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REDUCED-ORDER MODELING OF INCOMPRESSIBLE JET FLOW USING PROPER ORTHOGONAL DECOMPOSITION AND GALERKIN PROJECTION

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Abstract

Jet flows are complex and therefore challenging to be described accurately. This work aims at reduced-order modelling (ROM) of a three dimensional incompressible turbulent jet. The Galerkin approximation is utilized to project the governing partial differential equations (PDEs) onto a smaller size of ordinary differential equations (ODEs). A set of mesh grids are created in a cylindrical coordinate system with a nonlinear centerline in the streamwise direction to restore axial symmetry. Azimuthal decomposition is performed first as a pre-purification procedure, followed by the proper orthogonal decomposition (POD) with respect to kinetic energy norm and pressure norm. The capabilities of the POD modes to capture energy of the flow are presented in terms of eigenvalues. Different growing and rotating structures are found in the first few POD modes. The POD modes behaviors are further discussed in the spatial domain and the temporal domain.

The simulation database used in this work requires 50 $Mb$ of binary data to represent a single time snapshot of one flow quantity. Therefore computationally speaking, a significant challenge is the efficient processing of high volume data in a reasonable time with standard PC-capabilities. The method of snapshots is used in the POD procedure to reduce the computational cost.
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Boston, Massachusetts                                      Yiman Hou
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Chapter 1

Introduction

1.1 Motivation

Fluid flows are complex and therefore difficult to be described accurately. Although the fast development of computer hardware and software these days has made numerical simulations possible for complex flows, it is still impossible to solve the governing Navier-Stokes equations describing 3-dimensional flows with high Reynolds numbers. In addition to high computational costs, numerical solutions do not bring much understanding of the flow mechanisms.

It has been observed that the behavior of flows can be described in simple terms involving a lower number of degrees of freedom (DOF). The process of transferring a model with a high number of DOF, either from detailed numerical simulations (DNS) or experimental measurements, to a model with fewer DOF is termed reduced-order modelling (ROM). To achieve a lower order model, one can replace the governing partial differential equations with a smaller set of ordinary differential equations to approximate the original full sized model.

In our work we examine the structure of a 3-dimensional turbulent jet, aiming at
reduced order models. We utilize the Galerkin method to decompose the flow into spatial basis functions and time-dependent amplitudes, followed by an investigation of the temporal and spatial behavior of the lower order model.

1.2 Background

In this section, we state some background information about reduced order modelling and a literature review of the Proper Orthogonal Decomposition technique.

1.2.1 Reduced Order Modelling

There exists a well-established mathematical model for turbulent flows, the Navier-Stokes equations, which are a set of non-linear partial differential equations (PDEs). The nonlinearity makes most real situations impossible to solve, except through numerical simulations for given boundary and initial conditions. The direct numerical simulation (DNS) solver uses millions of mesh points and floating-point operations which require a tremendous amount of computational time and power. Therefore we can benefit from low-order approximate descriptions of the full model flows for less computational cost when studying their complicated behavior.

Reduced order modeling (ROM) is a way to achieve simplified representation of the behavior of a complicated full model with sufficient accuracy. The idea of ROM is to eliminate some degrees of freedom (DOF) of the full size \( n \) by transforming the high-dimensional model into a reduced model with lower DOF \( m \), with \( m \ll n \). The reduced-order model should retain maximum information from the full model among all possible models with the same DOF. Instead of considering the DOF of the full model, one can build a simplified model representing the essential behavior of the full
model with fewer DOF.

Most of the existing ROM methods are based on projection of the original high-dimensional space into a low-dimensional subspace. In this way, the governing PDEs of the full model can be replaced by a smaller set of ordinary differential equations (ODEs). During the past two decades, the method of Proper Orthogonal Decomposition (POD) and Galerkin projection have been widely used to perform model order reduction of the full model problem by projecting the original governing equations onto an optimal set of basis functions and representing the state space of the model using only the first few basis functions, i.e. the basis that contains the highest energy with respect to the inner product definition.

In this thesis, we focus on the method of Proper Orthogonal Decomposition (POD) and a subsequent Galerkin projection of the full sized differential equation model.

1.2.2 Literature Review of POD

The Proper Orthogonal Decomposition (POD) is a powerful and efficient method of high-dimensional data analysis with the aim of obtaining low-dimensional approximations that capture much of the phenomena of interest.


The POD procedure has been applied in various scientific fields ever since its incep-
tion, including fluid mechanics, image processing, pattern recognition, oceanography, weather prediction, heat flow.

The POD method was first introduced by Lumley [12] to identify low-dimensional behavior of the turbulent flow characterized by its coherent structures in 1967. Holmes et al. presented detailed discussion on the application of proper orthogonal decomposition in fluid dynamics [4]. A review of POD in the analysis of turbulent flows is given by Berkooz et al. [22]. The tutorial by Smith et al. [3] provides an overview on low-dimensional modelling of turbulence using POD.

Deane et al. [25] applied POD procedure to the flow in a periodically grooved channel and the flow around a circular cylinder. Their results showed that the models extrapolate reasonably well over a range of Reynolds number values for the grooved channel; however for the cylinder wake, due to the significant spatial wave number changes of the flow with the Reynolds number, the models are only valid in a small neighborhood of the decompositional Reynolds number.

Rajaee et al. [28] applied the method of POD to two-dimensional, two-component hotwire data from the region of an acoustically perturbed free shear layer that includes the first pairing process. It was found that a large fraction of the fluctuation energy is carried by the first few modes and the low-dimensional empirical engenfunction space is capable of describing the shear-flow coherent structures well.

Cazemier et al. [26] performed POD-based reduced order models for driven cavity flows. The POD modes educe the coherent structures in the original flow well and it was found that a closure model is needed to simulate the system of a reduced dimension over a certain Reynolds number.

Citriniti et al. [31], obtained dynamics of the structures from instantaneous realizations of the streamwise velocity field in a single plane and found that the most
energetic modes had azimuthal Fourier coefficient \( n = 5 \) in the shear-layer portion of the jet.

Rowley [27] performed different methods of model reduction on a linearized flow in a plane channel, including POD, balanced truncation and balanced POD. The results show that balanced truncation produces better reduced-order models than POD, but is not computationally tractable for very large systems. Balanced POD is a tractable method for computing approximate balanced truncations, that has computational cost similar to POD.

Lehmann et al. [29], used interpolated POD modes from a succession of low dimensional models from sections of a controled transient manifold on a wake flow behind a circular cylinder. Their results showed that the strategy improved the closed loop performance and optimized sensors location.

Luchtenburg et al. [30], used a tuned POD Galerkin models in the stabilization of the wake flow behind a circular cylinder. They achieved improved ability to suppress vortex shedding and an improved sensor performance over a wider transients range.

Schlegel et al. [33] performed reduced-order modelling strategy on a snapshot ensemble of an incompressible jet. In their research, the coherent flow structures of the jet were identified by the POD and wavelet analysis. They achieved a reduction of the degree of freedom by one order of magnitude.

1.2.3 Jet Flows

Our reduced-order modelling scheme is implemented on the flow quantities of a jet flow. The purpose of this section is to provide some basic understanding of jet flows.

The behavior of jet flows vary greatly, mainly depending on two parameters.
first is the Reynolds number $Re$, which is the ratio of inertial forces to viscous forces. For a given geometry, the character of a confined velocity field depends on this single dimensionless number. It has been found in the literature of fluid dynamics, that flow patterns change their character as the Reynolds number takes on different values. If the Reynolds number $Re$ is greater than 2300, the jet flow will behave dramatically with turbulent structures.

Another important parameter is the Mach number $Ma$, which is the ratio of the flow velocity to the speed of sound. If the Mach number of the flow is low, i.e. all velocities are small compared to the speed of sound, the flow is modeled as an incompressible flow, usually, $Ma < 0.3$ for incompressibility assumption. For compressible flow, its Mach number $Ma$ often exceeds 0.3. Gröschel et al. [6] compared the compressible jet flow and incompressible LES at the same Reynolds number, and their results showed that the incompressible flow is dynamically more complex.

In this thesis, we focus on reduced-order modelling of an incompressible jet flow at Reynolds number $Re = 3600$. Detailed description of the simulation database can be found in Chapter 2.

### 1.3 Organization of Thesis

This thesis focuses on the dimension reduction and model extraction of jet flow.

In Chapter 2, the simulation database considered in this thesis is presented. It is a three-dimensional incompressible jet at Reynolds number $Re = 3600$ quantified by flow velocities and pressure with uniformly distributed grid points along all three directions. The constant value surface visualizations of the streamwise velocity and pressure illustrating the physical dimensions of the jet are shown in Figure 2.1 and Figure 2.2.
Chapter 3 introduces the concept of Galerking Approximation. The procedure of Proper Orthogonal Decomposition (POD) and its discrete version, Singular Value Decomposition (SVD) are explained. The method of snapshots is discussed for handling large amount of data ensembles.

In Chapter 4, the reduced-order modelling procedure that has been performed on the jet data is discussed in details. The cylindrical representation of the simulation database is introduced to restore the axial symmetry and the azimuthal decomposition is performed as a pre-purification procedure followed by the POD procedure on the first 11 azimuthal modes. The kinetic energy norm and pressure norm are used to complete the approximation. A flow chart of the reduced-order modelling scheme is shown in section 4.5.

Chapter 5 presents the simulation results using the Galerkin approximation described in Chapter 4. The energies in the POD modes are listed in Table 5.1 and Table 5.2. The POD modes are visualized by constant surfaces of the dominant streamwise velocity and pressure for the most energetic azimuthal mode using both norm in Figure 5.5 - 5.8.

Finally in Chapter 6, a summary of work done in this thesis is presented as well as future directions as a closing phase.
Chapter 2

Simulation Database

In this section, we give a detailed description of the simulation database we used in this thesis.

The simulation database is an ensemble of the snapshots measurements of a 3-dimensional incompressible jet flow at Reynolds number $Re = 3600$. The $x$-direction of the database is the streamwise direction, the $y$-direction is the transverse direction, and the $z$-direction is the spanwise direction.

The ensemble of snapshots are interpolated from a Large Eddy Simulation (LES) by P. Comte with 6-th order compact difference scheme in the streamwise direction; spectral order accuracy in the transverse and spanwise direction; and 3-rd order in time. The details of the numerical simulation technique LES are illustrated in [8][9]. The database is a structure function subgrid model, simulated at a constant temperature with nondimensionalisation. The velocity field of the flow are measured in 3 directions as well as the pressure of the flow.

The jet has a grid interval of $[0, 28D] \times [-3.5D, 3.5D] \times [-3.5D, 3.5D]$ with $D$ being the jet diameter, which gives a computational domain of $28D$ in the $x$-direction.
Figure 2.1 – 3D iso-surface visualization of the streamwise velocity at snapshot 1000 showing the physical dimension of the flow. $D$ is the jet diameter.

and $7D$ in the $y$- and $z$-direction. The origin $(x, y, z) = (0, 0, 0)$ is the point of intersection of the inflow plane and the jet axis.

Figure 2.1 and Figure 2.2 are constant value surface visualizations of streamwise velocity and pressure of the 1000-th snapshot in the ensemble from different angle of view. From the two figures we can see clearly the size and physical dimensions of the jet in the Cartesian coordinate system. The dominant direction of the flow is the streamwise $x$-direction, but the fluctuation makes the space of the flow movement grow axially and temporally as shown in the figures.

There are $200 \times 128 \times 128$ uniform grid nodes in this simulation database, 200 in the streamwise $x$-direction and 128 in transverse $y$- and spanwise $z$-direction. These grid points figuratively form a “data box” of size $28D \times 7D \times 7D$ with equidistantly
Figure 2.2 – 2D iso-surface visualization of the pressure at snapshot 1000 showing physical dimension of the flow. $D$ is the jet diameter.

distributed mesh grids.

In the course of the simulation, the measurements of the pressure and the velocities in three directions were saved every 0.1 convective time. Each sample of these flow quantities is referred to as a snapshot. There are totally 5929 snapshots in this numerical simulation, of which the last 5400 snapshots are considered as a steady state ensemble equidistantly distributed over a time interval of 540 convective time units. This 5400 snapshot ensemble is used to compute the POD modes in this thesis.

Computationally, 50 $Mb$ of binary data represent a single time snapshot of the incompressible 3-dimensional jet simulation, which requires approximately 370 $Gb$ space for restoring the 5400 snapshot ensemble of the velocity field and the pressure. Therefore a significant challenge of the thesis is the efficient processing of such high volume data in reasonable time with standard PC-capabilities.
Chapter 3

Galerkin Approximation and Proper Orthogonal Decomposition

In this section, we discuss about the Galerkin method and the proper orthogonal decomposition technique that are utilized in the reduced-order modelling analysis in this thesis. In section 3.6, the decomposition steps are summarized.

3.1 Galerkin Projection

The procedure of Galerkin projection together with Proper Orthogonal Decomposition is widely used in the procedure of model reduction. The reduction procedure in this study is to project the original statespace onto a low-dimensional subspace based on POD and a subsequent Galerkin projection of the Navier-Stokes equations.

The Navier-Stokes equations describing the fluid motions are derived from Newton’s second law applied to fluid motion. For an incompressible flow, they are given
in terms of velocity $\mathbf{u}$, pressure $p$, density $\rho$ and Reynolds number $Re$ as follows

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{-1}{\rho} \nabla^2 p + \frac{1}{Re} \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$  \hspace{1cm} (3.1)

The Navier-Stokes equations are a set of Partial Differential Equations (PDEs), which have infinitely many Degrees of Freedom (DOF). The non-linear term in the equations makes them difficult to solve. We are seeking solutions on a finite spatial grid, which can be obtained via the method of Galerkin projection. In the Galerkin projection procedure, the governing PDEs of the full sized model are projected onto a subspace spanned by a set of orthogonal basis functions, which results in low-dimensional sets of Ordinary Differential Equations (ODEs).

To illustrate the procedure of Galerkin projection, we consider a simplified version of PDE as

$$\dot{\mathbf{u}}(t) = f(\mathbf{u}(t); \mu)$$  \hspace{1cm} (3.2)

where parameter $\mu$ represents system parameters. Assume $\mathbf{u}(t)$ are evaluated on some Hilbert space $\mathcal{H}$, and a set of orthogonal basis functions forms a subspace $\text{span}(\{\varphi_i\}_{i=1}^r) = \mathcal{H}_r \subset \mathcal{H}$. Then $\mathbf{u}(t)$ can be projected onto this subspace and expressed as a linear combination of the basis functions

$$\mathbf{u}_r(t) = \sum_{i=1}^r \alpha_i(t) \varphi_i$$  \hspace{1cm} (3.3)

Substituting equation (3.3) into equation (3.2), we can obtain
Using the orthogonality of the basis, we can get a set of ODEs by taking the inner product with $\varphi_j$ on both side of the equation

$$\sum_{i=1}^{r} \dot{\alpha}_i(t) \varphi_i = f(\sum_{i=1}^{r} \alpha_i(t) \varphi_i; \mu)$$

(3.4)

$$\dot{\alpha}_j(t) = (f(\sum_{i=1}^{r} \alpha_i(t) \varphi_i; \mu), \varphi_j)$$

(3.5)

In equations (3.2) - (3.5), if the number of basis functions $r$ goes up to its largest possible value, here, the largest $r$ is the number of grid points in the snapshot ensemble, the set of ODEs approximates the numerical solution of PDEs with very high accuracy. We are expecting that the ODE set can capture most of the phenomenon of the flow with a much lower value of $r$, therefore we truncate equation (3.2) to achieve a lower order model.

The set of orthogonal basis spanning the low-dimensional space are extracted by the POD procedure, and refered to as POD modes. Galerkin projection is then applied to the Navier-Stokes equations on the POD modes to obtain a reduced-order model.

### 3.2 Proper Orthogonal Decomposition

The Proper Orthogonal Decomposition, also known as the Karhunen-Loève Decomposition, has been applied in the procedure of reducing large amounts of high-dimensional data to low-dimensional discriptions by projecting the full model onto a set of basis functions then approximating the behavior of the model using only the first few basis functions.
Mathematically speaking, the procedure of POD seeks for an decomposed representation of a function over some domain of interest

\[ \Phi(x, t) = \sum_{i=1}^{N} \alpha_i(t) \varphi_i(x) \]  

(3.6)

where \( \Phi(x, t) \) is the data function we want to reduce, which is defined as a function of \( x \) and \( t \). This equation describes the function \( \Phi(x, t) \) in a way that \( x \) and \( t \) are in separate terms. The set \( \{ \varphi_i(x) \}_{i=1}^{N} \) forms a basis for the function, well \( \{ \alpha_i(t) \}_{i=1}^{N} \) are referred to as the coefficient functions.

If we can make \( N \) large enough, we can easily achieve the accuracy required. However, in most cases, we want to achieve maximum accuracy given a limited number \( N \). Therefore the basis functions extracted by POD have to be orthonormal and optimal. The orthonormality ensures that the limited basis functions selected do not contain redundant information with respect to each other, while optimality is required to capture maximum information on the reduced space. That is, the sequence of orthonormal basis functions \( \{ \varphi_i(x) \}_{i=1}^{N} \) must be in an order that the first \( N \) basis functions provides the best possible \( N \)-term approximation of the function \( \Phi(x, t) \).

Suppose the function \( \Phi(x, t) \) represents a flow quantity, we usually refer \( x \in \Omega \) as the spatial domain, typically a 2-dimensional or 3-dimensional vector field, and \( t \in T \) as the temporal domain. The POD basis functions, or POD modes are choosen according to the “energies”, which makes the set of POD modes the optimal solution of finding a low-order approximation that captures the highest kinetic energy if we are considering velocity measurement. We will discuss this more in section 3.2.

Once we obtained the basis functions, the coefficient functions \( \{ \alpha_i(t) \}_{i=1}^{N} \) can be determined accordingly. In the decomposed equation (3.6), there is no fundamental difference between \( x \) and \( t \), therefore we can choose one domain for the basis functions.
and the other for the coefficient functions for computational convenience, which means the decomposition can be considered as an expansion with spatial modes and temporal coefficients, or alternatively as an expansion with temporal modes and spatial coefficients. Therefore theoretically, one can do decomposition in the spatial domain or decomposition in the temporal domain.

3.3 POD of Flow Quantities

Now let’s discuss the decomposition of the velocity filed of the flow in the spatial domain. The velocity of the flow, which is evaluated over the spatial domain $x \in \Omega$ and the temporal domain $t \in T$, can be considered as a steady base of the flow and a fluctuating velocity

$$\Phi(x, t) = \Phi_0(x) + \tilde{\Phi}(x, t)$$  \hspace{1cm} (3.7)

where $\Phi_0$ is the steady base flow and $\tilde{\Phi}$ is the fluctuation. We are looking for a representation of the fluctuation via the proper orthogonal decomposition

$$\tilde{\Phi}(x, t) = \sum_{i=1}^{N} \alpha_i(t) \varphi_i(x)$$  \hspace{1cm} (3.8)

where $\{\varphi_i\}_{i=1}^{N} \subset L_2(\Omega)$ are the POD basis functions and $\alpha_i$ are the time coefficients.

With an appropriately defined inner product in the spatial domain, the optimal basis function can be obtained by solving the following eigenvalue problem for each $i$

$$\int_{\Omega} dx' R(x, x') \varphi_i(x') = \lambda_i \varphi_i(x)$$  \hspace{1cm} (3.9)
where $\varphi_i(x)$ is usually referred to as the $i$-th POD modes, well $\lambda_i$ is the $i$-th eigenvalue. $R(x, y)$ is the average autocorrelation function and defined in the book published by Holmes et al. [4] as

$$R(x, x') = \langle \tilde{\Phi}(x) \tilde{\Phi}^*(x') \rangle$$

(3.10)

where $\langle \cdot \rangle$ stands for the time average. According to Holmes et al., $R$ is self-adjoint and positive semidefinite, which give the POD modes the following properties:

1. The POD modes are orthogonal:

$$\int_{\Omega} dx \varphi_i(x) \varphi_j(x) = \delta_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ otherwise} \end{cases}$$

(3.11)

2. The POD modes are optimal: The eigenvalues $\lambda_i$ are real and positive, and the POD modes are in an order that the corresponding eigenvalues are decreasing $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0$.

Once the POD modes are obtained, the time-dependent coefficients can be determined accordingly

$$\alpha_i(t) = \int_{\Omega} \tilde{\Phi}(x, t) \varphi_i(x) dx$$

(3.12)

which implies that for the POD basis, the coefficient function $\alpha_i(t)$ depends only on $\varphi_i(x)$, not $\{\varphi_j(x)\}_{j \neq i}$.

Computationally, both the spatial domain $\Omega$ and the temporal domain $T$ are discrete domains, we should only refer to the discrete version of the problem, hence the integration is substituted by weighted matrix multiplication, and the procedure of POD is given in terms of the Singular Value Decomposition (SVD). A series of snapshots are taken at different time to represent the measurements of the flow quantities,
and SVD are applied to obtain a set of optimal basis functions. Then the governing PDEs of the flow are projected onto a space spanned by fewer POD basis functions, therefore the flow field is described by a smaller number of ODEs.

### 3.4 Singular Value Decomposition

In order to obtain the POD modes, we need to apply the discrete version of POD, the Singular Value Decomposition (SVD) for computational purpose. Given an ensemble of \( N_T \) snapshots representing the flow quantities \( \Phi(x, t_i) \), where \( i \in \{1 \ldots N_T\} \), we can extract the fluctuation by removing the ensemble average

\[
\tilde{\Phi}(x, t_i) = \Phi(x, t_i) - \frac{1}{N_T} \sum_{j=1}^{N_T} \Phi(x, t_j)
\]  

(3.13)

This means the \( i \)-th row of the matrix \( D \) contains the measurements of all the snapshots at the \( i \)-th position; well the \( j \)-th column of matrix \( D \) contains the measurements of the flow in the whole spatial domain at time \( t_j \).

The Singular Value Decomposition of matrix \( D \) is defined as

\[
D = U \Sigma V^T = \sum_{i=1}^{\min(N_T,m)} \sigma_i U_i V_i^T
\]  

(3.14)
where \( U \) is an orthogonal matrix of size \( m \times m \) with its columns \( U_i \) being referred to as the left-singular vectors of matrix \( D \); \( V \) is an orthogonal matrix of size \( N_T \times N_T \), and its columns \( V_i \) are the right-singular vectors. All these singular vectors are orthonormal vectors. \( \Sigma \) is an \( m \times N_T \) matrix with diagonal elements \( \sigma_i \) and other elements valued 0. The diagonal elements of matrix \( \Sigma \) are the singular values of matrix \( D \), which are in decreasing order \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_{\min(N_T,m)} \geq 0 \).

Let \( W^T = \Sigma V^T \), then we can get the discrete form of equation (3.8)

\[
D = UW^T = \sum_{i=1}^{\min(N_T,m)} U_i W_i^T
\]

(3.15)

where \( U_i^T \) represents the \( i \)-th POD mode \( \varphi_i(x) \), and \( W_i = \sigma_i V_i \) is the vector of time coefficient function \( \alpha_i(t) \).

Since matrix \( U \) is orthonormal, i.e. \( U^T U = I \), we can calculate the time coefficient matrix \( W \)

\[
W^T = U^T D
\]

\[
W_i^T = U_i^T D
\]

(3.16)

Equation (3.16) is the discrete form of equation (3.12), which implies that the \( i \)-th time coefficient depends only on the \( i \)-th POD basis in the spatial domain.

The POD approximation identifies a lower dimensional subspace \( \mathcal{H}_r \subset \mathcal{H} \) spanned by fewer POD modes

\[
\mathcal{H}_r = \text{span}(\{U_i\}_{i=1}^{N_r})
\]
where \( N_r \ll N_T \), \( m \) is the DOF of the reduced space. We can get the rank \( N_r \) approximation of the matrix \( D \) by setting all the singular values \( \sigma_i \) to 0 for \( i > N_r \).

\[
D_r = \sum_{i=1}^{N_r} \sigma_i U_i V_i^T = U \Sigma_r V^T
\]  

(3.17)

Computationally, we only need the first \( N_r \) columns of the matrices \( U \) and \( V \), and the non-zero singular values in matrix \( \Sigma_r \).

### 3.5 Method of Snapshots

The method of snapshots is introduced by Sirovich *et al.* [5] as a more efficient way to perform POD on huge data problems than directly solving the eigenvalue problem. In most cases, for a 3-dimensional flow, the number of grid points for each snapshot \( m = \text{size}(\Omega) \) is much larger than the number of snapshots in the ensemble \( N_T = \text{size}(T) \). In such situations, the right singular vectors \( V_i \) are computed first by solving an \( N_T \times N_T \) eigenvalue problem.

\[
D^T D = V \Sigma^2 V^T
\]  

(3.18)

The method of snapshots can be considered as a discretization of POD procedure in the temporal domain, which finds the time-dependent coefficients first by solving a smaller eigenvalue problem. The direct method of POD in the spatial domain involves solving an eigenvalue problem on the Hilbert space \( \mathcal{H} \), which has a dimension as the number of grid points in the simulation of the data, typically millions of them. By contrast, the method of snapshots only works with an eigenvalue problem of size as the number of snapshots taken, which is much smaller than the number of grid points involved in the simulation. Therefore the method of snapshots alleviates the
computational burden and has been widely used to efficiently determine the POD eigenfunctions for large problems.

3.6 POD Computation in This Thesis

In this section we briefly outline the notations and steps we used to perform POD on the simulation database.

As described before, the database is an ensemble of snapshots representing the flow quantities, which are functions of space and time \( \Phi(x, t) \), where \( x \in \Omega, \ t \in T \). Since the number of grid points for each snapshot \( m = \text{size}(\Omega) \), is much larger than the number of snapshots in the ensemble, \( N_T = \text{size}(T) \), the method of snapshots [5] is applied in this thesis.

The POD modes are computed as follows.

1. First, we compute the spatial covariance matrix \( M \) of the snapshots, whose elements \( M_{i,j} \) are the inner products of two snapshots

\[
M_{i,j} = \int_{\Omega} \tilde{\Phi}(x, t_i) \tilde{\Phi}(x, t_j) dx
\]

where \( \tilde{\Phi}(x, t_i) \) is the fluctuation of the \( i \)-th snapshot

\[
\tilde{\Phi}(x, t_i) = \Phi(x, t_i) - \frac{1}{N_T} \sum_{j=1}^{N_T} \Phi(x, t_j)
\]

2. Then we can solve the following eigenvalue problem via Singular Value Decomposition(SVD) of the matrix \( M \)

\[
Mv_n = \lambda_n v_n \quad n = 1, 2, \ldots, N_T
\]
where each \( \mathbf{v}_n = [v_{n1} \cdots v_{nN_T}]^T \), is an eigenvector of this \( N_T \times N_T \) matrix, and \( \lambda_n \) is the corresponding eigenvalue.

3. Next, the \( n \)-th POD mode can be computed as

\[
\varphi_n(\mathbf{x}) = \frac{1}{\lambda_n} \sum_{i=1}^{N_T} v_{ni} \tilde{\Phi}(\mathbf{x}, t_i)
\]  

The \( n \)-th time-dependent amplitude is then computed by taking the inner product of the snapshot fluctuation and the \( n \)-th POD mode

\[
\alpha_n(t) = \int_{\Omega} \tilde{\Phi}(\mathbf{x}, t) \varphi_n(\mathbf{x}) d\mathbf{x}
\]  

4. The POD modes \( \varphi^{(n)}(\mathbf{x}) \) form an optimally convergent series representation of the snapshot fluctuation with a decreasing order of the eigenvalues, \( \lambda^{(1)} \geq \lambda^{(2)} \geq \cdots \lambda^{(N_T)} \). Therefore the reconstructed fluctuation function can be expressed as a weighted sum of the first \( N_r \) (\( \ll N_T \)) POD modes

\[
\tilde{\Phi}_{N_r}(\mathbf{x}, t) = \sum_{n=1}^{N_r} \alpha_n(t) \varphi_n(\mathbf{x})
\]  

which is an optimal approximation given \( N_r \), in a sense that it can capture the most fluctuation energy than any other basis of size \( N_r \).
Galerkin Approximations of the Jet Flow

In this section, we present the procedure of Galerkin approximation on the jet flow database described in Chapter 2. A set of mesh grids are created in a cylindrical coordinate system with a nonlinear center line in the streamwise direction to restore symmetric structure of the flow. The azimuthal decomposition is performed on the simulation database followed by the method of POD with norms discussed in details. The flow chart in section 4.5 outlines the procedure of this reduced-order analysis.

4.1 Cylindrical Representation of the Database

Figure 4.1 shows the isolines of the pressure and velocities in 3 directions on the transverse section at $x = 7.0352D$ from snapshot 1000. We can see from the figures that there is no structural difference between the $y$ (spanwise) direction and the $z$ (transverse) direction, hence we can assume that the flow possesses symmetry in the
Therefore, to better capture the structure of the turbulent jet flow, it is helpful to restore symmetry as a preprocess of the flow quantities in the azimuthal direction. We can create new meshgrids in a polar coordinate system for each $x$ position, and therefore form a cylindrical coordinate system for the whole model. The new grid points should have similar density and distribution as the grids in the original Cartesian coordinate system to be capable of representing detailed flow structures. Then we can transform
the measurements of the flow quantities from the original Cartesian coordinate system into the cylindrical coordinate system.

The flow may shift in the $y$- and $z$-directions with respect to the streamwise direction, which means that the “spine” of the flow may not stay in a straight line in the $x$-direction. Hence the meshgrids in a cylindrical coordinate system with a nonlinear center line along the streamwise direction can better represent the flow movement. Therefore, for each transverse section, i.e. an $x$-position, we estimate the center of the flow given the measurements of the flow quantities on the section, then create grid points in a polar coordinate system with the pole being the center we found.

### 4.1.1 Spine of the Cylindrical Coordinates

For each transverse section slice of the flow, we can decide the center spot based on the flow quantity values of the grid points from the original rectangular coordinate under the assumption that the flow reaches the highest value at the center spot. Consider the measurement of the flow quantities at each grid point as the “probability” of the flow at that position, then the center of the flow on the transverse section can be decided as the spot with the highest weighted probability.

First, we create a probability density function on each transverse section slice according to the flow quantity values. Let $q(y, z)$ be the flow quantities on a transverse section slice at position $(y, z)$ in the rectangular coordinate system, then the probability density function $p(y, z)$ on this slice can be calculated as

$$p(y, z) = \frac{q(y, z) - \min_{y,z} (q(y, z))}{\sum_{y,z} (q(y, z) - \min_{y,z} (q(y, z)))}$$  \hspace{1cm} (4.1)
where $-3.5D \leq y, z \leq 3.5D$ are the computational domain in the two directions. And clearly, $\sum_{y,z} p(y, z) = 1$, which makes $p(y, z)$ a valid probability density function.

Then the position of the center $(c_y, c_z)$ can be defined as the expectations of the marginal probability functions in the two directions

$$
p(y) = \sum_z p(y, z) \quad c_y = \int_{-3.5D}^{3.5D} p(y) \times y \times dy
$$

$$
p(z) = \sum_y p(y, z) \quad c_z = \int_{-3.5D}^{3.5D} p(z) \times z \times dz
$$

By averaging the central positions based on pressure and the velocities in 3 directions, we can finally get the coordinates of the center in each transverse section slice. Connecting the centers of all the 200 slices in the streamwise direction, we can get a nonlinear center line, or the “spine” of the flow.

Figure 4.2 shows the center line of the flow from snapshot 1000. The maximum distance the “spine” is away from the streamwise axis of all the 5929 snapshots is
0.262 $D$, which is approximately 3.74% with respect to the computational domain in the $y$- and $z$- directions.

### 4.1.2 Axially Symmetric Grids

As stated before, to restore symmetry in the azimuthal direction, we can create axial symmetric grids in a polar coordinate system of each transverse section slice. Hence we can generate grid points on a series of concentric circles with respect to the center position calculated in section 4.1.1.

As shown in Figure 4.1, the flow quantities are generally more active near the center, therefore the cylindrical grids should have higher density close to the center than far away from it. For each transverse section, the mesh grid we generated has following structures:

- the mesh grid contains 64 concentric circles, the radii of which vary from $\frac{3}{64}D$ to $3D$ with equal increment;
- the number of grids on each concentric circle changes from 66 to 462.

There are, in all, 16878 grid points on each transverse section, which is comparable with the $128 \times 128 = 16384$ grid points in the original rectangular coordinate system.

After creating the new mesh grids in the cylindrical coordinate system with a non-linear center line, the flow quantity values in each new grid points can be determined based on the nearest 4 grid points from the original rectangular coordinate system by interpolation.

### 4.2 Azimuthal Decomposition

The azimuthal contributions to the flow can be related to structures with azimuth-
thall symmetry. According to [7], for periodic coordinate directions, Fourier modes are identical to POD modes. Therefore, to simplify the POD computation on the snapshots, we obtain the azimuthal modes first as a prepurification procedure.

The flow quantities $\Phi(x, r, \theta, t)$, are evaluated in the temporal domain $T$ and spatial domain $\Omega$, with $x \in [0, 28D], r \in [0, 3D], \theta \in [0, 2\pi], t \in T$. The flow can be considered as a combination of the steady base flow and the fluctuation contribution

$$\Phi(x, r, \theta, t) = \Phi_0(x, r, \theta) + \tilde{\Phi}(x, r, \theta, t) \quad (4.3)$$

where $\Phi_0(x, r, \theta)$ is a time-averaged field. Then the azimuthal decomposition of the flow fluctuation is given by

$$\tilde{\Phi}(x, r, \theta, t) = \sum_{k=-\infty}^{+\infty} \Phi^{(k)}(x, r, t) e^{jk\theta} \quad (4.4)$$

where $\Phi^{(k)}(x, r, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Phi}(x, r, \theta, t)e^{-jk\theta}d\theta$ are referred to as the azimuthal modes.

In practice on our simulation database, the fluctuation of the flow is obtained by removing the average of the snapshot ensemble, and then the Fourier decomposition is performed on each of the concentric circles in the cylindrical coordinate system we generated, in the form of Discrete Fourier Transform, That is we find the azimuthal modes of $\tilde{\Phi}$ at a fixed $x$ and $r$. Computationally, we use the symmetry property of the Fourier coefficients to reduce the computational cost, i.e. $\Phi^{(k)}(x, r, t) = \Phi^{(-k)*}(x, r, t)$, its complex conjugate.

Mathematically the Fourier decomposition gives us infinite number of azimuthal modes. In this thesis, we keep $N_\theta = 10$ Fourier modes. Let $\Phi^{(\pm k)}(x, r, t)$ be the $k$-th azimuthal compound mode, then we can get the approximation of the flow fluctuation using $N_\theta$ azimuthal compound modes
\[
\Phi_{N_\theta}(x, r, \theta, t) = \sum_{k=0}^{N_\theta} \Phi^{(\pm k)}(x, r, \theta, t) \ e^{\pm jk\theta} \tag{4.5}
\]

### 4.3 Proper Orthogonal Decomposition of Azimuthal Modes

After the decomposition in the azimuthal direction, the reduced-order representation using \(N_r\) POD modes is obtained for each \(k = 0, \ldots, N_\theta\).

\[
\Phi^{(k)}_{N_r}(x, r, t) = \sum_{i=1}^{N_r} \alpha^{(k)}_i(t) \varphi^{(k)}_i(x, r) \tag{4.6}
\]

where \(\alpha^{(k)}_i(t) = (\Phi^{(k)}(x, r, t), \varphi^{(k)}_i(x, r))_{X \times R}\) is the \(i\)-th time-dependent coefficient, i.e. the inner product of the fluctuation and the \(i\)-th POD mode. Here, \(x \in X\), \(r \in R\) are the computational domain of the streamwise direction and the radial direction.

Since we have over 3 million grid points in the spatial domain, and only 6 thousand snapshots, i.e. \(\text{size}(T) \ll \text{size}(\Omega)\), the indirect method of snapshots [5] can greatly alleviate the computational burden.

The matrix of the inner products of snapshots \(M^{(k)}\) contains the inner products of each pair of snapshots

\[
M^{(k)}_{ij} = (\Phi^{(k)}(x, r, t_i), \Phi^{(k)}(x, r, t_i))_{X \times R} \tag{4.7}
\]

which is an \(N_T \times N_T\) matrix. Note that after the azimuthal decomposition, the 3D integrals to calculate the inner products in the spatial domain \(\Omega\), equation (3.19), are
replaced by 2D integrals towards the streamwise and radial coordinates (shown in equation (4.14) and (4.15)).

The eigen decomposition of matrix $M$ is

$$M^{(k)}v^{(k)}_n = \lambda^{(k)}_n v^{(k)}_n$$

which can be solved via Singular Value Decomposition (SVD) from equation 3.18

$$M^{(k)} = V^{(k)} \Sigma^{(k)2} V^{(k)T} = \sum_{n=1}^{N_T} \sigma^{(k)2}_n v^{(k)}_n v^{(k)T}_n$$

with $\lambda^{(k)}_n = \sigma^{(k)2}_n$ being the $i$-th eigen value, and $v^{(k)}_n = [v^{(k)}_{n1} \cdots v^{(k)}_{nN_T}]^T$ being the corresponding eigen vector.

Then the POD modes are described as a linear combination of the snapshots

$$\varphi^{(k)}_n(x, r) = \frac{1}{\lambda^{(k)}_n} \sum_{i=1}^{N_T} v^{(k)}_{ni} \Phi^{(k)}(x, r, t_i)$$

Since each snapshot in the data ensemble is evaluated in $m = \text{size}(|\Omega|)$ grid points in the spatial domain, the discrete form of equation (4.10) is

$$\begin{bmatrix}
\varphi^{(k)}_n(x_1) \\
\vdots \\
\varphi^{(k)}_n(x_m)
\end{bmatrix} = \frac{1}{\lambda^{(k)}_n} \begin{bmatrix}
\Phi^{(k)}(x_1, t_1) & \cdots & \Phi^{(k)}(x_1, t_{N_T}) \\
\vdots & \ddots & \vdots \\
\Phi^{(k)}(x_m, t_1) & \cdots & \Phi^{(k)}(x_m, t_{N_T})
\end{bmatrix} \begin{bmatrix}
v^{(k)}_{n1} \\
\vdots \\
v^{(k)}_{nN_T}
\end{bmatrix}$$

where $\Phi^{(k)}(x, r, t_i) = [\Phi^{(k)}(x_1, t_i) \cdots \Phi^{(k)}(x_m, t_i)]^T$ represents the snapshot taken at time $t_i$ and vector $[\varphi^{(k)}_n(x_1) \cdots \varphi^{(k)}_n(x_m)]^T$ is the $n$-th POD mode.
The POD modes are sorted in a decreasing order with respect to the corresponding eigenvalues, i.e. $\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \ldots \geq \lambda_{NT}^{(k)} \geq 0$. In this way, the eigenvectors form an optimal basis in a sense that it provides more accurate approximation of the flow fluctuation than any other basis of the same size. Also, the POD modes are orthonormal with respect to the inner product used in the procedure, i.e. $(\varphi_i^{(k)}(x,r), \varphi_j^{(k)}(x,r))_{X \times R} = \delta_{ij}$.

Given the assumption of the rotation symmetry of the jet, it is known that POD modes are identical to Fourier modes with respect to the azimuthal angle. Gröschel et al. [6] showed that the Fourier decomposition can reduce the number of snapshots needed to converge the mode statistics.

4.4 Norms Used in This Thesis

4.4.1 Weighted Mesh Grids

As stated before, the integration is substituted by weighted matrix multiplication since both the spatial domain $\Omega$ and the temporal domain $T$ are discrete. The mesh grids we generated in our cylindrical coordinate system don’t have uniform density, therefore a reasonable weight for a grid point can be proportional to the “area” around it.

For each grid point in the cylindrical coordinate system, its position can be represented as $(x, r_n, \theta)$, where $r_n$ is the radius of the $n$-th concentric circle that the grid is on. Then the weight for this point is defined as

$$w_{(x,r_n,\theta)} = \frac{(\pi((r_n + \frac{1}{2}\Delta r)^2 - (r_n - \frac{1}{2}\Delta r)^2)) / N_n}{\pi \cdot 3^2} \quad (4.12)$$
where $\Delta r = r_{n+1} - r_n$ is the radius difference of the concentric circles, and $N_n$ is the number of grid points on the concentric circle with radius $r_n$. The "area" around the grid point is divided by $(\pi \cdot 3^2)$, the largest circular area, to normalize the weights. For the center spot of each transverse section slice, the weight is

$$w_{\text{center}} = \frac{\pi \cdot \left(\frac{1}{2} \Delta r\right)^2}{\pi \cdot 3^2}$$

(4.13)

Obviously, the weight of a grid point depends only on the radius of the concentric circle the grid is on, therefore we can simplify the notation into a function of $r$ only, i.e. $w(r) = w(x, r, \theta)$.

**4.4.2 Kinetic Energy Norm and Pressure Norm**

The measurements of flow quantities in our simulation database contain pressure and velocity in a 3-dimensional field. In this thesis, we use the Kinetic energy norm and pressure norm to perform Galerkin projection.

- **Kinetic energy norm**

  The snapshots are measurements of the 3-dimensional velocity field

  $$\Phi^{(k)}(x, r, t_i) = \begin{bmatrix} v_x^{(k)}(x, r, t_i) \\ v_y^{(k)}(x, r, t_i) \\ v_z^{(k)}(x, r, t_i) \end{bmatrix}$$

  The fluctuation kinetic energy norm is often used in incompressible flow, which is the integration of the 3-dimensional velocities over the spatial domain. The inner product of two snapshots with respect to the Kinetic energy is defined as
\begin{equation}
(\Phi^{(k)}(x, r, t_i), \Phi^{(k)}(x, r, t_j))_{X \times R} = 
\sum_{l=\{x,y,z\}} \int_{R} r \cdot dr \int_{X} w(r) \cdot v^{(k)}_l(x, r, t_i) \cdot v^{(k)}_l(x, r, t_j) \cdot dx \tag{4.14}
\end{equation}

- Pressure norm

The flow quantities of interest is the measurement of pressure

\[ \Phi^{(k)}(x, r, t_i) = p^{(k)}(x, r, t_i) \]

The pressure norm represents the pressure of the flow, and is computed by integrating the pressure of the jet over the spatial domain. The inner product of two snapshots with respect to the pressure is

\[ (\Phi^{(k)}(x, r, t_i), \Phi^{(k)}(x, r, t_j))_{X \times R} = \int_{R} r \cdot dr \int_{X} w(r) \cdot p^{(k)}(x, r, t_i) \cdot p^{(k)}(x, r, t_j) \cdot dx \tag{4.15} \]

Using the standard kinetic energy norm or pressure norm, we can get the POD mode via method of snapshots for each azimuthal component respectively.

32
4.5 Flowchart

snapshot ensemble

flow fluctuation

Azimuthal Decomposition

azimuthal modes

POD

POD

POD

POD

reconstruction of azimuthal modes

$\Phi(x, r, \theta, t_i)$

$\tilde{\Phi}(x, r, \theta, t_i)$
Chapter 5

Simulation Results

5.1 Energies in POD modes

When we use the kinetic energy norm for the POD procedure, the POD modes have a particularly intuitive meaning in a sense that they are associated with the most energetic structures of the flow in the velocity field.

The average fluctuation kinetic energy over the spatial domain $\Omega$ is given by

\[
K_{\Omega} = \frac{1}{2} \left\langle \int_{\Omega} \| \tilde{\Phi}(x, t) \|^2 dx \right\rangle
\]

\[
= \frac{1}{2} \left\langle \sum_{i,j}^{N_T} \alpha_i(t) \alpha_j^*(t) \int_{\Omega} \varphi_i(x) \varphi_j^*(x) dx \right\rangle
\]

\[
= \frac{1}{2} \sum_{i=1}^{N_T} \langle \alpha_i(t) \alpha_i^*(t) \rangle
\]

(5.1)

where $\langle \cdot \rangle$ stands for the time average. Therefore the average kinetic energy in the $n$-th POD mode is given by
\[ K_n = \frac{1}{2} \langle \alpha_n(t) \alpha_n^*(t) \rangle = \frac{1}{2} \lambda_n \quad (5.2) \]

The POD modes are ordered in a way that the eigenvalues \( \lambda_i \) are decreasing, therefore they form an optimal solution to the problem of finding a subspace spanned by \( n \) basis, the projection on which can capture the most possible kinetic energy on average. And the sum of all the eigenvalues is considered as the total kinetic energy in the snapshots.

Also, we can find the kinetic energy of the \( k \)-th azimuthal compound mode defined in equation (4.5)

\[ K^{(k)} = \frac{1}{2} \left( \int_{X \times R} \| \Phi^{(\pm k)}(x, r, t) \|^2 \right) = \frac{1}{2} \sum_{n=1}^{N_T} \lambda_n^{(k)} \quad (5.3) \]

Similarly, when we consider the pressure norm to perform POD, we can get an optimal reduced size approximation of the original flow that can capture the most possible pressure energy. The energy in the \( n \)-th POD mode at the \( k \)-th azimuthal component is represented by the corresponding eigenvalue

\[ P_n^{(k)} = \frac{1}{2} \lambda_n^{(k)} \quad (5.4) \]

The sum of all the eigenvalues at azimuthal number \( k \) is the total pressure energy in the \( k \)-th azimuthal component. The energy in terms of the flow pressure is usually considered as the potential energy of the flow.

Figure 5.1 shows the relative POD mode energies in terms of eigenvalues \( \lambda_n^{(k)} \) of the covariance matrices \( M^{(k)} \) using fluctuation kinetic energy norm. For each one of the first 11 compound azimuthal components, i.e. \( 0 \leq k \leq 10 \), the 10 largest eigenvalues \( \lambda_n^{(k)} \) are visualized, i.e. \( 1 \leq n \leq 10 \). We can tell from Figure 5.1, the POD modes
Figure 5.1 – Relative POD mode energies in terms of eigenvalues using kinetic energy norm.

Figure 5.2 – Relative POD mode energies in terms of eigenvalues using pressure norm.
for kinetic energy norm at the 2nd azimuthal component, i.e. $k = 2$, has the most energy, followed by $k = 1$ and $k = 0$. Also, the energies represented by eigenvalues decay with respect to $k$ for higher azimuthal components and fall off for both higher and lower $k$’s.

The relative POD mode energies using the pressure norm are shown in Figure 5.2, which indicates that the pressure mode energies decay much faster than the POD mode energies in the velocity field, both towards azimuthal numbers ($k$) and the eigenvalue numbers ($n$) from the same azimuthal component. The most energetic POD modes under the pressure norm are at $k = 0$, which is the axisymmetric azimuthal mode.

For the $k$-th azimuthal compound mode, the energy in this component that the first $L$ POD modes are able to capture can be represented by the $L$ largest eigenvalues

$$K_L^{(k)} = \frac{1}{2} \sum_{l=1}^{L} \lambda_l^{(k)}$$

(5.5)

Therefore we can evaluate the capability of the POD modes by considering the percentage of the energy in the $k$-th azimuthal component that these $L$ POD modes can obtain

$$\frac{\sum_{l=1}^{L} \lambda_l^{(k)}}{\sum_{l=1}^{N_T} \lambda_l^{(k)}}$$

(5.6)

where $N_T$ is the number of snapshots of interest.

Figure 5.3 shows the percentage of energy captured in different number of POD modes using the kinetic energy norm at each azimuthal component. On average of all 11 Fourier modes, the fluctuation kinetic energy norm can obtain 53.15% of the total
Figure 5.3 – Percentage of energy captured in different number of POD modes using fluctuation kinetic energy norm.

Figure 5.4 – Percentage of energy captured in different number of POD modes using pressure norm.
energy using 50 POD modes, 68.82% for 100 POD modes, and 88.93% for 300 POD modes.

For the pressure norm, the energies in eigenvalues decay much faster for lower Fourier numbers as indicated in Figure 5.2. When averaging among all 11 Fourier modes, the first 50 POD modes could capture 67.62% of the total potential energy, 76.56% for 100 POD modes, and 89.25% for 300 POD modes. The percentage of pressure energy captured in different number of POD modes at each azimuthal number is shown in Figure 5.4.

The number of POD modes required to capture certain percentage of energy in each azimuthal mode are listed in Table 5.1 and Table 5.2 for fluctuation kinetic energy norm and pressure norm respectively. As we can tell, the pressure norm is more capable of capturing energy of the flow using a small number of POD modes.
Table 5.1 – Number of kinetic energy POD modes needed to capture various percentages of energy in each azimuthal mode.

<table>
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<tr>
<th>k</th>
<th>90%</th>
<th>80%</th>
<th>70%</th>
<th>60%</th>
<th>50%</th>
<th>40%</th>
<th>30%</th>
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Table 5.2 – Number of pressure POD modes needed to capture various percentages of energy in each azimuthal mode.

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<th>60%</th>
<th>50%</th>
<th>40%</th>
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5.2 POD Modes Visualization

The POD modes for each azimuthal compound component can be reconstructed from the SVD results of the covariance matrix as described in Chapter 3. Figure 5.5 visualizes the constant value surfaces of the dominant streamwise velocity for the most energetic azimuthal mode, i.e. $k = 2$, using the fluctuation kinetic energy norm. Figure 5.6 is the corresponding pressure quantity visualization.

As shown in Figure 5.5, the flow components in the second Fourier mode have a doubled symmetry in the azimuthal direction. The downstream velocity is growing in the radiate direction towards the axis. Also, the axially symmetric components have long slowly rotating structures in the axial direction. However, the reconstructed pressure in Figure 5.6 is somehow not continuous along $x$ direction.

The structures of the POD modes using the pressure norm are different from the kinetic energy POD modes. Figure 5.7 and Figure 5.8 visualize the isosurfaces of the pressure component in the most energetic pressure POD mode at azimuthal number $k = 1$ and $k = 2$. The modes show axially symmetric structures rotating faster than the kinetic energy POD modes.

5.3 Temporal Analysis of the Jet Flow

The flow quantities are decomposed into different Fourier components $\Phi^{(k)}(x, r, t)$, then the POD modes for each azimuthal mode are computed as stated before. Therefore the flow in each azimuthal mode can be expressed as a linear combination of the POD modes.
Figure 5.5 – The streamwise velocity of the most energetic kinetic energy modes at azimuthal number $k = 2$: Iso-surfaces of positive (yellow) and negative (blue) components. Unit of the axes is the jet diameter $D$.

Figure 5.6 – The streamwise velocity of the most energetic kinetic energy modes at azimuthal number $k = 2$: Iso-surfaces of positive (yellow) and negative (blue) components. Unit of the axes is the jet diameter $D$. 

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Figure 5.7 – The pressure of the most energetic pressure modes at azimuthal number $k = 1$:
Iso-surfaces of positive (yellow) and negative (blue) components. ($D$ : jet diameter)

Figure 5.8 – The pressure of the most energetic pressure modes at azimuthal number $k = 2$:
Iso-surfaces of positive (yellow) and negative (blue) components. ($D$ : jet diameter)
\[ \Phi^{(k)}(x, r, t) = \sum_{n=1}^{N_T} \alpha^{(k)}_n(t) \varphi^{(k)}_n(x, r) \]  

(5.7)

The POD modes form an orthogonal basis for the original Hilbert space, hence the time-dependent coefficients can be obtained by taking the appropriately defined inner product of the snapshots fluctuation and the modes.

\[ \alpha^{(k)}_n = \langle \Phi^{(k)}(x, r, t), \varphi^{(k)}_n(x, r) \rangle_{X \times R} \]  

(5.8)

From the modes visualizations in section 5.2, we know that the flow has rotating structures in different modes. These modes are rotating around the axis both spatially and temporally. We could examine the frequencies of the rotating structures by taking the Fast Fourier Transform of the time-dependent amplitudes.

The FFT results indicate that there is no unique dominant frequency for a single POD mode. However, the range of the frequencies found result in a Stouhal number \( St \) between 0.2 and 1. A jet Strouhal number \( St \) is a dimensionless number describing oscillating flow mechanisms, defined as \( St = \frac{fD}{V} \), where \( f \) is the frequency, \( D \) is the jet diameter, and \( V \) is the jet core velocity. The FFT results suggest that the rotating structures of the POD modes visualized in section 5.2 are not rotating at a constant frequency, or can be separated by distinct rotating structures at different frequencies in the spatial domain.

5.4 Reconstruction and Spatial Analysis

The model reduction scheme implemented in this thesis is to perform POD on decomposed Fourier modes. Due to the orthogonality of the modes, the reconstructed
model is a linear combination of them. In section 5.1 we discussed about the average energy representation of the flow, and the energy captured in the selected POD modes, which is related to the eigenvalues. To better understand the spatially growing structure of the flow, we can calculate the reconstructed energy level of the flow at each grid point by considering the modes used in the reduced Hilbert space. Since we employed axially-symmetric mesh grids to restore symmetry for each \( r \), we should consider the energy level for each concentric circle discussed in section 4.1.2

\[
E(x, r) = \frac{1}{2} \sum_{k=-N_{\theta}}^{N_{\theta}} \sum_{n=1}^{L} \phi_n^{(k)}(x, r) \phi_n^{(k)}(x, r)
\] (5.9)

Therefore for the kinetic energy norm, the summation terms should include POD modes for velocity quantities in three directions, and for pressure norm, the POD modes used are based on the pressure measurements.

As mentioned in section 4.2, we set \( N_{\theta} = 10 \), so totally 11 azimuthal modes. Since the eigenvalues shown in Figure 5.1 and Figure 5.2 decay rapidly towards the azimuthal number, we compare the energy level contained in different numbers of POD modes with the energy in this 11 Fourier modes. Therefore we set \( N_{\theta} = 0 \) and vary \( L \), the number of POD modes in each azimuthal component.

Figure 5.9 shows the level of reconstructed kinetic energy at different \( x \) positions on the streamwise direction, from different number of POD modes. We can clearly see the most energetic points are moving away from the center as \( x \) increases, which can verify the growing structure from the modes visualizations in section 5.2. Similar observation can be obtained from Figure 5.10, which shows the level of reconstructed potential energy at different \( x \) position as a function of \( r \).
Figure 5.9 – Reconstruction of kinetic energy level at different $x$ positions using different number of POD modes. Horizontal axis is radius $r/D$, vertical axis is kinetic energy level.
Figure 5.10 – Reconstruction of pressure energy level at different $x$ positions using different number of POD modes. Horizontal axis is radius $r/D$, vertical axis is kinetic energy level.
Chapter 6

Conclusions

This thesis aims at reduced-order modelling of an incompressible jet flow. We performed the Galerkin method on a 3-dimensional incompressible jet flow via Fourier decomposition and proper orthogonal decomposition. The energy representation of different number of POD modes in terms of eigenvalues are discussed for each azimuthal component. POD modes computed for kinetic energy norm and pressure norm are visualized to show different structures captured in them, whose behaviors are discussed further both in spatial domain and in temporal domain.

In many flows one may expect to capture some dynamic features of the flow with only a few modes as a low order approximation. However, our POD procedure is performed after a Fourier decomposition, and the numbers of POD modes we keep focusing are for only one azimuthal component. Therefore if we compare the numbers of POD modes in Table 5.1 and Table 5.2 with 5400 snapshots and 16878 grid points, the reduced order results seem more impressive.

It has been demonstrated that the dynamical behavior of incompressible flows is more complex than compressible flows. Therefore our modes visualizations do not look
“clear”. Since we could not find a constant rotating frequency, more “purification” procedure should be utilized to get more neat structures.
Bibliography


