OPTIMAL STEP-STRESS PLANS FOR ACCELERATED LIFE TESTING CONSIDERING RELIABILITY/LIFE PREDICTION

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Abstract

Products or materials are often tested at higher levels of stress than operational conditions to quickly obtain information on the life distribution or product performance under normal use. Such testing could save much time and money. In step-stress accelerated life test (SSALT), the stress for survival units is generally changed to a higher stress level at a pre-determined time. Determination of the stress change times is one of the most important design problems in SSALT.

In this dissertation research, we focus on the SSALT design problem for Weibull failure data because of its broad application in industry. The optimal simple SSALT, which involves only two stress levels, is first derived. Log-linear life stress relationship is assumed. Two different types of optimization criteria are presented, considering life estimate and reliability estimate. Optimal SSALT plan is proposed by minimizing the Asymptotic Variance (AV) of the desired life/reliability estimate.

In many applications and for various reasons, it is desirable to use more than one accelerating stress variable. Integration of Weibull failure data with multiple stress variables results in more complexity in the Fisher information matrix, and a more complicated problem to solve. Two stress variables are considered first, leading to the bivariate SSALT model. Bivariate SSALT model is then extended to a more generalized model: multi-variate SSALT, which includes $k$ steps and $m$ stress variables.
In addition to log-linear life-stress relationship, proportional hazards (PH) model is another widely used life-stress relationship for multiple stress variables. In this dissertation research, the baseline intensity function is defined at the highest stress levels to obtain a quick initial estimate of the parameters. PH model is assumed for all other stress levels. A simple SSALT design is considered first. The results are extended to multiple SSALT, which considers multiple steps, but only one stress variable. Optimal stress change times for each step are obtained. A more generalized case, multi-variate SSALT based on PH model is then proposed, including $k$ steps and $m$ stress variables. Fisher information matrix and AV of maximum likelihood estimation (MLE) are constructed. Optimal plan is designed to minimize the AV of MLE.
I would like to express my deep gratitude to my dissertation advisor, Professor Nasser S. Fard, for his guidance throughout the development of this dissertation. His patience, confidence, insights, and expertise were foundational throughout the uncertainty faced in the completion of this effort. Without him, I would not have been able to finish this dissertation.

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List of Abbreviations

ALT Accelerated Life Test
CDF Cumulative Distribution Function
CE Cumulative Exposure
CISALT Continuously Increasing Stress Accelerated Life Test
CSALT Constant Stress Accelerated Life Test
FFL Failure-Free Life
GPH Generalized Proportional Hazards
HALT Highly Accelerated Life Testing
HASS Highly Accelerated Stress Screening
IPL Inverse Power Law
K-H Khamis-Higgins
LCEM Linear Cumulative Exposure Model
MLE Maximum Likelihood Estimation
MTTF Mean Time to Failure
pdf Probability Density Function
PH proportional hazards
SSALT Step-Stress Accelerated Life Test
T-H Temperature-Humidity
T-NT Temperature-Nonthermal
<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>TLS</td>
<td>Transformed Least Squares</td>
</tr>
<tr>
<td>AV</td>
<td>Asymptotic Variance</td>
</tr>
</tbody>
</table>
CHAPTER 1

Introduction

Due to the rapid improvement of the high technology, the products today become more and more reliable, and the products’ life gets longer and longer. It might take a long time, such as several years, for a product to fail, which makes it difficult or even impossible to obtain the failure information under usage condition for such highly reliable products. While running at higher stress level shortens the products’ life, the accelerated life test (ALT) is utilized to induce more failures and then derive the reliability information under usage condition. Step-stress ALT (SSALT) is one specific type of ALT, while the stress applied to the test product is increased in a specified discrete sequence. This dissertation is presented as a deep study of the data analysis and test design problems in SSALT.

1.1 Problem Background

Reliability and maintainability are important components in engineering design. Their growth has been motivated by several factors, which include the increased complexity and sophistication of systems, public awareness and insistence on product quality. Consumer protection laws and regulations concerning product liability, government contractual requirements to meet reliability and maintainability performance,
and profit considerations resulting from the high cost of failures, repairs, and warranty programs are among many other factors for insuring more reliable products.

Reliability is defined as the probability that a component or system will perform its required function without failure for a given period of time, when used under stated operating conditions. Reliability is also defined as the probability of a non-failure over time.

Accurate prediction and control of reliability plays an important role in scheduling preventive maintenance, determining warranty conditions and periods, and other aspects. Service costs for products within the warranty period or under a service contract are important factors during design and production process. Appropriate selection of warranty period needs the accurate information on reliability. And also, proper spare part stocking and personnel hiring for technical support also depend upon good reliability predictions. On the other hand, missing reliability targets may result in contractual penalties and may lead to a significant loss. Manufacturing systems that can economically design and market products meeting their customers' reliability expectations have a strong competitive advantage in today's marketplace.

An integrated product test program may consist of several types of tests, each having different objectives, such as functional or operational tests, environmental stress testing, reliability qualification tests, reliability life testing, safety testing, and reliability growth testing. These tests provide useful reliability information, and aggressive failure mode and effect analysis. The primary objective of reliability life testing is to obtain information concerning failures, thus to quantify reliability, determine satisfaction of reliability goals, and to improve product reliability. ALT method is a class of reliability
testing methods which utilizes failure test data from accelerated testing to analyze life under usage conditions. ALT consists of a variety of test methods, by which the stress levels are increased for hastening the product degradation and accelerating their failures.

1.2 Accelerated Life Testing

In the traditional life data analysis, times-to-failure data (of a product, system or component) obtained under usage conditions are analyzed in order to quantify the life characteristics of the product, system or component, and to make predictions about products' performance. In many situations, and for many reasons, it may be difficult or even impossible to obtain such a life data. The longer life times of today's products, the shorter period between product design and release, and the more challenges in testing products that are used continuously under usage conditions are among the difficulties. Therefore, in order to observe products' failures for analyzing their failure modes and understanding their life characteristics in a short time, ALTs are developed.

ALT is usually conducted by subjecting the product (or component) to severer conditions than those that the product will be experiencing at normal conditions or by using the product more intensively than in normal use without changing the normal operating conditions. Through ALT, stress levels which accelerate product failure are increased and life data for the product under accelerated stress conditions are captured. Those failure data under accelerated stress conditions are then utilized to derive the failure information under usage condition based on some life-stress relationship. ALT for a product or material is often used to quickly obtain information on the life
distribution or product performance under usage conditions. Figure 1-1 shows different types of ALT, which are described as follows.

1.2.1 Qualitative ALT vs. Quantitative ALT

In general, the accelerated life testing can be divided into two categories: qualitative ALT and quantitative ALT. Qualitative ALTs, such as Highly Accelerated Life Testing (HALT), Highly Accelerated Stress Screening (HASS), torture tests, shake and bake tests, are used primarily to reveal probable failure modes for the product so that product engineers can improve the product design. Quantitative ALTs are designed to quantify the life of the product and produce the data required for accelerated life data analysis. This analysis method uses life data obtained under accelerated conditions to extrapolate an estimated probability density function (pdf) for the product under normal use conditions.

In qualitative ALT, the objective is to identify failures and failure modes without attempting to make any predictions as to the product’s life under normal use conditions. Qualitative tests are performed on small samples with the specimens subjected to a single severe level of stress, to a number of stresses or to a time-varying stress (i.e. stress cycling, cold to hot, etc.). If the specimen survives, it passes the test. Otherwise, appropriate actions will be taken to improve the product's design in order to eliminate the cause(s) of failure. In general, qualitative tests do not quantify the life (or reliability) characteristics of the product under normal use conditions; however, they provide valuable information as to the types and level of stresses one may wish to employ during a subsequent quantitative test. Typical qualitative ALT includes HALT and HASS.
Accelerated Life Testing (ALT)

Qualitative ALT

- Highly-Accelerated Life Testing (HALT)
- Highly-Accelerated Stress Screening (HASS)

Quantitative ALT

- Step Stress ALT (SSALT)
- Constant Stress ALT (CSALT)
- Continuously Increasing Stress ALT (CISALT)

- Single Stress Variable
  - Complete Data
  - Exponential failure data
    - Miller and Nelson, 1983
    - Khamis and Haggins, 1996
  - Weibull Failure Data
    - Khamis and Haggins, 1998
    - Alhadded and Yang, 2002

- Censored Data
  - Exponential failure data

- Two Stress Variables
  - Weibull failure data,
    - Bivariant model
      - Li and Fard, 2007

- Multiple Stress Variables
  - Exponential failure data
    - Khamis, 1997

Figure 1-1 Different Types of ALT
HALT is used in the design phase to test the product in a severe condition and remove design related weaknesses. It exposes the products to a step-by-step cycling in environmental variables such as temperature, shock and vibration. The goal of the HALT is to break the product, find the weak components, and reinforce or improve the weak spots. HASS is an on-going screening test used to identify process and vendor problems during the production process. HASS differs from HALT in that it is a screening of the actual products being produced through manufacturing using a less stressful level of stimuli. We are not trying to break the product in this case but verify that the design is rugged enough to meet the actual specification limits agreed on by ourselves and our customer.

Quantitative ALT consists of tests designed to quantify the life characteristics of the product, component or system under normal use conditions and thereby provide reliability information. Reliability information includes the prediction of mean life and reliability of the product under normal use conditions, and projected returns and warranty costs. Units are tested at higher-than-usual levels of stress (e.g., temperature, voltage, pressure, vibration, and cycling rate) to induce early failure. Data obtained from Quantitative ALT are then analyzed based on models that relate the lifetime to stress. Finally, the results are extrapolated to estimate the life distribution at the normal use condition.
1.2.2 Constant Stress ALT vs. Step Stress ALT

Quantitative ALT can be classified as Constant Stress ALT (CSALT), Step Stress ALT (SSALT), Continuously Increasing Stress ALT (CISALT), and etc. These types of classification are according to the time dependency of the stress variables. Figure 1-2 shows the time dependency of the stress variables for CSALT, SSALT and CISALT.

![Different Types of Quantitative ALT](image)

In a CSALT, stress applied to the products is time-independent. Test units are subjected at a constant, higher-than-usual level of stress until either all units fail (without censoring) or the test is terminated, resulting in censored test data. In SSALT, the stress applied to the test product is increased in a specified discrete sequence. Test units are initially placed on a specified low stress, and then at a pre-determined time, the stress is changed to a higher level. Simple step-stress tests, which use 2 test-levels, have been widely studied. In CISALT, products are subjected to a stress as a continuous function.
of time. This study presents SSALT methods for both simple and multiple stress levels with censoring data.

1.2.3 Complete Data & Censored Data

Commonly all available test data obtained from ALT is used in analysis of life data. However, the obtained data may be incomplete or it may include uncertainty about the failure time. Therefore, life data could be separated into two categories: complete (all failure data are available) or censored (some of failure data are missing).

Complete data consist of the exact failure time of test units, which means that the failure time of each sample unit is observed or known. In many cases when life data are analyzed, all units in the sample may not fail. This type of data is commonly called censored or incomplete data. Due to different types of censoring, censored data can be divided into time-censored data and failure-censored data. Time censored data is also known as type I censored. This type of data is usually obtained when censoring time is fixed, and then the number of failures in that fixed time is a random variable. Data are failure censored (or type II censored) if the test is terminated after a specified number of failures. Where, the time to the fixed number of failures is a random variable. Considering the censoring type, step-stress ALT can also be divided into time-censored SSALT and failure-censored SSALT. In this study, only time-censored SSALT is considered.
1.3 **Step-Stress ALT Design Problem**

SSALT is often preferred to constant stress ALT because not only the test time and expense could be significantly decreased, but also it could avoid a high stress start point and possibly additional, unrelated failure modes. SSALT is particularly useful in new-product development when the appropriate stress levels for a constant stress ALT are unknown. Many studies have assumed exponential distribution, due to its simplicity in modeling and analysis of failure time. However, this assumption is not appropriate in many applications. Due to the flexibility, the Weibull distribution is one of the most widely used lifetime distributions in reliability engineering and life data analysis. In this study, two-parameter Weibull distribution is considered for the distribution of time to failure.

There are two major issues in the SSALT studies: data analysis and test design. Data analysis problem involves modeling data from SSALT and making inferences from such a data. Based on the underlying distribution assumption and life-stress relationship, SSALT models are developed. Point and interval estimation methods are applied to estimate the unknown parameters. The other issue is test design problem, which normally is to determine the optimal stress change times to achieve certain objective, such as getting the most precise estimation of the interested parameters. That is also the optimization criteria used in this research.
1.4 Main Objectives

The main objectives of this research are to develop SSALT which incorporates test data obtained under increased stress level for predicting reliability / mean time to failure under usage conditions. The objective of our SSALT design problem is to determine the optimal stress change time by minimizing the asymptotic variance of the MLE estimators. Models are developed considering two main criteria: 1) minimization of the asymptotic variance of reliability estimate at given life, 2) minimization of the asymptotic variance of log of the $p$-percentile life estimates.

Weibull distribution, which more accurately represents time to failure, is used instead of more commonly used exponential distribution. Although, incorporation of Weibull distribution in SSALT modeling adds to complexity of modeling and estimation, but due to its flexibility, it fits more accurately to life data than exponential distribution.

These models are then used to determine the optimum time at which the stress level are changed to higher levels. It is essential to obtain sufficient failure data at different stress level and during specified time periods. The test data are then used to predict reliability / $p$-percentile life under usage condition.

The method of maximum likelihood estimation (MLE) provides estimators that have both a reasonable intuitive basis and many desirable statistical properties. MLE method is often used in the SSALT parameter estimation. In order to evaluate and obtain consistent estimators, the minimum variance principle of estimator is obtained.
In order to get the asymptotic variance of the MLE estimator, the Fisher information matrix must be constructed. The Fisher information matrix plays a central role in parameter estimation of asymptotic variance of MLE. It summarizes the amount of information extracted from the data relative to the parameter estimation. Its elements are the negative second partial and mixed partial derivative of the log likelihood function with respect to unknown parameters. Integration of Weibull failure time in SSALT modeling leads to more complexity of Fisher information matrix, and more tedious problem to solve. The complexity even arises by considering more than one stress variable and multiple stress levels.

With the initial estimates of the unknown parameters, the optimal SSALT plan is obtained by minimizing the asymptotic variance of MLE of reliability/life. These preliminary values of parameters can be obtained from past experience from similar products and/or some target mean time to failure (MTTF) values to be achieved or a small sample experiment. The stress change time is the decision variable. It should be noted that different optimization criteria could lead to different optimal test design. Two types of criteria are considered in our studies as described below.

The first case considers the life estimate of a system/product as a primary criterion. Therefore the objective is to minimize the asymptotic variance of log of the \( p \)-percentile life estimates. The focus of the second case is on the probability that a product or a system will survive for a given time \( t_0 \), that is, the reliability estimate for a
given time $t_0$. Therefore, the optimization criterion is to minimize the asymptotic variance of reliability estimate at given time $t_0$.

Sensitivity analysis is performed to examine the effect of the preliminary estimates of parameters on the optimal values of hold time $\tau$. It will provide information about robustness of the optimal design. It will also identify the sensitive parameters, which need to be estimated with special care to minimize the risk of obtaining an erroneous optimal solution.

1.5 Dissertation Outline

The dissertation is organized as follows:

In Chapter 2, general design steps for SSALT are first presented, followed by the summary of the literature pertained to this dissertation research. The literature review falls into two categories: SSALT data analysis and optimal test design problems.

Chapter 3 presents the simple SSALT analysis and design. Simple SSALT models with exponential failure time assumption and Weibull failure time assumption are derived, followed by a numerical example and sensitivity analysis.

In Chapter 4, a bivariate SSALT model is presented. The test includes two stress variables, and each stress variable has two stress levels, and stress levels are changed at different times. Both exponential time-to-failure and Weibull time-to-failure are considered, and the optimal designs are derived.
Chapter 5 extends the results in Chapter 4 to the general case, multi-variate SSALT, which considers $k$ steps and $m$ stress variables. Censored Weibull failure data is assumed. The optimal test plan is developed to determine the stress change times for each step.

While the SSALT model in Chapter 3 to Chapter 5 all assume general log-linear relationship between life and stresses, Chapter 6 considers proportional hazards (PH) model and design an optimal simple SSALT based on PH model. It assumes a two parameter Weibull distribution for failure time at higher stress level, and PH model is applied for all other stress levels, using the intensity function at higher stress level as the baseline intensity function. The optimal test plan is developed to determine the hold time for the lower stress level. Numerical examples are given and sensitivity analysis is performed.

Chapter 7 extends the results in Chapter 6 to more general cases. First, multiple SSALT with only one stress variable is derived based on PH model. Optimal criteria are developed and numerical examples are given. It is then extended to $k$-step, $m$-variable SSALT design based on PH model.

The last chapter, Chapter 8, concludes with the major contributions of this dissertation and proposes future research problems that arise as extensions of what we currently achieved in this dissertation.
CHAPTER 2

General Design Steps and Literature Review

This chapter presents the design steps for SSALT, related models and literature review for our dissertation research. First, the general steps of optimal SSALT design problem are presented. And then, the research works pertained to this dissertation research is summarized.

2.1 SSALT Design Steps

Determining the pdf at normal use condition is the main purpose of life data analysis. Using this pdf, we can easily obtain all other reliability assessment, such as probability failing within warranty, risk analysis, and etc.

In traditional life data analysis, this pdf at normal use condition can be easily determined using regular failure data and an underlying distribution such as the Weibull, exponential and lognormal distributions.

In ALT data analysis, we need to determine this pdf at normal use condition from accelerated life test data instead of from regular failure data obtained under normal use condition. In order to do that, we must have an underlying life distribution and a life-stress relationship.
2.1.1 Choosing Appropriate Underlying Life Distribution

The first step in designing an SSALT is to choose an appropriate life distribution. The most commonly used life distributions are the exponential distribution, the Weibull distribution, and etc. The exponential distribution has been widely used in the past because of its simplicity, although it is rarely appropriate. The Weibull distribution, which requires more involved calculations, is more appropriate for most uses. Thus, most of this dissertation research focuses on the Weibull distributed failure data.

After selecting an appropriate underlying distribution, the second step is to select a proper life-stress relationship.

2.1.2 Selecting Proper Life-Stress Relationship

Figure 2-1 shows the relationship between actual stress and the life. The curve shows that at higher stress level less time is required for a failure occurrence and as the stress is reduced product life becomes longer. This relationship between stress and life provides an effective means for accelerating the test.

Table 2-1 lists some commonly used life-stress relationships[51].
Table 2-1  Commonly Used Life-Stress Relationship

<table>
<thead>
<tr>
<th>Relationship</th>
<th>Used Condition</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arrhenius Relationship</td>
<td>Used when the stimulus or acceleration variable (or stress) is thermal (i.e. temperature).</td>
<td>$\ln(L(V)) = \ln(C) + \frac{B}{V}$</td>
</tr>
<tr>
<td>Eyring Relationship</td>
<td>Most often used when thermal stress (temperature) is the acceleration variable, but also used for stress variables other than temperature, such as humidity.</td>
<td>$L(V) = \frac{1}{V} e^{-\left(\frac{A-B}{V}\right)}$</td>
</tr>
<tr>
<td>Inverse Power Law Relationship</td>
<td>Commonly used for non-thermal accelerated stresses.</td>
<td>$\ln(L) = - \ln(K) - n \ln(V)$</td>
</tr>
<tr>
<td>Temperature-Humidity Relationship</td>
<td>Often used when temperature and humidity are the accelerated stresses in a test.</td>
<td>$\ln(L(V, U)) = \ln(A) + \frac{\phi}{V} + \frac{b}{U}$</td>
</tr>
<tr>
<td>Temperature-Non thermal Relationship</td>
<td>Used when temperature and a second non-thermal stress (e.g. voltage) are the accelerated stresses of a test.</td>
<td>$\ln(L(U, V)) = \ln(C) - n \ln(U) + \frac{B}{V}$</td>
</tr>
<tr>
<td>General Log-Linear Relationship</td>
<td>Used when a test involves multiple accelerating stresses.</td>
<td>$\ln(L) = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_n X_n$</td>
</tr>
<tr>
<td>Proportional Hazards Model</td>
<td>Widely used in the biomedical field and recently increasing application in reliability engineering for multiple accelerating variables.</td>
<td>$\lambda(t; X) = \lambda_0(t) \cdot e^{\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n}$</td>
</tr>
</tbody>
</table>

Where $L$ represents a quantifiable life measure, $V$, $U$, $X_i$’s are the stress variables; $\lambda$ is the failure rate, and $A, B, C, K, n, \phi, b, \alpha_i$ are unknown parameters.
The Arrhenius life-stress model (or relationship) is probably the most common life-stress relationship utilized in accelerated life testing. It has been widely used when the stimulus or acceleration variable (or stress) is thermal (i.e. temperature). It is derived from the Arrhenius reaction rate equation proposed by the Swedish physical chemist Svandte Arrhenius in 1887. The Arrhenius life-stress model is formulated by assuming that life is proportional to the inverse reaction rate of the process, thus the Arrhenius life-stress relationship is given by:

\[ \ln(L(V)) = \ln(C) + \frac{B}{V} \]

The Eyring relationship was formulated from quantum mechanics principles and is most often used when thermal stress (temperature) is the acceleration variable. However, the Eyring relationship is also often used for stress variables other than temperature, such as humidity. The relationship is given by:

\[ L(V) = \frac{1}{V} e^{-\left(\frac{A}{V} + \frac{B}{V}\right)} \]

The inverse power law (IPL) model (or relationship) is commonly used for non-thermal accelerated stresses and is given by:

\[ \ln(L) = -\ln(K) - n \ln(V) \]

The temperature-humidity (T-H) relationship, a variation of the Eyring relationship, has been proposed for predicting the life at use conditions when temperature and humidity are the accelerated stresses in a test. This combination model is given by:

\[ \ln(L(V, U)) = \ln(A) + \frac{\phi}{V} + \frac{b}{U} \]
When temperature and a second non-thermal stress (e.g. voltage) are the accelerated stresses of a test, then the Arrhenius and the inverse power law relationships can be combined to yield the temperature-nonthermal (T-NT) relationship. This relationship is given by:

\[
\ln(L(U, V)) = \ln(C) - n \ln(U) + \frac{B}{V}
\]

When a test involves multiple accelerating stresses or requires the inclusion of an engineering variable, a general multivariable relationship is needed. Such a relationship is the general log-linear relationship, which describes a life characteristic as a function of a vector of \(n\) stresses, shown as follows:

\[
\ln(L) = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_n X_n
\]

Introduced by D. R. Cox [11], the PH model was developed in order to estimate the effects of different covariates influencing the times-to-failure of a system. According to the PH model, the failure rate of a system is affected not only by its operation time, but also by the covariates under which it operates. For example, a unit may have been tested under a combination of different accelerated stresses such as humidity, temperature, voltage, etc. It is clear then that such factors affect the failure rate of a unit. The failure rate of a unit is then given by:

\[
\lambda(t; X) = \lambda_0(t) \cdot e^{\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n}
\]

In this study, from Chapter 3 to Chapter 5, general log-linear relationship is assumed. And Chapter 6 and Chapter 7 considered proportional hazards model. After
determining the underlying life distribution and the life-stress relationship, a proper model is needed to analyze the SSALT data.

2.1.3 Cumulative Exposure (CE) Model – SSALT Data Analysis Model

In SSALT, failure data are obtained from the step-stress cumulative distribution function (CDF). However, commonly applied life distribution under normal operating stress, assumes constant stress. To analyze the data from SSALT, a model is needed to relate the distribution under step-stress to the distribution under constant stress. Thus, the third step is to select the proper model to relate the CDF under SSALT to the CDF under CSALT. The most commonly used model is cumulative exposure (CE) model, proposed by Nelson in 1980 [52].

The model assumes that the remaining life of specimens depend only on the current cumulative fraction failed and current stress — regardless of how the fraction accumulated. Moreover, if held at the current stress, survivors will fail according to the CDF of stress, but starting at the previously accumulated fraction failed.

Figure 2-2 describes the CE model, with a three step SSALT (a), and the three CDF’s for the constant stresses $V_1$, $V_2$, $V_3$ (b). It shows that the test units first follow the CDF for $V_1$ up to the first holding time $t_1$. When the stress is increased from $V_1$ to $V_2$, the survival units follow the CDF for $V_2$, starting at the accumulated fraction failed. Similar derivation can be obtained when the stress increases from $V_2$ to $V_3$. Figure
2-2-(c) shows the CDF of the SSALT, which includes the segments of the CDF’s for the CSALT.

The cumulative exposure model for k-step SSALT can be expressed as follows:

\[
F_k(t) = \begin{cases} 
F_1(t) & 0 \leq t \leq t_1 \\
F_2(t - t_1 + s_1) & t_1 \leq t \leq t_2 \\
F_3(t - t_2 + s_2) & t_2 \leq t \leq t_3 \\
\vdots \\
F_k(t - t_{k-1} + s_{k-1}) & t_{k-1} \leq t \leq t_k 
\end{cases} 
\]  

(2-1)

where \( s_0 = t_0 = 0 \); and \( s_i (i > 0) \) is the solution of: \( F_{i+1}(s_i) = F_i(t_i - t_{i-1} + s_{i-1}) \), for \( i = 1, \ldots, k-1 \).

There are also other models to analyze the SSALT failure data, which are shown in the following literature review section.
2.2 Literature Review

There are two popular and related issues in a SSALT: analysis of SSALT data and derivation of optimum test plans.

2.2.1 SSALT Data Analysis Models

In a SSALT, the data are usually obtained from the step-stress CDF. To obtain the life distribution under operating stress, a model is needed to relate the distribution under step-stress test to that under constant-stress test. Nelson [52] defined one such model, called cumulative exposure (damage) model (CE model). Most studies related to analysis of SSALT use the CE model, which assumes that the remaining life of specimens depends only on the current cumulative failure probability and current stress — regardless of how the probability is accumulated. Moreover, if held at the current stress, survivors will fail according to the CDF for that stress, but starting with the previously accumulated fraction failed.

Tang, Sun, Goh, and Ong [67] proposed a linear cumulative exposure model (LCEM) to analyze data from SSALT, in particular, those with failure-free life (FFL). When the life distribution contains an FFL characterized by a location parameter, the cumulative exposure model does not seem to apply. The LCEM is constructed to overcome this shortcoming. Under LCEM a general expression is derived for computing the MLE of stress-dependent distribution parameters under multiple censoring.
Khamis and Higgins [32] constructed another model (KH model), an alternative for Weibull cumulative exposure model. KH model is based on a time transformation of the exponential cumulative exposure model. The time-transformation enables the reliability engineer to use known results for multiple-steps, multiple-stress models that have been developed for the exponential step-stress model. KH model not only keeps the flexibility of CE model assuming Weibull failure time, but also its mathematical form makes it easier to obtain parameter estimates and the asymptotic variance.

Schmoyer [63] relates the ALT model with the proportional hazards model (PH model), suggested by Cox [11] for reliability analysis. He considered models of the form \[ \Pr(t; x) = 1 - e^{-g(x)h(t)}, \] where \( \Pr(t; x) \) denotes the probability of a failure by time \( t \) of an item subjected to a stress level \( x \). From this model, the nonparametric accelerated failure time model is constructed and confidence bounds are developed.

Bagdonavičius, Gerville-Réache, and Nikulin [3] combined the CE model (also called cumulative damage model) and PH model, and proposed generalized proportional hazards model (GPH model). The GPH is a generalization of CE and PH model. The new model is more reasonable than the PH model and wider than the CE model.

Van Dorp, Mazzuchi, Fornell and Pollock [73] considered the analysis of SSALT data from a Bayes viewpoint and developed a Bayes model for step-stress accelerated life testing. They discussed the incorporation of prior information into the analysis and developed Bayes point estimates and credibility intervals for the normal-using-conditions life parameters.
2.2.2 Derivation of Optimum Test Plans

For the literature of the SSALT planning problems, the commonly used optimization criterion is to minimize the asymptotic variance of the MLE of the log (mean life) or some percentile at the normal operating conditions.

Much attention has been focused on the problem of designing SSALT plans assuming exponentially distributed life. Miller and Nelson [49] first presented optimum simple SSALT plans for the case where test units have exponentially distributed life and all are run to failure. They also assumed that the mean life is a log-linear function of stress. Cumulative exposure model is used for data analysis, and MLEs are used for parameter estimation. Both time-step stress tests and failure-step stress tests are considered.

Bai, Kim and Lee [6] extends the results of Millers & Nelson [49] to the case where censoring is considered. They presented the optimum simple time-step and failure –step stress accelerated life tests for the case where a pre-specified censoring time is involved. An exponential life distribution with a mean that is a log-linear function of stress, and a cumulative exposure model are assumed. The optimum test plans are obtained to minimize the asymptotic variance of the MLE of the mean life at a normal use stress.

Khamis and Higgins [31] presented optimum 3-step, step-stress plans, which assume that a linear or quadratic relationship exists between the log (mean failure time)
and the stress. As an extension of the results for the linear model, the optimum test plan for a quadratic model is obtained. A compromise test plan is proposed as an alternative to the optimum linear & quadratic test plans, and its asymptotic properties are studied.

Khamis [30] proposed an optimal $m$-step SSALT design with $k$ stress variables, assuming complete knowledge of the stress-life relation with multiple stress variables. With the assumptions of exponential time to failure and cumulative exposure model, simple step-stress models are extended to include $k$ stress variables. Maximum likelihood estimators of the parameters are obtained, and the Fisher information matrix is derived. Optimal times at which to change stresses are obtained using the asymptotic variance of the MLE of log mean time to failure at normal use stress as the optimality criterion.

Yeo and Tang [85] and Tang [66] derived an optimal multiple-step SSALT, where not only the optimal hold time under low stress but also the optimum low stress level is determined by taking into consideration the target acceleration factor. They considered a simple SSALT where the high-stress level is fixed and there is a time constraint to complete the test. The s-expected failure proportion at high-stress level and the target acceleration factor are the necessary inputs to determine both the optimal hold time and optimal low stress. The simple SSALT plan is then generalized to multiple SSALT plans with optimal lower-stress levels and hold-times at all stress levels. The optimal plan for multiple SSALT is generated using a backward recursion method beginning from the simple SSALT. The procedure is as follows: (1) Solve for the
simple SSALT; (2) Split the lower stress into a simple SSALT with the optimal hold time as the censoring time.

Xiong [78], Xiong and Milliken [80] considered the simple step-stress model with type-II censored exponential model, where the stress change times are random variables. They assumed an exponential life distribution with a mean that is log-linear function of stress and a cumulative exposure model. They used MLE to estimate the parameters, and constructed confidence intervals for the unknown parameters.

Teng and Yeo [71] proposed a new approach for the data analysis of type II censored failure in a step-stress accelerated life tests with exponential failures, following an assumed log-linear life-stress relationship. This approach is a transformed least squares (TLS) approach.

Xiong [79] proposed a step-stress model with threshold parameter. He assumed a simple step-stress accelerated life-testing model based on a two-parameter exponential distribution with a threshold that depends on the stress level. This threshold acts as the ‘guarantee time’ or the minimum survival time. CE model is also assumed. The maximum likelihood estimators for the model parameters are obtained and exact confidence intervals of parameters are also derived.

Several other studies have focused on the Weibull SSALT. Bai and Kim [5] presented an optimum simple step-stress accelerated life test for the Weibull distribution under Type I censoring. They assumed that a log-linear relationship exists between the Weibull scale parameter and the (possibly transformed) stress and that a certain
cumulative exposure model for the effect of changing stress holds. The optimum plan—low stress and stress change time—is obtained, which minimizes the asymptotic variance of the MLE of a stated percentile at design stress.

CHAPTER 3

Simple SSALT Analysis and Design

In this chapter, the simple SSALT analysis and design are presented. The simple SSALT, which involves only two stress levels, is a relatively simple case in SSALT. The exponential distribution is assumed for the time-to-failure of test units in section 3.1. The MLE of unknown parameters are presented and the optimization criteria are then proposed. A numerical example is given to illustrate the derivation of the optimal simple SSALT, followed by the sensitivity analysis. In section 3.2, the failure time of test units is assumed to be Weibull distribution, which is more applicable in industry. A standardized model is proposed, giving the MLE and the expected Fisher information matrix. The AV of desired MLE is then obtained using the expected Fisher information matrix, and the optimal test design is derived. And then, numerical examples are given and different analyses are made.

In a simple SSALT, test units are initially placed at lower stress level $S_1$, and run until hold time $\tau$ (also called stress change time), when the stress is increased to $S_2$. Test is continued until all units fail or until a pre-specified censoring time $T$, whichever comes first. Test procedure is shown in Figure 3-1, where $S_0$ is usual stress level. In the test, total of $n_i$ failures are observed at time $t_{ij}$, $j = 1,2,\ldots,n_i$, while testing at stress level $S_i$, $i = 1,2$, and $n_c$ units remain un-failed and censored at time $T$, $n_c = n - n_1 - n_2$. 

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3.1 Simple Exponential SSALT Analysis and Design

The exponential distribution is a commonly used distribution in reliability engineering. Mathematically, it is a fairly simple distribution, which sometimes leads to its use in inappropriate situations. It is in fact a special case of the Weibull distribution where the shape parameter equals 1. The exponential distribution is used to model the behavior of units that have a constant failure rate. Many researchers have focused on the data analysis and test design under exponentially distributed lifetime assumption [4, 6, 8, 14, 24, 30, 31, 41, 42, 48, 49, 65-68].

3.1.1 Model Assumptions

The following basic assumptions are made:

1) For any level of stress, the life of test units is exponentially distributed.

2) The mean life \( \alpha_i \) of a test unit at stress \( S_i \) is a log-linear function of stress.

That is,

\[
\log(\alpha_i) = \beta_0 + \beta_1 S_i
\]
where $\beta_0$ and $\beta_i (< 0)$ are unknown parameters depending on the nature of the product and the method of test.

3) A cumulative exposure model holds.

From the assumptions of CE model and exponentially distributed life, the CDF of a test unit under simple SSALT is:

$$G(t) = 1 - \begin{cases} \exp\left(-\frac{t}{\alpha_1}\right) & \text{for } 0 \leq t \leq \tau \\ \exp\left(-\frac{t - \tau}{\alpha_2} - \frac{\tau}{\alpha_1}\right) & \text{for } \tau \leq t \leq \infty \end{cases} \quad (3-2)$$

Thus the likelihood function from observations $t_{ij}$, $i = 1, 2$, $j = 1, 2, \ldots, n_i$, is presented as follows [2, 14, 11]:

$$L(\alpha_1, \alpha_2) = \prod_{j=1}^{n_i} \left[ \frac{1}{\alpha_1} \cdot \exp\left(-\frac{t_{1,j}}{\alpha_1}\right) \right] \cdot \prod_{j=1}^{n_i} \left[ \frac{1}{\alpha_2} \cdot \exp\left(-\frac{t_{2,j} - \tau}{\alpha_2} - \frac{\tau}{\alpha_1}\right) \right] \cdot \prod_{j=1}^{n_i} \exp\left(-\frac{\tau}{\alpha_1} - \frac{T - \tau}{\alpha_2}\right) \quad (3-3)$$

where $n_c = n - n_1 - n_2$. The log likelihood function is then:

$$\log L(\alpha_1, \alpha_2) = -\left( n_1 \log \alpha_1 + n_2 \log \alpha_2 + \frac{U_1}{\alpha_1} + \frac{U_2}{\alpha_2} \right) \quad (3-4)$$

where $U_1 = \sum_{j=1}^{n_1} t_{1,j} + (n_2 + n_c) \cdot \tau$ and $U_2 = \sum_{j=1}^{n_2} (t_{2,j} - \tau) + n_c \cdot (T - \tau)$.

Then the maximum likelihood estimator is obtained by differentiating the log likelihood function.

$$\frac{\partial \log L(\alpha_1, \alpha_2)}{\partial \alpha_1} = -\frac{n_1}{\alpha_1} + \frac{U_1}{\alpha_1^2} = 0 \quad (3-5)$$

$$\frac{\partial \log L(\alpha_1, \alpha_2)}{\partial \alpha_2} = -\frac{n_2}{\alpha_2} + \frac{U_2}{\alpha_2^2} = 0$$
Thus we have the MLE of parameter $\alpha_1$ and $\alpha_2$: $\hat{\alpha}_1 = \frac{U_1}{n_1}$, $\hat{\alpha}_2 = \frac{U_2}{n_2}$.

### 3.1.2 Optimization Criteria

As an important property of estimation method, we would like the estimator to have minimum variance. To estimate the asymptotic variance, one needs to obtain the Fisher information matrix $\hat{\mathbf{F}}_1$ first. The Fisher information matrix plays a key role in the parameter estimation. It is a measure of the information content of the data relative to the parameters being estimated. Its elements are the negative second partial and mixed partial derivative of $\log[L(\alpha_1, \alpha_2)]$ with respect to parameters $\alpha_1$ and $\alpha_2$.

Then the observed Fisher information matrix $\hat{\mathbf{F}}_1$ is obtained as following:

$$
\hat{\mathbf{F}}_1 = n \cdot \begin{bmatrix}
A_1(\tau) & 0 \\
0 & A_2(\tau)
\end{bmatrix}
$$

(3-6)

where the elements of matrix are:

$$
A_1(\tau) = \frac{1 - \exp\left(-\frac{\tau}{\alpha_1}\right)}{\alpha_1^2}, \text{ and } A_2(\tau) = \frac{\exp\left(-\frac{\tau}{\alpha_1}\right) \cdot \left[1 - \exp\left(-\frac{T-\tau}{\alpha_2}\right)\right]}{\alpha_2^2}
$$

Using the Fisher information matrix, the asymptotic variance of the desired estimation can be obtained. Then by minimizing the asymptotic variance of the desired estimation, optimal stress change time (hold time) is determined which produces the optimal test design. It should be noted that different optimization criteria could lead to different optimal test design. Two types of criteria are considered as follows:
Chapter 3 Simple SSALT Test Design

A. Criterion I

The first case considers the life estimate of a system/product as a primary criterion. Therefore the objective is on the mean time to failure. Thus the optimization criterion is to minimize the asymptotic variance of log of the mean life estimates.

From the log-linear stress-life relationship (Assumption 2), we have:

\[ \log \alpha_1 = \beta_0 + \beta_1 S_1 \]  
(3-7)

\[ \log \alpha_2 = \beta_0 + \beta_1 S_2 \]  
(3-8)

We define: \( x_i = \frac{S_1 - S_0}{S_2 - S_0} \), then \( S_0 = \frac{S_1 - S_2 x_i}{1 - x_i} \).

Therefore,

\[ \log \alpha_0 = \beta_0 + \beta_1 S_0 = \beta_0 + \beta_1 \frac{S_1 - S_2 x_i}{1 - x_i} = \frac{\log \alpha_1 - x_i \cdot \log \alpha_2}{1 - x_i} \]  
(3-9)

The MLE of log of the mean life at normal operating condition is obtained as follows:

\[ \log \hat{\alpha}_0 = \frac{\log \hat{\alpha}_1 - x_i \cdot \log \hat{\alpha}_2}{1 - x_i} \]  
(3-10)

where, \( x_i = \frac{S_1 - S_0}{S_2 - S_0} \).

The AV of log of the mean life at normal operating condition is obtained:

\[ AV(\log \hat{\alpha}_0) = AV \left( \frac{\log \hat{\alpha}_1 - x_i \cdot \log \hat{\alpha}_2}{1 - x_i} \right) = H_1' \cdot \hat{F}_1^{-1} \cdot H_1 \]  
(3-11)

where \( \hat{F}_1 \) is the Fisher information matrix in Equation (3-6), and
\[ \mathbf{H}_1 = \left[ \frac{\partial \log \hat{\alpha}_0}{\partial \hat{\alpha}_1}, \frac{\partial \log \hat{\alpha}_0}{\partial \hat{\alpha}_2} \right]' = \left[ \frac{1}{\hat{\alpha}_1(1-x_i)}, \frac{-x_i}{\hat{\alpha}_2(1-x_i)} \right]' \]

### B. Criterion II

The focus of this case is on the probability that a product or a system will survive for a given time \( \varsigma \), that is, the reliability estimate for a given time \( \varsigma \). Therefore, the optimization criterion is to minimize the asymptotic variance of reliability estimate at given time \( \varsigma \).

The MLE of the reliability estimate at a given time \( \varsigma \) under normal operating condition is:

\[ \hat{R}_{0,\varsigma} = \exp(-\exp(\log(\varsigma) - \log \hat{\alpha}_0)) \]
\[ = \exp\left(-\exp\left(\log(\varsigma) - \frac{\log \hat{\alpha}_1 - x_i \cdot \log \hat{\alpha}_2}{1 - x_i}\right)\right) \] \hspace{1cm} (3-12)

The AV of reliability estimate of time \( \varsigma \) at normal operating condition is obtained:

\[ AV(\hat{R}_{0,\varsigma}) = AV\left(\exp\left(-\exp\left(\log(\varsigma) - \frac{\log \hat{\alpha}_1 - x_i \cdot \log \hat{\alpha}_2}{1 - x_i}\right)\right)\right) \]
\[ = \mathbf{H}_2' \cdot \hat{\mathbf{F}}_1^{-1} \cdot \mathbf{H}_2 \] \hspace{1cm} (3-13)

where \( \hat{\mathbf{F}}_1 \) is the Fisher information matrix in Equation (3-6), and

\[ \mathbf{H}_2 = \left[ \frac{\partial \hat{R}_{0,\varsigma}}{\partial \hat{\alpha}_1}, \frac{\partial \hat{R}_{0,\varsigma}}{\partial \hat{\alpha}_2} \right]' \]
3.1.3 Numerical Example

To illustrate the procedure of optimum test design, a numerical example is given as follows. Suppose that a simple step-stress test of cable insulation is run to estimate the reliability of life $\varsigma = 10,000$ minutes at the design voltage of 20 kV. Also suppose that the highest stress applicable to test units are 30 kV, the lower stress level of voltage is 24 kV, and the censoring time $T = 1000$ minutes. If from a previous experience based on similar data, or based on preliminary test, the failure time of the product follows exponential distribution with the initial estimates of parameters as: $\alpha_1 = 750$, $\alpha_2 = 600$. Then the optimum stress change time is determined by the following steps:

1. The stress level is $x_1 = 0.4$, and the standardized parameters are obtained as:
   \[ \eta_1 = \frac{\hat\alpha_1}{T} = 0.75, \quad \eta_2 = \frac{\hat\alpha_2}{T} = 0.6. \]

2. Since the test is run to estimate the reliability of life $\varsigma$, optimization criterion II is utilized. The problem is to design the optimum SSALT with decision variable $\tau_{\hat\eta_0}$, and the objective is to minimize the asymptotic variance of the reliability estimate at time $\varsigma$ under normal operating condition.

3. Using the pattern search method, the optimal solution is obtained by Maple 8:
   \[ \tau_{\hat\eta_0}^* = 0.5840. \]

4. The optimum stress change time is obtained as: \[ t^* = 584.0 \text{ minutes}. \]

3.1.4 Sensitivity Analysis

Sensitivity analysis is performed to examine the effect of the changes in the pre-estimated parameters $\hat\alpha_1$ and $\hat\alpha_2$ on the optimal hold time $\tau$. Its objective is to
identify the sensitive parameters, which need to be estimated with special care to minimize the risk of obtaining an erroneous optimal solution. Figure 3-2 shows the result of the sensitivity analysis.

![Figure 3-2 Optimal $\tau^*$ vs. changes in standardized parameters $\eta_1$ and $\eta_2$](image)

From Figure 3-2, we can conclude that $\eta_1$ and $\eta_2$ are not sensitive parameters. We also make 1% change in the parameters $\eta_1$ and $\eta_2$ at $\eta_1 = 0.6$, $\eta_2 = 0.4$, $x_1 = 0.4$. The results showed that the $\Delta \tau^*$ due to the change in $\eta_1$ is 0.56%, and $\Delta \tau^*$ due to the change in $\eta_2$ is -0.17%. Therefore, the test design obtained here is robust design.

### 3.2 Simple Weibull SSALT Analysis and Design

The Weibull distribution is one of the most widely used lifetime distributions in reliability engineering and life data analysis due to its versatility. Depending on the values of the shape parameter and scale parameter, the Weibull distribution can be used to model a variety of life behaviors. Thus, we focus on the test design under Weibull distribution assumption. In this section, we consider the simple Weibull SSALT, which
has only one stress variable and two stress levels, and derive the optimum test plan for
the simple Weibull SSALT considering different optimization criteria.

As we mentioned before, the purpose of SSALT experiment is to estimate the
product performance under normal use condition by analyzing the failure data from
accelerated stress condition. The more accurate our estimate is, the better the
experiment would be. Thus the problem objective of the SSALT experiment design is
to determine the optimal hold time which leads to the most accurate estimates.

3.2.1 Model Assumptions

The following assumptions are made:

(i) For a simple SSALT, only two stress-levels $S_1$ and $S_2$ ($S_1 < S_2$) are used in
the test. $S_0$ is the stress level under usual operating conditions. Test
procedure is shown in Figure 3-1.

(ii) Under any constant stress level, the time to failure of a test unit follows a Weibull
distribution with distribution function:

$$G_i(t) = 1 - \exp \left( - \frac{t^\delta}{\theta_i^\delta} \right), \quad 0 \leq t < \infty, \quad i = 0, 1, 2$$

where $\delta$ is the shape parameter, and $\theta_i$ is the scale parameter.

(iii) The cumulative distribution function of the time to failure of a test unit under
simple step-stress test follows the K-H model:

$$G(t) = \begin{cases} 
1 - \exp \left( - \frac{t^\delta}{\theta_1^\delta} \right), & 0 \leq t < \tau \\
1 - \exp \left( - \frac{t^\delta - \tau^\delta}{\theta_2^\delta - \theta_1^\delta} \right), & \tau \leq t < \infty
\end{cases}$$

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(iv) The scale parameter $\theta_i$ at stress level $i, i = 0, 1, 2$ is a log-linear function of stress. That is,
\[
\log(\theta_i) = \beta_0 + \beta_i S_i, \quad i = 0, 1, 2
\]  
(3-16)

where $\beta_0$ and $\beta_i (< 0)$ are unknown parameters depending on the nature of the product and the method of test.

Based on the above assumptions, we developed the following model.

### 3.2.2 Standardized Model

In this section, we proposed a standardized model. A standardized model is obtained by the following transformations. A standardized censoring time $T_0 = 1$ is assumed, and the standardized scale parameter $\eta_i$ are defined as the ratio of the scale parameters to the censoring time $\eta_i = \frac{\theta_i}{T}$. The standardized hold time $\tau_o$, is also defined as the ratio of the hold time to the censoring time $\tau_o = \frac{\tau}{T}$. Thus the value of $\tau_o$ that minimizes AV is the optimal standardized hold time, and the optimal hold time is derived from $\tau^* = \frac{\tau_o^*}{T}$. Using the standardized model, we eliminate one of the input values: the censoring time and embed it in the standardized scale parameters. The results of the optimal standardized hold time depends only on the standardized scale parameters and the shape parameter.

MLE is a powerful method for parameter estimations. The idea behind maximum likelihood parameter estimation is to determine the parameters that maximize
the probability (likelihood) of the sample data. From a statistical point of view, the
method of maximum likelihood is considered to be more robust (with some exceptions)
and yields estimators with good statistical properties. In other words, MLE methods are
versatile and apply to most models and to different types of data. In this section, MLE
method is used for data analysis and parameter estimation.

To simplify the model, let \( y = \log \left( \frac{t}{T} \right) \), \( \sigma = 1/\delta \), \( \mu_i = \log(\eta_i) = \log \left( \frac{\theta_i}{T} \right) \), and
\[ L_i = (\log(\tau_0) - \mu_i)/\sigma \], for \( i = 0, 1, 2 \). Then Equation (3-2) becomes:

\[
G(y) = \begin{cases}
1 - \exp \left( -\exp \left( \frac{y - \mu_1}{\sigma} \right) \right), & -\infty < y < \log(\tau_0) \\
1 - \exp \left( -\exp \left( \frac{y - \mu_2}{\sigma} \right) + \exp(L_2) - \exp(L_1) \right), & \log(\tau_0) \leq y < \infty
\end{cases}
\] (3-17)

By taking the derivative of equation (3-4) the probability density function of the
random variable \( Y \) is obtained as follows:

\[
f(y) = \begin{cases}
\frac{1}{\sigma} \exp \left( \frac{y - \mu_1}{\sigma} - \exp \left( \frac{y - \mu_1}{\sigma} \right) \right), & -\infty < y < \log(\tau_0) \\
\frac{1}{\sigma} \exp \left( \frac{y - \mu_2}{\sigma} - \left( \exp \left( \frac{y - \mu_2}{\sigma} \right) - \exp(L_2) + \exp(L_1) \right) \right), & \log(\tau_0) \leq y < \infty
\end{cases}
\] (3-18)

The maximum likelihood function for censored data is:

\[
L = \prod_{i=1}^r f(t_i) \cdot (1 - F(T))^{n-r}.
\] (3-19)

where \( r \) is the total number of failures, and \( n-r \) is the number of censored units.

Then from equations (3-4), (3-5) and (3-6), the likelihood function for type I
censored Weibull failure data \( Y_{ij} = \log \left( \frac{t_{ij}}{T} \right) \), \( i = 1, 2 \), \( j = 1, 2, \ldots, n_i \) is obtained:
\[ L(\mu_1, \mu_2, \sigma) = \prod_{j=1}^{n_1} \left[ \frac{1}{\sigma} \exp \left( \frac{y_{1j} - \mu_1}{\sigma} \right) - \exp \left( \frac{y_{1j} - \mu_1}{\sigma} \right) \right] \]
\[ \cdot \prod_{j=1}^{n_2} \left[ \frac{1}{\sigma} \exp \left( \frac{y_{2j} - \mu_2}{\sigma} \right) - \exp \left( \frac{y_{2j} - \mu_2}{\sigma} \right) + \exp(L_2) - \exp(L_1) \right] \]
\[ \cdot (\exp(-\exp(L_T) + \exp(L_2) - \exp(L_1)))^{n_c} \]

(3-20)

where \( L_T = \frac{\log(T_0) - \mu_2}{\sigma} \). The \( L(\mu_1, \mu_2, \sigma) \) is used for obtaining the derivatives of Fisher information matrix instead of \( L(\beta_0, \beta_1, \sigma) \), which would be more complicated. It is shown in appendix I that, the optimal solution will remain the same, by considering the likelihood function as either a function of \((\beta_0, \beta_1, \sigma)\) or as a function of \((\mu_1, \mu_2, \sigma)\).

Then the logarithm of the likelihood function of equation (3-7), \( \log[L(\mu_1, \mu_2, \sigma)] \) is obtained:

\[ \log[L(\mu_1, \mu_2, \sigma)] = -(n_1 + n_2) \log(\sigma) + \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{y_{ij} - \mu_i}{\sigma} - \exp \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \right) \]
\[ + (n_2 + n_c)(\exp(L_2) - \exp(L_1)) - n_c \exp(L_T) \]

(3-21)

Taking the partial derivative of the log-likelihood function in equation (3-21) with respect to each one of the parameters and setting them equal to zero yields the MLE for \((\mu_1, \mu_2, \sigma)\), shown in the following equations.

\[ \frac{\partial \log L}{\partial \mu_1} = -\frac{1}{\sigma} \left( n_1 - \sum_{j=1}^{n} \exp \left( \frac{y_{1j} - \mu_1}{\sigma} \right) \right) - (n_2 + n_c) \exp(L_1) = 0 \]

(3-22)

\[ \frac{\partial \log L}{\partial \mu_2} = -\frac{1}{\sigma} \left( n_2 - \sum_{j=1}^{n_2} \exp \left( \frac{y_{2j} - \mu_2}{\sigma} \right) \right) + (n_2 + n_c) \exp(L_2) - n_c \exp(L_T) = 0 \]

(3-23)

\[ \frac{\partial \log L}{\partial \sigma} = -\frac{1}{\sigma} \left( \left( n_1 + n_2 \right) + \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{y_{ij} - \mu_i}{\sigma} \right) - \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \exp \left( \frac{y_{ij} - \mu_2}{\sigma} \right) \right) \]
\[ + (n_2 + n_c)(L_2 \exp(L_2) - C_1 \exp(L_1)) - n_c C_T \exp(L_T) \]

(3-24)
The second partial and mixed partial derivatives of \( \log[L(\mu_1, \mu_2, \sigma)] \) with respect to \((\mu_1, \mu_2, \sigma)\) are then obtained as follows:

\[
\frac{\partial^2 \log L}{\partial \mu_1^2} = -\frac{1}{\sigma^2} \left( \sum_{j=1}^{n_1} \exp \left( \frac{y_{1j} - \mu_1}{\sigma} \right) + (n_2 + n_c) \exp(L_1) \right) = -\frac{n_1}{\sigma^2}, \tag{3-25}
\]

\[
\frac{\partial^2 \log L}{\partial \mu_1 \partial \mu_2} = \frac{\partial^2 \log L}{\partial \mu_2 \partial \mu_1} = 0, \tag{3-26}
\]

\[
\frac{\partial^2 \log L}{\partial \mu_2^2} = -\frac{1}{\sigma^2} \left( \sum_{j=1}^{n_2} \exp \left( \frac{y_{2j} - \mu_2}{\sigma} \right) - (n_2 + n_c) \exp(L_2) + n_c \exp(L_T) \right) = -\frac{n_2}{\sigma^2}, \tag{3-27}
\]

\[
\frac{\partial^2 \log L}{\partial \mu_1 \partial \sigma} = -\frac{n}{\sigma^2} \left( \sum_{j=1}^{n_1} \exp \left( \frac{y_{1j} - \mu_1}{\sigma} \right) \right) \left( \frac{1}{n} \sum_{j=1}^{n_1} \left( \frac{y_{1j} - \mu_1}{\sigma} \right) + \frac{(n_2 + n_c)}{n} L_1 \exp(L_1) \right), \tag{3-28}
\]

\[
\frac{\partial^2 \log L}{\partial \mu_2 \partial \sigma} = -\frac{1}{\sigma^2} \left( \sum_{j=1}^{n_2} \exp \left( \frac{y_{2j} - \mu_2}{\sigma} \right) \right) \left( -\frac{(n_2 + n_c)}{L_2} L_2 \exp(L_2) + n_c L_T \exp(L_T) \right), \tag{3-29}
\]

\[
\frac{\partial^2 \log L}{\partial \sigma^2} = \frac{n}{\sigma^2} \left( \sum_{j=1}^{n_1} \exp \left( \frac{y_{1j} - \mu_1}{\sigma} \right) \right) \left( \frac{1}{n} \sum_{j=1}^{n_1} \left( \frac{y_{1j} - \mu_1}{\sigma} \right)^2 \right) + \frac{(n_2 + n_c)}{n} \left( L_1^2 \exp(L_1) - L_2^2 \exp(L_2) + \frac{n_c}{n} L_T^2 \exp(L_T) \right), \tag{3-30}
\]

Equations (3-25) to (3-30) are used to calculate the expected Fisher information matrix, shown in Appendix I.

Recall that the objective is to design a test plan for an experiment subject to Type I censored failure data, and to determine the test duration for lower stress level. Where, the optimum test criterion is to minimize the AV of the MLE under usual operating conditions.
In order to obtain the AV, the Fisher information matrix is needed. The Fisher information matrix plays a central role in parameter estimation for measuring information. It summarizes the amount of information in the data relative to the parameters being estimated. The expected Fisher information matrix, \( \hat{F}_2 \), can be obtained by taking the expected value of the negative second partial and mixed partial derivatives of \( \log[L(\mu_1, \mu_2, \sigma)] \) with respect to \((\mu_1, \mu_2, \sigma)\) in equations (3-25) to (3-30):

\[
\hat{F}_2 = \frac{n}{\sigma^2} \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{12} & A_{22} & A_{23} \\
A_{13} & A_{23} & A_{33}
\end{bmatrix}
\]

where the elements of \( \hat{F}_2 \) are given below:

\[
A_{11} = 1 - \exp(-\exp(L_1)), \quad A_{12} = 0,
\]

\[
A_{13} = L_1 \exp(L_1 - \exp(L_1)) + \int_0^{\exp(L_1)} w \log(w) \exp(-w) dw
\]

\[
A_{22} = \exp(-\exp(L_1)) - \exp(-\exp(L_T) + \exp(L_2) - \exp(L_1))
\]

\[
A_{23} = \exp(\exp(L_2) - \exp(L_1)) \cdot \int_0^{\exp(L_2)} w \log(w) \exp(-w) dw - L_2 \exp(L_2 - \exp(L_1))
\]

\[
A_{33} = 1 - \exp(-\exp(L_T) + \exp(L_2) - \exp(L_1)) + L_1^2 \exp(L_1 - \exp(L_1)) - L_2^2 \exp(L_2 - \exp(L_2))
\]

\[
+ \int_0^{\exp(C_1)} w [\log(w)]^2 \exp(-w) dw + \exp(\exp(L_2) - \exp(L_1)) \cdot \int_0^{\exp(C_2)} w [\log(w)]^2 \exp(-w) dw
\]

Details of elements derivation are shown in Appendix I.

### 3.2.3 Optimum Test Design

As we mentioned before, the problem objective of the SSALT experiment design is to determine the optimal hold time which leads to the most accurate estimate. In practice, this estimate would be percentile life or reliability prediction. According to
different estimate, different optimization criterion is used for the test design problem. Criterion I is the commonly used optimization criterion, which is to minimize the asymptotic variance of the MLE of the logarithm of percentile life under usual operating condition. It is used when the percentile life is the desired estimate. Criterion II is used when considering reliability prediction. It is to minimize the asymptotic variance of reliability estimate at time $\zeta$ under usual operating condition.

A. Criterion I

As we mentioned before, the commonly used optimization criterion is to minimize the asymptotic variance of the MLE of the logarithm of percentile life under usual operating condition.

From equation (3-1), the reliability function at time $t$ under usual operating condition, $S_0$ is:

$$R_0(t) = 1 - G_0(t) = \exp\left(-\frac{t^\delta}{\theta_0^\delta}\right),$$

and for a specified reliability $R$, the 100(1-$R$) percentile life under usual operating condition $S_0$ is:

$$t_R = \theta_0 \cdot (-\log(R))^\sigma$$

From the assumption (iv), the scale parameter $\theta_i$ at stress level $i$, $i=0,1,2$ is a log-linear function of stress. That is,

$$\log(\theta_i) = \beta_0 + \beta_1 S_i, \quad i = 0,1,2$$

And also, from the definition of $x = \frac{S_1 - S_0}{S_2 - S_0}$, we obtain:
Chapter 3 Simple SSALT Test Design

\[ S_0 = \frac{S_1 - xS_2}{1 - x} \]

Thus,

\[
\log \left( \frac{\theta_0}{T} \right) = \beta_0 + \beta_1 S_0 - \log(T) = \beta_0 + \beta_1 \frac{S_1 - xS_2}{1 - x} - \log(T)
\]

\[
= \beta_0 + \beta_1 S_1 - \log(T) - x(\beta_0 + \beta_1 S_2 - \log(T)) = \frac{\mu_1 - x\mu_2}{1 - x}
\]

Therefore, the MLE of log of the 100(1-R) percentile life of the Weibull distribution with a specified reliability \( R \) under usual operating conditions \( S_0 \) is:

\[
\hat{Y}_{R,S_0} = \log \left( \frac{\hat{t}_g}{T} \right)
\]

\[
= \log \left( \frac{\hat{\theta}_0}{T} \right) + \log(-\log(R)) \cdot \hat{\sigma}
\]

\[
= \frac{\hat{\mu}_1 - x\hat{\mu}_2}{1 - x} + \log(-\log(R)) \cdot \hat{\sigma}
\]

where \( x = \frac{S_1 - S_0}{S_2 - S_0} \), \( \hat{\mu}_1, \hat{\mu}_2 \), and \( \hat{\sigma} \) are estimates of \( \mu_1, \mu_2 \), and \( \sigma \).

The optimality criterion used for the SSALT design is to minimize the AV of the MLE of the log of the 100(1-R) percentile life of the Weibull distribution at \( S_0 \) with a specified reliability \( R \). When \( R = 0.3679 \), \( \hat{Y}_{R,S_0} \) is the logarithm of the characteristic life at usual operating conditions. When \( R = 0.5 \), \( \hat{Y}_{R,S_0} \) is the logarithm of the median life at usual operating conditions with stress level \( S_0 \).

The optimal standardized hold time \( \tau^*_0 \) at which \( n \cdot AV[\hat{Y}_{R,S_0}(\tau_0)] \) reaches its minimum value, leads to the optimal plan. Where the optimal hold time is derived from \( \tau^* = \tau^*_0 \cdot T \).
where $\mathbf{H}_3$ is the row vector of the first derivative of $\hat{Y}_{R,S_0}$ with respect to $\mu_1, \mu_2, \sigma$.

Therefore, the test design problem using Criterion I is defined as:

$$
\min \quad n \cdot AV[\hat{Y}_{R,S_0}(\tau_0)]
$$

(3-34)

### B. Criterion II

Reliability prediction in a product design and during the developmental testing process is an important factor. In order to accurately estimate the product reliability, the test design criterion is defined to minimize the asymptotic variance of reliability estimate at a time $\varsigma$ under normal operating condition.

From Assumption (ii), under normal operating stress level $S_0$, the time to failure of a test unit follows a Weibull distribution:

$$
G(t) = 1 - \exp \left( - \frac{t^\delta}{\theta_0^\delta} \right), \quad 0 \leq t < \infty
$$

Thus, the MLE of reliability of life $\varsigma$ at normal operating stress level $S_0$ is:
\[ \hat{R}_{S_0, s} = 1 - G(\xi) = \exp \left( -\frac{\xi}{\theta_0^\delta} \right) = \exp \left( \frac{-\log(\xi) - \log(\theta_0)}{\hat{\sigma}} \right) = \exp \left( \frac{-\log \left( \frac{\xi}{T} \right) - \log \left( \frac{\theta_0}{T} \right)}{\hat{\sigma}} \right) \]  

(3-35)

Also, from the assumption (iv), the scale parameter \( \theta_i \) at stress level \( i, i=0,1,2 \) is a log-linear function of stress. That is,

\[ \log(\theta_i) = \beta_0 + \beta_1 S_i, \quad i = 0,1,2 \]

From \[ x = \frac{S_1 - S_0}{S_2 - S_0} \], we obtain:

\[ S_0 = \frac{S_1 - x S_2}{1 - x} \]

Therefore,

\[ \log \left( \frac{\theta_0}{T} \right) = \beta_0 + \beta_1 S_0 - \log(T) = \beta_0 + \beta_1 \frac{S_1 - x S_2}{1 - x} - \log(T) = \mu_1 - x \mu_2 \]

Then \( \hat{R}_{S_0, s} \), the MLE of reliability of life \( \xi \) at normal operating stress level \( S_0 \) in equation (3-35) becomes:

\[ \hat{R}_{S_0, s} = \exp \left( -\frac{\xi}{\theta_0^\delta} \right) = \exp \left( \frac{-\log(\xi) - \log(\theta_0)}{\hat{\sigma}} \right) = \exp \left( \frac{-\log \left( \frac{\xi}{T} \right) - \log \left( \frac{\theta_0}{T} \right)}{\hat{\sigma}} \right) \],

(3-36)
Then the asymptotic variance of reliability estimate at time $\varsigma$ under normal operating condition can be obtained as follows:

$$AV(\hat{R}_{s_0\varsigma}) = AV\left[ \exp \left( - \exp \left( \frac{\log \left( \frac{\varsigma}{T} \right) - \hat{\mu}_1 - x\hat{\mu}_2}{\hat{\sigma}} \right) \right) \right] = H_4 \cdot \hat{F}_2^{-1} \cdot H_4', \quad (3-37)$$

Where $\hat{H}_2$ is the row vector of the first derivative of $\hat{R}_{s_0\varsigma}$ with respect to $\mu_1, \mu_2, \sigma$

$$H_4 = \left[ \frac{\partial \hat{R}_{s_0\varsigma}(\varsigma)}{\partial \mu_1}, \frac{\partial \hat{R}_{s_0\varsigma}(\varsigma)}{\partial \mu_2}, \frac{\partial \hat{R}_{s_0\varsigma}(\varsigma)}{\partial \hat{\sigma}} \right]^T,$$

$$= \begin{bmatrix}
\exp \left( \frac{\log \left( \frac{\varsigma}{T} \right) - \hat{\mu}_1 - x\hat{\mu}_2}{\hat{\sigma}} \right) - \exp \left( \frac{\log \left( \frac{\varsigma}{T} \right) - \hat{\mu}_1 - x\hat{\mu}_2}{\hat{\sigma}} \right) / (1-x)\sigma \\
- x \cdot \exp \left( \frac{\log \left( \frac{\varsigma}{T} \right) - \hat{\mu}_1 - x\hat{\mu}_2}{\hat{\sigma}} \right) - \exp \left( \frac{\log \left( \frac{\varsigma}{T} \right) - \hat{\mu}_1 - x\hat{\mu}_2}{\hat{\sigma}} \right) / (1-x)\sigma \\
\exp \left( \frac{\log \left( \frac{\varsigma}{T} \right) - \hat{\mu}_1 - x\hat{\mu}_2}{\hat{\sigma}} \right) - \exp \left( \frac{\log \left( \frac{\varsigma}{T} \right) - \hat{\mu}_1 - x\hat{\mu}_2}{\hat{\sigma}} \right) / (1-x)\sigma \\
\end{bmatrix} \cdot \begin{bmatrix}
\log \left( \frac{\varsigma}{T} \right) - \hat{\mu}_1 - x\hat{\mu}_2 \\
\log \left( \frac{\varsigma}{T} \right) - \hat{\mu}_1 - x\hat{\mu}_2 \\
\log \left( \frac{\varsigma}{T} \right) - \hat{\mu}_1 - x\hat{\mu}_2 \\
\end{bmatrix}.$$

The value $\tau^*_0$ that minimizes $n \cdot AV[\hat{R}_{s_0\tau^*_0}]$, given by equation (3-37), leads to the optimal SSALT plan.

Therefore, the test design problem using Criterion II is defined as:

$$\text{minimize} \quad n \cdot AV[\hat{R}_{s_0\tau^*_0}] \quad (3-38)$$
3.2.4 Numerical Examples and Analysis

In this section, numerical examples are given for both criteria. The pattern search algorithm is developed to calculate the optimal standardized hold times $\tau_0^*$ of the simple SSALT for censoring time $T_0$. Since the derivative information of the AV is unavailable, a pattern search algorithm is used for the minimization problem.

Pattern search is an important class of direct search optimizers. Pattern search methods are closely related to experimental design and are provably convergent. It is widely believed that pattern searches are less likely than derivative-based methods to be trapped by local minimums. Pattern search can be customized to cast a wide range when searching for better points, so that the search may be able to escape a shallow basin that contains a local minimum and find a deeper basin that contains the global minimum. For more details on global optimization achieved from pattern search, the reader may consult the research report of Parker (1999).

3.2.4.1 Example I

For given values of $x, \eta_1, \eta_2, \delta$, and assuming $R = 0.5$, the algorithm determines the optimal standardized hold time $\tau_0^*$ using criterion I. The optimal standardized hold times are then estimated using the pattern search algorithm and the Maple program by solving equation (3-34). Table 3-1 presents the standardized optimal hold time for the following specified values of $x, \eta_1, \eta_2, \delta$:

$x = 0.2, 0.4, 0.6, 0.8$, $\delta = 1.0, 1.2$,

$\eta_1 = 0.3, 0.5, \ldots, 1.7$, $\eta_2 = 0.1, 0.3, \ldots, 1.5$. 
Table 3-1 Optimal Standardized Hold Time \( \tau_0^*(x, \eta_1, \eta_2, \delta) \)

<table>
<thead>
<tr>
<th>( \delta = 1 )</th>
<th>( \delta = 1.2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>( \eta_1 )</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
</tr>
<tr>
<td></td>
<td>1.1</td>
</tr>
<tr>
<td></td>
<td>1.3</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>1.7</td>
</tr>
</tbody>
</table>

| x  | \( \eta_1 \) | \( \eta_2 \) | 0.1  | 0.3  | 0.5  | 0.7  | 0.9  | 1.1  | 1.3  | 1.5  |
|-----------------|-----------------|
| 0.2 | 0.3 | .5120 | .4484 | 0.3 | .7646 | .6037 | 0.3 | .7221 | 0.3 | .6037 |
|  | 0.5 | .8313 | .6477 | 0.5 | .7626 | .5762 | 0.5 | .7221 | 0.5 | .6037 |
|  | 0.7 | .9238 | .7946 | 0.7 | .8544 | .6307 | 0.7 | .8304 | 0.7 | .7637 |
|  | 0.9 | .9520 | .8322 | 0.9 | .8718 | .7174 | 0.9 | .8718 | 0.9 | .7712 |
|  | 1.1 | .9661 | .8900 | 1.1 | .9045 | .7508 | 1.1 | .9140 | 1.1 | .7827 |
|  | 1.3 | .9739 | .9025 | 1.3 | .9246 | .7801 | 1.3 | .9260 | 1.3 | .7827 |
|  | 1.5 | .9793 | .9203 | 1.5 | .9353 | .8251 | 1.5 | .9400 | 1.5 | .7827 |
|  | 1.7 | .9824 | .9211 | 1.7 | .9400 | .8450 | 1.7 | .9400 | 1.7 | .7827 |
3.2.4.2 Example II

Suppose that a simple step-stress test of cable insulation is run to estimate the reliability at a specified time $\varsigma$ of 2000 minutes under a design voltage of 20 kV. The problem objective is to design such a test that achieves the best reliability estimates. Also suppose that the highest stress applicable to test units are 30 kV, the lower stress level of voltage is 24 kV, and the censoring time $T = 1000$ minutes, and from a previous experience based on a similar data, or based on a preliminary test result, the initial estimates of parameters are obtained as: $\theta_1 = 750$, $\theta_2 = 600$, $\delta = 2.2$.

Since the problem objective is to obtain the best reliability estimates, Criterion II is applied. That is to obtain the optimal stress change time $\tau$ by minimization of the asymptotic variance of the reliability estimate at time $\varsigma$ of 2000 minutes under a design voltage of 20 kV. Thus the optimum stress change time is determined by the following steps:

1) Scale parameters are standardized by using $\eta_i = \frac{\theta_i}{T}$. Thus we obtain $\eta_1 = 0.75$, $\eta_2 = 0.6$.

2) $x$ is determined by $x = \frac{S_1 - S_0}{S_2 - S_0}$, while $S_0 = 20$ kV, $S_1 = 24$ kV, $S_2 = 30$ kV. Therefore, we have $x = 0.4$.

3) The problem is to design the optimum SSALT with decision variable $\tau_0$, and the objective is to minimize the asymptotic variance of the reliability estimate at time $\varsigma = 2000$ under normal operating condition in equation (3-38).
4) Using pattern search algorithm, the problem is solved and the optimum non-scale stress change time is obtained: $\tau_0^* = 0.9016$.

5) The optimum stress change times is obtained as: $\tau^* = 901.6$ minutes.

### 3.2.4.3 Censoring Time Impact

To investigate the impact of the censoring time on the optimal test plans, the optimal hold times obtained using criterion I for an uncensored test (assuming the censoring time is infinity), and censored times of: 5, 10, and 20 are compared. The tests are run for six specified sets of $x$, $\theta_1$, $\theta_2$, and $\delta$ and $R=0.5$ as shown in Table 3-2.

**Table 3-2 Impact of Censoring Time on Optimal Simple SSALT Plan**

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Censoring Time</th>
<th>$\tau^*$</th>
<th>$n_{AV}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\rightarrow \infty$</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>$x = 0.4$, $\theta_1 = 8$, $\theta_2 = 5$, $\delta = 1.2$, $R = 0.5$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
<tr>
<td>$x = 0.6$, $\theta_1 = 8$, $\theta_2 = 5$, $\delta = 1.2$, $R = 0.5$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
<tr>
<td>$x = 0.4$, $\theta_1 = 10$, $\theta_2 = 5$, $\delta = 1.2$, $R = 0.5$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
<tr>
<td>$x = 0.4$, $\theta_1 = 8$, $\theta_2 = 3$, $\delta = 1.2$, $R = 0.5$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
<tr>
<td>$x = 0.4$, $\theta_1 = 8$, $\theta_2 = 5$, $\delta = 1.3$, $R = 0.5$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
<tr>
<td>$x = 0.6$, $\theta_1 = 10$, $\theta_2 = 3$, $\delta = 1.3$, $R = 0.5$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Censoring Time</th>
<th>$\tau^*$</th>
<th>$n_{AV}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0.4$, $\theta_1 = 8$, $\theta_2 = 5$, $\delta = 1.2$, $\zeta = 2$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
<tr>
<td>$x = 0.6$, $\theta_1 = 8$, $\theta_2 = 5$, $\delta = 1.2$, $\zeta = 2$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
<tr>
<td>$x = 0.4$, $\theta_1 = 10$, $\theta_2 = 5$, $\delta = 1.2$, $\zeta = 2$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
<tr>
<td>$x = 0.4$, $\theta_1 = 8$, $\theta_2 = 3$, $\delta = 1.2$, $\zeta = 2$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
<tr>
<td>$x = 0.4$, $\theta_1 = 8$, $\theta_2 = 5$, $\delta = 1.3$, $\zeta = 2$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
<tr>
<td>$x = 0.6$, $\theta_1 = 10$, $\theta_2 = 3$, $\delta = 1.3$, $\zeta = 2$</td>
<td>$\tau^*$</td>
<td>$n_{AV}$</td>
<td></td>
</tr>
</tbody>
</table>
From the experimental result, we can see that as censoring time increases, the optimal hold time converges to that with no censoring. And also, the AV changes significantly as censoring time changes.

### 3.2.4.4 Sensitivity Analysis

To examine the effect of changes in the initially estimated parameters $\eta_1, \eta_2,$ and $\delta$ on the optimal values of $\tau_0$ and nAV, sensitivity analysis is performed for the test result applying both criterion I and criterion II. Figure 3-3 presents the sensitivity analysis for the three parameters $\eta_1, \eta_2,$ and $\delta$ with criterion I, and the sensitivity analysis for the other two pre-determined parameters $R$ and $x$ is shown in Figure 3-4. Figure 3-5 presents the sensitivity analysis for the three parameters $\eta_1, \eta_2,$ and $\delta$ with criterion II, and the sensitivity analysis for the other two pre-determined parameters $\varsigma$ and $x$ is shown in Figure 3-6.

From Figure 3-3, it can be seen that:

1. As $\eta_1$ increases, the optimal value of $\tau_0$ slightly increases and nAV also increases.

2. As $\eta_2$ increases, the optimal value of $\tau_0$ slightly decreases, and nAV also decreases very slightly.

3. As $\delta$ increases, the optimal value of $\tau_0$ increases, and nAV decreases significantly for smaller $\delta$ value ($\delta<1$) and converges for larger $\delta$ value ($\delta>1$).
Figure 3-3  Optimal $\tau_0$ and $nAV$ vs. changes in $\eta_1$, $\eta_2$, and $\delta$ with criterion I
Figure 3-4 indicates that the optimal hold time is not too sensitive to parameters $R$ and $x$. As $R$ increases, the optimal value of $\tau_0$ slightly decreases, and as $x$ increases, the optimal value of $\tau_0$ also slightly decreases.

The conclusions from Figure 3-5 are:

1. As $\eta_1$ increases, the optimal value of $\tau_0$ slightly increases.

2. As $\eta_2$ increases, the optimal value of $\tau_0$ slightly decreases.

3. As $\delta$ increases, the optimal value of $\tau_0$ also increases.
Figure 3-5  Optimal $\tau_0$ and $nAV$ vs. changes in $\eta_1$, $\eta_2$, and $\delta$ with criterion II
Figure 3-6  Optimal $\tau_0$ and $n AV$ vs. changes in $\zeta$ and $x$ with criterion II

Figure 3-6 indicates that as $\zeta$ increases, the optimal value of $\tau_0$ slightly increases, and as $x$ increases, the optimal value of $\tau_0$ also decreases.
CHAPTER 4

Bivariate SSALT Analysis and Design

In previous chapter, the analysis and design of the simple SSALT with one stress variable under exponential and Weibull distribution assumption are presented. However, in many applications and for various reasons, it is desirable to use more than one accelerating stress variable. For example, an accelerated life test of capacitors could include temperature and voltage as two accelerated variables [52]. Other studies have also considered more than one stress variable [48, 70]. However, those studies did not focus on the SSALT. This chapter presents an SSALT model assuming two stress variables, subject to censored test data for exponential distribution and a 2-parameter Weibull distributed failure time.

The inclusion of two stress variables in a test design will lead to a better understanding of effect of two simultaneously operating stress variables, such as temperature, and humidity. Furthermore, sometimes accelerating one stress variable does not yield enough failure data. Thus, two stress variables may be needed for added acceleration. The integration of Weibull failure time in SSALT modeling leads to more complexity in the Fisher information matrix, and a more complicated problem to solve. The complexity even increases by considering more than one stress variable, and designing a test plan with two stress levels for each stress variable, subject to censored test time.
We consider the SSALT with two stress variables, and each stress variable has two stress levels. Let $S_{lk}$ be the $k$th stress level for variable $l$, for $l = 1, 2$, and for $k = 0, 1, 2$. The $S_{10}, S_{20}$ are stress levels at typical operating conditions. The test procedure of SSALT with two stress variables is shown as follows: $n$ test units are initially placed at first step with stress levels $(S_{11}, S_{21})$, and run until time $\tau_1$, when the stress variable one is increased from $S_{11}$ to $S_{12}$. The decision to choose one of the stress variables for a stress level change, while the other stress variable remains constant, will lead to a different experimental result, but has no effect on the model. The purpose of the experiment is to obtain failure test data at different stress level combinations.

Therefore, the two stress variables could be interchanged for an experiment, resulting in different sets of failure test data, and leading to the same approach, with similar results. The test is continued until time $\tau_2$, when the stress variable 2 is increased from $S_{21}$ to $S_{22}$. The test is then continued until all units fail, or until a predetermined censoring time $T$. The procedure is shown in Figure 4-1,

![Figure 4-1  Bivariate SSALT Test Procedure]
where $n_i$ failures are observed at time $t_{ij}$, for $j = 1, 2, \ldots, n_i$ in step $i$, for $i = 1, 2, 3$, and $n_c$ units are censored, for $n_c = n - n_1 - n_2 - n_3$.

4.1 Bivariate Exponential SSALT Analysis and Design

This section studies SSALT for two stress variables for an exponentially distributed failure times when the test is subject to a termination prior to failure of all experimental units, leading to a censored failure data. In section 4.1.1, test procedure and model assumptions are presented first, followed by the Fisher information matrix. Then in section 4.1.2, the optimality criterion and the optimal test design are presented. In section 4.1.3, numerical example is given, and sensitivity analysis is performed in section 4.1.4.

4.1.1 Model Assumptions

The following basic assumptions are made:

1) For any level of stress, the life of test units is exponentially distributed.

2) The mean life $\alpha_i$ of a test unit at step $i$ is a log-linear function of stresses, and there is no interaction between the two stresses. Thus, we proposed the following life-stress relationship:

\[
\begin{align*}
\text{Step 1: } & \quad \log(\alpha_1) = \beta_0 + \beta_1 S_{11} + \beta_2 S_{21} \\
\text{Step 2: } & \quad \log(\alpha_2) = \beta_0 + \beta_1 S_{12} + \beta_2 S_{21} \\
\text{Step 3: } & \quad \log(\alpha_3) = \beta_0 + \beta_1 S_{12} + \beta_2 S_{22}
\end{align*}
\]
where $\beta_0, \beta_1$, and $\beta_2$ are unknown parameters depending on the nature of the product, and the method of test. It is assumed that there are no interactions between the stress variables.

3) A cumulative exposure model holds.

Based on the above assumptions, the CDF of a test unit under bivariate SSALT is:

$$G(t) = 1 - \begin{cases} 
\exp \left( -\frac{t}{\alpha_1} \right) & \text{for } 0 \leq t \leq \tau_1 \\
\exp \left( -\frac{t - \tau_1}{\alpha_2} - \frac{\tau_1}{\alpha_1} \right) & \text{for } \tau_1 \leq t \leq \tau_2 \\
\exp \left( -\frac{t - \tau_2}{\alpha_3} - \frac{\tau_2 - \tau_1}{\alpha_2} - \frac{\tau_1}{\alpha_1} \right) & \text{for } \tau_2 \leq t \leq \infty
\end{cases} \tag{4-4}$$

Taking the first derivative of the CDF in equation (4-4), the pdf of the failure time is obtained:

$$f(t) = \frac{dG(t)}{dt} = \begin{cases} 
\frac{1}{\alpha_1} \exp \left( -\frac{t}{\alpha_1} \right) & \text{for } 0 \leq t \leq \tau_1 \\
\frac{1}{\alpha_2} \exp \left( -\frac{t - \tau_1}{\alpha_2} - \frac{\tau_1}{\alpha_1} \right) & \text{for } \tau_1 \leq t \leq \tau_2 \\
\frac{1}{\alpha_3} \exp \left( -\frac{t - \tau_2}{\alpha_3} - \frac{\tau_2 - \tau_1}{\alpha_2} - \frac{\tau_1}{\alpha_1} \right) & \text{for } \tau_2 \leq t \leq \infty
\end{cases} \tag{4-5}$$

Thus the likelihood function from observations $t_{ij}$, $i = 1, 2, 3$, $j = 1, 2, \ldots, n_i$ and $n_c$ censored items, is obtained from the CDF in equation (4-4) and the pdf in equation (4-5) as follows:

$$L(\beta_0, \beta_1, \beta_2) = \prod_{j=1}^{n_i} \left[ \frac{1}{\alpha_1} \cdot \exp \left( -\frac{t_{1,j}}{\alpha_1} \right) \right] \cdot \prod_{j=1}^{n_i} \left[ \frac{1}{\alpha_2} \cdot \exp \left( -\frac{t_{2,j} - \tau_1}{\alpha_2} - \frac{\tau_1}{\alpha_1} \right) \right] \cdot \prod_{j=1}^{n_i} \left[ \frac{1}{\alpha_3} \cdot \exp \left( -\frac{t_{3,j} - \tau_2}{\alpha_3} - \frac{\tau_2 - \tau_1}{\alpha_2} - \frac{\tau_1}{\alpha_1} \right) \right] \cdot \prod_{j=1}^{n_i} \left[ \frac{1}{\alpha_1} \cdot \exp \left( -\frac{T - \tau_2}{\alpha_1} - \frac{\tau_2 - \tau_1}{\alpha_2} - \frac{\tau_1}{\alpha_3} \right) \right]$$

where $n_c = n - n_1 - n_2$. The log likelihood function is then obtained:
\[
\log L(\beta_0, \beta_1, \beta_2) = -\left( n_1 \log(\alpha_1) + n_2 \log(\alpha_2) + n_3 \log(\alpha_3) + \frac{W_1}{\alpha_1} + \frac{W_2}{\alpha_2} + \frac{W_3}{\alpha_3} \right) \tag{4-6}
\]

where \( W_1 = \sum_{j=1}^{n_1} t_{1,j} + (n_2 + n_3 + n_e) \cdot \tau \), \( W_2 = \sum_{j=1}^{n_1} (t_{2,j} - \tau_1) + (n_3 + n_e) \cdot (\tau_2 - \tau_1) \), and \( W_3 = \sum_{j=1}^{n_1} (t_{3,j} - \tau_2) + n_e \cdot (T - \tau_2) \).

Then the maximum likelihood estimator is obtained by differentiating the log likelihood function in equation (4-6).

Then the observed Fisher information matrix \( \hat{F}_4 \) is obtained as following:

\[
\hat{F}_4 = \begin{bmatrix}
-\frac{\partial^2 \log L}{\partial \beta_0^2} & -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} \\
-\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_1^2} & -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} \\
-\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} & -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2}
\end{bmatrix}
= n \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{12} & A_{22} & A_{23} \\
A_{13} & A_{23} & A_{33}
\end{bmatrix} \tag{4-7}
\]

where the elements of matrix \( \hat{F}_4 \) are:

\[
A_{11} = E \left[ \frac{n_1}{n} \left( \frac{n_1}{n} + \frac{n_2}{n} + \frac{n_3}{n} \right) \right],
\]

\[
A_{12} = E \left[ \frac{n_1}{n} S_{11} + \frac{n_2}{n} S_{12} + \frac{n_3}{n} S_{12} \right],
\]

\[
A_{13} = E \left[ \frac{n_1}{n} S_{21} + \frac{n_2}{n} S_{21} + \frac{n_3}{n} S_{22} \right],
\]

\[
A_{23} = E \left[ \frac{n_1}{n} S_{11} S_{21} + \frac{n_2}{n} S_{12} S_{21} + \frac{n_3}{n} S_{12} S_{22} \right],
\]

\[
A_{22} = E \left[ \frac{n_1}{n} S_{11}^2 + \frac{n_2}{n} S_{12}^2 + \frac{n_3}{n} S_{12}^2 \right].
\]
\[ A_{33} = E \left[ \frac{n_1}{n} S_{21}^2 + \frac{n_2}{n} S_{21}^2 + \frac{n_3}{n} S_{22}^2 \right]. \]

The AV of the desired estimates is then obtained using the above expected Fisher information matrix, which leads to the optimization criteria.

### 4.1.2 Optimization Criteria

Considering different types of estimation objective, two types of criteria are presented as follows.

**A. Criterion I: Mean Life Estimation**

The MLE of log of the mean life at usual operating stress levels \((S_{10}, S_{20})\) is:

\[ \hat{Y}_{(S_{10}, S_{20})} = \log(\alpha_0) = \hat{\beta}_0 + \hat{\beta}_1 S_{10} + \hat{\beta}_2 S_{20} \]  

(4-8)

Then the first derivative of \(\hat{Y}_{(S_{10}, S_{20})}\) with respect to parameters \(\beta_0, \beta_1, \beta_2\) is obtained:

\[ \mathbf{H}_6 = \begin{bmatrix} \frac{\partial \hat{Y}_{(S_{10}, S_{20})}}{\partial \hat{\beta}_0}, & \frac{\partial \hat{Y}_{(S_{10}, S_{20})}}{\partial \hat{\beta}_1}, & \frac{\partial \hat{Y}_{(S_{10}, S_{20})}}{\partial \hat{\beta}_2} \end{bmatrix} = \begin{bmatrix} 1, & S_{10}, & S_{20} \end{bmatrix} \]  

(4-9)

Therefore, the AV of \(\hat{Y}_{(S_{10}, S_{20})}\) is derived from the first derivative in equation (4-9) and the expected information matrix in equation (4-7).

\[ AV(\hat{Y}_{(S_{10}, S_{20})}) = \mathbf{H}_6 \cdot \mathbf{F}_4^{-1} \cdot \mathbf{H}_6' \]  

(4-10)

Thus, the optimization criterion of mean life estimation is to minimize the AV of log of the mean life, which is shown as follows:

\[ \text{minimize} \quad n \cdot AV[\hat{Y}_{(S_{10}, S_{20})(\tau_1, \tau_2)}] \]  

(4-11)
B. Criterion II: Reliability Estimation

The MLE of reliability of life $\varsigma$ at normal operating stress level $S_0$ is:

$$
\hat{R}_{(s_{10}, s_{20})\varsigma} = 1 - G(\varsigma) = \exp\left(-\frac{\varsigma}{\alpha_0}\right) = \exp\left(-\exp\left(\log(\varsigma) - \log(\alpha_0)\right)\right)
$$

$$
= \exp\left(-\exp\left(\log(\varsigma) - \beta_0 - \beta_1 S_{10} - \beta_2 S_{20}\right)\right)
$$

(4-12)

Then the AV of $\hat{R}_{(s_{10}, s_{20})\varsigma}$ is derived from equation (4-12) and the expected information matrix in equation (4-7).

$$
AV(\hat{R}_{(s_{10}, s_{20})\varsigma}) = H_7 \cdot \hat{F}_4^{-1} \cdot H_7'
$$

(4-13)

Where $H_7$ is the row vector of the first derivative of $\hat{R}_{(s_{10}, s_{20})\varsigma}$ with respect to $\beta_0, \beta_1, \beta_2$, shown as follows:

$$
H_7 = \left[\frac{\partial \hat{R}_{(s_{10}, s_{20})\varsigma}}{\partial \beta_0}, \frac{\partial \hat{R}_{(s_{10}, s_{20})\varsigma}}{\partial \beta_1}, \frac{\partial \hat{R}_{(s_{10}, s_{20})\varsigma}}{\partial \beta_2}\right]
$$

Then the optimization criteria for the reliability estimate is to minimize the AV of $\hat{R}_{(s_{10}, s_{20})\varsigma}$ in equation (4-13):

$$\minimize n \cdot AV\left[\hat{R}_{(s_{10}, s_{20})\varsigma}(\tau_1, \tau_2)\right]
$$

(4-14)

4.1.3 Numerical Example

Suppose we want to design an optimum SSALT plan with two stress variables. From a pre-test or an existing field data, the pre-estimate of the model parameters in equation (4-1) to (4-3) are: $\beta_0 = 0$, $\beta_1 = 0.1$, $\beta_2 = 0.6$, and the stress levels are: $S_{11} = 0.3$, $S_{21} = 0.4$, $S_{12} = 1$, $S_{22} = 1$. 
Using the optimization criteria I for the mean life estimation, we obtained the optimal stress change times for different censoring time. Figure 4-2 shows the results of the optimal stress change time vs. the censoring time. From the figure, we can conclude that as censoring time increases, the optimal stress change times increase and converge to a fixed value: $t_1 = 1.24, \ t_2 = 1.67$.

![Figure 4-2 Optimal Stress Change Time vs. Censoring Time](image)

### 4.1.4 Sensitivity Analysis

The sensitivity analysis is performed in this section to examine the effect of the pre-estimated model parameters on the optimal stress change time. Results are shown in Figure 4-3. Figure 4-3 (a) shows that as $\beta_0$ increases from $-0.5$ to $0.5$, the optimal stress change times $\tau_1$ and $\tau_2$ also increase, and the ratio of $\tau_2$ and $\tau_1$ slightly decreases. Figure 4-3 (b) shows that as $\beta_1$ increases from 0 to 1, the optimal stress change times $\tau_1$ and $\tau_2$ have very slight change. Figure 4-3 (c) shows that as $\beta_2$ increases from $-0.1$ to 1.1, optimal stress change time $\tau_1$ slightly increases, and then
Figure 4-3  Optimal Stress Change Time vs. Model Parameters
has a slight decrease as $\beta_2$ increases from 1.2 to 1.5. And the optimal stress change time $\tau_2$ slightly increases as $\beta_2$ increases from $-0.1$ to 0.7, and has a slight decrease as $\beta_2$ increases from 0.8 to 1.5. Therefore, we can conclude that the parameters are not sensitive and the model is robust.

We also examine the impact of the stress levels on the optimal stress change times, and results are shown in Table 4-1. It is shown that the ratio of $\tau_2/\tau_1$ gets to the minimum value 1 when the stress level $S_{11}$ and $S_{12}$ are the same and the ratio increases as the difference between stress level $S_{11}$ and $S_{12}$ is increasing.
Table 4-1  Impact of Stress Levels on the Optimal Stress Change Times

<table>
<thead>
<tr>
<th>S_{11}</th>
<th>S_{12}</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
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<tbody>
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<td>τ_1</td>
<td>0.732</td>
<td>0.497</td>
<td>0.367</td>
<td>0.278</td>
<td>0.211</td>
<td>0.157</td>
<td>0.111</td>
<td>0.070</td>
<td>0.034</td>
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<tr>
<td></td>
<td>τ_2</td>
<td>0.732</td>
<td>0.614</td>
<td>0.543</td>
<td>0.496</td>
<td>0.462</td>
<td>0.436</td>
<td>0.417</td>
<td>0.402</td>
<td>0.390</td>
</tr>
<tr>
<td></td>
<td>τ_2/τ_1</td>
<td>1.00</td>
<td>1.23</td>
<td>1.48</td>
<td>1.78</td>
<td>2.19</td>
<td>2.78</td>
<td>3.76</td>
<td>5.71</td>
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<td>τ_2</td>
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<td>0.542</td>
<td>0.495</td>
<td>0.462</td>
<td>0.436</td>
<td>0.417</td>
<td>0.402</td>
<td>0.390</td>
</tr>
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<td>0.541</td>
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<td>0.402</td>
<td>0.390</td>
</tr>
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<td>0.494</td>
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<td>0.461</td>
<td>0.436</td>
<td>0.417</td>
<td>0.402</td>
<td>0.390</td>
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<td>1.00</td>
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<td>0.445</td>
<td>0.450</td>
<td>0.455</td>
<td>0.460</td>
<td>0.323</td>
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<td>0.779</td>
<td>0.681</td>
<td>0.578</td>
<td>0.460</td>
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<td>0.417</td>
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<td>0.390</td>
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<tr>
<td></td>
<td>τ_2/τ_1</td>
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<td>1.75</td>
<td>1.51</td>
<td>1.27</td>
<td>1.00</td>
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<td>1.90</td>
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<td>0.175</td>
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<td>τ_2</td>
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<td>0.402</td>
<td>0.390</td>
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<td>0.380</td>
<td>0.387</td>
<td>0.394</td>
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<td>0.875</td>
<td>0.828</td>
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<td>0.586</td>
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4.2 Bivariate Weibull SSALT Analysis and Design

This section studies SSALT for two stress variables with Weibull failure times when the test is subjected to termination prior to failure of all experimental units, leading to a censored failure data. The optimum test plan determines the test duration for each combination of stress levels.

4.2.1 Model Assumptions

The following assumptions are made:

i) Under any constant stresses, the time to failure of a test unit follows a Weibull distribution

\[
F(t) = 1 - \exp\left(-\frac{t^\delta}{\theta}\right), \quad 0 \leq t < \infty
\]

where \( \delta \) is the shape parameter, and \( \theta \) is the scale parameter.

ii) The cumulative distribution function of the time to failure of a test unit follows the K-H model

\[
F(t) = \begin{cases} 
1 - \exp\left(-\frac{t^\delta}{\theta_1^\delta}\right) & 0 \leq t < \tau_1 \\
1 - \exp\left(-\frac{t^\delta - \tau_1^\delta}{\theta_2^\delta} - \frac{\tau_1^\delta}{\theta_1^\delta}\right) & \tau_1 \leq t < \tau_2 \\
1 - \exp\left(-\frac{t^\delta - \tau_2^\delta}{\theta_3^\delta} - \frac{\tau_2^\delta - \tau_1^\delta}{\theta_2^\delta} - \frac{\tau_1^\delta}{\theta_1^\delta}\right) & \tau_2 \leq t < \infty
\end{cases}
\]

iii) We proposed the following life-stress relationship. The scale parameter \( \theta_i \) at test step \( i \), for \( i = 1, 2, 3 \), is assumed to be a log-linear function of stress levels. That is,
Step 1: \[ \log(\theta_1) = \beta_0 + \beta_1 S_{11} + \beta_2 S_{21} \] (4-17)

Step 2: \[ \log(\theta_2) = \beta_0 + \beta_1 S_{12} + \beta_2 S_{22} \] (4-18)

Step 3: \[ \log(\theta_3) = \beta_0 + \beta_1 S_{12} + \beta_2 S_{22} \] (4-19)

where \( \beta_0, \beta_1, \) and \( \beta_2 \) are unknown parameters depending on the nature of the product, and the method of test. It is assumed that there are no interactions between the stress variables.

iv) The shape parameter \( \delta \) is constant for all stress levels.

### 4.2.2 MLE and Fisher Information Matrix

MLE method is used for the parameter estimation as follows. Let \( y = \log(\beta), \sigma = 1/\delta, \) then the cumulative distribution function \( G(y) = F(e^y) \) of the random variable \( y \) is derived from (4-16) considering the three intervals.

Then the cumulative distribution function of the random variable \( y \) can be derived from (4-18). Details are shown in Appendix III.

\[
G(y) = \begin{cases} 
1 - \exp (-y_{\theta_1,\epsilon}), & -\infty < y < \log(\tau_1) \\
1 - \exp (-y_{\theta_2,\epsilon} + \tau_{1,\theta_1,\epsilon} - \tau_{1,\theta_1,\epsilon}), & \log(\tau_1) \leq y < \log(\tau_2) \\
1 - \exp (-y_{\theta_3,\epsilon} + \tau_{2,\theta_2,\epsilon} + \tau_{1,\theta_2,\epsilon} - \tau_{1,\theta_1,\epsilon}), & \log(\tau_2) \leq y < \infty 
\end{cases}
\]

Then, by taking the derivative of the CDF with respect to \( y \), the pdf of \( Y \) is obtained as

\[
f(y) = \begin{cases} 
\frac{1}{\sigma} \exp(y_{\theta_1} - y_{\theta_1,\epsilon}), & -\infty < y < \log(\tau_1) \\
\frac{1}{\sigma} \exp(y_{\theta_2} - y_{\theta_2,\epsilon} + \tau_{1,\theta_1,\epsilon} - \tau_{1,\theta_1,\epsilon}), & \log(\tau_1) \leq y < \log(\tau_2) \\
\frac{1}{\sigma} \exp(y_{\theta_3} - y_{\theta_3,\epsilon} + \tau_{2,\theta_2,\epsilon} + \tau_{1,\theta_2,\epsilon} - \tau_{1,\theta_1,\epsilon}), & \log(\tau_2) \leq y < \infty 
\end{cases}
\]
where \( y_{ij} = \frac{y_j - \log \theta_i}{\sigma} \), \( y_{ij,e} = \exp \left( \frac{y_j - \log \theta_i}{\sigma} \right) \), and \( \tau_{m_{ij},e} = \exp \left( \frac{\log \tau_m - \log \theta_i}{\sigma} \right) \), for \( m = 1, 2 \), and \( i = 1, 2, 3 \).

The MLE method is used to estimate the unknown parameters.

Using the CDF in equation (4-20) and the pdf in equation (4-21), the maximum likelihood function for censored observations \( Y_{ij} \), where \( Y_{ij} = \log(t_{ij}) \), \( i = 1, 2, 3 \), \( j = 1, 2, \ldots, n_i \) is obtained as:

\[
L(\beta_0, \beta_1, \beta_2, \sigma) = \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp \left( y_{ij,0} - y_{ij,\theta_i} \right) \right] \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp \left( y_{ij,\theta_i} - y_{ij,\theta_i,e} + \tau_{1,\theta_i,e} - \tau_{1,\theta_i,e} \right) \right] \\
\times \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp \left( y_{ij,\theta_i} - y_{ij,\theta_i,e} + \tau_{2,\theta_i,e} - \tau_{2,\theta_i,e} + \tau_{1,\theta_i,e} - \tau_{1,\theta_i,e} \right) \right] \\
\times \prod_{j=1}^{n_i} \exp \left( -T_{\theta_i,e} + \tau_{2,\theta_i,e} - \tau_{2,\theta_i,e} + \tau_{1,\theta_i,e} - \tau_{1,\theta_i,e} \right)
\]

where \( y_{ij,0} = \frac{y_j - \log \theta_i}{\sigma} \), \( y_{ij,\theta_i,e} = \exp \left( \frac{y_j - \log \theta_i}{\sigma} \right) \), for \( i = 1, 2, 3 \), \( j = 1, 2, \ldots, n_i \), and \( T_{\theta_i,e} = \exp \left( \frac{\log T_j - \log \theta_3}{\sigma} \right) \).

Then, the logarithm of the likelihood function is obtained as

\[
\log[L(\beta_0, \beta_1, \beta_2, \sigma)] = -(n_1 + n_2 + n_3) \log(\sigma) + \sum_{i=1}^{n_i} \sum_{j=1}^{n_j} (y_{ij,0} - y_{ij,\theta_i,e}) \\
+ (n_2 + n_3 + n_c)(\tau_{1,\theta_i,e} - \tau_{1,\theta_i,e}) + (n_3 + n_c)(\tau_{2,\theta_i,e} - \tau_{2,\theta_i,e}) - n_c T_{\theta_i,e}.
\]

(4-22)

The MLE for \( \beta_0, \beta_1, \beta_2, \sigma \) can be obtained from differentiating the log-likelihood function with respect to these parameters, and setting them equal to zero.

\[
\frac{\partial \log L}{\partial \beta_0} = -\frac{1}{\sigma} \left( n_1 + n_2 + n_3 - \sum_{i=1}^{n_i} \sum_{j=1}^{n_j} y_{ij,\theta_i,e} + (n_2 + n_3 + n_c)(\tau_{1,\theta_i,e} - \tau_{1,\theta_i,e}) \right) = 0
\]

(4-23)
Chapter 4 Bivariate SSALT Test Design

\[
\frac{\partial \log L}{\partial \beta_1} = -\frac{1}{\sigma} \left( n_1 S_{11} + n_2 S_{12} + n_3 S_{12} \right) = 0 \quad (4-24)
\]

\[
\frac{\partial \log L}{\partial \beta_2} = -\frac{1}{\sigma} \left( n_1 S_{21} + n_2 S_{21} + n_3 S_{22} \right) = 0 \quad (4-25)
\]

\[
\frac{\partial \log L}{\partial \sigma} = -\frac{1}{\sigma} \left( n_1 + n_2 + n_3 - \sum_{j=1}^{n} y_{j,\theta_1,e} \cdot y_{j,\theta_1,e} + \sum_{j=1}^{n} y_{j,\theta_1,e} + n_2 + n_3 + n_2 \right) = 0 \quad (4-26)
\]

The expected Fisher information matrix, \( \hat{F}_5 \), can be obtained by taking the expected values of the negative second partial, and mixed partial derivatives of \( \log[L(\beta_0, \beta_1, \beta_2, \sigma)] \) with respect to \( \beta_0, \beta_1, \beta_2, \sigma \). The second partial and mixed partial derivatives of \( \log[L(\beta_0, \beta_1, \beta_2, \sigma)] \) with respect to \( \beta_0, \beta_1, \beta_2, \sigma \) are obtained as follows:

\[
\frac{\partial^2 \log L}{\partial \beta_0^2} = -\frac{1}{\sigma^2} \left( \sum_{j=1}^{n} y_{j,\theta_1,e} + (n_2 + n_3) \tau_{1,\theta_1,e} + n_2 \tau_{2,\theta_1,e} \right) \quad (4-27)
\]

\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} = \frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_0} = -\frac{1}{\sigma^2} \left( S_{11} + n_2 + n_3 \tau_{1,\theta_1,e} - S_{12} (n_2 + n_3) \tau_{1,\theta_1,e} + S_{12} (n_3 + n_2) \tau_{1,\theta_1,e} \right) \quad (4-28)
\]
\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} = \frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} = -\frac{1}{\sigma^2} \left\{ S_{21} \sum_{j=1}^{n_1} y_{j, \beta_0, \epsilon} + S_{21} \sum_{j=1}^{n_2} y_{2, j, \beta_2, \epsilon} + S_{22} \sum_{j=1}^{n_3} y_{j, \beta_0, \epsilon} + S_{21} (n_2 + n_3 + n_\epsilon) (\tau_{1, \beta_0, \epsilon} - \tau_{1, \beta_2, \epsilon}) + S_{22} (n_3 + n_\epsilon) \tau_{2, \beta_2, \epsilon} + S_{22} n_\epsilon T_{\beta_0, \epsilon} \right\} 
\] (4-29)

\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \sigma} = \frac{\partial^2 \log L}{\partial \sigma \partial \beta_0} = -\frac{1}{\sigma^2} \left\{ \sum_{j=1}^{n_1} y_{j, \beta_0, \epsilon} \cdot y_{1, j, \beta_0, \epsilon} + \sum_{j=1}^{n_2} y_{2, j, \beta_2, \epsilon} \cdot y_{2, j, \beta_2, \epsilon} + \sum_{j=1}^{n_3} y_{j, \beta_1, \epsilon} \cdot y_{3, j, \beta_0, \epsilon} + (n_2 + n_3 + n_\epsilon) (\tau_{1, \beta_0, \epsilon} \cdot \tau_{1, \beta_1, \epsilon} - \tau_{1, \beta_2, \epsilon} \cdot \tau_{1, \beta_1, \epsilon}) + (n_3 + n_\epsilon) (\tau_{2, \beta_2, \epsilon} \cdot \tau_{2, \beta_2, \epsilon} - \tau_{2, \beta_1, \epsilon} \cdot \tau_{2, \beta_1, \epsilon}) + S_{12} n_\epsilon T_{\beta_0, \epsilon} \right\} 
\] (4-30)

\[
\frac{\partial^2 \log L}{\partial \beta_1^2} = -\frac{1}{\sigma^2} \left\{ S_{11} S_{21} \sum_{j=1}^{n_1} y_{j, \beta_0, \epsilon} + S_{12} S_{21} \sum_{j=1}^{n_2} y_{2, j, \beta_2, \epsilon} + S_{12} S_{22} \sum_{j=1}^{n_3} y_{j, \beta_0, \epsilon} + (n_2 + n_3 + n_\epsilon) (S_{11} S_{21} \tau_{1, \beta_0, \epsilon} - S_{12} S_{21} \tau_{1, \beta_2, \epsilon}) + (n_3 + n_\epsilon) (S_{12} S_{22} \tau_{2, \beta_2, \epsilon} - S_{12} S_{22} \tau_{2, \beta_2, \epsilon}) + S_{12} S_{22} n_\epsilon T_{\beta_0, \epsilon} \right\} 
\] (4-31)

\[
\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} = \frac{\partial^2 \log L}{\partial \beta_2 \partial \beta_1} = -\frac{1}{\sigma^2} \left\{ S_{11} \sum_{j=1}^{n_1} y_{j, \beta_0, \epsilon} \cdot y_{1, j, \beta_0, \epsilon} + S_{12} \sum_{j=1}^{n_2} y_{j, \beta_2, \epsilon} \cdot y_{2, j, \beta_2, \epsilon} + (n_2 + n_3 + n_\epsilon) (S_{11} \tau_{1, \beta_0, \epsilon} \cdot \tau_{1, \beta_0, \epsilon} - S_{12} \tau_{1, \beta_2, \epsilon} \cdot \tau_{1, \beta_2, \epsilon}) + (n_3 + n_\epsilon) (S_{12} \tau_{2, \beta_2, \epsilon} \cdot \tau_{2, \beta_2, \epsilon} - S_{12} \tau_{2, \beta_2, \epsilon} \cdot \tau_{2, \beta_2, \epsilon}) + S_{12} n_\epsilon T_{\beta_0, \epsilon} \right\} 
\] (4-32)

\[
\frac{\partial^2 \log L}{\partial \beta_1 \partial \sigma} = \frac{\partial^2 \log L}{\partial \sigma \partial \beta_1} = -\frac{1}{\sigma^2} \left\{ S_{11} \sum_{j=1}^{n_1} y_{j, \beta_0, \epsilon} \cdot y_{1, j, \beta_0, \epsilon} + S_{12} \sum_{j=1}^{n_2} y_{j, \beta_2, \epsilon} \cdot y_{2, j, \beta_2, \epsilon} + (n_2 + n_3 + n_\epsilon) (S_{11} \tau_{1, \beta_0, \epsilon} \cdot \tau_{1, \beta_0, \epsilon} - S_{12} \tau_{1, \beta_2, \epsilon} \cdot \tau_{1, \beta_2, \epsilon}) + (n_3 + n_\epsilon) (S_{12} \tau_{2, \beta_2, \epsilon} \cdot \tau_{2, \beta_2, \epsilon} - S_{12} \tau_{2, \beta_2, \epsilon} \cdot \tau_{2, \beta_2, \epsilon}) + S_{12} n_\epsilon T_{\beta_0, \epsilon} \right\} 
\] (4-33)
\[
\frac{\partial^2 \log L}{\partial \beta_2^2} = \frac{-1}{\sigma^2} \left( S_{21} \sum_{j=1}^{n_i} y_{1j, \theta_1} \cdot y_{1j, \theta_2, e} + S_{21} \sum_{j=1}^{n_2} y_{2j, \theta_1} \cdot y_{2j, \theta_2, e} + S_{22} \sum_{j=1}^{n_i} y_{1j, \theta_1} \cdot \tau_{1j, \theta_2, e} - \tau_{1j, \theta_1, e} \right) \\
+ \left( n_i + n_c \right) \left( S_{21} \tau_{1j, \theta_1} \cdot \tau_{1j, \theta_2, e} - \tau_{1j, \theta_1, e} \cdot \tau_{1j, \theta_1, e} \right) + S_{22} n_i T_{\theta_1, e} + S_{22} n_i T_{\theta_2, e} \\
\]

\[
\frac{\partial^2 \log L}{\partial \beta_2 \sigma} = \frac{\partial^2 \log L}{\partial \sigma \beta_2} = \frac{-1}{\sigma^2} \left( S_{21} \sum_{j=1}^{n_i} y_{1j, \theta_1} \cdot y_{1j, \theta_2, e} + S_{21} \sum_{j=1}^{n_2} y_{2j, \theta_1} \cdot y_{2j, \theta_2, e} + S_{22} n_i T_{\theta_1, e} \cdot \tau_{1j, \theta_2, e} \\
+ \left( n_i + n_c \right) \left( S_{21} \tau_{1j, \theta_1} \cdot \tau_{1j, \theta_2, e} - \tau_{1j, \theta_1, e} \cdot \tau_{1j, \theta_2, e} \right) \right) \\
\]

\[
\frac{\partial^2 \log L}{\partial \sigma^2} = \frac{-1}{\sigma^2} \left( \sum_{j=1}^{n_i} y_{1j, \theta_1} \cdot y_{1j, \theta_2, e} + S_{21} \sum_{j=1}^{n_2} y_{2j, \theta_1} \cdot y_{2j, \theta_2, e} \right) \\
+ \left( n_i + n_c \right) \left( \tau_{1j, \theta_1} \cdot \tau_{1j, \theta_2, e} - \tau_{1j, \theta_1, e} \cdot \tau_{1j, \theta_2, e} \right) + \left( n_i + n_c \right) \left( \tau_{2j, \theta_1} \cdot \tau_{2j, \theta_2, e} - \tau_{2j, \theta_1, e} \cdot \tau_{2j, \theta_2, e} \right) \]

where \( T_{\theta_1} = \frac{\log T - \log \theta_3}{\sigma} \).

From equations (4-23) to (4-26), we can get:

\[
n_1 = \sum_{j=1}^{n_i} y_{1j, \theta_1, e} + \left( n_2 + n_3 + n_c \right) \tau_{1j, \theta_1, e} \]

\[
n_2 = \sum_{j=1}^{n_2} y_{2j, \theta_2, e} - \left( n_2 + n_3 + n_c \right) \tau_{1j, \theta_2, e} + \left( n_3 + n_c \right) \tau_{2j, \theta_1, e} \]

\[
n_3 = \sum_{j=1}^{n_3} y_{3j, \theta_1, e} - \left( n_2 + n_c \right) \tau_{2j, \theta_1, e} + n_c T_{\theta_1, e} \]

Equations (4-37) to (4-39) are then used to simplify the second partial and mixed partial derivatives of \( \log[L(\beta_0, \beta_1, \beta_2, \sigma)] \) with respect to \( \beta_0, \beta_1, \beta_2, \sigma \) in equation (4-27).
to (4-36). And also, to simplify the second partial and mixed partial derivatives, the following definition are made:

\[
\Lambda_{jk} = \tau_{j,\theta_k} \tau_{j,\theta_k}, \quad \Omega_{jk} = \tau_{j,\theta_k}^2 \tau_{j,\theta_k}, \quad j = 1,2, \quad k = j, j+1, \quad \Lambda_T = T_{\theta_k} T_{\theta_k}, \quad \Omega_T = T_{\theta_k}^2 T_{\theta_k},
\]

and

\[
\Phi_i = E\left[ \frac{1}{n} \sum_{j=1}^{n} y_{ij,\theta_k} y_{ij,\theta_k} \right], \quad \Psi_i = E\left[ \frac{1}{n} \sum_{j=1}^{n} y_{ij,\theta_k}^2 y_{ij,\theta_k} \right], \quad i = 1,2,3, \quad C_i = E\left[ \frac{n_i}{n} \right], \quad i = 1,2,3.
\]

The detailed calculation of \( \Phi_i, \Psi_i \) and \( C_i \), \( i = 1,2,3 \) is presented in Appendix IV.

Therefore, the expected Fisher information matrix \( \hat{F}_5 \) is obtained by taking the expected values of the negative second partial and mixed partial derivatives of \( \log[L(\beta_0, \beta_1, \beta_2, \sigma)] \) with respect to \( \beta_0, \beta_1, \beta_2, \sigma \) in equations (4-27) to (4-36).

\[
\hat{F}_5 = \frac{n}{\sigma^2} \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{12} & A_{22} & A_{23} & A_{24} \\
A_{13} & A_{23} & A_{33} & A_{34} \\
A_{14} & A_{24} & A_{34} & A_{44}
\end{bmatrix}
\tag{4-40}
\]

where the elements of \( \hat{F}_5 \) are given as.

\[
A_{11} = E\left[ -\frac{\sigma^2}{n} \frac{\partial^2 \log L}{\partial \beta_0} \right] = C_1 + C_2 + C_3,
\]

\[
A_{12} = E\left[ -\frac{\sigma^2}{n} \frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} \right] = C_1 S_{11} + C_2 S_{11} + C_3 S_{21},
\]

\[
A_{13} = E\left[ -\frac{\sigma^2}{n} \frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} \right] = C_1 S_{21} + C_2 S_{21} + C_3 S_{22},
\]

\[
A_{14} = E\left[ -\frac{\sigma^2}{n} \frac{\partial^2 \log L}{\partial \beta_0 \partial \sigma} \right] = \Phi_1 + \Phi_2 + \Phi_3 + (1 - C_1)(\Lambda_{11} - \Lambda_{12}) + (1 - C_1 - C_2)(\Lambda_{22} - \Lambda_{23}) + (1 - C_1 - C_2 - C_3)\Lambda_T
\]
After the expected Fisher information matrix is obtained, preliminary estimates of the parameters $\theta_1, \theta_2, \theta_3, \delta$ are used to design the optimal test plan for a censored experiment with two stress variables. These preliminary estimates of parameters can be obtained from past experience of similar products, and/or from some target MTTF values or from a small sample experiment. To design the optimal test plan, first the optimal criterion must be determined. The optimality criterion used in the model is discussed next.
4.2.3 Optimality Criterion & Test Design

Considering the test objective of the model, our optimality criterion is defined to minimize the AV of the MLE of the logarithm of the life with the Weibull distribution, and a specified reliability \( R \).

The MLE of the logarithm of the Weibull distribution life with a specified reliability \( R \) at typical operating conditions with stress levels \((S_{10}, S_{20})\) is

\[
\hat{Y}_{R,(S_{10}, S_{20})} = \log(t_{R,(S_{10}, S_{20})}) = \log(\hat{\theta}_0) + \log(-\log(R)) \cdot \hat{\sigma} = \beta_0 + \beta_1 S_{10} + \beta_2 S_{20} + \log(-\log(R)) \cdot \hat{\sigma} \tag{4-41}
\]

When \( R = 0.3679 \), \( \hat{Y}_{R,(S_{10}, S_{20})} \) is the logarithm of the characteristic life at a typical operating conditions with stress levels \((S_{10}, S_{20})\). When \( R = 0.5 \), \( \hat{Y}_{R,(S_{10}, S_{20})} \) is the logarithm of the median life at typical operating conditions with stress levels \((S_{10}, S_{20})\). We use \( R = 0.5 \) for the following calculation & optimization.

The optimality criterion used for SSALT design is to determine the optimal values of \( \tau_1 \), and \( \tau_2 \), while minimizing the AV of \( \hat{Y}_{R,(S_{10}, S_{20})} \) in (4-41). This criterion is commonly used in the SSALT planning problems.

\[
n \cdot AV[\hat{Y}_{R,(S_{10}, S_{20})}] = n \cdot AV[\beta_0 + \beta_1 S_{10} + \beta_2 S_{20} + \log(-\log(R)) \cdot \hat{\sigma}] = n \cdot \left[ \frac{\partial \hat{Y}_{R,(S_{10}, S_{20})}}{\partial \beta_0} \quad \frac{\partial \hat{Y}_{R,(S_{10}, S_{20})}}{\partial \beta_1} \quad \frac{\partial \hat{Y}_{R,(S_{10}, S_{20})}}{\partial \beta_2} \quad \frac{\partial \hat{Y}_{R,(S_{10}, S_{20})}}{\partial \sigma} \right] \hat{F}_5^{-1} \\
= n \cdot \left[ \frac{\partial \hat{Y}_{R,(S_{10}, S_{20})}}{\partial \beta_0} \quad \frac{\partial \hat{Y}_{R,(S_{10}, S_{20})}}{\partial \beta_1} \quad \frac{\partial \hat{Y}_{R,(S_{10}, S_{20})}}{\partial \beta_2} \quad \frac{\partial \hat{Y}_{R,(S_{10}, S_{20})}}{\partial \sigma} \right]^T \hat{F}_5^{-1} \left[ \begin{array}{c} \beta_0 \\ \beta_1 \\ \beta_2 \\ \sigma \end{array} \right]
\]

\[
= n \cdot \left[ \begin{array}{cccc} 1 & S_{10} & S_{20} & \log(-\log(R)) \end{array} \right] \cdot \hat{F}_5^{-1} \cdot \left[ \begin{array}{cccc} \beta_0 \\ \beta_1 \\ \beta_2 \\ \sigma \end{array} \right]^T \tag{4-42}
\]

where \( \hat{F}_5^{-1} \) is the inverse of the expected Fisher information matrix in equation (4-40).
Then the problem objective is to

\[
\text{minimize } n \cdot AV\left[\hat{Y}_{r,(s_{i0},s_{j0})}(t_1, t_2)\right]. \tag{4-43}
\]

Because the derivative information of the asymptotic variance is unavailable, a pattern search algorithm is used for the minimization problem. Pattern search is an important class of direct search optimizers [15, 36, 37, 59, and 77]. Pattern search methods are closely related to experimental design, and are provably convergent. It is widely believed that pattern searches are less likely than derivative-based methods to be trapped by local minimums. Pattern search can be customized to cast a wide range when searching for better points, so that the search may be able to escape a shallow basin that contains a local minimum, and find a deeper basin that contains the global minimum. For more details on global optimization achieved from pattern search, the reader may consult [15, 59]. The next section presents a numerical example to demonstrate the calculation of the optimal hold times, and sensitivity analysis.

### 4.2.4 Numerical Examples and Sensitivity Analysis

In this section, numerical examples are given. The pattern search algorithm is developed to calculate the optimal hold times of the bivariate SSALT. Since the derivative information of the AV is unavailable, a pattern search algorithm is used for the minimization problem.

#### 4.2.4.1 Numerical Example

Teng & Kolarik [70] designed & performed a constant-stress accelerated life test on DC motors, comparing on/off cycling to continuous operation under multiple stresses. The
experiment was conducted for the motors to run until they ceased. The voltage, load, and operation mode were considered to be stress factors. The Weibull log-linear model with a constant shape parameter was used for the failure data analysis.

Mettas [48] illustrated similar experimental data for a typical three stress type CSALT. Censoring was considered in this experiment; and the temperature, voltage, and operation type were considered to be stress factors. The Weibull log-linear model with constant shape parameter was also used for the failure data analysis. Three scenarios were considered in their constant-stress tests: (i) only the temperature effect, (ii) two stress variables (temperature and voltage), and (iii) all three variables (temperature, voltage, and operation type) were considered.

The following example uses the experimental data of Mettas [48] for the two stress variables (temperature and voltage) to obtain the initial estimates of the unknown parameters. Table 4-2 shows the results of a constant-stress accelerated life test under two stress variables.

<table>
<thead>
<tr>
<th>Temperature (K)</th>
<th>Voltage (V)</th>
<th>Failure times</th>
</tr>
</thead>
<tbody>
<tr>
<td>358</td>
<td>12</td>
<td>445, 498, 586, 691, 750. 20 Units Suspended at 750.</td>
</tr>
<tr>
<td>378</td>
<td>12</td>
<td>176, 211, 252, 266, 298, 309, 343, 364, 387, 398. 14 Units Suspended at 445.</td>
</tr>
<tr>
<td>378</td>
<td>16</td>
<td>118, 145, 163, 192, 208, 210, 231, 249, 254, 293. 10 Units Suspended at 300.</td>
</tr>
</tbody>
</table>

It is also known that the typical operating temperature is 328K, and the typical voltage level is 10V.

Transformed stresses were used in the log-linear life-stress relationship. The Arrhenius relationship was used for transformation of the temperature stress, and the
Inverse Power for transformation of the voltage stress: \( S_1 = \frac{1}{Kelvin \ temperature} \), and \( S_2 = \log \) (voltage). Then the log-linear model with constant shape parameter was used:

\[
\log(\theta) = \beta_0 + \beta_1 S_1 + \beta_2 S_2;
\]

and the best fit values for the parameters were obtained [48]:

\[
\delta = 2.9479; \; \beta_0 = -5.9204; \; \beta_1 = 6071.43; \; \beta_2 = -1.5604.
\]

Note that these estimates are merely used as initial estimates of Weibull parameters. Therefore, the pre-estimated parameters are obtained from equations (4-17)-(4-19):

\[
\theta_1 = 1288.7826; \; \theta_2 = 525.3873; \; \theta_3 = 335.3708; \; \delta = 2.9479.
\]

Sensitivity of the optimal hold times, \( \tau_1^* \) & \( \tau_2^* \), with respect to these parameters, are analyzed in the following section. Having the pre-estimates of the scale parameters, and shape parameter, we can proceed to obtain the optimal SSALT plan with two stress variables from (4-43). Table 4-3 presents the optimal stress change times for different test censoring times \( T \).

**Table 4-3  Results for Optimal Bivariate SSALT Plan**

<table>
<thead>
<tr>
<th>Censoring Time T</th>
<th>Optimal Stress Change Time</th>
<th>nAV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \tau_1^* )</td>
<td>( \tau_2^* )</td>
</tr>
<tr>
<td>2500</td>
<td>1538.00</td>
<td>1558.31</td>
</tr>
<tr>
<td>1500</td>
<td>1461.43</td>
<td>1482.75</td>
</tr>
<tr>
<td>1000</td>
<td>984.14</td>
<td>995.77</td>
</tr>
</tbody>
</table>
4.2.4.2 Sensitivity Analysis

The sensitivity analysis identifies the sensitive parameters which need to be estimated with special care for the purpose of minimizing the risk of obtaining an erroneous optimal solution. The effect of a 1% change in the pre-estimated parameters \( \theta_1, \theta_2, \theta_3, \) and \( \delta \) on \( \tau_1^* \), and \( \tau_2^* \) are presented in Table 4-4.

<table>
<thead>
<tr>
<th></th>
<th>1% change in ( \theta_1 )</th>
<th>1% change in ( \theta_2 )</th>
<th>1% change in ( \theta_3 )</th>
<th>1% change in ( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( % \Delta \tau_1^* )</td>
<td>0.01144</td>
<td>-0.00042</td>
<td>0.00242</td>
<td>-0.00995</td>
</tr>
<tr>
<td>( % \Delta \tau_2^* )</td>
<td>0.01108</td>
<td>-0.00006</td>
<td>0.00249</td>
<td>-0.01081</td>
</tr>
</tbody>
</table>

To examine the sensitivity of optimal values of \( \tau_1^* \), and \( \tau_2^* \) with respect to changes in the pre-estimated parameters \( \theta_1, \theta_2, \theta_3, \) and \( \delta \), sensitivity analysis with respect to each parameter is demonstrated by Figure 4-4 for a censoring time of 2500.

The sensitivity analysis indicates, that because these parameters have a very small effect (less than 2%) on the optimal values, \( \tau_1^* \), and \( \tau_2^* \), they are not sensitive. Therefore, the proposed optimum plan is robust, and the initial estimates have a small effect on optimal values.
Figure 4-4  Sensitivity of Optimal $\tau_1$, and $\tau_2$, with respect to $\theta_1$, $\theta_2$, $\theta_3$, and $\delta$
CHAPTER 5

Multi-Variate SSALT Analysis and Design

In previous chapter, we discuss the analysis and design for bivariate SSALT under exponential and Weibull distribution assumption. Bivariate SSALT model considers only two stress variables, each with two stress levels, changing at different time. The bivariate SSALT model is extended to a more generalized model: multi-variate SSALT model.

In this chapter, we present an optimal k-step, m-variable SSALT design for censored Weibull failure data. The optimal test plan is developed to determine the stress change times for each step. The optimal stress change times are obtained by minimization of the AV of the MLE of the $p$-percentile life at usual operating stresses. Multiple steps and multiple stress variables are considered for a test plan with Weibull time to failure test data. In many applications, more than one stress variable may impact the performance and hence the life of a given product. Therefore, it is desirable to use more than one stress variable and more than one stress level change.

5.1 Multi-Variate SSALT Model

This section proposed the multi-variate SSALT model. Test procedure is presented first, followed by the model assumptions and parameter estimation.
5.1.1 Test Procedure

The following procedures are used in $k$-step, $m$-stress variable SSALT.

1) All $n$ test units are initially placed on a lower stress level ($S_{1,1}$, $S_{2,1}$, $\ldots$, $S_{m,1}$) and run until time $\tau_1$ when the stress level is changed to ($S_{1,2}$, $S_{2,2}$, $\ldots$, $S_{m,2}$).

2) The test is continued until time $\tau_2$ when the stress level is changed to ($S_{1,3}$, $S_{2,3}$, $\ldots$, $S_{m,3}$), and so on.

3) The test is continued until time $\tau_i$ when the stress level is changed to ($S_{1,i+1}$, $S_{2,i+1}$, $\ldots$, $S_{m,i+1}$), for $i = 1, 2, \ldots, k - 1$.

4) The test is continued at the step $k$, with stress level ($S_{1,k}$, $S_{2,k}$, $\ldots$, $S_{m,k}$), until all units fail or until a predetermined censoring time $T$, whichever occurs first. $n_i$ failures are observed at time $t_{ij}$, $j = 1, 2, \ldots, n_i$ in step $i$, $i = 1, 2, \ldots, k$, and $n_c$ units are censored, $n_c = n - \sum_{i=1}^{k} n_j$.

Figure 5-1 demonstrates the test procedure for the multi-variate SSALT model, where $S_{l,i}$ is the stress level for variable $l$ at step $i$, for $l = 1, 2, \ldots, m$, and $i = 1, 2, \ldots, k$, and ($S_{1,0}$, $S_{2,0}$, $\ldots$, $S_{m,0}$) are stress levels at usual operating conditions.
5.1.2 Model Assumptions

The following assumptions are made in multi-variate SSALT model:

(1) The time to failure of the test unit for a set of constant stress levels \((S_{1,i}, S_{2,i}, \ldots, S_{m,i})\), \(i=1,2,\ldots,k\), follows a Weibull distribution.

\[
F_i(t) = 1 - \exp \left( -\frac{t^{\delta}}{\theta_i^{\delta}} \right), \quad 0 \leq t < \infty.
\]  

Where \(\delta\) is the shape parameter, and \(\theta_i\) is the scale parameter, \(\delta > 1, \ \theta_i \geq 0\).

(2) The scale parameter \(\theta_i\) at step \(i\) is a log-linear function of stress \((S_{1,i}, S_{2,i}, \ldots, S_{m,i})\).

\[
\log(\theta_i) = \beta_0 + \beta_1 S_{1,i} + \beta_2 S_{2,i} + \cdots + \beta_m S_{m,i} = \beta_0 + \sum_{j=1}^{m} \beta_j S_{j,i}.
\]  

Where \(\beta_0, \beta_1, \beta_2, \ldots, \beta_m\) are unknown parameters depending on the nature of the product and the test method. It is assumed that there is no interaction between the stress variables.
(3) The shape parameter $\delta$ is assumed to be constant for all stress levels.

(4) The cumulative distribution function of the time to failure of a test unit follows the Khamis-Higgins model:

$$
F(t) = 1 - \left\{ \begin{array}{ll}
\exp \left( - \frac{t^\delta}{\theta_1^\delta} \right) & 0 \leq t < \tau_1 \\
\exp \left( - \frac{t^\delta - \tau_1^\delta}{\theta_2^\delta} - \frac{\tau_1^\delta}{\theta_1^\delta} \right) & \tau_1 \leq t < \tau_2 \\
\exp \left( - \frac{t^\delta - \tau_2^\delta}{\theta_3^\delta} - \frac{\tau_2^\delta - \tau_1^\delta}{\theta_2^\delta} - \frac{\tau_1^\delta}{\theta_1^\delta} \right) & \tau_2 \leq t < \tau_3 \\
\vdots & \vdots \\
\exp \left( - \frac{t^\delta - \tau_k^\delta}{\theta_k^\delta} - \sum_{i=2}^{k-1} \frac{\tau_i^\delta - \tau_{i-1}^\delta}{\theta_i^\delta} - \frac{\tau_{i-1}^\delta}{\theta_{i-1}^\delta} \right) & \tau_{k-1} \leq t < \infty
\end{array} \right.
$$

(5-3)

5.1.3 Parameter Estimation

The CDF of time to failure based on the Weibull distribution and Khamis-Higgins model is derived first. Then, from the likelihood function of time to failure, the unknown parameters are estimated.

Let $y = \log(t)$, $\sigma = 1/\delta$. The CDF of the random variable $y$ can then be derived from equation (5-3).

$$
G(y) = F(e^y) = 1 - \left\{ \begin{array}{ll}
\exp \left( - y_{\theta_1,e} \right) & -\infty < y < \log(\tau_1) \\
\exp \left( - y_{\theta_1,e} + \tau_{1,\theta_2,e} - \tau_{1,\theta_1,e} \right) & \log(\tau_1) \leq y < \log(\tau_2) \\
\exp \left( - y_{\theta_1,e} + \tau_{2,\theta_2,e} - \tau_{2,\theta_1,e} \right) & \log(\tau_2) \leq y < \log(\tau_3) \\
\vdots & \vdots \\
\exp \left( - y_{\theta_k,e} + \sum_{i=1}^{k-1} \tau_{i,\theta_{i+1},e} - \tau_{i,\theta_i,e} \right) & \log(\tau_{k-1}) \leq y < \infty
\end{array} \right.
$$

(5-4)

where $y_{\theta_i,e} = \exp \left( \frac{y - \log \theta_i}{\sigma} \right)$, and $\tau_{i,\theta_j,e} = \exp \left( \frac{\log \tau_i - \log \theta_j}{\sigma} \right)$
The pdf of the random variable $y$ is obtained by taking the first derivative of equation (5-4):

$$f(y) = \begin{cases} \frac{1}{\sigma} \exp \left( y_{\theta_1} - y_{\theta_1,e} \right) & -\infty < y < \log(\tau_1) \\ 1 \frac{1}{\sigma} \exp \left( y_{\theta_2} - y_{\theta_2,e} + \tau_{1,\theta_1,e} - \tau_{1,\theta_1,e} \right) & \log(\tau_1) \leq y < \log(\tau_2) \\ \vdots & \vdots \\ 1 \frac{1}{\sigma} \exp \left( y_{\theta_k} - y_{\theta_1,e} + \sum_{i=1}^{k-1} \tau_{i,\theta_{i+1},e} - \tau_{i,\theta_{i+1},e} \right) & \log(\tau_{k-1}) \leq y < \infty \end{cases}$$

(5-5)

where $y_{\theta_i} = \frac{y - \log \theta_i}{\sigma}$.

Next, the MLE method is used to estimate the unknown parameters.

The likelihood function from observations $y_{ij}$, where $Y_{ij} = \log(t_{ij})$, $i = 1, 2, \cdots, k$, $j = 1, 2, \ldots, n_i$, is derived as follows:

$$L(\beta_0, \beta_1, \beta_2, \cdots, \beta_m, \sigma) = \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp \left( y_{ij,\theta_1} - y_{ij,\theta_1,e} \right) \right]$$

$$\cdot \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp \left( y_{ij,\theta_2} - y_{ij,\theta_2,e} + \tau_{1,\theta_1,e} - \tau_{1,\theta_1,e} \right) \right]$$

$$\cdot \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp \left( y_{ij,\theta_3} - y_{ij,\theta_3,e} + \tau_{2,\theta_2,e} - \tau_{2,\theta_2,e} + \tau_{1,\theta_1,e} - \tau_{1,\theta_1,e} \right) \right]$$

$$\cdots$$

$$\cdot \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp \left( y_{ij,\theta_k} - y_{ij,\theta_1,e} + \sum_{i=1}^{k-1} \tau_{i,\theta_{i+1},e} - \tau_{i,\theta_{i+1},e} \right) \right]$$

$$\cdot \prod_{j=1}^{n_i} \exp \left( -T_{\theta_1,e} + \sum_{i=1}^{k-1} (\tau_{i,\theta_{i+1},e} - \tau_{i,\theta_{i+1},e}) \right)$$

where $y_{ij,\theta_i} = \frac{y_{ij} - \log \theta_i}{\sigma}$, $y_{ij,\theta_1,e} = \exp \left( \frac{y_{ij} - \log \theta_i}{\sigma} \right)$, $i = 1, 2, \cdots, k$, $j = 1, 2, \ldots, n_i$ and $T_{\theta_1,e} = \exp \left( \frac{\log T - \log \theta_k}{\sigma} \right)$. 
Chapter 5 Multi-Variate SSALT Analysis and Design

The logarithm of the likelihood function is obtained as follows:

\[
\log[L(\beta_0, \beta_1, \beta_2, \ldots, \beta_m, \sigma)] = -\sum_{i=1}^{k} n_i \log(\sigma) + \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij, \theta_i} - y_{ij, \theta_i,e}) - n \cdot T_{\theta_i,e} \\
+ \sum_{i=1}^{k-1} (\tau_{i, \theta_i,e} - \tau_{i, \theta_i,e,e}) \cdot \left(n - \sum_{r=1}^{i} n_r\right).
\]  

(5-6)

Taking the partial derivative of the log-likelihood function in equation (5-6) with respect to each one of the parameters and setting them equal to zero yields the MLE for \(\beta_0, \beta_1, \beta_2, \ldots, \beta_m, \sigma\), shown in the following equations.

\[
\frac{\partial \log L}{\partial \beta_0} = -\frac{1}{\sigma} \left(\sum_{i=1}^{k} n_i - \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij, \theta_i,e} - n \cdot T_{\theta_i,e} \right) \left(n - \sum_{r=1}^{i} n_r\right) = 0, \quad (5-7)
\]

\[
\frac{\partial \log L}{\partial \beta_1} = -\frac{1}{\sigma} \left(\sum_{i=1}^{k} S_{ii} n_i - \sum_{i=1}^{k} \sum_{j=1}^{n_i} S_{ii} y_{ij, \theta_i,e} - n \cdot S_{ik} T_{\theta_i,e} \right) \left(n - \sum_{r=1}^{i} n_r\right) = 0, \quad (5-8)
\]

\[
\frac{\partial \log L}{\partial \beta_2} = -\frac{1}{\sigma} \left(\sum_{i=1}^{k} \sum_{j=1}^{n_i} S_{ij} y_{ij, \theta_i,e} - n \cdot S_{ik} T_{\theta_i,e} \right) \left(n - \sum_{r=1}^{i} n_r\right) = 0, \quad (5-9)
\]

\[
\frac{\partial \log L}{\partial \beta_m} = -\frac{1}{\sigma} \left(\sum_{i=1}^{k} \sum_{j=1}^{n_i} S_{mj} y_{ij, \theta_i,e} - n \cdot S_{mk} T_{\theta_i,e} \right) \left(n - \sum_{r=1}^{i} n_r\right) = 0, \quad (5-10)
\]

\[
\frac{\partial \log L}{\partial \sigma} = -\frac{1}{\sigma} \left(\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij, \theta_i} - y_{ij, \theta_i,e}) - n \cdot T_{\theta_i} T_{\theta_i,e} \right) \left(n - \sum_{r=1}^{i} n_r\right) = 0, \quad (5-11)
\]
From equations (5-7) to (5-11), we can get:

\[ n_1 = \sum_{j=1}^{n_1} y_{1j,\theta_1,e} + (n - n_1)r_{1,\theta_1,e} \]  \hspace{1cm} (5-12)

\[ n_2 = \sum_{j=1}^{n_1} y_{2j,\theta_2,e} - (n - n_1)r_{1,\theta_1,e} + (n - n_1 - n_2)r_{2,\theta_2,e} \]  \hspace{1cm} (5-13)

\[ n_3 = \sum_{j=1}^{n_1} y_{3j,\theta_3,e} - (n - n_1 - n_2)r_{2,\theta_2,e} + (n - n_1 - n_2 - n_3)r_{3,\theta_3,e} \]  \hspace{1cm} (5-14)

\[ \vdots \]

\[ n_k = \sum_{j=1}^{n_k} y_{kj,\theta_k} - \left( n - \sum_{r=1}^{k-1} n_r \right)r_{k-1,\theta_k,e} + n_c T_{k,\theta_k,e} \]  \hspace{1cm} (5-15)

Equations (5-12) to (5-15) are then used to simplify the second partial and mixed partial derivatives of \( \log L(\beta_0, \beta_1, \beta_2, \ldots, \beta_m, \sigma) \) with respect to \( \beta_0, \beta_1, \beta_2, \ldots, \beta_m, \sigma \) for deriving the Fisher information matrix.

### 5.2 Optimal Multi-Variate SSALT Design

The optimal SSALT design with \( k \)-steps and \( m \)-variables is derived in this section. Fisher information matrix is first derived from the MLE and the second partial and mixed partial derivatives. The AV of MLE of the logarithm of the \( p \)-percentile life with the Weibull distribution at usual operating stress level \( (S_{1,0}, S_{2,0}, \ldots, S_{m,0}) \) is then obtained from the Fisher information matrix. Optimal test design is then derived by minimizing the AV.
5.2.1 Fisher Information Matrix

The expected Fisher information matrix, $\hat{F}_e$, can be obtained by taking the expected values of the negative second partial, and mixed partial derivatives of $\log[L(\beta_0, \beta_1, \beta_2, \ldots, \beta_m, \sigma)]$ with respect to $\beta_0, \beta_1, \beta_2, \ldots, \beta_m, \sigma$. The second partial and mixed partial derivatives of $\log[L(\beta_0, \beta_1, \beta_2, \ldots, \beta_m, \sigma)]$ with regard to parameters $\beta_0, \beta_1, \beta_2, \ldots, \beta_m, \sigma$ are obtained as follows:

\[
\frac{\partial^2 \log L}{\partial \beta_0^2} = -\frac{1}{\sigma^2} \sum_{i=1}^{k} n_i \tag{5-16}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} = -\frac{1}{\sigma^2} \sum_{i=1}^{k} n_i S_{1i} \tag{5-17}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} = -\frac{1}{\sigma^2} \sum_{i=1}^{k} n_i S_{2i} \tag{5-18}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_m} = -\frac{1}{\sigma^2} \sum_{i=1}^{k} n_i S_{mi} \tag{5-19}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \sigma} = \frac{1}{\sigma^2} \left[ \sum_{i=1}^{k} \sum_{j=1}^{q_i} y_{i,j,t_0} \cdot y_{i,j,t_0,e} + n \cdot t_{0,t_0,e} \right] \tag{5-20}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_1^2} = -\frac{1}{\sigma^2} \sum_{i=1}^{k} n_i S_{1i}^2 \tag{5-21}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} = -\frac{1}{\sigma^2} \sum_{i=1}^{k} n_i S_{1i} S_{2i} \tag{5-22}
\]
\[
\frac{\partial^2 \log L}{\partial \beta \partial \beta} = -\frac{1}{\sigma^2} \sum_{i=1}^{k} n_i S_{ii}\quad (5-23)
\]

\[
\frac{\partial^2 \log L}{\partial \beta \partial \sigma} = -\frac{1}{\sigma^2} \left[ \sum_{i=1}^{k} \sum_{j=1}^{n} S_{ij} y_{ij} \cdot y_{ij, e} + n_c S_{ik} T_{\theta_k} T_{\theta_e} \right]
\]

\[
\frac{\partial^2 \log L}{\partial \beta \partial \sigma} = -\frac{1}{\sigma^2} \left[ \sum_{i=1}^{k-1} \left( S_{u,i+1} \tau_{i,\theta_{u,i}} \tau_{i,\theta_{u,i+1}} - S_{u,i} \tau_{i,\theta_{u,i}} \tau_{i,\theta_{u,i}} \right) \left( n - \sum_{r=1}^{i} n_r \right) \right]
\]

In general, for \( u = 1, 2, \ldots, m \), \( v = 1, 2, \ldots, m \)

\[
\frac{\partial^2 \log L}{\partial \beta_u \partial \beta_v} = -\frac{1}{\sigma^2} \sum_{i=1}^{k} n_i S_{uv}\quad (5-26)
\]

\[
\frac{\partial^2 \log L}{\partial \beta_u \partial \sigma} = -\frac{1}{\sigma^2} \left[ \sum_{i=1}^{k} \sum_{j=1}^{n} S_{ui} y_{ij} \cdot y_{ij, e} + n_c S_{uk} T_{\theta_k} T_{\theta_e} \right]
\]

\[
\frac{\partial^2 \log L}{\partial \beta_u \partial \sigma} = -\frac{1}{\sigma^2} \left[ \sum_{i=1}^{k-1} \left( S_{u,i+1} \tau_{i,\theta_{u,i}} \tau_{i,\theta_{u,i+1}} - S_{u,i} \tau_{i,\theta_{u,i}} \tau_{i,\theta_{u,i}} \right) \left( n - \sum_{r=1}^{i} n_r \right) \right]
\]

\[
\frac{\partial^2 \log L}{\partial \sigma^2} = -\frac{1}{\sigma^2} \left[ \sum_{i=1}^{k} n_i + \sum_{i=1}^{k} \sum_{j=1}^{n} y_{ij, e}^2 + n_c T_{\theta_k} T_{\theta_e} \right]
\]

\[
\frac{\partial^2 \log L}{\partial \sigma^2} = -\frac{1}{\sigma^2} \left[ \sum_{i=1}^{k-1} \left( \tau_{i,\theta_{u,i}}^2 \tau_{i,\theta_{u,i+1}}^2 - \tau_{i,\theta_{u,i}}^2 \tau_{i,\theta_{u,i+1}}^2 \right) \left( n - \sum_{r=1}^{i} n_r \right) \right]
\]

Equations (5-16) to (5-28) are then used to derive the expected Fisher information matrix.

And also, to simplify the elements of the expected Fisher information matrix, the following definition are made:

\[
D_1 = C_r T_{\theta_k} T_{\theta_e} - \sum_{i=1}^{k-1} \left( \tau_{i,\theta_{u,i}} \tau_{i,\theta_{u,i+1}} - \tau_{i,\theta_e} \tau_{i,\theta_e} \right) \left( 1 - \sum_{r=1}^{i} C_r \right)\quad \text{for } u = 1, 2, \ldots, m
\]

\[
D_{u+1} = C_r S_{u,k} T_{\theta_k} T_{\theta_e} - \sum_{i=1}^{k-1} \left( S_{u,i+1} \tau_{i,\theta_{u,i}} \tau_{i,\theta_{u,i+1}} - S_{u,i} \tau_{i,\theta_e} \tau_{i,\theta_e} \right) \left( 1 - \sum_{r=1}^{i} C_r \right)
\]
\[ D_{m+2} = C_T T_{i,t}^2 T_{0,e} - \sum_{i=1}^{k-1} (\tau_{i,0,i}^2 \tau_{i,t,0,i}^2 - \tau_{i,0,t,i}^2 \tau_{i,t,0,i}^2) \left( 1 - \sum_{r=1}^{l} C_r \right) \]

\[ \Phi_i = E \left[ \frac{1}{n} \sum_{j=1}^{n} y_{j,0,i}^2 y_{j,0,e} \right], \quad \Psi_i = E \left[ \frac{1}{n} \sum_{j=1}^{n} y_{j,0,i}^2 y_{j,0,e} \right], \]

and \( C_i = E \left[ \frac{n_i}{n} \right], \) for \( i = 1, 2, \cdots, k \), \( C_T = E \left[ \frac{n_e}{n} \right] \).

For the detailed calculation of \( \Phi_i, \Psi_i, C_i, i = 1, 2, \cdots, k \), please refer to Appendix V.

The expected Fisher information matrix \( \hat{F}_6 \) is derived as following:

\[
\hat{F}_6 = \frac{n}{\sigma^2} \begin{bmatrix}
A_{1,1} & A_{1,2} & A_{1,3} & \cdots & A_{1,m+1} & A_{1,m+2} \\
A_{1,2} & A_{2,2} & A_{2,3} & \cdots & A_{2,m+1} & A_{2,m+2} \\
A_{1,3} & A_{2,3} & A_{3,3} & \cdots & A_{3,m+1} & A_{3,m+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
A_{1,m+1} & A_{2,m+1} & A_{3,m+1} & \cdots & A_{m+1,m+1} & A_{m+1,m+2} \\
A_{1,m+2} & A_{2,m+2} & A_{3,m+2} & \cdots & A_{m+1,m+2} & A_{m+2,m+2}
\end{bmatrix}
\]

(5-29)

where the elements of \( \hat{F}_6 \) are the expected values of the negative second partial and mixed partial derivatives of \( \log[L(\beta_0, \beta_1, \beta_2, \cdots, \beta_m, \sigma)] \) with respect to \( \beta_0, \beta_1, \beta_2, \cdots, \beta_m, \sigma \), shown as follows:

\[ A_{1,1} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \beta_0^2} \right) \right] = \sum_{i=1}^{k} C_i, \]

\[ A_{1,t+1} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_t} \right) \right] = \sum_{i=1}^{k} C_i S_{t,i}, \] for \( t = 1, 2, \cdots, m, \)

\[ A_{1,m+2} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \beta_0 \partial \sigma} \right) \right] = \sum_{i=1}^{k} \Phi_i + D_1, \]

And for \( u = 1, 2, \cdots, m, \) \( v = 1, 2, \cdots, m \)
\[ A_{u+1,v+1} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \beta_i \partial \beta_j} \right) \right] = \sum_{i=1}^{k} C_i S_{u,i} S_{v,j}, \]

\[ A_{u+1,m+2} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \beta_i \partial \sigma} \right) \right] = \sum_{i=1}^{k} S_{u,i} \Phi_i + D_{u+1}, \]

\[ A_{m+2,m+2} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \sigma^2} \right) \right] = \sum_{i=1}^{k} (C_i + \Psi_i) + D_{m+2} \]

The expected Fisher information matrix is then used in obtaining the AV of the MLE of the logarithm of the \( p \)-percentile life with the Weibull distribution at usual operating stress level \( (S_{1,0}, S_{2,0}, \ldots, S_{m,0}) \) in the following section. And the optimization criterion is proposed.

### 5.2.2 Optimization Criterion

The optimization criterion used here is to minimize the AV of the MLE of the logarithm of the \( p \)-percentile life with the Weibull distribution at usual operating stress level \( (S_{1,0}, S_{2,0}, \ldots, S_{m,0}) \).

The MLE of logarithm of the \( p \)-percentile life for the Weibull distribution at usual operating stress level \( (S_{1,0}, S_{2,0}, \ldots, S_{m,0}) \) is obtained as follows:

\[
\hat{\theta}_{p(S_{1,0}, S_{2,0}, \ldots, S_{m,0})} = \log(t_{p(S_{1,0}, S_{2,0}, \ldots, S_{m,0})}) = \log(\hat{\theta}_0) + \log(-\log(1-p)) \cdot \hat{\sigma} \\
= \hat{\beta}_0 + \hat{\beta}_1 S_{1,0} + \hat{\beta}_2 S_{2,0} + \cdots + \hat{\beta}_m S_{m,0} + \log(-\log(1-p)) \cdot \hat{\sigma} \\
= \hat{\beta}_0 + \sum_{i=1}^{m} \hat{\beta}_i S_{i,0} + \log(-\log(1-p)) \cdot \hat{\sigma} \quad (5-30)
\]

When \( p = 0.6321 \), \( \hat{\theta}_{p(S_{1,0}, S_{2,0}, \ldots, S_{m,0})} \) is equal to \( \log(\hat{\theta}_0) \): the logarithm of the characteristic life at typical operating conditions with stress levels \( (S_{10}, S_{20}) \). When
\( p = 0.5 \), \( \hat{Y}_{p(L|s_{i,0}, s_{2,0}, \ldots, s_{m,0})} \) is the logarithm of the median life at typical operating conditions with stress levels \((S_{10}, S_{20})\). In this section, we choose \( p = 0.5 \) for the following calculation & optimization.

The objective of the optimal \( k \)-step \( m \)-variable SSALT design is to determine the optimal values of \( \tau_i, i = 1, 2, \ldots, k - 1 \) by minimizing the AV of \( \hat{Y}_{p(L|s_{i,0}, s_{2,0}, \ldots, s_{m,0})} \) given in equation (5-30).

The AV multiplied by the sample size at usual operating stress level \((S_{1,0}, S_{2,0}, \ldots, S_{m,0})\) is then given by:

\[
\begin{align*}
n \cdot AV\left[ \hat{Y}_{p(L|s_{i,0}, s_{2,0}, \ldots, s_{m,0})} \right] &= n \cdot AV\left[ \hat{\beta}_0 + \sum_{i=1}^{m} \hat{\beta}_i S_{i,0} + \log\left(-\log(1 - p)\right) \cdot \hat{\sigma} \right] \\
&= n \cdot H_8 \cdot \hat{F}_6^{-1} \cdot H_8'
\end{align*}
\]

Where \( \hat{F}_6^{-1} \) is the inverse of the expected Fisher information matrix in equation (5-29), and \( H_8 \) is the row vector of the first derivative of \( \hat{Y}_{p(L|s_{i,0}, s_{2,0}, \ldots, s_{m,0})} \) with respect to the unknown parameters \( \beta_0, \beta_1, \beta_2, \ldots, \beta_m, \sigma \), shown as follows:

\[
H_8 = \begin{bmatrix}
\frac{\partial \hat{Y}_{p(L|s_{i,0}, s_{2,0}, \ldots, s_{m,0})}}{\partial \beta_0} & \frac{\partial \hat{Y}_{p(L|s_{i,0}, s_{2,0}, \ldots, s_{m,0})}}{\partial \beta_1} & \ldots & \frac{\partial \hat{Y}_{p(L|s_{i,0}, s_{2,0}, \ldots, s_{m,0})}}{\partial \beta_m} & \frac{\partial \hat{Y}_{p(L|s_{i,0}, s_{2,0}, \ldots, s_{m,0})}}{\partial \sigma}
\end{bmatrix}.
\]

Then the problem objective is the following:

\[
\text{minimize } n \cdot AV\left[ \hat{Y}_{p(L|s_{i,0}, s_{2,0}, \ldots, s_{m,0})}(\tau_1, \tau_2, \ldots, \tau_{k-1}) \right] \quad \text{(5-32)}
\]
The next section presents numerical examples and sensitivity analysis to demonstrate the calculation of the optimal hold times. Pattern search optimization algorithm is applied to obtain the minimum of $nAV$.

5.3 Numerical Examples and Sensitivity Analysis

In this section, numerical examples are given. The pattern search algorithm is developed to calculate the optimal hold times $\tau_1, \tau_2, \cdots, \tau_{k-1}$ for the multi-variate SSALT. Since the derivative information of the AV is unavailable, a pattern search algorithm is used for the minimization problem.

5.3.1 Numerical Example

Consider the bivariate model presented in previous chapter, as a special case of the $k$-step multi-variate SSALT model presented in this chapter, where $k = 3, m = 2$. The computational procedure is validated by using the failure data from section 4.2.4.1, shown in Figure 5-2. The optimal 3-steps, 2-variables SSALT is obtained.
As we stated in previous chapter, the normal operating temperature is 328K, and the use voltage level is 10V. We apply the same transformation to the stress variables as the previous chapter: the Arrhenius relationship was used for transformation of the temperature stress, and the Inverse Power for transformation of the voltage stress:

\[ S_1 = \frac{1}{Kevin \ temperat} \], and \[ S_2 = \log \text{(voltage)} \]. The transformed stresses were used in the log-linear life-stress relationship. And we use the same pre-estimate of unknown parameters as in previous chapter:

\[ \delta = 2.9479; \beta_0 = -5.9204; \beta_1 = 6071.43; \beta_2 = -1.5604 \].

Then the pre-estimated scale parameters and the shape parameter are obtained as follows:

\[ \theta_1 = 1288.7826; \theta_2 = 525.3873; \theta_3 = 335.3708; \delta = 2.9479 \]

Table 5-1 shows the optimal stress change times for different test censoring time \( T \).
Table 5-1  Optimal 3-Step 2-Variables SSALT Plan

<table>
<thead>
<tr>
<th>Censoring Time T</th>
<th>Multi-Variate Model</th>
<th>Bivariate Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \tau_1^* )</td>
<td>( \tau_2^* )</td>
</tr>
<tr>
<td>1500</td>
<td>1459.5</td>
<td>1480.5</td>
</tr>
<tr>
<td>1000</td>
<td>983.0</td>
<td>995.0</td>
</tr>
</tbody>
</table>

The computational results are shown in Table 5-1. There is only slight difference between the computational results of the bivariate model and the optimal design of the multi-variate SSALT model. The bivariate model is a special case of our model, and used to verify the computational accuracy of the multi-variate model.

5.3.2 Sensitivity Analysis

In order to examine the impact of the pre-estimated parameters on the optimal SSALT design, sensitivity analysis of \( \tau_1^* \), \( \tau_2^* \) and \( n \cdot AV\left[\hat{Y}_{p\left(S_{i,o,s_2,a,\ldots,s_{n,a}\right)}\right] \) with respect to variation of the pre-estimated parameters is performed.

As demonstrated by Figure 5-3:

- While \( \theta_1 \) increases from 1200 to 1400, the optimal stress change times; \( \tau_1^* \) and \( \tau_2^* \) both increase slightly, and the \( n \cdot AV\left[\hat{Y}_{p\left(S_{i,o,s_2,a,\ldots,s_{n,a}\right)}\right] \) also increases.
- While as \( \theta_2 \) increases from 420 to 600, first optimal stress change time \( \tau_1^* \) decreases slightly, and the second optimal stress change time \( \tau_2^* \) remains almost the same, while the \( n \cdot AV\left[\hat{Y}_{p\left(S_{i,o,s_2,a,\ldots,s_{n,a}\right)}\right] \) decreases.
- When \( \theta_3 \) increases from 230 to 450, both optimal stress change times \( \tau_1^* \) and \( \tau_2^* \) decrease slightly, and the \( n \cdot AV\left[\hat{Y}_{p\left(S_{i,o,s_2,a,\ldots,s_{n,a}\right)}\right] \) also decreases.
While increasing $\delta$ from 2.75 to 3 leads to slightly higher values for $\tau_1^*$ and $\tau_2^*$, and the $n \cdot AV\left[\hat{\gamma}_{p,\lbrace s_{0,0}, s_{2,0}, \ldots, s_{m,0}\rbrace}\right]$ decreases.

In other word, it is shown that the effect of the pre-estimates on the optimal SSALT design is very small, and the test design is robust in this case.

Another important factor that could influence the optimal design is stress level. Different stress levels have direct effect on the values of AV of $\hat{\gamma}_{p,\lbrace s_{0,0}, s_{2,0}, \ldots, s_{m,0}\rbrace}$. Also, if the stress level changes, the shape parameters at that stress level will change. Considering both effect, the changes of the optimal test design caused by the stress levels are shown in Fig. 3. It is shown from Fig. 3 that:

- The $n \cdot AV\left[\hat{\gamma}_{p,\lbrace s_{0,0}, s_{2,0}, \ldots, s_{m,0}\rbrace}\right]$ approaches minimum when $S_{11}$ equals to 355K. Both $\tau_1^*$ and $\tau_2^*$ decrease slightly when $S_{11}$ increases from 350 to 357, and they decrease much when $S_{11}$ is greater than 357.
- When $S_{21}$ is increased from 11 to 13, $\tau_1^*$ remains almost same, $\tau_2^*$ decreases slightly, and $n \cdot AV\left[\hat{\gamma}_{p,\lbrace s_{0,0}, s_{2,0}, \ldots, s_{m,0}\rbrace}\right]$ also decrease.
- When $S_{12}$ is increased from 363 to 378, $\tau_1^*$ increases, $\tau_2^*$ decreases slightly, and $n \cdot AV\left[\hat{\gamma}_{p,\lbrace s_{0,0}, s_{2,0}, \ldots, s_{m,0}\rbrace}\right]$ also decrease.
- When $S_{22}$ is increased from 13 to 13.3, there is almost no change in $\tau_1^*$, $n \cdot AV\left[\hat{\gamma}_{p,\lbrace s_{0,0}, s_{2,0}, \ldots, s_{m,0}\rbrace}\right]$, and $\tau_2^*$ increases slightly. For $S_{22}$ greater than 13.3, $\tau_1^*$ and $\tau_2^*$ both decreases, while $n \cdot AV\left[\hat{\gamma}_{p,\lbrace s_{0,0}, s_{2,0}, \ldots, s_{m,0}\rbrace}\right]$ slightly increases.
Figure 5-3  Effect of the Pre-Estimates on Optimal SSALT Plan
Figure 5-4  Effect of Stress Levels on Optimal SSALT Design
CHAPTER 6

Simple SSALT Design for Proportional Hazards Model

As was mentioned in chapter 2, there are different life-stress relationships, such as: Arrhenius relationship, Eyring relationship, inverse power law relationship, T-H relationship, T-NT relationship, and general log-linear relationship and PH model. The first five relationships have been either single stress relationships or two stress relationships. In most practical applications, however, life is a function of more than one or two variables (stress types). In addition, there are many applications where the life of a product as a function of stress and of some engineering variable other than stress is sought. The general log-linear relationship and the proportional hazards model are applied for the analysis of such cases where more than two accelerated stresses (or variables) need to be considered. All the models in previous chapters assume general log-linear relationship between the life and stresses. In this chapter, we will consider PH model and design an optimal simple SSALT based on PH model.

The PH model, introduced by D. R. Cox [11], has been primarily used in medical testing analysis, to model the effect of secondary variables on survival. The PH model assumes changing a stress variable (or explanatory variable) has the effect of multiplying the hazard rate by a constant. The model can be used to estimate the effects of different covariates influencing the times-to-failure of a system. It has been widely applied in the biomedical field and recently there has been an increasing interest in accelerated life
testing application. Elsayed & Chan [17] developed a proportional hazards model to estimate thin oxide dielectric reliability and time-dependent dielectric-breakdown hazard rates in a constant stress ALT. Newby [56] illustrated the advantage and some of the limitation of proportional hazards models through an example. Elsayed & Jiao [18], and Elsayed & Zhang [19] proposed optimal step-stress ALT methods based on the proportional hazards model for estimating reliability function at normal operating conditions, assuming the linear baseline intensity function.

This chapter presents a PH-CE model for a simple SSALT data analysis assuming a two parameter Weibull distribution for failure time at higher stress level. Cumulative exposure model is used to relate the CDF under constant-stress ALT to the CDF under SSALT. PH model is assumed for getting the failure density function under the lower stress level and the normal operating condition. The optimum test plan is developed to determine the optimal hold time for the lower stress level. The criterion is to minimize the AV of the MLE of the life under normal operating conditions with a specified reliability $R$.

Next section presents the well-known proportional hazards models in general. Section 6.2 presents the basic assumption and a revised model for the censored Weibull SSALT based on the PH model. In section 6.3, MLE method is used for the estimation. Fisher information matrix is then derived. The AV of MLE of the life under normal operating conditions with a specified reliability $R$ is obtained in section 6.4 and the optimization criterion is presented, followed by the numerical example and sensitivity analysis in section 6.5.
6.1 Proportional Hazards Model

The PH model was proposed D. R. Cox [11] to estimate the effects of different covariates influencing the times-to-failure of a system. The model has been widely used in the biomedical field and recently there has been an increasing interest in its application in reliability engineering. In its original form, the model is non-parametric, i.e. no assumptions are made about the nature or shape of the underlying failure distribution.

6.1.1 Non-Parametric Model Formulation

The PH model assumes that the failure rate of a system is affected not only by its operation time, but also by the covariates under which it operates. For example, a unit may have been tested under a combination of different accelerated stresses such as humidity, temperature, voltage, etc. It is clear then that such factors affect the failure rate of a unit.

The instantaneous failure rate (or hazard rate) of a unit is given by:

\[ \lambda(t) = \frac{f(t)}{R(t)} \]  

(6-1)

where: \( f(t) \) is the probability density function and \( R(t) \) is the reliability function.

The above failure rate depends only on time. Considering the case of the failure rate of a unit being dependent not only on time but also on other covariates, the above equation must be modified in order to be a function of time and of the covariates.

The PH model assumes that the intensity function of a unit is the product of:
Chapter 6 Simple SSALT Design for Proportional Hazards Model

- an unspecified baseline intensity function, $\lambda_0(t)$, which is a function of time only, and,
- a positive function $g(Z, A)$, independent of time, which incorporates the effects of a number of covariates such as humidity, temperature, pressure, voltage, etc.

The intensity function of a unit is then given by,

$$\lambda(t; Z) = \lambda_0(t) \cdot g(Z, A)$$  \hspace{1cm} (6-2)

where, $Z$ is a row vector consisting of stresses, $Z = (z_1, z_2, \cdots, z_m)$, $A$ is a column vector consisting of the unknown parameters of the model, $A = (\alpha_1, \alpha_2, \cdots, \alpha_m)^T$, $m$ is number of stresses.

Different functions of $g(Z, A)$ can be used. However, the exponential form is mostly used due to its simplicity as given by:

$$g(Z, A) = e^{ZA} = e^{\alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_m z_m}$$  \hspace{1cm} (6-3)

The proportional hazard model can then be developed from equations (6-2) and (6-3):

$$\lambda(t; Z) = \lambda_0(t) \cdot e^{\alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_m z_m}$$  \hspace{1cm} (6-4)

6.1.2 Parametric Model Formulation

A parametric form of the proportional hazards model can be developed by assuming an underlying distribution for time to failure. Two-parameter Weibull distribution for formulating the parametric proportional hazards model is considered.
The PH model assumes that the proportion of hazard rates of two units tested at two different stress levels remain constant over time.

We consider the Weibull distribution to formulate the parametric proportional hazards model. The exponential distribution case can be easily obtained from the Weibull distribution, by simply setting the shape parameter $\delta = 1$. In other words, it is assumed that the baseline failure rate in equation (6-2) is parametric and given by the Weibull distribution. In this case, the baseline failure rate is given by:

$$\lambda_0(t) = \frac{\delta}{\eta} \left( \frac{t}{\eta} \right)^{\delta-1}$$

The PH failure rate then becomes:

$$\lambda(t; X_i) = \frac{\delta}{\eta} \left( \frac{t}{\eta} \right)^{\delta-1} \cdot e^{a_1x_1 + a_2x_2 + \cdots + a_mx_m}$$

### 6.2 Assumptions and Model

This chapter considers simple SSALT with one stress variable, where $n$ test units are initially placed on a lower stress $S_1$, and run until time $\tau$ (also called hold time), when the stress is increased to $S_2$ and the test is continued until all units fail or until a predetermined censoring time $T$, whichever occurs first.

Test procedure is shown in Figure 6-1. Total of $n_i$ failures are observed at time $t_{ij}$, $j = 1, 2, \ldots, n_i$ while testing at stress level $S_i$, $i = 1, 2$, and $n_c$ units are censored. Where, $S_0$ is the stress level at normal operating condition.
The following assumptions are made:

1) The life time of the test units at higher stress level $S_2$ follows Weibull distribution with shape parameter $\delta$ and scale parameter $\eta$. The baseline failure intensity function is defined as:

$$\lambda_0(t) = \frac{\delta}{\eta} \left( \frac{t}{\eta} \right)^{\delta-1}$$  

(6-5)

2) The PH model is assumed for normal condition and for the lower stress level $S_1$. Then from equations (6-4) and (6-5), the PH failure intensity function under constant stress level becomes:

$$\lambda(t; X_i) = \frac{\delta}{\eta} \left( \frac{t}{\eta} \right)^{\delta-1} \cdot e^{\alpha X_i}$$  

(6-6)

where $X_i$ is standardized stress level $X_i = \frac{S_i - S_2}{S_0 - S_2}, 0 \leq X_i \leq 1, \ i = 0, 1, 2$ and $\alpha$ is an unknown parameter ($\alpha < 0$).
3) Cumulative exposure model [51] is utilized. Cumulative exposure model assumes that the remaining life of specimens depend only on the current cumulative fraction failed and current stress — regardless of how the fraction accumulated. Moreover, if held at the current stress, survivors will fail according to the CDF of stress, but starting at the previously accumulated fraction failed. The cumulative exposure model for simple SSALT can be expressed as follows:

\[
F(t) = \begin{cases} 
F_1(t) & 0 \leq t \leq \tau \\
F_2(t - \tau + s) & \tau \leq t < \infty 
\end{cases}
\]  

(6-7)

where \( s \) is the solution of: \( F_1(\tau) = F_2(s) \).

Based on the model assumptions, the revised cumulative density function of time to failure from equations (6-6) and (6-7) is obtained:

\[
F(t) = 1 - \begin{cases} 
\exp\left(-\frac{t^\delta \exp(aX_1)}{\eta^\delta}\right) & 0 \leq t \leq \tau \\
\exp\left(-\frac{t^\delta - \tau^\delta \exp(aX_2) - \tau^\delta \exp(aX_1)}{\eta^\delta}\right) & \tau \leq t < \infty
\end{cases}
\]  

(6-8)

To simplify the model, we let \( Y = \log(t), \ \sigma = \frac{1}{\delta}, \ \text{and} \ \theta = \log(\eta) \). Then the CDF of \( Y \) is:
Chapter 6 Simple SSALT Design for Proportional Hazards Model

\[
G(y) = \begin{cases} 
1 - \exp \left( -\exp \left( \frac{y - \theta}{\sigma} + aX_1 \right) \right) & -\infty < y \leq \log(\tau) \\
1 - \exp \left( \log(\tau) - \theta + aX_2 \right) & \log(\tau) \leq y < \infty \\
1 - \exp \left( \frac{y - \theta}{\sigma} + aX_2 \right) & -\infty < y \leq \log(\tau) \\
\end{cases}
\]

(6-9)

Please refer to the Appendix VI for detailed derivation of CDF of \( Y \).

Then the pdf of \( Y \) is obtained by taking the first derivative of \( G(y) \).

\[
f(y) = \begin{cases} 
\frac{1}{\sigma} \exp \left( \frac{y - \theta}{\sigma} + aX_1 \right) - \exp \left( \frac{y - \theta}{\sigma} + aX_1 \right) & -\infty < y \leq \log(\tau) \\
\frac{1}{\sigma} \exp \left( \log(\tau) - \theta + aX_2 \right) & \log(\tau) \leq y < \infty \\
\frac{1}{\sigma} \exp \left( \frac{y - \theta}{\sigma} + aX_2 \right) & -\infty < y \leq \log(\tau) \\
\end{cases}
\]

(6-10)

The CDF and pdf of \( Y \) are then used to derive the likelihood function of unknown parameters. The MLE and fisher information matrix are obtained from the likelihood function in the following section.

6.3 MLE and Fisher Information Matrix

Analysis of failure data will lead to the estimation of the parameters of failure time distribution under the normal operating condition. MLE method is used for the parameter estimation of the failure times \( t_y \) from the SSALT.

The likelihood function for \( n_i \) failures at \( Y_j, \ Y_j = \log(t_j), j = 1, 2, \ldots, n_i \) while testing at stress level \( S_i, \ i = 1, 2, \) and \( n_c \) censored test data is:
Chapter 6 Simple SSALT Design for Proportional Hazards Model

\[ L(\theta, a, \sigma) = \prod_{j=1}^{n} \left[ \frac{1}{\sigma} \exp \left( \frac{y_{ij} - \theta}{\sigma} + aX_j - \exp \left( \frac{y_{ij} - \theta}{\sigma} + aX_1 \right) \right) \right] \]

\[
\cdot \prod_{j=1}^{n} \frac{1}{\sigma} \exp \left[ \left( \frac{y_{2j} - \theta}{\sigma} + aX_j - \exp \left( \frac{y_{2j} - \theta}{\sigma} + aX_1 \right) \right) + \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_2 \right) \right] - \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_1 \right) \right] \right]
\]

\[
\cdot \prod_{j=1}^{n} \exp \left[ \left( -\exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) \right) \right] + \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) - \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_1 \right) \right] \right]
\]

(6-11)

And the log-likelihood function is derived as follows:

\[
\log L(\theta, a, \sigma) = -(n_1 + n_2)\log(\sigma) + \sum_{j=1}^{n} \left( \frac{y_{ij} - \theta}{\sigma} + aX_j - \exp \left( \frac{y_{ij} - \theta}{\sigma} + aX_1 \right) \right) \]

\[
+ \sum_{j=1}^{n} \left( \frac{y_{2j} - \theta}{\sigma} + aX_j - \exp \left( \frac{y_{2j} - \theta}{\sigma} + aX_1 \right) \right) \]

\[
+ \left( n_2 + n_c \right) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_2 \right) \]

\[
- \left( n_2 + n_c \right) \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_1 \right) - n_c \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) \]

(6-12)

The MLE for unknown parameters \( \theta, a, \sigma \) is obtained by solving the following partial derivative equations of the log-likelihood function.

\[
\frac{\partial \log L(\theta, a, \sigma)}{\partial \theta} = -\frac{1}{\sigma} (n_1 + n_2) + \frac{1}{\sigma} \sum_{j=1}^{n} \exp \left( \frac{y_{ij} - \theta}{\sigma} + aX_1 \right) + \frac{1}{\sigma} \sum_{j=1}^{n} \exp \left( \frac{y_{2j} - \theta}{\sigma} + aX_2 \right) \]

\[
- \frac{1}{\sigma} (n_2 + n_c) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_2 \right) + \frac{1}{\sigma} (n_2 + n_c) \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_1 \right) \]

\[
+ \frac{1}{\sigma} n_c \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) = 0 \]

(6-13)
\[
\frac{\partial \log L(\theta,a,\sigma)}{\partial a} = (n_1X_1 + n_2X_2) - X_1 \sum_{j=1}^{n_1} \exp \left( \frac{y_{1j} - \theta}{\sigma} + aX_1 \right) - X_2 \sum_{j=1}^{n_2} \exp \left( \frac{y_{2j} - \theta}{\sigma} + aX_2 \right) \\
+ X_2(n_2 + n_c) \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) - X_1(n_2 + n_c) \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_1 \right)
\]

(6-14)

\[
\frac{\partial \log L(\theta,a,\sigma)}{\partial \sigma} = - \frac{1}{\sigma} (n_1 + n_2) - \frac{1}{\sigma} \sum_{j=1}^{n_1} \left( \frac{y_{1j} - \theta}{\sigma} \right) - \frac{1}{\sigma} \sum_{j=1}^{n_2} \left( \frac{y_{2j} - \theta}{\sigma} \right) \\
+ \frac{1}{\sigma} \sum_{j=1}^{n_1} \exp \left( \frac{y_{1j} - \theta}{\sigma} + aX_1 \right) \frac{y_{1j} - \theta}{\sigma} + \frac{1}{\sigma} \sum_{j=1}^{n_2} \exp \left( \frac{y_{2j} - \theta}{\sigma} + aX_2 \right) \frac{y_{2j} - \theta}{\sigma} \\
- \frac{1}{\sigma} (n_2 + n_c) \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) \left( \frac{\log(T) - \theta}{\sigma} \right) \\
+ \frac{1}{\sigma} (n_2 + n_c) \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_1 \right) \left( \frac{\log(T) - \theta}{\sigma} \right) \\
+ \frac{1}{\sigma} n_c \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) \left( \frac{\log(T) - \theta}{\sigma} \right) = 0
\]

(6-15)

The second partial and mixed partial derivatives of \( \log L(\theta,a,\sigma) \) with respect to \( \theta, a, \sigma \) are then obtained as follows:

\[
\frac{\partial^2 \log L(\theta,a,\sigma)}{\partial \theta^2} = - \frac{1}{\sigma^2} \sum_{j=1}^{n_1} \left( \frac{y_{1j} - \theta}{\sigma} \right) + \sum_{j=1}^{n_2} \left( \frac{y_{2j} - \theta}{\sigma} \right) \\
- (n_2 + n_c) \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) + (n_2 + n_c) \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_1 \right)
\]

(6-16)

\[
\frac{\partial^2 \log L(\theta,a,\sigma)}{\partial \theta \partial a} = \frac{\partial^2 \log L(\theta,a,\sigma)}{\partial a \partial \theta} = \frac{1}{\sigma} \left( X_1 \sum_{j=1}^{n_1} \exp \left( \frac{y_{1j} - \theta}{\sigma} + aX_1 \right) \\
+ X_2 \sum_{j=1}^{n_2} \exp \left( \frac{y_{2j} - \theta}{\sigma} + aX_2 \right) - X_1(n_2 + n_c) \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) \\
+ X_1(n_2 + n_c) \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_1 \right) + X_2 n_c \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) \right)
\]

(6-17)
\[
\frac{\partial^2 \log L(\theta, a, \sigma)}{\partial a^2} = -\left( X_1^2 \sum_{j=1}^{n_1} \exp \left( \frac{y_{1j} - \theta}{\sigma} + aX_1 \right) + X_2^2 \sum_{j=1}^{n_2} \exp \left( \frac{y_{2j} - \theta}{\sigma} + aX_2 \right) \right) \\
- X_2^2 (n_2 + n_c) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_2 \right) + X_1^2 (n_2 + n_c) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_1 \right) \\
+ X_2^2 n_c \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) \quad (6-18)
\]

\[
\frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \theta \partial \sigma} = \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \sigma \partial \theta} = -\frac{1}{\sigma^2} \left( \sum_{j=1}^{n_1} \exp \left( \frac{y_{1j} - \theta}{\sigma} + aX_1 \right) \left( \frac{y_{1j} - \theta}{\sigma} \right) \right) \\
+ \sum_{j=1}^{n_2} \exp \left( \frac{y_{2j} - \theta}{\sigma} + aX_2 \right) \left( \frac{y_{2j} - \theta}{\sigma} \right) \\
- (n_2 + n_c) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_2 \right) \left( \frac{\log(\tau) - \theta}{\sigma} \right) \\
+ (n_2 + n_c) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_1 \right) \left( \frac{\log(\tau) - \theta}{\sigma} \right) \\
+ n_c \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) \left( \frac{\log(T) - \theta}{\sigma} \right) \quad (6-19)
\]

\[
\frac{\partial^2 \log L(\theta, a, \sigma)}{\partial a \partial \sigma} = \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \sigma \partial a} = \frac{1}{\sigma} \left( X_1 \sum_{j=1}^{n_1} \exp \left( \frac{y_{1j} - \theta}{\sigma} + aX_1 \right) \left( \frac{y_{1j} - \theta}{\sigma} \right) \right) \\
+ X_2 \sum_{j=1}^{n_2} \exp \left( \frac{y_{2j} - \theta}{\sigma} + aX_2 \right) \left( \frac{y_{2j} - \theta}{\sigma} \right) \\
- X_1 (n_2 + n_c) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_2 \right) \left( \frac{\log(\tau) - \theta}{\sigma} \right) \\
+ X_1 (n_2 + n_c) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_1 \right) \left( \frac{\log(\tau) - \theta}{\sigma} \right) \\
+ n_c X_2 \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) \left( \frac{\log(T) - \theta}{\sigma} \right) \quad (6-20)
\]
\[
\frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \sigma^2} = -\frac{1}{\sigma^2} \left( (n_1 + n_2) + \sum_{j=1}^{n_n} \exp \left( \frac{y_{1j} - \theta}{\sigma} + aX_1 \right) \left( \frac{y_{1j} - \theta}{\sigma} \right)^2 \right) \\
+ \sum_{j=1}^{n_n} \exp \left( \frac{y_{2j} - \theta}{\sigma} + aX_2 \right) \left( \frac{y_{2j} - \theta}{\sigma} \right)^2 \\
- (n_2 + n_c) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_2 \right) \left( \frac{\log(\tau) - \theta}{\sigma} \right)^2 \\
+ (n_2 + n_c) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_1 \right) \left( \frac{\log(\tau) - \theta}{\sigma} \right)^2 \\
+ n_c \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right) \left( \frac{\log(T) - \theta}{\sigma} \right)^2 
\].

(6-21)

From the equations (6-13) and (6-15), we get:

\[
n_1 = \sum_{j=1}^{n_n} \exp \left( \frac{y_{1j} - \theta}{\sigma} + aX_1 \right) + (n_2 + n_c) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_1 \right),
\]

(6-22)

\[
n_2 = \sum_{j=1}^{n_n} \exp \left( \frac{y_{2j} - \theta}{\sigma} + aX_2 \right) - (n_2 + n_c) \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_2 \right) \\
+ n_c \exp \left( \frac{\log(T) - \theta}{\sigma} + aX_2 \right).\]

(6-23)

Equations (6-22) and (6-23) are then used to simplify the second partial and mixed partial derivatives of \( \log[L(\theta, a, \sigma)] \) with respect to \( \theta, a, \sigma \) in equations (6-16) to (6-21). And also, to simplify the second partial and mixed partial derivatives, the following definition are made:

\[
\Phi_i = E \left[ \frac{1}{n} \exp(aX_i) \cdot \frac{n}{n} \sum_{j=1}^{n_n} \exp \left( \frac{y_{ij} - \theta}{\sigma} \right) \right],
\]

\[
\Psi_i = E \left[ \frac{1}{n} \left( \exp(aX_i) \right)^2 \sum_{j=1}^{n_n} \left( \frac{y_{ij} - \theta}{\sigma} \right)^2 \exp \left( \frac{y_{ij} - \theta}{\sigma} \right) \right], \quad C_i = E \left[ \frac{n_i}{n} \right], \quad i = 1, 2.
\]
From equations (6-22) and (6-23) and the above definition, the second partial and mix partial derivatives in equations (6-16) to (6-21) are simplified and the expected Fisher information matrix $\mathbf{\hat{F}}$ is derived as follows:

$$
\mathbf{\hat{F}}_\gamma = \frac{n}{\sigma^2} \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{12} & A_{22} & A_{23} \\
A_{13} & A_{23} & A_{33}
\end{bmatrix}
$$

(6-24)

where the elements of $\mathbf{\hat{F}}_\gamma$ are the expected value of the negative second partial and mixed partial derivative of $\log L(\theta, a, \sigma)$ with respect to parameters $\theta, a,$ and $\sigma$ in equations (6-16) to (6-21), shown as follows:

$$
A_{11} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \theta^2} \right) \right] = C_1 + C_2,
$$

$$
A_{12} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \theta \partial a} \right) \right] = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L(\theta, a, \sigma)}{\partial a \partial \theta} \right) \right]
= -C_1 X_1 \sigma - C_2 X_2 \sigma,
$$

$$
A_{22} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L(\theta, a, \sigma)}{\partial a^2} \right) \right] = C_1 X_1^2 \sigma^2 + C_2 X_2^2 \sigma^2,
$$

$$
A_{13} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \theta \partial \sigma} \right) \right] = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \sigma \partial \theta} \right) \right]
= \Phi_1 + \Phi_2 + C_T \exp(aX_2) \left( \frac{\log(T) - \theta}{\sigma} \right) \exp \left( \frac{\log(T) - \theta}{\sigma} \right)
+ \left( C_2 + C_T \left( \frac{\log(r) - \theta}{\sigma} \right) \exp \left( \frac{\log(r) - \theta}{\sigma} \right) \right) \left( \exp(aX_1) - \exp(aX_2) \right),
$$
\[ A_{23} = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial a \partial \sigma} \right) \right] = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \sigma \partial a} \right) \right] \]
\[ = -\sigma \left[ X_1 \Phi_1 + X_2 \Phi_2 + C_T X_2 \exp(aX_2) \left( \frac{\log(T) - \theta}{\sigma} \right) \exp \left( \frac{\log(T) - \theta}{\sigma} \right) \right] \]
\[ + \left( C_2 + C_T \right) \left( \frac{\log(\tau) - \theta}{\sigma} \right) \exp \left( \frac{\log(\tau) - \theta}{\sigma} \right) (X_1 \exp(aX_1) - X_2 \exp(aX_2)) \]

\[ A_{33} = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \sigma^2} \right) \right] \]
\[ = \left( C_1 + C_2 \right) + \Psi_1 + \Psi_2 + C_T \exp(aX_2) \left( \frac{\log(T) - \theta}{\sigma} \right)^2 \exp \left( \frac{\log(T) - \theta}{\sigma} \right) \]
\[ + \left( C_2 + C_T \right) \left( \frac{\log(\tau) - \theta}{\sigma} \right)^2 \exp \left( \frac{\log(\tau) - \theta}{\sigma} \right) (\exp(aX_1) - \exp(aX_2)) \]

Please refer to Appendix VII for the derivation of \( C_i, C_T, \Phi_i, \) and \( \Psi_i, i = 1, 2. \)

After the expected Fisher information matrix is obtained, preliminary estimates of the parameters \( \theta, a, \) and \( \sigma \) are used to design the optimal simple SSALT plan based on PH model. These preliminary estimates of parameters can be obtained from past experience of similar products, and/or from some target MTTF values or from a small sample experiment. To design the optimal test plan, the optimal criterion must be first determined. The optimality criterion used in the model is discussed next.

### 6.4 Optimality Criterion & Test Design

In order to evaluate and obtain consistent estimators, the minimum variance principle of estimator is applied. Considering the test objective of the model, the
optimality criterion is defined to minimize the AV of MLE of log of the life with Weibull distribution and reliability \( R \).

The failure intensity function at normal operating stress \( S_0 \) can be obtained from PH model in equation (6-6):

\[
\hat{\lambda}_o(t) = \frac{\hat{\delta}}{\hat{\eta}} \left( \frac{t}{\hat{\eta}} \right)^{\hat{\delta}-1} e^{{\hat{\delta} \cdot X_o}},
\]  
(6-25)

and the reliability at normal operating stress \( S_0 \) is:

\[
R_o(t) = \exp \left[ - \int_0^t \hat{\lambda}_o(t') dt' \right] = \exp \left[ - \left( \frac{t}{\hat{\eta}} \right)^{\hat{\delta}} \exp(\hat{a}X_o) \right].
\]  
(6-26)

Therefore, the log of MLE of the life with reliability \( R \) at normal operating stress \( S_0 \) is obtained by setting the equation (6-26) equal to \( R \) and solving for time \( t \), as shown below:

\[
\hat{Y}_{o,R} = \log(t_0(R)) = \log(\hat{\eta}) + \left[ \log(- \log(R)) - \hat{a} \right] \hat{\delta}
\]  
(6-27)

The AV of \( \hat{Y}_{o,R} \) is then derived from the expected Fisher information matrix in equation (6-24) as follows:

\[
AV(\hat{Y}_{o,R}) = AV(\hat{\theta} + (\log(- \log(R)) - \hat{a}) \cdot \hat{\sigma}) = H_o \cdot \hat{F}_7^{-1} \cdot H_o \]  
(6-28)

where \( H_o \) is the row vector of the first derivative of \( \hat{Y}_{o,R} \) in equation (6-27) with respect to parameters \( \theta, a, \) and \( \sigma \), shown as:

\[
H_o = \begin{bmatrix}
\frac{\partial \hat{Y}_o}{\partial \theta} & \frac{\partial \hat{Y}_o}{\partial a} & \frac{\partial \hat{Y}_o}{\partial \sigma}
\end{bmatrix} = \begin{bmatrix}
1 & -\hat{\sigma} & (\log(- \log(R)) - \hat{a})
\end{bmatrix}.
\]
Considering the test objective of the model, the optimality criterion is defined to minimize the AV of MLE of log of the life with Weibull distribution and reliability $R$ in equation (6-28). The decision variable is the time $\tau$ to change the stress level, referred as a hold time. Therefore, the objective is to:

$$\text{min. } n \cdot \text{AV}[\hat{Y}_{0,R}(\tau)]$$

(6-29)

6.5 Numerical Examples and Sensitivity Analysis

In this section, numerical examples are given. The pattern search algorithm is applied to solve the minimization problem in equation (6-29). The sensitivity analysis is performed to examine the effect of changes in the preliminary estimates of the unknown parameters.

6.5.1 Numerical Example

Having the initial estimates of the parameters $\eta, a, \delta$, the optimal test plan for censored Weibull SSALT data can be designed. These preliminary estimated values of parameters could be obtained from past experience with similar products and/or some target values for the product performance.

Suppose that we need to run a simple step-stress test to estimate the mean life of a system at a specified design voltage of 110V. Also suppose that the highest stress applicable to test units is given as 1000V, the lower stress level of voltage is 500 V, and the censoring time is $T = 10$ hours. In addition, based on a similar data from a previous experience, or based on a preliminary test the initial estimates of parameters are obtained
as: \( a = -2.0, \delta = 2.2 \), and the scale parameter for the higher stress level is estimated as 

3.5. Then the optimum stress change time is determined by the following steps:

1. Stress levels are standardized by using \( X_i = \frac{S_i - S_2}{S_0 - S_2} \). Therefore, we have \( X_0 = 1, X_1 = 0.56, X_2 = 0 \).

2. Scale parameter is standardized from dividing by censoring time \( T \). Thus we get \( \eta = 0.35 \).

3. The problem is to design the optimum SSALT with decision variable \( \tau \), and the objective is to minimize the asymptotic variance of mean life estimate under normal operating condition.

4. The optimal solution is then obtained by applying the pattern search algorithm in Maple 8 programming: \( \tau^* = 0.6824 \) and the optimum stress change times is:
   \( t^* = 6.824 \) hours.

6.5.2 Sensitivity Analysis

To examine the effect of changes in the preliminary estimates of parameters \( \eta, a, \delta \) on the optimal values of hold time \( \tau \) and AV, sensitivity analysis is performed. Figure 6-2 shows the results of sensitivity analysis of \( \eta, a, \delta \) on \( \tau^* \) and AV. The purpose of sensitivity analysis is to identify the sensitive parameters, which need to be estimated with special care to minimize the risk of obtaining an erroneous optimal solution.
Figure 6-2  Sensitivity Analysis of parameters $\eta, a, \delta$ on $\tau^*$ and AV
From Figure 6-2, it can be seen that parameters \( a \) and \( \beta \) are not sensitive parameters. For \( \eta \geq 0.3 \), optimal hold time \( \tau^\ast \) changes slightly as \( \eta \) changes. However, for \( \eta < 0.3 \), \( \tau^\ast \) is more sensitive to the change of \( \eta \). Thus, the initial value of \( \eta \) must be carefully estimated when it is less than 30% of the censoring time.
CHAPTER 7

Multi-Variate SSALT Plan for Proportional Hazards Model

In previous chapter, the simple SSALT plan for PH model is proposed. It is the relatively simple case in SSALT. Only two steps are included. In many cases and for many reasons, it is desired to perform the SSALT for more than two steps, which is called multiple SSALT, and also, sometimes it also includes more than one stress variables, which is multi-variate SSALT. This chapter extends the simple SSALT design based on PH model to multiple SSALT design, and then to multi-variate SSALT.

In this chapter we first consider multiple SSALT for the PH model by defining the baseline intensity function at the highest stress level. The reason for defining it at the highest stress level is to obtain the failure information easier and faster in the pre-test. The failure information is used for initial estimates of the unknown parameters. The assumption of Weibull distribution covers a broader range of failure data and has more application than the exponential distribution. We propose two optimization criteria for the \( k \)-step SSALT plan. Section 7.2 extends the results in the first section to the case where multiple variables are included, leading to the multi-variate SSALT for PH model.

7.1 Multiple SSALT for PH Model

This section presents a \( k \)-step SSALT considering Type I censoring. Test procedure is given first, followed by the model assumptions. Based on the model
assumption, the CDF of the failure time under multiple SSALT for PH model is derived and the pdf is obtained. Section 7.1.1 presents the assumptions and model, followed by the likelihood function and the Fisher information matrix in section 7.1.2. The optimization criteria are developed in section 7.1.3, and numerical examples are given in section 7.1.4.

7.1.1 Assumptions and Model

Multiple SSALT test involves \( k \) stress levels, shown in Figure 7-1. \( n \) test units are initially placed at a lower stress level \( S_i \). Test is continued and stress level is changed to a higher level \( S_{i+1} \) at time \( \tau_i, \ i = 1, 2, \ldots, k-1 \). Test is continued until all units fail or until a pre-determined censoring time \( T \) at stress level \( S_k \).

![k-step SSALT Test Procedure](image)

Figure 7-1 \( k \)-step SSALT Test Procedure

\( n_i \) failures are observed at stress level \( S_i \) between time \( (\tau_{i-1}, \tau_i] \), where \( \tau_0 = 0 \), \( i = 1, 2, \ldots, k-1 \). \( n_k \) failures are observed at stress level \( S_k \) between time \( (\tau_{k-1}, T] \) and
Chapter 7 Multi-Variate SSALT Plan for Proportional Hazards Model

$n_c$ units are censored at time $T$. Figure 7-1 presents the test, where $S_0$ is the stress level at the usual operating condition.

The following assumptions are made:

1) The life time of the test units at highest stress level $S_k$ follows Weibull distribution with shape parameter $\delta$ and scale parameter $\eta$. The baseline failure intensity function is defined as:

$$\lambda_0(t) = \frac{\delta}{\eta} \left( \frac{t}{\eta} \right)^{\delta-1}$$

(7-1)

The baseline failure intensity function is defined for test units at highest stress level to obtain a quick initial estimate of the parameters.

2) The PH model is assumed for other stress levels. The intensity function of a test unit is:

$$\lambda(t; X) = \lambda_0(t) \cdot e^{\alpha X}$$

(7-2)

Then from equations (7-1) and (7-2), the intensity function for stress level $X_i$, $i = 1, 2, \cdots, k$ becomes:

$$\lambda_i(t; X_i) = \frac{\delta}{\eta} \left( \frac{t}{\eta} \right)^{\delta-1} \cdot e^{\alpha X_i}$$

(7-3)

where $X_i$ is standardized stress level, $X_i = \frac{S_i - S_k}{S_0 - S_k}$, $0 < X_i < 1$, $i = 1, 2, \cdots, k$.

3) The cumulative exposure model [51] is utilized to relate the CDF of SSALT to the CDF of constant stress levels. The cumulative exposure model for $k$-step SSALT with censoring time $T$ can be expressed as follows:
Chapter 7 Multi-Variate SSALT Plan for Proportional Hazards Model

\[
F(t) = \begin{cases} 
F_1(t), & 0 \leq t < \tau_1 \\
\vdots \\
F_i(t - \tau_{i-1} + v_i), & \tau_{i-1} \leq t < \tau_i, \ i = 2, 3, \cdots, k - 1 \\
F_k(t - \tau_{k-1} + v_{k-1}), & \tau_{k-1} \leq t \leq T
\end{cases} \quad (7-4)
\]

where \( v_0 = \tau_0 = 0 \), and \( v_i \) is the solution of:

\[
F_{i+1}(v_i) = F_i(\tau_{i-1} - \tau_{i-1} + v_{i-1}), \ i = 1, 2, \cdots, k-1. \quad (7-5)
\]

To solve for \( v_i \), the test is analyzed step by step.

For stress level \( X_1 \), the failure intensity function under stress \( X_1 \) is obtained from equation (7-3):

\[
\lambda_1(t) = \delta \cdot t^{\delta-1} \eta^{-\delta} e^{ax_1}
\]

The CDF of failure data under the stress level \( X_1 \) is:

\[
F_1(t) = 1 - R_1(t) = 1 - e^{-\int_0^t \lambda_1(t) dt} = 1 - e^{-t^\delta \eta^{-\delta} e^{ax_1}}
\]

For stress level \( X_2 \), the failure intensity function under stress \( X_2 \) is obtained from equation (7-3):

\[
\lambda_2(t) = \delta \cdot t^{\delta-1} \eta^{-\delta} e^{ax_2}
\]

The CDF of failure data under the stress level \( X_2 \) is:

\[
F_2(t) = 1 - R_2(t) = 1 - e^{-\int_0^t \lambda_2(t) dt} = 1 - e^{-t^\delta \eta^{-\delta} e^{ax_2}}
\]

\( v_1 \) is the solution of \( F_2(v_1) = F_1(\tau_1 - \tau_0 + v_0) \).

\[
F_2(v_1) = F_1(\tau_1 - \tau_0 + v_0) = 1 - e^{-\tau_0^\delta \eta^{-\delta} e^{ax_0}} = 1 - e^{-\tau_1^\delta \eta^{-\delta} e^{ax_1}} \quad (7-6)
\]

\[
\Rightarrow v_1 = \tau_1 \cdot e^{\frac{\delta(a_1 - a_2)}{a_2}}
\]

Therefore, the CDF at the stress level \( X_2 \) in equation (7-3) is obtained:
Similarly, \( v_2 \) is the solution of

\[
F_3(v_2) = F_2(\tau_2 - \tau_1 + v_1)
\]

\[
\Rightarrow 1 - e^{-\bar{v}_2 e^{\alpha X_k}} = 1 - e^{-\left(\frac{\tau_2 - \tau_1}{\eta} + \frac{\tau_1}{\eta} e^{\alpha X_k}\right)^\delta}
\]

\[
\Rightarrow v_2 = \left(\tau_2 - \tau_1\right)e^{\frac{\alpha}{\eta} X_k} + \tau_1 e^{\frac{\alpha}{\eta} X_k} \frac{\alpha}{\eta} e^{\frac{\alpha}{\eta} X_k}
\]

Thus, the CDF at the stress level \( X_3 \) in equation (7-3) is then obtained:

\[
F_3(t - \tau_2 + v_2) = 1 - e^{-\left(\frac{t - \tau_2 + v_2}{\eta} e^{\alpha X_k}\right)^\delta}
\]

Therefore, we can derive the solution of equation (7-3) \( v_i \): \( F_{i+1}(v_i) = F_i(\tau_i - \tau_{i-1} + v_{i-1}) \)

\[
v_i = \left(\sum_{j=1}^{i} \left(\tau_j - \tau_{j-1}\right)e^{\frac{\alpha}{\eta} X_i}\right) \cdot e^{\frac{\alpha}{\eta} X_{i+1}} \quad \text{for} \quad i = 1, 2, \cdots, k - 1
\]

Proof of the above equation (7-8) is shown in Appendix VIII:

Thus, the CDF at the stress level \( X_k \) in equation (7-3) is then obtained:

\[
F_k(t - \tau_{k-1} + v_{k-1}) = 1 - e^{-\left(\frac{t - \tau_{k-1} + v_{k-1}}{\eta} e^{\alpha X_k}\right)^\delta}
\]

Therefore, the CDF of \( k \)-step Weibull PH-CE model is:

\[
F(t) = \begin{cases}
\exp\left(-\left(\frac{t}{\eta} \exp\left(\frac{a}{\delta} X_1\right)\right)^\delta\right) & 0 \leq t < \tau_1 \\
\exp\left(-\left(\frac{t - \tau_1}{\eta} \exp\left(\frac{a}{\delta} X_2\right) + \frac{\tau_1}{\eta} \exp\left(\frac{a}{\delta} X_1\right)\right)^\delta\right) & \tau_1 \leq t < \tau_2 \\
\vdots & \vdots \\
\exp\left(-\left(\frac{t - \tau_{k-1}}{\eta} \exp\left(\frac{a}{\delta} X_k\right) + \sum_{j=1}^{k-1} \frac{\tau_{j-1}}{\eta} \exp\left(\frac{a}{\delta} X_j\right)\right)^\delta\right) & \tau_{k-1} \leq t < \infty
\end{cases}
\]
Based on the model assumptions, the revised CDF of time to failure is proposed:

$$
F(t) = 1 - \begin{cases} 
\exp\left(-\left(\frac{t^\delta}{\eta^\delta}\exp(aX_1)\right)\right) & 0 \leq t < \tau_1 \\
\exp\left(-\left(\frac{t^\delta - \tau_1^\delta}{\eta^\delta}\exp(aX_2) + \frac{\tau_1^\delta}{\eta^\delta}\exp(aX_1)\right)\right) & \tau_1 \leq t < \tau_2 \\
\vdots & \\
\exp\left(-\left(\frac{t^\delta - \tau_{k-1}^\delta}{\eta^\delta}\exp(aX_k) + \sum_{l=1}^{k-1}\left(\frac{\tau_l^\delta - \tau_{l-1}^\delta}{\eta^\delta}\exp(aX_{l+1})\right)\right)\right) & \tau_{k-1} \leq t \leq T 
\end{cases}
$$

(7-9)

To simplify the model, we let $Y = \log(t)$, $\sigma = \frac{1}{\delta}$, and $\theta = \log(\eta)$. Then from equation (7-9) the CDF of $Y$ is obtained as follows:

$$
G(y) = F(\exp(y)) = 1 - \begin{cases} 
\exp\left(-y_{\theta,e} \cdot X_{1,e}\right) & -\infty < y \leq \log(\tau_1) \\
\exp\left(-y_{\theta,e} \cdot X_{2,e} + \tau_{1,\theta,e} \cdot (X_{2,e} - X_{1,e})\right) & \log(\tau_1) \leq y < \log(\tau_2) \\
\vdots & \\
\exp\left(-y_{\theta,e} \cdot X_{k,e} + \sum_{l=1}^{k-1}\tau_{l,\theta,e} \cdot (X_{l+1,e} - X_{l,e})\right) & \log(\tau_{k-1}) \leq y \leq \log(T) 
\end{cases}
$$

(7-10)

Where $y_{\theta,e} = \exp\left(\frac{y - \theta}{\sigma}\right)$, $X_{i,e} = \exp(aX_i)$, $\tau_{i,\theta,e} = \exp\left(\frac{\log(\tau_i) - \theta}{\sigma}\right)$, $i = 1, 2, \cdots, k - 1$.

Then the pdf of $Y$ is obtained by taking the first derivative of $G(y)$ in equation (7-10).

$$
f(y) = \begin{cases} 
\frac{1}{\sigma}\exp\left(\frac{y_{\theta} + aX_1 - y_{\theta,e} \cdot X_{1,e}}{\sigma}\right) & -\infty < y \leq \log(\tau_1) \\
\frac{1}{\sigma}\exp\left(\frac{y_{\theta} + aX_2 - y_{\theta,e} \cdot X_{2,e} + \tau_{1,\theta,e} \cdot (X_{2,e} - X_{1,e})}{\sigma}\right) & \log(\tau_1) \leq y < \log(\tau_2) \\
\vdots & \\
\frac{1}{\sigma}\exp\left(\frac{y_{\theta} + aX_k - y_{\theta,e} \cdot X_{k,e} + \sum_{l=1}^{k-1}\tau_{l,\theta,e} \cdot (X_{l+1,e} - X_{l,e})}{\sigma}\right) & \log(\tau_{k-1}) \leq y \leq \log(T) 
\end{cases}
$$

(7-11)
Chapter 7 Multi-Variate SSALT Plan for Proportional Hazards Model

Where \( y_\theta = \frac{y - \theta}{\sigma} \).

The \( f(y) \) is then used to obtain the likelihood function, and then MLE method is utilized to find the estimation of unknown parameters \( \hat{\theta}, \hat{a}, \) and \( \hat{\sigma} \). The Fisher information matrix is then obtained from the likelihood function.

7.1.2 MLE and Fisher Information Matrix

From the CDF in equation (7-10) and the pdf in equation (7-11), the likelihood function is constructed. The likelihood function for \( n_i \) failures at \( Y_{ij}, \)
\( Y_y = \log(t_y), \ j = 1, 2, \ldots n_i \) while testing at stress level \( S_i, \ i = 1, 2, \cdots, k \), and \( n_c \) censored test data is obtained:

\[
L(\theta, a, \sigma) = \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp \left( y_{1j,\theta} + aX_1 - y_{1j,\theta,e} \cdot X_{1,e} \right) \right] \\
\cdot \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp \left( y_{2j,\theta} + aX_2 - y_{2j,\theta,e} \cdot X_{2,e} + \tau_{1,\theta,e} \cdot (X_{2,e} - X_{1,e}) \right) \right] \\
\cdot \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp \left( y_{3j,\theta} + aX_3 - y_{3j,\theta,e} \cdot X_{3,e} + \tau_{1,\theta,e} \cdot (X_{3,e} - X_{2,e}) \right) \right] \\
\cdots \\
\cdot \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp \left( y_{kj,\theta} + aX_k - y_{kj,\theta,e} \cdot X_{k,e} + \sum_{i=1}^{k-1} \tau_{i,\theta,e} \cdot (X_{i+1,e} - X_{i,e}) \right) \right] \\
\cdot \prod_{j=1}^{n_i} \left[ \exp \left( -LT_{\theta,e} \cdot X_{k,e} + \sum_{i=1}^{k-1} \tau_{i,\theta,e} \cdot (X_{i+1,e} - X_{i,e}) \right) \right]
\]

(7-12)

Where \( y_{ij,\theta,e} = \exp \left( \frac{y_{ij} - \theta}{\sigma} \right), y_{ij,\theta} = \left( \frac{y_{ij} - \theta}{\sigma} \right), LT_{\theta,e} = \exp \left( \frac{\log(T) - \theta}{\sigma} \right) \).

And the log-likelihood function is derived as follows:
\[
\log L(\theta, a, \sigma) = -\sum_{i=1}^{k} n_i \cdot \log(\sigma) + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( y_{ij,\theta} + aX_i - y_{ij,\theta,e} \cdot X_{i,e} \right)
\]
\[
+ \sum_{j=1}^{n} (-IT_{\theta,e} \cdot X_{k,e}) + \sum_{i=1}^{k-1} \left( \tau_{i,\theta,e} \cdot (X_{i+1,e} - X_{i,e}) \left( \sum_{l=1}^{i} (n - n_l) \right) \right)
\]

(7-13)

The MLE of the unknown parameters \( \theta, a, \sigma \) can be obtained by solving the following differential equations from equation (7-13).

\[
\frac{\partial \log L}{\partial \theta} = -\frac{1}{\sigma} \sum_{i=1}^{k} n_i + \frac{1}{\sigma} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( y_{ij,\theta,e} \cdot X_{i,e} \right) + \frac{n_e}{\sigma} IT_{\theta,e} \cdot X_{k,e}
\]
\[
- \frac{1}{\sigma} \sum_{i=1}^{k-1} \left( \tau_{i,\theta,e} \cdot (X_{i+1,e} - X_{i,e}) \left( n - \sum_{l=1}^{i} n_l \right) \right) = 0
\]

(7-14)

\[
\frac{\partial \log L}{\partial a} = \sum_{i=1}^{k} n_i X_i - \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( X_i y_{ij,\theta,e} \cdot X_{i,e} \right) - n_e X_k IT_{\theta,e} \cdot X_{k,e}
\]
\[
+ \sum_{i=1}^{k-1} \left( \tau_{i,\theta,e} \cdot (X_{i+1,e} X_{i+1,e} - X_i X_{i,e}) \left( n - \sum_{l=1}^{i} n_l \right) \right) = 0
\]

(7-15)

\[
\frac{\partial \log L}{\partial \sigma} = -\frac{1}{\sigma} \sum_{i=1}^{k} n_i - \frac{1}{\sigma} \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij,\theta} + \frac{1}{\sigma} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( y_{ij,\theta,e} \cdot X_{i,e} \right)
\]
\[
+ \frac{n_e}{\sigma} IT_{\theta} \cdot IT_{\theta,e} \cdot X_{k,e} - \frac{1}{\sigma} \sum_{i=1}^{k-1} \left( \tau_{i,\theta} \cdot \tau_{i,\theta,e} \cdot (X_{i+1,e} - X_{i,e}) \left( n - \sum_{l=1}^{i} n_l \right) \right) = 0
\]

(7-16)

The second partial and mixed partial derivatives of the log-likelihood function \( \log L(\theta, a, \sigma) \) with regards to parameters \( \theta, a, \) and \( \sigma \) are obtained as follows:

\[
\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{1}{\sigma^2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( y_{ij,\theta,e} \cdot X_{i,e} \right) - \frac{n_e}{\sigma^2} IT_{\theta,e} \cdot X_{k,e}
\]
\[
+ \frac{1}{\sigma^2} \sum_{i=1}^{k-1} \left( \tau_{i,\theta,e} \cdot (X_{i+1,e} - X_{i,e}) \left( n - \sum_{l=1}^{i} n_l \right) \right)
\]

(7-17)
\[
= -\frac{1}{\sigma^2} \sum_{i=1}^{k} n_i
\]
\[ \frac{\partial^2 \log L}{\partial \theta \partial a} = \frac{\partial^2 \log L}{\partial a \partial \theta} = \frac{1}{\sigma} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( X_i \psi_j \cdot X_{i,e} \right) + \frac{1}{\sigma} n_c X_k IT_{\theta,e} \cdot X_{k,e} \]
\[ - \frac{1}{\sigma} \sum_{i=1}^{k} \left( \tau_{i,\theta,e} \cdot \left( X_{i+1,e} - X_{i,e} \right) \left( n - \sum_{l=1}^{i} n_l \right) \right) \]  
\[ = \frac{1}{\sigma} \sum_{i=1}^{k} n_i X_i \]

\[ \frac{\partial^2 \log L}{\partial \theta \partial \sigma} = \frac{\partial^2 \log L}{\partial \sigma \partial \theta} = -\frac{1}{\sigma^2} \left[ \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( y_{ij} \cdot y_{ij,e} \cdot X_{ij,e} \right) + n_c \cdot IT_{\theta,e} \cdot X_{k,e} \right] \]
\[ - \sum_{i=1}^{k} \left( \tau_{i,\theta,e} \cdot \left( X_{i+1,e} - X_{i,e} \right) \left( n - \sum_{l=1}^{i} n_l \right) \right) \] 
\[ = -\sum_{i=1}^{k} \left( \frac{1}{X_i} \right)^2 \]

\[ \frac{\partial^2 \log L}{\partial a \partial \sigma} = \frac{\partial^2 \log L}{\partial \sigma \partial a} = \frac{1}{\sigma} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( \psi_j \cdot X_{i,e} \right) + \frac{1}{\sigma} \sum_{i=1}^{k} \left( \tau_{i,\theta,e} \cdot \left( X_{i+1,e} - X_{i,e} \right) \left( n - \sum_{l=1}^{i} n_l \right) \right) \]
\[ = \frac{1}{\sigma} \sum_{i=1}^{k} \left( \frac{1}{X_i} \right)^2 \]

\[ \frac{\partial^2 \log L}{\partial \sigma^2} = \frac{1}{\sigma^2} \left[ \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( y_{ij} \cdot y_{ij,e} \cdot X_{ij,e} \right) + n_c \cdot IT_{\theta,e} \cdot X_{k,e} \right] \]
\[ - \sum_{i=1}^{k} \left( \tau_{i,\theta,e} \cdot \left( X_{i+1,e} - X_{i,e} \right) \left( n - \sum_{l=1}^{i} n_l \right) \right) \] 
\[ = -\sum_{i=1}^{k} \left( \frac{1}{X_i} \right)^2 \]

The expected Fisher information matrix \( \hat{F}_8 \) is obtained from the equations (7-17) to (7-22). Its element is the expected values of the negative second partial and mixed partial derivative of \( \log L(\theta, a, \sigma) \) with respect to parameters \( \theta, a, \) and \( \sigma \).
\[
\hat{F}_8 = \frac{n}{\sigma^2} \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{12} & A_{22} & A_{23} \\
A_{13} & A_{23} & A_{33}
\end{bmatrix}
\]

(7-23)

\[
A_{11} = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \theta^2} \right) \right] = \sum_{i=1}^{k} C_i,
\]

\[
A_{12} = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \theta \partial a} \right) \right] = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial a \partial \theta} \right) \right] = - \sum_{i=1}^{k} C_i \chi_i \hat{\sigma},
\]

\[
A_{22} = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial a^2} \right) \right] = \sum_{i=1}^{k} C_i \chi_i^2 \hat{\sigma}^2,
\]

\[
A_{13} = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \theta \partial \sigma} \right) \right] = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \sigma \partial \theta} \right) \right] = \sum_{i=1}^{k} \Phi_i + D_1,
\]

\[
A_{23} = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial a \partial \sigma} \right) \right] = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \sigma \partial a} \right) \right] = - \sum_{i=1}^{k} \chi_i \Phi_i \hat{\sigma} + D_2 \hat{\sigma},
\]

\[
A_{33} = E \left[ \frac{\sigma^2}{n} \left( - \frac{\partial^2 \log L(\theta, a, \sigma)}{\partial \sigma^2} \right) \right] = \sum_{i=1}^{k} C_i + \sum_{i=1}^{k} \Psi_i + D_3,
\]

where \( C_i = E \left[ \frac{n_i}{n} \right], \Phi_i = E \left[ \frac{1}{n} \sum_{j=1}^{n_i} (y_{ij, \theta} y_{ij, \theta, e} \cdot \chi_{i,e}) \right], \Psi_i = E \left[ \frac{1}{n} \sum_{j=1}^{n_i} (y_{ij, \theta}^2 y_{ij, \theta, e} \cdot \chi_{i,e}) \right].
\]

\[
D_1 = - \sum_{i=1}^{k-1} \tau_{i, \theta} \tau_{i, \theta, e} \cdot \left( \chi_{i+1,e} - \chi_{i,e} \right) \left( n - \sum_{i=1}^{k} n_i \right) + \frac{n_i}{n} \cdot IT_{\theta} \cdot IT_{\theta, e} \cdot \chi_{k,e}
\]

\[
D_2 = \sum_{i=1}^{k-1} \tau_{i, \theta} \tau_{i, \theta, e} \cdot \left( \chi_{i+1,e} \chi_{i+1,e} - \chi_{i,e} \chi_{i,e} \right) \left( n - \sum_{i=1}^{k} n_i \right) - \frac{n_i}{n} \cdot IT_{\theta} \cdot IT_{\theta, e} \cdot \chi_{k,e} \cdot \chi_{k,e},
\]

\[
D_3 = - \sum_{i=1}^{k-1} \tau_{i, \theta} \tau_{i, \theta, e} \cdot \left( \chi_{i+1,e} - \chi_{i,e} \right) \left( n - \sum_{i=1}^{k} n_i \right) + \frac{n_i}{n} \cdot IT_{\theta}^2 \cdot IT_{\theta, e} \cdot \chi_{k,e}.
\]
Details of derivation of $C_i$, $\Phi_i$, and $\Psi_i$, $i = 1, 2, \cdots, k$ are shown in Appendix IX.

### 7.1.3 Optimality Criterion & Test Design

In order to evaluate and obtain consistent estimators, the minimum variance principle of estimation is applied. The optimal stress change times are determined by minimizing the AV of the desired MLE estimates. Two types of optimization criteria are considered as follows.

#### A. Optimality Criterion I

The first case considers the life estimate of a product as the primary estimate. Thus the optimization criterion is to minimize the asymptotic variance of log of the percentile life under usual operating condition $S_0$.

The failure intensity function at normal operating stress $S_0$ can be obtained from PH model in equation (7-3):

$$\hat{\lambda}_0(t) = \frac{\hat{\delta}}{\hat{\eta}} \left( \frac{t}{\hat{\eta}} \right)^{\hat{\delta}-1} e^{\hat{\delta} \cdot x_0},$$

and the CDF at normal operating stress $S_0$ is derived as follows:

$$G_0(t) = 1 - \exp \left( - \int_0^t \lambda_0(t') dt' \right) = 1 - \exp \left( - \left( \frac{t}{\hat{\eta}} \right)^\delta \exp(\hat{\delta} X_0) \right).$$

Therefore, by letting the above CDF be equal to $p$, taking the logarithm of both sides of the equation twice, and solving for $\log(t)$, the log of the $p$-percentile life estimate at normal operating stress $S_0$ is obtained as follows:

$$\hat{Y}_{0,p} = \log(\hat{t}_{0,p}) = \hat{\theta} + \left[ \log(- \log(1 - p)) - \hat{\delta} X_0 \right] \cdot \hat{\sigma} \quad (7-24)$$

Where $\hat{t}_{0,p}$ is the $p$-percentile life estimate at normal operating stress $S_0$, and
\( \hat{\theta}, \hat{a}, \text{and} \ \hat{\sigma} \) are MLE of these parameters.

Next, the AV of \( \hat{Y}_{0,p} \) is derived as follows:

\[
AV(\hat{Y}_{0,p}) = AV[\hat{\theta} + (\log(-\log(1-p)) - \hat{\sigma}X_0) \cdot \hat{\sigma}]
\]

The column vector \( \mathbf{H}_{10} \) of the first partial derivative of \( Y_{0,p} \) with respect to each parameter \( \theta, a, \text{and} \ \sigma \) is calculated first,

\[
\mathbf{H}_{10} = \begin{bmatrix}
\frac{\partial \hat{Y}_{0,p}}{\partial \theta}, & \frac{\partial \hat{Y}_{0,p}}{\partial a}, & \frac{\partial \hat{Y}_{0,p}}{\partial \sigma}
\end{bmatrix} = [1, -\hat{\sigma}, (\log(-\log(1-p)) - \hat{\sigma}X_0)].
\]

The AV of \( \hat{Y}_{0,p} \) is obtained as follows:

\[
AV(\hat{Y}_{0,p}) = \mathbf{H}_{10} \cdot \hat{\mathbf{F}}^{-1}_{8} \cdot \mathbf{H}_{10}^\prime
\]  
(7-25)

where the prime “\(^\prime\)" denotes the transpose, and \( \hat{\mathbf{F}}^{-1}_{8} \) is the inverse of the expected Fisher information matrix in equation (7-23).

Therefore, the objective is to determine the optimal \( \tau_i, \ i = 1, 2, \cdots, k-1 \)

through:

\[
\text{minimize } n \cdot AV\left[\hat{Y}_{0,p}\right]
\]  
(7-26)

**B. Optimality Criterion II**

The focus of this case is on the probability that a product or a system will survive for a specified time \( \zeta \), that is, the reliability estimate for a given time \( \zeta \). Therefore, the optimization criterion is to minimize the AV of reliability estimate at a given time \( \zeta \).

The MLE of the reliability estimate at a given time \( \zeta \) under normal operating condition is:
\[
\hat{R}_{0,\sigma} = \exp\left[ - \int_0^\tau \lambda_0(t) \, dt \right] = \exp\left[ - \left( \frac{\tau}{\hat{\eta}} \right) \exp(\hat{a}X_0) \right]
\]

\[
= \exp\left[ - \exp\left( \frac{\log(\tau) - \hat{\theta}}{\hat{\sigma}} \right) + \hat{a}X_0 \right].
\]

(7-27)

The AV of reliability estimate at time \( \tau \) under normal operating condition is obtained:

\[
AV(\hat{R}_{0,\sigma}) = AV\left[ \exp\left[ - \exp\left( \frac{\log(\tau) - \hat{\theta}}{\hat{\sigma}} \right) + \hat{a}X_0 \right] \right]
\]

The column vector of the first partial derivative of \( R_{0,\sigma} \) with respect to each parameter \( \theta, a, \) and \( \sigma \) is calculated first,

\[
H_{11} = \left[ \frac{\partial \hat{R}_{0,\sigma}}{\partial \theta}, \frac{\partial \hat{R}_{0,\sigma}}{\partial a}, \frac{\partial \hat{R}_{0,\sigma}}{\partial \sigma} \right]'
\]

\[
= \left[ \begin{array}{c}
\frac{1}{\hat{\sigma}} \exp\left( \frac{\log(\tau) - \hat{\theta}}{\hat{\sigma}} + \hat{a}X_0 \right) - \exp\left( \frac{\log(\tau) - \hat{\theta}}{\hat{\sigma}} + \hat{a}X_0 \right)
- X_0 \exp\left( \frac{\log(\tau) - \hat{\theta}}{\hat{\sigma}} + \hat{a}X_0 \right) - \exp\left( \frac{\log(\tau) - \hat{\theta}}{\hat{\sigma}} + \hat{a}X_0 \right)
\frac{1}{\hat{\sigma}^2} \left( \log(\tau) - \hat{\theta} \right) \exp\left( \frac{\log(\tau) - \hat{\theta}}{\hat{\sigma}} + \hat{a}X_0 \right) - \exp\left( \frac{\log(\tau) - \hat{\theta}}{\hat{\sigma}} + \hat{a}X_0 \right) \end{array} \right].
\]

Thus, the AV of \( \hat{R}_{0,\sigma} \) is

\[
AV(\hat{R}_{0,\sigma}) = H_{11} \cdot \hat{F}_8^{-1} \cdot H_{11}'
\]

(7-28)

Therefore, the objective is to determine optimal \( \tau_i \), \( i = 1, 2, \ldots, k - 1 \) through:

\[
\text{minimize} \quad n \cdot AV(\hat{R}_{0,\sigma})
\]

(7-29)

Next section presents the numerical examples of the optimal design for \( k \)-step Weibull SSALT with censoring data using the above optimality criteria.
7.1.4  Numerical Examples and Sensitivity Analysis

Having the initial estimates of the parameters \( \hat{\theta}, \hat{a}, \) and \( \hat{\sigma} \), the optimal test for \( k \)-step Weibull SSALT with censoring data can be planned. These preliminary estimated values of parameters could be obtained from past experience with similar products and/or some target values for the product performance. Numerical examples are given to demonstrate the optimum test design. Sensitivity analysis is performed for each example.

A. Numerical Example 7-1

This example uses the experimental failure data of Mettas [48] to make the pre-estimates for the unknown parameters. The objective is to obtain optimal stress change time by minimization of the AV of logarithm of percentile life under usual operating conditions, based on optimality criterion I. Suppose we choose temperature as a stress factor. Table 7-1 shows the results of a constant-stress accelerated life test for two different temperatures: 358 K and 378 K. It is also known that the normal operating temperature is 328K.

<table>
<thead>
<tr>
<th>Temperature (K)</th>
<th>Failure times</th>
</tr>
</thead>
<tbody>
<tr>
<td>358</td>
<td>445, 498, 586, 691, 750. 20 units suspended at 750.</td>
</tr>
<tr>
<td>378</td>
<td>176, 211, 252, 266, 298, 309, 343, 364, 387, 398. 14 units suspended at 445.</td>
</tr>
</tbody>
</table>
There is only one acceleration factor (temperature) and only two stress levels are included, where $k = 2$ for this example. From the data of Table 1, the initial estimates of the parameters $\hat{\theta}, \hat{a},$ and $\hat{\sigma}$ are obtained:

$$\hat{\theta} = \log(\hat{\gamma}) = 6.17, \quad \hat{a} = -7.56, \quad \hat{\sigma} = 1/\hat{\delta} = 0.30,$$

and $X_1 = 0.4$.

Having the pre-estimates of parameters $\hat{\theta}, \hat{a},$ and $\hat{\sigma}$, we can proceed to obtain the optimal SSALT. In this example optimization criterion I of equation (7-26) is used to minimize the asymptotic variance of log of the percentile life under usual operating condition $S_0$. Table 7-2 shows the optimal hold time for different censoring time.

**Table 7-2  Results of Optimal Hold Time in Example 7-1**

<table>
<thead>
<tr>
<th>Censoring Time T</th>
<th>Optimal Hold Time $\tau^*$</th>
<th>nAV</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>176.40</td>
<td>768.242</td>
</tr>
<tr>
<td>500</td>
<td>296.00</td>
<td>141.506</td>
</tr>
<tr>
<td>800</td>
<td>790.32</td>
<td>23.423</td>
</tr>
</tbody>
</table>

The sensitivity analysis is performed in order to identify the sensitive parameters which must be estimated with special care. Table 7-3 presents the effect of a 1% increase in the pre-estimated parameters $\hat{\theta}, \hat{a},$ and $\hat{\sigma}$ on the optimal hold time $\tau^*$.

**Table 7-3  %Δ of $\tau^*$ due to 1% increase in $\hat{\theta}, \hat{a},$ and $\hat{\sigma}$ in Example 7-1**

<table>
<thead>
<tr>
<th></th>
<th>1% increase in $\hat{\theta}$</th>
<th>1% increase in $\hat{a}$</th>
<th>1% increase in $\hat{\sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>%Δ$\tau^*$</td>
<td>0.000169</td>
<td>-0.00135</td>
<td>0.004899</td>
</tr>
<tr>
<td>%ΔnAV</td>
<td>0.030373</td>
<td>-0.02062</td>
<td>-0.02091</td>
</tr>
</tbody>
</table>
From the result in Table 7-3, it can be seen that the effect of all three pre-estimates of parameters $\hat{\theta}, \hat{a},$ and $\hat{\sigma}$ on the optimal hold time $\tau^*$ are very small, therefore these parameters are not sensitive parameters.

**B. Numerical Example 7-2**

The results of an accelerated life-test on a plastic insulating system [25, 26] are shown in Table 7-4. It is assumed that the actual service stress is 3 kV/mm. The model given in [25, 26] assumes the Weibull inverse power law. Thus the stress values are transformed for this example by: $S = \log(V - 2.77)$. The objective is to find the optimal stress change time by minimization of the AV of reliability under usual operating conditions, based on optimality criterion II.

<table>
<thead>
<tr>
<th>Stress (kV/mm)</th>
<th>Failure times (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>397.4, 445.6, 592.3, 688.8, 707.0, 1642</td>
</tr>
<tr>
<td>10</td>
<td>25.2, 44.4, 44.5, 46.4, 58.1, 92.2</td>
</tr>
<tr>
<td>20</td>
<td>6.8, 7.4, 8.0, 11.7, 18.3, 23.0</td>
</tr>
</tbody>
</table>

In this example, there are three stress levels, thus $k = 3$. From the above data, the pre-estimates of the parameters from [25, 26] are first obtained as follows:

$$\hat{\theta} = \log(\hat{\gamma}) = 2.66, \quad \hat{\sigma} = \frac{1}{\hat{\beta}} = 0.488, \quad \hat{a} = -14.25, \quad \text{and} \quad X_1 = 0.61, X_2 = 0.20.$$  

Then by applying the pre-estimates and the optimization criterion II of equation (7-29), which is used to minimize the AV of reliability estimate at a given time $\zeta$, the
optimal SSALT plans are obtained. Table 7-5 shows the optimal stress change times for different \( \zeta \) and censoring time. The optimal standardized stress change time is defined as the ratio of the optimal stress change time to the censoring time \( T \). Results in Table 7-5 show that when censoring time increases, the optimal standardized change time \( \tau_1^* \) increases, while the \( \tau_2^* \) decreases. Also, the nAV decreases significantly as censoring time increases, since longer test time leads to more failure data and thus better estimates.

<table>
<thead>
<tr>
<th>Censoring Time ( T )</th>
<th>Given Time ( \zeta )</th>
<th>Optimal Standardized Stress Change Time</th>
<th>nAV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \tau_1^* )</td>
<td>( \tau_2^* )</td>
</tr>
<tr>
<td>10</td>
<td>1000</td>
<td>0.096</td>
<td>0.466</td>
</tr>
<tr>
<td></td>
<td>2000</td>
<td>0.082</td>
<td>0.451</td>
</tr>
<tr>
<td>20</td>
<td>1000</td>
<td>0.137</td>
<td>0.458</td>
</tr>
<tr>
<td></td>
<td>2000</td>
<td>0.121</td>
<td>0.439</td>
</tr>
</tbody>
</table>

For different stress levels \( X_1 \) and \( X_2 \), the optimal stress change times varies. Results are shown in Table 7-6. For a fixed \( X_1 = 0.6 \), the nAV is minimum at \( X_2 = 0.20 \). Thus, we choose \( X_2 = 0.20 \) and plan the SSALT for different \( X_1 \) values. Results show that the smaller \( X_1 \) values leads to the better estimate.

The sensitivity analysis for the three pre-estimate parameters \( \hat{\theta} = \log(\hat{\eta}) = 2.66, \hat{\sigma} = 1/\hat{\beta} = 0.488, \hat{a} = -14.25 \) for specified stress levels \( X_1 = 0.3 \) and \( X_2 = 0.2 \) are also performed.
Table 7-6  Optimal Hold Times for Different Stress Levels for Example 7-2

<table>
<thead>
<tr>
<th>Stress Level 1</th>
<th>Stress Level 2</th>
<th>Optimal Stress Change Time</th>
<th>nAV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\tau_1^*$</td>
<td>$\tau_2^*$</td>
</tr>
<tr>
<td>0.60</td>
<td>0.40</td>
<td>0.010</td>
<td>0.572</td>
</tr>
<tr>
<td></td>
<td>0.35</td>
<td>0.020</td>
<td>0.537</td>
</tr>
<tr>
<td></td>
<td>0.30</td>
<td>0.028</td>
<td>0.504</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.056</td>
<td>0.475</td>
</tr>
<tr>
<td></td>
<td><strong>0.20</strong></td>
<td><strong>0.085</strong></td>
<td><strong>0.451</strong></td>
</tr>
<tr>
<td></td>
<td>0.15</td>
<td>0.132</td>
<td>0.436</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.201</td>
<td>0.437</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.335</td>
<td>0.486</td>
</tr>
<tr>
<td>0.65</td>
<td></td>
<td>0.0810</td>
<td>0.4509</td>
</tr>
<tr>
<td>0.60</td>
<td></td>
<td>0.0850</td>
<td>0.4510</td>
</tr>
<tr>
<td>0.55</td>
<td></td>
<td>0.0900</td>
<td>0.4511</td>
</tr>
<tr>
<td>0.50</td>
<td></td>
<td>0.1033</td>
<td>0.4518</td>
</tr>
<tr>
<td>0.45</td>
<td></td>
<td>0.1190</td>
<td>0.4528</td>
</tr>
<tr>
<td>0.40</td>
<td></td>
<td>0.1389</td>
<td>0.4538</td>
</tr>
<tr>
<td>0.35</td>
<td></td>
<td>0.1699</td>
<td>0.4558</td>
</tr>
<tr>
<td>0.30</td>
<td></td>
<td>0.2131</td>
<td>0.4576</td>
</tr>
<tr>
<td>0.25</td>
<td></td>
<td>0.2738</td>
<td>0.4578</td>
</tr>
</tbody>
</table>

Figure 7-2 demonstrates that these parameters are not sensitive parameters:

- When $\theta$ increases, $\tau_1^*$ slightly decreases, and $\tau_2^*$ slightly increases, nAV decreases, and they all converge to a fixed value for a large $\theta$.
- When $\alpha$ increases, $\tau_1^*$ increases, $\tau_2^*$ slightly decreases, and nAV increases.
- When $\sigma$ increases, $\tau_1^*$ and $\tau_2^*$ decrease, and nAV decreases significantly.
Figure 7-2  Sensitivity Analysis of Parameters $\theta$, $a$, and $\sigma$ for Example 7-2
7.2 Multi-Variate SSALT for PH Model

This section extends the results of multiple SSALT in section 7.1 to the case where the test includes multiple stress variables, which is called the multi-variate SSALT for PH model. Assumptions and model are proposed in section 7.2.1. Section 7.2.2 presents the MLE and Fisher information matrix, followed by optimal criteria in section 7.2.3.

7.2.1 Assumptions and Model

In multi-variate SSALT, all \( n \) test units are initially placed on a lower stress level \((S_{1,1}, S_{2,1}, \ldots, S_{m,1})\) and run until time \( \tau_1 \) when the stress level is changed to \((S_{1,2}, S_{2,2}, \ldots, S_{m,2})\). The test is continued until time \( \tau_2 \) when the stress level is changed to \((S_{1,3}, S_{2,3}, \ldots, S_{m,3})\), and so on. The test is continued until time \( \tau_i \) when the stress level is changed to \((S_{1,i+1}, S_{2,i+1}, \ldots, S_{m,i+1})\), for \( i = 1, 2, \ldots, k - 1 \). The test is continued at the step \( k \), with stress level \((S_{1,k}, S_{2,k}, \ldots, S_{m,k})\), until all units fail or until a predetermined censoring time \( T \), whichever occurs first.

\( n_i \) failures are observed at time \( t_{ij}, j = 1, 2, \ldots, n_i \) in step \( i, i = 1, 2, \ldots, k \), and \( n_c \) units are censored, \( n_c = n - \sum_{i=1}^{k} n_i \). Figure 7-1 demonstrates the test procedure for the multi-variate SSALT model, where \( S_{l,i} \) is the stress level for variable \( l \) at step \( i \), for \( l = 1, 2, \ldots, m \), and \( i = 1, 2, \ldots, k \), and \((S_{1,0}, S_{2,0}, \ldots, S_{m,0})\) are stress levels at usual operating conditions.
Figure 7-3 describes the multi-variate SSALT test procedure, where $S_{i,l}$ is the stress level for variable $l$ at step $i$, for $l = 1, 2, \cdots, m$, and $i = 1, 2, \cdots, k$, and $(S_{1,0}, S_{2,0}, \cdots, S_{m,0})$ are stress levels at usual operating conditions.

**Figure 7-3  Test Procedure for Multi-Variate SSALT**

Similar assumptions are made as to the multiple SSALT with single stress variable.

1) The life time of the test units at highest stress level $(S_{1,k}, S_{2,k}, \cdots, S_{m,k})$ follows Weibull distribution with shape parameter $\delta$ and scale parameter $\eta$. The baseline failure intensity function is defined as:

$$
\lambda_0(t) = \frac{\delta}{\eta} \left( \frac{t}{\eta} \right)^{\delta-1}
$$

(7-30)

The baseline failure intensity function is defined for test units at highest stress level to obtain a quick initial estimate of the parameters.

2) The PH model is assumed for other stress levels. The intensity function of a test unit is:
\[ \lambda(t; X) = \lambda_0(t) \cdot e^{\mathbf{a}X}, \]  

(7-31)

where \( \mathbf{a} = [a_1, a_2, \ldots, a_m] \), and \( X_i = [X_{1,i}, X_{2,i}, \ldots, X_{m,i}]^T \),

\[
X_{l,i} = \frac{S_{l,i} - S_{l,k}}{S_{l,0} - S_{l,k}} \text{ is standardized stress level, } 0 < X_{l,i} < 1,
\]

\( l = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, k. \)

Then from equations (7-30) and (7-31), the intensity function for stress level \( X_i, \quad i = 1, 2, \ldots, k \) becomes:

\[
\lambda_i(t; X_i) = \frac{\delta}{\eta} \left( \frac{t}{\eta} \right)^{\delta-1} \cdot e^{\mathbf{a}X_i} \]  

(7-32)

3) The cumulative exposure model is utilized to relate the CDF of SSALT to the CDF of constant stress levels. The results of multiple SSALT PH-CE model can be easily used to obtain the multi-variate PH-CE model by setting \( \mathbf{a} \) as a row vector of unknown parameters and \( \mathbf{X}_i \) as a column vector of stress levels.

\[
F(t) = 1 - \left\{ \begin{array}{ll}
\exp \left( - \left( \frac{t}{\eta} \exp \left( \frac{\mathbf{a}}{\delta} \mathbf{X}_1 \right) \right)^{\delta} \right) & 0 \leq t < \tau_1 \\
\exp \left( - \left( \frac{t - \tau_1}{\eta} \exp \left( \frac{\mathbf{a}}{\delta} \mathbf{X}_2 \right) + \frac{\tau_1}{\eta} \exp \left( \frac{\mathbf{a}}{\delta} \mathbf{X}_1 \right) \right)^{\delta} \right) & \tau_1 \leq t < \tau_2 \\
\vdots & \\
\exp \left( - \left( \frac{t - \tau_{k-1}}{\eta} \exp \left( \frac{\mathbf{a}}{\delta} \mathbf{X}_k \right) + \sum_{j=1}^{k-1} \left( \tau_j - \tau_{j-1} \right) \exp \left( \frac{\mathbf{a}}{\delta} \mathbf{X}_j \right) \right)^{\delta} \right) & \tau_{k-1} \leq t < \infty
\end{array} \right.
\]

Based on the above assumptions, the revised CDF of time to failure is proposed:
Chapter 7 Multi-Variate SSALT Plan for Proportional Hazards Model

Let \( F(t) = 1 - \exp \left\{ - \frac{t^\delta}{\eta^\delta} \exp(aX_1) \right\} \) for \( 0 \leq t < \tau_1 \)

\[
F(t) = 1 - \begin{cases} 
\exp \left\{ - \frac{(t - \tau_1^\delta \exp(aX_2) + \tau_1^\delta \exp(aX_1))}{\eta^\delta} \right\} \quad \tau_1 \leq t < \tau_2 \\
\vdots \\
\exp \left\{ - \frac{(t - \tau_{k-1}^\delta \exp(aX_k) + \sum_{l=1}^{k-1} (\tau_l^\delta - \tau_{l-1}^\delta) \exp(aX_l))}{\eta^\delta} \right\} \quad \tau_{k-1} \leq t \leq T
\end{cases}
\] (7-33)

Let \( Y = \log(t), \sigma = \frac{1}{\delta}, \) and \( \theta = \log(\eta), \) then the CDF of random variable \( Y \) is obtained from equation (7-33) as follows:

\[
G(y) = F(\exp(y)) = 1 - \begin{cases} 
\exp \left( - \frac{y_{\theta, e} \cdot X_{1, e}}{\sigma} \right) \quad -\infty < y \leq \log(\tau_1) \\
\exp \left( - \frac{y_{\theta, e} \cdot X_{2, e} + \tau_{1, \theta, e} \cdot (X_{2, e} - X_{1, e})}{\sigma} \right) \quad \log(\tau_1) \leq y < \log(\tau_2) \\
\vdots \\
\exp \left( - \frac{y_{\theta, e} \cdot X_{k, e} + \sum_{l=1}^{k-1} \tau_{l, \theta, e} \cdot (X_{l+1, e} - X_{l, e})}{\sigma} \right) \quad \log(\tau_{k-1}) \leq y \leq \log(T)
\end{cases}
\] (7-34)

Where \( y_{\theta, e} = \exp \left( \frac{y - \theta}{\sigma} \right), X_{i, e} = \exp(aX_i), \tau_{l, \theta, e} = \exp \left( \frac{\log(\tau_l) - \theta}{\sigma} \right). \)

Then the pdf of \( Y \) is obtained by taking the first derivative of \( G(y) \) in equation (7-34).

\[
f(y) = \sigma \begin{cases} 
\frac{1}{\sigma} \exp \left( y_{\theta} + aX_1 - y_{\theta, e} \cdot X_{1, e} \right) \quad -\infty < y \leq \log(\tau_1) \\
\frac{1}{\sigma} \exp \left( y_{\theta} + aX_2 - y_{\theta, e} \cdot X_{2, e} + \tau_{1, \theta, e} \cdot (X_{2, e} - X_{1, e}) \right) \quad \log(\tau_1) \leq y < \log(\tau_2) \\
\vdots \\
\frac{1}{\sigma} \exp \left( y_{\theta} + aX_k - y_{\theta, e} \cdot X_{k, e} + \sum_{l=1}^{k-1} \tau_{l, \theta, e} \cdot (X_{l+1, e} - X_{l, e}) \right) \quad \log(\tau_{k-1}) \leq y \leq \log(T)
\end{cases}
\] (7-35)

Where \( y_{\theta} = \frac{y - \theta}{\sigma} \).
Then the likelihood function is constructed using the CDF and pdf of random variable \( Y \), and MLE method is applied for the estimation of unknown parameters \( \hat{\theta}, \hat{a}, \) and \( \hat{\sigma} \). And then, the expected Fisher information matrix is then obtained from the likelihood function and MLE.

### 7.2.2 MLE and Fisher Information Matrix

The likelihood function is then obtained from the CDF in equation (7-34) and the pdf in equation (7-35) of random variable \( Y \). The likelihood function for \( n_i \) failures at \( Y_j, \ Y_j = \log(t_j), j = 1, \ldots, n_i \) while testing at stress level \( X_i, \ i = 1, 2, \ldots, k \), and \( n_c \) censored test data is obtained:

\[
L(\theta, a_1, a_2, \ldots, a_m, \sigma) = \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma} \exp\left( y_{1j,\theta} + aX_1 - y_{1j,\theta,e} \cdot X_{1,e} \right) \right] \\
\cdot \prod_{j=1}^{n_2} \left[ \frac{1}{\sigma} \exp\left( y_{2j,\theta} + aX_2 - y_{2j,\theta,e} \cdot X_{2,e} + \tau_{1,\theta,e} \cdot (X_{2,e} - X_{1,e}) \right) \right] \\
\cdot \prod_{j=1}^{n_1} \left[ \frac{1}{\sigma} \exp\left( y_{3j,\theta} + aX_3 - y_{3j,\theta,e} \cdot X_{3,e} + \tau_{1,\theta,e} \cdot (X_{3,e} - X_{2,e}) \right) \right] \\
\ldots \\
\prod_{j=1}^{n_k} \left[ \frac{1}{\sigma} \exp\left( y_{kj,\theta} + aX_k - y_{kj,\theta,e} \cdot X_{k,e} + \sum_{i=1}^{k-1} \tau_{i,\theta,e} \cdot (X_{i+1,e} - X_{i,e}) \right) \right] \\
\cdot \prod_{j=1}^{n_c} \left[ \exp\left( -IT_{\theta,e} \cdot X_{k,e} + \sum_{i=1}^{k-1} \tau_{i,\theta,e} \cdot (X_{i+1,e} - X_{i,e}) \right) \right]
\]

(7-36)

Where \( y_{ij,\theta,e} = \exp\left( \frac{y_{ij} - \theta}{\sigma} \right) \), \( y_{ij,\theta} = \left( \frac{y_{ij} - \theta}{\sigma} \right) \), \( IT_{\theta,e} = \exp\left( \frac{\log(T) - \theta}{\sigma} \right) \).

And the log-likelihood function is derived as follows:
\[
\log L(\theta, a_1, a_2, \ldots, a_m, \sigma) = -\sum_{i=1}^{n} n_i \cdot \log(\sigma) + \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij,\theta} \cdot X_{i,j}) + \sum_{j=1}^{n} (-IT_{\theta,e} \cdot X_{k,e}) + \sum_{i=1}^{k} \left( \tau_{i,\theta,e} \cdot (X_{i+1,e} - X_{i,e}) \left(n - \sum_{r=1}^{i} n_r \right) \right)
\]

(7-37)

The MLE of the unknown parameters \((\theta, a_1, a_2, \ldots, a_m, \sigma)\) can be obtained by solving the following differential equations from equation (7-37).

\[
\frac{\partial \log L}{\partial \theta} = -\frac{1}{\sigma} \sum_{i=1}^{k} n_i + \frac{1}{\sigma} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij,\theta} \cdot X_{i,j}) + \frac{n_c}{\sigma} lT_{\theta,e} \cdot X_{k,e} - \frac{1}{\sigma} \sum_{i=1}^{k-1} \left( \tau_{i,\theta,e} \cdot (X_{i+1,i} - X_{i,i}) n - \sum_{r=1}^{i} n_r \right) = 0
\]

(7-38)

\[
\frac{\partial \log L}{\partial a_1} = \sum_{i=1}^{k} n_i X_{1,i} - \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{1,i} y_{ij,\theta} \cdot X_{i,j}) - n_c X_{1,e} lT_{\theta,e} \cdot X_{k,e} + \sum_{i=1}^{k-1} \left( \tau_{i,\theta,e} \cdot (X_{1,i+1,e} - X_{1,i,e}) n - \sum_{r=1}^{i} n_r \right) = 0
\]

(7-39)

\[
\frac{\partial \log L}{\partial a_2} = \sum_{i=1}^{k} n_i X_{2,i} - \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{2,i} y_{ij,\theta} \cdot X_{i,j}) - n_c X_{2,e} lT_{\theta,e} \cdot X_{k,e} + \sum_{i=1}^{k-1} \left( \tau_{i,\theta,e} \cdot (X_{2,i+1,e} - X_{2,i,e}) n - \sum_{r=1}^{i} n_r \right) = 0
\]

(7-40)

\[
\frac{\partial \log L}{\partial a_m} = \sum_{i=1}^{k} n_i X_{m,i} - \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{m,i} y_{ij,\theta} \cdot X_{i,j}) - n_c X_{m,e} lT_{\theta,e} \cdot X_{k,e} + \sum_{i=1}^{k-1} \left( \tau_{i,\theta,e} \cdot (X_{m,i+1,e} - X_{m,i,e}) n - \sum_{r=1}^{i} n_r \right) = 0
\]

(7-39)

\[
\frac{\partial \log L}{\partial \sigma} = -\frac{1}{\sigma} \sum_{i=1}^{k} n_i + \frac{1}{\sigma} \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij,\theta} + \frac{1}{\sigma} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij,\theta} \cdot y_{ij,\theta} \cdot X_{i,j}) + \frac{n_c}{\sigma} lT_{\theta,e} \cdot X_{k,e} - \frac{1}{\sigma} \sum_{i=1}^{k-1} \left( \tau_{i,\theta,e} \cdot \tau_{i,\theta,e} \cdot (X_{i+1,i,e} - X_{i,i,e}) n - \sum_{r=1}^{i} n_r \right) = 0
\]

(7-40)
The second partial and mixed partial derivatives of the log-likelihood function
\[ \log L(\theta, a_1, a_2, \ldots, a_m, \sigma) \] with regards to parameters \( (\theta, a_1, a_2, \ldots, a_m, \sigma) \) are obtained as follows:

\[
\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{1}{\sigma^2} \sum_{i=1}^{k} n_i \tag{7-41}
\]

\[
\frac{\partial^2 \log L}{\partial \theta \partial a_i} = \frac{\partial^2 \log L}{\partial a_i \partial \theta} = \frac{1}{\sigma} \sum_{i=1}^{k} n_i X_{1,i} \tag{7-42}
\]

\[
\frac{\partial^2 \log L}{\partial \theta \partial a_2} = \frac{\partial^2 \log L}{\partial a_2 \partial \theta} = \frac{1}{\sigma} \sum_{i=1}^{k} n_i X_{2,i} \tag{7-43}
\]

\[ : \]

\[
\frac{\partial^2 \log L}{\partial \theta \partial a_m} = \frac{\partial^2 \log L}{\partial a_m \partial \theta} = \frac{1}{\sigma} \sum_{i=1}^{k} n_i X_{m,i} \tag{7-44}
\]

\[
\frac{\partial^2 \log L}{\partial \theta \partial \sigma} = \frac{\partial^2 \log L}{\partial \sigma \partial \theta} = -\frac{1}{\sigma^2} \left[ \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( y_{ij,\theta} y_{ij,\theta,e} \cdot X_{i,e} \right) + n_c \cdot IT_{\theta} \cdot IT_{\theta,e} \cdot X_{k,e} \right] - \frac{k-1}{n} \sum_{i=1}^{k} \left( \tau_{i,\theta} \tau_{i,\theta,e} \left( X_{i+1,e} - X_{i,e} \left( n - \sum_{i=1}^{n} n_r \right) \right) \right) \tag{7-45}
\]

For \( u = 1, 2, \ldots, m, \ v = 1, 2, \ldots, m \)

\[
\frac{\partial^2 \log L}{\partial a_u^2} = -\sum_{i=1}^{k} n_i X_{u,i}^2 \tag{7-46}
\]

\[
\frac{\partial^2 \log L}{\partial a_u \partial a_v} = -\sum_{i=1}^{k} n_i X_{u,i} X_{v,i} \tag{7-47}
\]

\[ : \]

\[
\frac{\partial^2 \log L}{\partial a_u \partial \sigma} = \frac{\partial^2 \log L}{\partial \sigma \partial a_u} = \frac{1}{\sigma} \left[ \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( y_{ij,\theta} y_{ij,\theta,e} \cdot X_{u,i} \cdot X_{i,e} \right) + n_c \cdot IT_{\theta} \cdot IT_{\theta,e} \cdot X_{u,k} \cdot X_{k,e} \right] - \frac{k-1}{n} \sum_{i=1}^{k} \left( \tau_{i,\theta} \tau_{i,\theta,e} \left( X_{u+1,e} - X_{u,i} \left( n - \sum_{i=1}^{n} n_r \right) \right) \right) \tag{7-48}
\]
\[
\frac{\partial^2 \log L}{\partial \sigma^2} = -\frac{1}{\sigma^2} \left[ \sum_{i=1}^{k} n_i + \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij,0}^2 Y_{ij,0,e} \cdot X_{i,e}) + n_e I r^2 \cdot T_{\theta,e} \cdot X_{k,e} \right] - \sum_{i=1}^{k-1} \left( \tau_{i,\tau,0,e} (X_{i+1,e} - X_{i,e}) \left( n - \sum_{r=1}^{n_i} n_r \right) \right) \] (7-49)

The expected Fisher information matrix \( \hat{F}_9 \) is obtained from the equations (7-41) to (7-49). Its elements are the expected values of the negative second partial and mixed partial derivative of \( \log L(\theta, a_1, a_2, \cdots, a_m, \sigma) \) with respect to parameters \( (\theta, a_1, a_2, \cdots, a_m, \sigma) \).

\[
\hat{F}_k = \frac{n}{\sigma^2} 
\begin{bmatrix}
A_{1,1} & A_{1,2} & A_{1,3} & \cdots & A_{1,m+1} & A_{1,m+2} \\
A_{1,2} & A_{2,2} & A_{2,3} & \cdots & A_{2,m+1} & A_{2,m+2} \\
A_{1,3} & A_{2,3} & A_{3,3} & \cdots & A_{3,m+1} & A_{3,m+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
A_{1,m+1} & A_{2,m+1} & A_{3,m+1} & \cdots & A_{m+1,m+1} & A_{m+1,m+2} \\
A_{1,m+2} & A_{2,m+2} & A_{3,m+2} & \cdots & A_{m+1,m+2} & A_{m+2,m+2}
\end{bmatrix}
\] (7-50)

\[
A_{1,1} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \theta^2} \right) \right] = \sum_{i=1}^{k} C_i, \\
A_{1,2} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \theta \partial a_1} \right) \right] = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial a_1 \partial \theta} \right) \right] = -\sum_{i=1}^{k} C_i X_{1,i} \hat{\sigma}, \\
A_{1,l+1} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \theta \partial a_l} \right) \right] = -\sum_{i=1}^{k} C_i X_{l,i} \hat{\sigma}, \text{ for } l = 1, 2, \cdots, m \\
A_{1,m+2} = E \left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \theta \partial \sigma} \right) \right] = \sum_{i=1}^{k} \Phi_i + D_i,
\]

For \( u = 1, 2, \cdots, m \), \( v = 1, 2, \cdots, m \),
\[ A_{u+1,v+1} = E\left[ \sigma^2 \left( -\frac{\partial^2 \log L}{\partial a_u \partial a_v} \right) \right] = \sum_{i=1}^{k} C_i X_{u,i} X_{v,i} \hat{\sigma}^2, \]

\[ A_{u+1,m+2} = E\left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial a_u \partial \sigma} \right) \right] = E\left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \sigma \partial a_u} \right) \right] = -\sum_{i=1}^{k} X_{u,i} \Phi_i \hat{\sigma} + D_{u+1}\hat{\sigma}, \]

\[ A_{m+2,m+2} = E\left[ \frac{\sigma^2}{n} \left( -\frac{\partial^2 \log L}{\partial \sigma^2} \right) \right] = \sum_{i=1}^{k} C_i + \sum_{i=1}^{k} \Psi_i + D_{m+2}, \]

where \( C_i = E\left[ \frac{n_i}{n} \right], \Phi_i = E\left[ \frac{1}{n} \sum_{j=1}^{n_i} (y_{j,\theta} y_{j,\theta,e} X_{t,i,e}) \right], \Psi_i = E\left[ \frac{1}{n} \sum_{j=1}^{n_i} (y_{j,\theta}^2 y_{j,\theta,e} X_{t,i,e}) \right] \)

\[ D_1 = -\sum_{i=1}^{k-1} \left( \tau_{i,\theta} \tau_{i,\theta,e} \cdot (X_{i+1,e} - X_{i,e} \left( 1 - \sum_{r=1}^{i} C_r \right)) \right) + C_{\tau} \cdot I_{T,\theta} \cdot I_{T,\theta,e} \cdot X_{k,e} \]

\[ D_{u+1} = \sum_{i=1}^{k-1} \left( \tau_{i,\theta} \tau_{i,\theta,e} \cdot (X_{u,i+1} X_{i+1,e} - X_{u,i} X_{i,e} \left( 1 - \sum_{r=1}^{i} C_r \right)) \right) - C_{\tau} \cdot I_{T,\theta} \cdot I_{T,\theta,e} \cdot X_{u,k} \cdot X_{k,e}, \]

\[ D_{m+2} = -\sum_{i=1}^{k-1} \left( \tau_{i,\theta}^2 \tau_{i,\theta,e} \cdot (X_{i+1,e} - X_{i,e} \left( 1 - \sum_{r=1}^{i} C_r \right)) \right) + C_{\tau} I_{T,\theta}^2 \cdot I_{T,\theta,e} \cdot X_{k,e}. \]

Where the derivation of \( C_i, \Phi_i, \) and \( \Psi_i, \) \( i = 1, 2, \ldots, k \) are similar to that in Appendix IX.

### 7.2.3 Optimality Criterion & Test Design

The optimal stress change times are determined by minimizing the AV of the desired MLE estimates. Similar to section 7.1.3, two types of optimization criteria are considered as follows.
A. Optimality Criterion I

The optimality criterion I considers the percentile life estimate of a product as the primary estimate. Thus the optimization criterion is to minimize the asymptotic variance of log of the percentile life under usual operating condition \((S_{1,0}, S_{2,0}, \ldots, S_{m,0})\).

The failure intensity function at normal operating stress \((S_{1,0}, S_{2,0}, \ldots, S_{m,0})\) can be obtained from PH model in equation (7-32):

\[
\hat{\lambda}_0(t) = \frac{\delta}{\eta} \left( \frac{t}{\eta} \right)^{\delta-1} e^{\sum_{i=1}^{m} \hat{a}_i X_{i,0}},
\]

and the CDF at normal operating stress \((S_{1,0}, S_{2,0}, \ldots, S_{m,0})\) is derived as follows:

\[
G_0(t) = 1 - \exp \left[ - \int_0^t \hat{\lambda}_0(t') dt' \right] = 1 - \exp \left[ - \left( \frac{t}{\eta} \right)^{\delta} \exp \left( \sum_{i=1}^{m} \hat{a}_i X_{i,0} \right) \right].
\]

Therefore, by letting the above CDF be equal to \(p\), taking the logarithm of both sides of the equation twice, and then solving for \(\log(t)\), the log of the \(p\)-percentile life estimate at normal operating stress \((S_{1,0}, S_{2,0}, \ldots, S_{m,0})\) is obtained as follows:

\[
\hat{Y}_{0,p} = \log(\hat{t}_{0,p}) = \hat{\theta} + \left[ \log(-\log(1-p)) - \sum_{i=1}^{m} \hat{a}_i X_{i,0} \right] \cdot \hat{\sigma}
\]

(7-51)

Where \(\hat{t}_{0,p}\) is the \(p\)-percentile life estimate at normal operating stress \((S_{1,0}, S_{2,0}, \ldots, S_{m,0})\), and \(\{\hat{\theta}, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_m, \hat{\sigma}\}\) are MLE of unknown parameters.

Thus, the AV of \(\hat{Y}_{0,p}\) is derived as follows:

\[
AV(\hat{Y}_{0,p}) = AV \left[ \hat{\theta} + \left( \log(-\log(1-p)) - \sum_{i=1}^{m} \hat{a}_i X_{i,0} \right) \cdot \hat{\sigma} \right] = H_{12} \cdot \hat{F}_9^{-1} \cdot H_{12}'
\]

(7-52)
Where \( \hat{F}_g^{-1} \) is the inverse of the expected Fisher information matrix in equation (7-50) and the column vector \( \mathbf{H}_{12} \) is the first partial derivative of \( Y_{0,p} \) with respect to each parameter \( (\theta, a_1, a_2, \ldots, a_m, \sigma) \),

\[
\mathbf{H}_{12} = \left[ \begin{array}{c}
\frac{\partial \hat{Y}_{0,p}}{\partial \theta}, & \frac{\partial \hat{Y}_{0,p}}{\partial a_1}, & \frac{\partial \hat{Y}_{0,p}}{\partial a_2}, & \ldots, & \frac{\partial \hat{Y}_{0,p}}{\partial a_m}, & \frac{\partial \hat{Y}_{0,p}}{\partial \sigma} \\
\end{array} \right] = \left[ 1, -X_{1,0} \hat{\sigma}, -X_{2,0} \hat{\sigma}, \ldots, -X_{m,0} \hat{\sigma}, \left( \log(\log(1-p)) - \sum_{l=1}^{m} \hat{a}_l X_{l,0} \right) \right].
\]

Therefore, the objective is to determine the optimal \( \tau_i, i = 1, 2, \ldots, k - 1 \) through minimizing the nAV in equation (7-52):

\[
\text{minimize} \quad n \cdot \text{AV} \left[ \hat{Y}_{0,p} \right] \tag{7-53}
\]

**B. Optimality Criterion II**

In some occasions, we might want to focus on the probability that a product or a system will survive for a specified time \( \zeta \), that is, the reliability estimate for a given time \( \zeta \). Thus, the optimization criterion II is to minimize the AV of reliability estimate at a given time \( \zeta \).

The MLE of the reliability estimate at a given time \( \zeta \) under normal operating condition is:

\[
\hat{R}_{0,\zeta} = \exp \left[ -\int_0^\zeta \lambda_0(t')dt' \right] = \exp \left[ -\left( \frac{\zeta}{\hat{\eta}} \right) \exp \left( \sum_{l=1}^{m} \hat{a}_l X_{l,0} \right) \right] = \exp \left[ -\exp \left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) + \sum_{l=1}^{m} \hat{a}_l X_{l,0} \right]. \tag{7-54}
\]
The AV of reliability estimate at time $\zeta$ under normal operating condition is obtained:

$$AV(\hat{R}_{0,\zeta}) = AV\left[ \exp\left( - \exp\left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) + \sum_{i=1}^{m} \hat{a}_{i}X_{i,0} \right) \right]$$

Thus, the AV of $\hat{R}_{0,\zeta}$ is

$$AV(\hat{R}_{0,\zeta}) = H_{13} \cdot \hat{F}_{9}^{-1} \cdot H_{13}'$$ (7-55)

Where $H_{13}$ is the column vector of the first partial derivative of $R_{0,\zeta}$ with respect to each parameter $(\theta, a_{1}, a_{2}, \ldots, a_{m}, \sigma)$,

$$H_{13} = \left[ \frac{\partial \hat{R}_{0,\zeta}}{\partial \theta}, \frac{\partial \hat{R}_{0,\zeta}}{\partial a_{1}}, \frac{\partial \hat{R}_{0,\zeta}}{\partial a_{2}}, \ldots, \frac{\partial \hat{R}_{0,\zeta}}{\partial a_{m}}, \frac{\partial \hat{R}_{0,\zeta}}{\partial \sigma} \right]'$$

$$= \begin{bmatrix}
\frac{1}{\hat{\sigma}} \exp\left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) + \hat{a}X_{0} - \exp\left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) \\
-X_{1,0} \exp\left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) + \hat{a}X_{1,0} - \exp\left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) \\
-X_{2,0} \exp\left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) + \hat{a}X_{2,0} - \exp\left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) \\
\vdots \\
-X_{m,0} \exp\left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) + \hat{a}X_{m,0} - \exp\left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) \\
\frac{1}{\hat{\sigma}^2} \exp\left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) + \sum_{i=1}^{m} \hat{a}_{i}X_{i,0} - \exp\left( \frac{\log(\zeta) - \hat{\theta}}{\hat{\sigma}} \right) + \sum_{i=1}^{m} \hat{a}_{i}X_{i,0} \right]'
\right.$$

Therefore, the objective in this case is then to determine optimal $\tau_{i}, i = 1, 2, \ldots, k-1$ through minimizing the nAV in equation (7-55):

$$\text{minimize} \quad n \cdot AV(\hat{R}_{0,\zeta})$$ (7-56)
8.1 Dissertation Contribution

The contributions of this dissertation are summarized as follows:

1) Optimal simple SSALT design based on log-linear life stress relationship for censored Weibull failure data

The simple SSALT, which involves only two stress levels, is a relatively simple case in SSALT. Many researchers focused on exponential failure data because of its simplicity. As mentioned before, the primary feature of the exponential distribution is that it is used for modeling the behavior of items with a constant failure rate. It has a fairly simple mathematical form, which makes it quite easy to manipulate. Unfortunately, this fact also leads to the use of the model in situations where it is not appropriate. For example, it would not be appropriate to use the exponential distribution to model the reliability of an automobile. The constant failure rate of the exponential distribution would require the assumption that the automobile would be just as likely to experience a breakdown during the first mile as it would during the one hundred-thousandth mile. Clearly, this is not a valid assumption.

Weibull failure data is investigated in this dissertation research, which is more applicable in industry. The assumption of Weibull failure data from SSALT increases the complexity of the optimal design problem, leading to a complicated likelihood
function and Fisher information matrix. Two different types of optimization criteria are presented, considering life estimate and reliability estimate. Optimal SSALT plan is proposed by minimizing the AV of the desired life/reliability estimate.

II) Optimal bivariate and multi-variate SSALT design based on log-linear life stress relationship for censored Weibull failure data

In many applications and for various reasons, it is desirable to use more than one accelerating stress variable. The inclusion of more than one stress variable helps to understand the effect of multiple stress variables, and also, it gives added acceleration without changing the failure mode. Integration of Weibull failure data with multiple stress variables results in more complexity in the Fisher information matrix, and a more complicated problem to solve. Two stress variables are considered first, leading to the bivariate SSALT model. Bivariate SSALT model is then extended to a more generalized model: multi-variate SSALT, which includes \( k \) steps and \( m \) stress variables. Optimal stress change times are obtained by minimizing the AV of the MLE.

III) Optimal simple and multi-variate SSALT design based on PH model for Weibull baseline failure rate

The PH model has been widely used in the biomedical field and recently there has been an increasing interest in its application in reliability engineering. It assumes that the failure rate of a system is affected not only by its operation time, but also by the covariates under which it operates. The PH model assumes that the intensity function of a unit is the product of a baseline intensity function and a positive function of stress levels. In this dissertation research, the baseline intensity function is defined at the highest stress levels to obtain a quick initial estimate of the parameters. PH model is
assumed for all other stress levels. A simple SSALT design is first obtained by minimizing the nAV of the desired MLE. The results are extended to multiple SSALT, which considers multiple steps, but only one stress variable. Optimal stress change times for each step are obtained. A more generalized case, multi-variate SSALT based on PH model is then proposed, including $k$ steps and $m$ stress variables. Fisher information matrix and AV of MLE are constructed. Optimal plan is designed to minimize the AV of MLE.

8.2 Future Research

Potential future research work includes the following:

1) Failure-Censored Data Analysis and Design

All of our previous research is focused on time-censored failure data (Type I censored). It is obtained when censoring time is fixed, and then the number of failures in that fixed time is a random variable. There is another type of censoring, called failure-censored, or Type II censored. In this case, the test is stopped after a specified number of failures occur. The time to fixed number of failures is a random variable. Failure censoring can guarantee that enough failures are obtained. Although failure censoring cannot control the test time, it is still useful in many situations. Thus it is beneficial to propose the data analysis model for the Weibull SSALT under failure censoring. Using the order statistics of the data, we can construct the likelihood function and then get MLE and confidence interval of the concerned parameter at normal operating conditions. Optimum test can also be designed.
II) Test Design for Discrete Failure Time Problems

In some cases, it is difficult to get the exact failure time of a unit. Periodic inspection is scheduled at discrete times to reveal the failure of the unit. The failure time is then no longer continuous, but becomes discrete random variables. Under this condition, we will need to determine when the periodic inspection should be made, and how often it should be. One specific case is to assume that the inspection could only be done at times $k \cdot \zeta$, where $k = 1, 2, \cdots$, and $\zeta$ is the time interval for the periodic inspection, and also, the change of stress level could only be made at the inspection time $\mu \cdot \zeta$. Therefore, the design problem becomes to determine both the inspection time period $\zeta$ and the stress change time $\mu$.

III) Multiple Objective Model for SSALT Planning

All of the test design problems in this dissertation research only consider one objective, which is to minimize the asymptotic variance of the desired parameter at usual operating conditions. This objective is referred to the statistical precision of our estimate. It is the most commonly used criterion in the SSALT planning. However, the industrial might also want to minimize the cost for conducting the test, since the budget is limited. The cost for a test usually consists of fixed cost and variable cost. For optimization problem, only variable cost needs to be considered. In the accelerated life test, variable costs include the facility cost, cost of test units, labors cost, etc. Thus this becomes a multiple objective problem: one is to minimize the asymptotic variance of the estimate, and the other is to minimize the cost for conducting the test.
Appendix I Derivation of the Expected Fisher Information Matrix $\hat{F}_2$

The expected Fisher information matrix, $\hat{F}_2$, can be obtained by taking the expected value of the negative second partial and mixed partial derivatives of $\log[L(\mu_1, \mu_2, \sigma)]$ with respect to $(\mu_1, \mu_2, \sigma)$. Equations (3-25) to (3-30) present the second partial and mixed partial derivatives of $\log[L(\mu_1, \mu_2, \sigma)]$ with respect to $(\mu_1, \mu_2, \sigma)$. By taking the negative of the expected value in equations (3-25) to (3-30), the elements of $\hat{F}_2$ in equation (3-31) are obtained as follows.

$$A_{11} = E \left[ -\frac{\sigma^2}{n} \cdot \frac{\partial^2 \log L}{\partial \mu_1^2} \right] = C_1,$$
$$A_{12} = E \left[ -\frac{\sigma^2}{n} \cdot \frac{\partial^2 \log L}{\partial \mu_i \partial \mu_2} \right] = 0,$$
$$A_{13} = E \left[ -\frac{\sigma^2}{n} \cdot \frac{\partial^2 \log L}{\partial \mu_i \partial \sigma} \right] = \Lambda_1 + L_i \exp(L_i)(1 - C_1),$$
$$A_{22} = E \left[ -\frac{\sigma^2}{n} \cdot \frac{\partial^2 \log L}{\partial \mu_2^2} \right] = C_2,$$
$$A_{23} = E \left[ -\frac{\sigma^2}{n} \cdot \frac{\partial^2 \log L}{\partial \mu_2 \partial \sigma} \right] = \Lambda_2 - L_2 \exp(L_2)(1 - C_1) + L_T \exp(L_T)(1 - C_1 - C_2),$$
$$A_{33} = E \left[ -\frac{\sigma^2}{n} \cdot \frac{\partial^2 \log L}{\partial \sigma^2} \right] = C_1 + C_2 + \Omega_1 + \Omega_2 + (1 - C_1)(\exp(L_1) - \exp(L_2))$$
$$+ (1 - C_1 - C_2)\exp(L_T)$$

where $C_i = E \left[ \frac{n_i}{n} \right]$, $\Lambda_i = E \left[ \frac{1}{n} \sum_{j=1}^{n_i} \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \exp \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \right]$, and detailed calculation is shown next.
In order to calculate \( C_i = E \left[ \frac{n_i}{n} \right] \), \( i = 1, 2 \), the test procedure is analyzed.

At the first step, \( n \) iid new products are tested at stresses \( S_1 \) until time \( \tau \). The lifetime of each item follows the CDF in equation (3-17). Since \( n_i \) is the number of failures by time \( \tau \), it is a binomial random variable with parameters \( n \) and \( p_i \). From equation (3-17):

\[
p_1 = G \left( \log \left( \frac{\tau}{T} \right) \right) = 1 - \exp \left( - \exp \left( \frac{\log \tau_0 - \mu_i}{\sigma} \right) \right) = 1 - \exp(-\exp(L_i)), \text{ and}
\]

\[
N_1 = E \left[ \frac{n_1}{n} \right] = p_1 = 1 - \exp(-\exp(L_i))
\]

At the second step, \( n - n_1 \) iid unfailed items are then placed at stresses \( S_2 \) until pre-determined censoring time \( T \). The lifetime of each item follows the CDF of equation (3-17). Since \( n_2 \) is the number of failures between time \( \tau \) and \( T \), it is also a binomial random variable with parameters \( n - n_1 \) and \( p_2 \). From equation (3-17):

\[
p_2 = \Pr(\text{item fails in time } T \mid \text{it not fail in time } \tau \text{ in first step})
\]

\[
= 1 - \Pr(\text{item not fail in time } T \mid \text{item not fail in time } \tau) = 1 - \frac{\exp(-\exp(L_T) + \exp(L_1))}{\exp(-\exp(L_1))}
\]

\[
= 1 - \exp(-\exp(L_T) + \exp(L_2))
\]

\[
N_2 = E \left[ \frac{n_2}{n} \right] = E \left[ \frac{n_2}{n - n_1} \cdot \frac{n - n_1}{n} \right] = p_2 \cdot (1 - p_1)
\]

\[
= (1 - \exp(-\exp(L_T) + \exp(L_1))) \cdot \exp(-\exp(L_1)) = \exp(-\exp(L_1)) - \exp(-\exp(L_T) + \exp(L_2) - \exp(L_1))
\]

where \( G(\bullet) \) is the CDF of random variable \( Y \).
Calculation of $\Lambda_i$ and $\Omega_i$, $i=1,2,3$ is shown as follows:

From the definition of $\Lambda_i$ and $\Omega_i$, $i=1,2,3$, we have:

$$\Lambda_i = E\left[\frac{1}{n} \sum_{j=1}^{n_i} \left( \frac{y_{ij} - \mu_i}{\sigma} \exp\left(\frac{y_{ij} - \mu_i}{\sigma}\right) \right)\right],$$

$$\Omega_i = E\left[\frac{1}{n} \sum_{j=1}^{n_i} \left( \frac{(y_{ij} - \mu_i)^2}{\sigma} \exp\left(\frac{y_{ij} - \mu_i}{\sigma}\right) \right)\right].$$

In order to calculate $\Lambda_i$ and $\Omega_i$, the following transformation are made first:

Let $w = \begin{cases} \exp\left(\frac{y - \mu_1}{\sigma}\right) & -\infty < y \leq \log\tau_0 \\ \exp\left(\frac{y - \mu_2}{\sigma}\right) & \log\tau_0 < y \leq \infty \end{cases}$

For the first section $-\infty < y \leq \log\tau_0$, and $0 < w \leq \exp\left(\frac{\log\tau_0 - \mu_1}{\sigma}\right)$. Then

$$P(W \leq w) = P\left(\exp\left(\frac{y - \mu_1}{\sigma}\right) \leq w\right) = P(y \leq \log w \cdot \sigma + \mu_1) = G(\log w \cdot \sigma + \mu_1).$$

From the CDF of random variable $y$ in equation (3-17), we can obtain:

$$F(w) = 1 - \exp\left(-\exp\left(\frac{\log w \cdot \sigma + \mu_1 - \mu_1}{\sigma}\right)\right) = 1 - \exp(-w), \text{ for } 0 < w \leq \exp\left(\frac{\log\tau_0 - \mu_1}{\sigma}\right).$$

For the second section $\log\tau_0 < y \leq \infty$, and $\exp\left(\frac{\log\tau_0 - \mu_2}{\sigma}\right) < w < \infty$. Then

$$P(W \leq w) = P\left(\exp\left(\frac{y - \mu_2}{\sigma}\right) \leq w\right) = P(y \leq \log w \cdot \sigma + \mu_2) = G(\log w \cdot \sigma + \mu_2).$$

From the CDF of random variable $y$ in equation (3-17), we can obtain:

$$F(w) = 1 - \exp\left(-\exp\left(\frac{\log w \cdot \sigma + \mu_2 - \mu_2}{\sigma}\right) + \exp(L_2) - \exp(L_1)\right) = 1 - \exp(-w + \exp(L_2) - \exp(L_1)).$$
Therefore, the CDF of the random variable \( w \) is:

\[
F(w) = \begin{cases} 
1 - \exp(-w), & 0 < w \leq \exp\left(\frac{\log \tau_0 - \mu_1}{\sigma}\right) \\
1 - \exp(-w + \exp(L_2) - \exp(L_1)), & \exp\left(\frac{\log \tau_0 - \mu_2}{\sigma}\right) < w < \infty 
\end{cases}
\]

The pdf of \( w \) is obtained by taking the first derivative of the CDF.

\[
f(w) = \begin{cases} 
\exp(-w), & 0 < w \leq \exp\left(\frac{\log \tau_0 - \mu_1}{\sigma}\right) \\
\exp(\exp(L_2) - \exp(L_1))\cdot \exp(-w), & \exp\left(\frac{\log \tau_0 - \mu_2}{\sigma}\right) < w < \infty 
\end{cases}
\]

Let \( g(w) = \log(w) \cdot w \), then

\[
\Lambda_1 = E\left[\frac{1}{n} \sum_{j=1}^{n} \frac{y_{1j} - \mu_1}{\sigma} \exp\left(\frac{y_{1j} - \mu_1}{\sigma}\right)\right] = E\left[\frac{1}{n} \sum_{j=1}^{n} \log(w_j) \cdot w_j\right] = E\left[\frac{1}{n} \sum_{j=1}^{n} g(w_j)\right] \\
= \int_0^{\exp\left(\frac{\log \tau_0 - \mu_1}{\sigma}\right)} g(w) f(w) dw = \int_0^{\exp(L_1)} \log(w) \cdot w \cdot \exp(-w) dw
\]

Using similar method, we can have:

\[
\Lambda_2 = E\left[\frac{1}{n} \sum_{j=1}^{n} \frac{y_{2j} - \mu_2}{\sigma} \exp\left(\frac{y_{2j} - \mu_2}{\sigma}\right)\right] \\
= \int_{\exp(L_2)}^{\exp(L_1)} \log(w) \cdot w \cdot \exp(\exp(L_2) - \exp(L_1)) \cdot \exp(-w) dw
\]

\[
\Omega_1 = E\left[\frac{1}{n} \sum_{j=1}^{n} \left(\frac{y_{1j} - \mu_1}{\sigma}\right)^2 \exp\left(\frac{y_{1j} - \mu_1}{\sigma}\right)\right] = \int_{\exp(L_1)}^{\exp(L_1)} (\log(w))^2 \cdot w \cdot \exp(-w) dw
\]

\[
\Omega_2 = E\left[\frac{1}{n} \sum_{j=1}^{n} \left(\frac{y_{2j} - \mu_2}{\sigma}\right)^2 \exp\left(\frac{y_{2j} - \mu_2}{\sigma}\right)\right] \\
= \int_{\exp(L_2)}^{\exp(L_1)} (\log(w))^2 \cdot w \cdot \exp(\exp(L_2) - \exp(L_1)) \cdot \exp(-w) dw
\]
Appendix II  

Asymptotic Variance of \( \hat{Y} \) with respect to \((\beta_0, \beta_1, \sigma)\)

Optimal test plan is obtained from minimization of the AV of \( \hat{Y} \) with respect to \((\beta_0, \beta_1, \sigma)\). Results are compared with the optimal plan by minimizing 
\[ n \cdot AV[\hat{Y}_R(S_0)] \] of equation (3-33). The test results also support that optimal solutions obtained from the minimization of the AV \( \hat{Y} \) of with respect to \((\beta_0, \beta_1, \sigma)\) are the same as those obtained from the minimization of the AV \( \hat{Y} \) with respect to \((\mu_1, \mu_2, \sigma)\).

The AV of \( \hat{Y} \) with respect to \((\beta_0, \beta_1, \sigma)\) is:

\[
AV[\hat{Y}(\beta_0, \beta_1, \sigma)] = \mathbf{H}_5 \cdot \mathbf{F}_3^{-1} \cdot \mathbf{H}_5'
\]  
(A.II-1)

Where \( \mathbf{H}_5 \) is the row vector of the first derivative of \( Y \) with respect to \((\beta_0, \beta_1, \sigma)\),

\[
\mathbf{H}_5 = \begin{bmatrix}
\frac{\partial Y}{\partial \beta_0}, & \frac{\partial Y}{\partial \beta_1}, & \frac{\partial Y}{\partial \sigma}
\end{bmatrix},
\]
(A.II-2)

where \( \mathbf{F}_3 \) is the Fisher information matrix with respect to \((\beta_0, \beta_1, \sigma)\),

\[
\mathbf{F}_3 = \begin{bmatrix}
B_{11}, & B_{12}, & B_{13} \\
B_{12}, & B_{22}, & B_{23} \\
B_{13}, & B_{23}, & B_{33}
\end{bmatrix}
\]
(A.II-3)

\[
\begin{bmatrix}
\frac{\partial^2 \log L}{\partial \beta_0^2}, & \frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1}, & \frac{\partial^2 \log L}{\partial \beta_0 \partial \sigma} \\
\frac{\partial^2 \log L}{\partial \beta_1^2}, & \frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_0}, & \frac{\partial^2 \log L}{\partial \beta_1 \partial \sigma} \\
\frac{\partial^2 \log L}{\partial \sigma^2}, & \frac{\partial^2 \log L}{\partial \beta_0 \partial \sigma}, & \frac{\partial^2 \log L}{\partial \beta_1 \partial \sigma}
\end{bmatrix}
\]

Since the relationship between \((\mu_1, \mu_2)\) & \((\beta_0, \beta_1)\) is:

\[
\begin{cases}
\mu_1 = \beta_0 + \beta_1 S_1 - \log T \\
\mu_2 = \beta_0 + \beta_1 S_2 - \log T
\end{cases}
\]
(A.II-4)

then

\[
\begin{bmatrix}
\frac{\partial \mu_1}{\partial \beta_0} = 1, & \frac{\partial \mu_1}{\partial \beta_1} = S_1 \\
\frac{\partial \mu_2}{\partial \beta_0} = 1, & \frac{\partial \mu_2}{\partial \beta_1} = S_2
\end{bmatrix}
\]
(A.II-5)
Using the chain rules, we can calculate \( \mathbf{H}_3 \) & \( \hat{\mathbf{F}}_3 \) from \( \mathbf{H}_3 \) & \( \hat{\mathbf{F}}_2 \) with respect to \( (\mu_1, \mu_2, \sigma) \) in equation (3-31) & (3-33).

From equation (3-33), the AV of \( \hat{Y}_R(S_0) \) with respect to \( (\mu_1, \mu_2, \sigma) \) is:

\[
AV[\hat{Y}(\mu_1, \mu_2, \sigma)] = \mathbf{H}_3 \cdot \hat{\mathbf{F}}_2^{-1} \cdot \mathbf{H}_3'
\]

where \( \mathbf{H}_3 \) is the row vector of the first derivative of \( Y \) with respect to \( (\mu_1, \mu_2, \sigma) \),

\[
\mathbf{H}_3 = \begin{bmatrix}
\frac{\partial Y}{\partial \mu_1}, & \frac{\partial Y}{\partial \mu_2}, & \frac{\partial Y}{\partial \sigma}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{1-x}, & -\frac{x}{1-x}, & \log(-\log(R))
\end{bmatrix}
\]  

(A.II-6)

and \( \hat{\mathbf{F}}_2 \) is the expected Fisher information matrix with respect to \( (\mu_1, \mu_2, \sigma) \) in equation (3-31).

\[
\hat{\mathbf{F}}_2 = \frac{n}{\sigma^2} \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{12} & A_{22} & A_{23} \\
A_{13} & A_{23} & A_{33}
\end{bmatrix}
\]  

(A.II-7)

where the elements of \( \hat{\mathbf{F}}_2 \) are given below:

\[
A_{11} = 1 - \exp(-\exp(L_1)), \quad A_{12} = 0,
\]

\[
A_{13} = L_1 \exp(L_1 - \exp(L_1)) + \int_0^{\exp(L_1)} w \log(w) \exp(-w) dw
\]

\[
A_{22} = \exp(-\exp(L_1)) - \exp(-\exp(L_T) + \exp(L_2) - \exp(L_1))
\]

\[
A_{23} = L_T \exp(L_T - \exp(L_T) + \exp(L_2) - \exp(L_1)) - L_2 \exp(L_2 - \exp(L_1)) + \exp(L_2 - \exp(L_1)) \cdot \int_{\exp(L_2)}^{\exp(L_T)} w \log(w) \exp(-w) dw
\]

\[
A_{33} = 1 - \exp(-\exp(L_T) + \exp(L_2) - \exp(L_1)) + L_1^2 \exp(L_1 - \exp(L_1))
\]

\[
- L_2^2 \exp(L_2 - \exp(L_1)) + \int_0^{\exp(L_1)} w \log(w)^2 \exp(-w) dw
\]

\[
+ \exp(L_2 - \exp(L_1)) \cdot \int_{\exp(L_2)}^{\exp(L_T)} w \log(w)^2 \exp(-w) dw
\]

\[
+ L_T^2 \exp(L_T - \exp(L_T) + \exp(L_2) - \exp(L_1))
\]
Using the chain rules, from equations (A.II-2), (A.II-5) & (A.II-6), the first derivatives of $Y$ with respect to $(\beta_0, \beta_1, \sigma)$ are obtained as follows:

$$
\frac{\partial Y}{\partial \beta_0} = \frac{\partial Y}{\partial \mu_1} \cdot \frac{\partial \mu_1}{\partial \beta_0} + \frac{\partial Y}{\partial \mu_2} \cdot \frac{\partial \mu_2}{\partial \beta_0} = \frac{1}{1-x} \cdot 1 - \frac{x}{1-x} \cdot 1 = 1,
$$

$$
\frac{\partial Y}{\partial \beta_1} = \frac{\partial Y}{\partial \mu_1} \cdot \frac{\partial \mu_1}{\partial \beta_1} + \frac{\partial Y}{\partial \mu_2} \cdot \frac{\partial \mu_2}{\partial \beta_1} = \frac{1}{1-x} \cdot S_1 - \frac{x}{1-x} \cdot S_2 = \frac{S_1 - S_2 x}{1-x} = S_0, \text{ since }
$$

$$
x = \frac{S_1 - S_0}{S_2 - S_0}.
$$

$$
\frac{\partial Y}{\partial \sigma} = \log(-\log(R)).
$$

Thus, $\mathbf{H}_5 = \left[ \frac{\partial Y}{\partial \beta_0}, \frac{\partial Y}{\partial \beta_1}, \frac{\partial Y}{\partial \sigma} \right] = [1, S_0, \log(\log(R))]$ (A.II-8)

To calculate the Fisher information matrix $\hat{F}_3$, the first derivative of log-likelihood function with respect to $(\beta_0, \beta_1)$ is obtained first:

$$
\frac{\partial \log L}{\partial \beta_0} = \frac{\partial \log L}{\partial \mu_1} \cdot \frac{\partial \mu_1}{\partial \beta_0} + \frac{\partial \log L}{\partial \mu_2} \cdot \frac{\partial \mu_2}{\partial \beta_0} = \frac{\partial \log L}{\partial \mu_1} + \frac{\partial \log L}{\partial \mu_2},
$$

$$
\frac{\partial \log L}{\partial \beta_1} = \frac{\partial \log L}{\partial \mu_1} \cdot \frac{\partial \mu_1}{\partial \beta_1} + \frac{\partial \log L}{\partial \mu_2} \cdot \frac{\partial \mu_2}{\partial \beta_1} = S_1 \frac{\partial \log L}{\partial \mu_1} + S_2 \frac{\partial \log L}{\partial \mu_2}
$$

The second partial and mixed partial derivatives of log-likelihood function with respect to $(\beta_0, \beta_1, \sigma)$ is then derived.

$$
\frac{\partial^2 \log L}{\partial \beta_0^2} = \frac{\partial}{\partial \beta_0} \left( \frac{\partial \log L}{\partial \mu_1} \right) + \frac{\partial}{\partial \beta_0} \left( \frac{\partial \log L}{\partial \mu_2} \right)
$$

$$
= \frac{\partial^2 \log L}{\partial \mu_1^2} \cdot \frac{\partial \mu_1}{\partial \beta_0} + \frac{\partial^2 \log L}{\partial \mu_2^2} \cdot \frac{\partial \mu_2}{\partial \beta_0} + \frac{\partial^2 \log L}{\partial \mu_1 \partial \mu_2} \cdot \frac{\partial \mu_1}{\partial \beta_0} \frac{\partial \mu_2}{\partial \beta_0} + \frac{\partial^2 \log L}{\partial \mu_1^2} \cdot \frac{\partial^2 \log L}{\partial \mu_2^2} \frac{\partial \mu_1}{\partial \beta_0} \frac{\partial \mu_2}{\partial \beta_0},
$$

$$
= \frac{\partial^2 \log L}{\partial \mu_1^2} + 2 \frac{\partial^2 \log L}{\partial \mu_1 \partial \mu_2} + \frac{\partial^2 \log L}{\partial \mu_2^2}
$$
\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} = \frac{\partial}{\partial \beta_1} \left( \frac{\partial \log L}{\partial \mu_1} \right) + \frac{\partial}{\partial \beta_1} \left( \frac{\partial \log L}{\partial \mu_2} \right)
\]

\[
= \frac{\partial^2 \log L}{\partial \mu_1^2} \frac{\partial \mu_1}{\partial \beta_1} + \frac{\partial^2 \log L}{\partial \mu_2^2} \frac{\partial \mu_2}{\partial \beta_1} + \frac{\partial^2 \log L}{\partial \mu_1 \partial \mu_2} \frac{\partial \mu_1}{\partial \beta_1} \frac{\partial \mu_2}{\partial \beta_1} + \frac{\partial^2 \log L}{\partial \mu_1^2} \frac{\partial \mu_1}{\partial \beta_1} + \frac{\partial^2 \log L}{\partial \mu_2^2} \frac{\partial \mu_2}{\partial \beta_1}
\]

\[
= S_1 \frac{\partial^2 \log L}{\partial \mu_1^2} + (S_1 + S_2) \frac{\partial^2 \log L}{\partial \mu_1 \partial \mu_2} + S_2 \frac{\partial^2 \log L}{\partial \mu_2^2}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_1^2} = \frac{\partial}{\partial \beta_1} \left( \frac{\partial \log L}{\partial \mu_1} \right) + \frac{\partial}{\partial \beta_1} \left( \frac{\partial \log L}{\partial \mu_2} \right)
\]

\[
= S_1 \frac{\partial^2 \log L}{\partial \mu_1^2} + S_1 \frac{\partial^2 \log L}{\partial \mu_1 \partial \mu_2} + S_2 \frac{\partial^2 \log L}{\partial \mu_1 \partial \mu_2} + S_2 \frac{\partial^2 \log L}{\partial \mu_2^2} \frac{\partial \mu_2}{\partial \beta_1}
\]

\[
= S_1 \frac{\partial^2 \log L}{\partial \mu_1^2} + 2S_1S_2 \frac{\partial^2 \log L}{\partial \mu_1 \partial \mu_2} + S_2 \frac{\partial^2 \log L}{\partial \mu_2^2}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \sigma} = \frac{\partial^2 \log L}{\partial \mu_1 \partial \sigma} + \frac{\partial^2 \log L}{\partial \mu_2 \partial \sigma},
\]

\[
\frac{\partial^2 \log L}{\partial \beta_1 \partial \sigma} = S_1 \frac{\partial^2 \log L}{\partial \mu_1 \partial \sigma} + S_2 \frac{\partial^2 \log L}{\partial \mu_2 \partial \sigma}.
\]

Therefore, the Fisher information matrix with respect to \((\beta_0, \beta_1, \sigma)\) is obtained:

\[
\hat{F}_3 = \begin{bmatrix}
B_{11}, & B_{12}, & B_{13} \\
B_{12}, & B_{22}, & B_{23} \\
B_{13}, & B_{23}, & B_{33}
\end{bmatrix} = -E \begin{bmatrix}
\frac{\partial^2 \log L}{\partial \beta_0^2}, & \frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1}, & \frac{\partial^2 \log L}{\partial \beta_0 \partial \sigma} \\
\frac{\partial^2 \log L}{\partial \beta_1^2}, & \frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_0}, & \frac{\partial^2 \log L}{\partial \beta_1 \partial \sigma} \\
\frac{\partial^2 \log L}{\partial \sigma^2}, & \frac{\partial^2 \log L}{\partial \beta_0 \partial \sigma}, & \frac{\partial^2 \log L}{\partial \sigma^2}
\end{bmatrix}
\]

\[
B_{11} = -E \left[ \frac{\partial^2 \log L}{\partial \beta_0^2} \right] = \frac{n}{\sigma^2} \left( A_{11} + 2A_{12} + A_{22} \right) = \frac{n}{\sigma^2} \left( 1 - \exp(-\exp(L_1)) + \exp(L_2) - \exp(L_1) \right)
\]

\[
B_{12} = -E \left[ \frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_0} \right] = \frac{n}{\sigma^2} \left( S_1 A_{11} + (S_1 + S_2) A_{12} + S_2 A_{22} \right)
\]

\[
= \frac{n}{\sigma^2} \left( S_1 \left( 1 - \exp(-\exp(L_1)) \right) + S_2 \left( \exp(-\exp(L_1)) - \exp(-\exp(L_2)) + \exp(L_2) - \exp(L_1) \right) \right)
\]
The minimization of the $A V \hat{Y} \left( \beta_0, \beta_1, \sigma \right)$ given by equation (A.II-1) results in the optimal stress change time. These results are then compared with the optimal plans from minimization of $n \cdot A V \hat{Y} \left( S_0 \right)$ given by equation (3-33). The comparison indicates that both approaches lead to the same results.
Appendix III  Derivation of CDF of Random Variable $y$ in Equation (4-20)

From equation (4-16), we have the CDF of the failure time $t$:

$$F(t) = \begin{cases} 
1 - \exp\left(-\frac{t^{\delta}}{\theta_1^{\delta}}\right) & 0 \leq t < \tau_1 \\
1 - \exp\left(-\frac{t^{\delta} - \tau_1^{\delta} - \tau_1^{\delta}}{\theta_2^{\delta} - \theta_1^{\delta}}\right) & \tau_1 \leq t < \tau_2 \\
1 - \exp\left(-\frac{t^{\delta} - \tau_2^{\delta} - \tau_1^{\delta}}{\theta_3^{\delta} - \theta_2^{\delta} - \theta_1^{\delta}}\right) & \tau_2 \leq t < \infty
\end{cases}$$

Let random variable $y = \log(t)$, $\sigma = 1/\delta$, then the CDF of $y$ can be derived as follows:

1. For the first section $0 \leq t < \tau_1$, we have $-\infty < y \leq \log \tau_1$

$$P(Y \leq y) = P(\log(t) \leq y) = F(\exp(y)) = 1 - \exp\left(-\left(\frac{\exp(y)}{\theta_1}\right)^\delta\right)$$

$$= 1 - \exp\left(-\left(\exp(y - \log \theta_1)\right)^\delta\right) = 1 - \exp\left(-\frac{y - \log \theta_1}{\sigma}\right)$$

2. For the second section $\tau_1 \leq t < \tau_2$, we have $\log \tau_1 < y \leq \log \tau_2$

$$P(Y \leq y) = P(\log(t) \leq y) = F(\exp(y)) = 1 - \exp\left(-\left(\frac{\exp(y)}{\theta_2}\right)^\delta + \left(\frac{\tau_1}{\theta_2}\right)^\delta - \left(\frac{\tau_1}{\theta_1}\right)^\delta\right)$$

$$= 1 - \exp\left(-\left(\exp(y - \log \theta_2)\right)^\delta + \exp\left(\log \tau_1 - \log \theta_2\right)\right)^\delta - \left(\exp\left(\log \tau_1 - \log \theta_1\right)\right)^\delta$$

$$= 1 - \exp\left(-\frac{y - \log \theta_2}{\sigma} + \frac{\log \tau_1 - \log \theta_2}{\sigma} - \frac{\log \tau_1 - \log \theta_1}{\sigma}\right)$$

3. For the third section $\tau_2 \leq t < \infty$, we have $\log \tau_2 < y < \infty$
\[ P(Y \leq y) = P(\log(t) \leq y) = F(\exp(y)) \]
\[ = 1 - \exp \left( -\left( \frac{\exp(y)}{\theta_1} \right)^{\delta} + \left( \frac{\tau_2}{\theta_3} \right)^{\delta} - \left( \frac{\tau_1}{\theta_2} \right)^{\delta} - \left( \frac{\tau_1}{\theta_1} \right)^{\delta} \right) \]
\[ = 1 - \exp \left( -\left( \exp(y - \log \theta_1) \right)^{\delta} + \left( \exp(\log \tau_2 - \log \theta_3) \right)^{\delta} - \left( \exp(\log \tau_1 - \log \theta_2) \right)^{\delta} - \left( \exp(\log \tau_1 - \log \theta_1) \right)^{\delta} \right) \]
\[ = 1 - \exp \left( -\frac{y - \log \theta_1}{\sigma} \right) + \exp \left( \frac{\log \tau_2 - \log \theta_3}{\sigma} \right) - \exp \left( \frac{\log \tau_1 - \log \theta_2}{\sigma} \right) + \exp \left( \frac{\log \tau_1 - \log \theta_1}{\sigma} \right) \]

Thus, the CDF of \( y \) is shown as follows:

\[ G(y) = F(e^y) \]
\[ = \begin{cases} 
1 - \exp \left( -\frac{y - \log \theta_1}{\sigma} \right), & \text{for } -\infty < y < \log(\tau_1) \\
1 - \exp \left( -\frac{y - \log \theta_2}{\sigma} \right) + \exp \left( \frac{\log \tau_1 - \log \theta_2}{\sigma} \right) - \exp \left( \frac{\log \tau_1 - \log \theta_1}{\sigma} \right), & \log(\tau_1) \leq y < \log(\tau_2) \\
1 - \exp \left( -\frac{y - \log \theta_3}{\sigma} \right) + \exp \left( \frac{\log \tau_2 - \log \theta_3}{\sigma} \right) - \exp \left( \frac{\log \tau_2 - \log \theta_2}{\sigma} \right) + \exp \left( \frac{\log \tau_1 - \log \theta_3}{\sigma} \right) - \exp \left( \frac{\log \tau_1 - \log \theta_2}{\sigma} \right) - \exp \left( \frac{\log \tau_1 - \log \theta_1}{\sigma} \right), & \log(\tau_2) \leq y < \infty
\end{cases} \]

Let \( y_{\theta_i,e} = \exp \left( \frac{y - \log \theta_i}{\sigma} \right) \), and \( \tau_{m,\theta_i,e} = \exp \left( \frac{\log \tau_m - \log \theta_i}{\sigma} \right) \), for \( m = 1, 2 \), \( i = 1, 2, 3 \),

the above CDF is simplified as follows:

\[ G(y) = F(e^y) = \begin{cases} 
1 - \exp \left( -y_{\theta_1,e} \right), & \text{for } -\infty < y < \log(\tau_1) \\
1 - \exp \left( -y_{\theta_1,e} + \tau_{1,\theta_2,e} - \tau_{1,\theta_1,e} \right), & \log(\tau_1) \leq y < \log(\tau_2) \\
1 - \exp \left( -y_{\theta_1,e} + \tau_{2,\theta_2,e} - \tau_{2,\theta_1,e} + \tau_{1,\theta_2,e} - \tau_{1,\theta_1,e} \right), & \log(\tau_2) \leq y < \infty
\end{cases} \]
Appendix IV  Detailed Calculation of  $\Phi_i$,  $\Psi_i$,  $C_i$,  $i = 1, 2, 3$  in  $\hat{F}_3$

From the definition, we have:

$$
\Phi_i = E\left[\frac{1}{n} \sum_{j=1}^{n} y_{j,\theta_i,\theta,\varepsilon}\right], \quad \Psi_i = E\left[\frac{1}{n} \sum_{j=1}^{n} y_{j,\theta_i}^2 y_{j,\theta,\varepsilon}\right], \quad \text{and} \quad C_i = E\left[\frac{n_i}{n}\right], \quad i = 1, 2, 3
$$

To calculate  $\Phi_i$, and  $\Psi_i$, the following transformation is made.

Let

$$
\begin{align*}
\tau_1 &< y < \log \tau_1 & 0 < w < \tau_{1,\theta_i,\varepsilon} \\
\log \tau_1 &< y < \log \tau_2 & \tau_{1,\theta_i,\varepsilon} < w < \tau_{2,\theta_i,\varepsilon} \\
\log \tau_2 &< y < \infty & \tau_{2,\theta_i,\varepsilon} < w \leq \tau_{2,\theta_i,\varepsilon}
\end{align*}
$$

Next, the CDF, and the pdf of the random variable  $W$  are derived:

1) For the first interval  $-\infty < y < \log \tau_1$,  $0 < w < \tau_{1,\theta_i,\varepsilon}$.

$$
P(W \leq w) = P\left(\exp\left(\frac{y - \log \tau_1}{\sigma}\right) \leq w\right) = P\left(y \leq \log w \cdot \sigma + \log \theta_1\right) = G(\log w \cdot \sigma + \log \theta_1).
$$

The CDF of the random variable  $Y$,  $G(\bullet)$, from equation (4-20) is:

$$
G(y) = \begin{cases} 
1 - \exp(-y_{\theta_i,\varepsilon}), & -\infty < y < \log(\tau_1) \\
1 - \exp(-y_{\theta_i,\varepsilon} + \tau_{1,\theta_i,\varepsilon} - \tau_{1,\theta_i,\varepsilon}), & \log(\tau_1) < y < \log(\tau_2) \\
1 - \exp(-y_{\theta_i,\varepsilon} + \tau_{2,\theta_i,\varepsilon} - \tau_{1,\theta_i,\varepsilon} + \tau_{1,\theta_i,\varepsilon} - \tau_{1,\theta_i,\varepsilon}), & \log(\tau_2) \leq y < \infty,
\end{cases} \quad (A.IV-1)
$$

From the CDF of random variable  $Y$ in equation (A.IV-1), the CDF of the random variable  $W$  in the first interval is obtained as

$$
F(w) = 1 - \exp\left(-\exp\left(\frac{\log w \cdot \sigma + \log \theta_1 - \log \theta_1}{\sigma}\right)\right) = 1 - \exp(-w), \quad \text{for} \quad 0 < w \leq \tau_{1,\theta_i,\varepsilon}.
$$

2) For the second interval,  $\log \tau_1 < y < \log \tau_2$,  $\tau_{1,\theta_i,\varepsilon} < w \leq \tau_{2,\theta_i,\varepsilon}$.

$$
P(W \leq w) = P\left(\exp\left(\frac{y - \log \tau_2}{\sigma}\right) \leq w\right) = P\left(y \leq \log w \cdot \sigma + \log \theta_2\right) = G(\log w \cdot \sigma + \log \theta_2).
$$

Then the CDF of random variable  $W$  in the second interval is obtained as
\[ F(w) = 1 - \exp\left(-\exp\left(\frac{\log w \cdot \sigma + \log \theta_2 - \log \theta_2}{\sigma}\right) + \tau_{1,\theta_1,\epsilon} - \tau_{1,\theta_1,\epsilon} \right) \]

for \( \tau_{1,\theta_1,\epsilon} < w \leq \tau_{2,\theta_1,\epsilon} \).

3) For the third interval, \( \log \tau_2 < y < \infty \), \( \tau_{2,\theta_1,\epsilon} < w < \infty \).

\[ P(W \leq w) = P\left(\exp\left(\frac{y - \log \theta_3}{\sigma}\right) \leq w\right) = P\left(y \leq \log w \cdot \sigma + \log \theta_3\right) = G\left(\log w \cdot \sigma + \log \theta_3\right), \]

and the CDF of random variable \( W \) in the third interval is

\[ F(w) = 1 - \exp\left(-w + \tau_{2,\theta_1,\epsilon} - \tau_{2,\theta_1,\epsilon} + \tau_{1,\theta_1,\epsilon} - \tau_{1,\theta_1,\epsilon} \right) \quad \text{for} \quad \tau_{2,\theta_1,\epsilon} < w < \infty. \]

Thus, the CDF of the random variable \( W \) is:

\[
F(w) = \begin{cases} 
1 - \exp(-w) & 0 < w \leq \tau_{1,\theta_1,\epsilon} \\
1 - \exp(-w + \tau_{1,\theta_1,\epsilon} - \tau_{1,\theta_1,\epsilon}) & \tau_{1,\theta_1,\epsilon} < w \leq \tau_{2,\theta_1,\epsilon} \\
1 - \exp(-w + \tau_{2,\theta_1,\epsilon} - \tau_{2,\theta_1,\epsilon} + \tau_{1,\theta_1,\epsilon} - \tau_{1,\theta_1,\epsilon}) & \tau_{2,\theta_1,\epsilon} < w < \infty 
\end{cases}
\]

The pdf of the random variable \( W \) is obtained by taking the first derivative of the above CDF:

\[
f(w) = \begin{cases} 
\exp(-w) & 0 < w \leq \tau_{1,\theta_1,\epsilon} \\
\exp(\tau_{1,\theta_1,\epsilon} - \tau_{1,\theta_1,\epsilon}) \cdot \exp(-w) & \tau_{1,\theta_1,\epsilon} < w \leq \tau_{2,\theta_1,\epsilon} \\
\exp(\tau_{2,\theta_1,\epsilon} - \tau_{2,\theta_1,\epsilon} + \tau_{1,\theta_1,\epsilon} - \tau_{1,\theta_1,\epsilon}) \exp(-w) & \tau_{2,\theta_1,\epsilon} < w < \infty 
\end{cases}
\]

Let \( g(w) = \log(w) \cdot w \), then \( \Phi_i, \) and \( \Psi_i, \) \( i = 1, 2, 3 \) are evaluated as

\[
\Phi_1 = E\left[ \frac{1}{n} \sum_{j=1}^{n} y_{1,j,\theta_1} y_{1,j,\theta_1,\epsilon} \right] = E\left[ \frac{1}{n} \sum_{j=1}^{n} \log(w_j) w_j \right] = E\left[ \frac{1}{n} \sum_{j=1}^{n} g(w_j) \right] = \int_{0}^{\tau_{1,\theta_1,\epsilon}} g(w) f(w) dw = \int_{0}^{\tau_{1,\theta_1,\epsilon}} \log(w) \cdot w \cdot \exp(-w) dw
\]

Using a similar method, we can have

\[
\Phi_2 = E\left[ \frac{1}{n} \sum_{j=1}^{n} y_{2,j,\theta_1} y_{2,j,\theta_1,\epsilon} \right] = \int_{\tau_{1,\theta_1,\epsilon}}^{\tau_{2,\theta_1,\epsilon}} \log(w) \cdot w \cdot \exp(\tau_{1,\theta_1,\epsilon} - \tau_{1,\theta_1,\epsilon}) \cdot \exp(-w) dw
\]
Detailed calculations of $C_i = E \left[ \frac{n_i}{n} \right]$, $i = 1, 2, 3$ is demonstrated through the following three steps:

At the first step, $n$ new products are tested at stress levels $(S_{11}, S_{21})$ until time $\tau_1$, where the test units are assumed independent and identically distributed (iid). The log-life of items follows the CDF of $y$ in equation (A.IV-1). The number of failures $n_1$ in time $\tau_1$ is a binomial random variable with parameters $n$ and $p_1$. From the equation (A.IV-1), we have:

$$p_1 = G(\log \tau_1) = 1 - \exp(-\tau_{1,\theta_{1},e})$$

$$C_1 = E \left[ \frac{n_1}{n} \right] = p_1 = 1 - \exp(-\tau_{1,\theta_{1},e}) = G(\log \tau_1).$$

The second step starts with $n - n_1$ unfailed items, tested at stress levels $(S_{12}, S_{21})$ until time $\tau_2$. The log-life of items follows the CDF of $y$ given by the equation.
(A.IV-1), where the number of failures $n_2$ follows a binomial distribution with parameters $n - n_1$ and $p_2$. Then, from the equation (A.IV-14), we have:

$$p_2 = 1 - \Pr(\text{item not failing in time } \tau_2 \mid \text{item not failing in time } \tau_1) = 1 - \frac{\exp\left(-\tau_{2,\theta_2,e} + \tau_{1,\theta_1,e} - \tau_{1,\theta_1,e}\right)}{\exp(-\tau_{1,\theta_1,e})} = 1 - \exp\left(-\tau_{2,\theta_2,e} + \tau_{1,\theta_1,e}\right),$$

and

$$C_2 = E\left[\frac{n_2}{n}\right] = E\left[\frac{n_2}{n} \cdot \frac{n - n_1}{n}\right] = p_2 \cdot (1 - p_1) = (1 - \exp(-\tau_{2,\theta_2,e} + \tau_{1,\theta_1,e})) \cdot \exp(-\tau_{1,\theta_1,e}) = \exp(-\tau_{1,\theta_1,e}) - \exp(-\tau_{2,\theta_2,e} + \tau_{1,\theta_1,e} - \tau_{1,\theta_1,e}) = G(\log \tau_2) - G(\log \tau_1).$$

At the third step, $n - n_1 - n_2$ unfailed items are then placed at stress levels $(S_{12}, S_{22})$ until time $T$. The number of failures $n_3$ follows a binomial distribution with parameters $n - n_1 - n_2$ and $p_3$. Then,

$$p_3 = 1 - \Pr(\text{item not failing in third step} \mid \text{item not failing in first two steps}) = 1 - \frac{\exp\left(-T_{\theta_1,e} + \tau_{2,\theta_2,e} - \tau_{2,\theta_2,e} + \tau_{1,\theta_1,e} - \tau_{1,\theta_1,e}\right)}{\exp(-\tau_{2,\theta_2,e} + \tau_{1,\theta_1,e} - \tau_{1,\theta_1,e})} = 1 - \exp\left(-T_{\theta_1,e} + \tau_{2,\theta_2,e}\right),$$

and

$$C_3 = E\left[\frac{n_3}{n}\right] = E\left[\frac{n_3}{n} \cdot \frac{n - n_1 - n_2}{n}\right] = p_3 \cdot (1 - p_1 - p_2) = (1 - \exp(-T_{\theta_1,e} + \tau_{2,\theta_2,e})) \cdot \exp(-\tau_{2,\theta_2,e} + \tau_{1,\theta_1,e} - \tau_{1,\theta_1,e}) = \exp(-\tau_{2,\theta_2,e} + \tau_{1,\theta_1,e} - \tau_{1,\theta_1,e}) - \exp(-T_{\theta_1,e} + \tau_{2,\theta_2,e} - \tau_{2,\theta_2,e} + \tau_{1,\theta_1,e} - \tau_{1,\theta_1,e}) = G(\log T) - G(\log \tau_2),$$

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Appendix V  
Detailed Calculation of $\Phi_i$, $\Psi_i$, $C_i$, $i = 1, 2, 3$ in $\hat{F}_6$

From the definition, we have:

$$
\Phi_i = E\left[\frac{1}{n} \sum_{j=1}^{n} y_{ij,0} y_{ij,e}\right], \quad \Psi_i = E\left[\frac{1}{n} \sum_{j=1}^{n} y_{ij,0}^2 y_{ij,e}\right], \quad \text{and} \quad C_i = E\left[\frac{n_i}{n}\right], \text{for } i = 1, 2, \cdots, k.
$$

To calculate $\Phi_i$, and $\Psi_i$, the similar transformation is made as in Appendix IV.

Let $w = \begin{cases} y_{0,e} & -\infty < y \leq \log \tau_1 \\ y_{1,e} & \log \tau_1 < y \leq \log \tau_2 \\ \vdots & \vdots \\ y_{k,e} & \log \tau_{k-1} < y < \infty \end{cases}$.

Next, the CDF, and the pdf of the random variable $W$ are derived:

1) For the first two intervals $-\infty < y \leq \log \tau_1$ and $\log \tau_1 < y \leq \log \tau_2$, the derivation of the CDF of $W$ is same as in Appendix IV.

2) For the third interval, $\log \tau_2 < y \leq \log \tau_3$, $\tau_{2,0,e} < w \leq \tau_{3,0,e}$.

$$
P(W \leq w) = P\left(\exp\left(\frac{y - \log \theta_1}{\sigma}\right) \leq w\right) = P\left(y \leq \log w \cdot \sigma + \log \theta_1\right) = G\left(\log w \cdot \sigma + \log \theta_1\right),
$$

and the CDF of random variable $W$ in the third interval is:

$$
F(w) = 1 - \exp\left(-w + \tau_{2,0,e} - \tau_{2,0,e} + \tau_{1,0,e} - \tau_{1,0,e}\right), \quad \text{for } \tau_{2,0,e} < w \leq \tau_{3,0,e}.
$$

3) For the $i^{th}$ interval, $\log \tau_{i-1} < y \leq \log \tau_i$, $\tau_{i-1,0,e} < w \leq \tau_{i,0,e}$, for $i = 2, 3, \cdots, k-1$.

$$
P(W \leq w) = P\left(\exp\left(\frac{y - \log \theta_i}{\sigma}\right) \leq w\right) = P\left(y \leq \log w \cdot \sigma + \log \theta_i\right) = G\left(\log w \cdot \sigma + \log \theta_i\right),
$$

and the CDF of random variable $W$ in the $i^{th}$ interval is:
\[ F(w) = 1 - \exp\left( -w + \sum_{r=1}^{i-1} \left( r,\tilde{\theta}_{r,e} - r,\tilde{\theta}_{r,e} \right) \right), \quad \text{for } \tau_{r-1,e} < w \leq \tau_{r,e}. \]

4) For the \( k^{th} \) interval, \( \log \tau_{k-1} < y < \infty, \quad \tau_{k-1,\tilde{\theta}_{e}} < w < \infty. \)

\[ P(W \leq w) = P\left( \exp\left( \frac{y - \log \theta_k}{\sigma} \right) \leq w \right) = P(y \leq \log w \cdot \sigma + \log \theta_k) = G(\log w \cdot \sigma + \log \theta_k), \]

and the CDF of random variable \( W \) in the \( k^{th} \) interval is:

\[ F(w) = 1 - \exp\left( -w + \sum_{r=1}^{k-1} \left( r,\tilde{\theta}_{r,e} - r,\tilde{\theta}_{r,e} \right) \right), \quad \text{for } \tau_{k-1,\tilde{\theta}_{e}} < w < \infty. \]

Thus, the CDF of the random variable \( W \) is:

\[
F(w) = \begin{cases} 
1 - \exp(-w) & 0 < w \leq \tau_{1,\tilde{\theta}_{e}} \\
1 - \exp\left( -w + \tau_{1,\tilde{\theta}_{e}} - \tau_{1,\tilde{\theta}_{e}} \right) & \tau_{1,\tilde{\theta}_{e}} < w \leq \tau_{2,\tilde{\theta}_{e}} \\
1 - \exp\left( -w + \tau_{2,\tilde{\theta}_{e}} - \tau_{2,\tilde{\theta}_{e}} + \tau_{1,\tilde{\theta}_{e}} - \tau_{1,\tilde{\theta}_{e}} \right) & \tau_{2,\tilde{\theta}_{e}} < w \leq \tau_{3,\tilde{\theta}_{e}} \\
& \vdots \\
1 - \exp\left( -w + \sum_{r=1}^{k-1} \left( r,\tilde{\theta}_{r,e} - r,\tilde{\theta}_{r,e} \right) \right) & \tau_{k-1,\tilde{\theta}_{e}} < w < \infty \end{cases}
\]

The pdf of the random variable \( W \) is obtained by taking the first derivative of CDF with respect to \( w \), shown as follows:

\[
f(w) = \begin{cases} 
\exp(-w) & 0 < w \leq \tau_{1,\tilde{\theta}_{e}} \\
\exp\left( \tau_{1,\tilde{\theta}_{e}} - \tau_{1,\tilde{\theta}_{e}} \right) \cdot \exp(-w) & \tau_{1,\tilde{\theta}_{e}} < w \leq \tau_{2,\tilde{\theta}_{e}} \\
\exp\left( \tau_{2,\tilde{\theta}_{e}} - \tau_{2,\tilde{\theta}_{e}} + \tau_{1,\tilde{\theta}_{e}} - \tau_{1,\tilde{\theta}_{e}} \right) \exp(-w) & \tau_{2,\tilde{\theta}_{e}} < w \leq \tau_{3,\tilde{\theta}_{e}} \end{cases}
\]

\[ \vdots \]

\[ \exp\left( \sum_{r=1}^{k-1} \left( r,\tilde{\theta}_{r,e} - r,\tilde{\theta}_{r,e} \right) \right) \exp(-w) & \tau_{k-1,\tilde{\theta}_{e}} < w < \infty \]

Let \( g(w) = \log(w) \cdot w \), then \( \Phi_i \), and \( \Psi_i \), \( i = 1, 2, \cdots, k \) are evaluated as follows:

\[ \Phi_1 = E\left[ \frac{1}{n} \sum_{j=1}^{n} y_{1,j,\tilde{\theta}_{1,e}} y_{1,j,\tilde{\theta}_{1,e}} \right] = E\left[ \frac{1}{n} \sum_{j=1}^{n} \log(w_j) w_j \right] = E\left[ \frac{1}{n} \sum_{j=1}^{n} g(w_j) \right]. \]

\[ = \int_{\tau_{1,\tilde{\theta}_{e}}}^{\tau_{k,\tilde{\theta}_{e}}} g(w) f(w) dw = \int_{\tau_{1,\tilde{\theta}_{e}}}^{\tau_{k,\tilde{\theta}_{e}}} \log(w) \cdot w \cdot \exp(-w) dw \]
Using similar method, we can have:

\[ \Phi_2 = E \left[ \frac{1}{n} \sum_{j=1}^{n} y_{2j, \theta_2} y_{2j, \theta_2, e} \right] = \int_{\tau_{1, \theta_2, e}}^{\tau_{2, \theta_2, e}} \log(w) \cdot w \cdot \exp(\tau_{1, \theta_2, e} - \tau_{1, \theta_2, e}) \cdot \exp(-w) \, dw \]

\[ \Phi_3 = E \left[ \frac{1}{n} \sum_{j=1}^{n} y_{3j, \theta_3} y_{3j, \theta_3, e} \right] 
= \int_{\tau_{2, \theta_1, e}}^{\tau_{3, \theta_1, e}} \log(w) \cdot w \cdot \exp(\tau_{2, \theta_1, e} - \tau_{2, \theta_1, e} + \tau_{1, \theta_1, e} - \tau_{1, \theta_1, e}) \cdot \exp(-w) \, dw 
\]

\[ \Phi_k = E \left[ \frac{1}{n} \sum_{j=1}^{n} y_{kj, \theta_k} y_{kj, \theta_k, e} \right] 
= \int_{\tau_{k-1, \theta_k, e}}^{\tau_{k, \theta_k, e}} \log(w) \cdot w \cdot \exp(\tau_{k, \theta_k, e} - \tau_{k, \theta_k, e}) \cdot \exp(-w) \, dw \]

\[ \Psi_1 = E \left[ \frac{1}{n} \sum_{j=1}^{n} y_{1j, \theta_1} y_{1j, \theta_1, e} \right] = \int_{0}^{\tau_{1, \theta_1, e}} (\log(w))^2 \cdot w \cdot \exp(-w) \, dw. \]

\[ \Psi_2 = E \left[ \frac{1}{n} \sum_{j=1}^{n} y_{2j, \theta_2}^2 y_{2j, \theta_2, e} \right] = \int_{\tau_{1, \theta_2, e}}^{\tau_{2, \theta_2, e}} (\log(w))^2 \cdot w \cdot \exp(\tau_{1, \theta_2, e} - \tau_{1, \theta_2, e}) \cdot \exp(-w) \, dw \]

\[ \Psi_3 = E \left[ \frac{1}{n} \sum_{j=1}^{n} y_{3j, \theta_3}^2 y_{3j, \theta_3, e} \right] 
= \int_{\tau_{2, \theta_1, e}}^{\tau_{3, \theta_1, e}} (\log(w))^2 \cdot w \cdot \exp(\tau_{2, \theta_1, e} - \tau_{2, \theta_1, e} + \tau_{1, \theta_1, e} - \tau_{1, \theta_1, e}) \cdot \exp(-w) \, dw 
\]

\[ \Psi_k = E \left[ \frac{1}{n} \sum_{j=1}^{n} y_{kj, \theta_k}^2 y_{kj, \theta_k, e} \right] 
= \int_{\tau_{k-1, \theta_k, e}}^{\tau_{k, \theta_k, e}} (\log(w))^2 \cdot w \cdot \exp\left(\sum_{r=1}^{k-1} (\tau_{r, \theta_{r+1}, e} - \tau_{r, \theta_{r+1}, e})\right) \cdot \exp(-w) \, dw \]
For detailed calculations of $C_i = \frac{n_i}{n}, i = 1, 2, \cdots, k$, the first two steps are the same as in Appendix IV. For the third step, $n - n_1 - n_2$ iid survival items are then placed at stresses $(S_{1,3}, S_{2,3}, \ldots, S_{m,3})$ until time $\tau_3$. Since $n_3$ is the number of failures between time $\tau_2$ and $\tau_3$, it follows binomial distribution with parameters $n - n_1 - n_2$ and $p_3$.

From equation (5-4), we have:

$$p_3 = 1 - \Pr(\text{item not fail in third step} \mid \text{item not fail in first two steps})$$
$$= 1 - \frac{\exp\left(-\tau_{3,0,1} + \tau_{2,0,1} - \tau_{1,0,1} - \tau_{1,0,1}\right)}{\exp\left(-\tau_{2,0,1} + \tau_{1,0,1} - \tau_{1,0,1}\right)}$$
$$= 1 - \exp\left(-\tau_{3,0,1} + \tau_{2,0,1}\right)$$

$$C_3 = E\left[\frac{n_3}{n}\right] = E\left[\frac{n_2}{n - n_1 - n_2} \cdot \frac{n - n_1 - n_2}{n}\right] = p_3 \cdot (1 - p_1 - p_2)$$
$$= (1 - \exp\left(-\tau_{3,0,1} + \tau_{2,0,1}\right)) \cdot \exp\left(-\tau_{2,0,1} + \tau_{1,0,1} - \tau_{1,0,1}\right)$$
$$= \exp\left(-\tau_{2,0,1} + \tau_{1,0,1} - \tau_{1,0,1}\right) - \exp\left(-\tau_{3,0,1} + \tau_{2,0,1} - \tau_{2,0,1} + \tau_{1,0,1} - \tau_{1,0,1}\right)$$
$$= G(\log \tau_3) - G(\log \tau_2),$$

At the $i^{th}$ step, $i = 2, 3, \cdots, k-1$, $n - \sum_{r=1}^{i-1} n_r$ iid survival items are then placed at stresses $(S_{i,i}, S_{2,i}, \ldots, S_{m,i})$ until time $\tau_i$. Since $n_i$ is the number of failures between time $\tau_{i-1}$ and $\tau_i$, it follows Binomial distribution with parameters $n - \sum_{r=1}^{i-1} n_r$ and $p_i$.

From equation (5-4), we have:

$$p_i = 1 - \Pr(\text{item not fail in } i^{th} \text{ step} \mid \text{item not fail in first } i-1 \text{ steps})$$
$$= 1 - \frac{\exp\left(-\tau_{i,0,1} + \sum_{r=1}^{i-1} (\tau_{r,0,1} - \tau_{r,0,1})\right)}{\exp\left(-\tau_{i-1,0,1} + \sum_{r=1}^{i-2} (\tau_{r,0,1} - \tau_{r,0,1})\right)}$$
$$= 1 - \exp\left(-\tau_{i,0,1} + \tau_{i-1,0,1}\right).$$
\[ C_i = E \left[ \frac{n_i}{n} \right] = p_i \cdot \left( 1 - \sum_{r=1}^{i-1} p_r \right) = G(\log \tau_i) - G(\log \tau_{i-1}), \]

At the \( k \)th step, \( n - \sum_{r=1}^{k-1} n_r \) iid survival items are then placed at stresses \((S_{1,k}, S_{2,k}, \ldots, S_{m,k})\) until time \( T \). Since \( n_k \) is the number of failures between time \( \tau_{k-1} \) and \( T \), it follows binomial distribution with parameters \( n - \sum_{r=1}^{k-1} n_r \) and \( p_k \). From equation (5-4), we have:

\[
p_k = 1 - \Pr\{\text{item not fail in } k \text{th step} | \text{item not fail in first } k - 1 \text{ steps} \}
\]

\[
= 1 - \exp\left( -T_{\theta_k,e} + \sum_{r=1}^{k} (r_{\tau_{k-1},e} - r_{\tau_{k},e}) \right)
\]

\[
= 1 - \exp\left( -T_{\theta_k,e} + \tau_{k-1,\theta_k,e} + \sum_{r=1}^{k-2} (r_{\tau_{k-1},e} - r_{\tau_{k},e}) \right)
\]

\[
C_k = E \left[ \frac{n_k}{n} \right] = p_i \cdot \left( 1 - \sum_{r=1}^{i-1} p_r \right) = G(\log T) - G(\log \tau_{k-1}),
\]

Thus,

\[
C_r = E \left[ \frac{n_r}{n} \right] = 1 - \sum_{i=1}^{k} C_i
\]

\[
= 1 - G(\log T)
\]
Appendix VI  Derivation of CDF of Random Variable $Y$ in Equation (6-9)

From equation (6-8), we have the CDF of the failure time $t$:

$$
F(t) = 1 - \begin{cases} 
\exp \left( - \frac{t^\delta}{\eta^\delta} \exp(aX_1) \right) & 0 \leq t \leq \tau \\
\exp \left( - \frac{t^\delta - \tau^\delta}{\eta^\delta} \exp(aX_2) - \frac{\tau^\delta}{\eta^\delta} \exp(aX_1) \right) & \tau \leq t < \infty
\end{cases}
$$

Let random variable $Y = \log(t)$, $\sigma = \frac{1}{\delta}$, $\theta = \log(\eta)$, then the CDF of $Y$ is derived in the following two steps:

i) For the first section $0 \leq t \leq \tau$, we have $-\infty < y \leq \log \tau$

$$
G(y) = P(Y \leq y) = P(\log(t) \leq y) = F(\exp(y)) = 1 - \exp \left( - \left( \frac{\exp(y)}{\eta} \right)^\delta \exp(aX_1) \right)
$$

$$
= 1 - \exp \left( - \exp \left( \frac{y - \theta}{\sigma} + aX_1 \right) \right)
$$

(A.I.-1)

ii) For the second section $\tau \leq t < \infty$, we have $\log \tau \leq y < \infty$

$$
G(y) = P(Y \leq y) = P(\log(t) \leq y) = F(\exp(y))
$$

$$
= 1 - \exp \left( - \frac{(\exp(y))^{\delta} - \tau^\delta}{\eta^\delta} \exp(aX_2) - \frac{\tau^\delta}{\eta^\delta} \exp(aX_1) \right)
$$

$$
= 1 - \exp \left( - \exp \left( \frac{y - \theta}{\sigma} + aX_1 \right) + \exp \left( \frac{\log \tau - \theta}{\sigma} + aX_2 \right) - \exp \left( \frac{\log \tau - \theta}{\sigma} + aX_1 \right) \right)
$$

(A.I.-2)

Thus, from (A.I.-1) and (A.I.-2) the CDF of $y$ is shown as follows:

$$
G(y) = \begin{cases} 
1 - \exp \left( - \exp \left( \frac{y - \theta}{\sigma} + aX_1 \right) \right) & -\infty < y \leq \log(\tau) \\
1 - \exp \left( - \exp \left( \frac{y - \theta}{\sigma} + aX_2 \right) + \exp \left( \frac{\log(\tau) - \theta}{\sigma} + aX_2 \right) \right) & \log(\tau) \leq y < \infty
\end{cases}
$$
Appendix VII  Detailed Calculation of $C_i$, $\Phi_i$, $\Psi_i$, $i = 1, 2$ in $\hat{F}$

For $C_i = E\left[\frac{n_i}{n}\right]$, $i = 1, 2$, it is shown in Appendix I that:

$$C_i = E\left[\frac{n_i}{n}\right] = G(\log \tau), \quad C_2 = E\left[\frac{n_2}{n}\right] = G(\log T) - G(\log \tau),$$

and $C_T = E\left[\frac{n_T}{n}\right] = 1 - G(\log T)$

where $G(\cdot)$ is the CDF of random variable $Y$ in equation (6-9).

From the definition, we have:

$$\Phi_i = E[\Lambda_i] = E\left[\frac{1}{n} \exp(aX_i) \sum_{j=1}^{n} \frac{y_{ij} - \theta}{\sigma} \exp\left(\frac{y_{ij} - \theta}{\sigma}\right)\right],$$

$$\Psi_i = E[\Omega_i] = E\left[\frac{1}{n} \exp(aX_i) \sum_{j=1}^{n} \left(\frac{y_{ij} - \theta}{\sigma}\right)^2 \exp\left(\frac{y_{ij} - \theta}{\sigma}\right)\right], \quad i = 1, 2.$$

To calculate $\Phi_i$ and $\Psi_i$, we let $w = \exp \frac{y - \theta}{\sigma}$. The CDF of $w$ can be obtained.

$$P(W \leq w) = P\left(\exp \left(\frac{y - \theta}{\sigma}\right) \leq w\right) = P(y \leq \log w \cdot \sigma + \theta) = G(\log w \cdot \sigma + \theta). \quad (AIIV-1)$$

In equation (6-9), the CDF of $Y$ is:

$$G(y) = \begin{cases} 
1 - \exp\left(-\exp\left(\frac{y - \theta}{\sigma} + aX_1\right)\right) & -\infty < y \leq \log(\tau) \\
-\exp\left(\frac{y - \theta}{\sigma} + aX_2\right) & \\
1 - \exp\left(\frac{\log(\tau) - \theta}{\sigma} + aX_2\right) & \log(\tau) \leq y < \infty \\
-\exp\left(\frac{\log(\tau) - \theta}{\sigma} + aX_1\right) & 
\end{cases} \quad (AIIV-2)$$
From equations (AIV-1) and (AIV-2), we can obtain the CDF of $w$:

$$F(w) = \begin{cases} 
1 - \exp(-w \cdot \exp(aX_1)) & 0 < w \leq \exp\left(\frac{\log \tau - \theta}{\sigma}\right) \\
1 - \exp\left(-w \cdot \exp(aX_2) + \exp\left(\frac{\log(\tau) - \theta + aX_2}{\sigma}\right)\right) & \exp\left(\frac{\log \tau - \theta}{\sigma}\right) \leq w \leq \infty \\
1 - \exp\left(-\exp\left(\frac{\log(\tau) - \theta + aX_1}{\sigma}\right)\right) & \infty \leq w \leq \infty
\end{cases}$$

The pdf of $w$ is obtained by taking the derivative of the CDF:

$$f(w) = \begin{cases} 
\exp(aX_1) \cdot \exp(-w \cdot \exp(aX_1)) & 0 \leq w \leq \exp\left(\frac{\log \tau - \theta}{\sigma}\right) \\
\exp(aX_2) \cdot \exp\left(-w \cdot \exp(aX_2) + \exp\left(\frac{\log(\tau) - \theta + aX_2}{\sigma}\right)\right) & \exp\left(\frac{\log \tau - \theta}{\sigma}\right) \leq w \leq \infty \\
\exp(aX_2) \cdot \exp\left(-\exp\left(\frac{\log(\tau) - \theta + aX_1}{\sigma}\right)\right) & \infty \leq w \leq \infty
\end{cases}$$

Let $g(w) = \log(w) \cdot w$, then

$$\Phi_1 = E(\Lambda_1) = E\left[\frac{1}{n} \exp(aX_1) \cdot \sum_{j=1}^{n} \frac{y_{1j} - \theta}{\sigma} \exp\frac{y_{1j} - \theta}{\sigma}\right] = E\left[\frac{1}{n} \exp(aX_1) \cdot \sum_{j=1}^{n} \log(w_j) \cdot w_j\right]$$

$$= E\left[\frac{1}{n} \exp(aX_1) \cdot \sum_{j=1}^{n} g(w_j)\right] = \exp(aX_1) \cdot \int_0^{\exp\left(\frac{\log \tau - \theta}{\sigma}\right)} g(w) f(w) dw$$

$$= (\exp(aX_1))^2 \int_0^{\exp\left(\frac{\log \tau - \theta}{\sigma}\right)} \log(w) \cdot w \cdot \exp(-w \exp(aX_1)) dw$$

Using similar method, we can have:

$$\Phi_2 = E(\Lambda_2) = E\left[\frac{1}{n} \exp(aX_2) \cdot \sum_{j=1}^{n} \frac{y_{2j} - \theta}{\sigma} \exp\frac{y_{2j} - \theta}{\sigma}\right]$$

$$= (\exp(aX_2))^2 \cdot \exp\left[\exp\left(\frac{\log \tau - \theta + aX_2}{\sigma}\right) - \exp\left(\frac{\log \tau - \theta + aX_1}{\sigma}\right)\right] \cdot \int_0^{\exp\left(\frac{\log \tau - \theta}{\sigma}\right)} \log(w) \cdot w \cdot \exp(-w \exp(aX_2)) dw$$
\[
\Psi_1 = E(\Omega_1) = E \left[ \frac{1}{n} \exp(aX_1) \cdot \sum_{j=1}^{n} \left( \frac{y_{1j} - \theta}{\sigma} \right)^2 \exp \left( \frac{y_{1j} - \theta}{\sigma} \right) \right],
\]

\[
= (\exp(aX_1))^2 \cdot \int_{0}^{\exp\left(\frac{\log \tau - \theta}{\sigma}\right)} (\log(w))^2 \cdot w \cdot \exp(-w \exp(aX_1))dw
\]

\[
\Psi_2 = E(\Omega_2) = E \left[ \frac{1}{n} \exp(aX_2) \cdot \sum_{j=1}^{n} \left( \frac{y_{2j} - \theta}{\sigma} \right)^2 \exp \left( \frac{y_{2j} - \theta}{\sigma} \right) \right]
\]

\[
= (\exp(aX_2))^2 \cdot \exp \left[ \log \tau - \theta + aX_2 \right] \cdot \exp \left[ \frac{\log \tau - \theta + aX_2}{\sigma} + aX_1 \right] \cdot \exp \left[ \frac{\log \tau - \theta + aX_2}{\sigma} \right] \cdot \exp \left[ \frac{\log \tau - \theta + aX_2}{\sigma} \right] \cdot w \cdot \exp(-w \exp(aX_2))dw
\]
Appendix VIII  Proof of the Derivation of $v_i$ in Equation (7-8)

In equation (7-8), the value of $v_i$, the solution of equation (7-3):

$$F_{i+1}(v_i) = F_i(\tau_i - \tau_{i-1} + v_{i-1})$$

is shown as:

$$v_i = \left( \sum_{j=1}^{i} (\tau_j - \tau_{j-1})^{a_j} \right) e^{\frac{a_j}{\beta} x_{j+i}} \cdot e^{\frac{a_j}{\beta} x_{j+i-1}} \quad \text{for } i = 1, 2, \ldots, k - 1 \quad (A.VIII-1)$$

We prove this equation by induction as follows:

When $i = 1$, from equation (A.VIII-1), we obtain that

$$v_1 = (\tau_1 - \tau_0) e^{\frac{a_1}{\beta} x_1} \cdot e^{\frac{a_1}{\beta} x_{1+1}} = \tau_1 e^{\frac{a_1}{\beta} (x_1 - x_2)}$$

since $\tau_0 = 0$

The solution is same as what we get from equation (7-6).

When $i = 2$, from equation (A.VIII-1), we obtain that

$$v_2 = \left( \sum_{j=1}^{2} (\tau_j - \tau_{j-1})^{a_j} \right) e^{\frac{a_2}{\beta} x_2} = (\tau_2 - \tau_1) e^{\frac{a_2}{\beta} x_3} + \tau_1 e^{\frac{a_2}{\beta} x_1}$$

which is same as in equation (7-7).

Suppose that equation (A.VIII-1) holds for $i = k - 1$, thus:

$$v_{k-2} = \left( \sum_{j=1}^{k-2} (\tau_j - \tau_{j-1})^{a_j} \right) e^{\frac{a_j}{\beta} x_{j+k-1}}$$

Then the CDF at the stress level $X_{k-1}$ is then obtained:

$$F_{k-1}(t - \tau_{k-2} + v_{k-2}) = 1 - e^{-\frac{(t-\tau_{k-2})^{\frac{a_{k-1}}{\beta} x_{k-1}}}{\eta}}$$

For stress level $X_k$, the failure intensity function under stress $X_k$ is obtained from equation (7-3):
\[ \lambda_k(t) = \beta \cdot t^{\beta-1} \eta^{-\beta} e^{\alpha X_k} \]

The CDF of failure data under the stress level \( X_k \) is:

\[
F_k(t) = 1 - e^{-\int_0^t \lambda_k(t) dt} = 1 - e^{-t^\beta \eta^{-\beta} e^{\alpha X_k}}
\]

\( v_{k-1} \) is the solution of \( F_k(v_{k-1}) = F_{k-1}(\tau_{k-1} - \tau_{k-2} + v_{k-2}) \).

\[
F_k(v_{k-1}) = F_{k-1}(\tau_{k-1} - \tau_{k-2} + v_{k-2}) \]

\[
\Rightarrow 1 - e^{-v_{k-1}^\beta \eta^{-\beta} e^{\alpha X_k}} = 1 - e^{-\left(\frac{\tau_{k-1} - \tau_{k-2}}{\eta} \eta^{\alpha X_k} - \sum_{j=1}^{k-2} \left(\frac{\tau_j - \tau_{j-1}}{\eta} \eta^{\alpha X_k}\right)\right)^\beta} \]  
(A.VIII.-2)

\[
\Rightarrow v_{k-1} = \left(\sum_{j=1}^{k-1} (\tau_j - \tau_{j-1}) \eta^{\alpha X_j} \right) \cdot \eta^{-\frac{\alpha X_k}{\beta}}
\]

which is exactly the same as in equation (A.VIII-1).

Therefore, we can conclude that for equation (A.VIII-1) holds for all \( i = 1, 2, \cdots, k - 1 \).
APPENDIX IX Derivation of $C_i$, $\Phi_i$, and $\Psi_i$, $i = 1, 2, \ldots, k$ in $\hat{F}_8$

In the expected Fisher Information Matrix $\hat{F}_8$, $C_i$ are defined as:

$$C_i = E\left[ \frac{n_i}{n} \right], \quad i = 1, 2, \ldots, k.$$  

In order to calculate $C_i$, we made the following analysis.

At the stress level $S_1$, $n$ iid new products are tested at stresses $S_1$ until time $\tau_1$. The life time of any item follows the CDF in equation (7-10). Since $n_1$ is the number of failures in time $\tau_1$, it is a random variable following Binomial distribution with parameters $n$ and $p_1$, where $p_1$ is the probability of item failure in time $\tau_1$. From equation (7-10), we have:

$$p_1 = G(\log \tau_1) = 1 - \exp\left( - y_{0,e} \cdot X_{1,e} \right).$$

Thus, $C_1 = E\left[ \frac{n_1}{n} \right] = p_1 = 1 - \exp\left( - \tau_{1,0,e} \cdot X_{1,e} \right) = G(\log \tau_1)$.

where $G(\bullet)$ is the CDF of random variable $Y$.

At the second step, $n - n_1$ iid survival items are then placed at stresses $S_2$ until time $\tau_2$. The life time of any item follows the CDF in equation (7-10). Since $n_2$ is the number of failures between time $\tau_1$ and $\tau_2$, it follows Binomial distribution with parameters $n - n_1$ and $p_2$. From equation (7-10), we have:

$$p_2 = \Pr(\text{item fails in time } \tau_2 \mid \text{it not fail in time } \tau_1 \text{ in first step})$$
\[ C_2 = E \left[ \frac{n_2}{n} \right] = E \left[ \frac{n_2}{n-n_1} \cdot \frac{n-n_1}{n} \right] = p_2 \cdot (1 - p_1) \\
= (1 - \exp(-\tau_{2,\theta,e} \cdot X_{2,e} + \gamma_{1,\theta,e} X_{2,e})) \cdot \exp(-\gamma_{1,\theta,e} X_{1,e}) \cdot \exp(-\gamma_{1,\theta,e} X_{1,e}). \\
= G(\log \tau_2) - G(\log \tau_1). \\
\]

For the third step, \( n - n_1 - n_2 \) iid survival items are then placed at stresses \( S_3 \) until time \( \tau_3 \). The life time of any item follows the CDF in equation (7-10). Since \( n_3 \) is the number of failures between time \( \tau_2 \) and \( \tau_3 \), it follows Binomial distribution with parameters \( n - n_1 - n_2 \) and \( p_3 \). From equation (7-10), we have:

\[ p_3 = \Pr(\text{item fails in time } \tau_3 \mid \text{it not fail in time } \tau_2) \]

\[ = 1 - \Pr(\text{item not fail in time } \tau_3 \mid \text{item not fail in time } \tau_2) \]

\[ = 1 - \frac{1 - G(\log \tau_3)}{1 - G(\log \tau_2)} \]

\[ C_2 = E \left[ \frac{n_3}{n} \right] = E \left[ \frac{n_3}{n-n_1-n_2} \cdot \frac{n-n_1-n_2}{n} \right] = p_3 \cdot (1 - C_1 - C_2) \\
= \left(1 - \frac{1 - G(\log \tau_3)}{1 - G(\log \tau_2)} \right) \cdot (1 - G(\log \tau_1) - (G(\log \tau_2) - G(\log \tau_1))). \\
= G(\log \tau_3) - G(\log \tau_2). \\
\]

At the \((k - 1)\)th step, \( n - \sum_{i=1}^{k-2} n_i \) iid survival items are then placed at stresses \( S_{k-1} \) until time \( \tau_{k-1} \). The life time of any item follows the CDF in equation (7-10). Since
\( n_{k-1} \) is the number of failures between time \( \tau_{k-2} \) and \( \tau_{k-1} \), it follows Binomial distribution with parameters \( n - \sum_{i=1}^{k-2} n_i \) and \( p_{k-1} \). From equation (7-10), we have:

\[
p_{k-1} = \Pr(\text{item fails in time } \tau_{k-1} \mid \text{it not fail in time } \tau_{k-2})
\]

\[
= 1 - \Pr(\text{item not fail in time } \tau_{k-1} \mid \text{item not fail in time } \tau_{k-2})
\]

\[
= 1 - \frac{1 - G(\log \tau_{k-1})}{1 - G(\log \tau_{k-2})}
\]

\[
C_{k-1} = E\left[ \frac{n_{k-1}}{n} \right] = E \left[ \frac{n_{k-1}}{n} - \frac{n - \sum_{i=1}^{k-2} n_i}{n} \right] = p_{k-1} \cdot \left( 1 - \sum_{i=1}^{k-2} C_i \right)
\]

\[
= \left( 1 - \frac{1 - G(\log \tau_{k-1})}{1 - G(\log \tau_{k-2})} \right) (1 - G(\log \tau_{k-2}))
\]

For the \( k^{th} \) step, \( n - \sum_{i=1}^{k-1} n_i \) iid survival items are then placed at stresses \( S_{k-1} \) until all units fail or predetermined censoring time \( T \).

\[
C_k = E\left[ \frac{n_k}{n} \right] = G(\log T) - G(\log \tau_{k-1})
\]

Therefore,

\[
C_i = E\left[ \frac{n_i}{n} \right] = 1 - \sum_{i=1}^{k} n_i = 1 - G(\log T)
\]

Derivation of \( \Phi_i \) and \( \Psi_i \), \( i = 1, 2, \cdots, k \) is presented next. From the definition, we have:

\[
\Phi_i = E\left[ \frac{1}{n} \sum_{j=1}^{n} \left( y_{ij,0} y_{ij,\theta,0} x_{i,e} \right) \right], \quad \Psi_i = E\left[ \frac{1}{n} \sum_{j=1}^{n} \left( y_{ij,0}^2 y_{ij,\theta,0} x_{i,e} \right) \right].
\]
In order to calculate $\Phi_i$ and $\Psi_i$, we define $w = y_{\theta,e} = \exp\frac{y - \theta}{\sigma}$. Then the CDF of $w$ can be obtained.

$$P(W \leq w) = P\left(\exp\frac{y - \theta}{\sigma} \leq w\right) = P(y \leq \log w \cdot \sigma + \theta) = G(\log w \cdot \sigma + \theta).$$

From the above equation and the CDF of $y$ in equation (7-10), we can obtain the CDF of $w$:

$$F(w) = \begin{cases} 
0 < w \leq \tau_{1,\theta,e} & 1 - \exp(-w \cdot X_{1,e}) \\
\tau_{1,\theta,e} \leq w \leq \tau_{2,\theta,e} & 1 - \exp(-w \cdot X_{2,e} + \tau_{1,\theta,e}(X_{2,e} - X_{1,e})) \\
\vdots & \vdots \\
\tau_{k-1,\theta,e} \leq w < \infty & 1 - \exp(-w \cdot X_{k,e} + \sum_{j=1}^{k-1} \tau_{j,\theta,e} \cdot (X_{j+1,e} - X_{j,e}))
\end{cases}$$

The pdf of $w$ is obtained by taking the derivative of the CDF:

$$f(w) = \begin{cases} 
0 < w \leq \tau_{1,\theta,e} & X_{1,e} \cdot \exp(-w \cdot X_{1,e}) \\
\tau_{1,\theta,e} \leq w \leq \tau_{2,\theta,e} & X_{2,e} \cdot \exp(-w \cdot X_{2,e} + \tau_{1,\theta,e}(X_{2,e} - X_{1,e})) \\
\vdots & \vdots \\
\tau_{k-1,\theta,e} \leq w < \infty & X_{k,e} \exp(-w \cdot X_{k,e} + \sum_{j=1}^{k-1} \tau_{j,\theta,e} \cdot (X_{j+1,e} - X_{j,e}))
\end{cases}$$

Let $g(w) = \log(w) \cdot w$, then $\Phi_i = E\left[\frac{1}{n} \sum_{j=1}^{n_i} (y_{j,\theta,e} y_{j,\theta,e, \cdot \cdot} \cdot X_{1,e})\right]$.

$$\Phi_i = E\left[\frac{1}{n} \sum_{j=1}^{n_i} y_{j,\theta,e} y_{j,\theta,e, \cdot \cdot} \cdot X_{1,e}\right] = E\left[\frac{1}{n} X_{1,e} \cdot \sum_{j=1}^{n_i} \log(w) \cdot w\right]$$

$$= \frac{1}{n} X_{1,e} \cdot \sum_{j=1}^{n_i} g(w) = X_{1,e} \cdot \int_{0}^{\tau_{1,\theta,e}} g(w) f(w) dw$$

$$= X_{1,e} \cdot \int_{0}^{\tau_{1,\theta,e}} \log(w) \cdot w \cdot \exp(-w X_{1,e}) dw$$

Using similar method, we can have:
\[
\Phi_2 = E \left[ \frac{1}{n} \sum_{j=1}^{n_2} (y_{2j,0} y_{2j,\theta,e} \cdot X_{2,e}) \right] = E \left[ \frac{1}{n} X_{2,e} \cdot \sum_{j=1}^{n_2} \log(w_j) \cdot w_j \right] \\
= E \left[ \frac{1}{n} X_{2,e} \cdot \sum_{j=1}^{n_2} g(w_j) \right] = X_{2,e} \cdot \int_{r_{1,\theta,e}}^{r_{2,\theta,e}} g(w)f(w)dw \\
= X_{2,e}^2 \exp(\tau_{1,\theta,e} (X_{2,e} - X_{1,e})) \int_{r_{1,\theta,e}}^{r_{2,\theta,e}} \log(w) \cdot w \cdot \exp(-wX_{2,e})dw
\]

Thus,

\[
\Phi_k = E \left[ \frac{1}{n} \sum_{j=1}^{n_k} (y_{kj,0} y_{kj,\theta,e} \cdot X_{k,e}) \right] = E \left[ \frac{1}{n} X_{k,e} \cdot \sum_{j=1}^{n_k} \log(w_j) \cdot w_j \right] \\
= E \left[ \frac{1}{n} X_{k,e} \cdot \sum_{j=1}^{n_k} g(w_j) \right] = X_{k,e} \cdot \int_{r_{k-1,\theta,e}}^{r_{k,\theta,e}} g(w)f(w)dw \\
= X_{k,e}^2 \exp(\sum_{j=1}^{k-1} \tau_{j,\theta,e} (X_{j+1,e} - X_{j,e})) \int_{r_{k-1,\theta,e}}^{r_{k,\theta,e}} \log(w) \cdot w \cdot \exp(-wX_{k,e})dw
\]

\[
\Psi_1 = E \left[ \frac{1}{n} \sum_{j=1}^{n_1} (y_{1j,0} y_{1j,\theta,e} \cdot X_{1,e}) \right] \\
= X_{1,e} \int_0^{r_{1,\theta,e}} (\log(w))^2 \cdot w \cdot \exp(-wX_{1,e})dw
\]

\[
\Psi_2 = E \left[ \frac{1}{n} \sum_{j=1}^{n_2} (y_{2j,0} y_{2j,\theta,e} \cdot X_{2,e}) \right] \\
= X_{2,e}^2 \exp(\tau_{1,\theta,e} (X_{2,e} - X_{1,e})) \int_{r_{1,\theta,e}}^{r_{2,\theta,e}} (\log(w))^2 \cdot w \cdot \exp(-wX_{2,e})dw
\]

\[
\Psi_k = E \left[ \frac{1}{n} \sum_{j=1}^{n_k} (y_{kj,0} y_{kj,\theta,e} \cdot X_{k,e}) \right] \\
= X_{k,e}^2 \exp(\sum_{j=1}^{k-1} \tau_{j,\theta,e} (X_{j+1,e} - X_{j,e})) \int_{r_{k-1,\theta,e}}^{r_{k,\theta,e}} (\log(w))^2 \cdot w \cdot \exp(-wX_{k,e})dw
\]


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