Robust Stability Analysis for a Class of Extended Linearization Systems

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Sam Nazari
Boston, MA
15-July-2011
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Abstract

We consider analysis of nonlinear systems that can be brought into a state dependent representation known as extended linearization. Under this formulation, conventional linear analysis techniques may be adapted to study the stability, optimality, and robustness properties of nonlinear systems. When subject to system uncertainty, estimating the radius of stability for systems under extended linearization is difficult since the closed-loop system equations are not available explicitly. A method for obtaining the upper bound for the radius of stability in this class of systems is proposed. It is shown that the stability radius around a suitable domain can be obtained by computing the largest singular value of an overvalued matrix with special properties.

Additionally, a property of extended linearization is that it relies on a non-unique factorization of the system dynamics to bring the nonlinear system into a psuedo linear form referred to as the State Dependent Coefficient (SDC) parameterization. Under system uncertainty, each SDC parameterization will produce its own radius of stability in a region of interest in the state space. We propose a method to obtain the SDC parameterization which results in the maximum radius of stability for the original nonlinear system in the region of interest. It is shown that the problem of finding the maximum radius of stability from a hyperplane of SDC parameterizations can be reduced to constrained minimization of the spectral norm of a comparison system.
# Glossary of Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>The real numbers</td>
</tr>
<tr>
<td>$\mathbb{R}_+$</td>
<td>The positive real numbers</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>Field of scalars</td>
</tr>
<tr>
<td>$\mathbb{F}^n$</td>
<td>n-dimensional coordinate space</td>
</tr>
<tr>
<td>$V$</td>
<td>Lyapunov function</td>
</tr>
<tr>
<td>$C^1$</td>
<td>The class of continuous differentiable functions</td>
</tr>
<tr>
<td>$C^0$</td>
<td>The class of continuous functions</td>
</tr>
<tr>
<td>$x$</td>
<td>The system state vector</td>
</tr>
<tr>
<td>$V^*$</td>
<td>The value function</td>
</tr>
<tr>
<td>$u^*$</td>
<td>The optimal control</td>
</tr>
<tr>
<td>$x^*$</td>
<td>Optimal state trajectory corresponding to $u^*$</td>
</tr>
<tr>
<td>$Q$</td>
<td>State penalty</td>
</tr>
<tr>
<td>$R$</td>
<td>Control penalty</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Structured parametric uncertainty</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Parametric uncertainty value</td>
</tr>
<tr>
<td>$L$</td>
<td>Instantaneous cost</td>
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<tr>
<td>$H$</td>
<td>Hamiltonian function</td>
</tr>
<tr>
<td>$S$</td>
<td>Solution to Matrix Riccati Equation</td>
</tr>
<tr>
<td>$\sup$</td>
<td>Supremum</td>
</tr>
<tr>
<td>$\inf$</td>
<td>Infimum</td>
</tr>
<tr>
<td>$\max$</td>
<td>Maximum</td>
</tr>
<tr>
<td>$\min$</td>
<td>Minimum</td>
</tr>
<tr>
<td>$\rightarrow$</td>
<td>Tends to</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>Implies</td>
</tr>
<tr>
<td>$\rho(A)$</td>
<td>Spectral radius of a square matrix $A$</td>
</tr>
<tr>
<td>$\lambda(A)$</td>
<td>Eigenvalue of a square matrix $A$</td>
</tr>
<tr>
<td>$\sigma(B)$</td>
<td>Singular value of any matrix $B$</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

1.1 Background

Because of their complex dynamic properties, nonlinear systems pose a formidable challenge to many existing control design methodologies. Control system design becomes considerably more complicated as one progresses from analysis and control of linear systems to nonlinear systems. If the analysis is confined to a local operating point then linearization may be an acceptable strategy. On the other hand, linearization is an approximation and will generally not serve as a good cornerstone to develop a global control law. Analysis of nonlinear systems entails more sophisticated mathematics. The dynamics of a nonlinear system are richer; consequently the process of formulating a control law through a standard design process entails deeper understanding and more sophistication. Table 1.1, adapted from [29], documents some of the compounding characteristics of nonlinear systems that have made a systematic design paradigm elusive. Because of the rich spectrum of properties that each system may exhibit, finding a unified control methodology has eluded scientists. Rather, categories of control methodologies have emerged to address nonlinear systems. [29] [49].

Along these lines, a number of problems can be addressed by feedback linearization, or as it is sometimes referred to in the Aerospace community, dynamic inversion. This approach is mature with its foundations rigorously developed in differential geometry. Comprehensive treatments of feedback linearization are found in [25,29,49,50,54] with good example applications found in [18]. The premise of feedback linearization is to use a mapping, $z = T(x)$, from a nonlinear space, $x$, to a linear space, $z$. This
mapping renders the closed loop dynamics linear and opens the way for any of the existing linear control techniques to be applied to the system. After the control law is developed in the transformed coordinates, it must be transformed back into the original coordinates. Therefore, the transformation must have an inverse, \( T(\cdot)^{-1} \), that is well behaved. This means the derivatives of \( x \) and \( z \) should be \( C^1 \) (continuously differentiable) functions. A \( C^1 \) mapping with a \( C^1 \) inverse such that \( x, z \in \mathbb{R}^n \) with \( T \) proper is known as a global diffeomorphism \([29]\). It could be said that the crux of the feedback linearization approach lies in finding a global diffeomorphism to apply to the system to be controlled.

An unavoidable disadvantage of feedback linearization is that the global diffeomorphism removes beneficial nonlinearities from the system dynamics. Naturally, one prefers to preserve these nonlinearities. An additional consideration stems from the fact that a system may either possess the feedback linearization properties or not. The system states must be completely linearizable for the design to be effective. One scenario where feedback linearization cannot be effectively applied is when the zero dynamics of the system are unstable. The implication is that the zero dynamics are nonminimum phase and complications may arise if feedback linearization is used \([29]\). In this case application of another control methodology is required.

While feedback linearization emphasizes stability, receding horizon control (RHC) aims to achieve optimal performance. To this end, RHC solves a state trajectory optimization problem at every sample step. At each sample, the optimal state trajectory to be followed until a new state becomes available is computed. To do this, the RHC controller propagates the system dynamics along an optimal path. This procedure produces an open loop strategy for the duration of the sample interval. In more specific terms, RHC computes the current state, \( x(t) \), by an online computation of the optimal control, \( u^* \), in the interval \([t, t + T_s]\) in such a fashion so as to minimize the chosen cost function subject to the system dynamic constraints. The optimal control, \( u^*(t, t + T_s) \), is then a time history, \([u^*(t_1), u^*(t_2), u^*(t_3) \ldots u^*(T_s)]\), and at each \( t \), the corresponding control is appropriately applied. To close the loop in a feedback fashion, this procedure is repeated with state information periodically.

RHC is rooted in the Euler-Lagrange approach to optimal control. Both the Euler-Lagrange framework and the Pontryagin Minimum Principle (PMP) are based in the fundamental theorems of the calculus of variations. The calculus of variations
Table 1.1: Properties of Nonlinear Systems

<table>
<thead>
<tr>
<th>Summary of the Properties of Nonlinear Systems</th>
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<tbody>
<tr>
<td><strong>Finite Escape Time</strong></td>
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<td><strong>Multiple Equilibria</strong></td>
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<tr>
<td><strong>Limit Cycles</strong></td>
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<tr>
<td><strong>Periodic Oscillations</strong></td>
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<tr>
<td><strong>Chaos</strong></td>
</tr>
<tr>
<td><strong>Multimodal Behavior</strong></td>
</tr>
</tbody>
</table>

The problem considered by Pontryagin and his students in the 1960s was to minimize a performance integral subject to the constraints imposed by the system dynamics. The difficulty of this problem lies in the fact that an entire time history or optimal trajectory must be found that minimizes the chosen performance integral. This is the essence of a *dynamic optimization* problem. In contrast, for a traditional optimization problem a scalar value that represents the global minimum or maximum would be sufficient. Pontryagin’s contribution was to deduce the set of first order conditions which, when met, would reduce the dynamic optimization problem into a set of partial differential equations that would yield the optimal open loop trajectory for the state and control [1].

This simple approach has a major disadvantage. The Euler-Lagrange solution produces the optimal *open loop* trajectory over the interval *given* an initial condition. Controllers are more robust when formulated in a feedback configuration. The American mathematician Richard Bellman first recognized that by suitable application of the *principle of optimality*, the open loop trajectory optimization problem solved by Pontryagin can be fashioned into a closed loop feedback formulation. To apply the RHC technique in a useable feedback configuration, the optimal trajectory must be computed at every instance when a new state becomes available. That way a new control history can be formed. Solving this optimization problem at each instance when a new state becomes available is computationally expensive and wasteful. This is the reason that RHC applications are frequently restricted to systems with slow
process dynamics. In the special case of linear systems with quadratic cost criteria
the RHC problem reduces to a tractable quadratic problem which does not pose any
problems. In general, however, for nonlinear systems the optimization problem is
plagued by nonconvexity [41].

A more recent approach that ensures stability is the method known as recursive backstepping (RB). In [34], the authors point out the advantages of RB; noting strongly that RB is very well suited to address adaptive control of nonlinear systems. The method of RB is to proceed by designing a sequential control for each state of the nonlinear differential equation in the system [22]. As the design proceeds to the next state equation, the proceeding state is incorporated along with its control. Finally the control is found by carrying out this process to the last state equation. This recursive procedure is indicative of the methodologies nomenclature.

Because RB is based on a Lyapunov function each step of the way, closed loop stability is guaranteed. Performance tuning is accomplished by clever choice of intermediate controls by the designer. RB is limited and the method can yield several different designs depending on the designer and his/her knowledge and insight into the system as demonstrated by Doyle and Packard in the nonlinear control workshop at the 1997 American Control Conference [22].

Sliding mode is a Variable Structure Controller (VSC) documented in [29]. This method is well known and has been used with success in applications that can be found in [18]. The goal of the approach is to define a manifold in the state space that produces the desired state response. Once this desirable manifold is identified, the remainder of the task is to drive all other states that are not part of the manifold into the manifold. It is necessary to ensure that states in the manifold continue to remain there. Worthy of mention is the discontinuous nature of the control action which primarily acts on the feedback states to switch between different system structures in such a fashion that the new system (sliding mode) is the desired manifold. The high gain and discontinuous nature of this control methodology leads to superior system performance particularly under parameter variations and torque disturbances. When uncertainty is present in the system, this high gain becomes an advantage as it robustifies the controller. Unfortunately, this same property may cause the states to deviate from an optimal trajectory and from this point of view it is not an optimal control strategy. A serious disadvantage of this approach is its susceptibility to
“chattering”. Due to imperfections and time delays in switching devices, a state trajectory heading towards the sliding manifold may overshoot the manifold altogether. Consequently the control system will attempt to drive the state back into the sliding manifold. However, again, due to the imperfections and time delays of the switching device in the control system, the state trajectory will begin to oscillate or “chatter” around the sliding manifold. Chattering causes high heat losses in electrical circuits and undesirable wear and tear in mechanical parts. In complex mechanical devices, it can excite unmodeled modes of the system which will degrade the system performance and reduce the overall robustness.

Another notably successful Lyapunov based control methodology is documented in [50]. In the Control Lyapunov Function (CLF) approach, a $C^1$, proper, positive definite mapping, $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$, is found such that the condition, $\inf_u \left[ \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) u \right] < 0$, is met. The motivation for the methodology is conveyed through a hypothetical example: suppose we are provided a positive definite function, $V$. We wish to know if this function is a suitable Lyapunov function for a system to be stabilized. If we calculate the derivative of this function along all the trajectories of the system and it is possible to make the derivative negative along the entire trajectory by choosing an appropriate $u$, then the system can be stabilized with $V$ being the Lyapunov function for the controlled system.

It should be said that finding a CLF for a system in itself is an enormous feat. Generally, nonlinear systems do not lend themselves easily to construction of an appropriate Lyapunov function. Methods exist for feedback linearization and recursive backstepping. Once a suitable CLF has been chosen, the control system designer may apply Sontags formula. In [41] the connection between CLF and the value function of the HJB equation is demonstrated and the connection between Sontags formula and optimal control is introduced. Specifically, when $V$ has level curves that are the same as the value functions level curves, then Sontags formula produces the optimal control. Sontags formula uses the CLF $V$ as a replacement for the HJB value function. Therefore, a connection has been established between CLFs and the solution for the HJB value function.

Despite these advances in nonlinear control, many problems still remain unsolved. Today there remains a lack of unified control methodologies that address
stability, performance and overall robustness within a single design framework. The state-dependent Riccati equation (SDRE) or state-dependent LQR (SDLQR) method is a promising approach for systematic design of nonlinear controllers. The method allows for extremely effective nonlinear feedback control designs by preserving nonlinearities in the system states and simultaneously offering the flexibility of utilizing design matrices to the control designer. The procedure of SDLQR is to factorize (or parameterize) the nonlinear dynamics into the product of a state dependent matrix valued function and the state vector. The factorization approach fully captures the nonlinearities of the system. It brings the nonlinear system to a nonunique linear structure with state dependent coefficient (SDC) matrices. The goal of the controller is to minimize a performance index which may or may not possess state dependent design matrices. The state weighing matrix, $Q(x)$, provides the control designer with the opportunity to specify each state with a performance penalty. The control weighing matrix, $R(x)$, allows the control system designer to penalize control effort. The SDLQR control synthesis approach, therefore, provides a useful framework for assessing design tradeoffs between control effort and acceptable state errors. If the nonuniqueness of the SDC parameterization is exploited, an additional benefit can be realized by the control systems engineer. For $n$-dimensional systems it has been shown by Cloutier et al. [27], that at least two SDC parameterizations exist. Give two valid SDC parameterizations, $A_1(x)$ and $A_2(x)$, the convex set parameterization $A(x, \alpha)$ can be formed by:

$$A(x, \alpha) = \alpha A_1(x) + (1 - \alpha) A_2(x)$$

The parameterization $A(x, \alpha)$ introduces extra degrees of freedom that are not available in traditional methods. The extra degrees of freedom can be exploited to enhance state response, avoid controller singularities such as loss of controllability and to affect tradeoffs between optimality, robustness and disturbance rejection. An algebraic Riccati equation (ARE) with state dependent coefficients is then solved online to construct the pointwise optimal stabilizing control law.

Because of its design flexibility and ease of application to a broad class of nonlinear problems, today there is rekindled interest among SDLQR investigators. A special
Convex Set Parameterization

\[
A(x, \alpha) = \alpha A_1(x) + (1 - \alpha) A_2(x)
\]

Figure 1.1: Convex Set Parameterization of \(A(x)\)

session at the 17th IFAC Symposium on Automatic Control in Aerospace in 2007 reviewed the existing theory and applications of SDLQR. SDLQR investigators then noted that experimental and practical applications of SDLQR have exceeded the rate of theoretical results. Since then, a survey of existing results has been presented in [6] along with a comprehensive overview on the application of SDLQR in [7]. After the special session, a flurry of papers detailing the application of SDLQR to unmanned aerial vehicles, satellites and spacecraft control, autonomous underwater vehicles, automotive systems, biomedical systems with control of HIV dynamics, process control, robotics and various benchmark problems exhibiting interesting phenomena such as friction and backlash, unstable nonminimum-phase dynamics, time-delay, vibration and chaotic behavior [6,7] have appeared in the literature.

Although the method was first proposed by Pearson in 1962 and later revisited by Wernli and Cook in the 70’s, it was not until the 1990’s when the SDLQR approach was given deeper consideration by investigators. Alluded to by Friedland in his 1996 book, Advanced Control System Design, the SDLQR approach was first systematically presented by Cloutier, D’Souza and Mracek in the same year at the International Conference on Nonlinear Problems in Aviation and Aerospace. There,
it was shown that for the scalar case \((n = 1)\) the SDLQR closed loop solution is \(\text{globally asymptotically stable}\). For the \(n\)-dimensional case Cloutier, D’Souza and Mracek showed that if \(A_{cl}(x)\) is symmetric \(\forall x\), then the SDLQR closed loop solution is \(\text{globally asymptotically stable}\).

Along these lines in 1999 Cloutier, Stansbery & Sznaier investigated the problem of \(\text{recoverability}\) of nonlinear state feedback control laws using \(\text{extended linearization}\). A condition was derived that would indicate the existence of a SDLQR like controller for a given nonlinear control law. In 2004, Erdem & Alleyne showed that the origin can be \(\text{globally asymptotically stabilized}\) by arbitrary, constant, positive choices for \(Q(x)\) and \(R(x)\) in second order systems. Additionally, for \(n > 1\), Langson & Alleyne showed in 2002 that SDLQR produces \(\text{globally asymptotically stability}\) for a class of nonlinear systems with specific growth conditions. Another significant stability result was advanced in 2003 by Shamma et al who showed that if there exists any stabilizing feedback leading to a Lyapunov function with star-convex level sets, then there always exists a representation of the dynamics such that the SDLQR approach is stabilizing.

Investigators have also examined estimating the \(\text{region of attraction (ROA)}\) for SDLQR feedback control laws. In 2001, McCaffrey & Banks proposed a method to estimate the size of the ROA for a proposed SDLQR controller. The method entails evaluating a well known inequality along the Hamiltonian of the dynamical system without solving for the value function. The inequality used in their technique has previously been used to show that feedback controls sufficiently close to the optimal global stabilizing feedback are indeed stabilizing in the same domain themselves. McCaffrey & Banks result is not confined by the assumption that the Lyapunov function corresponding to the optimal feedback be smooth. The smoothness requirement, which severely limits the ROA, is surmounted by applying a geometric construct from Day (1998) along with convexity measures. The resulting ROA estimate is closer to the true domain of attraction in contrast to the conservative estimates that arise when smooth Lyapunov assumptions are used [6]. Similarly, in 2002 Erdem et al proposed a method to estimate the ROA for SDLQR in a domain of interest by utilizing a psuedo-overvalued comparison system. Extending their result to systems with parametric uncertainty is one of the contributions of this thesis.

Investigators have advanced a number of results to quantify the optimality of SDLQR controllers as well. Huang & Lu in 1996 provided conditions for the value
1.1. BACKGROUND

Figure 1.2: Nonlinear control method spectrum spanning design flexibility, robustness and performance.

A number of results relating to the capabilities of SDLQR have also been put forth in the literature. Practical problems involve limit stops on inputs and states. The SDLQR method offers the capability to impose limit stops on the control, the control rate and the control acceleration, as was demonstrated in [27,38]. The ability to satisfy state constraints and combined state and control constraints was addressed by Friedland in [19] and Cloutier in [8]. Examples of controlling nonminimum phase systems are provided in [9,10,14,26,35,36]. An often under utilized capability of SDLQR is the design flexibility afforded by the nonunique selection of the SDC parameterization. In [11] a rule of thumb is presented which aids the control designer in selecting the “best” SDC parameterization. The approach attempts to maximize function, $V(x)$, that guarantee SDLQR control is the optimal feedback controller. This result indicates that the global optimal controller can always be found if the gradient of the cost function has a prescribed form and the appropriate SDC factorization is chosen. An additional result revealed by Huang & Lu in 1996 is that provided $\frac{\partial}{\partial x} P(x)x$ is a symmetric matrix then the SDLQR control is optimal. While this is an easy condition to meet for $n = 1$, in higher order systems it is generally non-trivial. Later in 1998, Mracek & Cloutier showed that the SDLQR method is locally asymptotically stable and locally asymptotically optimal for the multi-dimensional case ($n > 1$). Additionally, with an appropriate choice for the SDC parameterization the First Order necessary Conditions for optimality are satisfied by an SDLQR controller.
the pointwise controllable and observable spaces of all possible control factorizations. In relation to choosing the state and weighing criteria, Hammet in [22] proposed a sufficient condition for convexity of the state component of the performance index. The implication of a globally convex performance index in $x$ and $u$ is realized in that numerical optimization algorithms rely on the convexity of the Hamiltonian function.

Although a formal robustness analysis has eluded investigators, simulation results suggest that SDLQR controllers possess attractive robustness properties against parametric uncertainty, unmodeled dynamics and disturbance attenuation. These results are provided in [31, 32, 36–39]. Furthermore, Hammet in 1997 [22] and Cloutier in 1996 [27] have included disturbance terms using the SDLQR nonlinear $H_{\infty}$ control formulation.

When implemented as an integral servomechanism, the SDLQR controller can perform tracking as illustrated by Cloutier in [10] and Stansbery in [52]. If only partial state feedback is available, the SDLQR method can again offer a framework for comprehensive design. As shown by Banks et al in [23] an estimator-compensated system can be constructed in which the estimated state asymptotically converges to the true state.

When the SDC parameterization is not a $C^0$ function, $f(x)$ must be approximated by a $C^0$ function. This was first discussed by Hull in [42]. If there is a bias in the state dependent terms, the requirement that the origin be stable may be violated. This can compromise the $C^1$ factorization of $f(x)$. Cloutier et al in [11] have proposed methods to handle this also. In the case when the state of a system is uncontrollable and unstable the Riccati equation cannot be solved appropriately, even for a bounded state. Hull in [42] and Cloutier in [8] addressed this issue by adding a stabilizing term to the dynamics of the unstable state. Systems that are nonaffine in control inputs have also been controlled with SDLQR. In [8] and [12] such a system was brought into the required structure with the introduction of an integral controller.
1.2 Thesis Objectives and Original Contributions

The analysis of systems subject to parameter perturbations has been a fruitful area of active research in the control community for some time. The stability of such systems in the linear case has attracted much attention with notable results. In the context of nonlinear systems that are put into the pseudo-linear form of Extended Linearization, on the other hand, there are no results in the literature for obtaining the radius of stability. Roughly speaking, the radius of stability problem amounts to finding the largest $\rho > 0$ such that $\| \Delta \| < \rho$ implies $\sigma(A + D\Delta E) \in \mathbb{C}^-$ where $\mathbb{C}^-$ is the open left half plane, $A \in F^{n\times n}$, $D \in F^{n\times m}$, $E \in F^{p\times n}$ are given matrices and $\Delta \in \Delta \subset F^{m\times p}$ is an unknown perturbation matrix confined to a region of interest. More specifically, the radius of stability, $r_C^F$, of a matrix, $A$, subject to the structured perturbation $(D, E)$, is defined by

$$r_C^F(A, D, E) = \inf \{ \| \Delta \| : \Delta \in \Delta \subset F^{m\times p}, \sigma(A + D\Delta E) \cap \mathbb{C}^+ \neq \emptyset \}$$

where $\mathbb{C}^+$ is the compliment of $\mathbb{C}^-$ and $\| \cdot \|$ is a matrix norm of interest. In the general case, this problem is difficult to solve in that it entails a minimization problem followed by a frequency sweep. However, if the class of systems being considered is Metzlerian, i.e. the matrix $A$ possess non-positive diagonal elements and non-negative off diagonal elements, then an explicit formula exists for obtaining the upper bound of the radius of stability. This thesis presents two new results pertaining to computing the upper bound for the radius of stability in systems under Extended Linearization that possess Metzler comparison systems.

Since Extended Linearization relies on a non-unique factorization of the system dynamics, each new factorization can be used to analyze the same nonlinear system under perturbations. While this is often noted in the literature, no theoretical results have been presented to date that utilize the non-unique factorizations to analyze the radius of stability of a nonlinear system under Extended Linearization. As a first contribution, Section 6.3 of this thesis introduces a new method for obtaining the maximally robust factorization with respect to a class of nonlinear system under Extended Linearization. Specifically, it is shown that maximizing the radius of stability for a class of systems under Extended Linearization corresponds to minimizing the
constrained spectral norm of a parameterized comparison system. This result addresses the problem associated with finding representation parameter, $\alpha$, that results in the factorization which is maximally robust with respect to parametric uncertainty. Secondly, in Section 6.2 of this thesis it is shown that the actual radius of stability for systems under Extended Linearization may be obtained by computing the largest singular value for an overvalued Metzler comparison system in a domain of interest. This result addresses the problem of obtaining the actual radius of stability.

For completeness, a survey of the current results pertaining to SDLQR and Extended Linearization is provided in Chapters 3, 4 and 5. As mentioned earlier in the chapter, at a special session of the IFAC Symposium on Automatic Control in Aerospace in 2007, the current theory of Extended Linearization and SDLQR was reviewed. Investigators noted that experimental and practical applications of SDLQR are also growing. Since then a flurry of papers has appeared in the literature pertaining to Extended Linearization and State-dependent control in general. Chapters 4 and 5 of this thesis provide a survey of all the existing results pertaining to SDLQR stability and optimality. Chapter 3 introduced the Extended Linearization formalism and shows how Extended Linearization provides the investigator with additional design degrees of freedom through the use of non-unique factorizations of the system dynamics.

Over the years, investigators have provided many results for global asymptotic stability of systems under Extended Linearization. While their work has been fruitful, it should be noted that global asymptotic stability of systems under Extended Linearization or SDLQR is still an open problem. This thesis will not directly address the problem of global asymptotic stability of systems under Extended Linearization. It should be noted that existing Lyapunov stability methods are equipped to address this theoretically challenging area and that existing local stability results for SDLQR have proven sufficient for practical designs. Furthermore, to date, no valid counter examples to SDLQR have been discovered to show that it is destabilizing.
Chapter 2

Preliminary Theory

The intention of this chapter is to present and elaborate on the theory of stability analysis and control synthesis of nonlinear systems. To this end, section 2.1 reviews Lyapunov concepts that provide a framework for assessing stability in nonlinear dynamical system. The material in that section has been adapted from [29] and [18]. After a deep grounding in these fundamentals, a brief review of optimal control is presented in section 2.2. The aim of this section is to present the theoretical cornerstones of optimal control in a concise fashion. Important concepts such as the Hamilton-Jacobi-Bellman (H-J-B) equation of dynamic programming and the Minimum Principle of Pontryagin (P-M-P) from the calculus of variations are presented. The aim here is to develop an understanding and an intuition for the “structures” of optimal controllers. This section serves yet another purpose; namely that the limitations of the dynamic programming solution and the calculus of variations solution reveal themselves. Hence, it could be said that the main value of this section is to motivate our study of the SDLQR methodology by examining the limitations of the existing control theoretic approaches.

2.1 Nonlinear System Analysis

This section considers results pertaining to the stability of equilibrium points. The basic differential equation considered is

\[ \dot{x} = f(x) + g(x)u, \quad f(0) = 0 \]  

(2.1)
where \( x(t) \in \mathbb{R}^n \) represent the state vector and \( u \in \mathbb{R}^m \) denotes the control. The mappings \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are continuous and locally Lipschitz in some domain \( D \subset \mathbb{R}^n \). If the equilibrium point in question is not the origin of the system, that is \( f(0) \neq 0 \), then the equilibrium point can be shifted to the origin without loss of generality. An equilibrium point is stable if all the solutions within a known neighborhood around it stay within that neighborhood. More specifically,

**Definition 2.1.** (Stability) The equilibrium point of Eqn (2.1) at \( x = 0 \) is

- **stable** if for any \( \epsilon > 0 \), there is \( \delta = \delta(\epsilon) > 0 \) such that
  \[
  ||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon, \quad \forall t \geq 0
  \]

- **asymptotically stable** if it is stable and \( \delta \) can be chosen to satisfy
  \[
  ||x(0)|| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0
  \]

- **and unstable** if not stable.

The solution, \( x(t) \), to the differential equation in Eqn (2.1) is needed for \( t \geq 0 \) in order to establish stability according to definition 2.1. Such a solution may be difficult to obtain for some differential equations. For this reason the stability of equilibrium points in nonlinear systems is primarily characterized according to methods derived from Lyapunov’s second method (referred to as “Lyapunov methods” in some contexts). In the next section, notions related to Lyapunov’s basic stability theorems are given.

**Lyapunov Stability**

Lyapunov’s second method is used to determine if the equilibrium points of Eqn (2.1) are stable without solving a (possibly) nonlinear differential equation. The method is based on the observation that if the rate of change of energy in a system is nonpositive along all its trajectories, then points originating near an equilibrium point will tend to stay in some known neighborhood around that equilibrium point. Lyapunov showed that any function besides the energy function of a system can be used to make this
determination as long as it satisfies two criteria. A candidate Lyapunov Function satisfying the two criteria is known as a Lyapunov Function and is endowed with notable properties. More precisely,

**Definition 2.2. (Lyapunov Function)** Let \( x = 0 \) be an equilibrium point of Eqn (2.1) and let \( D \subset \mathbb{R}^n \) be a domain containing \( x = 0 \). A \( C^1 \) mapping, \( V : D \to \mathbb{R} \), that satisfies

- **Condition 2.1.** \( V(0) = 0 \)
- **Condition 2.2.** \( V(x) > 0 \) \( \forall x \in D - \{0\} \)

is known as a Lyapunov Function.

A Lyapunov Function is a positive definite function when it satisfies both conditions (2.1) and (2.2). If the Lyapunov function satisfies only \( V(x) \geq 0 \) for \( x \neq 0 \) then it is said to be positive semidefinite. When \(-V(x)\) is positive definite or positive semidefinite then \( V(x) \) is said to be negative definite or negative semidefinite respectively. If \( V(x) \) is sign indeterminate then nothing can be inferred about the stability of the origin and the Lyapunov function is said to be sign indefinite.

When a candidate Lyapunov function is found that satisfies the conditions in definition (2.2), then it can be used to establish the stability or asymptotic stability of Eqn (2.1) with Lyapunov’s Second Method

**Theorem 2.1. (Lyapunov’s Second Method)** Let the origin be an equilibrium point of Eqn (2.1) and let \( D \subset \mathbb{R}^n \) be a domain containing the origin. Then the origin is stable if \( \exists V(x) \in C^1 \) such that \( \dot{V}(x) \) is negative semidefinite

\[ \dot{V}(x) \leq 0 \ \forall x \in D \]

and asymptotically stable if \( \exists V(x) \in C^1 \) such that \( \dot{V}(x) \) is negative definite

\[ \dot{V}(x) < 0 \ \forall x \in D \]
Table 2.1: Lyapunov Function Classification

<table>
<thead>
<tr>
<th>Lyapunov Function Outcomes</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive definite</td>
<td>$V(x) &gt; 0$</td>
</tr>
<tr>
<td>Positive semidefinite</td>
<td>$V(x) \geq 0$</td>
</tr>
<tr>
<td>Negative definite</td>
<td>$-V(x) &gt; 0$</td>
</tr>
<tr>
<td>Negative semidefinite</td>
<td>$-V(x) \geq 0$</td>
</tr>
<tr>
<td>Stability</td>
<td>$\dot{V}(x) \leq 0$</td>
</tr>
<tr>
<td>Asymptotic stability</td>
<td>$\dot{V}(x) &lt; 0$</td>
</tr>
</tbody>
</table>

Remark 2.1. Lyapunov’s stability theorem establishes the sufficient conditions for stability. Failure of a candidate Lyapunov function to satisfy the conditions for stability or asymptotic stability implies only that the stability property cannot be established using Lyaponuv methods. The stability of the equilibrium point can still be determined using other approaches.

2.2 Nonlinear Optimal Control

As pointed out by Sussmann in [53], the problem of optimal control first presented itself in a minimum time application in 1696 in the Netherlands as Johann Bernoulli challenged his colleagues with the following question: Given two points $A$ and $B$ constrained to the $xy$ plane, find the orbit $ACB$ of a free point $C$ that, when starting from $A$ and while acting on its own weight, terminates at $B$ in the shortest interval of time. Students of optimal control will recognize the problem posed by Johann Bernoulli as the branchystochrone problem.

Today’s optimal control formulations began with the problems considered by the American mathematician Richard Bellman and the Soviet mathematician Lev Pontryagin. The era of modern optimal control was conceived in the atmosphere of mutual distrust and competition; particularly during the space race. The symbolism of the polar nature of optimal control with one end being dominated by dynamic programming solutions and the other end dominated by solutions given by Portraying’s minimum principle is intriguing. Interestingly, Pontryagin’s minimum principle represents a generalization of the Euler-Lagrange equations from the classical calculus of variations. As pointed out by [41], this generalization may be viewed as an outgrowth
of the Hamiltonian approach to variational problems. The method of dynamic programming on the other hand can be viewed as an outgrowth of the Hamilton-Jacobi approach to variational problems. When viewed through this lens, many nonlinear optimal control approaches can be traced back to either “poles” of the existing design frameworks. This connection provides an easy way to identify inherited advantages and disadvantages.

The optimization problem to be considered in this thesis is the autonomous, infinite horizon, nonlinear regulator problem with a nonlinear differential constraint affine in control. Stated mathematically, we wish to minimize the cost function

$$J(x_0, u) = \frac{1}{2} \int_{t_0}^{\infty} (x'Q(x)x + u'R(x)u)dt$$

(2.2)

Subject to the differential constraint

$$\dot{x} = f(x) + g(x)u, \ x(0) = x_0$$

(2.3)

where $x(t) \in \mathbb{R}^n$ represents the state vector and $u \in \mathbb{R}^m$ denotes the control. Both $f(x)$ and $g(x)$ are $C^k$ functions with the mappings $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$. $Q$ is a positive semi-definite state weighing matrix, possibly state dependent, with the property that it is a $C^1$ function. The variable $R$ represents a scalar penalty associated with the control effort and is referred to as the control weighing coefficient. We assume that the system dynamics in Eq (2.3) are zero-state detectable. It should be mentioned that speaking strictly in mathematical terms, Eq (2.2) should be referred to as a cost functional. However, we chose to refrain from the strict mathematical parlance for convenience and readability.

Let the value function $V$ be defined as the greatest lower bound of the cost function in Eq (2.2). Then the dynamic mathematical optimization problem of Eq (2.2-2.3) can be written compactly as
and the desired solution is the optimal control law $u^*$. In the ensuing sections we discuss the dynamic programming and the Pontryagin Minimum Principle solution to the dynamic optimization problem presented in Eq (2.2-2.3).
Dynamic Programming Solution

Dynamic programming is a method pioneered by the American mathematician Richard Bellman [18]. The central concept of Bellman’s dynamic programming is the principle of optimality.

**Definition 2.3.** (Principle of Optimality) Let \( u^*(t) \) be the optimal control in the interval \([t_0, t_f]\). Then \( u^*(t) \) is also optimal over the subinterval \([t_0 + \Delta t, t_f]\) for any \( \Delta t \) such that \( t_0 \leq \Delta t \leq t_f \).

Figure (2.1) provides a visual illustration of this definition. The cost for the optimal path, \( J_{AD} \), is composed of the cost of to traverse \( AB \) and then \( BD \)

\[
J_{AD} = J_{AB} + J_{BD}
\]

By the principle of optimality, the costs \( J_{AB} \) and \( J_{BD} \) are minimal costs associated with traversing \( AB \) and \( BD \) respectively.

The aim of dynamic programming is to use the principle of optimality to decompose the dynamic optimization problem presented in Eqns (2.4-2.5) into a recurrence relation. Recurrence relations have the property that their subproblems possess the same structure as the previous subproblems. Mathematically, the dynamic programming solution to the nonlinear optimal control problem entails solving a partial differential equation known as the Hamilton-Jacobi-Bellman equation. To derive the solution in the most general terms, consider the cost function in Eqn (2.4), with a general instantaneous cost denoted by \( L(x, u) \) (modified from [20])

\[
V = \int_{t}^{T} L(x, u) d\tau,
\]

(2.6)

for this performance measure, the general process dynamics are

\[
\dot{x} = \frac{dx}{dt} = f(x, u)
\]

(2.7)

Using the chain rule
Figure 2.1: Optimal control $u^*$ leading to the optimal cost from $A$ to $D$, $J_{AD}$. Note that by the principle of optimality, both $J_{AB}$ and $J_{BD}$ are optimal costs.

\[
\dot{V} = V_x \frac{dx}{dt} = \frac{\partial V'}{\partial x} f(x, u) \tag{2.8}
\]

From the performance integral Eqn (2.6)

\[
\dot{V} = -L(x, u) \tag{2.9}
\]

Equating both expressions for $\dot{V}$, we obtain the partial differential equation:

\[
H = \frac{\partial V'}{\partial x} f(x, u) + L(x, u) = 0 \tag{2.10}
\]

Equation (2.10) is referred to as the Hamiltonian or H-function. The optimal control is:

\[
u^* = \arg\min_{u \in \Omega} H(V_x, x, u) \tag{2.11}\]

where $\Omega$ is the feasible control space. In the special case where Eqn (2.7) is affine in control and Eqn (2.6) is quadratic in state and control, the Hamiltonian Eqn (2.10) becomes

\[
H = \frac{\partial V'}{\partial x} (f(x) + g(x)u) + \frac{1}{2} [x'Q(x)x + u'R(x)u] \tag{2.12}
\]
Upon performing the minimization in Eqn (2.11), the solution to the mathematical optimization problem in Eqn (2.4-2.5) is

$$ u^* = \arg\min_u \left( \frac{\partial V'}{\partial x}(f(x) + g(x)u) + \left[ x^T Q(x)x + u^T R(x)u \right] \right) \quad (2.13) $$

After simplification

$$ u^* = -R^{-1}g(x)^{\prime}\frac{\partial V'}{\partial x} \quad (2.14) $$

Note that Eqn (2.13) is just another way to write the HJBE:

$$ \frac{\partial V}{\partial x}f(x) - \frac{1}{2} \frac{\partial V}{\partial x}g(x)R^{-1}g(x)^{\prime}\frac{\partial V}{\partial x} + x^T Q(x)x = 0 \quad (2.15) $$

The more common representation of the HJBE shown in Eqn (2.15) arises by substituting Eqn (2.14) into Eqn (2.13). In practice, Eqn (2.15) is solved for $V$, which is then substituted into Eqn (2.14) to yield the optimal control $u^*$. The optimal control is then applied to Eqn (2.7) and then integrated to produce the optimal trajectory of the state, $x$. The role of the Value Function, $V$, is interesting: for any given $x_0$, $V^*(t, x)$ gives the minimum cost among all feasible trajectories in the state space[13].

Eqn (2.15) is a first order, nonlinear partial differential equation which is difficult to solve analytically. There are several areas of research directed at approximating the solution to Eqn (2.15) [43]:

- **Method of Characteristics**

  In the method of characteristics, the approach is to forward integrate the problem and find the solution for one initial condition. This open loop strategy is sometimes used to train neural networks, providing a more comprehensive control framework [30] [45]. The main criticism against this approach is that it entails storing the open loop trajectory in memory to be used by a look up table. As one can imagine, there may be robustness issues also.

- **Series Approximation**
For a Series Approximation (SA) solution to the HJBE, one assumes that the systems equations are "analytic in the state". This assumption allows one to form a Taylor series expansion for the value function. The approximation is truncated after a suitable number of terms [13, 21]. The trouble with this approach is exactly this truncation: finite truncation can be the root cause of instability in the implementation. The Taylor series approximation also compromises the stability region of the system.

- **Regularization**

  - Regularization is just the art of adding terms to the cost function to "cancel" out the nonlinearities in the system dynamics [17, 33, 44, 47]. This approach is not suitable because the original control problem is not being addressed anymore.

When the system dynamics are linear and the state penalty function is quadratic, then the HJBE reduces to the standard differential Riccati equation of optimal control. Furthermore, if the upper limit on the cost integral in Eqn (2.6) is infinity, then the control problem being considered is the infinite horizon optimal control problem with quadratic state and control penalty functions which means the differential Riccati equation is further reduced to the steady state or algebraic Riccati equation.

**Dynamic Programming Solution Characteristics**

The important characteristics of the Dynamic Programming solution are summarized below:

**Property 2.1.** *(Global Minimum)* The solution obtained by the Dynamic Programming approach is the global minimum of the cost function, $J(x, u)$, for all $x$ in $\mathbb{R}^n$:

$$J(x, u^*) \leq J(x, u), \; \forall x \in \mathbb{R}^n$$

**Property 2.2.** *(Feedback Control Law)* The solution obtained by the Dynamic Programming approach can be implemented in feedback form.

**Property 2.3.** *(Curse of Dimensionality)* Let the dimensionality of $x$ be denoted by $\dim(x)$, the dimensionality of $u$ by $\dim(u)$ and the resolution of $x$ and $u$ by $R_x$ and
2.2. NONLINEAR OPTIMAL CONTROL

$R_u$ respectively. Then the computational cost of the Dynamic Programming approach is proportional to

$$R_x^{\dim(x)} \times R_u^{\dim(u)}$$

and the memory cost is proportional to

$$R_x^{\dim(x)}$$

This is known as the “curse of dimensionality”.

Pontryagin Minimum Principle

The HJB equation entails the gradient of $V$, and hence requires differentiability of $V$. The standard HJB equation solution approach is by the method of characteristics. Let:

$$p = -V \quad \text{(the “costate”) (2.16)}$$

Then Eqn. (2.10) becomes:

$$H = p^t f(x, u) - L(x, u) = 0 \quad (2.17)$$

The Pontryagin Minimum Principle (PMP), derived by methods of the calculus of variations, can be derived by considering the Hamiltonian function in Eqn. (2.17), without identifying the costate $p$ with $-V$. This eliminates the requirement that $V$ is differentiable. The relationship between the Dynamic Programming approach in Eqn. (2.10) and the Minimum Principle approach in Eqn. (2.17) is proven in [30].

Continuing with the Pontryagin formulation, let $u(\tau)^*$ be the optimal control in the sense that the value of $V$ using $u^*$ is smaller than it is for any other control signal $u(\tau)$. According to the Minimum Principle, the necessary conditions for the control signal $\{u^*(\tau) : \tau \in [t, T]\}$ to be optimal are:

1. The H-function in Eqn. (2.17) has an absolute minimum at $u(\tau) = u^*(\tau)$:

$$H(x^*, p^*, u^*) \leq H(x^*, p^*, u) \quad (2.18)$$
2. The value of the Hamiltonian function \( H \) at the terminal time \( t = T \) is zero:

\[
H(x^*, p^*, u^*) = 0 \tag{2.19}
\]

These observations lead to the dynamic and costate canonical representation [2,18,30]:

\[
\dot{x} = \frac{\partial H}{\partial p}(x, p, u) = f(x, u) \tag{2.20}
\]

\[
\dot{p} = -\frac{\partial H}{\partial x}(x, p, u) = -\frac{\partial L}{\partial x} - pf'(x, u) \tag{2.21}
\]

If Eqn. (2.7) is Linear Time Invariant (LTI) and the integrand of Eqn. (2.6) possesses a form that is quadratic in state and control with an infinite horizon time interval, i.e.,

\[
V = \int_t^\infty (x'Qx + u'Ru) d\tau,
\]

then Eqns. (2.20) and (2.21) are fashioned by:

\[
\dot{x} = \frac{\partial H}{\partial p}(x, p, u) = Ax^* + Bu^* \tag{2.22}
\]

\[
\dot{p} = -\frac{\partial H}{\partial x}(x, p, u) = -Qx^* - A'p \tag{2.23}
\]

\[
0 = \frac{\partial H}{\partial u}(p, u) = Ru^* + Bp^* \tag{2.24}
\]

where \( Q \) is an \( n \times n \) positive definite matrix known as the state weighting matrix, and \( R_u \) is a scalar known as the control weighting factor [30]. \( A \) and \( B \) are the LTI system matrices. Solving (2.24) for \( u^* \) and substituting into (2.22) yields a set of \( 2n \) linear homogenous differential equations:

\[
\begin{bmatrix}
\dot{x}^* \\
\dot{p}^*
\end{bmatrix} =
\begin{bmatrix}
A & -BR^{-1}B' \\
-Q & -A'
\end{bmatrix}
\begin{bmatrix}
x^* \\
p^*
\end{bmatrix} \tag{2.25}
\]

This system of differential equations can be solved for the optimal state trajectory. Using the boundary conditions, \( p^* \) can be eliminated and the optimal control \( u^* \) is:

\[
u^* = -R^{-1}B'Mx \tag{2.26}
\]
where $M$ is the solution to the matrix Riccati equation:

$$
\dot{M} = -MA - A'M - Q + MBR^{-1}B'M
$$

(2.27)

Note the dependence of $M$ on $t$ has been removed for convenience. $M$ is an $n \times n$ symmetric matrix; therefore $n(n+1)/2$ first-order differential equations must be solved to find $M$.

### Characteristics of the PMP Solution

**Property 2.4. (Open Loop)** The resulting optimal trajectory is explicitly solved for as a function of time $u(t)$, not as a feedback law.

**Property 2.5. (Local)** The resulting solution is only valid for the specified initial condition $x(0)$. If a new initial condition is specified, the problem must be solved again.

**Property 2.6. (Necessary)** The PMP equations are first order differential equations that specify the conditions for the existence of a stationary point. They represent necessary conditions for an optimal trajectory.

### Summary of Dynamic Optimization

The two basic paradigms of optimal control have been introduced. We have attempted to highlight the differences in each approach in order to distinguish the salient properties of their solutions. As noted earlier, the *dynamic programming* approach leads to a first order partial differential equation known as the Hamilton-Jacobi-Bellman equation. It provides a global control law that is suitable for feedback implementation. The *dynamic programming* approach is generally computationally intractable in that it entails the solution of the HJB equation.

The solution proposed by Pontryagin only requires the solution to a two-point boundary value ordinary differential equation. While this is a more computationally tractable problem to solve, the Pontryagin solution is not equivalent to the solution presented by Bellman. Instead, the PMP solves a trajectory optimization problem, i.e., it provides an open loop trajectory corresponding to a specified initial condition. Therefore, computational tractability is traded for the lack of a global solution.
These approaches are depicted in Figure (2.2). It is hoped that these two viewpoints toward the optimal control problem provide the background and context that is needed to interpret the SDLQR approach.
Chapter 3

State-Dependent Formulations

Time domain analysis and controller design for nonlinear systems that depend on coefficients which are pointwise functions of the system states are collectively known as state-dependent techniques. This chapter and the trilogy of chapters that follow consider state-dependent techniques. First a survey of existing results pertaining to state regulation or control is presented. Motivated by a real world application, this chapter opens by formulating the optimal nonlinear regulation problem for state-dependent techniques. The elegant theory of extended linearization, which encompasses an entire class of nonlinear control methods, is then presented. State-Dependent Coefficient (SDC) representations, which introduce additional degrees of freedom to the extended linearization method of control, are also examined. It will be seen that additional degrees of freedom provide performance and robustness tradeoffs to the designer.

Under extended linearization, the State-Dependent Riccati Equation (SDRE), also known as the State-Dependent LQR (SDLQR) controller, is developed. Through the non-uniqueness of the SDC matrix $A(x)$, controller performance or robustness can be enhanced, affording the designer extra degrees of freedom. This topic considered throughout the trilogy of chapters.

In the following chapter, theoretical aspects of SDLQR control such as existence of solutions, stability analysis and optimality analysis are considered. After this discussion, a chapter is devoted to stating the optimality properties of State Dependent Techniques.

In the final chapter of the trilogy, the Robust Extended Linearization problem is considered. A novel technique for computing the radius of stability of an SDLQR
system under parametric uncertainty is presented. An unique property of extended linearization is then exploited to optimize the radius of stability in order to obtain the maximally robust SDC parameterization.

Partly due to its ease of application and design flexibility, SDLQR has enjoyed widespread success in a number of applications. It has been utilized to explore optimal treatment modalities for HIV & AIDS, Chemotherapy administration, Missile Guidance, Satellite & Spacecraft design and to control the growing film thickness in high pressure chemical vapor deposition (HPCVD). Furthermore, SDLQR has been applied to the design of advanced flight control systems, utilized in automotive control systems and has applications in process control.

To motivate the state-dependent formulation, let us consider a typical aerospace application. Suppose the task is to precisely control an artifact traveling with high velocity in space. After examining the system, we discover the following plant characteristics:

- Nonlinear dynamics
- Limit-stops on the states and inputs
- Zero Dynamics of the plant are unstable
  - Implication: zero dynamics are nonminimum phase and complications will arise if feedback linearization is used.

A number of controller options exist, but each has its own drawbacks:

- Linearization with Integral Action
- Gain Scheduling
- P, PI, PID variants
- Feedback Linearization or Dynamic Inversion
  - Difficult to apply due to the presence of unstable nonminimum phase zero dynamics
- Sliding mode or some other variable structure control
• A CLF approach
• Backstepping
• Receding Horizon
• Pointwise Min-Norm
• Bang Bang Control

Each of the techniques listed above provide the control designer with its own flavor of advantages and disadvantages. One advantage of SDLQR is that it simultaneously offers the capability to address controller performance and overall system robustness. In SDLQR, system performance is addressed by approximating the HJB control law

\[ u^* = -R^{-1}g(x)'\frac{\partial V'}{\partial x} \]  \hspace{1cm} (3.1)

which requires the solution to the HJB equation

\[ \frac{\partial V}{\partial x} f(x) - \frac{1}{2} \frac{\partial V}{\partial x} g(x)R^{-1}g(x)'\frac{\partial V'}{\partial x} + x^TQ(x)x = 0 \]  \hspace{1cm} (3.2)

by the SDLQR control law

\[ u_{SDLQR} = -R^{-1}(x)B^T(x)S(x)x \]  \hspace{1cm} (3.3)

which only requires the online solution to a state dependent Matrix Riccati equation (SDMRE)

\[ S(x)A(x) + A^T(x)S(x) - S(x)B(x)R^{-1}(x)B^T(x)S(x) + Q(x) = 0 \]  \hspace{1cm} (3.4)

The HJBE is a partial differential equation. To obtain \( u^* \), the HJBE must be solved. This task is all but impossible given current technology. A computationally tractable solution, such as the one offered by SDLQR, is valuable because it approximates the HJBE solution locally by solving a SDMRE pointwise. Loosely speaking, the assumption is that in a local neighborhood of the solution, the HJBE and the SDMRE solution are roughly the same. This results in a suboptimal SDLQR control law, but the advantage of computational tractability is gained.
3.1 Problem Definition

Consider the autonomous, nonlinear regulator problem that is affine in input

\[ \dot{x} = f(x) + g(x)u. \]  

(3.5)

Here \( x \in \mathbb{R}^n \) represents the state vector and \( u \in \mathbb{R}^m \) denotes the control. Both \( f(x) \) and \( g(x) \) are \( C^k \) functions with the mappings \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m} \). We intend to find the control, \( u^* \), that minimizes an infinite horizon performance criterion with a convex integrand

\[ J(x) = \frac{1}{2} \int_{t_0}^{\infty} (x^T Q(x)x + u^T R(x)u) \, dt \]  

(3.6)

The matrix \( Q : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \) is a \( C^1 \) positive semi-definite state weighing matrix, possibly state dependent. The matrix \( R : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \) represents a penalty associated with the control effort. \( Q \) and \( R \) are the state and input design parameters which together provide the designer flexibility in assessing performance of the closed loop system.

Let \( \Omega \) be a bounded and open subset of \( \mathbb{R}^n \), \( \Omega \subset \mathbb{R}^n \), and let the origin of \( f(x) \) and the initial point \( x_0 \) be contained in \( \Omega \)

\[ x_0 \in \Omega \subset \mathbb{R}^n + \{0\}. \]

If \( f(x) \) and \( g(x) \) are matrix valued functions in \( \Omega \), then a stabilizing feedback control law in the form

\[ u(x) = -k(x)x \]  

(3.7)

is desired where \( k(\cdot) \in C^1(\Omega) \). This defines the control problem being considered.

3.2 Extended Linearization

There are various methods for analysis and design of nonlinear systems [29]. In the method of extended linearization [18], also known as State-Dependent-Coefficient (SDC) representation, Eq (3.5) is factored to bring it into a linear-like structure with state-dependent matrices. A continuous, nonlinear matrix-valued function, \( A : \Omega \rightarrow \)
3.2. EXTENDED LINEARIZATION

\( \mathbb{R}^{n \times n} \), can always be obtained by mathematical factorization if \( f(x) \) satisfies

**Condition 3.1.** The origin \( x = 0 \in \Omega \) is an equilibrium point of the system such that \( f(0) = 0 \).

**Condition 3.2.** The vector valued function, \( f(x) \), is at least continuously differentiable over \( \Omega \), that is

\[ f(\cdot) \in C^1(\Omega) \]

Under these conditions, \( f(x) \) can be factored as

\[ f(x) = A(x)x \quad (3.8) \]

**Definition 3.1 (SDC Matrix).** In the matrix valued representation of Eqn (3.8), the matrix \( A(x) \), which satisfies conditions (3.1) and (3.2), is known as the *SDC matrix*.

Depending on the system, a number of valid SDC matrices may exist. Because the SDC matrix, \( A(x) \), is non-unique the performance of the controller may vary. Later, it will be seen that this non-uniqueness of the SDC matrix is one advantage of SDLQR control. Unfortunately, in some circumstances it may be difficult to obtain a set of valid SDC matrices by factorization. For this reason, one way that \( f(x) \) can be recast into the form shown in Eqn (3.8) follows.

**Theorem 3.1 (Bass Factorization).** Let \( f : \Omega \rightarrow \mathbb{R}^n \) be a mapping that satisfies conditions (3.1) and (3.2) so that \( A : \Omega \rightarrow \mathbb{R}^{n \times n} \) exists for all \( x \in \Omega \). Then, an SDC matrix is given by

\[ A(x) = \int_0^1 \frac{\partial f_x(\lambda)}{\partial x} d\lambda d\lambda \quad (3.9) \]

where \( \lambda \in \mathbb{R} \) is a dummy variable.

**Proof.** Consider the collection of functions

\[ \{ f_x : \mathbb{R} \rightarrow \mathbb{R}^n \} \]

where \( f_x(\lambda) = f(\lambda x) \). For each \( x \in \Omega \), we have

\[ f(x) = f_x(1) = f_x(0) + \int_0^1 \left( \frac{df_x(\lambda)}{d\lambda} \right) d\lambda \]
By condition (3.1) we have \( f_x(0) = 0 \). Noting that \( \left( \frac{\partial f_x(\lambda)}{\partial \lambda} \right) = \left. \frac{\partial f}{\partial x} \right|_{x=\lambda} \) is the integrand of Eqn (3.9) completes the proof.

\[ \square \]

**Example 3.1.** Suppose

\[
f(x) = \begin{bmatrix} \sin(x_1 + x_2) \\ \sin(x_2) \end{bmatrix}
\]

Then

\[
\left. \frac{\partial f}{\partial x} \right|_{x=\lambda} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \big|_{x_1=\lambda} & \frac{\partial f_1}{\partial x_2} \big|_{x_2=\lambda} \\ \frac{\partial f_2}{\partial x_1} \big|_{x_1=\lambda} & \frac{\partial f_2}{\partial x_2} \big|_{x_2=\lambda} \end{bmatrix} = \begin{bmatrix} \cos(\lambda x_1 + \lambda x_2) & \cos(\lambda x_1 + \lambda x_2) \\ 0 & \cos(\lambda x_2) \end{bmatrix}
\]

and

\[
A(x) = \int_0^1 \begin{bmatrix} \cos(\lambda x_1 + \lambda x_2) & \cos(\lambda x_1 + \lambda x_2) \\ 0 & \cos(\lambda x_2) \end{bmatrix} d\lambda = \begin{bmatrix} \frac{\sin(x_1+x_2)}{x_1+x_2} & \frac{\sin(x_1+x_2)}{x_1+x_2} \\ \frac{\sin(x_2)}{x_2} & 0 \end{bmatrix}
\]

\[ \square \]

If more than one SDC matrix can be obtained from Eqn (3.5), it is beneficial to use the factorization that maximizes some meaningful metric, i.e., the controllability space of the problem or the radius of stability under parametric uncertainty. It has been shown [27] that for the multivariable case there always exists an infinite number of SDC matrices. By introducing the representation parameter \( \alpha \), every valid SDC matrix can be collectively represented by a hypersurface. The resulting SDC matrix is referred to as the parameterized SDC matrix.

**Theorem 3.2** (Parameterized SDC Matrix). Let \( f(x) = A_i(x)x, \ i = 1 \ldots k \) be \( k + 1 \) distinct SDC matrices \( \forall x \in \Omega \) and let \( \alpha \in \mathbb{R} \) be a real scalar parameter such that \( \alpha \in [0, 1] \). Then the parameterized SDC matrix, \( A(x, \alpha) \), corresponding to the continuum of SDC matrices on a hypersurface is given by

\[
A(x, \alpha) = (1 - \alpha_k)A_{k+1}(x) + \sum_{i=1}^{k} \left( \prod_{j=i}^{k} \alpha_j \right) (1 - \alpha_{i-1})A_i(x)
\]

(3.10)

where \( \alpha_0 \triangleq 0 \).
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Proof. This result is shown for any two SDC matrices $A_1(x)$ and $A_2(x)$. The general result of Eqn (3.10) follows immediately.

In the multivariable case the state vector $x$ has, by definition, at least two components $x_1$ and $x_2$. Suppose there exists a nonlinear scalar term $f_i(x)$ in one of the state equations. Then in that state equation at least two SDC matrices can be found corresponding to $f_i(x)/x_1$ and $f_i(x)/x_2$. Using any two valid SDC matrices $A_1(x)$ and $A_2(x)$, an infinite number of SDC matrices can be obtained from the convex set

$$A(x, \alpha) = \alpha A_1(x) + (1 - \alpha) A_2(x), \quad \alpha \in [0, 1] \quad (3.11)$$

In general, for $k + 1$ valid SDC matrices, $\alpha$ will be of dimension $k$ and $A(x, \alpha)$ will be the hypersurface of Eqn (3.10).

Remark 3.1. $A(x, \alpha)$ is a hypersurface since it is an $k$ dimensional surface embedded in an $k + 1$-dimension space. Therefore, given $k + 1$ SDC matrices, $k$ representation parameters $\alpha = [\alpha_0 \ldots \alpha_k]$ are needed.

Example 3.2. Suppose the two SDC matrices are $A_1(x)$ and $A_2(x)$. Then $k = 1$. The parameterized SDC matrix, $A(x, \alpha)$, is

$$A(x, \alpha) = (1 - \alpha_1) A_2(x) + \alpha_1 (1 - \alpha_0) A_1(x)$$

Upon letting $\alpha_0 = 0$

$$A(x, \alpha) = (1 - \alpha_1) A_2(x) + \alpha_1 A_1(x)$$

Note that this parameterization corresponds to a convex set with a line segment joining $A_1(x)$ to $A_2(x)$ as $\alpha$ is varied.

Example 3.3. Suppose there are three SDC matrices: $A_1(x)$, $A_2(x)$, and $A_3(x)$. Then $k = 2$ and $\alpha = [\alpha_0, \alpha_1, \alpha_2]$. The parameterized SDC matrix, $A(x, \alpha)$, obtained by using Eqn (3.10) is

$$A(x, \alpha) = (1 - \alpha_2) A_3(x) + \alpha_2 (1 - \alpha_3) A_2(x) + \alpha_1 A_1(x)$$
Figure 3.1: The surface produced by the parameterized SDC matric $A(x, \alpha)$ in example 3.3

The surface formed by $A(x, \alpha)$ is a polytope shown in figure 3.1 with the vertices

\[
\begin{align*}
\{ A_3(x) : \alpha_1 = \alpha_2 = 0 \} \\
\{ A_2(x) : \alpha_1 = 0, \alpha_2 = 1 \} \\
\{ A_1(x) : \alpha_1 = 1, \alpha_2 = 1 \} \\
\{ A_1(x) + A_3(x) : \alpha_1 = 1, \alpha_2 = 0 \}
\end{align*}
\]

By using extended linearization the system in Eq (3.5) can be represented in a pseudo-linear form with SDC matrices $A(x)$ and $B(x)$. The following definition makes this notion precise.

**Definition 3.2** (Parameterized SDC Representation). For the nonlinear system

\[ \dot{x} = f(x) + g(x)u \]

with parameterized SDC matrices $A(x, \alpha)$ and $B(x)$, the *parameterized SDC representation* is given by

\[ \dot{x} = A(x, \alpha)x + B(x)u \]  \hspace{1cm} (3.12)
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Under SDC representation, conventional linear control synthesis methods can be applied pointwise to stabilize the system in Eqn (3.5). Applying any linear control synthesis method to the system in Eqn (3.12) pointwise ultimately produces a closed-loop system matrix

\[
A(x)_{cl} = A(x) + B(x)K(x)
\]

that is pointwise Hurwitz \(\forall x \in \Omega\).

To apply linear control synthesis methods to Eqn (3.12), pointwise controllability and stabilizability conditions must be satisfied.

**Definition 3.3.** The SDC representation (3.12) is a stabilizable (respectively controllable) parameterization of the nonlinear system (3.5) in the region \(\Omega\) if the pair \(\{A(x), B(x)\}\) is pointwise stabilizable (respectively controllable) in the linear sense \(\forall x \in \Omega\).

**Definition 3.4.** The SDC representation (3.12) is pointwise Hurwitz in a region \(\Omega\) if the eigenvalues of \(A(x)\) are in the open left half complex plane, i.e. \(\Re\{\lambda_i(A(x))\} < 0 \ \forall x \in \Omega\).

The existence of a feedback stabilizing solution in the form of Eqn (3.7) for extended linearization has been addressed in [6] and [28]. The main result has to do with the concept of recoverability. Essentially, a recoverable control law is a control law of the form \(u = k(x), \ k(0) = 0\) which has been obtained from a given control design method. The necessary and sufficient conditions for extended linearization recoverability are stated in Theorem 3.3 [28].

**Definition 3.5.** A \(C^1\) control law \(u = k(x), \ k(0) = 0\) is said to be recoverable by extended linearization in the region \(\Omega\) if there exists a pointwise stabilizable SDC representation \(\{A(x), B(x)\}\) and an underlying linear control synthesis method which, based on \(\{A(x), B(x)\}\), is capable of producing a state dependent gain \(K(x)\) satisfying \(k(x) = K(x)x \ \forall x \in \Omega\).

**Theorem 3.3.** (EL Existence [28]) A \(C^1\) control law \(u = k(x), \ k(0) = 0\) is recoverable by Extended Linearization in a region \(\Omega\) if and only if there exists a control
law \( k(x) = K(x)x \) and a pointwise stabilizable SDC representation \( \{A(x), B(x)\} \) such that \( A(x) + B(x)K(x) \) is pointwise Hurwitz in \( \Omega \).

**Proof.** The proof is omitted for convenience. It can be found in [28] □

### 3.3 State-Dependent Regulation (SDLQR)

The extension of linear control synthesis methods to nonlinear systems under extended linearization forms a broad class of control design methods. One popular technique that mimics the LQR method of optimal linear control is referred to as the State-Dependent Riccati Equation (SDRE) approach or the State-Dependent LQR (SDLQR) control. Applying SDLQR requires that the following conditions be met in addition to conditions (3.1) and (3.2)

**Condition 3.3.** The design parameters \( R \) and \( Q \) are \( C^0(\Omega) \) matrix valued functions satisfying \( Q(x) = Q^T(x) \geq 0 \) and \( R(x) = R^T(x) > 0 \) for all \( x \in \Omega \).

**Condition 3.4 (Pointwise Minimum Phase).** The zeros of the loop gain \( K(x)[sI - A(x)]^{-1}B(x) \) must lie in the closed left half plane \( \Re(s) \leq 0 \).

The SDLQR control approach consists of the following three steps

1. Bring the system in Eq (3.5) under SDC representation (Eq 3.12).

2. Solve the state dependent matrix Riccati equation (SDMRE) online:

   \[
   S(x)A(x) + A^T(x)S(x) - S(x)B(x)R^{-1}(x)B^T(x)S(x) + Q(x) = 0
   \]

3. Form the control pointwise:

   \[
   u = -K(x)x \\
   K(x) = R^{-1}(x)B^T(x)S(x)
   \]

We conclude this chapter by noting that there are many theoretical issues to be resolved in SDLQR nonlinear regulation. However, researchers have found several
impressive examples of solved design problems recently. These examples show excellent agreement with solutions obtained by employing other well established control paradigms.
Chapter 4

Stability Analysis of State Dependent Systems

The stability properties of state dependent systems can be addressed using conventional Lyapunov theory. The results presented in this chapter establish the conditions that a solution for a stabilizing feedback controller may exist. Regarding stability, this chapter focuses on results that ensure Global Asymptotic Stability, Local Asymptotic Stability and estimating the region of attraction for an SDLQR controlled system. We note that Global Asymptotic Stability for the general SDLQR controlled system is still an open research area.

4.1 Existence of Solutions

Under conditions (3.1), (3.2) and (3.3), a unique solution to the algebraic Matrix Riccati equation (3.4) can be obtained. A given control law (if it is not SDLQR) must satisfy condition (3.4) to be SDLQR recoverable in accordance with the minimum phase property of the linear LQR synthesis method.

Theorem 4.1 (SDLQR Existence). A $C^1$ control law $u = k(x)$, $k(0) = 0$ is SDLQR recoverable in a region $\Omega$ if and only if

1. there exist a control law $k(x) = K(x)x$ and a pointwise stabilizable SDC representation $\{A(x), B(x)\}$, such that $A(x) + B(x)K(x)$ is pointwise Hurwitz in $\Omega$ and
2. the gain $K(x)$ satisfies the pointwise Minimum phase condition (3.4).

Application of SDLQR control results in the closed loop state trajectory

$$\dot{x} = [A(x) - B(x)R^{-1}(x)B^T(x)S(x)]x$$  \hspace{1cm} (4.1)

The resulting controller relies on pointwise application of the LQR control synthesis. The simplicity and apparent effectiveness of SDLQR is noteworthy. SDLQR does not attempt to solve the HJBE, making it computationally attractive. It is a true generalization of the LQR control synthesis methods of linear optimal control because, when the coefficient and weighting matrices are constant, the nonlinear regulator problem collapses to the LQR problem and the SDLQR control technique becomes the steady state linear regulator.

### 4.2 Stability

Stability analysis of nonlinear controllers typically entails applying Lyapunov’s second method to a candidate Control Lyapunov Function. In the case of nonlinear systems under extended linearization, however, this approach has not been very fruitful. The following useful result pertains to the local stability of equilibrium points for systems under SDLQR control.

**Theorem 4.2.** Assume that the SDC representation is chosen such that $\text{col}\{A(x)\} \in C^1$ in the neighborhood $\Omega$ about the origin and that the pair $\{A(x), B(x)\}$ is pointwise stabilizable in the linear sense for all $x \in \Omega$. Then the SDLQR nonlinear regulator produces a closed loop solution which is locally asymptotically stable.

**Proof.** Note that the closed loop solution is given by

$$\dot{x} = [A(x) - B(x)R^{-1}(x)B^T(x)S(x)]x = A_d(x)x$$  \hspace{1cm} (4.2)

and that the closed loop matrix $A_d(x) = A(x) - B(x)R^{-1}(x)B^T(x)S(x)$ is guaranteed to be stable at every point $x$ from Riccati equation theory. It is known from [38] and
references therein that $\text{col}\{S(x)\} \in C^1$ and $\text{col}\{A_{cl}(x)\} \in C^1$. Upon applying the Mean Value Theorem to $\text{col}\{A_{cl}(x)\}$

$$\text{col}^j\{A_{cl}(x)\} = \text{col}^j\{A_{cl}(0)\} + \frac{\partial\text{col}^j\{A_{cl}(z_j)\}}{\partial x} x, \ j = 1 \ldots n \hspace{1cm} (4.3)$$

where the vector $z_j$ is that point on the line segment joining the origin and $x$ that yields equality in the $j^{th}$ equation of Eqn (4.3). Upon substituting Eqn (4.3) into Eqn (4.2)

$$\dot{x} = A_{cl}(0)x + \left[ \frac{\partial\text{col}^1\{A_{cl}(z_1)\}}{\partial x} x; \frac{\partial\text{col}^2\{A_{cl}(z_2)\}}{\partial x}; \ldots; \frac{\partial\text{col}^n\{A_{cl}(z_n)\}}{\partial x} \right] x \hspace{1cm} (4.4)$$

$$= A_{cl}(0)x + \sum_{j=1}^{n} \sum_{i=1}^{n} x_i x_j \frac{\partial\text{col}^j\{A_{cl}(z_j)\}}{\partial x_i} \hspace{1cm} (4.5)$$

Multiplying Eqn () by $\|x\|\|x\|$ and defining

$$\Psi(x, z_1, z_2, \ldots, z_n) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{x_i x_j \partial\text{col}^j\{A_{cl}(z_j)\}}{\|x\|} \hspace{1cm} (4.6)$$

yields

$$\dot{x} = A_{cl}(0)x + \Psi(x, z_1, z_2, \ldots, z_n)\|x\| \hspace{1cm} (4.7)$$

where

$$\lim_{\|x\| \to 0} \Psi(x, z_1, z_2, \ldots, z_n) = 0 \hspace{1cm} (4.8)$$

In a neighborhood about the origin the linear term $A_{cl}(0)x$ dominates Eqn (4.7). This linear term has a constant stable coefficient, $A_{cl}(0)$, resulting in local asymptotic stability as desired.

Although Theorem 4.2 proves that all the eigenvalues of the closed loop dynamics matrix 3.13, in an SDLQR controlled system have negative real parts, this property alone does not imply global asymptotic stability for the SDLQR approach. To date, only a number of global stability results have been proposed for SDLQR. In the following, some of the notable results are presented below. We begin with a result which establishes global asymptotic stability of SDLQR in the scalar case and then move to results pertaining to the multivariable case.
4.2. STABILITY

Theorem 4.3 (Scalar GAS [27]). In the scalar case, or \( n = 1 \), the solution given by SDLQR is globally asymptotically stable.

Proof. By using the candidate Lyapunov function \( V(x) = \frac{1}{2}x^2 \) the result follows immediately.

Theorem 4.4 (Symmetric \( A_d \) GAS [27]). Let \( A(\cdot), B(\cdot), Q(\cdot) \) and \( R(\cdot) \) be \( C^1(\Omega) \) matrix valued functions and let the respective pairs \( \{A(x), B(x)\} \) and \( \{A(x), Q^{\frac{1}{2}}(x)\} \) be pointwise stabilizable and detectable SDC representations of the nonlinear system in Eqn (3.5). If the closed loop dynamics matrix \( A_d(x) \) is symmetric for all \( x \), then the SDLQR closed loop solution is globally asymptotically stable.

Proof. Let the candidate Lyapunov function be

\[
V(x) = x^T x
\]

Then

\[
\dot{V}(x) = x^T \dot{x} + \dot{x}^T x = x^T [A_d(x) + A_d^T(x)] x
\]

It is known that \( A_d(x) \) is stable for all \( x \in \Omega \). Hence if \( A_d(x) \) is symmetric then it will be negative-definite which implies that \( \dot{V} < 0 \ \forall x \in \Omega \).

Theorem 4.5 (Metzlerian \( A_d \) GAS [4]). Let \( A(\cdot), B(\cdot), Q(\cdot) \) and \( R(\cdot) \) be \( C^1(\Omega) \) matrix valued functions and let the respective pairs \( \{A(x), B(x)\} \) and \( \{A(x), Q^{\frac{1}{2}}(x)\} \) be pointwise stabilizable and detectable SDC representations of the nonlinear system in Eqn (3.5). If the closed loop dynamics system is Metzlerian stable and symmetric for all \( x \), then the SDLQR closed loop solution is globally asymptotically stable.

Proof. The proof of this theorem follows directly from Theorem (4.4).

Asymptotic stability of second order systems has been considered in [16] where conditions for global stability of second order systems were presented. The result takes advantage of a key representation of \( A(x) \) which allows the control law to be obtained in an analytical form. The stability analysis can then proceed along the lines of Lyapunov and LaSalle’s Invariance Principle.
**Theorem 4.6** (Second Order GAS [16]). For the second order nonlinear system

\[
\begin{align*}
\dot{x}_1 &= f_1(x) \\
\dot{x}_2 &= f_2(x) + g(x)u, \quad \forall x \in \mathbb{R}^2
\end{align*}
\]

let the SDLQR representation be

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} u(x)
\]

\[
\begin{bmatrix}
0 & f_1(x) \\
0 & f_2(x)
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
g(x)
\end{bmatrix} u(x)
\]

\[
A(x) \\
B(x)
\]

where the SDC matrices \(\{A(x), B(x)\}\) are stabilizable. If the following conditions hold

A1) The elements \(a_{12} = f_1/x_2\) and \(a_{22} = f_2/x_2\) are well defined \(\forall x \neq 0\) and have finite values \(a_{12}(0)\) and \(a_{22}(0)\) at \(x = 0\).

A2) \(\text{sgn}(a_{12}(x)b_2(x)) = 1\) \(\forall x\) where \(a_{12} = f_1(x)/x_2\).

A3) \(Q(x) = diag(q_1(x), q_2(x))\) and \(R(x) = r(x)\) where \(q_1(x) > 0, q_2(x) \geq 0, r(x) > 0\).

and the forms of \(q_1(x)\) and \(r(x)\) are chosen such

\[
\frac{a_{12}(x)}{b_2(x)} = k \sqrt[4]{\frac{q_1(x)}{r(x)}} \quad k > 0, \quad k \in \mathbb{R}, \quad \forall x \tag{4.9}
\]

the controlled closed loop system given by

\[
u = R^{-1}(x)B^T(x)S(x)x
\]

where \(S(x)\) is the solution to the SDMRE

\[
S(x)A(x) + A^T(x)S(x) - S(x)B(x)R^{-1}(x)B^T(x)S(x) + Q(x) = 0
\]
4.2. **STABILITY**

is globally asymptotically stable.

**Proof.** By assumption, the solution to the matrix Riccati equation

\[
S(x)A(x) + A^T(x)S(x) - S(x)B(x)R^{-1}(x)B^T(x)S(x) + Q(x) = 0
\]

is a symmetric, positive definite matrix

\[
S(x) = \begin{bmatrix} s_{11}(x) & s_{12}(x) \\ s_{21}(x) & s_{22}(x) \end{bmatrix}
\]

where

\[
s_{11} = \frac{\nu \sqrt{q_1 r} \sqrt{a_{22}^2 + q_2 b_2^2/r + 2a_{12}b_2 \sqrt{q_1/r}}}{a_{12}b_2}
\]

\[
s_{12} = \sqrt{q_1 r}/b_2
\]

\[
s_{22} = \frac{a_{22} + \nu \sqrt{a_{22}^2 + q_2 b_2^2/r + 2a_{12}b_2 \sqrt{q_1/r}}}{b_2^2/r}
\]

and the term

\[
\nu = \text{sgn}(a_{12}(x)b_2(x))
\]

is equal to 1 by $A2$. Hence, positive definiteness of $S(x)$ is assured. Substituting the above back into $u$, the feedback control law is obtained as

\[
u = \frac{\sqrt{q_1}}{r} x_1 - f_2/g - \frac{\text{sgn} x_2}{g} \sqrt{f_2^2 + q_2 x_2^2 g^2/r + 2f_1 x_2 g \sqrt{q_1/r}}
\]

Substituting this into the system equations gives the closed loop dynamics

\[
\dot{x}_1 = a_{12} x_2 = f_1(x)
\]

\[
\dot{x}_2 = -g \sqrt{q_1/r} x_1 - \text{sgn} x_2 \sqrt{f_2^2 + q_2 x_2^2 g^2/r + 2f_1 x_2 g \sqrt{q_1/r}}
\]

\[
= - b_2 \sqrt{q_1/r} x_1 - \left( a_{22}^2 + q_2 b_2^2/r + 2a_{12}b_2 \sqrt{q_1/r} \right) x_2
\]

Consider a quadratic Lyapunov function for stability analysis of Eqn (4.16) with
scaling coefficients $\alpha$ and $\xi$.

$$V(x) = \frac{1}{2}(\alpha x_1^2 + \xi x_2^2), \quad \alpha, \xi > 0 \quad (4.17)$$

Taking the derivative of Eqn (4.17), along the system trajectories of Eqn (4.16) results in

$$\dot{V} = \left(\alpha a_{12} - \xi b_2 \sqrt{\frac{q_1}{r}}\right) x_1 x_2 - \left(\xi \sqrt{a_{22}^2 + q_2 b_2^2/r} + 2a_{12} b_2 \sqrt{q_1/\sqrt{r}}\right) x_2^2 \quad (4.18)$$

The first term in Eqn (4.18) is sign indefinite while the second term is always positive. Using the coefficients $\alpha$ and $\xi$, the sign indefiniteness of the first term in Eqn (4.18) can be eliminated. For the sign indefiniteness to vanish, the following condition should be satisfied

$$\alpha a_{12}(x) - \xi b_2(x) \sqrt{\frac{q_1(x)}{r(x)}} = 0 \quad \forall x \quad (4.19)$$

Eqn (4.19) can be satisfied by defining $k \equiv \frac{\xi}{\alpha} > 0$ and utilizing $A3$. The resulting Lyapunov function derivative is

$$\dot{V}(x) = - \left(\xi \sqrt{a_{22}^2 + q_2 b_2^2/r} + 2a_{12} b_2 \sqrt{q_1/r}\right) x_2^2 \leq 0 \quad \forall x \quad (4.20)$$

Eqn (4.20) is a negative semidefinite function. It can be seen that

$$\dot{V}(x) = 0 \quad \forall x \rightarrow x_2(t) = 0 \rightarrow \dot{x}_2(t) = 0 \rightarrow x_1(t) = 0 \quad (4.21)$$

Therefore, $\dot{V} = 0$ can exist only at the origin. It follows from LaSalle’s Invariance Theorem that the origin of Eqn (4.16) is globally asymptotically stable. \qed

**Remark 4.1.** Generally, if $a_{12}$ and $b_2$ are functions of $x$, the choice of $q_1(x)$ and $r(x)$ are dependent.

**Remark 4.2.** When $a_{12}$ and $b_2$ are constants, the closed loop system can be made globally asymptotically stable with independent choices of $q_1$ and $r$ if and only if they are positive. The implication of this is that systems of the form

$$\dot{x}_1 = x_2 \quad (4.22)$$
4.2. **STABILITY**

\[ \dot{x}_2 = f_2(x) + u \]  

(4.23)

can be globally asymptotically stabilized by varying \( q_1, q_2 \) and \( r \) arbitrarily since Eqn (4.9) reduces to

\[ 1 = k\sqrt{\frac{q_1}{r}}, \; k > 0 \]  

(4.24)

**Remark 4.3.** In Eqn (4.9), the condition for global asymptotic stability is independent of \( q_2(x) \), hence \( q_2(x) \) can be chosen arbitrarily so long as it is \( > 0 \) \( \forall x \)

The following result from [48] establishes that for a class of stabilizable systems, a state-dependent feedback gain can always be found that can make the nonlinear system under the correct SDC representation pointwise Hurwitz.

**Theorem 4.7.** *(Stabilizing State-dependent Control law [48])* Let

\[ u = k(x) = K(x)x \]

be a stabilizing feedback law that admits a differentiable Lyapunov function, \( V(x) \), such that

\[ \frac{\partial V(x)}{\partial x} = P(x)x \]

for some matrix \( P(x) \), and

\[ \frac{\partial V(x)}{\partial x}(f(x) + B(x)k(x)) = x^TP(x)(A(x) + B(x)K(x))x \leq -x^TQ(x)x \]

with

\[ x^TQ(x)x > 0 \; \forall x \in \Omega - \{0\} \]

If

\[ x^TP(x)x > 0 \]

then there exists an \( E(x) \) such that

\[ E(x)x = 0 \]

and

\[ A(x) + E(x) + B(x)K(x) \]
is pointwise Hurwitz for all $x \in \Omega - \{0\}$

**Proof.** The proof can be found in [48] □

Note that the key assumption in Theorem 4.7 is on the Lyapunov function, $V(x)$, which must possess star-convexity. This can be seen from

$$x^T P(x)x = x^T \frac{\partial V(x)}{\partial x} > 0$$

which asserts this property. The theorem also assumes that two functions have quasi-linear form

$$\frac{\partial V(x)}{\partial x} = P(x)x$$

$$k(x) = K(x)x.$$ 

This poses no problem under mild regularity conditions such as condition 3.1 and 3.2.

To extend the useful result of Theorem 4.7 to SDLQR, we must show that the state-dependent feedback gain can be found by solving a Riccati equation pointwise.

**Theorem 4.8.** *(SDLQR Stabilization via Representation [48]*) Under the hypothesis of Theorem 4.7, there exists matrices $E(x)$ and $\tilde{Q}(x)$ and positive scalar $\gamma(x)$ such that

$$E(x)x = 0, \quad \tilde{Q}(x) > 0 \; \forall x \in \Omega - \{0\}$$

and SDMRE

$$S(x)(A(x) + E(x)) + (A(x) + E(x))^T S(x) + \tilde{Q}(x) - \gamma(x)S(x)B(x)B^T(x)S(x) = 0$$

whose solution leads to the stabilizing feedback

$$u = -\gamma(x)B^T(x)S(x)x.$$ 

**Proof.** By assumption, we have that

$$\inf_u x^T \left( S(x)A(x) + A^T(x)S(x) + 2Q(x) \right) x + 2x^T S(x)B(x)u \leq 0.$$
4.2. STABILITY

It is easy to show that the search over $u$ can be limited to $u$ of the form

$$u = -\gamma(x)B^T(x)P(x)x.$$  

Therefore

$$\inf_{\gamma > 0} x^T \left( S(x)A(x) + A^T(x)S(x) + 2Q(x) - 2\gamma S(x)B(x)B^T(x)S(x) \right) x \leq 0.$$  

Let $\gamma(x)$ denote the minimal $\gamma > 0$ that satisfies the aforementioned inequality. Then

$$x^T \left( S(x)A(x) + A^T(x)S(x) + 2Q(x) - \gamma(x)S(x)B(x)B^T(x)S(x) \right) x \leq 0$$

which implies that the feedback

$$u = -\gamma(x)B^T(x)S(x)x$$

is stabilizing. Following the same approach as in the proof Theorem 4.7 (found in [48]) leads to the construction of the desired matrices $E(x)$ and $\tilde{Q}(x)$.  

Global asymptotic stability, although desirable, may not be achievable with the results given above. For stability analysis of the closed loop SDLQR controlled system, one can alternatively seek an estimate of the region of attraction. The estimate of the region of attraction is the set in the state space that encloses all initial conditions such that when the system is excited, their trajectories will converge asymptotically into the origin. Defining the region of attraction

$$\mathcal{R}_\gamma = \{x_0 \in \mathbb{R}^n : x(t) \to 0 \text{ as } t \to \infty \text{ if } x(0) = x_0\} \quad (4.25)$$

where global asymptotic stability is equivalent to $\mathcal{R}_\gamma = \mathbb{R}$. Conversely, if the region of attraction, $\mathcal{R}_\gamma$ is very "small", then it is not hard to imagine that the SDLQR system may not be reliable when actually implemented.

Of the recent results regarding estimating the region of attraction for an SDLQR regulated system, the methods proposed by Seiler [46] and Erdem [15] are by far the most appealing in that their application is both intuitive and has the widest use cases. The method proposed by Seiler in [46] begins with the assumption that there
exist some Lyapunov function, $V(x)$, that proves the origin is locally asymptotically stable, i.e., that $\dot{V}(x) < 0$ for nonzero $x$ in some neighborhood of the origin. Then it is supposed that if a function $h(x)$ can be found such that

$$(V(x) - \gamma) x^T x \geq h(x) \frac{\partial V}{\partial x} f(x) \quad \forall x \in \mathbb{R}^n.$$ 

Now $\dot{V}(\bar{x}) = 0 \rightarrow \bar{x} = 0$ or

$$V(\bar{x}) \geq \gamma$$

Hence the function $h(x)$ is a function that proves

$$\dot{V}(x) \neq 0 \quad \forall x \in \mathcal{R}_\gamma$$

where

$$\mathcal{R}_\gamma = \{x \in \mathbb{R}^n : V(x) \leq \gamma\}$$

under some continuity assumptions. It is concluded that $\dot{V}(x) < 0 \quad \forall x \in \mathcal{R}_\gamma$ implying that $\mathcal{R}_\gamma$ is the region of attraction.
4.2. **STABILITY**

Seiler’s approach is interesting because it turns the problem of estimating the stability region of attraction into a search for a function, \( h(x) \), which satisfies an inequality constraint. For the class of nonlinear systems where \( f \) is a polynomial and \( h \) and \( V \) are restricted to be polynomials, there is a computationally tractable method to perform the required search for \( h(x) \) (See [46] and references therein). The method exploits ties between positive semidefinite matrices and sum of squares polynomials

\[
p(x) = \sum_{i=1}^{m} f_i^2(x)
\]

where \( \{f_i\}_{i=1}^{m} \) is some set of polynomials.

Before stating the Seiler estimate for the region of attraction, let us introduce some notation. The trace of the matrix \( M \in S^{n \times n} \) is denoted by \( \text{Tr}[M] = \sum_{k=1}^{n} m_{kk} \). The \( k \) leading principle minors of \( M \) are

\[
\Delta_k(M) = \det \begin{bmatrix} m_{11} & \cdots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \cdots & m_{kk} \end{bmatrix}, \ k = 1, \ldots, n
\]

Finally, we distinguish between the set in Eqn (4.26), which is the true region of attraction, and the estimated region of attraction, which denoted by

\[
\tilde{R}_\gamma = \{ x \in \mathbb{R}^n : V(x) < \gamma \}
\]

(4.27)

and its closure, which is denoted by

\[
cl(\tilde{R}_\gamma) = \{ x \in \mathbb{R}^n : V(x) \leq \gamma \}
\]

(4.28)

Theorem 4.9 provides a construction for an estimate, \( \tilde{R}_\gamma \subset R_\gamma \). This theorem requires the Lyapunov function, \( V(x) \), to guarantee local asymptotic stability of the origin.

**Theorem 4.9** (ROA by SOS Optimization [46]). If there exists \( \gamma \in \mathbb{R} \), functions \( r_1, s_1, \ldots, s_n : \mathbb{R}^n \times S^{n \times n} \rightarrow \mathbb{R} \) and a function \( R_2 : \mathbb{R}^n \times S^{n \times n} \rightarrow S^{n \times n} \) such that
CHAPTER 4. STABILITY ANALYSIS OF STATE DEPENDENT SYSTEMS

\[ s_k(x, P) \geq 0 \text{ for } k = 1, \ldots, n \]

\[
(V(x) - \gamma)x^T x + r_1(x, P) \frac{\partial V}{\partial x}(x) A_{cl}(x, P)x + \\
Tr[R_2(x, P)H(x, P)] - \sum_{k=1}^{n} s_k(x, P)\Delta_k(P) \geq 0
\]

(4.29)

hold for all \((x, P) \in \mathbb{R}^n \times S^{n \times n}\), then \(\tilde{\mathcal{R}}_{\gamma} \subset \mathcal{R}\).

Proof. Take any \(\bar{x}\) such that \(\dot{V}(\bar{x}) = 0\). First we show that either \(\bar{x} = 0\) or \(V(\bar{x}) \geq \gamma\).

At \(\bar{x}\), there exists a unique, positive semidefinite matrix, \(\bar{P}\), that satisfies the state dependent algebraic matrix Riccati equation: \(H(\bar{x}, \bar{P}) = 0\). Thus \(\text{Tr}[R_2(\bar{x}, \bar{P})H(\bar{x}, \bar{P})] = 0\) and hence Eqn (4.29) evaluated at \((\bar{x}, \bar{P})\) is

\[
(V(\bar{x}) - \gamma)\bar{x}^T \bar{x} \geq \sum_{k=1}^{n} s_k(\bar{x}, \bar{P})\Delta_k(\bar{P})
\]

(4.30)

For \(k = 1, \ldots, n\) Eqn (4.29) implies \(s_k(\bar{x}, \bar{P}) \geq 0\) and \(\bar{P} \geq 0\) implies \(\Delta_k(\bar{P}) \geq 0\). It follows from Eqn (4.30) that \((V(\bar{x}) - \gamma)\bar{x}^T \bar{x} \geq 0\). Therefore, \(\dot{V}(\bar{x}) = 0\) implies that either \(\bar{x} = 0\) or \(V(\bar{x}) \geq \gamma\).

This can be restated as: \(\dot{V}(x) \neq 0 \forall x \in \tilde{\mathcal{R}}_{\gamma} \{0\}\). By assumption, \(\dot{V}(x) < 0 \forall x\) in a neighborhood of the origin and \(V(x)\) is continuously differentiable. We conclude that \(\dot{V}(x) < 0 \forall x \in \tilde{\mathcal{R}}_{\gamma} \{0\}\).

The proof is completed by a standard Lyapunov argument. Take any \(x_0 \in \tilde{\mathcal{R}}_{\gamma}\). Then for some \(\epsilon > 0\), \(x_0 \in cl(\tilde{\mathcal{R}}_{\gamma-\epsilon})\). Since \(cl(\tilde{\mathcal{R}}_{\gamma-\epsilon}) \subset \tilde{\mathcal{R}}_{\gamma}\), \(\dot{V}(x) < 0 \forall x \in cl(\tilde{\mathcal{R}}_{\gamma-\epsilon})\). It follows from the proof of Theorem 3.1 in [29] that if \(x(0) = x_0\) then \(x(t) \to 0\) as \(t \to \infty\). Hence \(x_0 \in \mathcal{R}_{\gamma}\) and thus \(\tilde{\mathcal{R}}_{\gamma} \subset \mathcal{R}_{\gamma}\). \(\square\)

Alternatively, Erdem et al use vector norms in [15] to estimate the region of attraction for SDLQR controlled systems. The method is based on defining an overvaluing comparison system for the original nonlinear system using vector norms, an approach first proposed by Borne, Richard et al.

**Theorem 4.10. (ROA Erdem [15])** If it is possible to define a pseudo-overvaluing system for Eqn (3.12) in a neighborhood \(S \subset \mathbb{R}^n\) of the equilibrium point \(x = 0\) relative to a regular vector norm \(p(x)\)
\[ D^+ p(x) \leq Mp(x) \forall x \in S \quad (4.31) \]

for which the constant matrix \( M \) is irreducible and Hurwitz, then \( x = 0 \) is locally asymptotically stable and its region of attraction (ROA) includes the largest domain \( D \) which is strictly included in \( S \) and is the union of the following three sets

1. \( D_1 = \{ x \in \mathbb{R}^n : p^T(x)u_m(M^T) \leq \alpha(\text{constant}) \} \subset S \)

2. \( D_\infty = \{ x \in \mathbb{R}^n : p(x)\alpha u_m(M), (\alpha \text{ constant scalar}) \} \subset S \)

3. \( D_c = \{ x \in \mathbb{R}^n : p(x) \leq c, \ c \in \mathbb{R}^k, \ -Mc > 0 \} \subset S \)

Remark 4.4. If \( p(x) \) is of dimension 1, then this is like the second method of Lyapunov. If \( p(x) \) is of higher dimension, then it is dealt with via a vector-Lyapunov function.

This chapter examined the stability properties of state dependent systems under the extended linearization formulation. Results pertaining to the existence of a stabilizing feedback controller solution were presented. The local stability of equilibrium points were discussed and a result establishing local asymptotic stability was presented. Under specific conditions, it is also possible to prove global asymptotic stability for SDLQR systems. These results were also provided. Furthermore, a result from Shamma et al was presented that shows if there is a control possessing a Lyapunov function with star-convexity properties then there exists a parameterization for the SDLQR approach that is globally asymptotically stabilizing. Two methods that aide in estimating the Region of Attraction for an SDLQR controlled system were presented.
Chapter 5

Optimal Properties of Extended Linearization

In order to establish some of the results related to SDLQR optimality, let us assume the following conditions hold:

**Condition 5.1.** $A(x), B(x), S(x), Q(x)$ and $R(x)$ and their gradients $\frac{\partial A(x)}{\partial x}, \frac{\partial B(x)}{\partial x}, \frac{\partial S(x)}{\partial x}, \frac{\partial Q(x)}{\partial x}$ and $\frac{\partial R(x)}{\partial x}$ are bounded in a neighborhood $\Omega$ about the origin.

**Condition 5.2.** The value function, $V(x)$, previously defined in Eqn (2.4) by

$$V(x) = \inf_{u \in U} J(x, u)$$

in the HJB equation (3.2)

$$\frac{\partial V}{\partial x} f(x) - \frac{1}{2} \frac{\partial V}{\partial x} g(x) R^{-1} g(x) \frac{\partial V'}{\partial x} + x^T Q(x) x = 0$$

is locally Lipschitz in a region $\Omega$ around the origin.

**Theorem 5.1.** (Sub-optimality of SDLQR [38]) In the general multivariable case, the SDLQR nonlinear feedback solution and its associated state and costate trajectories satisfy the first necessary condition for optimality ($\frac{\partial H}{\partial u} = 0$) of the nonlinear optimal regulator problem in Eqn (3.5) and Eqn (3.6). Additionally, when condition 5.1 is satisfied, under asymptotic stability, as the state $x$ is driven to zero, the second necessary condition for optimality ($\dot{\lambda} = -\frac{\partial H}{\partial x}$) is asymptotically satisfied at a quadratic rate.
Proof. From Pontryagin’s Maximum Principle (PMP), the necessary conditions for optimality are:

\[ \frac{\partial H}{\partial u} = 0 \]
\[ \frac{\partial H}{\partial x} = -\dot{\lambda} \]
\[ \frac{\partial H}{\partial \lambda} = \dot{x} \]

where \( H \) is the Hamiltonian defined in Eqn (2.12)

\[ H = \frac{\partial V'}{\partial x}(f(x) + g(x)u) + \frac{1}{2}[x'Q(x)x + u'R(x)u] \]

and \( \lambda = \frac{\partial V}{\partial x} \). Using Eqn (3.3) and Eqn (2.12)

\[ \frac{\partial H}{\partial u} = B^T(x)\lambda + R(x)u \]
\[ = B^T(x)\lambda + R(x)R^{-1}(x)B^T(x)S(x)x \]
\[ = B^T(x)[\lambda - S(x)x] \] (5.1)

It can be shown that [6]

\[ \frac{\partial V(x)}{\partial x} = S(x)x \] (5.2)

Now since \( \lambda = \frac{\partial V}{\partial x} \), the costate vector satisfies

\[ \lambda = S(x)x \] (5.3)

By substituting the expression for \( \lambda \) in Eqn (5.3) into Eqn (5.1) we obtain \( \frac{\partial H}{\partial u} = 0 \) which is the desired result. Hence the SDLQQR feedback solution satisfies the first necessary condition of the nonlinear regulator problem.

To satisfy the second necessary condition, begin by taking the partial derivatives of the Hamiltonian with respect to \( x \) as usual

\[ \dot{\lambda} = -\frac{\partial f^T(x)}{\partial x}\lambda - u^T\frac{\partial B^T(x)}{\partial x}\lambda - Q(x)x - \frac{1}{2}x^T\frac{\partial Q(x)}{\partial x}x - \frac{1}{2}u^T\frac{\partial R(x)}{\partial x}u \] (5.4)
Differentiating Eqn (5.3) with respect to time gives

\[ \dot{\lambda} = \dot{S}(x)x + S(x) \dot{x} \quad (5.5) \]

Substituting Eqn (3.8), Eqn (3.3), Eqn (4.2), and Eqn (5.5) into Eqn (5.4) and rearranging terms gives

\[
\dot{S}x + \frac{1}{2} x^T \frac{\partial Q}{\partial x} x + \frac{1}{2} x^T SBR^{-1} \frac{\partial R}{\partial x} R^{-1} B^T Sx + x^T \frac{\partial A^T}{\partial x} Sx \\
- x^T SBR^{-1} \frac{\partial B^T}{\partial x} Sx + \left[ SA + A^T S + Q - SBR^{-1} B^T S \right] x = 0
\]

(5.6)

Since the control satisfies the SDMRE pointwise, the last term is zero. Therefore Eqn (5.6) reduces to

\[
\dot{S}x + \frac{1}{2} x^T \frac{\partial Q}{\partial x} x + \frac{1}{2} x^T SBR^{-1} \frac{\partial R}{\partial x} R^{-1} B^T Sx + x^T \frac{\partial A^T}{\partial x} Sx \\
- x^T SBR^{-1} \frac{\partial B^T}{\partial x} Sx = 0
\]

(5.7)

which is referred to as the SDLQR necessary condition for optimality [38]. Therefore, whenever this condition is satisfied, the closed loop solution satisfies all the first order necessary conditions of the PMP, since \( \frac{\partial H}{\partial x} = 0 \) is always satisfied.

In general, the SDLQR necessary condition for optimality is not satisfied for a given SDC matrix \( \lambda(x) \). However, a sub-optimality property can be revealed as follows. Expanding \( \dot{S} \) yields

\[
\dot{S}(x)x = \left( \sum_{i=1}^{n} \frac{\partial S(x)}{\partial x_i} \dot{x}_i \right) x = \sum_{i=1}^{n} \frac{\partial S(x)}{\partial x_i} [a_{CL}^i x]
\]

(5.8)

where \( a_{CL}^i \) is the \( i \)th row of the closed loop coefficient matrix \( A_{CL}(x) \). Eqn (5.8) can be rewritten as \( x^T N_i x \) where the elements of \( N_i \) are functions of the elements of \( \frac{\partial S(x)}{\partial x_j} \) and \( a_{CL}^j, j = 1, \ldots n \). Substituting this result into the necessary condition (5.7) yields

\[ x^T M_i x = 0 \]
where
\[ M_i = N_i + \frac{1}{2} \frac{\partial Q}{\partial x} + \frac{1}{2} SBR^{-1} \frac{\partial R}{\partial x} R^{-1} B^T S + \frac{\partial A^T}{\partial x} S x - x^T SBR^{-1} \frac{\partial B^T}{\partial x} S \]
whose elements are the functions of the elements of \( A(x), B(x), P(x), Q(x) \) and \( R(x) \) as well as their gradients. Under asymptotic stability, the state trajectories will eventually enter and remain in \( \Omega \). By condition 5.1, there exists a constant positive definite matrix such that
\[
\max_i |x^T M_i x| \leq x^T U x \quad \forall x \in \Omega \quad (5.9)
\]
Thus the \( \infty \)-norm of the left hand side of Eqn (5.7) is bounded above by a quadratic function of \( x \). This completes the proof.

Theorem 5.1 outlines the sub-optimality of SDLQR control. Because the second order necessary condition for optimality is only satisfied asymptotically, the results of the theorem only hold in the case when the state trajectories are near optimal to begin with. This means the optimality of the SDLQR controller is sensitive to initial conditions of the dynamic states. For this reason, it is desirable to know the conditions which lead to a globally optimal solution and the associated optimal cost. The following result will assist us in this endeavor

Lemma 5.1. Suppose a vector-valued function \( s : X \to \mathbb{R}^n \) is of class \( C^1(\mathbb{R}^n) \), and let \( s(x) = [s_1(x), \ldots, s_n(x)]^T \) for \( x \in X \). Then there exists \( V : X \to \mathbb{R} \) such that
\[
\frac{\partial V(x)}{\partial x} = s(x) \quad \text{iff} \quad \frac{\partial s_i(x)}{\partial x_j} = \frac{\partial s_j(x)}{\partial x_i} \quad \forall x \in X \quad i, j = 1, 2, \ldots, n. \quad (5.10)
\]
Moreover, if Eqn (5.10) holds, then \( V \) with \( V(0) = 0 \) is given by
\[
V(x) = x^T \int_0^1 s(tx) dt. \quad (5.11)
\]

Using this lemma, the optimality result follows.

Theorem 5.2. (Symmetric Positive Definite SDMRE [24]) Suppose the SDMRE
\[
S(x)A(x) + A^T(x)S(x) - S(x)B(x)R^{-1}(x)B^T(x)S(x) + Q(x) = 0
\]
has a positive-definite matrix valued solution $S : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$. If the vector-valued function
\[
\frac{\partial V(x)}{\partial x} = S(x)x
\]
satisfies
\[
\frac{\partial s_i(x)}{\partial x_j} = \frac{\partial s_j(x)}{\partial x_i} \quad \forall x \in X \quad i, j = 1, 2, \ldots, n.
\]
with $s(x) = S(x)x$, then the control law for the nonlinear optimal regulator problem is given by
\[
 u = -R^{-1}(x)B^T(x)S(x)x
\]
and the optimal cost is given by
\[
 V(x) = x^T \left( \int_0^1 tS(tx)dt \right) x, \quad x \geq 0.
\]

Hence, if $\frac{\partial}{\partial x} S(x)x$ is a symmetric matrix, then the SDLQR control law is the optimal control for the cost function in Eqn (3.6). This condition is not generally true in the multivariable case. However, as the next result proves, in the scalar case, the SDLQR solution is always globally optimal.

**Theorem 5.3.** (Scalar SDLQR Global Optimality [38]) For scalar systems, the globally asymptotically stabilizing SDLQR control law
\[
u(x) = -\frac{1}{b(x)} \left[ f(x) + sgn(x) \sqrt{f^2(x) + \frac{b^2(x)x^2q(x)}{r(x)}} \right]
\] (5.12)
always gives the globally optimal solution to the nonlinear optimal regulator problem.

**Proof.** In the scalar case $x$, from Theorem 4.3, the SDLQR control law is globally asymptotically stabilizing, and the symmetry condition in Eqn (5.10) is always satisfied. Therefore, the solution is optimal. Alternatively, optimality can be deduced using the SDLQR necessary condition for optimality with heavy algebraic manipulations. \qed

**Remark 5.1.** In the scalar case, even if the performance index is non-quadratic, the SDLQR method produces the optimal control law for the nonlinear optimal regulator
problem.

Huang and Lu have shown that an SDC matrix will exist such that the SDLQR feedback control law produces the optimal control law. Their result makes use of the following useful lemma.

**Lemma 5.2.** (Huang and Lu [24]) Suppose there exists a valid SDC matrix, $A(x)$, and some mapping $E : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ that satisfies

$$E(x)x = 0, \ \forall x \in \Omega.$$ 

Then a new SDC matrix, $A_0(x)$, may be formed and is given by

$$A_0(x) = A(x) + E(x) \quad (5.13)$$

Using this lemma, the SDMRE can be written as

$$S(x)(A(x) + E(x)) + (A(x) + E(x))^T S(x)$$
$$- S(x)B(x)R^{-1}(x)B^T(x)S(x) + Q(x) = 0 \quad (5.14)$$

**Theorem 5.4.** If the value function $V(x)$ has the gradient of the form

$$\frac{\partial V(x)}{\partial x} = S(x)x \quad (5.15)$$

for some positive-definite matrix valued function $S : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, then there always exists a SDC matrix

$$f(x) = A(x)x$$

such that $S(x)$ is the solution of the SDMRE

$$S(x)A(x) + A^T(x)S(x) - S(x)B(x)R^{-1}(x)B^T(x)S(x) + Q(x) = 0$$

which gives the optimal feedback controller.

**Proof.** Let $A_0(x)$ be a valid SDC matrix formed by application of lemma 5.2. Then the HJB equation becomes

$$x^T W(x)x = 0 \ \forall x \in \mathbb{R}^n \quad (5.16)$$
where

\[ W(x) = x^T[S(x)A(x) + A^T(x)S(x) - S(x)B(x)R^{-1}(x)B^T(x)S(x) + Q(x)]x \]  

(5.17)

with \( A(x) = A_0(x) - E(x) \). The SDMRE in Eqn (3.4) can be equal to the HJB equation in (3.2) if and only if for some \( E(x) \)

\[ W(x) + S(x)E(x) + E^T(x)S(x) = 0 \quad \forall x \in \mathbb{R}^n. \]

Because \( W(x) \) is symmetric it is possible to parameterize \( E(x) \) as

\[ E(x) = -\frac{1}{2}S^{-1}(x)[W(x) - T(x)], \]  

(5.18)

where \( T(x) \) is some skew-symmetric matrix. From this, and since \( E(x)x = 0 \), \( T(x) \) is chosen such that \( [W(x) - T(x)]x = 0 \). From Eqn (5.16) this is always possible.

One way to construct such a \( T(x) \) is as follows. If \( x = 0 \), simply choose \( T(0) = 0 \), and so \( E(0) = -\frac{1}{2}S^{-1}(0)S(0) \) from Eqn (5.18). For \( x \neq 0 \), since \( W(x) \) is symmetric the eigen-decomposition

\[ W(x) = U^T(x)D(x)U(x) \]

is used where \( U(x) \) is an orthogonal matrix and \( D(x) \) is a diagonal matrix of pointwise eigenvalues \( D(x) = \text{diag}[\lambda_1(x), \ldots, \lambda_n(x)] \). Now letting \( y(x) = U(x)x \triangleq [y_1(x), \ldots, y_n(x)]^T \), \( y(x) \) cannot be zero for any \( x \neq 0 \), since \( U(x) \) has full rank. Assuming, without loss of generality, that \( y_p \neq 0 \) for some \( p \leq n \) and \( y_i = 0 \) for \( p < i \leq n \), Eqn (5.16) implies \( \sum_{i=1}^{p} \lambda_i y_i^2 = 0 \).

Let

\[
F(x) = \begin{bmatrix}
0 & 0 & \ldots & \lambda_1 y_1/y_p & 0 & \ldots & 0 \\
0 & 0 & \ldots & \lambda_2 y_2/y_p & 0 & \ldots & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
\lambda_1 y_1/y_p & \lambda_2 y_2/y_p & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0
\end{bmatrix}
\]
so that $F(x)$ is a matrix with all zero entries except the first $p$ elements on the $(p+1)^{th}$ row and column, with $F(x) = -F(x)^T$. $T(x)$ is now

$$T(x) = U^T(x)F(x)U(x)$$

Which is skew-symmetric and satisfies $[W(x) - T(x)]x = 0$. Therefore, $E(x)$ can be obtained from Eqn (5.18) for all $x \in \mathbb{R}^n$. \hfill \Box

In summary, if the gradient of the cost function $V(x)$ has the form $S(x)x$ and the appropriate SDC matrix $A(x)$ is chosen, then the global optimal controller can be obtained from the SDMRE. Furthermore, although many solutions exist to the SDMRE due to the non-uniqueness of the SDC matrix, only one solution to the SDMRE will produce the optimal control. This is also the same control that is produced by the HJB solution. Therefore, if $S(x)$ in Eqn (5.2) is positive-definite then with the right SDC matrix the unique positive-definite solution to the SDMRE, which is thus $S(x)$, recovers the optimality.
Chapter 6

Robust Stability

The robust control problem has been motivated for years by the reality that most systems are nonlinear and time-varying. Therefore, any linear time invariant approximation of such a plant is an inadequate representation because changes in the original plant operating point inevitably lead to variations in the model parameters. This variation in the model parameters introduces uncertainty which a control system cannot appropriately compensate for, resulting in degraded performance or loss of stability. Furthermore, linearization introduces unstructured perturbations into the plant model. These perturbations arise in the linearization process from truncating a complicated model by retaining only its most dominant modes. Tolerance to uncertainty in the plant model is not the only consideration. Good disturbance rejection properties are also required. Control methods that stabilize the system for the entire range of expected variations of the plant parameters and are optimized to attenuate worst case disturbances are collectively referred to as robust control methods.

In classical control design, the problem is addressed through the gain or phase margin which can be easily ascertained from the Bode or Nyquist plots. Computing the gain margin entails replacing the open loop transfer function $G(s)$ by a perturbed version $kG(s)$. By varying $k$ one can determine the range of values that retain stability. Similarly, the phase margin calculation entails replacing $G(s)$ by $e^{-j\theta}G(s)$ and varying $\theta$ to determine the range where the system retains stability. These concepts provide a measure or margin of stability. In this way they are good indicators of the robustness of the system. Since they are easy to ascertain from the Nyquist or Bode plot, they have become ubiquitous in control system design. Unfortunately, however,
in realistic cases several independent parameters may be subject to variation and the classical notion of gain or phase margin is no longer a valid assessment of the stability margin. Furthermore, detailed analysis of the robust control system performance involves frequency dependent perturbations which cannot be captured by use of the gain and phase margin approach.

For these reasons, the decade of the 1960’s and 1970’s was a period of inactivity in the research of stability in the presence of large parametric uncertainty. At the time, the perception was that the parametric stability problem was nearly impossible to solve even with effective computational techniques. Accordingly, the only effective way to address the problem was to introduce imaginary components to the real parameters so the uncertainty problem could be addressed in the complex plane by overbounding appropriate uncertainty sets.

In 1978, the Russian theorist V.L. Kharitonov introduced a remarkable theorem that dramatically altered the landscape. Kharitonov showed that the left half plane stability of a family of polynomials of fixed but arbitrary degree corresponding to a box in the coefficient space could be verified by checking the stability of the four vertex polynomials. Kharitonov’s Theorem reduced the apparently impossible task of verifying the stability of countably infinite continuous polynomial functions to merely assessing the Routh-Hurwitz stability condition on four fixed polynomials. Since then, additional investigators, most notably John Doyle, have advanced the theory of robust analysis and synthesis.

This section considers the robust analysis and control of nonlinear systems under Extended Linearization (EL). Because the choice of $A(x)$ is non-unique, different factorizations may be considered which can lead to different robustness characteristics.

For disturbance rejection and/or command tracking, nonlinear $H_\infty$ control via SDLQR [22] has been proposed. To present these results, we begin with the problem definition and any assumptions. Then the sufficient conditions for achieving the control objective are presented under full state feedback and output feedback assumptions.
6.1 Parametric Uncertainty

For most practical control systems, the structure of the controller remains fixed while the plant parameters can vary over a wide range from some nominal value $P^0$. The ability of a control system to remain stable despite these drastic variations is referred to as *robust parametric stability*. This type of robustness is required since inherent uncertainties in the modeling process and parameter variations in system actuators are inevitable.

Parametric uncertainty in linear systems can be expressed as [51]:

$$\dot{x} = [A + D\Delta]x$$

Now let the parameterized SDC representation of a nonlinear system be

$$\dot{x} = A(x, \alpha)x + B(x)u.$$ 

then it can also be put into the robust formulation. If there is uncertainty in the system, the parameterized SDC representation simply includes a term to account for the system uncertainty.

**Definition 6.1.** Let the parameterized SDC representation with structured uncertainty be

$$\dot{x} = [A(x, \alpha) + D\Delta]x \quad \forall x \in \mathbb{R}^n$$

(6.1)

where it is assumed that $A(x, \alpha)$ is pointwise stable or stabilized according to Eqn (3.13). Then Eqn (6.1) is called the *parameterized robust SDC* representation of the nonlinear system, where $D$ and $E$ are $n \times r$ and $s \times n$ dimensional structured matrices, $\alpha \in [0, 1]$ and $\Delta$ is an unknown uncertainty matrix confined to a region of interest.

Later in this chapter a systematic approach for ascertaining the $\alpha$ that results in the SDC parameterization which is maximally robust with respect to parametric uncertainty is introduced.
6.2 Stability Radius

Under system uncertainty, estimating the radius of stability for an SDLQR controlled system is difficult since the closed-loop system equations are not available explicitly. Because of this constrain difficulties may arise for the designer wishing to assess the robustness and performance tradeoffs associated with each distinct parameterization.

To address this problem, the theory for computing the conservative radius of stability for SDLQR systems is introduced for a special class of systems known as Metzler systems. Under parametric uncertainty, Metzler systems reveal an explicit formula for computing the stability radius. By exploiting special properties of a Metzler matrix that is an overvalued comparison system for the SDLQR formulation of the original nonlinear system, the designer can obtain a reasonable estimate for the stability radius under system uncertainty without knowing the closed-loop system equations explicitly.

The cornerstone of this theory rests on the concept of a vector norm which was pioneered by Robert in the 1960's [5].

**Definition 6.2.** Let $E = \mathbb{R}^n$ be a vector space and $E_1, E_2, \ldots, E_k$ subspaces of $E$ which verify: $E = E_1 \cup E_2 \cup \ldots E_k$. Let $x \in E$ be an $n$ vector defined on $E$ with a projection in the subspace $E_i$ denoted by $x_i$, $x_i = P_i x$, where $P_i$ is a projection operator from $E$ into $E_i$. Then, $p_i$ is a scalar norm $(i = 1, 2 \ldots k)$ defined on the subspace $E_i$ and $p$ denotes a vector norm of dimension $k$ with $i$ the component, $p_i(x) = p_i(x_i), p_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}_+^k$, where $p_i(x_i)$ is a scalar norm of $x_i$.

The method for determining the stability of nonlinear systems that will be used here was first proposed by Borne, Richard et al [40]. Essentially the method is based on defining an overvaluing system for the original nonlinear system. It is assumed that the solution for the nonlinear differential equation exists and that adequate smoothness conditions are met. The overvaluing system will be defined by the use of the vector norm $p(x)$ of the state vector $x$ defined above. The notion of an upper Dini derivative is used to find the time derivative of the vector norm $p(x)$.
Theorem 6.1. The matrix $M : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ defines an overvaluing system of Eq (6.1) with respect to the vector norm $p(x)$ if and only if the following inequality is verified for each corresponding component:

$$D^+ p(x) \leq M(x, \alpha) p(x) \quad \forall x \in \mathbb{R}^n$$

Where $D^+$ is the right hand derivative operator.

Let

$$M(x, \alpha) = \mu_{ij}(x, \alpha), \quad \mu_{ij} : \mathbb{R}^n \to \mathbb{R}$$

Now allow a partition of the space $E$ define a block partition of a matrix $A$. Denote $I_i$ and $I_j$ as the sets of the indices of the rows and columns of the block $A_{ij}$ of any matrix $A(x, \alpha)$. If $p_i(x)$ is the "max norm", the maximum of the modulus of each component of $x_i$, then:

$$\mu_{ij}(x, \alpha) = \max_{s \in I_i} \left[ a_{ss} + \sum_{l \in I_i, l \neq s} |a_{sl}| \right] \quad (6.2)$$

$$\mu_{ij}(x, \alpha) = \max_{s \in I_j} \left[ \sum_{l \in I_j} |a_{sl}| \right] \quad (6.3)$$

When this operation is performed, the resulting $M(x, \alpha)$ system is an overvaluing matrix. The interpretation of these formula is that, first all of the off-diagonal elements of $A$ are replaced by their absolute value, leading to the overvaluing matrix. Then, $\mu_{ij}$ is the greatest sum of all components in each row of the block in $A$. The following lemmas will be of interest in determining the special properties of the system matrix, $A(x, \alpha)$. In the interest of brevity, the proofs are left out. They can be found in [40].

Lemma 6.1. Let a psuedo-overvalued matrix $M(x, \alpha)$ of $A(x, \alpha)$ be defined with respect to the vector norm $p(x)$. Then any one of the following conditions hold:

i) The off diagonal elements $\mu_{ij}(x, \alpha), (i \neq j)$ of $M(x, \alpha)$ are nonnegative.

ii) If $\lambda_A(x)$ denotes any of the $n$ eigenvalues of $A(x, \alpha)$, $\lambda_M(x)$ any of the $k$ eigenvalues of $M(x, \alpha)$ and $\lambda_m(x)$ the maximal real part of $\lambda_M(x)$, then the following
6.2. STABILITY RADIUS

holds:
\[ \Re(\lambda_A(x)) \leq \Re(\lambda_M(x)) \leq \lambda_m(x) \in \mathbb{R} \quad \forall x \in \mathbb{R} \]

iii) When all the real parts of the \( \lambda_M(x) \) are negative, the matrix \( M \) is the opposite of an \( M \)-matrix, i.e. it admits an inverse whose elements are all nonpositive.

iv) The pseudo-overvaluing matrix whose elements are defined Eqns (6.3-6.3) is the smallest pseudo-overvaluing matrix of \( A(x, \alpha) \) element wise with respect to the vector norm \( p \).

v) \( M(x, \alpha) \) verifies (iii) if and only if the Koteliansky conditions are verified, i.e. its successive principal minors are sign alternate:

\[ (-1)^i M(x, \alpha) \begin{bmatrix} 1, \ldots i \\ 1, \ldots i \end{bmatrix} > 0 \quad \forall i = 1, 2, \ldots k, \forall x \in E \]

vi) When the matrix \( M(x, \alpha) \) is the opposite of an \( M \)-matrix (or equivalently satisfies (iii) or (v)), and when its inverse is irreducible, then \( M(x, \alpha) \) admits an eigenvector \( u_m(x) \), called the importance vector of \( M(x, \alpha) \), whose components are strictly positive and which is associated with the real, maximal and negative eigenvalue \( \lambda_m(x) \).

Now a new system matrix can be formed. Since it facilitates the calculation of the radius of stability, we shall distinguish this new system matrix as the robust comparison system.

**Definition 6.3.** Let there exist a vector norm \( p(x) \) and a matrix \( M(x, \alpha) \) connected with Eq (6.1) such that the off diagonal elements of \( M(x, \alpha) \) are all nonnegative and that the equality in Theorem 6.1 is verified along the solution of Eq (6.1) . Then the system

\[ \dot{z} = (M(x, \alpha) + D\Delta E)z, \quad \forall z \in \mathbb{R}_+^k \]

is a robust comparison system of Eq (6.1) in the sense that \( z(t) \geq p(x(t)), \forall t \in \tau_0, \) holds as soon as \( z(t_0) \geq p(x_0) \).

**Definition 6.4.** The matrix \( A = [a_{ij}] \in \mathbb{R}^{n\times n} \) is a Metzlerian matrix if \( a_{ii} \leq 0 \quad \forall i \) and \( a_{ij} \geq 0 \quad i \neq j \) with \( i = j = 1, 2, \ldots, n \).
Remark 6.1. The necessary condition for stability of Metzlerian matrices is $a_{ii} < 0$ which is assumed throughout this paper.

This leads to the discussion of systems which possess this important structure.

Definition 6.5. A system

$$
\dot{z} = Mz + Nu \quad z(0) = z_0 \quad (6.4)
$$

$$
y = Wz \quad (6.5)
$$

is called a Metzlerian system if $M$ is a Metzler matrix, and $N \succeq 0$, $W \succeq 0$ are nonnegative matrices.

Theorem 6.2. A Metzlerian system is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

i) All eigenvalues of the Metzlerian matrix $M$ have negative real parts.

ii) All coefficients $a_i (i = 0, \ldots, n - 1)$ of the characteristic polynomial $\Delta(\lambda) = \det(\lambda I - M) = \lambda^k + a_{n-1}\lambda^{k-1} + \ldots + a_0\lambda + a_0$ are positive.

iii) All leading principal minors of the matrix $-M$ are positive.

iv) The matrix $M$ is nonsingular and $-M^{-1} > 0$.

v) There exists a diagonal matrix $P$ such that $M'P + PM < 0$.

By the above properties the overvalued system matrices $M(x, \alpha)$ and $M$ will possess Metzlerian structures. The following theorem, which is proven in [3], can be used to estimate the region of stability under system uncertainty.

Theorem 6.3. Suppose $\dot{z} = (M + D\Delta E)z$, where $M$ is Metzlerian stable, $D \geq 0$ and $E \geq 0$ are given structured matrices of appropriate dimensions, say $n \times r$ and $s \times n$, and $\Delta$ is an unknown uncertainty matrix confined to a certain set of interest. Then, the real and complex stability radii coincide and are given by the following formulas, depending on the characterization of $\Delta$: 
6.2. STABILITY RADIUS

(1) let \( \| \cdot \| = \| \cdot \|_2 \) be the Euclidean norm and let \( \Delta = \mathbb{R}^{r \times s} \). Then
\[
r_R(M, D, E) = r_C(M, D, E) = \frac{1}{\| EM^{-1}D \|_2} \quad (6.6)
\]

(2) let \( \Delta \) be defined by the set \( \Delta = \{ P \circ \Delta : p_{ij} \geq 0 \} \) with \( \| \Delta \| = \max \{ |\delta_{ij}| : \delta_{ij} \neq 0 \} \) where \( [P \circ \Delta]_{ij} = p_{ij}\delta_{ij} \) denotes the Schur product. Then
\[
r_R(M, D, E) = r_C(M, D, E) = \frac{1}{\rho(EM^{-1}DP)} \quad (6.7)
\]

It is assumed that the \( A(x, \alpha) \) system matrix used for overvaluing is stable. Note, however, that this assumption does not place a general constrain on the method. If necessary, stability can be achieved by closing the loop and using \( A_{cl}(x) \) in the overvaluing operation instead.

**Theorem 6.4.** Let \( A(x) \) be a valid SDC parameterization for the nonlinear system of the form given in Eq (2.3) with structured parametric uncertainty given by:
\[
\dot{x} = [A(x) + D\Delta E]x \quad \forall x \in \mathbb{R}^n
\]
and let:
\[
\Omega = \{ x \in \mathbb{R}^n : x \in (X_L, X_U) \}
\]
be the domain of interest where \( X_L \) and \( X_U \) define the lower and upper bound of the domain respectively.

If the psuedo-overvalued constant matrix, \( M \), representing the robust comparison system under consideration:
\[
\dot{z} = (M(x) + D\Delta E)z \quad \forall z \in \mathbb{R}_+^k
\]
is Metzlerian, then the stability radius for the SDC parameterized nonlinear system is bounded below by \( r_R(M, D, E) \) as given in Theorem 6.3.

**Proof.** We provide the proof of this theorem with respect to one parametric uncertainty. Its proof for general uncertainty structure follows immediately.

The single parametric uncertainty structure can be captured by:
\[ D = \{ d_{ij} : d_{11} = 1, d_{ij} = 0 \ \forall i, j > 1 \} \]
\[ E = \{ e_{ij} : e_{11} = 1, e_{ij} = 0 \ \forall i, j > 1 \} \]
\[ \Delta = \{ \delta_{ij} : \delta_{11} = 1, \delta_{ij} = 0 \ \forall i, j > 1 \} \]

where \( \delta \in [0, \delta_m] \) with \( \delta_m \) denoting the maximum allowable uncertainty for which the system remains stable. Assume that it is possible to define a pseudo-overvaluing system for Eq (6.1) in a neighborhood \( S \subset \mathbb{R}^n \) of the equilibrium point \( x = 0 \) relative to a regular vector norm \( p(x) \) with:

\[ D p(x) \leq M_{\delta} p(x) \quad \forall x \in \mathbb{R}^n \]

for which the constant matrix \( M_{\delta} \) is irreducible stable Metzlerian \( \forall \delta, \delta \in [0, \delta_m] \). Then \( x = 0 \) is locally asymptotically stable, and its region of attraction includes the largest domain \( D \) which is strictly included in \( S \) and is the union of the following three sets:

\[ D_1 = \{ x \in \mathbb{R}^n : p^T(x) u_m(M_{\delta}^T) \leq \alpha \} \subset S \]
\[ D_\infty = \{ x \in \mathbb{R}^n : p(x) \leq \alpha u_m(M_{\delta}) \subset S \}
\[ D_c = \{ x \in \mathbb{R}^n : p(x) \leq c, c \in \mathbb{R}_+^k, -M_{\delta} c > 0 \} \subset S \]

where \( \alpha \) is a constant.

If \( \delta = \mathbb{R}^n \), this stabilizing property is global. Moreover convergence in this domain is exponential in \( e^{\lambda_m(M_{\delta}) t} \). It is clear that \( \lambda_m(M_{\delta}) \) is the Perron-Frobenius root and \( \lambda_m(M_{\delta_m}) = 0 \) for which the stability radius is given by:

\[ r_R(M, D, E) = \frac{1}{\| E M_{\delta}^{-1} D \|_2} = \delta_m \]

This establishes the result for a single parametric uncertainty.

The application of this result is illustrated by two examples below.
Example 6.1. For the nonlinear system:

\[
\begin{align*}
\dot{x}_1 &= -x_1 + 2x_1^2 x_2 \\
\dot{x}_2 &= -x_2 + u
\end{align*}
\]

one valid SDC parameterization is given by:

\[
A(x) = \begin{bmatrix} -1 & 2x_1^2 \\ 0 & -1 \end{bmatrix}
\]

With \( B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( Q = I_{2 \times 2} \), \( R = 1 \). Although not necessary in general, here we consider overvaluing the closed loop system matrix, \( A_{cl}(x) \), to show that the method is applicable to the closed loop system matrix as well. Since \( A(x) = A(x_1) \), the feedback gains \( k_1 \) and \( k_2 \) depend on \( x_1 \) only. The closed loop matrix \( A_{cl}(x) \) is:

\[
A_{cl}(x) = \begin{bmatrix} -1 & 2x_1^2 \\ -k_1(x_1) & -1 - k_2(x_1) \end{bmatrix}
\]

The overvaluing matrix \( M(x) \) for \( A_{cl}(x) \) is:

\[
M(x) = \begin{bmatrix} -1 & \max_{x_1 \in S} |2x_1^2| \\ \max_{x_1 \in S} | - k_1(x_1)| & \max_{x_1 \in S} ( -1 - k_2(x_1)) \end{bmatrix}
\]

Suppose the region of interest is chosen as:

\[
S = \{x_1, x_2 \in \mathbb{R} : |x_1| < 1.2, |x_2| < \infty \}
\]

The maximum value of \( k_1 \) and \( k_2 \) can be deduced from their variation in Figure 6.4. Therefore, the constant overvaluing matrix \( M \) for the domain of interest is:

\[
M = \begin{bmatrix} -1 & 2.88 \\ 0.396 & -1.4142 \end{bmatrix}
\]

Note that \( M \) is a Metzlerian matrix. In constructing the robust comparison system,
we note that the system uncertainty can be fashioned by:

\[
D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

whereby the robust comparison system with system uncertainty is:

\[
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1 + \delta_1 & 2.88 \\ 0.396 & -1.4142 + \delta_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}
\]

Using Eq (6.5) the radius of stability is computed to be:

\[
r_R(M, D, E) = \frac{1}{\|EM^{-1}D\|_2} = 0.081
\]
Example 6.2. Consider the Lotka-Volterra equation:

\[
\begin{align*}
\dot{x} &= \alpha x - \beta xy \quad (6.9) \\
\dot{y} &= \delta xy - \gamma y + u \quad (6.10)
\end{align*}
\]

The obvious SDC parameterization:

\[
A(z) = \begin{bmatrix} \alpha & -\beta x \\ \delta y & -\gamma \end{bmatrix} \quad (6.11)
\]

\[
B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (6.12)
\]

\[
z = \begin{bmatrix} x \\ y \end{bmatrix}
\]

is controllable except at \(x = 0\) for the pair \(\{A(z), B\}\). Let the uncertainty be expressed by:

\[
D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
\Delta = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

In considering the domain of interest, let:

\[
S = \{z \in \mathbb{R}_+: x \in (0, \infty), y \in [0, \infty)\} \quad (6.13)
\]

\[
U = \{\alpha, \beta \in \mathbb{R}_- : \alpha \in (-\infty, 0), \beta \in (-\infty, 0)\} \quad (6.14)
\]

\[
V = \{\delta, \gamma \in \mathbb{R}_+ : \delta \in (0, \infty), \gamma \in (0, \infty)\} \quad (6.15)
\]

be the feasible sets for this problem. The space of the problem is now defined by Eqs (6.13-6.15).

Suppose we are interested in determining the radius of stability in a subset of \(S\).
Specifically, let
\[ S_1 = \{ z \in \mathbb{R}_+ : 1 \leq x \leq 1.5, 0 \leq y \leq 0.5 \} \] (6.16)
be the domain of interest in the state space. In this domain, let \( \alpha = -1, \beta = -2, \delta = 1, \gamma = 2 \). Hence we have:
\[ A(z) = \begin{bmatrix} -1 & 2x \\ y & -2 \end{bmatrix} \] (6.17)

By Definition 6.4 for \( x, y \in S_1 \), Eq (6.17) always possess the Metzlerian structure.

The closed loop system matrix, \( A_{cl} \), which is a function of \( z \), is:
\[ A_{cl}(z) = \begin{bmatrix} -1 & 2x \\ y - k_1(x, y) & -2 - k_2(x, y) \end{bmatrix} \] (6.18)

Upon overvaluing Eq (6.18) for \( z \in S_1 \):
\[ M_{S_1}(z) = \begin{bmatrix} -1 & \max_{z \in S_1} |2x| \\ \max_{z \in S_1} |y - k_1(x, y)| & \max_{z \in S_1} (-2 - k_2(x, y)) \end{bmatrix} \] (6.19)

Using MATLAB, the minimum and maximum values for \( k_1 \) and \( k_2 \) are found to be:
\[
\begin{align*}
    k_1^{\max}(x, y) &= 0.592 \\
    k_1^{\min}(x, y) &= 0.268 \\
    k_2^{\max}(x, y) &= 0.925 \\
    k_2^{\min}(x, y) &= 0.464
\end{align*}
\]

Now the expression in Eq (6.19) is evaluated to be:
6.3. STABILITY RADIUS OPTIMIZATION

\[
M_{S_1} = \begin{bmatrix}
-1 & 3 \\
0.092 & -2.46
\end{bmatrix}
\]

for which the radius of stability is found to be:

\[
r_R(M, D, E) = \frac{1}{\|EM^{-1}D\|_2} = 0.888
\]

The corresponding robust SDRE parameterization is therefore:

\[
A(z) = \begin{bmatrix}
-1 + \delta & 2x \\
y & -2
\end{bmatrix}
\]

suggesting that uncertainties of \(\delta \leq r_R\) will result in stable trajectories in the domain of interest defined by \(S_1\).

Figure 6.2 shows the phase portrait for the example problem with system uncertainty \(\delta = 0.25\). The region in Figure 6.2 bounded by \(S_1\) contains trajectories that converge to the stable equilibrium point at the origin as predicted by Theorem 6.4. It is expected that as the uncertainty is increased, the state trajectories that originate in \(S_1\) will converge asymptotically to the origin of the state space. As can be seen in Figure 6.3, this is indeed the case. Note that in Figure 6.3 the state trajectories seem to be shifting somewhat as if being drawn to another equilibrium point. This is expected also since as the system uncertainty \(\delta\) approaches the radius of stability for the domain of interest, the quantitative behavior of the phase portrait differs. Finally, we note that \(r_R\) is a conservative estimate and the actual radius of stability may be larger.

6.3 Stability Radius Optimization

One important property of SDLQR is that by varying the design parameter \(\alpha\), closed loop system robustness properties can be enhanced. Although there are some rules of thumb and guidelines when it comes to tuning \(\alpha\), a systematic approach would be preferred. In general, \(\alpha\) is chosen so that some meaningful property, such
as the pointwise controllability space of the problem, is maximized. By considering the variation in the radius of stability of a multivariable SDLQR controlled system as $\alpha$ is adjusted, it is possible to obtain the maximally robust parameterization.

In order to address the problem of maximizing the radius of stability for a family of SDC parameterizations under uncertainty using SDRE, a region of interest in the state space needs to be defined. By overvaluing the SDC parameterization in this region, the overall system is brought into a form that possess special properties. The resulting comparison system will be Metzlerian and will be used to ascertain the
6.3. **STABILITY RADIUS OPTIMIZATION**

parameter that robustifies the SDC parameterization.

The main result of this section is that maximizing the radius of stability for a multivariable EL controlled system corresponds to minimizing the constrained spectral norm of a parameterized comparison system. The result is based on the work in [4] and is intended to address the problem of choosing the most robust parameterization arising in a hyperplane defined by two or more valid SDC parameterizations.

For a family of robust comparison systems parameterized by $\alpha$, obtaining the parameter that corresponds with the maximally robust SDRE parameterization is a constrained minimization problem. Specifically, minimizing the spectral norm of the denominator of Eqn (6.5) with respect to $\alpha$ and subject to one of the equivalent stability conditions of Theorem 6.2 expressed in terms of $\alpha$. This produces the $\alpha^*$ that corresponds to the largest radius of stability.

$$
\det(C(x, \alpha)) \neq 0 \quad (6.23)
$$

produces the parameter $\alpha^*$ that leads to the largest radius of stability. This is captured in the following Theorem.

**Theorem 6.5.** For the nonlinear system of the form

$$
\dot{x} = f(x) + g(x)u
$$

with structured parametric uncertainty given by

$$
\dot{x} = [A(x, \alpha) + D\Delta E]x \quad \forall x \in \mathbb{R}^n
$$

where $A(x, \alpha)$ is a valid parameterized SDC representation on the hyperplane

$$
A(x, \alpha) = (1 - \alpha_k)A_{k+1}(x) + \sum_{i=1}^{k} (\prod_{j=i}^{k} \alpha_j)(1 - \alpha_{i-1})A_i(x)
$$

let

$$
\Omega = \{x \in \mathbb{R}^n : x \in (X_L, X_U)\}
$$

be the domain of interest with $X_L$ and $X_U$ defining the lower and upper bound of the
domain respectively and let the psuedo-overvalued constant matrix, \( M(\alpha) \), representing the robust comparison system

\[ \dot{z} = (M(\alpha) + D\Delta E)z \quad \forall z \in \mathbb{R}^k \]

be Metzlerian stable, then the SDC parameterization corresponding to the largest radius of stability

\[ r_R(M(\alpha), D, E) \leq r_R^*(M(\alpha^*), D, E) \]

in the original system is given by \( A(x, \alpha^*) \), where \( \alpha^* \) is the parameter that minimizes the spectral norm of \( EM(\alpha)^{-1}D \) subject to one of the equivalent stability conditions in Theorem 6.2 expressed in terms of \( \alpha \), e.g:

\[ \alpha^* = \arg\min_{\alpha \in [0,1]} \| EM(\alpha)^{-1}D \|_2 \]

Subject to

\[ \det(M(\alpha)) \neq 0 \]
\[ -M(\alpha)^{-1} > 0 \]

**Proof.** Note that maximizing the radius of stability with respect to the parameter \( \alpha \)

\[ \max_{\alpha} r_R(M(\alpha), D, E) \]

is equivalent to minimizing

\[ \min_{\alpha} \| EM(\alpha)^{-1}D \|_2 \]

It can be shown that

\[ \| EM(\alpha)^{-1}D \|_2 = \max \{ \sigma_i \} \]

where \( \sigma_i \) are the singular values of \( (EM(\alpha)^{-1}D)^T(EM(\alpha)^{-1}D) \). Now let,

\[ T = \left\{ \max_{\alpha} \sigma_i; i = 1 \ldots n, \alpha \in [0, 1] \right\} \]
be the set whose elements are the largest singular values of

\[(EM(\alpha)^{-1}D)^T(EM(\alpha)^{-1}D) \quad \forall \alpha \in [0, 1]\]

then we seek the desired \(\alpha\) according to

\[\alpha^* = \arg\inf_{\alpha} T\]

This establishes the proof for Theorem 6.5. \(\square\)

Remark 6.2. Note that \(\sigma_i\) is non-negative, since, by the definition of the matrix 2 norm, for any \(A \in M^{m \times n}\)

\[\|A\|_2^2 = \left\{ \max_{\|v\|=1} \|Av\| \right\}^2\]

\[= \max_{\|v\|=1} \|Av\|_2^2\]

\[= \max \limits_{v^T v = 1} v^T A^T Av\]

and by the method of Lagrange Multipliers

\[L = v^T A^T Av - \lambda(v^T v - 1)\]

we find that

\[A^T Av = \lambda v\]

Since \(v \neq 0\) it is clear that \(v A^T Av = \|Av\|_2^2 = \|v\|_2^2\) so \(\lambda > 0\) and \(\sigma\) is non-negative.

The following example illustrates the utility of this approach.

Example 6.3. For the nonlinear system:

\[
\begin{align*}
\dot{x}_1 &= -x_1 + 2x_1^2 x_2 \\
\dot{x}_2 &= -x_2 + u
\end{align*}
\]
Let the parametric uncertainty be captured by
\[
D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
and let:
\[
A_1(x) = \begin{bmatrix} -1 & 2x_1^2 \\ 0 & -1 \end{bmatrix} \quad A_2(x) = \begin{bmatrix} 0 & 2x_1^2 - \frac{x_1}{x_2} \\ 0 & -1 \end{bmatrix}
\]
with \( B = [0 \ 1]^T \), \( Q = I_{2 \times 2} \), \( R = 1 \). Using Eqn (3.10), the SDC parameterization is
\[
A(x, \alpha) = \begin{bmatrix} -\alpha & -2\alpha x_1^2 + (1 - \alpha)(2x_1^2 - \frac{x_1}{x_2}) \\ 0 & -1 \end{bmatrix}
\]
Although not necessary in general, here we consider overvaluing the closed loop system matrix, \( A_{cl}(x, \alpha) \), to show that the method is applicable to the closed loop system matrix as well. Since \( A(x, \alpha) = A(x_1, \alpha) \), the feedback gains \( k_1 \) and \( k_2 \) depend on \( x_1 \) only. The closed loop matrix \( A_{cl}(x, \alpha) \) is:
\[
A_{cl}(x, \alpha) = \begin{bmatrix} -\alpha & -2\alpha x_1^2 + (1 - \alpha)(2x_1^2 - \frac{x_1}{x_2}) \\ -k_1(x_1) & -1 - k_2(x_1) \end{bmatrix}
\]
Let the region of interest be:
\[
S = \{x_1, x_2 \in \mathbb{R} : |x_1| < 1.2, |x_2| < 1.2\}
\]
The overvaluing matrix \( M(x, \alpha) \) for \( A_{cl}(x, \alpha) \) is:
\[
M(x, \alpha) = \\
\begin{bmatrix}
\max_{x \in S}(-\alpha) & \max_{x \in S} \left| -2\alpha x_1^2 + (1 - \alpha)(2x_1^2 - \frac{x_1}{x_2}) \right| \\
\max_{x_1 \in S}(-k_1(x_1)) & \max_{x_1 \in S}(-1 - k_2(x_1))
\end{bmatrix}
\]
The maximum value of \( k_1 \) and \( k_2 \) can be deduced from their variation in Figure 6.4. Note that
\[
k_1^{max} = 0.396
\]
6.3. **STABILITY RADIUS OPTIMIZATION**

Figure 6.4: Variation of $k_1$ and $k_2$ in $S$

$$k_1^{\text{min}} = 0.00$$

$$k_1^{\text{max}} = 1.07$$

$$k_2^{\text{min}} = 0.414$$

The constant overvaluing matrix $M(\alpha)$ for the domain of interest becomes:

$$M(\alpha) = \begin{bmatrix} -\alpha & \alpha + 1.88 \\ 0.396 & -1.414 \end{bmatrix}$$

Note that $M(\alpha)$ is Metzlerian $\forall \alpha \in (0, 1]$ and it is Hurwitz stable for $\forall \alpha \in (0.731, 1]$.

This implies that for any $0.731 < \alpha < 1$ the stability radius can be computed using Theorem 6.3. The utility of this fact is that even though $\alpha^*$ yields the most robust SDC parameterization with respect to parametric uncertainty, the designer may wish to assess the tradeoff between robustness and closed loop performance by varying $\alpha$.

In this example, we seek the $\alpha$ that maximizes the radius of stability only. Therefore

$$\alpha^* = \arg\min_{\alpha \in (0.731, 1]} \|EM^{-1}(\alpha)D\|_2$$

Figure 6.5 plots the matrix norm of $EM^{-1}D$ and the corresponding radius of stability $r_R(M, D, E)$ as a function of $\alpha$. It can be seen that the maximum radius of stability
Figure 6.5: Thick solid line corresponds to radius of stability and thin line corresponds to the matrix norm. Both are plotted as a function of $\alpha$.

occurs at upper boundary of $\alpha$, so $\alpha^* = 1$ with a radius of stability of $r_R = 0.084$. Therefore, for this problem $A_1(x)$ corresponds to the parameterization that is maximally robust with respect to parametric uncertainty. Note that when additional constraints are added to the optimization problem, $\alpha^*$ may not necessarily be equal to the upper bound of $\alpha$. 
Bibliography


