Distributed Control of Multi-Agent Systems with Switching Topology, Delay, and Link Failure

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To My Family
Abstract

From ecology and evolutionary biology to social sciences, and from systems and control theory to aerospace and wireless sensor networks, researchers have been trying to develop an understanding of how a group of moving objects such as flocks of birds, schools of fish and crowds of people can perform collective tasks such as reaching a consensus or moving in a formation without centralized coordination. Researchers in the fields of robotics and control theory have also become interested in cooperative control of multi-agent systems such as a group of unmanned vehicles due to its vast variety of applications. This dissertation focuses on the study of distributed control of interconnected multi-agent systems. Research is conducted towards designing and analyzing distributed control algorithms for a fixed interconnection pattern as well as in the presence of switching network topology, interconnection delays between agents, and random link failure.

A decomposition approach is considered to design distributed controller for coupled dynamic systems for both fixed and switching network topology. A special similarity transformation constructed from interconnection pattern matrix along with the result of extended linear matrix inequality (LMI) formulation for continuous-time systems makes it possible to derive explicit expression for computing the parameters of distributed controller for both static state feedback and dynamic output feedback cases. The distributed controllers are designed under $H_2$, $H_\infty$ and $\alpha$-stability performances.

In the presence of interconnection delays, a distributed control design methodology is proposed, which realizes neighbor-based local controller for each agent. By transforming the original system into a decomposed structure and taking advantage of Lyapunov-Krasovskii functional, LMI based conditions are derived to design both
static state feedback and dynamic output feedback controllers.

Moreover, multi-agent dynamic systems with random link failure are modeled by a linear discrete-time system with multiplicative random coefficients. A distributed control algorithm which uses only local and available neighboring information is proposed to stabilize the system in mean-square sense. A sufficient condition for designing stabilizing distributed controller is provided that ensures prescribed disturbance attenuation in $L_2$ gain sense. The design is carried out using linear matrix inequalities.

The problem of designing decentralized PI-observer based controllers is also considered in the convex optimization context for interconnected systems with linear subsystems and nonlinear time-varying interconnections. When the interconnections are linear, distributed PI-observer is introduced for disturbance estimation and fault detection.

Finally, distributed controller based on linear quadratic regulator (LQR) is proposed for identical decoupled linear systems with a common objective. Two types of distributed dynamic output feedback algorithms are developed with local observers and distributed observers. The design provides a straightforward way to construct controller and observer gains that are decoupled from the communication graph structure.
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Chapter 1

Introduction

1.1 Background and Motivation

Modern technologies rely on the cooperation of many different machines, robots, transportation systems where their parts are linked by common resources through information networks. The analysis and control of such large-scale systems become very complex due to lack of full sensor information and limited computational capability. This crisis first was realized by the systems and control community in the 1970’s and numerous efforts have been made since then. Three different control architectures for large-scale systems are shown in Figure 1.1.

For the first architecture, Figure 1.1 (a), a centralized controller $K$ is used to control the dynamic system $S$. One direct way to solve the complexity crisis is model reduction [1]-[2]. Model reduction algorithms try to find reduced order systems both in time domain and frequency domain so that the complexity of the central controller can be moderated. The classical approaches to model reduction are summarized in [3]. After the publication of [4], and the development of balanced realization, researchers showed renewed interests in model and controller reduction techniques (see [5], [6] and references therein).
The second architecture in Figure 1.1 is decentralized control. In this scenario the dynamic system $S$ is regarded as an interconnection of subsystems, which may be known as physical entities or a purely mathematical artifice. Instead of using one centralized controller, separated controller $\{K_1, \ldots, K_n\}$ are introduced. The most definitive results on decentralized stabilization can be found in [7]- [10]. These references summarize the early research and provide details on how to use the vector Lyapunov method and weighted-sum method to test the stability of interconnected systems. The interpretation of their results is that if the gain of the local feedback
loop is smaller than 1 and the couplings between subsystems are weak enough, then the overall system is stable. Moreover, the concept of input-output stability for interconnected system is proposed in [11] and several exciting results are presented in [12], [13], and [14].

The control of large-scale interconnected systems based on a decentralized architecture has received considerable attention due to its merit upon centralized controllers. However, there are some key limitations in the use of a completely decentralized architecture. In general, the use of a decentralized architecture is efficient only for weakly interconnected systems. In the case of strong interconnection effects, the local controller is in most cases forced to generate unnecessary large control laws in order to compensate the unknown interconnection effects. As a consequence, there is an ever increasing interest in distributed control architectures, where each subsystem is allowed to exchange some information with the other subsystems.

The third architecture shown in Figure 1.1 (c), is distributed control. In this scenario the dynamic system is composed of multiple subsystems \( \{S_1, \ldots, S_n\} \) interaction with each others through an information exchange/interconnection network. Each subsystem is equipped with its own controller which uses the local and the neighbors’ information according to existing information exchange network. Thus, distributed control law are generated according to not only local feedback, but also the available information from other subsystems. Cooperative and coordinated control of networked multi-agent systems belongs to this structure.

In recent years, cooperative and coordinated control of multi-agent systems has received significant attention due to the development of communication and computation capabilities which facilitate the information exchange between spatially distributed systems. This enables each agent interacts autonomously with the environment and other agents to perform tasks through collaboration that may be too difficult
for an individual agent. Multi-agent systems have broad applications in many areas including space exploration [15]-[18], automated highway systems [19]-[22], oceanographic sampling [23]-[24], air traffic control [25], [26], flocking [27]-[29], formation control [30], [31], congestion control [32] and sensor networks [33], [34].

1.2 Previous Works

The study of cooperative control of multi-agent systems is largely inspired by the collective behavior of biological systems, such as colonies of ants, hives of honey bees, flocks of birds, and schools of fish. Usually, in these systems, each individual has very limited sensing, communication, and manipulation abilities. However, a well organized large collections of these elementary, simple individuals can produce remarkable capabilities and display highly complex behaviors by following some simple rules which require only local interactions among the individuals. Inspired by the collective behavior of biological systems, researchers have started focusing their attention on distributed control methods, where each agent makes decisions based solely on local information. The major objective of the research is to achieve better performance by designing a distributed control method which can assemble a number of simple, inexpensive machines rather than a single, complicated, expensive machine.

A considerable amount of research work has contributed to the analysis and implementation of the agreement problem or consensus. In the context of control systems, driving a certain state-dependent variable of each individual agent to a common value by distributed protocol is called consensus problem. The history behind this research line can be traced back to Reynolds “boids” model [35] and the work by Vicsek et al. [36]. In [35], each agent only reacts to its neighboring flock mates, i.e., those agents within a certain distance. In addition, they have to follow three ad/hoc protocols, namely, separation, alignment, and cohesion. As a special case, Vicsek et al. [36]
studied the situation where all the agents move at the same constant speed and update their headings following a nearest-neighbor rule. Velocity cohesion and flocking behavior were observed in both cases. In [38], the nearest-neighbor rule is characterized and a theoretical explanation for the observed behavior of Vicsek model has been given by exploiting the ideas from graph theory and stochastic matrix theory.

Suppose \( x_i \) is the state of agent \( i \) and the protocol can be summarized as

\[
x_i(k+1) = \sum_{j \in \mathcal{N}_i \cup \{i\}} \alpha_{ij}(k)x_j(k)
\]

where \( \mathcal{N}_i \) represents the set of agents whose state is available to agent \( i \) at step \( k \). We assume \( \alpha_{ij} \geq 0 \) and \( \sum_j \alpha_{ij}(k) = 1 \). In other words, agent state is updated as the weighted average of its current value and its neighbors’. Correspondingly, a continuous-time consensus protocol can be found in [41], [42] as

\[
\dot{x}_i(t) = -\sum_{j \in \mathcal{N}_i} \beta_{ij}(t)(x_i(t) - x_j(t))
\]

where \( \beta_{ij}(t) \) denotes the positive weight. The system has achieved a consensus if \( \|x_i - x_j\| \) converges to zero as \( t \to \infty \) for any \( i \neq j \). More specifically, it is called average consensus when \( x_i \to \sum x_i(0)/N \) as \( t \to \infty \) for any \( i \) where \( N \) is the number of agents.

In the last decade, numerous results were obtained [37] - [40] and consensus protocols has been employed in many engineering problems such as coordinated control to achieve vehicle formations, rendezvous problems to control agents arrive at a certain location simultaneously, spacecraft attitude alignment, distributed decision making, asynchronous peer-to-peer networks, and robot synchronization. Among them, formation control is a popular research topic which has been investigated in numerous applications such as coordination of multiple robots, formation of unmanned aerial vehicles (UAVs), underwater vehicles and satellite clusters. Although each application has its unique characteristics and challenges, there exist common features. In most of
the application, agents have identical dynamics and similar local controller structure. The control objective is letting agents maintain certain geometric formation by autonomously responding to other agents and the environment. Obviously, formation control is consistent with the third architecture in Figure 1.1; i.e. distributed control.

For agent $S_i$, we can present its dynamics as

$$\dot{x}_i = f(x_i, u_i(x_i, \bar{x}_i))$$  (1.3)

where $\bar{x}_i$ is the information collected by agent $i$ from the neighbors and determined by the interaction topology. The local controller $K_i$ generates the control law $u_i(x_i, \bar{x}_i)$. When agents have identical dynamics and local controllers, the interaction topology plays an important role for formation control. One powerful tool to study the interaction topology is graph theory, a mathematic theory on the properties of graphs. The objects of graph theory are a set of nodes and the edges between them. Basic ideas from algebraic graph theory has been summarized in appendix A.

A precise statement on formation stability for linear dynamics is presented in [39] where the formation stability is shown to be equivalent to the stability of a sequence of decoupled inhomogeneous subsystems. Here we briefly report their results. Let us consider a set of $N$ identical linear systems (agents, vehicles, etc.), whose dynamics is modeled by the equation:

$$\dot{x}_i = Ax_i + Bu_i$$  (1.4)

where $x_i \in \mathbb{R}^n$ are the agents states, $u_i \in \mathbb{R}^m$ their control inputs and $i = 1, ..., N$ is the index for the vehicles in the formation. From this it follows that the dynamics of the formation is described by the equation

$$\dot{x} = (I_N \otimes A)x + (I_N \otimes B)u$$  (1.5)

where we have $x = [x_1^T, x_2^T, ..., x_N^T]^T \in \mathbb{R}^{nN}$ and $u = [u_1^T, u_2^T, ..., u_N^T]^T \in \mathbb{R}^{mN}$. Let us now assume that each vehicle has a limited visibility with respect to the others; for
this purpose we define the set $\mathcal{N}_i \subset [1, \ldots, N] \setminus \{i\}$ of the vehicles that the $i$th vehicle can sense. Then, each vehicle has the following measurements available for feedback control:

$$y_i = C_a x_i$$

$$q_i = \frac{1}{|\mathcal{N}_i|} C_b \sum_{j \in \mathcal{N}_i} (x_i - x_j)$$

(1.6)

where $|\mathcal{N}_i|$ is the number of elements of the set $\mathcal{N}_i$ (assume all $|\mathcal{N}_i| \neq 0$: all agents can see at least one other agent). In this way, the global output function is equivalent to

$$y = (I_N \otimes C_a)x$$

$$q = (L \otimes C_b)x$$

(1.7)

where as before $y = [y_1^T, y_2^T, \ldots, y_N^T]^T \in \mathbb{R}^{r_y N}$ and $q = [q_1^T, q_2^T, \ldots, q_N^T]^T \in \mathbb{R}^{r_q N}$; thanks to the definition of $q$ in (1.7), it is possible to prove that $L$ is indeed the normalized Laplacian of the graph (see Appendix A) that describes the information flow in the formation (i.e., an edge connects node $i$ to node $k$ iff agent $k$ receives the output of agent $i$).

Let us now assume that each vehicle is locally controlled by identical local controllers $K$

$$\dot{v}_i = K_A v_i + K_{B_y} y_i + K_{B_q} q_i$$

$$u_i = K_C v_i + K_{D_y} y_i + K_{D_q} q_i$$

(1.8)

Then the following Theorem holds.

**Theorem 1.2.1.** A local controller $K$ as in (1.8) stabilizes the formation dynamics in (1.4), (1.6) if and only if it simultaneously stabilizes the following set of $N$ “modal” subsystems:

$$\dot{\hat{x}}_i = A \hat{x}_i + B \hat{u}_i$$

$$\hat{y}_i = C_a \hat{x}_i$$

$$\hat{q}_i = \lambda_i C_b \hat{x}_i$$

(1.9)

where the $\lambda_i$ are the eigenvalues of the matrix $L$ of (1.7).
The proof is based on the fact that for any square matrix $L$ there exists a Schur transformation such as:

$$L = T^{-1}UT$$

where $T$ is unitary and $U$ is upper diagonal, with the eigenvalues of $L$ on the diagonal; both $T$ and $U$ can be complex-valued. Through this observation it is possible to show that the formation in closed loop is equivalent to a block upper diagonal system, and so its stability depends on the stability of the diagonal blocks, which are equivalent to the $N$ systems of Theorem 1.2.1 in closed loop with the local controllers. This result is valid for any pattern matrix $L$ but the use of Laplacians will allow having information on the $\lambda_i$ without computing them.

The stability theorem proposed in [39] shows utility in analyzing the behavior of vehicle formations and in synthesizing control solutions. This useful framework in analysis of formation stability problems became a starting point for further research in this area [43], [44].

In [43], the authors extended the stability result of [39] to the case of more general systems, where a certain (limited) dynamic interaction between the subsystems is allowed. It should be noted that the interaction in this scenario follows the same pattern matrix. By assuming the pattern matrix $L$ to be diagonalizable, a variant of Theorem 1.2.1 is proved for the systems called decomposable systems. The precise definition of decomposable systems is presented in Chapter 2 where we follow this work and design distributed control for such systems in continuous-time domain.

The initial work of distributed control assumes fixed network topology with perfect interconnection links. However, this might not be the case in practical situations where interconnection links may change due to different reasons. For example, in formation control, some of the existing communication links can fail simply due to the existence of an obstacle. Moreover, the communication channels are imperfect and
subject to errors, packet drops and delays. Therefore, further insight to the problem of distributed control lead to variation of this problem under certain constraints. In this direction, many researcher consider the problem of distributed control under various conditions such as switching network topology, link failures and existence of time-delay in the network.

The distributed consensus problem has been studied in switching networks, where consensus is defined in a deterministic sense. [38] and [45] show that in undirected, switching communication networks, convergence is guaranteed if there is an infinitely occurring, contiguous sequence of bounded time intervals in which the network is jointly connected. The same condition also guarantees consensus in directed networks, as shown by [40] and [46]. [47] identifies a similar condition for consensus in directed networks based on an infinitely occurring sequence of jointly rooted graphs.

The impact of communication delays on consensus seeking are also studied based on various analysis methods. [41], [66]-[71]. In [41] and [66] a necessary and sufficient condition for a time-delay consensus of first-order dynamic agents was presented for undirected interconnection graph. Also, using tree-type transformation, some necessary and/or sufficient conditions are provided in [67] for consensus of multi-agent systems with non-uniform time-varying delays. For second-order multi-agent systems, the consensus conditions have been obtained under identical communication delay based on frequency domain analysis in [68] and [69]. For these systems, the consensus criteria are also obtained in the form of LMI using Lyapunov-Krasovskii and Lyapunov-Razumikhin functions in [70] and [71].

The imperfection of interconnection links in multi-agent systems can also be modeled by random variables. The randomness assumption on the existence of an information channel between agents, although in general conservative, offers a natural modeling framework for situations of practical interest [48]. The mathematical tools
that allows one to extend the analysis for these systems over static networks to random ones turns out to be stochastic stability. In this scenario, the convergence is defined in terms of the variance of deviation from average. For related works and survey on multi-agent coordination under random links assumption see [48], [49], [50] and the references therein.

Another important problem in multi-agent systems is fault detection and accommodation. As practical systems become large and complex, fault diagnosis and accommodation are becoming more critical issue. In practice, it is very difficult to address the problem of diagnosing faults in a large-scale system with a centralized architecture because of the constraints on computational power and communication bandwidth needed by real-time fault diagnosis. Decentralized fault diagnosis schemes have been captured a lot of interest in recent years [51]-[53]. On the other hand, a distributed architecture, where each subsystem is allowed to exchange limited information, avoids many of the disadvantages of totally decentralized architectures as it has already been emphasized. A distributed architecture can achieve the desired robustness for a greater class of interconnected systems subject to faults due to the exchange of information between subsystems. The distributed fault detection for interconnected systems is reported in [54], [55] and [56].

1.3 Organization

This dissertation is organized as follows. In Chapter 2, we consider the problem of designing distributed controller for a continuous-time system composed of a number of identical dynamically coupled subsystems called decomposable systems. The underlying mathematical derivation is based on Kronecker product and a special similarity transformation constructed from interconnection pattern matrix that decompose the system into modal subsystems. This along with the result of extended linear ma-
trix inequality (LMI) formulation for continuous-time systems makes it possible to derive explicit expression for computing the parameters of distributed controller for both static state feedback and dynamic output feedback cases. The main contribution of this chapter is the solution of distributed control problem for continuous-time systems under $H_2$, $H_\infty$ and $\alpha$-stability performances by solving a set of LMIs with non-common Lyapunov variables. The effectiveness of this method is demonstrated by means of a satellite formation example [95], [100].

In Chapter 3, the problem of designing distributed controller for decomposable systems with switching network topologies is considered. The modal decomposition approach is used to design a distributed controller that has the same interconnection structure as the networks in each instance; i.e. controller structure changes according to network topologies without any delay. An explicit expression for computing the controller parameters is derived using LMI approach under $H_2$ or $H_\infty$ criteria with $\alpha$-stability property. Moreover, a simple sufficient condition for the existence of such a controller is provided that ensures the stability of the network with any arbitrary switching provided that the networks switch among a finite set of admissible topologies [96].

In Chapter 4, we address the problem of designing distributed control of multi-agent systems in the presence of time-delay in the communication network. We extend decomposition-based design approach in Chapter 2 to incorporate time delay. Neighbor-based local controller is designed which uses the state or output information from neighbors under given communication constraints and network induced time delay. The system is modeled by delayed differential equations and Lyapunov-Krasovskii functional is used in the analysis with the help of matrix inequalities. Both static state feedback and dynamic output feedback controllers are considered with $H_2$ and $H_\infty$ performance with additional $\alpha$-stability constraint.
Chapter 5 considers the problem of designing distributed controller for networked dynamic systems with random link failure. We represent the network as a linear discrete-time system with multiplicative random coefficients which model link failures. A distributed control algorithm which uses only local and available neighboring information is proposed to stabilize the system in mean-square sense. We also provide a sufficient condition for designing stabilizing distributed controller that ensures prescribed disturbance attenuation in $L_2$ gain sense. The design is carried out using linear matrix inequalities (LMIs) [97].

In Chapter 6, the proportional integral (PI) observer based control and fault diagnosis is considered for multi-agent interconnected systems. First, the problem of designing decentralized PI-observer based controllers is considered in the convex optimization context for interconnected systems with linear subsystems and nonlinear time-varying interconnections. By taking advantage of additional degrees of freedom in PI observers, we enhance the design objective of maximizing the interconnection bounds [98]. Then distributed PI-observer is introduced for disturbance estimation and fault detection in large-scale multi-agent systems [99].

In Chapter 7, Distributed LQR-based control of multi-agent systems is presented. We consider a set of identical decoupled dynamical systems and a control problem where the performance index couples the behavior of the systems. Both distributed state variable feedback control and dynamic output feedback control are considered. Specifically, two types of dynamic output feedback algorithms are developed with local observers and distributed observers. A distributed control design method requires the solution of two algebraic Riccati equations with the size of the maximum vertex degree of the communication graph plus one. The design procedure provides a straightforward way to construct controller and observer gains that guarantee stabilization which is decoupled from the communication graph structure.
Chapter 2

Decomposition-Based Distributed Control

2.1 Introduction

In this chapter, we consider distributed control problem for a class of multi-agent systems called decomposable systems. Decomposable systems can be imagined as the result of the interconnection of a network of identical subsystems, which interact with each other following a pattern. The interaction between subsystems is in such a way that the states of each subsystem influence the states of its connected subsystems or subsystems share a common goal, or a combination of both. These systems have a nice property that can be decomposed into a set of independent modal subsystems with the aid of appropriate transformation. This enables one to design a disturbed controller in the frame of independent subsystems which are considerably smaller in size.

The modal decomposition technique allows one to use a set of LMIs to compute the controller parameters efficiently. However, it brings conservatism into the solution when one needs to equate the Lyapunov variables in all the LMIs. In [43] the authors
used extended LMI formulations for discrete-time systems to avoid common Lyapunov variables and solved the problem considering $H_2$ and $H_\infty$ performances. However, as stated in their paper, the continuous-time case was not solved due to the limitation of requiring a common Lyapunov variable. Here we provide a solution for the continuous-time case with the aid of dilated LMIs formulation for continuous-time systems to complete the gap in [43]. We solve the problem of designing distributed control for a set of agents with, $H_2$, $H_\infty$ and $\alpha$-stability performance for both cases of static state feedback and dynamic output feedback with non-common Lyapunov variable. We use a satellite formation scenario to illustrate the validity of our result [95], [100].

### 2.2 Preliminaries

In this section, we present preliminary notions and necessary background needed for our development. We denote by $\mathbb{R}$ the field of real numbers and $\mathbb{R}^{n \times m}$ the set of $n \times m$ real matrices. Furthermore, $A > 0$ means a positive definite matrix, $A \otimes B$ denotes the Kronecker product of $A$ and $B$, $\sigma(A)$ and $\text{tr}(A)$ are the singular value and trace of square matrix $A$ respectively, and $He\{A\}$ denotes $He\{A\} = A + A^T$.

**Definition 2.2.1.** Let us consider the $Nn$-th order continuous linear time-invariant system described by:

\[
\begin{align*}
\dot{x}(t) &= \bar{A}x(t) + \bar{B}_w w(t) + \bar{B}_u u(t) \\
z(t) &= \bar{C}_z x(t) + \bar{D}_{zw} w(t) + \bar{D}_{zu} u(t) \\
y(t) &= \bar{C}_y x(t) + \bar{D}_{yw} w(t)
\end{align*}
\]

(2.1)

where $y \in \mathbb{R}^{N_{r_y}}$ is the measurement output, $z \in \mathbb{R}^{N_{r_z}}$ is the performance output, $w \in \mathbb{R}^{N_{m_w}}$ is the disturbance and $u \in \mathbb{R}^{N_{m_u}}$ is the control input. The system (2.1) is called "decomposable system" if and only if $L$ is diagonalizable ($L = U\Lambda U^{-1}$) and
the system matrices have the form:

\[
\begin{align*}
\bar{A} &= I_N \otimes A_a + L \otimes A_b, \\
\bar{B}_w &= I_N \otimes B_{w,a} + L \otimes B_{w,b}, \\
\bar{B}_u &= I_N \otimes B_{u,a} + L \otimes B_{u,b}, \\
\bar{C}_z &= I_N \otimes C_{z,a} + L \otimes C_{z,b}, \\
\bar{C}_y &= I_N \otimes C_{y,a} + L \otimes C_{y,b}, \\
\bar{D}_{zw} &= I_N \otimes D_{zw,a} + L \otimes D_{zw,b}, \\
\bar{D}_{zu} &= I_N \otimes D_{zu,a} + L \otimes D_{zu,b}, \\
\bar{D}_{yw} &= I_N \otimes D_{yw,a} + L \otimes D_{yw,b}
\end{align*}
\] (2.2)

Moreover, if \( L \) is symmetric, we call the system “symmetric decomposable system”.

In this case \( U \) is real and orthogonal (\( \Lambda = U^{-1}LU \)) and \( \Lambda \) is also real.

**Lemma 2.2.2.** Consider the matrices \( M_a, M_b \in \mathbb{R}^{p \times q} \) and \( \bar{M} \in \mathbb{R}^{Np \times Nq} \) with the structure

\[
\bar{M} = I_N \otimes M_a + L \otimes M_b
\] (2.3)

and let \( U \) be a non-singular matrix such that \( \Lambda = U^{-1}LU \) is diagonal. Then the matrix \( M = (U \otimes I_p)^{-1}\bar{M}(U \otimes I_q) \) is block diagonal and has the structure

\[
M = I_N \otimes M_a + \Lambda \otimes M_b
\] (2.4)

where each of its block is in the form

\[
M_i = M_a + \lambda_i M_b
\] (2.5)

where \( \lambda_i \) is the \( i \)-th eigenvalue of \( L \). Moreover, for every matrix \( M \) in the form (2.4) we have

\[
\bar{M} = (U \otimes I_p)M(U \otimes I_q)^{-1} = I_N \otimes M_a + L \otimes M_b
\] (2.6)

**Proof.** The proof of this lemma can be shown by construction using the Kronecker product properties.

**Remark 2.2.3.** Throughout this chapter we follow the notations used in lemma 2.2.2 and whenever we show a matrix with a bar or a matrix with index \( i \), we mean that it has the structure of (2.3) or (2.5) respectively.
CHAPTER 2. DECOMPOSITION-BASED DISTRIBUTED CONTROL

Using the Kronecker product properties in lemma 2.2.2, we now present the following theorem which is the basis for the development of decomposition approach control synthesis.

**Theorem 2.2.4.** An $N_n$-th order decomposable system (2.1) as described in definition 2.2.1 is equivalent to $N$ independent subsystem of order $n$ where each of these subsystems has only $m_u$ input, $m_w$ disturbance, $r_z$ performance output and $r_y$ measurement output.

**Proof.** Using the following change of variable:

$$
\begin{align*}
    x &= (U \otimes I_n) \hat{x}(t), \quad w = (U \otimes I_{m_w}) \hat{w}(t) \\
    u &= (U \otimes I_{m_u}) \hat{u}(t), \quad z = (U \otimes I_{r_z}) \hat{z}(t) \\
    y &= (U \otimes I_{r_y}) \hat{y}(t)
\end{align*}
$$

and the property described in lemma 2.2.2, the system becomes

$$
\begin{align*}
    \hat{x}(t) &= A\hat{x}(t) + B_w\hat{w}(t) + B_u\hat{u}(t) \\
    \hat{z}(t) &= C_z\hat{x}(t) + +D_{zw}\hat{w}(t)D_{zu}\hat{u}(t) \\
    \hat{y}(t) &= C_y\hat{x}(t) + D_{yw}\hat{w}(t)
\end{align*}
$$

where the system matrices $A, B_w, B_u, C_z, \text{etc.}$ are all block diagonal and have the special structure (2.4). This is equivalent to the following set of $N$ independent $n$-th order systems:

$$
\begin{align*}
    \hat{x}_i(t) &= A_i\hat{x}_i(t) + B_{w,i}\hat{w}_i(t) + B_{u,i}\hat{u}_i(t) \\
    \hat{z}_i(t) &= C_{z,i}\hat{x}_i(t) + D_{zw,i}\hat{w}_i(t) + D_{zu,i}\hat{u}_i(t) \quad \text{for } i = 1, \ldots, N \\
    \hat{y}_i(t) &= C_{y,i}\hat{x}_i(t) + D_{yw,i}\hat{w}_i(t)
\end{align*}
$$

where $\hat{x}_i \in \mathbb{R}^{n \times 1}$ is the $i$-th block of $\hat{x}$, and $\hat{w}_i$, $\hat{u}_i$, $\hat{z}_i$ and $\hat{y}_i$ are similarly defined. According to lemma 2.2.2 the system matrices $A_i, B_{w,i}, \ldots, D_{yw,i}$ are defined as
follows:

\[ A_i = A_a + \lambda_i A_b \]

\[ B_{w,i} = B_{w,a} + \lambda_i B_{w,b} \]

\[ \vdots \]

\[ D_{yw,i} = D_{yw,b} + \lambda_i D_{yw,b} \text{ for } i = 1, \ldots, N \] (2.10)

We stress that these subsystems are different from the physical subsystems and for this reason we call them "modal subsystems".

Before closing this section, we state the following lemma that relates the norms of the decomposable system (2.1) to the associated modal subsystems (2.9). This results will be used in the process of designing the distributed controller.

**Lemma 2.2.5.** Let \( T_{wz} \) be the transfer function of the system (2.1) from disturbance \( w \) to output \( z \), and \( T_{\hat{w}\hat{z}} \) be the transfer function of the system (2.8) from the new disturbance \( \hat{w} \) to output \( \hat{z} \), after transforming the original system with (2.7). Then it holds

\[
\frac{\sigma(U)}{\sigma(U)} \| T_{\hat{w}\hat{z}} \|_{H_2} \leq \| T_{wz} \|_{H_2} \leq \frac{\sigma(U)}{\sigma(U)} \| T_{\hat{w}\hat{z}} \|_{H_2}
\]

\[
\frac{\sigma(U)}{\sigma(U)} \| T_{\hat{w}\hat{z}} \|_{H_\infty} \leq \| T_{wz} \|_{H_\infty} \leq \frac{\sigma(U)}{\sigma(U)} \| T_{\hat{w}\hat{z}} \|_{H_\infty}
\]

where \( \sigma(U) \) and \( \sigma(U) \) are respectively the maximum and minimum singular values of \( U \). Moreover let \( T_{\hat{w}_i\hat{z}_i} \) be the transfer function of each of the \( N \) modal subsystems from \( \hat{w}_i \) to \( \hat{z}_i \) in (2.9). Then

\[
\| T_{\hat{w}\hat{z}} \|^2_{H_2} = \sum_{i=1}^{N} \| T_{\hat{w}_i\hat{z}_i} \|^2_{H_2}
\]

\[
\| T_{\hat{w}\hat{z}} \|^2_{H_\infty} = \max_i \| T_{\hat{w}_i\hat{z}_i} \|^2_{H_\infty}
\]
Figure 2.1: A formation of 4 agents and their distributed controller: arrows indicate the interconnections among the agents as in the pattern matrix $L$. The circles represent local controllers implementing a distributed controller; the controller follows the pattern and thus uses only neighbor’s information.

**Remark 2.2.6.** If we have a symmetric decomposable system, then $U$ is orthogonal and $\sigma(U)/\sigma(U) = 1$. It can be concluded from lemma 2.2.5 that

\[
\| T_{\omega z} \|^2_{H_2} = \sum_{i=1}^{N} \| T_{\hat{\omega}_i \hat{z}_i} \|^2_{H_2} \quad (2.11)
\]

\[
\| T_{\omega z} \|^2_{H_{\infty}} = \max_{i} \| T_{\hat{\omega}_i \hat{z}_i} \|^2_{H_{\infty}} \quad (2.12)
\]

In subsequent sections, we will consider only symmetric decomposable systems since, in this case, the matrix $\Lambda$ will be real which avoids computation with complex numbers, and we can simply use (2.11) and (2.12) for system norms.

### 2.3 Controller Synthesis

According to Theorem 2.2.4, a decomposable system (2.1) is equivalent to a set of $N$ independent continuous-time modal subsystems (2.9). Therefore, for this class of systems, it is possible to solve the control problem in the frame of $N$ independent subsystems, which are considerably smaller in size. Once the solution has been obtained independently for each modal subsystem, one can retrieve the solution of the original problem using the inverse transformation.
The aim is to design either static state feedback controller

$$u = \bar{K}x, \quad \bar{K} = I_N \otimes K_a + L \otimes K_b$$  \hspace{1cm} (2.13)$$

or dynamic output feedback controller

$$\dot{x}_c(t) = \bar{A}_c x_c(t) + \bar{B}_c y(t)$$

$$u(t) = \bar{C}_c x_c(t) + \bar{D}_c y(t)$$  \hspace{1cm} (2.14)$$

with $\bar{A}_c = I_N \otimes A_{c,a} + L \otimes A_{c,b}$, $\bar{B}_c = I_N \otimes B_{c,a} + L \otimes B_{c,b}$, $\bar{C}_c = I_N \otimes C_{c,a} + L \otimes C_{c,b}$, $\bar{D}_c = I_N \otimes D_{c,a} + L \otimes D_{c,b}$ which indicate that the controllers have the same interconnection structures as the plant. Figure 2.1 illustrates this for 4 agents. The approach is to find $N$ controllers of the same type for the modal subsystems (2.9); i.e., static state feedback controller

$$\hat{u}_i = K_i \hat{x}_i \quad \text{for} \quad i = 1, \ldots, N$$  \hspace{1cm} (2.15)$$

or dynamic output feedback controller

$$\dot{\hat{x}}_{c,i}(t) = A_{c,i} \hat{x}_{c,i}(t) + B_{c,i} \hat{y}_i(t)$$

$$\hat{u}_i(t) = C_{c,i} \hat{x}_{c,i}(t) + D_{c,i} \hat{y}_i(t) \quad \text{for} \quad i = 1, \ldots, N$$  \hspace{1cm} (2.16)$$

and then retrieve the controller in the non-decomposed form.

The basic LMI approach to solve the $H_2$ control problem with state feedback for modal subsystems (2.9) with $D_{zw,i} = 0$ is to find a feasible solution to the following set of inequalities

$$\begin{bmatrix}
A_i X_i + X_i A_i^T + B_{u,i} W_i + W_i B_{u,i}^T & X_i C_{c,i}^T + W_i^T D_{zw,i}^T \\
* & -I
\end{bmatrix} < 0$$
where $A_i$, $B_{w,i}$, $B_{u,i}$, $C_{z,i}$, $D_{zu,i}$ are defined as (2.10) and the symbol $*$ is used to avoid repetition for symmetric matrix. The decision variables are $X_i = X_i^T$, $W_i$, and $Z_i = Z_i^T$. These LMIs have a solution if and only if a controller with $\|T_{wz}\|_{H^2} = \|\hat{T}_{wz}\|_{H^2} < \gamma_2$ exists. The static state feedback gains $K_i$’s can then be obtained by $K_i = W_iX_i^{-1}$. Using these $K_i$’s one can construct a block diagonal matrix $K$ with $K_i$ matrices in the diagonal blocks and the state feedback gain for the original untransformed system is obtained as:

$$K = (U \otimes I_{m_u})K(U^{-1} \otimes I_n)$$

In general $K$ has no sparsity, so the controller will be a full global controller, which will correspond to the solution of the control problem for the original untransformed system. According to lemma 2.2.2 and its corollary the matrix $K$ have the same sparsity as the original system if and only if $K_i$ matrices are parameterized as $K_i = K_a + \lambda_i K_b$. If we achieve this goal, a controller could be implemented locally, i.e. the computation of the control input for each agent would require only local information: the state of the agent itself and that of its neighbors according to pattern matrix $L$.

This objective can be easily achieved by the following constrains to the set of LMIs (2.17)

$$X_i = X, \quad W_i = W_a + \lambda_i W_b \quad \text{for} \quad i = 1, \ldots, N$$

which leads $K_i$ to be in the form $K_i = K_a + \lambda_i K_b$ where $K_a = W_aX^{-1}$, $K_b = W_bX^{-1}$. However, adding these constraints leads to a multiobjective optimization problem, where it is necessary to equate the matrix associated with the Lyapunov function $(X_i)$ in different LMIs. This can be quite conservative, considering that the number
of LMIs ($N$) may be large and all of them must have the same Lyapunov matrix. In [43], the authors used a result in discrete-time domain to reduce this conservatism and found a feasible solution for controller synthesis. However, establishing a similar result for the continuous-time case was not possible.

In the next section, we discuss the problem of multiobjective optimization and introduce dilated LMIs for continuous-time controller synthesis that reduces this conservatism and helps us in section 2.5 to find a distributed controller with non-common Lyapunov variables.

### 2.4 Multi-Objective Optimization

The authors in [57] facilitated a process for LMI-based controller synthesis under the discrete-time setting. They showed that a matrix $A$ is Schur stable if and only if there exist a Lyapunov variable $X$ and a multiplier $G$ satisfying

\[
\begin{bmatrix}
-X & AG \\
G^T A^T & X - G - G^T
\end{bmatrix} < 0
\]

This result has two nice properties with regards to the standard one $-X + A X A^T < 0$:

- The Lyapunov variable $X$ has no multiplication relations with $A$, whereas $G$ does.
- The dilated LMI reduces to the standard one by letting $(X, G) = (\mathcal{X}, \mathcal{X})$

By virtue of this parameterization, multiobjective controller synthesis with non-common Lyapunov variables and robust controller synthesis with parameter-dependent Lyapunov variables have been established. A lot of efforts have been done to find the corresponding results for continuous-time systems (see [59, 60, 61] and the references therein). Here we take advantage of the following results which will be used later for distributed controller synthesis.
Lemma 2.4.1. Let a matrix $A \in \mathbb{R}^{n \times n}$, scalars $\delta_1 > 0$, $\delta_2 > 0$, and a matrix $\Delta$ of column dimension $n$ be given. Then the following two conditions are equivalent, where $b = a^{-1} > 0$ is an arbitrary number.

1. There exists $X = X^T > 0$ such that

$$AX + XA^T + \delta_1 X + \delta_2 AXA^T + X \Delta^T \Delta X < 0 \quad (2.19)$$

2. There exist $X = X^T > 0$, and $G > 0$ such that

$$
\begin{bmatrix}
0 & -X & X & 0 & X \Delta^T \\
-X & 0 & 0 & -X & 0 \\
X & 0 & -\delta_1^{-1} X & 0 & 0 \\
0 & -X & 0 & -\delta_2^{-1} X & 0 \\
\Delta X & 0 & 0 & 0 & -I
\end{bmatrix}
+ H e \left\{ \begin{bmatrix} A \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix}, G \begin{bmatrix} I & -bI & bI & I & b \Delta^T \end{bmatrix} \right\} < 0 \quad (2.20)
$$

Moreover, for every solution $X > 0$, $(X, G) = (X, -a(A - aI)^{-1}X)$ is a solution of (2.20). Conversely, for every $X > 0$ such that (2.20) holds for some $G$, $X = X$ satisfies (2.19).

Proof. The equivalency of 1 and 2 can be shown by applying Schur complement technique and via some congruent transformations. Detailed proof can be found in [60].

Theorem 2.4.2. Let us consider the continuous-time system $T(s) = C(sI - A)^{-1}B$. Then, the following three conditions are equivalent, where $b = a^{-1} < 0$ is an arbitrary prescribed number:
1. The matrix $A$ is stable and the $H_2$ norm of $T$ is bounded by $\gamma_2 > 0$. Namely
   \[ \| T \|_{H_2} < \gamma_2. \]

2. There exist $X_2 = X_2^T > 0$ and $L_2 = L_2^T > 0$ such that
   \[ \begin{bmatrix} A & X_2 & X_2^T & C^T \end{bmatrix} X_2 < 0, \quad \begin{bmatrix} L_2 & B^T \\ B & X_2 \end{bmatrix} > 0, \quad tr(L_2) < \gamma_2^2 \quad (2.21) \]

3. There exist $X_2 = X_2^T > 0$, $Z_2 = Z_2^T > 0$, and $G_2 > 0$ such that
   \[ \begin{bmatrix} 0 & -X_2 & 0 \\ -X_2 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + H e \begin{bmatrix} A \\ I \\ C \end{bmatrix} G_2 \begin{bmatrix} I & -bI & 0 \end{bmatrix} < 0 \]
   \[ \begin{bmatrix} Z_2 & B^T \\ B & X_2 \end{bmatrix} > 0, \quad tr(Z_2) < \gamma_2^2 \quad (2.22) \]

Moreover, for every solution $(X_2, L_2)$ of (2.21), $(X_2, Z_2, G_2) = (X_2, L_2, -a(A-aI)^{-1})$ is a solution of (2.22). Conversely, for every pair of matrices $(X_2, Z_2)$ such that (2.22) holds for some $G_2$, $(X_2, L_2) = (X_2, Z_2)$ satisfy (2.21).

Proof. The equivalence of 1 and 2 is well-known. The equivalence of 2 and 3 and the latter assertion of the theorem immediately follow from lemma 2.4.1 by letting $\delta_1 = \delta_2 \to 0$ and $\Delta = \mathcal{C}$. 

Similarly, one can derive an Extended LMI for the $\alpha$-stability as following theorem by applying lemma 2.4.1 to the standard LMI.

**Theorem 2.4.3.** Let a matrix $A \in \mathbb{R}^{n\times n}$ given. Then, the following three conditions are equivalent, where $b = a^{-1} < 0$ is an arbitrary prescribed number:

1. The matrix $A$ satisfies $\sigma(A) \subset \Omega(\alpha)$, where $\Omega(\alpha) \triangleq \{ \lambda \in \mathbb{C} | \text{Re}(\lambda) < -\alpha \}$ $(\alpha > 0)$. 

2. There exist $X_2 = X_2^T > 0$ and $L_2 = L_2^T > 0$ such that
   \[ \begin{bmatrix} A & X_2 & X_2^T & C^T \end{bmatrix} X_2 < 0, \quad \begin{bmatrix} L_2 & B^T \\ B & X_2 \end{bmatrix} > 0, \quad tr(L_2) < \gamma_2^2 \quad (2.21) \]

3. There exist $X_2 = X_2^T > 0$, $Z_2 = Z_2^T > 0$, and $G_2 > 0$ such that
   \[ \begin{bmatrix} 0 & -X_2 & 0 \\ -X_2 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + H e \begin{bmatrix} A \\ I \\ C \end{bmatrix} G_2 \begin{bmatrix} I & -bI & 0 \end{bmatrix} < 0 \]
   \[ \begin{bmatrix} Z_2 & B^T \\ B & X_2 \end{bmatrix} > 0, \quad tr(Z_2) < \gamma_2^2 \quad (2.22) \]

Moreover, for every solution $(X_2, L_2)$ of (2.21), $(X_2, Z_2, G_2) = (X_2, L_2, -a(A-aI)^{-1})$ is a solution of (2.22). Conversely, for every pair of matrices $(X_2, Z_2)$ such that (2.22) holds for some $G_2$, $(X_2, L_2) = (X_2, Z_2)$ satisfy (2.21).
2. There exist $X_{\Omega} = X_{\Omega}^T > 0$ such that

$$AX_{\Omega} + X_{\Omega}A^T + 2\alpha X_{\Omega} < 0 \quad (2.23)$$

3. There exist $X_{\Omega} = X_{\Omega}^T > 0$ and $G_{\Omega}$ such that

$$\begin{bmatrix}
0 & -X_{\Omega} & X_{\Omega} \\
-X_{\Omega} & 0 & 0 \\
X_{\Omega} & 0 & -\frac{1}{2\alpha}X_{\Omega}
\end{bmatrix}
+ He \begin{bmatrix}
\mathcal{X} \\
I \\
0
\end{bmatrix}
G_{\Omega} \begin{bmatrix}
I \\
-bI \\
bI
\end{bmatrix}
< 0 \quad (2.24)$$

Moreover, for every solution $(X_{\Omega}, L_{\Omega})$ of (2.23), $(X_2, G_2) = (X_{\Omega}, -a(A - aI)^{-1}X_{\Omega})$ is a solution of (2.24). Conversely, for every matrix $(X_{\Omega} = X_{\Omega}^T > 0$ such that (2.24) holds for some $G_{\Omega}$, $X_{\Omega} = X_{\Omega}$ satisfies (2.23).

**Theorem 2.4.4.** Let us consider the continuous-time system $T(s) = C(sI - A)^{-1}B + D$. Then, the following three conditions are equivalent, where $b = a^{-1} < 0$ is a sufficiently small number:

1. The matrix $A$ is stable and the $H_\infty$ norm of $T$ is bounded by $\gamma_{\infty} > 0$. Namely

   $$\|T\|_{H_\infty} < \gamma_{\infty}.$$ 

2. There exist $X_{\infty} = X_{\infty}^T > 0$ such that

   $$\begin{bmatrix}
AX_{\infty} + X_{\infty}A^T & \mathcal{X}C^T & B \\
\ast & -\gamma_{\infty}I & D \\
\ast & \ast & -\gamma_{\infty}I
\end{bmatrix}
< 0 \quad (2.25)$$

3. There exist $X_{\infty} = X_{\infty}^T > 0$ and $G_{\infty} > 0$ such that

   $$\begin{bmatrix}
AG_{\infty} + G_{\infty}^TA^T & X_{\infty} - G_{\infty}^T + bAG_{\infty} & G_{\infty}^TC^T & B \\
\ast & -b(G_{\infty} + G_{\infty}^T) & bG_{\infty}^TC^T & 0 \\
\ast & \ast & -\gamma_{\infty}I & D \\
\ast & \ast & \ast & -\gamma_{\infty}I
\end{bmatrix}
< 0 \quad (2.26)$$
Moreover, for every solution \( X_\infty \) of (2.25), \((X_\infty, G_\infty) = (X_\infty, X_\infty)\) is a solution of (2.26). Conversely, for every matrix \( X_\infty \) such that (2.26) holds for some \( G_\infty \), \( X_\infty = X_\infty \) satisfy (2.25).

**Proof.** The equivalence of 1 and 2 is well-known. The equivalence of 2 and 3 and the latter assertion of the theorem is as follows. When a symmetric positive-definite matrix \( X_\infty \) satisfying (2.25) exists, a positive scalar \( b > 0 \) as \( b < 2\lambda_1/\lambda_2 \) can be found where

\[
\lambda_1 = \lambda_{\min} \left( - \begin{bmatrix}
AX_\infty + X_\infty A^T & X_\infty C^T & B \\
* & -\gamma_\infty I & D \\
* & * & -\gamma_\infty I
\end{bmatrix} \right)
\]

\[
\lambda_2 = \lambda_{\max} \left( \begin{bmatrix}
AX_\infty A^T & AX_\infty C^T & 0 \\
* & CAX_\infty C^T & 0 \\
* & * & 0
\end{bmatrix} \right)
\]

Then, applying Schur’s complement with respect to lmi (2.26) by choosing \( X_\infty = G_\infty = X_\infty \) yields

\[
\begin{align*}
\begin{bmatrix}
AX_\infty + X_\infty A^T & X_\infty C^T & B \\
* & -\gamma_\infty I & D \\
* & * & -\gamma_\infty I
\end{bmatrix} + \\
\frac{b}{2} \begin{bmatrix}
AX_\infty A^T & AX_\infty C^T & 0 \\
* & CAX_\infty C^T & 0 \\
* & * & 0
\end{bmatrix} & < 0 \\
\end{align*}
\]

Scalar \( b \) makes (2.26) always satisfy. Conversely, When a positive symmetric matrix \( X_\infty \), matrix \( G_\infty \) and a positive scalar \( b > 0 \) satisfying (2.26) exists, (2.25) can be obtained by applying congruent transformation

\[
T = \begin{bmatrix}
I & A & 0 & 0 \\
0 & C & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}
\]
to (2.26) and choosing $\mathcal{X}_\infty = X_\infty$.

Remark 2.4.5. The advantage of aforementioned dilated LMI’s is irrespective of parameter $b$. This means that in multi-objective synthesis problem the dilated-LMI-based approach with a common multiplier and a common prescribed scalar $b$ but with non-common Lyapunov variables always achieves an upper bound that is lower than or equal to the upper bound obtained in standard-LMI-based approach.

2.5 Distributed Control Design

Using the result of dilated linear matrix inequalities characterizations presented in section 2.4, we can parameterize controllers without involving the Lyapunov variables in the parameterization. Taking advantage of this feature, we can readily design our distributed controller with non-common Lyapunov variables for continuous-time case. In the following subsections, we will show how to synthesize distributed controller with $H_2$, $H_\infty$, and $\alpha$–stability performance for two cases of state and dynamic output feedback, avoiding the need for common Lyapunov variables in continuous-time decomposable systems.

2.5.1 Distributed Control with State Feedback

In this subsection, we state our main results at the following theorems for $H_2$, $H_\infty$, and $\alpha$–stability cases.

Theorem 2.5.1. Consider a continuous-time symmetric decomposable system (2.1). A sufficient condition for the existence of sparse static state feedback controller as in (2.13) of the kind $\hat{K} = I_N \otimes K_a + L \otimes K_b$ that yields $\| T_{wz} \|_{H_2} < \gamma_2$ is that the set of LMIs (2.28) has a feasible solution, where $X_{2,i} = X_{2,i}^T, W_i, G$, and $Z_i = Z_i^T$ are the
optimization variables in which \( W_i = W_a + \lambda_i W_b \) and the controller parameters are determined as \( K_a = W_a G^{-1}, \ K_b = W_b G^{-1} \).

\[
\begin{bmatrix}
He \{ A_i G + B_{u,i} W_i \} & -X_{2,i} - b(A_i G + B_{u,i} W_i) + G^T G^T C_{z,i}^T + W_i^T D_{z_u,i}^T \\
* & -b(G + G^T) & -b(G^T C_{z,i}^T + W_i^T D_{z_u,i}^T) \\
* & * & -I_{r_z}
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
Z_i & B_{w,i}^T \\
* & X_{2,i}
\end{bmatrix} < 0 \text{ for } i = 1, \ldots, N \text{ and } \sum_{i=1}^{N} \text{tr}(Z_i) < \gamma_2^2 \tag{2.28}
\]

\( \text{Proof.} \) In the state-feedback case, the system matrices for the closed-loop modal subsystems will be \( A = A_i + B_{u,i} K_i, \ B = B_{w,i}, \ C = C_{z,i} + D_{z_u,i} K_i, \ D = D_{z_w,i} \). Substituting these matrices in the extended LMI (2.22), applying the change-of-variable \( W_i = K_i G_i \), and introduction of the constraints \( G_i = G, \ W_i = W_a + \lambda_i W_b \) for \( i = 1, \ldots, N \) result to \( N \) sets of LMIs (2.28). Moreover, the equation (2.11) which relates the \( H_2 \) norm of the modal subsystems to the \( H_2 \) norm of the original system makes the coupling constraint \( \sum_{i=1}^{N} \text{tr}(Z_i) < \gamma_2^2 \) in (2.28). This leads to the sufficient condition for the existence of distributed state-feedback controller with \( H_2 \) performance.

**Theorem 2.5.2.** Consider a continuous-time symmetric decomposable system (2.1). A sufficient condition for the existence of sparse static state feedback controller as in (2.13) of the kind \( \bar{K} = I_N \otimes K_a + L \otimes K_b \) that yields \( \| T_{wz} \|_{H_\infty} < \gamma_\infty \) is that the set of LMIs (2.29) has a feasible solution, where \( X_{\infty,i} = X_{\infty,i}^T, W_i \) and \( G \) are the optimization variables in which \( W_i = W_a + \lambda_i W_b \) and the controller parameters are determined as \( K_a = W_a G^{-1}, \ K_b = W_b G^{-1} \).

\[
\begin{bmatrix}
He \{ A_i G + B_{u,i} W_i \} & X_{\infty,i} - G^T + b A_i G + b B_{u,i} W_i & G^T C_{z,i}^T + W_i^T D_{z_u,i}^T & B_{w,i} \\
* & -b(G + G^T) & bG^T C_{z,i}^T + W_i^T D_{z_u,i}^T & 0 \\
* & * & -\gamma_\infty I & D_{z_w,i} \\
* & * & * & -\gamma_\infty I
\end{bmatrix} < 0
\]

for \( i = 1, \ldots, N \) \tag{2.29}
Proof. Similar to the $H_2$ performance case one can establish the result by construction using the extended LMI (2.26).

According to theorem 2.2.4, the system (2.1) is equivalent to $N$ independent modal subsystems (2.9) which guarantees that the poles of the systems (2.1) are composed of all modal subsystems poles. Therefore, if the $i$-th controller places the poles of the $i$-th modal subsystem in region $\Omega_i$ then the poles of the resulting closed loop system in the original domain will be in region $\Omega = \bigcup \Omega_i$. This property allows us to develop the following result for the control design with $\alpha$-stability property (relative stability of degree $\alpha$).

**Theorem 2.5.3.** Consider the symmetric decomposable system (2.1). A sufficient condition for the existence of sparse static state feedback controller as in (2.13) of the kind $\tilde{K} = I_N \otimes K_a + L \otimes K_b$ that asymptotically stabilizes the system and places the poles of systems in region $\Omega(\alpha) \triangleq \{ p \in \mathbb{C} | \Re(p) < -\alpha \}$ is that the set of LMIs (2.30) has a feasible solution where $b > 0$ is an arbitrary prescribed number, $X_{\Omega,i} = X_{\Omega,i}^T$, $W_i$ and $G$ are the optimization variables in which $W_i = W_a + \lambda_i W_b$ and the controller parameters are determined as $K_a = W_a G^{-1}$, $K_b = W_b G^{-1}$.

\[
\begin{bmatrix}
He \{ A_i G + B_{u,i} W_i \} & -X_{\Omega,i} - b(A_i G + B_{u,i} W_i) + G^T X_{\Omega,i} + b(A_i G + B_{u,i} W_i) \\
* & -b(G + G^T) & bG \\
* & * & -\frac{1}{2\alpha} X_{\Omega,i}
\end{bmatrix} < 0
\]

for $i = 1, \ldots, N$ \hspace{1cm} (2.30)

Proof. Similar to the $H_2$ performance case one can establish the result by construction using the extended LMI (2.24).

**Remark 2.5.4.** In above theorems, there are situations when the set of LMIs can be reduced to only two sets instead on $N$. In these cases, considering only those associated with maximum and minimum values of $\lambda_i$ guarantees the feasibility of other LMIs. In
theorem 2.5.2 and 2.5.3, this happens if $B_{u,b} = 0$ and $D_{zu,b} = 0$. This also applies to theorem 2.5.1, by parameterizing $Z_i = Z_a + \lambda_i Z_b$ and replacing the last LMI in (2.28) by $\text{tr}(Z_a + \lambda_{ave} Z_b) < \frac{\gamma^2}{N}$ where $\lambda_{ave}$ is the average of the eigenvalues $\lambda_i$’s.

### 2.5.2 Distributed Control with Dynamic Output Feedback

One can use the same approach for the synthesis of distributed dynamic output-feedback controller and provide the equivalent theorems for $H_2$, $H_\infty$, and $\alpha$-stability cases. In order to avoid the lengthy proof for each case separately, here we show the necessary steps required to prove the main results for all cases. For the full-order dynamic output-feedback controller, the matrices $(A, B, C, D)$ are given by

\[
A = \begin{bmatrix}
A_i + B_{u,i}D_{c,i}C_{y,i} & B_{u,i}C_{c,i} \\
B_{c,i}C_{y,i} & A_{c,i}
\end{bmatrix}, \quad
B = \begin{bmatrix}
B_{w,i} + B_{u,i}D_{c,i}D_{yw,i} \\
B_{c,i}D_{yw,i}
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
C_{z,i} + D_{zu,i}D_{c,i}C_{y,i} & D_{zu,i}C_{c,i}
\end{bmatrix}, \quad
D = D_{zw,i} + D_{zu,i}D_{c,i}D_{yw,i}
\]

(2.31)

By substituting (2.31) in extended LMIs in theorem 2.4.2, 2.4.3, 2.4.4 for $i = 1, ..., N$ and using the following change-of-variable [58], the resulting bilinear matrix inequalities can be linearized. To simplify our notation in this process, we drop the subscript $i$ in $G_i$ multipliers and use a common multiplier $G$ for all sets of LMIs.

Let us first partition $G$ and its inverse $H$ as:

\[
G = \begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix}, \quad
H = \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}
\]

(2.32)

where $G_{11} \in \mathbb{R}^{n \times n}$, $H_{11} \in \mathbb{R}^{n \times n}$ and other matrices have compatible dimensions.

With (2.32) and the controller matrices given in (2.14), we further define the following matrices:

\[
\Xi_H = \begin{bmatrix}
H_{11} & I \\
H_{21} & 0
\end{bmatrix}, \quad
\Pi = H_{11}^T G_{11} + H_{21}^T G_{21}
\]

(2.33)
\[ R_i = H_{21}^T B_{c,i} + H_{11}^T B_{u,i} D_{c,i}, \quad S_i = C_{c,i} G_{21} + D_{c,i} C_{y,i} G_{11}, \quad U_i = D_{c,i} \]

\[ V_i = H_{21}^T A_{c,i} G_{21} + H_{11}^T (A_i + B_{u,i} D_{c,i} C_{y,i}) G_{11} + H_{21}^T B_{c,i} C_{y,i} G_{11} + H_{11}^T B_{u,i} C_{c,i} G_{21} \]

for \( i = 1, \ldots, N \) \hspace{1cm} (2.34)

Then, by applying appropriate congruent transformation for each case, the resulting constraints only involve the following terms.

\[ \Xi_H = \begin{bmatrix} P_{11}^{11} & P_{12}^{11} \\ P_{21}^{11} & P_{22}^{11} \end{bmatrix} \]

\[ \Xi_H^T G \Xi_H = \begin{bmatrix} H_{11}^T & \Pi \\ \Pi^T & I_n & G_{11} \end{bmatrix} \]

\[ \Xi_H^T B = \begin{bmatrix} H_{11}^T B_{w,i} + R_i D_{yw,i} \\ B_{w,i} + B_{u,i} D_{yw,i} \end{bmatrix} \]

\[ \Xi_H^T A \Xi_H = \begin{bmatrix} H_{11}^T A_i + R_i C_{y,i} \\ A_i + B_{u,i} C_{y,i} \end{bmatrix} \]

\[ \Xi_H^T A \Xi_H = \begin{bmatrix} H_{11}^T A_i + R_i C_{y,i} \\ A_i + B_{u,i} C_{y,i} \end{bmatrix} \]

(2.35)

Where the symbol \( \otimes \) is used to represent 2, \( \infty \), and \( \Omega \) corresponding to \( H_2, H_\infty \) and \( \alpha \)-stability cases. Substituting these terms, the resulting matrix inequalities can be expressed by LMIs. Introducing the constraints

\[ R_i = R_a + \lambda_i R_b, \quad S_i = S_a + \lambda_i S_b, \quad U_i = U_a + \lambda_i U_b, \quad V_i = V_a + \lambda_i V_b \]

(2.36)

and with additional assumptions (see Table 2.1) to avoid multiplication of \( \lambda_i \)'s in constructing the controller parameters, \((A_{c,i}, B_{c,i}, C_{c,i}, D_{c,i})\) is parameterized as

\[ A_{c,i} = A_{c,a} + \lambda_i A_{c,b}, \quad B_{c,i} = B_{c,a} + \lambda_i B_{c,b}, \quad C_{c,i} = C_{c,a} + \lambda_i C_{c,b}, \quad D_{c,i} = D_{c,a} + \lambda_i D_{c,b}. \]

This yields distributed dynamic output-feedback controller \((\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c)\) of the form (2.14), which has the same interconnection pattern as the one for the agents. Consequently, using the above development one can state the following theorems.
Table 2.1: Condition and Additional Constraints for Dynamic Output Feedback Case

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>Additional Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$C_{y,b} = 0$, $B_{u,b} = 0$</td>
<td>None</td>
</tr>
<tr>
<td>2</td>
<td>$C_{y,b} = 0$</td>
<td>$U_b = 0$</td>
</tr>
<tr>
<td>3</td>
<td>$B_{u,b} = 0$</td>
<td>$U_b = 0$, $R_b = 0$</td>
</tr>
</tbody>
</table>

Theorem 2.5.5. Consider a continuous-time symmetric decomposable system (2.1) associated with any one of the cases given in Table 2.1. A sufficient condition for the existence of distributed dynamic output-feedback controller $(\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c)$ described by (2.14) that yields $\|T_{wz}\|_{H_2} < \gamma_2$ is that the set of LMIs (2.37) has a feasible solution where the optimization variables are $P_{2,i} = P^T_{2,i}$, $G_{11}$, $H_{11}$, $R_i$, $S_i$, $V_i$, and $Z_i = Z^T_i$ in which $R_i = R_a + \lambda_i R_b$, $S_i = S_a + \lambda_i S_b$, $U_i = U_a + \lambda_i U_b$ and $V_i = V_a + \lambda_i V_b$, while Table 2.1 shows the additional constraints that might be needed in each case.
for \( i = 1, \ldots, N \),
\[
\sum_{i=1}^{N} \text{tr}(Z_i) < \gamma_2^2
\]  
(2.37)

The dynamic output-feedback controller (2.16) can be reconstructed by

\[
D_{c,a} = U_a, \\
D_{c,b} = U_b, \\
B_{c,a} = H_{21}^{-T} \{ R_a - H_{11}^T B_{u,a} U_a \}, \\
B_{c,b} = H_{21}^{-T} \{ R_a - H_{11}^T B_{u,a} U_b - H_{11}^T B_{u,b} U_a \}, \\
C_{c,a} = \{ S_a - U_a C_{y,a} G_{11} \} G_{21}^{-1}, \\
C_{c,b} = \{ S_b - U_a C_{y,b} G_{11} - U_b C_{y,a} G_{11} \} G_{21}^{-1}, \\
A_{c,a} = H_{21}^{-T} \{ V_a - H_{11}^T A_{a} G_{11} - R_a C_{y,a} G_{11} \} G_{21}^{-1} - H_{21}^{-T} H_{11}^T B_{u,a} C_{c,a} \\
A_{c,b} = H_{21}^{-T} \{ V_b - H_{11}^T A_{b} G_{11} - R_b C_{y,a} G_{11} - R_a C_{y,b} G_{11} \} G_{21}^{-1} \\
- H_{21}^{-T} H_{11}^T \{ B_{u,a} C_{c,b} + B_{u,b} C_{c,a} \}
\]  
(2.38)

where \( G_{21} \) and \( H_{21} \) are non-singular matrices satisfying \( H_{21} G_{21} = \Pi - H_{11}^T G_{11} \).

**Theorem 2.5.6.** Consider a continuous-time symmetric decomposable system (2.1) associated with any one of the cases given in table 2.1. A sufficient condition for the existence of distributed dynamic output-feedback controller \((\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c)\) described by (2.14) that yields \( \| T_{wz} \|_{\infty} < \gamma_\infty \) is that the set of LMIs (2.39) has a feasible solution where the optimization variables are \( P_{\infty,i} = P_{\text{inv},i}^T, G_{11}, H_{11}, R_i, S_i, U_i \) and \( V_i \) in which \( R_i = R_a + \lambda_i R_b, \ S_i = S_a + \lambda_i S_b, \ U_i = U_a + \lambda_i U_b \) and \( V_i = V_a + \lambda_i V_b \), while
Table 2.1 shows the additional constraints that might be needed in each case.

\[
\begin{align*}
He \{ H_{11}^T A_i + R_i C_{y,i} \} & \quad V_i + A_i^T + C_{y,i}^T U_i^T B_{u,i}^T P_{11}^{11} - H_{11} + b H_{11}^T A_i + b R_i C_{y,i} \\
* & \quad He \{ A_i G_{11} + B_{u,i} S_i \} \quad P_{11}^{21} - \Pi^T + b A_i + b B_{u,i} U_i C_{y,i} \quad -b(H_{11} + H_{11}^T) \\
* & \quad * \quad P_{22}^{12} - I_n + b V_i \quad C_{z,i}^T + C_{y,i}^T U_i^T D_{zu,i}^T \quad H_{11}^T B_{w,i} + R_i D_{yw,i} \quad < 0 \\
G_{11}^T - b(\Pi + I_n) & \quad b C_{z,i}^T + b C_{y,i}^T U_i^T D_{zu,i} \quad 0 \\
- \gamma_{\infty} I_{r_z} & \quad D_{zu,i} + D_{zu,i} U_i D_{yw,i} \\
* & \quad - \gamma_{\infty} I_{m_w}
\end{align*}
\]

The controller parameters can be obtained through dofparameters.

**Theorem 2.5.7.** Consider a continuous-time symmetric decomposable system (2.1) associated with any one of the cases given in Table 2.1. A sufficient condition for the existence of distributed dynamic output-feedback controller \((\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c)\) described by (2.14) that stabilizes the system and places the poles of systems in region \(\Omega(\alpha) \triangleq \{ p \in \mathbb{C} | \text{Re}(p) < -\alpha \} \) is that the following set of LMIs has a feasible solution where the decision variables are \(P_{11,i}, G_{11}, H_{11}, R_i, S_i, U_i, V_i\) in which \(R_i = R_a + \lambda_i R_b, S_i = S_a + \lambda_i S_b, U_i = U_a + \lambda_i U_b\) and \(V_i = V_a + \lambda_i V_b\), while Table 2.1 shows the additional constraints that might be needed in each case. The controller parameters
can be obtained through dofparameters.

\[
\begin{bmatrix}
  H e \{ H^T_{11} A_i + R_i C_{y,i} \} & V_i + A_i^T + C^T_{y,i} U_i^T B^T_{u,i} & -P^{11}_{\Omega,i} - b H^T_{11} A_i - b R_i C_{y,i} + H_{11} \\
  * & H e \{ A_i G_{11} + B_{u,i} S_i \} & -P^{21}_{\Omega,i} - b A_i - b B_{u,i} U_i C_{y,i} + \Pi^T \\
  * & * & -b (H_{11} + H^T_{11}) \\
  * & * & * \\
  * & * & * \\
  * & * & * \\
  * & * & * \\
\end{bmatrix} < 0
\]

\(2.40\)

**Remark 2.5.8.** As for Remark 2.5.4, the reduction into two sets of LMIs can be done for theorems 2.5.5, 2.5.6, 2.5.7 in the case of \(B_{u,b} = 0\), and \(D_{zu,b} = 0\) and \(D_{yw,b} = 0\). Again for the \(H_2\) case (Theorem 2.5.5), the Last LMI should be replaced by \(tr(Z_a + \lambda_{ave} Z_b) < \frac{\gamma^2}{N}\).

**Remark 2.5.9.** The reduction in the computational complexity of the solution in comparison with the centralized one is of the order of \(N\) similar to discrete-time counterpart. In centralized solution, the number of decision variables is in the order of the biggest decision matrix involved in an LMI that is \(X \in \mathbb{R}^{Nn \times Nn}\). So, the decision variables are \(\Theta(N^2 n^2)\). The number of constraints (size of the LMI) is \(\Theta(N^2 n^2)\) as well. On the other hand, for distributed controller; the biggest decision variables in LMIs associated with state feedback or dynamic output feedback are \(n \times n\) matrices.
which appear $N$ times. So, the decision variables are $\mathcal{O}(Nn^2)$. The constraints are $N$ LMIs of the order of $n \times n$, so the number of constraints is $\mathcal{O}(Nn^2)$ as well. Therefore, the computational complexity is reduced by the order of $N$. Moreover, if Remark 2.5.4 and 2.5.8 holds for state feedback or dynamic output feedback case respectively, the number of LMIs as well as the set of decision variables reduces to two. Therefore, the complexity is only $\mathcal{O}(n^2)$ and the computational complexity is reduced by the order of $N^2$.

2.6 Example

As an example, we solve the same satellite formation problem as in [43] in continuous-time domain without converting to discrete domain. Consider a swarm of satellites orbiting around earth on a circular orbit at an angular speed $w_n$ (See Figure 2.2). The small perturbations of their motion with respect to the nominal circular trajectory are described by the well-known Hill-Clohessy-Wiltshire (HCW) equations [62]:

\[
\begin{align*}
\ddot{x}_1 &= 3\omega_n^2 x_1 + 2\omega_n \dot{x}_2 + a_1 \\
\ddot{x}_2 &= -2\omega_n \dot{x}_1 + a_2 \\
\ddot{x}_3 &= -\omega_n^2 x_3 + a_3
\end{align*}
\]

where $x_1$, $x_2$ and $x_3$ are respectively the displacements in the radial, tangential and out-of-plane direction, and $a_1$, $a_2$ and $a_3$ are the accelerations of the spacecraft due to either propulsion or external disturbances.

Let us assume that $N$ satellites are uniformly distributed on the same circular orbit, and we want to design a controller that minimizes the error on their relative positions. As in this example there is no dynamic interaction between the satellites, all matrices in (2.1) will be block diagonal except $\bar{C}_2$. Since the goal is to minimize
the satellite’s relative positions, we can choose

\[ z_{x_k,i} = -x_{k,i-1} + 2x_{k,i} - x_{k,i+1} \quad \text{for } k = 1, 2, 3 \text{ and } i = 1, \ldots, N \]

as performance output which is the sum of position difference of \(i\)-th satellite with its neighbors. In addition, to restrict the use of actuator (the consumption of propellant) we can add the following performance output as well

\[ z_{a_k,i} = \omega a_{k,i} \quad \text{for } k = 1, 2, 3 \text{ and } i = 1, \ldots, N \]

where \(\omega\) is a weighting parameter. Assuming that each satellite can only communicate with its nearest neighbors, one can choose the following Laplacian matrix \(L\) as a pattern matrix:

\[
L = \begin{bmatrix}
2 & -1 & 0 & \cdots & 0 & -1 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & -1 & 2
\end{bmatrix}
\]

Representing the system in the continuous-time state space realization, one can apply Theorem 2.5.5 and hinfdof for designing dynamic output feedback controller with \(H_2\) and \(H_\infty\) performance respectively. The displacement of a satellite in the formation of 4 satellites in response to an impulse disturbance at \(t = 0.5\) in its radial
Table 2.2: $H_2$ and $H_\infty$ Norm of $T_{wz}$ For Different Number of Satellite

<table>
<thead>
<tr>
<th>Number of Satellite</th>
<th>$H_2/\alpha$-Stability</th>
<th>$H_\infty/\alpha$-Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Distributed</td>
<td>Centralized</td>
</tr>
<tr>
<td>1</td>
<td>0.1089</td>
<td>0.1089</td>
</tr>
<tr>
<td>2</td>
<td>0.2424</td>
<td>0.1886</td>
</tr>
<tr>
<td>3</td>
<td>0.2434</td>
<td>0.2115</td>
</tr>
<tr>
<td>4</td>
<td>0.3313</td>
<td>0.3042</td>
</tr>
</tbody>
</table>

direction is shown in Figure 2.3 and Figure 2.4. Figure 2.3 is the response corresponding to the dynamic output feedback controller under $H_2$ and $H_\infty$ criterion using LMIs (2.37) and (2.39) respectively. While Figure 2.4 is the response corresponding to the dynamic output feedback controller under $H_2$ and $H_\infty$ criterion with additional $\alpha$-stability constraint with degree $\alpha = 10$ using LMIs ((2.40). As it appears from these figures, the transient response of $H_2$ and $H_\infty$ controllers is very slow and oscillating while the controllers under $H_2/\alpha$-stability and $H_\infty/\alpha$-stability criteria has a nice transient response.

The simulation results for different number of satellites are summarized in Table 2.2. As can be seen, the distributed controller is of course suboptimal in comparison with an optimal centralized controller since it only communicates with its nearest neighbors. However, the performance of the distributed controller can be improved by allowing more communication links.

To certify Remark 2.4.5, we applied the approach under different values of $b$ for the problems of 4-satellite formation under $H_2/\alpha$-stability and $H_\infty/\alpha$-stability criteria and obtained the results in Fig. 2.5 and 2.6. In $H_2/\alpha$-stability case, the obtained upper bounds for any prescribed value of $b$ are always lower than the associated standard LMI which corresponds to LMI with and . In $H_\infty/\alpha$-stability case, the obtained upper bounds for any acceptable value of $b$ are always lower than the associated standard LMI which corresponds to LMI with $G_\infty = X_\infty$ and $b \to 0$. 
2.7 Conclusion

In this section, we presented the decomposition approach to design distributed controller for a coupled dynamic systems called decomposable systems. Motivated by a recent publication [43], we formulated and solved the distributed control problem for continuous-time systems with $H_2$, $H_{\infty}$ and $\alpha$-stability performance. With the aid of extended LMI formulation for continuous-time systems it was possible to derive explicit expression for computing the parameters of state and dynamic feedback by solving a set of LMIs with non-common Lyapunov Variables. The method is applied to a satellite formation problem and simulation results confirm the theoretical development.
Figure 2.3: Displacement of the satellite in response to the impulse disturbance in the radial direction at $t = 0.5$ (a) $H_2$ controller (b) $H_\infty$ controller.

Figure 2.4: Displacement of the satellite in response to the impulse disturbance in the radial direction at $t = 0.5$ (a) $H_\infty$ controller with $\alpha$-stability constraint (b) $H_\infty$ controller with $\alpha$-stability constraint.
CHAPTER 2. DECOMPOSITION-BASED DISTRIBUTED CONTROL

Figure 2.5: The $H_2$ costs achieved by the dilated LMIs (2.37)

Figure 2.6: The $H_\infty$ costs achieved by the dilated LMIs (2.39)
Chapter 3

Switching Network Topology

3.1 Introduction

In this chapter, we follow the work on previous chapter and consider distributed control problem for a decomposable multi-agent systems. Decomposable multi-agent systems can be imagined as the result of the interconnection of a network of identical systems, which interact with each other following the network topology. As it is shown in previous chapter, these systems have a nice property that can be decomposed into a set of independent subsystems with the aid of appropriate transformation which enables one to design a distributed controller in the frame of independent subsystems which are considerably smaller in size. Our main contribution in this chapter is to further develop the problem under the assumption of switching network topology and to provide a control design strategy for this scenario. This problem occurs frequently in various applications where interconnection links may change due to different reasons. For example, consider a network of mobile agents that communicate with each other and need to do a cooperative task. Since, the nodes of the network are moving, it is not hard to imagine that some of the existing communication links can fail simply due to the existence of an obstacle or in the opposite situation; new links between
nearby agents are created because the agents come to an effective range of detection with respect to each other. In terms of the network topology, this means that the interconnection pattern is not fixed but can change during the time. Here, we are interested to design a robust distributed controller that guarantees the stability of the overall dynamic network with switching topology.

Specifically, we design distributed dynamic output feedback controller under $H_2$ or $H_\infty$ criteria with $\alpha$-stability property that is robust to the change of network topology. We provide a sufficient condition for the existence of such a controller using LMI constraints. These are simple conditions since it is only necessary to check the feasibility of two sets of LMIs with relatively small size [96].

3.2 Problem Formulation

Decomposable systems can describe networked systems composed of interconnection of $N$ identical subsystems, each of order $n$. The network interconnections follow a pattern that is described by a “pattern matrix” $L$. This pattern can be fixed or varying during the time. While in the previous chapter only fixed interconnection pattern is analyzed, here we consider the case that this pattern is not fixed but can switch among a set of patterns $\mathcal{L} = \{L^s : s = 1, \ldots, m\}$ with any arbitrary switching signal $s(t) \in \mathcal{S} = \{1, 2, \ldots, m\}$. We use the superscript “$s$” to represent this time-varying interconnection situation and formally define a generalization of decomposable systems as follows.

**Definition 3.2.1.** Let us consider the $Nn$-th order continuous linear time-variant
system described by:

\[ \dot{x}(t) = \bar{A}^{s(t)} x(t) + \bar{B}_w^{s(t)} w(t) + \bar{B}_u^{s(t)} u(t) \]

\[ z(t) = \bar{C}_z^{s(t)} x(t) + \bar{D}_{zu}^{s(t)} u(t) + \bar{D}_{zw}^{s(t)} w(t) \]

\[ y(t) = \bar{C}_y^{s(t)} x(t) + \bar{D}_{yw}^{s(t)} w(t) \]

(3.1)

where \( y \in \mathbb{R}^{N_{ry}} \) is the measurement output, \( z \in \mathbb{R}^{N_{rz}} \) is the performance output, \( w \in \mathbb{R}^{N_{mw}} \) is the disturbance and \( u \in \mathbb{R}^{N_{mu}} \) is the control input. The system (1) is called “decomposable system” if the system matrices have the form:

\[ \bar{A}^{s(t)} = I_N \otimes A_a + L_s^{s(t)} \otimes A_b \]

\[ \bar{B}_w^{s(t)} = I_N \otimes B_{w,a} + L_s^{s(t)} \otimes B_{w,b} \]

\[ \bar{B}_u^{s(t)} = I_N \otimes B_{u,a} + L_s^{s(t)} \otimes B_{u,b} \]

\[ \bar{C}_z^{s(t)} = I_N \otimes C_{z,a} + L_s^{s(t)} \otimes C_{z,b} \]

\[ \bar{C}_y^{s(t)} = I_N \otimes C_{y,a} + L_s^{s(t)} \otimes C_{y,b} \]

\[ \bar{D}_{zu}^{s(t)} = I_N \otimes D_{zu,a} + L_s^{s(t)} \otimes D_{zu,b} \]

\[ \bar{D}_{zw}^{s(t)} = I_N \otimes D_{zw,a} + L_s^{s(t)} \otimes D_{zw,b} \]

\[ \bar{D}_{yw}^{s(t)} = I_N \otimes D_{yw,a} + L_s^{s(t)} \otimes D_{yw,b} \]

(3.2)

and \( L_s^{s(t)} \) is diagonalizable for every \( s(t) \in \mathcal{S} = \{1, 2, \ldots, m\} \) \( (L_s^s = U_s^s \Lambda_s^s (U_s^s)^{-1}) \).

In this representation, the diagonal part of the matrices (those with subscript “a”) represents the internal dynamics of the subsystems, while the part depending on the pattern matrix (with subscript “b”) represents the interactions between subsystems. Moreover, if \( L_s^s \)'s can be simultaneously diagonalized with a common \( U = U_s^s \), we call the system “simultaneously decomposable system”. It is worth pointing out that here the dynamics of the subsystems and their interactions are fixed while the changing pattern makes the system time-variant.

**Remark 3.2.2.** In this chapter we follow the notations used in lemma 2.2.2 and
whenever we show a matrix with a bar we mean it has the structure
\[ \bar{M}^s = I_N \otimes M_a + L^s \otimes M_b \]  
(3.3)
and a matrix with index \( i \) has the following structure:
\[ M_i^s = M_a + \lambda_i^s M_b \]  
(3.4)

Our goal is to design a robust controller that stabilizes the system with any switching between network topologies while minimizing system norms from disturbance \( w \) to performance output \( z \) for each topology. More specifically, we are going to design a dynamic output feedback controller
\[
\begin{align*}
\dot{\xi}(t) &= \bar{A}_K^s \xi(t) + \bar{B}_K^s y(t) \\
u(t) &= \bar{C}_K^s \xi(t) + \bar{D}_K^s y(t)
\end{align*}
\]  
(3.5)
where
\[
\begin{align*}
\bar{A}_K^s &= I_N \otimes A_{K,a} + L^{s(t)} \otimes A_{K,b} \\
\bar{B}_K^s &= I_N \otimes B_{K,a} + L^{s(t)} \otimes B_{K,b} \\
\bar{C}_K^s &= I_N \otimes C_{K,a} + L^{s(t)} \otimes C_{K,b} \\
\bar{D}_K^s &= I_N \otimes D_{K,a} + L^{s(t)} \otimes D_{K,b}
\end{align*}
\]  
(3.6)
This indicates that the controllers have the same interconnection structures as the plant in each instant, i.e. when the network topology between subsystems varies, the controllers immediately follow this change and use the new set of information without any delay.

The decomposable systems (3.1) have interesting property that allows them to be decomposed into a set of \( N \) independent “modal” subsystems for each pattern. The following theorem and its proof give details of such a decomposition which is essential for establishing our main results in subsequent sections.

**Theorem 3.2.3.** An \( Nn \)-th order decomposable system (3.1) as described in definition 3.2.1 for each pattern matrix \( L^s \) is equivalent to \( N \) independent subsystem of order \( n \)
where each of these subsystems has only \( m_u \) input, \( m_w \) disturbance, \( r_z \) performance output and \( r_y \) measurement output.

**Proof.** The line of proof follows directly from Theorem 2.2.4 adapted for varying \( L^s \) case. Using the following change of variable:

\[
x = (U^s \otimes I_n) \hat{x}^s(t), \quad w = (U^s \otimes I_{m_w}) \hat{w}^s(t)
\]

\[
u = (U^s \otimes I_{m_u}) \hat{u}^s(t), \quad z = (U^s \otimes I_{r_z}) \hat{z}^s(t) \quad \text{for} \quad s = 1, \ldots, m
\]

\[
y = (U^s \otimes I_{r_y}) \hat{y}^s(t)
\]

and the property described in lemma 2.2.2, the system becomes

\[
\dot{\hat{x}}^s(t) = A^s \hat{x}^s(t) + B^s_w \hat{w}^s(t) + B^s_u \hat{u}^s(t)
\]

\[
\dot{\hat{z}}^s(t) = C^s_z \hat{x}^s(t) + D^s_{zw} \hat{w}^s(t) + D^s_{zu} \hat{u}^s(t) \quad \text{for} \quad s = 1, \ldots, m
\]

\[
\hat{y}^s(t) = C^s_y \hat{x}^s(t) + D^s_{yw} \hat{w}^s(t)
\]

where the system matrices \( A^s, B^s_w, B^s_u, C^s_z \), etc. are all block diagonal and have the special structure \( M^s = I_N \otimes M_a + \Lambda^s \otimes M_b \). This is equivalent to the following set of \( N \) independent \( n \)-th order systems:

\[
\dot{\hat{x}}^s_i(t) = A^s_i \hat{x}^s_i(t) + B^s_{w,i} \hat{w}^s_i(t) + B^s_{u,i} \hat{u}^s_i(t)
\]

\[
\dot{\hat{z}}^s_i(t) = C^s_{z,i} \hat{x}^s_i(t) + D^s_{zw,i} \hat{w}^s_i(t) + D^s_{zu,i} \hat{u}^s_i(t) \quad \text{for} \quad i = 1, \ldots, N \text{ and } s = 1, \ldots, m
\]

\[
\hat{y}^s_i(t) = C^s_{y,i} \hat{x}^s_i(t) + D^s_{yw,i} \hat{w}^s_i(t)
\]

where \( \hat{x}^s_i \in \mathbb{R}^{n \times 1} \) is the \( i \)-th block of \( \hat{x}^s \), and \( \hat{w}_i^s, \hat{u}_i^s, \hat{z}_i^s \) and \( \hat{y}_i^s \) are similarly defined. According to lemma 2.2.2 the system matrices \( A^s_i, B^s_{w,i}, \ldots, D^s_{yw,i} \) are defined as follows:

\[
A^s_i = A^s_a + \lambda_i A^s_b
\]

\[
B^s_{w,i} = B^s_{w,a} + \lambda_i B^s_{w,b}
\]

\[
\vdots
\]

\[
D^s_{yw,i} = D^s_{yw,b} + \lambda_i D^s_{yw,b} \quad \text{for} \quad i = 1, \ldots, N
\]
where $\lambda_i^s$ is the $i$-th eigenvalues of $L_a$.

Before explaining the control synthesis procedure, let us first define the set of admissible pattern matrices which is considered to solve the distributed control problem. We assume that the pattern matrix $L_s$ belongs to a finite collection of symmetric matrices $\mathcal{L} = \{L^1, \ldots, L^m\}$ that can commute with each other. The following theorem gives us the nice property of this set that will be used in the next section to design a robust distributed controller.

**Theorem 3.2.4.** Let us consider a set of commuting symmetric matrices $\mathcal{L} = \{L^1, \ldots, L^m\}$ in which $L^i L^j = L^j L^i$ for $\forall i, j = 1, 2, \ldots, m$. Then there is a unitary matrix $U$ that simultaneously diagonalizes all the matrices in the set.

**Proof.** The proof of this theorem can be found in standard matrix analysis books (see [63]).

According to theorem 3.2.4, if $L^s \in \mathcal{L}$, then the system (3.1) will be symmetric simultaneously decomposable (3.2.1). In this case, $U$ is real and orthogonal ($U^{-1} = U^T$) and $\Lambda^s$’s are also real.

**Remark 3.2.5.** It is apparent from (3.3) that the same matrix $\bar{M}$ can be obtained with different $L$’s by adjusting $M_a$ and $M_b$. So the pattern matrix for specific topology is not unique. Therefore, for a system with finite network topologies, $L^s$’s can be easily chosen in such a way that they have the commuting property as long as the network interconnections are bidirectional.

### 3.3 Controller Synthesis

Before presenting our results for the output feedback controller case, let us elaborate on the design procedure and justification of system stability with the aid of finding a
stabilizing static state feedback

$$\bar{u} = \bar{K}^s x$$  \hspace{1cm} (3.11)

$$\bar{K}^s = I_N \otimes K_a + L^s \otimes K_b$$  \hspace{1cm} (3.12)

for the symmetric simultaneously decomposable system (3.1). The basic LMI approach to solve the problem is to find a feasible solution to the following inequalities

$$(\bar{A}^s + \bar{B}^s \bar{K}^s) \bar{P}^s + \bar{P}^s (\bar{A}^s + \bar{B}^s \bar{K}^s)^T < 0 \quad \text{for} \quad s = 1, \ldots, m$$  \hspace{1cm} (3.13)

to guarantee that the system is stable for any fixed topology. According to Theorem 3.2.3, it is possible to solve the control problem in the frame of $N$ independent subsystems, which are considerably smaller in size and then retrieve the solution of the original problem using the inverse transformation.

In the transformed domain, the above inequality can be expressed by the following set of independent LMIs:

$$A^s_i P^s_i + P^s_i (A^s_i)^T + B^s_{u,i} W^s_i + (W^s_i)^T (B^s_{u,i})^T < 0$$  \hspace{1cm} (3.14)

for $i = 1, \ldots, N$ and $s = 1, \ldots, m$

where $P^s_i = (P^s_i)^T > 0$ and $W^s_i \triangleq K^s_i P^s_i$ are decision variables. Then, the static state feedback gains $K^s_i$'s can then be obtained by $K^s_i = W^s_i (P^s_i)^{-1}$. Using these $K^s_i$'s one can construct a block diagonal matrix $K^s$ with $K^s_i$ matrices in the diagonal blocks and the state feedback gain for the original untransformed system is obtained as:

$$\bar{K}^s = (U \otimes I_{m_u}) K^s (U^{-1} \otimes I_n)$$

In general $\bar{K}^s$ has no sparsity, so the controller will be a full global controller. However, according to lemma 2.2.2 the matrix $\bar{K}^s$ have the same sparsity as the original system if and only if $K^s_i$ matrices are parameterized as $K^s_i = K_a + \lambda^s_i K_b$. This objective can be achieved by adding the following constrains to the set of LMIs (3.14)

$$P^s_i = P^s, \quad W^s_i = W_a + \lambda^s_i W_b \quad \text{for} \quad i = 1, \ldots, N$$  \hspace{1cm} (3.15)
which leads to $K_i^s = K_a + \lambda_i^s K_b$ where $K_a = W_a(P^s)^{-1}$ and $K_b = W_b(P^s)^{-1}$. However, adding these constraints requires

$$P_1^s = P_2^s = \cdots = P_N^s = P^s$$ (3.16)

This restriction brings conservatism into the design. Nevertheless, the resulting synthesis technique has valuable merits which enable us to prove the stability of the closed loop system for any arbitrary switching among the set of topologies.

A necessary condition for (asymptotic) stability of the overall system with arbitrary switching among communication topologies is that all systems with fixed topology are (asymptotically) stable. Indeed, if the $s$-th system is unstable, the overall system will be unstable for $s(t) \equiv s$. This condition is obviously satisfied from (3.14). However, the stability of each of these systems is not sufficient. The following theorem gives the sufficient condition for such stability.

**Theorem 3.3.1.** Consider the family of linear systems

$$\dot{x} = A^s(t)x, \quad s(t) \in \mathcal{S} = \{1, 2, \ldots, m\}$$ (3.17)

such that the matrices $A^s(t)$ are stable (i.e., with eigenvalues in the open left half of the complex plane) and the set $\{A^s : s \in \mathcal{S}\}$ is compact in $\mathbb{R}^{n \times n}$. If all systems in this family share a quadratic common Lyapunov function, the switched linear system (3.17) is uniformly asymptotically stable (the word uniformly is used here to describe uniformity with respect to switching signals). This means that if there exist two symmetric matrices and such that we have

$$A^s P + P (A^s)^T \leq Q, \quad \forall s \in \mathcal{S}$$

Then there exist positive constant $c$ and $\mu$ such that the solution of (3.17) for any initial state $x(0)$ and any switching signal $s(t)$ satisfies

$$\|x(t)\| \leq ce^{-\mu t} \|x(0)\| \quad \forall t \geq 0$$
According to this theorem if we find the common Lyapunov function for all the closed loop systems with different pattern matrix, which is equivalent to finding for the set of inequalities

\[(\bar{A}^s + \bar{B}^s_a \bar{K}^s)P + P(\bar{A}^s + \bar{B}^s_b \bar{K}^s)^T < 0 \quad s = 1, \ldots, m \] (3.18)

then we have globally uniformly asymptotically stability regardless of the switching. In what follows we show how this condition can be granted only if we can find a feasible solution for just two inequalities associate with minimum and maximum values of \(\lambda_i^s\).

**Theorem 3.3.2.** Consider symmetric simultaneously decomposable system (3.1) with \(B_{u,b} = 0\) (i.e. \(\bar{B}_u\) has only block diagonal part). Then the sufficient condition for the existence of distributed state feedback controller (3.11) that uniformly asymptotically stabilize system (3.1) for any switching signal \(s(t)\) is that two coupled LMIs

\[A_j P + PA^T_j + B_{u,a} W_j + W^T_j B^T_{u,a} < 0 \quad j = 1, 2 \] (3.19)

has a feasible solution where the decision variables are and \(P\). The matrices and have the structure of

\[
\begin{align*}
A_1 &= A_a + \lambda A_b \\
A_2 &= A_a + \bar{\lambda} A_b \\
W_1 &= W_a + \lambda W_b \\
W_2 &= W_a + \bar{\lambda} W_b
\end{align*}
\] (3.20)

with \(\lambda\) and \(\bar{\lambda}\) the minimum and maximum eigenvalues of \(L^s\) (\(\lambda = \min_{i,s} \lambda_i^s\) and \(\bar{\lambda} = \max_{i,s} \lambda_i^s\)). The controller parameters can be obtained through \(K_a = W_a P^{-1}\), \(K_b = W_b P^{-1}\).

**Proof.** if \(B_{u,b} = 0\), all the LMIs (3.14) can be expressed as a convex combination of two which contain the extreme values of \(\lambda_i^s\) (\(\lambda\) and \(\bar{\lambda}\)). Therefore, the feasibility of two inequalities (3.19) will automatically grant the feasibility of all the LMIs (3.14). In other words, \(P\) satisfies (3.14) for all \(i\) and \(s\). By constructing block diagonal
constraints with the inequalities (3.14) for \( i = 1, \ldots, N \) in diagonal blocks, we will reach the following set of constraints:

\[
A^s(I_N \otimes P) + (I_N \otimes P)(A^*)^T + B_u^s W^s + (W^s)^T (B_u^*)^T < 0 \quad \text{for } s = 1, \ldots, m \quad (3.21)
\]

Applying the congruent transformation \((U \otimes I_n)\) to all the matrices in inequalities (3.21) and considering the fact that \( U \) is unitary \((U^T U = I)\), it can be concluded from lemma 2.2.2 that LMIs (3.21) is equivalent to inequalities (3.18) with \( P = (I_N \otimes P) \). Therefore, according to theorem 3.3.1 the overall systems is asymptotically uniformly stable for any switching signal \( s(t) \).

\[\square\]

3.4 Distributed Dynamic Output Feedback Control

The method shown in section 3.3 stabilize a system can be generalized and used to find suboptimal dynamic output-feedback controller with respect to the system norms. We will explore the possibility of designing controllers with performance criteria based on their disturbance rejection ability. We denote by \( T_w^s \) the transfer function of the system (3.1) from disturbance \( w \) to the performance output \( z \) for the \( s \)-th topology and we are trying to minimize a system norm of this transfer function with an appropriate choice of the controller. As shown in previous section, the approach is to solve the synthesis problem in its decomposed from, so it is important to understand the relation between the norms of the systems in its original form and in the transformed one.

**Lemma 3.4.1.** Let \( T_w^s \) be the transfer function of the symmetric decomposable system (3.1) from disturbance \( w \) to output \( z \) for the \( s \)-th topology, and \( T^s_{\hat{w}_i \hat{z}_i} \) be the transfer function of its \( i \)-th modal subsystem (3.9) from the new disturbance \( \hat{w}_i^s \) to output \( \hat{z}_i^s \)
for that topology, after transforming the original system with (3.7). Then it holds
\[ \| T_{ws} \|_{H_{\infty}} = \max_i \| T_{ws_i^i} \|_{H_{\infty}} \quad \text{and} \quad \| T_{ws} \|_{H_2}^2 = \sum_{i=1}^{N} \| T_{ws_i^i} \|_{H_2}^2 \] (3.22)

**Proof.** These expressions can easily be obtained from the definitions of $H_2$ and $H_\infty$ norms using the properties of the Kronecker product. \qed

Following the procedure discussed in previous section, the approach is to find $N$ dynamic output feedback controllers for the modal subsystems (3.9)
\[ \dot{\hat{\xi}}_i(t) = A_{K,i}^{s} \hat{\xi}_i(t) + B_{K,i}^{s} \hat{y}_i(t) \quad \text{for} \ i = 1, \ldots, N \] (3.23)
and then retrieve the controller in the non-decomposed form using the back transformation.

In the state feedback case, all the inequalities are affine in $P$ and $KP$. It then only takes the change of variable $W \triangleq KP$ to turn all the constraints into LMI’s. However, the situation is not trivial for the output feedback case and the critical change of variable is needed. Here we adapt the result of [58] to the class of systems considered here to create the controller that minimize the $H_2$ and $H_\infty$ norms from $w$ to $z$ with $\alpha$-stability properties.

For the dynamic output feedback case, we can similarly evaluate the $N$ independent modal subsystems, and solve the $N$ independent LMIs. It is not difficult to see that under certain assumptions in the parameterization of the decision variables, resulting controller in the untransformed domain will be of the same structure as the plant.

Moreover, we need additional care to avoid the multiplicity of matrices of index $i$ in entries of LMI constraints and in constructing the controller parameters.

We summarize this result in the following theorems, the proofs of which can be established based on the results of the previous sections and following the line of proof of theorem 3.3.2.
Theorem 3.4.2. Consider the symmetric simultaneously decomposable system (3.1), with \( B_{u,b} = 0, \ C_{y,b} = 0, \ D_{zu,b} = 0 \) and \( D_{yw,b} = 0 \) (i.e. \( \bar{B}_u, \bar{C}_y, \bar{D}_{zu} \) and \( \bar{D}_{yw} \) have only block diagonal part). A sufficient condition for the existence of distributed dynamic output-feedback controller \((\bar{A}_K^*, \bar{B}_K^*, \bar{C}_K^*, \bar{D}_K^*)\) described by (3.5) and (3.6) that uniformly asymptotically stabilizes the system (3.1) for any switching signal \( s(t) \) and yields \( \| T^{s+}_w \|_{H_\infty} < \gamma_\infty \) is that two coupled LMIs (3.25) has a feasible solution where the decision variables are \( X,Y,R_j,S_j,V_j \) and \( W_j \). In (3.25) the matrices with subscript \( j \) like \( R_j \) have the structure of \( R_1 = R_a + \lambda R_b \) and \( R_2 = R_a + \bar{\lambda} R_b \) with \( \lambda \) and \( \bar{\lambda} \) the minimum and maximum eigenvalues of \( L^s \) (\( \lambda_i^s \) for all \( i = 1, \ldots, N \) and \( s = 1, \ldots, m \)). The controller parameters can be obtained through the following relations:

\[
\begin{align*}
D_{K,a} &= W_a, \quad D_{K,b} = W_b \\
C_{K,a} &= (S_a - D_{K,a} C_{y,a} X) M^{-T} \\
C_{K,b} &= (S_b - D_{K,b} C_{y,a} X) M^{-T} \\
B_{K,a} &= N^{-1}(R_a - Y B_{u,a} D_{K,a}) \\
B_{K,b} &= N^{-1}(R_b - Y B_{u,a} D_{K,b}) \\
A_{K,a} &= N^{-1}(V_a - N B_{K,a} C_{y,a} X - Y B_{u,a} C_{K,a} M^T \\
&\quad - Y (A_a + B_{u,a} D_{K,a} C_{y,a}) X) M^{-T} \\
A_{K,b} &= N^{-1}(V_b - N B_{K,b} C_{y,a} X - Y B_{u,a} C_{K,b} M^T \\
&\quad - Y (A_b + B_{u,a} D_{K,b} C_{y,a}) X) M^{-T}
\end{align*}
\]  

in which \( M \) and \( N \) are non-singular matrices that satisfy \( M N^T = I - X Y \).

\[
\begin{bmatrix}
A_j X + X A_j^T + B_{u,a} S_j + S_j^T B_{u,a}^T \\
V_j + (A_j + B_{u,a} W_j C_{y,a})^T \\
(B_{w,j} + B_{u,a} W_j D_{yw,a})^T \\
C_{z,j} X + D_{zu,a} S_j \\
C_{z,j}^T + D_{zu,a} W_j C_{y,a}
\end{bmatrix}^*
\begin{bmatrix}
A_j^T Y + Y A_j + R_j C_{y,a} + C_{y,a}^T R_j^T \\
Y B_{w,j} + R_j D_{yw,a} \\
C_{z,j} + D_{zu,a} W_j C_{y,a}
\end{bmatrix}
\]
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\[ \begin{bmatrix} * & * \\ * & * \\ -\gamma_{\infty} I & * \end{bmatrix} < 0 \quad \text{for } j = 1, 2 \quad (3.25) \]

Theorem 3.4.3. Consider the symmetric simultaneously decomposable system (3.1), with \( B_{u,b} = 0, C_{y,b} = 0, D_{zu,b} = 0 \) and \( D_{yw,b} = 0 \). A sufficient condition for the existence of distributed dynamic output-feedback controller \( (\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) \) described by (3.5) and (3.6) that uniformly asymptotically stabilizes the system (3.1) for any switching signal \( s(t) \) and yields \( \sum_s ||T_{wz}^s||_{\mathcal{H}_2}^2 < \gamma_2^2 \) is that two coupled LMIs (3.26) has a feasible solution where the decision variables are \( X, Y, R_j, S_j, V_j, W_j \) and \( Q_j \).

In (3.26) the matrices with subscript \( j \) have the same structure as in theorem 3.4.2 and the controller parameters can be obtained through (3.24). Moreover, \( \lambda_{av} \) is the average of the all eigenvalues \( \lambda_i^s \).

\[
\begin{bmatrix}
A_jX + XA_j^T + B_{u,a}S_j + S_j^TB_{u,a}^T & * & * \\
V_j + (A_j + B_{u,a}W_jC_{y,a})^T & A_j^TY + YA_j + R_jC_{y,a} + C_{y,a}^TR_j^T & * \\
(B_{w,j} + B_{u,a}W_jD_{yw,a})^T & (YB_{w,j} + R_jD_{yw,a})^T & -I
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
X & * & * \\
I & Y & * \\
C_{z,j}X + D_{zu,a}S_j & C_{z,j} + D_{zu,a}W_jC_{y,a} & Q_j
\end{bmatrix} > 0, \quad D_{zw,j} + D_{zu,a}W_jD_{yw,a} = 0
\]

for \( j = 1, 2 \) and \( tr(Q_a + \lambda_{av}Q_b) < \frac{\gamma_2^2}{mN} \) \quad (3.26)

Theorem 3.4.4. Consider the symmetric simultaneously decomposable system (3.1), with \( B_{u,b} = 0, C_{y,b} = 0, D_{zu,b} = 0 \) and \( D_{yw,b} = 0 \). A sufficient condition for the existence of distributed dynamic output-feedback controller \( (A_s^s, B_s^s, C_s^s, D_s^s) \) described by (3.5) and (3.6) that uniformly asymptotically stabilizes the system (3.1)
for any switching signal $s(t)$ and places the poles of systems $T_{wz}^s$ in region $\Omega(\alpha) \triangleq \{ p \in \mathbb{C} | \text{Re}(p) < -\alpha \}$ is that two coupled LMIs (3.27) has a feasible solution where the decision variables are $X, Y, R_j, S_j, V_j$ and $W_j$. The matrices with subscript $j$ have the same structure as in theorem 3.4.2 and the controller parameters can be obtained through (3.24).

$$
\begin{bmatrix}
A_jX + XA_j^T + B_{u,a}S_j + S_j^TB_{u,a}^T + 2\alpha X & * \\
V_j + (A_j + B_{u,a}W_jC_{y,a})^T + 2\alpha X & A_j^TY + YA_j + R_jC_{y,a} + C_{y,a}^TR_j^T + 2\alpha Y
\end{bmatrix} < 0
$$

for $j = 1, 2$ (3.27)

### 3.5 Simulation Results

As an example, here we consider the same satellite formation problem as in previous chapter, a swarm of satellites orbiting around earth on a circular orbit at an angular speed $w_n$. Let us assume that $N$ satellites are uniformly distributed on a circular orbit and we want to design a controller that minimizes the error on their relative positions with $H_2$ or $H_\infty$ criteria. We consider that the communication pattern between satellites is not fixed. It can change from the case that each satellite can only communicate with its immediate neighbors to the case that all the satellites can communicate with each other. If $s$-th topology is that each satellite can communicate with $s$ neighboring satellites in each of its side, then there are $[(N - 1)/2]$ communication patterns where $[\cdot]$ indicates floor function.

The goal is to design a distributed controller that stabilizes the system for any switching among these patterns while minimize the satellite’s relative positions:

$$
z_{x_k,i} = \frac{1}{2s} \sum_{l=1}^{s} (x_{k,i} - x_{k,i+l}) + (x_{k,i} - x_{k,i-l}) \quad \text{for} \quad k = 1, 2, 3 \text{ and } i = 1, ..., N \quad (3.28)$$

for the $s$-th topology. As in this example there is no dynamic interaction between the satellites, all matrices in (3.1) will be block diagonal except $C_z^s$ which have the
structure $L^s \otimes C_{z,b}$. With formulization (3.28), $L^s$’s are symmetric Toeplitz matrices with the first row:

$$
\begin{bmatrix}
1 & -1/2s & \cdots & -1/2s & 0 & \cdots & 0 & -1/2s & \cdots & -1/2s \\
\end{bmatrix} \quad \text{(3.29)}
$$

and clearly can commute with each other. Therefore, the system is symmetric simultaneously decomposable and we can apply our results. Representing the system in the continuous-time state space realization, and applying Theorem 3.4.2 and 3.4.3 by solving the sets of LMIs in (3.25) and (3.26) for the $H_\infty$ and $H_2$ case respectively, the simulation results are shown in Figure 3.2 for different number of satellites: (a) $H_\infty$ norm ($\gamma_\infty$) (b) normalized $H_2$ norm ($\frac{\gamma_2}{mN}$). Moreover, the behavior of the closed loop system for different random switching signals and arbitrary small switching times was observed which was consistent with the theoretical result of the stability of the overall systems.
3.6 Conclusion

In this chapter, we designed robust distributed controller for identical multi-agent systems called decomposable systems with varying network topologies. The controller structure follows the network topologies at each instance. The problem is formulated and solved for the continuous-time systems under $H_2$, $H_\infty$ and $\alpha$-stability criteria using LMI. Moreover, sufficient conditions for the existence of such a controller are provided that guarantee the overall system stability with any arbitrary switching among admissible topologies. The method is applied to a satellite formation problem and simulation results confirm the theoretical development.
Figure 3.2: $H_\infty$ and $H_2$ norms for different number of satellites; (a) $H_\infty$ norm ($\gamma_\infty$) (b) normalized $H_2$ norm ($\frac{2^2}{mN}$).
Chapter 4

Communication Delay

4.1 Introduction

In previous chapters, we addressed the problem of designing distributed control of multi-agent systems under different assumptions on the network topology; fixed and switching. In this chapter we further analyze this problem in the presence of time-delay in the communication network. Communication networks provide a large flexibility for control design of large-scale interconnected systems by allowing the information exchange between local controllers. However, it comes at the price on non-ideal signal transmission such as time delay which is a source of instability and deteriorates the control performance. Time delays resulting from communication links have been paid much attention to multi-agent systems because of its practical background. For example, for the coordination control of multiple autonomous agents, the communication delay that is related to the information flow between neighboring agents can not be negligible.

The consensus subject to communication delays has been extensively analyzed for the first-order multi-agent systems based on various analysis methods. In [41] a necessary and sufficient condition for a time-delay consensus of first-order dynamic agents
was presented for undirected interconnection graph. In [66], the authors generalized
the results of [41] by considering time-varying delays and also provides the bounds
on maximal delays. Using tree-type transformation, some necessary and/or sufficient
conditions are provided in [67] for consensus of multi-agent systems with non-uniform
time-varying delays.

For second-order multi-agent systems, the consensus conditions have been obtained
under identical communication delay based on frequency domain analysis in [68] and
[69]. For these systems, the consensus criteria are also obtained in the form of LMI
using Lyapunov-Krasovskii and Lyapunov-Razumikhin functions in [70] and [71].

In this chapter, we consider the problem of designing distributed controller for
general multi-agent \( n \)-th order dynamic systems in the presence of time-delay in the
communication network. The time delay is assumed to be identical and constant
for all communication links. We extend decomposition-based design approach in
Chapter 2 to incorporate time delay. Neighbor-based local controller is designed
which uses the state or output information from neighbors under given communication
constraints and network induced time delay. The system is modeled by delayed
differential equations and Lyapunov-Krasovskii functionals is used in the analysis
with the help of matrix inequalities. Both static state feedback and dynamic output
feedback controllers are considered with \( H_2 \) and \( H_\infty \) performance with additional
\( \alpha \)-stability constraint.

The content of this chapter is as following. In Section 4.2, we formulate the prob-
lem and model the system as a time-delay linear system. We extend the definition of
decomposable systems for time-delay interconnected systems and introduce our dis-
tributed control input law. Then, in section 4.3, we elaborate on stability, \( \alpha \)-stability
and bounds on the \( H_2 \) and \( H_\infty \) norms for delay systems and convert the conditions
into LMIs which are adapted later in designing the distributed controller. Section 4.4
includes our main results. Section 4.5 provides the simulation results to support the theoretical development. Finally, we conclude in Section 4.6.

### 4.2 Problem Formulation

The following definition is the extension of decomposable systems defined in Chapter 2 with additional time-delay terms to incorporate the communication delay between agents into the modeling.

**Definition 4.2.1.** Let us consider the $Nn$-th order continuous linear time-invariant system described by:

\[
\begin{align*}
\dot{x}(t) &= \bar{A}x(t) + \bar{A}^\tau x(t - \tau) + \bar{B}_w w(t) + \bar{B}_u u(t) \\
z(t) &= \bar{C}_z x(t) + \bar{C}^\tau_z x(t - \tau) + \bar{D}_{zw} w(t) + \bar{D}_{zu} u(t) \\
y(t) &= \bar{C}_y x(t) + \bar{C}^\tau_y x(t - \tau) + \bar{D}_{yw} w(t)
\end{align*}
\]

(4.1)

where $y \in \mathbb{R}^{N_{ry}}$ is the measurement output, $z \in \mathbb{R}^{N_{rz}}$ is the performance output, $w \in \mathbb{R}^{N_{mw}}$ is the disturbance, $u \in \mathbb{R}^{N_{mu}}$ is the control input and $\tau$ is the delay of any size. The system (4.1) is called “decomposable time-delay system” if and only if $L$ is diagonalizable ($L = U\Lambda U^{-1}$) and the system matrices have the form:

\[
\begin{align*}
\bar{A} &= I_N \otimes A_a + L \otimes A_b, \quad \bar{A}^\tau = I_N \otimes A_a^\tau + L \otimes A_b^\tau \\
\bar{B}_w &= I_N \otimes B_{w,a} + L \otimes B_{w,b}, \quad \bar{B}_u = I_N \otimes B_{u,a} + L \otimes B_{u,b} \\
\bar{C}_z &= I_N \otimes C_{z,a} + L \otimes C_{z,b}, \quad \bar{C}^\tau_z = I_N \otimes C_{z,a}^\tau + L \otimes C_{z,b}^\tau \\
\bar{C}_y &= I_N \otimes C_{y,a} + L \otimes C_{y,b}, \quad \bar{C}^\tau_y = I_N \otimes C_{y,a}^\tau + L \otimes C_{y,b}^\tau \\
\bar{D}_{zw} &= I_N \otimes D_{zw,a} + L \otimes D_{zw,b}, \quad \bar{D}_{zu} = I_N \otimes D_{zu,a} + L \otimes D_{zu,b} \\
\bar{D}_{yw} &= I_N \otimes D_{yw,a} + L \otimes D_{yw,b}
\end{align*}
\]

(4.2)

Moreover, if $L$ is symmetric, we call the system “symmetric decomposable time-delay system”. In this case $U$ is real and orthogonal ($\Lambda = U^{-1}LU$) and $\Lambda$ is also real.
The following theorem provides the relations between the system matrices of decomposable time delay system (4.1) in the original domain and the associated matrices of the decomposed system in the transformed domain.

**Theorem 4.2.2.** An $Nn$-th order decomposable time-delay system (4.1) as described in definition 4.2.1 is equivalent to $N$ independent subsystem of order $n$ where each of these subsystems has only $m_u$ input, $m_w$ disturbance, $r_z$ performance output and $r_y$ measurement output.

**Proof.** Using the following change of variable:

$$
x(t) = (U \otimes I_n) \hat{x}(t), \quad w = (U \otimes I_{m_w}) \hat{w}(t)\\
u(t) = (U \otimes I_{m_u}) \hat{u}(t), \quad z = (U \otimes I_{r_z}) \hat{z}(t)\\
y(t) = (U \otimes I_{r_y}) \hat{y}(t)
$$

and the property described in lemma 2.2.2, the system becomes

$$
\dot{\hat{x}}(t) = A\hat{x}(t) + A^\tau\hat{x}(t - \tau) + B_w\hat{w}(t) + B_u\hat{u}(t)\\
\dot{\hat{z}}(t) = C_z\hat{x}(t) + C_z^\tau\hat{x}(t - \tau) + D_{zw}\hat{w}(t) + D_{zu}\hat{u}(t)\\
\hat{y}(t) = C_y\hat{x}(t) + C_y^\tau\hat{x}(t - \tau) + D_{yw}\hat{w}(t)
$$

where the system matrices $A, A^\tau, B_w, etc.$ are all block diagonal and have the special structure (2.4). This is equivalent to the following set of $N$ independent $n$-th order time-delay systems:

$$
\dot{\hat{x}}_i(t) = A_i\hat{x}_i(t) + A_i^\tau\hat{x}_i(t - \tau) + B_{w,i}\hat{w}_i(t) + B_{u,i}\hat{u}_i(t)\\
\dot{\hat{z}}_i(t) = C_{z,i}\hat{x}_i(t) + C_{z,i}^\tau\hat{x}_i(t - \tau) + D_{zw,i}\hat{w}_i(t) + D_{zu,i}\hat{u}_i(t) \quad \text{for} \quad i = 1, \ldots, N\\
\hat{y}_i(t) = C_{y,i}\hat{x}_i(t) + C_{y,i}^\tau\hat{x}_i(t - \tau) + D_{yw,i}\hat{w}_i(t)
$$

where $\hat{x}_i \in \mathbb{R}^{n \times 1}$ is the $i$-th block of $\hat{x}$, and $\hat{w}_i$ , $\hat{u}_i$ , $\hat{z}_i$ and $\hat{y}_i$ are similarly defined.
According to lemma 2.2.2 the system matrices $A_i$, $D_{yw,i}$ are defined as follows:

$$A_i = A_a + \lambda_i A_b$$

$$A_i^\tau = A_a^\tau + \lambda_i A_b^\tau$$

$$B_{w,i} = B_{w,a} + \lambda_i B_{w,b}$$

$$\vdots$$

$$D_{yw,i} = D_{yw,b} + \lambda_i D_{yw,b} \quad \text{for} \quad i = 1, \ldots, N \quad (4.6)$$

According to Theorem 4.2.2, a decomposable time-delay system (4.1) is equivalent to a set of $N$ independent time-delay modal subsystems (4.5). Therefore, it is possible to solve the control problem in the frame of $N$ independent subsystems, which are considerably smaller in size, and then retrieve the solution of the original problem using the inverse transformation.

The aim is to design either static state feedback controller

$$u(t) = K x(t) + K^\tau x(t - \tau); \quad K = I_N \otimes K_a \quad \text{and} \quad K^\tau = L \otimes K_b^\tau \quad (4.7)$$

or dynamic output feedback controller

$$\dot{x}_c(t) = \bar{A}_c x_c(t) + \bar{A}_c^\tau x_c(t - \tau) + \bar{B}_c y(t)$$

$$u(t) = \bar{C}_c x_c(t) + \bar{C}_c^\tau x_c(t - \tau) + \bar{D}_c y(t) \quad (4.8)$$

with the system matrices

$$\bar{A}_c = I_N \otimes A_{c,a}; \quad \bar{A}_c^\tau = L \otimes A_{c,b}^\tau$$

$$\bar{C}_c = I_N \otimes C_{c,a}; \quad \bar{C}_c^\tau = L \otimes C_{c,b}^\tau$$

$$\bar{B}_c = I_N \otimes B_{c,a}; \quad \bar{D}_c = I_N \otimes D_{c,a} \quad (4.9)$$

such that
1. the closed-loop system is $\alpha$-uniformly asymptotically stable

2. the closed loop system guarantees, under zero initial conditions, $\|T_{zw}\|_\infty < \gamma_\infty$ or $\|T_{zw}\|_2 < \gamma_2$ for all non-zero $w(t) \in L_2$. In this case we say that $\gamma_\infty$ and $\gamma_2$ are $\alpha$-suboptimal.

It is easy to see that $\gamma_\infty$ and $\gamma_2$ are suboptimal in the usual sense if it is $\alpha$-suboptimal for $\alpha = 0$. Furthermore, if the closed-loop system is $\alpha$-suboptimal asymptotically stable, it is also uniformly asymptotically stable ($\alpha \equiv 0$).

It should be pointed out that the controllers is designed in such a way to have the same interconnection structures as the plant and use the available delayed neighboring information. The control design approach is to find $N$ controllers of the same type for the modal subsystems (4.5); i.e., static state feedback controller

$$\dot{\hat{u}}_i(t) = K_i\hat{x}_i(t) + K_i^T\hat{x}_i(t - \tau) \quad \text{for} \quad i = 1, \ldots, N \quad (4.10)$$

or dynamic output feedback controller

$$\begin{align*}
\dot{\hat{x}}_{c,i}(t) &= A_{c,i}\hat{x}_{c,i}(t) + A_{c,i}^T\hat{x}_{c,i}(t - \tau) + B_{c,i}\hat{y}_i(t) \\
\dot{\hat{u}}_i(t) &= C_{c,i}\hat{x}_{c,i}(t) + C_{c,i}^T\hat{x}_{c,i}(t - \tau) + D_{c,i}\hat{y}_i(t) \quad \text{for} \quad i = 1, \ldots, N \\
\end{align*} \quad (4.11)$$

and then retrieve the controller in the non-decomposed form.

In the next section, we discuss the problem of stability, $\alpha$-stability and bounds on the $H_2$ and $H_\infty$ norms for linear time-delay systems and introduce sufficient conditions in the form of LMIs for controller synthesis which helps us in section 4.4 to find distributed controller parameters efficiently.
4.3 Stability and Bounds on the $H_2$ And $H_\infty$ Norms

For Delay Systems

Large-scale systems include electrical power systems, computer communication systems, process control systems and ecological systems. In practices, due to the information transmission between subsystems, time delays naturally exist in large-scale systems. The existence of time delays is frequently a source of instability and encountered in various engineering systems [1], [2]. Hence, the problem of stability analysis of time-delay systems has been one of the main concerns of researchers wishing to inspect the properties of such systems. Stability criteria of time-delay systems so far have been approached in two main ways according to the dependence upon the size of delay. One direction is to contrive stability conditions that do not include information on the delay, while the other direction includes methods which take this into account. The former case is often referred to as delay-independent criteria which will be the focus of our development. Here, we summarize a few results on stability, $\alpha$-stability, $H_2$ and $H_\infty$ bounds of these systems based on the Lyapunov-Krasovskii functional [76] that will be used later in the controller design.

Consider the linear time-delay system

\begin{align*}
\dot{x}(t) &= \mathcal{A}x(t) + \mathcal{A}^r x(t - \tau) + B_w w(t) \\
z(t) &= \mathcal{C}_x x(t) + \mathcal{C}_w^r x(t - \tau) + D_{zw} w(t) \\
x(t) &= 0, \ \forall t \in [-\tau, 0]
\end{align*} \tag{4.12}

and the Lyapunov-Krasovskii functional

\begin{align*}
V(x(t)) &= x^T(t) \mathcal{P} x(t) + \int_{t-\tau}^{t} x^T(\theta) \mathcal{P}_1 x(\theta) d\theta \\
\end{align*} \tag{4.13}

where the matrices $\mathcal{P}$ and $\mathcal{P}_1$ are positive definite.
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Assuming that the exogenous input \( w(t) = 0 \) we have that

\[
\dot{V}(x(t)) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} \mathcal{A}^T \mathcal{P} + \mathcal{P} \mathcal{A} + \mathcal{P} \mathcal{A}^T \\ \mathcal{A}^T \mathcal{P} \\ -\mathcal{P} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}
\] (4.14)

Using \( V(x(t)) \) as a Lyapunov function, if \( \dot{V}(x(t)) < 0 \) for all \( t > 0 \), then the linear time-delay system (4.12) is asymptotically stable. A sufficient condition for this to happen is given in the next lemma.

**Lemma 4.3.1.** System (4.12) is asymptotically stable for any constant delay parameter \( \tau \geq 0 \) if there exist symmetric and positive definite matrices \( \mathcal{P} \) and \( \mathcal{P}_1 \) such that the following LMI has a feasible solution.

\[
\begin{bmatrix} \mathcal{A}^T \mathcal{P} + \mathcal{P} \mathcal{A} + \mathcal{P} \mathcal{A}^T \\ \mathcal{A}^T \mathcal{P} \\ -\mathcal{P} \end{bmatrix} < 0
\] (4.15)

**Proof.** According to (4.14), the negative definiteness of (4.15) makes \( \dot{V}(x(t)) < 0 \) which complete the proof.

The stability condition in the above lemma is delay independent, that is, it guarantees that the linear time-delay system is stable for any value of the delay parameter \( \tau \geq 0 \).

Now assume that (4.15) has a feasible solution. Since this inequality is strictly feasible, standard arguments can be used to show that there exist symmetric and positive definite matrices \( \mathcal{P} \) and \( \mathcal{P}_1 \) such that

\[
\begin{bmatrix} \mathcal{A}^T \mathcal{P} + \mathcal{P} \mathcal{A} + \mathcal{P} \mathcal{A}^T \\ \mathcal{A}^T \mathcal{P} \\ -\mathcal{P} \end{bmatrix} < -\begin{bmatrix} \mathcal{C}_z^T \\ \mathcal{C}^T \end{bmatrix}
\] (4.16)

also has a feasible solution. Assuming that the transfer function from \( w \) to \( z \) is strictly proper (\( \mathcal{D} = 0 \)), and multiplying (4.16) on the left by the vector \( \begin{bmatrix} x^T(t) \\ x^T(t-\tau) \end{bmatrix} \) and on the right by its transpose, we get

\[
\dot{V}(x(t)) = -z^T(t)z(t)
\]
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Assuming null initial conditions and successively applying unitary impulses $\delta(t)$ at each entry of the exogenous input vector $w(t)$, and integrating the above relation, on both sides, for each impulse response $z(t) = z_i(t), i = 1, \ldots, m$, it is possible to conclude that

$$\sum_{i=1}^{m} B_i^T P B_i = \text{trace}(B^T P B) > \sum_{i=0}^{\infty} z_i^T(\theta) z_i(\theta) d\theta$$

(4.17)

where $B_i, for i = 1, \ldots, m$, denote the columns of matrix $B$. Using Parseval's Theorem, the above inequality can be translated in terms of a bound on the $H_2$ norm of the non-rational transfer function $T_{wz}(s)$ as in the following lemma.

**Lemma 4.3.2.** Time-delay system (4.12) with $D = 0$ is asymptotically stable and $\|T_{wz}\|_2 < \gamma_2$ for any constant delay parameter $\tau \geq 0$ if there exist symmetric and positive definite matrices $Z$, $P$ and $P_1$ such that the following LMI has a feasible solution.

$$\begin{bmatrix}
A^T P + PA + P_1 & P A^T & C_2 Z^T
* & -P_1 & C_2^T Z \\
* & -P_1 & C_2 Z^T
* & * & -I
\end{bmatrix} < 0, \quad \begin{bmatrix}
Z & B^T P
* & P
\end{bmatrix} > 0 \quad \text{trace}(Z) < \gamma_2^2$$

(4.18)

**Proof.** The only missing point is the conversion of (4.16) and (4.17) into the LMI (4.18). This can be accomplished by introducing matrix $Z$ as $Z = B^T P P^{-1} P B$ and applying Schur complement on the resulting inequalities.

The following lemma provides the similar result for the bound on $H_\infty$ norm of linear time-delay systems.

**Lemma 4.3.3.** Time-delay system (4.12) is asymptotically stable and $\|T_{wz}\|_\infty < \gamma_\infty$ for any constant delay parameter $\tau \geq 0$ if there exist symmetric and positive definite
matrices $P$ and $P_1$ such that the following LMI has a feasible solution.

$$\begin{bmatrix}
A^TP + PA + P_1 & PA^T & C_z^T & PB \\
* & -P_1 & C_z^T & 0 \\
* & * & -\gamma_\infty I & D \\
* & * & * & -\gamma_\infty I
\end{bmatrix} < 0,$$

(4.19)

Proof. By applying the Schur complement to the LMI (4.19) we have

$$\begin{bmatrix}
A^TP + PA + P_1 & PA^T & P_1A^T & PB \\
A^TP & -P_1 & 0 & C_z^T \\
B^TP & 0 & -\gamma_\infty I & D^T
\end{bmatrix} < -\frac{1}{\gamma_\infty} \begin{bmatrix}
C_z^T \\
C_z^T \\
D^T
\end{bmatrix} \begin{bmatrix}
C_z & C_z^T & D
\end{bmatrix}$$

(4.20)

Multiplying the above inequality on the left by the vector $[x^T(t) \ x^T(t-\tau) \ w^T(t)]$ and on the right by its transpose, we obtain

$$\dot{V}(x(t)) = \gamma_\infty w^T(t)w(t) - \frac{1}{\gamma_\infty} z^T(t)z(t)$$

Assuming that $w(t) \in L_2$ and using the fact that the initial conditions are null, the integration of both sides from zero to infinity provides

$$\int_0^\infty z^T(\theta)z(\theta)d\theta < \gamma_\infty^2 \int_0^\infty w^T(\theta)w(\theta)d\theta$$

which completes the proof.

The following lemma gives a sufficient condition for $\alpha$-uniform asymptotic stability of the linear time-delay systems.

Lemma 4.3.4. Time-delay system (4.12) is $\alpha$-uniform asymptotically stable for constant delay parameter $\tau \geq 0$ if there exist symmetric and positive definite matrices $P$ and $P_1$ such that the following LMI has a feasible solution.

$$\begin{bmatrix}
A^TP + PA + P_1 + 2\alpha P & PA^T e^{\alpha\tau} \\
e^{\alpha\tau} A^TP & -P_1
\end{bmatrix} < 0$$

(4.21)
Proof. Let us consider the following functional differential equation:

\[ \dot{\eta}(t) = (A + \alpha I_n)\eta(t) + e^{\alpha \tau} A^T \eta(t - \tau) \]  

(4.22)

which corresponds to the differential equation (4.12) with \( w = 0 \) via the following state transformation

\[ x(t) = e^{-\alpha(t-t_0)}\eta(t), \quad t \geq t_0. \]  

(4.23)

From the \( \alpha \)-stability definition, it follows that the uniform asymptotic stability of the trivial solution of the functional differential equation (4.22) guarantees the \( \alpha \)-uniform asymptotic stability of the system (4.12) (free of disturbance inputs). Let us introduce the following Lyapunov-Krasovskii functional candidate:

\[ V(t, \eta(t)) = \eta^T(t) P \eta(t) + \int_{t-\tau}^{t} \eta^T(\theta) P_1 \eta(\theta) d\theta \]  

(4.24)

where the matrices \( P \) and \( P_1 \) are positive definite. The derivative of the Lyapunov-Krasovskii functional \( V(t, \eta(t)) \) along the trajectories of (4.22) can be rewritten after some algebraic manipulations in the following matrix form

\[ \dot{V}(t, \eta(t)) = \begin{bmatrix} \eta(t) \\ \eta(t-\tau) \end{bmatrix}^T \begin{bmatrix} A^T P + P A + P_1 + 2\alpha P & P A^T e^{\alpha \tau} \\ e^{\alpha \tau} A^T P & -P_1 \end{bmatrix} \begin{bmatrix} \eta(t) \\ \eta(t-\tau) \end{bmatrix} \]  

(4.25)

The strictly negative inequality (4.21) and the quadratic form of (4.25) leads to the negativity of the derivative \( V(t, \eta(t)) \) along the trajectories of (4.22). Thus, from Krasovskii Stability theorem, it follows the uniform asymptotic stability of the trivial solution of the functional differential equation (4.22), that is, the \( \alpha \)-uniform asymptotic stability of the system (4.12). Notice that \( \alpha \)-uniform asymptotic stability implies uniform asymptotic stability (which corresponds to the case \( \alpha = 0 \)) via the same Lyapunov-Krasovskii functional as shown before. \qed
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4.4 Distributed Control Design

Using the result of linear matrix inequality characterizations presented in Section 4.3, we can parameterize controllers efficiently with desired performances. In the following subsections, we will show how to synthesize distributed controller with $H_2$, $H_\infty$, and $\alpha$-stability performance for two cases of state and dynamic output feedback for continuous-time decomposable time-delay systems.

4.4.1 Distributed Control with State Feedback

In this subsection, we state our main results at the following theorems for $\alpha$-uniformly asymptotically stability with $H_2$ and $H_\infty$ performance.

Theorem 4.4.1. Consider a symmetric decomposable time-delay system (4.1) with communication delay $\tau \geq 0$. A sufficient condition for the existence of $\alpha$-suboptimal distributed static state feedback controller (4.7) that yields $\|T_{wz}\|_2 < \gamma_2$ is that the following set of LMIs

$$
\begin{bmatrix}
He \{ A_i X + B_{u,i} W \} + X_1 + 2\alpha X \ (A_i^T X + B_{u,i} W_i^\tau) e^{\alpha \tau} \\
\quad \star \\
\quad -X_1
\end{bmatrix} < 0 \quad (4.26)
$$

$$
\begin{bmatrix}
He \{ A_i X + B_{u,i} W \} + X_1 \ A_i^T X + B_{u,i} W_i^\tau \ X C_{z,i}^T + W_i^T D_i^T \quad X C_{z,i}^T + W_i^T D_i^T \\
\quad \star \\
\quad \star \\
\quad -I_{r_2}
\end{bmatrix} < 0
$$

$$
\begin{bmatrix}
Z_i & B_{w,i}^T \\
\star & X
\end{bmatrix} < 0 \quad \text{for} \quad i = 1, \ldots, N \quad \text{and} \quad \sum_{i=1}^N \text{tr}(Z_i) < \gamma_2^2 \quad (4.27)
$$

has a feasible solution, where $X = X^T$, $X_1 = X_1^T$, $W$, $W_i^\tau$, and $Z_i = Z_i^T$ are the optimization variables in which $W_i^\tau = \lambda_i W_b^\tau$ and the controller parameters are determined as $K_a = WX^{-1}$, $K_b^\tau = W_b^\tau X^{-1}$. 
Proof. Applying the congruent transformation \( \text{diag}(P^{-1}, P^{-1}) \) into the LMI (4.21) and the congruent transformation \( \text{diag}(P^{-1}, P^{-1}, I) \) and \( \text{diag}(I, P^{-1}) \) into the LMIs (4.18), the resulting inequalities can be rewritten as new LMIs

\[
\begin{bmatrix}
AX + XA^T + X_1 + 2\alpha X & A^TXe^{\alpha \tau} \\
* & -X_1
\end{bmatrix} < 0
\] (4.28)

\[
\begin{bmatrix}
AX + XA^T + X_1 & A^TX & X_{Cz}^T \\
* & -X_1 & XC_{z}^T \\
* & * & -I
\end{bmatrix} < 0,
\begin{bmatrix}
Z & B^T \\
X & *
\end{bmatrix} > 0 \quad \text{trace}(Z) < \gamma_2^2
\] (4.29)

by defining \( X = P^{-1} \) and \( X_1 = P^{-1}P_1P^{-1} \). Substituting the following system matrices of the closed-loop modal subsystems

\[
A = A_i + B_{u,i}K_i, \quad A^T = A_i^T + B_{u,i}K_i,
\]

\[
C = C_{z,i} + D_{zu,i}K_i, \quad C^T = C_{z,i}^T + D_{zu,i}K_i^T,
\]

\[
B = B_{w,i}, \quad D = D_{zu,i}.
\]

in the LMIs (4.29) and (4.28), applying the change-of-variable \( W = K_aX, W_i^T = K_i^T X \), and introduction of the constraints \( W_i^T = \lambda_i W_i^T \) for \( i = 1, \ldots, N \) result to \( N \) sets of LMIs (4.26) and (4.27). Moreover, the equation (2.11) which relates the \( H_2 \) norm of the modal subsystems to the \( H_2 \) norm of the original system makes the coupling constraint \( \sum_{i=1}^{N} \text{tr}(Z_i) < \gamma_2^2 \) in (4.27). This leads to the sufficient condition for the existence of distributed state-feedback controller with \( H_2 \) and \( \alpha \)-stability performance. 

\[\Box\]

**Theorem 4.4.2.** Consider a continuous-time symmetric decomposable system (4.1) with interconnection delay \( \tau \geq 0 \). A sufficient condition for the existence of \( \alpha \)-suboptimal distributed static state feedback controller (4.7) that yields \( \|T_{wz}\|_\infty < \gamma_\infty \)
is that the following set of LMIs

\[
\begin{bmatrix}
He \{A_iX + B_{u,i}W\} + X_1 + 2\alpha X & (A_i^TX + B_{u,i}W^T)e^{\alpha T} \\
* & -X_1
\end{bmatrix} < 0 \quad (4.31)
\]

\[
\begin{bmatrix}
He \{A_iX + B_{u,i}W\} + X_1 & A_i^TX + B_{u,i}W^T & XC_{z,i}^T & W^TD_{zu,i} & B_{w,i} \\
* & -X_1 & XC_{z,i}^T & W^TD_{zu,i} & 0 \\
* & * & -I_r & D_{zw,i} & \\
* & * & * & -\gamma I
\end{bmatrix} < 0
\]

for \( i = 1, \ldots, N \) \quad (4.32)

has a feasible solution, where \( X = X^T \), \( X_1 = X_1^T \), \( W \), and \( W^T \) are the optimization variables in which \( W^T_i = \lambda_i W_i^T \) and the controller parameters are determined as \( K_a = WX^{-1}, \quad K_b^r = W_b^r X^{-1} \).

**Proof.** Similar to the proof of Theorem 4.4.1 one can establish the result by construction using the LMI (4.19) and (4.21). \( \square \)

### 4.4.2 Distributed Control with Dynamic Output Feedback

In this subsection, we derive formulas for the design of distributed dynamic output feedback controllers for decomposable time-delay systems. The main contribution is to show that the \( \alpha \)-stability, \( H_2 \) and \( H_\infty \) performance control design problems based on the conditions developed in Section 4.3 can be cast and solved as LMI for the dynamic output feedback case as well. The result is based on the change-of-variables developed in [58], which we modify to cope with the inequalities given in Section 4.3 with additional delay terms.

**Theorem 4.4.3.** Consider a symmetric decomposable time-delay system (4.1) with interconnection delay \( \tau \geq 0 \). A sufficient condition for the existence of distributed dynamic output feedback controller (4.8) that makes the closed-loop system \( \alpha \)-uniformly asymptotically stable and yields \( \| T_{wz} \|_2 < \gamma_2 \) is that the following set of LMIs

\[
\begin{bmatrix}
He \{A_iX + B_{u,i}W\} + X_1 + 2\alpha X & (A_i^TX + B_{u,i}W^T)e^{\alpha T} \\
* & -X_1
\end{bmatrix} < 0 \quad (4.31)
\]

\[
\begin{bmatrix}
He \{A_iX + B_{u,i}W\} + X_1 & A_i^TX + B_{u,i}W^T & XC_{z,i}^T & W^TD_{zu,i} & B_{w,i} \\
* & -X_1 & XC_{z,i}^T & W^TD_{zu,i} & 0 \\
* & * & -I_r & D_{zw,i} & \\
* & * & * & -\gamma I
\end{bmatrix} < 0
\]

for \( i = 1, \ldots, N \) \quad (4.32)

has a feasible solution, where \( X = X^T \), \( X_1 = X_1^T \), \( W \), and \( W^T \) are the optimization variables in which \( W^T_i = \lambda_i W_i^T \) and the controller parameters are determined as \( K_a = WX^{-1}, \quad K_b^r = W_b^r X^{-1} \).
He\{A_i(X_0, Y_0, F, R, Q_a, L_a)\} + X_1 + 2\alpha X(X_0, Y_0) e^{\alpha \tau A_i^T(X_0, Y_0, F, R, Q_{b_i}^r, L_{b_i}^r)} - X_1 < 0 \\
(4.33)

He\{A_i(X_0, Y_0, F, R, Q_a, L_a)\} + X_1 A_i^T(X_0, Y_0, F, R, Q_{b_i}^r, L_{b_i}^r) C_i^T(X_0, R, L_a) - X_1 C_i^{\tau T}(X_0, R, L_{b_i}^r) < 0 \\
(4.34)

with the matrices $A_i$, $A_i^T$, $B_i$, $C_i$, $C_i^T$, $D_i$ and $X$ defined as

$$A_i(X_0, Y_0, F, R, Q_a, L_a) = \begin{bmatrix} A_i X_0 + B_{u,a} L_a & A_i + B_{u,a} R C_{y,i} \\ Q_a & Y_0 A_i + F C_{y,i} \end{bmatrix}$$

$$A_i^T(X_0, Y_0, F, R, Q_{b_i}^r, L_{b_i}^r) = \begin{bmatrix} A_i^T X_0 + B_{u,a} L_{b_i}^r & A_i^T + B_{u,a} R C_{y,i}^{\tau} \\ \lambda_i Q_{b_i}^r & Y_0 A_i^T + F C_{y,i}^{\tau} \end{bmatrix}$$

$$B_i(Y_0, F, R) = \begin{bmatrix} B_{w,i} + B_{u,a} R D_{yw,i} \\ Y_0 B_{w,i} + F D_{yw,i} \end{bmatrix}$$

$$C_i(X_0, R, L_a) = \begin{bmatrix} C_{z,i} X_0 + D_{z_{u,a}} L_a & C_{z,i} + D_{z_{u,a}} R C_{y,i} \end{bmatrix}$$

$$C_i^{\tau}(X_0, R, L_{b_i}^r) = \begin{bmatrix} C_{z,i}^{\tau} X_0 + \lambda_i D_{z_{u,a}} L_{b_i}^r & C_{z,i}^{\tau} + D_{z_{u,a}} R C_{y,i}^{\tau} \end{bmatrix}$$

$$D_i(R) = D_{z_{w,i}} + D_{z_{u,a}} R D_{yw,i}$$

$$X(X_0, Y_0) = \begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix}$$
has a feasible solution where the optimization variables are $Z_i = Z_i^T$, $X_0 = X_0^T$, $Y_0 = Y_0^T$, $X_1$, $F$, $R$, $Q_a$, $Q_b^\tau$, $L_a$ and $L_b^\tau$. Furthermore, the controller parameters can be reconstructed by

\begin{align*}
D_{c,a} &= R \\
B_{c,a} &= V_0^{-1} \{ F - Y_0 B_{u,a} R \} \\
C_{c,a} &= \{ L_a - RC_{y,a} X_0 \} U_0^{-1} \\
C_{c,b}^\tau &= \{ L_b^\tau - RC_{y,b} X_0 \} U_0^{-1} \\
A_{c,a} &= V_0^{-1} \{ Q_a - Y_0 A_a X_0 - Y_0 B_{u,a} L_a - FC_{y,a} X_0 \} U_0^{-1} \\
A_{c,b}^\tau &= V_0^{-1} \{ Q_b^\tau - Y_0 A_b^\tau X_0 - Y_0 B_{u,a} L_b^\tau - FC_{y,b} X_0 \} U_0^{-1}
\end{align*}

where $U_0$ and $V_0$ are non-singular matrices satisfying $V_0 U_0 = I - Y_0 X_0$.

**Proof.** The closed-loop connection of modal subsystem (4.5) and full-order dynamic output-feedback controller (4.11) yeilds the following linear time-delay system with the augmented state vector $\tilde{x}_i = [\hat{x}_i^T, \hat{x}_{c,i}^T]^T$,

\begin{align*}
\dot{\hat{x}}_i(t) &= A_i \hat{x}_i(t) + \bar{A}_i \hat{x}_i(t - \tau) + B_{w,i} \hat{w}_i(t) \\
\dot{\hat{z}}_i(t) &= C_{z,i} \hat{x}_i(t) + \bar{C}_{z,i} \hat{x}_i(t - \tau) + D_{z,w,i} \hat{w}_i(t) \\
\dot{\tilde{x}}_i(t) &= 0, \forall t \in [-\tau, 0]
\end{align*}

where the closed loop system $\mathcal{A}_i$, $\bar{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$, $\bar{C}_i$, $\mathcal{D}_i$ is described by

\begin{align*}
\mathcal{A}_i &= \begin{bmatrix}
A_i + B_{u,i} D_{c,i} C_{y,i} & B_{u,i} C_{c,i} \\
B_{c,i} C_{y,i} & A_{c,i}
\end{bmatrix}, & \bar{A}_i &= \begin{bmatrix}
A_i^\tau + B_{u,i} D_{c,i} C_{y,i}^\tau & B_{u,i} C_{c,i}^\tau \\
B_{c,i} C_{y,i}^\tau & A_{c,i}^\tau
\end{bmatrix} \\
\mathcal{B}_i &= \begin{bmatrix}
B_{w,i} + B_{u,i} D_{c,i} D_{y,w,i} \\
B_{c,i} D_{y,w,i}
\end{bmatrix}, & \mathcal{D}_i &= D_{z,w,i} + D_{z,u,i} D_{c,i} D_{y,w,i} \\
\mathcal{C}_i &= \begin{bmatrix}
C_{z,i} + D_{z,u,i} D_{c,i} C_{y,i} \\
D_{z,u,i} C_{c,i}
\end{bmatrix}, & \bar{C}_i &= \begin{bmatrix}
C_{z,i}^\tau + D_{z,u,i} D_{c,i} C_{y,i}^\tau \\
D_{z,u,i} C_{c,i}^\tau
\end{bmatrix}
\end{align*}

(4.38)
CHAPTER 4. COMMUNICATION DELAY

Substituting (4.38) in LMIs (4.18) and (4.21) for \( i = 1, \ldots, N \), we are performing the following change-of-variable, adapted from [58], to obtained the sufficient conditions in the form of LMI. Let us first define the partitions associated with the symmetric matrix \( P \) and its inverse as:

\[
P = \begin{bmatrix} Y_0 & V_0 \\ V_0^T & \hat{Y}_0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} X_0 & U_0^T \\ U_0 & \hat{P}_0 \end{bmatrix}
\]

where \( X_0 \in \mathbb{R}^{n \times n}, Y_0 \in \mathbb{R}^{n \times n} \) and other matrices have compatible dimensions. With (4.39) and the controller matrices given in (4.11), we introduce the change of variable

\[
\begin{bmatrix} Q_i & Q_i^T & F_i \\ L_i & L_i^T & R_i \end{bmatrix} = \begin{bmatrix} V_0 & Y_0 B_{u,a} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{c,i} & A_{c,i}^T & B_{c,i} \\ C_{c,i} & C_{c,i}^T & D_{c,i} \end{bmatrix} \begin{bmatrix} U_0 & 0 & 0 \\ 0 & U_0 & 0 \end{bmatrix} \begin{bmatrix} C_{g,i} X_0 & C_{g,i}^T X_0 & I \end{bmatrix}
\]

\[
+ \begin{bmatrix} Y_0 \\ 0 \end{bmatrix} \begin{bmatrix} A_i & A_i^T & 0 \end{bmatrix}
\]

Let us define the transformation matrix

\[
\mathcal{F} = \begin{bmatrix} X_0 & I \\ U_0 & 0 \end{bmatrix}
\]

Applying congruent transformation \( \text{diag}(\mathcal{F}, \mathcal{F}) \) into the resulting inequality for \( \alpha \)-stability and the congruent transformation \( \text{diag}(\mathcal{F}, \mathcal{F}, I) \) and \( \text{diag}(I, \mathcal{F}) \) into the resulting inequalities for \( H_2 \) performance, yields to LMIs (4.33) and (4.34) where they are all affine in the variables

\[
A_i(X_0, Y_0, F, R, Q_a, L_a) = \mathcal{F}^T \mathcal{P} A_i \mathcal{F}
\]

\[
A_i^T(X_0, Y_0, F, R, Q_b^T, L_b^T) = \mathcal{F}^T \mathcal{P} A_i^T \mathcal{F}
\]

\[
C_i(X_0, R, L_a) = C_i \mathcal{F}, \quad C_i^T(X_0, R, L_b^T) = C_i^T \mathcal{F}
\]

\[
B_i(Y_0, F, R) = \mathcal{F}^T \mathcal{P} B_i, \quad D_i(R) = D_i
\]
\[ \mathbf{X}(X_0, Y_0) = \mathcal{F}^T \mathcal{P} \mathcal{F}, \quad X_1 = \mathcal{F}^T \mathcal{P}_1 \mathcal{F} \]

as defined in (4.35) with the constraints \( Q_i = Q_a, Q_i^\tau = \lambda_i Q_b^\tau, L_i = L_a, L_i^\tau = \lambda_i L_b^\tau, \)
\( R_i = R, \) and \( F_i = F \) to parametrize the controller parameters \((A_{c,i}, A_{c,i}^\tau, B_{c,i}, C_{c,i}, C_{c,i}^\tau, D_{c,i})\)
as defined in \( A_{c,i} = A_{c,a} \), \( A_{c,i}^\tau = \lambda_i A_{c,b}^\tau \), \( B_{c,i} = B_{c,a} \), \( C_{c,i} = C_{c,a} \), \( C_{c,i}^\tau = \lambda_i C_{c,b}^\tau \), \( D_{c,i} = D_{c,a} \).

This yields distributed dynamic output-feedback controller \((\bar{A}_c, \bar{A}_c^\tau, \bar{B}_c, \bar{C}_c, \bar{C}_c^\tau, \bar{D}_c)\) of the form (4.8).

\[ \square \]

The above derivation for \( H_2 \) performance can also be established for the \( H_\infty \) case with additional \( \alpha \)-stability performance using LMI characterizations in (4.18) and (4.21). Since the essential equations have been derived for the \( H_2 \) case, here we state the main results as follows.

**Theorem 4.4.4.** Consider a symmetric decomposable system (4.1) with interconnection delay \( \tau \geq 0 \). A sufficient condition for the existence of distributed dynamic output feedback controller (4.8) that makes the closed-loop system \( \alpha \)-uniformly asymptotically stable and yields \( \| T_{wx} \|_\infty < \gamma_\infty \) is that the following set of LMIs

\[ \begin{bmatrix} \mathcal{H}\{A_i(X_0, Y_0, F, R, Q_a, L_a)\} + X_1 + 2\alpha \mathbf{X}(X_0, Y_0) & e^{\alpha \tau} A_i^\tau(X_0, Y_0, F, R, Q_b^\tau, L_b^\tau) \\ * & -X_1 \end{bmatrix} < 0 \]

(4.43)

\[ \begin{bmatrix} \mathcal{H}\{A_i(X_0, Y_0, F, R, Q_a, L_a)\} + X_1 & A_i^\tau(X_0, Y_0, F, R, Q_b^\tau, L_b^\tau) \quad C_i^T(X_0, R, L_a) & B_i(Y_0, F, R) \\ * & -X_1 & C_i^T(X_0, R, L_b) & 0 \\ * & * & -I & D_i(R) \\ * & * & * & -\gamma_\infty I \end{bmatrix} < 0 \]

\[ \mathbf{X}(X_0, Y_0) > 0 \quad \text{for} \quad i = 1, \ldots, N \]

(4.44)
with the matrices $A_i$, $A_i^T$, $B_i$, $C_i$, $C_i^T$, $D_i$ and $X$ defined in (4.35) has a feasible solution where the optimization variables are $X_0 = X_0^T$, $Y_0 = Y_0^T$, $X_1$, $F$, $R$, $Q_a$, $Q_b^T$, $L_a$ and $L_b^T$. Furthermore, the controller parameters can be reconstructed by (4.36).

Proof. The line of proof is similar to the Theorem 4.4.4 by using LMIs (4.18) and (4.21).

\[ \square \]

### 4.5 Simulation Results

As an example, here we consider the same satellite formation problem as in previous chapter, a swarm of satellites orbiting around earth on a circular orbit at an angular speed $\omega_n$. The trajectory of $i$-th satellite around their nominal circular orbit can be described by:

\[
\begin{align*}
\ddot{\xi}_{i,1} &= 3\omega_n^2 \xi_{i,1} + 2\omega_n \dot{\xi}_{i,2} + \phi_{i,1} \\
\ddot{\xi}_{i,2} &= -2\omega_n \dot{\xi}_{i,1} + \phi_{i,2} \\
\ddot{\xi}_{i,3} &= -\omega_n^2 \xi_{i,3} + \phi_{i,3}
\end{align*}
\]

where $\xi_1$, $\xi_2$ and $\xi_3$ are respectively the displacements in the radial, tangential and out-of-plane direction, and $\phi_1$, $\phi_2$ and $\phi_3$ are the accelerations of the spacecraft due to either propulsion or external disturbances. The state representation of these dynamic systems can be written as:

\[
\begin{align*}
\dot{x}_i(t) &= Ax_i(t) + B_u u_i(t) + B_w w_i(t) \\
y_i(t) &= Cx_i(t) \quad \text{for } i = 1, \ldots, N \quad (4.45)
\end{align*}
\]

where $x_i = [\xi_{i,1}, \dot{\xi}_{i,1}, \xi_{i,2}, \dot{\xi}_{i,2}, \xi_{i,3}, \dot{\xi}_{i,3}]^T$ is the state, $u_i = [u_{i,1}, u_{i,2}, u_{i,3}]^T$ is the propulsion and $w_i = [w_{i,1}, w_{i,2}, w_{i,3}]^T$ is the external disturbance and the system matrices are...
defined as follows

\[
A = \begin{bmatrix}
    0 & 1 & 0 & 0 & 0 & 0 \\
    3w_n^2 & 0 & 0 & 2w_n & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 2w_n & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 & -w_n^2 & 0 \\
\end{bmatrix}, \quad B_u = B_w = \begin{bmatrix}
    0 & 0 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 1 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Let us assume that \( N \) satellites are uniformly distributed on a circular orbit and there exists communication delay \( \tau \) between satellites. The aim is to design distributed controller that keeps the formation and minimizes the error on their relative positions with \( H_2 \) or \( H_\infty \) criteria using delayed information. Choosing the performance output as the sum of position difference of \( i \)-th satellite with its neighbors with additional performance output to restrict the use of actuator (as we did in Chapter 2), the dynamics of the entire system can be described by:

\[
\dot{x}(t) = \bar{A}x(t) + \bar{B}_w w(t) + \bar{B}_u u(t) \\
z(t) = \bar{C}_z x(t) + \bar{C}_z^\tau x(t - \tau) + \bar{D}_{zu} u(t) \\
y(t) = \bar{C}_y x(t) + \bar{C}_y^\tau x(t - \tau) 
\] (4.46)

where the system matrices have the form:

\[
\bar{A} = I_N \otimes A, \quad \bar{B}_w = I_N \otimes B_w, \quad \bar{B}_u = I_N \otimes B_u \\
\bar{C}_z = I_N \otimes C, \quad \bar{C}_z^\tau = L \otimes C, \quad \bar{D}_{zu} = I_N \otimes D_{zu} \\
\bar{C}_y = I_N \otimes C, \quad \bar{C}_y^\tau = L \otimes C_{y,b}^\tau 
\]
with the following pattern matrix

\[
L = \begin{bmatrix}
2 & -1 & 0 & \cdots & 0 & -1 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & -1 & 2
\end{bmatrix}
\]

It is easy to see that the entire system (4.46) is a decomposable time-delay system and we can apply our result for designing distributed state feedback controller (4.7) or dynamic output feedback controller (4.8). The displacement of a satellite in the formation of 4 satellites in response to an impulse disturbance at \( t = 0.5 \) in its radial direction is shown in Figure 4.1 and Figure 4.2. Figure 4.1 is the response corresponding to the state feedback controller under \( H_2 \) criterion with additional \( \alpha \)-stability constraint with degree \( \alpha = 1 \) and communication delay \( \tau = 0.1 \) using LMIs (4.26) and (4.27). While Figure 4.2 shows the response corresponding to the dynamic output feedback controller for the same parameters using LMIs (4.33) and (4.34).

Furthermore, the upper bound on \( H_2 \) norm of the closed-loop system as a function of \( \alpha \) is depicted in Figure 4.3 for different values of time-delay \( \tau \). The results clearly show the trade-off between \( \alpha \)-stability performance and \( H_2 \) performance. As it is expected, if \( \alpha \to 0 \), the upper bounds for different value of \( \tau \) converge to \( 8.82 \times 10^{-2} \) which corresponds to the case of delay-independent stability with \( H_2 \) performance.

### 4.6 Conclusion

In this Chapter, we presented the decomposition approach to design distributed controller for interconnected dynamic systems in the presence of time-delay in the communication network. The time delay is assumed to be identical and constant for all communication links. We formulated and solved the distributed control problem with \( H_2 \) and \( H_\infty \) performance with additional \( \alpha \)-stability constraint for both cases of state
feedback and dynamic output feedback controllers. The special change of variable adapted from [58] enabled us to derive explicit expression for computing the parameters of dynamic output feedback controller by solving a set of LMIs. The method is applied to a satellite formation problem and simulation results confirm the theoretical development.

Figure 4.1: Displacement of the satellite in response to an impulse disturbance - state feedback $H_2$ controller with $\alpha$-stability constraint.
Figure 4.2: Displacement of the satellite in response to an impulse disturbance - dynamic output feedback $H_2$ controller with $\alpha$-stability constraint.
Figure 4.3: $H_2$ bound dynamic output feedback $H_2$ controller with $\alpha$-stability constraint.
Chapter 5

Random Link Failure

5.1 Introduction

In chapter 2, we designed distributed controller for multi-agent systems called decomposable systems based on decomposition approach. In Chapter 3, we further developed the problem under the assumption of switching network topology where the network topology can switch among a finite set of admissible topologies and provided a control design strategy for this scenario. Moreover, the problem of designing distributed controller for multi-agent systems in the presence of time-day in communication network was considered in Chapter 5. The optimal control solution which guarantees the stability and system performance in the presence of delay was provided.

In this Chapter we tackle the problem under the assumption of random link failures. Link Failures occur frequently in networked multi-agent systems due to node mobility or errors in communication among subsystems and in certain cases cause system instability and deteriorate the network performance if one does not consider them in control design.

Here, we consider a network with an underlying topology that is an arbitrary undi-
rected graph where links fail with independent but not necessarily identical probability. Our goal is to design a distributed controller for dynamic networks under this scenario which stabilizes the system and at the same time secures the system performance by attenuating the disturbance under the assumption of stochastic link failures. To this end, we consider each link as a random process, extract them as a stochastic uncertainty and model the entire network as a system with multiplicative random coefficients. Then by incorporating the same random variables in the control law, we introduce a distributed control algorithm that uses only the information of local subsystem and neighbors in which their connections do not fail during that time interval. It is noteworthy to emphasize that with the proposed algorithm, empha priori knowledge of network link failures is not necessary and each local controller only uses available neighboring information.

We model the links as Bernoulli random variables and provide sufficient condition for mean-square stability of the overall system with proposed distributed control rule. In order to apply the convex optimization algorithm, the sufficient stability condition is converted into a linear matrix inequality (LMI) problem, and the control parameters are obtained by solving this LMI. Moreover, the problem of stabilizing distributed control with disturbance attenuation is formulated as an LMI and the optimal controller is found with respect to the network performance.

The content of this chapter is organized as following. In Section 5.2, we formulate the problem and show how we model the network as a linear system with multiplicative random coefficients. Then, in section 5.3, we elaborate on the mean-square stability of these systems and convert the stability condition into LMI that are adapted later in designing the distributed controller. Section 5.4 includes our main results. Section 5.5 provides a numerical example to support the theoretical results. Finally, we conclude in Section 5.6.
5.2 Problem Formulation

In this work, we consider a dynamic network of $N$ interconnected systems where the interconnections links may fail. We model the network by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ is the set of $N$ nodes representing subsystems and $\mathcal{E} = \{e_1, e_2, \ldots, e_L\}$ is the set of $L$ edges representing the interconnection links between them. Each link $e_l = (v_i, v_j)$ connects $v_i$ and $v_j$ in both directions; i.e. $(v_i, v_j) = (v_j, v_i)$. We assume that each link $e_l = (v_i, v_j)$ fails with independent probability of failing $p_l = p_{(i,j)}$ in each time interval. If a link fails in a time interval, no interactions takes place through that link in either direction during that interval.

If we consider each link as an independent random process, we can model a system consisting of $N$ interconnected subsystems as a system with multiplicative random coefficients. The model can be described by the following discrete-time state space representation

$$x_i(k+1) = A_i x_i(k) + \sum_{j \in \mathcal{N}_i(k)} \delta_{(i,j)}(k) A_{ij} x_j(k) + B_{u,i} u_i(k) + B_{w,i} w_i(k) \quad (5.1)$$

for $i = 1, \ldots, N$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^m$ is the control input, $w_i \in \mathbb{R}^{m_w}$ is the disturbance, and $A_i$, $A_{ij}$, $B_{u,i}$ and $B_{w,i}$ are constant matrices of appropriate dimensions. $\mathcal{N}_i$ is the neighbor set of node $i$ in $k$-th time interval. $\delta_l = \delta_{(i,j)}$ is a Bernoulli random variable with

$$\delta_l(k) = \delta_{(i,j)}(k) = \begin{cases} 
0 & \text{with probability } p_l \\
1 & \text{with probability } 1 - p_l 
\end{cases} \quad (5.2)$$

When $\delta_l = 0$, the link $e_l = (v_i, v_j)$ has failed.

The entire interconnected system (5.1) can equivalently be written in a compact form.
as follows:

\[
    x(k + 1) = \left( A + \sum_{l=1}^{L} \delta_l(k) \tilde{A}_l \right) x(k) + B_u u(k) + B_w w(k)
\]

\[
    z(k) = C_z x(k) + D_{zu} u(k) + D_{zw} w(k)
\]

where \( x = [x_1^T, \ldots, x_N^T]^T \in \mathbb{R}^{nN} \), \( u = [u_1^T, \ldots, u_N^T]^T \in \mathbb{R}^{mN} \), \( w = [w_1^T, \ldots, w_N^T]^T \in \mathbb{R}^{m_w N} \) and \( z \in \mathbb{R}^{r_z} \) are the state, control input, disturbance and performance output of the entire network respectively. \( z(k) \) is defined to evaluate the performance of the system and is related to the network goal. \( A = \text{diag}(A_1, \ldots, A_N) \), \( B_u = \text{diag}(B_{u,1}, \ldots, B_{u,N}) \), \( B_w = \text{diag}(B_{w,1}, \ldots, B_{w,N}) \) and \( C_z \in \mathbb{R}^{r_z \times nN} \) are constant matrices, and \( \tilde{A}_l \) is defined by \( \tilde{A}_l = S_{ij} \otimes A_{ij} \) in which \( S_{ij} \in \mathbb{R}^{N \times N} \) are symmetric structured matrices with nonzero elements at \((i, j)\) and \((j, i)\) entries; i.e. \( [s_{ij}] = [s_{ji}] = 1 \). Moreover, let us define zero-mean random variable \( \mu_l(k) = \delta_l(k) - (1 - p_l) \). Then the dynamics of the networked system (5.3) can be expressed by

\[
    x(k + 1) = \left( \tilde{A} + \sum_{l=1}^{L} \mu_l(k) \tilde{A}_l \right) x(k) + B_u u(k) + B_w w(k)
\]

\[
    z(k) = C_z x(k) + D_{zu} u(k) + D_{zw} w(k)
\]

in which \( \tilde{A} = A + \sum_{l=1}^{L} (1 - p_l) \tilde{A}_l \) and \( \mu_l(k) \)'s are random variables with zero mean and variance \( \sigma^2_l = p_l(1 - p_l) \).

Remark 5.2.1. In above formulation we consider all the subsystems to be of the same size for the simplicity in notations. In general, the formulation can be extended to the systems composed of subsystems with different sizes.

The objective is to design a distributed controller for such system that use only local and neighbor’s information to stabilize the system and achieve the network goal. Here, since we are dealing with stochastic systems, by the term stability we mean mean-square stability which is a strong notion of stability for stochastic systems. In particular, it implies stability of the mean \( \mathbb{E}x(k) \) and that all the trajectories converge.
with probability one. In the next section, we elaborate on the mean-square stability of the systems with multiplicative noise and give theorems that will be adapted later in the design of distributed controllers.

5.3 Stability of systems With Multiplicative Noise

Consider the discrete-time stochastic system

\[ x(k + 1) = \left( A_0 + \sum_{l=1}^{L} \mu_l(k) A_l \right) x(k) \]  

(5.5)

where \( \mu_l(0), \mu_l(1), \mu_l(2), \ldots \) are independent, identically distributed random variables with

\[ \mathbb{E}\mu(0) = 0, \quad \mathbb{E}\mu(k)\mu(k)^T = \Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_L^2) \]

Then the state correlation matrix \( M(k) \) defined as \( \mathbb{E}x(k)x(k)^T \) satisfies the linear recursion

\[ M(k + 1) = A_0 M(k) A_0^T + \sum_{l=1}^{L} \sigma_l^2 A_l M(k) A_l^T, \quad M(0) = \mathbb{E}x(0)x(0)^T \]  

(5.6)

If this linear recursion is stable, i.e. regardless of \( x(0) \), \( \lim_{k \to \infty} M(k) = 0 \), we say the system is mean-square stable. Mean-square stability implies, for example, that \( x(k) \to 0 \) almost surely. It can be shown that the mean-square stability is equivalent to the existence of a matrix \( P = P^T > 0 \) satisfying the following LMI [64]:

\[ A_0^T PA_0 - P + \sum_{l=1}^{L} \sigma_l^2 A_l^T PA_l < 0 \]  

(5.7)

or equivalently the following dual inequality condition holds for some \( Q = Q^T > 0 \)

\[ A_0 Q A_0^T - Q + \sum_{l=1}^{L} \sigma_l^2 A_l Q A_l^T < 0. \]  

(5.8)

Remark 5.3.1. For \( P = P^T > 0 \) satisfying (5.7), we can interpret the function \( V(x) = x^T P x \) as a stochastic Lyapunov function where \( \mathbb{E}V(k) \) decreases along trajectories of (5.5).
Next consider the following stochastic system with additive exogenous input \( w(k) \) with the same assumption on \( \mu_l(k) \)

\[
x(k + 1) = A_0 x(k) + B_w w(k) + \sum_{l=1}^{L} \mu_l(k) A_l x(k) \tag{5.9}
\]

\[
z(k) = C_z x(k) + D_z w(k) + \sum_{l=1}^{L} \mu_l(k) C_{z,l} x(k)
\]

We assume that \( w(k) \) is deterministic and define the \( L_2 \) gain \( \eta \) of this system as

\[
\eta^2 = \sup \left\{ \mathbb{E} \sum_{k=0}^{\infty} z(k)^T z(k) \left| \sum_{k=0}^{\infty} w(k)^T w(k) \leq 1 \right. \right\} \tag{5.10}
\]

and suppose that \( V(x) = x^T P x \), with \( P = P^T > 0 \), satisfies

\[
\mathbb{E} V(x(k + 1)) - \mathbb{E} V(x(k)) \leq \gamma^2 w(k)^T w(k) - \mathbb{E} z(k)^T z(k) \tag{5.11}
\]

where \( \gamma \geq \eta \). The condition (5.11) can be shown to be equivalent to the following LMI:

\[
\begin{bmatrix}
A & B_w \\
C_z & D_z w
\end{bmatrix}^T
\begin{bmatrix}
P & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A & B_w \\
C_z & D_z w
\end{bmatrix}
- \begin{bmatrix}
P & 0 \\
0 & \gamma^2 I
\end{bmatrix}
\]

\[
+ \sum_{l=1}^{L} \sigma_l^2
\begin{bmatrix}
A_l & 0 \\
C_{z,l} & 0
\end{bmatrix}^T
\begin{bmatrix}
P & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A_l & 0 \\
C_{z,l} & 0
\end{bmatrix} \leq 0 \tag{5.12}
\]

Minimizing \( \gamma \) subject to LMI (5.12) yields an upper bound on system gain \( \eta \).

### 5.4 Distributed Control Design

In this section, we introduce the distributed control algorithm in which the local control actions for each subsystem consist of two parts: the part with subscript “a” which simply is the internal state feedback within each subsystem, and the part with subscript “b” which uses the state information from neighboring subsystems that
their link does not fail. Using this control input, we formulate the problem of system stability and performance as an LMI problem and then provide a distributed control design procedure for system (5.3) by solving an LMI.

We consider the following distributed control algorithm

\[ u_i(k) = K_{a,i}x_i(k) + \sum_{j \in N_i(k)} \delta_{i,j}(k)K_{b,j}x_j(k) \]  

(5.13)

where the control input in each node can be constructed from the state of that node and the state of neighbors with which it has communication link. It is worthy to point out that this control law can be implemented without any \textit{a priori} knowledge of link failures.

With these control actions, the entire input vector can be written with zero-mean random variables as following:

\[ u(k) = \left( K_a + \sum_{l=1}^{L} (1-p_l)\tilde{S}_lK_b \right)x(k) + \sum_{l=1}^{L} \mu_l(k)\tilde{S}_lK_bx(k) \]  

(5.14)

where \( \tilde{S}_l = S_{ij} \otimes I_{m_u} \), \( K_a = diag(K_{a,1}, \ldots, K_{a,N}) \), and \( K_b = diag(K_{b,1}, \ldots, K_{b,N}) \).

By putting the control action (5.14) into system equation (5.4), the dynamics of the closed-loop system can be obtained as

\[ x(k+1) = \left( \tilde{A} + B_uK_a + \tilde{B}_uK_b \right)x(k) + B_uw(k) + \sum_{l=1}^{L} \mu_l(k)\left( \tilde{A}_l + \tilde{B}_{u,l}K_b \right)x(k) \]

\[ z(k) = \left( C_z + D_{zu}K_a + \tilde{D}_{zu}K_b \right)x(k) + D_{zu}w(k) + \sum_{l=1}^{L} \mu_l(k)\tilde{D}_{zu,l}K_bx(k) \]  

(5.15)

where \( \tilde{B}_{u,l} = B_{u,l}\tilde{S}_l \), \( \tilde{B}_u = \sum_{l=1}^{L} (1-p_l)\tilde{B}_{u,l} \), \( \tilde{D}_{zu,l} = D_{zu,l}\tilde{S}_l \) and \( \tilde{D}_{zu} = \sum_{l=1}^{L} (1-p_l)\tilde{D}_{zu,l} \).

### 5.4.1 Stability

Let us consider \( w = 0 \) in (5.15). The goal is to find distributed controller parameters \( K_a \) and \( K_b \) such that the overall system becomes mean-square stable. Adapting the
inequality (5.8) for the closed loop system (5.15) with \( w = 0 \), we arrive at the stability condition
\[
(\tilde{A} + B_u K_a + \tilde{B}_u K_b) Q (\tilde{A} + B_u K_a + \tilde{B}_u K_b)^T - Q 
+ \sum_{i=1}^{L} \sigma_i^2 (\tilde{A}_i + \tilde{B}_{u,i} K_b) Q (\tilde{A}_i + \tilde{B}_{u,i} K_b)^T < 0
\]
and we seek \( K_a, K_b \) and \( Q = \text{diag}(Q_1, \ldots, Q_N) > 0 \) such that (5.16) holds. With the change of variables \( Y_a = K_a Q \) and \( Y_b = K_b Q \), we obtain the equivalent condition
\[
(\tilde{A} Q + B_u Y_a + \tilde{B}_u Y_b) Q^{-1} (\tilde{A} Q + B_u Y_a + \tilde{B}_u Y_b)^T - Q 
+ \sum_{i=1}^{L} \sigma_i^2 (\tilde{A}_i Q + \tilde{B}_{u,i} Y_b) Q^{-1} (\tilde{A}_i Q + \tilde{B}_{u,i} Y_b)^T < 0
\]
which can be easily written as an LMI in \( Y_a, Y_b \) and \( Q \).

**Theorem 5.4.1.** Consider the interconnected system (5.3) in which the interconnection links between subsystems fail with probability \( p_l \) as in (5.2) and assume \( w = 0 \). The system is mean-square stabilizable with the distributed controller (5.14) under link failures if and only if the following LMI is feasible
\[
\begin{bmatrix}
1/\sigma_i^2 Q & 0 & \cdots & 0 & \tilde{A}_L Q + \tilde{B}_{u,L} Y_b \\
* & \ddots & \ddots & \vdots & \vdots \\
* & * & 1/\sigma_i^2 Q & 0 & \tilde{A}_1 Q + \tilde{B}_{u,1} Y_b \\
* & * & * & Q & \tilde{A} Q + B_u Y_a + \tilde{B}_u Y_b \\
* & * & * & * & Q
\end{bmatrix} > 0
\]
where the optimization variables are \( Y_a, Y_b \) and \( Q \). Then, the stabilizing state-feedback gains can be obtained by \( K_a = Y_a Q^{-1} \) and \( K_b = Y_b Q^{-1} \).

As an extension, one can also maximize the closed-loop mean-square stability margin using the following corollary.

**Corollary 5.4.2.** The system (5.3) is mean-square stable for \( \Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_L^2) \leq \).


\[ \sigma^2 I \text{ if and only if} \]
\[
\left( \bar{A}Q + B_uY_a + \bar{B}_uY_b \right) Q^{-1} \left( \bar{A} + B_uY_a + \bar{B}_uY_b \right)^T - Q
\]
\[+ \sigma^2 \sum_{i=1}^{L} \left( \bar{A}_iQ + \bar{B}_{u,i}Y_b \right) Q^{-1} \left( \bar{A}_iQ + \bar{B}_{u,i}Y_b \right)^T < 0 \]
\[(5.19)\]
holds for some \( Y_a, Y_b \) and \( Q > 0 \). Thus, finding \( K_a \) and \( K_b \) that maximize \( \bar{\sigma} \) reduces to the following generalized eigenvalue problem:

Maximize \( \bar{\sigma} \)

Subject to \( Q > 0 \) and (5.19)

### 5.4.2 Minimizing the bound on \( L_2 \) Gain

Using the results given in previous section, we can design a distributed controller that not only stabilize the interconnected system but also guarantees an allowable level of network performance despite of random link failures.

Now consider the system (5.3) with additional input \( w \). We seek state-feedback gains \( K_a \) and \( K_b \) which minimize the bound on \( L_2 \) gain \( \eta \) from the exogenous input \( w \) to network performance \( z \) as defined by (5.10). According to condition (5.12), \( \gamma \) is an upper bound for the \( L_2 \) gain system (5.15) if there exist \( P > 0, K_a \) and \( K_b \) such that the following inequality holds

\[
\begin{bmatrix}
\bar{A} + B_uK_a + \bar{B}_uK_b & B_w \\
C_z + D_{zu}K_a + \bar{D}_{zu}K_b & D_{zw}
\end{bmatrix}
\end{bmatrix}^T \tilde{P}
\begin{bmatrix}
\bar{A} + B_uK_a + \bar{B}_uK_b & B_w \\
C_z + D_{zu}K_a + \bar{D}_{zu}K_b & D_{zw}
\end{bmatrix}
\begin{bmatrix}
\bar{A}_i + \bar{B}_{u,i}K_b & 0 \\
\bar{D}_{zu,i}K_b & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{P} & \tilde{P} & P & 0 & 0 & \gamma^2 I
\end{bmatrix}
\leq 0
\]
\[(5.20)\]
where \( \tilde{P} = \text{diag}(P, I) \). Defining the change of variables \( Q = P^{-1} \), \( Y_a = K_aQ \) and \( Y_b = K_bQ \) and applying a congruence transformation with \( \tilde{Q} = \text{diag}(Q, I) \), we obtain
the following inequality

\[
\begin{bmatrix}
\tilde{A}Q + B_u Y_a + \tilde{B}_u Y_b & B_w \\
C_z Q + D_{zu} Y_a + \tilde{D}_{zu} Y_b & D_{zw}
\end{bmatrix}
+ \sum_{l=1}^{L} \sigma_l^2
\begin{bmatrix}
\tilde{A}_l Q + \tilde{B}_{u,l} Y_b & 0 \\
\tilde{D}_{zu,l} Y_b & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{Q}^{-1} \\
\tilde{Q}^{-1}
\end{bmatrix}
\begin{bmatrix}
\tilde{A}_l Q + \tilde{B}_{u,l} Y_b & 0 \\
\tilde{D}_{zu,l} Y_b & 0
\end{bmatrix}
\begin{bmatrix}
Q & 0 \\
0 & \gamma^2 I
\end{bmatrix}
\leq 0
\]

(5.21)

which can be easily written as an LMI in $Y_a$, $Y_b$ and $Q$. This leads to the following theorem that is the main result of this paper.

**Theorem 5.4.3.** Consider the interconnected system (5.3) with stochastic link failures as in (5.2). A sufficient condition for the existence of distributed feedback controller (5.14) that guarantees the $L_2$ gain of the system from $w$ to $z$ be less than $\gamma$ is that the following LMI optimization problem has a feasible solution

\[
\text{Minimize } \gamma \\
\text{subject to } Q > 0 \text{ and }
\]

\[
\begin{bmatrix}
1/\sigma_L^2 \tilde{Q} & 0 & \ldots & 0 \\
* & \ddots & \ddots & \vdots \\
* & \ast & 1/\sigma_1^2 \tilde{Q} & 0 \\
* & \ast & \ast & \tilde{Q}
\end{bmatrix}
\begin{bmatrix}
\tilde{A}_L Q + \tilde{B}_{u,L} Y_b & 0 \\
\tilde{D}_{zu,L} Y_b & 0 \\
\tilde{A}_1 Q + \tilde{B}_{u,1} Y_b & 0 \\
\tilde{D}_{zu,1} Y_b & 0 \\
\tilde{A}Q + B_u Y_a + \tilde{B}_u Y_b & B_w \\
C_z Q + D_{zu} Y_a + \tilde{D}_{zu} Y_b & D_{zw}
\end{bmatrix}
\begin{bmatrix}
Q & 0 \\
0 & \gamma^2 I
\end{bmatrix}
> 0
\]

(5.22)
where $Y_a$, $Y_b$ and $Q$ are the optimization variables with $\tilde{Q} = \text{diag}(Q, I)$ and the controller parameters are determined as $K_a = Y_a Q^{-1}$ and $K_b = Y_b Q^{-1}$.

## 5.5 Example

As an example, we consider a system composed of four coupled subsystems connected to each other with four links as in Fig. 5.1.

![Networked system consisting of four subsystems](image.png)

Figure 5.1: A networked system consisting of four subsystems

Let the dynamic of each subsystem be described by:

$$
x_i(k+1) = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} x_i(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_i(k)
$$

$$
+ \sum_{j \in N_i(k)} \delta_{i,j}(k) \begin{bmatrix} 0.5 & 0 \\ 1 & 0.5 \end{bmatrix} x_j(k)
$$

$$
y_i(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i(k) \quad \text{for} \quad i = 1, 2, \ldots, 4
$$

and assume each link has 0.5 chance of failure. The aim is to design a distributed controller that stabilizes the system while minimizing the effect of disturbance $w = [w_1^T, \ldots, w_4^T]^T$ on the system performance i.e. to keep the output of neighboring subsystems close to each other in each instance.
To achieve this objective, we define the performance outputs as the sum of the differences between neighboring systems’ outputs:

\[ z_{x_i} = (y_i - y_{i+1}) + (y_i - y_{i-1}) \]

Moreover, to confine the energy of control signals, we define additional terms:

\[ z_{u_i} = a_i u_i \]

Using the above definitions, the performance output \( z = [z_{x_i}^T z_{u_i}^T]^T \) can be written as \( z = C_z x + D_{zu} u \) where

\[ C_z = \begin{bmatrix} \bar{C}_z \\ 0 \end{bmatrix} \quad \text{and} \quad D_{zu} = \begin{bmatrix} 0 \\ \bar{D}_{zu} \end{bmatrix} \]

in which \( \bar{D}_{zu} = \text{diag}(a_1, \ldots, a_4) \) and \( \bar{C}_z = L \otimes C \) with \( C = \begin{bmatrix} 1 & 0 \end{bmatrix} \) and

\[ L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \]

By applying theorem 5.4.3, we find optimal distributed controller (5.14) with local controller gain \( K_{a,i} = [-0.12, -0.20] \) and \( K_{b,i} = [-2.00, -3.15] \) for \( i = 1, \ldots, 4 \) which stabilizes the network and provides disturbance attenuation less than \( \gamma = 5.64 \).

## 5.6 Conclusion

In this chapter, we have developed a distributed control design approach for interconnected system with random link failures in which the interconnection links are assumed to fail with independent probability. Each link is considered to be an independent random variable and the interconnected system is modeled as a discrete-time...
system with multiplicative random coefficients. This allowed us to formulate and solve the problem as an LMI. Distributed controllers, which use only local and available neighboring information in each time interval, is designed to ensure the system stability despite link failures. Furthermore, we developed results characterizing the system performance and provided a sufficient condition for designing distributed controller that guarantees prescribed disturbance attenuation in the network. A simple numerical example was provided to support the theoretical results.
Chapter 6

Decentralized PI Observer-Based Control

6.1 Introduction

It has been recognized that certain limitation of Luenberger-type P observer can be overcome by using proportional Integral (PI) observer. This type of observer has an additional integration path, which can improve robustness against parameters variations and disturbances, and can offer design flexibility to achieve other objectives such as steady-state accuracy and stability robustness. The additional degrees of freedom provided by the integral gain has shown to be advantageous in LTR design [78]-[80], attenuation or decoupling disturbances [81], [82], and detecting sensor or actuator fault [83]. PI observer has also been used for disturbance attenuation in a class of nonlinear systems [77]. The application of PI observer for robust fault diagnosis of descriptor system has been reported in [83]. The adaptive version of PI observer was equally successful for simultaneous parameters, state, and disturbance estimation [84].

In this Chapter, we study the benefits of PI observers in connection with decentral-
ized and distributed control and fault diagnosis of large-scale interconnected systems. In section 6.2 the problem of designing decentralized PI-Observer controllers is considered in the convex optimization context for interconnected systems with linear subsystems and nonlinear time-varying interconnections. We take advantage of additional degrees of freedom in PI observer to enhance the design objective of maximizing the interconnection bounds. Finally, Distributed PI-observer is introduced in section 6.3 for disturbance estimation and fault detection in large-scale multi-agent systems.

### 6.2 Decentralized PI-Observer based Control

In this section we consider the problem of designing decentralized PI-Observer controllers in the convex optimization context for interconnected systems with linear subsystems and nonlinear time-varying interconnections. The pioneer work in this context is [85] where the state-feedback controller is designed to guarantee robust stability of the overall systems while maximizing the bounds of unknown interconnection terms. Subsequently, several algorithms are proposed for designing decentralized proportional (P) observer-based controller to robustly stabilize the overall systems [86]-[88]. In [86] and [87] a sequential two-step design procedure has been introduced to solve the non-convex optimization problem and provide a way to maximize the interconnection bounds. In [88] the problem is solved by avoiding sequential procedure and applied to an industrial utility boiler. The main contribution of this section is a modification in terms of a proportional-integral (PI) observer and its solution using an extended LMI. We take advantage of additional degrees of freedom in PI observer to enhance the design objective of maximizing the interconnection bounds. We show that the solution of the decentralized PI observer-based control problem reduces to an extended LMI setup which can easily be solved with the advantage that the feasible region is extended by proportional-integral gains leading to a robust solution.
We also reformulate the original problem of robust stabilization by including a disturbance term and provide a sufficient condition to guarantee prescribed disturbance attenuation performance while maximizing the interconnection bounds. Using this formulation, it is also possible to minimize the effect of disturbance in $L_2$ gain sense when a-priori bound on the interconnection terms is given [98].

6.2.1 Preliminaries

Consider a large-scale interconnected system composed of $N$ subsystems represented by

$$
\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + h_i(t, x) \\
y_i(t) = C_i x_i(t)
$$

for $i = 1, \ldots, N$ (6.1)

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^{p_i}$ are the state, input, and output vector of the $i$th subsystem, respectively. $x = [x_1^T, \ldots, x_N^T]^T$ ($n = \sum_{i=1}^N n_i$) is the state of the overall system, and $h_i(t, x) : \mathbb{R}^{n_i+1} \rightarrow \mathbb{R}^{n_i}$ is nonlinear interconnection function of the $i$-th subsystem. The exact expression of $h_i(t, x)$ is unknown but it is assumed to satisfy the following quadratic constraint:

$$
h_i^T(t, x)h_i(t, x) \leq \alpha_i^2 x^T H_i^T H_i x
$$

(6.2)

where $\alpha_i > 0$ are interconnection bounds and $H_i \in \mathbb{R}^{l_i \times n_i}$ are constant bounding matrices. The entire interconnected system (6.1) can be written as:

$$
\dot{x}(t) = Ax(t) + Bu(t) + h(t, x) \\
y(t) = Cx(t)
$$

(6.3)

where $A = diag(A_1, \ldots, A_N)$, $B = diag(B_1, \ldots, B_N)$, $C = diag(C_1, \ldots, C_N)$, $u = [u_1^T, \ldots, u_N^T]^T \in \mathbb{R}^m$ ($m = \sum_{i=1}^N m_i$), $y = [y_1^T, \ldots, y_N^T]^T \in \mathbb{R}^p$ ($p = \sum_{i=1}^N p_i$), and $h(t, x) = [h_1^T(t, x), \ldots, h_N^T(t, x)]^T$ for which we have the constraint

$$
h^T(t, x)h(t, x) \leq x^T H^T \Psi^{-1} H x
$$

(6.4)
where $H^T = [H^T_1, \ldots, H^T_N]$ is an $l \times n$ matrix ($l = \sum_{i=1}^{N} l_i$), $\Psi = \text{diag}(\psi_1 I_{l_1}, \ldots, \psi_N I_{l_N})$ and $\psi_i = \alpha_i^{-2}$.

The following theorem by Siljak and Stipanovic [85] on general LMI-based formulation of the robust stabilization problem represents a basis for the further elaboration on decentralized PI observer-based control design in following.

**Theorem 6.2.1.** The nonlinear system $\dot{z} = A_f z + h_f(t, x), C_f z$ where $z \in \mathbb{R}^n$ is the state of the system and $h_f(t, z) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is the nonlinearity which satisfies the quadratic inequality

$$h_f^T(t, z)h_f(t, z) \leq z^T H_f^T \Psi^{-1} H_f z$$

(6.5)

is robustly stable with degree $\alpha = [\alpha_1, \ldots, \alpha_N]$ if the following problem has a feasible solution.

Minimize $Tr \Psi$

subject to $X_f > 0$

$$
\begin{bmatrix}
X_f A_f + A_f^T X_f & X_f & H_f^T \\
X_f & -I & 0 \\
H_f & 0 & -\Psi
\end{bmatrix} < 0
$$

(6.6)

**Proof.** This result can be proved using the $S$-procedure i.e. to combine the Lyapunov inequality and constraint (6.6) in one LMI followed by applying Schur complement formula. The complete proof can be found in [85].

6.2.2 Robust Stability

The objective is to design a decentralized observer based linear controller that robustly regulates the state of the overall system without any information exchange between subsystems, i.e. the local controller $u_i$ is constrained to use only local output signal $y_i$. This problem is investigated in [86], [87] and [88] with Luenberger-type $P$ observer

$$\dot{x}(t) = A\dot{x}(t) + Bu(t) + L(C\dot{x}(t) - y(t))$$

(6.7)
In [87] three approaches are used to solve the problem. As pointed out in the introduction, an observer with additional integral path known as PI observer found useful in various estimation and control scenarios. We apply this type of observer in connection to the decentralized framework to enhance the control objective. Although we can generalize and formulate three equivalent problems based on PI observer, we only concentrate on the third problem and provide a complete solution for it.

We assume that the controller is composed of a PI-observer and a state feedback control law, i.e.

\[ \dot{x}(t) = A\dot{x}(t) + Bu(t) + L_p(C\dot{x}(t) - y(t)) + Bv(t) \]
\[ \dot{v}(t) = L_I(C\dot{x}(t) - y(t)) \]
\[ u(t) = K\dot{x}(t) \]

(6.8)

where \([\dot{x}^T, v^T]^T \in \mathbb{R}^{n+m}\) is the augmented observer state. \(\dot{x} = [\dot{x}_1^T, \ldots, \dot{x}_N^T]^T, \dot{x}_i \in \mathbb{R}^{n_i}, v = [v_1^T, \ldots, v_N^T]^T, v_i \in \mathbb{R}^{m_i}, K = \text{diag}(K_1, \ldots, K_N), L_I = \text{diag}(L_{I1}, \ldots, L_{IN})\) and \(L_p = \text{diag}(L_{P1}, \ldots, L_{PN})\) represent the global controller parameter matrices, while triplet \((L_{Pi}, L_{Ii}, K_i)\) determine the local dynamic controllers.

Combining (6.8) and global system (6.3), the resulting closed-loop system can be written as:

\[ \dot{z}(t) = A_fz(t) + h_f(t, x), \quad y(t) = C_fz(t) \]

(6.9)

where \(z\) is the state of the closed-loop system. Defining \(z = [z_1^T, z_2^T, z_3^T]^T, z_1 = x, z_2 = \dot{x} - x\) and \(z_3 = v\), we have

\[ A_f = \begin{bmatrix} A + BK & BK & 0 \\ 0 & A + L_pC & B \\ 0 & L_IC & 0 \end{bmatrix}, \quad h_f = \begin{bmatrix} h(t, z_1) \\ -h(t, z_1) \\ 0 \end{bmatrix}, \quad C_f = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \]

(6.10)
in which \( A_f \) and \( c_f \) can be rephrased as

\[
A_f = \begin{bmatrix}
A + BK & BK_X \\
0 & A_X + L_X C_X
\end{bmatrix},
\]

\[
c_f = \begin{bmatrix}
C_X \\
0
\end{bmatrix}
\]

(6.11)

where

\[
A_X = \begin{bmatrix}
A & B \\
0 & 0
\end{bmatrix},
B_X = \begin{bmatrix}
B \\
0
\end{bmatrix},
C_X = \begin{bmatrix}
C & 0
\end{bmatrix},
\]

(6.12)

Moreover, the quadratic inequality (6.5) will be satisfied with

\[
H_f = \begin{bmatrix}
2H & 0 & 0
\end{bmatrix}
\]

(6.13)

Let us define \( Q = \text{diag}(Q_1, \ldots, Q_N) \), \( P = \text{diag}(P_1, \ldots, P_N, \bar{P}_1, \ldots, \bar{P}_N) \), \( W = KQ = \text{diag}(W_1, \ldots, W_N) \) and \( V^T = L_X^T P^T = [\text{diag}(V_1, \ldots, V_N)^T, \text{diag}(\bar{V}_1, \ldots, \bar{V}_N)^T]^T \), where \( Q_i, P_i, \bar{P}_i, W_i, V_i \) and \( \bar{V}_i \) are \( n_i \times n_i \), \( n_i \times n_i \), \( m_i \times m_i \), \( m_i \times n_i \), \( n_i \times p_i \) and \( m_i \times p_i \) matrices respectively. Substituting (6.11) and \( X_f = \text{diag}(Q^{-1}, P) \) into (6.6), after appropriate congruent transformation, we can obtain

\[
\begin{bmatrix}
S_1 & BK_X & I & 0 & 2QH^T \\
K_X^T B^T & S_2 & 0 & P & 0 \\
I & 0 & -I & 0 & 0 \\
0 & P & 0 & -I & 0 \\
2QH & 0 & 0 & 0 & -\Psi
\end{bmatrix} < 0
\]

(6.14)

where \( S_1 = AQ + QA^T + BW + W^T B^T \) and \( S_2 = PA_X + A_X^T P + V C_X + C_X^T V^T \) with \( W = KQ \) and \( V = PL_X \). By taking into account the independency between \( K_X \) and \( L_X \) in (6.14), we propose the following two steps algorithm to solve this optimization problem.

Step 1:
Minimize $Tr\Psi$

subject to $Q > 0$, and

$$
\begin{bmatrix}
S_1 & 0 & 2QH^T \\
0 & -I & 0 \\
2HQ & 0 & -\Psi
\end{bmatrix} < 0
$$

(6.15)

Step 2:

Minimize $Tr\Delta$

subject to $P > 0$, and

$$
\begin{bmatrix}
-I & 0 & I & 0 & 0 \\
0 & S_2 & K_X^TB^T & P & 0 \\
I & BK_X & S_1 & 0 & 2QH^T \\
0 & P & 0 & -I & 0 \\
0 & 0 & 2HQ & 0 & -\Psi \Delta
\end{bmatrix} < 0
$$

(6.16)

where $\Delta = diag(\delta_1I_t, \ldots, \delta_NI_N)$ ($\delta_i > 0$), while $Q, S_1, |Psi|$, and $K_X = \begin{bmatrix} K & 0 \end{bmatrix}$ ($K = WQ^{-1}$) are obtained in step 1.

**Theorem 6.2.2.** The nonlinear interconnected system $eqref{nonlinsys}$ is robustly stabilized with degree $\alpha = [\alpha_1, \ldots, \alpha_N]$ by the decentralized controller $eqref{dc3}$ if steps one and two have feasible solutions. Controller parameter are given by $K = WQ^{-1}$ and $L_X = P^{-1}V$. The robustness degree bounds are $\alpha_i = \frac{1}{\sqrt{\psi_i\delta_i}}$.

**Proof.** The proof follows simply the fact that second LMI in (6.16) is obtained by applying appropriate permutation into LMI (6.14) with $\Psi$ replaced by $\Psi \Delta$. Notice that step 2 has to be performed after step 1 and not simultaneously.

The LMI optimization Problems given by (6.15) and (6.16) do not pose any restric-
tions on the size of the matrix variables $Q$, $W$, $P$ and $V$. Consequently, the results of these two optimization problems may yield very large controller and observer gain matrices $K$ and $L_X$. One can restrict $K$ and $L_X$ by posing constraints on the matrices $Q$, $W$, $P$ and $V$ as

$$Q_i^{-1} < \kappa_{Q_i}, \quad W_i W_i^T < \kappa_{W_i}$$

$$P_i^{-1} < \kappa_{P_i}, \quad V_i V_i^T < \kappa_{V_i}$$

Equations (6.17) and (6.18) are, respectively, equivalent to

$$\begin{bmatrix} -Q_i & -I \\ -I & -\kappa_{Q_i} \end{bmatrix} < 0, \quad \begin{bmatrix} -\kappa_{W_i} I & W_i \\ W_i^T & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} -P_i & -I \\ -I & -\kappa_{P_i} \end{bmatrix} < 0, \quad \begin{bmatrix} -\kappa_{V_i} I & V_i^T \\ V_i & -I \end{bmatrix} < 0$$

Combining (6.15) with (6.19) and (6.16) with (6.20), and changing the optimization objective to the minimization of $\sum_{i=1}^N (\psi_i + \kappa_{Q_i} + \kappa_{W_i})$ and $\sum_{i=1}^N (\delta_i + \kappa_{P_i} + \kappa_{V_i})$ results in the following two optimization problems:

**Step 1:**

Minimize $\sum_{i=1}^N (\psi_i + \kappa_{Q_i} + \kappa_{W_i})$  

Subject to (6.15) and (6.19)

**Step 2:**

Minimize $\sum_{i=1}^N (\delta_i + \kappa_{P_i} + \kappa_{V_i})$  

Subject to (6.16) and (6.20)

### 6.2.3 Disturbance Attenuation

Up to know we concentrated on robust stability. However, there are many instances that one would like to attenuate the disturbance while preserving the robust stability.
In this section, we formulate the problem in terms of an LMI and provide the complete solution for it. Let us consider the nonlinear interconnected system with external disturbance as follows:

\[
\dot{x}_f(t) = Ax_f(t) + h_f(t, x_f) + B_ww(t)
\]
\[
z_f(t) = C_zx_f(t) + D_zww(t)
\]

where \( A = \text{diag}(A_1, \ldots, A_N) \), \( B_w = \text{diag}(B_{w1}, \ldots, B_{wN}) \), \( C_z = \text{diag}(C_{z1}, \ldots, C_{zN}) \), \( D_zw = \text{diag}(D_{zw1}, \ldots, D_{zwN}) \), \( w = [w_1^T, \ldots, w_N^T]^T \in \mathbb{R}^{m_w} \) \( (m_w = \sum_{i=1}^N m_{wi}) \), \( z = [z_1^T, \ldots, z_N^T]^T \in \mathbb{R}^p \) \( (p_z = \sum_{i=1}^N p_{zi}) \), and \( h_f(t, x_f) = [h_{f1}^T(t, x_f), \ldots, h_{fN}^T(t, x_f)]^T \) for which we have the constraint

\[
h_{f1}^T(t, x_f)h_{f1}(t, x_f) \leq x_f^T H_f^T\Psi^{-1} H_f x_f
\]

where \( H_f = [H_{f1}^T, \ldots, H_{fN}^T] \) is an \( l \times n \) matrix \( (l = \sum_{i=1}^N l_i) \), \( \Psi = \text{diag}(\psi_1I_{l_1}, \ldots, \psi_NI_{l_N}) \) and \( \psi_i = \alpha_i^{-2} \).

We define the \( L_2 \) gain of the system (6.23) from \( w \) to \( z \) as the quantity

\[
\sup_{\|w\|_2 \neq 0} \frac{\|z\|_2}{\|w\|_2}
\]

where the \( L_2 \) norm of \( w \) is \( \|w\|_2^2 = \int_0^\infty w^T w \, dt \). Now suppose there exists the quadratic function \( V(\xi) = \xi^T X_f \xi, X_f > 0 \), and \( \gamma \geq 0 \) such that for all \( t \)

\[
\frac{d}{dt} V(x_f) + z^T z - \gamma^2 w^T w \leq 0
\]

Then it can be shown that the \( L_2 \) gain of the system (6.23) is less than \( \gamma \). Condition (6.25) is equivalent to

\[
x_f^T (A^T X_f + X_f A)x_f + 2x_f^T X_f h_f + z^T z - \gamma^2 w^T w \leq 0
\]

For all \( x_f \) and \( h_f \) satisfying

\[
h_f^T h_f \leq x_f^T H_f^T \Psi^{-1} H_f x_f
\]
Using the S-procedure one can show that this is true if and only if there exists $\alpha > 0$ such that

$$
\begin{bmatrix}
    \bar{A}\bar{Y}_f + \bar{Y}_f \bar{A}^T + aH_f^T \Psi^{-1} H_f & \bar{Y}_f C_z^T & B_w & \bar{Y}_f \\
    C_z \bar{Y}_f & -\gamma I & D_{zw} & 0 \\
    B_w^T & D_{zw}^T & -\gamma I & 0 \\
    \bar{Y}_f & 0 & 0 & -aI
\end{bmatrix} < 0
$$

where $\bar{Y}_f = X_f^{-1}$. It is further equivalent to

$$
\begin{bmatrix}
    AY_f + Y_f A^T + H_f^T \Psi^{-1} H_f & Y_f C_z^T & bB_w & Y_f \\
    C_z Y_f & -\gamma bI & bD_{zw} & 0 \\
    bB_w^T & bD_{zw}^T & -\gamma bI & 0 \\
    Y_f & 0 & 0 & -I
\end{bmatrix} < 0 \quad (6.27)
$$

where $b = a^{-1}$ and $Y_f = b\bar{Y}_f$. Relying on Schur complement formula, (6.27) can be written as

$$
\begin{bmatrix}
    AY_f + Y_f A^T & Y_f C_z^T & bB_w & Y_f & H_f^T \\
    C_z Y_f & -\gamma bI & bD_{zw} & 0 & 0 \\
    bB_w^T & bD_{zw}^T & -\gamma bI & 0 & 0 \\
    Y_f & 0 & 0 & -I & 0 \\
    H_f & 0 & 0 & 0 & -\Psi
\end{bmatrix} < 0 \quad (6.28)
$$

If one is interested in maintaining robust stability with maximum nonlinear interconnection while attenuate the disturbance by a prescribed factor $\gamma$, the following theorem can be employed.

**Theorem 6.2.3.** The nonlinear interconnected system (6.23) is robustly stable with degree $\alpha = [\alpha_1, \ldots, \alpha_N]$ and disturbance attenuation less than $\gamma$ in the $L_2$ gain sense if the following problem is feasible.

Minimize $\text{Tr} \Psi$
subject to $Y_f > 0$, $b > 0$ and (6.28)

It is interesting to point out that from LMI (6.28), one can also be able to formulate the problem of minimizing $L_2$ gain of the system from $w$ to $z$ for the case of known bound on nonlinear the interconnection terms.

Using theorem 6.2.3 one can follow the procedure discussed in last section and design robust decentralized controller with disturbance attenuation performance.

**Theorem 6.2.4.** Consider the nonlinear interconnected system

$$\dot{x}(t) = Ax(t) + Bu(t) + h(t, x) + B_w w(t)$$
$$z(t) = C_z x(t) + D_{zw} w(t)$$

where $A = \text{diag}(A_1, \ldots, A_N)$, $B = \text{diag}(B_1, \ldots, B_N)$, $B_w = \text{diag}(B_{w1}, \ldots, B_{wN})$, $C_z = \text{diag}(C_{z1}, \ldots, C_{zN})$, $D_{zw} = \text{diag}(D_{zw1}, \ldots, D_{zwN})$, $w = [w_1^T, \ldots, w_N^T]^T \in \mathbb{R}^{m_w}$

$(m_w = \sum_{i=1}^N m_{wi})$, $z = [z_1^T, \ldots, z_N^T]^T \in \mathbb{R}^p$ ($p_z = \sum_{i=1}^N p_{zi}$), and $h(t, x) = [h_{w1}^T(t, x), \ldots, h_{wN}^T(t, x)]^T$

for which we have the constraint

$$h^T(t, x) h(t, x) \leq x^T H^T \Psi^{-1} H x$$

(6.30)

where $H^T = [H_{11}^T, \ldots, H_{NN}^T]$ is an $l \times n$ matrix ($l = \sum_{i=1}^N l_i$), $\Psi = \text{diag}(\psi_1 I_l, \ldots, \psi_N I_N)$

and $\psi_i = \alpha_i^{-2}$. Then the decentralized state-feedback control

$$u = K_D x$$

(6.31)

where $K_D = \text{diag}(K_1, \ldots, K_N)$ with $K_i \in \mathbb{R}^{m_i \times n_i}$ robustly stabilizes the nonlinear interconnected system (6.29) with degree $\alpha$ and disturbance attenuation less than $\gamma$ in
the $L_2$ gain sense if the following problem has a feasible solution.

\[
\begin{bmatrix}
AY + YA^T + BW + WTBT & YC_2^T bB_w & Y & HT \\
C_2Y & -\gamma bI & bD_{zw} & 0 & 0 \\
bB_w^T & bD_{zw}^T & -\gamma bI & 0 & 0 \\
Y & 0 & 0 & -I & 0 \\
H & 0 & 0 & 0 & -\Psi \\
\end{bmatrix} < 0 \quad (6.32)
\]

where $Y = \text{diag}(Y_1, \ldots, Y_N)$, $W = \text{diag}(W_1, \ldots, W_N)$, $\Psi$ and scalar $b$ are the optimization variables. Furthermore, the controller gain is obtained by $K_D = Y^{-1}W$.

This LMI optimization problems may yield very large controller gain matrices $K_D$. Similar to the approach outlined before, One can restrict $K_D$ by posing constraints on the matrices $Y$ and $W$ as

\[
Y_i^{-1} < \kappa_{Y_i}, \quad W_iW_i^T < \kappa_{W_i} \quad (6.33)
\]

or equivalently

\[
\begin{bmatrix}
-Y_i & -I \\
-I & -\kappa_{Y_i} \\
\end{bmatrix} < 0, \quad \begin{bmatrix}
-\kappa_{W_i}I & W_i \\
W_i^T & -I \\
\end{bmatrix} < 0 \quad (6.34)
\]

and solve the following optimization problem:

\[
\text{Minimize } \sum_{i=1}^{N} (\psi_i + \kappa_{Y_i} + \kappa_{W_i}) \quad (6.35)
\]

Subject to (6.32) and (6.34)

**Remark 6.2.5.** It should be pointed out that in certain situation where disturbance attenuation is not the objective; rather its estimation places a critical role, the use of PI observer can be beneficial. The advantage of PI observer in connection to fault detection has been demonstrated in [89] and certainly can be employed here in decentralized framework as well. This will be the subject of future paper.
6.2.4 Example

As an example, we consider the system composed of two coupled inverted pendulums as shown in Figure 6.1. The pendulums are coupled by a sliding spring, the position of which is uncertain. We allow the sliding of the spring to be discontinuous function of both time and state of the system. The normalized model for the motion of two pendulums as in Figure 6.1 can be described by:

\[ \dot{x}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 + e_1 x_1 + e_1 x_1 + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_1 \\
\dot{x}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2 + e_2 x_1 + e_2 x_1 + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_2 \\
y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_1 \quad y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_2 \]

Figure 6.1: Coupled Inverted Pendulums

Using the optimization problem (6.21) and (6.22), the decentralized PI observer-based controller (6.8) yields \( \alpha_i = 1.40 \) with local controller gain \( K_i = [583.29 \ 2.97] \) and local observer gain \( L_{Pi} = [-71.09 \ -120.87]^T \) and \( L_{Li} = -17.65 \) for \( i = 1, 2 \).

In the presence of disturbance, one can use optimization problem (6.35) to design a
robust decentralized controller to attenuate the disturbance. For inverted pendulums example, the resulting local controllers provide $\alpha_i = 0.34$ with control gains $K_i = [-13.19, -12.10]$ for $i = 1, 2$ with desired disturbance attenuation $\gamma = 0.1$.

### 6.3 Distributed PI-Observer Design

In this section, we consider the problem of designing distributed PI observer with prescribed degree of convergence for a set of linearly interconnected systems. Under the assumption of linear interactions, we provide a direct design procedure for the PI observer which can effectively be used in simultaneous state and disturbance/fault estimation. The main contribution of this section is a modification in terms of a proportional-integral (PI) observer and its solution using an extended LMI. We take advantage of additional integral term in PI observer to estimate disturbance or fault [99].

Consider a linearly interconnected system composed of $N$ subsystems represented by

\[
\begin{align*}
\dot{x}_i(t) & = A_i x_i(t) + B_i u_i(t) + h_i(t, x) + w_i(t) \\
y_i(t) & = C_i x_i(t)
\end{align*}
\]

for $i = 1, \ldots, N$ (6.36)

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^{p_i}$ are the state, input, and output vector of the $i$th subsystem, respectively. $w_i \in \mathbb{R}^{m_{wi}}$ is denoted disturbance or fault and $h_i(t, x)$ is represented linear time-invariant interconnection function described by:

\[
h_i(t, x) = \sum_{j=1}^{N} x_j(t)
\]

(6.37)

We shall assume the pair $(A_i, B_i)$ is controllable and the pair $(A_i, C_i)$ is observable. The entire interconnected system (6.36) can equivalently be written in composite form as

\[
\begin{align*}
\dot{x}(t) & = Ax(t) + Bu(t) + h(t, x) + w(t) \\
y(t) & = Cx(t)
\end{align*}
\]

(6.38)
where $A = \text{diag}(A_1, \ldots, A_N)$, $B = \text{diag}(B_1, \ldots, B_N)$, $C = \text{diag}(C_1, \ldots, C_N)$, $x = [x_1^T, \ldots, x_N^T]^T \in \mathbb{R}^n$ ($n = \sum_{i=1}^N n_i$), $u = [u_1^T, \ldots, u_N^T]^T \in \mathbb{R}^m$ ($m = \sum_{i=1}^N m_i$), $w = [w_1^T, \ldots, w_N^T]^T \in \mathbb{R}_w^m$ ($m_w = \sum_{i=1}^N m_{wi}$), $y = [y_1^T, \ldots, y_N^T]^T \in \mathbb{R}^p$ ($p = \sum_{i=1}^N p_i$), and $h(t, x) = [h_1^T(t, x), \ldots, h_N^T(t, x)]^T$. When the subsystems are decoupled, i.e., $h_i(t, x) = 0 \forall i = 1, \ldots, N$, and the disturbance or fault is absent, i.e. $w_i(t) = 0$, it is easy to construct $N$ independent Luenberger observer for the subsystems. However, such proportional type observers (P-observers) have shortcoming when disturbance or fault is present in the above system. Therefore, we take advantage of the proportional-integral observer (PI-observer), which has proven advantage in various scenarios, and evaluate its robust performance in the interconnected systems framework. Let the decoupled subsystems incorporate disturbance terms $w_i = E_i d_i$ where $d_i \in \mathbb{R}^{m_i}$ represent constant disturbances and $E_i \in \mathbb{R}^{n_i \times m_i}$ is a known distribution matrix associated with the $i$th disturbance. Thus, we have

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + E_i d_i(t) \quad \text{for } i = 1, \ldots, N \tag{6.39}$$

$$y_i(t) = C_i x_i(t)$$

and we define the PI-observer for (6.39) as

$$\dot{\hat{x}}_i = (A_i - L_{Pi} C_i) \hat{x}_i + L_{Pi} y_i + B_i u_i + E_i \hat{d}_i \quad \text{for } i = 1, \ldots, N \tag{6.40}$$

$$\dot{\hat{d}}_i = L_{Ii}(y_i - C_i \hat{x}_i)$$

which can be written in the augmented form

$$\dot{z}_i = (A_{Xi} - L_{Xi} C_{Xi}) z_i + L_{Xi} y_i + B_{Xi} u_i \tag{6.41}$$

where

$$z_i = \begin{bmatrix} \hat{x}_i \\ \hat{d}_i \end{bmatrix}, \quad L_{Xi} = \begin{bmatrix} L_{Pi} \\ L_{Ii} \end{bmatrix}, \quad A_{Xi} = \begin{bmatrix} A_i & E_i \\ 0 & 0 \end{bmatrix}, \quad B_{Xi} = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad C_{Xi} = \begin{bmatrix} C_i & 0 \end{bmatrix} \tag{6.42}$$
Denoting the state equation error by $e_i = x_i - \hat{x}_i$ and the disturbance estimation error by $\varepsilon_i = d_i - \hat{d}_i$, the error dynamics of the PI-observer is given by

$$\dot{\varphi}_i = (A_{Xi} - L_{Xi}C_{Xi})\varphi_i$$  \hspace{1cm} (6.43)

where $\varphi_i = [e_i^T \ v_i^T]^T$. Since the pair $(A_{Xi}, C_{Xi})$ is observable one can use any eigenvalue assignment techniques to obtain $L_{Xi}$, which specifies both the proportional and integral gains of the observer $L_{Pi}$ and $L_{Ii}$. At this point, we show that one can design the PI-observer with a desired degree of convergence $\alpha > 0$ i.e. $\varphi(t)$ decays by the rate at least $\exp(-\alpha t)$ and $\lim_{t \to \infty} \varphi(t) = 0$. To do this, $L_{Xi}$ must be chosen by

$$L_{Xi} = P_{Xi}C_{Xi}^T$$  \hspace{1cm} (6.44)

where $P_{Xi} \in \mathbb{R}^{(n_i+m_i) \times (n_i+m_i)}$ is the symmetric positive definite solution of the Riccati equation

$$P_{Xi}(A_{Xi}^T + \alpha I_{Xi}) + (A_{Xi} + \alpha I_{Xi})P_{Xi} - P_{Xi}C_{Xi}^T C_{Xi}P_{Xi} + Q_{Xi} = 0$$  \hspace{1cm} (6.45)

where $Q_{Xi} \in \mathbb{R}^{(n_i+m_i) \times (n_i+m_i)}$ is an arbitrary symmetric nonnegative definite matrix, and $I_{Xi}$ is an identity matrix of the same size as the matrix $A_{Xi}$. This result follows directly from optimal control theory as applied to our augmented structure.

It is evident that the local observers under the assumption $h_i(t, x) \neq 0$ will not be satisfactory. This is expected since the interconnection function or the output of the other subsystems is not used in constructing the observer for subsystems. Hence, the problem of interest is to seek modification of the above observer design, which results in the same degree of convergence when $h_i(t, x) \neq 0$.

**Theorem 6.3.1.** Consider the large-scale system (6.36) with linear interactions (6.37) and constant $d_i$, i.e.

$$\dot{x}_i(t) = A_ix_i(t) + B_iu_i(t) + \sum_{j=1}^{N} H_{ij}x_j(t) + E_id_i$$  \hspace{1cm} (6.46)
Let the composite matrix \( H = [H_{ij}], i, j = 1, \ldots, N \) satisfy
\[
\text{rank} \begin{bmatrix} C \\ H \end{bmatrix} = \text{rank} [C] = p = \sum_{i=1}^{N} p_i \tag{6.47}
\]

Then a set of distributed PI-observers
\[
\dot{z}_i = (A_{Xi} - L_{Xi}C_{Xi})z_i + L_{Xi}y_i + B_{Xi}u_i + \sum_{i=1}^{N} \tilde{L}_{Xij}y_j \tag{6.48}
\]

where \( L_{Xi} \) is given by (6.44) and \( \tilde{L}_{Xij} = \begin{bmatrix} \tilde{L}_{ij} \\ 0 \end{bmatrix} \in \mathbb{R}^{(n_i+m_i) \times p_j} \) are the elements of the composite matrix \( \tilde{L}_X \) given by
\[
\tilde{L}_X = H_XC_X^T(C_XC_X^T)^{-1} \tag{6.49}
\]

where \( H_X = [H_{Xij}] \) with \( H_{Xij} = \begin{bmatrix} H_{ij} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n_i+m_i) \times (n_i+m_i)} \) ensures the degree of convergence \( \alpha \) for the observer.

**Proof.** Let the gain \( L_{Xi} \) be selected as (6.44) such that the distributed PI observer written in a composite form
\[
\dot{z} = (A_X - L_XC_X)z + L_Xy + B_Xu \tag{6.50}
\]

has degree of convergence \( \alpha \). Under the rank condition (6.47), \( H_X \) can be expressed as \( H_X = \tilde{L}_XC_X \) with \( \tilde{L}_X \) given by (6.49) and
\[
\rho \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \rho \begin{bmatrix} C \\ C(A+H) \\ \vdots \\ C(A+H)^{n-1} \end{bmatrix} \tag{6.51}
\]
or equivalently
\[
\begin{bmatrix}
C_X \\
C_X A_X \\
\vdots \\
C_X A_X^{n-1}
\end{bmatrix} = \begin{bmatrix}
C_X \\
C_X(A_X + H_X) \\
\vdots \\
C_X(A_X + H_X)^{n-1}
\end{bmatrix}
\]  

(6.52)

where \( \rho(.) \) denote the rank of a matrix. Since the pair \((A_i, C_i)\) is observable, the corresponding block diagonal pair \((A, C)\) is observable and the left side of (6.51) is full column rank. Therefore the pair \((A_X + H_X, C_X)\) is observable and a gain matrix \(G_X\) can be selected to guarantee the stability of the following observer

\[
\dot{z} = (A_X + H_X - G_X C_X)z + G_X y + B_X u
\]  

(6.53)

A choice of \(G_X = L_X + \tilde{L}_X\) would result in

\[
\sigma(A_X + H_X - G_X C_X) = \sigma(A_X - L_X C_X)
\]

where \(\sigma(.)\) denotes the spectrum of a matrix. This establishes the equivalence of (6.53) and (6.48).

The distributed PI-observer (6.48) has the potential of simultaneously estimating the state and disturbance/fault. In fact, \(\hat{x}_i(t) \to x_i(t)\) and \(\hat{d}_i(t)\) converges to \(d_i\) for \(i = 1, ..., N\) as \(t \to \infty\).
Chapter 7

Distributed LQR-Based Control

7.1 Introduction

Linear Quadratic regulator (LQR) with available state information has been successfully used in vast variety of scenarios due its impressive robustness properties. In multi-agent systems, LQR-based optimal control approach was used for Distributed design for identically decoupled systems in [30] and synchronization of cooperative systems in [90]. In these works like most work on cooperative control static consensus protocols are considered that rely on receiving full state information from ones neighbors. However, in many applications, full state is not always available for controller design, thus the state estimation and/or output feedback design is required. Consensus using observer-based feedback was given in [91], and consensus region was analyzed. In [92], distributed observers were designed at each node to estimate the state of the leader node. A low gain approach to dynamic output feedback compensator design for consensus was given in [93].

In this Chapter we follow the design approach proposed in [30] for distributed LQR-based control of multi-agent systems. We focus on the problem of distributed control design for identical decoupled linear time-invariant systems which can be for-
mulated as follows. A dynamical system is composed of (or can be decomposed into) distinct dynamical subsystems that can be independently actuated. The subsystems are dynamically decoupled but have common objectives, which make them interact with each other. The interaction is local, i.e., the goal of a subsystem is a function of only a subset of other subsystems states. The interaction will be represented by an interaction graph, where the nodes represent the subsystems and an edge between two nodes denotes a coupling term in the controller associated with the nodes. Also, it is assumed that the exchange of information has a special structure, i.e., it is assumed that each subsystem can sense and/or exchange information with only a subset of other subsystems. We will assume that the interaction graph and the information exchange graph coincide.

A distributed control design method is presented which requires the solution of a single linear quadratic regulator (LQR) problem. The size of the LQR problem is equal to the maximum vertex degree of the communication graph plus one. Both distributed state variable feedback control and dynamic output feedback control are considered. Specifically, two types of dynamic output feedback algorithms are proposed. It is seen that LQR based optimal design provides a straightforward way to construct controller and observer gains that guarantee stabilization. The design shows how stability of the large-scale system is related to the robustness of local controllers, observers and the spectrum of a matrix representing the desired sparsity pattern of the distributed controller design problem. It is also worth mentioning that by using this approach, the control gain and observer gain design are decoupled from the communication graph structure.

The content of this Chapter is organized as follows. First in Section 7.2 we introduce the notation, bring preliminary results, and study the solution properties of the LQR for a set of identical, decoupled linear systems. Such properties will be
used later to construct a stabilizing distributed controller for arbitrary interconnection structures. Section 7.3 presents the distributed state-feedback controller design procedure using the local LQR solution properties. In Section 7.4 different design procedures for distributed dynamic output feedback controller are presented. The proposed distributed control design method and the effect of the free tuning parameters in the local LQR design is illustrated by a simulation example in Section 7.5.

7.2 Preliminaries

**Notation 1**: Let $M \in \mathbb{R}^{m \times n}$, then $M[i : j, k : l]$ denote a matrix of dimension $(j - i + 1, l - k + 1)$ obtained by extracting rows $i$ to $j$ and columns $l$ to $k$ form the matrix $M$, with $m \geq j \geq i \geq 1, n \geq k \geq l \geq 1$.

**Notation 2**: Let $\lambda_i(M)$ denotes the $i$th eigenvalue of matrix $M \in \mathbb{R}^{n \times n}$. The spectrum of $M$ will be denoted by $\rho(M) = \{\lambda_1(M), \ldots, \lambda_n(M)\}$.

**Proposition 7.2.1.** Consider two matrices $A = \alpha I_n$ and $B \in \mathbb{R}^{n \times n}$. Then $\lambda_i(A + B) = \alpha + \lambda_i(B)$ for $i = 1, \ldots, n$

**Proof.** Take any eigenvalue $\lambda_i(B)$ and the corresponding eigenvector $v_i \in \mathbb{C}^n$. Then $(A + B)v_i = Av_i + Bv_i = \alpha v_i + \lambda_i v_i = (\alpha + \lambda_i)v_i$.

**Proposition 7.2.2.** Given $A, C \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{n \times n}$, consider two matrices $\bar{A} = I_n \otimes A$ and $\bar{C} = B \otimes C$, where $\bar{A}, \bar{C} \in \mathbb{R}^{nm \times nm}$. Then $\rho(\bar{A} + \bar{C}) = \bigcup_{i=1}^{N} \rho(A + \lambda_i(B)C)$ where $\lambda_i(B)$ is the $i$th eigenvalue of $B$.

**Proof.** This property can be shown using the eigenvector argument and Kronecker product properties.

Now consider a set of $N_L$ identical, decoupled linear time-invariant dynamical
systems, the \( i \)th system being described by the continuous-time state equation:

\[
\dot{x}_i(t) = A x_i(t) + B u_i(t) \tag{7.1}
\]

\[
y_i(t) = C x_i(t), \quad x_i(0) = x_{i0}
\]

where \( x_i \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^m \) and \( y_i \in \mathbb{R}^r \) are states, input and measurement output of the \( i \)th system at time \( t \), respectively. Let \( \tilde{x} \in \mathbb{R}^{N_L} \), \( \tilde{u} \in \mathbb{R}^{mN_L} \) and \( \tilde{y} \in \mathbb{R}^{rN_L} \) be the vectors which collect the states, inputs and outputs of the \( N_L \) systems at time \( t \)

\[
\dot{\tilde{x}}(t) = \tilde{A} \tilde{x}(t) + \tilde{B} \tilde{u}(t) \tag{7.2}
\]

\[
\tilde{y}(t) = \tilde{C} \tilde{x}(t), \quad \tilde{x}(0) = \tilde{x}_0 = [x_{10}, \ldots, x_{N_L0}]^T
\]

with

\[
\tilde{A} = I_{N_L} \otimes A' \quad \tilde{B} = I_{N_L} \otimes B, \quad \tilde{C} = I_{N_L} \otimes C \tag{7.3}
\]

We consider an LQR control problem for the set of \( N_L \) systems where the following cost function

\[
J(\tilde{u}, \tilde{x}_0) = \int_0^\infty \left( \sum_{i=1}^{N_L} x_i^T(t) Q_{ii} x_i(t) + u_i^T(t) R_{ii} u_i(t) + \sum_{i=1}^{N_L} \sum_{j \neq i} (x_i(t) - x_j(t))^T Q_{ij} (x_i(t) - x_j(t)) \right) dt
\]

(7.4)

couples the dynamic behavior of individual systems with

\[
R_{ii} = R_{ii}^T > 0, Q_{ii} = Q_{ii}^T \geq 0 \quad \forall i \tag{7.5}
\]

\[
Q_{ij} = Q_{ij}^T = Q_{ji} \geq 0 \quad \forall i \neq j
\]

The cost function (7.4) contains terms which weigh the \( i \)th system states and inputs, as well as the difference between the \( i \)th and the \( j \)th system states and can be rewritten using the following compact notation:

\[
J(\tilde{u}, \tilde{x}_0) = \int_0^\infty \left( \tilde{x}^T(t) \tilde{Q} \tilde{x}(t) + \tilde{u}^T(t) \tilde{R} \tilde{u}(t) \right) dt \tag{7.6}
\]

where the matrices \( \tilde{Q} \) and \( \tilde{R} \) have the following special structure with \( N_L^2 \) blocks of
dimension \( n \times n \) and \( m \times m \) respectively:

\[
\tilde{Q} = \begin{bmatrix}
\tilde{Q}_{11} & \tilde{Q}_{12} & \cdots & \tilde{Q}_{1N_L} \\
\vdots & \ddots & \vdots & \vdots \\
\tilde{Q}_{N_L1} & \cdots & \cdots & \tilde{Q}_{N_LN_L}
\end{bmatrix}, \quad \tilde{R} = I_{N_L} \otimes R
\]  

(7.7)

with

\[
\tilde{Q}_{ii} = Q_{ii} + \sum_{k=1, k \neq i}^{N_L} Q_{ik}, \quad i = 1, \ldots, N_L
\]

(7.8)

\[
\tilde{Q}_{ij} = -Q_{ij}, \quad i, j = 1, \ldots, N_L, \ i \neq j
\]

Let \( \tilde{K} \) and \( \tilde{x}_0^T \tilde{P} \tilde{x}_0 \) be the optimal controller and the value function corresponding to the following LQR problem:

\[
\min_k J(\tilde{u}, \tilde{x}_0) \quad \text{subject to} \quad \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t) \quad \tilde{x}(0) = \tilde{x}_0
\]

(7.9)

We will assume that a stabilizing solution to the LQR problem (7.9) with finite performance index exists and is unique:

**Assumption 1:** System \( \tilde{A}, \tilde{B} \) is stabilizable and system \( \tilde{A}, \tilde{C} \) is observable, where \( \tilde{C} \) is any matrix such that \( \tilde{C}^T \tilde{C} = \tilde{Q} \).

We will also assume local stabilizability and observability:

**Assumption 2:** System \( A, B \) is stabilizable and system \( A, C \) is observable, where \( C \) is any matrix such that \( C^T C = Q \).

It is well known that

\[
\tilde{K} = -\tilde{R}^{-1} \tilde{B}^T \tilde{P}
\]

(7.10)

where \( \tilde{P} \) is the symmetric positive definite solution to the following algebraic Riccati equation (ARE):

\[
\tilde{A}^T \tilde{P} + \tilde{P} \tilde{A} - \tilde{P} \tilde{B} \tilde{R}^{-1} \tilde{B}^T \tilde{P} + \tilde{Q} = 0
\]

(7.11)

We decompose \( \tilde{K} \) and \( \tilde{P} \) into \( N_L^2 \) blocks of dimension \( m \times n \) and \( n \times n \), respectively. Denote by \( \tilde{K}_{ij} \) and \( \tilde{P}_{ij} \) the \((i, j)\) block of the matrix \( \tilde{K} \) and \( \tilde{P} \), respectively. In the
following theorems we show that $\tilde{K}_{ij}$ and $\tilde{P}_{ij}$ satisfy certain properties which will be critical for the design of stabilizing distributed controllers and distributed observer. These properties stem from the special structure of the LQR problem (7.9). Next, the matrix $X$ is defined as $X = BR^{-1}B^T$.

**Theorem 7.2.3.** Assume the weighting matrices (7.8) of the LQR problem (7.9) are chosen as

\[
Q_{ii} = Q_1, \quad \forall i = 1, \ldots, N_L
\]

\[
Q_{ij} = Q_2, \quad \forall i, j = 1, \ldots, N_L, \ i \neq j
\]  

(7.12)

Let $\tilde{x}_0^T \tilde{P} \tilde{x}_0$ be the value function of the LQR problem (7.9) with weights (7.12), and the blocks of the matrix $\tilde{P}$ be denoted by $\tilde{P}_{ij} = \tilde{P}[(i-1)n : in, (j-1)n : jn]$ with $i, j = 1, \ldots, N_L$. Then

1. $\sum_{j=1}^{N_L} \tilde{P}_{ij} = P$ for all $i = 1, \ldots, N_L$, where $P$ is the symmetric positive definite solution of the ARE associated with a single node local problem:

\[
A^T P + PA - PBR^{-1}B^T P + Q = 0
\]  

(7.13)

2. $\sum_{j=1}^{N_L} \tilde{K}_{ij} = K$ for all $i = 1, \ldots, N_L$, where $K = -R^{-1}B^T P$.

3. $\tilde{P}_{ij} = \tilde{P}_{lm} \triangleq \tilde{P}_2 \forall i \neq j$ and $\forall l \neq m$ is a symmetric negative semidefinite matrix.

**Proof.** The proof can be extracted from [30]

Under the hypothesis of Theorem 7.2.3, because of symmetry and equal weights $Q_2$ on the neighboring state differences and equal weights $Q_1$ on absolute states, the LQR optimal controller will have the following structure:

\[
\tilde{K} = \begin{bmatrix}
K_1 & K_2 & \ldots & K_2 \\
K_2 & K_1 & \ldots & K_2 \\
\vdots & \ddots & \ddots & \vdots \\
K_2 & \ldots & \ldots & K_1
\end{bmatrix}
\]  

(7.14)
with $K_1$ and $K_2$ functions of $N_L$, $A$, $B$, $Q_1$, $Q_2$ and $R$, and the symmetric positive definite solution $\tilde{P}$ have the following structure:

$$\tilde{P} = \begin{bmatrix} P - (N_L - 1)\tilde{P}_2 & \tilde{P}_2 & \cdots & \tilde{P}_2 \\ \tilde{P}_2 & P - (N_L - 1)\tilde{P}_2 & \cdots & \tilde{P}_2 \\ \vdots & \ddots & \ddots & \vdots \\ \tilde{P}_2 & \cdots & \cdots & P - (N_L - 1)\tilde{P}_2 \end{bmatrix} \quad (7.15)$$

The following corollaries of Theorem 7.2.3 follow from the stability and the robustness of the LQR controller $-R^{-1}B^T(-N_L\tilde{P}_2)$ for system $A - XP$.

**Corollary 7.2.4.** $A - XP + N_LX\tilde{P}_2$ is a Hurwitz matrix.

From the gain margin properties of LQR controller we have:

**Corollary 7.2.5.** $A - XP + \alpha N_LX\tilde{P}_2$ is a Hurwitz matrix for all $\alpha > 1/2$, with $\alpha \in \mathbb{R}$.

**Remark 7.2.6.** $A - XP$ is a Hurwitz matrix, thus the system in Corollary 7.2.5 is stable for $\alpha = 0$ as well. ($A - XP = A + BK$ with $K$ being the LQR gain for system $(A,B)$ with weights $(Q_1,R)$).

The following condition defines a class of systems and LQR weighting matrices which will be used in later sections to extend the set of stabilizing distributed controller structures.

**Condition 1 :** $A - XP + \alpha N_LX\tilde{P}_2$ is a Hurwitz matrix for all $\alpha \in [0,1/2]$, with $\alpha \in \mathbb{R}$.

Essentially, Condition 1 characterizes systems for which the LQR gain stability margin described in Corollary 7.2.5 is extended to any positive $\alpha$. The fact that $A - XP$ is already stable, does not necessarily guarantee this property.
Checking the validity of Condition 1 for a given tuning of $P$ and $\tilde{P}_2$ may be performed as a stability test for an affine parameter-dependent model

$$\dot{x} = (A_0 + \alpha A_1)x$$  \hspace{1cm} (7.16)

where $A_0 = A - XP$, $A_1 = N_L X \tilde{P}_2$ and $0 \leq \alpha \leq 1/2$. This test can be posed as an LMI problem (Proposition 5.9 in [94]) searching for quadratic parameter-dependent Lyapunov functions.

### 7.3 Distributed LQR-Based Control Design

Consider a set of $N$, linear, identical and decoupled dynamical systems, described by the continuous-time state equation:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t)$$  \hspace{1cm} (7.17)

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ are states and inputs of the $i$th system at time $t$, respectively. Let $\hat{x} \in \mathbb{R}^{nN}$ and $\hat{u} \in \mathbb{R}^{mN}$ be the vectors which collect the states and inputs of the $N$ systems at time $t$

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t)$$  \hspace{1cm} (7.18)

$$\hat{y}(t) = \hat{C}\hat{x}(t), \hspace{0.5cm} \hat{x}(0) = \hat{x}_0 = [x_{10}, \ldots, x_{N0}]^T$$

with

$$\hat{A} = I_N \otimes A, \hspace{0.5cm} \hat{B} = I_N \otimes B, \hspace{0.5cm} \hat{C} = I_N \otimes C$$  \hspace{1cm} (7.19)

**Remark 7.3.1.** Systems (7.2) and (7.18) differ only in the number of subsystems. We will use system (7.2) with $N_L$ subsystems when referring to local problems, and system (7.18) with $N$ subsystems when referring to the global problem. Accordingly, tilded matrices will refer to local problems and hatted matrices will refer to the global problem.
We use a graph to represent the coupling in the control objective and the communication in the following way. We associate the $i$th system with the $i$th node of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. If an edge $(i, j)$ connecting the $i$th and $j$th node is present, then 1) the $i$th system has full information about the state of the $j$th system and, 2) the $i$th system control law minimizes a weighted distance between the $i$th and the $j$th system states.

The class of matrices $\mathcal{K}^N_{n,m}(\mathcal{G})$ is defined as follows:

**Definition 7.3.2.** $\mathcal{K}^N_{n,m}(\mathcal{G}) = \{ M \in \mathbb{R}^{nN \times mN} | M_{ij} = 0 \text{ if } (i, j) \notin \mathcal{E}, M_{ij} = M[(i-1)n : in, (j-1)m : jm], i, j = 1, \ldots, N \}$

The distributed optimal control problem is defined as follows:

$$
\min_{\hat{K}} \quad J(\hat{u}, \hat{x}_0) = \int_0^{\infty} \left( \hat{x}^T(t) \hat{Q} \hat{x}(t) + \hat{u}^T(t) \hat{R} \hat{u}(t) \right) dt \\
\text{subject to } \quad \dot{\hat{x}}(t) = \hat{A} \hat{x}(t) + \hat{B} \hat{u}(t) \\
\hat{u}(t) = \hat{K} \hat{x}(t) \\
\hat{K} \in \mathcal{K}^N_{m,n}(\mathcal{G}) \\
\hat{Q} \in \mathcal{K}^N_{n,n}(\mathcal{G}), \quad \hat{R} = I_N \otimes R \\
\hat{x}(0) = \hat{x}_0
$$

(7.20)

with $\hat{Q} = \hat{Q}^T \geq 0$ and $\hat{R} = \hat{R}^T \geq 0$. In general, computing the solution to problem (7.20) is an NP-hard problem. Next, we propose a suboptimal control design leading to a controller $\hat{K}$ with the following properties:

1. $\hat{K} \in \mathcal{K}^N_{m,n}(\mathcal{G})$

2. $\hat{A} + \hat{B}\hat{K}$ is Hurwitz.

3. Simple tuning of absolute and relative state errors and control effort within $\hat{K}$.

Such controller will be referred to as distributed suboptimal controller. The following theorem will be used to propose a distributed suboptimal control design procedure.
Theorem 7.3.3. Consider the LQR problem (7.9) with \( N_L = d_{\text{max}}(\mathcal{G}) + 1 \) and weights chosen as in (7.12) and its solution (7.14) and (7.15). Let \( M \in \mathbb{R}^{N \times N} \) be a symmetric matrix with the following property:

\[
\lambda_i(M) > \frac{N_L}{2}, \quad \forall \lambda_i(M) \in \rho(M) \setminus \{0\}
\]

(7.21)

and construct the feedback controller:

\[
\hat{K} = -I_N \otimes R^{-1}B^TP + M \otimes R^{-1}B^T\tilde{P}_2
\]

(7.22)

Then, the closed loop system

\[
A_{cl} = I_N \otimes A + (I_N \otimes B)\hat{K}
\]

(7.23)

is asymptotically stable.

Proof. Consider the eigenvalues of the closed-loop system \( A_{cl} \)

\[
\rho(A_{cl}) = \rho \left( I_N \otimes (A - XP) + M \otimes (X\tilde{P}_2) \right).
\]

By Proposition 7.2.2:

\[
\rho \left( I_N \otimes (A - XP) + M \otimes (X\tilde{P}_2) \right) = \bigcup_{i=1}^{N} \rho \left( A - XP + \lambda_i(M)X\tilde{P}_2 \right)
\]

We will prove that \( A - XP + \lambda_i(M)X\tilde{P}_2 \) is a Hurwitz matrix \( \forall i = 1, \ldots, N \), and thus prove the theorem. If \( \lambda_i(M) = 0 \) then \( A - XP + \lambda_i(M)X\tilde{P}_2 \) is Hurwitz based on Remark 7.2.6. If \( \lambda_i(M) \neq 0 \), then from Corollary 7.2.5 and from condition (7.21), we conclude that \( A - XP + \lambda_i(M)X\tilde{P}_2 \) is Hurwitz.

This Theorem have the following consequences for designing distributed controller.

If \( M \in \mathcal{K}^N_{1,1}(\mathcal{G}) \), then \( \hat{K} \) in (7.22) is an asymptotically stabilizing distributed controller. Therefore, one can find a stabilizing distributed controller with a desired sparsity pattern (which is in general a formidable task by itself) by solving a low-dimensional problem (characterized by \( d_{\text{max}}(\mathcal{G}) \)) compared to the full problem size.
(7.20). This feature relies on the special structure of the defined problem. It should be pointed out that the result is independent of the local LQR tuning. Thus $Q_1$, $Q_2$ and $R$ can be used in order to influence the compromise between minimization of absolute and relative terms, and the control effort in the global performance.

For the special class of systems defined by Condition 1, the hypothesis of Theorem 7.3.3 can be relaxed as follows:

**Theorem 7.3.4.** Consider the LQR problem (7.9) and weights chosen as in (7.12) and its solution (7.14) and (7.15). Assume that the Condition 1 holds. Let $M \in \mathbb{R}^{N \times N}$ be a symmetric matrix with the following property:

$$\lambda_i(M) \geq 0, \quad \forall \lambda_i(M) \in \rho(M) \quad (7.24)$$

Then, the closed loop system (7.23) is asymptotically stable when $\hat{K}$ is constructed as in (7.22).

*Proof.* If Condition 1 holds, then $\left( A - XP + \lambda_i(M)X\tilde{P}_2 \right)$ is Hurwitz for all $\lambda_i(M) \geq 0$ which proves the Theorem. \(\square\)

### 7.4 Distributed LQR-Based Output Feedback Controller

In many practical applications, full state information is not always available for controller design. Therefore, the use of observer for estimating the state is necessary. In this section, we propose distributed output feedback controller for the system (7.18). Two protocols are considered: (i) The LQR based control gain (7.22) with local observers (ii) The LQR based control gain (7.22) with distributed LQR based observer.
7.4.1 Distributed Controller and Local Observer

This type of control protocol uses the local and neighborhood state estimation information for the controller design and local output information for the observer design. The feedback controller is designed as

$$\hat{u}(t) = \hat{K}\hat{x}_{est}(t)$$  \hspace{1cm} (7.25)

with

$$\hat{K} = -I_N \otimes R^{-1}B^TP + M \otimes R^{-1}B^T\bar{P}_2$$  \hspace{1cm} (7.26)

where $\hat{x}_{est} = [x_{est,1}^T, \ldots, x_{est,N}^T]^T$ is the estimated state information obtained by the local observers:

$$\dot{x}_{est,i}(t) = Ax_{est,i}(t) + Bu(t) + L(y_i - Cx_{est,i}).$$  \hspace{1cm} (7.27)

The observer gain $L$ is given by

$$L = P_OC^TR^{-1}$$  \hspace{1cm} (7.28)

where $P_O$ is the unique positive definite solution of the observer ARE

$$AP_O + P_OA^T + Q - P_OC^TR^{-1}CP_O = 0$$  \hspace{1cm} (7.29)

in which $Q = Q^T \in \mathbb{R}^{n \times n}$ and $R = R^T \in \mathbb{R}^{r \times r}$ are arbitrary positive definite matrices.

Defining $\hat{e} = \hat{x} - \hat{x}_{est}$, it can be shown that dynamic of the closed-loop system is:

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{e}} \end{bmatrix} = \begin{bmatrix} A_{cl} & \hat{B}\hat{K} \\ 0 & I_N \otimes (A + LC) \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{e} \end{bmatrix}$$  \hspace{1cm} (7.30)

It is clear from dynamic equation (7.30), that the entire system is stable if both $A_{cl} = I_N \otimes A + (I_N \otimes B)\hat{K}$ and $I_N \otimes (A + LC)$ are Hurwitz. The former is guaranteed by picking $\hat{K}$ as in (7.26) according to Theorem 7.3.3, and the latter can also be guaranteed by choosing $L$ as in (7.28), since $(A, C)$ is assumed to be observable.
7.4.2 Distributed Controller and Distributed Observer

This type of control protocol uses the local and neighborhood state estimation information for the controller design and local and neighborhood output information for the observer design. The distributed observer gains are obtained by solving a low-dimensional ARE similar to the controller design counterpart. The following Theorem will be used to propose a distributed observer design procedure.

**Theorem 7.4.1.** Consider the dynamic system (7.2) with \( N_L = d_{\text{max}}(\mathcal{G}) + 1 \) and positive definite weights chosen as in (7.12). Then the solution of the ARE:

\[
\tilde{A}\tilde{P}_O + \tilde{P}_O\tilde{A}^T + \tilde{Q} - \tilde{P}_O\tilde{C}^T\tilde{R}^{-1}\tilde{C}\tilde{P}_O = 0
\]

(7.31)

has the following structure:

\[
\tilde{P}_O = \begin{bmatrix}
    P_O - (N_L - 1)\tilde{P}_{O2} & \tilde{P}_{O2} & \ldots & \tilde{P}_{O2} \\
    \tilde{P}_{O2} & P_O - (N_L - 1)\tilde{P}_{O2} & \ldots & \tilde{P}_{O2} \\
    \vdots & \ddots & \ddots & \vdots \\
    \tilde{P}_{O2} & \ldots & \ldots & P_O - (N_L - 1)\tilde{P}_{O2}
\end{bmatrix}
\]

(7.32)

Let \( M \in \mathbb{R}^{N \times N} \) be a symmetric matrix with the following property:

\[
\lambda_i(M) > \frac{N_L}{2}, \quad \forall \lambda_i(M) \in \rho(M) \setminus \{0\}
\]

(7.33)

and construct the observer gain:

\[
\hat{L} = -I_N \otimes P_O C^T R^{-1} + M \otimes \tilde{P}_{O2} C^T R^{-1}
\]

(7.34)

Then, the system matrix

\[
A_O = I_N \otimes A + \hat{L}(I_N \otimes C)
\]

(7.35)

is Hurwitz.
Proof. This result can be proved similar to Theorem 7.3.3 by considering the linear system \((A^T, C^T)\) and using the results developed in the previous Section.

Theorem 7.4.2. Consider the dynamic system (7.2) with \(N_L = d_{\text{max}}(\mathcal{G}) + 1\) and positive definite weights chosen as in (7.12). Assume that the Condition 1 holds. Let \(M \in \mathbb{R}^{N \times N}\) be a symmetric matrix with the following property:

\[
\lambda_i(M) \geq 0, \quad \forall \lambda_i(M) \in \rho(M)
\] (7.36)

Then, the system matrix (7.35) is asymptotically stable when \(\hat{L}\) is constructed as in (7.34).

Proof. If Condition 1 holds, then the result of Theorem 7.4.1 can be extended for all \(\lambda_i(M) \geq 0\) which proves the Theorem.

With these results, Distributed dynamic output feedback controller for the system (7.18) can be written as:

\[
\dot{x}_{\text{est}}(t) = \hat{A}x_{\text{est}}(t) + \hat{B}u(t) + \hat{L}(\hat{y} - \hat{C}x_{\text{est}})
\] (7.37)

\[
\hat{u}(t) = \hat{K}x_{\text{est}}
\]

with the observer and controller gains

\[
\hat{L} = -I_N \otimes P_O C^T R^{-1} + M \otimes \hat{P}_{O_2} C^T R^{-1}
\] (7.38)

\[
\hat{K} = -I_N \otimes R^{-1} B^T P + M \otimes R^{-1} B^T \hat{P}_2
\] (7.39)

Theorem 7.4.3. Consider the dynamic system (7.18). Then the dynamic output feedback controller (7.37) with the observer and controller gains (7.38) and (7.39), respectively, stabilizes the system.

Proof. The dynamic of the closed-loop system can be written as

\[
\begin{bmatrix}
\dot{x} \\
\dot{e}
\end{bmatrix} =
\begin{bmatrix}
I_N \otimes A + (I_N \otimes B)\hat{K} & \hat{B}\hat{K} \\
0 & I_N \otimes A + \hat{L}(I_N \otimes C)
\end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix}
\] (7.40)
The entire system is stable if both $A_{cl} = I_N \otimes A + (I_N \otimes B) \hat{K}$ and $A_O = I_N \otimes A + \hat{L}(I_N \otimes C)$ are Hurwitz. By choosing $\hat{K}$ as in (7.39), Theorem 7.3.3 guarantees the stability of $A_{cl}$. Moreover, according to Theorem 7.4.1, $\hat{L}$ as in (7.38) makes $A_O$ stable. This complete the proof.

7.5 Example

Consider a $5 \times 5$ mesh interconnection of $N = 25$ identical, dynamically decoupled and independently actuated linear systems moving in a plane with double integrator dynamics in both spatial dimensions

$$
\ddot{x}_i = u_{x,i} \quad \ddot{y}_i = u_{y,i} \quad i = 1, \ldots, 25
$$

(7.41)

The interconnection structure is depicted in Figure 7.1. The control objective is to move each subsystem to a desired absolute position corresponding to its location in the pre-specified rectangular grid, which has equal separation distances defined between each orthogonal neighbor. The maximum vertex degree of such an interconnection graph is 4. A stabilizing distributed controller is designed for the mesh interconnection of systems, by solving the following, simple LQR problem involving only $N_L = d_{max} +$
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1 = 5 nodes.

\[
\min_{\tilde{u}} J(\tilde{u}, \tilde{x}_0) \quad \text{subject to} \quad \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t) \quad \tilde{x}(0) = \tilde{x}_0
\]  

(7.42)

where

\[
\tilde{A} = I_5 \otimes A, \quad \tilde{B} = I_5 \otimes B, \quad \tilde{C} = I_5 \otimes C
\]

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad
B = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}, \quad
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

(7.43)

The cost function defined in (7.4) uses the following weights for absolute and relative state information

\[
Q_{ii} = Q_a, \quad Q_{ij} = Q_r, \forall i \neq j
\]  

(7.44)

The weight on the control effort was kept constant in the following examples \((R = 1)\) for each control input in both spatial dimensions. The subsystem model and chosen weights satisfy Condition 1.

The solution of the above local LQR problem will yield a controller matrix of the following form in this specific example

\[
\tilde{K} = \begin{bmatrix}
K_a & K_r & K_r & K_r & K_r \\
K_r & K_a & K_r & K_r & K_r \\
K_r & K_r & K_a & K_r & K_r \\
K_r & K_r & K_r & K_a & K_r \\
K_r & K_r & K_r & K_r & K_a
\end{bmatrix}
\]

(7.45)

The distributed controller for the grid of interconnected systems has the same structure as the underlying graph as depicted in Figure 7.1 and is constructed by

\[
\hat{K} = I_N \otimes K_a + A \otimes K_r
\]  

(7.46)
where \( A \) denotes the adjacency matrix of the mesh grid interconnection of systems. This control gain corresponds to \( \hat{K} = -I_N \otimes R^{-1}B^T P + M \otimes R^{-1}B^T \tilde{P}_2 \) with \( M = 4I_N - A \). Since \( \lambda_i(M) \geq 0 \) according to the property of Adjacency matrix \( A \) and the condition 1 holds, the stability of the closed-loop system is guaranteed.

Two simulation results, starting from the same initial conditions, are presented with different choices for the parameters \( Q_a \) and \( Q_r \). The first simulation weighs the absolute state information in the local LQR solution more heavily than the second simulation and uses \( Q_a(1,1) = 1 \) and \( Q_r(1,1) = 1 \) values for position states in both spatial dimensions. Velocity states were not weighted in these simulation examples. Snapshots of the simulation are shown in Figure 7.2, indicating that the subsystems converge to their desired absolute positions along fairly straight lines starting from their initial conditions. The second simulation illustrates the behavior of the large-scale distributed control system, when much more emphasis is put on the relative state information in the local LQR cost function, by selecting \( Q_a(1,1) = 0.001 \) and \( Q_r(1,1) = 1000 \). As the snapshots in Figure 7.3 demonstrate, the behavior of the overall interconnected system has changed fundamentally. Although the distributed controller is still stabilizing, the dynamic behavior has changed drastically and shows wave-like oscillations due to the high weights on relative state information.

Next consider the case that the state information are not available and output measurements can be obtained according to matrix \( C \) (we only have information of the positions rather than positions and velocities.) The simulation results for two proposed protocol of dynamic output feedback controller are presented in Figures 7.4 and 7.5. In both simulations more emphasis is put on the relative state information in the local LQR cost function, by selecting \( Q_a(1,1) = 0.001 \) and \( Q_r(1,1) = 1000 \). Figures 7.4 shows the behavior of the mesh grid with local observer and distributed controller. As can be seen from the figure, the dynamic behavior of the grid does
not show wave-like oscillations in spite of high weights on relative state information. It is observed that if the observer gains increased enormously to achieve very fast convergence, then the wave-like oscillations can be recovered. Figures 7.5 depicts the behavior of the mesh grid with distributed observer and distributed controller. In this case, the dynamic behavior shows wave-like oscillations with low observer and controller gains.

7.6 Conclusion

In this Chapter the problem of distributed LQR-based control design for multi-agent systems is considered. Both distributed state variable feedback control and dynamic output feedback control are presented. Specifically, two types of distributed dynamic output feedback algorithms are developed with local observers and distributed observers. The design procedure provides a straightforward way to construct controller and observer gains that guarantee stabilization. Using this approach, the control gain and observer gain design are decoupled from the communication graph structure.
Figure 7.2: Snapshots of the mesh grid simulation with state feedback controller - larger weight on absolute position references.
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Figure 7.3: Snapshots of the mesh grid simulation with state feedback controller - larger weight on relative position references.
Figure 7.4: Snapshots of the mesh grid simulation with dynamic feedback controller - local observer and distributed controller.
Figure 7.5: Snapshots of the mesh grid simulation with dynamic feedback controller - distributed observer and distributed controller.
Appendix A

Graph Theory

1.1 Basic Concepts

A directed graph $G = (V, E)$ is composed by a finite vertex set $V$ and edge set $E \subseteq V \times V$. Suppose there are $N$ vertices in $V$, each vertex is labelled by an integer $i \in \{1, 2, \ldots, N\}$ and $N$ is the order of the graph. Each edge can be denoted by a pair of distinct vertices $(v_i, v_j)$ where $v_i$ is the head and $v_j$ is the tail. If $(v_i, v_j) \in E \Leftrightarrow (v_j, v_i) \in E$, the graph is called undirected. A graph $G$ is said to be complete if every possible edge exists.

For directed graph $G$, the number of edges whose head is $v_i$ is called the out-degree of node $v_i$. The number of edges whose tail is $v_i$ is called the in-degree of node $v_i$. If edge $(v_i, v_j) \in E$, then $v_j$ is one of the neighbors of $v_i$. The set of neighbors of $v_i$ is denoted by $N(i) = \{v_i \in V : (v_i, v_j) \in E\}$.

A strong path in a directed graph is a sequence of distinct vertices $[v_0, \ldots, v_r]$ where $(v_{i-1}, v_i) \in E$ for any $i \in \{1, 2, \ldots, r - 1\}$ and $r$ is called the length of the path. A weak path is also a sequence of distinct vertices $[v_0, \ldots, v_r]$ as long as either $(v_{i-1}, v_i)$ or $(v_i, v_{i-1})$ belongs to $E$.

Directed graphs over a vertex set can be categorized according to their connectivity
properties. A directed graph is *weakly connected* if any two ordered vertices in the graph can be joined by a weak path, and is *strongly connected* if any two ordered vertices in the graph can be joined by a strong path. If a graph is not connected, then it is called *disconnected*. An undirected graph is either connected or disconnected.

A directed graph $G$ is called *acyclic* if it does not contain any edge cycle. In a directed asyclic graph, there exist at least one vertex that has zero out-degree. A *rooted directed spanning tree* for directed graph $G$ is a subgraph $G_r = (V, E_r)$ where $E_r$ is a subset of $E$ that connects, without any cycle, all vertices in $G$ so that each vertex, except the root, has one and only one outcoming edge. Thus, a rooted directed spanning tree is acyclic.

### 1.2 Matrices in Algebraic Graph Theory

Algebraic graph theory studies graphs in connection with linear algebra. Various matrices are naturally associated with the vertices and edges in directed graph. Properties of graphs can be reflected in the algebraic analysis of these matrices. A couple of important matrices are introduced below.

An *adjacency matrix* $A = \{a_{ij}\}$ of a directed graph $G$ with order $N$ is an $N \times N$ matrix that is defined as

$$a_{ij} = \begin{cases} 
1 & \text{if } (v_i, v_j) \in E \\
0 & \text{otherwise.}
\end{cases} \quad (A.1)$$

Thus, $A$ is symmetric when $G$ is undirected.

An *out-degree matrix* $D = \{d_{ij}\}$ of a directed graph $G$ with order $N$ is an $N \times N$ diagonal matrix defined as

$$d_{ij} = \sum_{j \neq i} a_{ij}. \quad (A.2)$$

It is clear that every diagonal element in $D$ equals the out-degree of the corresponding vertex. When $G$ is undirected, the out-degree for each vertex equals the in-degree
and $D$ is also called the degree matrix.

In general, a weighted adjacency matrix $A = \{a_{ij}\}$ is defined as

$$a_{ij} = \begin{cases} w_{ij} & \text{if } (v_i, v_j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \quad (A.3)$$

where $w_{ij}$ is the positive weight associated with edge $(v_i, v_j)$. Then, the out degree of node $v_i$ is the sum of the weights of the edges whose head is $v_i$ and the in-degree of node $v_i$ is the sum of the weights of the edges whose tail is $v_i$. So the aforementioned adjacency matrix can be treated as a special case of weighted adjacency matrix where all weights equal to 1.

A Laplacian matrix $L$ of a directed graph $G$ with order $N$ is an $N \times N$ matrix that is defined as

$$L = D - A \quad (A.4)$$

If we normalized each row of adjacency matrix by corresponding out-degree, we get the normalized adjacency matrix as

$$\tilde{A} = D^{-1}A \quad (A.5)$$

In order to complete the definition, we set $d_{ii}^{-1} = 0$ if a vertex $v_i$ has zero out-degree. Moreover, we define the normalized Laplacian matrix as

$$\tilde{L} = D^{-1}L = I - \tilde{A} \quad (A.6)$$
Bibliography


