mathbb{T}_n \). For
1 \leq j \leq n$, $X_{j,t_0} = M_0(e_j)$ and $\tilde{g}_j^{\ell_0,t_0} = e_j$, so take $F_{j,t_0} = 1$ and $\lambda_{j,t_0} = 0$. Now suppose that for some vertex $t \in \mathbb{T}_n$, each $F_{j,t}$ has been defined satisfying (6.9) and (6.10). Let $t' \in \mathbb{T}_n$ such that $t \to k \to t'$. If $j \neq k$, then $X_{j,t'} = X_{j,t}$ and $\tilde{g}_j^{\ell_0,t_0} = \tilde{g}_j^{\ell_0,t_0}$, so put $F_{j,t'} = F_{j,t}$. Now consider the cluster variable $X_{k,t'}$. Using Proposition 5.9 and the fact that $[b]_+ = b + [-b]_+$ for any $b \in \mathbb{Z}$,

\begin{equation}
X_{k,t'} = M_t(-e_k + \sum_{i=1}^{m} [-b^i_{ik}]_+ e_i + \tilde{b}^k_t) + M_t(-e_k + \sum_{i=1}^{m} [-b^i_{ik}]_+ e_i)
\end{equation}

\begin{equation}
= q^{\frac{1}{2} d_k} M_t(\tilde{b}^k_t) M_t(-e_k + \sum_{i=1}^{m} [-b^i_{ik}]_+ e_i) + M_t(-e_k + \sum_{i=1}^{m} [-b^i_{ik}]_+ e_i)
\end{equation}

\begin{equation}
= (q^{\frac{1}{2} d_k} \tilde{Y}_{k,t} + 1) M_t(-e_k + \sum_{i=1}^{m} [-b^i_{ik}]_+ e_i).
\end{equation}

The second equality follows from the fact that for $j \in [1,n]$, $i \in [1,m]$

\begin{equation}
\Lambda_t(\tilde{b}^{j,t}, e_i) = \delta_{ij} d_j,
\end{equation}

which is a restatement of the compatibility condition between $\tilde{B}^t$ and $\Lambda_t$.

Some notation that will be used throughout this section and the next: let $G_1, \ldots, G_p$ be elements of a skew-field. Then define

\begin{equation}
\prod_{i \in [1,p]} G_i = G_1 \ldots G_p.
\end{equation}

The elements $G_1, \ldots, G_p$ are not necessarily commutative, so this product notation establishes a fixed order in which the elements are to be multiplied.

Let $\rho_1 \in \frac{1}{2} \mathbb{Z}$ be such that $M_t(-e_k + \sum_{i=1}^{m} [-b^i_{ik}]_+ e_i)$ is equal to

\begin{equation}
q^{\rho_1} M_t(-e_k) \left( \prod_{i \in [1,n]} M_t([-b^i_{ik}]_+ e_i) \right) M_0( \sum_{i=n+1}^{m} [-b^i_{ik}]_+ e_i)
\end{equation}

Thus,

\begin{equation}
X_{k,t'} = q^{\rho_1}(q^{\frac{1}{2} d_k} \tilde{Y}_{k,t} + 1) X_{k,t}^{-1} \left( \prod_{i \in [1,n]} X_{k,t}^{[-b^i_{ik}]_+} \right) \left( M_0( \sum_{i=n+1}^{m} [-b^i_{ik}]_+ e_i) \right).\end{equation}

Using the expressions given at (6.10) for cluster variables at the vertex $t$,
\begin{align}
X_{k,t'} &= q^{\rho_1 + \lambda'}(q^{\frac{1}{2}d_k} \hat{Y}_{k,t} + 1)(F_{k,t}(\hat{Y}) M_0(\tilde{B}^{t_0}))(\hat{Y})^{-1} \\
&\quad \times \left( \prod_{i \in [1,n]} (F_{i,t}(\hat{Y}) M_0(\tilde{B}^{t_0}))^{-b^i_{tk}} \right) M_0(\sum_{i=n+1}^{m} [-b^i_{tk} + e_i]),
\end{align}

where \( \lambda' \in \frac{1}{2} \mathbb{Z} \) is given by

\begin{equation}
\lambda' = -\lambda_k + \sum_{i=1}^{n} [-b^i_{tk} + \lambda_{i;t}].
\end{equation}

Observe that \( \lambda' \) does not depend on \( \mathcal{A} \), only on the choice of \( \tilde{B}^0 \), and that \( \lambda' = 0 \) if \( \tilde{B}^0 \) is principal.

By (6.14), the elements \( \tilde{Y}_j \) and \( X_i \) obey the following quasi-commutation relations for \( j \in [1,n], i \in [1,m] \):

\begin{equation}
\tilde{Y}_j X_i = q^{\delta_{ij} d_j X_i \tilde{Y}_j}.
\end{equation}

Observe that this quasi-commutation relation only depends on the entries of \( D \). Using (6.20), we can move all \( M_0(\tilde{B}^{t_0}) \) to the right in (6.18), from which it follows that

\begin{align}
X_{k,t'} &= q^{\rho_1 + \lambda'}(q^{\frac{1}{2}d_k} \hat{Y}_{k,t} + 1)P_{k,t'}(\hat{Y}) M_0(\tilde{B}^{t_0})(\hat{Y})^{-1} \\
&\quad \times \left( \prod_{i \in [1,n]} M_0(\tilde{B}^{t_0})^{-b^i_{tk}} \right) M_0(\sum_{i=n+1}^{m} [-b^i_{tk} + e_i]),
\end{align}

where \( P_{k,t'} \) is some element of \( \mathcal{F}(R) \). Observe that \( P_{k,t'} \) does not depend on \( \tilde{B}^0 \) or the choice of quantum cluster algebra \( \mathcal{A} \) for the following reasons: for each \( j \in [1,n], F_{j,t} \) does not depend on \( \mathcal{A} \) or \( \tilde{B}^0 \), by the inductive hypothesis; the first \( n \) coordinates of each \( \tilde{g}_{j,t}^{\tilde{B}^{t_0}} \) are equal to \( \tilde{g}_{j,t} \), which only depend on \( \tilde{B}^0 \) and \( D \), and the remaining \( m - n \) coordinates correspond to coefficient variables \( X_{n+1}, \ldots, X_m \), which commute with the \( \tilde{Y}_j \) elements; finally, the powers \( [-b^i_{tk}]_+ \) of the \( M_0(\tilde{B}^{t_0}) \) are in the principal part of \( \tilde{B}^t \), which is the same for every quantum cluster algebra presently under consideration.

Write
(6.22) \[ \hat{g}'_{k,t'} = -\hat{g}_{k,t} + \sum_{i=1}^{m} [-b_{ik} + \hat{g}_{i,t}] \]

and let \( \rho_2 \in \frac{1}{2} \mathbb{Z} \) such that

\[
M_0(\hat{g}^{'}_{k,t'}) = q^{\rho_2} M_0(-\hat{g}_{k,t}) \left( \prod_{i \in [1,n]} M_0(\hat{g}^{0}_{i,t}[-b'_{ik}]) \right) \times M_0(\sum_{i=n+1}^{m} [-b'_{ik} + e_i])
\]

(6.23) \]

Put \( \rho = \rho_1 - \rho_2 \). Then

\[
X_{k,t'} = q^{\rho + \lambda'} (q^{\frac{d_e}{2}} \hat{Y}_{k,t} + 1) P_{k,t'}(\hat{Y}) M_0(\hat{g}^{'}_{k,t'}). \tag{6.25}
\]

To bring this expression closer to (6.10), we must show that \( \hat{Y}_{k,t} \) can be expressed as a subtraction-free rational expression in \( \hat{Y}_1, \ldots, \hat{Y}_n \).

**Lemma 6.4.** Let \( t \xrightarrow{k} t' \) be vertices of \( \mathbb{T}_n \), let \( j \in [1,n] \), and write \( b = b'_{kj} = -b''_{kj} \). Then

\[
\hat{Y}_{j,t'} = \begin{cases} 
\hat{Y}_{j,t} \prod_{p=0}^{b-1} (1 + q^{-d_e p - \frac{d_e}{2}} \hat{Y}_{k,t}) & \text{if } b \leq 0, j \neq k \\
\hat{Y}_{j,t} \hat{Y}_{k,t} \prod_{p=0}^{b-1} (\hat{Y}_{k,t} + q^{-d_e p - \frac{d_e}{2}})^{-1} & \text{if } b > 0, j \neq k \\
\hat{Y}_{k,t} & \text{if } j = k
\end{cases}
\]

(6.26)

**Proof.**

**Case 1:** \( b \leq 0, j \neq k \).

Using (5.16),

\[
\hat{Y}_{j,t'} = \sum_{p=0}^{[b]} \binom{|b|}{p} q^{p/2} M_t(E_- \vec{b}^{j,t'} - p\vec{b}^{k,t}),
\]

(6.27)

where \( E_- = (e_{il}) \) is the \( m \times m \) matrix whose entries are given at (5.12). For each \( p = 1, \ldots, |b| \),

\[
E_- \vec{b}^{j,t'} = -b''_{kj} e_k + \sum_{i \in [1,m], i \neq k} (b''_{ij} + b''_{ik} [b''_{ik} + ] e_i
\]

(6.28)

\[ = b'_{kj} e_k + \sum_{i \in [1,m], i \neq k} (b'_{ij} + b'_{ik} [b'_{ik} + ] sgn(b) - b[b_{ik} + ] e_i
\]
Since \( b \leq 0 \), one can check that

\[
(6.29) \quad E_{-\bar{b}^{jt'}} = b_{kj}^t e_k + \sum_{i \in [1,m], i \neq k} (b_{ij}^t - b_{ik}^t) e_i = \bar{b}^{jt} - \bar{b}^{kt}.
\]

It follows that

\[
(6.30) \quad \hat{Y}_{j; t'} = \frac{\sum_{p=0}^{\vert b \vert} \binom{\vert b \vert}{p} q^{d_k/2} M_t(\bar{b}^{jt} - p\bar{b}^{kt})}{\sum_{p=0}^{\vert b \vert} \binom{\vert b \vert}{p} q^{d_k/2} M_t(\bar{b}^{jt} + p\bar{b}^{kt})} = \frac{\sum_{p=0}^{\vert b \vert} \binom{\vert b \vert}{p} q^{-d_k|b|p/2} M_t(\bar{b}^{jt}) M_t(\bar{b}^{kt})^p}{\sum_{p=0}^{\vert b \vert} \binom{\vert b \vert}{p} q^{d_k/2} M_t(\bar{b}^{jt})} = M_t(\bar{b}^{jt}) \prod_{p=0}^{\vert b \vert-1} \left[ 1 + (q^{d_k/2})(|b|-1-2p)(q^{-d_k|b|/2} M_t(\bar{b}^{kt})) \right]
\]

The last equality follows from the \( t \)-binomial formula. The proposition in this case follows after simplifying this last expression.

**Case 2: \( b \geq 0 \), \( j \neq k \).**

In this case, \( \hat{Y}_{j; t'} = M_t(-\bar{b}^{jt'})^{-1} \), and

\[
(6.31) \quad M_t(-\bar{b}^{jt'}) = \sum_{p=0}^{\vert b \vert} \binom{\vert b \vert}{p} q^{d_k/2} M_t(-E_{-\bar{b}^{jt'}} - p\bar{b}^{kt})
\]

Using the expression at \( 6.28 \),

\[
(6.32) \quad E_{-\bar{b}}^{jt'} = b_{kj}^t e_k + \sum_{i \in [1,m], i \neq k} b_{ij}^t e_i = \bar{b}^{jt}.
\]
Thus,

\[
\hat{Y}_{j;t'} = \left[ \sum_{p=0}^{\lfloor b \rfloor} \binom{\lfloor b \rfloor}{p} q^{dk/2} M_t(-\tilde{b};t - p\tilde{b};t) \right]^{-1}
\]

\[
= \sum_{p=0}^{\lfloor b \rfloor} \binom{\lfloor b \rfloor}{p} q^{dk/2} M_t(\tilde{b};t)^{-p} M_t(-\tilde{b};t)^{-1} \]

\[
= M_t(\tilde{b};t) \prod_{p=0}^{\lfloor b \rfloor-1} \left[ 1 + (q^{dk/2})^{\lfloor b \rfloor - 1 - 2p} (q^{dk/2} M_t(\tilde{b};t)^{-1}) \right]^{-1}
\]

where the last expression follows from the \(t\)-binomial formula (5.15). Additional simplification yields the proposition in this case.

**Case 3: \( j = k \)**

In this case, \( b = 0 \). Using (5.16) again,

\[
\hat{Y}_{k;t'} = M_t(E - \tilde{b};t) = M_t(-\tilde{b};t) = M_t(\tilde{b};t)^{-1}.
\]

\[
\square
\]

**Remark 6.5.** The recurrence relations for the \( \hat{Y}_{i;t} \) which appear in Lemma 6.4 are essentially the same as certain quantum mutation maps which occur in quantum spaces \( X_q \) as defined in [6] (see Lemma 3.4 in loc. cit.).

By Lemma 6.4, one may show that there exists a unique \( Y_{k;t} \in \mathcal{F}(R) \) such that \( Y_{k;t}(\tilde{Y}) = \tilde{Y}_{k;t} \); furthermore, \( Y_{k;t} \) can be expressed as a subtraction-free rational expression in \( Z_1, \ldots, Z_n \) depending only on \( B^0 \) and \( D \), and not on \( \tilde{B}^0 \) or \( A \). Set

\[
G_{k;t} = q^{1/2} Y_{k;t} + 1 \in \mathcal{F}(R).
\]

By (3.19),

\[
\tilde{g}_{k;t}^{0_{t_0}} = g_{k;t'}^{I} - \sum_{i=1}^{n} [b_{ik}] + \tilde{b}^i.
\]

Let \( \lambda'' \) be an element of \( \frac{1}{2} \mathbb{Z} \) satisfying
Thus, $\lambda''$ depends on $\tilde{\rho}$ in the choice of $\tilde{\rho}$ define $\lambda$. Note that the proof of the existence of $F \in \mathcal{F}(R)$ and Proposition 3.8, $\lambda''$ can be written explicitly as

\[
(6.37) \quad M_0(\tilde{g}_{k';t'}) = q^{\lambda''} M_0(-[-b_{nk}^{t}] + \tilde{b}^{t}_n) \ldots M_0(-[-b_{nk}^{t} + \tilde{b}^{1,t_0}] M_0(\tilde{g}_{k';t'}) = q^{\lambda''} \tilde{Y}_n \ldots \tilde{Y}_1 M_0(\tilde{g}_{k';t'})
\]

Using (6.3) and Proposition 3.8, \( \lambda'' \) can be written explicitly as

\[
(6.38) \quad \frac{1}{2} \Lambda_0 \left( \sum_{i=1}^{n} [-b_{ik}^{t} + \tilde{b}^{i}] \tilde{g}_{k';t'} + \cdots \right) + \frac{1}{2} \sum_{1 \leq i < j \leq n} \Lambda_0([-b_{ik}^{t} + \tilde{b}^{i}], [-b_{jk}^{t} + \tilde{b}^{j}]) = \frac{1}{2} \Lambda_0 \left( \sum_{i=1}^{n} [-b_{ik}^{t} + \tilde{b}^{i}], \tilde{g}_{k';t'} + \cdots \right) + \frac{1}{2} \sum_{1 \leq i < j \leq n} \left([-b_{ik}^{t} + \tilde{b}^{i}] + [-b_{jk}^{t} + \tilde{b}^{j}] \right)
\]

Thus, $\lambda''$ will only depend on $B^0$, $D$, not on $\tilde{B}^0$ or on $A$. Let $\rho_\bullet$ be the value of $\rho$ as obtained above when $\tilde{B}^0$ is the principal matrix corresponding to $B^0$. Let

\[
(6.39) \quad F_{k';t'} = q^{\alpha - \lambda''} \tilde{G}_{k';t'} P_{k';t'} \tilde{Z}_1 [-b_{nk}^{t}] \ldots \tilde{Z}_n [-b_{nk}^{t}],
\]

\[
(6.40) \quad \lambda_{k';t'} = \rho - \rho_\bullet + \lambda'.
\]

By (6.25),

\[
(6.41) \quad X_{k';t'} = q^{\rho + \lambda'} (q^{\frac{1}{2} \hat{d}_k} \tilde{Y}_{k',t} + 1) P_{k';t'} (\tilde{Y}) M_0(\tilde{g}_{k';t'}) = q^{\rho - \rho_\bullet + \lambda'} F_{k';t'} (\tilde{Y}) \tilde{Y}_n [-b_{nk}^{t}] \ldots \tilde{Y}_1 [-b_{nk}^{t}] M_0(\tilde{g}_{k';t'}) = q^{\rho - \rho_\bullet + \lambda'} F_{k';t'} (\tilde{Y}) M_0(\tilde{g}_{k';t'})
\]

The proof of the existence of $F_{j';t} \in \mathcal{F}(R)$ satisfying (6.9) and (6.10) now follows by induction.

Note that $\lambda_{k';t'} = 0$ if $\tilde{B}^0$ is the principal matrix corresponding to $B^0$. To check that $F_{k';t'}$ is independent of the choice of $\tilde{B}^0$ or $A$, and that $\lambda_{k';t'}$ depends on $\tilde{B}^0$ but not $A$, we must prove that $\rho$ depends on $\tilde{B}^0$, but not on the choice of quantum cluster algebra $A$. For $t \in \mathbb{T}_n$, $i, j \in [1, m]$, define

\[
(6.44) \quad \rho_{ij}^t (\tilde{B}^0) = \Lambda_t (e_i, e_j) - \Lambda_0 (\tilde{g}_{i,t}^{g^0_i,t_0}, \tilde{g}_{j,t}^{g^0_i,t_0}).
\]
With this notation and the definitions of $\rho_1, \rho_2$ given at (6.16), (6.23), it follows that

\begin{equation}
(6.45) \quad \rho = \rho_1 - \rho_2 \\
= -\frac{1}{2} \sum_{i=1}^{m} [-b_{ik}^t + \rho_{ik}^t(\tilde{B}^0)] + \frac{1}{2} \sum_{i=1}^{m} [-b_{ik}^t + -b_{jk}^t + \rho_{ij}^t(\tilde{B}^0)],
\end{equation}

where the last summation ranges over $j \in [1, n], i \in [1, m]$ such that $j < i$. Thus, the assertion about $\rho$ follows from the next lemma.

**Lemma 6.6.** Let $i, j \in [1, m]$, and let $\rho_{ij}^t = \rho_{ij}^t(\tilde{B}^0)$. The following properties hold:

\begin{align}
(6.46) & \quad \rho_{ij}^{t_0} = 0 \\
(6.47) & \quad \rho_{ij}^t = -\rho_{ji}^t \\
(6.48) & \quad \rho_{ii}^t = 0 \\
(6.49) & \quad \rho_{ij}^t = 0 \quad \text{if } i \in [n+1, m] \text{ or } j \in [n+1, m].
\end{align}

If $t \xrightarrow{k} t'$ in $\mathbb{T}_n$, then

\begin{align}
(6.50) & \quad \rho_{ij}^{t'} = \rho_{ij}^t \quad \text{if } i, j \neq k \\
(6.51) & \quad \rho_{ik}^{t'} = -\rho_{ik}^t + \sum_{\ell=1}^{m} [-b_{ik}^t + \rho_{i,\ell}^t] - \sum_{\ell=1}^{n} [-b_{i,n+1,k}^t + \rho_{i,\ell}^t] + g_{\ell}d_{\ell} \quad \text{if } i \neq k, i \in [1, n]
\end{align}

where $g_{i,t}^{B_0; t_0} = (g_1, \ldots, g_n)$.

Consequently, $\rho_{ij}^t(\tilde{B}^0)$ depends only on $\tilde{B}^0$, not on the choice of quantum cluster algebra $\mathcal{A}$.

**Proof.** Equations (6.46), (6.50) follow from (3.17), (3.18), respectively, while (6.47), (6.48) follow from the fact that $\Lambda_0$, $\Lambda_t$ are skew-symmetric. Also, (6.49) is clear when $i, j \in [n+1, m]$. Let $i \in [1, m], i \neq k$. From (5.13),

\begin{equation}
(6.52) \quad \Lambda_{t'}(e_i, e_k) = -\Lambda_{t}(e_i, e_k) + \sum_{\ell=1}^{m} [-b_{ik}^t] + \Lambda_{t}(e_i, e_{\ell})
\end{equation}
Using (3.19), \( A_0(\tilde{g}_{i;t'}, \tilde{g}_{k;t'}) \) equals

\[
(6.53) \quad A_0(\tilde{g}_{i;t}, \tilde{g}_{k;t'}) = \sum_{\ell=1}^{m} \left[ b_{\ell+1}^t + \tilde{g}_{\ell+1}^t - \sum_{\ell=1}^{n} \left[ \rho_{\ell+n, k}^t + \tilde{b}_{\ell+n, k}^t \right] \right] \\
- \sum_{\ell=1}^{n} \left[ \rho_{\ell+n, k}^t + \Lambda_0(\tilde{g}_{\ell+n, k}^t, \tilde{b}_{\ell+n, k}^t) \right] \\
= -\Lambda_0(\tilde{g}_{i;t}, \tilde{g}_{k;t'}) + \sum_{\ell=1}^{m} \left[ -b_{\ell+1}^t + \rho_{\ell+1}^t + \sum_{\ell=1}^{n} \left[ \rho_{\ell+n, k}^t + \Lambda_0(\tilde{g}_{\ell+n, k}^t, \tilde{b}_{\ell+n, k}^t) \right] \right] \\
- \sum_{\ell=1}^{n} \left[ -b_{\ell+n, k}^t + \Lambda_0(\tilde{g}_{\ell+n, k}^t, \tilde{b}_{\ell+n, k}^t) \right].
\]

First, consider the case where \( i \in [1, n] \). Then

\[
(6.54) \quad \rho_{ik}^t = -\rho_{ik}^t + \sum_{\ell=1}^{m} \left[ -b_{\ell+1}^t + \rho_{\ell+1}^t + \sum_{\ell=1}^{n} \left[ \rho_{\ell+n, k}^t + \Lambda_0(\tilde{g}_{\ell+n, k}^t, \tilde{b}_{\ell+n, k}^t) \right] \right] \\
- \sum_{\ell=1}^{n} \left[ -b_{\ell+n, k}^t + \Lambda_0(\tilde{g}_{\ell+n, k}^t, \tilde{b}_{\ell+n, k}^t) \right].
\]

By Proposition 3.8, for \( i \in [1, n] \), the first \( n \) coordinates of \( \tilde{g}_{i;t}^t \) are given by \( g_{i;t}^t \). By (6.14),

\[
(6.55) \quad \sum_{\ell=1}^{n} \left[ -b_{\ell+n, k}^t + \Lambda_0(\tilde{g}_{\ell+n, k}^t, \tilde{b}_{\ell+n, k}^t) \right] = -\sum_{\ell=1}^{n} \left[ -b_{\ell+n, k}^t + \Lambda_0(\tilde{g}_{\ell+n, k}^t, \tilde{b}_{\ell+n, k}^t) \right].
\]

This proves (6.51).

Now suppose that \( i \in [n + 1, m] \). Then

\[
(6.56) \quad \rho_{ik}^t = -\rho_{ik}^t + \sum_{\ell=1}^{m} \left[ -b_{\ell+1}^t + \rho_{\ell+1}^t \right] + \rho_{ik}^t.
\]

To prove that \( \rho_{ij}^t = 0 \) when \( i \) or \( j \) is in \([n + 1, m]\), use induction on the distance of \( t \) from \( t_0 \), with (6.46) as the base of the induction, and (6.47), (6.48), (6.49) (known when both \( i, j \in [n + 1, m] \)), (6.50), and (6.56) as recurrence relations.

Setting \( q = 1 \) in (6.9) yields

\[
(6.57) \quad x_{j;t} = F_{j;t}|_{q=1}(\hat{y}_1, \ldots, \hat{y}_n) x_1^{\eta_1} \ldots x_m^{\eta_m} \in A_*(B^0, t_0),
\]

where \( g_{j;t} = (g_1, \ldots, g_n) \). Comparing this with (3.14), it follows that

\[
(6.58) \quad F_{j;t}|_{q=1}(\hat{y}_1, \ldots, \hat{y}_n) = F_{j;t}^0(\hat{y}_1, \ldots, \hat{y}_n).
\]

Since \( \hat{y}_1, \ldots, \hat{y}_n \) are algebraically independent, part (3) of Theorem 6.3 follows.

For the uniqueness of \( F_{j;t} \), observe that if another \( F'_{j;t} \in R \) satisfies equation (6.9), then \( F_{j;t}(\hat{Y}) = F'_{j;t}(\hat{Y}) \). Since \( \hat{Y}_1, \ldots, \hat{Y}_n \) are algebraically independent, it follows that \( F_{j;t} = F'_{j;t} \).
We conclude the proof of part (1) of Theorem 6.3 by showing that $F_{j;t}$ is a polynomial in $Z_1, \ldots, Z_n$ with coefficients in $\mathbb{Z}[q^{\pm \frac{1}{2}}]$. The quantum Laurent phenomenon (Theorem 5.11) implies that each cluster variable $X_{j;t}$ in $A_*(B_0, D, \Lambda, t_0)$ can be expressed as a Laurent polynomial in $X_1, \ldots, X_{2n}$ with coefficients in $\mathbb{Z}[q^{\pm \frac{1}{2}}]$, which implies the same is true for $F_{j;t}(\hat{Y})$. Let $F_{j;t}(\hat{Y}) = P(X_1, \ldots, X_{2n})$ for some Laurent polynomial $P$ with coefficients in $\mathbb{Z}[q^{\pm \frac{1}{2}}]$, and write $F_{j;t} = A^{-1}C$ for some elements $A, C \in R$. Then

$$C(\hat{Y}) = A(\hat{Y})P(X_1, \ldots, X_{2n}).$$

If $F$ is a Laurent polynomial in $X_1, \ldots, X_{2n}$, then let the Newton polytope Newt($F$) of $F$ be the convex hull in $\mathbb{R}^n$ of the set

$$\{(a_1, \ldots, a_{2n}) \in \mathbb{Z}^{2n} : X_1^{a_1} \cdots X_{2n}^{a_{2n}} \text{ has nonzero coefficient in } F\}.$$

Taking the Newton polytope of both sides of (6.59),

$$\text{Newt}(C(\hat{Y})) = \text{Newt}(A(\hat{Y})) + \text{Newt}(P(X_1, \ldots, X_{2n})),
$$

where the sum above is a Minkowski sum of convex sets in $\mathbb{R}^n$. This implies that Newt($P(X_1, \ldots, X_{2n})$) is contained in the $\mathbb{Q}$-linear span of $\tilde{b}^1, \ldots, \tilde{b}^n$. Thus, each exponent vector of $P(X_1, \ldots, X_{2n})$ can be expressed as a $\mathbb{Q}$-linear combination of $\tilde{b}^1, \ldots, \tilde{b}^n$. However, the last $n$ coordinates of the vector $\tilde{b}^i$ is given by the vector $e_i \in \mathbb{Z}^n$, so each exponent vector must in fact be a $\mathbb{Z}$-linear combination of $\tilde{b}^1, \ldots, \tilde{b}^n$. This proves $P(X_1, \ldots, X_{2n}) = F_{j;t}(\hat{Y})$ is a Laurent polynomial in $\hat{Y}_1, \ldots, \hat{Y}_n$. Since $\hat{Y}_1, \ldots, \hat{Y}_n$ are algebraically independent, this implies $F_{j;t}$ is a Laurent polynomial in $Z_1, \ldots, Z_n$ with coefficients in $\mathbb{Z}[q^{\pm \frac{1}{2}}]$.

Now consider the Newton polytope $N(F_{j;t})$ of $F_{j;t}$ with respect to $Z_1, \ldots, Z_n$. That is, $N(F_{j;t})$ is the convex hull in $\mathbb{R}^n$ of the set

$$\{a \in \mathbb{Z}^n : Z^a \text{ has nonzero coefficient in } F_{j;t}\}.$$

Denote by $N(F)$ the Newton polytope of the polynomial $F(u_1, \ldots, u_n)$ with respect to $u_1, \ldots, u_n$, which is defined in a analogous way $N(F_{j;t})$. That is, $N(F)$ is the convex hull of the set

$$\{(a_1, \ldots, a_n) \in \mathbb{Z}^n : u_1^{a_1} \cdots u_n^{a_n} \text{ has nonzero coefficient in } F\}.$$
By following the proof of the existence of $F_{j:t}$, it is easy to see that $F_{j:t}$ is obtained from $Z_1, \ldots, Z_n$ via a series of subtraction-free rational transformations. This means that for each vertex $c \in N(F_{j:t})$, the monomial in $F_{j:t}$ with exponent vector $c$ will have a coefficient that can be expressed as a subtraction-free rational expression in $\mathbb{Z}[q^{\pm \frac{1}{2}}]$. Thus, setting $q = 1$ will not shrink the Newton polytope. Consequently,

$$N(F_{j:t}) = N(F_{j:t}^{B_0:t_0}),$$

(6.64)

where $F_{j:t}^{B_0:t_0}$ is the nonquantum $F$-polynomial. Since $F_{j:t}$ is a polynomial in $u_1, \ldots, u_n$, the polytope $N(F_{j:t}^{B_0:t_0})$ does not contain any points with negative coordinates. The same is true of $N(F_{j:t})$, forcing $F_{j:t}$ to be a polynomial in $Z_1, \ldots, Z_n$.

\[ \square \]

**Example 6.7.** For all $j \in [1,n]$, $F_{j:t_0} = 1$. If $t_0 \xleftarrow{k} t$ in $T_n$, then the $g$-vector recurrences (Proposition 3.9) imply that

$$g_{k:t} = -e_k + \sum_{i=1}^{n} (-b_{ik}^0) + e_i.$$

(6.65)

From equation (6.13), it follows that

$$F_{k:t} = q^{d_k/2}Z_k + 1.$$  

(6.66)

For $j \neq k$, $F_{j:t} = 1$.

### 7. Properties of Quantum $F$-polynomials

We continue to use the same notation as in the previous section. Part (2) of Theorem 6.3 may be strengthened under a certain condition which is conjectured to be true in general.

**Theorem 7.1.** If the nonquantum $F$-polynomial $F_{j:t}^{B_0:t_0}$ has nonzero constant term for all $j \in [1,n]$, $t \in T_n$, then for any cluster variable $X_{j:t} \in A$,

$$X_{j:t} = F_{j:t}(\hat{Y})M_0(\tilde{g}_{j:t}^{B_0:t_0}).$$

(7.1)

**Proof.** We need to show that $\lambda_{j:t} = 0$ (see (6.10)). Proceed by induction on the distance of the vertex $t$ from $t_0$ in $T_n$, with $\lambda_{j:t_0} = 0$ already established in the proof of Theorem 6.3. Let $t \xleftarrow{k} t'$ in $T_n$, and suppose that $\lambda_{j:t} = 0$ for all $j \in [1,n]$. From the proof of Theorem 6.3, $\lambda_{j:t'} = \lambda_{j:t} = 0$
for \( j \neq k \). Note that \( \lambda_{k:t'} \) is given by equations (6.19) and (6.40). To show that \( \lambda_{k:t'} = 0 \), it suffices to prove that \( \rho = \rho_0 = 0 \). The proof follows by induction after the next lemma is proven.

\[ \square \]

**Lemma 7.2.** Suppose that for all \( j \in [1, n], t \in T_n \), the nonquantum \( F \)-polynomial \( F_{j:t}^{\hat{B}^0,t} \) has nonzero constant term. Then \( \Lambda_t(e_i, e_j) = \Lambda_0(g_{i;t}, g_{j;t}) \), and consequently, \( \rho_{ij}(\hat{B}^0) = 0 \) for all \( i, j \in [1, m] \).

**Proof.** First, assume that \( \hat{B}^0 \) is principal. Let

\[
\begin{align*}
(7.2) & \quad \lambda = \Lambda_t(e_i, e_j) \\
(7.3) & \quad \lambda' = \Lambda_0(g_{i;t}, g_{j;t})
\end{align*}
\]

Use the convention that \( F_{\ell,t} = 1 \) if \( \ell \in [n + 1, m] \). The expressions for \( X_{i,t}, X_{j,t} \) from (6.9) imply that

\[
(7.4) 
F_{i,t}(\hat{Y})M_0(g_{i;t})F_{j,t}(\hat{Y})M_0(g_{j;t}) = q^{\lambda}F_{j,t}(\hat{Y})M_0(g_{j;t})F_{i,t}(\hat{Y})M_0(g_{i;t}).
\]

For some \( P_i, P_j \in F(R) \),

\[
(7.5) 
F_{i,t}(\hat{Y})P_j(\hat{Y})M_0(g_{i;t})M_0(g_{j;t}) = q^{\lambda}F_{j,t}(\hat{Y})P_i(\hat{Y})M_0(g_{j;t})M_0(g_{i;t}).
\]

Since \( M_0(g_{i;t})M_0(g_{j;t}) = q^{\lambda'}M_0(g_{j;t})M_0(g_{i;t}) \), it follows that

\[
(7.6) 
q^{\lambda'}F_{i,t}(\hat{Y})P_j(\hat{Y}) = q^{\lambda}F_{j,t}(\hat{Y})P_i(\hat{Y}).
\]

Since \( \hat{B}^0 \) has full rank, \( \hat{Y}_1, \ldots, \hat{Y}_n \) are algebraically independent. Thus,

\[
(7.7) 
q^{\lambda'}F_{i,t}P_j = q^{\lambda}F_{j,t}P_i.
\]

By (6.64), the fact that \( F_{i,t}^{\hat{B}^0,t} \) has nonzero constant term implies that same thing about \( F_{i,t}^{B_0,D;\ell} \).

Observe that \( F_{j,t} \) (resp. \( F_{i,t} \)) has the same constant term as \( P_j \) (resp. \( P_i \)). Considering the constant term of both sides of (7.7), we conclude that \( \lambda = \lambda' \). As a consequence, (6.51) implies that for any \( t \in T_n, i, k \in [1, n], i \neq k \),

\[
(7.8) 
\sum_{\ell=1}^{n} [-b_{\ell+n,k}] g_{j} \epsilon_d = 0,
\]

where \( g_{i;t}^{B_0,t_0} = (g_1, \ldots, g_n) \).
To prove that $\rho^t_{ij}(\tilde{B}^0) = 0$ in the general case where $\tilde{B}^0$ is any exchange matrix with principal part $B^0$, apply induction on the distance of $t$ from $t_0$ in $T_n$, and use Lemma 6.6 and (7.8). □

**Remark 7.3.** It was conjectured that nonquantum $F$-polynomials always have constant term 1 in [10, Conjecture 5.4]. This conjecture was proven in [4] for the case when $B^0$ is skew-symmetric, and also in [11] in the situation where the cluster algebra admits a certain categorification.

Clearly, one can prove the existence and uniqueness of analogous “right” quantum $F$-polynomials by similar arguments as given in the proof of Theorem 6.3. However, the next proposition demonstrates that these polynomials can easily be computed once the “left” $F$-polynomials are known.

Following [1, Section 6], define the $\mathbb{Z}$-linear *bar-involution* $X \mapsto \bar{X}$ on a quantum cluster algebra $A$ with initial quantum seed $(M_0, \tilde{B}^0)$ by setting

$$q^{r/2}M_0(c) = q^{-r/2}M_0(c) \quad (r \in \mathbb{Z}, c \in \mathbb{Z}^m) \quad (7.9)$$

We may also define a $\mathbb{Z}$-linear *bar-involution* $X \mapsto \bar{X}$ on $R_{B^0,D}$ by

$$q^{r/2}Z^c = q^{-r/2}Z^c \quad (r \in \mathbb{Z}, c \in \mathbb{Z}^n) \quad (7.10)$$

For $X, Y \in A$ (resp. $X, Y \in R_{B^0,D}$), it’s easy to verify that $\bar{XY} = \bar{Y} \bar{X}$.

**Proposition 7.4.** Let $j \in [1,n]$, $t \in T_n$. Then $F_{j,t}$ is the unique polynomial in $Z_1, \ldots, Z_n$ with coefficients in $\mathbb{Z}[q^{\pm 1/2}]$ such that $X_{j,t} \in A_*(B^0, D, \Lambda, t_0)$ is given by

$$X_{j,t} = M_0(g_{j,t})\bar{F}_{j,t}(\hat{Y}). \quad (7.11)$$

Furthermore, for any quantum cluster algebra $A$ whose initial exchange matrix is $\tilde{B}^0$, the cluster variable $X_{j,t} \in A$ is given by

$$X_{j,t} = q^{-\Lambda_{j,t}}M_0(\bar{g}_{j,t})\bar{F}_{j,t}(\hat{Y}). \quad (7.12)$$

Setting $q = 1$ and $Z_i = u_i$ in $F_{j,t}$ yields the nonquantum $F$-polynomial $F_{j,t}^{B^0,t_0}$.

**Proof.** By [1, Proposition 6.2], if $(M, \tilde{B})$ is some quantum seed associated with $A$, then $M(c)$ is invariant under the bar-involution for any $c \in \mathbb{Z}^{2n}$. In particular, cluster variables are invariant
under the bar-involution. Now apply the bar-involution to both sides of (6.10):

\[ (7.13) \quad X_{j;t} = q^{-\lambda_{j;t}} M_0(\tilde{g}^0_{j;t})F_{j;t}(\hat{Y}) = q^{-\lambda_{j;t}} M_0(\tilde{g}^0_{j;t})\tilde{F}_{j;t}(\hat{Y}). \]

Equation (7.11) follows from the fact that \( \lambda_{j;t}^{B^0} = 0 \) and \( \tilde{g}^0_{j;t} = g_{j,t} \) when \( B^0 \) is principal. Applying the bar-involution to \( F_{j;t} \) merely multiplies the coefficient of each monomial \( Z^a \) in \( F_{j;t} \) by \( q^c \) for some \( c \in \frac{1}{2}\mathbb{Z} \), so the last assertion is proven. \( \square \)

For \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n), c = (c_1, \ldots, c_n) \in \mathbb{Z}^n \), define the notation

\[ (7.14) \quad a \cdot b \cdot c = \sum_{i=1}^{n} a_ib_ic_i \in \mathbb{Z}. \]

We obtain the following result relating the \( g \)-vector \( g_{j;t} \) and the coefficients of \( F_{j;t} \).

**Corollary 7.5.** Let \( j \in [1, n], t \in \mathbb{T}_n \). For \( a \in \mathbb{Z}^n \), let \( P_a \in \mathbb{Z}[x, x^{-1}] \) such that \( P_a(q^{\frac{1}{2}}) \) is the coefficient of \( Z^a \) in the quantum \( F \)-polynomial \( F_{j;t} \). Also, let \( d = (d_1, \ldots, d_n) \).

Then

\[ (7.15) \quad P_a(q^{\frac{1}{2}}) = q^{-g_{j;t} \cdot a \cdot d} P_a(q^{\frac{1}{2}}). \]

In particular, if \( P_a(q) = q^c \) for some \( c \in \frac{1}{2}\mathbb{Z} \), then \( c = -\frac{g_{j;t} \cdot a \cdot d}{2} \).

**Proof.** Using (7.11) with (6.20),

\[ (7.16) \quad M_0(g_{j;t})\tilde{F}_{j;t}(\hat{Y}) = \sum_{a \in \mathbb{Z}^n} M_0(g_{j;t})P_a(q^{-\frac{1}{2}})Z^a \]

\[ (7.17) \quad = \left( \sum_{a \in \mathbb{Z}^n} q^{-g_{j;t} \cdot a \cdot d} P_a(q^{-\frac{1}{2}})Z^a \right) M_0(g_{j;t}). \]

The uniqueness of the quantum \( F \)-polynomial implies that

\[ (7.18) \quad F_{j;t} = \sum_{a \in \mathbb{Z}^n} q^{-g_{j;t} \cdot a \cdot d} P_a(q^{-\frac{1}{2}})Z^a. \]

The coefficient of \( Z^a \) in this expression is equal to \( q^{-g_{j;t} \cdot a \cdot d} P_a(q^{-\frac{1}{2}}) \). \( \square \)

The last result of this section is a recurrence relation for the quantum \( F \)-polynomials. For the remainder of this section, assume that \( A_* = A_*(B^0, D, \Lambda, t_0) \) for some \( n \times n \) skew-symmetric integer matrix \( \Lambda \). Let \( \mathcal{R} = \mathbb{Z}[q^{\frac{1}{2}}, Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}] \). For \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \), define a \( \mathbb{Z}[q^{\frac{1}{2}}] \)-linear
operator \( L[a] : \mathcal{R} \to \mathcal{R} \) by setting

\[
L[a](Z^b) = q^{-a \cdot b}d Z^b,
\]

for \( b = (b_1, \ldots, b_n) \in \mathbb{Z}^n \).

The next lemma gives some basic properties of the operator \( L[a] \).

**Lemma 7.6.** Let \( a, c \in \mathbb{Z}^n \), \( F, G \in \mathcal{R} \). Then

1. \( F' = L[a](F) \) is the unique element of \( \mathcal{R} \) such that
   \[
   \text{M}_0(a)F(\hat{Y}) = F'(\hat{Y})\text{M}_0(a).
   \]

2. \( L[a + c] = L[a] \circ L[c] = L[c] \circ L[a] \)

3. \( L[a](FG) = L[a](F) \cdot L[a](G) \)

**Proof.** For (1), use (6.20) to show first that (7.20) holds when \( F = Z^b \), then extend linearly. Uniqueness follows from the fact that \( \hat{Y}_1, \ldots, \hat{Y}_n \) are algebraically independent when \( \tilde{B}^0 \) is principal. Statement (2) is a straightforward check. For (3), let \( H = L[a](FG) \), \( F' = L[a](F) \), and \( G' = L[a](G) \). Applying part (1) two different ways gives

\[
H(\hat{Y})\text{M}_0(a) = \text{M}_0(a)F(\hat{Y})G(\hat{Y}) = F'(\hat{Y})G'(\hat{Y})\text{M}_0(a).
\]

Thus, \( H(\hat{Y}) = F'(\hat{Y})G'(\hat{Y}) \), which implies (3). \( \square \)

Suppose \( t \xrightarrow{k} t' \) in \( \mathbb{T}_n \). Write \( \tilde{B}' = (b_{ij}) \) for the exchange matrix at \( t \). For \( j \in [1, n], j \neq k, \epsilon \in \{+,-\} \), let

\[
G_j^{(\epsilon)} = \prod_{i \in [1,|eb_{jk}|]} L[(i - 1)\mathbf{g}_{j;\epsilon}](F_{j;\epsilon})
\]

if \( \epsilon b_{jk} \geq 1; G_j^{(\epsilon)} = 1 \) otherwise. Next, for \( j \neq k \), let

\[
\hat{F}_{j;t}^{\{[eb_{jk}]^+\}} = L \left[ -\mathbf{g}_{k;t} + \sum_{i \in [1,j-1]} [eb_{ik}]^+ \mathbf{g}_{i;t} \right] (G_j^{(\epsilon)}).
\]

Also, let

\[
\hat{F}_{k;t}^{(-1)} = (L[-\mathbf{g}_{k;t}(F_{k;t}))^{-1}.
\]
For $\epsilon \in \{+, -, \}$, let $\rho^\epsilon$ equal

$$\frac{1}{2} \left( - \sum_{i=1}^{2n} [eb^\epsilon_{ik} \rho^\epsilon_{ik} + \sum_{1 \leq i < j \leq n} [eb^\epsilon_{jk} + \rho^\epsilon_{ij} + \sum_{1 \leq i, j \leq n} [eb^\epsilon_{ik} + \rho^\epsilon_{jn} + \rho^\epsilon_{ji} \right).$$

Finally, let

$$\lambda^\epsilon = \frac{1}{2} \sum_{i=1}^{n} [eb^\epsilon_{n+i, k}] + g'_i d_i, \quad (7.25)$$

where $g_{k:t'} = (g'_1, \ldots, g'_n)$.

**Theorem 7.7.** The quantum $F$-polynomials $F_{j:t}$ are given by the following recurrence relations:

The initial quantum $F$-polynomials are given by

$$F_{j:t_0} = 1. \quad (7.26)$$

Suppose $t \longrightarrow k \longrightarrow t'$ in $\mathbb{T}_n$. Then

$$F_{j:t'} = F_{j:t} \text{ if } j \neq k. \quad (7.27)$$

Using the notation given above, the quantum $F$-polynomial $F_{k:t'}$ is

$$F_{k:t'} = q^{(\rho^+ - \lambda^+)} \hat{F}^{\{1\}}_{k:t} \left( \prod_{i \in [1,n]} \hat{F}^{[b_{ik}]_+}_{i:t} \right) Z^{(\sum_{\ell=1}^{n} [b_{n+E_k} + \epsilon_t)}}$$

$$+ q^{(\rho^- - \lambda^-)} \hat{F}^{\{-1\}}_{k:t} \left( \prod_{i \in [1,n]} \hat{F}^{[-b_{ik}]_+}_{i:t} \right) Z^{(\sum_{\ell=1}^{n} [-b_{n+E_k} + \epsilon_t)}}. \quad (7.28)$$

**Remark 7.8.**

1. By Lemma 7.2, if the $F$-polynomial $F_{j:t_0}^{B^0,t_0}$ has nonzero constant term for every $j \in [1, n], t \in \mathbb{T}_n$, then $\rho^\epsilon = 0$.

2. Setting $q = 1$ and $Z_i = u_i$ in $\hat{F}^{(r)}_{j:t}$ yields $(F_{j:t}^{B^0,t_0})^*$; under this specialization, the quantum $F$-polynomial recurrence above becomes the $F$-polynomial recurrence given in Proposition 3.4.
Proof. (7.26) and (7.27) are already known from the proof of Theorem 6.3. Let \( t^k t' \) in \( \mathbb{T}_n \).

The cluster variable \( X_{k; t'} \in \mathcal{A}_n \) can be expressed as

\[
X_{k; t'} = \sum_{\epsilon \in \{+, -\}} M_t(-e_k + \sum_{i=1}^{2n}[\epsilon b_{ik}] + e_i).
\]

(7.29)

For \( \epsilon \in \{+, -\} \), let \( \rho_1^\epsilon \in \frac{1}{2} \mathbb{Z} \) such that

\[
M_t(-e_k + \sum_{i=1}^{2n}[\epsilon b_{ik}] + e_i)
= q^{\rho_1^\epsilon} M_t(-e_k) \left( \prod_{i \in [1, n]} M_t([\epsilon b_{ik}] + e_i) \right) M_0(\sum_{i=1}^{n}[\epsilon b_{n+i,k}] + e_{n+i}).
\]

(7.30)

Now \( X_{k; t'} \) may be rewritten as

\[
\sum_{\epsilon \in \{+, -\}} q^{\rho_1^\epsilon} M_t(-e_k) \left( \prod_{i \in [1, n]} M_t([\epsilon b_{ik}] + e_i) \right) M_0(\sum_{i=1}^{n}[\epsilon b_{n+i,k}] + e_{n+i}).
\]

(7.31)

Using (6.9), \( X_{k; t'} \) can be further rewritten as

\[
\sum_{\epsilon \in \{+, -\}} q^{\rho_1^\epsilon} \left( F_{k; t}(\hat{Y}) M_0(g_{k; t}) \right)^{-1} \left( \prod_{i \in [1, n]} \left( F_{i; t}(\hat{Y}) M_0(g_{i; t}) \right)^{[\epsilon b_{ik}]} \right) M_0(\sum_{i=1}^{n}[\epsilon b_{n+i,k}] + e_{n+i}).
\]

Let

\[
g_{k; t}^{(\epsilon)} = -g_{k; t} + \sum_{i=1}^{n}[\epsilon b_{ik}] + g_{i; t} + \sum_{i=1}^{n}[\epsilon b_{n+i,k}] + e_{i+n}.
\]

(7.32)

Pushing all the quantum \( F \)-polynomials to the left in the above expression for \( X_{k; t'} \), the cluster variable may further be rewritten as

\[
X_{k; t'} = \sum_{\epsilon \in \{+, -\}} q^{\rho_1^\epsilon} \frac{1}{\hat{F}_{k; t}^{(-1)}(\hat{Y})} \left( \prod_{i \in [1, n]} \hat{F}_{i; t}^{[\epsilon b_{ik}]}(\hat{Y}) \right) M_0(g_{k; t}^{(\epsilon)}).
\]

(7.33)

From the recurrence relation for \( g \)-vectors (Proposition 3.9), it follows that

\[
- g_{k; t'} + g_{k; t}^{(\epsilon)} = \sum_{i=1}^{n}[\epsilon b_{n+i,k}] + b^i \quad (\epsilon \in \{+, -\})
\]

(7.34)
where $\tilde{b}^i$ is the $i$th column of $\tilde{B}^0$, the principal matrix with respect to $B^0$. For $\epsilon \in \{+, -\}$, observe that $\lambda^\epsilon \in \frac{1}{2} \mathbb{Z}$ satisfies

\begin{equation}
\lambda^\epsilon = \frac{1}{2} \Lambda_0(-g_{k;t'}, \sum_{i=1}^n [\epsilon b_{n+i,k}]_+ \tilde{b}^i + g_{k;t'}),
\end{equation}

since $\Lambda_0(e_j, \tilde{b}^i) = -\delta_{ij} d_i$ for $i \in [1, 2n]$, $j \in [1, n]$. This means that

\begin{equation}
M_0(-g_{k;t'} + g_{k;t'}^{(\epsilon)}) = q^{\lambda^\epsilon} M_0(g_{k;t'}^{(\epsilon)}) M_0(-g_{k;t'}).
\end{equation}

Consequently,

\begin{equation}
F_{k;t'}(\hat{Y}) = X_{k;t'} M_0(-g_{k;t'})
= \sum_{\epsilon \in \{+, -\}} q^{(\rho - \lambda^\epsilon)} \hat{F}_{k;t'}^{(-1)}(\hat{Y}) \left( \prod_{i \in [1, n]} \hat{F}_{i;t'}^{[\epsilon b_{n+i,k}]_+} (\hat{Y}) \right) M_0(\sum_{i=1}^n [\epsilon b_{n+i,k}]_+ \tilde{b}^i).
\end{equation}

The theorem follows from the fact that $\hat{Y}_1, \ldots, \hat{Y}_n$ are algebraically independent. \hfill \Box

**Example 7.9** (Type $A_2$; cf. [8, Section 6]). For $n = 2$, the tree $T_2$ is an infinite chain. We call the vertices $\ldots, t_{-1}, t_0, t_1, t_2, \ldots$, and label the edges as follows:

\begin{equation}
\begin{array}{cccccccc}
& & & & & 2 & \rightarrow & & \\
& & & & & 1 & \rightarrow & & \\
& & & & & t_{-1} & \rightarrow & t_0 & \rightarrow \\
& & & & & t_1 & \rightarrow & t_2 & \rightarrow \\
& & & & & t_2 & \rightarrow & t_3 & \rightarrow \\
& & & & & & & & \\
\end{array}
\end{equation}

Let $B^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ be the initial $n \times n$ exchange matrix, and $D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Let $\tilde{B}^0$ be the principal matrix corresponding to $B^0$. Then we may use matrix mutation (equation (2.2)) and the $g$-vector recurrences (Proposition 3.9) to compute $\tilde{B}^t$ and the $g$-vectors $g_{j;t} = g_{j;t}^{B_0^0;0}$. Let $\mathbf{F}^0(t_0)$ and $\mathbf{F}^{(\epsilon)}(t_1)$ be computed using Example 6.7. For the remaining quantum $F$-polynomials, we use recurrence relations given in Theorem 7.7.

To compute $F_{1;t_2}$, let $t = t_1$, $t' = t_2$, $k = 1$. In this case, $\hat{F}_{1;t_1}^{(-1)} = 1$, $\hat{F}_{2;t_1}^{(1)} = qZ_2 + 1$, and the recurrence (7.28) becomes

$$F_{1;t_2} = q \hat{F}_{2;t_1}^{(1)} Z_1 + 1 = q(qZ_2 + 1)Z_1 + 1 = qZ^{e_1+e_2} + qZ_1 + 1.$$
Table 1. Type $A_2$, quantum $F$-polynomials

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\tilde{B}^t$</th>
<th>$g_1; t$</th>
<th>$g_2; t$</th>
<th>$F_{1; t}$</th>
<th>$F_{2; t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\begin{bmatrix} 0 &amp; 1 \ -1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$e_1$</td>
<td>$e_2$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\begin{bmatrix} 0 &amp; -1 \ 1 &amp; 0 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>$e_1$</td>
<td>$-e_2$</td>
<td>1</td>
<td>$qZ_2 + 1$</td>
</tr>
<tr>
<td>2</td>
<td>$\begin{bmatrix} 0 &amp; 1 \ -1 &amp; 0 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>$-e_1$</td>
<td>$-e_2$</td>
<td>$qZ^{e_1+e_2} + qZ_1 + 1$</td>
<td>$qZ_2 + 1$</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{bmatrix} 0 &amp; -1 \ -1 &amp; 0 \ 1 &amp; 1 \end{bmatrix}$</td>
<td>$-e_1$</td>
<td>$e_2 - e_1$</td>
<td>$qZ^{e_1+e_2} + qZ_1 + 1$</td>
<td>$qZ_1 + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$\begin{bmatrix} 0 &amp; 1 \ -1 &amp; 0 \ 1 &amp; 0 \end{bmatrix}$</td>
<td>$e_2$</td>
<td>$e_2 - e_1$</td>
<td>1</td>
<td>$qZ_1 + 1$</td>
</tr>
<tr>
<td>5</td>
<td>$\begin{bmatrix} 0 &amp; -1 \ 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$e_2$</td>
<td>$e_1$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

To compute $F_{2; t_3}$, let $t = t_2, t' = t_3, k = 2$. Then $\hat{F}_{2; t_2}^{(-1)} = (q^{-1}Z_2 + 1)^{-1}, \hat{F}_{1; t_2}^{(1)} = q^{-1}Z^{e_1+e_2} + qZ_1 + 1$.

The recurrence (7.28) in this case yields

$$F_{2; t_3} = \hat{F}_{2; t_2}^{(-1)}(\hat{F}_{1; t_2}^{(1)} + q^{-1}Z_2) = qZ_1 + 1.$$ 

For $F_{1; t_4}$, let $t = t_3, t' = t_4, k = 1$. Then $\hat{F}_{1; t_3}^{(-1)} = (q^{-1}Z^{e_1+e_2} + q^{-1}Z_1 + 1)^{-1}, \hat{F}_{2; t_3}^{(1)} = q^{-1}Z_1 + 1$.

The recurrence in this case is

$$F_{1; t_4} = \hat{F}_{1; t_3}^{(-1)}(\hat{F}_{2; t_3}^{(1)} + q^{-1}Z^{e_1+e_2}) = 1.$$ 

Finally, for $F_{2; t_5}$, let $t = t_4, t' = t_5, k = 2$. Then $\hat{F}_{2; t_4}^{(-1)} = (q^{-1}Z_1 + 1)^{-1}, \hat{F}_{1; t_4}^{(1)} = 1$, and the recurrence gives

$$F_{2; t_5} = \hat{F}_{2; t_4}^{(-1)}(\hat{F}_{1; t_4}^{(1)} + q^{-1}Z_1) = 1.$$ 

For the purpose of computing quantum $F$-polynomials in the remaining sections, we will use the following simplified formula which may be proven by similar reasoning as in Theorem 7.7.

**Theorem 7.10.** Let $k_1, \ldots, k_\ell \in [1, n]$ be a sequence of mutations such that $k_\ell \notin \{k_1, \ldots, k_{\ell-1}\}$, and set $k = k_\ell$. Write $t \in \mathbb{T}_n$ for the vertex obtained from $t_0$ along the path with edges labeled by
Let \( k_1, \ldots, k_\ell, \) and let \( t' \in \mathbb{T}_n \) such that \( t \xrightarrow{k} t' \). Let \( B = (b_{ij}) \) be the principal part of the exchange matrix at \( t \). For \( \epsilon \in \{+, -\} \), let

\[ I_\epsilon = \{ j : j = k_i \text{ for some } i, \text{ and } \epsilon b_{jk} > 0 \} \]

Let \( k^+, \ldots, k^p \) be a list of the elements of \( I_+ \) in some order, and let \( k^-, \ldots, k^-r \) be a list of the elements of \( I_- \). For \( \epsilon \in \{+, -\} \), let

\[ G_\epsilon(j) = \prod_{i \in [1, j-1]} L[(i - 1)g_{k^\epsilon j, t}](F_{k^\epsilon j, t}) \]

for \( j \in [1, p] \) if \( \epsilon = + \) and for \( j \in [1, r] \) if \( \epsilon = - \). Also, let

\[ \hat{F}_\epsilon(j) = L \left[ \sum_{i \in [1, j-1]} [\epsilon b_{k^\epsilon j, k}]_+ g_{k^\epsilon j, t} \right] (G_\epsilon(j)) \]

Then there exist \( \lambda^+, \lambda^- \in \mathbb{Z}^n \) and \( a^+, a^- \in \mathbb{Z}^n \) such that

\[ F_{k,t'} = q^{\lambda^+} \left( \prod_{j \in [1, p]} \hat{F}_\epsilon(j)^{+} \right) Z^{a^+} + q^{\lambda^-} \left( \prod_{j \in [1, r]} \hat{F}_\epsilon(j)^{-} \right) Z^{a^-} \]

8. Quantum F-polynomials corresponding to trees

We resume using the same notation as in Section 4.1. Let \( B^0 \) be any \( n \times n \) skew-symmetric integer matrix, and let \( D = dI_n \), where \( I_n \) is the \( n \times n \) identity matrix and \( d \) is a positive integer.

For \( S \subset [1, n] \), let \( \phi(S) \) be the number of components in the subgraph of \( Q^0 \) induced by \( S \). Let \( F_T = F_T^{B^0; D; t_0} \) be the quantum F-polynomial corresponding to \( T \).

**Theorem 8.1.** The quantum F-polynomial \( F_T \) is given by

\[ F_T = \sum_S q^{\phi(S)} Z^{es} \]

where the summation ranges over subsets \( S \subset T \) such that (4.6) is satisfied. In particular, any F-polynomial of type A\(_n\) may be computed using (8.1).

**Proof.** First, we need to show that the quantum F-polynomial \( F_T \) is given by

\[ F_T = \sum_S q^{-\frac{d}{2}(g_T es)} Z^{es}, \]

where the summation ranges over \( S \subset T \) satisfying (4.6).
Apply induction on \( \ell \), the number of vertices in \( T \). If \( T = \{1\} \), then Proposition 4.3 implies that the first component of \( g_T \) is \(-1\). Since \( F_T = F_{1,i_1}^{B_0,D_{j_0}} \), (8.2) follows from Example 6.7. Next, assume that (8.2) is known for \( F_{[1,i]} \), where \( i = 1, \ldots, \ell - 1 \). It suffices to prove that

\[
F_T = \sum_{S} P_S(q^{\frac{1}{2}})Z^{e_S},
\]

where the summation ranges over \( S \subset T \) such that (4.6) holds, and each \( P_S(q^{\frac{1}{2}}) \) is of the form \( q^\lambda \) for some \( \lambda \in \frac{1}{2}\mathbb{Z} \). Then (8.2) follows from Corollary 7.5 and induction.

Using \( t = t_{\ell-1} \) and \( t' = t_\ell \), Theorem 7.10 implies that there exists some \( a, a' \in \mathbb{Z}^n \) and \( \lambda, \lambda' \in \frac{1}{2}\mathbb{Z} \) such that

\[
F_T = q^\lambda \hat{F}_1 \cdots \hat{F}_r Z^a + q^{\lambda'} \hat{F}'_1 \cdots \hat{F}'_s Z^{a'},
\]

where the \( \hat{F}_j \) and \( \hat{F}'_k \) are each of the form \( L[h](F_{[1,p]}) \) for some \( h \in \mathbb{Z}^n, p \in [1, \ell] \). By (7.19) and the induction hypothesis, the coefficients of any monomial \( Z^c \) \( (c \in \mathbb{Z}^n) \) in the \( \hat{F}_j \) and \( \hat{F}'_k \) are powers of \( q \). Thus, (8.4) implies that \( F_T = \sum P_S(q^{\frac{1}{2}})Z^{e_S} \), where the sum ranges over \( S \subset [1, n] \), and each \( P_S \in \mathbb{Z}[x, x^{-1}] \) is a subtraction-free Laurent polynomial.

Now set \( q = 1 \) and \( Z_i = u_i \) for all \( i \in [1, n] \) in the equation \( F_T = \sum P_S(q^{\frac{1}{2}})Z^{e_S} \). By the last part of Theorem 6.3, this gives an expression for \( F_T^{cl} \). Note that no monomial in \( F_T \) with nonzero coefficient disappears under this specialization. By Proposition 4.2, it follows that \( Z^{e_S} \) occurs with nonzero coefficient in \( F_T \) if and only if \( S \subset T \) and (4.6) is satisfied; furthermore, in this case, we have \( P_S(1) = 1 \), which forces \( P_S(q^{\frac{1}{2}}) \) to be a power of \( q \), as desired. This finishes the proof of (8.2).

To conclude the proof of the theorem, we need to prove that

\[
-g_T \cdot e_S = \phi(S)
\]

for any \( S \subset T \) such that \( S \) satisfies (4.6). Fix such a subset \( S \). By Proposition 4.3,

\[
-g_T \cdot e_S = \sum_{k \in S} (1 - |I_{out}(k)|) = |S| - \sum_{k \in S} |I_{out}(k)|.
\]

Since (4.6) holds for \( S \), we have that for \( k \in S \), \( |I_{out}(k)| \) is the number of \( i \in S \) such that \( k \to i \) in \( Q^0 \). Thus, \( \sum_{k \in S} |I_{out}(k)| \) is equal to the number of edges whose endpoints are both in \( S \), which is also equal to the number of edges in the subgraph of \( Q^0 \) induced by \( S \). The number of vertices in a
Table 2.

<table>
<thead>
<tr>
<th>Denominator vector</th>
<th>g-vector</th>
<th>Quantum F-polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_1)</td>
<td>(-e_1 + e_2)</td>
<td>(qZ_1 + 1)</td>
</tr>
<tr>
<td>(e_2)</td>
<td>(e_3 - e_2)</td>
<td>(qZ_2 + 1)</td>
</tr>
<tr>
<td>(e_3)</td>
<td>(e_1 - e_3)</td>
<td>(qZ_3 + 1)</td>
</tr>
<tr>
<td>(e_4)</td>
<td>(e_3 - e_4)</td>
<td>(qZ_4 + 1)</td>
</tr>
<tr>
<td>(e_1 + e_2)</td>
<td>(-e_1)</td>
<td>(qZ^{e_1+e_2} + qZ_1 + 1)</td>
</tr>
<tr>
<td>(e_1 + e_3)</td>
<td>(-e_3)</td>
<td>(qZ^{e_1+e_3} + qZ_3 + 1)</td>
</tr>
<tr>
<td>(e_2 + e_3)</td>
<td>(-e_2)</td>
<td>(qZ^{e_2+e_3} + qZ_2 + 1)</td>
</tr>
<tr>
<td>(e_3 + e_4)</td>
<td>(e_1 - e_4)</td>
<td>(qZ^{e_3+e_4} + qZ_4 + 1)</td>
</tr>
<tr>
<td>(e_1 + e_3 + e_4)</td>
<td>(-e_4)</td>
<td>(qZ^{e_1+e_3+e_4} + qZ^{e_3+e_4} + qZ_4 + 1)</td>
</tr>
<tr>
<td>(e_2 + e_3 + e_4)</td>
<td>(-e_2 + e_3 - e_4)</td>
<td>(qZ^{e_2+e_3+e_4} + qZ^{e_2+e_4} + qZ_2 + qZ_4 + 1)</td>
</tr>
</tbody>
</table>

tree minus the number of edges is 1, so \(|S| - \sum_{k \in S} |I_{out}(k)|\) must equal the number of components in the subgraph of \(Q^0\) induced by \(S\). \(\square\)

**Example 8.2.** Let \(B^0\) be the following initial exchange matrix of type \(A_4\):

\[
B^0 = \begin{pmatrix}
0 & 1 & -1 & 0 \\
-1 & 0 & 1 & 0 \\
1 & -1 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

Then \(Q^0 = Q(B^0)\) is the quiver below:

```
2
\downarrow
1 \longrightarrow 3 \longrightarrow 4
```

Also, let \(D = 2I_4\), where \(I_4\) is the \(4 \times 4\) identity matrix. The complete list of denominator vectors corresponding to cluster variables not in the initial cluster is given in Table 2. The corresponding g-vectors and quantum \(F\)-polynomials are computed using Proposition 4.12 and Theorem 8.1. To obtain the \(F\)-polynomial, plug \(q = 1\) and \(Z_i = u_i\) for \(i \in [1, 4]\) into the corresponding quantum \(F\)-polynomial.
9. Quantum F-polynomials in Classical Types for Acyclic Initial Matrix

Let $B^0 = (b_{ij}^0)$ be an acyclic $n \times n$ exchange matrix of type $A_n$, $B_n$, $C_n$, or $D_n$. We resume using the same notation established in Section 4.3. For type $A_n$ and type $D_n$, let $\delta = (1, \ldots, 1) \in \mathbb{Z}^n$. For type $C_n$, let $\delta = (1, \ldots, 1, 2) \in \mathbb{Z}^n$. For type $B_n$, let $\delta = (2, \ldots, 2, 1) \in \mathbb{Z}^n$. Let $d$ be a positive integer, and let $D$ be the $n \times n$ diagonal matrix whose entries are given by the vector $d\delta$. Thus, $DB^0$ is a skew-symmetric matrix.

Write $F_d = F_d^{B^0;D;0}$, $F_{cl}^d = F_d^{B^0;D;0}$, $g_d = g_d^{B^0;0}$ for the quantum $F$-polynomial, $F$-polynomial, and $g$-vector corresponding to $d \in \Phi_+(B^0)$. By convention, set $F_0 = 1$, $g_0 = 0$.

For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n_\geq$ such that $u_1^{a_1} \ldots u_n^{a_n}$ has nonzero coefficients in $F_{cl}^d$, let $\phi_d(a) \in \mathbb{Z}_{\geq 0}$ such that $q^{\phi_d(a)}$ is the coefficient of $u_1^{a_1} \ldots u_n^{a_n}$ in $F_{cl}^d$ (i.e., $\phi_d(a) = c$, where $c$ is given in Theorem 4.16). Also, let $\rho_d(a) = 1$ if $B^0$ is of type $B_n$, the $n$th component of $a$ is 1, and $\phi_d(a) \geq 1$, and let $\rho_d(a) = 0$ otherwise.

**Theorem 9.1.** Let $d \in \Phi_+(B^0)$. In types $A_n$, $C_n$ and $D_n$, the quantum $F$-polynomial $F_d$ is given by

\[
F_d = \sum q^{-\frac{d}{2}g_d^d a}(q^\frac{d}{2} + q^{-\frac{d}{2}})^{\phi_d(a)}Z^a.
\]

In type $B_n$,

\[
F_d = \sum q^{-\frac{d}{2}g_d^d a}(q^{-\frac{d}{2}} + q^{d})^{\rho_d(a)}(q^{-d} + q^{d})^{\phi_d(a)-\rho_d(a)}Z^a
\]

In either case, each summation ranges over $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n_\geq$ such that $u_1^{a_1} \ldots u_n^{a_n}$ has nonzero coefficient in $F_{cl}^d$.

**Example 9.2.** Use the same $B^0$ as in Example 4.19, and let $D$ be the $4 \times 4$ diagonal matrix with diagonal entries $2\delta = (4, 4, 4, 2)$. Then the quantum $F$-polynomials for the same denominator vectors are given below.
(9.3) \[ F_{e_2} = q^2Z_2 + 1 \]

(9.4) \[ F_{e_2 + e_3} = q^2Z^{e_2 + e_3} + q^2Z_2 + 1 \]

(9.5) \[ F_{e_2 + e_3 + e_4} = qZ^{e_2 + e_3 + e_4} + q^3Z^{e_2 + e_4} + qZ_4 + q^2Z_2 + 1 \]

(9.6) \[ F_{e_2 + 2e_3 + 2e_4} = q^2Z^{e_2 + 2e_3 + 2e_4} + q^2Z^{e_3 + 2e_4} + q^4Z^{2e_4} \]
\[ + (q + q^3)Z^{e_2 + e_3 + e_4} + (q^2 + q^6)Z^{e_2 + e_3 + 2e_4} \]
\[ + q^6Z^{e_2 + 2e_4} + (q^3 + q^5)Z^{e_2 + e_4} \]
\[ + (q + q^3)Z_4 + q^2Z_2 + 1 \]

(9.7) \[ F_{e_1 + 2e_2 + 2e_3 + 2e_4} = q^2Z^{e_1 + 2e_2 + 2e_3 + 2e_4} + q^4Z^{2e_2 + 2e_3 + 2e_4} \]
\[ + (q^4 + q^8)Z^{e_1 + 2e_2 + e_3 + 2e_4} + (q^6 + q^{10})Z^{2e_2 + e_3 + 2e_4} \]
\[ + q^{10}Z^{e_1 + 2e_2 + 2e_4} + q^{12}Z^{2e_2 + 2e_4} \]
\[ + (q^3 + q^5)Z^{e_1 + 2e_2 + e_3 + e_4} + (q^7 + q^9)Z^{e_1 + 2e_2 + e_4} \]
\[ + (q^5 + q^7)Z^{2e_2 + e_3 + e_4} + q^6Z^{e_1 + 2e_2} \]
\[ + (q^9 + q^{11})Z^{2e_2 + e_4} + q^8Z^{2e_2} + q^2Z^{e_1 + e_2 + e_3 + 2e_4} \]
\[ + q^6Z^{e_1 + e_2 + 2e_4} + (q^2 + q^6)Z^{e_2 + e_3 + 2e_4} \]
\[ + (q^6 + q^{10})Z^{e_2 + 2e_4} + q^4Z^{2e_4} + (q^3 + q^5)Z^{e_1 + e_2 + e_4} \]
\[ + (q + q^3)Z^{e_2 + e_3 + e_4} + (q^3 + q^5 + q^7 + q^9)Z^{e_2 + e_4} \]
\[ + q^2Z^{e_1 + e_2} + (q + q^3)Z_4 + (q^2 + q^6)Z_2 + 1 \]

For example, let \( d = e_1 + 2e_2 + 2e_3 + 2e_4 \). To verify the coefficient of \( Z^{e_2 + e_4} \) in \( F_d \) is \( q^3 + q^5 + q^7 + q^9 = q^6(q + q^{-1})(q^2 + q^{-2}) \), observe that Example 4.19 shows that \( \phi_d(e_2 + e_4) = 2 \) and \( \rho_d(e_2 + e_4) = 1 \). Using the \( g \)-vector \( g_d = e_1 - 2e_2 + 2e_3 - 2e_4 \) from the same example, it is easy to check that \( -\frac{d}{2} \cdot g_d \cdot (e_2 + e_4) = 6 \), as desired.

In type \( A_n \), Theorem 9.1 follows from Proposition 4.6 and Theorem 8.1. The proof in the remaining types will be given in the subsections below.
9.1. Combinatorial realizations of cluster algebras of types $B_n$, $C_n$, $D_n$. For each of these types, we recall from [9] certain combinatorial realizations of cluster variables and clusters which will be useful in the proofs below.

For types $B_n$ and $C_n$, the realization is given in terms of diagonals of the regular $(2n + 2)$-gon $P_{2n+2}$. In order to deal with both types at once, set $r = 1$ for type $B_n$, and $r = 2$ for type $C_n$. Let $\theta$ be the $180^\circ$ rotation of $P_{2n+2}$. Then $\theta$ also acts on the diagonals of $P_{2n+2}$. If $a, b$ are endpoints of a diagonal, then we write $[ab]$ for the diagonal. If $a$ is a vertex in $P_{2n+2}$, then write $a = \theta(a)$ for the opposite vertex on $P_{2n+2}$.

The cluster variables are in bijective correspondence with the $\theta$-orbits of diagonals of $P_{2n+2}$. Two cluster variables are in some cluster together if and only if the diagonals in their corresponding $\theta$-orbits do not cross each other.

Definition 9.3. Call $D$ a maximal diagonal set if $D$ is a set of diagonals of $P_{2n+2}$ such that

- no two elements of $D$ cross;
- $D$ is closed under the action of $\theta$;
- every diagonal outside of $D$ crosses at least one of the elements of $D$.

Maximal diagonal sets of $P_{2n+2}$ are in bijective correspondence with clusters. The initial cluster corresponding to $B_0$ is represented by the set of diagonals $\alpha_1, \ldots, \alpha_n$ and their $\theta$-orbits, which are constructed as follows: Let $\alpha_1$ be any diagonal of shortest length. Then we construct $\alpha_2, \ldots, \alpha_n$ in order so that $\alpha_{i+1}$ shares an endpoint with $\alpha_i$, the other endpoints of $\alpha_i$, $\alpha_{i+1}$ are the endpoints of a side of $P_{2n+2}$, and $\alpha_{i+1}$ is clockwise (by an acute angle) to $\alpha_i$ if $i \rightarrow i + 1$ in $Q_0$ and counterclockwise to $\alpha_i$ otherwise. Note that $\alpha_n$ is a diameter of $P_{2n+2}$. Let $D^0 = \{\alpha_1, \ldots, \alpha_n, \theta(\alpha_1), \ldots, \theta(\alpha_{n-1})\}$.

In type $B_n$ (resp. $C_n$), the cluster variable with denominator vector $\sum_{i \in [1,n]} a_i e_i \in \Phi_+(B^0)$ is represented by the unique $\theta$-orbit $\{\beta, \theta(\beta)\}$ such that for each $i \in [1,n]$, the diagonal $\alpha_i$ (resp. $\beta$) crosses the diagonals in $\{\beta, \theta(\beta)\}$ (resp. $\{\alpha_i, \theta(\alpha_i)\}$) at a total of $a_i$ points.

Next, we describe all possible cluster mutations in terms of maximal diagonal sets. Consider a cluster which corresponds to a maximal diagonal set $D$, and let $B = (b_{ij})$ be the principal part of the exchange matrix. Let $k \in [1,n]$. Suppose that the $k$th cluster variable corresponds to a diagonal $[ac]$ in $D$. Then there exist vertices $b, f$ of $P_{2n+2}$ such that $[ab], [bc], [cf], \text{ and } [af]$ are each either a diagonal in $D$ or side of $P_{2n+2}$, and $a, b, c, f$ are distinct vertices in counterclockwise
order on $\mathbb{P}_{2n+2}$. Mutation in direction $k$ then corresponds to replacing the diagonals $[ac], \theta([ac])$ by $[bf], \theta([bf])$.

Let $i \in [1, n]$. If $i$ corresponds to a diameter, then

$$b_{ik} = \begin{cases} 
2/r & \text{if } i \text{ corresponds to } [cf] \text{ or } [ab] \\
-2/r & \text{if } i \text{ corresponds to } [bc] \text{ or } [af] \\
0 & \text{otherwise}
\end{cases}$$

If $k$ corresponds to a diameter, then

$$b_{ik} = \begin{cases} 
r & \text{if } i \text{ corresponds to } [cf] \text{ or } [ab] \\
-r & \text{if } i \text{ corresponds to } [bc] \text{ or } [af] \\
0 & \text{otherwise}
\end{cases}$$

Otherwise,

$$b_{ik} = \begin{cases} 
1 & \text{if } i \text{ corresponds to } [cf] \text{ or } [ab] \\
-1 & \text{if } i \text{ corresponds to } [bc] \text{ or } [af] \\
0 & \text{otherwise}
\end{cases}$$

**Example 9.4.** Let

$$B^0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix}$$

Then $B^0$ is of type $B_3$, and the corresponding maximal diagonal set in $\mathbb{P}_8$ is given in Figure 2. The diameter $[ae]$ and $\theta$-orbit $\{[af], [be]\}$ correspond to the cluster variables with denominator vector $e_1 + e_2 + e_3$ and $e_1 + 2e_2 + 2e_3$, respectively.

**Example 9.5.** Let

$$B^0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}$$
Then $B^0$ is of type $C_3$, and the maximal diagonal set given in Figure 2 also corresponds to this $B^0$. In this case, the diameter $[ae]$ and $\theta$-orbit $\{[af], [be]\}$ correspond to the cluster variables with denominator vector $2e_1 + 2e_2 + e_3$ and $e_1 + 2e_2 + e_3$, respectively.

For type $D_n$, the cluster variables are in bijective correspondence with the $\theta$-orbits of diagonals of $\mathbb{P}_{2n}$, where diameters may be of two different “colors,” which are denoted by $[a\overline{a}]$ and $[\overline{a}a]$. Two cluster variables are in some cluster together if and only if the diagonals in their corresponding $\theta$-orbits do not cross each other, with the additional assumption that diameters of the same color do not cross.

Define maximal diagonal sets in an analogous manner to Definition 9.3 so that $\mathcal{D}$ is a subset of diagonals and (colored) diameters of $\mathbb{P}_{2n}$, and “crossing” takes into account the convention just given. Maximal diagonal sets of $\mathbb{P}_{2n}$ are in bijective correspondence with clusters.

The initial cluster corresponding to $B^0$ is represented by the set of diagonals $\alpha_1, \ldots, \alpha_n$ and their $\theta$-orbits, which are constructed as follows: Construct $\alpha_1, \ldots, \alpha_{n-1}$ in $\mathbb{P}_{2n}$ in a similar manner as in the type $C_n$ setting. Note that $\alpha_{n-1}$ is a diameter of the first color in $\mathbb{P}_{2n}$. If the arrow between $n-2$ and $n$ in $Q^0$ is in the same direction as the arrow between $n-2$ and $n-1$, then let $\alpha_n$ be the diagonal $\tilde{\alpha}_{n-1}$; otherwise, let $\alpha_n$ be the diameter with the same color as $\alpha_{n-1}$ which shares the other endpoint of $\alpha_{n-2}$. Let $\mathcal{D}^0 = \{\alpha_1, \ldots, \alpha_n, \theta(\alpha_1), \ldots, \theta(\alpha_{n-2})\}$.

The cluster variable with denominator vector $\sum_{i \in [1,n]} a_i e_i \in \Phi_+(B^0)$ is represented by the unique $\theta$-orbit such that for each $i \in [1,n]$, the diagonals in this orbit cross the diagonals representing $\alpha_i$ at
pairs of centrally symmetric points (where we count an intersection of two diameters of different colors and location as one such pair).

Next, we describe certain cluster mutations in terms of maximal diagonal sets. Consider a cluster which corresponds to a maximal diagonal set \( \mathcal{D} \), and let \( B = (b_{ij}) \) be the principal part of the exchange matrix. Let \( k \in [1, n] \). Suppose that the \( k \)th cluster variable corresponds to a diagonal \( \beta = [ac] \in \mathcal{D} \), or to \( \beta = [ac] \in \mathcal{D} \), where \( c = \overline{a} \). Assume there exist vertices \( b, f \) of \( \mathbb{P}_2^{n+2} \) such that \([ab],[bc],[cf] \), and \([af] \) are each either a diagonal in \( \mathcal{D} \) or side of \( \mathbb{P}_2^{n+2} \), and \( a, b, c, f \) are distinct vertices in counterclockwise order on \( \mathbb{P}_2^{n+2} \). Furthermore, assume that if \([ab] \) is a diameter, then \( \overline{[ab]} \) is also in \( \mathcal{D} \), and assume similar conditions with \([bc],[cf] \), and \([af] \). If \( \beta = [ac] \) (resp. \( \beta = [ac] \)), and \( \beta \) is a diameter, then \( c = \overline{a}, b = \overline{f} \), and mutation in direction \( k \) corresponds to replacing \( \beta \) by the diameter \([bf] \) (resp. \( \overline{[bf]} \)). If \( \beta \) is not a diameter, then mutation in direction \( k \) corresponds to replacing the \( \theta \)-orbit \( \{\beta, \theta(\beta)\} \) by \( \{[bf], \theta([bf])\} \).

Let \( i \in [1, n] \). Then

\[
b_{ik} = \begin{cases} 
1 & \text{if } i \text{ corresponds to } [cf], [ab], [\overline{cf}], \text{ or } [\overline{ab}] \\
-1 & \text{if } i \text{ corresponds to } [bc], [af], [\overline{bc}], \text{ or } [\overline{af}] \\
0 & \text{otherwise}
\end{cases}
\] (9.13)

Note that in the above, if \( i \) corresponds to \( [ab] \), for example, then we assume that \( [ab] \) is a diameter, i.e., \( b = \overline{a} \). Likewise, the same convention applies to \( [cf], [bc], \) and \( [af] \).

**Example 9.6.** Let

\[
B^0 = \begin{pmatrix} 
0 & -1 & 0 & 0 \\
1 & 0 & -1 & 1 \\
0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\] (9.14)

Then \( B^0 \) is of type \( \text{D}_4 \), and the corresponding maximal diagonal set in \( \mathbb{P}_8 \) is given on the left in Figure 3. The diameters \([ae] \) and \( [\overline{ae}] \) correspond to the cluster variables with denominator vector \( e_1 + e_2 \) and \( e_1 + e_2 + e_3 + e_4 \), respectively, while the \( \theta \)-orbit \( \{[af], [be]\} \) corresponds to the cluster variable with denominator vector \( e_1 + 2e_2 + e_3 + e_4 \).
Example 9.7. Let

\[
B^0 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\]

Then $B^0$ is of type $D_4$, and the corresponding maximal diagonal set in $\mathbb{P}_8$ is given on the right in Figure 3. The diameters $[ae]$ and $[\overline{ae}]$ correspond to the cluster variables with denominator vector $e_1 + e_2 + e_4$ and $e_1 + e_2 + e_3$, respectively.

Example 9.8. Let

\[
B^0 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 1 & -1 \\
0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

Then $B^0$ is of type $D_4$, and the corresponding maximal diagonal set in $\mathbb{P}_8$ is given on the left in Figure 3. The diameters $[ae]$ and $[\overline{ae}]$ correspond to the cluster variables with denominator vector $e_1 + e_2$ and $e_1 + e_2 + e_3 + e_4$, respectively.
Remark 9.9. In [7], Fomin, Shapiro, and Thurston associate cluster algebras to punctured Riemann surfaces. In particular, cluster algebras of type $D_n$ may be realized in terms of triangulations of a once-punctured $n$-gon; see [7, Example 6.7] for additional details.

9.2. Type $D_n$. Let $B^0 = (b^0_{ij})$ be an acyclic $n \times n$ exchange matrix of type $D_n$.

The theorem follows from Theorem 8.1 for $d \in \Phi_+(B^0)$ such that the components of $d$ are all either 0 or 1. Thus, it remains to prove the theorem for $d$ of the form $\sum_{i=p}^n e_i + \sum_{j=r}^{n-2} e_j$ ($1 \leq p < r \leq n - 1$). Proceed by induction on $n - 2 - p$. The base of the induction occurs when $n - 2 - p = 0$, so that $d = e_{n-2} + e_{n-1} + e_n$, and in this case, the theorem has already been proven.

Fix $d = \sum_{i=p}^n e_i + \sum_{j=r}^{n-2} e_j$ ($1 \leq p < r \leq n - 2$), and suppose that the theorem has been proven for all denominator vectors $\sum_{i=p'}^n e_i + \sum_{j=r'}^{n-2} e_j$, where $p < p' < r' \leq n - 2$. Write $d = (d_1, \ldots, d_n)$.

Lemma 9.10. At least one of the two statements below is true:

1. For some $\lambda', \lambda'' \in \frac{1}{2}\mathbb{Z}$, $a', a'' \in \mathbb{Z}^n$, and $d', d'' \in \Phi_+(B^0) \cup \{0\}$ such that $d', d'' < d$ and the $p$th component of both vectors is 0, the quantum $F$-polynomial $F_d$ satisfies

\[
F_d = q^{\lambda'} F_{d'} Z^{a'} + q^{\lambda''} F_{d''} Z^{a''}
\]

2. Let

\[
d'' = e_r + \cdots + e_{n-1}, \quad d''' = e_r + \cdots + e_{n-2} + e_n
\]

if the arrow between $n - 2$ and $n$ and the arrow between $n - 2$ and $n - 1$ in $Q^0$ are in the same direction, and let

\[
d'' = e_r + \cdots + e_{n-1}, \quad d''' = e_r + \cdots + e_n
\]

otherwise. Also, let

\[
F_{d'''} = L[g_{d'''}](F_{d''}).
\]

For some $a', a'' \in \mathbb{Z}^n$, $\lambda', \lambda'' \in \frac{1}{2}\mathbb{Z}$, and $d' \in \Phi_+(B^0) \cup \{0\}$ such that $d' < d$ and the $p$th component of $d'$ is 0, we have

\[
F_d = q^{\lambda'} F_{d'} Z^{a'} + q^{\lambda''} F_{d'''} F_{d''} Z^{a''}
\]
Proof. We will show that there exists a sequence of cluster mutations which yields the cluster variable corresponding to $d$ by using the combinatorial representation given above. Then Theorem 7.10 will guarantee that at least one of the equations for $F_d$ occurs.

Note that the diagonals $\alpha_1, \ldots, \alpha_{n-1}$ divide half of $\mathbb{P}_{2n}$ into $n-1$ triangles. For $i = 1, \ldots, n-2$, let $\Delta_i$ be the triangle containing $\alpha_i$ and $\alpha_{i+1}$ as sides, and let $\Delta_0$ be the triangle with $\alpha_1$ as a side but not $\alpha_2$. Let $v$ be the vertex opposite $\alpha_r$ in $\Delta_{r-1}$, and for $i \in [1, n-2]$, let $v_i$ be the vertex opposite $\alpha_i$ in $\Delta_i$.

First, if $\alpha_{n-1}$ and $\alpha_n$ are diameters of the same color, then we perform a mutation on $D^0$ in direction $n$ so that $\alpha_n$ is mutated to $\tilde{\alpha}_{n-1}$. Then mutate in directions $k = r, \ldots, n$. Under this sequence of mutations, the diagonal $\alpha_i$ ($i \in [r, n-2]$) is mutated to $\alpha'_i = [v v_i]$, and $\alpha_{n-1}$ and $\tilde{\alpha}_{n-1}$ are mutated to the diagonals $[v v']$ and $[v v]$.

Let $r \leq j_s < \ldots < j_1 \leq n-2$ be indices such that the arrow between $j_{\ell}$ and $j_{\ell} + 1$ is opposite the arrow between $r - 1$ and $r$ for $\ell = 1, \ldots, s$. Next, we continue mutating in directions $k = j_1, \ldots, j_s$. Let $w$ be the endpoint of the diagonal $\alpha_{r-1}$ which is not equal to $v$. Then $\alpha'_{j_1}$ is mutated to $\alpha''_{j_1} = [\tilde{v} v_{j_1+1}]$ (for $\ell \in [1, s-1]$) and $\alpha'_{j_s}$ is mutated to $[\tilde{v} w]$.

Let $w_i$ be the vertex opposite $\alpha_i$ in the triangle $\Delta_{i-1}$ for $i \in [p, r-1]$. Continue mutating in directions $k = r - 1, \ldots, p$. Then $\alpha_k$ is mutated to $\alpha'_k = [\tilde{v} w_k]$ for $k = p, \ldots, r - 1$. Note that $\alpha'_p$ corresponds to the cluster variable with denominator vector $d$ since it crosses the diagonals $\alpha_p, \ldots, \alpha_n, \theta(\alpha_r), \ldots, \theta(\alpha_{n-2})$.

Let $a_1, a_2$ be the endpoints of $\alpha_p$. The diagonal $\alpha'_p$ is the diagonal of a quadrilateral whose sides are each in $D'$ or sides of $\mathbb{P}_{2n}$. To be more precise, the sides of the quadrilateral are $\beta_1 = [\tilde{v} a_1]$, $\beta_2 = [\tilde{v} a_2]$, $\alpha_{p-1}$ (if $p - 1 \geq 1$), and $[\tilde{v} v]$ (if one of the $a_i$ equals $v$); the remaining sides (if any) of the quadrilateral are sides of $\mathbb{P}_{2n}$.

Observe that the only elements of $D^0$ that the $\beta_i$ can intersect come from the list

$$\alpha_{p+1}, \ldots, \alpha_n, \theta(\alpha_r), \ldots, \theta(\alpha_{n-2}).$$

Thus, each $\beta_i$ corresponds to some $d' \in \Phi_+(B^0)$ such that $d' < d$, and the $p$th component of $d'$ is equal to 0.

If $a_i \neq v$ for $i = 1, 2$, then the diagonals $\beta_1, \beta_2$ are not diameters. It follows from Theorem 7.10 in conjunction with (9.13) that there exists a mutation of the form (9.17), where the $d', d'' \in \Phi_+(B^0)$ in this equation correspond to the diagonals $\beta_1, \beta_2$. If $a_i = v$ for some $i$, then observe that $[v v]$
intersects the interiors of \( \alpha_r, \ldots, \alpha_n \), which means that \([\nu\overline{\nu}]\) and \([\overline{\nu}\overline{\nu}]\) correspond to denominator vectors \( d'', d''' \) as at (9.18) or (9.19). In this case, we get an equation for \( F_d \) of the form (9.21). □

The equations for \( F_d \) given in Lemma 9.10 show that the terms of \( F_d \) are subtraction-free Laurent polynomials in \( Z_1, \ldots, Z_n \) with coefficients in \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \). By Theorem 6.3, setting \( q = 1 \) and \( Z_i = u_i \) (\( i \in [1,n] \)) in \( F_d \) yields \( F_d^d \), so it follows that for \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \), the monomial \( Z^a \) occurs with nonzero coefficient in \( F_d \) if and only if \( u_1^{a_1} \ldots u_n^{a_n} \) occurs with nonzero coefficient in \( F_d^d \).

To prove Theorem 9.1, it suffices to show that all nonzero coefficients of \( F_d \) are of the form

\[
q^{c_1}(1 + q^d)^{c_2}
\]

for some \( c_1, c_2 \in \frac{1}{2}\mathbb{Z} \). Once this fact has been established, the proof of the theorem in type \( D_n \) may be finished in the following way. Suppose that the coefficient of \( Z^a \) is of the form given at (9.22). This expression may be rewritten as \( q^{c_1}(q^{-\frac{1}{2}} + q^{\frac{1}{2}})^{c_2} \) for some \( c'_1 \in \frac{1}{2}\mathbb{Z} \). Since setting \( q = 1 \) and \( Z_i = u_i \) for \( i \in [1,n] \) in \( F_d \) yields \( F_d^d \), it follows that \( c_2 = \phi_d(a) \). Using Corollary 7.5, we have that

\[
q^{c'_1}(q^{-\frac{1}{2}} + q^{\frac{1}{2}})^{\phi_d(a)} = q^{-d\delta \cdot g_d \cdot a} q^{-c'_1}(q^{-\frac{1}{2}} + q^{\frac{1}{2}})^{\phi_d(a)}.
\]

Thus, \( c'_1 = -\frac{d\delta \cdot g_d \cdot a}{2} \), as desired.

By Theorem 4.16, \( Z^d \) occurs with nonzero coefficient in \( F_d \), and \( F_d \) has nonzero constant term. Thus, in the expressions (9.17) and (9.21), one of \( a' \) or \( a'' \) must have \( p \)th component 1, and the other must be equal to the 0-vector. In (9.17), this means that one of the expansions of \( q^Z F_d' Z^a' \), \( q^{\lambda''} F_d'' Z^a'' \) contains all monomials \( Z^a \) in \( F_d \) with \( Z_p \), and the other contains all monomials which do not have \( Z_p \). Consequently, if (9.17) holds, then it follows immediately from the induction hypothesis that all monomials in \( F_d \) have coefficient of the form at (9.22). By similar reasoning, in the expression (9.21), one of the expansions for \( q^Z F_d' Z^a' \), \( q^{\lambda''} F_d'' F_{d''}^d Z^a'' \) contains all monomials with \( Z_p \), and the other contains all monomials without \( Z_p \). By the induction hypothesis, it is clear that coefficients of the monomials in the expansion of \( q^Z F_d' Z^a' \) have the form (9.22). The remainder of the proof of Theorem 9.1 is devoted to showing the same statement is true of the monomials in \( F_d'' F_{d''}^d \).

Fix the denominator vectors \( d'', d''' \) as at (9.18) or (9.19), and fix \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) such that the monomial \( Z^a \) occurs with nonzero coefficient in \( F_{d''} F_{d''}^d \). Let \( S_a \) denote the set of all possible pairs \((v, w) \in \mathbb{Z}^n \times \mathbb{Z}^n \) such that \( a = v + w \), and such that \( Z^v \) (resp. \( Z^w \)) occurs with
nonzero coefficient in \( F_{d^u} \) (resp. \( F'_{d^u} \)). The next goal is to describe the elements that can occur in \( S_a \).

Let \( S \) be the subgraph of \( Q^0 \) induced by \( \{ i \in [r, n - 2] : a_i = 1 \} \). Let \( C(S) \) denote the set of components of \( S \), excluding the component which contains \( n - 2 \) if at least one of the arrows between \( n - 2 \) and \( n \) and between \( n - 2 \) and \( n - 1 \) in \( Q^0 \) is critical with respect to \((d, a)\).

**Lemma 9.11.**

1. \( 0 \leq a_i \leq 2 \) for \( i = [r, n - 2] \),
   
   \begin{align*}
   &0 \leq a_i \leq 1 \text{ for } i = n - 1, n, \\
   &a_i = 0 \text{ for } i \in [1, r - 1].
   \end{align*}
2. If \( i \to j \) in \( Q^0 \) with \( r \leq i, j \leq n - 2 \), then \( a_j \geq a_i \).
3. If \( n - 2 \to n \) in \( Q^0 \) (resp. \( n - 2 \to n - 1 \)) and \( a_{n-2} = 2 \), then \( a_n = 1 \) (resp. \( a_{n-1} = 1 \)).
4. If \( n \to n - 2 \) (resp. \( n - 1 \to n - 2 \)), then \( a_{n-2} \geq a_n \) (resp. \( a_{n-2} \geq a_{n-1} \)).
5. At least one of the arrows between \( n - 2 \) and \( n - 1 \) and between \( n - 2 \) and \( n \) in \( Q^0 \) is not critical with respect to \((d, a)\).

**Proof.** Let \((v, w) \in S_a\), and write \( v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n)\). Statement (1) follows immediately from part (1) of Theorem 4.16. Part (2) of the same theorem implies that if \( i \to j \) in \( Q^0 \) and \( i, j \in [r, n - 2] \), then \( v_i \leq v_j \) and \( w_i \leq w_j \), so statement (2) of the lemma also follows. If \( n - 2 \to n \) in \( Q^0 \) and \( a_{n-2} = 2 \), then \( v_{n-2} = w_{n-2} = 1 \), so using part (2) of Theorem 4.16 again, it follows that \( 1 = v_{n-2} \leq v_n \leq a_n \) if (9.18) holds and \( 1 = w_{n-2} \leq w_n \leq a_n \) if (9.19) holds. Similar reasoning applies to the case when \( n - 2 \to n - 1 \) in \( Q^0 \). This proves statement (3) of the lemma. For statement (4), if \( n \to n - 2 \) in \( Q^0 \), then parts (1) and (2) of Theorem 4.16 imply that \( v_n = 0 \) and \( w_n \leq w_{n-2} \leq a_{n-2} \), so \( a_n \leq a_{n-2} \). Similar reasoning applies to the situation where \( n - 1 \to n - 2 \) in \( Q^0 \). Finally, for the final statement of the lemma, suppose for the sake of contradiction that both of the arrows between \( n - 2 \) and \( n - 1 \) and between \( n - 2 \) and \( n \) are critical. We consider the case when (9.18) holds and leave the other case as an easy check. If \( n - 2 \to n \) and \( n - 2 \to n - 1 \) in \( Q^0 \), then \( v_{n-2} \leq v_{n-1} \) and \( w_{n-2} \leq w_n \). The fact that the stated arrows are critical implies that \( a_{n-2} = 1 \) and \( a_n = a_{n-1} = 0 \). This means that \( v_{n-1} = 0 \) and \( w_n = 0 \), and either \( v_{n-2} = 1 \) or \( w_{n-2} = 1 \), contradiction. Now suppose that \( n \to n - 2 \) and \( n - 1 \to n - 2 \) in \( Q^0 \). Then \( v_{n-1} \leq v_{n-2} \) and \( w_n \leq w_{n-2} \). The fact that the arrows are critical means that \( a_{n-2} = 1 \) and \( a_n = a_{n-1} = 1 \), so \( v_{n-1} = w_n = 1 \), and either \( v_{n-2} = 0 \) or \( w_{n-2} = 0 \), contradiction. \( \square \)
Let

\[
\begin{align*}
v^{(0)} &= e_{n-1} + \sum_{i \in [r,n-2], a_i \geq 1} e_i \\
w^{(0)} &= a_ne_n + \sum_{i \in [r,n-2], a_i = 2} e_i
\end{align*}
\]

if \( d'' , d''' \) are as at (9.18) and \( a_{n-1} = 1 \); let

\[
\begin{align*}
v^{(0)} &= \sum_{i \in [r,n-2], a_i = 2} e_i \\
w^{(0)} &= a_ne_n + \sum_{i \in [r,n-2], a_i \geq 1} e_i
\end{align*}
\]

if \( d'' , d''' \) are as at (9.18) and \( a_{n-1} = 0 \); let

\[
\begin{align*}
v^{(0)} &= \sum_{i \in [r,n-2], a_i = 2} e_i \\
w^{(0)} &= a_{n-1}e_{n-1} + a_ne_n + \sum_{i \in [r,n-2], a_i \geq 1} e_i
\end{align*}
\]

if \( d'' , d''' \) are as at (9.19) and \( a_{n-1} = 1 \); finally, let

\[
\begin{align*}
v^{(0)} &= \sum_{i \in [r,n-2], a_i \geq 1} e_i \\
w^{(0)} &= a_{n-1}e_{n-1} + a_ne_n + \sum_{i \in [r,n-2], a_i = 2} e_i
\end{align*}
\]

if \( d'' , d''' \) are as at (9.19) and \( a_{n-1} = 0 \).

**Lemma 9.12.** If \( v^{(0)} , w^{(0)} \) are as at (9.24) or (9.27), then

\[
\begin{align*}
S_a &= \{(v^{(0)} - a', w^{(0)} + a') : J \subset C(S), a' = \sum_{C \in J} e_C \}.
\end{align*}
\]

Otherwise,

\[
\begin{align*}
S_a &= \{(v^{(0)} + a', w^{(0)} - a') : J \subset C(S), a' = \sum_{C \in J} e_C \}.
\end{align*}
\]

**Proof.** Assume that the arrows between \( n-2 \) and \( n-1 \) and between \( n-2 \) and \( n \) are in the same direction, so that \( d'' , d''' \) are as at (9.18). (The proof when \( d'' , d''' \) are as at (9.19) is similar.)
Let $a' = \sum_{C \in J} e_C$ for some $J \subset \mathcal{C}(S)$, and let $v = v^{(0)} \pm a' = (v_1, \ldots, v_n)$, $w = w^{(0)} \pm a' = (w_1, \ldots, w_n)$ (taking the top signs in the situation of (9.24) and the bottom signs otherwise). First, we will show that $Z^v$ occurs with nonzero coefficient in $F_{d^v}$. Let $i, j \in [r, n - 1]$ with $i \to j$ in $Q^0$. Then we need to show that $v_j \geq v_i$.

Case 1: $i, j \in S$

Then $i, j$ are in the same component of $S$, which means that the $i$th and $j$th components of $v^{(0)}$ and hence of $v$ are equal.

Case 2: $i \in S$, $j \in [r, n - 2] - S$

Then $a_i = 1$, so $a_j = 2$ by (1) and (2) of Lemma 9.11. In this case, $v_i \leq 1 = v_j$.

Case 3: $i \in [r, n - 2] - S$, $j \in S$

Then $a_j = 1$, so $a_i = 0$ by (1) and (2) of Lemma 9.11. Thus, $v_i = 0 \leq v_j$.

Case 4: $i, j \in [r, n - 2] - S$

Then $a_i = 0$ or $a_i = 2$. If $a_i = 0$, then $v_i = 0$, so $v_i \leq v_j$. If $a_i = 2$, then $a_j = 2$ by (2) of Lemma 9.11, which means that $v_j = 1 \geq v_i$.

Case 5: $n - 2 \to n - 1$ in $Q^0$.

If $a_{n-2} = 0$, then $v_{n-2} = 0 \leq v_{n-1}$. If $a_{n-2} = 2$, then $a_{n-1} = 1$ by (3) of Lemma 9.11, so $v_{n-1} = 1 \geq v_{n-2}$. If $a_{n-2} = 1$ and $a_{n-1} = 1$, then the $(n - 1)$th component of $v^{(0)}$ and hence of $v$ is equal to 1. If $a_{n-2} = 1$ and $a_{n-1} = 0$, then the $(n - 1)$th and $(n - 2)$th components of $v^{(0)}$ are both equal to 0. Since $n - 2 \to n - 1$ in $Q^0$ is critical with respect to $(d, a)$, the statement for $v^{(0)}$ also holds for $v$.

Case 6: $n - 1 \to n - 2$ in $Q^0$.

If $a_{n-1} = 0$, then $v_{n-1} = 0 \leq v_{n-2}$. Suppose that $a_{n-1} = 1$. Then $a_{n-2} \geq 1$ by (4) of Lemma 9.11, so the $(n - 1)$th and $(n - 2)$th components of $v^{(0)}$ are both equal to 1. If $a_{n-2} = 1$, then $n - 1 \to n - 2$ in $Q^0$ is critical with respect to $(d, a)$, so the $(n - 1)$th and $(n - 2)$th components of $v$ are also equal to 1. If $a_{n-2} = 2$, then $v_{n-2} = 1$, so $v_{n-1} \leq v_{n-2}$. 
Next, we need to prove that \( Z^w \) occurs with nonzero coefficient in \( F_{d''}^l \). This means proving that if \( i, j \in [r, n - 2] \cup \{n\} \) with \( i \rightarrow j \) in \( Q^0 \), then \( w_j \geq w_i \). If \( i, j \in [r, n - 2] \), then the proof is similar to the proof that \( v_j \geq v_i \) above given in Cases 1 to 4.

Suppose \( n - 2 \rightarrow n \) in \( Q^0 \). If \( a_{n-2} = 0 \) or 2, then the proof is similar to Case 5 above. If \( a_{n-2} = 1 \) and \( a_n = 0 \), then (5) of Lemma 9.11 implies that \( n - 2 \rightarrow n - 1 \) is not critical, so \( a_{n-1} = 1 \). Thus, \( w_{n-2} = 0 \leq w_n \). If \( a_n = 1 \), then \( w_n = 1 \geq w_{n-2} \).

Suppose \( n \rightarrow n - 2 \) in \( Q^0 \). If \( a_n = 0 \), then \( w_n = 0 \leq w_{n-2} \). Suppose that \( a_n = 1 \). Then \( w_n = 1 \), and \( a_{n-2} \geq 1 \) by (4) of Lemma 9.11. If \( a_{n-2} = 2 \), then \( w_{n-2} = 1 \geq w_n \). If \( a_{n-2} = 1 \), then \( n \rightarrow n - 2 \) is critical, so by (5) of Lemma 9.11, we must have \( a_{n-1} = 0 \), which means that \( w_{n-2} = 1 \geq w_n \), as desired. This proves that \( (v, w) \in S_a \).

Now suppose that \( (v, w) \in S_a \) with \( v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \). Clearly, the \( i \)th component of \( v^{(0)} \) is equal to the \( i \)th component of \( v \) whenever \( a_i = 0 \) or 2, and for \( i = n - 1 \) and \( i = n \). Note that if \( i, j \in S \) and \( i \rightarrow j \) in \( Q^0 \), then \( v_j \geq v_i \) and \( w_i \geq w_j \); since \( v_i + w_i = 1 \) and \( v_j + w_j = 1 \), one may show that \( v_i = v_j, w_i = w_j \). It follows that if \( i, j \) are in the same component of \( S \), then \( v_i = v_j \).

It remains to show that if either the arrow between \( n - 2 \) and \( n - 1 \) in \( Q^0 \) or the arrow between \( n - 2 \) and \( n \) in \( Q^0 \) is critical, then the \( (n - 2) \)th components of \( v \) and \( v^{(0)} \) are equal. Suppose that \( n - 2 \rightarrow n - 1 \) and \( n - 2 \rightarrow n \) in \( Q^0 \). If \( n - 2 \rightarrow n - 1 \) is critical, then \( n - 2 \rightarrow n \) is not critical by (5) of Lemma 9.11, which means that \( a_{n-2} = 1, a_{n-1} = 0, \) and \( a_n = 1 \). Since \( v_{n-2} \leq v_{n-1} \), this means that \( v_{n-2} = 0 \). If \( n - 2 \rightarrow n \) is critical, then \( n - 2 \rightarrow n - 1 \) is not critical, and this means that \( a_{n-2} = 1, a_{n-1} = 1, \) and \( a_n = 0 \). Since \( w_{n-2} \leq w_n = 0 \), it follows that \( w_{n-2} = 0 \) and hence, \( v_{n-2} = 1 \). The case where \( n - 1 \rightarrow n - 2 \) and \( n \rightarrow n - 2 \) in \( Q^0 \) is similar. \( \square \)

For \( (v, w) \in S_a \), let

\[
\psi_{d''}(v) = -\frac{d}{2} (g_{d''} \cdot v)
\]
\[
\psi'_{d''}(w) = -\frac{d}{2} (g_{d''} \cdot w) - d(g_{d''} \cdot w).
\]

The coefficients of \( Z^v \) in \( F_{d''} \) and \( Z^w \) in \( F'_{d''} \) are given by \( q_{d''}(v) \) and \( q'_{d''}(w) \), respectively.
Define a skew-symmetric bilinear form \( \langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \frac{1}{2} \mathbb{Z} \) by

\[
\langle e_i, e_j \rangle = \frac{d_b^{0}_{ij}}{2}.
\]

Write

\[
\tilde{\psi}(v, w) = \psi_{d''}(v) + \psi'_{d'''}(w) + \langle v, w \rangle.
\]

By (6.7),

\[
q^{\psi_{d''}(v)} Z^v \cdot q^{\psi'_{d'''}(w)} Z^w = q^{\tilde{\psi}(v, w)} Z^a.
\]

**Lemma 9.13.** Let \([s_1, s_2] \in \mathcal{C}(S)\), and let \((v, w) \in S_a\) with \(v = (v_1, \ldots, v_n)\) and \(w = (w_1, \ldots, w_n)\) such that \(v_i = 1\) for all \(i \in [s_1, s_2]\). Write \(a' = \sum_{i \in [s_1, s_2]} e_i\). Then

\[
\tilde{\psi}(v - a', w + a') = d + \tilde{\psi}(v, w).
\]

**Proof.** Note that

\[
\tilde{\psi}(v - a', w + a') - \tilde{\psi}(v, w) = - \frac{d}{2} (g_{d''} + g_{d'''}) \cdot a' + \langle a, a' \rangle.
\]

We show that the right hand side of this expression is equal to \(d\). Assume that \(d'', d'''\) are as at (9.18). (The proof when \(d'', d'''\) are as at (9.19) is similar.) Also, assume that \(s_1 \geq 2\). (The proof below may be easily modified in the case that \(s_1 = 1\) by omitting any expression that contains \(s_1 - 1\) as a subscript.) First, suppose that \(s_2 < n - 2\). By Theorem 4.18,

\[
(-g_{d''} - g_{d'''}) \cdot a' = 2(s_2 - s_1 + 1) - \sum_{i \in [r, n-1], j \in [s_1, s_2]} [-b^0_{ji}]_+ - \sum_{i \in [r, n-1], j \in [s_1, s_2]} [-b^0_{ji}]_+
\]

where the last summation on the right hand side of the equation ranges over \(i \in [r, n-2] \cup \{n\}, j \in [s_1, s_2]\).

Since \([-b^0_{i,i+1}]_+ = 1\) or \([-b^0_{i+1,i}]_+ = 1\) (but not both) for each \(i \in [s_1, s_2 - 1]\), we have

\[
\sum_{i, j \in [s_1, s_2]} [-b^0_{ji}]_+ = s_2 - s_1.
\]

Thus,

\[
(-g_{d''} - g_{d'''}) \cdot a' = 2 - 2[-b^0_{s_1,s_2-1}]_+ + 2[-b^0_{s_2,s_2+1}]_+.
\]
where \( \epsilon_{s_1} = 1 \) if \( s_1 - 1 \geq r \), and \( \epsilon_{s_1} = 0 \) otherwise. Next, note that if \( s_2 \to s_2 + 1 \) in \( Q^0 \) (i.e., \( b_{s_2+1,s_2} = 1 = -b_{s_2,s_2+1}^0 \)), then \( a_{s_2} = 1 \) implies \( a_{s_2+1} = 2 \) by (2) of Lemma 9.11. On the other hand, if \( s_2 + 1 \to s_2 \) in \( Q^0 \) (i.e., \( b_{s_2+1,s_2} = -1 = -b_{s_2,s_2+1}^0 \)), then \( a_{s_2} = 1 \) implies \( a_{s_2+1} = 0 \) by the same property. In either case, we have \( a_{s_2+1} b_{s_2+1,s_2}^0 = 2[-b_{s_2,s_2+1}^0]^+ \). Likewise, it is easy to show that \( a_{s_1-1} b_{s_1-1,s_1}^0 = 2[-b_{s_1,s_1-1}^0]^+ \) if \( s_1 - 1 \geq r \) and \( a_{s_1-1} = 0 \) otherwise. Thus,

\[
\begin{align*}
\langle a, a' \rangle &= \frac{d}{2} (a_{s_1-1} a_{s_1} b_{s_1-1,s_1}^0 \epsilon_{s_1} + a_{s_2} a_{s_2+1} b_{s_2,s_2+1}^0) \\
(9.40) &= d[-b_{s_1,s_1-1}^0]^+ \epsilon_{s_1} + d[-b_{s_2,s_2+1}^0]^+. \\
(9.41) &\end{align*}
\]

This proves that the right hand side of (9.36) is equal to \( d \) in this case.

Next, we consider the case where \( s_2 = n - 2 \). By reasoning as in the previous case, one may show that

\[
\begin{align*}
(9.42) \quad ( -g_{d''} - g_{d''} ) \cdot a' &= 2 - 2[-b_{s_1,s_1-1}^0]^+ \epsilon_{s_1} - [-b_{n-2,n}^0]^+ - [-b_{n-2,n-1}^0]^+. \\
\end{align*}
\]

Note that the arrows between \( n - 2 \) and \( n \) and between \( n - 2 \) and \( n - 1 \) in \( Q^0 \) are not critical with respect to \( (d,a) \) since \( [s_1, n - 2] \in C(S) \). Using the fact that \( a_{n-2} = 1 \), one may show that

\[
\begin{align*}
(9.43) \quad a_n b_{n,n-2}^0 &= [-b_{n-2,n}^0]^+ \\
(9.44) \quad a_{n-1} b_{n,n-1}^0 &= [-b_{n-2,n-1}^0]^+. \\
\end{align*}
\]

Therefore,

\[
\begin{align*}
(9.45) \quad \langle a, a' \rangle &= \frac{d}{2} (a_{s_1-1} a_{s_1} b_{s_1-1,s_1}^0 \epsilon_{s_1} + a_{n-2} a_{n} b_{n,n-2}^0 + a_{n-1} a_{n} b_{n,n-1}^0) \\
(9.46) &= d[-b_{s_1,s_1-1}^0]^+ \epsilon_{s_1} + \frac{d}{2} [-b_{n-2,n}^0]^+ + \frac{d}{2} [-b_{n-2,n-1}^0]^+. \\
\end{align*}
\]

This proves the desired assertion.

\( \square \)

Now we are ready to complete the proof of the theorem for type \( D_n \). Assume that \( v^{(0)}, w^{(0)} \) are as at (9.24). (The proofs in the other cases are similar.) By Lemma 9.12, the coefficient of \( Z^a \) in the expansion of \( F_{q^v} F_{d''} \) is

\[
\sum_{(v,w) \in S_a} q^{\tilde{v}(v,w)} = \sum_{(v,w) \in S_a} q^{\tilde{v}(v^{(0)} - a',w^{(0)} + a')} ,
\]
where the summation on the right hand side ranges over \( a' = \sum_{C \in J} e_C \) such that \( J \subset \mathcal{C}(S) \). Using Lemma 9.13 and induction on the cardinality of \( J \), it is easy to show that

\[
\tilde{\psi}(v^{(0)} - a', w^{(0)} + a') = \tilde{\psi}(v^{(0)}, w^{(0)}) + d|J|.
\]

Using the binomial theorem, the right hand side of (9.47) may be rewritten as

\[
\sum_{i=0}^{|[\mathcal{C}(S)]|} \binom{|[\mathcal{C}(S)]|}{i} q^{i\tilde{\psi}(v^{(0)}, w^{(0)}) + di} = q^{\tilde{\psi}(v^{(0)}, w^{(0))}}(1 + q^{d}|\mathcal{C}(S)|).
\]

This proves that the coefficient of \( Z^a \) in \( F_d v F_d' w \) has the form at (9.22), as desired.

9.3. Type \( B_n \). Let \( B^0 = (b^0_{ij}) \) be an acyclic \( n \times n \) exchange matrix of type \( B_n \).

First, we will prove the theorem for \( d \) of the form \( e_p + \ldots + e_r \), where \( 1 \leq p \leq r \leq n \). It suffices to show that for \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \), \( Z^a \) occurs with nonzero coefficient in \( F_d \) if and only if \( u_1^{a_1} \ldots u_n^{a_n} \) occurs with nonzero coefficient in \( F_d' \), and in this case, the coefficient of \( Z^a \) in \( F_d \) is of the form \( q^c \) for some \( c \in \frac{1}{2}\mathbb{Z} \). Then Theorem 9.1 follows from Corollary 7.5.

Lemma 9.14. Let \( d = e_p + \ldots + e_r \in \Phi_+(B^0) \). Then

\[
F_d = q^{\lambda'} F_d' Z^{a'} + q^{\lambda''} F_d'' Z^{a''}
\]

for some \( \lambda', \lambda'' \in \frac{1}{2}\mathbb{Z}, a', a'' \in \mathbb{Z}_2^n \), and \( d', d'' \in \Phi_+(B^0) \cup \{0\} \) such that \( d', d'' < d \) and the \( r \)th component of \( d' \) and \( d'' \) are both 0.

Proof. A similar strategy as in the proof of Lemma 9.10 will be used. Note that the diagonals \( \alpha_1, \ldots, \alpha_n \) divide half of \( \mathbb{P}_{2n+2} \) into \( n \) triangles. For \( i = 1, \ldots, n - 1 \), let \( \Delta_i \) be the triangle containing \( \alpha_i \) and \( \alpha_{i+1} \) as sides, and let \( \Delta_0 \) be the triangle with \( \alpha_1 \) as a side but not \( \alpha_2 \). For \( i \in [1, n-1] \), let \( v_i \) be the vertex opposite \( \alpha_i \) in \( \Delta_i \). Let \( v_n \) be the endpoint of \( \theta(\alpha_{n-1}) \) which is not on the diagonal \( \alpha_n \). Let \( w_i \) be the vertex opposite \( \alpha_i \) in the triangle \( \Delta_{i-1} \) for \( i \in [1, n] \).

Mutate the initial cluster in directions \( k = p, \ldots, r \), and consider the corresponding sequence of mutations of \( \mathcal{D}^0 \). When the mutation in direction \( k \in [p, r] \) occurs, the new diagonal obtained is \([w_p v_k]\) if \( k < n \) and \([w_p v_k]\) if \( k = n \). Note that the elements of \( \mathcal{D}^0 \) that the diagonal \([w_p v_k]\) \( (k < n) \) intersects are precisely \( \alpha_p, \ldots, \alpha_k \), and the elements that the diagonal \([w_p v_k]\) intersects are precisely \( \alpha_p, \ldots, \alpha_n, \theta(\alpha_{n-1}), \ldots, \theta(\alpha_p) \). In either case, the \( k \)th cluster variable in the final cluster has denominator vector \( e_p + \ldots + e_k \). Observe that \([w_p v_k]\) is not a diameter unless \( k = r = n \). By
using (9.9) or (9.10) in conjunction with Theorem 7.10, it is easy to show that an equation of the form (9.50) holds.

Proceed by induction on the difference \( r - p \). It is easy to show that \( F_{e_i} = q^d Z_i + 1 \) for \( i \in [1, n-1] \) and \( F_{e_n} = q^{d/2} Z_n + 1 \) using Theorem 7.10. Now suppose that \( r - p > 1 \), and assume the theorem has been proven for all \( \mathbf{e}_{p'} + \ldots + \mathbf{e}_{r'} \in \Phi_+(B^0) \) such that \( r' - p' < r - p \). Let \( \mathbf{d} = \mathbf{e}_p + \cdots + \mathbf{e}_r \).

By reasoning as in the type \( D_n \) case, one may show that for \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \), \( Z^\mathbf{a} \) occurs with nonzero coefficient in \( F\mathbf{d} \) if and only if \( u_1^{a_1} \ldots u_n^{a_n} \) occurs with nonzero coefficient in \( F_{\mathbf{d}} \).

Furthermore, one of the expressions \( q^\lambda F_{\mathbf{d}'} Z^\mathbf{a}', q^\lambda' F_{\mathbf{d}''} Z^\mathbf{a}'' \) in Lemma 9.14 contains all of the terms in \( F\mathbf{d} \) with \( Z_r \) in it, and the other expression contains all of the terms in \( F\mathbf{d} \) without \( Z_r \). It is known that the coefficient of every monomial in \( F\mathbf{d}' \) and \( F_{\mathbf{d}''} \) is a power of \( q \), so the same statement is true of \( F\mathbf{d} \). This finishes the proof of the theorem in the case where \( d = \mathbf{e}_p + \cdots + \mathbf{e}_r \).

For the remainder of the proof of the theorem, we turn our attention to denominator vectors of the form \( \mathbf{d} = \sum_{i=p}^n \mathbf{e}_i + \sum_{j=r}^n \mathbf{e}_j \in \Phi_+(B^0) \), where \( p < r \). The base of the induction occurs when \( n - p = 0 \), in which case, \( \mathbf{d} = \mathbf{e}_n \), and the theorem has already been proven.

Now let \( 1 \leq p < r \leq n \), and assume that Theorem 9.1 has been proven for \( \mathbf{d} = \sum_{i=p}^n \mathbf{e}_i + \sum_{j=r}^n \mathbf{e}_j \) for \( 1 \leq p' < r' \leq n \) such that \( n - p' < n - p \). It suffices to show that for \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n \), \( Z^\mathbf{a} \) occurs with nonzero coefficient in \( F\mathbf{d} \) if and only if \( u_1^{a_1} \ldots u_n^{a_n} \) occurs with nonzero coefficient in \( F_{\mathbf{d}} \), and in this case, the coefficient of \( Z^\mathbf{a} \) in \( F\mathbf{d} \) is of the form

\[
q^{c_1} (1 + q^d)^{\rho_{\mathbf{d}}(\mathbf{a})} (1 + q^{2d})^{c_2}.
\]

Once this is done, it is easy to finish the proof of the theorem as follows. By Theorem 6.3, setting \( q = 1 \) and \( Z_i = u_i \) for \( i \in [1, n] \) in \( F\mathbf{d} \) yields \( F_{\mathbf{d}}^{\mathbf{d}} \). This forces \( c_2 = \phi_{\mathbf{d}}(\mathbf{a}) - \rho_{\mathbf{d}}(\mathbf{a}) \). Note that the expression at (9.51) can be rewritten as

\[
q^{c_1'} (q^{-d/2} + q^{d/2})^{\rho_{\mathbf{d}}(\mathbf{a})} (q^{-d} + q^{d})^{\phi_{\mathbf{d}}(\mathbf{a}) - \rho_{\mathbf{d}}(\mathbf{a})}
\]

for some \( c_1' \in \frac{1}{2} \mathbb{Z} \). By applying Corollary 7.5, it follows that \( c_1' = -\frac{d}{2} \cdot g_{\mathbf{d}} \cdot \mathbf{a} \), as desired.

**Lemma 9.15.** Let \( \mathbf{d} = \mathbf{e}_p + \cdots + 2 \mathbf{e}_r + \cdots + 2 \mathbf{e}_n \in \Phi_+(B^0) \). At least one of the two statements below is true:

1. For some \( \lambda', \lambda'' \in \frac{1}{2} \mathbb{Z} \), \( \mathbf{a}', \mathbf{a}'' \in \mathbb{Z}^n \), and \( \mathbf{d}', \mathbf{d}'' \in \Phi_+(B^0) \cup \{0\} \) such that \( \mathbf{d}', \mathbf{d}'' < \mathbf{d} \) and the \( p \)th component of both vectors is 0, the quantum \( F \)-polynomial \( F_{\mathbf{d}} \) satisfies
\begin{equation}
F_d = q^\lambda F_{d'} Z^a' + q^{\lambda''} F_{d''} Z^a'' \tag{9.53}
\end{equation}

(2) For some \(a', a'' \in \mathbb{Z}^n\), \(\lambda', \lambda'' \in \frac{1}{2} \mathbb{Z}\), and \(d' \in \Phi_+(B^0) \cup \{0\}\) such that \(d' < d\) and the \(p\)th component of \(d'\) is 0, we have

\begin{equation}
F_d = q^\lambda F_{d'} Z^a' + q^{\lambda''} F_{d''} F_{d'''} Z^a'' \tag{9.54}
\end{equation}

where

\begin{equation}
d'' = e_r + \cdots + e_n \tag{9.55}
\end{equation}

\begin{equation}
F_{d''} = L[g_{d''}] (F_{d''}) \tag{9.56}
\end{equation}

\textbf{Proof.} Similar reasoning as in Lemma 9.10 will be used. Use the same notation as established in the proof of Lemma 9.14. First, mutate in directions \(k = r, \ldots, n\).

Let \(r \leq j_1 < \cdots < j_s \leq n - 1\) be indices such that the arrow between \(j_{\ell}\) and \(j_{\ell} + 1\) in \(Q^0\) is opposite the arrow between \(r - 1\) and \(r\) for \(\ell = 1, \ldots, s\). Next, we continue mutating in directions \(k = j_1, \ldots, j_s\). Let \(w\) be the endpoint of the diagonal \(\alpha_{r-1}\) which is not equal to \(w_r\). Then \(\alpha'_{j_\ell}\) is mutated to \(\alpha''_{j_\ell} = [\overline{w_r} \nu_{j_{\ell}+1}]\) (for \(\ell \in [1, s - 1]\)) and \(\alpha'_{j_s}\) is mutated to \([\overline{w_r} w]\).

Continue mutating in directions \(k = r - 1, \ldots, p\). Then \(\alpha_k\) is mutated to \(\alpha_k' = [\overline{w_r} w_k]\) for \(k = p, \ldots, r - 1\). Note that \(\alpha_p'\) corresponds to the cluster variable with denominator vector \(d\) since \(\alpha_p'\) crosses the diagonals \(\alpha_p, \ldots, \alpha_n, \theta(\alpha_r), \ldots, \theta(\alpha_{n-1})\), and \(\theta(\alpha_p')\) crosses the diagonals \(\theta(\alpha_p), \ldots, \theta(\alpha_{n-1}), \alpha_n, \alpha_r, \ldots, \alpha_{n-1}\).

Let \(D'\) be the final collection of diagonals obtained through this process. Let \(c_1, c_2\) be the endpoints of \(\alpha_p\). The diagonal \(\alpha_p'\) is the diagonal of a quadrilateral whose sides are each in \(D'\) or sides of \(\mathbb{P}_{2n+2}\). To be more precise, the sides of the quadrilateral are \(\beta_1 = [\overline{w_r} c_1], \beta_2 = [\overline{w_r} c_2]\), and \(\alpha_{p-1}\) (if \(p - 1 \geq 1\); the remaining sides of the quadrilateral (if any) are sides of \(\mathbb{P}_{2n+2}\).

Observe that the only elements of \(D^0\) that the \(\beta_i\) can intersect come from the list

\[\alpha_{p+1}, \ldots, \alpha_n, \theta(\alpha_r), \ldots, \theta(\alpha_{n-1})\]

and \(\theta(\beta_i)\) can only intersect diagonals in the list \(\theta(\alpha_{p+1}), \ldots, \theta(\alpha_n), \alpha_n, \alpha_r, \ldots, \alpha_{n-1}\). Thus, each \(\beta_i\) corresponds to some \(d' \in \Phi_+(B^0)\) such that \(d' < d\), and the \(p\)th component of \(d'\) is equal to 0.
Use Theorem 7.10 in conjunction with (9.8) or (9.10). If \( c_i \neq w_r \) for \( i = 1, 2 \), then the diagonals \( \beta_1, \beta_2 \) are not diameters, and we get an equation for \( F_d \) of the form (9.54) where the \( d', d'' \in \Phi_\perp(B^0) \) correspond to the diagonals \( \beta_1, \beta_2 \). If \( c_i = w_r \) for some \( i \), then observe that \([w_r \overline{w}_r]\) intersects the interiors of \( \alpha_r, \ldots, \alpha_n \). In this case, (9.55) holds. \( \square \)

Using Lemma 9.15 and a similar argument as in the type \( D_n \), it is easy to show that for \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \), \( Z^a \) occurs with nonzero coefficient in \( F_d \) if and only if \( u_1^{a_1} \ldots u_n^{a_n} \) occurs with nonzero coefficient in \( F'_d \).

We will consider the coefficients of \( F_{d'} F_{d''} \) for \( d', d'' \) as in Lemma 9.15. Let \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) such that \( Z^a \) occurs with nonzero coefficient in the expansion of \( F_{d'} F_{d''} \), and let \( \rho(a) = 1 \) if the \( n \)th component of \( a \) is 1, and \( \rho(a) = 0 \) otherwise. The next goal is to show that the coefficient of \( Z^a \) in \( F_{d'} F_{d''} \) is of the form

\[
q^{c_1} (1 + q^d)^{\rho(a)} (1 + q^{2d})^{c_2}
\]

for some \( c_1, c_2 \in \mathbb{Z}/2\mathbb{Z} \).

Let \( S \) be the subgraph of \( Q^0 \), induced by \( \{ i \in [1, n] : a_i = 1 \} \). Let \( \mathcal{C}(S) \) denote the set of components of the graph \( S \). Let

\[
v^{(0)} = \sum_{i \in [1, n], a_i \geq 1} e_i
\]

\[
w^{(0)} = \sum_{i \in [1, n], a_i = 2} e_i.
\]

Let \( S_a \) denote the set of \((v, w) \in \mathbb{Z}^n \times \mathbb{Z}^n \) such that \( v + w = a \), and \( Z^v, Z^w \) each occur with nonzero coefficient in \( F_{d'} \) (and thus in \( F'_d \)).

We will need to use the following properties of \( a \), which may be easily proven using the known formula for \( F_{d'}^{-1} \).

**Lemma 9.16.**

1. \( 0 \leq a_i \leq 2 \) for \( i \in [r, n] \), \( a_i = 0 \) otherwise.
2. If \( i \rightarrow j \) in \( Q^0 \), \( i, j \in [r, n] \), then \( a_i \leq a_j \).

**Proof.** Let \((v, w) \in S_a \) with \( v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \). Part (1) of Theorem 4.16 implies that \( 0 \leq v_i, w_i \leq 1 \) for \( i \in [r, n] \) and \( v_i, w_i = 0 \) for \( i \in [1, r - 1] \), so statement (1) of the lemma follows. Part (2) of Theorem 4.16 implies that if \( i, j \in [r, n] \) with \( i \rightarrow j \) in \( Q^0 \), then \( v_i \leq v_j \) and \( w_i \leq w_j \). The second part of the lemma follows immediately. \( \square \)
The next lemma explicitly gives the elements of the set $S_a$.

**Lemma 9.17.**

\[(9.60)\quad S_a = \{ (v^{(0)} - a', w^{(0)} + a') : J \subset \mathcal{C}(S), a' = \sum_{C \in J} e_C \}.\]

**Proof.** Let $(v, w) \in \mathbb{Z}^n \times \mathbb{Z}^n$, and write $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$.

First, suppose that $(v, w) = (v^{(0)} - a', w^{(0)} + a')$ with $a'$ as in the right hand side of \((9.60)\). It suffices to prove that if $i \rightarrow j$ in $Q^0$, $i, j \in [r, n]$, then $v_i \leq v_j$ and $w_i \leq w_j$. If $a_i = 0$, then $v_i = 0 \leq v_j$. If $a_i = 2$, then $a_i \leq a_j = 2$, so $v_j = 1 \geq v_i$. Now suppose that $a_i = 1$. Then $i, j$ are in the same component in $S$, so $v_i = v_j$. Similarly, one shows that $w_i \leq w_j$.

Next, suppose that $(v, w) \in S_a$, and write $v^{(0)} = (v_1^{(0)}, \ldots, v_n^{(0)})$. Let $a' = v^{(0)} - v = (a'_1, \ldots, a'_n)$. It is easy to see that $v_i = v_i^{(0)}$ whenever $a_i = 0$ or $a_i = 2$, so $a'_i = 0$ for such indices $i$. If $i \rightarrow j$ in $Q^0$ and $i, j$ are vertices in $S$, then $a_i = a_j = 1$, and one may show that $v_i = v_j = 1$ and $w_i = w_j = 0$, or $v_i = v_j = 0$ and $w_i = w_j = 1$. Thus, if $i, j$ are vertices in $S$ connected an edge, then $a'_i = a'_j$. Consequently, the same equality holds if $i, j$ are vertices in the same component in $S$. Thus, $a'$ is a sum of elements $e_C$, where $C \in \mathcal{C}(S)$, as desired. \[\square\]

For $(v, w) \in S_a$, let

\[(9.61)\quad \psi_{d^\nu}(v) = -\frac{d}{2}(\delta \cdot g_{d^\nu} \cdot v)\]

\[(9.62)\quad \psi'_{d^\nu}(w) = -\frac{3d}{2}(\delta \cdot g_{d^\nu} \cdot w).\]

The coefficients of $Z^v$ in $F_{d^\nu}$ and $Z^w$ in $F'_{d^\nu}$ are given by $q^{\psi_{d^\nu}(v)}$ and $q^{\psi'_{d^\nu}(w)}$, respectively.

Define a skew-symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \frac{1}{2} \mathbb{Z}$ by

\[(9.63)\quad \langle e_i, e_j \rangle = d \cdot \text{sgn}(b_{ij}^0)\]

Write

\[(9.64)\quad \tilde{\psi}(v, w) = \psi_{d^\nu}(v) + \psi'_{d^\nu}(w) + \langle v, w \rangle.\]

From \((6.7)\), it follows that

\[(9.65)\quad q^{\psi_{d^\nu}(v)} Z^v \cdot q^{\psi'_{d^\nu}(w)} Z^w = q^{\tilde{\psi}(v, w)} Z^a.\]
Lemma 9.18. Let \([s_1, s_2] \in \mathcal{C}(S)\), and let \((v, w) \in S_a\) with \(v = (v_1, \ldots, v_n)\) such that \(v_i = 1\) for all \(i \in [s_1, s_2]\). Write \(a' = \sum_{i \in [s_1, s_2]} e_i\). If \(s_2 < n\), then

\[
\tilde{\psi}(v - a', w + a') = 2d + \tilde{\psi}(v, w).
\]

(9.66)

Otherwise, if \(s_2 = n\), then

\[
\tilde{\psi}(v - a', w + a') = d + \tilde{\psi}(v, w).
\]

(9.67)

Proof. Note that

\[
\tilde{\psi}(v - a', w + a') - \tilde{\psi}(v, w) = -d \delta \cdot g_{d'} \cdot a' + \langle a, a' \rangle.
\]

(9.68)

We show that the right hand side of this expression is equal to \(2d\) if \(s_2 < n\) and equal to \(d\) if \(s_2 = n\). Assume that \(s_1 \geq 2\). (The proof below may be easily modified in the case that \(s_1 = 1\) by omitting any expression that contains \(s_1 - 1\) as a subscript.) First, suppose that \(s_2 \leq n - 1\). By Theorem 4.18,

\[
-\delta \cdot g_{d'} \cdot a' = 2(s_2 - s_1 + 1) - \sum_{i \in [r, n]} \sum_{j \in [s_1, s_2]} [-b_{0,i}]_+.
\]

(9.69)

Since \([-b_{0,i+1}]_+ = 1\) or \([-b_{0,i}]_+ = 1\) (but not both) for each \(i \in [s_1, s_2 - 1]\), we have

\[
\sum_{i,j \in [s_1, s_2]} [-b_{0,i}]_+ = s_2 - s_1.
\]

(9.70)

Thus,

\[
-\delta \cdot g_{d'} \cdot a' = 2 - 2[-b_{0,s_1,s_1-1}] + \epsilon_{s_1} - 2[-b_{0,s_2,s_2+1}] +,
\]

(9.71)

where \(\epsilon_{s_1} = 1\) if \(s_1 - 1 \geq r\), and \(\epsilon_{s_1} = 0\) otherwise.

Note that if \(s_2 \rightarrow s_2 + 1\) in \(Q^0\) (i.e. \(b_{s_2,s_2+1} = -1 = -b_{s_2+1,s_2}\)), then \(a_{s_2} = 1\) implies that \(a_{s_2+1} = 2\) by Lemma 9.16. If \(s_2 + 1 \rightarrow s_2\) in \(Q^0\) (i.e. \(b_{s_2+1,s_2+1} = 1 = -b_{s_2+1,s_2}\)), then \(a_{s_2} = 1\) implies that \(a_{s_2+1} = 0\), again by the same lemma. In either case, this means that \(a_{s_2+1} \text{sgn}(b_{s_2+1,s_2}) = 2[-b_{s_2,s_2+1}]_+\). Likewise, it is easy to show that \(a_{s_1-1} \text{sgn}(b_{s_1-1,s_1}) = 2[-b_{s_1,s_1-1}]_+\) when \(s_1 - 1 \geq r\).
It follows that

\begin{align}
\langle a, a' \rangle &= d(a_{s_1}a_{s_1-1} \text{sgn}(b^0_{s_1-1,s_1})\epsilon_{s_1} + a_{s_2}a_{s_2+1} \text{sgn}(b^0_{s_2+1,s_2})) \\
&= 2d[-b^0_{s_1,s_1-1}]+\epsilon_{s_1} + 2d[-b^0_{s_2,s_2+1}]+. 
\end{align}

This proves that the right hand side of (9.68) is equal to 2d.

Now suppose that \( s_2 = n \). In this case, \( -\delta \cdot \text{g}_{d''} \cdot a' \) equals

\begin{align}
2(n - s_1) + 1 - 2[-b^0_{n-1,n}]+ &= \sum_{i \in [r, n-1]} 2[-b^0_{ij}]+ \\
&= 3 - 2[-b^0_{n-1,n}]+ - 2[-b^0_{n,n-1}] + [-b^0_{n,n-1}]+\epsilon_{s_1}. 
\end{align}

Either \( b^0_{n-1,n} = -1 \) and \( b^0_{n,n-1} = 2 \), or \( b^0_{n-1,n} = 1 \) and \( b^0_{n,n-1} = -2 \), which means that \( -2[-b^0_{n-1,n}]+ - [-b^0_{n,n-1}] = -2 \). Thus,

\begin{equation}
-\delta \cdot \text{g}_{d''} \cdot a' = 1 - 2[-b^0_{s_1,s_1-1}]+\epsilon_{s_1}.
\end{equation}

On the other hand,

\begin{equation}
\langle a, a' \rangle = da_{s_1}a_{s_1-1} \text{sgn}(b^0_{s_1-1,s_1})\epsilon_{s_1} = 2d[-b^0_{s_1,s_1-1}]+\epsilon_{s_1}.
\end{equation}

Thus, in the case that \( s_2 = n \), we have that the right hand side of (9.68) is equal to \( d \). \( \square \)

Observe that the coefficient of \( Z^a \) in \( F_{d''}F_{d''} \) is

\begin{equation}
\sum_{(v,w) \in S_a} q\tilde{\psi}(v,w) = \sum q\tilde{\psi}(v^{(0)}-a',w^{(0)}+a'),
\end{equation}

where the summation on the right hand side ranges over \( a' = \sum_{C \in J} e_C \) such that \( J \subset C(S) \). Using Lemma 9.18 and induction on the cardinality of \( J \), it is easy to show that

\begin{equation}
\tilde{\psi}(v^{(0)}-a',w^{(0)}+a') = \tilde{\psi}(v^{(0)},w^{(0)}) + 2d|J|
\end{equation}

if the \( n \)th component of \( a' \) is 0, and

\begin{equation}
\tilde{\psi}(v^{(0)}-a',w^{(0)}+a') = \tilde{\psi}(v^{(0)},w^{(0)}) + 2d(|J| - 1) + d
\end{equation}
if the \( n \)th component of \( \mathbf{a}' \) is 1.

If the \( n \)th component of \( \mathbf{a} \) is 0 or 2, then the right hand side of (9.78) may be rewritten as

\[
(9.81) \quad \sum_{i=0}^{|C(S)|-1} (q^i \widetilde{\psi}(\mathbf{v}(0),\mathbf{w}(0)+2di) \left( \begin{array}{c} |C(S)| \\ i \end{array} \right)) = q^i \widetilde{\psi}(\mathbf{v}(0),\mathbf{w}(0))(1 + q^{2d}|C(S)|)
\]

If the \( n \)th component of \( \mathbf{a} \) is 1, then the summation in the right hand side of (9.78) may be split into two summations, one which ranges over all \( \mathbf{a}' \) as above such that the \( n \)th component of \( \mathbf{a}' \) is 1, and the other over \( \mathbf{a}' \) where the \( n \)th component is 1. Then the right hand side of (9.78) equals

\[
(9.82) \quad \sum_{i=0}^{|C(S)|-1} q^i \widetilde{\psi}(\mathbf{v}(0)-\mathbf{a}',\mathbf{w}(0)+\mathbf{a}') + q^d \sum_{i=0}^{|C(S)|-1} q^i \widetilde{\psi}(\mathbf{v}(0)-\mathbf{a}',\mathbf{w}(0)+\mathbf{a}')
\]

\[
(9.83) \quad = (1 + q^d) \sum_{i=0}^{|C(S)|-1} (q^i \widetilde{\psi}(\mathbf{v}(0),\mathbf{w}(0)+2di) \left( \begin{array}{c} |C(S)|-1 \\ i \end{array} \right))
\]

\[
(9.84) \quad = q^i \widetilde{\psi}(\mathbf{v}(0),\mathbf{w}(0))(1 + q^{2d}|C(S)|^{-1})(1 + q^d).
\]

This proves that the coefficient of \( Z^a \) in \( F_{d^r}F_{d^r}' \) has the form at (9.57), as desired.

Now we complete the proof of the theorem. Suppose that (9.54) holds. (The reasoning when (9.53) holds is similar.) By reasoning as in the type \( D_n \) case, one may show that one of the expansions of \( q^X F_{d^r} Z^{a'} \), \( q^X F_{d^r} F_{d^r}' Z^{a''} \) contains all the terms in \( F_d \) with \( Z_p \), while the other contains all terms in \( F_d \) without \( Z_p \). Thus, it suffices to show that the coefficients in the expansions of \( q^X F_{d^r} Z^{a'} \), \( q^X F_{d^r} F_{d^r}' Z^{a''} \) are of the form (9.51).

First, consider the expression \( q^X F_{d^r} F_{d^r}' Z^{a''} \). Note that the \( n \)th component of \( a'' \) is 0. Thus, for \( \mathbf{a} \in \mathbb{Z}^n \) such that \( Z^{a-a''} \) occurs with nonzero coefficient in \( F_{d^r} F_{d^r}' \), we have \( \rho_d(\mathbf{a}) = \rho(\mathbf{a} - a'') \).

Since it is known that the coefficient of \( Z^{a-a''} \) in \( F_{d^r} F_{d^r}' \) is of the form \( q^{c_1} (1 + q^d) \rho(\mathbf{a} - a'')(1 + q^{2d}) \), it follows that the coefficient of \( Z^a \) in the expansion \( q^X F_{d^r} F_{d^r}' Z^{a''} \) is of the form at (9.51).

Consider the coefficients in the expression \( q^X F_{d^r} Z^{a'} \). Suppose that is known that the coefficients of \( F_{d^r} \) are of the form given at (9.51). Either \( d' = e_{p'} + \cdots + 2e_{p'} + \cdots + e_{p'} \) for some \( p < p' < r' \leq n \), or \( d' = e_{p'} + \cdots + e_{p'} \) for some \( p < p' < r' \leq n \). In the former case, the \( n \)th component of \( \mathbf{a}' \) is equal to 0. Thus, for all \( \mathbf{a} \in \mathbb{Z}^n \) such that \( Z^{a-a'} \) occurs with nonzero coefficient in \( F_{d^r} \), it follows that \( \rho_{d'}(\mathbf{a} - \mathbf{a}') = \rho_d(\mathbf{a}) \). In the latter case, all coefficients of \( F_{d^r}^{cl} \) are 0 or 1, which means that
\[ \rho_d(a) = 0, \phi_d(a) = 0 \text{ for all } a \in \mathbb{Z}^n \text{ such that } Z^{a-a'} \text{ occurs with nonzero coefficient in } F_{d'}. \] This proves that the coefficients of \( F_d \) has the form at (9.51), as desired.

### 9.4. Type \( C_n \)

Let \( B^0 = (b^0_{ij}) \) be an acyclic \( n \times n \) exchange matrix of type \( C_n \).

First, we consider the case when \( d = e_p + \cdots + e_r \in \Phi_+(B^0) \) (\( 1 \leq p \leq r \leq n \)). Once the lemma below is proven, the proof for this type of denominator vector follows by similar reasoning as in the type \( B_n \) case.

**Lemma 9.19.** Let \( d = e_p + \cdots + e_r \in \Phi_+(B^0) \). Then

\[
F_d = q^{\lambda'} F_{d'} Z^{a'} + q^{\lambda''} F_{d''} Z^{a''}
\]

for some \( \lambda', \lambda'' \in \frac{1}{2} \mathbb{Z}, a', a'' \in \mathbb{Z}^n_{\geq 0} \), and \( d', d'' \in \Phi_+(B^0) \cup \{0\} \) such that \( d', d'' < d \) and the \( p \)th component of \( d' \) and \( d'' \) are both 0.

**Proof.** We use a similar strategy to Lemma 9.10. Use the same notation as established in the proof of Lemma 9.14. Mutate the initial cluster in directions \( k = r, \ldots, p \), and consider the corresponding sequence of mutations applied to \( D^0 \). When the mutation in direction \( k \in [p, r] \) occurs, the new diagonal obtained is \([w_k v_r]\). This diagonal intersects the diagonals \( \alpha_k, \ldots, \alpha_r \) and no other elements in \( D^0 \). Thus, the new \( k \)th cluster variable obtained has denominator vector \( e_k + \cdots + e_r \). The lemma follows from (9.10) and Theorem 7.10. \qed

For the remaining \( d \in \Phi_+(B^0) \), it suffices to prove that for \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n, Z^a \) occurs with nonzero coefficient in \( F_d \) if and only if \( u_1^{a_1} \cdots u_n^{a_n} \) occurs with nonzero coefficient in \( F_{d'}^d \), and in this case, the coefficient of each monomial \( Z^a \) is of the form

\[
q^{c_1} (1 + q^d)^{c_2}
\]

for some \( c_1, c_2 \in \frac{1}{2} \mathbb{Z} \). After this is done, the rest of the proof of Theorem 9.1 is similar to the type \( D_n \) case.

First, we will consider denominator vectors of the form \( d = 2e_p + \cdots + 2e_{n-1} + e_n \).

**Lemma 9.20.** Let \( d = 2e_p + \cdots + 2e_{n-1} + e_n \in \Phi_+(B^0) \). Then there exist \( \lambda', \lambda'' \in \frac{1}{2} \mathbb{Z}, a', a'' \in \mathbb{Z}^n \), and \( d', d'' \) with each of \( d', d'' \) either equal to 0 or of the form \( e_\ell + \cdots + e_r \) for some \( p \leq \ell \leq r \leq n-1 \), such that

\[
F_d = q^{\lambda_1} F_{d'} Z^{a'} + q^{\lambda_2} F_{d''} Z^{a''},
\]
where

\begin{align}
F_{d'} &= L[g_{d'}](F_{d'}) \\
F_{d''} &= L[g_{d''}](F_{d''})
\end{align}

Consequently, for \( a = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \), \( Z^a \) occurs with nonzero coefficient in \( F_d \) if and only if \( u_1^{a_1} \cdots u_n^{a_n} \) occurs with nonzero coefficient in \( F_{d'} \).

**Proof.** Use the same notation as in Lemma 9.14. Mutate the initial cluster in directions \( k = p, \ldots, n \).

In this case, the \( k \)th cluster variable in the final cluster has denominator vector \( e_p + \cdots + e_k \) \((k \in [p, n-1])\), while the \( n \)th cluster variable has denominator vector \( 2e_p + \cdots + 2e_{n-1} + e_n \). The lemma follows from Theorem 7.10 and (9.9). \( \square \)

We need to show that the coefficients of \( F_{d'} F'_{d} \) and \( F_{d''} F'_{d''} \) are of the form given at (9.86). Let \( d'' = e_p + \cdots + e_r \), where \( p \leq r \leq n - 1 \), and let \( a = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \) such that \( Z^a \) occurs with nonzero coefficient in \( F_d \). Using the vectors \( d'' \) and \( a \), define \( S, \mathbf{v}^{(0)}, \mathbf{w}^{(0)}, S_a, \mathcal{C}(S), \psi_{d''}, \psi'_{d''} \) as in the previous subsection. Observe that Lemmas 9.16 and 9.17 hold in this setting using the same proof.

Define a skew-symmetric bilinear form \( \langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \frac{1}{2} \mathbb{Z} \) by

\begin{equation}
\langle e_i, e_j \rangle = \frac{d}{2} b^0_{ij}
\end{equation}

if \( (i, j) \neq (n, n - 1) \), and

\begin{equation}
\langle e_n, e_{n-1} \rangle = d b^0_{n,n-1}.
\end{equation}

Define \( \tilde{\psi} \) as at (9.64) using this bilinear form. Then (9.65) holds in this setting as well.

**Lemma 9.21.** Let \( [s_1, s_2] \in \mathcal{C}(S) \), and let \( (\mathbf{v}, \mathbf{w}) \in S_a \) with \( \mathbf{v} = (v_1, \ldots, v_n) \) and \( \mathbf{w} = (w_1, \ldots, w_n) \) such that \( v_i = 1 \) for all \( i \in [s_1, s_2] \). Write \( a' = \sum_{i \in [s_1, s_2]} e_i \). Then

\begin{equation}
\tilde{\psi}(\mathbf{v} - a', \mathbf{w} + a') = d + \tilde{\psi}(\mathbf{v}, \mathbf{w}).
\end{equation}

**Proof.** The proof will use similar reasoning to the proof of Lemma 9.18. It suffices to show that

\begin{equation}
-d \delta \cdot g_{d''} \cdot a' + \langle a, a' \rangle = d.
\end{equation}
Note that $-\delta \cdot g_{d'} \cdot a' = -g_{d'} \cdot a'$ since the $n$th component of $a'$ is 0. First, consider the case where $s_2 < n - 1$. Then one can show that

$$- \delta \cdot g_{d'} \cdot a' = (s_2 - s_1 + 1) - \sum_{i \in [p,r], j \in [s_1,s_2]} [-b_{ij}^0]_+$$

(9.94)

$$= 1 - [-b_{s_1,s_1-1}^0]_+ \epsilon_{s_1} - [-b_{s_2,s_2+1}^0]_+ \epsilon_{s_2},$$

(9.95)

where $\epsilon_{s_1} = 0$ if $s_1 - 1 < r$ and $\epsilon_{s_1} = 1$ otherwise, and $\epsilon_{s_2} = 0$ if $s_2 + 1 > r$ and $\epsilon_{s_2} = 1$ otherwise. Furthermore,

$$\langle a, a' \rangle = a_{s_1-1}a_{s_1} \epsilon_{s_1} \cdot \frac{d}{2} b_{s_1-1,s_1}^0 + a_{s_2+1}a_{s_2} \epsilon_{s_2} \cdot \frac{d}{2} b_{s_2+1,s_2}^0.$$  

(9.96)

It is easy to show that $a_{s_1-1}b_{s_1-1,s_1}^0 = 2[-b_{s_1,s_1-1}^0]_+$ if $s_1 - 1 \geq p$ and $a_{s_2+1}b_{s_2+1,s_2}^0 = 2[-b_{s_2,s_2+1}^0]_+$ if $s_2 + 1 \leq r$. Thus, (9.93) holds in this case.

Now suppose that $s_2 = n - 1$. Then $r = n - 1$, and

$$- \delta \cdot g_{d'} \cdot a' = (n - s_1) - \sum_{i \in [p,n-1], j \in [s_1,n-1]} [-b_{ij}^0]_+$$

(9.97)

$$= 1 - [-b_{s_1,s_1-1}^0]_+ \epsilon_{s_1},$$

(9.98)

Also,

$$\langle a, a' \rangle = a_{s_1-1}a_{s_1} \epsilon_{s_1} \cdot \frac{d}{2} b_{s_1-1,s_1}^0.$$  

(9.99)

Again, (9.93) holds.  

By reasoning as in the type $D_n$ case, it is easy to show that the coefficient of $Z^a$ in $F_{d''}F_{d'\nu}$ has the form at (9.86). It then follows from Lemma 9.20 that the same is true of the coefficients of $F_d$.

Next, we consider denominator vectors of the form

$$\sum_{i=p}^{n} e_i + \sum_{j=r}^{n-1} e_j = e_p + \cdots + 2e_r + \cdots + 2e_{n-1} + e_n \in \Phi_+(B^0),$$

(9.100)

where $p < r$. Proceed by induction on $n - p$. The base of the induction occurs when $n - p = 1$, in which case $d = e_{n-1} + e_n$, and Theorem 9.1 is already proven. Now suppose that $d = e_p + \cdots + 2e_r + \cdots + 2e_{n-1} + e_n$, and the theorem has been proven for denominator vectors $e_{p'} + \cdots + 2e_{r'} + \cdots + 2e_{n-1} + e_n$ such that $p < p' < r' \leq n - 1$. 

Lemma 9.22. Let \( d = e_p + \cdots + 2e_r + \cdots + 2e_{n-1} + e_n \in \Phi_+(B^0) \). Then there exist \( \lambda_1, \lambda_2 \in \frac{1}{2}Z \), \( c', c'' \in Z^n_{\geq 0} \), and \( d', d'' \in \Phi_+(B^0) \cup \{0\} \) satisfying \( d', d'' < d \) and the \( p \)th component of \( d' \) and \( d'' \) are 0, such that

\[
F_d = q^{\lambda_1} F_{d'} Z^{c'} + q^{\lambda_2} F_{d''} Z^{c''}.
\]

Consequently, for \( a = (a_1, \ldots, a_n) \in Z^n_{\geq 0} \), \( Z^a \) occurs with nonzero coefficient in \( F_d \) if and only if \( u_1^{a_1} \cdots u_n^{a_n} \) occurs with nonzero coefficient in \( F_d' \).

Proof. The proof is similar to that of Lemma 9.15. Use the same notation as defined in that proof and the same sequence of mutations, the diagonal \( \alpha_p \) obtained now corresponds to the denominator vector \( d \), and the diagonals \( \beta_1, \beta_2 \) correspond to \( d', d'' \in \Phi_+(B^0) \) such that \( d', d'' < d \) and the \( p \)th component of each vector is equal to 0. To finish the lemma, use (9.8) or (9.10) together with Theorem 7.10.

Use the expression for \( F_d \) given by (9.22). By arguing as in the type D\(_n\) case, one can show that all the terms in \( F_d \) containing \( Z_p \) are contained in one of the expansions of \( q^{\lambda_1} F_{d'} Z^{c'} , q^{\lambda_2} F_{d''} Z^{c''} \), and all of the terms not containing \( Z_p \) are contained in the other expansion. By induction or by using the theorem in the already established cases above, the coefficients of \( F_{d'} \), \( F_{d''} \) are of the form (9.86). Therefore, the same is true of the coefficients of \( F_d \). This completes the proof of the theorem for type C\(_n\).
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