Robust Stability, Constrained Stabilization, and Observer-Based Controller Designs for Time Delay Systems

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To My Wife and My Children
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ABSTRACT OF DISSERTATION

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Abstract

This thesis initially presents the problem of stability pertaining to delay systems and provides useful results on stability analysis using linear matrix inequalities (LMIs). The connection between recently published results on stability robustness in the domain of time delays and robust stability with respect to structured perturbation of system matrices is investigated. Consequently, computable stability radius formulas for multi-delay systems using Rekasius substitution are derived. Using the concepts of kernel and offspring curves, which describe the complete portrait of possible imaginary characteristic roots of the nominal system, the fundamental stability region is defined among a set of disjoint stability regions for two independent time-delays. This makes it possible to give a necessary and sufficient condition for stability robustness of two time-delay systems with structured uncertainties in terms of two distinct LMIs. We also provide an explicit formula for stability radius of a special class of time-delay systems and generalize the result for multi-delay case. Important results on stabilization and robust stabilization using state feedback and dynamic output feedback controller for both cases of delay-independent and delay-dependent are provided. These results are essential in the development of constrained stabilization, which is explored in later chapter. Subsequently, design of observer-based controller for robust stabilization of time-delay systems is explored. We present design methods for Proportional Observer (PO) and Proportional-Integral Observer (PIO) and compare their robust performance. By using Lyapunov-based stability condition for time-delay systems, we establish the stability and convergence of the observer. A method for PIO design is proposed to attenuate the disturbance to a pre-specified level while estimating the state of the delay system. The method guarantees the stability of the observer and
minimizes the $H_\infty$ norm between the disturbance and the estimated error. All designs require solving certain modified algebraic Riccati equations. Due to the fact that attenuation is not the only objective for the designer, it is also shown that PIO has the capability of making simultaneous estimation of states and unknown constant disturbance, which can reliably be used in robust fault detection. Finally, the class of positive systems for continuous-time delay systems, defined as Metzlerian delay systems, is considered. The stability and robust stability of this class of systems are analyzed and direct formula for robust stability radius is derived. The stabilization and robust stabilization of Metzlerian delay systems are also explored and design methods are provided. A by-product of the results is the design procedures of state feedback and dynamic output feedback controllers for general linear time-delay systems such that the closed-loop system is constrained to be Metzlerian delay stable. The thesis includes several illustrative examples to support the theoretical results.
I wish to express my gratitude and indebtedness to Professor Bahram Shafai for suggesting the problem tackled in this thesis and for his helpful guidance, encouragement and stimulating discussions during its preparation and for his fruitful comments and continuous follow up of this work during its development.

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Notations

Throughout this Thesis the following notations will be used:

- $t$ time-variable.
- $\tau$ time-delay.
- $\bar{\tau}$ upper bound on time-delay.
- $x(t)$ state variable at time $t$.
- $n$ dimension of the state variable.
- $\dot{x}(t)$ time-derivative of the state variable at time $t$.
- $u(t)$ input at time $t$.
- $y(t)$ output at time $t$.
- $V$ Lyapunov function.
- $\bullet$ symmetric block in a symmetric matrix, i.e.,
  \[
  \begin{bmatrix}
  X & Y \\
  \bullet & Z
  \end{bmatrix}
  =
  \begin{bmatrix}
  X & Y \\
  Y^T & Z
  \end{bmatrix}.
  \]
- $I_n$ denotes the $(n \times n)$ identity matrix.
- $A^{-1}$ inverse of the matrix $A$.
- $A^T$ transpose of the matrix $A$.
- $\det(A)$ determinant of the matrix $A$.
- $A > 0$ $A$ is symmetric and positive definite.
- $A \geq 0$ $A$ is symmetric and positive semidefinite.
- $A < 0$ $A$ is symmetric and negative definite.
- $A \leq 0$ $A$ is symmetric and negative semidefinite.
- $A_\perp$ orthogonal complement of the matrix $A$.
- $\sigma(A)$ denotes the singular value of $A$.
- $\rho(A)$ denotes the spectral radius of $A$.
- $j$ imaginary number, $j = \sqrt{-1}$.
- $w$ frequency, imaginary part of eigenvalue.
CONTENTS

$s$  Laplace variable.

$\lambda(A)$  eigenvalue of a matrix $A$.

$tr(A)$  trace of a matrix $A$.

$\lambda(A, B)$  generalized eigenvalue of matrix pair $(A, B)$.

$\mathcal{C}$  set of complex numbers.

$\mathcal{C}_g$  Left half complex plane

$\mathcal{C}_b$  Right half complex plane

$\mathcal{R}$  set of real numbers.

$\mathcal{C}([a,b], \mathcal{R}^n)$  set of all continuous functions mapping $[a, b] \rightarrow \mathcal{R}^n$.

$\mathcal{R}^n(\mathcal{C}^n)$  set of real (complex) $n$-dimensional vectors.

$\mathcal{R}^{n \times m}(\mathcal{C}^{n \times m})$  set of real (complex) $n \times m$ matrices.

$\mathcal{R}_+$  set of positive real numbers.

$\mathcal{R}_+^n$  set of positive real $n$-dimensional vectors.

$\mathcal{R}_+^{n \times m}$  set of positive real $n \times m$ matrices.

$\operatorname{Re}(\cdot), \operatorname{Im}(\cdot)$  real, imaginary part of a complex number.

$\| \cdot \|$  norm of vectors and matrices

$\|X(s)\|_2$  $H_2$ norm of $X(s)$, $\|X(s)\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} X^T(-jw)X(jw) \ dw \right)^{\frac{1}{2}}$.

$\|X(s)\|_{\infty}$  $H_{\infty}$ norm of $X(s)$, $\|X(s)\|_{\infty} = \sigma_{\max}(X(jw))$.

$r^*_R$  real stability radius.

$r^*_c$  complex stability radius.

**Remark** Note that the notations $A > 0$ and $A \geq 0$ in chapter 6 are also used to define positive and non-negative matrices. One can easily distinguish them from positive definite and positive semidefinite matrices by the associated subject material.
Chapter 1

Introduction

1.1 Brief Overview

In many applications one assumes that the system under consideration is governed by a principle of causality, which means that the future state of the system is independent of the past states and determined solely by the present. Moreover, if one assumes that the system is described by an equation where the rate of change of the state is determined by its current value, one is considering ordinary differential equations. However, under a closer look it becomes apparent that the principle of causality is often just a first approximation of the true situation and a more realistic model would include also some dependency on the past states of the system, leading to a modeling with delay differential equations (DDEs).

Examples of DDEs can be found in many applications, see [29], [47], [48]. Indeed, it is now generally recognized that delays are natural components of dynamical processes in physics, biology and engineering, for example population dynamics, epidemiology, elasticity, aeronautics and aerospace engineering, just to mention a few. Even when the delays are not present in the model of the system itself, they may be introduced when some control act on the system is undertaken, due to e.g. comput-
tational delays, AD-DA conversion. Finally, delay equations also arise as approximations of high dimensional systems, including systems described by partial differential equations.

Also Volterra, in his research on population dynamics and viscoelasticity [114], [115], already introduced the concept of the evolution of an energy-type function along the solutions of delay equations in the thirties. The systematic study of stability and stabilization of DDEs goes back to the work of Karsovskii in the sixties, who generalized the second method of Lyapunov in [45]. In the following decades, the stability of DDEs has been studied by many authors, just to mention the monograph of Kolmanovskii and Nosov [48], and the work of R. Datko [14], [15] and E. Verriest [113]. In the mid nineties an intensifying of this research took place, initiated by the work of Dambrine [13] and Niculescu [68].

The literature on stability and stabilization of DDEs is extensive. A collection of concepts and methods can be found in [47], [55], [73], [82], [88] and some new trends in [6], [25], [60], [71], and [120]. This thesis presents some major research directions, based on these references and provide new results to enhance the development of robust stability and control of time-delay systems.

In the following sections we introduce delay differential equations emphasizing stability. First we make a classification of DDEs and discuss general properties of their solutions. Then we define various boundedness and stability properties and discuss some stability results using Lyapunov framework. The connection between solution and stability of linear-time invariant delay system is also established. Finally, we outline research topics covered in the subsequent chapters.

The aim of this chapter is not to provide a complete characterization of stability of DDEs, but rather to explain some basic concepts, which are necessary for a better understanding of the results developed throughout the thesis.
1.2 Delay Differential Equations

Delay differential equations are a type of differential equation where the time derivatives at the current time depend on the solution, and possibly its derivatives, at previous times. A class of such equations, that involve derivatives with delays as well as the solution itself has historically been called neutral DDEs [29]. In this dissertation only retarded DDEs where there is no time-delay in the derivative terms are considered. The basic theory concerning stability and works on fundamental theory, e.g., existence and uniqueness of solutions, was presented in [4], [47]. Since then, DDEs have been extensively studied in many years and a great number of monographs and books have been published including significant works on dynamics of DDEs by Hale and Lunel, [29], Gorecki et. al. [22], Richard [87], and Zavarei and Jamshidi [123], on stability by Gu, Kharitonov, and Chen [25], Niculescu [68] and the references therein. The reader is also referred to many papers published thereafter. The interest in study of DDEs is caused by the fact that many processes have time-delays and have been modeled for better representations by systems of DDEs in science, engineering, economics. Such systems, however, are still challenging to precisely analyze and control, thus, the study of systems of DDEs has actively been conducted during the past decades.

Consider a simple linear functional differential equation of the form

\[
\dot{x}(t) = f(t, x(t), x(t - \tau)) = f(t, x_t) \quad (1.1)
\]

where \(x_t\) is the function mapping \([t - \tau, t]\) to \(\mathcal{R}^n\), \(x(t) \in \mathcal{R}^n\) are the system state, \(f : \mathcal{R} \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n\) and \(\tau\) is the delay magnitude. Note that the space of continuous-time functions mapping \([t - \tau, t]\) to \(\mathcal{R}^n\) is denoted by \(\mathcal{C}_t = \mathcal{C}(\mathcal{R}^n)\) and \(\mathcal{C}_0 = \mathcal{C}(\mathcal{R}^n)\) is also defined by \(\mathcal{C}[-\tau, 0]\). For simplicity we use \(\mathcal{C}\) to represent such functions.
Equation (1.1) has only one fixed, discrete delay. Without attempting to make a complete classification, we now give an overview of other types of delay differential equations. One can have multiple delays

\[ \dot{x}(t) = f(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_r)) \]  

(1.2)

bounded or unbounded disturbed delay,

\[ \dot{x}(t) = f(t, \int_{-\tau}^{0} g(x(t + s)) \, ds) \]  

(1.3)

respectively

\[ \dot{x}(t) = f(t, \int_{-\infty}^{0} g(x(t + s)) \, ds) \]  

(1.4)

and time and/or state-dependent delay

\[ \dot{x}(t) = f(t, x(t), x(t - \tau(t, x(t)))) \]  

(1.5)

The above equations are all of retarded type. If there is also a dependency of the right-hand side on the derivative of the state,

\[ \dot{x}(t) = f(t, x(t), x(t - \tau), \dot{x}(t - \tau)) \]  

(1.6)

The equation is of neutral type. Several combinations of the above types are also possible. All these equations can be written in a general form

\[ \dot{x}(t) = F(t, x_t), \]

where the right-hand side depends on the profile \( x : \mathcal{R} \to \mathcal{R}^n \) and \( F \) is a functional which maps into \( \mathcal{R}^n \). This class of so-called functional differential equations (FDE) is so broad that very little can be said about its solutions in general. Hence the need arises to define subclasses which share enough properties to allow some analysis, but there are broad enough to encompass models from a number of applications.

In this thesis we will mostly restrict ourselves to retarded functional differential equations (RFDE) with one delay and two delays. Occasionally, we generalize our
results for multiple delays. Therefore, we start to discuss properties of equation (1.1) in the rest of this section.

1.3 Stability and Boundedness

We consider some stability and boundedness properties of the solutions of DDE (1.1). They are defined in a similar fashion as in the Ordinary Differential Equations (ODEs) case.

**Definition 1.3.1** (Uniform ultimate boundedness) The solutions of (1.1) are uniformly ultimate bounded (UUB) if there exist a $b_0 > 0$ such that for all $\alpha > 0$ there is $T(\alpha) > 0$ such that $\| x(\sigma, \varphi)(t) \| \leq b_0$ for all $t \geq \sigma + T(\alpha)$, $\sigma \in \mathcal{R}$ and $\varphi \in \mathcal{C}$ with $\| \varphi \|_s \leq \alpha$. The real number $b_0 > 0$ is called a radius of convergence.

**Definition 1.3.2** (Uniform boundedness) The solutions of (1.1) are uniformly bounded (UB) if for all $\alpha > 0$ there is $\beta(\alpha) > 0$ such that $\| x(\sigma, \varphi)(t) \| \leq \beta(\alpha)$ for all $t \geq \sigma, \sigma \in \mathcal{R}$ and $\varphi \in \mathcal{C}$ with $\| \varphi \|_s \leq \alpha$.

**Definition 1.3.3** (Uniform asymptotic stability) Suppose $f(t, 0, 0) = 0$ for all $t$. The null solution of (1.1) is uniformly asymptotically stable (UAS) if the following conditions hold simultaneously:

1. Uniform stability: for all $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that $\| x(\sigma, \varphi)(t) \| < \epsilon$ for all $t \geq \sigma, \sigma \in \mathcal{R}$ and $\varphi \in \mathcal{C}$ with $\| \varphi \|_s < \delta(\epsilon)$.

2. Uniform attractivity: There exists $c_1 > 0$ such that for all $c_2 > 0$ there is $T(c_1, c_2) > 0$ such that $\| x(\sigma, \varphi)(t) \| < c_2$ for all $t \geq \sigma + T(c_1, c_2), \sigma \in \mathcal{R}$ and $\varphi \in \mathcal{C}$ with $\| \varphi \|_s < c_1$.

When, in addition, the second condition holds for all $c_1 > 0$ and the solution of (1.1) are UB, the zero-solution is globally uniformly asymptotically stable (GUAS).
In this definition of GUAS, the uniformity property not only concerns the behavior of the solutions with respect to the initial time $\sigma$, but also with respect to the size of the initial condition, $\| \varphi \|_s$. Without the uniformly requirement, i.e. for global asymptotic stability (GAS), only the stability property (where $\delta$ may depend on $\sigma$) is needed and $\lim_{t \to \infty} x(\sigma, \varphi)(t) = 0$, for all $\sigma \in \mathcal{R}$ and $\varphi \in \mathcal{C}$.

### 1.4 Stability in a Lyapunov Framework

In this section we provide sufficient conditions for the stability in a Lyapunov context, based on Hale and Lunel [29], Kolmanovskii and Myshkis [47] and [25].

Because the state space of a delay differential equations is infinite dimensional, a direct application of the second method of Lyapunov for ordinary differential equations is inappropriate: requiring the time-derivative of a Lyapunov function $V : \mathcal{R} \times \mathcal{R}^n \to \mathcal{R}$ along the solutions of an asymptotically stable delay differential equation to be negative definite is mostly not possible and only applicable to delay differential equations which are very similar to ordinary differential equations. However, there are two approaches to generalize Lyapunov stability theory to the delay dependent equation case. The first approach, the method of Lyapunov functionals, can be considered as a natural generalization. Instead of Lyapunov functions over $\mathcal{R} \times \mathcal{R}^n$, Lyapunov functionals over $\mathcal{R} \times \mathcal{C}$ are used. The second approach, the method of Razumikhin, make use of Lyapunov functions, but tries to reduce the conservatism induced by infinite dimensional nature of delay differential equations by relaxing $\dot{V} \leq 0$. Because of the actual state space of the DDE (1.1) is not $\mathcal{R}^n$ but $\mathcal{C}$, it is natural to derive stability and boundedness results by considering the evolution of so-called Lyapunov - Karasovskii functionals over $\mathcal{R} \times \mathcal{C}$ along the solutions. In this way most classical results for ODEs can be directly generalized to the DDE case. When a functional $V : \mathcal{R} \times \mathcal{C} \to \mathcal{R}$ is continuous, we define its upper right hand
derivative along the solution of (1.1) at \((t, \varphi)\) as
\[
\dot{V}(t, \varphi) = \lim \sup_{\tau \to 0+} \frac{1}{\tau} \left[ V(t + \tau, x_{t+\tau}(t, \varphi)) - V(t, \varphi) \right].
\]

The following theorem provides a necessary and sufficient con-ditions for uni-form asymptotic stability of delay system (1.1).

**Theorem 1.4.1** Let the delay system be defined by (1.1) and suppose \(f(t, 0, 0) = 0\). Then the solution of (1.1) is uniformly asymptotically stable if and only if there exist a continuous functional \(V : \mathcal{R} \times \mathcal{C} \to \mathcal{R}\), a number \(\alpha > 0\) and continuous nondecreasing functions \(u, v, w : \bar{\mathcal{R}}_+ \to \bar{\mathcal{R}}_+\) with \(u(0) = v(0) = w(0) = 0\) and \(u(s) > 0\), \(v(s) > 0\), \(w(s) > 0\) for \(s > 0\), such that for all \(\varphi \in \mathcal{C}\) with \(\| \varphi \|_s \leq \alpha\) and for all \(t\),
\[
u(\| \varphi \|_s) \leq V(t, \varphi) \leq v(\| \varphi \|_s),
\]
and
\[
\dot{V}(t, \varphi) \leq -w(\| \varphi \|_s).
\]

A difficulty in applying Theorem 1.4.1 consists in the fact that in practice, one often obtains upper bounds on \(\dot{V}\) which only depend on \(\| \varphi(0) \|\). For such cases, the following theorem is useful. It provides sufficient conditions for \((G)UAS\) of (1.1).

**Theorem 1.4.2** (Lyapunov - Karasovskii Stability Theorem) Let the delay system be defined by (1.1) and suppose \(f(t, 0, 0) = 0\). Then the solution of (1.1) is uniformly asymptotically stable if there exist a continuous functional \(V : \mathcal{R} \times \mathcal{C} \to \mathcal{R}\) and continuous nondecreasing functions \(u, v, w : \bar{\mathcal{R}}_+ \to \bar{\mathcal{R}}_+\) with \(u(0) = v(0) = w(0) = 0\) and \(u(s) > 0\), \(v(s) > 0\), \(w(s) > 0\) for \(s > 0\), such that for all \(t\),
\[
u(\| \varphi(0) \|_s) \leq V(t, \varphi) \leq v(\| \varphi \|_s),
\]
and
\[
\dot{V}(t, \varphi) \leq -w(\| \varphi(0) \|_s)
\]
Furthermore, if \(u(s) \to \infty\) as \(s \to \infty\), then the delay system is GUAS.
Many variations of the above theorem exist in the literature (see for example [82], [16], [17], [25], and [47]).

Instead of using Lyapunov functionals, there is another way to take the infinite dimensional nature of DDEs into account. Lyapunov functionals over $\mathbb{R} \times \mathbb{R}^n$ become useful when the condition $\dot{V}(x) \leq 0$ is relaxed, leading to the so-called Lyapunov - Razumikhin-type theorem.

The key idea behind the Razumikhin theorem focuses on a function $V(x)$ representative of the size of $x(t)$. For such a function,

$$V(x_\tau) = \max_{\theta \in [-\tau,0]} V(x(t + \theta))$$

serves to measure the size of $x_\tau$. If $V(x(t)) < V(x_\tau)$, then $\dot{V}(x) > 0$ does not make $\bar{V}(x_\tau)$ grow. Indeed, for $\bar{V}(x_\tau)$ to not grow, it is only necessary that $\dot{V}(x(t))$ is not positive whenever $V(x(t)) = \bar{V}(x_\tau)$.

**Theorem 1.4.3 (Lyapunov - Razumikhin Stability Theorem)** Let the delay system be defined by (1.1) and suppose $f(t,0,0) = 0$. Then the solution of (1.1) is UAS, if there exist a continuous functional $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and continuous nondecreasing functions $u,v,w : \bar{R}_+ \to \bar{R}_+ \text{ with } u(0) = v(0) = w(0) = 0 \text{ and } u(s) > 0, v(s) > 0, w(s) > 0 \text{ for } s > 0$, such that for all $t$,

$$u(\|x\|_s) \leq V(t,x) \leq v(\|x\|_s),$$

and

$$\dot{V}(t,x(t)) \leq -w(\|x(t)\|_s)$$

whenever

$$V(t + \theta, x(t + \theta)) < \pi(V(t,x(t))), \quad \forall \theta \in [-\tau,0], \quad (1.7)$$

Furthermore, if $u(s) \to \infty$ as $s \to \infty$, then the delay system is GUAS.
It is important to note that Razumikhin-type theorems only provide sufficient conditions for stability.

Finally, the DDE (1.1) is called autonomous (time-invariant) when its right hand side does not explicitly depend on time,

\[ \dot{x}(t) = f(x(t), x(t - \tau)). \]  

(1.8)

The definitions and the stability results of the previous section remain valid. When the autonomous DDE (1.8) is linear we have a linear time invariant (LTI) delay system, i.e. when it has the form,

\[ \dot{x}(t) = A_0 x(t) + A_1 x(t - \tau), \quad A_0, A_1 \in \mathbb{R}^n, \]  

(1.9)

in the case asymptotic stability is equivalent with the fact that all the eigenvalues, the roots of the characteristic equation

\[ \det(\lambda I - A_0 - A_1 e^{-\lambda \tau}) = 0 \]  

(1.10)

are in the open left half plane. Condition (1.10) is equivalent to requiring the existence of a nonzero vector \( v \in \mathbb{C}^{n \times 1} \) such that \( (\lambda I - A_0 - A_1 e^{-\lambda \tau})v = 0 \). The function segment \([-\tau, 0] \ni \theta \rightarrow ve^{-\lambda \theta} \in \mathbb{C}\) is the eigenfunction corresponding to the eigenvalue \( \lambda \).

### 1.5 Solution and Stability of Time-Delay Systems

A fundamental assumption on a system modeled by ordinary differential equations in the form of state variable representation

\[ \dot{x}(t) = f(t, x(t)), \quad x(t) \in \mathbb{R}^n \]  

(1.11)

with the initial condition \( x(t_0) = x_0 \) is that the future evolution of the system is completely determined by the current value of the state variables. In other words,
the value of the state variable $x(t)$, $t_0 \leq t < \infty$, for any $t_0$, can be formed once the initial condition $x(t_0) = x_0$ is known.

In many practical situations dynamical systems cannot be modeled by an ordinary differential equation as represented above. In particular, for many systems, the future evolution of the state variables $x(t)$ not only depends on their current values $x(t_0)$, but also on their past values, say $x(\xi)$, $t_0 - \tau \leq \xi \leq t_0$. Such a system is called a time-delay system. One can use functional differential equations to describe time-delay systems. Let $C([a, b], \mathbb{R}^n)$ be the set of continuous functions mapping the interval $[a, b]$ to $\mathbb{R}^n$. In most cases that we wish to identify a maximum time delay $\tau$ of a system, we are interested in mapping $[-\tau, 0]$ to $\mathbb{R}^n$, for which we simplify the notation to $C = C([\tau, 0], \mathbb{R}^n)$. Then, the general form of a retarded functional differential equation (RFDE) is

$$\dot{x}(t) = f(t, x_t)$$ (1.12)

where $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R} \times C \to \mathbb{R}^n$. Equation (1.12) indicates that the derivative of the state variables $x(t)$ at time $t$ depends on $t$ and $x(\xi)$ for $t - \tau \leq \xi \leq t$. As such, it is necessary to specify the initial state variables $x(t)$ in a time interval of length $\tau$, say from $t_0 - \tau$ to $t_0$, i.e.,

$$x_0 = \varphi, \quad \varphi \in C$$ (1.13)

Examples of RFDEs are

\[
\begin{align*}
\dot{x}(t) &= ax(t) + bx(t - \tau) \quad (i) \\
\dot{x}(t) &= (a + \cos wt)x(t) + \sin wt \quad (ii) \\
\dot{x}(t) &= \int_{-\tau}^{0} ax(\theta)x(t + \theta) \, d\theta \quad (iii) \\
\dot{x}(t) &= ax(t)x(t - \tau) \quad (iv)
\end{align*}
\]

where (i) is a homogeneous linear time invariant system with a single delay, (ii) is a nonhomogenous linear time-varying system without delay, (iii) is a system with
distributed delay and (iv) represent a nonlinear system with a single delay. It is important to point out that in functional differential equation of retarded type (RFDE), the highest order derivative does not contain any delayed variables. When such a term exists, then we have a functional differential equation of neutral type (NFDE). For example,

\[ a \dot{x}(t) + b \dot{x}(t - \tau) + cx(t) + d x(t - \tau) = 0 \]

is a NFDE. This thesis consider only RFDE.

A fundamental result for all types of differential equations is the condition under which guarantees the existance and uniqueness of solution. Here we state the following Lipschetz theorem pertinent to RFDEs without proof.

**Theorem 1.5.1** Suppose \( \Omega \) is an open set in \( \mathcal{R} \times \mathcal{C} \), and \( f(t, \varphi) : \Omega \rightarrow \mathcal{R}^n \) is Lipschitzian in \( \varphi \) such that there exists a constant \( L \) satisfying

\[ \|f(t, \varphi_1) - f(t, \varphi_2)\| \leq L \|\varphi_1 - \varphi_2\| \]  

(1.14)

for any \( (t, \varphi_1) \in \Omega \) and \( (t, \varphi_2) \in \Omega \). If \( (t_0, \varphi) \in \Omega \), then there exists a unique solution of (1.12) with \( (t_0, \varphi) \in \Omega \).

In what follows, we assume that the condition in the above theorem are ensured in order to establish the solution of time-delay systems and relevant analysis in subsequent sections. Furthermore, we assume that the delay systems is linear time invariant., i.e. LTI-RFDE or LTI-RDDE.

Let us illustrate this with a simple example of scaler time-delay system:

\[ \dot{x}(t) = a_0 x(t) + a_1 x(t - \tau) + u(t) \]  

(1.15)

where \( a_0 \), \( a_1 \), and \( \tau \) are real numbers with \( \tau > 0 \) as its delay time. In order to calculate \( \dot{x}(0) \) to advance the solution beyond \( t = 0 \), we need the value of \( x(0) \) and \( x(-\tau) \). Similarly, in order to calculate \( \dot{x} \) at the instant \( t = \xi \), \( 0 \leq \xi < \tau \), we need
$x(\xi)$ and $x(\xi - \tau)$. For such $\xi$, since $-\tau \leq \xi - \tau < 0$, $x(\xi - \tau)$ cannot be generated in the solution process. It is, therefore, clear that for the solution to be uniquely defined, it is necessary to specify the value of $x(t)$ for the entire interval $-\tau \leq t \leq 0$; in other words, we need to specify the initial condition

$$x(t) = \varphi(t), \quad t \in [-\tau, 0]$$  \hspace{1cm} (1.16)

with some continuous function $\varphi : [-\tau, 0] \to \mathcal{R}$. Once the initial condition (1.16) is given, the solution is well defined. Indeed, for $t \in [0, \tau]$, since $x(t - \tau)$ is already known in this interval, one can treat (1.15) as ordinary differential equation, so we can obtain

$$x(t) = e^{a_0 t} x(0) + \int_0^t e^{a_0 (t-\alpha)} [a_1 x(\alpha - \tau) + u(\alpha)] d\alpha$$

upon obtaining $x(t)$ for $t \in [0, \tau]$, we can calculate $x(t)$ for $[\tau, 2\tau]$ similarly, i.e.

$$x(t) = e^{a_0 (t-\tau)} x(\tau) + \int_\tau^t e^{a_0 (t-\alpha)} [a_1 x(\alpha - \tau) + u(\alpha)] d\alpha$$

continuing this process will allow us to obtain the solution $x(t)$, $-\tau \leq t < \infty$. Such a process is known as the method of steps.

Assuming that the forcing function $u(t)$ is exponentially bounded, i.e.

$$|u(t)| \leq K e^{ct} \quad \text{for some } K > 0, \ c \in \mathcal{R}$$

we have its Laplace transform as

$$U(s) = \mathcal{L}[u(t)]$$

and it can be shown that the solution $x(t)$ is bounded. Now, if we take the Laplace transform of (1.15) together with the initial condition (1.16), we obtain

$$sX(s) - \varphi(0) = a_0 X(s) + a_1 \left[ e^{-s\tau} X(s) + \int_{-\tau}^0 e^{-s(\alpha+\tau)} \varphi(\alpha) \, d\alpha \right] + U(s)$$

where $X(s)$ is the Laplace transform of $x(t)$. Solving for $X(s)$, we obtain

$$X(s) = \frac{1}{\Delta(s)} \left[ \varphi(0) + a_1 \int_{-\tau}^0 e^{-s(\alpha+\tau)} \varphi(\alpha) \, d\alpha + U(s) \right]$$
where
\[ \Delta(s) = s - a_0 - a_1 e^{-s\tau} \]
is the characteristic quasipolynomial of the system and
\[ \Phi(t) = \mathcal{L}^{-1} \left[ \frac{1}{\Delta(s)} \right] \]
is the fundamental solution of (1.15). Given the fundamental solution \( \Phi(t) \), using the convolution theorem, the solution (1.15) with initial condition \( \varphi(t) \) and the forcing function \( u(t) \) can be expressed as
\[
x(t) = \Phi(t)\varphi(0) + a_1 \int_{-\tau}^{0} \Phi(t - \alpha - \tau)\varphi(\alpha) \, d\alpha + \int_{0}^{t} \Phi(t - \alpha)u(\alpha) \, d\alpha \tag{1.17}
\]
Note that the poles of the system can be obtained from
\[ \Delta(s) = 0 \]
which has an infinite number of solutions for time-delay systems.

A general LTI delay systems without input delays can be described by
\[
\begin{align*}
\dot{x}(t) &= \sum_{i=0}^{k} A_i x(t - \tau_i) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*} \tag{1.18, 1.19}
\]
where \( A_i \)'s are given \( n \times n \) real constant matrices and \( \tau_i \)'s are delay constants ordered such that
\[ 0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k = \tau \tag{1.20} \]
If we take the Laplace transform of (1.18) together with the initial condition vector
\[ x(t) = \varphi(t), \quad t \in [-\tau, 0] \tag{1.21} \]
we obtain
\[
x(s) - \varphi(0) = A_0 X(s) + \sum_{i=1}^{k} A_i \left[ e^{-s\tau_i} X(s) + \int_{-\tau_i}^{0} e^{-s(\alpha+\tau_i)} \varphi(\alpha) \, d\alpha \right] + BU(s) \tag{1.22}
\]
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solving for \( X(s) \) yields

\[
X(s) = \Phi(s)\varphi(0) + \Phi(s) \left[ \sum_{i=1}^{k} A_i \int_{-\tau_i}^{0} e^{-s(\alpha+\tau_i)} \varphi(\alpha) \, d\alpha + BU(s) \right]
\]  \tag{1.23}

where

\[
\Phi(s) = \left[ sI - \sum_{i=0}^{k} A_i e^{-s\tau_i} \right]^{-1}
\]  \tag{1.24}

Defining the general state transition matrix \( \Phi(t) \) as

\[
\Phi(t) = \mathcal{L}^{-1}[\varphi(s)]
\]  \tag{1.25}

we can express the solution of (1.18) in terms of initial condition and input as follows

\[
x(t) = \Phi(t)\varphi(0) + \sum_{i=1}^{k} A_i \int_{-\tau_i}^{t} \Phi(t - \alpha - \tau_i)\varphi(\alpha) \, d\alpha + \int_{0}^{t} \Phi(t - \alpha)Bu(\alpha) \, d\alpha
\]  \tag{1.26}

For such systems, the characteristic quasi-polynomial \( \Delta(s) \) takes the form

\[
\Delta(s) = p(s; e^{-\tau_1 s}, \ldots, e^{-\tau_k s}) = \det \left( sI - \sum_{i=0}^{k} A_i e^{-s\tau_i} \right)
\]  \tag{1.27}

\[
= p_0(s) + \sum_{i=1}^{l} p_i(s) e^{-h_is}
\]  \tag{1.28}

where \( p_i(s), i = 0, 1, \ldots, l \) are polynomials, and \( h_i, i = 1, 2, \ldots, l \) are sums of the delay parameters \( \tau_i \). It is evident that this class of systems can also be described by

the scaler differential-difference equation

\[
y^{(n)}(t) + \sum_{j=0}^{n-1} \sum_{i=0}^{l} p_{ij} y^{(j)}(t - h_i) = 0
\]  \tag{1.29}

which admits the same quasi-polynomials with

\[
p_0(s) = s^n + \sum_{j=0}^{n-1} p_{0j} s^j
\]  \tag{1.30}

\[
p_i(s) = \sum_{j=0}^{n-1} p_{ij} s^j, \quad i = 1, 2, \ldots, l
\]

In general the ratio between the delays \( \frac{\tau_i}{\tau_j} \) may be irrational numbers, in which case the delays are said to be incommensurate. When all such ratios are rational numbers,
on the other hand, we say that the delays are commensurate. In this case, the delays \( \tau_i \) become integer multiples of a certain positive \( \tau \). Concentrating on the LTI delay systems with commensurate delays i.e.

\[
\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - i\tau)
\]  

(1.31)

the characteristic quasi-polynomial \( \Delta(s) \) of such systems can then be written

\[
\Delta(s) = a(s, e^{-\tau s}) = \det \left( sI - \sum_{i=0}^{k} A_i e^{-is\tau} \right) = \sum_{i=0}^{k} a_i(s)e^{-is\tau}
\]

(1.32)

where

\[
a_o(s) = s^n + \sum_{j=0}^{n-1} a_{o_j} s^j, \quad a_i(s) = \sum_{j=0}^{n-1} a_{ij} s^j, \quad i = 1, 2, \ldots, k
\]

Theorem 1.5.2 Consider the system described by (1.31). Define

\[
\alpha_0 = \sup \left\{ Re \{s\} : \det \left( \Delta(s) \right) = 0 \right\}
\]

(1.33)

Then the following statements are true

(i) System (1.31) is stable if and only if \( \alpha_0 < 0 \)

(ii) For any \( \alpha > \alpha_0 \), there exists an \( L > 0 \) such that any solution \( x(t) \) of (1.31) with the initial condition \( x(t) = \varphi(t), \ t \in [-\tau, 0] \) is bounded by

\[
\|x(t)\| \leq Lm_\varphi e^{\alpha t}
\]

where

\[
m_\varphi = \max_{-\tau \leq t \leq 0} \|\varphi(t)\|
\]

(iii) \( \Delta(s) = a(s, e^{-\tau s}) \neq 0, \ \forall s \in C_+ \)

We say that the delay system (1.30) is stable independent of delay if the stability persists with respect to nonnegative delay \( \tau \), i.e. for all \( \tau > 0 \), on the other hand, if it is stable only for a range of nonnegative delay, we say that the system is delay-dependent stable.
We note that \( p(s, e^{-\tau s}) \) may also be the characteristic quasi-polynomial of the system
\[
y^{(n)}(t) + \sum_{j=0}^{n-1} \sum_{i=0}^{l_a} a_{ij} y^{(j)}(t - i\tau) = 0
\] (1.34)
which is equivalent to (1.30). It is convenient to introduce the variable \( z = e^{-\tau s} \), and write (1.32) as a bivariate polynomial
\[
a(s, z) = \sum_{i=0}^{k} a_i(s) z^{-i}, \quad z = e^{-\tau s}
\] (1.35)

We assume that the system (1.30) is stable at \( \tau = 0 \), or equivalently \( a(s, 1) \) in (1.35) is stable. The smallest deviation of \( \tau \) from \( \tau = 0 \) such that the system becomes unstable can be determined as
\[
\bar{\tau} = \min \left\{ \tau \geq 0 : a(jw, e^{-j\omega_{\tau}}) = 0 \text{ for some } w \in \mathbb{R} \right\}
\]
We call \( \bar{\tau} \) the delay margin of the system. For any \( \tau \in [0, \bar{\tau}) \), the system is stable, and whenever \( \bar{\tau} = \infty \), the system is stable independent of delay. Note that for any finite \( \bar{\tau} \), the frequency at which \( a(jw, e^{-j\omega_{\tau}}) = 0 \) represents the first contact or crossing of the characteristic roots from the stable region to the unstable one. Note also that multiple crossings may exist at some \( \tau \).

Since all roots of quasi-polynomial \( a(s, e^{-\tau s}) \) appear in complex conjugate pairs, it satisfies the conjugate symmetry property and it is sufficient to consider only the zero crossings at positive frequencies. Thus, let
\[
a(jw_i, e^{-j\theta_i}) = 0, \quad w_i > 0, \quad \theta_i \in [0, 2\pi], \quad i = 1, 2, \ldots, N
\]
Furthermore, define \( \eta_i = \frac{\theta_i}{w_i} \) and we have
\[
\bar{\tau} = \min_{1 \leq i \leq N} \eta_i = \min_{1 \leq i \leq N} \left\{ \frac{\theta_i}{w_i} : w_i > 0 \right\}
\]
for computing the delay margin.
There are some classical stability tests available using the bivariate polynomial \( a(s, z) \) given in (1.35). These tests are commonly referred to as a two-variable criterion, in which one attempts to eliminate one variable and seeking the solution of polynomials of one single variable.

To elaborate on this, consider the bilinear transformation

\[
s = \frac{1 + \lambda}{1 - \lambda}
\]

which maps \( s \) from open right half complex plane \( C_+ \) to \( \lambda \) in the open unit disk \( D \). Then construct the 2-D polynomial

\[
b(\lambda, z) = (1 - \lambda)^n a\left(\frac{1 + \lambda}{1 - \lambda}, z\right)
\]

and investigate the stability of 2-D polynomial which is a well-established topic in 2-D systems. However, in broader terms, one may view the bivariate polynomial \( a(s, z) \) as a 2-D polynomial and tackle the stability problem by defining the conjugate polynomial

\[
\bar{a}(s, z) = z^k a(-s, z^{-1})
\]

and solve simultaneously polynomial equations

\[
a(s, z) = 0, \quad \bar{a}(s, z) = 0
\]

when no solution exists, and when the system is stable in the delay-free case, it must also be stable independent of delay. Otherwise, when the two equations do admit a common solution, it is possible to eliminate one variable, resulting in a polynomial of one single variable and then check the stability for delay independent or dependent case.

Such classical stability results have shortcomings and computationally intensive. They work for low order systems with one or two delays and its generalization to higher order case is cumbersome.
1.6 Examples of systems with time-delays

Rolling System

Let us consider a system for metal rolling. Its goal is to produce metal sheets of desired thicknesses. Figure 1.1 illustrates the different components of a metal rolling system. In principle, ribbons of hot metal are drawn through rollers at high speed. The lower roller is fixed and the upper one is used to adjust the spacing between the two rollers, which controls the thickness of the metal sheet. This can be done by controlling the position of the upper roller.

For obvious reasons, the thickness sensor cannot be placed at the rollers, so it is placed down stream which results in delay in the measurement. This delay will depend on the distance $d$ between the rollers and the sensor and the speed $v$ at which the ribbons are fed into the system. If we denote by $y(t)$ the real thickness of the produced sheet, the measured one will be $y(t - \tau)$, where $\tau = \frac{d}{v}$. To control the position of the upper roller, we can use a drive motor. Based on classic control theory we can get the block diagram of Figure 1.2 of the corresponding system with $G(s)$ the transfer function between the position of the upper roller and the input voltage of the drive motor; $H(s)$ the transfer function of the sensor ($H(s) = e^{-\tau s}$), and $C(s)$ the transfer
function of the used controller. The input, $P(s)$, is introduced to represent the effect of the exogenous disturbance on the system. In the rest of this example, we assume that the controller is a proportional one with a gain $k_p$, and we consider the case of modeling the group drive motor and upper roller when the system is disturbance free, i.e., $p(t)$ for all $t \geq 0$.

Let us assume that the transfer function $G(s)$ is given by

$$G(s) = \frac{k_m}{s(Ts + 1)}.$$

Based on the block diagram and classical control theorem, we get

$$\ddot{y}(t) + \dot{y} + k_p k_m y(t - \tau) = k_p k_m y_d(t).$$

with $y_d(t)$ the desired thickness. If we let $y(t) = x_1(t)$, $x_2(t) = \dot{y}(t)$, we get the following state space representation:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + Bu(t), \quad x(0) = x_0 \text{ initial thickness}$$
$$y(t) = C x(t)$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{k_p k_m}{T} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{k_p k_m}{T} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ and } u(t) = y_d(t).$$

(1.36)
Heating System

Let us consider the heating system (Figure 1.3. This system uses hot air to control the temperature of a given room that is located a little bit far from the heater. Due to this long distance that separates the heater and the space that we would like to control its temperature, a time delay will result. If the distance is $d$ and the velocity of the hot air in the pipe is $v$, the time delay is then given by $\tau = \frac{d}{v}$.

![Heating System Diagram](image)

Figure 1.3: Heating System

Let us denote by $q_1(t)$ and $q_2(t)$ the heat flow rate from the and the one acting on the room, respectively. Due to the long pipe, we have the following relationship between these two flow rates

$$q_1(t) = q_2(t - \tau).$$

From classical control theory, we can show that the transfer function between the temperature $\theta(t)$ of the room and the flow rate $q_2(t)$ is approximated by

$$G(s) = \frac{\theta(t)}{Q_2(s)} = \frac{k}{Ts + 1}$$
where $T$ and $k$ are given constants that depend on the system.

A block diagram that represents the control system is illustrated by Figure 1.4

![Block Diagram of Heating System](image)

Figure 1.4: Block Diagram of Heating System

If we define $y(t) = \theta(t) = x(t)$ and $u(t) = r(t)$ where $r(t)$ is the desired temperature, we get

$$
\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + B u(t-\tau), \quad x(0) \text{ given}
$$

where $A_0 = -\frac{1}{T}$, $A_1 = -\frac{k_p k}{T}$, $B = \frac{k_p k}{T}$ and $C = 1$.

**Control Feeding of Production System**

Let us consider the control of a feeder of a production system (see Figure 1.5). The goal consists of feeding the downstream machine with a constant flow of mass. The feed is brought to the control of the opening of a mechanism and allows the mass flow. The sensor (for the same reasons as for the rolling system) is placed at a distance, $d$, from the opening. If the material is flowing with a speed $v$ the time delay is $\tau = \frac{d}{v}$, and the corresponding block diagram is similar to the rolling system.

If we choose a proportional controller and we assume that the transfer function of the group amplifier-motor and the mechanical part that controls the opening is given by:

$$
G(s) = \frac{L(s)}{Z(s)} = \frac{k}{s(Ts + 1)}
$$

where $l(t)$ is the position of the mechanical part, $z(t)$ is the applied voltage to the group amplifier, motor, and mechanical part, $k$ and $T$ are given constants, then the
state space representation of this system becomes

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + Bu(t), \quad x(0) = x_0
\]

\[
y(t) = Cx(t)
\]

where

\[
A_0 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{T} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ -\frac{k_p k_m}{T} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{k_p k_m}{T} \end{bmatrix}, \quad C = [1 \ 0] \quad \text{and} \quad u(t) = r(t).
\]

(1.37)

with \( y(t) = x_1(t) = l(t), \quad x_2(t) = \dot{l}(t) \).

The Williams-Otto process

The Williams-Otto process [117] has many characteristics of a typical chemical process and is therefore frequently discussed in the literature, especially in journals of chemical engineering. Here, common operations such as separation (through decanting and distillation) and reaction are involved. The system considered is a model of a refining plant. The schematic of the flow sheet for the Williams-Otto process is shown in Figure 1.6.

Upon entering the chemical reactor two kinds of raw materials take part in three chemical reactions which produce the desired product, along with some by-products. The feed rates of the raw materials are denoted by \( F_A \) and \( F_B \). A heat exchanger is required to cool the reactants to a temperature at which an undesirable by-product

![Figure 1.5: Control Feeding of Production System](image-url)
(\(F_{W_1}\)) will settle out of the reactant mixture. This settling takes place in the decanter. Subsequently the material enters a distillation column. The material contains the desired product, impurities, and a certain percentage of the raw material with some by-products of the chemical reaction. The valuable product (\(F_P\)) is removed in the overhead of the distillation column. At the bottom of the column the purge (\(F_{W_2}\)) is led off, whereas the raw material with the by-products is recycled to the chemical reactor, where it is reprocessed. The recycle loop ensures that useful products will not be discarded.

The recycle loop represents a significant transport lag. In practical situations, it is not at all unusual for material to take ten minutes to travel from the chemical reactor through the cooler, the decanter, the distillation column, and the recycling to the reactor.

The differential equations governing this chemical process are nonlinear. However, for the determination of proper corrections of the feed rates \(F_A\) and \(F_B\) at the desired operating point, a corresponding linearized model is useful. For a recycle time
of 10 minutes, the linearized and time-scaled (one time unit is 10 min) equations are:

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - 1) + Bu(t) \tag{1.38}
\]

where

\[
A_0 = \begin{bmatrix}
-4.93 & -1.01 & 0 & 0 \\
-3.20 & -5.30 & -12.8 & 0 \\
6.40 & 0.347 & -32.5 & -1.04 \\
0 & 0.833 & 11.0 & -3.96
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
1.92 & 0 & 0 & 0 \\
0 & 1.92 & 0 & 0 \\
0 & 0 & 1.87 & 0 \\
0 & 0 & 0 & 0.724
\end{bmatrix}
\tag{1.39}
\]

and

\[
B = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

The dimensionless components \(x_1, x_2, x_3, \) and \(x_4\) of the state vector \(x\) represents the deviations in the weight compositions of the raw materials \(A\) and \(B\), of an intermediate product, and of the desired product, respectively, from their nominal values. The control inputs \(u_1\) and \(u_2\) are defined to be equal to \(\frac{\delta F_A}{V_R}\) and \(\frac{\delta F_B}{V_R}\), respectively, where \(V_R\) is the volume of the chemical reactor \((V_R = 92.8 \text{ ft}^3 \approx 2.628 \text{ m}^3)\), and \(\delta F_A\) and \(\delta F_B\) are the deviations in the feed rates (in pounds per hour; 1 pound = 0.453592 kg) of the raw materials \(A\) and \(B\), respectively, from their nominal values.

**Wind Tunnel**

At the NASA Langley Research Center in Hampton, VA, a wind tunnel was constructed to achieve Reynolds numbers of one order of magnitude higher than those in existing tunnels. The desired test chamber temperatures are maintained at cryogenic levels by injection of liquid nitrogen into the airstream near the fan section of the tunnel. Fine control of the Mach number in the test chamber is effected through changes in the inlet guide vane angle setting in the fan section. Schematically, the tunnel can be depicted as in Figure 1.7.
Figure 1.7: Wind tunnel

Modeling this system based on the Navier-Stokes theory does not lead to useful equations for the design of a control law. A simple model for the Mach number control loop was proposed in [2]. In order to take into account the flow times through sections of the tunnel, a transport lag was included in the model. The proposed equations for this system are as follows

\[
\dot{x}(t) = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -w^2 & -2\xi w \end{bmatrix} x(t) + \begin{bmatrix} 0 & ka & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t-1) + \begin{bmatrix} 0 \\ 0 \\ -w^2 \end{bmatrix} u(t) \quad (1.40)
\]

where \(\frac{1}{a} = 1.964 \text{ [sec}^{-1}\), \(w = 6.0 \text{ [rad/sec]}, \xi = 0.8 [-]\), and \(k = -0.0117 \text{ [deg}^{-1}\). The state vector \(x = [\delta M, \delta \theta, \delta \dot{\theta}]^T\) consists of the variation in Mach number \(\delta M\), the variation in guide vane angle \(\delta \theta\), and the variation in guide vane angle velocity \(\delta \dot{\theta}\). The control \(u(t)\) represents the guide vane angle actuator input.
CHAPTER 1. INTRODUCTION

Gasoline Engine

A modern engine test bench was developed at the Measurement and Control Laboratory of the Swiss Federal Institute of Technology (ETH) in Zurich. This test bench is used for various purposes, e.g. the emulation of the load dynamics of the drivetrain of the target vehicle for the engine under test, the development of system identification methods for SI engines, or the testing of multivariable model-based controllers for SI engines to control the air-to-fuel ratio and the speed.

In the work of Onder (1993), [78] an efficient method for the off-line identification of an engine model is presented. The general nonlinear, delayed model contains continuous-time and discrete-time subsystems. It turns out that delays have to be taken into account in order to describe this system in an appropriate way. The dynamic model can be partitioned into the following five subsytems:

- throttle actuator
- intake manifold
- torque generation and rotational inertia
- air-to-fuel ratio sensor
- wall-wetting dynamics.

In [78], the engine model is not given in the form of a linear delayed, state space model. However, from the linear model described in [78] on page 137, together with the information about the delays presented on page 131, one immediately obtains a state-space model with seven different delays in the state and three delays in the control. Simulations show that for our purposes an appropriate single-delay system is a sufficiently good approximation to describe the behavior of the engine. (All delays in the control are neglected, whereas the delays in the states are rounded to the
maximal delay.) Following in this way the results of [78], the corresponding model for a six-cylinder 3.4-liter BMW engine with sequential injection working at idle speed,

\[
\begin{align*}
\alpha_{\text{throttle}} &= 6^\circ \\
P_{\text{manifold}} &= 0.45 \text{ bar} \\
 n &= 900 \text{ rpm} \\
\lambda &= 1 \\
\alpha_{\text{ignition}} &= 18^\circ \\
M_{\text{load}} &= 38 \text{ Nm} \\
T_{\text{exhaust}} &= 693^\circ \text{ K}
\end{align*}
\]

is

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + B_0 u(t) + B_d d(t)
\]

(1.41)

where \( \tau = \frac{2}{9} \) [sec] and

\[
A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -3.00 \cdot 10^1 & 0 & 2.05 \cdot 10^2 & -4.96 \cdot 10^{-2} & 0 \\ 2.62 \cdot 10^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & -8.60 \cdot 10^{-1} \end{bmatrix},
\]

\[
A_1 = \begin{bmatrix} 1.61 \cdot 10^2 & 1.17 \cdot 10^3 & -6.68 \cdot 10^{-1} & 0 & 9.18 \cdot 10^2 \\ -3.51 \cdot 10^{-1} & 4.83 & 2.76 \cdot 10^{-3} & 0 & -2.00 \\ 2.45 \cdot 10^{-2} & 0 & -1.33 \cdot 10^{-4} & 0 & 0 \end{bmatrix},
\]
The variables $x(t), u(t),$ and $d(t)$ represent the deviation from their nominal values (idle speed) with respect to the following physical meanings:

- $x_1$: Throttle position [degree]
- $x_2$: Intake manifold pressure [bar]
- $x_3$: Engine speed [rpm]
- $x_4$: Lambda signal [-]
- $x_5$: State of the wall-wetting model [g/min]
- $u_1$: Commanded throttle position [degree]
- $u_2$: The base value of the metered fuel is multiplied by $u_2$ [-]
- $u_3$: Difference between demanded spark angle to static calibration [degree]
- $d$: External load torque [Nm]. Intake manifold pressure [bar]

Note that the load torque is considered as a disturbance. The corresponding model for a four-cylinder 1.8-liter BMW engine have recently been analyzed. The model is of the form (1.41). Of course, the numerical values of the system matrices are different.

At idle speed,

$$
\alpha_{\text{throttle}} = 10^\circ \\
P_{\text{manifold}} = 0.48 \text{ bar} \\
n = 900 \text{ rpm} \\
\lambda = 1 \\
$$

$$
m_{\text{air flow}} = 470 \text{ g/min} \\
\alpha_{\text{ignition}} = 18^\circ \\
M_{\text{load}} = 20 \text{ Nm} \\
T_{\text{exhaust}} = 693^\circ \text{ K}
$$

the corresponding matrices are as follows:

$$
\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + B_0 u(t) + B_d d(t) 
$$
where $\tau = \frac{2}{9}$ [sec] and

\[
A_0 = \begin{bmatrix}
1.68 \cdot 10^{-1} & -2.35 & -1.63 \cdot 10^{-3} & 0 & 0 \\
0 & 1.23 \cdot 10^2 & -6.36 \cdot 10^{-2} & 0 & 0 \\
0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & -1.50 \cdot 10^1 \\
\end{bmatrix},
\]

\[
A_1 = \begin{bmatrix}
2.87 \cdot 10^1 & 1.83 \cdot 10^3 & -8.87 \cdot 10^{-2} & 0 & 2.59 \cdot 10^2 \\
-2.22 \cdot 10^{-1} & 3.97 & 2.22 \cdot 10^{-3} & 0 & -2.00 \\
4.16 \cdot 10^{-1} & 0 & -3.47 \cdot 10^{-3} & 0 & 0 \\
35 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
B_0 = \begin{bmatrix}
0 & 2.07 \cdot 10^2 & 1.57 \cdot 10^1 \\
0 & -1.60 & 0 \\
0 & 3.00 & 0 \\
\end{bmatrix},
B_d = \begin{bmatrix}
0 \\
-16 \\
0 \\
0 \\
\end{bmatrix}.
\]

The delays of the models mainly represent retarded influences of some states on the torque generation and on the air-to-fuel ratio. The subsystem of the lambda sensor additionally contains a transport delay. The non-negligible influence of all these delays can be demonstrated with the models given above. The values of time constants for the corresponding delay-free systems are between 1 and 0.03 [sec]. The time-delay $\tau = \frac{2}{9}$ [sec] is of the same order of magnitude.

The first four states can be measured. Some (or all) of these measured signals can be used to control the air-to-fuel ratio and the speed. The design of controllers which show a good disturbance rejection as well can be based on linear models. These controllers have been tested with success.

**Application to Diesel Engine Control**

Let us consider a diesel engine with an exhaust gas recirculation (EGR) valve and a turbo compressor with a variable geometry turbine (VGT) was modeled in [41]
with 3 state variables, \( x(t) = \{x_1 \ x_2 \ x_3\}^T \): intake manifold pressure \((x_1)\), exhaust manifold pressure \((x_2)\), and compressor power \((x_3)\).

![Diego Engine System](image.png)

**Figure 1.8: Diesel Engine System**

The model includes intake-to-exhaust transport delay \((h = 60 \text{ ms} \text{ when engine speed, } N, \text{ is } 1500 \text{ RPM})\). Thus, a linearized system of DDEs was introduced in [42], for a specific operating point \((N = 1500 \text{ RPM})\) whose coefficients are given by: Because of the time-delay, which is caused by the fact that the gas in the intake manifold enters the exhaust manifold after transport time, \(h\), the system can be represented by

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - 1) + Bu(t)
\]

where

\[
A_0 = \begin{bmatrix} -27 & 3.6 & 6 \\ 9.6 & -12.5 & 0 \\ 0 & 9 & -5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 21 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.26 & 0 \\ -0.9 & -0.8 \\ 0 & 0.18 \end{bmatrix}.
\]

The number of eigenvalues is infinite and one of them is positive real. Thus, the response of this linearized system shows unstable behavior (see the eigenspectrum
and the response in Figure 1.8). The system with the coefficients in (1.44) satisfies the conditions for pointwise controllability and observability.

1.7 Outline of the Thesis

The outline of the thesis is as follows.

Chapter 2 provides the existing results for stability methods for time-delay systems.

Chapter 3 describes in detail a computable stability radius formula for multi-delay systems using Rekasius substitution. Also, we provide a necessary and sufficient condition for stability robustness of two time-delay systems with structured uncertainties in terms of two distinct linear matrix inequalities. Contributions in this chapter were published and can be found in [92] and [93].

Chapter 4 summarizes the most relevant results on stabilization and robust stabilization based on LMIs, which will be used in connection to the development of subsequent chapters. Since the aim of this thesis is emphasized on stability and stabilization of time-delay systems, it is intuitively assumed that the time-delay systems admit controllability and observability conditions as it is well known from various studies (see [123] and [52] and the references therein).

In chapter 5, the time delay Proportional-Integral observer designs for linear systems are considered. The first design uses Lyapunov-based stability condition for time-delay systems to establish the stability and convergence of the observer. The second design involves attenuating of the disturbance to a pre-specified level while estimating the states of the delay system, the design requires solving a certain algebraic Riccati equation. Then an alternative design procedure which also requires solving a certain modified algebraic Riccati equation, is also provided. The methods guarantee the stability of the observer and minimizes the $H_\infty$ norm between the
disturbance and the estimated error. Contributions in this chapter were published and can be found in [89], [90] and [91].

Chapter 6 considers a special class of time-delay systems defined as Metzlerian delay systems. The stability and robust stability of Metzlerian delay systems are analyzed and closed form expression for robust stability radius has been derived. The control of this class of system is also investigated and design procedures for state feedback and dynamic output feedback controllers are given. The constrained stabilization of general time-delay system is also provided. This allows one to design controller for the general time-delay systems in such a way that the closed-loop system becomes Metzlerian delay stable. Contributions in this chapter were published and can be found in [104].

Finally, Chapter 7 briefly states the conclusions from this research work and possible avenues for future work.
Chapter 2

Stability Analysis of Time-Delay Systems

Analysis and control problems associated with time-delay systems have been the subject matter of numerous publications, most of which are summarized in several books [6], [25], [60], [68], [64], and [123]. Stability analysis and control design of time-delay systems have attracted the attention of numerous investigators as it can be seen from the extensive list of references in this thesis. In this regard, stability criteria for linear state-delay systems can be broadly classified into two categories:

- Delay-independent, which are applicable to delays of arbitrary size, and

- Delay-dependent, which include information on the size of the delay.

*Delay-Independent Stability:* When considering delay-independent stability (DIS) tests, one wants to check for a given system whether it can preserve its stability in spite of the presence of a delay of any size. It is hoped that the magnitude of the delay term is very small relative to the current state and the value of the delayed state can take any value. The DIS test tries to check if the delayed term’s value is significant/insignificant to change the original system stability. No information about
the delay is needed and only the values of the matrices of the current state and the
delayed states are considered. Clearly, this direction does not require any informa-
tion about the nature of the delay and when it yields positive results, meaning that
a system is found to be stable independent of the delay value, then it can be used
regardless of what is the magnitude of the delay or how fast it changes.

From the published results in this area, it was found that generally this type
of test is relatively easier to be derived and some system can satisfy its condition. On
the other hand, it was concluded that it suffers from some degree of conservatism due
to the following:

- Not all systems have delayed states with small magnitude and, in these cases,
systems will not satisfy the test conditions,

- In many cases, the delay is fixed and the system is time-invariant, and applying
delay-independent stability test yields unnecessary conditions on the system,

- When the delay is not fixed but bounded by some relatively small values, then
the delay-independent test is unsuitable, and

- Delay-independent conservatism carry over in feedback stabilization.

For these reasons many researchers shift their interest to the delay-dependent stability
tests.

*Delay-Dependent Stability:* In contrast to the DIS test, in the delay-dependent
stability (DDS) tests some a priori information about the delay is needed to check the
system stability. Depending on the delay pattern and related information required,
appropriate stability results can be readily obtained. This information can be used in
either one of the following scenarios:

Given a dynamical system with some delay information, check whether the
system is stable or not, or
CHAPTER 2. STABILITY ANALYSIS OF TIME-DELAY SYSTEMS

Given a dynamical system, check for at what delay limit the system still remains stable.

Generally, the second scenario is used in qualifying the stability theorems, because as we will see later, in some cases, we have to make a test of sufficient (not necessary) conditions type to check for the system stability. This means that it is for a stable system to satisfy these conditions. If a system succeed in satisfying them, then the system is stable. All the methods attempt to reduce the conservatism as much as possible and produce measures or criteria to judge with how much delay the system remains stable.

Literature reports many different stability results pertaining to time-delay systems \[68\], \[25\] and the references therein. Our initial introduction to stability based on the classical tests in chapter 1 should have sufficiently demonstrated the need for more efficient, computationally oriented stability criteria.

With the availability of efficient interior-point minimization methods, all the recent results have been cast in linear matrix inequalities (LMIs) format \[5\]. Two distinct features can be viewed in this direction which led to the improved methods pertaining to the problems of determining robust stability criteria and robust control design of uncertain time-delay systems (see \[50\], \[55\], \[60\], \[61\], \[81\], \[120\] and many references used in these papers). The first feature concerns the choice of an appropriate Lyapunov - Krasovskii functional (LKF) for stability and performance analysis within the framework of LMIs. General LKF forms might lead to a complicated system of inequalities and the selection of new and effective LKF forms is becoming crucial for deriving less conservative stability criteria. Secondly, effective stability methods based on LMI were employed to be applicable in the problems of determining robust stability criteria and robust control design of uncertain time-delay systems.
2.1 Frequency-Sweeping Tests

Let us first start with frequency-sweeping tests for retarded and commensurate delay systems. To highlight the main idea, it will be easier to consider systems with a single delay. Extension to multiple-delay case is straightforward. Thus, we consider the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau), \quad x(t) = \varphi(t), \quad -\tau \leq t \leq 0$$

where \( x(t) \in \mathbb{R}^n \) is the state; \( \varphi(t) \) is the continuous initial condition. In (2.1), the scalar \( \tau \) is the constant delay for system, \( A_0 \in \mathbb{R}^{n \times n} \) and \( A_1 \in \mathbb{R}^{n \times n} \) are known real constant matrices.

The characteristic polynomial of (2.1) is given by

$$a(s, e^{-s\tau}) = \det(sI - A_0 - A_1 e^{-s\tau})$$

**Theorem 2.1.1** The system (2.1) is stable independent of delay if and only if

(i) \( A_0 \) is stable

(ii) \( A_0 + A_1 \) is stable, and

(iii) \( \rho((jwI - A_0)^{-1}A_1) < 1, \forall w > 0 \)

where \( \rho(\cdot) \) denotes the spectral radius of a matrix.

Furthermore, if we define \( \rho(A, B) = \min\{|\lambda|, \det(A - \lambda B) = 0\} \) with respect to the generalized eigenvalue of the pair of matrices \((A, B)\). Then, condition (iii) can be replaced by \( \rho(jwI - A_0, A_1) > 1, \forall w > 0 \).

It is interesting to point out that the frequency dependent matrix pencil \((jwI - A_0 - \lambda A_1)\) associated with the replacement condition \( \rho(jwI - A_0, A_1) > 1, \forall w > 0 \) of Theorem 2.1.1 can be used to state the following theorem for delay dependent case.

The idea is to use the frequency-dependent generalized eigenvalues \( \lambda_i(jwI - A_0, A_1) \) and replace it with \( \lambda_i((jwI - A_0)^{-1}A_1) \), which may not exist at certain frequencies.
Theorem 2.1.2 Suppose the system (2.1) is stable at \( \tau = 0 \). Let \( \text{rank}(A_1) = q \) and define

\[
\bar{\tau}_i = \begin{cases} 
\min_{1 \leq k \leq n} \frac{\theta_k^i}{w_k^i} & \text{if } \lambda_i(jw_k^iI - A_0, A_1) = e^{-j\theta_k^i} \\
\infty & \text{if } \rho(jwI - A_0, A_1) > 1, \ \forall \ w > 0.
\end{cases}
\]

Then the system (2.1) is stable for all \( \tau \in [0, \bar{\tau}) \) where \( \bar{\tau} = \min_{1 \leq i \leq q} \bar{\tau}_i \), but becomes unstable at \( \tau = \bar{\tau} \). Furthermore, if \( \bar{\tau}_i = \infty, \ \forall \ i \). Then the condition \( \rho(jwI - A_0, A_1) > 1, \ \forall \ w > 0 \) guarantees stability independent of delay.

As pointed out, one can generalize the above theorems for multiple commensurate delays. Specifically, consider

\[
\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - i\tau), \ \tau \geq 0
\] (2.2)

Then using Schur complement identity for determinant one can establish equivalent conditions as in Theorems 2.1.1 and 2.1.2. However, sweeping frequencies make applicability of these results restrictive. There are however constant matrix tests that can reliably be used when (2.2) is considered. The following theorem summarizes the result for both delay independent and delay dependent cases.

Theorem 2.1.3 Suppose that the characteristic quasi-polynomial (1.32) is stable at \( \tau = 0 \). Let \( H_n = 0, \ T_n = I \) and

\[
H_j = \begin{bmatrix} a_{k,j} & a_{k-1,j} & \ldots & a_{1,j} \\ 0 & a_{k,j} & \ldots & a_{2,j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{k,j} \end{bmatrix}, \quad T_j = \begin{bmatrix} a_{0,j} & 0 & \ldots & 0 \\ a_{1,j} & a_{0,j} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-1,j} & a_{k-2,j} & \ldots & a_{0,j} \end{bmatrix}, \ j = 0, 1, \ldots, n-1
\]

\[
P_j = \begin{bmatrix} (\text{Im } j)^j T_j & (\text{Im } j)^j H_j \\ (-\text{Im } j)^j H_j^T & (-\text{Im } j)^j T_j^T \end{bmatrix}, \quad j = 0, 1, \ldots, n \quad \text{with } (\text{Im } j)^2 = -1.
\]

Furthermore, define
Then \( \bar{\tau} = \infty \) if \( \sigma(P) \cap \mathcal{R}_+ = \phi \) or \( \sigma(P) \cap \mathcal{R}_+ = \{0\} \). Additionally, let

\[
F(s) = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \\
\vdots & \vdots & \ddots & \vdots \\ \\
0 & 0 & \ldots & 1 \\
-a_0(s) & -a_1(s) & \ldots & -a_{k-1}(s) \end{bmatrix}, \quad G(s) = \text{diag}(1, 1, \ldots, a_k(s)).
\]

Then \( \bar{\tau} = \infty \) if \( \sigma(F(jw_k), G(jw_k)) \cap \partial \mathcal{D} = \phi \) for all \( 0 \neq w_k \in \sigma(P) \cap \mathcal{R}_+ \). In these cases the quasi-polynomial (1.32) or equivalently (2.2) is stable independent of delay. Otherwise, \( \bar{\tau} = \min_{1 \leq k \leq 2nq} \frac{\theta_k}{w_k} \) where \( e^{-j\theta_k} \in \sigma(F(jw_k), G(jw_k)) \) and (1.32) or (2.2) is stable for all \( \tau \in [0, \bar{\tau}) \), but is unstable at \( \tau = \bar{\tau} \).

## 2.2 LMI Stability Conditions

For system (2.1), by selecting the Lyapunov-Krasovskii functional

\[
V(t, x) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)Qx(s) \, ds
\]

and invoking the Lyapunov-Krasovskii theorem, the following stability condition can be derived.

**Theorem 2.2.1** The system (2.1) is asymptotically stable independent of delay if there exist matrices \( P > 0 \) and \( Q > 0 \) such that

\[
\begin{bmatrix}
PA_0 + A_0^T P + Q & PA_1 \\
\bullet & -Q
\end{bmatrix} < 0
\]

(2.4)
One can also use the Razumikhin theorem to obtain a simple stability condition using the Lyapunov function

\[ V(x) = x^T P x \]

**Theorem 2.2.2** The system described by (2.1) is asymptotically stable independent of delay if there exist a scalar \( \alpha > 0 \) and a real matrix \( P \) such that

\[
\begin{bmatrix}
PA_0 + A_0^T P + \alpha P & PA_1 \\
\bullet & -\alpha P
\end{bmatrix} < 0
\]  

(2.5)

or equivalently there exist a scalar \( \alpha > 0 \) and a real symmetric matrix \( Q \) such that

\[
\begin{bmatrix}
A_0 Q + Q A_0^T + \alpha Q & A_1 Q \\
\bullet & -\alpha Q
\end{bmatrix} < 0
\]  

(2.6)

In control problems it is preferable to use (2.6) which can be defined from (2.5) by defining \( Q = P^{-1} \) and left multiply (2.5) by \( \text{diag}(P^{-1}, P^{-1}) \) and right multiply by its transpose.

It has been recognized that stability condition based on Lyapunov - Krasovskii is less conservative. This motivated numerous researchers to adopt the Lyapunov - Krasovskii theorem in conducting research seeking improved delay dependent stability and stabilization conditions. We follow this trend throughout the thesis unless otherwise considered beneficial to apply the Lyapunov - Razumikhin theorem.

We start our development using explicit and implicit model transformation as suggested originally by the experts studying the stability in terms of LMI. The idea of model transformation is a series of steps that leads to a distributed delay format, which includes the original delay system as a special case. Then by applying stability condition on transformed disturbed delay system one can arrive at sufficient LMI stability condition for the original delay system. Using the explicit transformation technique one can arrive at the following result.
Theorem 2.2.3 The system (2.1) is asymptotically stable dependent of delay if there exists real symmetric matrices \( P, S_0 \) and \( S_1 \) such that \( P > 0 \) and

\[
\begin{bmatrix}
M & -PA_0 & -PA_1 \\
\bullet & -S_0 & 0 \\
\bullet & \bullet & -S_1 \\
\end{bmatrix} < 0
\] (2.7)

where

\[
M = \frac{1}{\tau} \left[ P(A_0 + A_1) + (A_0 + A_1)^T P \right] + S_0 + S_1.
\]

Note that by defining \( S_0 = \alpha_0 P \) and \( S_1 = \alpha_1 P \), one can only search for \( P > 0 \) and scalers \( \alpha_0 \) and \( \alpha_1 \) such that (2.7) is satisfied.

On the other hand using implicit model transformation one arrives at the following result.

Theorem 2.2.4 The system (2.1) is asymptotically stable dependent of delay if there exists real matrices \( X^T = X, Y \) and \( P > 0 \) such that

\[
\begin{bmatrix}
\hat{N} & PA_1 - Y & -A_0^T Y^T \\
\bullet & -S & -A_1^T Y^T \\
\bullet & \bullet & -\frac{1}{\tau} X \\
\end{bmatrix} < 0
\] (2.8)

where

\[
\hat{N} = PA_0 + A_0^T P + S + \tau X + Y + Y^T
\] (2.9)

Note that if we set \( Y = 0 \) and \( X = 0 \), we obtain the delay independent stability result of Theorem 2.2.1. On the other hand, if we choose \( Y = PA_1 \), it can be shown that the criterion is equivalent to delay-dependent stability condition in Theorem 2.2.3 with the following variable transformation from \((S, X)\) to \((S_0, S_1)\):

\[
S = \tau S_1, \quad X = -\frac{1}{\tau} \left[ P(A_0 + A_1) + (A_0 + A_1)^T P \right] - S_0 - S_1.
\]

Finally, we state a simple stability condition, using a Razumikhin-Lyapunov approach which requires solving a Lyapunov equation.
The following theorem guarantees the uniform asymptotic stability of the time-delay system (2.1) for any positive time-delay in a neighborhood of 0, that is \([0, \tau^*]\).

**Theorem 2.2.5** The system (2.1), satisfying Hurwitz stability of \(A_0 + A_1\), is uniformly asymptotically stable for any constant delay \(\tau\) in the interval \(0 < \tau \leq \tau^*\), with

\[
\tau^*(Q, P, \beta_1, \beta_2) = \left\| Q^{-\frac{1}{2}} \left[ \beta_1 P A_1 A_0 P^{-1} A_0^T A_1^T P + \beta_2 B(A_1)^2 P^{-1} (A_1^T)^2 P + (\beta_1^{-1} + \beta_2^{-1}) P \right] Q^{-\frac{1}{2}} \right\|^{-1}
\]

where \(\beta_1, \beta_2\) are positive real numbers and \((P, Q)\) is a pair of symmetric and positive-definite real matrices satisfying the Lyapunov equation:

\[
(A_0 + A_1)^T P + P(A_0 + A_1) + Q = 0.
\]

It should be pointed out that the above theorems can also be generalized to multiple commensurate delays. To conclude this section let us consider the following theorem which is stated for delay independent case of section 2.2 as a generalization of Theorem 2.2.1.

**Theorem 2.2.6** If there exist a set of symmetric positive-definite matrices \(P > 0, Q_i > 0, i = 1, 2, \ldots, k\) satisfying the LMI

\[
\begin{bmatrix}
A_0^T P + P A_0 + \sum_{i=1}^{k} Q_i & PA_1 & \ldots & PA_k \\
\cdot & -Q_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\cdot & \cdot & \ldots & -Q_k
\end{bmatrix}
< 0.
\]

Then the system (2.2) is asymptotically stable.
Finally, the following theorem provides a delay-dependent stability condition for multiple-delay systems

**Theorem 2.2.7** If there exist symmetric positive-definite matrices \( P > 0, Q_l > 0 \), \( 0 \leq l \leq k \) such that the following holds

\[
\begin{bmatrix}
\bar{A}P + PA + \sum_{i=1}^{k} \sum_{l=0}^{l} \tau_l Q_l & M^T \\
\cdot & -\bar{Q}
\end{bmatrix} < 0.
\]

where

\[
\bar{A} = \sum_{i=0}^{k} A_i
\]

\[
M^T = (\tau_1 PA_1 A_0 \ldots \tau_l PA_l A_k; \ldots ; \tau_k PA_k A_0 \ldots \tau_k PA_k A_k)
\]

\[
\bar{Q} = \text{diag}(\tau_1 Q_0 \ldots \tau_1 Q_k; \ldots ; \tau_k Q_0 \ldots \tau_k Q_k).
\]

Then the system (2.2) is asymptotically stable.

### 2.3 Alternative Delay-Dependent Stability Techniques

#### 2.3.1 Newton-Leibniz Formula

Initial efforts made to get a delay-dependent criteria were using model transformation for the time-delay system (2.1). On applying the fundamental Newton-Leibniz formula

\[
x(t - \tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(s) \, ds
\]

\[
= x(t) - \int_{t-\tau}^{t} [A_0 x(s) + A_1 x(s - \tau)] \, ds
\]

to get around the delayed state, then system (2.1) becomes

\[
\dot{x}(t) = A_0 x(t) + A_1 \left[ x(t) - \int_{t-\tau}^{t} \dot{x}(s) \, ds \right]
\]

\[
= (A_0 + A_1) x(t) - A_1 \int_{t-\tau}^{t} \dot{x}(s) \, ds
\]

\[
= (A_0 + A_1) x(t) - A_1 \int_{t-\tau}^{t} [A_0 x(s) + A_1 x(s - \tau)] \, ds
\]

(2.10)
It should be observed that the asymptotic stability of the time-delay system (2.10) implies that of the system (2.1). For this reason, we focus on studying the stability of (2.10). To this end, we choose a Lyapunov-Krasovskii functional of the form

\[ V = V_0 + V_1 + V_2 \]

\[ V_0 = x^T(t)Px(t), \quad V_1 = \int_{t-\tau}^{t} x^T(s)Qx(s) \, ds \]

\[ V_2 = \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)A_s^TZA_sA_s^T \dot{x} \, ds \, d\theta \]  \tag{2.11}

where \( P > 0, \ Q > 0, \ Z > 0. \)

**Remark 2.3.1** In terms of dynamic models with generalized coordinates, one can interpret the first component \( V_0 \) of the LKF, \( V \) as a measure of the internal energy of system 2.1, the second term \( V_1 \) is intuitively seen to provide a measure of the signal energy during the delay period \([\tau, 0]\), and the third term \( V_2 \) gives a measure of the energy corresponding to the difference between the state \( x(t) \), the signal sought for feedback stabilization, and the delayed state \( x(t-\tau) \), the one that might be available for feedback processing, that is given by the Newton-Leibniz formula \( x(t) - x(t-\tau) = \int_{t-\tau}^{t} \dot{x}(s) \, ds \).

On the other hand, by selecting a Lyapunov-Krasovskii functional of the form

\[ \hat{V} = V_0 + V_1 + V_2 \]

\[ V_0 = x^T(t)P^{-1}x(t), \quad V_1 = \int_{-\tau}^{0} \int_{t+\theta}^{t} x^T(s)A_s^TQ^{-1}A_s x(s) \, d\theta \]

\[ V_2 = \int_{-\tau}^{0} \int_{t-\tau+\theta}^{t} \dot{x}^T(s)A_s^TZA_sA_s^T \dot{x} \, ds \]

the following stability condition can be derived:

**Theorem 2.3.2** The system (2.10) is asymptotically stable for any constant delay \( \tau \) satisfying \( 0 < t \leq \bar{\tau} \), if there exist matrices \( P > 0 \) and \( Q > 0 \) such that

\[ \begin{bmatrix} \Pi & \bar{\tau}PA_0^T & \bar{\tau}PA_1^T \\ \cdot & -Q & 0 \\ \cdot & \cdot & -Z \end{bmatrix} < 0 \]  \tag{2.12}
where
\[ \Pi = P(A_0 + A_1) + (A_0 + A_1)^TP + A_1(Q + Z)A_1^T \]

### 2.3.2 Cross-Product Terms

One of the main purposes in the study of delay-dependent stability for time-delay systems is to develop methods to reduce conservatism of existing delay-dependent stability conditions. On taking the time derivative of \( V \) or \( \hat{V} \), the following crossproduct term
\[ -2x^T(t)PA_1 \int_{t-\tau}^{t} \dot{x}(s) \, ds \] (2.13)
appears, and since it is neither positive nor negative definite, it may lead to a complication in establishing the negative definiteness of \( V \) or \( \hat{V} \). It is known that the finding of better bounds on some weighted cross products arising in the analysis of the delay-dependent stability problem plays a key role in reducing conservatism. There was a common practice to resolve this complication by majorizing the term and replacing it by upper-bound terms which are of either positive or negative definite nature. In some effort, they used the following algebraic inequality
\[ -2a^Tb < a^TXa + b^TX^{-1}b, \quad X > 0 \] (2.14)
for some vectors \( a, b \), and matrix \( X \). This solves the problem; however, it makes the result more conservative because we are adding positive terms in \( \hat{V} \) and it has smaller chance to become negative. To get the smallest possible upper bound, matrix \( X \) should be selected such that the term \(-2a^Tb\) is replaced by \( M \), given by
\[ M = \inf_{X > 0} (a^TXa + b^TX^{-1}b) \] (2.15)
which means that select a \( X > 0 \) that gives the minimum \( M \).
2.3.3 Bounding Inequalities

By focusing on the case of constant but unknown delay and seeking an alternative way, it is suggested to employ the following inequality

$$-2a^Tb < (a + Mb)^TX(a + Mb) + b^TX^{-1}b + 2b^TMb, \quad X > 0 \quad (2.16)$$

Here $M$ can take any value and if one puts $M = 0$, equation (2.14) is obtained. Therefore, (2.14) is a special case of (2.16) and at the worst case one is able to find the same result as in (2.14) by setting $M = 0$. The following result stands out:

**Theorem 2.3.3** The system (2.1) is asymptotically stable for any delay $\tau$ satisfying $0 < \tau \leq \bar{\tau}$, if there exist matrices $R > 0$, $S > 0$, $M > 0$ and $N > 0$ such that

$$\begin{bmatrix}
\tilde{\Pi} & -NA^T_1 & A^T_0 A^T_1 M & \bar{\tau}(S + N^T) \\
\cdot & -S & A^T_1 A^T_1 M & 0 \\
\cdot & \cdot & -M & 0 \\
\cdot & \cdot & \cdot & -M \\
\end{bmatrix} < 0 \quad (2.17)$$

where $\tilde{\Pi} = R^T(A_0 + A_1) + (A_0 + A_1)^T R + N^T A_1 + A^T_1 N + S$.

A summary of the features of the method developed as follows:

- Employs first-order transformation given in (2.10),
- Incorporates the bounding technique (2.16),
- Deals with unknown fixed delay pattern,
- Introduces LKF with three terms,
- Manipulates three Lyapunov matrices and two free-weighting matrices, and
- Considers nominal time-delay models only.
Further improvement over the inequality in (2.14) to reduce the conservatism can be attained by replacing the cross-product term mentioned in (2.13) with its upper bound by using the following inequality

\[-2\int a^T(s)Nb(s) \, ds \leq \int \begin{bmatrix} a(s)^T & X & Y - N^T \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \\ b(s) \end{bmatrix} \, ds\]

such that

\[\begin{bmatrix} X & Y \\ \bullet & Z \end{bmatrix} \geq 0 \tag{2.18}\]

It is not difficult to show that with suitable substitution for matrices \(Z\) and \(Y\), one can show that inequalities (2.14) and (2.16) are special cases of (2.18). Therefore, so it is expected to give less conservative stability results. The corresponding stability condition is established below based on the LKF (2.30):

**Theorem 2.3.4** The system (2.1) is asymptotically stable for any constant delay \(\tau\) satisfying \(0 < \tau \leq \bar{\tau}\), if there exist matrices \(P > 0\), \(Q > 0\), \(X\), \(Y\) and \(Z\) such that

\[
\begin{bmatrix}
\tilde{\Pi} & PA_1 - Y & \bar{\tau}A_0^T Z \\
\bullet & -Q & \bar{\tau}A_1^T Z \\
\bullet & \bullet & -\tau Z
\end{bmatrix} < 0 \tag{2.19}
\]

\[
\begin{bmatrix} X & Y \\ \bullet & Z \end{bmatrix} \geq 0 \tag{2.20}
\]

where \(\tilde{\Pi} = PA_0 + A_0^T P + \bar{\tau}X + Y + Y^T + Q\).

When dealing with state feedback stabilization, it turns out that the foregoing method determines the feedback gain matrices using iterative computational procedure. By and large, the method is capable of accommodating norm-bounded uncertainties. It is important to note that the iterative method should start with stable system for some delay factor \(\tau > 0\), which means this method is not applicable for unstable systems. In addition, for relatively large systems the iterative method takes quite a long time to yield the desired results.
Along another direction by using the Jensen’s integral inequality and choosing the LKF

\[
\dot{V} = V_0 + V_1 + V_2
\]

\[
V_0 = x^T(t)P^{-1}x(t), \quad V_1 = \int_{t-\tau}^t x^T(s)Qx(s) \, ds
\]

\[
V_2 = \tau \int_{-\tau}^0 \int_{t+\theta}^t \dot{x}^T(s)Z\dot{x}(s) \, ds d\theta
\]

(2.21)

the following results can be obtained.

**Theorem 2.3.5** The system (2.1) is asymptotically stable for any constant delay \( \tau \) satisfying \( 0 < \tau \leq \bar{\tau} \), if there exist matrices \( P > 0 \), \( Q > 0 \) and \( Z \) such that

\[
\begin{bmatrix}
\tilde{E} & PA_1 - Z & \bar{\tau}A_0^T Z \\
\bullet & -Q - Z & \bar{\tau}A_1^T Z \\
\bullet & \bullet & -Z
\end{bmatrix} < 0
\]

(2.22)

where

\[
\tilde{E} = A_0P + A_0^TP + Q - Z.
\]

We note that Theorem 2.3.5 establishes that the time-delay system (2.1) is asymptotically stable for any constant delay \( \tau \) satisfying \( 0 < \tau \leq \bar{\tau} \) when the LMI (2.22) has a feasible solution, which implies that for \( \tau \) satisfying \( 0 < \tau \leq \frac{\bar{\tau}}{2} \), the time-delay system (2.1) is asymptotically stable too. A way to reduce the conservatism is by introducing the half-delay into system (2.1) to gain more information. Consequently, we consider the augmented system

\[
\dot{x}(t) = A_0x(t) + A_1x(t - \tau)
\]

\[
\dot{x}\left(t + \frac{\tau}{2}\right) = A_0x\left(t + \frac{\tau}{2}\right) + A_1x\left(t - \frac{\tau}{2}\right)
\]

(2.23)

In terms of

\[
y(t) = [x^T(t + \frac{\tau}{2}) \quad x^T(t)]^t
\]
system (5.83) can be cast into the form

\[ \dot{y}(t) = \begin{bmatrix} A_0 & 0 \\ 0 & A_0 \end{bmatrix} y(t) + \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix} y(t - \tau) \]
\[ = \tilde{A}_0 y(t) + \tilde{A}_1 y(t - \tau) \]

Choosing the LKF

\[ \bar{V}(y_t) = \bar{V}_0 + \bar{V}_1 + \bar{V}_2 \]
\[ \bar{V}_0 = y^T(t) P y(t), \quad \bar{V}_1 = \sum_{j=1}^{2} \int_{t-j\tau}^{t} y^T(s) Q_j y(s) \, ds \]
\[ \bar{V}_2 = \tau \sum_{j=1}^{2} \int_{t-j\tau}^{t} \int_{0}^{\tau} \dot{y}^T(s) Z_j \dot{y}(s) \, ds \, d\theta \]

the following result can be obtained

**Theorem 2.3.6** The system (2.1) is asymptotically stable for any constant delay \( \tau \) satisfying \( 0 < \tau \leq \bar{\tau} \), if there exist matrices \( P > 0, Q_1 > 0, Q_2 > 0, Z_1 > 0 \) and \( Z_2 > 0 \) such that

\[ \beta^\top W \beta^T < 0 \]

where \( \beta^\top \) is an orthogonal complement of \( \beta \) given by

\[ \beta = \begin{bmatrix} I & \tilde{A}_0 & 0 & \tilde{A}_1 & 0 & 0 \\ 0 & -I & I & 0 & I & 0 \\ 0 & I & 0 & I & 0 & I \\ 0 & I & I & I & 0 & 0 \\ \tau^2 Z_1 + \tau^2 Z_2 & P & 0 & 0 \end{bmatrix} \]

\[ W = \begin{bmatrix} P & Q_1 + Q_2 & 0 & 0 \\ 0 & 0 & -Q & 0 \\ 0 & 0 & 0 & -Z \end{bmatrix} \]

where

\[ I_r = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad I_f = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}, \quad I_s = \begin{bmatrix} 0 & 0 \\ -I & 0 \end{bmatrix} \]
\[ Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \]
2.3.4 Descriptor System Approach

To introduce the descriptor system approach, we first consider the time-delay system (2.1) and represent (2.1) in the following form:

\[
\dot{x}(t) = y(t)
= (A_0 + A_1)x(t) - A_1 \int_{t-\tau}^{t} \dot{x}(s) \, ds
\]

\[
0 = -y(t) + (A_0 + A_1)x(t) - A_1 \int_{t-\tau}^{t} y(s) \, ds
\]  

(2.27)

which can be rewritten in the compact form

\[
E \dot{\tilde{x}} = \bar{A}_0 \tilde{x}(t) + \bar{A}_1 \tilde{x}(t - \tau)
\]

\[
E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} 0 \\ A_1 \end{bmatrix}
\]

\[
\bar{A}_0 = \begin{bmatrix} 0 & I \\ A_0 + A_1 & -I \end{bmatrix}
\]  

(2.28)

where the time delay has the pattern

\[
0 \leq \tau \leq \bar{\tau}, \quad \hat{\tau} \leq \mu < 1
\]  

(2.29)

and the system is subjected to either norm bounded or polytopic uncertainty. Given that the descriptor model (2.27) is equivalent to system (2.1), it was concluded that improved stability results can be obtained by choosing the LKF

\[
\hat{V}(\tilde{x}) = \hat{V}_0 + \hat{V}_1 + \hat{V}_2
\]

\[
\hat{V}_0 = \tilde{x}^T(t)EP\tilde{x}(t), \quad \hat{V}_1 = \int_{t-\tau}^{t} x^T(s)Qx(s) \, ds
\]

\[
\hat{V}_2 = \int_{-\tau}^{0} \int_{t+\theta}^{t} y^T(s)A_1^TZA_1y(s) \, dsd\theta
\]

(2.30)

and deploying the bounding inequality (2.18), the following result can be obtained:
Theorem 2.3.7 The system (2.1) is asymptotically stable for any delay $\tau$ satisfying $0 < \tau \leq \bar{\tau}$, if there exist matrices $P_1 > 0$, $P_2$, $P_3$, $Z > 0$, $Y_{11}$, $Y_{12}$, $S_{11}$, $S_{12}$, and $S_{13}$; such that

$$
\begin{bmatrix}
\bar{\Pi} + \tau S_1 & P^T \bar{A}_1 - Y_{11}^T \\
\cdot & -Q
\end{bmatrix} < 0 \quad (2.31)
$$

$$
\begin{bmatrix}
Z & Y_{11} \\
\cdot & S_1
\end{bmatrix} \geq 0 \quad (2.32)
$$

where

$$
\bar{\Pi} = P^T \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix} + \begin{bmatrix} 0 & I \end{bmatrix}^T P + \begin{bmatrix} Q & 0 \\ 0 & \tau Z \end{bmatrix} + \begin{bmatrix} Y_{11} \\ Y_{12} \end{bmatrix} + \begin{bmatrix} Y_{11} \\ \cdot \end{bmatrix}^T \quad (2.33)
$$

$$
P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} Y_{11} & Y_{12} \end{bmatrix}, \quad S_1 = \begin{bmatrix} S_{11} & S_{12} \\ \cdot & S_{13} \end{bmatrix} \quad (2.34)
$$

An equivalent form of Theorem 2.3.7 is stated below.

Theorem 2.3.8 The system (2.1) is asymptotically stable for any delay $\tau$ satisfying $0 < \tau \leq \bar{\tau}$, if there exist matrices $P_1 > 0$, $P_2$, $P_3$, $Z > 0$, $Y_{11}$, $Y_{12}$ such that

$$
\begin{bmatrix}
\bar{\Pi} & P^T \bar{A}_1 - Y_{11}^T \\
\cdot & -Q & 0
\end{bmatrix} < 0 \quad (2.35)
$$

where $\bar{\Pi}$, $P$, and $Y_1$ are given in (2.33) and (2.34).

2.3.5 Free-Weighting Matrices Method

Subsequent research studies focused on further reduction of conservatism. It becomes clear that new methods should be developed which do not arise from model transformation nor upper-bounding. A new approach was developed through the introduction of free-weighting matrices (slack matrix variables) method, which is based on adding zero-valued equations to the linear matrix inequality (LMI) under consideration plus...
incorporating the Newton-Leibniz formula. Candidate examples of this type are

\[ 2x^T(t) \left[ x(t) - x(t-\tau) - \int_{t-\tau}^{t} x(s) \, ds \right] \] (2.36)

\[ 2x^T(t-\tau)W \left[ x(t) - x(t-\tau) - \int_{t-\tau}^{t} x(s) \, ds \right] \] (2.37)

Here \( Y \) and \( T \) are free matrices to be manipulated to reach a feasible solution. Furthermore, it can be seen that

\[
\tau \begin{bmatrix}
    x(t) \\
    x(t-\tau)
\end{bmatrix}^T
\begin{bmatrix}
    X_{11} & X_{12} \\
    \bullet & X_{22}
\end{bmatrix}
\begin{bmatrix}
    x(t) \\
    x(t-\tau)
\end{bmatrix}
- \int_{t-\tau}^{t} \begin{bmatrix}
    x(t) \\
    x(t-\tau)
\end{bmatrix}^T
\begin{bmatrix}
    X_{11} & X_{12} \\
    \bullet & X_{22}
\end{bmatrix}
\begin{bmatrix}
    x(t) \\
    x(t-\tau)
\end{bmatrix} \, ds = 0 \] (2.38)

Now by using the LKF (2.21) in addition to (2.36), (2.37), and (2.38), the following theorem summarizes the main delay-dependent stability result:

**Theorem 2.3.9** The system (2.1) is asymptotically stable for any delay \( \tau \) satisfying \( 0 < \tau \leq \bar{\tau} \), if there exist matrices \( P > 0, Q > 0, Z > 0, X_{11}, X_{12}, X_{22}, Y \) and \( W \) such that

\[
\begin{bmatrix}
    \Psi_a & \Psi_c & \bar{\tau}A_0^T Z \\
    \bullet & -\Psi_e & \bar{\tau}A_1^T Z \\
    \bullet & \bullet & -\bar{\tau}Z
\end{bmatrix}
\begin{bmatrix}
    X_{11} & X_{12} & Y \\
    \bullet & X_{22} & W \\
    \bullet & \bullet & Z
\end{bmatrix}
\geq 0 \] (2.39)

where

\[
\Psi_a = PA_0 + A_0^T P + Y + Y^T + Q + \tau X_{11} \\
\Psi_c = PA_1 - Y + W^T + \tau X_{12} \\
\Psi_e = Q + W + W^T - \tau X_{22} \] (2.41)

In the case that equation (2.38) was not utilized, then simple manipulations can show that Theorem 2.3.9 reduces to
Theorem 2.3.10 The system (2.1) is asymptotically stable for any delay $\tau$ satisfying $0 < \tau \leq \bar{\tau}$, if there exist matrices $P > 0$, $Q > 0$, $Z > 0$, $Y$ and $W$ such that

$$
\begin{bmatrix}
\varphi_a & \varphi_c & \bar{\tau}Y & \bar{\tau}A_0^T Z \\
\bullet & -\varphi_e & -\bar{\tau}W & \bar{\tau}A_1^T Z \\
\bullet & \bullet & -\bar{\tau}Z & 0 \\
\bullet & \bullet & \bullet & -\bar{\tau}Z
\end{bmatrix} < 0
$$

where

$$
\varphi_a = PA_0 + A_0^T P + Y + Y^T + Q \\
\varphi_c = PA_1 - Y + W^T \\
\varphi_e = Q + W + W^T
$$
Chapter 3

Robust Stability and Stability Radius

3.1 Introduction

The stability analysis of time-delay systems has been extensively studied in the literature as we explored in the previous chapter. The reported results can generally be classified into two types: delay-dependent stability results which take the size of the delay into account, and delay-independent stability results, which can be applied to delays with arbitrary size. It is known that delay-dependent stability results are less conservative than the delay-independent ones. Therefore, considerable attention has been focused on the later case and various approaches have been proposed. A recent paper [120] provides a proper perspective about several delay-dependent stability results in terms of linear matrix inequalities (LMIs). This reference establishes the equivalence of seven stability criteria published in the literature (see [120] and the references therein). At the same time, effort has been devoted on stability radius computation for regular and time-delay systems ([36], [37], [38], [51], [76], [83], [84], and [100]). One can refer to the extensive list of publications cited in these references.
for additional results. Motivated by the delay-dependent stability result of [120], it is of particular interest to compute the stability radius of time-delay systems in terms of delay interval. This establishes a connection between stability radius and time-delay for various types of uncertainties and different class of delay systems. It turns out that one can switch from robust stability results based on LMI to stability radius computation as reported in this section. Based on ([36] and [83]), an alternative and more convenient procedure for stability radius computation is given. Using Rekasius substitution enables an efficient computation of stability radius since the frequency sweep can directly be incorporated in the algorithm. Here we look into the robust stability against uncertainties in delays by using the Cluster Treatment of Characteristic Roots (CTCR) methodology, as proposed in [86]. Then, by characterizing robust stability region, we provide a convenient way to assess stability and robust stability of delay-dependent systems with structured uncertainties. It is evident from the literature that the stability robustness of multi-delay systems in terms of both time-delays and structured perturbations remain an open problem. For simplicity, we concentrate on systems with two delays. CTCR detects all the stability regions precisely, in the space of time delays. To achieve this, the so-called kernel and offspring curves, which describe the complete portrait of the possible imaginary characteristic roots are constructed. The boundary of these curves determine the exact display of the robustness region against delay uncertainties. Consequently, one can determine the stability robustness of time-delay systems under structured uncertainties in any specified stability region using stability radius formula which has been derived for delay systems. We present computable stability radius formula for multi-delay systems inspired by the features of Rekasius substitution. The by-product of this computation is a surface of stability radius with respect to the domain of time-delays. The portrait of such surfaces for two time-delay systems can be displayed in three dimen-
sional coordinate systems. We also define the fundamental stability region among a set of disjoint stability regions for two time-delay systems. Consequently, in this fundamental stability region a necessary and sufficient condition for stability robustness of two time-delay systems with structured uncertainties in terms of two distinct linear matrix inequalities becomes possible. Several examples are included to demonstrate the validity of theoretical results.

### 3.2 Robust Stability of Time Delay Systems

Consider a nominal linear state delayed system with one delay $\tau = \tau_1$

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_1), \quad \tau_1 \geq 0 \quad (3.1)$$

where $x \in \mathbb{R}^n$ is the state vector. The system (3.1) is said to be stable if the zeros of

$$\det(sI - A_0 - A_1 e^{-s \tau_1}) = 0 \quad (3.2)$$

are contained in $\mathcal{C}_g$, where $\mathcal{C}_g = \{s \in \mathbb{C} : \text{Re}(s) < 0\}$.

Now, let the time delay system (3.1) be subjected to structured perturbation as follows

$$\dot{x}(t) = (A_0 + E_0 \Delta_0 F_0)x(t) + (A_1 + E_1 \Delta_1 F_1)x(t - \tau_1) \quad (3.3)$$

where $\Delta_0 \in \mathbb{R}^{m \times p_0}$, $\Delta_1 \in \mathbb{R}^{m \times p_1}$ denote a perturbation matrices and $E_0 \in \mathbb{R}^{n \times m}$, $E_1 \in \mathbb{R}^{n \times m}$, $F_0 \in \mathbb{R}^{p_1 \times n}$ and $F_1 \in \mathbb{R}^{p_1 \times n}$, are known matrices defining the structure of perturbations. Without loss of generality, we assume that $E_0 = E_1 = E$, and define $\Delta = [\Delta_0 \Delta_1]$.

The characteristic equation of the perturbed systems (3.3) can similarly be written as in (3.2) with the perturbed matrices to determine the stability. However, equivalent stability criteria in terms of linear matrix inequalities (LMI) for uncertain time delay systems are available [50], [53] and [60]. One possible result is the following theorem which can conveniently be used for the perturbed system above.
Theorem 3.2.1 If there exist \( P > 0 \), \( Q > 0 \), \( X \), \( Y \) and \( Z \) and scalars \( e_1 \) and \( e_2 \) such that

\[
\begin{bmatrix}
Y_{11} & -Y + PA & \bar{\tau}_1 A_0^T Z & PE_0 & PE_1 \\
-Y^T + A_1^T P & -Q_{22} & \bar{\tau}_1 A_1^T Z & 0 & 0 \\
\bar{\tau}_1 Z A_0 & \bar{\tau}_1 Z A_1 & -\bar{\tau}_1 Z & \bar{\tau}_1 Z E_0 & \bar{\tau}_1 Z E_1 \\
E_0^T P & 0 & \bar{\tau}_1 E_0^T Z & -e_1 I & 0 \\
E_1^T P & 0 & \bar{\tau}_1 E_1^T Z & 0 & -e_2 I
\end{bmatrix} < 0 \quad (3.4)
\]

\[
\begin{bmatrix}
X & Y \\
Y^T & Z
\end{bmatrix} \geq 0 \quad (3.5)
\]

where \( Y_{11} = A_0^T P + PA_0 + \bar{\tau}_1 X + Y + Y^T + Q + e_1 F_0^T F_0 \) and \( Q_{22} = Q - e_2 F_1^T F_1 \), then the system (3.3) is asymptotically stable for any time delay \( \tau_1 \) satisfying \( 0 \leq \tau_1 \leq \bar{\tau}_1 \) and all admissible uncertainties. Furthermore, if the uncertainty is captured on the matrix \( A_0 \) alone (\( E_1 = 0 \) or \( F_1 = 0 \)), then the inequality (3.4) will be reduced to a \( 4 \times 4 \) block matrix by deleting the fifth block row and column. Similarly, if the uncertainty is captured on the matrix \( B \) alone (\( E_1 = 0 \) or \( F_1 = 0 \)), then the inequality (3.4) will be reduced to a \( 4 \times 4 \) block matrix by deleting the fourth block row and column. Finally, when there are no uncertainties, then the inequality (3.4) will be reduced to a \( 3 \times 3 \) block matrix by deleting the fourth and fifth block rows and columns.

The above result provides a computationally convenient way to check the stability of uncertain time delay systems (3.3). We will use it in connection to the stability radius of time delay systems in the next section.
3.3 Stability Radius

In the absence of time delay, the stability of regular dynamical system with perturbation

\[ \dot{x}(t) = (A + E\Delta F)x(t) \] (3.6)

has been studied by many researchers in the past. An interesting problem in this analysis was to compute the stability radius associated with the above system. Let us partition the complex plane \( \mathbb{C} \) into two disjoint subsets \( \mathbb{C}_g \) and its complement \( \mathbb{C}_b \) such that \( \mathbb{C} = \mathbb{C}_g \cup \mathbb{C}_b \), and denoting the singular values of a matrix \( M \in \mathbb{C}^{p \times m} \), ordered non-increasingly, by \( \sigma_i(M), \ i = 1, 2, \ldots, \min\{p, m\} \). Also denoting \( \sigma_1(M) \) by \( \bar{\sigma}(M) \), the structured real stability radius of a matrix triple \((A, E, F) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}\) is defined by

\[ r_R(A, E, F) = \inf_{\Delta \in \mathbb{R}^{m \times p}} \{ \bar{\sigma}(\Delta) : \text{and } A + E\Delta F \text{ is unstable} \} \]

which leads to

\[ r_R(A, E, F) = \left( \sup_{s \in \partial \mathbb{C}_g} \mu_R[F(sI - A)^{-1}E] \right)^{-1} \] (3.7)

where

\[ \mu_R(M) := \inf_{\gamma \in (0, 1]} \sigma_2 \left( \begin{bmatrix} \text{Re}(M) & -\gamma \text{Im}(M) \\ \gamma^{-1} \text{Im}(M) & \text{Re}(M) \end{bmatrix} \right) \] (3.8)

and \( \partial \mathbb{C}_g \) denotes the boundary of \( \mathbb{C}_g \). The computational on \( \mu_R(M) \) for a given \( M \) can be carried out at low computational cost through minimization of a unimodal function on \((0, 1]\) (see [36]).

It is also important to point out that the structured complex stability radius can be obtained through

\[ r_C(A, E, F) = \left( \sup_{s \in \partial \mathbb{C}_g} \bar{\sigma}[F(sI - A)^{-1}E] \right)^{-1}, \] (3.9)

since it is well known that \( \mu_R(M) = \bar{\sigma}(M) \). For special class of systems the complex and real stability radius coincide. This will be elaborated later in this thesis.
Following the development of regular system, we define the real structured stability radius of uncertain time delay system (3.1) by

\[
R_r = \inf_{s \in \partial C} \inf \{ \sigma(\Delta) : \Delta \in \mathbb{R}^{m \times (p_0 + p_1)} \text{ and } \left| sI - A_0 - A_1 e^{-s\tau} - E\Delta_0 F_0 - E\Delta_1 F_1 e^{-s\tau} \right| = 0 \}
\]

\[
= \inf_{s \in \partial C} \inf \{ \sigma(\Delta) : \Delta \in \mathbb{R}^{m \times (p_0 + p_1)} \text{ and } \left| I - \left[ \Delta_0 F_0 (sI - A_0 - A_1 e^{-s\tau})^{-1} E + \Delta_1 F_1 (sI - A_0 - A_1 e^{-s\tau})^{-1} e^{-s\tau} E \right] \right| = 0 \}
\]

\[
= \inf_{s \in \partial C} \inf \{ \sigma(\Delta) : \Delta \in \mathbb{R}^{m \times (p_0 + p_1)} \text{ and } |I - \Delta M| = 0 \}
\]

where

\[
M = \begin{bmatrix}
F_0(sI - A_0 - A_1 e^{-s\tau})^{-1} E \\
F_1(sI - A_0 - A_1 e^{-s\tau})^{-1} e^{-s\tau} E
\end{bmatrix},
\]

(3.10)

In this chapter, we simplify the expression given in (3.10) by using an alternative expression called Rekasius substitution [77] which is given by

\[
e^{-s\tau} = \frac{1 - sT}{1 + sT}, \quad \tau \in \mathbb{R}^+, \quad T \in \mathbb{R},
\]

(3.11)

and defined only for \( s = jw, \ w \in \mathbb{R} \). This is an exact substitution not an approximation, with the obvious mapping condition of:

\[
\tau = \frac{2}{w} |\tan^{-1}(wT) \mp l\pi|, \ l = 0, 1, 2, \ldots, \infty.
\]

(3.12)

This equation describes an asymmetric mapping in which one \( T \) is mapped into infinitely many \( \tau' \)s for a given \( w \). Inversely for the same \( w \), one particular \( \tau \) corresponds to one \( T \) only. The reason for this modification is due to the fact that recent developments of time delay systems provided new stability results for multiple delays in which the above substitution played an important role, [77]. This makes it convenient to tie the stability radius computation via the aforementioned substitution as well. Furthermore, the computational efficiency is improved by this substitution as it can be shown in the illustrative examples.
Theorem 3.3.1 Let the linear time delay system (3.1) be stable. The real stability radius for the perturbed system (3.3) is given by

\[
r_R = \inf_{s \in \partial C} \inf \{ \sigma(\Delta) : \Delta \in \mathcal{R}^{m \times (p_0+p_1)} \text{ and } |I - \Delta M| = 0 \}
\]

\[
= \left\{ \sup_{s \in \partial C} \inf_{\gamma \in (0,1]} \sigma_2 \left( \begin{bmatrix} \text{Re}(M) & -\gamma \text{Im}(M) \\ \gamma^{-1} \text{Im}(M) & \text{Re}(M) \end{bmatrix} \right) \right\}^{-1}
\]

(3.13)

where

\[
M(w, \tau) = \begin{bmatrix} M_0(w, \tau) \\
M_1(w, \tau) \end{bmatrix},
\]

(3.14)

with

\[
M_0(w, \tau) = [1 + j \tan(\frac{\tau w}{2} - l\pi) ] F_0 \times 
\left( [jwI - (A_0 + A_1)] + j \tan(\frac{\tau w}{2} - l\pi) [jwI - (A_0 - A_1)] \right)^{-1} E,
\]

\[
M_1(w, \tau) = [1 - j \tan(\frac{\tau w}{2} - l\pi) ] F_1 \times 
\left( [jwI - (A_0 + A_1)] + j \tan(\frac{\tau w}{2} - l\pi) [jwI - (A_0 - A_1)] \right)^{-1} E,
\]

\[ l = 0, 1, \ldots, \infty, \ 0 \leq \tau \leq \bar{\tau}, \ E \in \mathcal{R}^{n \times m}, \ F_0 \in \mathcal{R}^{p_0 \times n} \text{ and } F_1 \in \mathcal{R}^{p_1 \times n}, \text{ are known scaling matrices, and } \sigma_2(\cdot) \text{ denote the second largest singular value is } (\cdot).
\]

**Proof** The proof can be established by simple substitution and algebraic manipulations.

We show this only for the term $M_0$. From the definition above, we have

\[
|sI - A_0 - A_1 e^{-st} - E\Delta_0 F_0 - E\Delta_1 F_1 e^{-st}| = 0,
\]

which implies

\[
\left| (sI - A_0 - A_1 e^{-st}) \times \left( I - \Delta_0 F_0(sI - A_0 - A_1 e^{-st})^{-1} E \right. 
- \Delta_1 F_1(sI - A_0 - A_1 e^{-st})^{-1} E e^{-st} \big) \right| = 0,
\]

since system (3.1) is stable, using eqn.(3.2) we get
\begin{equation}
\left| I - \Delta_0 F_0(sI - A_0 - A_1 e^{-s\tau})^{-1} E - \Delta_1 F_1(sI - A_0 - A_1 e^{-s\tau})^{-1} E e^{-s\tau} \right| = 0,
\end{equation}

which can be set in the form

\begin{equation}
|I - \Delta M| = 0, \quad \Delta = [\Delta_0 \Delta_1], \quad M = \begin{bmatrix} M_0 \\ M_1 \end{bmatrix}
\end{equation}

where

\begin{align*}
M_0 &= F_0 \left( sI - A_0 - A_1 e^{-s\tau} \right)^{-1} E = F_0 \left( sI - A_0 - A_1 \frac{1 - sT}{1 + sT} \right)^{-1} E \\
&= (1 + sT) \times F_0 \left( (1 + sT)(sI - A_0) - (1 - sT)A_1 \right)^{-1} E \\
&= (1 + sT) \times F_0 \left( (sI - (A_0 + A_1)) + sT(sI - (A_0 - A_1)) \right)^{-1} E \\
&= \left( 1 + j \tan(\frac{\tau w}{2} - l\pi) \right) F_0 \times \left( [jwI - (A_0 + A_1)] + j \tan\left( \frac{\tau w}{2} - l\pi \right) \times \\
&\quad (jwI - (A_0 - A_1)) \right)^{-1} E.
\end{align*}

By applying similar steps, the expression $M_1$ can be derived.

Note that in driving the expression (3.10) or (3.14) we take advantage of the well-known equality

$$\det(I + A_0 A_1) = \det(I + A_1 A_0)$$

and the fact that $\det(H(s) + \delta H(s)) = 0$ is equivalent to $\det(I + \delta H(s)H^{-1}(s)) = 0$.

This allows interchanging the structured matrices such that the term $I - \Delta M$ is identified. In order to achieve this when both uncertainty structures are present in [36], it is required to assume that $E_0 = E_1 = E$, which makes the separation transparent. However, when $E_0 \neq E_1$, one can still perform the separation and derive a modified expression in which the stability radius is computable. This requires an alternative assumption relating $E_0$ and $E_1$. Suppose that $E_1 = E_0 K$, where $K$ is invertible. Then $E_1 \Delta_1 F_0 = E_0 K \Delta_1 F_1 = E_0 \tilde{\Delta}_1 F_1$ and one can define $\tilde{\Delta} = [\Delta_0 \tilde{\Delta}_1]$. The computation of real stability radius reduces to the one for $I - \Delta \tilde{M}$. To obtain a smallest real perturbation $\tilde{\Delta}$ such that $I - \Delta \tilde{M}$ is singular requires sweeping the
frequency. At the frequency $w^*$, the minimum of the second singular value of $\bar{M}$ occurs at $\gamma^*$. Then its corresponding left and right singular vectors $U$ and $V$ are easily obtained which leads to a smallest real $\Delta$ given by $\Delta = \bar{r}_R[v_1 v_2][u_1 u_2]^T$. The next step is to partition $\bar{\Delta} = [\Delta_0 \bar{\Delta}_1]$ and determine $\Delta_1$ from $\Delta_1 = K^{-1} \bar{\Delta}_1$. Finally, $r_R$ for the original problem can be computed by

$$r_R = \| \Delta \| = \| [\Delta_0 \Delta_1] \| .$$

It is evident that an alternative characterization of robust stability can be stated in terms of robust stability radius. Consequently, robust stability radius for linear time delay system exists for any time delay $\tau$, $0 \leq \tau \leq \bar{\tau}$ if and only if the LMI in Theorem 3.2.1 has a feasible solution.

### 3.4 Robust Stability of Linear Multi-Delay Systems

For the multi-delay retarded type linear system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) \quad (3.15)$$

there are many stability conditions available for time delay systems of retarded type (see [6], [25], [66], and [68]). Among such results, linear matrix inequalities (LMIs) offer sufficient delay-dependent and delay-independent stability conditions for similar class of delay systems considered here. One such possible delay dependent result is given in the following theorem which is the generalization of [66].

**Theorem 3.4.1** The system (3.15) is stable if there exists $0 < P_1, P_2, P_3$, and
$R_i = R_i^T$, $i = 1, 2, \ldots, k$ that satisfy the following LMI:

$$\begin{pmatrix}
Y_{11} & Y_{12} & \tau_1 P_2^T A_1 & \tau_2 P_2^T A_2 & \ldots & \tau_k P_2^T A_k \\
Y_{12}^T & Y_{22} & \tau_1 P_3^T A_1 & \tau_2 P_3^T A_2 & \ldots & \tau_k P_3^T A_k \\
\tau_1 A_1^T P_2 & \tau_1 A_1^T P_3 & -\tau_1 R_1 & 0 & \ldots & 0 \\
\tau_2 A_2^T P_2 & \tau_2 A_2^T P_3 & 0 & -\tau_2 R_2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\tau_k A_k^T P_2 & \tau_k A_k^T P_3 & 0 & 0 & \ldots & -\tau_k R_k
\end{pmatrix} < 0 \quad (3.16)$$

where

$$Y_{11} = \left( \sum_{i=0}^{k} A_i^T \right) P_2 + P_2^T \left( \sum_{i=0}^{k} A_i \right),$$

$$Y_{12} = P_1 - P_2 + \left( \sum_{i=0}^{k} A_i^T \right) P_3 \quad \text{and}$$

$$Y_{22} = -P_3 - P_3^T + \sum_{i=1}^{k} \tau_i R_i.$$

The above theorem is an elegant result if one is interested to check the stability for a set of time delays $\tau_1, \tau_2, \ldots, \tau_k$.

There are also Lyapunov-based results available for robust stability with respect to time delays which can be formulated as LMI’s in terms of upper bounds of delays, see for instance [26], [25], [40], [68], and [124] and the references therein. However there is no result available to extend such results under structured perturbations on system matrices $A_i$’s except for the case of single delay [66]. This motivates us to investigate the connection between the stability robustness in terms of time-delays and stability robustness under structured perturbations. The departure point is to consider the above results for single delay case, which will be explored here [92].

In this section we take advantage of previously obtained results and generalize them non-trivially for two time-delays. This is achieved primarily by the help of CTCR methodology which helps reveal stability robustness of the dynamics against delay uncertainties in the domain of multiple delays. Once such stability regions are
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at hand, robustness measure of the multiple delay problem against structured uncertainties becomes functional and interpretable. Therefore, we first depart from kernel and offspring curves as proposed in [77] under the framework of CTCR, and define a particularly important region called fundamental stability region. This enables us to provide necessary and sufficient conditions in terms of two LMI’s for two time-delay systems with uncertainties both in delay and structured perturbations. A major stepping stone to investigate the robust stability of systems with structured perturbations is the concept of stability radius. In the next section we provide a formula to compute the stability radius for two time-delay systems. In this section we provide an overview of the precise determination of the stability regions in the domain of multiple time delays using the “Cluster Treatment of Characteristic Roots” (CTCR) is given in [107]. For simplicity we concentrate on the class of systems with two time delays. The main idea behind the construction of these regions is the framework called the “Cluster Treatment of Characteristic Roots”, CTCR, which characterizes the existence of the so-called “kernel curves”, which carries the complete information concerning the imaginary characteristic roots of the dynamics at hand. CTCR not only enables non-trivially the robust stability regions of the dynamics against uncertainties in delays, but it also allows one to interconnect its results with stability radius computation against multi-structured uncertainties, as we will present in the following section.

The linear two time-delay system

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_1) + A_2 x(t - \tau_2), \quad 0 \leq \tau_i \leq \bar{\tau}_i.
\]  

(3.17)

where \(x \in \mathbb{R}^n\) is the system state, \(A_i \in \mathbb{R}^{n \times n}\). The system (3.17) is said to be stable if the zeros of

\[
\det(sI - A_0 - A_1 e^{-s\tau_1} - A_2 e^{-s\tau_2}) = 0 \quad \text{lie in } C_g.
\]  

(3.18)
Observation 3.4.2 If there exists an imaginary root of (3.18) at \( s = \pm w_c j \) (\( c \) stands for crossing) for \( \{ \tau_0 \} = (\tau_{10}, \tau_{20}) \), it is trivial to show that the same imaginary root will also exist at all the countably infinite grid points of

\[
\{ \tau \} = (\tau_{10} + \frac{2\pi}{w_c} i, \tau_{20} + \frac{2\pi}{w_c} k), \; i = 1, 2, \ldots, \; k = 1, 2, \ldots
\]

\[
\tau_{10} - \frac{2\pi}{w_c} \leq 0 \quad \tau_{20} - \frac{2\pi}{w_c} k \leq 0. \quad (3.19)
\]

Furthermore, the grid points defined in (3.19) are all identified by one simple parameter, \( w_c \). Nevertheless, more than one set of \((\tau_1, \tau_2)\) may render the same \( w_c \), or vice versa, one set of \((\tau_1, \tau_2)\) may cause more than one \( w_c \). It is not difficult to observe that if one considers all possible variations of \( w_c \), one obtains the complete set of continuous curves in \((\tau_1, \tau_2)\) space passing through the grid points of (3.19). Again, notice that all these curves are derived from a fundamental curve traced by \((\tau_{10}, \tau_{20})|_{w_c}\)

where the notation implies the imaginary characteristic root of \( w_c j \) corresponding to the minimum delays of \((\tau_{10}, \tau_{20})\).

Definition 3.4.3 Assume that the complete set of trajectories of \((\tau_{10}, \tau_{20})|_{w_c}\) are determined in \( \{ \tau \} = (\tau_1, \tau_2) \) space for all possible \( w_c \) values satisfying (3.19). Then we define this set of trajectories as the “kernel curves set” of the system described by the characteristic equation (3.18) and denote it by

\[
P_0(\tau_1, \tau_2) = \Omega_\tau \left( (\tau_1, \tau_2) \big| CE(s, \tau_1, \tau_2) = 0, \; s = w_c j, \; \tau_i > 0, \; \tau_i - \frac{2\pi}{w} \leq 0, \; i = 1, 2, \; w_c \in \Omega_w \right)
\]

(3.20)

where \( \Omega_\tau \) represents the complete set of kernel points in \((\tau_1, \tau_2)\) space, and \( \Omega_w \) is the set of all possible \( w_c \)s.

Furthermore, all the curves which are obtained from the “kernel curves set” using the point-wise nonlinear mapping of equation (3.19) constitute the so-called
“offspring curves set” of the dynamics described in (3.17). The “offspring curves set” carry the identical frequency signatures of the “kernel curves set” since equation (3.19) preserves $w_c$ from the kernel to offspring. These offspring curves are denoted by $P_{ik}(\tau_1, \tau_2), i = 1, 2, \ldots; k = 1, 2, \ldots$. Thus, the “kernel curves” and their “offspring curves”, by definition, contain all possible points in $\{\tau\}$ space, which renders at least one imaginary characteristic root. A crucial problem is the determination of all the “kernel curves” exhaustively for all possible $w_c \in \Omega_w$. In order to achieve this we use the Rekasius substitution (3.11), which describes an asymmetric mapping in which one $T_1$ (or $T_2$) is mapped into infinitely many $\tau_1^T s$ (or $\tau_2^T s$) for a given $w$. Inversely for the same $w_c$, one particular $\tau_1$ (or $\tau_2$) corresponds to one $T_1$ (or $T_2$) only. It is important to point out that the same substitution is used in computing the stability radius, which ties with the development of this section and improve computational efficiency of the stability radius.

The Rekasius substitution holds for $s \in C_0$ where $s = w_c j$ exactly with the companion condition (3.12) relating $\tau_i$ and $T_i$ which are distributed with a periodicity of $\frac{2\pi}{w_c}$. The substitution of (3.11) into (3.18) results in a new polynomial equation $\tilde{C}E(s, T_1, T_2)$, which can further be simplified to an alternative characteristic equation $\overline{CE}(s, T_1, T_2)$. We define the following four sets based on the above development: $\Omega_1 = \{s|CE(s, \tau_1, \tau_2) = 0, (\tau_1, \tau_2)\text{fixed} \in \mathbb{R}^{2+}\}$ a set with countably infinite members for a given $(\tau_1, \tau_2)$. $\bar{\Omega}_1 = \{\Omega_1 | (\tau_1, \tau_2) \in \mathbb{R}^{2+}\}$ the complete topology of $\Omega_1^T s$ for the entire $2-D$ quadrant $(\tau_1, \tau_2) \in \mathbb{R}^{2+}$. $\Omega_2 = \{s|\overline{CE}(s, T_1, T_2) = 0, (T_1, T_2)\text{fixed} \in \mathbb{R}^2\}$ a countably set of nine members for a given $(T_1, T_2)$. $\bar{\Omega}_2 = \{\Omega_2 | (T_1, T_2) \in \mathbb{R}^2\}$ the complete topology of $\Omega_2^T s$ for the entire $2-D$ quadrant $(T_1, T_2) \in \mathbb{R}^2$.

**Property 3.4.4** All purely imaginary members of the topology $\bar{\Omega}_1$ for $\{\tau\} \in \mathbb{R}^{2+}$ and $\bar{\Omega}_2$ for $\{T\} \in \mathbb{R}^2$ are identical, i.e. $\bar{\Omega}_1 \cap C_0 \equiv \bar{\Omega}_2 \cap C_0$ and furthermore for one such root, $s = w_c j$, (3.12) is automatically satisfied (see [85] for more details). It is
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evident that determining $\bar{\Omega}_2 \cap C_0$ for $(T_1, T_2) \in \mathbb{R}^2$ is considerably easier mission than determining $\bar{\Omega}_1 \cap C_0$ for $(\tau_1, \tau_2) \in \mathbb{R}^{2+}$. This is the crucial reason for using Rekasius substitution.

Using the Routh’s array for $CE(s, T_1, T_2)$, it is not difficult to show that any $s \in \bar{\Omega}_2 \cap C_0$ requires that the corresponding $(T_1, T_2)$ satisfy

$$R_1(T_1, T_2) = 0$$  \hspace{1cm} (3.21)

where $R_1$ corresponds to $s^1$-row of Routh’s table. Thus $R_1(T_1, T_2)$ represents a well-defined curve in $(T_1, T_2)$ space. The auxiliary equation

$$R_{21}(T_1, T_2)s^2 + R_{22}(T_1, T_2) = 0$$  \hspace{1cm} (3.22)

identifies imaginary parts and should have the sign agreement between coefficients, i.e.

$$R_{21}(T_1, T_2)R_{22}(T_1, T_2) > 0$$  \hspace{1cm} (3.23)

Consequently, (3.21) and (3.23) define a curve in $(T_1, T_2)$ space which constitutes the complete trajectory of $(T_1, T_2)$ points yielding the entire set of imaginary characteristic roots $s \in \bar{\Omega}_2 \cap C_0$ and the corresponding crossing frequencies are found from (3.22). Let us denote the resultant curve by

$$\overline{R}_1(T_1, T_2) = \left\{ R_1(T_1, T_2) = 0 \right\} \cap \left[ R_{21}(T_1, T_2)R_{22}(T_1, T_2) > 0 \right]$$  \hspace{1cm} (3.24)

and call it the core curve. In fact, $\overline{R}_1(T_1, T_2)$ represents the projection of all $P_{ik}(\tau_1, \tau_2)$ on the $(T_1, T_2)$ space. Using equation (3.12) point by point along $\overline{R}_1(T_1, T_2)$ and the corresponding $w^T_s$ one can construct the respective kernel curves $P_0(\tau_1, \tau_2)$ in $(\tau_1, \tau_2)$ space. The kernel curves set, $P_0(\tau_1, \tau_2)$ and the offspring curves set $P_{ik}(\tau_1, \tau_2)$ are the only locations where possible stability switching can take place. Thus, the rest of the domain $(\tau_1, \tau_2) \in \mathbb{R}^{2+}$ is of no concern from the point of stability switching.
Using the CTCR, we can completely characterize the region of stability in 
\((\tau_1, \tau_2)\) space.

Figure 3.1 shows a possible scenario of stable regions for two time delay sys-
tems. It is clear that we are able to depict geometrically the robust stability regions 
in terms of \((\tau_1, \tau_2)\). Theorem 3.4.1 provides a tool to check the stability of any point 
\((\tau_1, \tau_2)\) and the robust stability radius can be obtained in specified stable subregion 
using (3.26) when applied for \(r = 2\). It is easy to prove that the delay-independent 
stability property of time-delay systems is not robust with respect to structured per-
turbations of the parameters \(A_i\) of the form \(A_i + E_i \Delta_i F_i\). Therefore, it is of particular 
interest to obtain robust stability results for time-delay systems under parameter 
perturbation of system matrices \(A_i^T\). Before starting our main result in this section 
we need to define a special region of stability which plays an important role in our 
subsequent development.

![Figure 3.1: A Scenario for Stability Regions of Two Time-Delay Systems](image)

**Definition 3.4.5** Let the two time-delay system (3.17) be zero delay stable. Then 
a subset of kernel and offspring curves such that any point on these curves can be 
connected to the origin by a continuous path in the stable region is called fundamental 
kernel and offspring curves. Furthermore, the region of stability bounded by \(\tau_1, \tau_2\) axes
and the fundamental kernel and offspring curves is called the fundamental stability region.

Figure 3.1 shows a possible fundamental stability region for two time delay system. It should be pointed out that among all complicated stability boundaries which have infinitely many possibilities, the boundary of fundamental stability region can only have four distinct possible shapes with respect to $\tau_1$, $\tau_2$ axes. The fundamental stability region of Figure 3.2 has one side open along $\tau_1$-axis and closed at $\tau_2$-axis. Conversely, the second possible shape is when the opening is along $\tau_2$-axis and closed at $\tau_1$-axis. The third and fourth possibilities are when the fundamental kernel and offspring curves along $\tau_1$ and $\tau_2$ are either open or closed.

![Figure 3.2: Fundamental Stability Region.](image)

The main result for two time-delay systems requires the above results for single delay system.

Let the time-delay system (3.17) be subjected to structured perturbation as follows

$$\dot{x}(t) = (A_0 + E\Delta_0 F_0)x(t) + \sum_{i=1}^{2} (A_i + E\Delta_i F_i)x(t - \tau_i),$$

where $\Delta_i \in \mathbb{R}^{m \times p_i}$, $i = 0, 1, 2$ denote perturbation matrices and $E \in \mathbb{R}^{n \times m}$, $F_i \in \mathbb{R}^{p_i \times n}$, $i = 0, 1, 2$ are known matrices defining the structure of perturbations.
Then we can state our main theorem as follows:

**Theorem 3.4.6** Let the two time delay-system be zero delay robustly stable and let the box defined by $B = \{(\tau_1, \tau_2) : 0 \leq \tau_1 \leq \bar{\tau}_1, \ 0 \leq \tau_2 \leq \bar{\tau}_2\}$ be a subset of the fundamental stability region defined for (3.17). Then the perturbed system (3.25) is robustly stable if and only if the following two LMI’s corresponding to two separate single delay systems are satisfied:

**System 1 defined when $\tau_2 = 0$:**

$$\dot{x}(t) = (A_{02} + E\Delta_{02}F_{02})x(t) + (A_1 + E\Delta_1F_1)x(t - \tau_1),$$

where $A_{02} = A_0 + A_2, \ \Delta_{02} = [\Delta_0 \ \Delta_2], \text{ and } F_{02} = \begin{bmatrix} F_0 \\ F_2 \end{bmatrix}$, i.e., if there exist $P > 0, \ Q > 0, \ X, \ Y$ and $Z$ and scalars $e_1$ and $e_2$ such that

$$
\begin{bmatrix}
Y_{11} & -Y + PA_1 & \bar{\tau}_1 A_{02}^T Z & PE & PE \\
-Y^T + A_1^T P & -Q_{22} & \bar{\tau}_1 A_1^T Z & 0 & 0 \\
\bar{\tau}_1 Z A_{02} & \bar{\tau}_1 Z A_1 & -\bar{\tau}_1 Z & \bar{\tau}_1 Z E & \bar{\tau}_1 Z E \\
E^T P & 0 & \bar{\tau}_1 E^T Z & -e_1 I & 0 \\
E^T P & 0 & \bar{\tau}_1 E^T Z & 0 & -e_2 I
\end{bmatrix} < 0
$$

where $Y_{11} = A_{02}^T P + PA_{02} + \bar{\tau}_1 X + Y + Y^T + Q + e_1 F_0^T F_0$ and $Q_{22} = Q - e_2 F_1^T F_1$.

**System 2 defined when $\tau_1 = 0$:**

$$\dot{x}(t) = (A_{01} + E\Delta_{01}F_{01})x(t) + (A_2 + E\Delta_2F_2)x(t - \tau_2),$$

where $A_{01} = A_0 + A_1, \ \Delta_{01} = [\Delta_0 \ \Delta_1], \text{ and } F_{01} = \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}$, i.e., if there exist $P > 0, \ Q > 0, \ X, \ Y$ and $Z$ and a positive scalars $e_1$ and $e_2$ such...
that
\[
\begin{bmatrix}
  Y_{11} & -Y + PA_2 & \tilde{\tau}_2 A_{01}^T Z & PE & PE \\
  -Y^T + A_2^T P & -Q_{22} & \tilde{\tau}_2 A_2^T Z & 0 & 0 \\
  \tilde{\tau}_2 Z A_{01} & \tilde{\tau}_2 Z A_2 & -\tilde{\tau}_2 Z & \tilde{\tau}_2 Z E & \tilde{\tau}_2 Z E \\
  E^T P & 0 & \tilde{\tau}_2 E^T Z & -e_1 I & 0 \\
  E^T P & 0 & \tilde{\tau}_2 E^T Z & 0 & -e_2 I
\end{bmatrix} < 0
\]

where \( Y_{11} = A_0^T P + PA_0 + \tau_2 X + Y + Y^T + Q + e_1 F_0^T F_0 \) and \( Q_{22} = Q - e_2 F_2^T F_2 \).

\textbf{Proof} The proof of this theorem follows directly from Theorem 3.2.1 and the fact that the box \( B \) is contained in the fundamental stability region. \qed

\textbf{Theorem 3.4.7} Let the linear time delay system (3.17) be stable. The real stability radius for the perturbed system (3.25) is given by

\[
\begin{align*}
\rho_R &= \inf_{s \in \mathcal{C}_0} \sigma_2\left( \begin{bmatrix}
  Re(M) & -\gamma \text{Im}(M) \\
  \gamma^{-1} \text{Im}(M) & Re(M)
\end{bmatrix}\right) \\
&= \left\{ \sup_{s \in \mathcal{C}_0, \gamma \in [0,1]} \inf_{\epsilon_0 \in [0,1]} \sigma_2\left( \begin{bmatrix}
  Re(M) & -\gamma \text{Im}(M) \\
  \gamma^{-1} \text{Im}(M) & Re(M)
\end{bmatrix}\right) \right\}^{-1}
\end{align*}
\]

where

\[
M(w, \tau_1, \tau_2) = \begin{bmatrix}
  M_0(w, \tau_1, \tau_2) \\
  M_1(w, \tau_1, \tau_2) \\
  M_2(w, \tau_1, \tau_2)
\end{bmatrix},
\]

with

\[
\begin{align*}
M_i &= \left[ 1 + (-1)^i j \tan\left( \frac{\tau_2 w}{2} - l_1 \pi \right) \right] \left[ 1 + j \tan\left( \frac{\tau_2 w}{2} - l_2 \pi \right) \right] F_i \times \left[ j \omega I - \sum_{i=0}^{2} A_i \right] \\
&\times \prod_{i=1}^{2} \left[ 1 + j \tan\left( \frac{\tau_2 w}{2} - l_i \pi \right) \right] - 2j \tan\left( \frac{\tau_2 w}{2} - l_1 \pi \right) \left[ 1 + j \tan\left( \frac{\tau_2 w}{2} - l_2 \pi \right) \right] A_1 \\
&- 2j \tan\left( \frac{\tau_2 w}{2} - l_2 \pi \right) \times \left[ 1 + j \tan\left( \frac{\tau_2 w}{2} - l_1 \pi \right) \right] A_2 \times E,
\end{align*}
\]

\( i = 0, 1, 2; \ l_i = 0, 1, \ldots, \infty; \ 0 \leq \tau_i \leq \tilde{\tau}_i, \ E \in \mathbb{R}^{n \times m}, \ F_i \in \mathbb{R}^{p_i \times m} \) are known scaling matrices, and \( \sigma_2(\cdot) \) denote the second largest singular value.
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**Proof** The proof can be established by simple algebraic manipulations and using Rekasius substitution (3.11). We show this only for the term $M_0$.

From the definition above, we have

$$\det(sI - A_0 - E\Delta_0 F_0 - \sum_{i=1}^2 (A_i + E\Delta_i F_i) e^{-s\tau_i}) = 0,$$

which implies

$$\det\left((sI - A_0 - \sum_{i=1}^2 A_i e^{-s\tau_i})\left(I - \Delta_0 F_0(sI - A_0 - \sum_{i=1}^2 A_i e^{-s\tau_i})^{-1} E - \Delta_1 F_1\right.\right.$$

$$\times\left.(sI - A_0 - \sum_{i=1}^2 A_i e^{-s\tau_i})^{-1} E e^{-s\tau_1} - \Delta_2 F_2(sI - A_0 - \sum_{i=1}^2 A_i e^{-s\tau_i})^{-1} E e^{-s\tau_2}\right)\right) = 0,$$

since system (3.1) is stable, using Eq. (3.18), we get

$$\det\left(I - \Delta_0 F_0(sI - A_0 - \sum_{i=1}^2 A_i e^{-s\tau_i})^{-1} E - \Delta_1 F_1(sI - A_0 - \sum_{i=1}^2 A_i e^{-s\tau_i})^{-1} E e^{-s\tau_1}\right.$$

$$- \Delta_2 F_2(sI - A_0 - \sum_{i=1}^2 A_i e^{-s\tau_i})^{-1} E e^{-s\tau_2}\right) = 0,$$

which can be set in the form

$$|I - \Delta M| = 0, \quad \Delta = [\Delta_0 \Delta_1 \Delta_2], \quad M = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \end{bmatrix}$$

where

$$M_0 = F_0(sI - A_0 - A_1 e^{-s\tau_1} - A_2 e^{-s\tau_2})^{-1} E = F_0\left(sI - A_0 - A_1 \frac{1 - sT_1}{1 + sT_1} - A_2 \frac{1 - sT_2}{1 + sT_2}\right)^{-1} E$$

$$= (1 + sT_1)(1 + sT_2)F_0\left((1 + sT_1)(1 + sT_2)(sI - A_0) - (1 - sT_1)(1 + sT_2)A_1\right.$$\n
$$- (1 + sT_1)(1 - sT_2)A_2\right)^{-1} E = (1 + sT_1)(1 + sT_2)F_0\left((1 + sT_1)(1 + sT_2)(sI - \sum_{i=0}^2 A_i)\right.$$\n
$$- 2sT_1(1 + sT_2)A_1 - 2sT_2(1 + sT_1)A_2\right)^{-1} E$$

$$= [1 + j \tan(\frac{\tau w}{2} - l_1\pi)][1 + j \tan(\frac{\tau w}{2} - l_2\pi)]F_0\left[j\omega I - \sum_{i=0}^2 A_i\right]$$

$$\times \prod_{i=1}^2[1 + j \tan(\frac{\tau w}{2} - l_i\pi)] - 2j \tan(\frac{\tau w}{2} - l_1\pi)[1 + j \tan(\frac{\tau w}{2} - l_2\pi)]A_1$$

$$- 2j \tan(\frac{\tau w}{2} - l_2\pi)[1 + j \tan(\frac{\tau w}{2} - l_1\pi)]A_2\right)^{-1} E.$$

By applying similar steps, the expressions $M_1$ and $M_2$ can be derived. \qed

Let the system (3.15) be subjected to multi-structure perturbations as follows

$$\dot{x}(t) = (A_0 + E_0\Delta_0 F_0)x(t) + \sum_{i=1}^k (A_i + E_i\Delta_i F_i)x(t - \tau_i) \quad (3.27)$$
where \( \Delta_i \in \mathcal{R}^{m \times p_i} \) denote perturbation matrices and \( E_i \in \mathcal{R}^{n \times m}, \ F_i \in \mathcal{R}^{p_i \times n}, \ i = 0, 1, 2, \ldots, k \) are known matrices defining the structure of perturbations. Without loss of generality, we assume that \( E_i = E, \ i = 0, 1, 2, \ldots, k \) and define \( \Delta = [\Delta_0 \ \Delta_1 \ldots \ \Delta_k] \).

Following the development of regular systems, we define the real structured stability radius of uncertain time delay system (3.27) by

\[
\begin{align*}
\rho_R &= \inf_{s \in \mathbb{C}_0} \inf_{\Delta \in \mathcal{R}^{m \times p}, \ E, \ F_i \in \mathcal{R}^{n \times m}} \{ \bar{\sigma}(\Delta) : \det(sI - \sum_{i=0}^{k} (A_i + E\Delta_i F_i)e^{-s\tau_i}) = 0 \}, \ \tau_0 = 0, \\
&= \inf_{s \in \mathbb{C}_0} \inf_{\Delta \in \mathcal{R}^{m \times p}} \{ \bar{\sigma}(\Delta) : \det(I - (\Delta_0 \ \Delta_1 \ldots \ \Delta_k)M(s)) = 0 \} \\
&= \inf_{s \in \mathbb{C}_0} \inf_{\Delta \in \mathcal{R}^{m \times p}} \{ \bar{\sigma}(\Delta) : \det(I - \Delta M(s)) = 0 \}, \ \ P = \sum_{i=0}^{k} p_i,
\end{align*}
\]

where

\[
M(s) = \begin{bmatrix} F_0 \\ F_1 e^{-\tau_1 s} \\ \vdots \\ F_k e^{-\tau_k s} \end{bmatrix} (sI - A_0 - \sum_{i=1}^{k} A_i e^{-\tau_i s})^{-1} E,
\]

Then the real stability radius of (3.27) is given by

\[
\rho_R = \left\{ \sup_{s \in \partial \mathbb{C}} \inf_{\gamma \in (0, 1]} \sigma_2(\begin{bmatrix} \operatorname{Re}(M) & -\gamma \operatorname{Im}(M) \\ \gamma^{-1} \operatorname{Im}(M) & \operatorname{Re}(M) \end{bmatrix}) \right\}^{-1}
\]

**Theorem 3.4.8** Let the multi-delay system (3.15) be stable. The real stability radius for the perturbed system (3.27) is given by (3.29), where

\[
M(w, \tau_1, \ldots, \tau_k) = \begin{bmatrix} M_0(w, \tau_1, \ldots, \tau_k) \\ M_1(w, \tau_1, \ldots, \tau_k) \\ \vdots \\ M_k(w, \tau_1, \ldots, \tau_k) \end{bmatrix}
\]

where

\[
M_i = F_i[1 - j \tan(\frac{\tau_i w}{2} - l_i \pi)] \prod_{r=0 \atop r \neq i}^{k} [1 - j \tan(\frac{\tau_r w}{2} - l_i \pi)] \left\{ sI \prod_{i=0}^{k} [1 - j \tan(\frac{\tau_i w}{2} - l_i \pi)] \right\}^{-1} E,
\]

for the perturbed system (3.27), where

\[
M_i = F_i[1 - j \tan(\frac{\tau_i w}{2} - l_i \pi)] \prod_{r=0 \atop r \neq i}^{k} [1 - j \tan(\frac{\tau_r w}{2} - l_i \pi)] \left\{ sI \prod_{i=0}^{k} [1 - j \tan(\frac{\tau_i w}{2} - l_i \pi)] \right\}^{-1} E
\]
for \( l_i = 0, 1, \ldots, \infty, \ 0 \leq \tau_i \leq \bar{\tau}, \ \tau_0 = 0, \ E \in \mathbb{R}^{n \times m}, \ F_i \in \mathbb{R}^{p_i \times n}, \ i = 0, 1, \ldots, k, \) are known scaling matrices, and \( \sigma_2(\cdot) \) denote the second largest singular value is \( (\cdot) \).

**Proof** The proof can be established by simple substitution and algebraic manipulations.

Substitute (3.11) into (3.28), we get

\[
M = \text{diag}(1, \frac{1 - sT_1}{1 + sT_1}, \ldots, \frac{1 - sT_k}{1 + sT_k}) \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_k \end{bmatrix} \{ sI - A_0 \sum_{i=1}^k A_i \frac{1 - sT_i}{1 + sT_i} \}^{-1} E, \quad T_0 = 0,
\]

We show this for the general term \( M_i, \ i = 0, 1, \ldots, k, \)

\[
M_i = F_i \frac{1 - sT_i}{1 + sT_i} \prod_{t=0}^k (1 + sT_i) \left\{ sI \prod_{i=0}^k (1 + sT_i) - \sum_{i=0}^k (1 - sT_i)A_i \prod_{r=0}^k (1 + sT_r) \right\}^{-1} E, \quad T_0 = 0,
\]

using (3.12), we get

\[
M_i = F_i [1 - j \tan(\frac{\tau_i w}{2} - l_i \pi)] \prod_{r=0}^k [1 - j \tan(\frac{\tau_r w}{2} - l_i \pi)] \left\{ sI \prod_{i=0}^k [1 - j \tan(\frac{\tau_i w}{2} - l_i \pi)] - \sum_{i=0}^k [1 - j \tan(\frac{\tau_i w}{2} - l_i \pi)]A_i \prod_{r=0}^k [1 - j \tan(\frac{\tau_r w}{2} - l_i \pi)] \right\}^{-1} E, \quad \tau_0 = 0. \quad \square
\]
3.5 Illustrative Examples

Example 3.5.1 Consider the system

\[
\dot{x}(t) = (A_0 + E_0 \Delta_0 F_0)x(t) + (A_1 + E_1 \Delta_1 F_1)x(t - \tau),
\]

where

\[
A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad E_0 = E_1 = I_2 = F_0 = F_1.
\]

It is known that the system without uncertainties is stable when \( \tau = 0 \) and unstable when \( \tau > \pi \). This can directly be obtained by solving the LMI (3.4) and (3.5). In this case by applying the real stability radius result given in Theorem 3.3.1, we obtain the plot of the real stability radius with respect to delay \( \tau \), \( 0 \leq \tau \leq \bar{\tau} \) in Figure 3.3, and it is observed that the plot obtained is consistent with that in [10] and [36].

![Figure 3.3: Real stability radius with respect to delay for a linear-delay system.](image-url)
Example 3.5.2 In this example we consider the uncertainty associated with a pure delay term. It is desired to find $r_R(B, E_1, F_1)$ for the pure delay system:

$$\dot{x}(t) = (B + E_1 \Delta F_1) x(t - \tau)$$

which is known to be stable in the case when $\tau = 0$ and $\Delta = 0$. Here

$$B = \begin{bmatrix} 79 & 20 & -30 & -20 \\ -41 & -12 & 17 & 13 \\ 167 & 40 & -60 & -38 \\ 33.5 & 9 & -14.5 & -11 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.2190 & 0.9347 \\ 0.0470 & 0.3835 \\ 0.6789 & 0.5194 \\ 0.6793 & 0.8310 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0.0346 & 0.5297 & 0.0077 & 0.0668 \\ 0.0535 & 0.6711 & 0.3834 & 0.4175 \end{bmatrix}.$$

Applying the LMI, we found that the maximum time delay $\bar{\tau} = 0.56$. Applying the real stability radius of (3.13) for each time delay $\tau$, $0 \leq \tau \leq \bar{\tau}$ leads to Figure 3.4, which shows the correspondence between allowable stability radius and the associated time delay.

![Figure 3.4: Real stability radius with respect to delay for a pure delay system](image-url)
Example 3.5.3 Consider the system

\[ \dot{x}(t) = (A_0 + E\Delta_0 F_0)x(t) + (A_1 + E\Delta_1 F_1)x(t - \tau_1) + (A_2 + E\Delta_2 F_2)x(t - \tau_2), \]

where \( A_0 = A_2 = \begin{bmatrix} 0 & 0.5 \\ -0.5 & -0.5 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \) and \( E = F_0 = F_1 = F_2 = I_2 \).

It is known that the system without uncertainties is stable when \( \tau_1 = \tau_2 = 0 \) and unstable when \( \tau_1 > \pi \) and \( \tau_2 > 5.71 \) along \( \tau_2 \) and \( \tau_1 \) axis, respectively. It is evident, by applying Theorem 3.4.6, that the box \( B_1 = \{ (\tau_1, \tau_2) : 0 \leq \tau_1 \leq 0.75, \ 0 \leq \tau_2 \leq 3 \} \) (dotted lines in Figure 3.6) is entirely in the fundamental stability region (shaded region in Figure 3.6). It is sufficient to check the two LMI’s in Theorem 3.4.7, which are satisfied for \( \bar{\tau}_1 = 0.75 \), and \( \bar{\tau}_2 = 3 \), therefore the above system is robustly stable in the box \( B_1 \). It is clear that the largest possible box within the fundamental stability region is \( B_2 = \{ (\tau_1, \tau_2) : 0 \leq \tau_1 \leq 1.15, \ 0 \leq \tau_2 \leq 5.71 \} \), see dashed lines in Figure 3.6. By applying the real stability radius result of (3.26), we obtain the surface of the real stability radius in Figure 3.5 with respect to the fundamental stability region (shaded in Figure 3.6). It is interesting to see that the superposition of this surface with the fundamental stability region in Figure 3.7 perfectly complement each other. Obviously, stability radius exists only for those \((\tau_1, \tau_2)\) points which fall into the fundamental stability region. Stability radius computations for \((\tau_1, \tau_2)\) points that do not belong to fundamental stability regions (although they turn out to be positive quantities) however are meaningless and should be ignored. The results of CTCR revealing the fundamental stability regions is obviously appreciated as they allow one to successfully determine at which parts on the delay domain stability radius exists. Once this link is established, computable formula of stability radius merged with Rekasius substitution is revisited for efficient computations.
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Figure 3.5: Surface of real stability radius for two time-delay systems.

Figure 3.6: Fundamental stability region for two time-delay systems.

Figure 3.7: Fundamental kernel and offspring curves (cyan curves) superposed with gray scale color-coded stability radius.
Chapter 4

Stabilization and Robust Stabilization

4.1 Stabilization

Different kind of controllers will be addressed in this chapter. Among the controllers we will use to stabilize our class of time-delay linear systems we consider the state feedback controller (memoryless and memory) in both cases: delay-independent and delay-dependent systems without/with parameter uncertainty. Also, we consider the output feedback controller for linear systems with a constant time-delay in the state. We first consider the stabilization problem for linear time-delay systems without uncertainty. We obtain conditions to insure the stabilizability of delay systems via state feedback controller then via dynamic output feedback controller and provide techniques for both cases. The proposed methods are based on the solution of linear matrix inequalities (LMIs). These approaches are then extended to the problems of robust stabilization of linear state delayed systems with norm-bounded parameter uncertainty in all the matrices of the system state equation.

To study the stabilizability of linear time systems with time-delay using linear matrix
inequalities LMIs, we consider the following continuous linear time-delayed system:

\[ \dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + Bu(t), \quad x(t) = \varphi(t), \quad -\tau \leq t \leq 0 \quad (4.1) \]

where \( x(t) \in \mathbb{R}^n \) is the state; \( \varphi(t) \) is the continuous initial condition. In (4.1), the scalar \( \tau \) is the constant delay for the system, \( A_0 \in \mathbb{R}^{n \times n} \) and \( A_1 \in \mathbb{R}^{n \times n} \) are known real constant matrices.

Our goal is to establish the conditions under which the unforced system (4.1), with \( u(t) = 0 \) will be stabilizable. Then using these conditions are establish results by which one can design state feedback and output feedback controllers to stabilize time-delay systems. The conditions are defined for both delay-independent and delay-dependent cases.

A parallel treatment is considered for the robust stabilization of (4.1) when the matrices \( A_0 \) and \( A_1 \) encounter uncertainties. The reported results in this chapter are well-known in the literature (see [6], [53], [54] and [60] for more details). However, a careful summary is provided to understand the available tools for stabilization, which will be used in connection to the new results in chapter 5 for observer-based controller design and in chapter 6 for constrained stabilization of delay system.

### 4.1.1 State Feedback Stabilization

**Delay-independent state feedback stabilization:** We assume that the state of the system is perfectly available for state feedback and restrict our study to the case of linear state feedback control, that is

\[ u(t) = K_0 x(t) + K_1 x(t - \tau) \quad (4.2) \]

where \( K_0 \) and \( K_1 \) are constant matrices of appropriate dimension.

First, let us consider the stabilizability problem using memoryless controller \( K_1 = 0 \). The stabilizability of system (4.1) using a linear memoryless controller con-
consists of finding a gain matrix $K_0$ such that the closed-loop system

$$\dot{x}(t) = (A_0 + BK_0)x(t) + A_1x(t - \tau), \quad x(t) = \varphi(t), \quad -\tau \leq t \leq 0 \quad (4.3)$$

is asymptotically stable.

The following theorem provides an LMI-based design technique for finding a stabilizing gain matrix $K_0$ in light of stability result of Theorem 2.2.1 of chapter 2.

**Theorem 4.1.1** If there exist a symmetric, positive definite matrices $X$, $U > 0$ and the matrix $Y$ such that the following LMI holds:

$$
\begin{bmatrix}
A_0X + XA_0^T + BY + Y^T B^T + A_1^T UA_1 & A_1X \\
\bullet & -U
\end{bmatrix} < 0,
$$

then the system (4.1) is asymptotically stable independent of delay under the state feedback control law

$$u(t) = K_0x(t) \quad \text{with} \quad K_0 = YX^{-1}.$$  

Theorem 4.1.1 provides a memoryless state feedback controller design technique.

When the time delay $\tau$ is known constant, a controller (4.2) that contains a delayed state can be constructed in the same way. Using the controller (4.2) the closed-loop system becomes

$$\dot{x}(t) = (A_0 + BK_0)x(t) + (A_1 + BK_1)x(t - \tau). \quad (4.5)$$

The following theorem develops a delay-independent stabilizability condition and provides a method to design a stabilizing controller.

**Theorem 4.1.2** If there exist a symmetric, positive definite matrices $X$, $U > 0$ and the matrices $Y$, $Y_1$ such that the following LMI holds:

$$
\begin{bmatrix}
A_0X + XA_0^T + BY + Y^T B^T + U & A_1X + BY_1 \\
\bullet & -U
\end{bmatrix} < 0.
$$

Then the system (4.1) is asymptotically stable independent of delay under control (4.2) with the gains $K_0 = YX^{-1}$ and $K_1 = Y_1X^{-1}$.  

The above theorem provides a sufficient delay-independent condition for system (2.1) to be asymptotically stable under control law (4.2) and provides a control design algorithm. Obviously LMI (4.5) contains variables $X, Y, Y_1$, which are to be determined. If the variables $Y, Y_1$ can be eliminated, the computation burden can be reduced and one can obtain stabilizability condition with control law (4.2). The following theorem establishes this result.

**Theorem 4.1.3** System (4.1) is asymptotically stabilizable with controller (4.2) if one of the following conditions is satisfied:

1. There exist symmetric, positive-definite matrices $X > 0, U > 0$, such that
   \[
   \begin{bmatrix}
   A_0X + XA_0^T + U + \sigma BB^T & A_1X \\
   \bullet & -U
   \end{bmatrix} < 0,
   \]
   holds for some scalar $\sigma > 0$.

2. There exist symmetric, positive-definite matrices $X > 0, U > 0$, such that
   \[
   \hat{B}^T \begin{bmatrix}
   A_0X + XA_0^T + U & A_1X \\
   \bullet & -U
   \end{bmatrix} \hat{B} < 0,
   \]
   holds, where $\hat{B} = \text{diag}\{B_\perp, I\}$ and $B_\perp$ is the orthogonal complement of $B$, i.e. $B^T B_\perp = 0$.

**Remark 4.1.4** All the results we developed so far are delay-independent and therefore conservative. We overcome this conservatism by providing delay-dependent sufficient conditions.

**Delay-dependent state feedback stabilization**: First, in light of Delay-Dependent Theorems 2.2.3, 2.2.4 and in particular Theorem 2.2.7 for $k = 1$, we state the following equivalent stability condition which is convenient for the development of state feedback stabilization.
Theorem 4.1.5 If there exists symmetric, positive definite matrices $P > 0$, $Q_0 > 0$ and $Q_1 > 0$ such that the following LMI holds:

$$
\begin{bmatrix}
(A_0 + A_1)^T P + P(A_0 + A_1) + \tau(Q_0 + Q_1) & \tau PA_0 & \tau PA_1 \\
\bullet & -\tau Q_0 & 0 \\
\bullet & \bullet & -\tau Q_1
\end{bmatrix} < 0.
$$ (4.9)

Then the system (2.1) is asymptotically stable dependent of delay.

When the delay is known, the condition (4.9) can be used to check the stability of the system (4.1) when $u(t) = 0$. In case the size of delay is not known, one can take advantage of this result and derive an upper bound on the delay for which the system will remain stable. To do this, one can equivalently write (4.9) as

$$
\begin{bmatrix}
Q_0 + Q_1 & PA_0 & PA_1 \\
\bullet & -Q_0 & 0 \\
\bullet & \bullet & -Q_1
\end{bmatrix} < \begin{bmatrix}
J & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$ (4.10)

where $J = -(A_0 + A_1)^T P - P(A_0 + A_1)$ and $\eta = \frac{1}{\tau}$. Thus, one can cast the problem into the framework of the generalized eigenvalue problem (GEVP) and formulate the optimization problem by maximizing $\tau$ or minimizing $\eta$ subject to the above LMI constraint. i.e,

$$
\min_{\eta > 0, P > 0, Q_0 > 0, Q_1 > 0} \eta
$$ (4.11)

s.t. (4.10)

Apparently (4.11) can not be solved directly using GEVP and one needs to introduce an auxiliary matrix $\Gamma > 0$ and rewrite (4.11) as

$$
\begin{bmatrix}
Q_0 + Q_1 & PA_0 & PA_1 \\
\bullet & -Q_0 & 0 \\
\bullet & \bullet & -Q_1
\end{bmatrix} < \begin{bmatrix}
\Gamma & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$ (4.12)

$$
\Gamma < \eta[-(A_0 + A_1)^T P - P(A_0 + A_1)]
$$ (4.13)
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Since $Q_0 > 0$, $Q_1 > 0$, to make $Q_0$, $Q_1$, and $P$ satisfy (4.9), we need $(A_0 + A_1)^T P + P(A_0 + A_1) < 0$. Therefore, we can require the right hand side of (4.13) to be positive definite. In this way the optimization problem becomes a standard GEVP, which can be solved numerically using LMI toolbox.

**Theorem 4.1.6** Let $\eta^*$ be the optimal solution of the following GEVP

$$\min_{\eta>0,P>0,Q_0>0,Q_1>0 \text{ and } \tau>0} \eta,$$

s.t. (4.12) and (4.13)

Then for any $\tau \in [0, \tau^*]$ system (4.1) with $u(t) = 0$ is asymptotically stable, where $\tau^* = \frac{1}{\eta^*}$ is the upper bound of time delay.

A simple example using the following parameters

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.91 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1.1 \end{bmatrix}$$

was used and applied to LMI toolbox of Matlab. The GEVP was solved and $\tau^* = 0.8979$ was obtained.

Let us now continue with the design of a delay-dependent stabilizing controller for the delay system (4.1) using a memoryless state feedback controller (4.2) with $K_1 = 0$, i.e. $u(t) = K_0 x(t)$.

Using Theorem 4.1.5 and applying the memoryless state feedback controller we get the following result.

**Theorem 4.1.7** If there exists symmetric, positive definite matrices $X > 0$, $U_0 > 0$, $U_1 > 0$ and a matrix $Y$ with appropriate dimensions such that

$$\begin{bmatrix} U & \tau A_1(A_0 X + BY) & \tau A_1 A_1 X \\ \cdot & -\tau U_0 & 0 \\ \cdot & \cdot & -\tau U_1 \end{bmatrix} < 0,$$

where $U = (A_0 + A_1) X + X(A_0 + A_1)^T + BY + Y^T B^T + \tau (U_0 + U_1)$. Then the
system (4.1) is asymptotically stable dependent of delay under the feedback control law
\( u = K_0 X \) and \( K_0 = Y X^{-1} \) is a stabilizing gain.

It is important to point out that similar procedure leading to Theorem 4.1.7 can be employed for computing a control law that stabilizes the system when the delay is maximized.

**Theorem 4.1.8** Let \( X > 0 \), \( Y \), \( \Gamma > 0 \) and \( \eta^* = \frac{1}{\tau^*} \) be the solution to the following GEVP

\[
\begin{align*}
\min_{\eta > 0, X > 0, Y, \Gamma > 0} \eta, \\
\text{s.t.} \\
\begin{bmatrix}
U_0 + U_1 & A_1(A_0X + BY) & A_1A_1X \\
\cdot & -U_0 & 0 \\
\cdot & \cdot & -U_1
\end{bmatrix} < 
\begin{bmatrix}
\Gamma & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

\[
\Gamma < \eta[-(A_0 + A_1)X - X(A_0 + A_1)^T - BY - Y Y^T B^T]
\]

Then, the control law \( u(t) = K_0 x(t) \) with \( K_0 = Y X^{-1} \) stabilizes the system (4.1) for any \( \tau \in [0, \tau^*] \), where \( \tau^* = \frac{1}{\eta^*} \) is the upper bound of the time delay.

Note that (4.15) and (4.16) are equivalent expressions as in (4.12) and (4.13) incorporating the control law.

Unfortunately, unlike the delay-independent case, the delay-dependent stability of Theorem 4.1.5 cannot provide the delay dependent stabilizability by replacing \( A_1 \) by \( A_1 + BK_1 \) and obtain the control law (4.2). However, one of the alternative delay-dependent stability result can be used to establish the following theorem.

**Theorem 4.1.9** If there exists symmetric, positive definite matrix \( X > 0 \) and matrices \( Y \) and \( Y_1 \) satisfying

\[
\begin{bmatrix}
M_{11} + M_{11}^T & M_{12} & M_{13} \\
\cdot & -J_2 & 0 \\
\cdot & \cdot & -J_3
\end{bmatrix} < 0,
\]

incorporating the control law.
where

\[
M_{11} = (A_0 + A_1)X + BY + BY_1, \quad J_2 = \frac{1}{\tau}X
\]
\[
M_{12} = (X A_0^T + Y^T B^T, X A_1^T + Y^T B^T), \quad J_3 = \frac{1}{2\tau}X
\]
\[
M_{13} = A_1 X + BY_1
\]

Then the system (4.1) is asymptotically stable dependent of delay under control law (4.2) with \( K_0 = YX^{-1}, \; K_1 = Y_1X^{-1}. \)

Suppose the delay system (4.1) has the following parameters

\[
A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.5 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix}
\]

Using Theorem 4.1.9 we get the following gains for different time delays (\( \tau = 1, \tau = 2 \)).

\[
\tau_1 : K_0 = [0.2055 \; -0.4683], \quad K_1 = [-2.0377 \; -0.8619]
\]
\[
\tau_2 : K_0 = [1.9825 \; -2.3857], \quad K_1 = [-1.7443 \; -0.7324]
\]

Furthermore, by employing Theorem 4.1.8, we can solve the GEVP and obtain \( \tau^* = 24.1899 \) with the corresponding gains

\[
K_0 = [3.4653 \; 5.3012], \quad K_1 = [1.5362 \; -0.3998]
\]

4.1.2 Alternative Delay-Dependent Stabilization Method

Recall that chapter 2 provides various alternative delay-dependent stability condition. Depending on each stability condition one can derive its corresponding LMI based stabilization process. Here, in light of Theorem 2.3.4 we provide an alternative delay-dependent stabilization for (4.1). For convenience we repeat the result of Theorem 2.3.4 below.
Theorem 4.1.10 If there exist matrices $P > 0$, $Q > 0$, $X$, $Y$ and $Z$ such that

\[
\begin{bmatrix}
PA_0 + A_0^r P + \bar{\tau} X + Y + Y^T + Q & PA_1 - Y & \bar{\tau} A_0^T Z \\
\cdot & -Q & \bar{\tau} A_1^T Z \\
\cdot & \cdot & -\bar{\tau} Z
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
X & Y \\
\cdot & Z
\end{bmatrix} \geq 0.
\]

Then unforced time-delay the system (4.1) is asymptotically stable for any-delay \(\tau\) satisfying \(0 < \tau \leq \bar{\tau}\).

Extending on Theorem 4.1.10 to design a stabilizing memoryless controller \(u = K_0 x(t)\) for the system (4.1), the following result has been obtained:

Theorem 4.1.11 If there exist matrices $L > 0$, $M$, $N$, $R$, $V$ and $W$ such that

\[
\begin{bmatrix}
\bar{\Pi} & A_1 L - N & \bar{\tau}(LA_0^T + V^T B) \\
\cdot & -W & \bar{\tau}LA_1^T \\
\cdot & \cdot & -\bar{\tau} R
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
M & N \\
\cdot & LR^{-1} L
\end{bmatrix} \geq 0
\]

where

\[
\bar{\Pi} = A_0 L + L A_0^T + BV + V^T B^T + \bar{\tau} M + N + N^T + W.
\]

Then the system (4.1) with control

\[
u(t) = VL^{-1} x(t)
\]

is asymptotically stable for any constant delay \(\tau\) satisfying

\[0 < \tau \leq \bar{\tau}.
\]

For a special case where the information on the size of delay \(\tau\) is available we also consider a delayed feedback controller of the form

\[
u(t) = K_0 x(t) + K_1 x(t - \tau)
\]
In this case, the delayed feedback controller (4.22) can be used instead of the memoryless controller. By setting \( A_1 = A_1 + BK_1 \) in (4.23) and applying the change of variable \( V_1 = K_1L \), we can obtain a stabilization condition for a delay feedback controller (4.22) as follows:

**Theorem 4.1.12** Suppose that \( \tau \) is known. If there exist matrices \( L > 0, M, N, R, V, V_1 \) and \( W > 0 \) such that

\[
\begin{bmatrix}
\tilde{\Pi} & A_1L - N + BV_1 & \tau(LA_0^T + V_1^TB^T) \\
\cdot & -W & \tau(LA_1^T + V_1B^T) \\
\cdot & \cdot & -\tau R
\end{bmatrix} < 0
\]

(4.23)

\[
\begin{bmatrix}
M \\
N \\
\cdot \\
LR^{-1}L
\end{bmatrix} \geq 0
\]

(4.24)

where

\( \tilde{\Pi} = A_0L + LA_0^T + BV + V^T B^T + \tau M + N + N^T + W \).

Then the system (4.1) with control

\[ u(t) = VL^{-1}x(t) + V_1L^{-1}x(t - \tau) \]

is asymptotically stable.

### 4.1.3 Output Feedback Stabilization

**Delay-independent output feedback stabilization:**

Let us consider the following dynamic output feedback control law

\[
\begin{align*}
\dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\
u(t) &= C_c x_c(t) + D_c y(t)
\end{align*}
\]

(4.25)

for the system 4.1 where \( x_c(t) \in \mathcal{R}^n \), and \( A_c, B_c, C_c \) and \( D_c \) are matrices of appropriate dimensions. Combining (4.25) and (4.1), we get the augmented closed loop system

\[
\dot{z} = \tilde{A}z(t) + \tilde{A}_1I_0z(t - \tau), \quad z = \begin{bmatrix} x \\ x_c \end{bmatrix}
\]

(4.26)
where
\[ I_0 = \begin{bmatrix} I & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A_0 + BD_c C & BC_c \\ B_c C & A_c \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \] (4.27)

**Theorem 4.1.13** For given matrices \( A_c, B_c, C_c, D_c \) if there exists symmetric and positive definite matrix \( Q_1 \) such that
\[
\begin{bmatrix}
\bar{A}P + P\bar{A} + I_0^T Q_1 I_0 & P I_0^T A_1 \\
\bullet & -Q_1
\end{bmatrix} < 0 \quad (4.28)
\]
has a feasible solution \( P > 0 \), the closed-loop system is asymptotically stable independent of delay.

Obviously, (4.28) is nonlinear with respect to \( P \) and \( A_c, B_c, C_c, D_c \) when seeking the control design. To cast the control design problem into the LMI framework, we decompose \( \bar{A} \) and define the notation
\[
\bar{A} = \bar{A}_0 + B_0 K C_0 \quad (4.29)
\]
where
\[
\bar{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad K = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}.
\]
In view of LMI Theory, \( P \) and \( Q_1 \) satisfy (4.28) if and only if
\[
P \bar{A} + \bar{A}^T P + I_0^T Q_1 I_0 + P I_0^T A_1 Q_1^{-1} A_1^T I_0 P < 0 \quad (4.30)
\]
or equivalently
\[
W + P B_0 K C_0 + C_0^T K^T B_0^T P < 0 \quad (4.31)
\]
where \( W = P \bar{A}_0 + \bar{A}_0^T P + I_0^T Q_1 I_0 + P I_0^T A_1 Q_1^{-1} A_1^T I_0 P. \)
Then it follows that there exists a matrix \( K \) satisfying (4.31) if and only if
\[
B_{0\perp}^T P^{-1} W P^{-1} B_{0\perp} < 0 \quad (4.32)
\]
where \( (PB_0)_{\perp} = P^{-1} B_{0\perp} \) is intuitively used to obtain (4.32).
CHAPTER 4. STABILIZATION AND ROBUST STABILIZATION

Let us now suppose that the matrices $P$ and $P^{-1}$ are partitioned as follows

$$
P = \begin{bmatrix} Y & N \\ \bullet & W_1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} X & M \\ \bullet & W_2 \end{bmatrix} \quad (4.33)$$

where all sub-matrices are of size $n \times n$. Then from $PP^{-1} = I$, one can easily obtain

$$
YX + NM^T = I
$$

$$
YM + NW_2 = 0
$$

$$
N^TX + W_1M^T = 0
$$

and we get

$$
P^{-1}WP^{-1} = \bar{A}P^{-1} + P^{-1}A^T_0 + \begin{bmatrix} \bar{A}_1Q_1A_1^T & \begin{bmatrix} X \\ M^T \end{bmatrix} Q_1 \begin{bmatrix} X & M \end{bmatrix} \end{bmatrix} \quad (4.35)
$$

and

$$
W = \begin{bmatrix} Y \bar{A}_0 + A_0^TY & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Y \\ N^T \end{bmatrix} A_1Q_1^{-1}A_1^T \begin{bmatrix} Y & N \end{bmatrix} + \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} \quad (4.36)
$$

By noting that $B_{0\perp} = \begin{bmatrix} B_\perp \\ 0 \end{bmatrix}$ and $C_{0\perp} = \begin{bmatrix} C_\perp & 0 \end{bmatrix}$, we can obtain (4.35) and (4.36) to obtain an equivalent expression for (4.32) as follows

$$
B_{\perp}^T[AX + XA^T + \bar{A}_1Q_1^{-1}A_1^T + XQ_1X]B_\perp < 0 \quad (4.37)
$$

$$
C_{\perp}^T[YA + A^TY + YA_1Q_1^{-1}A_1^TY + Q_1]C_\perp < 0.
$$

Using the properties from LMI Theory (4.38) is equivalent to

$$
B^T \begin{bmatrix} A & X \\ \bullet & -Q_1^{-1} \end{bmatrix} B < 0 \quad (4.38)
$$

$$
\bar{C} \begin{bmatrix} Y & YA_1 \\ \bullet & -Q_1 \end{bmatrix} \bar{C}^T < 0. \quad (4.39)
$$

where

$$
R = AX + XA^T + \bar{A}_1Q_1^{-1}A_1^T
$$

$$
T = YA + AY + Q_1
$$

$$
\bar{B} = \text{diag}(B_{\perp}, I)
$$

$$
\bar{C} = \text{diag}(C_{\perp}, I).
$$

The above development leads to the following Theorem.
**Theorem 4.1.14** If there exists symmetric and positive definite matrices $P > 0$, $Q > 0$ and the matrices $A_c$, $B_c$, $C_c$, $D_c$ satisfying (4.28), which means that the closed-loop system (4.26) is asymptotically stable, then the LMIs (4.38) and (4.39) and

$$
\begin{bmatrix}
X & I \\
I & Y
\end{bmatrix} \geq 0
$$

(4.40)

has a set of feasible solution $X > 0$, $Y > 0$. Furthermore, if the LMIs (4.28) with (4.29), and (4.40) have a set of feasible solutions $X > 0$, $Y > 0$ for given $Q_1 > 0$, then there exist an output feedback controller of the form (4.25).

The condition (4.40) of the above theorem can be recast as follows. Let us define the following two matrices

$$
\Phi_1 = \begin{bmatrix}
X & I \\
M^T & 0
\end{bmatrix},
\Phi_2 = \begin{bmatrix}
I & Y \\
0 & N^T
\end{bmatrix}
$$

(4.41)

Since $M$ is of full rank, $\Phi_1$ is invertible. Moreover, direct manipulation gives

$$
\Phi_1^T P \Phi_1 = \begin{bmatrix}
X & I \\
I & Y
\end{bmatrix} > 0.
$$

(4.42)

With the aid of (4.41) and (4.42), and Theorem 4.1.14, one can obtain a stabilizing output feedback controller by the following algorithm.

**Algorithm 1**

step 1. Solve the feasible problem (4.38) and (4.39) to get the matrices $X$, $Y$.

step 2. Compute two full rank matrices $M$ and $N \in \mathcal{R}^{n \times n}$ such that

$$
MN^T = I - XY
$$

step 3. Solve the matrix equation

$$
\begin{bmatrix}
I & Y \\
0 & N^T
\end{bmatrix} = P \begin{bmatrix}
X & I \\
M^T & 0
\end{bmatrix}
$$

to get $P > 0$
step 4. With $P$ defined in step 3, solve the feasible problem (4.28) and a set of control parameters $K$ be obtained.

A delay system with the following parameters

$$
A_0 = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \ 0]
$$

was used to apply the algorithm 1. With $Q = I$, the LMIs were solved and led to the solution

$$
X = \begin{bmatrix} 2.0367 & 0.18712 \\ 0.18712 & 5.333 \end{bmatrix}, \quad Y = \begin{bmatrix} 11.8185 & -4.6526 \\ -4.6526 & 2.7491 \end{bmatrix}
$$

$$
M = \begin{bmatrix} -23.8867 & 1.6989 \\ 25.9218 & 1.5651 \end{bmatrix}, \quad N = \begin{bmatrix} 0.8984 & -0.4392 \\ -0.4392 & -0.8984 \end{bmatrix}
$$

$$
A_c = \begin{bmatrix} -41.5886 & -2.0227 \\ -2.0227 & -4.4046 \end{bmatrix}, \quad B_c = \begin{bmatrix} -316.2049 \\ -13.8679 \end{bmatrix}
$$

$$
C_c = [ -7.2781 \ -1.6916 ], \quad D_c = -59.6668.
$$

**Delay-dependent output feedback stabilization:**

In this subsection, we provide a delay-dependent method for designing linear dynamic output feedback controller for state delayed systems such that the resulting closed-loop system is globally uniformly asymptotically stable for any constant time-delay $\tau$ no longer than a given positive scaler $\bar{\tau}$.

Now, consider the following linear time-delay system which is rewritten here for convenience

$$
\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + B u(t), \quad (4.43)
$$

$$
x(t) = \varphi(t), \quad -\tau \leq t \leq 0 \quad (4.44)
$$

$$
y(t) = C_0 x(t) \quad (4.45)
$$
where \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R}^m \) is the control input, \( y(t) \in \mathbb{R}^p \) is the output, \( \tau > 0 \) is the time-delay of the system, \( \varphi(\cdot) \) is the continuous initial condition, and \( A_0, A_1, B, C \) are real constant matrices of appropriate dimensions. Furthermore, we assume that \( (A_0 + A_1, B) \) is stabilizable and \( (A_0 + A_1, C) \) is detectable.

We restrict our study to the case of linear dynamic output feedback controller for the system (4.43) - (4.45), that is

\[
\begin{align*}
\dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\
u(t) &= C_c x_c(t) + D_c y(t)
\end{align*}
\]

where \( x_c \in \mathbb{R}^n \) and \( A_c, B_c, C_c, D_c \) are matrices with appropriate dimensions.

The closed-loop control system can be described by the augmented state equation

\[
\dot{z} = \tilde{A}_0 z + \tilde{A}_1 z(t - \tau), \quad z = \begin{bmatrix} x \\ x_c \end{bmatrix}
\]

where

\[
\begin{align*}
\tilde{A}_0 &= \bar{A}_0 + \bar{B} K \bar{C}, \quad K = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}, \quad \bar{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} \\
\bar{A}_1 &= \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}
\end{align*}
\]

The following two lemmas are essential in the derivation of the main result.

**Lemma 4.1.15** Given a scaler \( \bar{\tau} > 0 \), the unforced system is globally uniformly asymptotically stable for any constant time-delay \( \tau \) satisfying \( 0 \leq \tau \leq \bar{\tau} \) if there exist a matrix \( X > 0 \) and a scaler \( \beta > 0 \) solving the following LMI

\[
\begin{bmatrix} M & X A_0^T & X A_1^T & A_1 \\ A_0 X & -\frac{1}{\tau} \beta I & 0 & 0 \\ A_1 X & 0 & -\frac{1}{\tau} (1 - \beta) I & 0 \\ A_1^T & 0 & 0 & -\frac{1}{\tau} I \end{bmatrix} < 0
\]

where

\[
M = X (A_0 + A_1)^T + (A_0 + A_1) X
\]
Lemma 4.1.16  Given matrices \( G = G^T \in \mathbb{R}^{m \times m}, \ Y \in \mathbb{R}^{r \times m} \) and \( Z \in \mathbb{R}^{s \times m} \), then there exists a matrix \( K \in \mathbb{R}^{r \times s} \) satisfying

\[
G + Y^T K Z + Z^T K^T Y < 0 \tag{4.49}
\]

if and only if

\[
N_Y^T G N_Y < 0, \quad N_Z^T G N_Z < 0
\]

where \( N_Y \) and \( N_Z \) are any matrices whose columns form bases of the null spaces of \( Y \) and \( Z \), respectively.

The results of designing linear dynamic output feedback stabilizing controllers can be summarized as follows:

Theorem 4.1.17  Consider the system \((4.43)-(4.45)\) satisfying the assumption stated above. Given a scaler \( \bar{\tau} > 0 \), this system is stabilizable via an output feedback controller \((4.46)\) for any constant time-delay \( \tau \) satisfying \( 0 \leq \tau \leq \bar{\tau} \), if there exist \( n \times n \) matrices \( R, S \) and \( P \), and a scaler \( \beta > 0 \) satisfying the following LMIs:

\[
\begin{bmatrix}
N_{B_1} & 0 \\
N_{B_2} & I
\end{bmatrix}^T H_S(S, P, \beta)
\begin{bmatrix}
N_{B_1} & 0 \\
N_{B_2} & I
\end{bmatrix} < 0 \tag{4.50}
\]

\[
\begin{bmatrix}
N_C & 0 \\
0 & I
\end{bmatrix}^T H_R(R, \beta)
\begin{bmatrix}
N_C & 0 \\
0 & I
\end{bmatrix} < 0 \tag{4.51}
\]

\[
\begin{bmatrix}
R & I \\
I & S
\end{bmatrix} > 0 \tag{4.52}
\]

where \( \begin{bmatrix} N_{B_1} \\ N_{B_2} \end{bmatrix} \) and \( N_C \) are any matrices whose columns form bases of the null space.
of $\begin{bmatrix} B^T & B^T \end{bmatrix}$ and $C$, respectively, and

\[
H_S(S, P, \beta) = \begin{bmatrix}
Q_S & S A_0^T & A_0 + A_1 & S A_1^T & A_1 \\
A_0 S & -\frac{1}{\tau} \beta I & A_0 & 0 & 0 \\
A_0^T + A_1^T & A_0^T & -\frac{1}{\tau} P & A_1^T & 0 \\
A_1 S & 0 & A_1 & -\frac{1}{\tau}(1 - \beta) I & 0 \\
A_1^T & 0 & 0 & 0 & -\frac{1}{\tau} I
\end{bmatrix}
\] (4.53)

\[
H_R(R, \beta) = \begin{bmatrix}
Q_R & A_0^T & A_1^T & R A_1 \\
A_0^T & -\frac{1}{\tau} \beta I & 0 & 0 \\
A_1^T & 0 & -\frac{1}{\tau} \beta I & 0 \\
A_1^T R & 0 & 0 & -\frac{1}{\tau} I
\end{bmatrix}
\] (4.54)

where

\[
Q_S = S(A_0 + A_1)^T + (A_0 + A_1)S,
\] (4.55)

\[
Q_R = (A_0 + A_1)^T R + R(A_0 + A_1).
\] (4.56)

In the case when the conditions of Theorem 4.1.10 are fulfilled, an output feedback controller that solves the stabilization problem can be easily obtained by the following steps:

1. Compute an $n \times n$ nonsingular matrix $N$ such that

\[
N N^T = \beta P^{-1}
\] (4.57)

2. Find the matrix $X > 0$ from

\[
X = \begin{bmatrix}
S & N \\
\cdot N^T (S - R^{-1})^{-1} N
\end{bmatrix}
\] (4.58)

3. Compute the controller matrices $A_c$, $B_c$, $C_c$, $D_c$ by solving the LMI for

\[
K = \begin{bmatrix} A_c & B_c \\
C_c & D_c \end{bmatrix}
\] from

\[
G_X + Y^T K Z_X + Z_X^T K^T Y < 0
\] (4.59)
where
\[
G_x = \begin{bmatrix}
Q_x & X\bar{A}_0^T & X\bar{A}_1^T & \bar{A}_1 \\
\bullet & -\frac{1}{\tau}\beta I & 0 & 0 \\
\bullet & \bullet & -\frac{1}{\tau}(1 - \beta)I & 0 \\
\bullet & \bullet & \bullet & -\frac{1}{\tau}I
\end{bmatrix}
\]

\[
Q_x = X(\bar{A}_0 + \bar{A}_1)^T + (\bar{A}_0 + \bar{A}_1)X \\
Y = [B^T \quad B^T \quad 0 \quad 0] \\
Z_x = [\bar{C}X \quad 0 \quad 0 \quad 0]
\]

with \(K, \bar{A}_0, \bar{A}_1, B, C\) as defined in (4.48).

**Remark 4.1.18** The problem of finding the largest \(\bar{\tau}\) which insures output feedback stabilization using the method of Theorem (4.1.17) can be easily solved without the need of carrying out iterations for increasing \(\bar{\tau}\). Indeed, the largest \(\bar{\tau}\) can be computed by the following quasi-convex optimization problem in

\[
\text{minimize} \quad \bar{\eta} \\
\text{subject to} \quad R > 0, \quad S > 0, \quad P > 0, \quad \beta > 0, \quad \bar{\eta} > 0, \quad \text{and (4.50)} - (4.52).
\]

The largest value of \(\tau\), namely \(\bar{\tau}^*\), is given by \(\bar{\tau}^* = \frac{1}{\bar{\eta}^*}\), where \(\bar{\eta}^*\) is the optimal value of \(\bar{\eta}\). Note that the above optimization problem has the form of a generalized eigenvalue problem, which is known to be solvable numerically very efficiently.

### 4.2 Robust Stabilization of Time-Delay Systems

In this subsection, we provide delay-independent and delay-dependent robust stabilization conditions for uncertain time-delay systems in term of LMI.
4.2.1 Robust State Feedback stabilization

Consider the following linear model of uncertain time-delay systems

\[
\dot{x}(t) = (A_0 + \Delta A_0(t))x(t) + (A_1 + \Delta A_1(t))x(t - \tau) + (B + \Delta B(t))u(t)
\]

\[
x(t) = \varphi(t), \quad \forall t \in [-\tau, 0]
\]

\[
y(t) = Cx(t).
\]

where the admissible uncertainties are assumed to be of the form

\[
[\Delta A_0(t) \quad \Delta B(t)] = D_0\Delta_0(t) [E_0 \quad E_b], \quad \Delta A_1(t) = D_1\Delta_1(t)E_1
\]

with

\[
\Delta_i^T(t)\Delta_i(t) \leq I
\]

and the matrices \(D_0, D_1, E_0, E_b\) of appropriate sizes.

**Definition 4.2.1** The uncertain time delay system (4.60) - (4.62) is said to be robustly stable if its solution with \(u(t) = 0\) is asymptotically stable for all admissible uncertainties \(\Delta A_0(t)\) and \(\Delta A_1(t)\). Furthermore, (4.60) - (4.62) is said to robustly stabilizable if there exists a control law \(u(t)\) such that the closed-loop system is robustly stable.

**Definition 4.2.2** The system (4.60) - (4.62) is said to be quadratically stable if there exists symmetric positive definite matrices \(P > 0\), \(Q_1 > 0\) such that

\[
\begin{bmatrix}
(A_0 + \Delta A_0)^T P + P(A_0 + \Delta A_0) + Q_1 & P(A_1 + \Delta A_1) \\
& -Q_1
\end{bmatrix} < 0,
\]

holds for all admissible uncertainties.

It can be shown that if (4.60) is quadratically stable, then it is robustly stable. However, the condition of definition (4.2.2) contains the system uncertainties \(\Delta_0(t)\) and \(\Delta_1(t)\) which are unknown and thus, it is not convenient to verify the quadratic stability using (4.63). The following theorem gives a convenient way to check quadratic stability avoiding the appearance of uncertainty matrices.
Theorem 4.2.3 The system (4.60) - (4.62) with \( u = 0 \) is quadratically stable if there exist a symmetric positive definite matrices \( P > 0 \), \( Q_1 > 0 \) and a positive constant \( \epsilon > 0 \) such that
\[
\begin{bmatrix}
A_0^T P + PA_0 + Q_1 + \epsilon E_0^T E_0 & PA_1 & PD_0 D_1 \\
\epsilon E_1^T E_1 - Q_1 & 0 \\
-\epsilon I & -\epsilon I & -\epsilon I
\end{bmatrix} < 0.
\] (4.64)

Delay-Independent Stabilization With Memoryless controller

Applying the memoryless control law \( u = K_0 X \) to the system (4.60) - (4.62), the closed-loop system becomes
\[
\dot{x}(t) = [A_0 + BK_0 + D_0 \Delta_0 (E_0 + E_b K_0)] x(t) + [A_1 + D_1 \Delta_1 E_1] x(t - \tau)
\] (4.65)

Then the following theorem gives an LMI-based method for the design of a memoryless controller that quadratically stabilize the uncertain delay system.

Theorem 4.2.4 If there exists positive constant \( \epsilon > 0 \) and matrices \( X > 0 \), \( U > 0 \), and \( Y \) such that the following LMI is feasible
\[
\begin{bmatrix}
M & A_1 X &XE_0^T + Y^T E_b^T & 0 \\
\cdot & -U & 0 & X E_1 \\
\cdot & \cdot & -\epsilon I & 0 \\
\cdot & \cdot & \cdot & -\epsilon I
\end{bmatrix} < 0.
\] (4.66)

where \( M = X A_0^T + A_0 X + B Y + Y^T B^T + U + \epsilon (D_0 D_0^T + D_1 D_1^T) \). Then the system (4.60) - (4.62) is robust quadratically stabilizable, and a memoryless stabilizing controller is given by \( u = K_0 x(t) \) with \( K_0 = Y X^{-1} \).

Delay-Independent Stabilization With Memory controller

In this case we assume that \( \Delta_1 = \Delta_0 \), \( D_1 = D_0 \) and the control law \( u = K_0 x + K_1 x(t - \tau) \) gives the closed-loop system
\[
\dot{x}(t) = [A_0 + BK_0 + D_0 \Delta_0 (E_0 + E_b K_0)] x(t) + [A_1 + BK_1 + D_0 \Delta_0 (E_1 + E_b K_1)] x(t - \tau)
\] (4.67)
Then the following theorem establish a design method for the controller.

**Theorem 4.2.5** If there exists positive constant $\epsilon > 0$ and symmetric positive definite matrices $X > 0$, $V > 0$, and matrices $Y$ and $Y_1$ such that the following LMI is feasible

$$
\begin{bmatrix}
N & A_1X + BY_1 & XE_0^T + Y_0^T E_b^T \\
- & -V & XE_1^T + Y_1^T E_b^T \end{bmatrix} < 0.
$$

where

$$
N = XA_0^T + A_0X + BY + Y^T B^T + V + \epsilon D_0D_0^T.
$$

Then the system (4.60) - (4.62) is robust quadratically stabilizable, and a memory stabilizing controller is given by

$$
u = K_0x(t) + K_1x(t - \tau) \text{ with } K_0 = YX^{-1} \text{ and } K_1 = Y_1X^{-1}.
$$

The validity of the above theorem can be shown by a delay system with the parameters

$$
A_0 = \begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 1 \\ 0.3 \end{bmatrix}, \quad E_0 = [0 \quad 1]
$$

$$
E_1 = [0.1 \quad 0.2], \quad E_b = 0.3.
$$

For this set of parameters we get the feasible solution to the LMI (4.68) as

$$
X = \begin{bmatrix} 19.8571 & -10.4726 \\ -10.4726 & 35.2969 \end{bmatrix}, \quad V = \begin{bmatrix} 60.7259 & 0.3028 \\ 0.3028 & 73.2009 \end{bmatrix}, \quad Y = \begin{bmatrix} -220.9053 & -10.9895 \\ 16.1332 & 10.7613 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 1.1538 & 0.6472 \end{bmatrix}, \quad \epsilon = 108.3
$$

and the control gains are obtained by

$$
K_0 = YX^{-1} = [ -13.3831 \quad -4.2821 ],
$$

$$
K_1 = Y_1X^{-1} = [ 1.1538 \quad 0.6472 ].
$$

**Delay-Dependent Robust Stabilization By State feedback**

Although, there exist several delay-dependent stabilization results. We use the alternative design approach of section 4.1.2 and extend it for the robust stabilization.
Consider the following uncertain time-delay systems

\[ \dot{x}(t) = (A_0 + D_0F_0E_0)x(t) + (A_1 + D_1F_1E_1)x(t - \tau) + (B + D_0F_0E_0)u(t) \]
\[ x(t) = \varphi(t), \quad \forall t \in [-\tau, 0]. \]  

(4.69)

with control

\[ u(t) = K_0x(t) + K_1x(t - \tau) \]

Theorem 4.2.6 If there exist matrices \( P > 0 \), \( Q > 0 \), \( X \), \( Y \), \( Z \), and scalars \( e_1 \) and \( e_2 \) such that

\[
\begin{bmatrix}
  Y_{11} & PA_1 - Y & \bar{\tau}A_0^TP & PD_0 & PD_1 \\
  \bullet & -Q + e_2E_1^TE_1 & \bar{\tau}A_1Z & 0 & 0 \\
  \bullet & \bullet & -\bar{\tau}Z & \bar{\tau}ZD_0 & \bar{\tau}ZD_1 < 0 \\
  \bullet & \bullet & \bullet & -e_1I & 0 \\
  \bullet & \bullet & \bullet & \bullet & -e_2I \\
\end{bmatrix}
\begin{bmatrix}
  X \\
  Y \\
  Z
\end{bmatrix} \geq 0
\]

(4.70)

where

\[
Y_{11} = A_0^TP + PA_0 + \tau A + Y + Y^T + Q + e_1E_0^TE_0.
\]

Then the unforced system (4.1) is asymptotically stable for any time delay \( \tau \) satisfying \( 0 \leq \tau \leq \bar{\tau} \) and all admissible uncertainties.

Next, we extend Theorem 4.2.6 to design a robust stabilizing memoryless controller \( u(t) = K_0x(t) \) for the system (4.69) in the following theorem.
**Theorem 4.2.7** If there exist matrices $L > 0$, $M$, $N$, $R$, $V$, $W > 0$ and scalars $e_1$ and $e_2$ such that

\[
\begin{bmatrix}
Y_{11} & A_1 L - N & Y_{13} & LF_0^T + V F_b^T & Y_{15} & 0 & 0 \\
- W & \tau L A_1^T & 0 & 0 & LF_1^T & \tau L F_1^T \\
\bullet & \bullet & Y_{33} & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & - e_1 I & - e_3 I & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & - e_2 I & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & - e_4 I & - e_6 I \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & - e_3 I \\
\end{bmatrix} < 0 \quad (4.72)
\]

where

\[
Y_{11} = LA_0^T + A_0 L + BV + V^T B^T + \tau M + N + N^T + W + e_1 D_0 D_0^T + e_4 D_1 D_1^T
\]
\[
Y_{13} = \tau (LA_0^T + V^T B^T) + e_3 D_0 D_0^T + e_6 D_1 D_1^T
\]
\[
Y_{15} = \tau (LE_0^T + V^T E_b^T)
\]
\[
Y_{33} = - \tau R + e_2 D_0 D_0^T + e_5 D_1 D_1^T.
\]

Then the system (4.69) with the control

\[ u(t) = VL^{-1}x(t) \]

is asymptotically stable for any time delay $\tau$ satisfying $0 \leq \tau \leq \bar{\tau}$ and all admissible uncertainties.

If we consider a case where the size of the delay $\tau$ is measurable, a stabilization condition for a delayed feedback controller (4.22) can be contained as follows:
Theorem 4.2.8 Suppose that \( \tau \) is known. If there exist matrices \( L > 0, M, N, R, V, V_1, W > 0 \) and scalars \( e_1, e_2, \ldots, e_6 \) such that

\[
\begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} & LE_0^T + V^T E_b^T & \tau (LE_0^T + V^T E_b^T) & 0 & 0 \\
- W & Y_{23} & V_1^T E_b^T & \tau V_1^T E_b^T & LE_1^T & \tau LE_1^T \\
- e_1 I & - e_3 I & 0 & 0 & 0 & 0 \\
- e_2 I & 0 & 0 & 0 & 0 \\
- e_4 I & - e_6 I & 0 & 0 & 0 \\
- e_3 I & 0 & 0 & 0 & 0
\end{bmatrix} < 0 \quad (4.74)
\]

\[
\begin{bmatrix}
M & N \\
LR^{-1} L
\end{bmatrix} \geq 0 \quad (4.75)
\]

where

\[
Y_{11} = LA_0^T + A_0 L + BV + V^T B^T + \tau M + N + N^T + W + e_1 D_0 D_0^T + e_4 D_1 D_1^T
\]

\[
Y_{12} = - N - A_1 L + BV_1
\]

\[
Y_{13} = \tau (LA_0^T + V^T B^T) + e_3 D_0 D_0^T + e_6 D_1 D_1^T
\]

\[
Y_{23} = \tau (LA_1^T + V_1^T B^T)
\]

\[
Y_{33} = - \tau R + e_2 D_0 D_0^T + e_5 D_1 D_1^T
\]

Then the system (4.69) with the control \( u(t) = VL^{-1} x(t) + V_1 L^{-1} x(t - \tau) \) is asymptotically stable.

4.2.2 Robust Output Feedback Stabilization:

This subsection is devoted to the robust stabilization of time-delay system using output feedback controller. Due to consideration of most delay independent output feedback stabilization, we only provide a delay dependent robust output feedback stabilization method. We follow the result of Theorem 4.2.8 and generalize it for robust stabilization case.
Consider the following linear model of uncertain time-delay system

\[
\dot{x}(t) = (A_0 + \Delta A_0(t))x(t) + (A_1 + \Delta A_1(t))x(t - \tau) + (B + \Delta B(t))u(t) \tag{4.76}
\]

\[
x(t) = \varphi(t), \quad \forall \ t \in [-\tau, 0] \tag{4.77}
\]

\[
y(t) = Cx(t). \tag{4.78}
\]

where \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R}^m \) is the control input, \( y(t) \in \mathbb{R}^p \) is the output, \( \tau > 0 \) is the time delay of the system, \( \varphi(\cdot) \) is the initial condition, \( A_0, A_1, B \) and \( C \) are known real constant matrices of appropriate dimensions which describe the nominal system of (4.76) – (4.78), and \( \Delta A_0(\cdot), \Delta A_1(\cdot) \) and \( \Delta B(\cdot) \) are unknown real norm bounded matrix functions which represent time-varying parameter uncertainties. The admissible uncertainties are assumed to be of the form

\[
\begin{bmatrix}
\Delta A_0(t) \\
\Delta B(t)
\end{bmatrix} = D_0 \Delta_0(t) \begin{bmatrix}
E_0 \\
E_b
\end{bmatrix}, \quad \Delta A_1(t) = D_1 \Delta_1(t) E_1 \tag{4.79}
\]

where \( \Delta_0(t) \in \mathbb{R}^{i \times j} \) and \( \Delta_1(t) \in \mathbb{R}^{i_1 \times j_1} \) are unknown real time-varying matrices with Lebesgue measurable elements satisfying

\[
\|\Delta_0(t)\| \leq 1; \quad \|\Delta_1(t)\| \leq 1, \quad \forall \ t \tag{4.80}
\]

Next, the robust stabilization in this subsection is as follows: Given a scaler \( \bar{\tau} > 0 \), find a controller of the form (4.25) for the system (4.76) – (4.78) such that the resulting closed-loop system is globally uniformly asymptotically stable for any constant time-delay satisfying \( 0 \leq \tau \leq \bar{\tau} \) and all admissible uncertainties \( \Delta A_0, \Delta A_1 \) and \( \Delta B \).

In order to derive a solution to the robust output feedback stabilization problem, the following delay-dependent robust stability result will be needed.

**Lemma 4.2.9** Consider the system (4.76), (4.77) with \( u(t) = 0 \). Given a scaler \( \bar{\tau} > 0 \), this system is robustly stable for any constant time-delay \( \tau \) satisfying \( 0 \leq \tau \leq \bar{\tau} \).
and for all admissible uncertainties $\Delta A_0$ and $\Delta A_1$ if there exist a matrix $X > 0$ and scalars $\alpha_i > 0$, $i = 1, \ldots, 5$, solving the following LMI:

$$
\begin{bmatrix}
Q(X) & X \hat{M}^T & L^T(\alpha_1, X) & \hat{N} \\
\hat{M}X & -\frac{1}{\tau} \hat{U}_1 & 0 & 0 \\
L(\alpha_1, X) & 0 & -\alpha_1 I & 0 \\
\hat{N}^T & 0 & 0 & -\frac{1}{\tau} \hat{U}_2
\end{bmatrix} < 0
$$

where

$$
Q(X) = X(A_0 + A_1)^T + (A_0 + A_1)X
$$

$$
L^T(\alpha_1, X) = [\alpha_1 D_0 \quad \alpha_1 D_1 \quad X E_0^T \quad X E_1^T]
$$

$$
\hat{M}^T = [A_0^T \quad E_0^T \quad A_1^T \quad E_1^T]
$$

$$
\hat{N} = [A_1 \quad D_1]
$$

$$
\hat{U}_1 = \text{diag}\{\alpha_2 I - \alpha_3 D_0 D_0^T, \alpha_3 I, (1 - \alpha_2) I - \alpha_4 D_1 D_1^T, \alpha_4 I\}
$$

$$
\hat{U}_2 = \text{diag}\{I - \alpha_5 E_1^T E_1, \alpha_5 I\}.
$$

The result follows immediately from Theorem 3.1 of [53].

**Theorem 4.2.10** Consider the system (4.76) – (4.78) satisfying the above Assumption. Given a scaler $\bar{\tau} > 0$, this system is stabilizable via an output feedback controller (4.46) for any constant time-delay $\tau$ satisfying $0 \leq \tau \leq \bar{\tau}$. If there exist $n \times n$ symmetric positive definite matrices $R$, $S$ and $P$, and a scalar $\alpha_1 > 0$, $i = 1, \ldots, 5$ satisfying the following LMIs:

$$
\begin{bmatrix}
N_S & 0 \\
0 & I
\end{bmatrix}^T
\begin{bmatrix}
Q_S & S \hat{F}_1^T & \hat{F}_2 & \hat{F}_3^T \\
\hat{F}_1 S & -\hat{T}_1 & \hat{F}_1 & 0 \\
\hat{F}_2 & \hat{F}_1^T & -\frac{1}{\tau} P & \hat{F}_4^T \\
\hat{F}_3^T & 0 & \hat{F}_4 & -\hat{T}_2
\end{bmatrix}
\begin{bmatrix}
N_S & 0 \\
0 & I
\end{bmatrix} < 0
$$
where $N_S$ and $N_C$ are any matrices whose columns form bases of the null spaces of
\[
\begin{bmatrix}
B^T & B^T & E_b^T & E_b^T
\end{bmatrix}
\text{ and } C, \text{ respectively, } Q_S \text{ and } Q_R \text{ are as in (4.55) and (4.56), and}
\]
\[
\begin{align*}
\hat{F}_1^T &= \begin{bmatrix} A_0^T & E_1^T & E_1^T \end{bmatrix} \\
\hat{F}_2^T &= A_0 + A_1 \\
\hat{F}_3^T &= \begin{bmatrix} S A_1^T & S E_1^T & \alpha_1 D_0 & \alpha_1 D_1 & S E_1^T & A_1 & D_1 \end{bmatrix} \\
\hat{F}_4^T &= \begin{bmatrix} A_1^T & E_1^T & 0 & 0 & E_1^T & 0 & 0 \end{bmatrix} \\
\hat{F}_5^T &= \begin{bmatrix} A_1^T & E_1^T & \alpha_1 RD_0 & \alpha_1 RD_1 & E_1^T & E_1^T & RA_1 & RD_1 \end{bmatrix} \\
\hat{T}_1^T &= \text{diag}\left\{ \frac{1}{\tau} J_1, \alpha_3 \frac{1}{\tau} I, \alpha_1 I \right\} \\
\hat{T}_2^T &= \text{diag}\left\{ \frac{1}{\tau} J_2, \alpha_4 \frac{1}{\tau} I, \alpha_1 I, \alpha_1 I, \alpha_1 I, \frac{1}{\tau} J_3, \alpha_5 \frac{1}{\tau} I \right\} \\
\hat{T}_3^T &= \text{diag}\left\{ \frac{1}{\tau} J_2, \alpha_4 \frac{1}{\tau} I, \alpha_1 I, \alpha_1 I, \alpha_1 I, \frac{1}{\tau} J_3, \alpha_5 \frac{1}{\tau} I \right\} \\
J_1 &= \alpha_2 I - \alpha_3 D_0 D_0^T; \quad J_2 = (1 - \alpha_2) I - \alpha_4 D_1 D_1^T \\
J_3 &= I - \alpha_3 E_1^T E_1.
\end{align*}
\]

By Lemma (4.2.9), the closed-loop system
\[
\dot{z} = [\hat{A}_0 + D_0 F_0(t) \hat{E}_0] z(t) + [\hat{A}_1 + D_1 F_1(t) E_1] z(t - \tau), \quad z = \begin{bmatrix} x \\ x_c \end{bmatrix}
\]
where
\[ \hat{A}_0 = \bar{A}_0 + BK\bar{C}, \quad K = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}, \quad \bar{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \]
\[ B_0 = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}, \quad \bar{D}_0 = \begin{bmatrix} \bar{D}_0 \\ 0 \end{bmatrix}, \quad \bar{D}_1 = \begin{bmatrix} \bar{D}_1 \\ 0 \end{bmatrix} \]
\[ \hat{E}_0 = \bar{E}_0 + \bar{E}_1K\bar{C}, \quad \bar{E}_0 = [E_0 \ 0], \quad \bar{E}_1 = [E_1 \ 0], \quad \bar{E}_b = [0 \ E_b] \]
is robustly stable for any constant time-delay \( \tau \) satisfying \( 0 \leq \tau \leq \bar{\tau} \) if there exist a matrix \( X > 0 \) and scalars \( \alpha_i > 0, \ i = 1, \ldots, 5 \), satisfying the following inequality
\[ G_X + Y^T KZ_X + Z_X^T K^T Y < 0 \] (4.85)
where
\[
\begin{align*}
\hat{Y} &= \begin{bmatrix} \bar{B}^T & \bar{B}^T & \bar{E}_b^T & 0 & 0 & 0 & \bar{E}_b^T & 0 & 0 & 0 \end{bmatrix} \\
\bar{Z}_X &= \begin{bmatrix} \bar{C}X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]
where
\[
\hat{G}_X = \begin{bmatrix} \bar{Q}(X) & XM^T & L^T(\alpha_1, X) & N^Y \\ \bullet & -\frac{1}{\bar{\tau}}U_1 & 0 & 0 \\ \bullet & \bullet & -\alpha_1I & 0 \\ \bullet & \bullet & \bullet & -\frac{1}{\bar{\tau}}U_2 \end{bmatrix}
\]
\[
\begin{align*}
\bar{Q}(X) &= X(\bar{A}_0 + \bar{A}_1)^T + (\bar{A}_0 + \bar{A}_1)^T \\
L^T(\alpha_1, X) &= [\alpha_1\bar{D}_0 \ \alpha_1\bar{D}_1 \ X\bar{E}_0^T \ X\bar{E}_1^T] \\
M^T &= [\bar{A}_0^T \ \bar{E}_0^T \ \bar{A}_1^T \ \bar{E}_1^T] \\
N^T &= [\bar{A}_1 \ \bar{D}_1] \\
U_1 &= \text{diag}\{\alpha_2I - \alpha_3\bar{D}_0\bar{D}_0^T, \ \alpha_3I, \ (1 - \alpha_2)I - \alpha_4\bar{D}_1\bar{D}_1^T, \ \alpha_4I\} \\
U_2 &= \text{diag}\{I - \alpha_5\bar{E}_1^T\bar{E}_1, \ \alpha_5I\}.
\end{align*}
\]

The computation of a robust output feedback controller can be carried out using a procedure similar to that of Section 4.1.3 for the controller design. More
specifically, when the LMIs (4.81) - (4.83) are satisfied for some matrices $R$, $S$ and $P$, and scaler $\alpha_i$, $i = 1, \ldots, 5$, an output feedback controller that solves the robust stabilization problem can be obtained as follows:

1. Compute an $n \times n$ nonsingular matrix $N$ such that
   $$NN^T = \alpha_2 P^{-1}$$

2. Find the matrix $X > 0$ from
   $$X = \begin{bmatrix} S & N \\ N^T(S - R^{-1})^{-1}N \end{bmatrix}$$

3. Compute the controller matrices $A_c, B_c, C_c, D_c$ by solving the LMI (4.85) for $K = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$.

To illustrate the above procedure, we consider the following delay system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 & -0.5 \\ 0 & -1 \end{bmatrix} x(t - \tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \ 1] x(t).$$

We observe that the above system with $\tau = 0$ is not asymptotically stable.

It should be noted that the delay-independent output feedback stabilization methods cannot be applied for the system (4.86) as the pair $(A, B)$ is not stabilizable. On the other hand, applying Theorem 4.1.17 to the system (4.86), it was found by using the software package MATLAB-LMI Lab that this system is stabilizable via linear dynamic output feedback for any constant time-delay $\tau$ satisfying $0 \leq \tau \leq 0.2650$. Moreover, a stabilizing output feedback controller is given by (4.46) where

$$A_c = \begin{bmatrix} -40.672 & -110.271 \\ -20.876 & -60.7423 \end{bmatrix}, \quad B_c = \begin{bmatrix} 10.158 \\ 5.455 \end{bmatrix}, \quad C_c = [4.311 \ -12.778], \quad D_c = -0.61.$$

We emphasize that the smaller controller gain could be obtained by reducing the maximum allowed time-delay.
Chapter 5

Observers for Time-Delay Systems

5.1 Introduction

State observer design plays an essential role in the design of linear control systems [79]. Several observers have been proposed and successfully applied in different scenarios. Among different types of observers, the PI observer [74], [98] was introduced in order to provide the designer with additional degrees of freedom through the insertion of an integration path. Using the added degrees of freedom, PI observers have been designed with relatively low gain and facilitated the LTR design. The PI observer was also found useful in disturbance estimation and fault detection [75], [103]. Therefore, it is also important to be generalized for time-delay systems.

The design of observer for time-delay systems is not as trivial as in the case of regular systems. The time-delay often significantly complicates the stability analysis and control design. As a result the design of observer-based controller for time-delay systems becomes very challenging. There are several books published to capture different approaches to the stability and design of time-delay systems [25], [64], [123].

The problem of designing an asymptotic observer for time-delay systems has
been solved by many authors in the past for different purposes see [85], [79], [94], [116], [121] and the references therein.

Some papers have developed observer-based controllers for time-delay systems, [11], but the delay is considered in state variables only and the observer design requires the solution of a pair of Algebraic Riccati-like equations. The problem of $H_\infty$ observer design for a linear time-delay systems is provided in [20].

Since the work of Lee et al. [49], few works have considered the problem of an asymptotic observer design for time-delay systems.

In this chapter, an asymptotic PI observer for linear time-delay systems is proposed to extend the result of [20]. A delay-dependent method based on the Algebraic Riccati equation is developed in order to stabilize the observer and to minimize the $H_\infty$ norm between the disturbance and the estimated error. A “weak” sufficient condition for the existence of such an observer is given. An alternative design procedure based on modified algebraic Riccati equations is include to provide the allowable size of delay.

It should be pointed out that the PI observer has additional advantage of estimating the disturbance or default and it is more desirable to be used in robust control design. Although the PI observer design is considered for systems with one delay, it can also be generalized for multi-delay systems. A brief connection between PIO and UIO is given to show their applications in disturbance estimation and fault detection rather than disturbance attenuation.
5.2 PI Observer Design for Linear Time-Delay Systems

Consider the continuous linear time-delayed system described by

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + B_0 u(t) + B_1 u(t - \tau) \tag{5.1}
\]
\[
y(t) = C_0 x(t) + C_1 x(t - \tau) \tag{5.2}
\]

where \( x(t) \in \mathbb{R}^n \) is the state variable, \( u(t) \in \mathbb{R}^m \) is the control input, \( y(t) \in \mathbb{R}^p \) is the output. \( A_0, A_1, B_0, B_1, C_0, C_1 \) are constant matrices with appropriate dimensions, and \( \tau \geq 0 \) is the time delay.

The PI observer for the system (5.1), (5.2) can be given by the following dynamical system

\[
\dot{\hat{x}}(t) = A_0 \hat{x}(t) + A_1 \hat{x}(t - \tau) + B_0 u(t) + B_1 u(t - \tau) - L_p \left[ C_0 \hat{x}(t) + C_1 \hat{x}(t - \tau) - y(t) \right] \tag{5.3}
\]
\[
\dot{w}(t) = L_i \left[ C_0 \hat{x}(t) + C_1 \hat{x}(t - \tau) - y(t) \right] + w(t) + w(t - \tau) \tag{5.4}
\]

with the properties

\[
\lim_{t \to \infty} e(t) = 0 \tag{5.5}
\]
\[
\lim_{t \to \infty} w(t) = 0 \tag{5.6}
\]

for which the state estimation error is defined as

\[
e(t) = x(t) - \hat{x}(t) \tag{5.7}
\]

where \( \hat{x}(t) \in \mathbb{R}^n \) is the state estimation, \( L_p \) is the proportional gain and \( L_i \) is the integral gain.
The observer in (5.3), (5.4) can be represented as an augmented state observer in the form

\[
\dot{\xi}(t) = \hat{A}_0 \xi(t) + \hat{A}_1 \xi(t - \tau)
\]

where

\[
\hat{A}_i = \bar{A}_i - L \bar{C}_i, \quad \bar{A}_i = \begin{bmatrix} A_i & B_i \\ 0 & I \end{bmatrix}, \quad \bar{C}_i = [C_i \ 0], \quad i = 0, 1,
\]

\[
L = \begin{bmatrix} L_P \\ L_I \end{bmatrix}, \quad \text{and} \quad \xi(t) = \begin{bmatrix} e(t) \\ w(t) \end{bmatrix}.
\]

The purpose of this section is to design a constant gain \(L\) for the PI observer (5.3), (5.4).

To prove Theorem 5.2.2, the following lemma is needed [46].

**Lemma 5.2.1** Let \(S = \begin{bmatrix} X_{n \times n} & Y_{n \times m} \\ Y_{m \times n}^T & Z_{m \times m} \end{bmatrix}\) where \(X\) and \(Z\) are symmetric. Then \(S > 0\) if and only if \(Z > 0\) and \(X - YZ^{-1}Y^T > 0\).

**Theorem 5.2.2** Let the observer gain matrix \(L\) (yet to be determined) be such that \(\hat{A}\) is a stability matrix. Then the solution of (5.8) is uniformly asymptotically stable, if the symmetric matrices \(P\) and \(Q\) associated with the Lyapunov equation \(\hat{A}_0^T P + P \hat{A}_0 = -2Q\) satisfy the inequalities \(Q > 0\) and \(Q - P \hat{A}_1 Q^{-1} \hat{A}_1^T P > 0\).

**Proof** Define the Lyapunov functional

\[
V(\xi(t), t) = \xi^T(t) P \xi(t) + \int_{t-\tau}^t \xi^T(s) Q \xi(s) \, ds.
\]

Then, the sufficient conditions for (5.8) to be uniformly asymptotically stable for all \(t, \xi \neq 0\) are

(a) \(K_1 \| \xi \| \leq V(\xi) \leq K_2 \| \xi \|, \quad K_1, \ K_2 > 0,\) and

(b) \(\dot{V}(\xi) \leq -K_3 \| \xi \|^2, \quad K_3 > 0\)
where \( V(0) = 0 \).

Since \( \hat{A} \) is stability matrix and \( Q > 0 \), the matrix \( P > 0 \), which establishes (a).

To show (b), the proof proceeds as follows. Differentiating \( \dot{V}(\xi(t), t) \) along the motion starting at \((\xi(t), t)\) yields

\[
\dot{V} = \dot{\xi}^T(t) P \xi(t) + \xi(t) P \dot{\xi}^T(t) + \xi^T(t) Q \xi(t) - \xi^T(t - \tau) Q \xi(t - \tau)
\]

\[
= \xi^T(t) \{ \hat{A}_0^T P + P \hat{A}_0 + Q \} \xi(t) + \xi^T(t - \tau) \hat{A}_1^T P \xi(t) + \xi^T(t) P \hat{A}_1 \xi(t - \tau)
\]

\[
- \xi^T(t - \tau) Q \xi(t - \tau)
\]

\[
= \begin{bmatrix}
\xi(t) \\
\xi(t - \tau)
\end{bmatrix}^T
\begin{bmatrix}
Q & -P \hat{A}_1 \\
-\hat{A}_1^T P & Q
\end{bmatrix}
\begin{bmatrix}
\xi(t) \\
\xi(t - \tau)
\end{bmatrix}.
\]

Thus the hypothesis and Lemma 5.2.1 imply

\[
\dot{V}(\xi(t), t) \leq -K_3 \| \xi \|^2,
\]

and the Theorem is proved.

Using Theorem 5.2.2, the gain \( L \) can be designed such that the system (5.3), (5.4) is a PI observer for the system (5.1), (5.2) as follows.

**Theorem 5.2.3** Consider the time-delay system (5.1), (5.2) and the PI observer (5.3), (5.4). If the following algebraic Riccati equation

\[
\hat{A}_0^T P + P \hat{A}_0 + \frac{2}{\alpha} \hat{A}_1 \hat{A}_1^T P - 2 \bar{C}_0^T \left( I_p - \frac{1}{\alpha} \bar{C}_1 \bar{C}_1^T \right) \bar{C}_0 + \alpha I_n = 0
\]

(5.9)

has a symmetric positive definite solution \( P \), then for all \( \tau \geq 0 \)

1. observer (5.3), (5.4) is asymptotically stable,

2. \( \lim_{t \to \infty} \xi(t) = 0 \).

In this case the required observer gain is given by

\[
L = P^{-1} \bar{C}_0^T.
\]

(5.10)
**Proof** Suppose that Riccati equation (5.9) has a symmetric positive definite solution $P$. Choosing $L = P^{-1} \tilde{C}_0^T$, Riccati equation (5.9) can be written as follows

\[
\dot{\tilde{A}}_0^T P + P \dot{\tilde{A}}_0 + \frac{2}{\alpha} P \tilde{A}_1 \tilde{A}_1^T P - 2 P L \left( I_p - \frac{1}{\alpha} \tilde{C}_1 \tilde{C}_1^T \right) L^T P + \alpha I_n = 0
\]

\[
\dot{\tilde{A}}_0^T P + P \dot{\tilde{A}}_0 + \frac{2}{\alpha} P \tilde{A}_1 \tilde{A}_1^T P - 2 P L L^T P + \frac{2}{\alpha} P L \tilde{C}_1 \tilde{C}_1^T L^T P + \alpha I_n = 0
\]

\[
\dot{\tilde{A}}_0^T P + P \dot{\tilde{A}}_0 + \frac{2}{\alpha} P \tilde{A}_1 \tilde{A}_1^T P - P L \tilde{C}_0 - \tilde{C}_0^T L^T P + \frac{2}{\alpha} P L \tilde{C}_1 \tilde{C}_1^T L^T P + \alpha I_n = 0
\]

\[
(\dot{\tilde{A}}_0^T - \tilde{C}_0^T L^T)P + P(\tilde{A}_0 - L \tilde{C}_0) + \frac{2}{\alpha} P \tilde{A}_1 \tilde{A}_1^T P + \frac{2}{\alpha} P L \tilde{C}_1 \tilde{C}_1^T L^T P + \alpha I_n = 0
\]

\[
\dot{\tilde{A}}_0^T P + P \dot{\tilde{A}}_0 + \frac{1}{\alpha} P \tilde{A}_1 \tilde{A}_1^T P + \frac{1}{\alpha} P L \tilde{C}_1 \tilde{C}_1^T L^T P + \alpha I_n = -\frac{1}{\alpha} P \tilde{A}_1 \tilde{A}_1^T P - \frac{1}{\alpha} P L \tilde{C}_1 \tilde{C}_1^T L^T P.
\]

Using the following inequality

\[-XX^T - YY^T \leq XY^T + YX^T\]

where $X$ and $Y$ are any two matrices with suitable dimensions, we can write

\[
\dot{\tilde{A}}_0^T P + P \dot{\tilde{A}}_0 + \frac{1}{\alpha} P \tilde{A}_1 \tilde{A}_1^T P + \frac{1}{\alpha} P L \tilde{C}_1 \tilde{C}_1^T L^T P + \alpha I_n \leq \frac{1}{\alpha} P \tilde{A}_1 \tilde{C}_1^T L^T P + \frac{1}{\alpha} P L \tilde{C}_1 \tilde{A}_1^T P
\]

\[
\dot{\tilde{A}}_0^T P + P \dot{\tilde{A}}_0 + \frac{1}{\alpha} P (\tilde{A}_1 \tilde{A}_1^T - \tilde{A}_1 \tilde{C}_1^T L^T - L \tilde{C}_1 \tilde{A}_1^T + L \tilde{C}_1 \tilde{C}_1^T L^T) P + \alpha I_n \leq 0.
\]

Using the bounded real Lemma, which states that, the following statements are equivalent

1. There exists a symmetric positive definite matrix $\tilde{P}$ such that

\[A^T \tilde{P} + \tilde{P} A + \tilde{P} B B^T \tilde{P} + C^T C < 0\]
2. The Riccati equation

\[ A^T P + PA + PBB^T P + CC^T = 0 \]

has a solution \( P > 0 \). Furthermore, if these statements hold then \( P < \bar{P} \),

leads to

\[
\dot{A}_0^T P + P \dot{A}_0 + \frac{1}{\alpha} P(\bar{A}_1 - L\bar{C}_1)(\bar{A}_1 - L\bar{C}_1)^T P + \alpha I_n < 0
\]

\[
\dot{A}_1^T P + P \dot{A}_1 + \frac{1}{\alpha} P \bar{A}_1 \bar{A}_1^T P + \alpha I_n < 0,
\]

which is equivalent to

\[
\begin{bmatrix}
\dot{A}_0^T P + P \dot{A}_0 + \alpha I_n & P \dot{A}_1 \\
\dot{A}_1^T P & -\alpha I_n
\end{bmatrix} < 0,
\]

and by comparison to the inequality of Theorem 5.2.2 with \( Q = \alpha I_n \), the asymptotic

stability of the observer is guaranteed. \( \Box \)

5.3 PI Observer Design for Multi-Delay Systems

The PI observer design for systems with one delay can easily be generalized for multi-
delay systems. The derivations are similar as in above section with minor modifications.

Consider the continuous multi-delayed system described by

\[
\begin{align*}
\dot{x}(t) & = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) + B_0 u(t) + \sum_{i=1}^{k} B_i u(t - \tau_i) \\
y(t) & = C_0 x(t) + \sum_{i=1}^{k} C_i x(t - \tau_i)
\end{align*}
\] (5.11) (5.12)

where \( x(t) \in \mathcal{R}^n \) is the state variable, \( u(t) \in \mathcal{R}^m \) is the control input, \( y(t) \in \mathcal{R}^p \)
is the output. \( A_i, B_i, C_i; \ i = 0, 1, 2, \ldots, k \) are constant matrices with appropriate
dimensions, and \( 0 < \tau_1 < \tau_2 \ldots < \tau_k < \infty \).
The PI observer for the system (5.11), (5.12) can be given by the following dynamical system

\[ \dot{\hat{x}}(t) = A_0 \hat{x}(t) + \sum_{i=1}^{k} A_i \hat{x}(t-i\tau) + B_0 u(t) + \sum_{i=1}^{k} B_i u(t-i\tau) - B_0 w(t) - \sum_{i=1}^{k} B_i w(t-i\tau) \]
\[ + L_p \left( y(t) - C_0 \hat{x}(t) - \sum_{i=1}^{k} C_i \hat{x}(t-i\tau) \right) \]  
\[ \dot{w}(t) = -L_I \left( y(t) - C_0 \hat{x}(t) - \sum_{i=1}^{k} C_i \hat{x}(t-i\tau) \right) + w(t) + \sum_{i=1}^{k} w(t-i\tau) \]  

where \( \hat{x}(t) \in \mathbb{R}^n \) is the state estimation, \( L_p \) is the proportional gain and \( L_i \) is the integral gain.

The observer in (5.13), (5.14) can be represented as an augmented state observer in the form

\[ \dot{\xi}(t) = \hat{A}_0 \xi(t) + \sum_{i=1}^{k} \hat{A}_i \xi(t-i\tau) \]  

where

\[ \hat{A}_i = \bar{A}_i - \bar{C}_i, \quad \bar{A}_i = \begin{bmatrix} A_i & B_i \\ 0 & I \end{bmatrix}, \quad \bar{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix}, \quad i = 0, 1, \ldots, k, \]

\[ L = \begin{bmatrix} L_p \\ L_i \end{bmatrix}, \quad \text{and} \quad \xi(t) = \begin{bmatrix} e(t) \\ w(t) \end{bmatrix}. \]

**Theorem 5.3.1** Let the observer gain matrix \( L \) (yet to be determined) be such that \( \hat{A}_0 \) is a stability matrix. Then the solution of (5.15) is uniformly asymptotically stable, if the symmetric matrices \( P \) and \( Q \) associated with the Lyapunov equation \( \hat{A}_0^T P + P \hat{A}_0 = -(k+1)Q \) satisfy the inequalities \( Q > 0 \) and

\[ Q - \sum_{i=1}^{k} P \bar{A}_i Q^{-1} \bar{A}_i^T P > 0. \]

**Proof** Define the Lyapunov functional

\[ V(\xi(t), t) = \xi^T(t) P \xi(t) + \sum_{i=1}^{k} \int_{t-\tau_i}^{t} \xi^T(s) Q \xi(s) \, ds. \]

Then, the sufficient conditions for (5.15) to be uniformly asymptotically stable for all \( t, \xi \neq 0 \) are
(a) \( K_1 \| \xi \| \leq V(\xi) \leq K_2 \| \xi \|, \quad K_1, K_2 > 0, \) and

(b) \( \dot{V}(\xi) \leq -K_3 \| \xi \|^2, \quad K_3 > 0 \)

where \( V(0) = 0. \)

Since \( \hat{A}_0 \) is stability matrix and \( Q > 0, \) the matrix \( P > 0, \) which establishes (a).

To show (b), the proof proceeds as follows. Differentiating \( \dot{V}(\xi(t), t) \) along the motion starting at \((\xi(t), t)\) yields

\[
\frac{dV}{dt}(\xi(t), t) = -\begin{pmatrix}
\xi(t) \\
\xi(t - \tau_1) \\
\xi(t - \tau_2) \\
\vdots \\
\xi(t - \tau_k)
\end{pmatrix}^T \begin{pmatrix}
Q & -P\hat{A}_1 & -P\hat{A}_2 & \ldots & -P\hat{A}_k \\
-\hat{A}_1^T P & Q & 0 & \ldots & 0 \\
-\hat{A}_2^T P & 0 & Q & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\hat{A}_k^T P & 0 & 0 & \ldots & Q
\end{pmatrix} \begin{pmatrix}
\xi(t) \\
\xi(t - \tau_1) \\
\xi(t - \tau_2) \\
\vdots \\
\xi(t - \tau_k)
\end{pmatrix}
\]

Thus the hypothesis and Lemma 5.2.1 imply

\[
\frac{dV}{dt}(\xi(t), t) \leq -K_3 \| \xi \|^2,
\]

and the Theorem is proved.

Using theorem 5.3.1, the gain \( L \) can be designed such that the system (5.13), (5.14) is a PI observer for the system (5.11), (5.12) as follows:

**Theorem 5.3.2** Consider the time-delay system (5.11), (5.12) and the PI observer (5.13), (5.14). If the following algebraic Riccati equation

\[
\hat{A}_0^T P + P\hat{A}_0 + \frac{2}{\alpha} P \left( \sum_{i=1}^{k} \hat{A}_i \hat{A}_i^T \right) P - 2C_0^T \left( I_p - \frac{1}{\alpha} \sum_{i=1}^{k} C_i C_i^T \right) C_0 + \alpha(1 + k)I_n = 0 \quad (5.16)
\]

has a symmetric positive definite solution \( P, \) then for all \( \tau \geq 0, \)

1. observer (5.13), (5.14) is asymptotically stable,

2. \( \lim_{t \to \infty} \xi(t) = 0. \)
In this case the required observer gain is given by

\[ L = P^{-1} \bar{C}_0^T. \]  

(5.17)

**Proof** Suppose that Riccati equation (5.56) has a symmetric positive definite solution \( P \). Choosing \( L = P^{-1} \bar{C}_0^T \), Riccati equation (5.56) can be written as follows

\[
\begin{align*}
\dot{X} P + P \dot{X} + \frac{2}{\alpha} P \left( \sum_{i=1}^{k} \bar{A}_i \bar{A}_i^T \right) P - 2PL \left( I_p - \frac{1}{\alpha} \sum_{i=1}^{k} \bar{C}_i \bar{C}_i^T \right) L^T P + \alpha(k+1)I_n &= 0 \\
\dot{X} P + P \dot{X} + \frac{2}{\alpha} P \left( \sum_{i=1}^{k} \bar{A}_i \bar{A}_i^T \right) P - 2PLL^T P + \frac{2}{\alpha} P \left( \sum_{i=1}^{k} \bar{C}_i \bar{C}_i^T \right) L^T P + \alpha(k+1)I_n &= 0 \\
\dot{X} P + P \dot{X} + \frac{2}{\alpha} P \left( \sum_{i=1}^{k} \bar{A}_i \bar{A}_i^T \right) P - P L \bar{C}_0 - \bar{C}_0^T L^T P + \frac{2}{\alpha} P \left( \sum_{i=1}^{k} \bar{C}_i \bar{C}_i^T \right) L^T P + \alpha(k+1)I_n &= 0 \\
\dot{X} P + P \dot{X} + \frac{2}{\alpha} P \left( \sum_{i=1}^{k} \bar{A}_i \bar{A}_i^T \right) P + \frac{2}{\alpha} P \left( \sum_{i=1}^{k} \bar{C}_i \bar{C}_i^T \right) L^T P + (k+1) \alpha I_n &= 0 \\
\dot{X} P + P \dot{X} + \frac{2}{\alpha} P \left( \sum_{i=1}^{k} \bar{A}_i \bar{A}_i^T \right) P + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \bar{C}_i \bar{C}_i^T \right) L^T P + \alpha(k+1)I_n &= 0 \\
\dot{X} P + P \dot{X} + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \bar{A}_i \bar{A}_i^T \right) P + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \bar{C}_i \bar{C}_i^T \right) L^T P + (k+1) \alpha I_n &= 0 \\
\dot{X} P + P \dot{X} + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \bar{A}_i \bar{A}_i^T \right) P + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \bar{C}_i \bar{C}_i^T \right) L^T P + \alpha(k+1)I_n &= 0 \\
\dot{X} P + P \dot{X} + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \bar{A}_i \bar{A}_i^T \right) P - \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \bar{C}_i \bar{C}_i^T \right) L^T P. 
\end{align*}
\]

Using the following inequality

\[-XX^T - YY^T \leq XY^T + YX^T\]

where \( X \) and \( Y \) are any two matrices with suitable dimensions, we can write

\[
\begin{align*}
\dot{X} P + P \dot{X} + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \bar{A}_i \bar{A}_i^T \right) P + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \bar{C}_i \bar{C}_i^T \right) L^T P + \alpha(k+1)I_n \\
\leq \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \bar{A}_i \bar{A}_i^T \right) L^T P + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \bar{C}_i \bar{C}_i^T \right) L^T P \\
\dot{X} P + P \dot{X} + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \left( \bar{A}_i \bar{A}_i^T - \bar{A}_i \bar{C}_i L^T - LC_i \bar{A}_i^T + LC_i \bar{C}_i L^T \right) \right) P + \alpha(k+1)I_n &\leq 0.
\end{align*}
\]
Using the Bounded Real Lemma, which states that, the following statements are equivalent

1. There exists a symmetric positive definite matrix $\bar{P}$ such that

$$A^T \bar{P} + \bar{P}A + \bar{P}BB^T \bar{P} + C^T C < 0$$

2. The Riccati equation

$$A^T P + PA + PBB^T P + CC^T = 0$$

has a solution $P > 0$. Furthermore, if these statements hold then $P < \bar{P}$, leads to

$$\dot{A}_0^T P + P \dot{A}_0 + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} (\dot{A}_i - LC_i)(\dot{A}_i - LC_i)^T \right) P + \alpha(k + 1)I_n < 0$$

$$\dot{A}_0^T P + P \dot{A}_0 + \frac{1}{\alpha} P \left( \sum_{i=1}^{k} \dot{A}_i \dot{A}_i^T \right) P + \alpha(k + 1)I_n < 0,$$

which is equivalent to

$$\begin{pmatrix} A_0^T P + P \dot{A}_0 + k\alpha I_n & P \dot{A}_1 & P \dot{A}_2 & \ldots & P \dot{A}_k \\ \dot{A}_1^T P & -\alpha I_n & 0 & \ldots & 0 \\ \dot{A}_2^T P & 0 & -\alpha I_n & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dot{A}_k^T P & 0 & 0 & \ldots & -\alpha I_n \end{pmatrix} < 0$$

and by comparison to the inequality of Theorem 5.3.1 with $Q = \alpha I_n$, the asymptotic stability of the observer is guaranteed. \qed
5.4 PI $H_\infty$ Observer Design for Disturbance Attenuation and Fault Detection in Linear Time-Delay Systems

Consider the continuous linear time-delay system described by

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + B_0 u(t) + B_1 u(t - \tau) + Ed(t) \tag{5.18}
\]
\[
y(t) = C_0 x(t) + C_1 x(t - \tau) + Fd(t) \tag{5.19}
\]
\[
x(t) = \varphi(t), \quad t \in [-\tau, 0]
\]

where $x(t) \in \mathcal{R}^n$ is the state variable, $u(t) \in \mathcal{R}^m$ is the control input, $y(t) \in \mathcal{R}^p$ is the output, $d(t) \in \mathcal{R}^q$ is the square-integrable disturbance vector, $\varphi(t) \in C[-\tau; 0]$ is the continuous initial value function vector, $A_0$, $A_1$, $B_0$, $B_1$, $C_0$, $C_1$, $E$ and $F$ are constant matrices with appropriate dimensions, and $0 \leq \tau < \infty$ is the constant known time-delay duration.

It should be noted that matrices $E$ and $F$ may include parameter uncertainties and/or modeling errors.

The PI observer for the system (5.18), (5.19) can be given by the following dynamical system

\[
\dot{\hat{x}}(t) = A_0 \hat{x}(t) + A_1 \hat{x}(t - \tau) + B_0 u(t) + B_1 u(t - \tau) - B_0 w(t) - B_1 w(t - \tau) + L_P \left( y(t) - C_0 \hat{x}(t) - C_1 \hat{x}(t - \tau) \right) \tag{5.20}
\]
\[
\dot{w}(t) = -L_I \left( y(t) - C_0 \hat{x}(t) - C_1 \hat{x}(t - \tau) \right) + w(t) + w(t - \tau) \tag{5.21}
\]

where $\hat{x}(t) \in \mathcal{R}^n$ is the state estimation, $L_P$ is the proportional gain and $L_I$ is the integral gain.

The estimated error, defined as

\[
e(t) = x(t) - \hat{x}(t) \tag{5.22}
\]
obeys the following error dynamics.

\[
\dot{e}(t) = (A_0 - L_p C_0) e(t) + (A_1 - L_p C_1) e(t - \tau) + B_0 w(t) + B_1 w(t - \tau) \\
+ (E - L_p F) d(t)
\]  

(5.23)

and

\[
\dot{w}(t) = - L_i C_0 e(t) - L_i C_1 e(t - \tau) + w(t) + w(t - \tau) - L_i F d(t)
\]  

(5.24)

Then the PI observer can be represented as an augmented state observer in the form

\[
\dot{\xi}(t) = \hat{A}_0 \xi(t) + \hat{A}_1 \xi(t - \tau) + \hat{D} d(t)
\]  

(5.25)

where

\[
\hat{A}_i = \bar{A}_i - L \bar{C}_i, \quad \bar{A}_i = \begin{bmatrix} A_i & B_i \\ 0 & I \end{bmatrix}, \quad \bar{C}_i = [C_i \ 0], \ i = 0, 1;
\]

\[
L = \begin{bmatrix} L_p \\ L_i \end{bmatrix}, \quad \hat{D} = \bar{E} - L \bar{F}, \quad \bar{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \quad \bar{F} = F \text{ and } \xi(t) = \begin{bmatrix} e(t) \\ w(t) \end{bmatrix}.
\]

**Definition 5.4.1** Given a positive scaler \( \gamma \), the system (5.20), (5.21) is said to be a \( \gamma \)-PI observer for the associated system (5.18), (5.19) if the solution of the differential equation (5.25) with \( d(t) \equiv 0 \) converges to zero asymptotically and, under zero initial condition, the \( H_\infty \) norm of the transfer function between the disturbance and the estimated error is bounded by \( \gamma \), that is:

1. \( \lim_{t \to \infty} \xi(t) \to 0 \) for \( d(t) = 0 \),

2. \( \| (sI_n - \hat{A} - \hat{A}_1 e^{-s\tau})^{-1} \hat{D} \|_\infty \leq \gamma \),

under zero initial condition.

The purpose of this section is to design a constant gain \( L \) such that the system (5.20), (5.21) is a \( \gamma \)-PI observer for the associated system (5.18), (5.19) for some positive scaler \( \gamma \).
Theorem 5.4.2 Given a positive scalar \( \gamma \), if there exists two positive definite matrices \( Q \) and \( P \) such that

\[
\hat{A}^T P + P \hat{A} + P \hat{A}_1 Q^{-1} \hat{A}_1^T P + Q + \frac{1}{\gamma^2} I_n + P \hat{D} \hat{D}^T P < 0 \tag{5.26}
\]

then the system (5.25) is asymptotically stable for any value of the delay and the following inequality holds:

\[
\left\| \left( s I_n - \hat{A} - \hat{A}_1 e^{-s \tau} \right)^{-1} \hat{D} \right\|_\infty \leq \gamma \tag{5.27}
\]

**Proof** Define the Lyapunov functional \( V \) as follows:

\[
V = \xi^T(t) P \xi(t) + \int_{t-\tau}^{t} \xi^T(\theta) Q \xi(\theta) \, d\theta
\]

then

\[
\dot{V} = [\xi(t) \, \xi(t-\tau)] \begin{bmatrix} \hat{A}^T P + P \hat{A} + Q & P \hat{A}_1 \\ \hat{A}_1^T P & -Q \end{bmatrix} \begin{bmatrix} \xi(t) \\ \xi(t-\tau) \end{bmatrix}.
\]

Since \( \frac{1}{\gamma^2} I_n + P \hat{D} \hat{D}^T P > 0 \), using the Schur complement and (5.26) it follows that the matrix in (5.28) is negative definite. This implies that the asymptotic stability of the system (5.25) by the Krasovskii theorem [28].

Let us define a positive definite matrix \( S \) as follows:

\[
S := -\left( \hat{A}^T P + P \hat{A} + P \hat{A}_1 Q^{-1} \hat{A}_1^T P + Q + \frac{1}{\gamma^2} I_n + P \hat{D} \hat{D}^T P \right)
\]

then we have

\[
\hat{A}^T P + P \hat{A} + P \hat{A}_1 Q^{-1} \hat{A}_1^T P + Q + \frac{1}{\gamma^2} I_n + P \hat{D} \hat{D}^T P + S = 0
\]

and

\[
(-jw I_n - \hat{A}^T - e^{jw \tau} \hat{A}_1) P + P(jw I_n - \hat{A} - e^{-jw \tau} \hat{A}_1^T) - P \hat{A}_1 Q^{-1} \hat{A}_1^T P - Q - P \hat{D} \hat{D}^T P
\]

\[
- \frac{1}{\gamma^2} I_n - S = -e^{jw \tau} \hat{A}_1^T P - e^{-jw \tau} P \hat{A}_1 \tag{5.28}
\]
for all $w \in \mathcal{R}$. Let us define $W(jw)$ and $X(jw)$ as follows:

$$
W(jw) = P \hat{A}_1 Q^{-1} A_1^T P + Q - \hat{A}_1^T P e^{jw \tau} - \hat{A}_1 P e^{-jw \tau}
$$

$$
X(jw) = (jw I_n - \hat{A}_0 - \hat{A}_1 e^{-jw \tau})^{-1}
$$

Note that $W(jw)$ is a non-negative definite for all $w \in \mathcal{R}$.

Using the matrices $W(jw)$ and $X(jw)$, (5.28) may be rewritten as

$$
\left( X^T(-jw) \right)^{-1} P + PX^{-1}(jw) - W(jw) - \frac{1}{\gamma^2} I_n - P \hat{D} \hat{D}^T P - S = 0
$$

which leads to

$$
\hat{D}^T X^T(-jw) P \hat{D} + \hat{D}^T PX(jw) \hat{D} - \hat{D}^T X^T(-jw) P \hat{D} \hat{D}^T PX(jw) \hat{D} - I_n
$$

$$
= -I_n + \hat{D}^T X^T(-jw) \{ W(jw) + \frac{1}{\gamma^2} I_n + S \} X(jw) \hat{D}.
$$

It follows that

$$
-\left( I_n - \hat{D}^T PX(-jw) \hat{D} \right)^T \left( I_n - \hat{D}^T PX(-jw) \hat{D} \right) = -I_n + \frac{1}{\gamma^2} \hat{D}^T X^T(jw) X(jw) \hat{D}
$$

$$
+ \hat{D}^T X^T(-jw) \{ W(jw) + S \} X(jw) \hat{D}
$$

(5.31)

The left hand side of the above equation is non-negative definite for all $w \in \mathcal{R}$.

Define $T(jw) := X(jw) \hat{D}$, then equation (5.31) implies

$$
-I_n + \hat{D}^T X^T(-jw) \{ W(jw) + S \} X(jw) \hat{D} + \frac{1}{\gamma^2} T^T(-jw) T(jw) \leq 0
$$

and

$$
T^T(-jw) T(jw) \leq \gamma^2 I_n - \gamma^2 \hat{D}^T X^T(-jw) \{ W(jw) + S \} X(jw) \hat{D}
$$

for all $w \in \mathcal{R}$.

Since $W(jw)$ is non-negative definite matrix and $S$ is positive definite matrix for all $w \in \mathcal{R}$, then

$$
\|T(jw)\|_\infty = T^T(-jw) T(jw) \leq \gamma^2 I_n
$$

for all $w \in \mathcal{R}$, i.e.,

$$
\left\| \left( sI_n - \hat{A} - \hat{A}_1 e^{-st} \right)^{-1} \hat{D} \right\|_\infty \leq \gamma.
$$
Theorem 5.4.3 Consider the time-delay system (5.18), (5.19). Given a positive scalar \( \gamma \), there exists a \( \gamma \)-PI observer of the form (5.20), (5.21) for the system (5.18), (5.19), if the following algebraic Riccati equation

\[
\dot{A}_0^T P + PA_0 + 2P \left\{ \gamma^2 \bar{A}_1 \bar{A}_1^T + \bar{E} \bar{E}^T \right\} P - \frac{2}{\gamma} \tilde{C}_0 \left\{ I_p - \frac{1}{\gamma} \left( \gamma^2 \bar{C}_1 \bar{C}_1 + \bar{F} \bar{F}^T \right) \right\} \tilde{C}_0 + 2\gamma^2 I_n = 0
\]

(5.32)

has a symmetric positive definite solution \( P \), for some positive scaler \( \epsilon \). In this case the required observer gain is given by

\[
L = \frac{1}{\epsilon} P^{-1} \tilde{C}_0^T.
\]

(5.33)

Proof Suppose that Riccati equation (5.32) has a symmetric positive definite solution \( P \). Choosing \( L = \frac{1}{\epsilon} P^{-1} \tilde{C}_0^T \), Riccati equation (5.32) can be written as follows

\[
\dot{A}_0^T P + PA_0 + 2P \left\{ \gamma^2 \bar{A}_1 \bar{A}_1^T + \bar{E} \bar{E}^T \right\} P - 2\epsilon PL \left\{ I_p - \left\{ \gamma^2 \bar{C}_1 \bar{C}_1 + \bar{F} \bar{F}^T \right\} \right\} L^T P + \frac{2}{\gamma^2} I_n = 0
\]

\[
\dot{A}_0^T P + PA_0 + 2P \left\{ \gamma^2 \bar{A}_1 \bar{A}_1^T + \bar{E} \bar{E}^T \right\} P - 2\epsilon PL \bar{L}^T P + 2\gamma^2 PLC_1 \bar{C}_1^T L^T P
\]

\[
+ 2PL \bar{F} \bar{F}^T L^T P + \frac{2}{\gamma^2} I_n = 0
\]

\[
\dot{A}_0^T P + PA_0 + 2P \left\{ \gamma^2 \bar{A}_1 \bar{A}_1^T + \bar{E} \bar{E}^T \right\} P - PLC_0 - \bar{C}_0^T L^T P + 2\gamma^2 PLC_1 \bar{C}_1^T L^T P
\]

\[
+ 2PL \bar{L} \bar{F} \bar{F}^T L^T P + \frac{2}{\gamma^2} I_n = 0
\]

\[
(\dot{A}_0^T - C_0^T L^T) P + P(\bar{A}_0 - LC_0) + P \left\{ \gamma^2 \bar{A}_1 \bar{A}_1^T + \bar{E} \bar{E}^T \right\} P + \gamma^2 PLC_1 \bar{C}_1^T L^T P + PL \bar{F} \bar{F}^T L^T P
\]

\[
+ 2\gamma^2 I_n = -\gamma^2 P \bar{A}_1 \bar{A}_1^T P + \gamma^2 PLC_1 \bar{C}_1^T L^T P + PE \bar{E} \bar{E}^T P - PL \bar{F} \bar{F}^T L^T P
\]

Using the following inequality

\[
-XX^T - YY^T \leq XY^T + YX^T
\]

where \( X \) and \( Y \) are any two matrices with suitable dimensions we have,

\[
\dot{A}_0^T P + PA_0 + \gamma^2 P \bar{A}_1 \bar{A}_1^T P + PE \bar{E} \bar{E}^T P + \gamma^2 PLC_1 \bar{C}_1^T L^T P + PL \bar{F} \bar{F}^T L^T P
\]
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\[ + \frac{2}{\gamma^2} I_n \leq \gamma^2 P \hat{A}_1 \hat{C}_1^T L^T P + \gamma^2 P \hat{L} \hat{C}_1 \hat{A}_1^T P + P \hat{E} \hat{E}^T L^T P + P \hat{F} \hat{F}^T L^T P \]

\[ \hat{A}_0^T P + P \hat{A}_0 + \gamma^2 P \left( \hat{A}_1 \hat{A}_1^T - \hat{A}_1 \hat{C}_1^T L - \hat{L} \hat{C}_1 \hat{A}_1^T + \hat{L} \hat{C}_1 \hat{C}_1^T L^T \right) P \]

\[ + P \left( \hat{E} \hat{E}^T - \hat{L} \hat{F} \hat{E} L^T - \hat{E} \hat{E}^T L^T + \hat{L} \hat{F} \hat{F}^T L^T \right) P + \frac{2}{\gamma^2} I_n \leq 0. \]

\[ \hat{A}_0^T P + P \hat{A}_0 + \gamma^2 P (\hat{A}_1 - \hat{L} \hat{C}_1)(\hat{A}_1 - \hat{L} \hat{C}_1)^T P + \frac{2}{\gamma^2} I_n + P(\hat{E} - \hat{L} \hat{F})(\hat{E} - \hat{L} \hat{F})^T P \leq 0. \]

By using the bounded real lemma, the following statements are equivalent

1. There exists a symmetric positive definite matrix $\bar{P}$ such that

\[ A^T \bar{P} + \bar{P} A + \bar{P} B B^T \bar{P} + C^T C < 0 \]

2. The Riccati equation

\[ A^T P + P A + P B B^T P + C C^T = 0 \]

has a solution $P > 0$. Furthermore, if these statements hold then $P < \bar{P}$, leads to

\[ \hat{A}_0^T P + P \hat{A}_0 + \frac{1}{\gamma^2} P \hat{A}_1 \hat{A}_1^T P + 2 \gamma^2 I_n + P \hat{D} \hat{D}^T P < 0, \]

with $Q = \gamma^2 I_n$, and we obtain

\[ \hat{A}_0^T P + P \hat{A}_0 + P \hat{A}_1 Q^{-1} \hat{A}_1^T P + Q + P \hat{D} \hat{D}^T P + \gamma^2 I_n < 0, \quad (5.35) \]

for some $\bar{P} = \bar{P}^T > 0$ (note that $\bar{P} = P$ if (5.58) is a strict inequality).

Using Theorem 5.4.2, it follows that the system (5.25) is asymptotically stable independent of the delay and $\| (s I_n - \hat{A}_0 - \hat{A}_1 e^{s \tau})^{-1} \hat{D} \|_\infty \leq \gamma.$
5.5 Alternative Design

The purpose of this section is to design a constant gain $L$ for the PI observer (5.20), (5.21) with the following properties as in definition 5.4.1:

1. $\lim_{t \to \infty} \xi(t) \to 0$ for $d(t) = 0$,
2. $\| \xi \|_2 \leq \gamma \| d \|_2$ for $d(t) \neq 0$, $\forall t \leq 0$

and for any non zero $d(t) \in \mathcal{L}_2$

where $\mathcal{L}_2$ is the space of square integrable functions on $[0; \infty)$, $\| \cdot \|_2$ is the $\mathcal{L}_2$-norm, $\gamma > 0$ is the disturbance attenuation level to be minimized. Furthermore, it is of interest to specify the allowable size of delay.

In order to solve the above problem, equation (5.25) can be rewritten as follows:

$$\dot{\xi}(t) = \hat{A}_0 \xi(t) + \hat{A}_1 \xi(t - \tau) + \hat{D} \bar{d}(t)$$

(5.36)

where $\bar{d}(t) := [d^T \; \bar{\gamma} d^T(t)]^T$, $\bar{\gamma}$ is some positive constant, $\hat{D} := [\bar{E} - \frac{1}{\bar{\gamma}} L \bar{F}]$, with $\bar{E} = \begin{pmatrix} E \\ 0 \end{pmatrix}$, and $\bar{F} = F$.

Consider now the following output:

$$z(t) = P \xi(t)$$

(5.37)

where $P = P^T > 0$ is an $n \times n$ matrix to be determined.

After some algebraic manipulations, one can easily find that:

$$\text{if } \| z \|_2 \leq \| \bar{d} \|_2, \text{ then } \| \xi \|_2 \leq \| P^{-1} \|_2 \sqrt{1 + \bar{\gamma}^2} \| d \|_2$$

Let us now consider the following Lemma.

**Lemma 5.5.1** [20] Consider the system:

$$\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau) + Dd(t) \\
z(t) &= Cx(t)
\end{align*}$$

(5.38)
where $z(t) \in \mathbb{R}^p$ is the measured output vector, $A_0$, $A_1$, $D$ and $C = C^T > 0$ are matrices with appropriate dimensions. Given a scalar $\delta > 0$, system (5.38) is asymptotically stable and $\| z \|_2 \leq \delta \| d \|_2$ for any constant time-delay $0 \leq \tau < \bar{\tau}$ if there exist two symmetric positive definite matrices $P$ and $Q$ satisfying the following Algebraic Riccati Equation:

$$(A_0 + A_1^T P + P(A_0 + A_1) + C^T C + \frac{1}{\delta^2} P D D^T P + Q = 0)$$

where $\bar{\tau}$ is given by

$$\bar{\tau} = \left[ Q^{-\frac{1}{2}} \left[ \beta_1 P A_1 A_0 P^{-1} A_0^T A_1^T P + \beta_2 A_1 A_1 P^{-1} A_1^T A_1^T P + (\beta_1^{-1} + \beta_2^{-1}) P \right] Q^{-\frac{1}{2}} \right]^{-1}$$

for any positive real numbers $\beta_1$ and $\beta_2$.

With the aid of the above Lemma as applied to (5.36), (5.37), we can state our main result.

**Theorem 5.5.2** Consider the time-delay system (5.18), (5.19) and the PI observer of the form (5.20), (5.21) for the system (5.18), (5.19) If the following algebraic Riccati equation

$$(\bar{A} + \bar{A}_1)^T P + P(\bar{A}_0 + \bar{A}_1) + P\{\bar{E}\bar{E}^T + (1 + \epsilon)I_p\} P

-(\bar{C}_0 + \bar{C}_1)^T \{2I_p - \frac{1}{\bar{\gamma}^2} \bar{F}\bar{F}^T \}(\bar{C}_0 + \bar{C}_1) = 0$$

has a symmetric positive definite solution $P$, for some positive scalers $\bar{\gamma}$ and $\epsilon$, then for all $0 \leq \tau < \bar{\tau}$:

1. observer (5.20), (5.21) is asymptotically stable,

2. $\| \xi \|_2 \leq \gamma \| d \|_2$.
where

\[ \tau = \left\| (\epsilon PP)^{-\frac{1}{2}} [\beta_1 P \hat{A}_0 \hat{A}_0 P^{-1} \hat{A}_0^T \hat{A}_0^T P + \beta_2 \hat{A}_1 \hat{A}_1 P^{-1} \hat{A}_1^T \hat{A}_1^T P + (\beta_1^{-1} + \beta_2^{-1}) P] (\epsilon PP)^{-\frac{1}{2}} \right\|^{-1} \]

\[ \gamma = \| P^{-1} \| \sqrt{1 + \gamma^2}. \]

In this case the required observer gain is given by

\[ L = P^{-1} (\bar{C}_0 + \bar{C}_1)^T. \] (5.40)

**Proof**  
By applying the above lemma on the modified system (5.36), (5.37) for \( \delta = 1 \) and \( Q = \epsilon PP \), one can derive the modified Algebraic Riccati equation (5.39) as follows:

\[(\hat{A}_0 + \hat{A}_1)^T P + P(\hat{A}_0 + \hat{A}_1) + PP + P\bar{D}\bar{D}^T P + \epsilon PP = 0\]

\[(\hat{A}_0 + \hat{A}_1)^T P + P(\hat{A}_0 + \hat{A}_1) + (1 + \epsilon)PP + P(EE^T + \frac{1}{\gamma^2} LFF^T L^T) P = 0\]

\[(\hat{A}_0 + \hat{A}_1)^T P + P(\hat{A}_0 + \hat{A}_1) + P[EE^T + (1 + \epsilon)I_n] P + \frac{1}{\gamma^2} (\bar{C}_0 + \bar{C}_1)^T FF^T (\bar{C}_0 + \bar{C}_1) = 0\]

\[(\bar{A}_0 - L\bar{C}_0 + \bar{A}_1 - L\bar{C}_1)^T P + P(\bar{A}_0 - L\bar{C}_0 + \bar{A}_1 - L\bar{C}_1) + P[EE^T + (1 + \epsilon)I_n] P + \frac{1}{\gamma^2} (\bar{C}_0 + \bar{C}_1)^T FF^T (\bar{C}_0 + \bar{C}_1) = 0\]

\[(\bar{A}_0 + \bar{A}_1)^T P + P(\bar{A}_0 + \bar{A}_1) - (\bar{C}_0 + \bar{C}_1)^T L^T P - PL(\bar{C}_0 + \bar{C}_1) + P[EE^T + (1 + \epsilon)I_n] P + \frac{1}{\gamma^2} (\bar{C}_0 + \bar{C}_1)^T FF^T (\bar{C}_0 + \bar{C}_1) = 0\]

\[(\bar{A}_0 + \bar{A}_1)^T P + P(\bar{A}_0 + \bar{A}_1) - 2(\bar{C}_0 + \bar{C}_1)^T (\bar{C}_0 + \bar{C}_1) + P[EE^T + (1 + \epsilon)I_n] P + \frac{1}{\gamma^2} (\bar{C}_0 + \bar{C}_1)^T FF^T (\bar{C}_0 + \bar{C}_1) = 0\]

\[(\bar{A}_0 + \bar{A}_1)^T P + P(\bar{A}_0 + \bar{A}_1) + P[EE^T + (1 + \epsilon)I_n] P - (\bar{C}_0 + \bar{C}_1)^T (2I_p - \frac{1}{\gamma^2} FF^T) (\bar{C}_0 + \bar{C}_1) = 0. \]

\[\square\]
**Proposition 5.5.3** For a given time-delay system of the form (5.18), (5.19), if the pairs \([(C_0+C_1), (A_0+A_1)]\) is detectable, then there exists PI $H_\infty$ observer of the form (5.20), (5.21) with certain disturbance attenuation level.

It should be noted that the detectability condition given above is weaker than the strong and the hyper observability (see [49] for the definitions).

### 5.6 PI $H_\infty$ Observer for Multi-Delayed Systems

Consider the multi-delayed system described by

\[
\dot{x}(t) = A_0x(t) + \sum_{i=1}^{k} A_i x(t - i\tau) + B_0u(t) + \sum_{i=1}^{k} B_i u(t - i\tau) + Ed(t) \quad (5.41)
\]
\[
y(t) = C_0x(t) + \sum_{i=1}^{k} C_i x(t - i\tau) + Fd(t) \quad (5.42)
\]
\[
x(t) = \varphi(t), \quad t \in [-k\tau, 0] \quad (5.43)
\]

where $x(t) \in \mathcal{R}^n$ is the state variable, $u(t) \in \mathcal{R}^m$ is the control input, $y(t) \in \mathcal{R}^p$ is the output, $d(t) \in \mathcal{R}^q$ is the square-integrable disturbance (or fault) vector, $\varphi(t) \in \mathcal{C}[-k\tau, 0]$ is the continuous initial value function vector, $A, A_1, \ldots, A_k, B_0, B_1, \ldots, B_k, C_0, C_1, \ldots, C_k, E$ and $F$ are constant matrices with appropriate dimensions, and $0 \leq \tau < \infty$ is the constant known time-delay duration.

It should be noted that matrices $E$ and $F$ may include parameter uncertainties and/or modeling errors.

The PI observer for the system (5.41),(5.43) can be given by the following dynamical system.

\[
\dot{\hat{x}}(t) = A_0\hat{x}(t) + \sum_{i=1}^{k} A_i \hat{x}(t - i\tau) + B_0u(t) + \sum_{i=1}^{k} B_i u(t - i\tau) - B_0w(t)
\]
\[
- \sum_{i=1}^{k} B_i w(t - i\tau) + L_p [y(t) - C_0\hat{x}(t) - \sum_{i=1}^{k} C_i \hat{x}(t - i\tau)] \quad (5.44)
\]
\[
\dot{w}(t) = -L_I [y(t) - C_0\hat{x}(t) - \sum_{i=1}^{k} C_i \hat{x}(t - i\tau)] + w(t) + \sum_{i=1}^{k} w(t - i\tau) \quad (5.45)
\]
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where \( \hat{x}(t) \in \mathbb{R}^n \) is the state estimation, \( L_p \) is the proportional gain and \( L_i \) is the integral gain.

The estimated error, defined as

\[
e(t) = x(t) - \hat{x}(t)
\]

obeys the following error dynamics,

\[
\dot{e}(t) = (A_0 - L_p C_0)e(t) + \sum_{i=1}^{k} (A_i - L_p C_i)e(t - i\tau) + B_0 w(t) \\
+ \sum_{i=1}^{k} B_i w(t - i\tau) + (E - L_p F)d(t)
\]

and

\[
\dot{w}(t) = -L_i C_0 e(t) - L_i \sum_{i=1}^{k} C_i e(t - i\tau) + w(t) + \sum_{i=1}^{k} w(t - i\tau) - L_i F d(t)
\]

Then the PI observer can be represented as an augmented state observer in the form

\[
\dot{\xi}(t) = \hat{A}_0 \xi(t) + \sum_{i=1}^{k} \hat{A}_i \xi(t - i\tau) + \hat{D}d(t)
\]

where \( \xi(t) = \begin{pmatrix} e(t) \\ w(t) \end{pmatrix} \),

\[\hat{A}_i = \bar{A}_i - L \bar{C}_i, \quad \bar{A}_i = \begin{pmatrix} A_i & B_i \\ 0 & I \end{pmatrix}, \quad \bar{C}_i = \begin{pmatrix} C_i & 0 \end{pmatrix}, \quad i = 0, 1, \ldots, k,\]

\[\hat{D} = \bar{E} - L \bar{F}, \quad L = \begin{pmatrix} L_p \\ L_i \end{pmatrix}, \quad \bar{E} = \begin{pmatrix} E \\ 0 \end{pmatrix}, \quad \text{and} \quad \bar{F} = F.\]

**Definition 5.6.1** Given a positive scaler \( \gamma \), the system (5.44),(5.45) is said to be a \( \gamma \)-PI observer for the associated system (5.41),(5.43) if the solution of the differential equation (5.49) with \( d(t) \equiv 0 \) converges to zero asymptotically and, under zero initial condition, the \( H_\infty \) norm of the transfer function between the disturbance and the estimated error is bounded by \( \gamma \), that is:

1. \( \lim_{t \to \infty} \xi(t) \to 0 \) for \( d(t) = 0 \),
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2. \[ \left\| \left( sI_n - \hat{A}_0 - \sum_{i=1}^{k} \hat{A}_i e^{-si\tau} \right)^{-1} \hat{D} \right\|_{\infty} \leq \gamma, \]

under zero initial condition.

The purpose of this paper is to design a constant gain \( L \) such that the system (5.44), (5.45) is a \( \gamma \)-PI observer for the associated system (5.41), (5.43) for some positive scaler \( \gamma \).

The above development leads to the following theorems which are the generalization of [20] for the case of PI-observer.

**Theorem 5.6.2** Given a positive scalar \( \gamma \), if there exists two positive definite matrices \( Q \) and \( P \) such that

\[
\hat{A}_0^T P + P \hat{A}_0 + P \left( \sum_{i=1}^{k} \hat{A}_i Q^{-1} \hat{A}_i^T \right) P + rQ + \frac{1}{\gamma^2} I_n + P \hat{D} \hat{D}^T P < 0, \tag{5.50}
\]

then the system (5.49) is asymptotically stable for any value of the delay and the following inequality holds:

\[
\left\| \left( sI_n - \hat{A}_0 - \sum_{i=1}^{k} \hat{A}_i e^{-si\tau} \right)^{-1} \hat{D} \right\|_{\infty} \leq \gamma \tag{5.51}
\]

**Proof** Define the Lyapunov functional \( V \) as follows:

\[ V = \xi^T(t) P \xi(t) + \sum_{i=1}^{k} \int_{t-\tau}^{t} \xi^T(\theta) Q \xi(\theta) \, d\theta \]

then

\[
\frac{dV}{dt} = \begin{pmatrix}
\xi(t) \\
\xi(t - \tau) \\
\xi(t - 2\tau) \\
\vdots \\
\xi(t - k\tau)
\end{pmatrix}^T \begin{pmatrix}
Z_{11} & P \hat{A}_1 & P \hat{A}_2 & \cdots & P \hat{A}_k \\
\hat{A}_1^T P & -Q & 0 & \cdots & 0 \\
\hat{A}_2^T P & 0 & -Q & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\hat{A}_k^T P & 0 & 0 & \cdots & -Q
\end{pmatrix} \begin{pmatrix}
\xi(t) \\
\xi(t - \tau) \\
\xi(t - 2\tau) \\
\vdots \\
\xi(t - k\tau)
\end{pmatrix} \tag{5.52}
\]
where $Z_{11} = \hat{A}_0^T P + P \hat{A}_0 + kQ$.

Since $\frac{1}{\gamma^2} I_n + P \hat{D} \hat{D}^T P > 0$, inequality (5.50) implies that

$$\hat{A}_0^T P + P \hat{A}_0 + P[\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k] \text{diag}\{ Q^{-1} \}_k[\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k]^T P + kQ < 0$$  

where $\text{diag}\{ Q^{-1} \}$ denotes a $r \times r$ block-diagonal matrix with diagonal matrices $Q^{-1}$.

Using the Schur complement and (5.53) it follows that the matrix in (5.52) is negative definite. This implies that the asymptotic stability of the system (5.49) by the Krasovskii theorem [29].

Let us define a positive definite matrix $S$ as follows:

$$S := - \left( \hat{A}_0^T P + P \hat{A}_0 + P[\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k] \text{diag}\{ Q^{-1} \}_k[\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k]^T P + kQ + \frac{1}{\gamma^2} I_n + P \hat{D} \hat{D}^T P \right)$$

then we have

$$\hat{A}_0^T P + P \hat{A}_0 + P[\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k] \text{diag}\{ Q^{-1} \}_k[\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k]^T P + rQ + \frac{1}{\gamma^2} I_n + P \hat{D} \hat{D}^T P + S = 0$$

It follows that

$$\left( X^T (-j\omega) \right)^{-1} P + PX^{-1}(j\omega) - W(j\omega) - \frac{1}{\gamma^2} I_n - P \hat{D} \hat{D}^T P - S = 0$$

where

$$W(j\omega) = P[\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k] \text{diag}\{ Q^{-1} \}_k[\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k]^T P + kQ$$

$$- \sum_{i=1}^k \hat{A}_i^T P e^{j\omega r} - \sum_{i=1}^k \hat{A}_i P e^{-j\omega r}$$

$$X(j\omega) = \left( j\omega I_n - \hat{A}_0 - \sum_{i=1}^k \hat{A}_i e^{-j\omega r} \right)^{-1}$$

Note that $W(j\omega)$ is a non-negative definite for all $\omega \in \mathbb{R}$.

This implies that

$$\hat{D}X^T (-j\omega) P \hat{D} + \hat{D}^T PX(j\omega) \hat{D} - \hat{D}^T X^T (-j\omega) P \hat{D} \hat{D}^T PX(j\omega) \hat{D} - I_n$$

$$= -I_n + \hat{D}^T X^T (-j\omega) \left[ W(j\omega) + \frac{1}{\gamma^2} I_n + S \right] X(j\omega) \hat{D}$$
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It follows that

\[- \left( I_n - \hat{D}^T PX(-jw) \hat{D} \right)^T \left( I_n - \hat{D}^T PX(-jw) \hat{D} \right) = -I_n + \hat{D}^T X^T(-jw) \left[ W(jw) + S \right] X(jw) \hat{D} + \frac{1}{\gamma^2} \hat{D}^T X^T(jw) X(jw) \hat{D} \]  \hspace{1cm} (5.55)

The left hand side of the above equation is non-negative definite for all \( w \in \mathbb{R} \).

Define \( T(jw) := X(jw) \hat{D} \), then equation (5.55) implies

\[-I_n + \hat{D}^T X^T(-jw) \left[ W(jw) + S \right] X(jw) \hat{D} + \frac{1}{\gamma^2} \hat{D}^T X^T(-jw) T(jw) \leq 0 \]

and

\[T^T(-jw) T(jw) \leq \gamma^2 I_n - \gamma^2 \hat{D}^T X^T(-jw) \left[ W(jw) + S \right] X(jw) \hat{D}\]

for all \( w \in \mathbb{R} \).

Since \( W(jw) \) is non-negative definite matrix and \( S \) is positive definite matrix for all \( w \in \mathbb{R} \), then

\[\|T(jw)\|_\infty \leq \gamma^2 I_n\]

for all \( w \in \mathbb{R} \), i.e., \[\left\| \left( sI_n - \hat{A}_0 - \sum_{i=1}^k \hat{A}_i e^{-s\tau_i} \right)^{-1} \hat{D} \right\|_\infty \leq \gamma. \] \hspace{1cm} \( \square \)

Theorem 5.6.3 Consider the Multi-delay system (5.41), (5.43). Given a positive scalar \( \gamma \), there exists a \( \gamma \)-PI observer of the form (5.44), (5.45) for the system (5.41), (5.43) if the following algebraic Riccati equation

\[\hat{A}_0^T P + P \hat{A}_0 + 2P \left[ \gamma^2 \sum_{i=1}^k \hat{A}_i \hat{A}_i^T + \bar{E} \bar{E}^T \right] P - \frac{2}{\epsilon} \bar{C}_0^T I_p - \frac{1}{\epsilon} \left( \gamma^2 \sum_{i=1}^k \bar{C}_i \bar{C}_i + \bar{F} \bar{F}^T \right) \bar{C}_0 + \frac{1 + k}{\gamma^2} I_n = 0 \]  \hspace{1cm} (5.56)

has a symmetric positive definite solution \( P \), for some positive scaler \( \epsilon \).

In this case the required observer gain is given by

\[L = \frac{1}{\epsilon} P^{-1} \bar{C}_0^T \]  \hspace{1cm} (5.57)

Proof Suppose that Riccati equation (5.56) has a symmetric positive definite solution \( P \). Choosing \( L = \frac{1}{\epsilon} P^{-1} \bar{C}_0^T \), Riccati equation (5.56) can be written as follows
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\[ \dot{A}_0^T P + P \dot{A}_0 + 2P \left[ \gamma^2 \sum_{i=1}^{k} \dot{A}_i A_i^T + \dot{E} \dot{E}^T \right] P - 2 \epsilon P L \left[ \epsilon I_p - \left( \gamma^2 \sum_{i=1}^{k} \dot{C}_i C_i^T + \dot{F} \dot{F}^T \right) \right] L^T P \]

\[ + \frac{1+k}{\gamma^2} I_n = 0 \]

\[ \dot{A}_0^T P + P \dot{A}_0 + 2P \left[ \gamma^2 \sum_{i=1}^{k} \dot{A}_i A_i^T + \dot{E} \dot{E}^T \right] P - 2 \epsilon PLL^T P + 2 \gamma^2 PL \left[ \sum_{i=1}^{k} \dot{C}_i C_i^T \right] L^T P \]

\[ + 2PLL\dot{F}^T L^T P + \frac{1+k}{\gamma^2} I_n = 0 \]

\[ \dot{A}_0^T P + P \dot{A}_0 + 2P \left[ \gamma^2 \sum_{i=1}^{k} \dot{A}_i A_i^T + \dot{E} \dot{E}^T \right] P - PL \dot{C} - \dot{C}^T L^T P + 2 \gamma^2 PL \left[ \sum_{i=1}^{k} \dot{C}_i C_i^T \right] L^T P \]

\[ + 2PLL\dot{F}^T L^T P + \frac{1+k}{\gamma^2} I_n = 0 \]

\[ (\dot{A}_0^T - \dot{C}_0^T L^T )P + P(\dot{A}_0 - LC_0) + P \left[ \gamma^2 \sum_{i=1}^{k} \dot{A}_i A_i^T + \dot{E} \dot{E}^T \right] P + \frac{1+k}{\gamma^2} I_n + 2 \gamma^2 PL \left[ \sum_{i=1}^{k} \dot{C}_i C_i^T \right] L^T P \]

\[ + PLL\dot{F}^T L^T P = -\gamma^2 PL \left[ \sum_{i=1}^{k} \dot{C}_i C_i^T \right] L^T P - \gamma^2 P \left[ \sum_{i=1}^{k} \dot{A}_i A_i^T \right] P - PLL\dot{F}^T L^T P - P\dot{E} \dot{E}^T P \]

Using the following inequality

\[-XX^T - YY^T \leq XY^T + YX^T\]

where \( X \) and \( Y \) are any two matrices with suitable dimensions we have,

\[ \dot{A}_0^T P + P \dot{A}_0 + \gamma^2 P \left[ \sum_{i=1}^{k} \dot{A}_i A_i^T \right] P + PEE^T P + \gamma^2 PL \left[ \sum_{i=1}^{k} \dot{C}_i C_i^T \right] L^T P + PLL\dot{F}^T L^T P \]

\[ -\gamma^2 P \left[ \sum_{i=1}^{k} \dot{A}_i \dot{C}_i^T \right] L^T P - \gamma^2 PL \left[ \sum_{i=1}^{k} \dot{A}_i A_i^T \right] P - PLL\dot{F}^T L^T P - P\dot{E} \dot{E}^T P \cdot \frac{1+k}{\gamma^2} I_n \leq 0 \]

\[ \dot{A}_0^T P + P \dot{A}_0 + \gamma^2 P \left[ \sum_{i=1}^{k} \dot{A}_i A_i^T - \sum_{i=1}^{k} \dot{A}_i \dot{C}_i^T - L \sum_{i=1}^{k} \dot{C}_i \dot{A}_i^T + L \sum_{i=1}^{k} \dot{C}_i C_i^T L^T \right] P \]

\[ + P \left[ \dot{C}^T L^T - L \dot{F} \dot{E}^T L^T - L \dot{E} \dot{F}^T L^T + L \dot{F} \dot{F}^T L^T \right] P + \frac{1+k}{\gamma^2} I_n \leq 0. \]

\[ \dot{A}_0^T P + P \dot{A}_0 + \gamma^2 P \left[ \sum_{i=1}^{k} (\dot{A}_i - LC_i)(\dot{A}_i - LC_i)^T \right] P + P(E - LF)(E - LF)^T P + \frac{1+k}{\gamma^2} I_n \leq 0. \]

By using the bounded real lemma, the following statements are equivalent

1. There exists a symmetric positive definite matrix \( \hat{P} \) such that

\[ \dot{A}_0^T \hat{P} + \hat{P} \dot{A}_0 + \hat{P}BB^T \hat{P} + C^T C < 0 \]
2. The Riccati equation

\[ A^T P + PA + PBB^T P + CC^T = 0 \]

has a solution \( P > 0 \). Furthermore, if these statements hold then \( P < \bar{P} \),
leads to

\[ \hat{A}_0^T P + P\hat{A}_0 + \gamma^2 P \left( \sum_{i=1}^{k} \hat{A}_i \hat{A}_i^T \right) P + \frac{k}{\gamma^2} I_n + \frac{1}{\gamma^2} I_n + P\hat{D}\hat{D}^T P < 0, \]

with \( Q = \frac{1}{\gamma^2} I_n \), and we obtain

\[ \hat{A}_0^T P + P\hat{A}_0 + \gamma^2 P \left( \sum_{i=1}^{k} \hat{A}_i Q^{-1} \hat{A}_i^T \right) P + kQ + \frac{1}{\gamma^2} I_n + P\hat{D}\hat{D}^T P < 0 \]  \hspace{1cm} (5.59)

for some \( \bar{P} = \bar{P}^T > 0 \) (note that \( \bar{P} = P \) if (5.58) is a strict inequality).

Using Theorem 5.3.1, it follows that the system (5.49) is asymptotically stable
independent of the delay and \( \left\| \left( sI_n - \hat{A}_0 - \sum_{i=1}^{k} \hat{A}_ie^{-si}\right)^{-1}\hat{D} \right\|_\infty \leq \gamma. \) \hspace{1cm} \Box

5.7 PI Observer in Fault Detection and Its Connection to DO and UIO

The PI observer can be used in disturbance estimation and fault detection. For simplicity of our discussion let us first consider the system without delay described by

\[ \dot{x}(t) = Ax(t) + Bu(t) + Ed \]  \hspace{1cm} (5.60)
\[ y(t) = Cx(t) \]  \hspace{1cm} (5.61)

where \( d \) represents either the disturbance or fault.

A PI observer for (5.60), (5.61) can be written as

\[ \dot{x}(t) = (A - L_p C)\dot{x}(t) + L_p y + Bu(t) - Ew \]  \hspace{1cm} (5.62)
\[ \dot{w}(t) = -L_i (y - C\dot{x}(t)) \]  \hspace{1cm} (5.63)
If the estimation error is defined as $e = x - \hat{x}$, then the error dynamics of PI observer is given by

$$\dot{z} = \hat{A}z + \hat{B}d$$

where $z = \begin{pmatrix} e \\ w \end{pmatrix}$, $\hat{A} = \bar{A} - L\bar{C}$ with $\bar{A} = \begin{pmatrix} A & E \\ 0 & 0 \end{pmatrix}$, $\bar{C} = \begin{pmatrix} C & 0 \end{pmatrix}$, $L = \begin{pmatrix} L_P \\ L_I \end{pmatrix}$ and $\hat{B} = \begin{pmatrix} E \\ 0 \end{pmatrix}$.

The observability of the pair $(\bar{A}, \bar{C})$ guarantees the stability of $\hat{A}$ by proper selection of $L$. Under the assumption of constant disturbance and stability of $\hat{A}$, as $t \to \infty$, we have $z(\infty) = -\hat{A}^{-1}\hat{B} \hat{d}$, which leads to $e(\infty) = 0$ and $\hat{d} = L_I \int_0^\infty C e \, d\tau$.

The above analysis can be generalized to time delay systems and one can outline the design procedure for the fault detection purpose. This will be the subject of a future work.

The PI observer has a direct connection to the disturbance observer. If the disturbance model is available, one can combine its dynamic representation with the system model to construct an augmented system and then design a regular observer for it.

It is clear that under the assumption of constant disturbance $\dot{d} = 0$, the construction of disturbance observer matches with the PI observer as derived above. The only difference is that the dynamic of PI observer is pre-specified and can also be modified to include a fading term (PIOF). This allows to use PI observer and its extension for various purposes. In this sense the PI observer is more general than the disturbance observer.

To compare PI observer with UIO, it is sufficient to say that the PI observer is capable of simultaneously estimate the states and the disturbance. However, an UIO has the capability of making state estimation completely decoupled from the unknown input. Once the state estimates are obtained, the unknown input or fault can be estimated by using the estimated states through an equation which relates them. This will also be established in a separate paper for time delay systems. Since we outlined the
Consider the time delay system with the disturbance model
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + A_1 x(t-\tau) + Bu(t) + Ed(t), \quad 0 < \tau < \bar{\tau} \quad (5.64) \\
y(t) &= Cx(t) \quad (5.65) \\
\dot{d}(t) &= 0 \quad (5.66)
\end{align*}
\]
Combining the above dynamics leads to
\[
\dot{z}(t) = \bar{A}z(t) + \bar{A}_1 z(t-\tau) + \bar{B}u(t) \quad (5.67)
\]
where \( z = \begin{pmatrix} x \\ d \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} A & E \\ 0 & 0 \end{pmatrix}, \quad \bar{A}_1 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \) and \( \bar{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}. \)

**Assumption 1:** \((\bar{A}, \bar{C})\) observable pair

**Assumption 2:** \( \| H \| \leq \alpha, \) where \( H = Ed \) and \( \alpha \) is a positive constant.

Note that when \( d \) is a constant disturbance or fault, then the assumption 2 is automatically satisfied by a proper choice of \( \alpha. \)

Let
\[
\begin{align*}
\dot{\hat{z}}(t) &= \bar{A}\hat{z}(t) + \bar{A}_1 \hat{z}(t-\tau) + \bar{B}u(t) + L(\hat{y} - y) \quad (5.68) \\
\hat{y}(t) &= C\hat{z}(t) \quad (5.69)
\end{align*}
\]
where \( \bar{C} = (C \quad 0) \) and \( L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \) represent an observer for the system (5.67).

Then one can state the following result.

**Theorem 5.7.1** Let \( L \) be an \((n+q) \times p\) matrix such that \( \bar{A} - L\bar{C} \) is stable then (5.69) is a stable observer for (5.2), (5.66) with bounded estimation error \( e_z = z - \hat{z} \) given by
\[
\| e_z \| \leq 2 \frac{\lambda_{\max}(P)^{\frac{1}{2}} \alpha}{\lambda_{\min}(P)^{\frac{1}{2}} \beta}
\]
provided that
\[ \beta = \lambda_{\min}(Q) - 2 \| P \bar{A}_1 \| > 0 \]
where \( P \) is positive definite matrix solution of
\[ P(\bar{A} - L\bar{C}) + (\bar{A} - L\bar{C})^T = -Q, \quad \text{with } Q > 0. \]

**Proof** Using Lyapunov and Razumikhin theorems with some algebraic manipulations, one can establish the result.

### 5.8 Observer-Based Controller Design for Time-Delay Systems

In this section, the design of observer-based controller for time delay system is considered. We start by investigating the separation principal and then provide a design procedure for one delay case followed by a generalization for multiple delay case. Finally, we include a guaranteed cost control design for uncertain time delay systems.

#### 5.8.1 Separation Principle for Time-Delay Systems

Let us consider the delay system with single delay
\[
\begin{align*}
\dot{x}(t) &= A_0x(t) + A_1x(t-\tau) + Bu(t), \quad \tau \geq 0 \\
y(t) &= Cx(t) \\
x(t) &= \varphi(t), \quad -\tau \leq t \leq 0
\end{align*}
\]  
(5.70)

Then an observer-based controller for the above delay system can be written as
\[
\begin{align*}
\hat{x}(t) &= A_0\hat{x}(t) + A_1\hat{x}(t-\tau) + Bu(t) + L_0[y(t) - C\hat{x}(t)] \\
&\quad + L_1[y(t-\tau) - C\hat{x}(t-\tau)] \\
u(t) &= K_0\hat{x}(t) + K_1\hat{x}(t-\tau)
\end{align*}
\]  
(5.71)
Defining the state estimation error as

\[ e(t) = x(t) - \hat{x}(t) \]

The closed-loop system can be written by combining (5.70) and (5.71) or equivalently it can be represented through error dynamics by proper transformation as follows

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{e}(t)
\end{bmatrix} =
\begin{bmatrix}
A_0 + BK_0 & -BK_0 \\
0 & A_0 - L_0 C
\end{bmatrix}
\begin{bmatrix}
x(t) \\
e(t)
\end{bmatrix} +
\begin{bmatrix}
A_1 + BK_1 & -BK_1 \\
0 & A_1 - L_1 C
\end{bmatrix}
\begin{bmatrix}
x(t-\tau) \\
e(t-\tau)
\end{bmatrix}
\]

(5.72)

By defining

\[
z(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix}, \quad \bar{A}_0 = \begin{bmatrix} A_0 + BK_0 & -BK_0 \\ 0 & A_0 - L_0 C \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} A_1 + BK_1 & -BK_1 \\ 0 & A_1 - L_1 C \end{bmatrix}, \quad \bar{C} = [C \ 0]
\]

(5.73)

One can rewrite (5.70) as

\[
\begin{align*}
\dot{z}(t) &= \bar{A}_0 z(t) + \bar{A}_1 z(t-\tau) \\
y(t) &= \bar{C} z(t)
\end{align*}
\]

(5.74)

In view of the structures of \( \bar{A}_0 \) and \( \bar{A}_1 \), the characteristic equation of the system (5.74) can be written as

\[
\det[sI - \bar{A}_0 - \bar{A}_1 e^{-\tau s}] = 0 \\
\det[sI - (A_0 + BK_0) - e^{-\tau s}(A_1 + BK_1)] \cdot \det[sI - (A_0 - L_0 C) - e^{-\tau s}(A_1 - L_1 C)] = 0.
\]

This shows that the separation principle also holds for time-delay systems and one can design the controller and the observer gains independently.

Thus, it is possible to design an observer-based state feedback controller for the delay system (5.70). Here, we introduce a design method based on Lyapunov functional when \( K_1 = 0 \) and \( L_1 = 0 \). In this case (5.74) is defined by

\[
\bar{A}_0 = \begin{bmatrix} A_0 + BK_0 & -BK_0 \\ 0 & A_0 - L_0 C \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}
\]

(5.75)
Theorem 5.8.1 If there exist symmetric positive definite matrices $P$ and $Q$ such that
\[
\begin{bmatrix}
\bar{A}^T_0 P + P \bar{A}_0 + Q & -P \bar{A}_1 \\
-\bar{A}_1 & -Q
\end{bmatrix} < 0.
\] (5.76)
Then the observer-based state feedback controller (5.71) stabilizes the delay system (5.70) and the estimation error $e(t)$ converges asymptotically to zero.

Proof The proof follows by considering the Lyapunov functional
\[
V(t) = z^T P z + \int_{-\tau}^{0} z^T (\alpha) Q z (\alpha) d\alpha
\]
and taking the derivative of $V$ i.e. $\dot{V}$ along the trajectory of error dynamics which leads to $\dot{V} < 0$ if inequality (5.76) is satisfied. \qed

Now, let $P$ and $Q$ be defined as $P = \text{diag}(P_0, P_1)$ and $Q = \text{diag}(Q_0, Q_1)$. Then expanding (5.76) shows clearly that the inequality involves the unknown matrices $P_0$, $P_1$, $K_0$, and $L_0$ which are coupled. This is not an LMI feasible problem. In order to apply LMI technique, one should set $W = P_1 L_0$ and by fixing the feedback gain $K_0$ such that $A_0 + BK_0$ is Hurwitz stable, the following theorem can be stated.

Theorem 5.8.2 If there exist symmetric positive definite matrices $P_0$, $P_1$, $Q_0$, $Q_1$ and a matrix $W$ such that for a given stabilizable gain $K_0$ (i.e. $A_0 + BK_0$ is Hurwitz stable) the following inequality holds
\[
\begin{bmatrix}
J_0 & -P_0 BK_0 & -P_0 A_1 & 0 \\
\bullet & J_1 & 0 & -P_1 A_1 \\
\bullet & \bullet & -Q_0 & 0 \\
\bullet & \bullet & \bullet & -Q_1
\end{bmatrix} < 0,
\] (5.77)
where $J_0 = A_0^T P_0 + P_0 A_0 + K_0^T B_0^T P_0 + P_0 B K_0 + Q_0$ and $J_1 = A_0^T P_1 + P_1 A_0 - C^T W^T - W C + Q_1$. Then the observer-based state feedback controller (5.71) stabilizes the delay system (5.70) and the estimation error $e(t)$ converges asymptotically to zero. Furthermore, the observer gain $L_0$ is given by $L_0 = P_1^{-1} W$. 

5.8.2 Observer-Based Output Feedback Stabilization

Let the observer-based state feedback control system be represented by

\[
\begin{align*}
\dot{x}(t) &= (A + BK - LC)\dot{x}(t) + \sum_{i=1}^{k} A_i \dot{x}(t - \tau_i) + Ly(t) \\
u(t) &= K\dot{x}(t)
\end{align*}
\] (5.78)

for the multi-delay system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\] (5.79)

where we used \(A = A_0\) for simplicity of notation in the subsequent development.

Let \(z^T(t) = [x^T(t) \ \dot{x}^T(t)]\) and combine (5.78) and (5.79) in the augmented format described by

\[
\dot{z}(t) = \bar{A}z(t) + \sum_{i=1}^{k} \bar{A}_i z(t - \tau_i)
\]

where

\[
\bar{A} = \begin{bmatrix} A & BK \\ LC & A + BK - LC \end{bmatrix}, \quad \bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & A_i \end{bmatrix}.
\]

Then the following theorem provides a method to design an observer-based control system for multi-delay system (5.79).

**Theorem 5.8.3** If there exists symmetric positive definite matrices \(X, Y \in \mathbb{R}^{n \times n}\), \(U_i \in \mathbb{R}^{2n \times 2n}, 1 \leq i \leq k\) and \(\bar{A}, \bar{B} \in \mathbb{R}^{m \times n}\), such that the LMI

\[
\begin{bmatrix}
J_{11} + \sum_{i=1}^{k} U_i & M_i & \ldots & M_k \\
\vdots & -U_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ldots & -U_k
\end{bmatrix} < 0.
\] (5.80)

is feasible and the matrix equation

\[
\begin{align*}
\bar{A} &= BK\bar{Y} + XC^r L^T \\
\bar{B} &= BK\bar{Y} - LC\bar{Y}
\end{align*}
\] (5.81)
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has a set of solutions $K, L$, where

$$
J_{11} = \begin{bmatrix}
AX + XA^T & \tilde{A} \\
A_iX & \bullet \\
\end{bmatrix},
$$

$$
M_i = \begin{bmatrix}
A_iX & \bullet \\
\bullet & A_iY \\
\end{bmatrix}.
$$

(5.82)

Then the observer-based state feedback control system (5.78) with gain $K, L$ is asymptotically stable.

5.8.3 Guaranteed Cost Control of Uncertain Time-Delay Systems

Consider a class of linear uncertain time-delay systems described by the following state space equation:

$$
\dot{x}(t) = (A_0 + \Delta A_0)x(t) + (A_1 + \Delta A_1)x(t - \tau) + (B + \Delta B)u(t)
$$

$$
x(t) = \varphi(t), \quad t \in [-\tau, 0].
$$

(5.83)

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector, $A_0, A_1$ and $B$ are known constant real matrices of appropriate dimensions $\Delta A_0, \Delta A_1$ and $\Delta B$ are matrix-valued functions representing time-varying parameter uncertainties in the system model, $\tau > 0$ is the delay constant, and $\varphi(t)$ is a given continuous vector valued function.

The parameter uncertainties considered here are assumed to be norm bounded and of the form

$$
[\Delta A_0 \quad \Delta A_1 \quad \Delta B] = D\Delta(t) [E_0 \quad E_1 \quad E_2]
$$

(5.84)

where $D, E_0, E_1$ and $E_2$ are known constant real matrices of appropriate dimensions, which represent the structure of uncertainties, and $\Delta(t)$ is an unknown matrix function with Lebesgue measurable elements and satisfies

$$
\Delta^T(t)\Delta(t) \leq I
$$

(5.85)
in which $I$ denotes the identity matrix of appropriate dimension.

Associate with system (5.83) is the cost function

$$J = \int_{0}^{\infty} [x^T Q x(t) + u^T R u(t)] \, dt,$$  \hspace{1cm} (5.86)

where $Q$ and $R$ are given positive-definite symmetric matrices.

**Definition 5.8.4** Consider the uncertain system (5.83), if there exist a control law $u^*(t)$ and a positive $J^*$ such that for all admissible uncertainties, the closed-loop system is stable and the closed-loop value of the cost function (5.86) satisfies $J \leq J^*$, then $J^*$ is said to be a guaranteed cost and $u^*(t)$ is said to be a guaranteed cost control law for the uncertain system (5.83).

The objective of this section is to develop a procedure to design memoryless state feedback guaranteed cost control law $u(t) = K x(t)$ for the uncertain time-delay system (5.83).

We first present a sufficient condition for the existence of memoryless state feedback guaranteed cost control laws for the uncertain system (5.83)

**Theorem 5.8.5** $u = K x(t)$ is a guaranteed cost controller if there exist symmetric positive-definite matrices $P, S \in \mathbb{R}^{n \times n}$ such that for all uncertain matrices $\Delta$ satisfying (5.85),

$$\begin{bmatrix} \Sigma & P(A_1 + D \Delta E_1) \\ \bullet & -S \end{bmatrix} < 0$$  \hspace{1cm} (5.87)

where

$$\Sigma = Q + K^T R K + S + P [A_0 + B K + D \Delta (E_0 + E_2 K)]$$

$$+ [A_0 + B K + D \Delta (E_0 + E_2 K)]^T P$$  \hspace{1cm} (5.88)

**Proof** Let $u(t) = K x(t)$ in (5.83), the result closed-loop system is

$$\dot{x}(t) = [A_0 + B K + D \Delta (E_0 + E_2 K)] x(t) + (A_1 + D \Delta E_1) x(t - \tau).$$  \hspace{1cm} (5.89)
Suppose now there exist symmetric matrix $P > 0$, $S > 0$ such that the matrix inequality (5.87) holds for all admissible uncertainties. Define

$$V(x(t)) = x^T P x(t) + \int_{t-\tau}^{t} x^T(\tau) S x(\tau) \, d\tau.$$  \hspace{1cm} (5.90)$$

Then the derivative of $V(\cdot)$ along the trajectory of the closed-loop system (5.89) is given by

$$\dot{V}(x(t)) = 2 x^T [A_0 + B K + D \Delta (E_0 + E_2 K)] x(t) + 2 x^T(t) P (A_1 + D \Delta E_1) x(t - \tau)$$

$$+ x^T(t) S x(t) - x^T(t - \tau) S x(t - \tau)$$

$$= \begin{bmatrix} x(t) \\
 x(t - \tau) \end{bmatrix}^T \begin{bmatrix} \Sigma - Q - K^T R K & P (A_1 + D \Delta E_1) \\
 & -S \end{bmatrix} \begin{bmatrix} x(t) \\
 x(t - \tau) \end{bmatrix}$$

where $\Sigma$ is given in (5.88). (5.87) implies that

$$\dot{V}(x(t)) < x^T(t)(-Q - K^T R K) x(t) < 0.$$  \hspace{1cm} (5.91)$$

Therefore, the closed-loop system (5.89) is asymptotically stable. Furthermore, by integrating both sides of the inequality (5.91) from 0 to $T$ and using the initial condition, we obtain

$$- \int_0^T x^T(t)(Q + K^T R K) x(t) \, dt > x^T(T) P x(T) - x^T(0) P x(0) + \int_{T-\tau}^{T} x^T(\tau) S x(\tau) \, d\tau$$

$$- \int_{-\tau}^{0} x^T(\tau) S x(\tau) \, d\tau.$$  

As the closed-loop system (5.89) is asymptotically stable, when $T \to \infty$,

$$x^T(T) P x(T) \to 0, \quad \int_{T-\tau}^{T} x^T(\tau) S x(\tau) \, d\tau \to 0.$$  

Hence, we get

$$\int_0^\infty x^T(t)(Q + K^T R K) x(t) \, dt \leq \varphi^T(0) P \varphi(t) + \int_{-\tau}^{0} \varphi^T(\tau) S \varphi(\tau) \, d\tau.$$  

It follows from the definition that the result of the theorem is true. This complete the proof. \qed
The following theorem shows that the above sufficient condition for the existence of guaranteed cost controllers is equivalent to the solvability of a system of LMIs.

**Theorem 5.8.6** For system (5.83), there exist symmetric positive-definite matrices $P, S$ such that the matrix inequality (5.87) holds if and only if there exist a scalar $\epsilon > 0$, a matrix $W \in \mathcal{R}^{m \times n}$ and symmetric positive-definite matrices $X, Y \in \mathcal{R}^{n \times n}$ such that the following LMI is satisfied.

$$
\begin{bmatrix}
\bar{A} & A_1Y & (E_0X + E_2W)^T & X & W^T & X \\
\bullet & -Y & YE_1^T & 0 & 0 & 0 \\
\bullet & \bullet & -\epsilon I & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & -Q^{-1} & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & -R^{-1} & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & -Y
\end{bmatrix} < 0.
$$

(5.92)

where $\bar{A} = A_0X + BW + (A_0X + BW)^T + \epsilon DD^T$. Furthermore, if (5.92) has a feasible solution $\epsilon, W, X > 0, Y > 0$, then the state feedback control law

$$
u^*(t) = WX^{-1}x(t)
$$

(5.93)

is guaranteed cost control law and

$$
J^* = \varphi^T(0)X^{-1}\varphi(0) + \int_{-\tau}^{0} \varphi^T(\tau)Y^{-1}\varphi(\tau) \, d\tau
$$

(5.94)

is a guaranteed cost for the uncertain system.

**Proof** Define

$$
Z = \begin{bmatrix}
Q + K^T RK + S + P(A_0 + BK) + (A_0 + BK)^T P & PA_1 \\
\bullet & -S
\end{bmatrix},
$$

then (5.87) is equivalent to

$$
Z + \begin{bmatrix}
PD \\
0
\end{bmatrix} \Delta \begin{bmatrix}
E_0 + E_2K & E_1 \\
E_0 + E_2K & E_1
\end{bmatrix}^T \Delta^T \begin{bmatrix}
PD \\
0
\end{bmatrix}^T < 0
$$
by applying Lemma 2.4 in [119], the above matrix inequality holds for all \( \Delta \) satisfying 
\[ \Delta \Delta^T \leq I \] 
if and only if there exists a constant \( \epsilon > 0 \) such that

\[
Z + \epsilon \begin{bmatrix} PD & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} PD \end{bmatrix}^T + \epsilon^{-1} \begin{bmatrix} E_0 + E_2K & E_1 \end{bmatrix}^T \begin{bmatrix} E_0 + E_2K & E_1 \end{bmatrix} < 0
\]

that is

\[
\begin{bmatrix}
\Omega & PA_1 + \epsilon^{-1}(E_0 + E_2K)^t E_1 \\
-\epsilon \begin{bmatrix} E_0 + E_2K \end{bmatrix}^t E_1 & -S + \epsilon^{-1}E_1^t E_1
\end{bmatrix} < 0.
\] (5.95)

where

\[
\Omega = Q + K^T RK + S + P(A_0 + BK) + (A_0 + BK)^t P + \epsilon PDD^T P + \epsilon^{-1}(E_0 + E_2K)^t (E_0 + E_2K).
\]

It follows from the Schur complement that (5.95) is equivalent to

\[
\begin{bmatrix}
\hat{A} & PA_1 & (E_0 + E_2K)^t & I & K^t & I \\
-\epsilon \begin{bmatrix} E_0 + E_2K \end{bmatrix}^t & E_1^t & 0 & 0 & 0 \\
-\epsilon I & -Q^{-1} & 0 & 0 & 0 \\
-\epsilon \begin{bmatrix} E_0 + E_2K \end{bmatrix}^t & -R^{-1} & 0 & 0 & 0 \\
\end{bmatrix} < 0.
\] (5.96)

where \( \hat{A} = P(A_0 + BK) + (A_0 + BK)^t P + \epsilon PDD^T P \). Pre- and post-multiplying both side of (5.96) by

\[
\begin{bmatrix}
P^{-1} & 0 & 0 & 0 & 0 \\
S^{-1} & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 \\
\end{bmatrix} < 0.
\] (5.97)

and denoting \( X = P^{-1} \), \( W = KP^{-1} \), \( Y = S^{-1} \) yield the matrix inequality (5.92). It can be further concluded from the proof of Theorem 5.8.5 that the remainder of this theorem is true. This proves the theorem. \(\square\)
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Since (5.92) is a linear matrix inequality in $\epsilon$, $W$, $X$, $Y$, in (5.92) defines a convex solution set of $(\epsilon, W, X, Y)$, and therefore various efficient convex optimization algorithms can be used to test whether the LMI is solvable and to generate particular solutions. Moreover, its solutions parametrize the set of guaranteed cost controller. This parameterized representation can be exploited to design the guaranteed cost controllers with some additional requirements. In particular, the optimal guaranteed cost control law which minimizes the value of the guaranteed cost for the closed-loop uncertain system can be determined by solving a certain optimization problem. This is the following theorem:

**Theorem 5.8.7** Consider system (5.83) with cost function (5.86), if the following optimization problem

$$
\min_{\epsilon, \alpha, W, Y, M} \alpha + \text{tr}(M)
$$

subject to: (i) (5.92) being satisfied

(ii) \[ \begin{bmatrix}
-\alpha & \phi^T(0) \\
0 & -X
\end{bmatrix} < 0 \]

(iii) \[ \begin{bmatrix}
-\epsilon & N^T(0) \\
-M & -Y
\end{bmatrix} < 0 \]

has a solution $\epsilon$, $\alpha$, $W$, $Y$, $M$ where $\text{tr}(\cdot)$ denotes the trace of the matrix $(\cdot)$. Then the control law of the form (5.93) is an optimal memoryless state feedback guaranteed cost control law which ensures the minimization of the guaranteed cost (5.94) for the uncertain time-delay system (5.83), where $\int_{-\tau}^{0} \psi(\tau) \varphi^T(\tau) = NN^T$.

**Proof** By Theorem 5.8.6, the control law (5.93) constructed in terms of any feasible solution $\epsilon$, $\alpha$, $W$, $X$, $Y$, $M$ is a guaranteed cost controller system (5.83).

It follows from Schur complement that (ii) in (5.98) is equivalent to $\phi^T(0)X^{-1}\phi(0) < \alpha$, (iii) in (5.98) is equivalent to $N^T(0)Y^{-1}N(0) < M$. On the other hand,

$$
\int_{-\tau}^{0} \varphi^T(\tau)Y^{-1}\varphi(\tau) \, d\tau = \int_{-\tau}^{0} \text{tr}(\varphi^T(\tau)Y^{-1}\varphi(\tau)) \, d\tau
$$

$$
= \text{tr}(NN^T Y^{-1})
$$
\[ = \text{tr}(N^T Y^{-1} N) < \text{tr}(M). \]

So it follows from (5.93) that

\[ J^* < \alpha + \text{tr}(M). \]

Thus, the optimization of \( \alpha + \text{tr}(M) \) implies the minimization of the guaranteed cost for the uncertain system (5.83). The optimality of the solution of the optimization problem (5.98) follows from the convexity of the objective function and of the constraints. This completes the proof. \( \square \)
5.9 Illustrative Examples

Example 5.9.1 Consider the two time-delay system

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau) + B_0 u(t) + B_1 u(t - \tau) \\
y(t) &= C_0 x(t) + C_1 x(t - \tau)
\end{align*}
\]

where,

\[
A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 10 \end{bmatrix}, \quad C_0 = C_1 = [1 \ 1],
\]

then

\[
\bar{A}_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 10 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad C_0 = C_1 = [1 \ 1 \ 0].
\]

For \( \alpha = 1.4 \), the algebraic Riccati equation (5.9) has the solution

\[
P = \begin{bmatrix} 1.5369 & 0.1556 & -1.8455 \\ 0.1556 & 0.3064 & -1.9143 \\ -1.8455 & -1.9143 & 21.0651 \end{bmatrix},
\]

and the corresponding gain is given by

\[
L = \begin{bmatrix} 0.7944 \\ 7.6233 \\ 0.7624 \end{bmatrix},
\]

Also, the matrix \( \hat{A} \) is obtained as

\[
\hat{A} = \begin{bmatrix} -0.7944 & 0.2056 & 0.0000 \\ -7.6233 & -7.6233 & 10.0000 \\ -0.7624 & -0.7624 & 1.0000 \end{bmatrix},
\]

which is stable and its eigenvalues are \(-6.3406, -1.0770, \text{and} -0.0001\).
Example 5.9.2 Consider the time-delay system

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau) + B_0 u(t) + B_1 u(t - \tau) + Ed(t) \\
y(t) &= C_0 x(t) + C_1 x(t - \tau) + Fd(t)
\end{align*}
\]

where,

\[
A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

\[
E = [1 \ 1], \quad F = [1], \quad C_0 = [0 \ 1], \quad C_1 = [1 \ 1].
\]

For \( \gamma = 1 \), and \( \epsilon = 0.1 \) the algebraic Riccati equation (5.9) has the solution

\[
P = \begin{bmatrix}
0.5582 & -0.3571 & -0.0751 \\
-0.3571 & 9.9186 & -0.1068 \\
-0.0751 & -0.1068 & 0.6724
\end{bmatrix},
\]

and the corresponding gain is given by

\[
L = \begin{bmatrix} 0.6953 \\ 1.0358 \\ 0.2422 \end{bmatrix}.
\]
Figure 5.1: Estimated errors for $\tau = 1$ sec.

Figure 5.2: $\| T(jw) \|_\infty$ versus $w$. 
Example 5.9.3 Consider the two time-delay system

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + A_2 x(t-2\tau) + B_0 u(t) + B_1 u(t-\tau) + B_2 u(t-2\tau) + E d(t) \\
y(t) &= C_0 x(t) + C_1 x(t-\tau) + C_2 x(t-2\tau) + F d(t)
\end{align*}
\]

where,

\[
A_0 = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},
\]

\[
B_0 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}
\]

\[
C_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \end{bmatrix}, \text{ and } F = \begin{bmatrix} 1 \end{bmatrix}.
\]

For \(\gamma = 0.1\), and \(\epsilon = 0.1\) the algebraic Riccati equation (5.9) has the solution

\[
\]

and the corresponding gain is given by

\[
L = \begin{bmatrix} 0.0965 & 0 \\ 0.0812 & 0 \\ 0.0133 & 0 \\ 0.0265 & 0 \end{bmatrix}.
\]
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Figure 5.3: Estimated errors for $\tau = 1$ sec.

Figure 5.4: $\|T(jw)\|_{\infty}$ versus $w$. 
Example 5.9.4 The purpose of this example is to use the alternative design based on Theorem (5.5.2) which specifies the upper bound of delay $\bar{\tau}$.

Consider the time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + B_0 u(t) + B_1 u(t - \tau) + Ed(t)$$

$$y(t) = C_0 x(t) + C_1 x(t - \tau) + Fd(t)$$

with time delay $\tau = 0.2$, where

$$A_0 = \begin{pmatrix} -10 & 10 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$C_0 = (0 \ 10), C_1 = (1 \ 1), \quad E = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$  

Note that this time-delay system is weakly observable but is not strongly observable as $(C_0, A_0)$ is not observable (see [49] for the definitions). For this example, note that the pairs $[(C_0 + C_1), (A_0 + A_1)]$ and $[(\bar{C}_0 + \bar{C}_1), (\bar{A}_0 + \bar{A}_1)]$ are detectable.

For $\gamma = 2.1$, and $\epsilon = 1$, the algebraic Riccati equation (5.39) has the solution

$$P = \begin{pmatrix} 6.8875 & -3.3324 & -1.1597 \\ -3.3324 & 9.6519 & 0.6336 \\ -1.1597 & 0.6336 & 2.1978 \end{pmatrix},$$

and the corresponding gain is given by (5.40) as

$$L = \begin{pmatrix} 0.8416 \\ 1.4281 \\ 0.0324 \end{pmatrix}.$$
where $\bar{\tau} = 0.2589$ sec (for $\beta_1 = 0.18$, $\beta_2 = 1$) and $\gamma_{\text{min}} = 1.21$ is obtained. The estimated errors are shown in Figure 5.5 for $\tau = 0.2$ sec. The disturbance applied at $t = 4$ sec. is of magnitude 0.5, see Figure 5.7.

In Figure 5.6, the maximum singular value of $T(jw)$ is traced as a function of the frequency. It shows that $\|T(jw)\|_\infty \leq \gamma_{\text{min}}$, $\forall w \in \mathcal{R}$. Also the maximum delay $\bar{\tau}$ obtained is conservative, since additional simulation results showed that the observer retains its properties for some values of $\tau$ larger than $\bar{\tau}$. 

Figure 5.5: Estimated errors for $\tau = 0.2$ sec.

Figure 5.6: Maximum SV of $T(jw)$, $\gamma_{\text{min}} = 1.21$. 
Example 5.9.5 Consider the system (5.70), where

\[
A_0 = \begin{bmatrix} 12.5 & -0.5 \\ 0 & -8 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -4 & -0.2 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1.5 & 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -2 & -0.5 \end{bmatrix}, \quad \tau = 2.
\]

A controller gain \( K_0 = \begin{bmatrix} -8 & -1 \\ 6 & -5 \end{bmatrix} \) makes \( A_0 + BK_0 \) Hurwitz stable.

Applying LMI method to (5.77) yields

\[
P_0 = \begin{bmatrix} 1.03 & 0.86 \\ 0.861 & 0.97 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 2.74 & -0.08 \\ -0.08 & 4.96 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 5.08 & 4.52 \\ 4.52 & 5.4 \end{bmatrix},
\]

\[
Q_1 = \begin{bmatrix} 19.5 & 0.21 \\ 0.21 & 18.19 \end{bmatrix}, \quad W = \begin{bmatrix} -1.1366 & 2.1179 \\ 2.1179 & 0.2879 \end{bmatrix}
\]

and the observer gain is obtained by

\[
L_0 = P_1^{-1}W = \begin{bmatrix} -24.88 & -28.71 \\ -2.31 & 10.53 \end{bmatrix}
\]
Example 5.9.6 Consider the following time-delay system

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0.2 & 0.1 \end{bmatrix} x(t - \tau) + \begin{bmatrix} 0 & 0.1 \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t)
\]

Solving the LMI (5.80) with respect to \( X, Y, U \) = \( U \) and \( \tilde{A}, \tilde{B} \); we get

\[
X = \begin{bmatrix} 151.9746 & -53.0935 \\ -53.0935 & 59.9963 \end{bmatrix},
Y = \begin{bmatrix} -78.1362 & 51.9118 \\ 51.9118 & 7.4424 \end{bmatrix},
\]

\[
U = \begin{bmatrix} 53.0935 & -7.1043 & 0 & 0 \\ -7.1043 & 66.8991 & 0 & 0 \\ 0 & 0 & 69.664 & 0 \\ 0 & 0 & 0 & 69.664 \end{bmatrix},
\]

\[
\tilde{A} = 0, \quad \tilde{B} = \begin{bmatrix} 173.4875 & -18.245 \\ -18.245 & -63.929 \end{bmatrix}.
\]

Then by substituting the above computed matrix parameters into the coupled matrix equation (5.81), one can obtain the gain matrices \( K \) and \( L \) as follows

\[
K = \begin{bmatrix} 0.3404 & -0.8427 \\ -30.4744 & 13.9397 \end{bmatrix},
L = \begin{bmatrix} -0.9197 & -2.3643 \\ 0.9709 & 1.3119 \end{bmatrix}.
\]
Example 5.9.7 Consider the following uncertain time delay system

\[
\dot{x}(t) = (A_0 + r A_{00}) x(t) + (A_1 + p A_{11}) x(t-1) + (B + q B_1) u(t) \tag{5.99}
\]

and performance index (5.86), where

\[
A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \quad A_{00} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

\[|r| \leq 0.1, \quad |p| \leq 0.1, \quad |q| \leq 0.1, \quad Q = I, \quad R = I, \quad x_1(t) = e^{t+1}, \quad x_2(t) = 0, \quad t \in [-1,0].\]

Define

\[
D = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Delta = 10 \times \text{diag}\{r, p, q\}
\]

the system (5.99) can be rewritten into

\[
\dot{x}(t) = (A_0 + D \Delta E_0) x(t) + (A_1 + D \Delta E_1) x(t-1) + (B + D \Delta E_2) u(t)
\]

By applying Theorem 5.8.7 and solving the corresponding optimization problem, we obtain the optimal memoryless state feedback guaranteed cost control law

\[
u^*(t) = -\begin{bmatrix} 1.0046 & 3.3408 \end{bmatrix} x(t)
\]

the guaranteed cost of the uncertain closed-loop system is \(J^* = 45.4437\).
Chapter 6

Positive Delay Systems

Positive systems have found application in many fields (see [43], [18] and the references therein). This chapter gives a summary of most important results on positive systems and provide its generalization to delay systems.

6.1 Continuous-Time Positive (Metzlerian) Systems

Definition 6.1.1 A matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is called a Metzler matrix if all of its diagonal entries are negative and all of its off-diagonal elements are nonnegative, i.e. $a_{ii} < 0$, $a_{ij} \geq 0$, $\forall i \neq j$, $i, j = 1, 2, \ldots, n$.

Metzler matrices are closely related to the so-called $M$-matrices. An $M$-matrix has positive diagonal entries and negative off-diagonal entries. Thus, if $A$ is a Metzler matrix, then $-A$ is an $M$-matrix.

Definition 6.1.2 A real square matrix $M$ is called a nonsingular $M$-matrix if the following two conditions are satisfied

1. $m_{ii} > 0$ for all $i$ and $m_{ij} \leq 0$ for are $i \neq j$.

2. $M^{-1} \geq 0$ (elementwise nonnegative matrix)
Properties of Nonsingular $M$-matrix

(P1) Let $Q \in \mathbb{R}^{n \times n}$ with $Q \geq 0$. The matrix $(\lambda I - Q)$ is a nonsingular $M$-matrix if and only if $\rho(Q) < \lambda$ where $\rho(\cdot)$ denotes the spectral radius of the matrix $(\cdot)$.

(P2) The matrix $M$ with $m_{ii} > 0$ and $m_{ij} \leq 0$, $i \neq j$ is a nonsingular $M$-matrix if and only if all the leading principal minors of $M$ are positive.

(P3) The characteristic polynomial of an $M$-matrix is anti Hurwitz, i.e., all the eigenvalues have positive real parts.

(P4) For any two $M$-matrices $Y \geq X$, $\rho(Y) \geq \rho(X)$.

Note that $M$ is a nonsingular $M$-matrix if there exists a matrix $N \in \mathbb{R}^{n \times n}_+$ with maximal eigenvalue $\lambda$ such that $M = cI - N$ where $c \geq \lambda$.

From the above properties of nonsingular $M$-matrices one can deduce that stable Metzler matrices admit similar properties, i.e., if $A$ is a stable Metzler matrix then $-A$ is nonsingular $M$-matrix. Also note that every Metzler matrix has a real eigenvalue $\alpha = \max \text{ Re } \lambda_i$ and $\text{ Re } \lambda_i < 0$ for $i = 1, 2, \ldots, n$ if $\alpha < 0$, where $\lambda_i$, $i = 1, 2, \ldots, n$ are the eigenvalues of $A$.

The underlying theory of such matrices stems from the theory of nonnegative (positive) matrices based on Frobenius-Perron theorem [18]. Due to the connection of these matrices with the corresponding models of continuous and discrete time systems, one can similarly define continuous-time positive (Metzlerian) systems and discrete-time positive (nonnegative) systems. Here we only concentrate on continuous-time case.

Consider the linear continuous time system described by

\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \quad (6.1) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p \) are state, input and output vectors with system matrices \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n} \) and \( D \in \mathbb{R}^{p \times m} \).

Let \( \mathbb{R}^{l \times q}_+ \) be the set of \( l \times q \) matrices with nonnegative entries and \( \mathbb{R}^{l \times 1}_+ = \mathbb{R}^{l}_+ \) represent the corresponding notation for the nonnegative vectors.

**Definition 6.1.3** The system (6.1) is called externally positive if and only if for every input \( u \in \mathbb{R}^m_+ \) and \( x_0 = 0 \), the output \( y \in \mathbb{R}^p_+ \) for all \( t \geq 0 \).

**Theorem 6.1.4** The system (6.1) is externally positive if and only if its impulse response matrix is nonnegative, i.e., \( g(t) \in \mathbb{R}^{p \times m}_+ \) for all \( t \geq 0 \).

The sufficiency proof of the above theorem follows immediately from the fact that \( u \in \mathbb{R}^m_+ \) and \( g \in \mathbb{R}^{p \times m}_+ \) implies that 
\[
y(t) = \int_0^t g(t-\tau)u(\tau) \, d\tau \in \mathbb{R}^p_+ \quad \text{for all} \quad t \geq 0.
\]
The necessity is obvious from the definition 6.1.3. Note that for single input single output system the condition reduces to \( g(t) \in \mathbb{R}^+_1 \). In this case if the coefficient of the corresponding transfer function
\[
G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0} \quad (6.2)
\]
satisfy \( a_i \leq 0 \) and \( b_i \geq 0 \) for \( i = 0, 1, \ldots, n-1 \), then \( g(t) \in \mathbb{R}^+_1 \). This can be established by expanding \( G(s) \) as
\[
G(s) = g_1s^{-1} + g_2s^{-2} + \ldots \quad (6.3)
\]
and equating (6.2) with (6.3). Comparing the coefficients of like powers lead to
\[
g_1 = b_{n-1}, \quad g_2 = b_{n-2} - a_{n-1}g_1, \quad \ldots, \quad g_k = b_{n-k} - a_{n-1}g_{k-1} - a_{n-2}g_{k-2} - \ldots - a_{n-k+1}g_1
\]
and \( g_k \in \mathbb{R}^+_1 \) for \( k = 1, 2, \ldots \) if \( a_i \leq 0 \) and \( b_i \geq 0 \).

**Corollary 6.1.5** Let the single input single output system with transfer function (6.2) satisfy \( a_i > 0, \quad b_i > 0 \). Then the system is externally positive if and only if \( g_k > 0 \), for \( k = 1, 2, \ldots \).
Definition 6.1.6 The system \((6.1)\) is called internally positive if and only if for any \(x_0 \in \mathcal{R}_+^n\) and for every \(u \in \mathcal{R}_+^m\) we have \(x \in \mathcal{R}_+^n\) and \(y \in \mathcal{R}_+^p\) for all \(t \geq 0\).

Theorem 6.1.7 The continuous-time system \((6.1)\) is internally positive if and only if the system matrix \(A\) is a Metzler matrix and \(B \in \mathcal{R}_+^{n \times m}, C \in \mathcal{R}_+^{p \times n}, D \in \mathcal{R}_+^{p \times m}\).

Proof The sufficiency is established based on the result that \(e^{At} \in \mathcal{R}_+^{n \times n}\) if and only if \(A\) is a Metzler matrix.

Note that
\[
x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau
\]
Therefore, if \(A\) is Metzler matrix and \(B \in \mathcal{R}_+^{n \times m}, x_0 \in \mathcal{R}_+^n, u(t) \in \mathcal{R}_+^m\) for \(t \geq 0\), then \(x(t) \in \mathcal{R}_+^n\) for \(t \geq 0\) and \(y(t) \in \mathcal{R}_+^p\) since \(C \in \mathcal{R}_+^{p \times n}\) and \(D \in \mathcal{R}_+^{p \times m}\). The necessity is obvious from definition 6.1.6 and the interpretation of dynamical equation \((6.1)\) with positivity constraints.

Note also that \(g(t) = Ce^{At}B + D\delta(t)\) with \(A\) being a Metzler matrix and \(B \in \mathcal{R}_+^{n \times m}, C \in \mathcal{R}_+^{p \times n}, D \in \mathcal{R}_+^{p \times m}\) becomes \(g(t) \in \mathcal{R}_+^{p \times m}\) for all \(t \geq 0\). This means that the internal positive systems implies external positivity.

6.2 Stability of Continuous Time Positive (Metzlerian) Systems

Consider the continuous time internally positive systems \((6.1)\) with \(x(0) = x_0\) and \(u(t) = 0\) where \(A\) is a Metzler matrix. Then asymptotic stability implies that
\[
\lim_{t \to \infty} x(t) = 0
\]
for every \(x_0 \in \mathcal{R}_+^n\) where \(x(t) = e^{At}x(0)\) is the zero input response of the system.

The continuous-time internally positive system \((6.1)\) with \(A\) being a Metzler matrix is simply called a Metzlerian system.
Theorem 6.2.1 The Metzlerian system (6.1) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All eigenvalues of $A$ have negative real parts.

2. All coefficients of the characteristic equation

$$\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$$

are positive, i.e., $a_i > 0$ for $i = 0, 1, 2, \ldots, n - 1$.

3. All principal minors of the matrix $-A$ are positive.

4. The matrix $A$ is nonsingular and $-A^{-1} > 0$

5. There exist a positive definite diagonal matrix $P$ such that $A^TP + PA < 0$.

6. There exists a positive vector $v \in \mathbb{R}^n_+$ such that $Av < 0$

It is not difficult to see that if at least one diagonal entry of the Metzler matrix $A$ (corresponding to an internally positive system) is positive, i.e., $a_{ii} > 0$ for some $i \in (1, 2, \ldots, n)$, then the system is unstable. Also, it is interesting to point out that if $A$ is a Metzler matrix of the internally positive asymptotically stable system, then $-A^{-1} > 0$.

Note that asymptotic stability and Lyapunov stability of Metzlerian systems are integrated in the above theorem and one can refer to stability in terms of the equilibrium point $x_e$ which satisfies the condition $Ax_e = 0$. Now consider the single input Metzlerian systems $\dot{x}(t) = Ax + Bu$ with a positive constant input $\bar{u} > 0$. Then we have $Ax_e + b\bar{u} = 0$ and it can be shown that for asymptotically stable system $x_e > 0$ since $\det(A) \neq 0$ and we have $x_e = -A^{-1}b\bar{u}$ which is positive due to $-A^{-1} > 0$. 
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6.3 Input-Output Stability

Consider the internally positive continuous-time single-input single-output system (SISO Metzlerian system)

\[ \dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0 \]  
\[ y(t) = cx(t) + du(t) \]

where \( x \in \mathbb{R}^n, \ u \in \mathbb{R}, \ y \in \mathbb{R}, \ A \in \mathbb{R}^{n \times n} \) is a Metzler matrix, and \( b, c \in \mathbb{R}_+^n, \ d \in \mathbb{R}_+ \).

**Definition 6.3.1** The SISO Metzlerian system (6.4) is called Bounded Input Bounded Output (BIBO) stable if and only if its output is bounded for any bounded input for all \( t \in [0, \infty) \).

**Theorem 6.3.2** The SISO Metzlerian system (6.4) is BIBO stable if and only if

\[ \int_0^t g(\tau) \, d\tau < \infty \quad \text{for all} \quad t \in [0, \infty) \]  

Unlike absolute integrability condition for general system we have the above integral condition for internally positive system since \( u(t) \in \mathbb{R}_+ \) and its bounded assumption \( u(t) \leq \bar{u} \).

**Theorem 6.3.3** The SISO Metzlerian system (6.4) is BIBO stable if and only if the denominator of the transfer function

\[ G(s) = c(sI - A)^{-1}b + d = \frac{N(s)}{D(s)} \]

where \( D(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0, \) has all positive coefficients, i.e. \( a_i > 0, \ i = 0, 1, \ldots, n - 1. \)

The proof of the above theorem follows from the fact that \( g(t) = L^{-1}[G(s)] = ce^{At}b + d\delta(t) \) and the impulse response is bounded if and only if \( Re(p_k) < 0 \) for \( k = 1, 2, \ldots, n \) where \( p_k \)'s are poles of \( G(s) \). Also, note that the internally positive
SISO system is BIBO stable if and only if its unit step response is bounded. This is true since 
\[ g(t) = \frac{dh(t)}{dt} \] and 
\[ h(t) = \int_0^t g(\tau) d\tau \] which should be bounded based on (6.6).

Finally, the above theorems can simply be generalized for MIMO case with proper adjustments. It is worth pointing out that asymptotic stability of Metzlerian systems implies BIBO stability but the converse is not true in general.

**Remark 6.3.4** An interesting and useful result for interconnection of positive system (continuous and discrete) is the fact that series, parallel, and feedback connection of positive systems remain positive. this can be shown by construction. For simplicity assume \( \{A_i, B_i, C_i, D_i\}, i = 1, 2 \) for two positive systems and establish the result.

### 6.4 Continuous-Time Positive Delay Systems (Metzlerian Delay Systems)

Consider the continuous-time linear systems with delays

\[ \dot{x}(t) = \sum_{i=0}^{k} A_i x(t - \tau_i) + \sum_{j=0}^{l} B_j u(t - \tau_j) \]  \hspace{1cm} (6.7)

\[ y(t) = Cx(t) + Du(t) \]  \hspace{1cm} (6.8)

where \( x \in \mathcal{R}^n, u \in \mathcal{R}^m, y \in \mathcal{R}^p \) are the state, input and output vectors; respectively and \( A_i \in \mathcal{R}^{n \times n}, i = 0, 1, 2, \ldots, k; \ B_j \in \mathcal{R}^{n \times m}, j = 0, 1, 2, \ldots, l; \ C \in \mathcal{R}^{p \times n}, \ D \in \mathcal{R}^{p \times m} \) and \( \tau_i > 0, i = 1, 2, \ldots, k, \tau_j > 0, j = 1, 2, \ldots, l \) represent delays.

We usually assume that \( \tau_i = i\tau, d = \max_i \tau_i, i = 1, 2, \ldots, k, \) and \( \tau_j = j\tau, j = 1, 2, \ldots, l \), so (6.7) - (6.8) can alternatively be written as

\[ \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - i\tau) + B_0 u(t) + \sum_{j=1}^{l} B_j u(t - j\tau) \]  \hspace{1cm} (6.9)

\[ y(t) = Cx(t) + Du(t) \]  \hspace{1cm} (6.10)

with initial conditions \( x_0(t) \) for \( t \in [-k\tau, 0] \) and \( u_0(t) \) for \( t \in [-l\tau, 0] \).
Definition 6.4.1 The system (6.9), (6.10) is called internally positive delay system or Metzlerian delay system if for every initial conditions \( x_0(t), u_0(t) \) and all input \( u(t) \in \mathcal{R}^m_+, t \geq 0 \), we have \( x(t) \in \mathcal{R}^n_+ \) and \( y(t) \in \mathcal{R}^p_+ \) for \( t \geq 0 \).

Theorem 6.4.2 The system (6.9), (6.10) is internally positive delay system (Metzlerian delay system) if and only if \( A_0 \) is a Metzler matrix and the matrices \( A_i, i = 1, 2, \ldots, k; B_j, j = 0, 1, 2, \ldots, l; C, D \) have nonnegative entries, i.e. \( A_i \in \mathcal{R}^{n \times n}_+, B_j \in \mathcal{R}^{n \times m}_+, C \in \mathcal{R}^{p \times n}_+, D \in \mathcal{R}^{p \times m}_+ \) are nonnegative matrices.

Proof To simplify the notation, the proof will be shown for \( k = l = 1 \). Using the step method and defining the vectors

\[
\tilde{x} = \begin{bmatrix} x(t) \\ x(t + \tau) \\ \vdots \\ x(t + q\tau) \end{bmatrix}, \quad \tilde{u} = \begin{bmatrix} u(t) \\ u(t + \tau) \\ \vdots \\ u(t + q\tau) \end{bmatrix}, \quad z_0 = \begin{bmatrix} A_1x(t - \tau) + B_1u(t - \tau) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

we may write (6.9), (6.10) as

\[
\dot{x}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t) + z_0, \quad t \in [0, \tau] \tag{6.11}
\]

\[
y(t) = \tilde{C}\tilde{x}(t) + \tilde{D}\tilde{u}(t) \tag{6.12}
\]

where \( \tilde{A} = \begin{bmatrix} A_0 & 0 & 0 & \ldots & 0 & 0 \\ A_1 & A_0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & A_1 & A_0 \end{bmatrix} \), \( \tilde{B} = \begin{bmatrix} B_0 & 0 & 0 & \ldots & 0 & 0 \\ B_1 & B_0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & B_1 & B_0 \end{bmatrix} \)

\[
\tilde{C} = \begin{bmatrix} C & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} D & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.
\]

It is clear that the system (6.11), (6.12) is internally positive if and only if the matrix \( \tilde{A} \) is a Metzler matrix and the matrices \( \tilde{B}, \tilde{C}, \tilde{D} \) have nonnegative entries.

From the structure of the above matrices, it follows that the system (6.9), (6.10)
is internally positive if and only if $A_0$ is a Metzler matrix and $A_i \in \mathbb{R}^{n \times n}_+$; $B_j \in \mathbb{R}^{n \times m}_+$, $C \in \mathbb{R}^{p \times n}_+$, $D \in \mathbb{R}^{p \times m}_+$ are nonnegative matrices.

**Definition 6.4.3** The system (6.9), (6.10) is called strictly Metzlerian system if $A_0$ and $\sum_{i=0}^{k} A_i$ are Metzlerian.

The rational for defining the above strict assumption is due to the convenience of mathematical analysis when $\tau$ approaches or set equal to 0.

### 6.5 Stability of Metzlerian Delay Systems

Consider the continuous-time linear systems with $k$ delays in state

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) + Bu(t) \quad (6.13)$$

$$y(t) = Cx(t) + Du(t) \quad (6.14)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors; $A_i$, $i = 0, 1, \ldots, k$; $B, C, D$ are real matrices of appropriate dimensions and $\tau_i = i \tau; \ i = 1, 2, \ldots, k$ are delays. The initial conditions for (6.13), (6.14) has the form

$$x(t) = x_0(t) \quad \text{for} \quad t \in [-d, 0], \ d = \max_{i} \tau_i$$

where $x_0(t)$ is a given vector function. Note that (6.13), (6.14) is a simplified form of (6.7), (6.8) or equivalently (6.9), (6.10) with no delay in input. Furthermore, we assume that the linear continuous-time delay system (6.13), (6.14) is internally positive i.e. $x(t) \in \mathbb{R}_+^n$, $y(t) \in \mathbb{R}_+^p$ for any $x_0(t) \in \mathbb{R}_+^n$ and for all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$. Based on Theorem 6.4.2, internally positive delay system (6.13), (6.14) or Metzlerian delay system admits the property that $A_0$ is a Metzler matrix and matrices $A_i$, $i = 1, 2, \ldots, k$; $B, C, D$ are nonnegative matrices.
Definition 6.5.1 Let (6.13), (6.14) be Metzlerian delay system. Then the system
or equivalently its equilibrium point is called asymptotically stable if and only if the
solution of (6.13) with \( u(t) = 0 \) satisfies the condition
\[
\lim_{t \to \infty} x(t) = 0, \quad \text{for } x_0(t) \in \mathcal{R}_+^n, \ t \in [-d, 0].
\]

Definition 6.5.2 Let a constant input \( u(t) = u \in \mathcal{R}_+^n \) be applied to an asymptotically
stable system (6.13), (6.14), then a vector \( x_e \in \mathcal{R}_+^n \) satisfying the equality
\[
\sum_{i=0}^{k} A_i x_e + B u = 0
\]
is called the equilibrium point of the system corresponding to the input \( u \).

If the Metzlerian delay system (6.13), (6.14) is asymptotically stable then the matrix
\[
A = \sum_{i=0}^{k} A_i
\]
is a nonsingular Metzler matrix and we have
\[
x_e = - A^{-1} B u
\]

Theorem 6.5.3 The equilibrium point \( x_e \) corresponding to strictly positive \( B u > 0 \)
of an asymptotically stable Metzlerian system (6.13), (6.14) is strictly positive, i.e.
\( x_e > 0 \).

Proof Since \( A \) is a nonsingular Metzler matrix, it is equivalently a stable Metzler
matrix and by Theorem 6.2.1 \(- A^{-1} > 0\) or \(- A^{-1} \in \mathcal{R}_+^{n \times n}\), and with \( B u > 0 \) it follows
that \( x_e > 0 \). \( \square \)

Theorem 6.5.4 The system (6.13), (6.14) is asymptotically stable if and only if there
exists a strictly positive vector \( v \in \mathcal{R}_+^n \) satisfying the inequality
\[
A v < 0, \quad \text{with} \quad A = \sum_{i=0}^{k} A_i
\]  

(6.15)
Proof From the previous results, we conclude that for asymptotically stable Metzlerian systems (6.13), (6.14), the matrix $A$ is Metzlerian and stable. Thus, one can use properties of this class of matrices and conclude the inequality (6.15). A direct proof can also be established by using (6.13) as follows. First we show that if (6.13) is asymptotically stable then there exists a strictly positive vector $v \in \mathcal{R}^n_+$ satisfying (6.15).

Let $B = 0$ and integrate (6.13) in the interval $[0, \infty)$ we obtain

$$
\int_0^\infty \dot{x}(t) \, dt = A_0 \int_0^\infty x(t) \, dt + \sum_{i=1}^k A_i \int_0^\infty x(t - \tau_i) \, dt
$$

and we have

$$
x(\infty) - x(0) - \sum_{i=1}^k A_i \int_0^0 x(t) \, dt = A \int_0^\infty x(t) \, dt
$$

For asymptotically stable Metzlerian system

$$
x(\infty) = 0, \quad x(0) + \sum_{i=1}^k A_i \int_{-\tau_i}^0 x(t) \, dt > 0
$$

Since $A$ is a Metzlerian stable matrix, we have

$$
v = \int_0^\infty x(t) \, dt > 0
$$

Now, we shall show that if (6.15) holds then the Metzlerian system (6.13) is asymptotically stable.

It is well-known that (6.13) is asymptotically stable if and only if the corresponding transpose system

$$
\dot{x}(t) = A_0^T x(t) + \sum_{i=1}^k A_i^T x(t - \tau_i)
$$

is asymptotically stable. As a candidate for a Lyapunov function for (6.16) we choose the function

$$
V(x) = x^T v + \sum_{i=1}^k \int_{t-\tau_i}^t x^T(\tau_i) \, d\tau \ A_i v
$$
which is positive for any nonzero $x(t) \in \mathbb{R}_n^+$. Using (6.16) and (6.17) we obtain

$$
\dot{V}(x) = \dot{x}^T(t)v + \sum_{i=1}^{k}(x^T(t) - x^T(t - \tau_i)) A_i v
$$

$$
= x^T(t)A_0 v + \sum_{i=1}^{k} x^T(t) A_i v + \sum_{i=1}^{k}(x^T(t) - x^T(t - \tau_i)) A_i v
$$

$$
= x^T(t)A_0 v + \sum_{i=1}^{k} x^T(t) A_i v
$$

$$
= x^T(t)Av
$$

(6.18)

and if the condition (6.15) holds, i.e. $Av < 0$, we have $\dot{V}(x) < 0$ and the system (6.13) is asymptotically stable Metzlerian system. Note also that as a strictly positive vector $v$, one can choose the equilibrium point $v = -A^{-1}Bu$ since $Av = A(-A^{-1}Bu) = -Bu < 0$ for $Bu > 0$.

\begin{proof}

The above theorem implies that all properties and stability results of Metzlerian matrices can be employed for Metzlerian delay systems without considering its delays (see Theorem 6.4.2 of previous section regarding Metzlerian matrices).

\end{proof}

\begin{lemma}

Let the system (6.7), (6.8) be Metzlerian delay system and let $A = \sum_{i=0}^{k} A_i$, then the Metzlerian delay system is asymptotically stable if and only if any one of the equivalent conditions of Theorem 6.2.1 is satisfied.

\end{lemma}
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Proof Using Theorem 6.5.5, we know that the asymptotic stability of Metzlerian delay systems is equivalent to the asymptotic stability of Metzlerian system without delay. Thus, the equivalence of 1) - 6) of Theorem 6.2.1 is obvious from the properties of positive systems and Theorem 6.5.4.

The above Lemma enables one to consider the constrained stabilization problem for a general time-delay systems, which will be discussed in section 6.7.

6.6 Robust Stability of Metzlerian Delay Systems

Consider the class of positive delay dynamic systems

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau), \quad (6.20)$$

where $A_0$ is a Metzlerian matrix and the matrix $A_1$ is either nonnegative or Metzlerian such that $A_0 + A_1$ remains Metzlerian.

Theorem 6.6.1 The Metzlerian delay system (6.20) is stable for all $\tau > 0$ if any one of the following equivalent conditions is satisfied:

1. All roots of $\text{det}(sI - A_0 - A_1 e^{-\tau s}) = 0$ in $\mathbb{C}$ (left half of s plane).

2. $\mu(A_0) + \min\{\mu(|A_1|), \| A_1 \|\} < 0$, where

$$\mu(M) = \lim_{\epsilon \to 0} \frac{\| I + \epsilon M \| - 1}{\epsilon}$$

is defined as the matrix measure of a matrix $M$.

3. All leading principle minors of $-(A_0 + A_1)$ is positive, i.e. $A_0 + A_1$ is Hurwitz.

4. There exist $P > 0$, $Q > 0$ such that

$$\begin{pmatrix}
A_0^T P + PA_0 + Q & -Q + PA_1 \\
-Q^T + A_1^T P & -Q
\end{pmatrix} < 0,$$

i.e. the above LMI has a feasible solution.
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Theorem 6.6.2 The uncertain Metzlerian delay system

\[
\dot{x}(t) = (A_0 + D_0 \Delta_0 E_0)x(t) + (A_1 + D_1 \Delta_1 E_1)x(t - \tau_1)
\]  \hspace{1cm} (6.21)

is stable for all structured perturbations as defined previously, if there exist

\( P > 0, \ Q > 0, \ X \geq 0 \) and \( Y \geq 0 \) and scalar \( e_1 \) and \( e_2 \) such that

\[
\begin{pmatrix}
Y_{11} & -Y + PA_1 & PD_0 & PD_1 \\
-Y^T + A_1^T P & -Q_{22} & 0 & 0 \\
D_0^T P & 0 & -e_1 I & 0 \\
D_1^T P & 0 & 0 & -e_2 I
\end{pmatrix} < 0
\]

where \( Y_{11} = A_0^T P + PA_0 + Y + Y^T + Q + e_1 E_0^T E_0 \) and \( Q_{22} = Q - e_2 E_1^T E_1 \).

The following result establishes the computation of the stability radius for the Metzlerian delay systems. Obviously one can use the general results of chapter 4 for this class of systems as well. However, we provide a direct formula for the stability radius as we have reported for regular Metzlerian system in [100]. We consider the simple special case of (6.21) where the perturbation is captured in the matrix \( A_0 \) alone.

It is well-known that the asymptotic stability condition independent of delay

\[
\det(sI_n - A_0 - A_1 e^{-s \tau_1}) \neq 0, \ \forall s \in \mathbb{C}, \ Re(s) \geq 0 \ \forall \tau_1 \geq 0
\]

is equivalent to

\[
\det(sI_n - A_0 - A_1 z) \neq 0, \ \forall (s, z) \in \mathbb{C}^2, \ Re(s) \geq 0 \ \forall s \neq 0,
\]

\(|z| \leq 1 \text{ or } s = 0, \ z = 1\).

Denoting

\[
H(s, z) = sI - A_0 - zA_1,
\]

the delay-independent stability condition above is called strongly stable [73] if

\[
\det H(s, z) \neq 0, \ \forall (s, z) \in \mathbb{C}^2, \ Re(s) \geq 0, \ |z| \leq 1.
\]

Let us also define \( G(s, z) = E_0 H(s, z) D_0 \), then we have the following result.
Theorem 6.6.3 Let the Metzlerian delay system (6.20) be strongly delay-independently stable. Then the real and complex stability radii of the uncertain Metzlerian delay system

\[
\dot{x}(t) = (A_0 + D_0\Delta_0 E_0)x(t) + A_1x(t - \tau_1)
\]

coincide and it is given by

\[
r_R = \frac{1}{\|G(0,1)\|} = \frac{1}{\|E_0(A_0 + A_1)^{-1}D_0\|}. \quad (6.22)
\]

Consider the class of positive delay dynamic systems

\[
\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{2} A_i x(t - \tau_i), \quad (6.23)
\]

where \(A_0\) is a Metzlerian matrix and the matrix \(A_1\) and/or \(A_2\) are either nonnegative or Metzlerian such that \(\sum_{i=0}^{2} A_i\) remains Metzlerian, then we call this class of systems “Strongly Metzlerian delay systems” however, for simplicity we use “Metzlerian delay systems” as it is commonly used.

It is obvious that Metzlerian delay systems admit the general stability condition of two time-delay systems, namely all roots of \(\text{det} \left(sI - A_0 - \sum_{i=1}^{2} A_i e^{-\tau_i s}\right) = 0\) in \(\mathbb{C}_g\).

It is not difficult to show that the asymptotic stability condition independent of delay is equivalent to strongly delay independent stability (as defined in [73]) for this class of systems.

Theorem 6.6.4 Let the Metzlerian delay system (6.23) be strongly delay-independent stable. Then All leading principle minors of \(-\sum_{i=0}^{2} A_i\) is positive, i.e. \(\sum_{i=0}^{2} A_i\) is Hurwitz. Furthermore, the real and complex stability radii of the uncertain Metzlerian delay system

\[
\dot{x}(t) = (A_0 + D_0\Delta_0 E_0)x(t) + \sum_{i=1}^{2} A_i x(t - \tau_i)
\]

coincide and it is given by

\[
r_C = r_R = \frac{1}{E_0 \left(\sum_{i=0}^{2} A_i\right)^{-1} D_0}. 
\]
CHAPTER 6. POSITIVE DELAY SYSTEMS

Remark 6.6.5 The above theorem can be generalized with any number of delays, say \( k \) delays. In this case the above formula will be

\[
 r_C = r_R = \frac{1}{\left\| E_0 \left( \sum_{i=0}^{k} A_i \right)^{-1} D_0 \right\|}.
\]

Now we consider the general delay systems (6.7) with parametric uncertainties described by

\[
 \dot{x}(t) = A_0(\delta)x(t) + \sum_{i=1}^{k} A_i(\delta)x(t - \tau_i) + Bu(t) \tag{6.24}
\]

where \( A_i(\delta), \ \forall \ i = 0, 1, 2, \ldots, k \) are uncertain matrices defined by affine multiple parameter perturbations of the form

\[
 A_i(\delta) = A_i + \sum_{r=1}^{q_i} \delta_i^r E_i^r, \quad r = 1, \ldots, q_i; \ i = 0, \ldots, k \tag{6.25}
\]

where \( \delta_i^r \) are unknown scalars and \( E_i^r \) are given matrices specifying the structure of the perturbations, and \( \delta_i = [\delta_i^1 \ \delta_i^2 \ \ldots \ \delta_i^{q_i}]^T \) is the \( i \)th sub-vector of uncertain parameter \( \delta_i^r, \ r = 1, \ldots, q_i \), confined within a prescribed set of interest \( \Omega_i \), i.e. \( \delta_i \in \Omega_i, \ i = 0, 1, 2, \ldots, k. \)

When \( r = 1 \), we have an affine single perturbation structure and \( A_i(\delta) \) reduces to \( A_i(\delta) = A_i + \delta_i E_i, \ \forall \ i = 0, 1, \ldots, k \); where \( \delta_i, E_i \) are redefined as \( \delta_i, E_i \) respectively. In this case the number of uncertain parameters \( \delta_i \) matches with the number of system matrices \( A_i \) and the associated vector set of unknown parameters becomes \( \delta = [\delta_0 \ \delta_1 \ \ldots \ \delta_k]^T \). Concentrating on (6.25) and assuming for simplicity \( q_i = q \) and \( \delta_i^r = \delta_r \) for all \( i = 0, 1, \ldots, k \) in, then

\[
 A_i(\delta) = A_i + \sum_{r=1}^{q} \delta_r E_i^r, \quad i = 0, 1, 2, \ldots, k \tag{6.26}
\]

where \( \delta = [\delta_1 \ \delta_2 \ \ldots \ \delta_q]^T \). Thus, for single and multiple perturbations, \( \delta \) can be represented as the vector of uncertain parameters confined within a prescribed set of interest \( \Omega \), i.e. \( \delta \in \Omega \).
Note that robust stability results are available for general time-delay systems using LMI conditions [68], [64]. However, simple results can be obtained when (6.24) is assumed to be Metzlerian delay systems.

So, let us impose the constraint that the uncertain delay system (6.24) is Metzlerian delay system i.e. $A_0(\delta)$ is a Metzler matrix and the matrices $A_i(\delta) \in \mathcal{R}^{n \times n}_+$, $i = 1, 2, \ldots, k$ are non-negative for all $\delta$, and $B \in \mathcal{R}^{n \times m}_+$. Then we have the following result, which can directly be stated with the aid of Theorem 6.5.5.

**Theorem 6.6.6** Let the system (6.24) be uncertain Metzlerian delay system. Then it is robustly stable independent of delays if and only if the corresponding system without delays, i.e.

$$
\dot{x}(t) = A(\delta)x(t) + Bu, \quad \delta \in \Omega
$$

is robustly stable, where

$$
A(\delta) = \sum_{i=0}^{l} A_i(\delta)
$$

is Metzler matrix for all $\delta \in \Omega$.

Now by substituting $A_i(\delta)$ from (6.26) in (6.28) we have

$$
A(\delta) = \sum_{i=0}^{k} A_i + \sum_{r=1}^{q} \delta_r \sum_{i=0}^{k} E_{ir}
$$

whereby various robust stability results can be developed using classical technique [100], [39]. It is also possible to provide robust stability bounds on uncertain parameters assuming the system without uncertainty is stable. One particular situation is when $\delta_r = \delta$ for all $r = 1, \ldots, q$; then (6.29) can be written as $A(\delta) = A + \delta E$, where $A = \sum A_i$ and $E = \sum E_{ir}$ from which direct formula for $\delta$ can be obtained. Obviously, this is also true for single perturbation case by setting $r = 1$.

Since the interval matrices comprise of a closed and bounded convex set, they can be represented as a convex hull of its vertex matrices. Therefore, one can relate interval and affine uncertainties. For Metzlerian delay system with affine un-
certainty structure and non-negative perturbation matrices $E_i^r \in \mathbb{R}_{+}^{n \times n}$ we have $\delta_r E_i^r \in [\delta_r E_i^r, \bar{\delta}_r E_i^r]$ for all fixed $\delta_r \in [\delta_r, \bar{\delta}_r]$. This indicates that $A_i(\delta) \in [A_i(\delta), \bar{A}_i(\delta)]$ with $A_i(\delta) = A_i + \sum_{r=1}^{q} \delta_r E_i^r$ and $\bar{A}_i(\delta) = A_i + \sum_{r=1}^{q} \bar{\delta}_r E_i^r$, $\forall i = 0, 1, \ldots, k$; where $A_i$’s are nominal Metzler matrices and $E_i^r$’s represent the non-negative perturbation matrices. Thus, $A_0(\delta) \in [\underline{A}_0, \overline{A}_0]$ is a Metzler interval matrix and $A_i(\delta) \in [\underline{A}_i, \overline{A}_i]$, $i > 0$ is a non-negative interval matrix. Consequently, $A(\delta)$ can be represented as interval uncertainty $A(\delta) = [\underline{A}, \overline{A}]$ where $\underline{A} = \sum_{i=0}^{k} \underline{A}_i$, $\overline{A} = \sum_{i=0}^{k} \overline{A}_i$. So, if $A(\delta)$ is represented as interval uncertainty through the affine perturbation structure above leading to interval Metzlerian delay system or the uncertain Metzlerian delay system is defined as interval Metzlerian delay system, we can write (6.24) as

$$\dot{x}(t) = [\underline{A}_0, \overline{A}_0]x(t) + \sum_{i=1}^{k} [\underline{A}_i, \overline{A}_i]x(t - \tau_i) + Bu(t)$$

(6.30) then the following result can be stated

**Theorem 6.6.7** The Metzlerian interval delay system (6.30) is robustly stable if and only if the following single Metzlerian system without delay, i.e.

$$\dot{x} = \overline{A}x + Bu, \quad \overline{A} = \sum_{i=0}^{k} \overline{A}_i$$

is asymptotically stable, and the stability can be verified by checking the positivity of leading principal minors of $-\overline{A}$.

**Proof** With the aid of Theorem 6.5.5 and Theorem 6.6.6, we know that (6.30) is asymptotically stable if and only if the Metzlerian interval system without delay $\dot{x} = [\underline{A}, \overline{A}]x(t)$ is asymptotically stable. It has also been established [97], [102] that the robust stability of Metzlerian interval systems $\dot{x} = [\underline{A}, \overline{A}]x$ is equivalent to the asymptotic stability of the system with the upper interval, namely $\dot{x} = \overline{A}x$. This establishes the proof of the Theorem. $\square$

To conclude this section we also provide a closed form expression for stability radius of Metzlerian delay systems which can be considered as a generalization of
Let the general continuous-time delay systems (6.7) with a more general type of uncertainty structure be described by

\[ \dot{x}(t) = \sum_{i=0}^{k} A_i(\Delta_i)x(t - \tau_i) + Bu(k) \]  (6.31)

where \( A_i \)'s in (6.7) are subjected to affine perturbations of the form

\[ A_i(\Delta_i) = A_i + D_i \Delta_i E_i \]  (6.32)

where \( D_i \in \mathbb{R}^{n \times d_i} \), \( E_i \in \mathbb{R}^{e_i \times n} \) represent the structure of the uncertainties and the matrices \( \Delta_i \in \mathbb{R}^{d_i \times e_i} \) are unknown uncertainty matrices. The robust stability results for (6.31) and (6.32) is not trivial when general type of continuous-time delay systems is considered. Although, stability and robust stability tests in terms of LMIs are available, the problem of computing robust stability radius is more complex even in case of conventional delay free systems. In general, the computation of complex and real stability radii, \( r_C \) and \( r_R \) requires the solution of a complicated global optimization problem [83]. However, for the class of positive systems (continuous and discrete cases), the complex and real stability radii coincide and can be computed by closed form expression [100], [39].

Here we concentrate on continuous-time Metzlerian delay systems and show that the stability radii can simply be computed with the aid of Theorem 6.5.4, which does not require long derivation as in [76]. It is also important to point out that in computing stability radius, one should always assume that the system without uncertainty is stable.

**Theorem 6.6.8** Let the system (6.31) with uncertainty structure (6.32) be an uncertain Metzlerian delay system and assume that it is asymptotically stable independent of delay without uncertainty. Then the real and complex stability radii of the uncertain Metzlerian system (6.32) coincide and it is given by the following formula if
$D_i = D_j \in \mathcal{R}^{n \times d_i}$ or $E_i = E_j \in \mathcal{R}^{e_i \times n}$ for all $i, j = 0, 1, \ldots, k$

$r_C = r_R = \frac{1}{\max_i \| E_i (-\sum_{i=0}^{k} A_i)^{-1} D_i \|}$  \hspace{1cm} (6.33)

Furthermore, with respect to the affine uncertainty (6.26), we have

$r_C = r_R = \frac{1}{\rho((-\sum_{i=0}^{k} A_i)^{-1}(\sum E_i'))}$  \hspace{1cm} (6.34)

**Proof** The asymptotic stability of Metzlerian delay system without uncertainty is equivalent to the asymptotic stability of Metzlerian system without delay based on Theorem 6.5.5. So, imposing uncertainty structure on Metzlerian system without delay

\[ \dot{x}(t) = (A + \sum D_i \Delta_i E_i) x(t) + Bu(k) \]  \hspace{1cm} (6.35)

where $A = \sum_{i=0}^{k} A_i$, one can use the result of stability radius in [100], [39] to conclude (6.33) and (6.34). We also state the following theorem when the uncertainty is only on $A_0$.

**Theorem 6.6.9** Let the Metzlerian delay system without uncertainty (6.7) be asymptotically stable independent of delay. Then, the real and complex stability radii of the uncertain Metzlerian delay system

\[ \dot{x}(t) = (A_0 + D_0 \Delta E_0) x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) \]

coincide and it is given by following formulas depending on the characterization of $\Delta$

1. Let $\| \cdot \|_2$ denote the Euclidean norm and let $\Delta = \mathcal{R}^{m \times p}$ then

$$r_C = r_R = \frac{1}{\| E_0 (\sum_{i=0}^{k} A_i)^{-1} D_0 \|}.$$  \hspace{1cm}

2. Let $\Delta$ be defined by the set $\Delta = \{ S \circ \Delta : S_{ij} \geq 0 \}$ with $\| \Delta \| = \max \{ |\delta_{ij} : \delta_{ij} \neq 0 \}$ where $[S \circ \Delta]_{ij} = s_{ij} \delta_{ij}$ represents the Schur product. Then

$$r_C = r_R = \frac{1}{\rho(E_0 (\sum_{i=0}^{k} A_i)^{-1} D_0 S)}.$$  \hspace{1cm}

where $\rho(\cdot)$ denotes the spectral radius of a matrix.
6.7 Positive Stabilization of Delay Systems

The stabilization and robust stabilization techniques of chapter 4 can equally be applied for the special class of Metzlerian delay systems. However, the goal of this section is to consider the general time-delay system and not only to stabilize it but also make it structurally constrained in such a way that the closed-loop system is Metzlerian delay stable with desirable robustness properties.

6.7.1 Constrained Stabilization of Delay Systems

The problem of constrained positive stabilization for delay-free systems was introduced in [101] and developed for delay systems in [35] and [104]. Here we further elaborate on it.

Lemma 6.7.1 Let the system (6.7), (6.8) be Metzlerian delay system and let $A = \sum_{i=0}^{k} A_i$, then the Metzlerian delay system is asymptotically stable if and only if any one of the equivalent conditions of Theorem 6.2.1 is satisfied.

Proof Using Theorem 6.5.5, we know that the asymptotic stability of Metzlerian delay systems is equivalent to the asymptotic stability of Metzlerian system without delay. Thus, the equivalence of 1) - 6) of Theorem 6.2.1 is obvious from the properties of positive systems and Theorem 6.5.4.

The above Lemma enables one to consider the constrained stabilization problem for a general time-delay systems.

Let the state feedback control law of the form

$$u(t) = K_0 x(t) + \sum_{i=1}^{k} K_i x(t - \tau_i), \quad t \geq 0$$

be applied to the system (6.7), (6.8). Then we get the following dynamical equation for the closed-loop system.
\[
\dot{x}(t) = (A_0 + BK_0)x(t) + \sum_{i=1}^{k}(A_i + BK_i)x(t - \tau_i), \quad t \geq 0
\]
\[
x(t) = \varphi(t), \quad t \in [-k\tau, 0]
\] (6.37)

**Lemma 6.7.2** Consider the closed-loop delay system (6.37). Then the closed-loop system without delay is asymptotically stable Metzlerian system if the following LMI has a feasible solution with respect to the variables \(Q\) and \(Y\)

\[
\begin{cases}
Q > 0, & (AQ + BY)_{ij} \geq 0, \quad i \neq j \\
QA^T + YT^B + AQ + BY < 0
\end{cases}
\] (6.38)

where \(A = \sum_{i=0}^{k} A_i\) and the feedback gain \(K = \sum_{i=0}^{k} K_i\) is obtained from \(K = YQ^{-1}\).

**Proof** Assuming that the Metzlerian feedback system (6.37) is asymptotically stable, then the equivalent conditions of Theorem 6.2.1 are satisfied with respect to the Metzlerian feedback systems without delay. Using the property that the leading principal minors of \(M = -(A + BK)\) to be positive, i.e. \(|M(\alpha)| > 0\) where \(\alpha \in N\) defines the order of the minors and \(M_{ii} > 0\) and \(M_{ij} \leq 0\), one can find the gain matrix \(K = \sum_{i=0}^{k} K_i\). Alternatively, we seek a diagonal matrix \(Q > 0\) and the matrix \(K\) such that \(Q(A + BK)^T + (A + BK)Q < 0\), with off-diagonal being non-negative. Since \(Q > 0\) is diagonal, the last condition holds if and only if all the off-diagonal entries of \((A + BK)Q\) are non-negative. Therefore, with the change of variable \(Y = KQ\), the asymptotic stability of the closed-loop Metzlerian system without delay is equivalent to the LMI (6.38) with the variables \(Q\) and \(Y\).

The above Lemma provides a solution to the constrained Metzlerian stabilization without delay. Unfortunately, the feedback gain \(K\) does not provide a solution to \(K_i\)’s. However with the aid of the above Lemma one can state the following Theorem whereby \(K_i\)’s are exactly determined. Once \(K_i\)’s are obtained, (6.38) serves as a test for validity of the result. Note that when \(K_i = 0\) for \(i > 0\), (6.38) determines \(K_0\) for delay free case.
Theorem 6.7.3 Let the feedback control law (6.36) be applied to the general time-delay system (6.7), (6.8). Then the closed-loop system (6.37) is asymptotically stable and Metzlerian if and only if one of the following equivalent conditions is satisfied:

1. $A_0 + BK_0$ and $\sum_{i=0}^{k}(A_i + BK_i)$ are stable Metzler matrices, and $A_i + BK_i \geq 0$, $\forall i = 1, 2, \ldots, k$.

2. There exists a vector $w \in \mathbb{R}^n_+$ and the gain matrices $K_i$’s such that

$$\left[\sum_{i=0}^{k}(A_i + BK_i)\right]w < 0, \quad K_i \in \mathbb{R}^{m \times n}.$$ 

3. There exists a positive definite diagonal matrix $P$ such that

$$\left[\sum_{i=0}^{k}(A_i + BK_i)\right]^TP + P\left[\sum_{i=0}^{k}(A_i + BK_i)\right] < 0$$

Furthermore, the feedback control law (6.36) is determined if the following LMI has a feasible solution for $Q = P^{-1} \in \mathbb{R}^{n \times n}$ and $G_i \in \mathbb{R}^{m \times n}$,

$$\left(\sum_{i=0}^{k}A_i\right)Q + Q\left(\sum_{i=0}^{k}A_i\right)^T + B\left(\sum_{i=0}^{k}G_i\right) + \left(\sum_{i=0}^{k}G_i\right)^TB^T < 0 \quad (6.39)$$

$$(A_iQ + BG_i)_{jr} \geq 0, \quad \forall j \neq r, \quad j, r = 1, \ldots, n,$$

$$i = 1, 2, \ldots, k. \quad (6.40)$$

$$(A_0Q + BG_0)_{jr} > 0, \quad \forall j \neq r, \quad j, r = 1, \ldots, n \quad (6.41)$$

$$(A_0Q + BG_0)_{jj} < 0$$

where $(M)_{jr}$ means $jr$ element of a matrix $M$, and the feedback gain matrices $K_i$, $i = 0, 1, \ldots, k$ are obtained by $K_i = G_iQ^{-1}$.

Proof The equivalent conditions of Theorem 6.7.3 are obvious from Lemma 6.7.1. Taking advantage of LMI formulation of Lemma 6.7.2 and the equivalent conditions of Theorem 6.7.3, it is sufficient to fulfill either conditions 1) and 2) or conditions
1) and 3). Concentrating on 1) and 3), we need to write them in an LMI format for seeking the feasible solution. Condition 3) can alternatively be written as
\[ Q \left[ \sum_{i=0}^{k} A_i + BK_i \right]^T + \left[ \sum_{i=0}^{k} A_i + BK_i \right] Q < 0 \]
using congruent transformation with \( Q = P^{-1} \) and by simple manipulation one can obtain (6.39). Furthermore, if we multiply \( A_i + BK_i \geq 0 \) from right by \( Q \) and use \( G_i = K_i Q \), the constraint inequalities (6.40) and (6.41) corresponding to the condition 1) regarding the positivity and Metzlerian constraints can be established. \( \square \)

### 6.7.2 Constrained Stabilization of Interval Delay Systems

Let (6.24) be an unstable interval delay system i.e. \( A_i(\delta) = [\underline{A}_i, \bar{A}_i], \ i = 0, 1, \ldots, k \). Then we seek to obtain a control law (6.36) such that the closed-loop interval system is asymptotically stable and constrained to be Metzlerian. We state the following theorem without proof since the steps of the proof is constructive and follows directly from the proof of Theorem 6.7.3.

**Theorem 6.7.4** Let the feedback control law (6.36) be applied to the general unstable interval time-delay system (6.24), where \( A_i(\delta) = [\underline{A}_i, \bar{A}_i], \ i = 0, 1, \ldots, k \). Then the closed-loop system is asymptotically stable and interval Metzlerian if and only if one of the following equivalent conditions is satisfied:

1. \( \bar{A}_0 + BK_0, \ \sum_{i=0}^{k} (\bar{A}_i + BK_i) \) are stable Metzler matrices, and \( \bar{A}_i + BK_i \geq 0, \ \underline{A}_i + BK_i \geq 0, \ \forall i = 1, \ldots, k \).

2. There exists a vector \( w \in \mathbb{R}^n_+ \) and the gain matrices \( K_i \)'s such that
\[
\left[ \sum_{i=0}^{k} (\bar{A}_i + BK_i) \right] w < 0, \ K_i \in \mathbb{R}^{m \times n}.
\]

3. There exists a positive definite diagonal matrix \( P \) such that
\[
\left[ \sum_{i=0}^{k} (\bar{A}_i + BK_i) \right]^T P + P \left[ \sum_{i=0}^{k} (\bar{A}_i + BK_i) \right] < 0
\]
Furthermore, the feedback control law (6.36) is determined if the following LMI has a feasible solution for \( Q = P^{-1} \in \mathcal{R}^{n \times n} \) and \( G_i \in \mathcal{R}^{m \times n} \),

\[
(\sum_{i=0}^{k} \bar{A}_i)Q + Q(\sum_{i=0}^{k} \bar{A}_i)^T + B(\sum_{i=0}^{k} G_i) + (\sum_{i=0}^{k} G_i)B^T < 0 \tag{6.42}
\]

\[
(\bar{A}_iQ + BG_i)_{jr} \geq 0, \quad \forall \ j \neq r, \ j, r = 1, \ldots, n, \quad i = 1, 2, \ldots, k. \tag{6.43}
\]

\[
(\bar{A}_0Q + BG_0)_{jr} > 0, \tag{6.44}
\]

where \( (M)_{jr} \) means \( jr \) element of a matrix \( M \), and the feedback gain matrices \( K_i, i = 0, 1, \ldots, k \) are obtained by \( K_i = G_iQ^{-1} \).

### 6.8 Positive Observer Design

Chapter 5 considered the observer design for time-delay systems. The proposed methods can equally be applied to the special class of continuous time positive delay systems (Metzlerian delay systems). However, similar to the constrained stabilization problem of previous section, it is also of interest to design observer with structural constraint.

From practical point of view and based on observer theory, constrained observer design makes sense when the original system itself is constrained. Thus, this section is devoted to the design of positive observer (Metzlerian observer) for Metzlerian delay systems.

Consider the following Metzlerian delay system

\[
\dot{x}(t) = Ax(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) + Bu(t)
\]

\[
y(t) = Cx(t)
\]

\[
x(t) = \varphi(t) \in \mathcal{R}^n_+, \quad \text{for} \quad -\tau \leq t \leq 0
\]

\[
\tag{6.45}
\]
where $A$ is a Metzler matrix and $A_i$'s are positive matrices.

An observer for the above system can be represented by

$$
\dot{x}(t) = (A - LC)\dot{x}(t) + \sum_{i=1}^{k} A_i \dot{x}(t - \tau_i) + Ly(t) + Bu(t) \tag{6.46}
$$

where $L$ is the observer gain to be determined. The goal is to give condition on the above observer such that the nonnegative estimates of the states and the asymptotic stability of the observer is guaranteed. This requires that we impose structural constraint on (6.46) so that it represents a Metzlerian delay observer for Metzlerian delay system (6.45).

**Lemma 6.8.1** There exist a Metzlerian delay observer (6.46) for Metzlerian delay system (6.45) if and only if the following conditions holds

(i) $A - LC$ is a Metzler matrix

(ii) $LC \geq 0$

(iii) $A - LC + \sum_{i=1}^{k} A_i$ is a stable Metzler matrix

**Proof** The proof of this lemma follows immediately from the properties of Metzlerian system and its stability conditions (see sections 6.4 and 6.5) when applied to the augmented system

$$
\begin{bmatrix}
\dot{x}(t) \\
\dot{\hat{x}}(t)
\end{bmatrix} =
\begin{bmatrix}
A & 0 \\
LC & A - LC
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\hat{x}(t)
\end{bmatrix} + 
\sum_{i=1}^{k} 
\begin{bmatrix}
A_i & 0 \\
0 & A_i
\end{bmatrix}
\begin{bmatrix}
x(t - \tau_i) \\
\hat{x}(t - \tau_i)
\end{bmatrix} + 
\begin{bmatrix}
B \\
B
\end{bmatrix} u(t). \square
$$

The following theorem provides a computational approach for the design of Metzlerian delay observer using linear programming (LP) adopted from [1]. It should be pointed out that alternative LMI based method similar to Theorem 6.7.3 can also be established. This can be done by transposing the conditions (i)-(iii) of the above lemma and set up the equivalent LMI constraints with $A = A_0^T$, $L = -K_0^T$ and $C = B^T$. Since we deal with a single gain $L$, the associated dual LMI leads to (6.38) of Lemma 6.7.2 with the additional condition of $(LC)^T \geq 0$. 
Theorem 6.8.2 The following statements are equivalent

1. There exist a Metzlerian delay observer (6.46) for Metzlerian delay system (6.45).

2. There exists a matrix $L \in \mathbb{R}^{n \times p}$ such that $A - LC + \sum_{i=1}^{k} A_i$ is stable Metzler matrix.

3. There exist a vector $v \in \mathbb{R}^n_+$ and the gain matrix $L$ such that

\[(A - LC + \sum_{i=1}^{k} A_i)v < 0\]

4. There exists a positive diagonal matrix $P$ such that

\[\begin{bmatrix} A - LC + \sum_{i=1}^{k} A_i \end{bmatrix}^T P + P \begin{bmatrix} A - LC + \sum_{i=1}^{k} A_i \end{bmatrix} < 0\]

Furthermore, the Metzlerian delay observer (6.46) is determined if the following LP problem in the variable $v \in \mathbb{R}^n_+$ and $X \in \mathbb{R}^{p \times n}$ is feasible

\[
\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} < 0
\]

\[A^T \text{ diag } (v) - C^T X + I \geq 0 \]

\[C^T X \geq 0 \]

\[v > 0 \]

where $\text{diag } (v)$ is the diagonal matrix whose diagonal elements are components of vector $v$, and the gain matrix $L$ is obtained by

\[L = [\text{diag } (v)]^{-1} X^T.\]
6.9 Positive Observer-based Control Design

In this section we show that the stabilization of Metzlerian delay systems by using Metzlerian delay observer is not possible. Consider the unstable Metzlerian delay system with control input $u$

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) + Bu(t)$$
$$y(t) = Cx(t)$$

(6.47)

and let the observer be represented by

$$\dot{\hat{x}}(t) = (A - LC)\hat{x}(t) + \sum_{i=1}^{k} A_i \hat{x}(t - \tau_i) + Ly(t) + Bu(t)$$

(6.48)

where state feedback control law $u = K\hat{x}$ is used for the purpose of stabilization.

Theorem 6.9.1 Assume that the zero input response of the system (6.47) is not asymptotically stable, or equivalently $A + \sum_{i=1}^{k} A_i$ is not stable matrix. Then it does not exist a Metzlerian delay observer of the form (6.48) for (6.47) together with a feedback control law $u = K\hat{x}$ that is both asymptotically stabilizing and ensuring the nonnegativity of the observer state, i.e. the separation principle fails.

Proof Assume that $A + \sum_{i=1}^{k} A_i$ is not stable and that there exist matrices $L$ and $K$ which meet the positivity of the closed-loop system under the feedback control $u = K\hat{x}$. The following argument shows that such statement leads to a contradiction.

Since the augmented system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & BK \\ LC & A - LC + BK \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \sum_{i=1}^{k} \begin{bmatrix} A_i & 0 \\ 0 & A_i \end{bmatrix} \begin{bmatrix} x(t - \tau_i) \\ \hat{x}(t - \tau_i) \end{bmatrix}.$$ 

must be positive, then it is easy to see that necessarily $BK \geq 0$ must hold.

Now, by using the following transformation $z_1 = x - \hat{x}$ and $z_2 = \hat{x}$, one can rewrite the augmented system as

$$\dot{z}(t) = \begin{bmatrix} A - LC & 0 \\ LC & A + BK \end{bmatrix} z + \sum_{i=1}^{k} \begin{bmatrix} A_i & 0 \\ 0 & A_i \end{bmatrix} z(t - \tau_i).$$
Thus, if the augmented system is asymptotically stable, then it is necessary that
$A + \sum_{i=1}^{k} A_i + BK$ must be a stable Metzler matrix, which is impossible.

To see this, from the stability properties of Metzler matrix, one can imply the
existence of a vector $v > 0$ such that $(A + \sum_{i=1}^{k} A_i + BK)v < 0$. Consequently, as
$BK$ is positive, it requires that $(A + \sum_{i=1}^{k} A_i)v < 0$. This means that $A + \sum_{i=1}^{k} A_i$ is
necessarily a stable matrix which contradicts the original assumption. $\square$

The above theorem motivates one to seek for an alternative solution to this
problem. Although the separate design of state feedback and observer with posi-
tivity constraint can not be achieved, the possibility of the combined solution may
be possible by employing functional observer design. This will be explored in near
future.
6.10 Illustrative Examples

Example 6.10.1 Consider a Metzlerian system

\[ \dot{x}(t) = (A_0 + D_0 \Delta_0 E_0) x(t) + A_1 x(t - \tau), \]

where 

\[
A_0 = \begin{pmatrix}
-22.3578 & 1.6015 & 0.0138 & 2.3068 \\
1.0000 & -8.0777 & 1.7000 & 4.0013 \\
2.4217 & 3.6563 & -9.9694 & 2.0000 \\
13.2160 & 3.5504 & 0.0519 & -11.6325 \\
-1.0000 & 1.0000 & 0.5000 & 2.0000
\end{pmatrix}, \quad \text{and} \quad D_0 = E_0 = I.
\]

\[
A_1 = \begin{pmatrix}
0.0000 & -1.0000 & 1.0323 & 1.7200 \\
2.4000 & 2.2000 & -1.0000 & 0.5648 \\
1.2000 & 1.0000 & 3.0000 & -1.0000
\end{pmatrix}
\]

A direct computation shows from Theorem 6.6.3 that

\[ r_R = \frac{1}{\| E_0(A_0 + A_1)^{-1} D_0 \|_2} = 0.4488. \]

Applying the real stability radius (3.13) this time, after some iterations, we obtain the same result \( r_R = 0.4488 \). Figure 6.1 shows clearly that for this class of systems the stability radius remains constant independent of delay.

![Figure 6.1: Real stability radius with respect to delay for a Metzlerian system.](image-url)
Example 6.10.2 Consider the system

\[ \dot{x}(t) = (A_0 + D_0 E_0) x(t) + A_1 x(t - \tau_1) + A_2 x(t - \tau_2), \]

where

\[ A_0 = \begin{pmatrix} -3 & 0.5 \\ 0.25 & -2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.4 \end{pmatrix}, \]

and \( D_0 = E_0 = I_2 \).

Applying the real stability radius (3.26) for this case, we obtain that \( r_R = 1.3165 \).

Figure 6.2 clearly shows that the stability surface (shaded) is parallel to \( \tau_1 \) vs. \( \tau_2 \)-plane. By applying Theorem 6.6.4, one can directly obtain the same result.

Figure 6.2: Surface of real stability radius for two time delay Metzlerian system.
Example 6.10.3 Consider the following uncertain delay system

\[
\dot{x} = (A_0 + D_0 \Delta E_0)x(t) + \sum_{i=1}^{2} A_i x(t - \tau_i) + Bu(t)
\]

where
\[
A_0 = \begin{pmatrix} 1.5 & 1.75 \\ 4.75 & -0.75 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and}
\]
\[D_0 = E_0 = I_2.\]

The system without uncertainty can be stabilized using the approach of Theorem 6.7.3 which leads to the feedback gain \(K_0 = [ -4.5 \ -1.25 ]\) and the resulting closed-loop system matrix becomes \(A_{0c} = A_0 + BK_0 = \begin{pmatrix} -3 & 0.5 \\ 0.25 & -2 \end{pmatrix}\). Now, consider the stabilized system with the structured uncertainty as defined above. Then the stability radius can be obtained by using Theorem 6.6.9. A direct computation yields

\[
r_R(A_{0c}, D_0, E_0) = \frac{1}{\| E_0(A_{0c} + A_1 + A_2)^{-1} D_0 \|_2} = 1.32.
\]

Example 6.10.4 Consider the following interval delay system

\[
\dot{x}(t) = [A_{0i}, \bar{A}_{0i}] x(t) + A_1 x(t - \tau_1) + A_2 x(t - \tau_2) + Bu
\]

where
\[
A_0 = \begin{pmatrix} 1.4 & 1.7 \\ 4.75 & -1 \end{pmatrix}, \quad \bar{A}_0 = \begin{pmatrix} 1.5 & 1.75 \\ 4.75 & -0.75 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.4 \end{pmatrix}\] and
\[B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

Using Theorem 6.7.4, the feedback gain \(K_0 = [ -4.5 \ -1.25 ]\) robustly stabilizes the unstable interval delay systems and the closed-loop system becomes Metzlerian stable interval delay system.
Example 6.10.5 Consider the Diesel Engine control system given in chapter 1

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + Bu(t)
\]

where

\[
A_0 = \begin{bmatrix} -27 & 3.6 & 6 \\ 9.6 & -12.5 & 0 \\ 0 & 9 & -5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 21 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.26 & 0 \\ -0.9 & -0.8 \end{bmatrix}.
\]

As it is evident, this system is quasi Metzlerian since \( A_0 \) is Metzlerian, \( A_1 \) is nonnegative but \( B \) is not positive. One can use constrained stabilization discussed in chapter 6 to make this system Metzlerian stable with the feedback gains

\[
K_0 = \begin{bmatrix} 3.1 & 1.27 & 6.32 \\ 6.31 & 0.82 & -7.5 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 18.30 & 0 & 0 \\ 3.19 & 0 & 0 \end{bmatrix}.
\]

The resulting closed-loop system matrices become

\[
A_{0c} = A_0 + BK_0 = \begin{bmatrix} -26.194 & 3.930 & 7.643 \\ 1.762 & -14.299 & 0.312 \\ 1.136 & 9.148 & -6.350 \end{bmatrix},
\]

\[
A_{1c} = A_1 + BK_1 = \begin{bmatrix} 4.758 & 0 & 0 \\ 1.978 & 0 & 0 \\ 0.574 & 0 & 0 \end{bmatrix}.
\]

Now, consider the stabilized system with structured uncertainty

\[
\dot{x} = [(A_0 + BK_0) + D_0 \Delta E_0] x(t) + [A_1 + BK_1] x(t - \tau)
\]

where \( D_0 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \) and \( E_0 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \).

Then the stability radius can be obtained by using Theorem 6.6.9. A direct computation yields

\[
r_R(A_{0c}, A_{1c}, D_0, E_0) = \frac{1}{\| E_0 (A_{0c} + A_{1c})^{-1} D_0 \|_2} = 2.44357.
\]
Chapter 7

Conclusions

In this thesis, the connection between robust stability of multiple time delay systems against delay uncertainties and multi-structured uncertainties is established. A recently introduced stability analysis technique, Cluster Treatment of Characteristic Roots, against multiple delays is utilized to successfully enable the interpretations on the measure of such multi-structured uncertainties in the form of stability radius computations in the multiple delay parametric space. The established connection between these two robustness studies enables, interestingly, a set of two LMIs, feasibility of which guarantees the stability robustness against multiple delays residing in a box and existence of stability radius in this given box. Furthermore, an interesting transformation forming the backbone of CTCR is shown to render successfully and efficiently the computation of the stability radius in the domain of delays. Finally, an explicit formula is defined in place of the stability radius computation algorithm for some specific formations of state and control matrices. Various example studies are provided to present the obtained results.

Also, methods for PI observer and $H_\infty$ asymptotic PI observer design for linear time-delay systems has been developed. The design method not only stabilizes the observer but also minimizes the $H_\infty$ norm between the disturbance and the estimated error.
The design requires solving certain algebraic Riccati equation, which can be tied to an LMI for reliable computation. The proposed method has been generalized to the case of multiple delays in state and output equations. An alternative method for PI observer is also provided to quantify the size of delay.

It should be pointed out that minimizing certain performance criterion allows to find a reasonable solution in many cases. Finally the thesis provides a complete analysis of Metzlerian delay systems. The stability, robust stability, stability radius, stabilization and robust stabilization of this class of systems are studied. As a consequence of this study it was possible to solve the problem of constrained stabilization of delay systems. This allows one to design controller for the general time-delay systems in such a way that the closed-loop system becomes Metzlerian delay stable with desirable properties. The positive observer design for Metzlerian delay systems has also been included in the thesis. A natural outcome of the positive observer design incorporating state feedback controller revealed that the separation principle fails for Metzlerian delay systems. This motivates one to further study this issue in future.

It should also be pointed out that most of the results in this thesis can equally be derived for discrete time-delay systems by careful adjustment imposed by characteristic and stability properties of discrete-time delay systems.
Appendix A

Linear Algebra

In this section, we present the linear algebra that is used throughout the thesis. We begin by providing some useful facts about matrices including vector and matrix norms as well as singular values and their properties. We then present a set of useful linear matrix inequality (LMI) Lemmas, including the Schur complement.

First, consider the following facts about an \( n \times n \) matrix \( A \) without giving any derivations

\[
\begin{align*}
\text{tr}(A) &= \sum_{i=1}^{n} a_{ii}, \\
\hat{\rho}(A) &= \max_{1 \leq i \leq n} |\lambda_i(A)|, \quad \underline{\rho}(A) = \min_{1 \leq i \leq n} |\lambda_i(A)| \\
\sum_{i=1}^{n} \lambda_i(A) &= \text{tr}(A), \quad \prod_{i=1}^{n} \lambda_i(A) = \det(A)
\end{align*}
\]

If \( A, B, \) and \( D \) are square matrices, then we have

\[
\det(AB) = \det(A) \cdot \det(B)
\]

and

\[
\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det(A) \cdot \det(D).
\]

Using the above and assuming \( A \) to be nonsingular, one can write

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}
\]
to obtain
\[
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \cdot \det(D - CA^{-1}B),
\]
which is known as Schur (determinant) complement. Similarly, if \(D\) is nonsingular, we have
\[
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(D) \cdot \det(A - BD^{-1}C).
\]
Note that whenever \(A = I\) and \(D = I\) one arrives at another useful result providing that \(CB\) is square
\[
\det(I - CB) = \det(I - BC),
\]
and if \(c\) and \(b\) are column vectors, we have
\[
\det(I - cb^T) = 1 - b^T c.
\]
We also have a matrix inversion formula if \(A\) or \(D\) is nonsingular,
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} M & -A^{-1}B\Delta_1^{-1} \\ -\Delta_1^{-1}CA^{-1} & \Delta_1^{-1} \end{bmatrix}
\]
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} \Delta_2^{-1} & -\Delta_2^{-1}BD^{-1} \\ -D^{-1}C\Delta_2^{-1} & N \end{bmatrix}
\]
where \(\Delta_1 = D - CA^{-1}B\) (\(\Delta_2 = A - BA^{-1}C\)) is the Schur complement of \(A(D)\),
\(M = A^{-1} + A^{-1}B\Delta_1^{-1}CA^{-1}\) and \(N = D^{-1} + D^{-1}C\Delta_2^{-1}BD^{-1}\). In case \(A\) and \(D\) are both nonsingular, one can deduce the following result,
\[
(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B\Delta_1^{-1}CA^{-1}
\]
\[
(D - CA^{-1}B)^{-1} = D^{-1} + D^{-1}C\Delta_2^{-1}BD^{-1}
\]

**Definition A.1** If \(A\) is an \((n \times n)\) complex matrix, then its singular values are defined as the positive square roots of the eigenvalues of \(A^*A\), where \(A^*\) is the complex conjugate transpose of \(A\). We denote by \(\tilde{\sigma}(A)\) and \(\underline{\sigma}(A)\) the largest and smallest singular values of \(A\).
**Lemma A.1** Singular values, defined in Definition A.1, have the following properties:

\[ \sigma(A - B) \geq \sigma(A) - \bar{\sigma}(B) \]

\[ \sigma(A + B) \geq \sigma(A) + \bar{\sigma}(B) \]

We denote by \( \| \cdot \| \) a norm, which is a function defined on \( \mathbb{C}^n \). The following definition applies to vector norms.

**Definition A.2** The norm function, \( \| \cdot \| \), satisfies the following properties for vectors \( v, v_1 \) and \( v_2 \)

1. \( \|v\| = 0 \) if and only if \( v = 0 \).
2. \( \|\alpha v\| = |\alpha| \cdot \|v\| \).
3. \( \|v_1 + v_2\| \leq \|v_1\| + \|v_2\| \).

Specifically, we denote by \( \| \cdot \|_2 \) to be the Euclidean norm, or the 2-norm,

\[ \|v\|_2 = \sqrt{\sum_{i=1}^{n} |v_i|^2} \]

where \( v_i \) is the \( i^{th} \) element of the \( v \) vector. We define the \( k \)-norm, as

\[ \|v\|_k = \sqrt[k]{\sum_{i=1}^{n} |v_i|^k} \]

The 1-norm and \( \infty \)-norm are also defined as

\[ \|v\|_1 = \sum_{i=1}^{n} |v_i| \quad \text{and} \quad \|v\|_\infty = \max_{1 \leq i \leq n} |v_i| \]

We also define matrix norms by the same symbol, \( \| \cdot \| \), where the argument in the norm function determines if it is either a vector norm or a matrix norm. Thus, we define the induced matrix norm in terms of the vector norm as follows.
Definition A.3  The induced matrix norm, or matrix $k$-norm, $\| \cdot \|_k$, for a matrix $A \in \mathbb{C}^{m \times n}$ is defined as

$$\| A \|_k = \max_{\| v \|_k \leq 1} \| Av \|_k = \max_{\| v \|_k \neq 0} \frac{\| Av \|_k}{\| v \|_k}$$

where the matrix $k$-norm satisfies the norm properties in Definition A.2.

The matrix norm satisfies the property

$$\| AB \|_k \leq \| A \|_k \cdot \| B \|_k$$ \hspace{1cm} (A.1)

The matrix 2-norm, which derives directly out of Definition A.3, is

$$\| A \|_2 = \sqrt{\lambda_{\text{max}}(A^H A)}$$ \hspace{1cm} (A.2)

where $\lambda_{\text{max}}(\cdot)$ is the maximum eigenvalue. Thus, $\| A \|_2 = \sigma(A)$.

Similar to vector norm case, the 1-norm and $\infty$-norm of a matrix is defined by

$$\| A \|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \quad \text{and} \quad \| A \|_\infty = \max_{1 \leq i \leq n} \sum_{i=1}^{n} |a_{ij}|$$

Next, we present a common result to define bounds on a quadratic function. This inequality proves useful for Lyapunov based stability analysis.

Lemma A.2  Consider the quadratic function

$$V(v) = v^T P v$$ \hspace{1cm} (A.3)

where $P$ is a real, symmetric matrix of size $(n \times n)$, and $v$ is a vector of size $n$. Then, (A.3) has the property

$$\lambda_{\text{min}}(P) v^T v \leq v^T P v \leq \lambda_{\text{max}}(P) v^T v$$ \hspace{1cm} (A.4)

where $\lambda_{\text{min}}(P)$ and $\lambda_{\text{max}}(P)$ represent the minimum and maximum eigenvalues of the matrix $P$.

The proof of Lemma A.2 is immediate by diagonalizing $P$ since $P$ is symmetric.
Linear Matrix Inequality (LMI)

First, we define positive definite matrices. Positive definite matrices play a significant role in defining LMIs as well as the Lyapunov stability theory.

Definition A.4 A real \((n \times n)\) matrix, \(A\), is positive (negative) definite if \(v^T Av > 0\) \((v^T Av < 0)\) for all \(v \neq 0\). We denote by \(A > 0\) \((A < 0)\) to indicate \(A\) is positive (negative) definite. Semi-definiteness can similarly be defined by \(A \geq 0\) \((A \leq 0)\).

Definition A.4 is used in the following LMI definition.

Definition A.5 An LMI has the following form

\[
F(v) = F_0 + \sum_{i=1}^{m} v_i F_i > 0
\]

where \(v \in \mathbb{R}^m\) has elements \(v_i\) for \(i = 1, 2, \ldots, m\). and \(F_i = F_i^T \in \mathbb{R}^{n \times n}\) for \(i = 0, 1, \ldots, m\).

The following properties are useful to rewrite LMIs into more transparent forms.

Lemma A.3 For any nonsingular matrix \(A\) of compatible dimension, the matrix inequality

\[
F > 0
\]

is satisfied if and only if

\[
AFA^T > 0
\]

Lemma A.3 is immediate based on the definition of positive definite matrices given in Definition A.4.

A useful tool to break-up an LMI into a set of LMIs of smaller dimension, or vice versa, is presented below. It is known as the Schur complement. It derives out of Lemma A.3.

Lemma A.4 For matrices \(A, B, C\), the inequality

\[
\begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix} > 0
\]

(A.5)
is equivalent to

\[ A > 0 \]  \hspace{1cm} \text{(A.6)}

\[ C - B^T A^{-1} B > 0 \]

or,

\[ C > 0 \]  \hspace{1cm} \text{(A.7)}

\[ A - B C^{-1} B^T > 0 \]

**Proof** We obtain the LMI (A.6) by applying Lemma A.3 to (A.5) as follows,

\[
\begin{bmatrix}
 I & 0 \\
 -B^T A^{-1} & I
\end{bmatrix}
\begin{bmatrix}
 A & B \\
 B^T & C
\end{bmatrix}
\begin{bmatrix}
 I & -A^{-T} B \\
 0 & I
\end{bmatrix} > 0
\]  \hspace{1cm} \text{(A.8)}

and since LMIs are symmetric by Definition A.5, we have \( A = A^T \) in (A.8), which reduces to

\[
\begin{bmatrix}
 A & 0 \\
 0 & C - B^T A^{-1} B
\end{bmatrix} > 0
\]

which is exactly (A.6). Similarly, to obtain the LMI (A.7), we apply Lemma A.3 to (A.5) as follows,

\[
\begin{bmatrix}
 I & -B C^{-1} \\
 0 & I
\end{bmatrix}
\begin{bmatrix}
 A & B \\
 B^T & C
\end{bmatrix}
\begin{bmatrix}
 I & 0 \\
 -C^{-T} B^T & I
\end{bmatrix} > 0
\]  \hspace{1cm} \text{(A.9)}

and since LMIs are symmetric by Definition A.5, we have \( C = C^T \) in (A.9), which reduces to

\[
\begin{bmatrix}
 A - B C^{-1} B^T & 0 \\
 0 & C
\end{bmatrix} > 0
\]

which is exactly (A.7). \( \square \)

Finally, the following lemmas are used to prove various results related to Chapter 2 and Chapter 4. The following lemma is used to eliminate a common matrix variable along the diagonal of two LMIs.

**Lemma A.5** There exists a symmetric matrix \( X \) such that

\[
\begin{bmatrix}
 P_1 - L XL^T & Q_1 \\
 \bullet & R_1
\end{bmatrix} > 0
\]  \hspace{1cm} \text{(A.10)}
if and only if
\[
\begin{bmatrix}
P_1 + LP_2L^T & Q_1 & LQ_2 \\
\bullet & R_1 & 0 \\
\bullet & \bullet & R_2
\end{bmatrix} > 0 \quad (A.12)
\]

**Proof** By applying the Schur complement, defined in Lemma A.4, to (A.12), we have
\[
R_1 > 0 \quad (A.13)
\]
\[
R_2 > 0 \quad (A.14)
\]
and
\[
\hat{\Delta} = P_1 + LP_2L^T - Q_1 R_1^{-1} Q_1^T - LQ_2R_2^{-1} Q_2^T L^T > 0 \quad (A.15)
\]
We can again use the Schur complement to rewrite (A.10) and (A.11) as (A.13), (A.14) and the following
\[
\hat{\Delta}_1 = P_1 - LX L^T - Q_1 R_1^{-1} Q_1^T > 0 \quad (A.16)
\]
\[
\hat{\Delta}_2 = P_2 + X - Q_2 R_2^{-1} Q_2^T > 0 \quad (A.17)
\]
To show that (A.10) and (A.11) are equivalent to (A.12), we show that given (A.13) and (A.14), there exists an \(X\) satisfying (A.16) and (A.17) if and only if (A.15) is satisfied.

Notice that \(\hat{\Delta} = \hat{\Delta}_1 + L\hat{\Delta}_2 L^T\), so if \(X\) satisfies (A.16) and (A.17) then (A.15) is satisfied.

If (A.15) is satisfied, then let \(X = Q_2 R_2^{-1} Q_2^T - P_2 + cI\) for scalar \(c > 0\) sufficiently small gives,
\[
\hat{\Delta}_2 = cI > 0
\]
\[
\hat{\Delta}_1 = \hat{\Delta} - L\hat{\Delta}_2 L^T = \hat{\Delta} - cLL^T > 0.
\]
\[\square\]
The following lemma shows how to eliminate a matrix variable from an LMI

**Lemma A.6** There exist a matrix $X$ such that

$$
\begin{bmatrix}
  P & Q & X \\
  \bullet & R & V \\
  \bullet & \bullet & S
\end{bmatrix} > 0
$$

if and only if

$$
\begin{bmatrix}
P & Q \\
\bullet & R
\end{bmatrix} > 0 \quad \text{and} \quad
\begin{bmatrix}
R & V \\
\bullet & S
\end{bmatrix} > 0.
$$

The following lemmas are also useful

**Lemma A.7** Let $X, Y$ be real constant matrices of compatible dimensions. Then

$$
X^T Y + Y^T X \leq \frac{1}{\epsilon} X^T X + \frac{1}{\epsilon} Y^T Y
$$

holds for any $\epsilon > 0$.

This lemma is critical for transferring a nonsingular problem into the framework of LMI. The proof follows from the inequality

$$
0 \leq \left( \sqrt{\epsilon} X^T - \frac{1}{\sqrt{\epsilon}} Y^T \right) \left( \sqrt{\epsilon} X - \frac{1}{\sqrt{\epsilon}} Y \right)
$$

**Lemma A.8** Let $G, U, V$ be given matrices with $G$ being symmetric.

(i) then there exists a matrix $X$ such that

$$
G + UXV^T + VX^T U^T > 0
$$

if and only if

$$
U^T GU \succ 0, \quad V^T GV \succ 0
$$

where $U_\perp$ and $V_\perp$ are orthogonal complements of $U$ and $V$ respectively.

(ii) $U^T GU \succ 0$ holds if and only if there exist a scalar $\sigma$ such that

$$
G - \sigma UU^T > 0.
$$
Lemma A.9 Let $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{n \times n}$ be symmetric positive definite matrices.

Then there exists a symmetric, positive definite matrix $P > 0$ satisfying

$$P = \begin{bmatrix} Y & \ast \\ \ast & \ast \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} X & \ast \\ \ast & \ast \end{bmatrix}$$

if and only if $X - Y^{-1} \geq 0$. 
Appendix B

Ancillary Theory Proofs

Proof of Theorem 2.2.1:

Considering a Lyapunov functional candidate as

$$V(x(t)) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(\alpha)Qx(\alpha) \, d\alpha$$

where $x(t) = (x(\alpha), t - \bar{\tau} \leq \alpha < t)$. Then

$$\dot{V}(x(t)) = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) + x^T(t)Qx(t) - x^T(t - \tau)Qx(t - \tau)$$

which can be rewritten as

$$\dot{V}(x(t)) = x^T(A_0^TP + PA_0 + Q)x + 2x^T(t - \tau)A_1Px(t) - x^T(t - \tau)Qx(t - \tau)$$

and implies that $\dot{V}(x(t)) < 0$ when (2.4) is negative definite. \( \square \)

The proof of Theorem 2.2.6 for multiple delay case is similar to the proof of Theorem 2.2.1 by considering the Lyapunov - Krasovskii functional

$$V(x(t)) = x^T(t)Px(t) + \sum_{i=1}^{k} \int_{t-\tau_i}^{t} x^T(\alpha)Q_i x(\alpha) \, d\alpha$$

and taking its derivative along the trajectory of (2.2), which leads to

$$\dot{V}(x(t)) = x^T(A_0^TP + PA_0 + \sum_{i=1}^{k} Q_i)x + 2 \sum_{i=1}^{k} x^T(t - \tau_i)A_i^TPx(t)$$
\[-\sum_{i=1}^{k} x^T (t - \tau_i) Q_i x(t - \tau_i)\]

Thus, the stability is guaranteed by imposing $\dot{V}(x(t)) < 0$ or equivalently satisfying the above LMI condition.

**Proof of Theorem 2.2.2:**

Take Lyapunov function $V$ as

$$V = x^T P x \quad \text{(B.1)}$$

where $P > 0$ is a real symmetric matrix. Since the Lyapunov function has a quadratic form, the first condition of Razumikhin Theorem is satisfied by Lemma A.2. To satisfy the second condition, take the time derivative of $V$ along the trajectory of (2.1)

$$\frac{dV(x(t))}{dt} = 2x^T P [A_0 x(t) + A_1 x(t - \tau)] \quad \text{(B.2)}$$

if for some chosen $\pi = 1 + c$, where $c > 0$ sufficiently small, $\exists \ t$, $\forall \ \xi$, such that

$$V(x(t + \xi)) < \pi V(x(t)) \quad \text{(B.3)}$$

For any $\alpha > 0$, we subtract $V(x(t + \xi))$ from both sides of (B.3), and add it to (B.2) to obtain the inequality

$$\frac{dV(x(t))}{dt} \leq 2x^T P [A_0 x(t) + A_1 x(t - \tau)] + \alpha \pi^2 [x^T(t)Px(t) - x^T(t - \tau)Px(t - \tau)] \quad \text{(B.4)}$$

which we rewrite as

$$\frac{dV(x(t))}{dt} \leq [x^T \ x^T(x(t - \tau)) \begin{bmatrix} PA_0 + A_0^T P + \alpha \pi P & PA_1 \\ \bullet & -\alpha P \end{bmatrix}] \begin{bmatrix} x \\ x(t - \tau) \end{bmatrix} \quad \text{(B.5)}$$

which then along with (2.5), implies that the RHS of (B.5) is negative. Finally, since (B.5) is in a quadratic form, then using Lemma A.2, we know that for a sufficiently small $\bar{\psi} > 0$, we satisfy

$$\frac{dV(x(t))}{dt} \leq -\bar{\psi} \|x(t)\|^2 - \bar{\psi} \|x(t - \tau)\|^2 \quad \text{(B.6)}$$
which then implies
\[ \frac{dV(x(t))}{dt} \leq -\bar{\psi} \|x(t)\|^2 \]  
(B.7)
as well, which is the second condition of Razumikhin theorem Hence, all conditions of Razumikhin theorem are satisfied. \qed

Proof of Theorem 2.2.3:

We begin the proof by applying a transformation to (2.1) to obtain a distributed-delay system. We then obtain stability conditions for the distributed-delay system. The following procedures are rewritten and expanded from [25].

PART 1: Transformation

We can immediately obtain a stability condition for (2.1) using Razumikhin theorem, however, the conditions are time-independent, and hence, conservative. One solution to this problem is to transform (2.1) into a distributed delay system:

\[ \dot{x}(t) = \bar{A}_0 x(t) + \int_{-2\tau}^{0} \bar{A}(\theta) x(t + \theta) \, d\theta \]  
(B.8)

where

\[ \begin{align*}
\bar{A}_0 &= A_0 + A_1 \\
\bar{A}(\theta) &= -A_1 A_0, \quad \theta \in [-\tau, 0], \\
\bar{A}(\theta) &= -A_1 A_1, \quad \theta \in [-2\tau, -\tau)
\end{align*} \]  
(B.9)

with initial condition

\[ x(\theta) = \begin{cases} 
\varphi(\theta), & -2\tau \leq \theta \leq \tau, \\
\text{solution of (2.1) with I.C. } \varphi \tau \leq \theta \leq 0
\end{cases} \]  
(B.10)

Notice that the initial condition interval has been expanded from \([-\tau, 0]\) in (2.1) to \([-2\tau, 0]\) in (B.8) which is twice as large. The advantage to such a transformation is that the time-delay value \(\tau\) will appear explicitly in the resulting stability conditions.

To see how we transform (2.1) into (B.8), observe that

\[ \int_{-\tau}^{0} \frac{dx}{dt}(t + \theta) \, d\theta = x(t) - x(t - \tau) \]
Then we can rewrite the expression to obtain \( x(t - \tau) \) as

\[
x(t - \tau) = x(t) - \int_{-\tau}^{0} \frac{dx}{dt}(t + \theta) \, d\theta
\]

and substituting (2.1) for \( \frac{dx}{dt} \), we get

\[
x(t - \tau) = x(t) - \int_{-\tau}^{0} \left[ A_0(t + \theta) + A_1(t - \tau + \theta) \right] d\theta
\]

we then substitute this back into (2.1) to obtain

\[
\frac{dx}{dt} = [A_0 + A_1]x(t) + \int_{-\tau}^{0} \left[ -A_1A_0(t + \theta) - A_1A_1(t - \tau + \theta) \right] d\theta \quad (B.11)
\]

We can then make the assignments (B.9) to obtain (B.8) with initial condition (B.10).

The result of this transformation is that the stability of (B.8) - (B.10) implies the stability of (2.1), but the converse is not true. This is a result of the expanded initial condition (B.10) that does not appear for system (2.1). An effect of the transformation of (2.1) into (B.8) - (B.10) is the introduction of extra dynamics not present in (2.1) [25]. Hence, one of these extra dynamics may be unstable even though the original system (2.1) is stable.

**PART 2: Delay-Dependent Stability Conditions**

We now derive conditions for stability of (B.8) by applying Razumikhin theorem. We choose the Lyapunov function \( V(x) = x^T P x \) for the ensuing analysis. Thus, the first condition of Razumikhin theorem is satisfied by Lemma A.2. Select \( \pi > 1 \).

If \( \exists t, \forall \theta \in [-2\tau, 0] \) such that

\[
V(x(t + \theta)) < \pi V(x(t)),
\]

is satisfied, we calculate the time derivative of \( V \) along the system trajectory (B.8). Thus,

\[
\frac{dV(x(t))}{dt} = 2x^T P \left[ \bar{A}_0 x(t) + \int_{-2\tau}^{0} \bar{A}(\theta) x(t + \theta) \, d\theta \right]
\]
\[ \begin{align*}
\leq & \quad 2x^T P \left[ \tilde{A}_0 x(t) + \int_{-2\tau}^{0} \tilde{A}(\theta) x(t + \theta) \, d\theta \right] \\
& + \int_{-2\tau}^{0} \alpha(\theta) \left[ \pi x^T(t)Px(t) - x^T(t + \theta)Px(t + \theta) \right] \, d\theta
\end{align*} \]

where \( \alpha > 0 \) is a scalar, and we used (B.12). Next, after adding and subtracting 
\[ \int_{-2\tau}^{0} x^T(t)R(\theta)\dot{x}(t) \, d\theta \]
for a symmetric matrix function \( R(\theta) \), and after a little manipulation, we obtain,

\[ \frac{dV(x(t))}{dt} = x^T(t)P \left[ \tilde{A}_0 x(t) + \int_{-2\tau}^{0} \tilde{A}(\theta) x(t + \theta) \, d\theta \right] \\
+ \left[ x^T(t) \tilde{A}_0^T + \int_{-2\tau}^{0} x^T(t + \theta) \tilde{A}^T(\theta) \, d\theta \right] Px(t) \\
+ \int_{-2\tau}^{0} x^T(t)R(\theta)\dot{x}(t) \, d\theta - \int_{-2\tau}^{0} x^T(t)R(\theta)x(t) \, d\theta \\
+ \int_{-2\tau}^{0} \alpha \left[ \pi x^T(t)Px(t) - x^T(t + \theta)Px(t + \theta) \right] \, d\theta
\]

Therefore,

\[ \frac{dV(x(t))}{dt} \leq x^T(t) \left[ PA_0 + A_0^T P + \int_{-2\tau}^{0} R(\theta) \, d\theta \right] x(t) \\
+ \int_{-2\tau}^{0} \left[ x^T(t) x^T(t + \theta) \right] \begin{bmatrix} \alpha(\theta)\pi P - R(\theta) & PA(\theta) \\ \alpha(\theta)P & -\alpha(\theta)P \end{bmatrix} \begin{bmatrix} x(t + \theta) \\ x(t) \end{bmatrix} \, d\theta
\]

Note that \( \pi \) can be arbitrary close to 1. Thus, we see that if the following conditions hold,

\[ \begin{bmatrix} \alpha(\theta)\pi P - R(\theta) & PA(\theta) \\ \alpha(\theta)P & -\alpha(\theta)P \end{bmatrix} < 0, \quad -2\tau \leq \theta \leq 0 \quad \text{(B.13)} \]

then (B.12) and (B.13) imply

\[ \frac{dV(x(t))}{dt} \leq -b\|x(t)\|^2 \quad \text{(B.14)} \]

for some sufficiently small \( b > 0 \). Thus, by Razumikhin theorem, system (B.8) is asymptotically stable. By applying the Schur complement to (B.13) we also get \( P > 0 \).
To summarize, the system with distributed delays described by (B.8) is asymptotically stable if there exists a symmetric matrix $P$ such that (B.12) and (B.13) are satisfied for a symmetric matrix function $R(\theta)$ and a scalar function $\alpha(\theta) > 0$ for $-2\tau \leq \theta \leq 0$.

Next we find more explicit sufficient conditions for (B.12) and (B.13) to hold. If we choose the piecewise constant functions

$$\alpha(\theta) = \begin{cases} 
\alpha_0, & -\tau \leq \theta < 0 \\
\alpha_1, & -2\tau \leq \theta < -\tau 
\end{cases}$$

$$R(\theta) = \begin{cases} 
R_{10}, & -\tau \leq \theta < 0 \\
R_{11}, & -2\tau \leq \theta < -\tau 
\end{cases}$$

where $\alpha_0 > 0$ and $\alpha_1 > 0$ are real scalars, and $R_{10}$, and $R_{11}$ are real symmetric matrices, using (B.9) we see (B.12) and (B.13) hold, if we satisfy

$$P(A_0 + A_1) + (A_0 + A_1)^T P + \tau (R_{10} + R_{11}) < 0 \quad (B.15)$$

$$\begin{bmatrix} 
\alpha_j P - R_{1j} & -PA_1A_j \\
\bullet & -\alpha_j P 
\end{bmatrix} < 0, \quad j = 0, 1. \quad (B.16)$$

Thus sufficient stability conditions are (B.15) and (B.16) where $R_{10} > 0$ and $R_{11} > 0$ are real and symmetric.

**PART 3: Alternative Stability Conditions**

We can rewrite (B.15) and (B.16) without relying on the matrices $R_{10}$ and $R_{11}$ by applying Lemma A.5 twice. Firstly, divide (B.15) by $\tau$, to get

$$\frac{1}{\tau}[P(A_0 + A_1) + (A_0 + A_1)^T P] + R_{10} + R_{11} < 0 \quad (B.17)$$

Notice that we can augment (B.17) into the form that we will use to eliminate $R_{11}$,

$$\begin{bmatrix} 
P_2 + X & Q_2 \\
\bullet & R_2 
\end{bmatrix} < 0 \quad (B.18)$$
where \( R_2 < 0 \) is any arbitrary symmetric, negative definite matrix, and

\[
P_2 = \frac{1}{\tau} [P(A_0 + A_1) + (A_0 + A_1)^T P] + R_{10}
\]

\[X = R_{11}\]

\[Q_2 = 0\]

We can also write (B.16) for \( j = 1 \) as

\[
\begin{bmatrix}
P_1 - LXL^T & Q_1 \\
\bullet & R_1
\end{bmatrix} < 0
\]

(B.19)

where

\[P_1 = \alpha_1 P\]

\[LXL^T = R_{11} \Rightarrow L = I\]

\[Q_1 = -PA_1^2\]

\[R_1 = -\alpha_1 P\]

Then, using Lemma A.5, we eliminate \( R_{11} \) from (B.18) and (B.19) to get

\[
\begin{bmatrix}
P_1 + P_2 & Q_1 & 0 \\
\bullet & R_1 & 0 \\
\bullet & \bullet & R_2
\end{bmatrix} < 0
\]

(B.20)

we can then eliminate \( R_2 \) from (B.20) immediately to get

\[
\begin{bmatrix}
P_1 + P_2 & Q_1 \\
\bullet & R_1
\end{bmatrix} < 0
\]

(B.21)

or equivalently, replacing back using the notation in (B.18) and (B.19)

\[
\begin{bmatrix}
\frac{1}{\tau} [P(A_0 + A_1) + (A_0 + A_1)^T P] + R_{10} + \alpha_1 P & -PA_1^2 \\
\bullet & -\alpha_1 P
\end{bmatrix} < 0
\]

(B.22)

Finally, apply again Lemma A.5 to eliminate \( R_{10} \) from (B.22) and (B.16) for \( j = 0 \). We can write (B.22) in the form of (B.18) by choosing

\[P_2 = \frac{1}{\tau} [P(A_0 + A_1) + (A_0 + A_1)^T P] + \alpha_1 P\]

\[X = R_{10}\]

\[Q_2 = -PA_1^2\]

\[R_2 = \alpha_1 P\]
Appendix B.

We can also write \((B.16)\) for \(j = 0\) in the form of \((B.19)\) by choosing

\[
P_1 = \alpha_0 P
\]

\[
LXL^T = R_{10}
\]

\[
Q_1 = -PA_1A_0
\]

\[
R_1 = -\alpha_0 P
\]

then by Lemma A.5, we get an inequality of the same form as \((2.8)\). Thus, \((2.9)\) is a sufficient condition for the stability of \((2.1)\).

The following upper bound for the inner product of two vectors plays an important role

\[
-2a^T b \leq \inf_{X,Y,Z} \left[ \begin{array}{c} a^T \\ b \end{array} \right]^T \left[ \begin{array}{cc} X & Y - I \\ Y^T - I & Z \end{array} \right] \left[ \begin{array}{c} a \\ b \end{array} \right] \geq 0 \quad \text{(B.23)}
\]

where \(I\) denotes an identity matrix with an appropriate dimension.

A special choice of \(Y\) and \(Z\) such that \(Y = I\) and \(Z = X^{-1}\) in \((B.23)\) provide a well known upper bound

\[
-2a^T b \leq \inf_{X>0,M} \{a^T Xa + b^T X^{-1}b\}
\]

also choosing \(Y = I + XM\) and \(Z = (M^T X + I)X^{-1}(XM + I)\) gives an upper bound

\[
-2a^T b \leq \inf_{X>0,M} \{(a + Mb)^T X(a + Mb) + b^T X^{-1}b + 2b^T Mb\}
\]

Lemma B.1 Assume that \(a(\cdot) \in R^{m_a}\), \(b(\cdot) \in R^{m_b}\) and \(N(\cdot) \in R^{m_a \times m_b}\) are defined on the interval \(\Omega\). Then, for any matrices \(X \in R^{m_a \times m_b}\), \(Y \in R^{m_a \times m_b}\) and \(Z \in R^{m_a \times m_b}\), the following holds

\[
-2 \int_{\Omega} a^T(\alpha) N b(\alpha) \ d\alpha \leq \int_{\Omega} \left[ \begin{array}{c} a(\alpha) \\ b(\alpha) \end{array} \right]^T \left[ \begin{array}{cc} X & Y - N \\ Y^T - N^T & Z \end{array} \right] \left[ \begin{array}{c} a(\alpha) \\ b(\alpha) \end{array} \right] \ d\alpha \quad \text{(B.25)}
\]

where

\[
\left[ \begin{array}{cc} X & Y \\ Y^T & Z \end{array} \right] \geq 0, \quad \text{(B.26)}
\]
Proof of Theorem 4.1.10:

Choose a Lyapunov functional as

\[
V(x(t - \alpha), \alpha \in [0, \bar{\tau}]) = V_1 + V_2 + V_3
\]  

(B.27)

where

\[
V_1 \doteq x^T(t)Px(t)
\]

\[
V_2 \doteq \int_{t-\tau}^{t} \int_{t+\beta}^{t} \dot{x}^T(\alpha)Z\dot{x}(\alpha) \, d\alpha \, d\beta
\]

\[
V_3 \doteq \int_{t-\tau}^{t} x^T(\alpha)Q\dot{x}(\alpha) \, d\alpha
\]

Since it holds that

\[
x(t) - x(t - \tau) \equiv \int_{t-\tau}^{t} \dot{x}(\zeta) \, d\zeta = \int_{t-\tau}^{t} [A_0x(\zeta) + A_1x(\zeta - \tau)] \, d\zeta
\]

the system (2.1) can be written as [29]

\[
\dot{x}(t) = (A_0 + A_1)x(t) - \int_{t-\tau}^{t} [A_0x(\alpha) + A_1x(\alpha - \tau)] \, d\alpha
\]

and thus the derivative of \( V_1 \) satisfies the relation

\[
\dot{V}_1 = 2x^T(t)P(A_0 + A_1)x(t) - 2x^T(t)PA_1\int_{t-\tau}^{t} \dot{x}(\alpha) \, d\alpha
\]

Defining \( a(\cdot), \ b(\cdot), \) and \( N \) in (B.25) as \( a(\cdot) \doteq x(t), \ b(\cdot) \doteq \dot{x}(\alpha), \) and \( N \doteq PA_1 \) for all \( \alpha \in [t - \tau, t] \) and applying Lemma B.1 will supply (B.26) and

\[
\dot{V}_1 \leq 2x^T(t)P(A_0 + A_1)x(t) + \tau x^T(t)Xx(t) + 2x^T(t)(Y - PA_1)\int_{t-\tau}^{t} \dot{x}(\alpha) \, d\alpha + \int_{t-\tau}^{t} \dot{x}^T(\alpha)Z\dot{x}(\alpha) \, d\alpha
\]

\[
\leq x^T(t)(A_0^T P + PA_0 + \bar{\tau}X + Y + Y^T)x(t) + 2x^T(t)(PA_1 - Y)x(t - \tau) + \int_{t-\tau}^{t} \dot{x}^T(\alpha)Z\dot{x}(\alpha) \, d\alpha
\]
Since \( \dot{V}_2 \) and \( \dot{V}_3 \) yield the relation
\[
\dot{V}_2 = \tau [A_0 x(t) + A_1 x(t - \tau)]^T Z [A_0 x(t) + A_1 x(t - \tau)] - \int_{t-\tau}^{t} \dot{x}(\alpha) Z \dot{x}(\alpha) \, d\alpha
\]
\[
\dot{V}_3 = x^T(t) Q x(t) - x^T(t - \tau) Q x(t - \tau)
\]
we have the derivative of \( V \) as
\[
\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3
\]
\[
\leq \begin{bmatrix}
  x(t) \\
  x(t - \tau)
\end{bmatrix}^T \begin{bmatrix}
  Y_{11} & PA_1 - Y + \bar{\tau} A_0^T Z A_1 \\
  A_0^T P - Y^T + \bar{\tau} A_0^T Z A_0 & -Q + \bar{\tau} A_1^T Z A_1
\end{bmatrix} \begin{bmatrix}
  x(t) \\
  x(t - \tau)
\end{bmatrix}
\]
where
\[
Y_{11} = A_0^T P + PA_0 + \bar{\tau} X + Y + Y^T + \bar{\tau} A_0^T Z A_0 + Q.
\]

Then using the Lyapunov - Krasovskii stability theorem and Schur complement [5], we can conclude that the system (2.1) is asymptotically stable if (2.19) and (2.20) hold. This completes the proof. \( \square \)

**Proof of Theorem 4.1.1:**

Using Theorem 2.2.1 it is sufficient to show the asymptotic stability of the closed loop system by
\[
\left[ [A_0 + BK_0]^T P + P [A_0 + BK_0] + Q \begin{bmatrix} PA_1 & \cdot \\ \cdot & -Q \end{bmatrix} \right] < 0
\]
pre- and post- multiplying the above inequality by \( \text{diag}(P^{-1}, P^{-1}) \) and letting \( X = P^{-1}, Y = K_0 P^{-1}, U = P^{-1} Q P^{-1} \) yields (4.4). Therefore, if \( X > 0, U > 0, \) and \( Y \) satisfy (4.4). Then \( P = X^{-1}, Q = X^{-1} U X^{-1} \) and \( K_0 = Y X^{-1} \) satisfy the above inequality. This completes the proof. \( \square \)

**Proof of Theorem 4.1.2:**

The proof of Theorem 4.1.2 is similar to the proof of Theorem 4.1.1 and can be established in light of Theorem 2.2.1 when \( \tau \) is known constant. \( \square \)
Proof of Theorem 4.1.11:

In view of the closed-loop system of (4.1) with the control \( u = K_0 x(t) \), we replace \( A_0 \) in (4.18) with \( A_0 + BK_0 \). Now pre- and post- multiply \( \text{diag}(P^{-1}, P^{-1}, Z^{-1}) \) and \( \text{diag}(P^{-1}, P^{-1}) \) to (4.18) and (4.19), respectively and apply the change of variables such that \( L = P^{-1} \), \( M = P^{-1} XP^{-1} \), \( N = P^{-1} Y P^{-1} \), \( R = Z^{-1} \), \( W = P^{-1} Q P^{-1} \), and \( V = K_0 P^{-1} \), then we obtain (4.23) and (4.24). This completes the proof. \( \square \)

Proof of Theorem 4.1.17:

The closed-loop system of (4.43) - (4.45) with the control (4.46) can be described by the following state-space model

\[
\dot{z} = \hat{A}_0 z(t) + \hat{A}_1 z(t - \tau) \tag{B.28}
\]

where

\[
z = [x^T \ x_c^T]^T, \quad \hat{A}_0 = \bar{A}_0 + \bar{B}K\bar{C}, \quad K = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}, \tag{B.29}
\]

\[
\bar{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}. \tag{B.30}
\]

Applying Lemma 4.1.15 to the system (B.28), it follows that the controller (4.46) solves the stabilization problem for a given \( \bar{\tau} > 0 \) if there exist a \( 2n \times 2n \) matrix \( X > 0 \) and a scaler \( \beta > 0 \) such that

\[
\bar{G}_x + \bar{Y}^T K\bar{Z}_x + \bar{Z}_x^T K^T \bar{Y} < 0 \tag{B.32}
\]

where

\[
\bar{G}_x = \begin{bmatrix} Q_x & XA^T & XA_1^T & \bar{A}_1 \\ \bullet & -\frac{1}{\bar{\tau}} \beta I & 0 & 0 \\ \bullet & \bullet & -\frac{1}{\bar{\tau}} (1 - \beta) I & 0 \\ \bullet & \bullet & \bullet & -\frac{1}{\bar{\tau}} I \end{bmatrix}. \tag{B.33}
\]
\[ Q_X = X(A_0 + A_1)^T + (\bar{A}_0 + \bar{A}_1)X, \]
\[ Y = \begin{bmatrix} \bar{B}^T & \bar{B}^T & 0 & 0 \end{bmatrix}, \]
\[ Z_X = \begin{bmatrix} CX & 0 & 0 & 0 \end{bmatrix}. \]

By Lemma 4.1.16, the inequality (B.32) is equivalent to

\[ N_Y^T G_X N_Y < 0, \quad N_{Z_X}^T G_X N_{Z_X} < 0 \quad \text{(B.34)} \]

where \( N_Y \) and \( N_{Z_X} \) are any matrices whose columns from bases of the null spaces of \( Y \) and \( Z_X \), respectively.

Note that by defining

\[ Z = \begin{bmatrix} \bar{C} & 0 & 0 & 0 \end{bmatrix} \quad \text{(B.35)} \]

we have

\[ Z = Z_X^{-1} \]

where

\[ J_X = \begin{bmatrix} X & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \]

Hence, the columns of

\[ N_Z = J_X N_{Z_X} \]

from a basis of the null space of \( Z \). This implies that

\[ N_{Z_X}^T G_X N_{Z_X} = N_{Z_X}^T J_X G_{X^{-1}} J_X N_{Z_X} = N_{Z_X}^T G_{X^{-1}} N_{Z_X} \]

where

\[ \bar{G}_{X^{-1}} = \begin{bmatrix} Q_{X^{-1}} & A_0^T & A_1^T & X^{-1}A_1 \\ \bullet & -\frac{1}{\tau} \beta I & 0 & 0 \\ \bullet & \bullet & -\frac{1}{\tau}(1 - \beta)I & 0 \\ \bullet & \bullet & \bullet & -\frac{1}{\tau} I \end{bmatrix} \quad \text{(B.36)} \]
\[ Q_{x^{-1}} = (\bar{A}_0 + \bar{A}_1)^T X^{-1} + X^{-1} (\bar{A}_0 + \bar{A}_1) \]

Hence, it follows that \( N_{x}^T G_x N_{x} < 0 \) is equivalent to \( N_{x}^T G_{x^{-1}} N_{x} < 0 \).

Next, we shall express the conditions \( N_{x}^T G_x N_{x} < 0 \) and \( N_{x}^T G_{x^{-1}} N_{x} < 0 \) in terms of the plant parameters. To this end, we shall partition \( X \) and \( X^{-1} \) as

\[
    X = \begin{bmatrix} S & N \\ \bullet & V \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} R & M \\ \bullet & U \end{bmatrix}
\]

where \( R, S, M, N, U \) and \( V \) are \( n \times n \) real matrices. Note that in view of (4.45) and considering that \( R, S, M \) and \( N \) satisfy

\[ MN^T = I - RS \]

it results that \( M \) and \( N \) are non-singular matrices. Moreover, it can be easily established that given any non-singular matrices \( S > 0 \), \( R > 0 \) and \( N \), there exist unique matrices \( M, U \) and \( V \) such that \( X > 0 \). Indeed, we have that

\[
    M = (I - RS)N^{-T}, \quad N^{-1} (SRS - S)N^{-T}, \quad V = N^T (S - R^{-1})^{-1} N.
\]

Time-delay systems with the partition as in (B.37) and considering (B.31), \( G_x \) and \( G_{x^{-1}} \) of (B.33) and (B.36), respectively, can be rewritten as:

\[
    \tilde{G}_x = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ \bullet & Q_1 & 0 & 0 \\ \bullet & \bullet & Q_2 & 0 \\ \bullet & \bullet & \bullet & Q_3 \end{bmatrix}
\]

\[
    \tilde{G}_{x^{-1}} = \begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} & \hat{G}_{13} & \hat{G}_{14} \\ \bullet & Q_1 & 0 & 0 \\ \bullet & \bullet & Q_2 & 0 \\ \bullet & \bullet & \bullet & Q_4 \end{bmatrix}
\]
where
\[
G_{11} = \begin{bmatrix} Q_s & (A_0 + A_1)N \\ \bullet & 0 \end{bmatrix}, \quad G_{12} = \begin{bmatrix} SA_0^T \\ N^T A_0^T \end{bmatrix}, \quad G_{13} = \begin{bmatrix} SA_1^T \\ N^T A_1^T \end{bmatrix}, \quad G_{14} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}, \quad \bar{G}_{11} = \begin{bmatrix} Q_s R & M^T (A_0 + A_1)N \\ \bullet & 0 \end{bmatrix}, \quad \bar{G}_{12} = \begin{bmatrix} A_0^T \\ 0 \end{bmatrix}, \quad \bar{G}_{13} = \begin{bmatrix} A_1^T \\ N^T A_1^T \end{bmatrix}, \quad \bar{G}_{14} = \begin{bmatrix} M^T A_1 \\ 0 \end{bmatrix},
\]

\[
Q_1 = \begin{bmatrix} -\frac{1}{\tau} \beta I & 0 \\ 0 & -\frac{1}{\tau} \beta I \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -\frac{1}{\tau} (1 - \beta) I & 0 \\ 0 & -\frac{1}{\tau} (1 - \beta) I \end{bmatrix},
\]

and the matrices
\[
Q_s = S (A_0 + A_1)^T + (A_0 + A_1) S
\]

and
\[
Q_R = (A_0 + A_1)^T R + R (A_0 + A_1).
\]

On the other hand, considering that
\[
Y = \begin{bmatrix} \bar{B}^T & \bar{B}^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & I & 0 & 0 & 0 & 0 \\ B^T & 0 & B^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

and
\[
Z = \begin{bmatrix} \bar{C} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]
it follows that $\mathcal{N}_Y$ and $\mathcal{N}_Z$ are of the form as below:

$$
\mathcal{N}_Y = \begin{bmatrix}
N_{B_1} & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
N_{B_2} & 0 & 0 & 0 & 0 & 0 \\
0 & -I & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I \\
\end{bmatrix},
$$

(B.41)

and

$$
\mathcal{N}_Z = \begin{bmatrix}
N_C & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & 0 & I \\
\end{bmatrix},
$$

(B.42)

where $\begin{bmatrix} N_{B_1} \\ N_{B_2} \end{bmatrix}$ and $N_C$ are any matrices whose columns form bases of the null spaces of $[B^T \quad B^T]$ and $C$, respectively.

By considering (B.39) and (B.41), it can be easily established that the condi-
tion $N_y^T G_x N_y < 0$ is equivalent to

$$
\begin{bmatrix}
N_{b_1} & 0 & 0 & 0 \\
N_{b_2} & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
\end{bmatrix}^T
\begin{bmatrix}
N_{b_1} & 0 & 0 & 0 \\
N_{b_2} & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
\end{bmatrix} < 0 \quad \text{(B.43)}
$$

where

$$\bar{H}_s = \begin{bmatrix}
Q_s & S A_0^T (A_0 + A_1)N & S A_1^T & A_1^T \\
-\frac{1}{\tau} \beta I & A_0 N & 0 & 0 \\
-\frac{1}{\tau} \beta I & N^T A_1^T & 0 & 0 \\
-\frac{1}{\tau} (1 - \beta) I & 0 & 0 & 0 \\
\end{bmatrix},$$

Now, multiplying (B.43) on the left and on the right by $H^T$ and $H$, respectively, where

$$H = \begin{bmatrix}
I & 0 & 0 & 0 \\
0 & N^{-1} & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
\end{bmatrix},$$

and introduce the new variable

$$P = \beta (NN^T)^{-1} > 0,$$

on the other hand, by considering (B.40) and (B.42) it can be easily shown that the condition $N_x^T G_{x^{-1}} N_x < 0$ is equivalent to the inequality of (4.82) which completes the proof. \qed

**Proof of Theorem 4.2.6:**

Replace $A_0$ and $A_1$ in (2.1) with $A_0 + E_0 \Delta_0 E_0$ and $A_1 + E_1 \Delta_1 E_1$, respectively and multiply both sides of the resulting matrix by vectors $x_i, \ i = 1, 2, 3$.

Next define

$$p = \Delta_0 F_0 x_1, \quad q = \Delta_1 F_1 x_2$$
then we have the condition

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    p \\
    q
\end{bmatrix}
\begin{bmatrix}
    Y_{11} & -Y + PA_1 & \bar{\tau}A_0^TZ & PE_0 & PE_1 \\
    -Y^T + A_1^TP & -Q & \bar{\tau}A_1^TZ & 0 & 0 \\
    \bar{\tau}ZA_0 & \bar{\tau}ZA_1 & -\bar{\tau}Z & \bar{\tau}ZE_0 & \bar{\tau}ZE_1 \\
    E_0^TP & 0 & \bar{\tau}E_0^TZ & 0 & 0 \\
    E_1^TP & 0 & \bar{\tau}E_1^TZ & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    p \\
    q
\end{bmatrix} < 0 \quad (B.44)
\]

where \( Y_{11} = A_0^TP + PA_0 + \bar{\tau}X + Y + Y^T + Q \), for all admissible \( \Delta_0 \) and \( \Delta_1 \).

Since the conditions \( \|\Delta_0\| \leq 1 \), and \( \|\Delta_1\| \leq 1 \) can be replaced with the existance conditions of \( e_1 > 0 \) and \( e_2 > 0 \) such that

\[
e_1p^T p \leq e_1x_1^TF_0^TF_0x_1, \quad e_2q^T q \leq e_2x_3^TF_1^TF_1x_3
\]

Applying the S-procedure [5] allows us to obtain (4.70). This completes the proof. \( \square \)
Bibliography


