Cluster Algebras of Finite Type via Semisimple Groups and Generalized Minors

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ABSTRACT OF DISSERTATION

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Abstract

In this thesis, we give a uniform geometric realization for the cluster algebra of an arbitrary finite type with principal coefficients at an arbitrary acyclic seed. This algebra is realized as the coordinate ring of a certain reduced double Bruhat cell in the simply connected semisimple algebraic group of the same Cartan-Killing type. In this realization, the cluster variables appear as certain (generalized) principal minors.

Based on this realization, we give combinatorial formulas for $F$-polynomials in cluster algebras of classical types in terms of the weighted paths in certain directed graphs. As a consequence we prove the positivity of $F$-polynomials in cluster algebras of classical types.
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Chapter 1

Introduction

The original motivation for the theory of cluster algebras, first introduced and studied by S. Fomin and A. Zelevinsky in [7], lay in the desire to create an algebraic framework for total positivity and (dual) canonical bases in semisimple algebraic groups ([5], [18], [19]). Since its inception, the theory of cluster algebras has found many connections and applications: Poisson geometry, Teichmüller theory, discrete integrable systems, quiver representations, preprojective algebras, Calabi-Yau algebras and categories, etc.

The primary subject of this dissertation is the theory of cluster algebras of finite type with principal coefficients at an arbitrary acyclic seed. In this chapter we collect some basic definitions and facts on cluster algebras of geometric type. Most of the material in this chapter is taken from [27, 10].

Before giving precise definitions, let us present some of the main features of cluster algebras. For any positive integer $n$, a cluster algebra $\mathcal{A}$ of rank $n$ is a commutative algebra with unit and no zero divisors equipped with a distinguished family of gener-
ators called *cluster variables*. The set of cluster variables is the (non-disjoint) union of a distinguished collection of \( n \)-subsets called *clusters*. These clusters have the following *exchange property*: for any cluster \( x \) and any element \( x \in x \), there is another cluster obtained from \( x \) by replacing \( x \) with an element \( x' \) related to \( x \) by a binomial exchange relation

\[
x x' = M_1 + M_2,
\]

where \( M_1 \) and \( M_2 \) are disjoint monomials in the \( n - 1 \) variables \( x - \{x\} \). Furthermore, any two clusters can be obtained from each other by a sequence of exchanges of this kind.

Examples of cluster algebras include the homogeneous coordinate rings of Grassmannians, Schubert varieties, and other related varieties (after a minor adjustment).

The prototypical example of a cluster algebra of rank 1 is the coordinate ring \( A = \mathbb{C}[SL_2] \) of the group \( SL_2 \), viewed in the following way. Writing a generic element of \( SL_2 \) as

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

we consider the entries \( a \) and \( d \) as cluster variables, and the entries \( b \) and \( c \) as scalars. There are two clusters \( \{a\} \) and \( \{d\} \), and \( A \) is the algebra over the ground ring \( \mathbb{C}[b, c] \) generated by the cluster variables \( a \) and \( d \) subject to the binomial exchange relation

\[
ad = 1 + bc.
\]

A nice example of a cluster algebra of an arbitrary rank \( n \) is the homogeneous coordinate ring \( \mathbb{C}[Gr_{2,n+3}] \) of the Grassmannian of 2-dimensional subspaces in \( \mathbb{C}^{n+3} \). This ring is generated by the Plücke coordinates \( x_{ij} \), for \( 1 \leq i < j \leq n + 3 \), subject
to the relations
\[ x_{ik}x_{jl} = x_{ij}x_{kl} + x_{il}x_{jk}, \]
for all \( i < j < k < l \). It is convenient to identify the indices \( 1, \ldots, n + 3 \) with the vertices of a convex \((n+3)\)-gon, and the Plücker coordinates with its sides and diagonals. Let us view the sides of the polygon as scalars, and the diagonals as cluster variables. The clusters are the maximal families of pairwise non-crossing diagonals; thus, they are in a natural bijection with the triangulations of the polygon.

## 1.1 Cluster Algebras of Geometric Type

For simplicity, we restrict ourselves, for now, to a special class of cluster algebras called \textit{cluster algebras of geometric type} (general definition of cluster algebras is given in Section 2.4).

Let \( m \) and \( n \) be two positive integers such that \( m \geq n \), and let \( \mathcal{F} \) be the field of rational functions over the \( \mathbb{Q} \) in \( m \) independent variables.

**Definition 1.1.1 (Labeled seeds of geometric type).** A (skew-symmetrizable) \textit{labeled seed} in \( \mathcal{F} \) is a pair \( (\tilde{x}, \tilde{B}) \), where

- \( \tilde{x} = \{x_1, \ldots, x_m\} \) is a set of algebraically independent generators for \( \mathcal{F} \), and
- \( \tilde{B} = (b_{i,j}) \) is an \( m \times n \) integer matrix such that the \( n \times n \) submatrix \( B = (b_{i,j}) \) for \( i, j \in [1, n] = \{1, 2, \ldots, n\} \) is \textit{skew-symmetrizable}, that is, \( d_ib_{i,j} = -d_jb_{j,i} \) for some positive integers \( d_1, \ldots, d_n \).

We refer to \( x = \{x_1, \ldots, x_n\} \) as the \textit{cluster}, to its elements \( x_1, \ldots, x_n \) as \textit{cluster
variables (of the seed \((\tilde{x}, \tilde{B})\)), to \(c = \{x_{n+1}, \ldots, x_m\}\) as the coefficient tuple, and to \(B\) as the exchange matrix. We will sometimes refer to labeled seeds of geometric type simply as seeds, when there is no risk of confusion.

We will use the notation \([b]_+ = \max(b, 0)\).

**Definition 1.1.2 (Seed mutations).** Let \((\tilde{x}, \tilde{B})\) be a seed in \(F\), and \(k \in [1, n]\). The seed mutation in direction \(k\) transforms \((\tilde{x}, \tilde{B})\) into another seed \((\tilde{x}', \tilde{B}')\) defined as follows:

- The entries of \(\tilde{B}' = (b'_{i,j})\) are given by

\[
b'_{i,j} = \begin{cases} 
- b_{i,j} & \text{if } i = k \text{ or } j = k; \\
 b_{i,j} + [b_{i,k}]_+ [b_{k,j}]_+ - [-b_{i,k}]_+ [-b_{k,j}]_+ & \text{otherwise.} 
\end{cases}
\]  

(1.1)

- The cluster \(x'\) is given by \(x'_j = x_j\) for \(j \neq k\), whereas \(x'_k \in F\) is determined by the exchange relation

\[
x_k' x_k = \prod_{i=1}^{m} x_i^{[b_{i,k}]_+} + \prod_{i=1}^{m} x_i^{[-b_{i,k}]_+}. 
\]  

(1.2)

Note that during the mutation process we only exchange the cluster variables and leave the coefficient tuple unchanged.

It is easy to see that \((\tilde{x}', \tilde{B}')\) is indeed a seed. Furthermore, the seed mutation \(\mu_k\) is involutive, that is, it transforms \((\tilde{x}', \tilde{B}')\) back into \((\tilde{x}, \tilde{B})\). This makes the following equivalence relation on the seeds (as well as on the matrices \(\tilde{B}\)) well defined: we say that two seeds \((\tilde{x}, \tilde{B})\) and \((\tilde{x}', \tilde{B}')\) are mutation equivalent, and write \((\tilde{x}, \tilde{B}) \sim (\tilde{x}', \tilde{B}')\).
1.2 Principal Coefficients

if \((\tilde{x}', \tilde{B}')\) can be obtained from \((\tilde{x}, \tilde{B})\) by a sequence of seed mutations. All seeds \((\tilde{x}', \tilde{B}')\) mutation equivalent to a given seed \((\tilde{x}, \tilde{B})\) share the same set \(c = \tilde{x} \setminus x\).

We denote by \(\mathcal{X} = \mathcal{X}(\tilde{x}, \tilde{B})\) the union of clusters of all seeds in the mutation equivalence class of seeds containing \((\tilde{x}, \tilde{B})\). We refer to the elements in \(\mathcal{X}\) as cluster variables.

**Definition 1.1.3 (Cluster algebra of geometric type).** Let \(\mathbb{Z}[c^{\pm 1}]\) be the ring of Laurent polynomials in the variables from \(c\). The cluster algebra \(\mathcal{A} = \mathcal{A}(\tilde{x}, \tilde{B}) = \mathcal{A}(x, \tilde{B})\) (of rank \(n\)) with an initial seed \((\tilde{x}, \tilde{B})\) is the \(\mathbb{Z}[c^{\pm 1}]\)-subalgebra of the ambient field \(\mathcal{F}\) generated by all cluster variables, that is by the union of all clusters obtained from the initial cluster by iterating seed mutations in all directions.

Note that the definition of a cluster algebra does not depend on the choice of an initial seed and depends only on its mutation equivalence class. That is, \(\mathcal{A}(\tilde{x}, \tilde{B})\) is naturally identified with \(\mathcal{A}(\tilde{x}', \tilde{B}')\) for any \((\tilde{x}', \tilde{B}')\) in the mutation equivalence class of seeds containing \((\tilde{x}, \tilde{B})\). We sometimes denote the cluster algebra \(\mathcal{A}(\tilde{x}, \tilde{B})\) simply as \(\mathcal{A}(\tilde{B})\), because \(\tilde{B}\) determines this algebra uniquely up to an automorphism of \(\mathcal{F}\).

## 1.2 Principal Coefficients

One of the important general properties of cluster algebras is the following Laurent phenomenon.

**Theorem 1.2.1 (Laurent phenomenon [7, Theorem 3.1]).** Any cluster variable viewed as a rational function in the variables of any given cluster is a Laurent polynomial whose coefficients are integer Laurent polynomials in the coefficient variables.
1.2. Principal Coefficients

\[ x_{n+1}, \ldots, x_m. \]

This means that even though the numerators of these Laurent polynomials may contain a huge number of monomials, at every stage of the exchange process – moving the numerator for a cluster variable \( x \) into the denominator when we compute the cluster variable \( x' \) obtained from \( x \) by an exchange relation – a cancelation will inevitably occur, leaving a single monomial in the denominator.

It was shown in \([10]\) that with any exchange matrix one can associate an important system of principal coefficients which in a certain sense controls all the other choices of coefficients.

**Definition 1.2.2 (Principal Coefficients \([10, \text{Definition 3.1}]\)).** We say that a seed \((\tilde{x}, \tilde{B})\) has principal coefficients if \(\tilde{B}\) is a \(2n \times n\) integer matrix and the bottom part (the row set indexed by \(\{n+1, \ldots, 2n\}\)) of \(\tilde{B}\) is the \(n \times n\) identity matrix. In this case we denote the coefficient tuple \(c = \{x_{n+1}, \ldots, x_{2n}\}\) by \(y = \{y_1, \ldots, y_n\}\) and the seed \((\tilde{x}, \tilde{B})\) is then denoted by \((x, y, B)\).

Let \(\mathcal{A} = \mathcal{A}(x, y, B)\) be the cluster algebra with principal coefficients at an initial seed \((\tilde{x}, \tilde{B})\). In this case, the Laurent phenomenon can be sharpened so that every element of \(\mathcal{A}\) is a Laurent polynomial in \(x_1, \ldots, x_n\) whose coefficients are integer polynomials in \(y_1, \ldots, y_n\). More precisely, for \(j = 1, \ldots, n\), we set

\[ \hat{y}_j = y_j \prod_{i=1}^{n} (x_i)^{b_{i,j}} \in \mathcal{F}. \] (1.3)

As a special case of \([10, \text{Corollary 6.3}]\), every cluster variable \(z \in \mathcal{A}\) can be (uniquely) written in the form

\[ z = (x_1)^{g_1} \cdots (x_n)^{g_n} F_{z;\mathbf{x}}(\hat{y}_1, \ldots, \hat{y}_n) \] (1.4)
for some integer vector
\[ g_{z:x} = (g_1, \ldots, g_n) \in \mathbb{Z}^n, \] (1.5)
and some integer polynomial \( F_{z:x}(t_1, \ldots, t_n) \in \mathbb{Z}[t_1, \ldots, t_n] \) not divisible by any variable \( t_i \). The vector \( g_{z:x} \) (resp. the polynomial \( F_{z:x} \)) is called the \( g \)-vector (resp. the \( F \)-polynomial) of \( z \) with respect to \( x \).

According to \([10, Corollary 6.3]\) (see also Section 2.4), for any choice of coefficients, expressing any cluster variable \( z \) in terms of an initial cluster \( x \) only requires knowing the \( g \)-vector \( g_{z:x} \) and the \( F \)-polynomial \( F_{z:x} \).

The \( F \)-polynomials are conjectured (and in many cases proved) to have positive coefficients. In Chapter 3 of this thesis, we give a combinatorial proof of this conjecture for cluster algebras of classical type with an arbitrary acyclic initial seed.

1.3 Classification of Cluster Algebras of Finite Type

The theory of cluster algebras has a lot in common with the theory of Kac-Moody algebras. In both instances, the structure of an algebra in question is encoded by a square integer matrix: a generalized Cartan matrix \( A \) in the Kac-Moody case, and an exchange matrix \( B \) in the case of cluster algebras. (Note an important distinction between the two cases: the sign pattern of matrix entries is symmetric for \( A \) but skew-symmetric for \( B \).) In both instances, there is a natural notion of finite type.

**Definition 1.3.1 (Cluster algebra of finite type).** We say that a cluster algebra is of finite type if it has finitely many (distinct) cluster variables.

As shown in \([9]\), the cluster algebras of finite types are classified by the same
1.3. Classification of Cluster Algebras of Finite Type

Cartan-Killing types as semisimple Lie algebras. To be more specific, to every skew-symmetrizable matrix $B = (b_{i,j})$ one can associate a generalized Cartan matrix $A = A(B) = (a_{i,j})$ of the same size by setting

$$a_{i,j} = \begin{cases} 
2 & \text{if } i = j; \\
-|b_{i,j}| & \text{if } i \neq j.
\end{cases}$$

(1.6)

The following theorem was given in [9].

**Theorem 1.3.2.** Let $\tilde{B}$ be an integer $m \times n$ matrix whose principal part $B$ is skew-symmetrizable. The cluster algebra $\mathcal{A}(\tilde{B})$ is of finite type if and only if $B$ is mutation equivalent to a matrix $B'$ such that $A(B')$ is a Cartan matrix of finite type, i.e., is a direct sum of the matrices from the Cartan-Killing list $A_n, B_n, \ldots, G_2$.

By Theorem 1.3.2, the property of a cluster algebra $\mathcal{A}$ to be of finite type does not depend on the choice of a coefficient system.

It is expected and in many cases proved that, for any simply-connected semisimple complex group $G$, the coordinate rings of many interesting varieties related to $G$, have a natural structure of a cluster algebra. This occasionally leads to an intriguing collision of two types of symmetry: the coordinate ring of a variety associated with $G$ may carry a natural cluster algebra structure whose Cartan-Killing type is completely different from that of $G$. For example, the base affine space $G/N$ for $G = \text{SL}_5$ inherits the symmetry of type $A_4$ from $G$, but the coordinate ring $\mathbb{C}[G/N]$ has a natural cluster algebra structure of type $D_6$. (here $N$ is the group of unipotent upper triangular matrices, and we extend the scalars of the cluster algebra from $\mathbb{Z}$ to $\mathbb{C}$).

Here are some other instances of this phenomenon (see [2, Proposition 2.26]).
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\[ \mathbb{C}[\text{Gr}_{2,n+3}] \quad A_n \quad \mathbb{C}[\text{SL}_3/N] \quad A_1 \]

\[ \mathbb{C}[\text{Gr}_{3,6}] \quad D_4 \quad \mathbb{C}[\text{SL}_4/N] \quad A_3 \]

\[ \mathbb{C}[\text{Gr}_{3,7}] \quad E_6 \quad \mathbb{C}[\text{Sp}_4/N] \quad B_2 \]

\[ \mathbb{C}[\text{Gr}_{3,8}] \quad E_8 \quad \mathbb{C}[\text{SL}_3] \quad D_4. \]

In each case, a cluster algebra is accompanied by its “cluster type” given by Theorem 1.3.2.

One way to relate the two classifications to each other is by exhibiting a uniform construction of a variety associated with a semisimple group \( G \) whose coordinate ring naturally carries a cluster algebra structure of the same Cartan-Killing type. Such a construction was given in [2, Example 2.24] with the variety in question being the double Bruhat cell \( G^{c,c^{-1}} \), where \( G \) is a simply connected semisimple complex Lie group and \( c \) is a Coxeter element in the Weyl group of \( G \).

In Chapter 2, based on a joint work with A. Zelevinsky [26], we improve the construction introduced in [2] in the following two aspects.

First, there are many non-isomorphic cluster algebras of the same type differing from each other by the choice of a coefficient system. From this perspective, there is nothing especially distinguished about the coefficient system for \( \mathbb{C}[G^{c,c^{-1}}] \). We realize the cluster algebra of finite type with principal coefficients at an arbitrary acyclic initial cluster by replacing the double cell \( G^{c,c^{-1}} \) with its reduced version \( L^{c,c^{-1}} \) introduced in [3].

Second and perhaps more importantly, in [2] only the initial cluster in \( \mathbb{C}[G^{c,c^{-1}}] \) was given explicitly, while no information was obtained about the rest of the cluster variables. In [26], we computed explicitly all cluster variables in \( \mathbb{C}[L^{c,c^{-1}}] \), and they
turn out to be an interesting special family of regular functions on $G$ called principal generalized minors.

In Chapter 3, based on the realization of cluster variables as principal generalized minors, we give combinatorial formulas for $F$-polynomials in cluster algebras of classical types in terms of the weighted paths in certain directed graphs. These combinatorial formulas extend the well-known result due to B. Lindström (see [17], [11], [12], [5] and [6]) on ordinary minors.
Chapter 2

Cluster Algebras of Finite Type via Coxeter Elements and Principal Minors

This chapter is based on joint work with A. Zelevinsky [26] where we gave a uniform geometric realization for the cluster algebra of an arbitrary finite type with principal coefficients at an arbitrary acyclic seed. This algebra is realized as the coordinate ring of a certain reduced double Bruhat cell in the simply connected semisimple algebraic group of the same Cartan-Killing type. In this realization, the cluster variables appear as certain (generalized) principal minors.
2.1 Summary of Main Results

Before stating the general results, we present a motivating example. Let $G = SL_{n+1}(\mathbb{C})$ be of type $A_n$, and let $c = s_1 \cdots s_n$, in the standard numbering of simple roots. In this case $L^{c,c-1}$ is the subvariety of $SL_{n+1}(\mathbb{C})$ consisting of tridiagonal matrices of the form

$$
M = \begin{pmatrix}
v_1 & y_1 & 0 & \cdots & 0 \\
v_2 & y_2 & \ddots & \vdots \\
0 & 1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & y_n \\
0 & \cdots & 0 & 1 & v_{n+1}
\end{pmatrix}
$$

with all $y_1, \ldots, y_n$ non-zero. Let $x_{[i,j]} \in \mathbb{C}[L^{c,c-1}]$ denote the regular function on $L^{c,c-1}$ given by the minor with rows and columns $i, i+1, \ldots, j$, with the convention that $x_{[i,j]} = 1$ unless $1 \leq i \leq j \leq n + 1$. For instance, we have $x_{[i,i]} = v_i$, and $x_{[1,n+1]} = \det(M) = 1$.

**Theorem 2.1.1.** 1. The algebra $\mathcal{A} = \mathbb{C}[L^{c,c-1}]$ is the cluster algebra of type $A_n$ with principal coefficients at the initial seed $(\mathbf{x}, \mathbf{y}, B)$ given by

$$
\mathbf{x} = (x_{[1,1]}, x_{[1,2]}, \ldots, x_{[1,n]}),
$$

and

$$
\mathbf{y} = (y_1, \ldots, y_n),
$$
2.1. Summary of Main Results

The set of cluster variables in $A$ is \{x_{i,j} : 1 \leq i \leq j \leq n+1, (i,j) \neq (1,n+1)\}.

The exchange relations in $A$ are:

$$x_{i,k}x_{j,\ell} = y_{j-1}y_j \cdots y_k x_{[i,j-2]} x_{[k+2,\ell]} + x_{[i,\ell]}x_{[j,k]} \quad (2.2)$$

for $1 \leq i \leq j - 1 \leq k \leq \ell - 1 \leq n$.

Note that among the relations (2.2), there are the exchange relations from the initial cluster, which can be rewritten as follows:

$$x_{[1,k+1]} = u_{k+1}x_{[1,k]} - y_k x_{[1,k-1]} \quad (k = 1, \ldots, n). \quad (2.3)$$

These relations play a fundamental part in the classical theory of orthogonal polynomials in one variable. Thus, the cluster algebra in Theorem 2.1.1 can be viewed as some kind of “enveloping algebra” for this theory. Note also that the variety $L^{c,c-1}$ is very close to the variety of tridiagonal matrices used by B. Kostant [16] in his study of a generalized Toda lattice.

The main result in this chapter is a generalization of Theorem 2.1.1 to the case of an arbitrary simply connected semisimple complex group $G$ and an arbitrary Cox-
eter element $c$ in its Weyl group $W$. We use the terminology and notation in [3, Sections 4.2, 4.3] (more details will be given in Sections 2.2 and 2.3). Recall that $W$ is a finite Coxeter group generated by the simple reflections $s_i (i \in I)$, and that $c = s_{i_1} \cdots s_{i_n}$ for some permutation $(i_1, \ldots, i_n)$ of the index set $I$. For $i, j \in I$, we will write $i \prec_c j$ if $i$ and $j$ are joined by an edge in the Coxeter graph (that is, if the Cartan matrix entry $a_{i,j}$ is non-zero), and $s_i$ precedes $s_j$ in the factorization of $c$. We associate to $c$ a skew-symmetrizable matrix $B(c) = (b_{i,j})_{i,j \in I}$ by setting

$$b_{i,j} = \begin{cases} -a_{i,j} & \text{if } i \prec_c j; \\ a_{i,j} & \text{if } j \prec_c i; \\ 0 & \text{otherwise}. \end{cases} \quad (2.4)$$

Recall from [5, 3] the finite family of (generalized) minors $\Delta_{\gamma,\delta} \in \mathbb{C}[G]$ labeled by pairs of weights $\gamma, \delta$ belonging to the $W$-orbit of the same fundamental weight $\omega_i (i \in I)$. We call a minor $\Delta_{\gamma,\delta}$ principal if $\gamma = \delta$; note that this terminology differs from that in [5], where “principal” referred only to the minors $\Delta_{\omega_i,\omega_i}$. We will use the following notation:

$$x_{\gamma,c} \in \mathbb{C}[L^{c,c^{-1}}] \text{ is the restriction of } \Delta_{\gamma,\gamma} \text{ to the reduced double cell } L^{c,c^{-1}}. \quad (2.5)$$

As shown in [5, 3], each of the minors $\Delta_{\omega_i,\omega_i}$ vanishes nowhere on $L^{c,c^{-1}}$, so can be viewed as an invertible element of $\mathbb{C}[L^{c,c^{-1}}]$. For $j \in I$, we use the following notation:

$$y_{j,c} \in \mathbb{C}[L^{c,c^{-1}}] \text{ is the restriction to } L^{c,c^{-1}} \text{ of the product } \Delta_{\omega_j,\omega_j} \prod_{i \prec_c j} \Delta_{\omega_i,\omega_i}. \quad (2.6)$$
Now we are ready to generalize Part (1) of Theorem 2.1.1.

**Theorem 2.1.2.** The coordinate ring $\mathcal{A}(c) = \mathbb{C}[L^{c,c^{-1}}]$ is the cluster algebra with principal coefficients at the initial seed $(x, y, B)$ given by

$$x = (x_{\omega_i c} : i \in I), \quad y = (y_{i c} : i \in I), \quad B = B(c).$$

Furthermore, $\mathcal{A}(c)$ is of finite type, and its Cartan-Killing type is the same as that of $G$.

**Remark 2.1.1.** We call an exchange matrix $B$ acyclic if there is a linear ordering $\prec$ of $I$ such that $b_{i,j} \geq 0$ for all $i, j \in I$ with $i \prec j$. We also call a seed acyclic if its exchange matrix is acyclic. Following [9, Theorems 1.8, 7.1], it is easy to see that every acyclic exchange matrix in a cluster algebra of finite type is of the form $B(c)$ for some Coxeter element $c$. Thus Theorem 2.1.2 provides a realization of any cluster algebra of finite type with principal coefficients at an arbitrary acyclic seed.

We next generalize Part (2) of Theorem 2.1.1, by showing that the set of all cluster variables in $\mathcal{A}(c)$ is of the form $\{x_{\gamma c} : \gamma \in \Pi(c)\}$ for some set of weights $\Pi(c)$. To describe $\Pi(c)$, we need to develop some combinatorics related to the action on weights by the cyclic group $\langle c \rangle$ generated by $c$. Interestingly, this combinatorics turns out to be very close to the one developed in the classical paper [15] by B. Kostant, and used recently in [14] for (seemingly) completely different purposes.

We consider the usual partial order on weights: $\gamma \geq \delta$ if $\gamma - \delta$ is a nonnegative integer linear combination of simple roots. We also denote by $i \mapsto i^*$ the involution on $I$ given by

$$\omega_{i^*} = -w_i(\omega_i),$$

(2.7)
where, as usual, \( w_\circ \) is the longest element of \( W \).

**Proposition 2.1.3.** For every \( i \in I \), there is a positive integer \( h(i; c) \) such that

\[
\omega_i > c\omega_i > c^2\omega_i > \cdots > c^{h(i; c)}\omega_i = -\omega_i. \tag{2.8}
\]

**Theorem 2.1.4.** The set of cluster variables in \( A(c) \) is \( \{x_\gamma : \gamma \in \Pi(c)\} \), where the set of weights \( \Pi(c) \) is given by

\[
\Pi(c) = \{c^m\omega_i : i \in I, 0 \leq m \leq h(i; c)\}. \tag{2.9}
\]

As for Part (3) of Theorem 2.1.1, in the general case we give explicit formulas not for all exchange relations but only for the primitive ones, that is, those in which one of the products of cluster variables on the right-hand side is equal to 1.

**Theorem 2.1.5.** The primitive exchange relations in \( A(c) \) are exactly the following:

\[
x_{-\omega_k; c} x_{\omega_k; c} = y_{k; c} \prod_{i < k} x^{-a_{i,k}} x^{-a_{k,i}} + 1; \tag{2.10}
\]

\[
x^{c^m-1}\omega_{k; c} x^{c^m}\omega_{k; c} = \prod_{i < k} x^{-a_{i,k}} \prod_{k < i} x^{-a_{i,k}} + \prod_{j \in I} [c^{m-1}\omega_k - c^m\omega_k : \alpha_j], \tag{2.11}
\]

where \( k \in I, 1 \leq m \leq h(k; c) \), and \([\gamma : \alpha_j]\) stands for the coefficient of \( \alpha_j \) in the expansion of a weight \( \gamma \) in the basis of simple roots.

The fact that the weights \( c^{m-1}\omega_i \) and \( c^m\omega_i \) appearing in the right hand side of (2.11) belong to \( \Pi(c) \), is guaranteed by the following key property of the numbers \( h(i; c) \).
Proposition 2.1.6. If \( i \prec_c j \) then we have

\[
h(i; c) - h(j; c) = \begin{cases} 
1 & \text{if } j^* \prec_c i^*; \\
0 & \text{if } i^* \prec_c j^*.
\end{cases}
\] (2.12)

Note that the numbers \( h(i; c) \) are uniquely determined by the relations (2.12) combined with the following property.

Proposition 2.1.7. For every \( i \in I \), the sum \( h(i; c) + h(i^*; c) \) is equal to the Coxeter number of the connected component of \( I \) containing \( i \) and \( i^* \).

In what follows, we identify the index set \( I \) with \([1, n] = \{1, \ldots, n\}\) so that the (fixed) Coxeter element \( c \) has the form \( c = s_1 \cdots s_n \). By the Laurent phenomenon (Theorem 1.2.1), every cluster variable \( z \) can be uniquely written in the form

\[
z = \frac{N(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}},
\] (2.13)

where \( N(x_1, \ldots, x_n) \) is a polynomial not divisible by any cluster variable \( x_i \in \mathbf{x} \). We call the integer vector \( d_{z; \mathbf{x}} = (d_1, \ldots, d_n) \in \mathbb{Z}^n \) the denominator vector of \( z \) with respect to a cluster \( \mathbf{x} \).

The following result gives an explicit formula for the denominator vector of any cluster variable \( z \in \mathcal{A}(c) \) with respect to the initial cluster.

Theorem 2.1.8. Let \( \mathbf{x} \) be the initial cluster in Theorem 2.1.2, and let \( z = x_{\gamma; c} \) be a cluster variable not belonging to \( \mathbf{x} \). Identifying \( \mathbb{Z}^n \) with the root lattice by means of the basis \( \alpha_1, \ldots, \alpha_n \) of simple roots, the denominator vector \( d_{z; \mathbf{x}} \) gets identified with \( c^{-1}\gamma - \gamma \).
2.1. Summary of Main Results

The following corollary agrees with [10, Conjecture 7.4 (i),(ii)].

**Corollary 2.1.9.** In the situation of Theorem 2.1.8, all the components $d_i$ of the denominator vector $d_{z;x}$ are nonnegative. Furthermore, $d_i = 0$ if and only if $z$ and the initial cluster variable $x_{\omega_i;c}$ belong to the same cluster in $A(c)$.

Our realization of cluster variables allows us to obtain explicit expressions for their $g$-vectors and $F$-polynomials (cf. (1.4)). We start with $g$-vectors.

**Theorem 2.1.10.** Let $x$ be the initial cluster in Theorem 2.1.2, and let $z = x_{\gamma;c}$ for some $\gamma \in \Pi(c)$. Identifying $\mathbb{Z}^n$ with the weight lattice by means of the basis $\omega_1, \ldots, \omega_n$ of fundamental weights, the $g$-vector $g_{z;x}$ gets identified with $\gamma$.

**Remark 2.1.2.** An alternative description of these $g$-vectors was given in [21, Theorem 10.2]. It is stated in different terms, and proved by a quite different method, relying on [10, Conjecture 7.12]. It is not difficult to check the equivalence of the two descriptions.

The following corollary is an easy consequence of Theorems 2.1.8 and 2.1.10. It describes the relationship between the denominator vector and the $g$-vector of the same cluster variable.

**Corollary 2.1.11.** In a cluster algebra of finite type, suppose that the initial seed has an acyclic exchange matrix $B$. Then the $g$-vector $g = (g_1, \ldots, g_n)$ and the denominator vector $d = (d_1, \ldots, d_n)$ of the same cluster variable not belonging to the initial cluster are related by

$$g_j = -d_j + \sum_i [-b_{j,i}]_+ d_i. \quad (2.14)$$
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The following corollary agrees with [10, Conjecture 7.12] for mutations between two acyclic clusters giving a nice transition rule between $g$-vectors.

**Corollary 2.1.12.** In a cluster algebra of finite type, let $B^0$ and $B^1$ be two acyclic exchange matrices with clusters $x^0$ and $x^1$ respectively, and let $B^1 = \mu_k(B^0)$. Then for any cluster variable $z$, the $g$-vectors $g_{z,x^0} = (g_1, \ldots, g_n)$ and $g_{z,x^1} = (g'_1, \ldots, g'_n)$ are related as follows:

$$g'_j = \begin{cases} -g_k & \text{if } j = k; \\ g_j + [b^0_{j,k}] + g_k - b^0_{j,k} \min(g_k, 0) & \text{if } j \neq k. \end{cases} \quad (2.15)$$

To give a formula for the $F$-polynomials, recall that, for each $i \in [1, n]$, there are one-parameter root subgroups in $G$ given by

$$x_i(t) = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \bar{x}_i(t) = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

where $\varphi_i : SL_2 \to G$ denotes the canonical embedding corresponding to the simple root $\alpha_i$.

**Theorem 2.1.13.** Under the assumptions of Theorem 2.1.10, the $F$-polynomial $F_{z;x}$ is given by

$$F_{z;x}(t_1, \ldots, t_n) = \Delta_{\gamma,\gamma}(x_1(1) \cdots x_n(1)x_n(t_n) \cdots x_1(t_1)). \quad (2.17)$$

The following corollary agrees with [10, Conjecture 5.4].

**Corollary 2.1.14.** Every $F$-polynomial in Theorem 2.1.13 has constant term 1.
Example 2.1.3. Let $c$ be as in Theorem 2.1.1 (for $G$ of type $A_n$). Then we have $h(k; c) = n + 1 - k$ for $k = 1, \ldots, n$. By Theorem 2.1.10, the cluster variable $x_{c^m \omega_k; c} = x_{[m+1, m+k]}$ has the $g$-vector $c^m \omega_k = \omega_{m+k} - \omega_m$ (with the convention $\omega_0 = \omega_{n+1} = 0$). By Theorem 2.1.8, for $m \geq 1$, the denominator vector of $x_{c^m \omega_k; c}$ is equal to

$$c^{m-1} \omega_k - c^m \omega_k = \alpha_m + \alpha_{m+1} + \cdots + \alpha_{m+k-1}.$$ 

By computing the determinant in Theorem 2.1.13, we conclude that $x_{c^m \omega_k; c}$ has the $F$-polynomial

$$F(t_1, \ldots, t_n) = 1 + t_m + t_m t_{m+1} + \cdots + t_m t_{m+1} \cdots t_{m+k-1}$$

(with the convention $t_0 = 0$).

The rest of this chapter is devoted to the proofs of the above results. Most of our proofs rely on the following essentially combinatorial fact:

**every Coxeter element can be obtained from any other by a sequence of operations sending $c = s_{i_1} \cdots s_{i_n}$ to $\tilde{c} = s_{i_1} cs_{i_1}$.**

Thus to prove some statement for an arbitrary Coxeter element $c$, it is enough to check that it holds for some special choice of $c$, and that it is preserved by the above operations. As this special choice, we take a bipartite Coxeter element $t$, for which most of the above results were essentially established in earlier papers (see Section 2.2 for more details).

In Section 2.2, after a short reminder on root systems, we develop the combinatorics of the action of a Coxeter element on roots and fundamental weights, in
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particular, proving Propositions 2.1.3, 2.1.6, and 2.1.7.

In Section 2.3 we recall necessary background on (reduced) double Bruhat cells and generalized minors, and prove the relations (2.10) and (2.11) from Theorem 2.1.5.  

Both sections 2.2 and 2.3 are totally independent of the theory of cluster algebras, which are not even mentioned there.

In Section 2.4 we recall basic definitions and facts about cluster algebras, following mainly [10], then prove Theorem 2.1.2.

In Sections 2.5 and 2.6 we prove Theorems 2.1.4 and 2.1.5. In the course of the proof we obtain some results of independent interest. Proposition 2.5.1 transfers the compatibility degree function introduced in [8, Section 3] to each set Π(c). This leads to a combinatorial description of all the clusters in A(c) given in Corollary 2.6.3. Another important result is Proposition 2.6.1, which provides, for every choice of a Coxeter element c, a realization of the cluster algebra with universal coefficients introduced in [10, Section 12], with respect to the initial seed with the exchange matrix B(c). As a corollary of the technique developed in these sections, we describe explicit isomorphisms between various algebras A(c) (Corollary 2.6.4) as well as corresponding geometric isomorphisms between various reduced double cells L_{c,c^{-1}} (Remark 2.6.2).

In Section 2.7, we prove Theorems 2.1.8, 2.1.10 and 2.1.13, as well as Corollaries 2.1.9, 2.1.11 and 2.1.14. As explained in Remark 2.7.1, these results allow us to give an alternative combinatorial description of the c-Cambrian fans studied in [21].

In Section 2.8, we illustrate the above results by an example dealing with type A_n and the Coxeter element c = s_1· · ·s_n in the standard numbering of simple roots. In particular, we prove Theorem 2.1.1 and present an example making more explicit the
2.2 Combinatorics of Coxeter Elements

We start by laying out the basic terminology and notation related to root systems. In what follows, $A = (a_{i,j})_{i,j \in I}$ is an indecomposable $n \times n$ Cartan matrix, i.e., one of the matrices $A_n, B_n, \ldots, G_2$ in the Cartan-Killing classification (the general case in Propositions 2.1.3, 2.1.6, and 2.1.7 easily reduces to the case of $A$ indecomposable).

Let $\Phi$ be the corresponding rank $n$ root system with the set of simple roots $\{\alpha_i : i \in I\}$. Let $W$ be the Weyl group of $\Phi$, i.e., the group of linear transformations of the root space, generated by the simple reflections $s_i (i \in I)$ whose action on simple roots is given by

$$s_i(\alpha_j) = \alpha_j - a_{i,j}\alpha_i.$$  \hfill (2.18)

Let $\{\omega_i : i \in I\}$ be the set of fundamental weights related to simple roots via

$$\alpha_j = \sum_{i \in I} a_{i,j}\omega_i.$$ \hfill (2.19)

The action of simple reflections on the fundamental weights is given by

$$s_i\omega_j = \begin{cases} 
\omega_i - \alpha_i & \text{if } j = i; \\
\omega_j & \text{if } j \neq i.
\end{cases}$$ \hfill (2.20)

For future use, here are a couple of useful lemmas.
Lemma 2.2.1. Suppose $I = \{1, \ldots, n\}$ and $c = s_1 \cdots s_n$. For $i = 1, \ldots, n$, let

\[ \beta_i = s_1 \cdots s_{i-1} \alpha_i, \quad (2.21) \]

so that $\{\beta_1, \ldots, \beta_n\}$ is the set of positive roots $\beta$ such that $c^{-1} \beta$ is negative. Then we have

\[ \omega_i - c\omega_i = \beta_i, \quad (2.22) \]

and

\[ \beta_j + \sum_{i=1}^{j-1} a_{i,j} \beta_i = \alpha_j. \quad (2.23) \]

Proof. The identity (2.22) is an immediate consequence of (2.20):

\[
\beta_i = s_1 \cdots s_{i-1} \alpha_i \\
= s_1 \cdots s_{i-1} (\omega_i - s_i \omega_i) \\
= \omega_i - s_1 \cdots s_i \omega_i \\
= \omega_i - c\omega_i.
\]
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As for (2.23), we have

$$\beta_j = s_1 \cdots s_{j-1} \alpha_j$$

$$= s_1 \cdots s_{j-2} (\alpha_j - a_{j-1,j} \alpha_{j-1})$$

$$= s_1 \cdots s_{j-2} \alpha_j - a_{j-1,j} \beta_{j-1}$$

$$\vdots$$

$$= \alpha_j - \sum_{i=1}^{j-1} a_{i,j} \beta_i.$$  

□

**Lemma 2.2.2** ([4, Proposition VI.1.33]). In the situation of Lemma 2.2.1, let \(\langle c \rangle\) denote the cyclic subgroup of \(W\) generated by \(c\). Then every \(\langle c \rangle\)-orbit in \(\Phi\) contains exactly one positive root \(\beta\) such that \(c^{-1} \beta\) is negative (i.e., one of the roots \(\beta_1, \ldots, \beta_n\)) and exactly one positive root \(\alpha\) such that \(c \alpha\) is negative (i.e., one of the roots \(-c^{-1} \beta_1, \ldots, -c^{-1} \beta_n\)).

A reduced word for \(w \in W\) is a sequence of indices \(i = (i_1, \ldots, i_\ell)\) of shortest possible length \(\ell = \ell(w)\) such that \(w = s_{i_1} \cdots s_{i_\ell}\). The group \(W\) possesses a unique longest element denoted by \(w_0\).

The Coxeter graph associated to \(\Phi\) has the index set \(I\) as the set of vertices, with \(i\) and \(j\) joined by an edge whenever \(a_{i,j} a_{j,i} > 0\). Since we assume that \(A\) is indecomposable, the root system \(\Phi\) is irreducible, and the Coxeter graph \(I\) is a tree (one of the familiar Coxeter-Dynkin diagrams \(A_n, D_n, E_6, E_7, E_8\)).

Recall that a Coxeter element is the product of all simple reflections \(s_i\) (for \(i \in I\)) taken in an arbitrary order. Thus, we have \(\ell(c) = n\). It is well-known that all Coxeter
elements are conjugate to each other in $W$; in particular, they have the same order $h$
called the \textit{Coxeter number} of $W$.

Note that Coxeter elements are in a natural bijection with orientations of the Coxeter graph: we orient an edge $j \to i$ if $i \prec_c j$, i.e., if $i$ precedes $j$ in some (equivalently, any) reduced word for $c$. We introduce the following elementary operation on Coxeter elements:

\begin{equation}
\text{replace } c \text{ with } \tilde{c} \text{ if } c = s_i c' \text{ and } \tilde{c} = c' s_i \text{ for some } i \in I \text{ and } c' \in W \text{ with } \ell(c') = n - 1. \tag{2.24}
\end{equation}

We will use the following well-known fact:

any Coxeter element can be reached from any other one by a sequence of moves (2.24).

(Passing from Coxeter elements to corresponding orientations of the Coxeter graph, (2.25) translates into the statement that any orientation of a tree can be obtained from any other orientation by a repeated application of the following operation: reversing all arrows at some sink; this is easily proved by induction on the size of a tree.)

Our proofs of the needed properties of Coxeter elements will use the same strategy: show that the property in question is preserved under any operation (2.24), and prove that it holds for some particular choice of a Coxeter element. As this particular choice, we use the \textit{bipartite} Coxeter element defined as follows.

Since the Coxeter graph is a tree, it is bipartite, so the set of vertices $I$ is a
disjoint union of two parts $I_+$ and $I_-$ such that every edge joins two vertices from different parts. Note that $I_+$ and $I_-$ are determined uniquely up to renaming. We write $\varepsilon(i) = \varepsilon$ for $i \in I_\varepsilon$.

Now we define the bipartite Coxeter element $t$ by setting

$$t = t_+ t_-,$$  (2.26)

where

$$t_\pm = \prod_{\varepsilon(i) = \pm 1} s_i.$$  (2.27)

Note that the order of factors in (2.27) does not matter because $s_i$ and $s_j$ commute whenever $\varepsilon(i) = \varepsilon(j)$. Let $i_-$ and $i_+$ be some reduced words for the elements $t_-$ and $t_+$, respectively.

**Lemma 2.2.3** ([4, Exercise V.6.2]). We have $t_- t_+ \cdots t_\pm t_\mp \overline{h \text{ factors}} = w_0$. Moreover, the word

$$i_o \overset{\text{def}}{=} i_+ i_- i_+ \cdots i_\pm i_\mp \overline{h \text{ segments}}$$  (2.28)

(concatenation of $h$ segments) is a reduced word for $w_0$.

Regarding Lemma 2.2.3, recall from the tables in [4] that $h$ is even for all types except $A_n$ with $n$ even; in the exceptional case of type $A_{2e}$, we have $h = 2e + 1$. If $h$ is even, then Lemma 2.2.3 says that $t^{h/2} = w_0$.

Now everything is in place for the proofs of Propositions 2.1.3, 2.1.6, and 2.1.7.

**Proof of Proposition 2.1.3.** We start by observing that Proposition 2.1.3 is a conse-
2.2. Combinatorics of Coxeter Elements

sequence of the following seemingly weaker statement:

for every Coxeter element \( c \) and \( i \in I \), the weight \( w_\circ \omega_i \) belongs to the \( \langle c \rangle \)-orbit of \( \omega_i \).

Indeed, assuming (2.29), we can define \( h(i; c) \) as the smallest positive integer \( n \) such that \( c^m \omega_i = w_\circ \omega_i \). To prove the inequalities in (2.8), note that, in view of (2.22), we have \( c^m \omega_i - c^{m+1} \omega_i = c^m \beta_i \) for \( m \in \mathbb{Z} \), and so the differences \( c^m \omega_i - c^{m+1} \omega_i \) form the \( \langle c \rangle \)-orbit of \( \beta_i \) in \( \Phi \). Now recall Lemma 2.2.2. Since \( \omega_i \) is the maximal element of its \( W \)-orbit, while \( w_\circ \omega_i \) is the minimal element of the same orbit, we see that \( \beta = \omega_i - c \omega_i \) (resp. \( \alpha = c^{-1} w_\circ \omega_i - w_\circ \omega_i \)) is the unique positive root in the \( \langle c \rangle \)-orbit of \( \beta_i \) such that \( c^{-1} \beta \) (resp. \( c \alpha \)) is negative. It follows that all the roots \( \gamma = c^m \omega_i - c^{m+1} \omega_i \) with \( 0 \leq m < h(i; c) \) are positive, as required.

As an easy consequence of Lemma 2.2.3, the statement (2.29) (hence Proposition 2.1.3) holds for the bipartite Coxeter element \( t \); furthermore, we have

\[
h(i; t) = \begin{cases} \left\lfloor \frac{h}{2} \right\rfloor & \text{for } \varepsilon(i) = -1 ; \\
\left\lceil \frac{h}{2} \right\rceil & \text{for } \varepsilon(i) = +1 . \end{cases}
\]

(2.30)

The fact that \( t \) satisfies Propositions 2.1.6 and 2.1.7, follows at once from (2.30) together with an observation that, for every \( i \in I \), we have

\[
\varepsilon(i^*) = \begin{cases} \varepsilon(i) & \text{if } h \text{ is even}; \\
-\varepsilon(i) & \text{if } h \text{ is odd}. \end{cases}
\]

(2.31)
In view of (2.25), to finish the proofs of Propositions 2.1.3, 2.1.6, and 2.1.7, it suffices to assume that they hold for some Coxeter element \( c \) and show that the same is true for the element \( \tilde{c} \) obtained from \( c \) via (2.24). Without loss of generality, we assume that \( I = \{1, \ldots, n\} \), \( c = s_1 s_2 \cdots s_n \), and \( \tilde{c} = s_2 \cdots s_n s_1 \).

The key statement to prove is the following: \( \tilde{c} \) satisfies (2.29) (hence Proposition 2.1.3), and we have

\[
h(i; \tilde{c}) = \begin{cases} 
  h(i; c) - 1 & \text{if } i = 1 \text{ and } i^* \neq 1; \\
  h(i; c) + 1 & \text{if } i \neq 1 \text{ and } i^* = 1; \\
  h(i; c) & \text{otherwise.}
\end{cases}
\]  

(2.32)

We split the proof of (2.32) into four cases:

**Case 1.** Let \( i \neq 1 \) and \( i^* \neq 1 \). Since \( \tilde{c}^m = s_1 c^m s_1 \) for all \( m > 0 \), using (2.20), we obtain

\[
\tilde{c}^m \omega_i = -\omega_i^* \iff s_1 c^m s_1 \omega_i = -\omega_i^* \iff c^m \omega_i = -\omega_i^*,
\]

implying that \( h(i; \tilde{c}) = h(i; c) \).

**Case 2.** Let \( i = i^* = 1 \). Writing \( \tilde{c}^m = s_2 \cdots s_n c^m s_n \cdots s_2 \) and using (2.20), we obtain

\[
\tilde{c}^m \omega_1 = -\omega_1 \iff c^m \omega_1 = -\omega_1,
\]

again implying that \( h(1; \tilde{c}) = h(1; c) \).

**Case 3.** Let \( i = 1 \) and \( i^* \neq 1 \). Writing \( \tilde{c}^m = s_1 c^{m+1} s_n \cdots s_2 \) and using (2.20), we
obtain
\[ \tilde{c}^m \omega_1 = -\omega_i^* \iff c^{m+1} \omega_1 = -\omega_i^*, \]
implying that \( h(i; \tilde{c}) = h(i; c) - 1 \).

**Case 4.** Finally, let \( i \neq 1 \) and \( i^* = 1 \). Writing \( \tilde{c}^m = s_2 \cdots s_n c^{m-1} s_1 \) and using (2.20), we obtain
\[ \tilde{c}^m \omega_i = -\omega_1 \iff c^{m-1} \omega_i = -\omega_1, \]
implying that \( h(i; \tilde{c}) = h(i; c) + 1 \).

This concludes the proof of Proposition 2.1.3 and the relation (2.32). □

The proofs of Propositions 2.1.6 and 2.1.7 now become purely combinatorial exercises.

**Proof of Proposition 2.1.7.** Let again \( c = s_1 s_2 \cdots s_n \) and \( \tilde{c} = s_2 \cdots s_n s_1 \), so the numbers \( h(i; c) \) and \( h(i; \tilde{c}) \) are related via (2.32). To prove Proposition 2.1.7, it suffices to show that \( h(i; \tilde{c}) + h(i^*; \tilde{c}) = h(i; c) + h(i^*; c) \) for all \( i \), which is immediate from (2.32). This concludes the proof of Proposition 2.1.7. □

**Proof of Proposition 2.1.6.** To prove Proposition 2.1.6, it suffices to show the following: if the numbers \( h(i; c) \) satisfy (2.12), and the \( h(i; \tilde{c}) \) are given by (2.32), then the \( h(i; \tilde{c}) \) also satisfy (2.12) with \( c \) replaced by \( \tilde{c} \). There are several cases to consider.

**Case 1.** Suppose that \( i \prec \tilde{c} j \), and \( i^* \prec \tilde{c} j^* \). Then we have \( i \neq 1 \) and \( i^* \neq 1 \), hence \( h(i; \tilde{c}) = h(i; c) \). This case breaks into four subcases according to which of the two indices \( j \) and \( j^* \) are equal to 1. In each of these subcases, the desired equality \( h(j; \tilde{c}) = h(i; \tilde{c}) \) is seen by a direct inspection. For instance, if \( j = 1 \neq j^* \), we have
2.3. Generalized Minors and Determinantal Identities

\[ h(j; c) = h(j; \tilde{c}) - 1 \text{ by (2.32)}; \text{ on the other hand, in this case we have } j \prec_c i, \text{ and } i^* \prec_c j^*, \text{ hence } h(j; c) - 1 = h(i; c) \text{ by (2.12)}. \]

**Case 2.** Now suppose that \( i \prec \tilde{c} j \), and \( j^* \prec \tilde{c} i^* \). Then we have \( i \neq 1 \) and \( j^* \neq 1 \). This case also breaks into four subcases according to which of the two indices \( j \) and \( i^* \) are equal to 1. Again, in each of these subcases, the desired equality \( h(j; \tilde{c}) = h(i; \tilde{c}) - 1 \) is seen by a direct inspection. For instance, if \( j = 1 \neq i^* \), we have \( h(i; \tilde{c}) = h(i; c) \), and \( h(j; \tilde{c}) = h(j; c) - 1 \text{ by (2.32)}; \text{ on the other hand, in this case we have } j \prec_c i, \text{ and } j^* \prec_c i^*, \text{ hence } h(j; c) = h(i; c) \text{ by (2.12)}. \)

This concludes the proof of Proposition 2.1.6. \( \square \)

### 2.3 Generalized Minors and Determinantal Identities

In this section, we briefly recall necessary facts about generalized minors and reduced double cells; more details can be found in [5, 3].

Let \( g \) be a complex semisimple Lie algebra of rank \( n \) with Chevalley generators \( f_i, \alpha_i^\vee, \text{ and } e_i \) for \( i \in I \), where \( I \) is an \( n \)-element index set, which will be often identified with \( \{1, \ldots, n\} \). The elements \( \alpha_i^\vee \) are *simple coroots* of \( g \); they form a basis of a Cartan subalgebra \( h \) of \( g \). The *simple roots* \( \alpha_i (i \in I) \) form a basis in the dual space \( h^* \) such that \([h, e_i] = \alpha_i(h)e_i, \text{ and } [h, f_i] = -\alpha_i(h)f_i \) for any \( h \in h \) and \( i \in I \). The structure of \( g \) is uniquely determined by the *Cartan matrix* \( A = (a_{i,j}) \) given by \( a_{i,j} = \alpha_j(\alpha_i^\vee) \).

Let \( G \) be the simply connected complex semisimple Lie group with the Lie algebra \( g \). For every \( i \in I \), let \( \varphi_i : SL_2 \to G \) denote the canonical embedding corresponding
to the simple root $\alpha_i$. We use the notation

$$x_i(t) = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp (te_i), \quad x_i(t) = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp (tf_i). \quad (2.33)$$

We also set

$$t^{\alpha_i^\vee} = \varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in H$$

for any $i \in I$ and any $t \neq 0$. Let $N$ (resp. $N_-$) be the maximal unipotent subgroup of $G$ generated by all $x_i(t)$ (resp. $x_i(t)$). Let $H$ be the maximal torus in $G$ with the Lie algebra $\mathfrak{h}$. Let $B = HN$ and $B_- = HN_-$ be the corresponding two opposite Borel subgroups.

The Weyl group $W$ associated to $A$ is naturally identified with $\text{Norm}_G(H)/H$; this identification sends each simple reflection $s_i$ to the coset $\overline{s_i}H$, where the representative $\overline{s_i} \in \text{Norm}_G(H)$ is defined by

$$\overline{s_i} = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.34)$$

The elements $\overline{s_i}$ satisfy the braid relations in $W$; thus the representative $\overline{w}$ can be unambiguously defined for any $w \in W$ by requiring that $\overline{uv} = \overline{u} \cdot \overline{v}$ whenever $\ell(uv) = \ell(u) + \ell(v)$.

With some abuse of notation, we identify the weight lattice $P$ in $\mathfrak{h}^*$ with the group of rational multiplicative characters of $H$, here written in the exponential notation: a weight $\gamma \in P$ acts by $a \mapsto a^\gamma$. Under this identification, the fundamental weights
$\omega_1, \ldots, \omega_n$ act in $H$ by $(t^\alpha)^{\omega_i} = t^{\delta_{ij}}$.

Recall that the set $G_0 = N_-HN$ of elements $x \in G$ that have Gaussian decomposition is open and dense in $G$; this (unique) decomposition of $x \in N_-HN$ is written as $x = [x]_-[x]_0[x]_+$.

We now recall some basic properties of generalized minors introduced in [5]. For $u, v \in W$ and $i \in I$, the generalized minor $\Delta_{u\omega_i,v\omega_i}$ is the regular function on $G$ whose restriction to the open set $uG_0v^{-1}$ is given by

$$
\Delta_{u\omega_i,v\omega_i}(x) = ([x^{-1}xv]_0)^{\omega_i}.
$$

As shown in [5], $\Delta_{u\omega_i,v\omega_i}$ depends on the weights $u\omega_i$ and $v\omega_i$ alone, not on the particular choice of $u$ and $v$. In the special case $G = SL_{n+1}$, the generalized minors are nothing but the ordinary minors of a matrix. More precisely, the Weyl group is identified with the symmetric group $S_{n+1}$ and it acts on the index set $[1, n+1]$ in the natural way. For permutations $u, v \in S_{n+1}$, the evaluation of $\Delta_{u\omega_i,v\omega_i}$ at a matrix $x \in SL_{n+1}$ is equal to the determinant of the submatrix of $x$ whose rows (resp., columns) are labeled by the elements of the set $u([1, i])$ (resp., $v([1, i])$).

Generalized minors have the following properties (see [5, (2.14), (2.25)]):

$$
\Delta_{\gamma,\delta}(a_1 xa_2) = a_1^\gamma a_2^\delta \Delta_{\gamma,\delta}(x) \quad (a_1, a_2 \in H; \ x \in G); \quad (2.36)
$$

$$
\Delta_{\gamma,\delta}(x) = \Delta_{\delta,\gamma}(x^T), \quad (2.37)
$$

where $x \mapsto x^T$ is the "transpose" involutive antiautomorphism of $G$ acting on genera-
2.3. Generalized Minors and Determinantal Identities

We will also use the involutive antiautomorphism \( \iota \) of \( G \) introduced in [5, (2.2)]; it is defined by

\[
a^\iota = a^{-1} \quad (a \in H) , \quad x_i(t)^\iota = x_i(t) , \quad x_i^\iota(t) = x_i(t) .
\] (2.39)

By [5, (2.25)], we have

\[
\Delta_{\gamma,\delta}(x) = \Delta_{-\delta,-\gamma}(x^\iota)
\] (2.40)

for any generalized minor \( \Delta_{\gamma,\delta} \), and any \( x \in G \).

The following important identity was obtained in [5, Theorem 1.17].

**Proposition 2.3.1.** Suppose \( u, v \in W \) and \( k \in I \) are such that \( us_k \omega_k < u \omega_k \) and \( vs_k \omega_k < v \omega_k \). Then

\[
\Delta_{u \omega_k,v \omega_k} \Delta_{us_k \omega_k,vs_k \omega_k} = \Delta_{us_k \omega_k,v \omega_k} \Delta_{u \omega_k,vs_k \omega_k} + \prod_{i \neq k} \Delta_{u \omega_i,v \omega_i}^{-a_{i,k}} .
\] (2.41)

The group \( G \) has two **Bruhat decompositions**, with respect to opposite Borel subgroups \( B \) and \( B_- \):

\[
G = \bigcup_{u \in W} BuB = \bigcup_{v \in W} B_- v B_- .
\]

The **double Bruhat cells** \( G^{u,v} \) are defined by \( G^{u,v} = BuB \cap B_- v B_- \). These varieties were introduced and studied in [5]. Each double Bruhat cell can be defined inside \( G \) by a collection of vanishing/non-vanishing conditions of the form \( \Delta(x) = 0 \) and \( \Delta(x) \neq 0 \), where \( \Delta \) is a generalized minor. The following description can be found in [2, Proposition 2.8].
Proposition 2.3.2. A double Bruhat cell $G^{u,v}$ is given inside $G$ by the following conditions, for all $i \in I$:

\begin{align*}
\Delta_{u' \omega_i, \omega_i} &= 0 \text{ whenever } u' \omega_i \not\leq u \omega_i \text{ in the Bruhat order,} \\
\Delta_{\omega_i, v' \omega_i} &= 0 \text{ whenever } v' \omega_i \not\leq v^{-1} \omega_i \text{ in the Bruhat order,} \\
\Delta_{u \omega_i, \omega_i} &\neq 0, \quad \Delta_{\omega_i, v^{-1} \omega_i} \neq 0 .
\end{align*}

(2.42, 2.43, 2.44)

In this thesis we concentrate on the following subset $L^{u,v} \subset G^{u,v}$ introduced in [3] and called a reduced double Bruhat cell:

$$L^{u,v} = N \bar{u} N \cap B_- v B_-. \quad (2.45)$$

The equations defining $L^{u,v}$ inside $G^{u,v}$ look as follows.

Proposition 2.3.3 ([3, Proposition 4.3]). An element $x \in G^{u,v}$ belongs to $L^{u,v}$ if and only if $\Delta_{u \omega_i, \omega_i}(x) = 1$ for $i \in I$.

The variety $L^{u,v}$ (resp. $G^{u,v}$) is biregularly isomorphic to a Zariski open subset of an affine space of dimension $\ell(u) + \ell(v)$ (resp. $\ell(u) + \ell(v) + n$).

Local coordinates in $L^{u,v}$ and $G^{u,v}$ were constructed in [5, 3]. To simplify the notation, we will describe them only in the following special case.

In the rest of the section we assume that the index set $I$ is $\{1, \ldots, n\}$, and work with the fixed Coxeter element $c = s_1 \cdots s_n$, and with the varieties $L^{c,c^{-1}}$ and $G^{c,c^{-1}}$.

In accordance with [5, Theorem 1.2], a generic element $x \in G^{c,c^{-1}}$ can be uniquely factored as

$$x = x_1(u_1) \cdots x_n(u_n) a x_n(t_n) \cdots x_1(t_1) \quad (2.46)$$
for some \( a \in H \) and \( u_i, t_i \in \mathbb{C}^* \) (see (2.33)).

In the case of the reduced double cell \( L^{c,c^{-1}} \), the factorization (2.46) can be modified as follows. For any nonzero \( u \in \mathbb{C} \) and any \( i = 1, \ldots, n \), denote

\[
x_{-i}(u) = x_i(u)u^{-a_i^\gamma} = \varphi_i \begin{pmatrix} u^{-1} & 0 \\ 1 & u \end{pmatrix}.
\]

(2.47)

In accordance with [3, Proposition 4.5], a generic element \( x \in L_c^{c,c^{-1}} \) can be uniquely factored as

\[
x = x_{-1}(u_1) \cdots x_{-n}(u_n) x_n(t_n) \cdots x_1(t_1),
\]

(2.48)

where \( u_i, t_i \in \mathbb{C}^* \).

After this preparation, let us turn to the proof of (2.11).

**Proof of relation (2.11).** Let \( k = 1, \ldots, n \) and \( 1 \leq m \leq h(k; c) \). Specializing (2.41) for \( u = v = c^{m-1}s_1 \cdots s_{k-1} \) (and taking into account (2.20)), we get

\[
\Delta_{c^{m-1}\omega_k, c^{m-1}\omega_k} \Delta_{c^{m}\omega_k, c^{m}\omega_k} = \Delta_{c^{m}\omega_k, c^{m-1}\omega_k} \Delta_{c^{m-1}\omega_k, c^{m}\omega_k} + \prod_{i<k} \Delta_{c^{m}\omega_i, c^{m}\omega_i} \prod_{i>k} \Delta_{c^{m-1}\omega_i, c^{m-1}\omega_i}.
\]

(2.49)

Remembering (2.5) and (2.6), and comparing (2.49) with the desired equality (2.11), we see that it remains to prove the following lemma.

**Lemma 2.3.1.** For every \( k = 1, \ldots, n \) and \( 1 \leq m \leq h(k; c) \), the restriction to \( L_c^{c,c^{-1}} \)
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of the function \( \Delta^{c m \omega_k, c m -1 \omega_k} \Delta^{c m -1 \omega_k, c m \omega_k} \) is equal to that of the product

\[
\prod_{j=1}^{n} (\Delta_{\omega_j, \omega_j} \prod_{i<j} \Delta_{\omega_j, \omega_i}^{a_{i,j, \omega_i}})^{[c m -1 \omega_k - c m \omega_k : \alpha_j]}, \tag{2.50}
\]

**Proof of Lemma 2.3.1.** First of all, in view of (2.22), we have

\[
c^{m-1} \omega_k - c^m \omega_k = c^{m-1} \beta_k. \tag{2.51}
\]

Using (2.23), we can rewrite the product in (2.50) as

\[
\prod_{j=1}^{n} (\Delta_{\omega_j, \omega_j} \prod_{i<j} \Delta_{\omega_j, \omega_i}^{a_{i,j, \omega_i}})^{[c m -1 \omega_k - c m \omega_k : \alpha_j]} = \prod_{i=1}^{n} \Delta_{\omega_i, \omega_i}^{[c m -1 \beta_k : \beta_i]}, \tag{2.52}
\]

where the exponent \([c m -1 \beta_k : \beta_i]\) stands for the coefficient of \( \beta_i \) in the expansion of \( c^{m-1} \beta_k \) in the basis \( \{ \beta_1, \ldots, \beta_n \} \).

Now let us deal with the functions \( \Delta^{c m \omega_k, c m -1 \omega_k} \) and \( \Delta^{c m -1 \omega_k, c m \omega_k} \). Let \( x \) be a generic element of \( G^{c, c^{-1}} \) expressed as in (2.46). We claim that

\[
\Delta^{c m -1 \omega_k, c m \omega_k}(x) = a^{c m -1 \omega_k} \prod_{j=1}^{n} t_{j}^{[c m -1 \beta_k : \alpha_j]}, \tag{2.53}
\]

Our proof of (2.53) uses a little representation theory. Consider the ring of regular functions \( \mathbb{C}[G] \) as a \( G \times G \)-representation under the action \( (g_1, g_2) f(x) = f(g_1^T x g_2) \) (see (2.38) for the definition of \( g_1^T \)). We denote by \( f \mapsto (v_1, v_2) f \) the corresponding action of \( U(\mathfrak{g}) \times U(\mathfrak{g}) \), where \( U(\mathfrak{g}) \) is the universal enveloping algebra of \( \mathfrak{g} \). For every \( f \in \mathbb{C}[G] \) and \( x \in G^{c, c^{-1}} \) as above, \( f(x) \) is a polynomial in \( u_i \) and \( t_i \), and the coefficient of each monomial \( u_1^{h_1} \cdots u_n^{h_n} t_1^{d_1} \cdots t_n^{d_n} \) is equal to \( ((e_1^{(h_1)} \cdots e_n^{(h_n)}, e_1^{(d_1)} \cdots e_1^{(d_1)} f)(a), \)
where \( e^{(d)}_i \) stands for the divided power \( e^d_i/d! \) (cf. [1, Lemma 3.7.5]). Furthermore, if \( f \) is a weight vector of weight \((\gamma, \gamma')\) then \( f(a) \) can be nonzero only if \( \gamma = \gamma' \).

Now recall that, in view of (2.36), each generalized minor \( \Delta_{\gamma, \delta} \) with \( \gamma, \delta \in W\omega_k \) is a weight vector of weight \((\gamma, \delta)\) in the \(G \times G\)-irreducible representation with highest weight \((\omega_k, \omega_k)\). Recall also the following standard facts from the representation theory of \( G \): the weight polytope of the fundamental \( G \)-representation \( V_{\omega_k} \) with highest weight \( \omega_k \) has \( W\omega_k \) as the set of vertices, and for each weight \( \delta \in W\omega_k \), the corresponding weight subspace \( V_{\omega_k}(\delta) \) is one-dimensional. Together with basic facts about the representations of \( SL_2 \), this implies the following: if \( \delta \in W\omega_k \), and \( s_i \delta \geq \delta \) for some \( i \) then \( s_i \delta - \delta = d_i \alpha_i \), where \( e^{(d_i)}_i(V_{\omega_k}(\delta)) = \{0\} \) for \( d > d_i \); furthermore, \( e^{(d_i)}_i \) establishes the same isomorphism \( V_{\omega_k}(\delta) \to V_{\omega_k}(s_i \delta) \) as the group element \( s_i^{-1} \in G \).

Applying all this to \( \gamma = c^{m-1}\omega_k \) and \( \delta = c^m\omega_k \) with \( 1 \leq m \leq h(k; c) \), we conclude that the only tuple of nonnegative integers \((h_1, \ldots, h_n; d_1, \ldots, d_n)\) such that \((e^{(h_1)}_1 \cdots e^{(h_n)}_n, e^{(d_n)}_n \cdots e^{(d_1)}_1)(\Delta_{\gamma, \delta}) \neq 0 \) is the one with all \( h_i \) equal to 0, and \( \sum_i d_i \alpha_i = \gamma - \delta \). Furthermore, for this tuple we have

\[
((e^{(h_1)}_1 \cdots e^{(h_n)}_n, e^{(d_n)}_n \cdots e^{(d_1)}_1)\Delta_{\gamma, \delta})(a) = \Delta_{\gamma, \gamma}(a) = a^\gamma,
\]

implying (2.53).

We now show that

\[
\Delta_{c^{m-1}\omega_k, c^m\omega_k}(x) = \prod_{i=1}^{n} \Delta_{\omega_i, c\omega_i}(x)^{[c^{m-1}\beta_k; \beta_i]} \quad (2.54)
\]

for any \( x \in G^{c,c^{-1}} \). It is enough to show that (2.54) holds for \( x \) of the form (2.46).
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Then both sides of (2.54) are given by (2.53). Comparing the contributions of \( a \in H \) and of each \( t_j \) on both sides, we only have to prove the following two identities:

\[
\begin{align*}
  c^{m-1}\omega_k &= \sum_{i=1}^{n} [c^{m-1}\beta_k : \beta_i]\omega_i; \\
  [c^{m-1}\beta_k : \alpha_j] &= \sum_{i=1}^{n} [c^{m-1}\beta_k : \beta_i][\beta_i : \alpha_j].
\end{align*}
\]

The identity (2.56) is immediate, so let us prove (2.55). Abbreviating \([c^{m-1}\beta_k : \beta_i] = b_i\) and recalling (2.51), we can rewrite the equality

\[
c^{m-1}\beta_k = \sum_{i=1}^{n} b_i\beta_i
\]

in the form

\[
c^{m-1}\omega_k - \sum_{i=1}^{n} b_i\omega_i = c(c^{m-1}\omega_k - \sum_{i=1}^{n} b_i\omega_i).
\]

To finish the proof of (2.55), it remains to use the well-known property that no Coxeter element has 1 as an eigenvalue (see, e.g., [15, Lemma 8.1]).

In view of (2.54) and (2.52), to finish the proof of Lemma 2.3.1 (hence that of the relation (2.11)), it remains to show that the restriction to \( L^{c,c-1} \) of each \( \Delta_{c^{m}\omega_k,c^{m-1}\omega_k} \) is equal to 1. In view of (2.37), the identity (2.54) implies that

\[
\Delta_{c^{m}\omega_k,c^{m-1}\omega_k}(x) = \prod_{i=1}^{n} \Delta_{\omega_{i},\omega_{i}}(x)^{[c^{m-1}\beta_k : \beta_i]}
\]

for any \( x \in G^{c,c-1} \). Now the fact that the right-hand side of this equality is equal to 1 for \( x \in L^{c,c-1} \), follows from Proposition 2.3.3, and we are done. \( \square \)
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Let us now turn to the proof of (2.10).

Proof of relation (2.10). It is enough to show that both sides of this identity take the same values at a general element \( x \in L^{c,c-1} \) expressed in the form (2.48).

To compute \( \Delta_{\omega_i,\omega_i}(x) = ([x]_0)^{\omega_i} \) (see (2.35)), express each factor \( x_{-j}(u_j) \) in \( x \) as \( x_j(u_j) u_j^{-\alpha_j^\vee} \), and move the factors \( u_j^{-\alpha_j^\vee} \) all the way to the “center”, using the commutation relations in [5, (2.5)]:

\[
ax_j(t) = x_j(a^{\alpha_j t}) a, \quad ax_j(u) = x_j(a^{-\alpha_j} u) a \quad (a \in H). \tag{2.57}
\]

We obtain

\[
x = x_1(r_1) \cdots x_n(r_n) \prod_{j=1}^n u_j^{-\alpha_j^\vee} x_n(t_n) \cdots x_1(t_1),
\]

where the precise expression for \( r_j \) is immaterial for our current purposes. It follows that \( [x]_0 = \prod_{j=1}^n u_j^{-\alpha_j^\vee} \), hence

\[
\Delta_{\omega_i,\omega_i}(x) = \prod_{j=1}^n u_j^{\omega_i(-\alpha_j^\vee)} = u_i^{-1}. \tag{2.58}
\]

In view of (2.53), we also have

\[
\Delta_{\omega_i,\omega_i}(x) = u_i^{-1} \prod_{j=1}^n t_j^{[\beta_j;\alpha_j]}.
\]

Remembering the definition (2.6) and using (2.23), we get

\[
y_{k:c}(x) = \Delta_{\omega_k,\omega_k}(x) \prod_{i<k} \Delta_{\omega_i,\omega_i}(x)^{a_{i,k}} = t_k u_k^{-1} \prod_{j<k} u_j^{-a_{j,k}}. \tag{2.59}
\]
Substituting the expressions in (2.58) and (2.59) into (2.10), we can simplify it as follows:

\[
\Delta_{-\omega_k,-\omega_k}(x) = t_k \prod_{j > k} \Delta_{-\omega_j,-\omega_j}(x)^{-a_{j,k}} + u_k.
\] (2.60)

To prove (2.60), note that, according to (2.40), we have

\[
\Delta_{-\omega_k,-\omega_k}(x) = \Delta_{\omega_k,\omega_k}(x^t).
\]

Here \(x \mapsto x^t\) is an involutive antiautomorphism of the group \(G\) defined in (2.39). As an easy consequence of this definition, we have

\[
x_{-j}(u^t) = x_{-j}(u^{-1}),
\]

hence

\[
x^t = x_1(t_1) \cdots x_n(t_n)x_{-n}(u_n^{-1}) \cdots x_{-1}(u_1^{-1}).
\]

To compute \(\Delta_{\omega_k,\omega_k}(x^t)\) we use the commutation relations in [3, Proposition 7.2], which we rewrite as follows:

\[
x_j(t)x_{-i}(u^{-1}) = x_{-i}(u^{-1})x_j(tu^{-a_{i,j}}) \quad (i \neq j),
\] (2.61)

\[
x_j(t)x_{-j}(u^{-1}) = x_{-j}((t + u)^{-1})x_j(w),
\] (2.62)

where the precise expression for \(w\) is immaterial for our purposes. Iterating these relations, we can rewrite \(x^t\) in the form

\[
x^t = x_{-n}(v_n^{-1}) \cdots x_{-1}(v_1^{-1})x_1(w_1) \cdots x_n(w_n),
\]
where the parameters $v_1, \ldots, v_n$ satisfy the recurrence relations

$$v_k = t_k \prod_{j>k} v_j^{-a_{j,k}} + u_k.$$ 

It remains to observe that $\Delta_{\omega_k, \omega_k}(x^\epsilon) = v_k$ (cf. (2.58)), so the last formula implies (2.60). This completes the proof of (2.10).

\[\square\]

## 2.4 Cluster Algebras

We now present a cluster algebra setup more general than that developed earlier in Section 1.1.

We start by recalling basic definitions and facts on cluster algebras following mostly [10]. The definition of a cluster algebra $\mathcal{A}$ starts with introducing its ground ring. Let $\mathbb{P}$ be a semifield, i.e., an abelian multiplicative group endowed with a binary operation of (auxiliary) addition $\oplus$ which is commutative, associative, and distributive with respect to the multiplication in $\mathbb{P}$. The multiplicative group of $\mathbb{P}$ is torsion-free [7, Section 5], hence its group ring $\mathbb{Z}\mathbb{P}$—which will be used as a ground ring for $\mathcal{A}$—is a domain. Let $\mathbb{Q}(\mathbb{P})$ denote the field of fractions of $\mathbb{Z}\mathbb{P}$.

Every finite family $\{u_j : j \in J\}$ gives rise to a tropical semifield $\text{Trop}(u_j : j \in J)$ defined as follows. As a multiplicative group, $\text{Trop}(u_j : j \in J)$ is an abelian group freely generated by the elements $u_j (j \in J)$; and the addition $\oplus$ in $\text{Trop}(u_j : j \in J)$ is given by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j,b_j)}.$$ (2.63)
Thus, the group ring of $\text{Trop}(u_j : j \in J)$ is the ring of Laurent polynomials in the variables $u_j$.

As an ambient field for a cluster algebra $\mathcal{A}$ of rank $n$, we take a field $\mathcal{F}$ isomorphic to the field of rational functions in $n$ independent variables, with coefficients in $\mathbb{Q}(\mathbb{P})$. Note that the definition of $\mathcal{F}$ ignores the auxiliary addition in $\mathbb{P}$.

**Definition 2.4.1** (Seeds). A (labeled) seed in $\mathcal{F}$ is a triple $(x, y, B)$, where

- $x = \{x_i : i \in I\}$ is an $n$-tuple of elements of $\mathcal{F}$ forming a free generating set for $\mathcal{F}$, that is, the elements $x_i$ for $i \in I$ are algebraically independent over $\mathbb{Q}(\mathbb{P})$, and $\mathcal{F} = \mathbb{Q}(\mathbb{P})(x)$.
- $y = \{y_i : i \in I\}$ is an $n$-tuple of elements of $\mathbb{P}$.
- $B = (b_{i,j})_{i,j \in I}$ is an $n \times n$ integer matrix which is skew-symmetrizable, that is, $d_i b_{i,j} = -d_j b_{j,i}$ for some positive integers $d_i$ ($i \in I$).

We refer to $x$ as the cluster of a seed $(x, y, B)$, to the tuple $y$ as the coefficient tuple, and to the matrix $B$ as the exchange matrix.

We will use the notation $[b]_+ = \max(b, 0)$.

**Definition 2.4.2** (Seed mutations). Let $(x, y, B)$ be a seed in $\mathcal{F}$, and $k \in I$. The seed mutation $\mu_k$ in direction $k$ transforms $(x, y, B)$ into the seed $\mu_k(x, y, B) = (x', y', B')$ defined as follows:

- The entries of $B' = (b'_{i,j})$ are given in the same way as in 1.1.
• The coefficient tuple $y'$ is given by
\[
y'_j = \begin{cases} 
  y^{-1} & \text{if } j = k; \\
  y_j y_k^{[b_{j,k}]} (y_k \oplus 1)^{-b_{k,j}} & \text{if } j \neq k. 
\end{cases}
\] (2.64)

• The cluster $x'$ is given by $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the exchange relation
\[
x'_k = \frac{y_k \prod x_i^{[b_{i,k}]} + \prod x_i^{-b_{i,k}}}{(y_k \oplus 1)x_k}. 
\] (2.65)

The mutation equivalence relation on seeds and the set of cluster variables are defined (and denoted) in the same way as in Section 1.1.

**Definition 2.4.3 (Cluster algebra).** The cluster algebra $\mathcal{A}$ associated with a mutation equivalence class of seeds $\mathcal{S}$ is the $\mathbb{Z} \mathbb{P}$-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A} = \mathbb{Z} \mathbb{P}[\mathcal{X}]$. We denote $\mathcal{A} = \mathcal{A}(\mathcal{S}) = \mathcal{A}(x, y, B) = \mathcal{A}(y, B)$, where $(x, y, B)$ is any seed in $\mathcal{S}$.

With Definition 1.1.3 in mind, a cluster algebra of geometric type is nothing but a cluster algebra associated to a tropical semifield.

The following definition of a system of principal coefficients is equivalent to Definition 1.2.2.

**Definition 2.4.4 (Principal coefficients).** We say that a seed $(x^o, y^o, B)$ has principal coefficients if the coefficient semifield $\mathbb{P}^o$ is the tropical semifield $\mathbb{P}^o = \text{Trop}(y^o_i : i \in I)$ generated by the coefficient tuple $y^o$. 
To emphasize the significance of principal coefficients, we sometimes use the notation $A^\circ = A(x^\circ, y^\circ, B)$ to denote the cluster algebra with principal coefficients at an initial seed $(x^\circ, y^\circ, B)$. We now describe in more details on how the system of principal coefficients controls all the other choices of coefficients through the $g$-vectors and $F$-polynomials (cf. (1.4)).

For convenience, let us use $I = \{1, \ldots, n\}$ as the index set. Recall that the $F$-polynomials are conjectured (and in many cases proved) to have positive coefficients. However, even with this conjecture not yet proved in complete generality, one can still evaluate every $F_{z; x^\circ}$ at an $n$-tuple of elements of an arbitrary semifield $\mathbb{P}$, since, as shown in [10], $F_{z; x^\circ}(u_1, \ldots, u_n)$ can be expressed as a subtraction-free rational expression in $u_1, \ldots, u_n$. We denote such an evaluation as $F_{z; x^\circ}|_{\mathbb{P}}$. Using this notation, one can use $g$-vectors and $F$-polynomials to compute all the cluster variables in an arbitrary cluster algebra $A = A(x, y, B)$ with the coefficients in an arbitrary semifield $\mathbb{P}$. Namely, consider again the cluster algebra $A^\circ = A(x^\circ, y^\circ, B)$ with principal coefficients at the initial seed with the same exchange matrix $B$. Then, as shown in [10, Corollary 6.3], every cluster variable $z^\circ \in A^\circ$ gives rise to a cluster variable $z \in A$ given by

$$z = x_1^{\hat{y}_1} \cdots x_n^{\hat{y}_n} \frac{F_{z; x^\circ}(\hat{y}_1, \ldots, \hat{y}_n)}{F_{z^\circ; x^\circ}|_{\mathbb{P}}(y_1, \ldots, y_n)},$$

(2.66)

and all the cluster variables in $A$ are of this form; here the elements $\hat{y}_j$ of the ambient field for $A$ have the same meaning as in (1.3). Comparing (2.66) and (1.4), we obtain the following assertion.

**Proposition 2.4.5.** In the above notation, suppose that the elements $y_1, \ldots, y_n \in \mathbb{P}$
are multiplicatively independent, i.e., the correspondence \( y_j^\circ \mapsto y_j \) identifies \( \mathbb{P}^\circ \) with a multiplicative subgroup of \( \mathbb{P} \). Then the cluster algebra \( \mathcal{A}^\circ \) can be identified with a \( \mathbb{ZP}^\circ \)-subalgebra of \( \mathcal{A} \) via the correspondence sending each cluster variable \( z^\circ \in \mathcal{A}^\circ \) to \( F_z^\circ, x^\circ |_{\mathbb{F}(y_1, \ldots, y_n)} z \in \mathcal{A} \). With this identification, \( \mathcal{A} \) is obtained from \( \mathcal{A}^\circ \) by the extension of scalars from \( \mathbb{ZP}^\circ \) to \( \mathbb{ZP} \).

Recall that a cluster algebra is of finite type if it has finitely many cluster variables. As mentioned before, a classification of cluster algebras of finite type was given in [9]. We present it in the form convenient for our current purposes.

**Theorem 2.4.6 ([9]).** A cluster algebra \( \mathcal{A} \) is of finite type if and only if the exchange matrix at some seed of \( \mathcal{A} \) is of the form \( B(c) \) (see (2.4)) for some Coxeter element in the Weyl group associated to a Cartan matrix \( A \) of finite type. Furthermore, the type of \( A \) in the Cartan-Killing nomenclature is uniquely determined by the mutation equivalence class of seeds in \( \mathcal{A} \), and, for a given \( A \), all the matrices \( B(c) \) associated to different Coxeter elements are mutation equivalent to each other.

By Theorem 2.4.6, the property of a cluster algebra \( \mathcal{A} \) to be of finite type does not depend on the choice of a coefficient system. As shown in [9], the same is true for the structure of the set \( \mathcal{X} \) of cluster variables and its division into clusters. To be more precise, fix a Cartan matrix \( A \) of the same type as \( \mathcal{A} \), and choose an initial seed \((x, y, B)\) with the exchange matrix \( B = B(t) \), where \( t = t_\uparrow t_\downarrow \) is the bipartite Coxeter element (see (2.26)). As before, let \( \Phi \) be the root system associated with \( A \). Then the cluster variables in \( \mathcal{A} \) are in a natural bijection \( \alpha \mapsto x[\alpha] \) with the set \( \Phi_{\geq 1} \) of “almost positive” roots (the union of the set of positive roots and the set of negative simple roots \(-\alpha_i\)). (Recall that, for each \( \alpha \in \Phi_{\geq 1} \), the integer vector in \( \mathbb{Z}' \) with
the components $d_i = [\alpha : \alpha_i]$ is the denominator vector in the Laurent expansion of the cluster variable $x[\alpha]$ with respect to the initial cluster.) In particular, each initial cluster variable takes the form $x_i = x[-\alpha_i]$.

To describe the clusters in $\mathcal{A}$, we recall the two involutive permutations $\tau_+, \tau_-$ of $\Phi_{\geq -1}$ given by

$$
\tau_{\varepsilon}(\alpha) = \begin{cases} 
\alpha & \text{if } \alpha = -\alpha_j \text{ with } \varepsilon(j) = -\varepsilon; \\
t_{\varepsilon}(\alpha) & \text{otherwise.}
\end{cases}
$$

As shown in [8, Section 3.1], for any $\alpha, \beta \in \Phi_{\geq -1}$, there is a well-defined nonnegative integer $(\alpha \parallel \beta)$ (compatibility degree) uniquely characterized by the following two properties:

$$
(\alpha \parallel \beta) = [[\beta : \alpha_i]]_+,
$$

$$
(\tau_{\varepsilon} \alpha \parallel \tau_{\varepsilon} \beta) = (\alpha \parallel \beta),
$$

for any $\alpha, \beta \in \Phi_{\geq -1}$, any $i \in I$, and any sign $\varepsilon$. We say that $\alpha$ and $\beta$ are compatible if $(\alpha \parallel \beta) = 0$ (this is in fact a symmetric relation). With this terminology in place, the following was shown in [9].

**Proposition 2.4.7.** Under the above parameterization of cluster variables by almost positive roots, the clusters in $\mathcal{A}$ are exactly the families $x(C) = \{x[\alpha] : \alpha \in C\}$, where $C$ runs over all maximal by inclusion subsets of $\Phi_{\geq -1}$ consisting of mutually compatible roots.

Note that every cluster $x(C)$ is included in a unique seed $(x(C), y(C), B(C))$ in $\mathcal{A}$. To emphasize the dependence on $\mathcal{A}$, we use the notation $(x(C), y(C), B(C)) =$
(\(x(C)^{(A)}, y(C)^{(A)}, B(C)\)); note that the exchange matrix \(B(C)\) is independent of \(A\), and it is explicitly described in [9, Definition 4.2].

Now we introduce a tool to relate to each other two cluster algebras of the same finite type but with different coefficient systems. This is a version of the coefficient specialization discussed in [10, Section 12]. We will deal only with cluster algebras of geometric type. Let us start with a simple general proposition.

**Proposition 2.4.8.** Let \(\mathbb{P} = \text{Trop}(u_j : j \in J)\) and \(\tilde{\mathbb{P}} = \text{Trop}(u_{\tilde{j}} : \tilde{j} \in \tilde{J})\) be tropical semifields, and \(\varphi : \tilde{\mathbb{P}} \to \mathbb{P}\) a homomorphism of multiplicative groups. Then \(\varphi\) is a homomorphism of semifields if and only if each \(\varphi(u_{\tilde{j}}) \in \mathbb{P}\) is a monomial with nonnegative exponents in the generators \(u_j\) of \(\mathbb{P}\), and no two such monomials have a common factor \(u_j\) for some \(j \in J\).

**Proof.** This follows at once from the equalities \(u_{\tilde{j}} \oplus 1 = 1\) and \(u_{\tilde{j}} \oplus u_{\tilde{j}}' = 1\) for \(\tilde{j} \neq \tilde{j}'\). \(\square\)

Now let \(\tilde{A}\) be a cluster algebra of finite type as above, with the coefficient semifield \(\tilde{\mathbb{P}} = \text{Trop}(u_{\tilde{j}} : \tilde{j} \in \tilde{J})\). In the above notation, the seeds of \(\tilde{A}\) are of the form \((x(C)^{(\tilde{A})}, y(C)^{(\tilde{A})}, B(C))\). Let \(\varphi : \tilde{\mathbb{P}} \to \mathbb{P}\) be a homomorphism of tropical semifields as in Proposition 2.4.8. Using \(\varphi\), we can form a cluster algebra \(A\) of the same type as \(\tilde{A}\), but with coefficients in \(\mathbb{P}\): its seeds are of the form \((x(C)^{(A)}, y(C)^{(A)}, B(C))\), where the coefficient tuple \(y(C)^{(A)}\) at some (equivalently, any) seed is taken as \(\varphi(y(C)^{(\tilde{A})})\). The following relationship between \(A\) and \(\tilde{A}\) is immediate from the definitions.

**Proposition 2.4.9.** The semifield homomorphism \(\varphi\) induces an isomorphism of \(\mathbb{Z}[\mathbb{P}]-\)algebras

\[\varphi_* : \tilde{A} \otimes_{\mathbb{Z}[\tilde{\mathbb{P}}]} \mathbb{Z}[\mathbb{P}] \to A\]
given by $\varphi_*(x[\alpha](\tilde{A}) \otimes 1) = x[\alpha](A)$ for $\alpha \in \Phi_{\geq -1}$.

We are now ready to prove Theorem 2.1.2.

**Proof of Theorem 2.1.2.** We will show that Proposition 2.4.9 applies to $\tilde{A}_C = \mathbb{C}[G_{c,c}^{-1}]$ and $A_C = \mathbb{C}[L_{c,c}^{-1}]$, where $\tilde{A}_C$ and $A_C$ are obtained from $\tilde{A}$ and $A$ by extension of scalars from $\mathbb{Z}$ to $\mathbb{C}$. First of all, the fact that the coordinate ring of $G_{c,c}^{-1}$ carries the structure of a (complexified) geometric type cluster algebra $\tilde{A}_C$ of the same finite type as $G$, was established in [2, Example 2.24]. We will use a slightly modified description of its initial seed by choosing a different reduced word for $(c, c^{-1})$. Namely, we have:

- the ambient field for $\tilde{A}_C$ is the field of rational functions on $G_{c,c}^{-1}$;
- the coefficient semifield $\tilde{P}$ is the tropical semifield with $2n$ generators: $\Delta_{c\omega_i, \omega_i} |_{G_{c,c}^{-1}}$ and $\Delta_{\omega_i, c\omega_i} |_{G_{c,c}^{-1}}$ for $i \in I$;
- the initial cluster consists of the functions $\Delta_{\omega_i, \omega_i} |_{G_{c,c}^{-1}}$;
- the initial coefficient tuple consists of the restrictions to $G_{c,c}^{-1}$ of the functions $\Delta_{\omega_j, c\omega_j} \Delta_{c\omega_j, \omega_j} \prod_{i \prec j} (\Delta_{\omega_j, c\omega_j} \Delta_{c\omega_j, \omega_i})^{a_{i,j}}$;
- the initial exchange matrix is $B(c)$.

Now recall that by Proposition 2.3.3, the reduced double cell $L_{c,c}^{-1}$ is obtained from $G_{c,c}^{-1}$ by specializing each invertible regular function $\Delta_{\omega_i, \omega_i}$ to 1. Algebraically, this means that the coordinate ring $\mathbb{C}[L_{c,c}^{-1}]$ can be described as follows. Let $\overline{\mathbb{P}}$ be the tropical semifield with the $n$ generators $\Delta_{\omega_i, \omega_i} |_{L_{c,c}^{-1}}$ for $i \in I$, and let $\varphi : \overline{\mathbb{P}} \to \mathbb{P}$ be the restriction homomorphism from $G_{c,c}^{-1}$ to $L_{c,c}^{-1}$. Then $\varphi$ is as in Proposition 2.4.8,
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acting on the generators by

$$\varphi(\Delta_{\omega_i,\omega_i}|_{G_<c^{-1}}) = 1, \quad \varphi(\Delta_{\omega_i,\omega_i}|_{G_<c^{-1}}) = \Delta_{\omega_i,\omega_i}|_{L_<c^{-1}};$$

and we have

$$\mathbb{C}[L_{c,c^{-1}]} = \mathbb{C}[G^c_{c^{-1}}] \otimes_{\mathbb{Z}_{\tilde{P}}} \mathbb{Z}_{\tilde{P}}.$$

Comparing this with Proposition 2.4.9, we conclude that $\mathbb{C}[L_{c,c^{-1}}]$ is the complexified cluster algebra $\mathcal{A}_c$ with the coefficient semifield $\mathbb{P}$ and the initial seed as in Theorem 2.1.2.

This is still not quite what we need to prove Theorem 2.1.2, since the auxiliary addition in $\mathbb{P}$ is different from the one in $\mathbb{P}^0 = \text{Trop}(y_{j;c} : j \in I)$. However, the elements $y_{j;c} \in \mathbb{P}$ are related with the generators $\Delta_{\omega_i,\omega_i}|_{L_<c^{-1}}$ by a triangular (hence invertible) monomial transformation. Therefore, as a multiplicative group, $\mathbb{P}^0$ coincides with $\mathbb{P}$.

By Proposition 2.4.5, the cluster algebra $\mathcal{A}^0$ with principal coefficients at the initial seed can be identified with $\mathcal{A}$ (with the cluster variables in $\mathcal{A}^0$ obtained by rescaling from those in $\mathcal{A}$). This concludes the proof of Theorem 2.1.2. \qed

### 2.5 $c$-compatibility degree

In this section we fix an indecomposable Cartan matrix $A$ of finite type, and freely use the root system formalism developed in Section 2.2, and the cluster algebra formalism developed in Section 2.4 (as before, lifting the restriction that $A$ is indecomposable presents no problem).

Let us recap a little. For a Coxeter element $c$ in the Weyl group of $A$, let $\mathcal{A}^0(c)$
be the cluster algebra with principal coefficients at an initial seed \((x^\circ, y^\circ, B(c))\) as defined in Definition 2.4.4. We have already shown in Theorem 2.1.2 that the complexification of \(A^\circ(c)\) can be identified with the coordinate ring \(\mathbb{C}[L^{c,c^{-1}}]\), and under this identification, the initial cluster variables \(x^0_i = x_{\omega_i,c}\) and the initial coefficients \(y^0_j = y_{j,c}\) are given by (2.5) and (2.6), respectively. We have also proved that the elements \(x_{\gamma,c}\) (for \(\gamma \in \Pi(c)\)) and \(y_{j,c}\) satisfy algebraic relations (2.10) and (2.11) in \(A^\circ(c)\). Thus, to prove Theorem 2.1.4 and to complete the proof of Theorem 2.1.5, we can forget about the geometric realization of \(A^\circ(c)\), and it remains to prove the following two statements:

The elements \(x_{\gamma,c} \in A^\circ(c)\) (for \(\gamma \in \Pi(c)\)) satisfying \((2.70)\) and \((2.71)\) are exactly the cluster variables in \(A^\circ(c)\).

The relations \((2.10)\) and \((2.11)\) are exactly \((2.71)\) the primitive exchange relations in \(A^\circ(c)\).

For \((2.71)\) recall from [10, Section 12] that an exchange relation \((2.65)\) is called *primitive* if one of the products of cluster variables in the right hand side is empty, understood to be equal to 1.

We start with some combinatorial preparation. First we transfer the compatibility degree function given by \((2.68)\) and \((2.69)\) from \(\Phi_{\geq-1}\) to each \(\Pi(c)\). Let \(\tau_c\) be the
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permutation of $\Pi(c)$ given by

$$
\tau_c(c^{m-1}\omega_i) = c^m\omega_i \quad (1 \leq m \leq h(i; c)), \quad \tau_c(-\omega_i) = \omega_i.
$$

\[\text{(2.72)}\]

**Proposition 2.5.1.** There is a unique assignment of a nonnegative integer $(\gamma\parallel\delta)_c$ to any pair of weights $\gamma, \delta \in \Pi(c)$, satisfying the following properties:

1. $(\omega_i\parallel\omega_j)_c = 0 \quad (i, j \in I)$,

2. $(\omega_i\parallel\gamma)_c = [c^{-1}\gamma - \gamma : \alpha_i] \quad (\gamma \in \Pi(c) - \{\omega_j : j \in I\})$,

3. $(\tau_c\gamma\parallel\tau_c\delta)_c = (\gamma\parallel\delta)_c \quad (\gamma, \delta \in \Pi(c))$.

We call $(\gamma\parallel\delta)_c$ the $c$-compatibility degree of $\gamma$ and $\delta$.

**Proof.** The uniqueness of the $c$-compatibility degree with desired properties is clear. Note also that, in view of (2.22), condition (2) in Proposition 2.5.1 can be rewritten as

$$
(\omega_i\parallel c^m\omega_j)_c = [c^{m-1}\beta_j : \alpha_i] \quad (i, j \in I; 1 \leq m \leq h(j; c)),
$$

\[\text{(2.73)}\]

where the root $\beta_j$ is given by (2.21).

In view of (2.25), to prove the existence it suffices to do it for the bipartite Coxeter element $t = t_+t_-$, and then to show that the existence of the $c$-compatibility degree with desired properties implies that of $\tilde{c}$-compatibility degree, where $\tilde{c}$ is obtained from $c$ as in (2.24).

We start by dealing with $\Pi(t)$. Consider the permutation $\tau = \tau_+\tau_-$ of $\Phi_{\geq-1}$, where $\tau_+$ and $\tau_-$ are given by (2.67).
Lemma 2.5.1. There is a bijection \( \psi : \Pi(t) \to \Phi_{\geq -1} \) uniquely determined by the properties that \( \psi(\omega_i) = -\alpha_i \) for all \( i \in I \), and \( \psi \circ \tau_i = \tau \circ \psi \). Explicitly, \( \psi \) is given by

\[
\psi(\gamma) = \begin{cases} 
-\alpha_i & \text{if } \gamma = \omega_i \text{ for some } i \in I; \\
t^{-1}\gamma - \gamma & \text{otherwise.}
\end{cases}
\] (2.74)

Proof. The uniqueness of \( \psi \) with the desired properties is clear. The fact that (2.74) defines a bijection between \( \Pi(t) \) and \( \Phi_{\geq -1} \) follows from Lemmas 2.2.1 and 2.2.2. It remains to prove that the map \( \psi \) given by (2.74) satisfies the property that \( \psi(\tau_i \gamma) = \tau \psi(\gamma) \) for \( \gamma \in \Pi(t) \). This is clear if both \( \psi(\gamma) \) and \( \psi(\tau_i \gamma) \) fall into the second case in (2.74), i.e., if \( \gamma \) is not of the form \( \pm \omega_i \). By the definitions, we also have

\[
\psi(\tau_i \omega_i) = \tau \psi(\omega_i) = \begin{cases} 
\alpha_i & \text{if } \varepsilon(i) = +1; \\
t_i \alpha_i & \text{if } \varepsilon(i) = -1,
\end{cases}
\]

and \( \psi(\tau_i (-\omega_i)) = \tau \psi(-\omega_i) = -\alpha_i \) for all \( i \in I \), finishing the proof. \( \square \)

By Lemma 2.5.1, one can define the \( t \)-compatibility degree by setting

\[
(\gamma \parallel \delta)_t = (\psi(\gamma) \parallel \psi(\delta)) \quad (\gamma, \delta \in \Pi(t)),
\] (2.75)

and it satisfies the desired properties.

Now suppose that \( \tilde{c} \) is obtained from \( c \) via (2.24). Without loss of generality, we assume that \( I = \{1, \ldots, n\} \), \( c = s_1 s_2 \cdots s_n \), and \( \tilde{c} = s_2 \cdots s_n s_1 \). It remains to show that the existence of the \( c \)-compatibility degree with the desired properties implies
that of the $\tilde{c}$-compatibility degree. The following lemma is an easy consequence of (2.32).

**Lemma 2.5.2.** There is a bijection $\psi_{c,\tilde{c}} : \Pi(\tilde{c}) \to \Pi(c)$ uniquely determined by the properties that

\[
\psi_{c,\tilde{c}}(\omega_i) = \begin{cases} 
\omega_i & \text{if } i \neq 1; \\
\alpha_i & \text{if } i = 1,
\end{cases} \quad (2.76)
\]

and $\psi_{c,\tilde{c}} \circ \tau_{\tilde{c}} = \tau_c \circ \psi_{c,\tilde{c}}$. Explicitly, for $\gamma \in \Pi(\tilde{c})$ we have

\[
\psi_{c,\tilde{c}}(\gamma) = \begin{cases} 
\omega_1 & \text{if } \gamma = -\omega_1; \\
s_1 \gamma & \text{otherwise}.
\end{cases} \quad (2.77)
\]

Now we define the $\tilde{c}$-compatibility degree by setting

\[
(\gamma\parallel\delta)_{\tilde{c}} = (\psi_{c,\tilde{c}}(\gamma)\parallel\psi_{c,\tilde{c}}(\delta))_c \quad (\gamma, \delta \in \Pi(\tilde{c})). \quad (2.78)
\]

This makes condition (3) in Proposition 2.5.1 obvious. It remains to check conditions (1) and (2).

The equality $(\omega_i\parallel\omega_j)_{\tilde{c}} = 0$ is obvious if $i, j \neq 1$. Let us show that $(\omega_1\parallel\omega_i)_{\tilde{c}} = 0$ and $(\omega_i\parallel\omega_1)_{\tilde{c}} = 0$ for $i \neq 1$. Indeed, we have

\[
(\omega_1\parallel\omega_i)_{\tilde{c}} = (\omega_1\parallel\omega_i)_c = (\omega_1 - \omega_i)_c \\
= [\omega_i - c^{-1}\omega_1 : \alpha_1] = [s_ns_{n-1}\cdots s_{i+1}\alpha_i : \alpha_1] = 0,
\]

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and

$$(\omega_i \parallel \omega_1)_{\tilde{c}} = (\omega_i \parallel c\omega_1)_{\tilde{c}} = [\omega_i - c\omega_1 : \alpha_i] = [\alpha_1 : \alpha_i] = 0.$$  

It remains to prove that $(\omega_i \parallel c^m \omega_j)_{\tilde{c}}$ for $i, j \in \{1, \ldots, n\}$ and $1 \leq m \leq h(j; \tilde{c})$ is given by (2.73), i.e., we have

$$(\omega_i \parallel c^m \omega_j)_{\tilde{c}} = [c^{m-1} s_2 \cdots s_{j-1} \alpha_j : \alpha_i],$$  

(2.79)

with the convention that for $j = 1$, the index $j - 1$ is understood to be equal to $n$.

In checking (2.79), we will repeatedly use the following obvious property: the coefficient $[\alpha : \alpha_i]$ does not change under replacing $\alpha$ with $s_k\alpha$ for $k \neq i$. Now let us consider four separate cases.

**Case 1.** Let $i, j \neq 1$. Then we have

$$(\omega_i \parallel c^m \omega_j)_{\tilde{c}} = (\omega_i \parallel c^m \omega_j)_{\tilde{c}} = [c^{m-1} s_1 \cdots s_{j-1} \alpha_j : \alpha_i]$$

$$= [s_1 c^{m-1} s_2 \cdots s_{j-1} \alpha_j : \alpha_i] = [c^{m-1} s_2 \cdots s_{j-1} \alpha_j : \alpha_i],$$

proving (2.79).

**Case 2.** Let $i = 1$ and $j \neq 1$. If $m \geq 2$ then we have

$$(\omega_1 \parallel c^m \omega_j)_{\tilde{c}} = (\omega_1 \parallel c^m \omega_j)_{\tilde{c}} = [c^{m-2} s_1 \cdots s_{j-1} \alpha_j : \alpha_1]$$

$$= [s_2 \cdots s_n c^{m-2} s_1 \cdots s_{j-1} \alpha_j : \alpha_1] = [c^{m-1} s_2 \cdots s_{j-1} \alpha_j : \alpha_1],$$
proving (2.79). And if \( m = 1 \) then

\[
(\omega_1||\tilde{c}\omega_j)_{\tilde{c}} = (\omega_1||\omega_j)_{c} = 0 = [s_2 \cdots s_{j-1}\alpha_j : \alpha_1],
\]

again proving (2.79).

**Case 3.** Let \( i \neq 1 \) and \( j = 1 \). Then we have

\[
(\omega_i||\tilde{c}^m\omega_1)_{\tilde{c}} = (\omega_i||c^{m+1}\omega_1)_c = [c^m\alpha_1 : \alpha_i]
\]

\[
= [s_1c^{m-1}s_2 \cdots s_n\alpha_1 : \alpha_i] = [\tilde{c}^{m-1}s_2 \cdots s_n\alpha_1 : \alpha_i],
\]

proving (2.79).

**Case 4.** Let \( i = j = 1 \). Then we have

\[
(\omega_1||\tilde{c}^m\omega_1)_{\tilde{c}} = (\omega_1||c^m\omega_1)_c = [c^{m-1}\alpha_1 : \alpha_1]
\]

\[
= [s_2 \cdots s_n\alpha_1 \alpha_1 : \alpha_i] = [\tilde{c}^{m-1}s_2 \cdots s_n\alpha_1 : \alpha_i],
\]

proving (2.79) in this case as well.

This completes the proof of Proposition 2.5.1.

We now show that the \( c \)-compatibility degree satisfies “Langlands duality.” Let \( \Phi^\vee \) denote the dual root system to \( \Phi \); it corresponds to the transpose Cartan matrix \( A^T \). The Weyl group of \( A^T \) is identified with the Weyl group \( W \) of \( A \), and there is a canonical \( W \)-equivariant bijection \( \alpha \mapsto \alpha^\vee \) between \( \Phi \) and \( \Phi^\vee \). This bijection restricts
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to a bijection between \( \Phi_{\geq -1} \) and \( \Phi_{\geq -1}^\vee \). As shown in [8, Proposition 3.3], we have

\[
(\alpha \parallel \beta) = (\beta^\vee \parallel \alpha^\vee) \quad (\alpha, \beta \in \Phi_{\geq -1}).
\]  

(2.80)

Now consider the set \( \Pi(c)^\vee \) defined in the same way as \( \Pi(c) \) but associated with the dual root system. Thus, the elements of \( \Pi(c)^\vee \) are coweights of the form \( c^m \omega_i^\vee \), where the \( \omega_i^\vee \) are fundamental coweights, and \( 0 \leq m \leq h(i; c) \) (it easily follows from Lemmas 2.2.1 and 2.2.2 that the numbers \( h(i; c) \) for the dual root system \( \Phi^\vee \) are the same as for the root system \( \Phi \)). The correspondence \( \omega_i \mapsto \omega_i^\vee \) uniquely extends to the \( \tau_c \)-equivariant bijection \( \gamma \mapsto \gamma^\vee \) between \( \Pi(c) \) and \( \Pi(c)^\vee \). By the definition, for every \( \gamma \in \Pi(c) - \{ \omega_i : i \in I \} \), we have

\[
(\tau_c^{-1} \gamma - \gamma)^\vee = \tau_c^{-1} \gamma^\vee - \gamma^\vee.
\]  

(2.81)

With some abuse of notation, we denote by \( (\gamma^\vee \parallel \delta^\vee)_c \) the \( c \)-compatibility degree on \( \Pi(c)^\vee \).

**Proposition 2.5.2.** For \( \gamma, \delta \in \Pi(c) \), we have

\[
(\gamma \parallel \delta)_c = (\delta^\vee \parallel \gamma^\vee)_c.
\]  

(2.82)

**Proof.** The equality (2.82) follows at once from (2.80) and an obvious fact that each of the bijections \( \psi \) and \( \psi_{c, \tilde{c}} \) (see Lemmas 2.5.1 and 2.5.2) commutes with passing to the dual roots and weights. \( \square \)

We illustrate the use of (2.82) by the following lemma to be used later.
Lemma 2.5.3. For any index $j \in I$, and $\gamma \in \Pi(c) - \{\omega_j, c\omega_j\}$, we have

$$\langle \gamma \| \omega_j \rangle_c + \sum_{i < c, j} a_{i,j} \langle \gamma \| \omega_i \rangle_c = -\langle \gamma \| \omega_j \rangle_c + \sum_{j < c} a_{i,j} \langle \gamma \| \omega_i \rangle_c. \quad (2.83)$$

Furthermore, for $\gamma = \omega_j$, the right hand side of (2.83) is equal to 0, while the left hand side is equal to 1; and for $\gamma = c\omega_j$, the left hand side of (2.83) is equal to 0, while the right hand side is equal to $-1$.

Proof. First of all, if $\gamma = \omega_k$ for some $k$, then the right hand side of (2.83) is equal to 0 by condition (1) in Proposition 2.5.1. As for the left hand side, by condition (2) it is equal to

$$[\beta_j + \sum_{i < c, j} a_{i,j} \beta_i] : \alpha_k = [\alpha_j : \alpha_k] = \delta_{jk}$$

(see (2.23)). The case $\gamma = c\omega_k$ is treated similarly: now the left hand side of (2.83) is equal to 0, while the right hand side is equal to $-\delta_{jk}$ (by applying (2.23) with $c$ replaced by $c^{-1}$).

It remains to consider the case when $\gamma$ is not of the form $\omega_k$ or $c\omega_k$. Let $\alpha = c^{-1}\gamma - \gamma$. Using (2.82), we can rewrite the right hand side of (2.83) as

$$-\langle [\alpha^\vee : \alpha_j^\vee] + \sum_{j < c, i} a_{i,j} [\alpha^\vee : \alpha_i^\vee] \rangle = -\langle \alpha^\vee, \omega_j + \sum_{j < c} a_{i,j} \omega_i \rangle,$$

where $\langle ?, ? \rangle$ is the standard $W$-invariant pairing between the coweights and weights. Similarly, the left hand side of (2.83) can be rewritten as

$$\langle c^{-1} \alpha^\vee, \omega_j + \sum_{i < c, j} a_{i,j} \omega_i \rangle = \langle \alpha^\vee, c(\omega_j + \sum_{i < c} a_{i,j} \omega_i) \rangle.$$
It remains to check the identity
\[ c(\omega_j + \sum_{i \prec j} a_{i,j} \omega_i) = -(\omega_j + \sum_{j \prec i} a_{i,j} \omega_i). \]

Using (2.19), (2.22) and (2.23), we obtain
\[ -\left(\omega_j + \sum_{j \prec i} a_{i,j} \omega_i\right) = -\alpha_j + \omega_j + \sum_{i \prec j} a_{i,j} \omega_i \]
\[ = (\omega_j - \beta_j) + \sum_{i \prec j} a_{i,j} (\omega_i - \beta_i) = c(\omega_j + \sum_{j \prec i} a_{i,j} \omega_i), \]
finishing the proof. \(\square\)

2.6 Universal Coefficients

Returning to the proofs of (2.70) and (2.71), we will deduce these statements from the following proposition of independent interest.

**Proposition 2.6.1.** For every Coxeter element \(c\), there exists a cluster algebra \(\tilde{A}(c)\) of geometric type, satisfying the following properties:

1. The coefficient semifield of \(\tilde{A}(c)\) is \(\tilde{P}(c) = \text{Trop}(p[\gamma] : \gamma \in \Pi(c))\).

2. The cluster variables in \(\tilde{A}(c)\) are labeled by the set \(\Pi(c)\), the variable corresponding to \(\gamma \in \Pi(c)\) being denoted \(x[\gamma]\).

3. The initial seed of \(\tilde{A}(c)\) is of the form \((x, y, B(c))\), where \(x = (x[\omega_i] : i \in I)\),
and \( y = (y_j : j \in I) \) with

\[
y_j = p[\omega_j]p[c\omega_j]^{-1} \prod_{\gamma \in \Pi(c) \setminus \{\omega_j, c\omega_j\}} p[\gamma]^{(\gamma \parallel c\omega_j)c + \sum_{i < j} a_{i,j}(\gamma \parallel \omega_i)c}.
\] (2.84)

4. The primitive exchange relations in \( \tilde{\mathcal{A}}(c) \) are exactly the following (for \( k \in I \) and \( 1 \leq m \leq h(k; c) \)):

\[
x[-\omega_k] x[\omega_k] = p[\omega_k] \prod_{i < k} x[\omega_i]^{-a_{i,k}} \prod_{k < i} x[-\omega_i]^{-a_{i,k}} + \prod_{\gamma \in \Pi(c)} p[\gamma]^{(\gamma \parallel \omega_k)c};
\] (2.85)

\[
x[c^{m-1}\omega_k] x[c^m\omega_k] = p[c^m\omega_k] \prod_{i < k} x[c^m\omega_i]^{-a_{i,k}} \prod_{k < i} x[c^{m-1}\omega_i]^{-a_{i,k}} \prod_{\gamma \in \Pi(c)} p[\gamma]^{(\gamma \parallel c^m\omega_k)c}.
\] (2.86)

Before proving Proposition 2.6.1, let us show that it implies the desired statements (2.70) and (2.71). Indeed, let \( \mathbb{P}^\circ = \text{Trop}(y_i^\circ : i \in I) \) be the coefficient semifield of the cluster algebra with principal coefficients \( \mathcal{A}^\circ(c) \) (see Definition 2.4.4). Let \( \varphi : \mathbb{P}(c) \to \mathbb{P}^\circ \) be the tropical semifield homomorphism acting on the generators as follows:

\[
\varphi(p[\gamma]) = \begin{cases} y_i & \text{if } \gamma = \omega_i; \\ 1 & \text{otherwise.} \end{cases}
\] (2.87)

Applying Proposition 2.4.9 to this homomorphism, we see that the cluster variables in \( \mathcal{A}^\circ(c) \) are in a natural bijection with those in \( \tilde{\mathcal{A}}(c) \), and so can be labeled by the same set \( \Pi(c) \). If we denote by \( x_{\gamma,c} \) the cluster variable in \( \mathcal{A}^\circ(c) \) corresponding
to \(x[\gamma] \in \tilde{\mathcal{A}}(c)\), then in view of Proposition 2.4.9, both (2.70) and (2.71) become consequences of the following statement: under the homomorphism \(\varphi\), the relations (2.85) and (2.86) specialize respectively to (2.10) and (2.11). But this is immediate from the definition of the \(c\)-compatibility degree.

**Proof of Proposition 2.6.1.** We obtain the required assertion as a consequence of the following two statements:

1. For the bipartite Coxeter element \(t\), the cluster algebra \(\tilde{\mathcal{A}}(t)\) with required properties can be identified (by renaming the cluster variables and the generators of the coefficient semifield) with the cluster algebra with *universal coefficients* constructed in [10, Theorem 12.4].

2. If \(\tilde{\mathcal{A}}(c)\) is the cluster algebra with required properties for \(c = s_1 \cdots s_n\), and \(\tilde{c} = s_2 \cdots s_n s_1\), then \(\tilde{\mathcal{A}}(\tilde{c})\) can be identified (as above) with \(\tilde{\mathcal{A}}(c)\).

Thus, each \(\tilde{\mathcal{A}}(c)\) for different choices of a Coxeter element \(c\) is a realization of the same cluster algebra with universal coefficients, but with a different choice of an acyclic initial seed (with the exchange matrix \(B(c)\)).

To prove (1), let us first recall the nomenclature of cluster variables in [10]. Following [10, Definition 9.1], for every \(i \in I\) and \(m \in \mathbb{Z}\) such that \(\varepsilon(i) = (-1)^m\), we define a root \(\alpha(i; m)\), by setting, for all \(r \geq 0\):

\[
\alpha(i; r) = t_{-t_+ \cdots t_{\varepsilon(i)}}(-\alpha_i) \quad \text{for } \varepsilon(i) = (-1)^r; \quad (2.88)
\]

\[
\alpha(j; -r - 1) = t_{+t_- \cdots t_{\varepsilon(j)}}(-\alpha_j) \quad \text{for } \varepsilon(j) = (-1)^{r-1}. \quad (2.89)
\]
In particular, we have $\alpha(i; m) = -\alpha_i$ for $m \in \{0, -1\}$.

The following proposition is a consequence of a classical result of R. Steinberg [22] (cf. [8, Lemma 2.1, Proposition 2.5]).

**Proposition 2.6.2.** The roots $\alpha(i; m)$ with $m \in [-h - 1, -2]$, and $\varepsilon(i) = (-1)^m$ are positive, distinct, and every positive root is of such a form. Furthermore, for $m \in [0, h + 1]$, and $\varepsilon(i) = (-1)^m$, we have $\alpha(i; m) = \alpha(i^*; m - h - 2)$.

In [10], the cluster variables are denoted by $x_{i;m}$ with $i \in I$, $m \in \mathbb{Z}$ and $\varepsilon(i) = (-1)^m$. If we parameterize them by the set $\Phi_{\geq -1}$ as in Section 2.4 above (that is, using denominator vectors), then by [10, Theorem 10.3], each $x_{i;m}$ for $m \in [-h - 1, h + 1]$ will correspond to the root $\alpha(i; m)$, and so we will write $x_{i;m} = x[\alpha(i; m)]$. Comparing (2.89) with the description of the bijection $\psi : \Pi(t) \to \Phi_{\geq -1}$ given in Lemma 2.5.1, it is easy to see that, for $0 \leq m \leq h(i; t)$, we have

$$
\psi(t^m \omega_i) = \begin{cases} 
\alpha(i; -2m) & \text{if } \varepsilon(i) = +1; \\
\alpha(i; -2m - 1) & \text{if } \varepsilon(i) = -1.
\end{cases} \tag{2.90}
$$

According to [10, (8.12), (10.11)], the primitive exchange relations are those having the product $x_{j;m-1}x_{j;m+1}$ in the left hand side. Renaming each cluster variable $x_{i;m}$ in these relations as $x[\gamma]$, where $\gamma \in \Pi(t)$ is such that $\psi(\gamma) = \alpha(i; m)$, and ignoring the coefficients, an easy check using (2.90) shows that the cluster variables appear in these relations in exactly the same way as in (2.85) and (2.86).

To deal with coefficients, recall that the generators of the coefficient semifield in

$\text{1 Unfortunately, the corresponding result in [10, Proposition 9.3] was stated incorrectly.}$
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[10, Theorem 12.4] are of the form \( p[\alpha^\gamma] \) with \( \alpha^\gamma \in \Phi_{\geq -1}^\gamma \). We identify \( \Phi_{\geq -1}^\gamma \) with \( \Pi(t) \) by means of the following modification of Lemma 2.5.1:

**Lemma 2.6.1.** There is a bijection \( \psi^\gamma : \Pi(t) \to \Phi_{\geq -1}^\gamma \) uniquely determined by the properties that \( \psi^\gamma(\omega_i) = \varepsilon(i)\alpha_i^\gamma \) for all \( i \in I \), and \( \psi^\gamma \circ \tau_t = \tau^{-1} \circ \psi^\gamma \). Explicitly, \( \psi^\gamma \) is given by

\[
\psi^\gamma(\gamma) = (\tau_+ \psi(\gamma))^\gamma,
\]

(2.91)

where \( \psi \) is as in Lemma 2.5.1.

**Proof.** The uniqueness statement is clear, as well as the fact that (2.91) defines a bijection \( \Pi(t) \to \Phi_{\geq -1}^\gamma \). Using Lemma 2.5.1 and (2.67), we obtain that this bijection satisfies

\[
\psi^\gamma(\omega_i) = (\tau_+(-\alpha_i))^\gamma = \varepsilon(i)\alpha_i^\gamma,
\]

and

\[
\psi^\gamma\tau_t(\gamma) = (\tau_+ \tau \psi(\gamma))^\gamma = (\tau^- \psi(\gamma))^\gamma = (\tau^- \tau_+ \psi(\gamma))^\gamma = \tau^{-1} \psi^\gamma(\gamma),
\]

as desired. \( \square \)

As a consequence of (2.75), (2.69) and (2.91), we have

\[
(\gamma\|\delta)_t = (\psi^\gamma(\delta)\|\psi^\gamma(\gamma)) \quad (\gamma, \delta \in \Pi(t)).
\]

(2.92)

Now everything is ready to check the following: replacing \( p[\gamma] \) with \( p[\psi^\gamma(\gamma)] \) for all \( \gamma \in \Pi(t) \) transforms each element \( y_j \) given by (2.84) (with \( c = t \)) into the element

\[
y_{j;0} = \prod_{\alpha^\gamma \in \Phi_{\geq -1}^\gamma} p[\alpha^\gamma]^\varepsilon(j)[\alpha^\gamma;\alpha^\gamma]
\]
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in [10, (12.5)].

**Case 1.** Let $\varepsilon(j) = +1$. Then we have $\psi^\vee(\omega_j) = \alpha_j^\vee$ and $\psi^\vee(t\omega_j) = -\alpha_j^\vee$. Since there are no indices $i$ with $i \prec_t j$, if $\psi^\vee(\gamma) = \alpha^\vee \neq \pm \alpha_j^\vee$, then the exponent of $p[\gamma]$ in the right hand side of (2.84) is equal to

$$(\gamma||t\omega_j)_t = (-\alpha_j^\vee||\alpha^\vee) = [\alpha^\vee : \alpha_j^\vee].$$

Thus, the replacement of $p[\gamma]$ with $p[\alpha^\vee]$ indeed transforms $y_j$ into $y_{j;0}$.

**Case 2.** Let $\varepsilon(j) = -1$. Then we have $\psi^\vee(\omega_j) = -\alpha_j^\vee$ and $\psi^\vee(t\omega_j) = \alpha_j^\vee$. In this case, there are no indices $i$ with $j \prec_t i$, so by Lemma 2.5.3, if $\psi^\vee(\gamma) = \alpha^\vee \neq \pm \alpha_j^\vee$, then the exponent of $p[\gamma]$ in the right hand side of (2.84) is equal to

$$-(\gamma||\omega_j)_t = -(-\alpha_j^\vee||\alpha^\vee) = -[\alpha^\vee : \alpha_j^\vee],$$

proving our claim in this case as well.

To finish the proof of (1), it remains to check that the same replacement of each $p[\gamma]$ with $p[\psi^\vee(\gamma)]$ transforms the coefficients in the relations (2.85) and (2.86) into the coefficients of the corresponding relations in the cluster algebra from [10, Theorem 12.4]. Pick some $\delta \in \Pi(t)$, and let $\alpha^\vee = \psi^\vee(\delta)$. Recall from [10, Section 12] that $p[\alpha^\vee]$ is the primitive coefficient (the one at the non-trivial product of cluster variables) in the relation between $x[\tau_-\alpha]$ and $x[\tau_+\alpha]$. This agrees with the fact that $p[\delta]$ is the primitive coefficient in the relation (of the form (2.85) or (2.86)) between $x[\tau_t^{-1}\delta]$ and $x[\delta]$. It remains to observe that the constant term in the same relation (2.85) or (2.86), that is, the element $\prod_{\gamma \in \Pi(t)} p[\gamma]^{(\gamma||\delta)_t}$, transforms, in view of (2.92),
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into $\prod_{\beta \in \Phi} p[\beta^\vee] p[\alpha^\vee]$, which is the constant term in the corresponding relation in $[10]$ (see $[10$, (12.18)], where unfortunately there is an annoying typo: the correct exponent on the right is $(\alpha^\vee \| \beta^\vee)$ instead of $(\beta^\vee \| \alpha^\vee)$). This completes the check of property (1) above.

To prove (2), it is enough to show the following: after replacing, for each $\gamma \in \Pi(\bar{c})$, the element $x[\gamma] \in \tilde{A}(\bar{c})$ by $x[\psi_{c,\bar{c}}(\gamma)] \in \tilde{A}(c)$, and the coefficient $p[\gamma] \in \tilde{P}(\bar{c})$ by $p[\psi_{c,\bar{c}}(\gamma)] \in \tilde{P}(c)$ (see Lemma 2.5.2), the properties (3) and (4) for $\tilde{A}(\bar{c})$ in Proposition 2.6.1 become identical to the corresponding properties for $\tilde{A}(c)$.

Starting with property (4), it is enough to check it for the relations (2.85), since the relations (2.86) are obtained from them by repeatedly applying the transformation $\tau_{\bar{c}}$ (resp. $\tau_c$) to all occurring labeling weights $\gamma \in \Pi(\bar{c})$ (resp. $\gamma \in \Pi(c)$), and the bijection $\psi_{c,\bar{c}} : \Pi(\bar{c}) \to \Pi(c)$ intertwines $\tau_{\bar{c}}$ with $\tau_c$. The check that every relation (2.85) for $\tilde{A}(\bar{c})$ turns into one of the (2.85) or (2.86) for $\tilde{A}(c)$ after applying $\psi_{c,\bar{c}}$ is immediate from the definitions.

To deal with the remaining property (3), we note that $\psi_{c,\bar{c}}$ sends the initial cluster $(x[\omega_1], \ldots, x[\omega_n])$ in $\tilde{A}(\bar{c})$ into the cluster $(x[\omega_1], x[\omega_2], \ldots, x[\omega_n])$ in $\tilde{A}(c)$ obtained from the initial one by the mutation $\mu_1$. Remembering (2.4) and (1.1), we see that $\mu_1$ sends the exchange matrix $B(c)$ into $B(\bar{c})$. It remains to show that the transformation (2.64) (for $k = 1$) applied to the coefficient tuple in $\tilde{P}(c)$ given by (2.84) produces the elements $y'_j$ obtained from the corresponding $\tilde{y}_j \in \tilde{P}(\bar{c})$ by applying $\psi_{c,\bar{c}}$ to all the labeling weights.
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For \( j = 1 \), we can express \( y_1 \) as

\[
y_1 = p[\omega_1]^{-1} \prod_{\gamma \in \Pi(c)} p[\gamma]^{(\gamma || c\omega_1)c},
\]

therefore, we have

\[
y'_1 = y_1^{-1} = p[\omega_1] \prod_{\gamma \in \Pi(c)} p[\gamma]^{-(\gamma || c\omega_1)c}.
\]

On the other hand, using (2.83), we can express \( \tilde{y}_1 \in \tilde{P}(\tilde{c}) \) as

\[
\tilde{y}_1 = p[\omega_1] \prod_{\gamma \in \Pi(\tilde{c})} p[\gamma]^{-(\gamma || c\omega_1)c},
\]

which indeed transforms into \( y'_1 \) by applying \( \psi_{c,\tilde{c}} \) to all its labeling weights.

Finally, for \( j > 1 \), we have \( b_{1,j} = -a_{1,j} \geq 0 \), hence (2.64) results in

\[
y'_j = y_j y_1^{-a_{1,j}} p[\omega_1]^{-a_{1,j}} = y_j \prod_{\gamma \in \Pi(c)} p[\gamma]^{-a_{1,j}(\gamma || c\omega_1)c}
\]

\[
= p[\omega_j] p[\omega_j]^{-1} \prod_{\gamma \in \Pi(c) - \{\omega_j, c\omega_j\}} p[\gamma]^{(\gamma || c\omega_j)c + \sum_{1 \neq i \neq j} a_{1,j}(\gamma || c\omega_i)c},
\]

which is again obtained from \( \tilde{y}_j \) by applying \( \psi_{c,\tilde{c}} \) to all its labeling weights.

This finishes the proof of Proposition 2.6.1, hence also the proofs of Theorems 2.1.4 and 2.1.5.

In the rest of the section we give some corollaries of the developed techniques. First, since the \( c \)-compatibility degree for every Coxeter element \( c \) is obtained by a chain of bijections from the compatibility degree in [8, Section 3.1] (see (2.75) and (2.78)), combining the above results with Proposition 2.4.7 yields the following
Corollary 2.6.3. For an arbitrary Coxeter element \( c \) and any \( \gamma, \delta \in \Pi(c) \), the cluster variables \( x_{\gamma;c} \) and \( x_{\delta;c} \) in \( A(c) \) belong to the same cluster if and only if \( (\gamma \parallel \delta)_c = 0 \) (in particular, the latter condition is symmetric in \( \gamma \) and \( \delta \)).

Second, combining the arguments in the proof of Proposition 2.6.1 with Proposition 2.4.5, we conclude that all the algebras with principal coefficients \( A(c) \) for different choices of the Coxeter element \( c \) are isomorphic to each other. To be more specific, let us again suppose that \( c = s_1s_2 \cdots s_n \) and \( \tilde{c} = s_2 \cdots s_n s_1 \). Then the isomorphism \( \tilde{A}(\tilde{c}) \to \tilde{A}(c) \) constructed above gives rise to the isomorphism \( A(\tilde{c}) \to A(c) \) given as follows.

Corollary 2.6.4. The correspondence in Proposition 2.4.5 establishes an isomorphism of cluster algebras \( \varphi : A(\tilde{c}) \to A(c) \) acting on the initial cluster variables \( x_{\omega_1;\tilde{c}} \) and the initial coefficient tuple \( (y_{1;\tilde{c}}, \ldots, y_{n;\tilde{c}}) \) in \( A(\tilde{c}) \) as follows:

\[
\varphi(x_{\omega_1;\tilde{c}}) = x_{\omega_1;c}, \quad \varphi(y_{i;\tilde{c}}) = y_{i;c}y_{1;c}^{-a_{1,i}} \quad (i \neq 1),
\]

\[
\varphi(x_{\omega_1;\tilde{c}}) = x_{\omega_1;c}^{-1}(y_{1;c} + \prod_{i \neq 1} x_{\omega_i;c}^{-a_{i,1}}), \quad \varphi(y_{1;\tilde{c}}) = y_{1;c}^{-1}.
\]

Remark 2.6.2. Since, in view of Theorem 2.1.2, the algebra \( A(c) \) is the coordinate ring of the variety \( L^{c,c^{-1}} \), the isomorphism \( \varphi : A(\tilde{c}) \to A(c) \) gives rise to a biregular isomorphism \( \varphi^* : L^{c,c^{-1}} \to L^{\tilde{c},\tilde{c}^{-1}} \). This isomorphism can be described as follows. Note that the definition (2.45) and the standard properties of double Bruhat cells imply the following statement: if in the Weyl group \( W \) we have factorizations \( u = u_1 \cdots u_k \) and \( v = v_1 \cdots v_k \) such that \( \ell(u) = \ell(u_1) + \cdots + \ell(u_k) \) and \( \ell(v) = \ell(v_1) + \cdots + \ell(v_k) \),
then the product map in $G$ induces an open embedding

$$L^{u_1,v_1} \times \cdots \times L^{u_k,v_k} \hookrightarrow L^{u,v}.$$ 

In particular, denoting $c_\circ = s_2 \cdots s_n$, we have open embeddings

$$L^{s_1,e} \times L^{c_\circ,c_\circ^{-1}} \times L^{e,s_1} \hookrightarrow L^{c,c^{-1}}, \quad L^{e,s_1} \times L^{c_\circ,c_\circ^{-1}} \times L^{s_1,e} \hookrightarrow L^{e,c^{-1}}.$$ 

Note also that

$$L^{s_1,e} = \{x_{-1}(u) : u \in \mathbb{C}^*\}, \quad L^{e,s_1} = \{x_1(t) : t \in \mathbb{C}^*\}.$$ 

Now we claim that $\varphi^*$ restricts to an isomorphism

$$L^{s_1,e} \cdot L^{c_\circ,c_\circ^{-1}} \cdot L^{e,s_1} \to L^{e,s_1} \cdot L^{c_\circ,c_\circ^{-1}} \cdot L^{s_1,e}$$

given by

$$\varphi^*(x_{-1}(u)x_0x_1(t)) = x_1(u)x_0x_{-1}(t^{-1}) \quad (u, t \in \mathbb{C}^*, \ x_0 \in L^{c_\circ,c_\circ^{-1}}). \quad (2.93)$$

It suffices to prove (2.93) for $x_0$ of the form $x_{-2}(u_2) \cdots x_{-n}(u_n)x_n(t_n) \cdots x_2(t_2)$ with all $u_i$ and $t_i$ nonzero complex numbers, since such elements form an open dense subset of $L^{c_\circ,c_\circ^{-1}}$.

More precisely, let $x_0 = x_{-2}(u_2) \cdots x_{-n}(u_n)x_n(t_n) \cdots x_2(t_2)$ with all $u_i$ and $t_i$ nonzero complex numbers, to prove (2.93), we need to check the following identities.
\[ x_{\omega_{i};c}(x_1(u)x_0x_{-1}(t^{-1})) = x_{\omega_{i};c}(x_{-1}(u)x_0x_1(t)) = u_{i}^{-1} \quad \text{for } i \neq 1, \]

\[ y_{i;\tilde{c}}(x_1(u)x_0x_{-1}(t^{-1})) = (y_{i;\tilde{c}}y_{1;\tilde{c}}^{-a_{1,1}})(x_{-1}(u)x_0x_1(t)) = t^{-a_{1,i}}t_iu_{i}^{-1} \prod_{j=2}^{i-1} u_j^{-a_{j,i}} \quad \text{for } i \neq 1, \]

\[ x_{\omega_{1};c}(x_1(u)x_0x_{-1}(t^{-1})) = (x_{\omega_{1};c}(y_{1;\tilde{c}} + \prod_{i \neq 1} x_{\omega_{i};\tilde{c}}^{-a_{i,1}}))(x_{-1}(u)x_0x_1(t)) = t + u \prod_{i \neq 1} u_i^{a_{i,1}}, \]

\[ y_{1;\tilde{c}}(x_1(u)x_0x_{-1}(t^{-1})) = y_{1;\tilde{c}}^{-1}(x_{-1}(u)x_0x_1(t)) = t^{-1}u. \]

All the above four identities follow immediately from (2.58), (2.59) and the following expression for \( x_1(u)x_0x_{-1}(t^{-1}) \) obtained by the commutation relations in [3, Proposition 7.2], which we rewrite in (2.61) and (2.62). Note that the precise expression for \( w \) in (2.62) is \( u^{-1}(1 + t^{-1}u)^{-1} \).

\[ x_1(u)x_0x_{-1}(t^{-1}) = x_{-2}(u_2) \cdots x_{-n}(u_n)x_{-1}(p)x_1(q)x_n(t_n t^{-a_{1,n}}) \cdots x_2(t_2 t^{-a_{1,2}}), \]

where

\[ p = (u \prod_{i \neq 1} u_i^{a_{i,1}} + t)^{-1} \quad \text{and} \quad q = t^{-1}u(u + t \prod_{i \neq 1} u_i^{-a_{i,1}})^{-1}. \]
2.7 Proofs of Theorems 2.1.8, 2.1.10 and 2.1.13, and their corollaries

In this section we work with the cluster algebra $\mathcal{A}(c)$ from Theorem 2.1.2. When it is convenient, we assume without further warning that the index set $I$ is $[1,n]$, and the Coxeter element $c$ under consideration is $s_1 \cdots s_n$.

Proof of Theorem 2.1.8. For $\gamma \in \Pi(c)$, let $d(\gamma)$ denote the denominator vector of the cluster variable $x_{\gamma;c}$ with respect to the initial cluster $(x_{\omega_i;c} : i \in I)$. We identify $\mathbb{Z}^n$ with the root lattice $\mathcal{Q}$ using the basis of simple roots, and so assume that $d(\gamma) \in \mathcal{Q}$.

In view of $[10, (7.6), (7.7)]$, the vectors $d(\gamma)$ are uniquely determined from the initial conditions

$$d(\omega_i) = -\alpha_i, \quad (2.94)$$

and the recurrence relations

$$d(c^m \omega_j) + d(c^{m-1} \omega_j) = \left[ -\sum_{i<j} a_{i,j} d(c^m \omega_i) - \sum_{j<i} a_{i,j} d(c^{m-1} \omega_i) \right]_+ \quad (2.95)$$

(for all $j \in I$ and $1 \leq m \leq h(j;c)$), which follow from (2.11); here the notation $[v]_+$ for $v \in \mathcal{Q}$ is understood component-wise, i.e.,

$$[\sum a_i \alpha_i]_+ = \sum [a_i]_+ \alpha_i.$$

We need to show that the solution of the relations (2.95) with the initial conditions...
(2.94) is given by

\[ d(c^m \omega_j) = c^{m-1} \omega_j - c^m \omega_j = c^{m-1} \beta_j \quad (j \in I, 1 \leq m \leq h(j; c)) \]  

(2.96)

(see Lemma 2.2.1).

First let us check that the values given by (2.94) and (2.96) satisfy (2.95) for \( m = 1 \). Indeed, the right hand side of (2.95) is equal to

\[ [-\sum_{i < c j} a_{i,j} \beta_i + \sum_{j < c i} a_{i,j} \alpha_i]_+ = -\sum_{i < c j} a_{i,j} \beta_i, \]

since each \( \beta_i = s_1 \cdots s_{i-1} \alpha_i \) is a positive linear combination of the roots \( \alpha_{i'} \) with \( i' < i \) (in any total order on the index set \( I \) compatible with the relation \( i' \prec_c i \)). On the other hand, the left hand side of (2.95) is equal to \( \beta_j - \alpha_j \). So the two sides are equal to each other by (2.23).

For \( m \geq 2 \), the desired identity (2.95) takes the form

\[ c^{m-1} \beta_j + c^{m-2} \beta_j = -\sum_{i < c j} a_{i,j} c^{m-1} \beta_i - \sum_{j < c i} a_{i,j} c^{m-2} \beta_i, \]

which can be simplified to

\[ \beta_j + \sum_{i < c j} a_{i,j} \beta_i = -c^{-1} (\beta_j + \sum_{j < c i} a_{i,j} \beta_i). \]  

(2.97)

Now the left hand side of (2.97) is equal to \( \alpha_j \) by (2.23). On the other hand, since

\[ -c^{-1} \beta_i = -c^{-1} s_1 \cdots s_{i-1} \alpha_i = -s_n \cdots s_{i+1} s_i \alpha_i = s_n \cdots s_{i+1} \alpha_i, \]
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the right hand side of (2.97) is also equal to \( \alpha_j \) by the same equality (2.23) with the Coxeter element \( c \) replaced by \( c^{-1} \). This completes the proof of Theorem 2.1.8. \( \square \)

Proof of Corollary 2.1.9. The first assertion is clear, and the second one follows at once by combining the expression for denominator vectors given in Theorem 2.1.8 with Corollary 2.6.3 and formula (2) in Proposition 2.5.1. \( \square \)

Proof of Theorem 2.1.10. Recall that the \( g \)-vector \( g_{z;x} = (g_1, \ldots, g_n) \in \mathbb{Z}^n \) and the \( F \)-polynomial \( F_{z;x}(t_1, \ldots, t_n) \in \mathbb{Z}[t_1, \ldots, t_n] \) of a cluster variable \( z \in \mathcal{A}^\circ \) with respect to the initial cluster \( x \) are defined by (1.4). As prescribed by Theorem 2.1.10, we now identify \( \mathbb{Z}^n \) with the weight lattice \( P \) using the basis of fundamental weights. Following [10], we introduce the \( P \)-(multi)grading in the Laurent polynomial ring \( \mathbb{Z}[x^\pm 1, y^\pm 1] \) by setting

\[
\deg(x_{\omega_j;c}) = \omega_j, \quad \deg(y_{j;c}) = -\sum_{i \in I} b_{i,j} \omega_i = \sum_{i < c,j} a_{i,j} \omega_i - \sum_{j < c,i} a_{i,j} \omega_i.
\]

(2.98)

Then by (1.3), all elements \( \hat{y}_j \) are homogeneous of degree 0, hence the same is true for the factor \( F_{z;x^\circ}(\hat{y}_1, \ldots, \hat{y}_n) \) in (1.4), so the \( g \)-vector is equal to \( g_{z;x} = \deg(z) \).

Following (2.48), let us write a generic element of \( L^{c;c^{-1}} \) in the form

\[
x_{-1}(u_1) \cdots x_{-n}(u_n)x_n(t_n) \cdots x_1(t_1),
\]

and view all \( t_j \) and \( u_j \) as rational functions on \( L^{c;c^{-1}} \). In view of (2.58) and (2.59), the functions \( t_j \) and \( u_j \) are Laurent monomials in the initial cluster variables \( x_{\omega_i;c} \) and
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the coefficients \( y_{i,c} \), and we have

\[
\begin{align*}
\deg(u_j) &= -\deg(x_{\omega_j;c}) = -\omega_j, \quad (2.99) \\
\deg(t_j) &= \deg(y_{j;c}) + \deg(u_j) + \sum_{i < j} a_{i,j} \deg(u_i) = -(\omega_j + \sum_{j < i} a_{i,j} \omega_i).
\end{align*}
\]

Remembering (2.5), Theorem 2.1.10 can be restated as follows: with respect to the \( P \)-grading given by (2.99), we have

\[
\deg(\Delta_{\gamma,\gamma}(x_{-1}(u_1) \cdots x_{-n}(u_n)x_n(t_n) \cdots x_1(t_1))) = \gamma. \quad (2.100)
\]

To prove (2.100), we write each factor \( x_{-j}(u_j) \) as \( x_j(u_j)u_j^{-\alpha_j^\gamma} \) (see (2.47)), and move all the factors \( u_j^{-\alpha_j^\gamma} \) all the way to the right, using the commutation relations in (2.57). We obtain

\[
x_{-1}(u_1) \cdots x_{-n}(u_n)x_n(t_n) \cdots x_1(t_1) = x_1(w_1) \cdots x_n(w_n)x_n(v_n) \cdots x_1(v_1) \prod_{i \in I} u_i^{-\alpha_i^\gamma},
\]

where the \( w_j \) and \( v_j \) are given by

\[
w_j = u_j \prod_{i < j} u_i^{a_{i,j}}, \quad v_j = t_j \prod_{i=1}^n u_i^{-a_{i,j}}. \quad (2.101)
\]
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Expressing $\gamma$ in the form $\gamma = \sum_i g_i \omega_i$ and using (2.36), we obtain

\[
\Delta_{\gamma,\gamma}(x_{-1}(u_1) \cdots x_{-n}(u_n) x_n(t_n) \cdots x_1(t_1)) \\
= \Delta_{\gamma,\gamma}(x_1(w_1) \cdots x_n(w_n) x_n(v_n) \cdots x_1(v_1)) \cdot (\prod_i u_i^{-g_i})^\gamma \\
= \Delta_{\gamma,\gamma}(x_1(w_1) \cdots x_n(w_n) x_n(v_n) \cdots x_1(v_1)) \prod_i u_i^{-g_i},
\]

hence

\[
\deg(\Delta_{\gamma,\gamma}(x_{-1}(u_1) \cdots x_{-n}(u_n) x_n(t_n) \cdots x_1(t_1))) \\
= \gamma + \deg(\Delta_{\gamma,\gamma}(x_1(w_1) \cdots x_n(w_n) x_n(v_n) \cdots x_1(v_1))).
\]

Thus it remains to prove that

\[
\deg(\Delta_{\gamma,\gamma}(x_1(w_1) \cdots x_n(w_n) x_n(v_n) \cdots x_1(v_1))) = 0.
\]

Using (2.101) and (2.99), we deduce that

\[
\deg(w_j) = -\omega_j - \sum_{i < j} a_{i,j} \omega_i = -\deg(v_j).
\]

Now for every nonzero complex numbers $w_1, \ldots, w_n$ there is $a \in H$ such that $w_j = a^{\alpha_j}$
for all \( j \). Using (2.36) and (2.57), we conclude that

\[
\Delta_{\gamma,\gamma}(x_1(w_1) \cdots x_n(w_n)x_n(v_n) \cdots x_1(v_1)) \\
= \Delta_{\gamma,\gamma}(ax_1(w_1) \cdots x_n(w_n)x_n(v_n) \cdots x_1(v_1)a^{-1}) \\
= \Delta_{\gamma,\gamma}(x_1(1) \cdots x_n(1)x_n(w_nv_n) \cdots x_1(w_1v_1)).
\]

Since each product \( w_jv_j \) has degree 0, this completes the proof of Theorem 2.1.10. \( \square \)

**Proof of Corollary 2.1.11.** Assume that the simple roots are ordered so that \( c = s_1 \cdots s_n \) and \( B = B(c) \). Then (2.14) takes the form

\[
-g_j = d_j + \sum_{i<j} a_{j,i}d_i.
\]

Let \( \gamma = \sum_j g_j\omega_j \), then Theorem 2.1.8 and Theorem 2.1.10 imply the following identity:

\[
c^{-1}\gamma - \gamma = \sum_j d_j\alpha_j.
\]

On one side of the identity, we have

\[
c^{-1}\gamma - \gamma = -\sum_j g_j(\omega_j - c^{-1}\omega_j)
\]

\[
= -\sum_j g_j s_n \cdots s_{j+1}\alpha_j.
\]
By applying (2.23) to the Coxeter element $c^{-1}$, we obtain
\[
\sum_j d_j \alpha_j = \sum_j d_j (s_n \cdots s_{j+1} \alpha_j + \sum_{i > j} a_{i,j} s_n \cdots s_{i+1} \alpha_i)
= \sum_j (d_j + \sum_{i < j} a_{j,i} d_i) s_n \cdots s_{j+1} \alpha_j.
\]
Comparing these two expansions concludes the proof of Corollary 2.1.11. \qed

**Proof of Corollary 2.1.12.** Without loss of generality, we assume that $c = s_1 s_2 \cdots s_n$ and $\tilde{c} = s_2 \cdots s_n s_1$ such that $B^0 = B(c)$, $B^1 = B(\tilde{c})$ and $B(\tilde{c}) = \mu_1(B(c))$. Let $\sum g'_j \omega_j \in \Pi(\tilde{c})$. According to the bijection (2.77), if $\sum g'_j \omega_j \neq -\omega_1$ we have
\[
\sum g_j \omega_j = s_1 (\sum g'_j \omega_j)
= g'_1 (\omega_1 - \alpha_1) + \sum_{j \neq 1} g'_j \omega_j
= -g'_1 \omega_1 + \sum_{j \neq 1} (g'_j - a_{j,1} g'_1) \omega_j
= -g'_1 \omega_1 + \sum_{j \neq 1} (g'_j - b^0_{j,1} g'_1) \omega_j,
\]
that is, $g'_1 = -g_1$ and $g'_j = g_j - b^0_{j,1} g_1$ for $j \neq 1$. Note that we always have $[b^0_{j,1}]_+ = 0$, hence, to prove Corollary 2.1.12 in the case when $\sum g'_j \omega_j \neq -\omega_1$ (that is $\sum g_j \omega_j \neq -\omega_1$), it is enough to show that for every $\gamma = \sum_j g_j \omega_j \in \Pi(c)$, we have $g_1 \leq 0$ unless $\gamma = \omega_1$. This property follows immediately from Corollary 2.1.11 and the fact that the entries of any denominator vector are always greater or equal to 0 unless the corresponding cluster variable belongs to the initial cluster.
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The case when \( \sum g_j^i \omega_j = -\omega_1 \) can be checked easily. This completes the proof of Corollary 2.1.12. \( \square \)

*Proof of Theorem 2.1.13.* We use the notation in the proof of Theorem 2.1.10. In view of (1.4), the \( F \)-polynomial \( F_{z,x}(t_1, \ldots, t_n) \) is obtained from the expansion of the cluster variable \( z = x_{\gamma_1 c} \) by specializing all initial cluster variables \( x_{\omega_i c} \) to 1, and each coefficient \( y_{i c} \) to \( t_i \). In view of (2.58) and (2.59), this amounts to specializing all \( u_i \) to 1 in \( \Delta_{x, \gamma}(x_{-1}(u_1) \cdots x_{-n}(u_n)x_n(t_n) \cdots x_1(t_1)) \), which is exactly the desired assertion. \( \square \)

*Proof of Corollary 2.1.14.* The constant term of the polynomial \( F_{z,x}(t_1, \ldots, t_n) \) is obtained by specializing all \( t_i \) to 0 in (2.17), i.e., is equal to \( \Delta_{\gamma, \gamma}(x_1(1) \cdots x_n(1)) \). The fact that the latter is equal to 1 follows at once by the representation-theoretic arguments in the above proof of (2.53). \( \square \)

**Remark 2.7.1.** We have already mentioned in Remark 2.1.2 that the set \( \Pi(c) \) is identical with the set of generators of the \( c \)-Cambrian fan studied in [21]. In particular, our Theorem 2.1.10 implies [21, Theorem 10.2], proved there only modulo [10, Conjecture 7.12]. Comparing our Corollary 2.6.3 with [21, Theorem 10.1], we obtain the following description of the cones in the \( c \)-Cambrian fan: they are exactly the cones generated by subsets \( \{\gamma_1, \ldots, \gamma_k\} \subset \Pi(c) \) such that \( (\gamma_i \| \gamma_j)_c = 0 \) for all \( i, j = 1, \ldots, k \).

### 2.8 Proof of Theorem 2.1.1

In this section we assume that \( G = SL_{n+1}(\mathbb{C}) \) is of type \( A_n \), and the Coxeter element \( c \) is equal to \( s_1 \cdots s_n \) in the standard numbering of simple roots. As usual, the Weyl
2.8. Proof of Theorem 2.1.1

group $W$ is identified with the symmetric group $S_{n+1}$, so that $s_i$ becomes a simple transposition $(i, i+1)$. Then $c$ is the cycle $(1, 2, \cdots, n+1)$.

In this case, the generalized minors $\Delta_{\gamma, \delta}$ specialize to the ordinary minors (i.e., determinants of square submatrices). The weights in $W\omega_k$ are in bijection with the $k$-subsets of $[1, n+1] = \{1, \ldots, n+1\}$, so that $W = S_{n+1}$ acts on them in a natural way, and $\omega_k$ corresponds to $[1, k]$. If $\gamma$ and $\delta$ correspond to $k$-subsets $I$ and $J$, respectively, then $\Delta_{\gamma, \delta} = \Delta_{I, J}$ is the minor with the row set $I$ and the column set $J$.

Now let $L$ denote the subvariety of $G$ consisting of tridiagonal matrices $M$ of the form (2.1) (with all $y_1, \ldots, y_n$ non-zero). We start our proof of Theorem 2.1.1 by showing that $L$ is indeed the reduced double cell $L^{c,c-1}$. We use the characterization of $L^{c,c-1}$ given by Propositions 2.3.2 and 2.3.3. It is well-known that the Bruhat order on $k$-elements subsets of $[1, n+1]$ is component-wise: if $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$ then $I \leq J$ means that $i_\nu \leq j_\nu$ for $\nu = 1, \ldots, k$. Thus, we can rewrite Propositions 2.3.2 and 2.3.3 as follows.

**Corollary 2.8.1.** A matrix $M \in SL_{n+1}(\mathbb{C})$ belongs to the reduced double Bruhat cell $L^{c,c-1}$ for $c = s_1 \cdots s_n$ if and only if it satisfies the following conditions:

1. $\Delta_{I, [1,k]} = \Delta_{[1,k], I} = 0$ for $k = 1, \ldots, n$, and all subsets $I = \{i_1 < \cdots < i_k\} \subset [1, n+1]$ such that $i_\nu > \nu + 1$ for some $\nu = 1, \ldots, k$.

2. $\Delta_{[1,k],[2,k+1]} \neq 0$, and $\Delta_{[2,k+1],[1,k]} = 1$ for $k = 1, \ldots, n$.

To prove that $L = L^{c,c-1}$, we start with the inclusion $L \subseteq L^{c,c-1}$. Let $M = (m_{i,j}) \in L$. Since $M$ is tridiagonal, it satisfies condition (1) in Corollary 2.8.1 because every term in the expansion of $\Delta_{I, [1,k]}(M)$ or $\Delta_{[1,k],I}(M)$ (with $I$ as in this condition)
contains at least one matrix entry \( m_{i,j} \) with \(|i - j| > 1\). To prove condition (2), note that any tridiagonal matrix \( M \) becomes triangular after removing the first column and the last row (or the first row and the last column), implying that

\[
\Delta_{[1,k], [2,k+1]}(M) = m_{1,2} m_{2,3} \cdots m_{k,k+1} \tag{2.102}
\]

\[
\Delta_{[2,k+1], [1,k]}(M) = m_{2,1} m_{3,2} \cdots m_{k+1,k}.
\]

So for \( M \in L \), we have \( \Delta_{[1,k], [2,k+1]}(M) = y_1 \cdots y_k \neq 0 \), and \( \Delta_{[2,k+1], [1,k]}(M) = 1 \), as required.

To prove the reverse inclusion \( L^{c,c^{-1}} \subseteq L \), first let us show that every matrix in \( L^{c,c^{-1}} \) is tridiagonal. It is enough to check this for a generic element of the form (2.48). Note that in our situation the matrix \( x_{-k}(u) \) (resp. \( x_k(t) \)) is obtained from the identity matrix by replacing the \( 2 \times 2 \) submatrix with rows and columns \( k \) and \( k + 1 \) by

\[
\begin{pmatrix}
    u^{-1} & 0 \\
    1 & u
\end{pmatrix}
\quad \text{(resp. } \begin{pmatrix}
    1 & t \\
    0 & 1
\end{pmatrix}\).
\]

Then one can check that the product

\[
x_{-1}(u_1) \cdots x_{-n}(u_n)x_n(t_n) \cdots x_1(t_1)
\]

is tridiagonal by a direct matrix computation (or better yet, by using the graphical formalism for computing determinants of such products developed in [6, Proof of Theorem 12]).

Once we know that every matrix \( M \in L^{c,c^{-1}} \) is tridiagonal, the remaining conditions \( m_{k,k+1} \neq 0 \) and \( m_{k+1,k} = 1 \) follow at once from (2.102) and condition (2) in Corollary 2.8.1. This concludes the proof of the equality \( L = L^{c,c^{-1}} \).

We continue the proof of Theorem 2.1.1. Part (1) of Theorem 2.1.1 is a special
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We need only to observe that the expression for $y_{j:c}$ in (2.6) specializes to the restriction to $L$ of the function $\Delta_{[1,j],[2,j+1]}/\Delta_{[1,j-1],[2,j]}$, which, in view of (2.102) is equal to the above-diagonal matrix entry $y_j$ (see (2.1)).

Part (2) of Theorem 2.1.1 is a special case of Theorem 2.1.4. We need only to observe that in our case $h(k;c) = n + 1 - k$, and, for $0 \leq m \leq n + 1 - k$, the weight $c^m \omega_k \in \Pi(c)$ corresponds to the subset $[m + 1, m + k] \subset [1, n + 1]$. Turning to Part (3), we first note that the primitive exchange relations (2.10) and (2.11) specialize to the following relations, which are among those in (2.2):

$$x_{[1,k]} x_{[k+1,n+1]} = y_k x_{[1,k-1]} x_{[k+2,n+1]} + 1; \quad (2.103)$$

$$x_{[m,m+k-1]} x_{[m+1,m+k]} = y_m y_{m+1} \ldots y_{m+k-1} + x_{[m,m+k]} x_{[m+1,m+k-1]}, \quad (2.104)$$

where $k \in [1, n]$, $1 \leq m \leq h(k;c) = n + 1 - k$. To prove Theorem 2.1.1, it remains to show that the exchange relations in $\mathcal{A}(c)$ are exactly those in (2.2).

We start by recalling the geometric interpretation of the cluster variables and clusters in type $A_n$ given in [9, Section 12.2]. Namely, the cluster variables can be associated with the diagonals of a regular $(n + 3)$-gon $P_{n+3}$, and the clusters are all maximal collections of mutually non-crossing diagonals, which are naturally identified with the triangulations of $P_{n+3}$. We label the vertices of $P_{n+3}$ by numbers $1, \ldots, n+3$ in the counter-clockwise order, and denote by $x_{(i,j)}$ the cluster variable associated with the diagonal $\langle i, j \rangle$ connecting vertices $i$ and $j$ (with the convention that $x_{(i,j)} = 1$ if $i$ and $j$ are two adjacent vertices of $P_{n+3}$). According to [9, (12.2)], in this realization
the exchange relations have the form

\[ x_{(i,k)} x_{(j,\ell)} = p_{i,k,j,\ell} x_{(i,j)} x_{(k,\ell)} + p_{i,k,j,\ell} x_{(i,\ell)} x_{(j,k)}, \]  

(2.105)

where \( i, j, k, \ell \) are any four vertices of \( P_{n+3} \) taken in counter-clockwise order, and \( p_{i,k,j,\ell}^{\pm} \) are some elements of the coefficient semifield.

Clearly, the set of labels \( \Pi(c) \) is in a bijection with the set of all diagonals of \( P_{n+3} \) via the correspondence

\[ [i, j] \leftrightarrow [i, j + 2] \quad (1 \leq i \leq j \leq n + 1). \]  

(2.106)

Renaming each cluster variable \( x_{[i,j]} \) in Theorem 2.1.1 by \( x_{(i,j+2)} \), we see that, if we ignore the coefficients, then the desired relations (2.2) turn into (2.105). Note that, under this relabeling, the initial cluster gets associated with the triangulation of \( P_{n+3} \) by all the diagonals from the vertex 1, see Figure 2.1 (ignoring for the moment the segments labeled by \( y_1, y_2 \) and \( y_3 \)).

To complete the proof, it remains to verify the coefficients in the relations (2.2). To do this, we use the geometric realization of the universal coefficients (cf. Proposition 2.6.1 and [10, Theorem 12.4]) due to S. Fomin and A. Zelevinsky (the proof will appear elsewhere).

Consider the dual \((n+3)\)-gon \( P'_{n+3} \) whose vertices are the midpoints of the sides of \( P_{n+3} \). For \( i = 2, \ldots, n + 3 \), we denote by \( i' \) the midpoint of the side with vertices \( i - 1 \) and \( i \); we also denote by \( 1' \) the midpoint of the side connecting 1 and \( n + 3 \). We refer to the diagonals of \( P'_{n+3} \) as dual diagonals.
Proof of Theorem 2.1.1

Proposition 2.8.2 (S. Fomin, A. Zelevinsky). After relabeling the cluster variables and the generators of the coefficient semifield in the cluster algebra $\tilde{A}(c)$ in Proposition 2.6.1 by the diagonals of $P_{n+3}$ via the correspondence (2.106), the coefficient $p_{ik,j\ell}^+$ (resp., $p_{ik,j\ell}^-$) in an exchange relation (2.105) turns into the product of the generators $p\langle a, b \rangle$ such that the dual diagonal $\langle a', b' \rangle$ is contained in the strip formed by $\langle i, j \rangle$ and $\langle k, \ell \rangle$ (resp., by $\langle i, \ell \rangle$ and $\langle j, k \rangle$).

Example 2.8.1. In type $A_3$, the exchange relation between $x\langle 2, 5 \rangle$ and $x\langle 4, 6 \rangle$ has the form

$$x\langle 2, 5 \rangle x\langle 4, 6 \rangle = p\langle 1, 5 \rangle p\langle 2, 5 \rangle x\langle 2, 4 \rangle x\langle 5, 6 \rangle + p\langle 3, 6 \rangle p\langle 4, 6 \rangle x\langle 2, 6 \rangle x\langle 4, 5 \rangle,$$

as illustrated by Figure 2.2 (where we show only those dual diagonals that contribute to the exchange relation); note that by our convention, $x\langle 4, 5 \rangle = x\langle 5, 6 \rangle = 1$.

As shown in Section 2.6, the cluster algebra with principal coefficients $A(c)$ is obtained from $\tilde{A}(c)$ by the coefficient specialization (2.87). Using the relabeling in
Proposition 2.8.2, this specialization is described as follows: it sends $p(1, j + 2)$ to $y_j$ for $j = 1, \ldots, n$, and the rest of the generators $p(i, j)$ are sent to 1. (This specialization is illustrated by Figure 2.1 showing the dual diagonals associated with $y_1, y_2, y_3$.)

Now consider any relation (2.2). Rewriting it in the form (2.105), we obtain

$$x_{i,k+2} x_{j,\ell+2} = p^+ x_{i,j+2} x_{k,\ell+2} + p^- x_{i,\ell+2} x_{j,k+2},$$

where the coefficients $p^+$ and $p^-$ are obtained from those given in Proposition 2.8.2 by the above specialization. By inspection of the corresponding dual diagonals, we conclude that $p_+ = y_{j-1}y_j \cdots y_k$ and $p_- = 1$, verifying the coefficients in (2.2) and finishing the proof of Theorem 2.1.1.

We conclude this section by an example illustrating the correspondence between various reduced double cells given by (2.93).

Example 2.8.2. Let $c = s_1s_2$ and $\tilde{c} = s_2s_1$ be the two Coxeter elements in type $A_2$
and let \( L = L^{c,c} \) and \( \tilde{L} = \tilde{L}^{\tilde{c},\tilde{c}} \) be the corresponding reduced double Bruhat cells in \( G = SL_3(\mathbb{C}) \). The variety \( L \) has been already described: it consists of matrices

\[
M = \begin{pmatrix}
v_1 & y_1 & 0 \\
1 & v_2 & y_2 \\
0 & 1 & v_3
\end{pmatrix}
\]

with nonzero \( y_1, y_2, y_3 \). Using Propositions \( 2.3.2 \) and \( 2.3.3 \), the variety \( \tilde{L} \) can be described as the set of matrices \( \tilde{M} = (m_{i,j}) \in G \) satisfying the conditions

- \( \Delta_{[1,2],[2,3]}(\tilde{M}) = \Delta_{[2,3],[1,2]}(\tilde{M}) = 0; \)
- \( m_{1,3} \neq 0, \Delta_{[1,2],[1,3]}(\tilde{M}) \neq 0; \)
- \( m_{3,1} = \Delta_{[1,3],[1,2]}(\tilde{M}) = 1. \)

The correspondence \( L \rightarrow \tilde{L} \) in (2.93) sends a generic element

\[
M = x_{-1}(u_1)x_{-2}(u_2)x_2(t_2)x_1(t_1) \in L
\]

to

\[
\tilde{M} = x_1(u_1)x_{-2}(u_2)x_2(t_2)x_{-1}(t_1^{-1}) \in \tilde{L}.
\]

Performing the matrix multiplication, we see that \( M \) and \( \tilde{M} \) are given by

\[
M = \begin{pmatrix}
\frac{1}{u_1} & \frac{1}{u_2}t_1 & 0 \\
1 & \frac{1}{u_2} + t_1 & \frac{1}{u_2}t_2 \\
0 & 1 & u_2 + t_2
\end{pmatrix}, \quad \tilde{M} = \begin{pmatrix}
\frac{u_1 u_2^{-1}}{u_1 u_2^{-1} + t_1} & \frac{u_1 u_2^{-1}}{u_1 u_2^{-1} + t_1} & \frac{u_1 u_2^{-1}}{u_1 u_2^{-1}} \\
\frac{u_1 u_2^{-1}}{u_1 u_2^{-1} + t_1} & \frac{u_1 u_2^{-1}}{u_1 u_2^{-1} + t_1} & \frac{u_1 u_2^{-1}}{u_1 u_2^{-1}} \\
1 & \frac{u_1 u_2^{-1}}{u_1 u_2^{-1} + t_1} & \frac{u_1 u_2^{-1}}{u_1 u_2^{-1} + t_1}
\end{pmatrix}.
\]
An easy calculation shows that the correspondence $M \mapsto \tilde{M}$ extends to an isomorphism $L \to \tilde{L}$ given by

$$M = \begin{pmatrix} v_1 & y_1 & 0 \\ 1 & v_2 & y_2 \\ 0 & 1 & v_3 \end{pmatrix} \mapsto \tilde{M} = \begin{pmatrix} v_2 & v_1 v_2 y_1^{-1} - 1 & y_2 \\ v_1 v_2 - y_1 & v_1 (v_1 v_2 y_1^{-1} - 1) & v_1 y_2 \\ 1 & v_1 y_1^{-1} & v_3 \end{pmatrix},$$

while the inverse isomorphism $\tilde{L} \to L$ is given by

$$\tilde{M} = \begin{pmatrix} m_{1,1} & m_{1,1} m_{3,2} - 1 & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ 1 & m_{3,2} & m_{3,3} \end{pmatrix} \mapsto M = \begin{pmatrix} m_{2,3} m_{1,3}^{-1} & m_{1,1} m_{2,3} m_{1,3}^{-1} - m_{2,1} & 0 \\ 1 & m_{1,1} & m_{1,3} \\ 0 & 1 & m_{3,3} \end{pmatrix}.$$
Chapter 3

Combinatorial Expressions for

$F$-polynomials in Classical Types

This chapter is based on the work in [25]. We give combinatorial formulas for $F$-polynomials in cluster algebras of classical types in terms of the weighted paths in certain directed graphs. As a consequence we prove the positivity of $F$-polynomials in cluster algebras of classical types.

3.1 Introduction and Summary

The main result of this chapter is the following theorem.

Theorem 3.1.1. In the cluster algebra of classical type with an arbitrary acyclic initial seed, explicit combinatorial expressions for the $F$-polynomials are given. The descriptions for the types $A_n$, $D_n$, $B_n$ and $C_n$ are given in Propositions 3.1.2, 3.1.4, 3.1.6 and 3.1.8 respectively. Furthermore, in all these cases, the coefficients of the $F$-
polynomials are manifestly positive.

There are other formulas for the $F$-polynomials and proofs for the positivity conjecture in the literature. In particular, S. Fomin and A. Zelevinsky’s work in [8] together with [10] gave explicit formulas and proved the positivity for the $F$-polynomials in classical types for a bipartite initial cluster; G. Musiker, R. Schiffler and L. Williams’s work in [20] deals with cluster algebras from surfaces; the results in [23, 24] by T. Tran have the same generality as in this chapter. The answers in this chapter are given in very different terms and obtained by totally different methods.

According to Theorem 2.1.13, The $F$-polynomials are given by certain generalized minors. In the type $A_n$ case, the generalized minors specialize to the ordinary minors and our combinatorial formula is a well-known result due to B. Lindström (see [17], [11], [12], [5] and [6]). For the convenience of the reader, we will recall the type $A_n$ theory in Section 3.1.1.

3.1.1 Type $A_n$

Type $A_n$: Let $E_{i,j}$ denote the $(n + 1) \times (n + 1)$ matrix whose $(i,j)$-entry is equal to 1 while all other entries are 0, and let $\text{Id} \in G$ denote the identity matrix. For $i = 1, \ldots, n$, let
and

$$x_i(t) = \text{Id} + tE_{i,i+1} = \begin{pmatrix}
1 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 1 & t & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}$$

(3.1)

and

$$x_\bar{i}(t) = \text{Id} + tE_{i+1,i}$$

(3.2)

(the transpose of $x_i(t)$).

For any $i \in [1, n] \cup [\bar{1}, \bar{n}]$, where $[\bar{1}, \bar{n}] = \{\bar{1}, \ldots, \bar{n}\}$, we construct an “elementary chip” corresponding to $x_i(t)$ to be a weighted directed graph of one of the kinds shown in Figure 3.1.

Figure 3.1: “Elementary chips” of type $A_n$.

Note that in each chip, the horizontal levels are labeled by $1, \ldots, n+1$ starting from the bottom. The chip corresponding to $x_i(t)$ or $x_\bar{i}(t)$ has a diagonal edge connecting
the horizontal levels \(i\) and \(i + 1\) with weight \(t\). All other (unlabeled) edges have weight 1 and all edges are presumed to be oriented from right to left. The directed graph \(\Gamma(A_n, c)\) associated with \(c = s_{i_1} \cdots s_{i_n}\) is constructed as a concatenation of elementary chips \(x_{i_1}(1), \ldots, x_{i_n}(1), x_{i_1}(t_{i_n}), \ldots, x_{i_1}(t_{i_1})\) (in this order). We number the \(n + 1\) sources and \(n + 1\) sinks of the graph \(\Gamma(A_n, c)\) bottom-to-top, and define the weight of a path in \(\Gamma(A_n, c)\) to be the product of the weights of all edges in the path. We also define the weight of a family of paths to be the product of the weights of all paths in the family.

The Weyl group of type \(A_n\) is identified with the symmetric group \(S_{n+1}\), and it acts on the index set \([1, n + 1]\) as permutations. The simple reflections are \(s_i = (i, i + 1)\) for \(i \in [1, n]\). Then the \(F\)-polynomials in type \(A_n\) are computed as follows:

**Proposition 3.1.2.** The \(F\)-polynomial \(F_{c^n \omega_k}(t_1, \ldots, t_n)\) equals the sum of weights of all collections of vertex-disjoint paths in \(\Gamma(A_n, c)\) with the sources and sinks labeled by \(c^n \cdot [1, k]\).

**Example 3.1.3.** Type \(A_3\): Let \(c = s_1 s_3 s_2 = (1, 2, 4, 3)\), then \(c \cdot [1, 2] = \{2, 4\}\), hence

\[
F_{c^2 \omega_2}(t_1, t_2, t_3) = 1 + t_1 + t_3 + t_1 t_3 + t_1 t_2 t_3.
\]

In Figure 3.2, we give all families of vertex-disjoint paths in \(\Gamma(A_3, s_1 s_3 s_2)\) with the sources and sinks labeled by \(\{2, 4\}\) and each family of paths is depicted by thick lines.

### 3.1.2 Type \(D_n\)

**Type \(D_n\) (\(n \geq 4\))**: We use the standard numbering of simple roots as in [4]. For each \(i \in [1, n] \cup [\overline{1}, \overline{n}]\), the elementary chip corresponding to \(x_i(t)\) is shown in Figure 3.3.
In each chip, the vertices consist of all the endpoints of the horizontal edges and all of the edges are oriented from right to left. We number the horizontal levels from bottom to top in the order $1, \ldots, n, \bar{n}, \ldots, 1$. The numbering of the horizontal levels for the first (resp., last) two chips in Figure 3.3 is shown on the left (resp., right) of the figure. The two diagonal edges in each chip have weight $t$, all other unlabeled edges have weight $1$.

The directed graph $\Gamma(D_n, c)$ associated with $c = s_{i_1} \cdots s_{i_n}$ is constructed as a concatenation of elementary chips $x_{\bar{n}}(1), \ldots, x_{\bar{n}}(1), x_{i_n}(t_{i_n}), \ldots, x_{i_1}(t_{i_1})$ (in this order). We number the $2n$ sources and the $2n$ sinks of the graph $\Gamma(D_n, c)$ bottom-to-top in the order $1, \ldots, n, \bar{n}, \ldots, 1$.

Note that in the chips corresponding to $x_n(t)$ and $x_n(t)$, the intersections of the diagonal edges and the horizontal edges in the middle of each horizontal edge are not vertices. We call a family of paths bundled if within each elementary chip, either both
of the diagonal edges belong to the family of paths, or neither belong to the family of
count. The weight of a family of paths is defined in the same way as in the type $A_n$.

The Weyl group of type $D_n$ acts on the index set $[1, n]$ as permutations with even
number of “bar” changes. When written as permutations on $[1, n] \cup \{1, n\}$, the simple
reflections are $s_i = (i, i+1)(i+1, i)$ for $i = 1, \ldots, n-1$, and $s_n = (n-1, n)(n, n-1)$. Then the $F$-polynomials in type $D_n$ are computed as follows:

**Proposition 3.1.4.** In type $D_n$:

1. For $k = 1, \ldots, n-2$, $F_{c^m \omega_k}(t_1, \ldots, t_n)$ equals the sum of weights of all collections
   of vertex-disjoint paths in $\Gamma(D_n, c)$ with the sources and sinks labeled by $c^m \cdot [1, k]$;

2. $F_{c^m \omega_n^1}(t_1, \ldots, t_n)$ equals the sum of square roots of weights of all collections of
   bundled vertex-disjoint paths in $\Gamma(D_n, c)$ with the sources and the sinks labeled
   by $c^m \cdot \{1, 2, \ldots, n-1, n\}$;

3. $F_{c^m \omega_n}(t_1, \ldots, t_n)$ equals the sum of square roots of weights of all collections of
   bundled vertex-disjoint paths in $\Gamma(D_n, c)$ with the sources and the sinks labeled
3.1. Introduction and Summary

by \( c^n \cdot \{1, 2, \ldots, n - 1, n\} \).

All the proofs of the results in this section will be given in Section 3.2. Here is an example to illustrate this proposition.

**Example 3.1.5.** Type \( D_4 \): Let \( c = s_1 s_2 s_3 s_4 = (1, 2, 3, \bar{1}, \bar{2}, \bar{3})(4, \bar{4}) \), then \( c^2 \cdot [1, 2] = \{3, \bar{1}\} \). We have (see Figure 3.4)

\[
F_{c^2 \omega_2}(t_1, t_2, t_3, t_4) = 1 + t_1 + t_2 + 2t_1t_2 + t_1t_2t_3 + t_1t_2t_4 + t_1t_2^2 + t_1t_2t_3 + t_1t_2t_4 + t_1t_2^2t_3t_4.
\]

We also have \( c^2 \cdot \{1, 2, 3, \bar{4}\} = \{3, \bar{4}, \bar{2}, \bar{1}\} \), hence \( F_{c^2 \omega_3}(t_1, t_2, t_3, t_4) = 1 + t_2 + t_2t_4 \) (see Figure 3.5). Remember that in this case we require bundled families of paths and only square roots of their weights contribute to the \( F \)-polynomial.

3.1.3 Type \( B_n \)

**Type \( B_n \) (\( n \geq 2 \)):** For each \( i \in [1, n] \cup [\bar{1}, \bar{n}] \), the elementary chip corresponding to \( x_i(t) \) is shown in Figure 3.6. In each chip, the vertices consist of all the endpoints of the \( 2n + 1 \) horizontal edges. All the edges are oriented from right to left, we number the horizontal levels from bottom to top in the order \( 1, \ldots, n, 0, \bar{n}, \ldots, \bar{1} \). In the chips corresponding to \( x_{\pi}(t) \) and \( x_n(t) \), the intersections of the diagonal edges and the horizontal edges in the middle of the diagonal edges on the horizontal level 0 are not vertices. The numbering of the horizontal levels for the first (resp., last) two chips in Figure 3.6 is shown on the left (resp., right) of the figure. All unlabeled edges have weight 1. The directed graph \( \Gamma(B_n, c) \) associated with \( c = s_{i_1} \cdots s_{i_n} \) is constructed as a concatenation of elementary chips \( x_{i_1}(1), \ldots, x_{i_n}(1), x_{i_n}(t_{i_n}), \ldots, x_{i_1}(t_{i_1}) \) (in this
order). We number the $2n + 1$ sources and the $2n + 1$ sinks of the graph $\Gamma(B_n, c)$ bottom-to-top in the order $1, \ldots, n, 0, \bar{n}, \ldots, \bar{1}$.
To finish the type $B_n$ case, we need to introduce another graph $\Gamma_S(B_n, c)$ (it corresponds to the spin representation). For each $i \in [1, n] \cup [\bar{1}, \bar{n}]$, the elementary chip corresponding to $x_i(t)$ in $\Gamma_S(B_n, c)$ is shown in Figure 3.7. The vertices for each elementary chip consist of all the endpoints of the $2n$ horizontal edges. We label the $2n$ horizontal levels from bottom to top by $1, \ldots, n, \bar{n}, \ldots, \bar{1}$. All the edges are oriented from right to left with their weights shown in the figure, all unlabeled edges have weight 1.

The directed graph $\Gamma_S(B_n, c)$ associated with $c = s_{i_1} \cdots s_{i_n}$ is constructed as a concatenation of elementary chips $x_{\bar{1}}(1), \ldots, x_{\bar{n}}(1), x_{i_n}(t_{i_n}), \ldots, x_{i_1}(t_{i_1})$ (in this order).
We number the $2n$ sources and the $2n$ sinks of the graph $\Gamma_S(B_n, c)$ bottom-to-top in the order $1, \ldots, n, \bar{n}, \ldots, \bar{1}$.

As before, we call a family of paths in $\Gamma_S(B_n, c)$ bundled if within each elementary chip that corresponds to $x_i(t)$, for $i \in [1, n-1] \cup [\bar{1}, \bar{n}-1]$, either both of the diagonal edges belong to the family of paths, or neither belong to the family of paths. We will only need the bundled families of vertex-disjoint paths in $\Gamma_S(B_n, c)$.

The Weyl group of type $B_n$ acts on the index set $[1, n] \cup [\bar{1}, \bar{n}]$ by permutations and “bar” changes. When written as permutations on $[1, n] \cup [\bar{1}, \bar{n}]$, the simple reflections are $s_i = (i, i+1)(\bar{i+1}, \bar{i})$ for $i = 1, \ldots, n-1$, and $s_n = (n, \bar{n})$. The definition of the weight of paths is the same as before. Then the $F$-polynomials in type $B_n$ are computed as follows:

**Proposition 3.1.6.**

1. For $k = 1, \ldots, n-1$, $F_{c^m \omega_k} (t_1, \ldots, t_n)$ equals the sum of weights of all collections of vertex-disjoint paths in $\Gamma(B_n, c)$ with the sources and sinks labeled by $c^m \cdot [1, k]$;

2. $F_{c^m \omega_n} (t_1, \ldots, t_n)$ equals the sum of square roots of weights of all collections
of bundled vertex-disjoint paths in $\Gamma_S(B_n, c)$ with the sources and sinks labeled by $c^m \cdot [1, n]$.

Example 3.1.7. Type $B_2$: Let $c = s_2s_1 = (2, 1, \overline{2}, \overline{1})$, then $c \cdot [1] = \{ \overline{2} \}$ and $c^2 \cdot [1, 2] = \{ \overline{2}, \overline{1} \}$. We have

$$F_{c\omega_1}(t_1, t_2) = 1 + 2t_2 + t_2^2 + t_1t_2^2 \quad \text{and} \quad F_{c^2\omega_2}(t_1, t_2) = 1 + t_2 + t_1t_2.$$

Figure 3.8: $F_{c\omega_1}(t_1, t_2)$ and $F_{c^2\omega_2}(t_1, t_2)$ in type $B_2$ with $c = s_2s_1$. 
3.1.4 Type $C_n$

**Type $C_n$ ($n \geq 2$)**: For each $i \in [1, n] \cup [\overline{1}, \overline{n}]$, the elementary chip corresponding to $x_i(t)$ is shown in Figure 3.9. The vertices for each elementary chip consist of all the endpoints of the $2n$ horizontal edges. We number the horizontal levels from bottom to top by $1, \ldots, n, \overline{n}, \ldots, 1$. All the edges are oriented from right to left with weights shown in the figure, all unlabeled edges have weight 1.

![Elementary chips in $\Gamma(C_n, c)$](image)

Figure 3.9: Elementary chips in $\Gamma(C_n, c)$ $(i = 1, \ldots, n - 1)$.

The directed graph $\Gamma(C_n, c)$ associated with $c = s_{i_1} \cdots s_{i_n}$ is constructed as a concatenation of elementary chips $x_{i-1}(1), \ldots, x_{i-n}(1), x_{i-n}(t_{i-n}), \ldots, x_{i-1}(t_{i-1})$ (in this order). We number the $2n$ sources and the $2n$ sinks of the graph $\Gamma(C_n, c)$ bottom-to-top in the order $1, \ldots, n, \overline{n}, \ldots, \overline{1}$. The definition of the weight of paths is the same as before.

The Weyl group of type $C_n$ acts on the index set $[1, n] \cup [\overline{1}, \overline{n}]$ in the same way as the Weyl group of type $B_n$. We then have the following proposition for computing the $F$-polynomials of type $C_n$.

**Proposition 3.1.8.** For $k \in [1, n]$ , the $F$-polynomials $F_{c^m\omega_k}(t_1, \ldots, t_n)$ equals the sum of weights of all collections of vertex-disjoint paths in $\Gamma(C_n, c)$ with the sources
and sinks labeled by $e^m \cdot [1, k]$.

3.2 Proofs of the main results

In this section, let $\mathfrak{g}$ be a complex simple Lie algebra of rank $n$ with Chevalley generators $f_i, \alpha_i^\vee$, and $e_i$ for $i \in [1, n]$, and let $G$ be the simply connected simple complex Lie group with Lie algebra $\mathfrak{g}$. Let $V_{\omega_k}$ be the fundamental representation of $G$ and $v$ be a highest weight vector with highest weight $\omega_k$, then for $u, v \in W$, the Weyl group of $G$, the two vectors $\pi v$ and $\tau v$ are two weight vectors with weights $u\omega_k$ and $v\omega_k$, respectively (see (2.34) for the definition of $\pi$). From the definition of generalized minors, it is not hard to see that $\Delta_{u\omega_k, v\omega_k}(x)$ is the coefficient of $\pi v$ in the expression of $x \cdot \tau v$ (the action of the group element $x$ on $V_{\omega_k}$) in terms of a weight basis containing both $u v$ and $v v$.

Let $c = s_{i_1} \cdots s_{i_n}$ be a Coxeter element $W$. Remembering Theorem 2.1.13, let $x_c = x_{i_1}(1) \cdots x_{i_n}(1)x_{i_n}(t_{i_n}) \cdots x_{i_1}(t_{i_1})$. We compute the generalized minors of the form $\Delta_{e^m \omega_k, e^m \omega_k}(x_c)$ (hence the $F$-polynomials) by explicitly computing the action by $x_c$ on each fundamental representation $V_{\omega_k}$; recall that $x_i(t)$ and $x_i^\tau(t)$ act in every finite-dimensional representation of $\mathfrak{g}$ by

$$x_i(t) = \sum_{n \geq 0} \frac{t^n}{n!} e_i^n \quad \text{and} \quad x_i^\tau(t) = \sum_{n \geq 0} \frac{t^n}{n!} f_i^n. \quad (3.3)$$

For the type $A_n$ case (i.e., when $G = \text{SL}_{n+1}$), the generalized minors specialize to the ordinary minors. The Weyl group $W$ is identified with the symmetric group $S_{n+1}$, and $V_{\omega_k} = \wedge^k \mathbb{C}^{n+1}$, the $k$-th exterior power of the standard representation. All the
weights of $V_{\omega_k}$ are extremal, and are in bijection with the $k$-subsets of $[1, n+1]$, so that $W$ acts on them in a natural way, and $\omega_k$ corresponds to $[1, k]$. If $\gamma$ and $\delta$ correspond to $k$-subsets $I$ and $J$, respectively, then $\Delta_{\gamma,\delta} = \Delta_{I,J}$ is the minor with the row set $I$ and the column set $J$.

Note that the directed graph $\Gamma(A_n, c)$ provides a combinatorial model for the action of $x_c$ in each $\bigwedge^k \mathbb{C}^{n+1}$, in the sense that each of the elementary chips corresponding to $x_i(t)$ captures the action of $x_i(t)$ on the fundamental representations. For example, let $G = \text{SL}_3$ and $V$ be its standard representation with basis $v_1, v_2, v_3$. We have $x_1(t) \cdot (v_2 \wedge v_3) = v_2 \wedge v_3 + t v_1 \wedge v_3$, where the coefficient 1 (resp., $t$) of $v_2 \wedge v_3$ (resp., $v_1 \wedge v_3$) is the product of the weights of the edges connecting the sources labeled \{2,3\} and the sinks labeled by \{2,3\} (resp., \{1,3\}). The directed graphs $\Gamma(D_n, c)$, $\Gamma(B_n, c)$, $\Gamma_S(B_n, c)$ and $\Gamma(C_n, c)$ are designed and constructed to serve the same purpose, that is to capture the action of $x_c$ on the fundamental representations. This will become clear after we recall the Lie algebra action on the corresponding fundamental representations in each of the classical types (c.f. [13]).

**Proof of Proposition 3.1.4.** Let $\mathfrak{g}$ be the simple Lie algebra of type $D_n$ for $n \geq 4$, that is, the even special orthogonal Lie algebra $\mathfrak{so}_{2n}$. Then the action of generators in the standard $2n$-dimensional representation $V$ with respect to the standard basis
3.2. Proofs of the main results

$v_1, \ldots, v_n, v_{\pi}, \ldots, v_{\tau}$ can be written as:

$$e_i \cdot v_j = \begin{cases} 
  v_i, & \text{if } i \neq n \text{ and } j = i + 1; \\
  v_{i+1}, & \text{if } i \neq n \text{ and } j = i; \\
  v_n, & \text{if } i = n \text{ and } j = n - 1; \\
  v_{n-1}, & \text{if } i = n \text{ and } j = n; \\
  0, & \text{otherwise},
\end{cases} \quad (3.4)$$

$$f_i \cdot v_j = \begin{cases} 
  v_{i+1}, & \text{if } i \neq n \text{ and } j = i; \\
  v_\tau, & \text{if } i \neq n \text{ and } j = i + 1; \\
  v_\pi, & \text{if } i = n \text{ and } j = n - 1; \\
  v_{n-1}, & \text{if } i = n \text{ and } j = n; \\
  0, & \text{otherwise}.
\end{cases}$$

for $i \in [1, n]$ and $j \in [1, n] \cup [1, n]$. The group elements $x_i(t)$ and $x_\tau(t)$ act as $I + te_i$ and $I + tf_i$ respectively on $V$. We associate each vertex of $\Gamma(D_n, c)$ on the horizontal level $j$ with the basis vector $v_j \in V$ for $j \in [1, n] \cup [1, n]$, then the action of $x_i(t)$ and $x_\tau(t)$ on $V$ can be read from the corresponding elementary chips. For instance, the fragment shown in Figure 3.10 expresses the action $x_i(t) \cdot v_i = 1v_i + tv_{i+1}$ for $i \neq n$. Note that this fragment is part of the elementary chip corresponding to $x_i(t)$ for $i \neq n$. Therefore

$$x_i(t) \cdot v_i = 1v_i + tv_{i+1}$$

Figure 3.10: $x_i(t) \cdot v_i = 1v_i + tv_{i+1}$ for $i \neq n$.

the graph $\Gamma(D_n, c)$ (constructed by concatenation of the elementary chips) provides
3.2. Proofs of the main results

A combinatorial model for the action of $x_c$ on $V$, that is, the coefficient of $v_j$ in the expression of $x_c \cdot v_i$ is equal to the sum of the weights of all paths in $\Gamma(D_n, c)$ with source labeled by $i$ and sink labeled by $j$.

This observation can be generalized to the exterior powers of $V$ and used to compute the generalized minors. Recall that in the type $D_n$ case, the fundamental representation $V_{\omega_k}$ for $k = 1, \ldots, n - 2$ is realized as $\bigwedge^k V$ with the highest weight vector $v_1 \wedge \cdots \wedge v_k$. Each extremal weight $u\omega_k$ of $V_{\omega_k}$ corresponds to a $k$-subset $u \cdot [1, k]$ in $[1, n] \cup [\overline{1}, \overline{n}]$. Note that $i$ and $\overline{i}$ do not appear simultaneously in $u \cdot [1, k]$ for any $i \in [1, n]$ and $u \in W$. We define a linear ordering on the index set $[1, n] \cup [\overline{1}, \overline{n}]$ by $1 < \cdots < n < \overline{n} < \cdots < 1$. Let $I = \{i_1 < \cdots < i_k\}$ be a $k$-subset in $[1, n] \cup [\overline{1}, \overline{n}]$ corresponding to an extremal weight $\gamma$, and define a basis vector $v_I = v_{i_1} \wedge \cdots \wedge v_{i_k}$ in $\bigwedge^k V$. Then the principal minor $\Delta_{\gamma, \gamma}(x_c)$ equals to the coefficient of $v_I$ in the expression of $x_c \cdot v_I$ (in terms of the standard basis in $\bigwedge^k V$). It can be computed as follows:

$$\Delta_{\gamma, \gamma}(x_c)$$

equals the sum of signed-weights of all collections of vertex-disjoint paths in $\Gamma(D_n, c)$ with sources and sinks labeled by $I$.

The requirement of the paths to be vertex-disjoint is because $v \wedge v = 0$ for any $v \in V$.

To define the signed-weight of a family of paths, we first recall that in the chips corresponding to $x_\pi(t)$ and $x_n(t)$, the intersections of the diagonal edges and the horizontal edges in the middle of each horizontal edge are not vertices. Hence two vertex-disjoint paths in $\Gamma(D_n, c)$ can cross each other at such points (see Figure 3.3). One crossing of this kind is shown in Figure 3.11 and the two paths crossing each other
are depicted by thick lines (Note that there are four kinds of crossings in $\Gamma(D_n, c)$, see Figure 3.14). It represents that the expression of $x_n(t) \cdot (v_\pi \land v_{n-1})$ contains the term $t v_\pi \land v_n = -t v_n \land v_\pi$. This negative coefficient leads to the definition of signed-weight. We define the \textit{signed-weight} of a family of paths to be the weight of the family of paths if there are an even number of such crossings in the paths, and to be the \textit{negative} of the weight of the family of paths if there are an odd number of such crossings.

The crossing can only happen in the elementary chips corresponding to $x_\pi(t)$ and $x_n(t)$ and this two chips appear in $\Gamma(D_n, c)$ exactly once, therefore at most two crossings can appear in a family of vertex-disjoint paths in $\Gamma(D_n, c)$. Also when the sources and the sinks are labeled by the same index set, we always have a family of vertex-disjoint paths consisting of the horizontal levels connecting the sources and the sinks. This family has (signed-)weight 1. Hence, to prove part (1) of Propositions 3.1.4, it is enough to show that there does not exist a family of vertex-disjoint paths in $\Gamma(D_n, c)$ with the sources and sinks labeled by $c^m \cdot [1, k]$ for $k \in [2, n-2]$ such that there is exactly one crossing among its paths.

Suppose for the sake of contradiction that such a family of paths exists. We use $x_i$ to represent the corresponding element chip if there is no danger of confusion. All families of paths are assumed to be vertex-disjoint. We first consider the case that
the (only one) crossing appears in \( x_n \) on the level \( \pi \).

![Figure 3.12: Two paths cross each other in the chip \( x_n \) on level \( \pi \).](image)

Let \( P \) be a family of paths and \( p_1, p_2 \) be two paths in \( P \) crossing each other in \( x_n \) on the level \( \pi \). Let \( p_1 \) be the path that passes through the (upper) diagonal edge of \( x_n \), and let \( p_2 \) be the path that passes through the horizontal edge of \( x_n \) on level \( \pi \). We claim that the paths \( p_1 \) and \( p_2 \) appear partially as illustrated in Figure 3.12. Recall that all paths travel from right to left. In Figure 3.12, if \( i \geq 2 \) then \( p_1 \) must stay on level \( n - 1 \) between \( x_{n-1} \) and \( x_{n-1} \). When \( i = 1 \), it is possible for \( p_1 \) to go down after \( x_{n-1} \), then go up to the level \( n - 1 \) before \( x_{n-1} \).

Since \( p_1 \) and \( p_2 \) are the only paths crossing each other, it is easy to see that the label of the source of \( p_1 \) (resp., \( p_2 \)) will be the label of the sink of \( p_2 \) (resp., \( p_1 \)). Also the source of \( p_1 \) is “higher” than the source of \( p_2 \) (i.e, the label of the source of \( p_1 \) is bigger than the label of the source of \( p_2 \) in the linear order on \([1, n] \cup [\bar{T}, \pi]\) defined before), since for \( i \in [1, n - 1] \), all the edges of the chip \( x_i \) either keep the same horizontal level or bring the level down by 1. Each \( x_i \) appears exactly once in \( \Gamma(D_n, c) \), hence the labels of the sources of \( \{p_1, p_2\} \) must be \( \{n - i, \pi\} \) or \( \{n - 1 - i, n - 1\} \) for some \( i \geq 1 \). To see the later case cannot happen, we assume that the labels of the sources of \( \{p_1, p_2\} \) are \( \{n - i, n - 1\} \) for some \( i \geq 2 \): In this case, the chips \( x_{n-1}, x_{n-2}, x_{\pi}, x_n, x_{n-2}, x_{n-1} \) must appear in \( \Gamma(D_n, c) \) in this order. See Figure 3.13.
3.2. Proofs of the main results

Now consider the path $p_2$. It stays on the horizontal level $\overline{n}$ in the chip $x_n$. There is no edge connecting the horizontal levels $\overline{n}$ and $n$, and the chip $x_{\overline{n-2}}$ appears on the left of $x_{\overline{n}}$: therefore, we conclude that $p_2$ must stay on the horizontal level $\overline{n}$ when it arrives in the chip $x_{\overline{n}}$. Then it is clearly impossible for the path $p_2$ to reach its sink $\overline{n-i}$. Therefore the sources of $\{p_1, p_2\}$ must be $\{\overline{n-i}, \overline{n}\}$ for some $i \geq 1$ and it is easy to see that the paths $p_1$ and $p_2$ appear (partially) as shown in Figure 3.12.

By a similar argument, we obtain in Figure 3.14 all cases of paths that have exactly one crossing in $\Gamma(D_n, c)$. In all cases, we denote $p_1$ and $p_2$ to be the two paths crossing each other, where $p_1$ is the path having higher source.

In all the cases either $\{\overline{n}, \overline{n-i}\} \in I$ or $\{n, \overline{n-i}\} \in I$ for some $i \geq 1$. Note that if $s_{n-2}$ appears in between $s_n$ and $s_{n-1}$ in the expression of the Coxeter element $c$, that is, $c$ can be written as one of the following forms: $\cdots s_n \cdots s_{n-2} \cdots s_{n-1} \cdots$ or $\cdots s_{n-1} \cdots s_{n-2} \cdots s_n \cdots$, then the source of $p_1$ must be $\overline{n-1}$. However, in this case, the indices $n-1$ and $\overline{n-1}$ form a single two cycle when $c$ is written as a permutation on the index set $[1, n] \cup [\overline{1}, \overline{n}]$ which implies that $\overline{n-1}$ does not belong to $c^m \cdot [1, k]$ for any $k \in [2, n-2]$ and $m \in \mathbb{Z}$. On the other hand, if $s_{n-2}$ does
not appear in between $s_n$ and $s_{n-1}$ in the expression of the Coxeter element $c$ then the indices $n$ and $\bar{n}$ form a single two cycle when $c$ is written as a permutation on the index set $[1, n] \cup [\bar{1}, \bar{n}]$. This implies that neither the index $n$ nor the index $\bar{n}$ belongs to $c^m \cdot [1, k]$ for any $k \in [2, n-2]$ and $m \in \mathbb{Z}$. This shows that there does not exist a family of vertex-disjoint paths in $\Gamma(D_n, c)$ with the sources and sinks labeled by $c^m \cdot [1, k]$ for $k \in [2, n-2]$ such that there is exactly one crossing among its paths. This completes the proof of part (1) of Proposition 3.1.4. In fact, it can be shown in a similar argument that no crossing (one or two) can appear in any family of vertex-disjoint paths in $\Gamma(D_n, c)$ with the sources and sinks labeled by $c^m \cdot [1, k]$ for $k \in [2, n-2]$.

Part (2) and part (3) of Proposition 3.1.4 will become clear after we recall the corresponding spin representations. Let $T$ be an $n$-subset of $[1, n] \cup [\bar{1}, \bar{n}]$, then the spin representations $V_{\omega_{n-1}}$ and $V_{\omega_n}$ can be realized as the vector spaces span by basis

Figure 3.14: All cases of paths that have exactly one crossing in $\Gamma(D_n, c)$.
3.2. Proofs of the main results

vectors as follows:

\[ V_{\omega_{n-1}} = \langle T \mid i \text{ and } \bar{i} \text{ do not appear simultaneously in } T, \text{ there are an odd number of } \bar{i}'s \text{ appearing in } T. \rangle, \]

\[ V_{\omega_n} = \langle T \mid \text{ with } i \text{ and } \bar{i} \text{ do not appear simultaneously in } T, \text{ there are an even number of } \bar{i}'s \text{ appearing in } T. \rangle. \]  

(3.5)

The so\(_{2n}\)-actions on \( V_{\omega_{n-1}} \) and \( V_{\omega_n} \) are given as follows:

\[ e_i \cdot T = \begin{cases} 
T \setminus \{ i + 1, \bar{i} \} \cup \{ i, i + 1 \}, & \text{if } i \neq n \text{ and } i + 1, \bar{i} \in T; \\
T \setminus \{ \bar{i}, \bar{n} - 1 \} \cup \{ n - 1, n \}, & \text{if } i = n \text{ and } \bar{n}, \bar{n} - 1 \in T; \\
0, & \text{otherwise},
\end{cases} \]

\[ f_i \cdot T = \begin{cases} 
T \setminus \{ i, i + 1 \} \cup \{ i + 1, \bar{i} \}, & \text{if } i \neq n \text{ and } i, i + 1 \in T; \\
T \setminus \{ n - 1, n \} \cup \{ \bar{n}, \bar{n} - 1 \}, & \text{if } i = n \text{ and } n - 1, n \in T; \\
0, & \text{otherwise}.
\]  

(3.6)

Hence \( x_i(t) \) and \( x_{\bar{i}}(t) \) act as \( I + t e_i \) and \( I + t f_i \) on \( V_{\omega_{n-1}} \) and \( V_{\omega_n} \) respectively. The fundamental representations \( V_{\omega_{n-1}} \) and \( V_{\omega_n} \) have highest weight vectors \( \{ 1, 2, \ldots, n - 1, \bar{n} \} \) and \( \{ 1, 2, \ldots, n - 1, n \} \) respectively.

The combinatorial meaning of the graph \( \Gamma(D_n, c) \) in these cases is completely analogous to the one before. The requirement of the paths being bundled is due to that the non-trivial actions of \( e_i \) and \( f_i \) require two specified indices to appear simultaneously in \( T \). In this case, the coefficient of the corresponding basis vector should be \( t \) instead of \( t^2 \), therefore we take the square root of the weight of a family
of bundled vertex-disjoint paths. This completes the proof of Proposition 3.1.4. □

The proofs of the Propositions 3.1.6, 3.1.8 are similar to the proof of Proposition 3.1.4.

Proof of Proposition 3.1.6. Let \( g \) be the simple Lie algebra of type \( B_n \) for \( n \geq 2 \), that is, the odd special orthogonal Lie algebra \( so_{2n+1} \). Then the action of generators in the standard \( 2n+1 \)-dimensional representation \( V \) with respect to the standard basis \( v_1, \ldots, v_n, v_0, v_{\pi}, \ldots, v_\bar{\pi} \) can be written as:

\[
e_i \cdot v_j = \begin{cases} 
  v_i, & \text{if } i \neq n \text{ and } j = i + 1; \\
  v_{i+1}, & \text{if } i \neq n \text{ and } j = i; \\
  \sqrt{2} v_{\pi}, & \text{if } i = n \text{ and } j = 0; \\
  \sqrt{2} v_0, & \text{if } i = n \text{ and } j = \pi; \\
  0, & \text{otherwise,} \\
\end{cases}
\]

\[
f_i \cdot v_j = \begin{cases} 
  v_{i+1}, & \text{if } i \neq n \text{ and } j = i; \\
  v_{\bar{i}}, & \text{if } i \neq n \text{ and } j = \pi + 1; \\
  \sqrt{2} v_0, & \text{if } i = n \text{ and } j = n; \\
  \sqrt{2} v_{\pi}, & \text{if } i = n \text{ and } j = 0; \\
  0, & \text{otherwise,} \\
\end{cases}
\]

for \( i \in [1, n] \) and \( j \in [1, n] \cup \{0\} \cup [\bar{1}, \bar{n}] \).

It is easy to see that \( x_i(t) \) and \( x_{\pi}(t) \) act as \( I + te_i + \frac{t^2}{2} e_i^2 \) and \( I + tf_i + \frac{t^2}{2} f_i^2 \) respectively on \( V \). The fundamental representation \( V_{\omega_k} \) for \( k = 1, \ldots, n-1 \) is realized as \( \bigwedge^k V \) with highest weight vector \( v_1 \wedge \cdots \wedge v_k \).

To prove part (1) of Proposition 3.1.6, it is enough to show that there is no crossing
among any family of vertex-disjoint paths in $\Gamma(\mathcal{B}_n, c)$ with the sources and sinks labeled by $c^m \cdot [1, k]$ for $k \in [2, n - 1]$. Note in $\Gamma(\mathcal{B}_n, c)$, the crossing can only happen on the horizontal level 0 in $x_n$ or $x_\pi$ (see Figure 3.6). The index 0 does not belong to any index set corresponding to an extremal weight, and the only diagonal edges connecting the horizontal level 0 are within the chips $x_n$ and $x_\pi$ themselves. Together with the fact that $x_n$ and $x_\pi$ only appear once in $\Gamma(\mathcal{B}_n, c)$, we conclude that such a crossing can not happen. This completes the proof of part (1) of Proposition 3.1.6.

To prove part (2) of Proposition 3.1.6, we first recall the spin representation in this case. Let $T$ be an $n$-subset of $[1, n] \cup [\overline{1}, \overline{n}]$. Then the spin representation can be realized as a vector space span by basis vectors as follows:

$$V_{\omega_n} = \langle T \mid i \text{ and } \overline{i} \text{ do not appear simultaneously in } T \rangle. \quad (3.8)$$

The $\mathfrak{so}_{2n+1}$-action on $V_{\omega_n}$ is given as follows:

$$e_i \cdot T = \begin{cases} 
T \setminus \{i + 1, \overline{i}\} \cup \{i, \overline{i + 1}\}, & \text{if } i \neq n \text{ and } i + 1, \overline{i} \in T; \\
T \setminus \{\pi\} \cup \{n\}, & \text{if } i = n \text{ and } \pi \in T; \\
0, & \text{otherwise},
\end{cases} \quad (3.9)$$

$$f_i \cdot T = \begin{cases} 
T \setminus \{i, \overline{i + 1}\} \cup \{i + 1, \overline{i}\}, & \text{if } i \neq n \text{ and } i, \overline{i + 1} \in T; \\
T \setminus \{n\} \cup \{\overline{n}\}, & \text{if } i = n \text{ and } n \in T; \\
0, & \text{otherwise}.
\end{cases}$$

Hence $x_i(t)$ and $x_\overline{i}(t)$ act as $I + te_i$ and $I + tf_i$ on $V_{\omega_n}$ respectively. The fundamental representation $V_{\omega_n}$ has highest weight vector $\{1, 2, \ldots, n - 1, n\}$. The reasons
for requiring the bundled condition and taking the square root of the weight of a collection of vertex-disjoint paths are the same as those in the part (2) and (3) of Proposition 3.1.4. This completes the proof of Proposition 3.1.6. □

Proof of Proposition 3.1.8. Let \( g \) be the simple Lie algebra of type \( C_n \) for \( n \geq 2 \), that is, the symplectic Lie algebra \( \text{sp}_{2n} \). Then the action of generators in the standard \( 2n \)-dimensional representation \( V \) with respect to the standard basis \( v_1, \ldots, v_n, v_{\pi}, \ldots, v_{\bar{\pi}} \) can be written as:

\[
\begin{align*}
  e_i \cdot v_j &= \begin{cases} 
    v_i, & \text{if } i \neq n \text{ and } j = i+1; \\
    v_{i+1}, & \text{if } i \neq n \text{ and } j = \bar{i}; \\
    v_n, & \text{if } i = n \text{ and } j = \bar{n}; \\
    0, & \text{otherwise},
  \end{cases} \\
  f_i \cdot v_j &= \begin{cases} 
    v_{i+1}, & \text{if } i \neq n \text{ and } j = i; \\
    v_{\bar{i}}, & \text{if } i \neq n \text{ and } j = \bar{i} + 1; \\
    v_{\pi}, & \text{if } i = n \text{ and } j = n; \\
    0, & \text{otherwise},
  \end{cases}
\end{align*}
\] (3.10)

for \( i \in [1, n] \) and \( j \in [1, n] \cup [\bar{\pi}, \bar{n}] \).

It is easy to see that \( x_i(t) \) and \( x_{\bar{i}}(t) \) act as \( I + te_i \) and \( I + tf_i \) respectively on \( V \). Although the fundamental representation \( V_{\omega_k} \) is not isomorphic to the exterior power \( \Lambda^k V \) for \( k > 1 \), it can be realized as a subrepresentation in \( \Lambda^k V \) with highest weight vector \( v_1 \wedge \cdots \wedge v_k \). Hence, for our purpose, it makes no difference to work inside \( \Lambda^k V \). Proposition 3.1.8 clearly holds since there is no crossing in any family of vertex-disjoint paths in \( \Gamma(C_n, c) \). This completes the proof. □
Bibliography


