Cramer Rao Bound Study of Scattering Systems in One-Dimensional Space

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by

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Abstract

A Cramer-Rao bound (CRB) is developed to characterize the information content about scattering parameters through the wave scattering information contained in reflective (R), transmissive (T), and combined R plus T data. The analysis is developed for scalar wave scattering systems in one-dimensional space, paying particular attention to elastic scatterers, which simplifies the signal model (relative to more general scattering systems), thereby allowing closed-form expressions for the CRB for special cases. The derived CRB results quantify the effect of multiple scattering, be it in enhancing or diminishing imaging capabilities, relative to the Born approximation model which is customarily used as the standard reference for resolution limits (the so-called “diffraction limit”). The work also discusses the role of the measurement configuration for scatterer information extraction (e.g., reflective data applicable to monostatic radar, versus transmissive data applicable to bistatic radar). In particular, we show that certain natural parameters control the enhancement in the estimation performance from scattering data. Additionally, we discuss how specific sensing configurations enhance the estimation associated to those parameters.
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CHAPTER 1

Introduction

This research is concerned with the estimation-theoretic quantification of the “information content” about a wave scatterer that is contained in scattered field data. Motivation is provided by the recent interest on the role of the physical phenomenon of multiple scattering in either enhancing [1] or diminishing [2] the imaging capabilities relative to the classical reference provided by diffraction theory and, in particular, inverse scattering in the Born approximation, where the well-known “$\lambda/2$ limit” applies [3]. It is important to gain further insight into conditions under which the intrinsic estimation performance limits associated to the two models exhibit strong disagreement, as well as the counterpart conditions under which, on the contrary, the mismatch is not significant whereby the readily tractable Born approximation estimates are good. Our research builds on recent work, in particular, the investigations carried out by Brancaccio et al. [4] and Pierri et al. [5], which emphasize the question of the number of degrees of freedom of the linear mapping from the scattering potential function to the scattering field data or “essential dimensionality” of the Born-approximated matrix. Our work uses a different approach based on the fundamental Cramer-Rao bound (CRB) [6], that addresses both Born-approximable conditions and the more general multiple scattering case, focusing on canonical systems of multiply-scattering point-like scatterers. In particular, the information about point-like scatterers that is contained in scattering data is quantified, for different sensing configurations and single- versus multi-frequency conditions, via the CRB benchmark, which measures under mild statistical conditions the lowest achievable error in the unbiased estimation from noisy scattering data of sought-after scattering parameters (in the present context, the scattering strengths and
positions of point-like scatterers). It is important to emphasize that the CRB holds for the given signal model, in other words, the bound on estimation error under multiple scattering applies to an estimator that uses the correct signal model including multiple scattering, while the bound for a fictitious or approximate system based on the Born approximation correspondingly applies to an estimator that uses the Born approximation model. The important issue of mismodeling, e.g., assuming Born in the inversion while in reality multiple scattering was significant, is also partially addressed in this work, in the sense that applying the Born approximation to multiple-scattering data cannot lead to unbiased estimation which are superefficient, relative to the CRB for the multiple-scattering model.

To facilitate analytical and computational handling of a variety of interesting questions, this work focuses only on the simple case of scatterers or inhomogeneities in one-dimensional (1D) space. This particular scenario simulates, e.g., a transmission line environment where the small scatterer can represent a local inhomogeneity or iris (see figure 2.1(a)), or a canonical monostatic or bistatic radar or lidar model with “in-line-of-sight” target (see figure 2.1(b)). In the latter context this model is suitable for (transversely) large, (longitudinally) thin planar targets that exhibit negligible variation of constitutive properties in the transverse coordinates and non-negligible variation mostly in the direction of propagation of the probing fields, as applicable to far-field transmissive or reflective tomographic measurements.

The use of Cramer-Rao analysis as a tool to assess imaging systems is not new, and has been a topic of important investigations in the past 40 years. Of much interest is the work by Shahram and Milanfar [7] who focused on the rigorous information-theoretic determination of the resolution of incoherent optical imaging systems by formulating a detection problem consisting of identifying single-source versus two-source systems and who also studied the CRB in the estimation of source parameters. However, unlike this seminal work, which focuses on an incoherent radiation system involving a
1. INTRODUCTION

linear signal model, we address a coherent scattering system where the signal model is generally nonlinear. Of greater relevance to the present investigation are a number of more recent papers by Shi and Nehorai [8], Sentenac et al. [2], and Simonetti et al. [1]. Shi and Nehorai [8] developed a Cramer-Rao analysis of multiply-scattering point targets in three-dimensional (3D) space and showed via exhaustive numerical computation that multiple scattering can enhance the estimation and also the possibility of adding artificial scatterers to enhance estimation of parameters associated to sought-after scatterers. The role of scatterers to enhance imaging has also been studied in [9], and this issue is, in fact, part of the broader area of radiation and imaging enhancements via substrate media including metamaterials [10]. The enhancement in imaging thanks to multiple scattering as a physical resource has been studied in Sentenac et al. [2] and Simonetti et al. [1] via the CRB benchmark. Unlike this previous work, we pay special attention to the influence of each parameter of the system in the performance of the multiple scattering and how it is connected with the sensing configuration used in the wave scattered field measurements.

In particular, recently there has been much interest in addressing the resolution limits in imaging when multiple scattering is significant. This has been investigated in [1, 2, 8] in connection with the forward point of view, where comparison is made between two different physical scattering models, one where multiple scattering is negligible (Born approximation), applicable to weakly scattering objects describable by the Born approximation, versus the more general one where multiple scattering is significant and cannot be neglected. This has been investigated also in connection with the companion inverse point of view or imaging from given data, where one compares algorithms valid only in the Born approximation versus more general methods applicable in the exact scattering framework [1, 11, 12]. It has been shown that so-called fundamental “resolution limits” in imaging such as the Rayleigh resolution limit hold only in the Born or linearizing approximation regime, and that they do not generally
hold under active inverse-scattering-based imaging if significant multiple scattering is involved. Enhancement of resolution via multiple scattering is possible. However, our work shows clearly that the enhancement may depend on the particular remote sensing configuration and particular scattering parameters, and, in fact, multiple scattering can either enhance [8] or diminish resolution [2]. Both possibilities are further demonstrated in the present study in which we highlight the connection between the parameters of the system (the position of target, the target separation and the target strength) and the definition of those regions of interest. Additional, we gain further understanding of the connection of the sensing configuration with the fact that multiple scattering improves or deteriorates the inverse scattering estimation performance.

It is important to emphasize that the real physical model is, of course, the multiple scattering one. The Born approximation is valid for weakly scattering objects only. However, in comparing the two models one gains insight into the possible gain in resolution due to multiple scattering. Clearly if there are scattering configurations for which the Born approximation model outperforms the multiple scattering one, this is clearly a fictitious and non-physically important result, since in the same physical situation only the multiple scattering model is exact. Thus for fixed scattering parameters, if it is found that the CRB using the Born approximation is less than that using the multiple scattering model, this would mean that the Born-approximation resolution limit actually is too optimistic for those particular scattering parameters. A completely different interpretation of the same analysis holds if, on the other hand, one compares two different systems where the first (multiple scattering) system has given scattering strengths (say $\tau_1, \tau_2, \cdots$) while the other (Born approximation system) has a scaled (reduced) value of the same scattering strengths, say $\tau_1^{(B)} = \alpha \tau_1, \tau_2^{(B)} = \alpha \tau_2, \cdots$ where $\alpha \in (0, 1)$ where $\alpha$ is chosen small enough such that the Born approximation holds for the given positions of the scatterers. In this other interpretation the comparison of multiple scattering versus Born approximation models is meant to apply to the comparison
of estimation error for target strengths (as a percentage of error relative to the value of the scattering strength) in strongly scattering versus weakly scattering targets under the same signal-to-noise ratio (SNR). Clearly both interpretations above are important in practical application of the results of the present and past CRB studies for scattering systems.

The overall goal of this research is the investigation of certain fundamental aspects not fully covered by work in this area, mostly in the role of different measurement configurations for scatterer information extraction. In the scenarios depicted in figure 2.1, they reduce to 3 canonical arrangements: reflective (R) system (like a monostatic radar or light detection and ranging (lidar) system), transmissive (T) system (like a bistatic radar or lidar system), and combined transmissive and reflective setups (like a more general multistatic system). The aspect of sensing configuration is also treated by Gustafsson and Nordebo [13] who employed the CRB as a way to assess the ill-posedness of the electromagnetic inverse scattering problem for nonmagnetic multilayer media, in terms of resolution, as measured by the length of the grid cells adopted in inversions, versus estimation accuracy, associated to the inversion of the sought-after constitutive property (permittivity) at each cell. The study considered time-domain reflective (R) and transmissive (T) data. Some of the results complement the pioneering work in [13]. For example, among other results, it was found that transmissive data give a rank-1 Fisher information matrix. But complementary aspects not addressed in [13] are also considered in chapter 3 and 4, such as the Born approximation versus multiple scattering comparison, more sensing configurations (in particular, besides R, T, and the one-sided reflective and transmissive data (RT), it also explore two-sided reflective data plus single-sided transmissive data (RRT), and two-sided reflective and transmissive data (RRTT)), as well as the role of a priori information on the scatterer strengths or scatterer positions. A preliminary account of the present work has been presented in a recent conference [14,15].
Chapter 2 reviews the representation of the general forward scattering analysis in 1D space. The scattering data matrix $K$ is defined in Section 2.2, under different sensing configurations to extract the information content of the wave scattering measurements. The forward scattering problem is presented for each reflective and transmissive configuration under the Born approximation model in Section 2.3 and the multiple scattering model in Section 2.4, follow by the most important aspects in common and differences between both models in Section 2.5. The CRB for general type of scattering regime is developed in Chapter 3 corresponding of different type of signal model parameters in Section 3.1. The Fisher Information matrix (FIM) is computed in Section 3.2. This is followed by some numerical experiments in Section 3.3. Discussion of the conditions in which the multiple scattering enhance the estimation is presented in Chapter 4 where we make emphasis on the elastic scattering regime and focus on the diagonal entries of the Fisher information matrix. Closed-form expressions for FIM are derived. Numerical illustration is presented in Sections 4.2 and 4.3. The elastic scattering regime reduces the number of required scattering parameters whose scattering strengths are of the reduced parametric form Eq.(A.47) in Appendix A of this work. Furthermore, although the focus of this study is the 1D case, analogous reduced parametric models apply in the 2D and 3D cases [1]. Therefore, the same general techniques used in this work have the potential of being expanded in the future to the more exciting 3D case where more conclusive analytical results are feasible thanks to the emphasis in the diagonal entries of the FIM.
CHAPTER 2

Scattering Model

2.1. Scattering Formulation

Consider in 1D free space, primary scalar sources ($\rho$) and scalar fields (incident fields) ($\psi_i$) related by the scalar Helmholtz equation

\begin{equation}
\left( \frac{\partial^2}{\partial x^2} + k^2 \right) \psi_i(x) = \rho(x)
\end{equation}

so that the incident fields

\begin{equation}
\psi_i(x) = \int G_0(x, x') \rho(x') dx'
\end{equation}

where the free space Green function (see [16], p. 912)

\begin{equation}
G_0(x, x') = -\frac{i}{2k} \exp(i k |x - x'|).
\end{equation}

It obeys

\begin{equation}
\left( \frac{\partial^2}{\partial x^2} + k^2 \right) G_0(x, x') = \delta(x - x')
\end{equation}

and outgoing wave conditions at infinity. In these results, $k = \omega/c$ is the wavenumber of the field at angular oscillation frequency $\omega$ where $c$ is the free space speed of light.

For more general media, the differential equation relating sources $\rho$ and (total) fields $\psi_t$ is

\begin{equation}
\left( \frac{\partial^2}{\partial x^2} + k^2 \right) \psi_t(x) = V(x) \psi_t(x) + \rho(x)
\end{equation}
where $V(x)$ denotes the scattering potential, so that $V(x) = \kappa^2(x) - k^2$. The quantity $\kappa(x)$ is the space-dependent wavenumber of the field in the (total) medium.

In this new medium, radiation is governed by

$$\psi_t(x) = \int G(x, x') \rho(x') dx'$$

where $G$ is the total Green function in the new medium. It obeys

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) G(x, x') = V(x) G(x, x') + \delta(x - x')$$

To relate this Green’s function to the free space one, we note that Eqs.(2.4,2.7) yield (after some manipulations involving Green’s theorem)

$$G(x, x') - G_0(x, x') = \int dx'' G_0(x, x'') V(x'') G(x'', x') = \int dx'' G(x, x'') V(x'') G_0(x'', x')$$

which coincidentally implies from Eqs.(2.2,2.6) the following result valid for rather general excitations (transmit sources $\rho$):

$$\psi_t(x) - \psi_i(x) = \int dx' G_0(x, x') V(x') \psi_t(x') = \int dx' G(x, x') V(x') \psi_t(x').$$

For the particular case of point-like scatterers of individual strengths or reflectivities $\tau_m, m = 1, 2, ..., M$, and positions $X_m, m = 1, 2, ..., M$, we have $V(x) = \sum_{m=1}^{M} \tau_m \delta(x - X_m)$. Then (2.8) gives

$$G(x, x') - G_0(x, x') = \sum_{m=1}^{M} G_0(x, X_m) \tau_m G(X_m, x')$$

$$= \sum_{m=1}^{M} G(x, X_m) \tau_m G_0(X_m, x'),$$

a result to be used later, with emphasis on the particular two-scatterer case. In the
2.1. SCATTERING FORMULATION

Figure 2.1. (a) Transmission line depiction of the problem where $\tau_m, m = 1, 2, ...$ are the scattering strengths of small, point-like scatterers or inhomogeneities embedded to the propagation medium, and $X_m, m = 1, 2, ...$ are the respective positions of these scatterers. (b) Schematization of radar or lidar system with “in-line-of-sight” target which is modeled by a collection of scattering centers with strengths $\tau_m, m = 1, 2, ...$ at positions $X_m$.

The remainder of this work it will be assumed that the scatterers are all in the region $(0, D)$, and that scattering experiments are done involving point sources and receivers at $x \leq 0$ and/or $x \geq D$. Also if scattering is elastic (non-dissipative), then $\tau_m$ must obey the condition $\Re(\tau_m)^2 + \Re(\tau_m) + [\Im(\tau_m)]^2 = 0$ (for more details see Appendix A). This condition will be used in selecting values for the computer study; however, we do not require it in the CRB calculations which hold for arbitrary $\tau_m$. The particular case studies of interest are detailed next.
2.2. Scattering Data

In this research, the wave scattered field measurements are performed at the transceivers “-” and “+”, (refer to figure 2.1). It is assumed that transceivers “-” and “+” are identical and small compared with the wavelength so they are approximated at point elements at $x = 0$ and $x = D$ respectively. The scattering data for the inverse scattering problem case are the outputs of this transceivers and it is known as the “multistatic data matrix” or “scattering data matrix” $K$.

For the purposes of analysis, it will suffice without loss of generality to assume the following scattering experiments:

- “Reflective $-, -$”: a point source at $x = 0$ ($\rho(x) = \delta(x - X_T)$) and a point-receiver also at $x = 0$. The measured scattered field for this experiment will be denoted $K(-, -)$. This is a particular entry in the scattering matrix $K$.
- “Reflective $+, +$”: a point source at $x = D$ and a point-receiver also at $x = D$. The entry in the scattering matrix $K$ associated to this experiment will be denoted as $K(+, +)$.
- “Transmissive $+, -$”: a point source at $x = 0$ and a point-receiver at $x = D$. Due to reciprocity this experiment has the same signal content as the reciprocal counterpart involving $x = D$ as source point and $x = 0$ as receiver point. The entries in the scattering matrix $K$ of these experiments will be denoted as $K(+, -)$ and $K(-, +)$. These entries are, apart from noise, identical. However, in the presence of noise there can be additional information in taking the complementary entry (larger SNR thanks to more samples).

In our Cramer-Rao analysis it shall consider different sensing configurations, for single frequency and the more general multi-frequency case, like for example:

- Case R: Reflective $-, -$.
- Case T: Transmissive $+, -$.
• Case RT: Reflective $-, -$ and transmissive $+, -$.
• Case RRT: Reflective $-, -, +$ and transmissive $+, -$.
• Case RRTT: Reflective $-, -, +, +, -$ and transmissive $+, -, +$.

Each sensing configuration above generates a particular scattering matrix $K$ which is the key to study the information content through the CRB study. It also allows us to identify the next optimal experiment in the estimation process.

2.3. The Forward Scattering Map in the Born Approximation

The scattered field (total minus incident field) for the above experiments is defined by $G - G_0$, which according to Eqs. (2.8,2.10) gives:

• $K(-, -) = G(0, 0) - G_0(0, 0) = \int dx''G_0(0, x'')V(x'')G(x'', 0)$ for the case “reflective $-, -$”.
• $K(+, +) = G(D, D) - G_0(D, D) = \int dx''G_0(D, x'')V(x'')G(x'', D)$ for the case “reflective $+, +$”.
• $K(+, -) = G(D, 0) - G_0(D, 0) = \int dx''G_0(D, x'')V(x'')G(x'', 0) = K(-, +)$ for the cases “transmissive $+, -$” or the reciprocal equivalent, case “transmissive $-, +$”.

Let emphasize next the particular case of two point-like scatterers of individual strengths or reflectivities $\tau_1$ and $\tau_2$ and positions $X_1$ and $X_2$, so that $V(x) = \tau_1 \delta(x - X_1) + \tau_2 \delta(x - X_2)$.

In the Born approximation for which $G \simeq G_0$ we then obtain from (2.3,2.10)

\begin{equation}
K_B(-, -) = \left(-\frac{i}{2k}\right)^2 \exp(i2kX_1) [\tau_1 + \tau_2 \exp(i2kd)]
\end{equation}

where it is introduced the target separation $d = X_2 - X_1$.

It is also obtain

\begin{equation}
K_B(+, +) = \left(-\frac{i}{2k}\right)^2 \exp[i2k(D - X_1 - d)] [\tau_2 + \tau_1 \exp(i2kd)]
\end{equation}
and

\[
K_B(\pm, -) = \left( -\frac{i}{2k} \right)^2 \exp(\pm 2kD) (\tau_1 + \tau_2) = K_B(-, \pm)
\]

which completes the picture of scattering by two point-like scatterers within the Born approximation. Let consider next the more general multiple scattering case.

2.4. The Forward Scattering Map Including Multiple Scattering

The more general case of multiply scattering point targets can be readily handled by carefully following the successive multiple scattering events as is done, e.g., in [17], p. 220-223 (see, in particular, equations 5.67-d, 5.68 in [17]). (Remark: However, care must be exercised in noting that the electromagnetic discussion in [17] applies to a slightly different physical situation, in particular, one of a three-layered medium, unlike the present medium formed by free space as background and two point-like scatterers or reflectivities embedded to that background. Thus the results in [17] reduce to the results presented in this work after certain substitutions: in this case, scattering by the point-like inhomogeneities is isotropic, hence the reflectivity by coming from the right is the same as by coming from the left, unlike in the layered medium where the reflectivity varies in sign when applied to the opposite direction of travel. But besides this issue, the results in [17] and the present results outlined below are equivalent.)

One obtains the multiple scattering generalization of (2.11)

\[
K(-, -) = \left( -\frac{i}{2k} \right) \exp(\pm 2kX_1) \left[ \hat{\tau}_1 + \frac{(1 + \hat{\tau}_1)^2 \hat{\tau}_2 \exp(\pm 2kd)}{1 - \hat{\tau}_1 \hat{\tau}_2 \exp(2kd)} \right]
\]

where the expression of \( \hat{\tau}_m \) have introduced for notational compactness

\[
\hat{\tau}_m = \left( -\frac{i}{2k} \right) \tau_m \quad m = 1, 2.
\]
Similarly, the multiple scattering generalization of (2.12) is found to be

\[ K(+, +) = \left( -\frac{i}{2k} \right) \exp[i2k(D - X_1 - d)] \left[ \hat{\tau}_2 + (1 + \hat{\tau}_2)^2 \hat{\tau}_1 \exp(i2kd) \frac{1 - \hat{\tau}_1 \hat{\tau}_2 \exp(i2kd)}{1 - \hat{\tau}_1 \hat{\tau}_2 \exp(i2kd)} \right]. \]

Finally, the multiple scattering generalization of (2.13) is

\[ K(+, -) = \left( -\frac{i}{2k} \right) \exp(i2kD) \left[ \frac{(1 + \hat{\tau}_1)(1 + \hat{\tau}_2)}{1 - \hat{\tau}_1 \hat{\tau}_2 \exp(i2kd)} - 1 \right] = K(-, +). \]

### 2.5. Born Versus Multiple Scattering Cases

Some differences become obvious in terms of the information available in the multiple scattering case, due to greater contrast giving rise to multiple scattering, versus in the Born approximation case for weakly scattering targets. For example, the Born approximation scattering matrix entry \( K_B(+, -) \) in (2.13) has \textit{no dependence} on the target separation \( d \), while the multiple scattering counterpart (2.17) does exhibit such a dependence. Thus in the Born approximation the transmissive experiment giving \( K_B(+, -) \) provides no information about the target separation, but instead gives only (apart from a constant) the sum of the two target strengths (corresponding to low-frequency information about the scattering potential as a whole). In contrast, in the more general multiply scattering case the same transmissive experiment does provide such information thanks to multiple scattering events between the two scatterers.

Note, however, that the dependence on \( d \) is of the form \( \exp(i2kd) \) which implies that \( d \) and \( d + n\lambda/2, n = 1, 2, ... \) (where \( \lambda = 2\pi/k \)) exhibit the same value of \( K(+, -) \) so that even in the multiple scattering case show limitations in resolution ability under a single frequency experiment (we cannot \textit{distinguish} between distances separated by half wavelength). (Remark: This problem does not hold in the 2D or 3D case since then there is wave amplitude attenuation in propagation, which is not present in the 1D case). However, this is an \textit{aliasing} issue, and it does not mean that the subwavelength scatterer separations can not be estimated. As we will show next, \textit{we can}, in the multiple
scattering signal model. But, subject to the ambiguity whether the estimated distance is, say, $d$, or $d + n\lambda/2, n = 1, 2, ...$. It is in the latter interpretation that we will speak in the following about subwavelength imaging resolution (with the clear understanding that we are referring to estimation of the “principal value” of $d$ only). In fact, in our computer simulations we will from the onset restrict attention to $d \in (0, \lambda/2)$.

Our next goal is to quantify and better understand the information content for a two-scatterer system characterized by scattering strengths $\tau_1$ and $\tau_2$, and scatterer position $X_1$ and target separation $d$, from different scattering data sets. This will be done via the fundamental Cramer-Rao bound pertinent to the variance (error) of unbiased estimators for the scattering parameters $\tau_1, \tau_2, d$ and $X_1$, under the two signal models above (Born versus multiple scattering), under different configurations (data sets R, T, RT, RRT, RRTT), and for single- versus multi-frequency data.
CHAPTER 3

Generalized Cramer Rao Study for Two Point Scatterer Systems

The Cramer-Rao study is developed directly in the multi-frequency case under the assumption that the scattering strengths are frequency-independent (nondispersive media) for the frequency bands used in the interrogation experiments, thus in the multi-frequency case we consider $\tau_m(\omega) = \tau_m, m = 1, 2$. The latter assumption simplifies the analysis and applies to many practical situations. For the more general case of dispersive scatterers, see [13]. The single-frequency case is discussed as an important special case.

3.1. Signal Model Parameters

In practice, one collects noisy data which we take into account via the signal model:

\begin{equation}
\tilde{K}(\xi) = \bar{K}(\xi) + W
\end{equation}

where $\bar{K}(\xi)$ is the noise-free data vector formed by the available entries of the response matrix $K$, which depend on the remote sensing configuration (R, T, RT, etc.), $\tilde{K}(\xi)$ is the noisy realization of that matrix, $W$ is white Gaussian noise with variance $\sigma_0^2$ (which is handled next as nuisance parameter that needs to be estimated from the data), and $\xi$ is the vector of the scattering plus noise parameters that ones wishes to estimate from the data. We consider different vectors $\xi$ corresponding to different combinations of the following parameters: The position $X_1$ of target 1, the target separation $d$, the real and imaginary parts $\tau_r(m)$ and $\tau_i(m)$ of target strength $\tau_m, m = 1, 2$, and the unknown noise.
3.1. SIGNAL MODEL PARAMETERS

variance $\sigma_0^2$. Thus we consider the case of no a priori knowledge, where the parameter vector

\begin{equation}
\xi = [X_1, d, \tau_{(r)1}, \tau_{(r)2}, \tau_{(i)1}, \tau_{(i)2}, \sigma_0^2]^T
\end{equation}

(where $T$ denotes transpose) as well as the special cases where $\tau_1, \tau_2$ are known (modeling a priori knowledge of the scatterer’s materials or “Known material case”), so that

\begin{equation}
\xi = [X_1, d, \sigma_0^2]^T
\end{equation}

and where the positions $X_1, X_2$ (and thus also $d$) are known (modeling a priori knowledge of the support of the scatterers or “Known support case”), so that

\begin{equation}
\xi = [\tau_{(r)1}, \tau_{(r)2}, \tau_{(i)1}, \tau_{(i)2}, \sigma_0^2]^T.
\end{equation}

In the following we shall refer to the cases in (3.20) and (3.21) as “known material” and “known support”, respectively. The three cases in (3.19, 3.20, 3.21) are studied next for both single- and multi-frequency problems.

Consider first the single-frequency case. Then for the different remote sensing cases, the noise-free data vector $\bar{K}$ is as follows (with $T$ denoting transpose):

- **Case R**: $\bar{K} = [K(-, -)]$
- **Case T**: $\bar{K} = [K(+, -)]$
- **Case RT**: $\bar{K} = [K(-, -) \ K(+, -)]^T$
- **Case RRT**: $\bar{K} = [K(-, -) \ K(+, -) \ K(+, +)]^T$
- **Case RRTT**: $\bar{K} = [K(-, -) \ K(+, -) \ K(-, +) \ K(+, +)]^T$

If data are gathered for a number $N_f$ of frequencies $\omega_1, ..., \omega_{N_f}$, then in place of the scalar $K(-, -)$ we have the vector $[K(-, -; \omega_1) \ ... \ K(-, -; \omega_{N_f})]^T$, and so on. Then the noise-free data vector for each case is as follows:

- **Case R**: $\bar{K} = \bar{K}_{-, -, N_f}^T \equiv [K(-, -; \omega_1) \ ... \ K(-, -; \omega_{N_f})]^T$
3.2. Fisher Information Matrix and Cramer-Rao Bound

The corresponding Fisher information matrix \( I(\xi) \) is given by [6, eq.15.52]

\[
(3.22) \quad I(\xi)_{ij} = \text{tr} \left[ C^{-1}_K(\xi) \frac{\partial C_K(\xi)}{\partial \xi_i} C^{-1}_K(\xi) \frac{\partial C_K(\xi)}{\partial \xi_j} \right] + 2\Re \left[ \frac{\partial K^H(\xi)}{\partial \xi_i} C^{-1}_K(\xi) \frac{\partial K(\xi)}{\partial \xi_j} \right]
\]

where \( H \) denotes the conjugate transpose, and \( C_K \) denotes the covariance matrix [6, pp. 501] which in our case is simply \( C_K = \sigma_0^2 I \). Here \( I \) denotes the identity matrix, whose size is determined by the length of the data vector \( \tilde{K} \) which we will denote next as \( L \). Clearly \( L \) is equal to \( N_f \) for cases R and T, to \( 2N_f \) for case RT, to \( 3N_f \) for case RRT, and to \( 4N_f \) for case RRTT. The CRB (\( \text{CRB}[\xi(i)] \)) is the lower bound for the variance \( \text{var}[\xi(i)] \) of any unbiased estimator for the parameter \( \xi(i) \), and is defined in terms of the Fisher information matrix \( I(\xi) \) by [6, eq.3.20]

\[
(3.23) \quad \text{var}[\xi(i)] \geq [I^{-1}(\xi)]_{ii} = \text{CRB}[\xi(i)].
\]

Under the parameter vector \( \xi \) in (3.19) which is of length 7, the Fisher information matrix associated to the signal model Eq.(3.18) reduces (for both Born and multiple
3.2. FISHER INFORMATION MATRIX AND CRAMER-RAO BOUND

scattering cases) to

$$ I(\xi) = \left[ \frac{L}{\sigma_0^4} \delta_{i,j} \delta_{i,7} \right] + \frac{2}{\sigma_0^2} \Re \left[ \frac{\partial \tilde{K}^H (\xi)}{\partial \xi_i} \frac{\partial \tilde{K} (\xi)}{\partial \xi_j} \right] $$

(3.24)

$$ = \left[ \frac{L}{\sigma_0^4} \delta_{i,j} \delta_{i,7} \right] + \frac{2}{\sigma_0^2} \Re \left[ \begin{array}{ccccccc}
\frac{\partial \tilde{K}^H}{\partial X_1} & \frac{\partial \tilde{K}}{\partial \tau_{(i)}^2} & \cdots & \frac{\partial \tilde{K}^H}{\partial \tau_{(i)2}} & \frac{\partial \tilde{K}}{\partial \tau_{(i)2}} & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial \tilde{K}^H}{\partial \tau_{(i)2}} & \frac{\partial \tilde{K}}{\partial \tau_{(i)2}} & \cdots & \frac{\partial \tilde{K}^H}{\partial \tau_{(i)2}} & \frac{\partial \tilde{K}}{\partial \tau_{(i)2}} & : \\
0 & \cdots & \cdots & 0 & 0 & 0
\end{array} \right] $$

The results for the parameter vectors $\xi$ in (3.20) and (3.21) are of the same general form, and are constructed by suitable reduction of the Fisher information matrix above, thus, for example, for the $\xi$ vector in (3.20) ("known material" case) one obtains

$$ I(\xi) = \left[ \frac{L}{\sigma_0^4} \delta_{i,j} \delta_{i,3} \right] + \frac{2}{\sigma_0^2} \Re \left[ \begin{array}{ccc}
\frac{\partial \tilde{K}^H}{\partial X_1} & \frac{\partial \tilde{K}}{\partial \tau_{(i)}^2} & \frac{\partial \tilde{K}^H}{\partial \tau_{(i)2}} \\
\frac{\partial \tilde{K}^H}{\partial \tau_{(i)}^2} & \frac{\partial \tilde{K}}{\partial \tau_{(i)2}} & \frac{\partial \tilde{K}^H}{\partial \tau_{(i)2}} \\
0 & \cdots & 0
\end{array} \right] $$

(3.25)

Let us highlight salient aspects of each case (R, T, RT, etc., and Born versus multiple scattering) separately. Rather than providing the analytical expressions for the Fisher information matrix for each case, which would account for a lengthy discussion, we limit to highlighting the general structure and rank of this matrix for each case. Tables of the rank of the Fisher information matrix for representative cases of interest are provided in Tables 1, 2, and 3, for the parameter vectors in Eq.(3.19), (3.20) and (3.21), respectively.

- **Case R:** In this case $\tilde{K} = K(-, -) + W$, where in general $K(-, -)$ is given by (2.14) while in the Born approximation it is approximated by $K_B(-, -)$ in (2.11). Consider the single frequency case $N_f = 1$. In this case the Fisher information matrix has at most rank 3 so that in the no a priori knowledge and "known support" cases the Fisher information matrix is singular and it is not possible to estimate all the desired parameters from the data. However, the "known material" case which has only
3 unknown parameters admits a non-singular Fisher information matrix, so that the CRB can be studied in that case. The rank is 3 since the signal is complex-valued so that $K(-, -)$ contains two independent data points and the noise is independent from the signal which gives up to 2 scattering system features plus 1 noise feature ($\sigma_0^2$) that can be estimated from the data, hence the rank 3. In the multi-frequency case this reflective configuration achieves a full-rank, invertible Fisher information matrix (meaning that all parameters can be estimated with finite error) with at least 3 different data frequencies for the no a priori knowledge case and 2 different frequencies for the “known support” case. The rank remains 3 even for multiple frequencies in the “known material” case. This was validated with several numerical examples during the course of this investigation (refer to Tables 1, 2, and 3).

• Case T: Consider first the single-frequency regime, in the Born approximation. From (2.13), clearly the signal does not depend on $X_1$ or $d$, i.e., $\frac{\partial \tilde{K}}{\partial X_1} = 0 = \frac{\partial \tilde{K}}{\partial d}$, so that the general expression (3.24) takes the following reduced form

\[
I[(\xi)] = \left[ \frac{1}{\sigma_0^4} \delta_{i,j} \delta_{i,7} \right] + \frac{2}{\sigma_0^2} \mathcal{R} \begin{bmatrix}
0 & \ldots & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & 0 \\
0 & 0 & \frac{\partial \tilde{K}_H}{\partial \tau_i(r_1)} \frac{\partial \tilde{K}}{\partial \tau_j(r_1)} & \ldots & \frac{\partial \tilde{K}_H}{\partial \tau_i(r_1)} \frac{\partial \tilde{K}}{\partial \tau_j(r_1)} & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \frac{\partial \tilde{K}_H}{\partial \tau_i(r_2)} \frac{\partial \tilde{K}}{\partial \tau_j(r_2)} & \ldots & \frac{\partial \tilde{K}_H}{\partial \tau_i(r_2)} \frac{\partial \tilde{K}}{\partial \tau_j(r_2)} & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & 0
\end{bmatrix}
\]

which means that, as outlined earlier, there is no information in the data about the target positions. As illustrated in Tables 1 and 3, the rank is 3 in the no a priori knowledge and “known support” cases. But in the “known material” case the rank is 1. These results hold for both single and multiple frequencies. The rank 1 is associated only to the noise since there is no information in the T data about the target positions. The information content associated to the transmissive experiment is only on the scattering
strengths, and we will see that transmissive data in combination with reflective data (e.g., in the RRT and RRTT cases to be studied in further detail next) does enhance the estimation performance pertinent to the scattering strengths.

In the more general multiple scattering regime addressed in (2.17) the data still does not depend on $X_1$ (hence $\frac{\partial K}{\partial X_1} = 0$) but unlike the Born approximation the data now depends on the target separation $d$, so that (in contrast with (3.26)) the Fisher information has the more general structure

$$(3.27) \quad I[\xi] = \left[ \frac{N_f}{\sigma_0^2} \delta_{i,j} \delta_{i,7} \right] + \frac{2}{\sigma_0^2} \Re \begin{pmatrix} 0 & \ldots & \ldots & \ldots & 0 \\ \vdots & \frac{\partial \tilde{K}^H}{\partial d} \frac{\partial \tilde{K}}{\partial d} & \ldots & \frac{\partial \tilde{K}^H}{\partial d} \frac{\partial \tilde{K}}{\partial \tau_{(1)2}} & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \frac{\partial \tilde{K}^H}{\partial d} \frac{\partial \tilde{K}}{\partial \tau_{(1)2}} & \ldots & \frac{\partial \tilde{K}^H}{\partial d} \frac{\partial \tilde{K}}{\partial \tau_{(1)2}} & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix}$$

In this case the rank is 3 in the no a priori knowledge and “known support” cases, and it is 2 in the “known material” case for a single frequency. The latter is an enhancement due to multiple scattering, relative to the Born model where the rank is only 1.

Note that for the three situations associated to the parameter vector $\xi$, the Fisher information matrix is singular under the transmissive data set alone, for a single frequency. In the multi-frequency case, the rank is at most 6 for the no a priori knowledge case and remains 2 in the “known material” case. However, in the “known support” case the rank is 5 (full rank), consequently finite CRB for the five parameters (strengths and noise) can be computed in this case.

• Case RT: The Fisher information matrix for this case has the general form Eq.(3.24) with $L = 2N_f$. In the single-frequency regime, under the Born approximation, $\frac{\partial \tilde{K}}{\partial d} = \left[ \frac{\partial K_B(-,-)}{\partial d} 0 \right]^T$ and a similar expression for $X_1$. Under multiple scattering, there is greater dependence on $d$ but $\frac{\partial \tilde{K}}{\partial X_1} = \left[ \frac{\partial K(-,-)}{\partial X_1} 0 \right]^T$. 

3.2. FISHER INFORMATION MATRIX AND CRAMER-RAO BOUND

In this case the rank of the Fisher information matrix is 5 in the \textit{no a priori} knowledge and “known support” cases, and 3 for the “known material” case for single frequency case. Therefore in the case RT, and under the “known material” and “known support” cases the Fisher information matrix is invertible (hence all the parameters can be estimated with finite error). In the multi-frequency case the rank of the Fisher information matrix increases for the \textit{no a priori} knowledge case, and the Fisher information matrix becomes invertible for at least two different data frequencies.

- Case RRT: The Fisher information matrix has the general form Eq.(3.24) with \( L = 3N_f \). In the single-frequency regime, under the Born approximation, \( \frac{\partial \tilde{K}}{\partial d} = \left[ \frac{\partial K_B(\cdot, \cdot)}{\partial d} \quad 0 \quad \frac{\partial K_B(\cdot, \cdot)}{\partial d} \right]^T \) and a similar expression for \( X_1 \). In the multiple scattering model there is greater dependence on \( d \) but \( \frac{\partial \tilde{K}}{\partial X_1} = \left[ \frac{\partial K_B(\cdot, \cdot)}{\partial X_1} \quad 0 \quad \frac{\partial K_B(\cdot, \cdot)}{\partial X_1} \right]^T \).

The rank for the \textit{no a priori} knowledge case is now at most 7, and this value is achieved if \( \tau_1 \neq \tau_2 \). However for \( \tau_1 = \tau_2 \) the rank was found to be 6 in the single frequency case (refer to Table 1). But for multiple frequency one recovers the rank 7.

For the “known material” and “known support” cases the Fisher information matrix is full rank, as is illustrated in Tables 2 and 3, hence all parameters can be estimated with finite error. These results hold, for both single and multiple frequency conditions, and for both Born and multiple scattering models.

- Case RRTT: Since the signals \( K(\cdot, \cdot, \cdot) \) and \( K(\cdot, \cdot, \cdot) \) are identical, one expects that the additional data entry will provide only marginal extra information about the scattering parameters relative to case RRT. This will be illustrated in the computer simulations section. In this case the Fisher information matrix is of the general form Eq.(3.24) with \( L = 4N_f \) and, in the Born approximation model \( \frac{\partial \tilde{K}}{\partial d} = \left[ \frac{\partial K_B(\cdot, \cdot)}{\partial d} \quad 0 \quad \frac{\partial K_B(\cdot, \cdot)}{\partial d} \right]^T \) and a similar expression for \( X_1 \). In the multiple scattering model, there is greater dependence on \( d \), and \( \frac{\partial \tilde{K}}{\partial X_1} = \left[ \frac{\partial K_B(\cdot, \cdot)}{\partial X_1} \quad 0 \quad \frac{\partial K_B(\cdot, \cdot)}{\partial X_1} \right]^T \). The Fisher information matrix is full rank in this case (see Tables 1, 2, and 3), except in the single frequency
3.3. Numerical Experiments

In this section several CRB results are presented and discussed that are pertinent for both the Born approximation signal model, and the more general multiple scattering signal model. We consider scatterers separation $d$ in the interval $(0, \lambda/2)$ where the wavelength $\lambda = 2\pi/k$ which entails the aliasing issue addressed earlier, by which in the 1D case it is not possible to differentiate between distances $d$ and $d + n\lambda/2$, $n = 1, 2, \ldots$.

Thus in the following we shall focus on distances $d \in (0, \lambda/2)$, in the motivational context of the question of subwavelength resolution in the 1D case, and related questions. In the multi-frequency case, the condition becomes $d \in (0, \lambda_{\text{max}}/2)$ where $\lambda_{\text{max}}$ is the largest wavelength used for probing. As particular values of the scattering strengths, we adopt for the illustrations $\tau_1 = -0.5 + i0.5$ and $\tau_2 = -0.9 + i0.3$, which obey the elastic scattering condition presented in Appendix A.1

Remark: As explained earlier, in employing the Born approximation for these values, one of the two interpretations is that the associated results hold for reduced scattering strengths $\alpha \tau_1$ and $\alpha \tau_2$ where

---

### Table 3.1. Rank of the 7 × 7 Fisher information matrix under single-frequency (SF) and multi-frequency (MF) conditions for parameter vector $\xi = [X_1, d, \tau_{(r)1}, \tau_{(r)2}, \tau_{(i)1}, \tau_{(i)2}, \sigma_0^2]^T$.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$\tau_1 = -0.5 + i0.5$</th>
<th>$\tau_1 = -0.5 + i0.5$</th>
<th>$\tau_1 = -0.5 + i0.5$</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>MR</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>BT</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>MT</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>BRT</td>
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<td>5</td>
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</tr>
<tr>
<td>MRT</td>
<td>5</td>
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</tr>
<tr>
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<td>7</td>
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</tr>
<tr>
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<td>7</td>
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</tr>
<tr>
<td>MRRRTT</td>
<td>6</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>

---

1Remark: As explained earlier, in employing the Born approximation for these values, one of the two interpretations is that the associated results hold for reduced scattering strengths $\alpha \tau_1$ and $\alpha \tau_2$ where
### Table 3.2. Rank of the $3 \times 3$ Fisher information matrix under single-frequency (SF) and multi-frequency (MF) conditions for parameter vector $\xi = [X_1, d, \sigma_0^2]^T$.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$\tau_1 = -0.5 + i0.5$</th>
<th>$\tau_1 = -0.5 + i0.5$</th>
<th>$\tau_1 = -0.5 + i0.5$</th>
<th>$\tau_1 = -0.5 + i0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tau_2 = -0.5 + i0.5$</td>
<td>$\tau_2 = -0.5 + i0.5$</td>
<td>$\tau_2 = -0.5 + i0.5$</td>
<td>$\tau_2 = -0.5 + i0.5$</td>
</tr>
<tr>
<td></td>
<td>$\tau_2 = -0.9 + i0.3$</td>
<td>$\tau_2 = -0.5 + i0.3$</td>
<td>$\tau_2 = -0.5 + i0.3$</td>
<td>$\tau_2 = -0.5 + i0.3$</td>
</tr>
<tr>
<td>SF</td>
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<td>3</td>
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<td>MR</td>
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<tr>
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<td>MRRRT</td>
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<td>3</td>
<td>3</td>
</tr>
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</table>

### Table 3.3. Rank of the $5 \times 5$ Fisher information matrix under single-frequency (SF) and multi-frequency (MF) conditions for parameter vector $\xi = [\tau(r)_1, \tau(r)_2, \tau(i)_1, \tau(i)_2, \sigma_0^2]^T$.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$\tau_1 = -0.5 + i0.5$</th>
<th>$\tau_1 = -0.5 + i0.5$</th>
<th>$\tau_1 = -0.5 + i0.5$</th>
<th>$\tau_1 = -0.5 + i0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tau_2 = -0.5 + i0.5$</td>
<td>$\tau_2 = -0.5 + i0.5$</td>
<td>$\tau_2 = -0.5 + i0.5$</td>
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<tr>
<td></td>
<td>$\tau_2 = -0.9 + i0.3$</td>
<td>$\tau_2 = -0.9 + i0.3$</td>
<td>$\tau_2 = -0.9 + i0.3$</td>
<td>$\tau_2 = -0.9 + i0.3$</td>
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<tr>
<td>SF</td>
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<td>MR</td>
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</tr>
<tr>
<td>MRRRT</td>
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</tbody>
</table>
The computer results are organized into experiment sets 1, 2 and 3, corresponding to the no a priori knowledge, “known material”, and “known support” cases, respectively. Within each experiment, the results are elaborated for the two physical models of interest (Born approximation versus multiple scattering), and provide the associated comparative analysis. Particular attention is given to the CRB dependence on noise level, and on scatterer separation \( d \).

3.3.1. Experiment Set 1: No a priori Knowledge Case. In this experiment, the wavenumber is considered \( k = 2\pi/\lambda \), for unit-value lambda \( \lambda = 1 \). Particular attention is given to the CRB for estimation of \( d \) and \( \tau_1 \), for different values of noise level \( (\sigma_0^2) \) and \( d \). As explained earlier, in the no a priori knowledge case the Fisher information matrix is non-singular only for cases RRT and RRTT, therefore those are the only cases to be discussed next. In our study of the estimation performance versus noise we present results for the particular cases \( d = \lambda/3 \), and \( \lambda/8 \).

Figures 3.1 and 3.2 illustrate CRB(\( d \)) and CRB(\( \tau_{(r)1} \)), respectively, as function of \( \sigma_0^2 \). Figure 3.1 shows that CRB(\( d \)) is lower in the multiple scattering model (relative to the Born approximation) for \( d = \lambda/8 \). However, the same figure also reveals that CRB(\( d \)) is lower for the Born model for larger separation \( d = \lambda/3 \). This observation naturally opens the question of in which intervals of \( d \) the Born or multiple scattering model outperforms the other. This issue is addressed in figures 3.3 and 3.4. But continuing with figure 3.1, we also see that CRB(\( d \)) for \( d = \lambda/3 \) is smaller for both MRRTT (multiple scattering case RRTT) and BRRTT (Born model case RRTT) than for \( d = \lambda/8 \).

By looking at figure 3.2 we see that CRB(\( \tau_{(r)1} \)) is consistently smaller for Born than for the multiple scattering model. Similar findings were obtained for CRB(\( \tau_{(i)1} = \Im(\tau_1) \)), CRB(\( \tau_{(r)2} \)), and CRB(\( \tau_{(i)2} \)) (results not shown).
3.3. NUMERICAL EXPERIMENTS

Figure 3.1. CRB(d) for cases RRTT vs noise level for distances $d = \lambda/3$, $\lambda/8$ for the no a priori knowledge case.

Figure 3.2. CRB($\tau(\tau)_1$) for case RRTT vs noise level for distance $d = \lambda/3$, $\lambda/8$ for the no a priori knowledge case.
Figure 3.3 shows plots of CRB($d$) versus $d$ for the sensing configurations MRRT, MRRRTT, BRRT and BRRTT. An important observation is that for all these physical models and sensing configurations there appears to be enhanced estimation (of minimal achievable error) around $d \simeq \lambda/4$. For $d$ smaller than this optimal distance, the multiple scattering model functions better than the Born model, while for the complementary situation ($d \gtrsim \lambda/4$) the Born model performs better.

We also learn that (as expected) the enhancement due to the extra measurement T in RRTT yields only slightly better estimation relative to RRT. Figure 3.4 shows the corresponding behavior for CRB($\tau(r)_{1}$). The general characteristics noted in connection with figure 3.3 still apply but in this case the CRB is too large to enable reliable estimation of the scattering strengths for most of the values of $d$. The lowest CRB($\tau(r)_{1}$) value occurs around $d \simeq \lambda/4$ for both cases (RRT and RRTT), but with different optimal values (unlike CRB($d$) which has the same lowest CRB for all cases).
Figure 3.4. CRB((τ(r)1)) for case RRT and RRTT vs scatterers separation d with \( \sigma_0^2 = 10^{-4} \) for the no a priori knowledge case.

3.3.2. Experiment Set 2: “Known Material” Case. In this experiment set, let us compute the CRB of d and \( X_1 \) under the assumption that the scattering strengths are known a priori (“known material” case). We will employ the same conditions of experiment set 1. All the sensing configurations (R, RT, etc.) described above will be used with the exception of the case T which has singular Fisher information matrix for both models as it was mentioned in section 3.2.

Figure 3.5 shows plots of CRB(d) versus d for the R, RT, RRT and RRTT sensing schemes and under the Born approximation. First of all, the results for R versus RT, and for RRT versus RRTT are respectively identical, as expected since under the Born approximation the T experiments render no information about the scatterer positions. Clearly the RR results (RRT and RRTT) show improvement in estimation accuracy relative to the R results (R and RT), as expected.
The enhancement in resolution (from single R to double RR) is particularly noticeable particularly for $d$ around 0.2 and 0.25. It is important to note that the CRB dependence on $d$ is relatively flat, i.e., the CRB remains within the same order of magnitude for all the values of $d$ considered for RRTT, in clear contrast with figure 3.3 representing analogous results for the no a priori knowledge case. Figure 3.6 shows the corresponding plots of $\text{CRB}(d)$ versus $d$ for the R, RT, RRT and RRTT sensing schemes, under the multiple scattering model. The CRB values are comparable to those of the Born approximation, as is further illustrated in the companion figure 3.7 (provided to facilitate comparison of Born versus multiple scattering).

We note that the difference between the R and RT plots, and the RRT and RRTT plots, is quite minor under multiple scattering. Thus, unlike in the Born case, where the R and RT, and the RRT and RRTT plots exhibit no difference, in the multiple scattering case, however, there is some difference which represents marginal information in the T
Figure 3.6. CRB\((d)\) for cases R, RT, RRT, and RRTT under multiple scattering model vs scatterers separation distance \(d\) with \(\sigma_0^2 = 10^{-4}\) for the “known material” case.

Data. It is important to note from figures 3.5 and 3.6 that in the single R data set, reliable estimation of \(d\) is limited to certain ranges of \(d\). Thus there are ranges of \(d\) for which CRB\((d)\) is of the same order of magnitude or larger than the distance \(d\). The RR experiments clearly render much better CRB, and reliable estimation is possible for all values of \(d\) of interest.

3.3.3. Experiment Set 3: “Known Support” Case. Figure 3.8 shows plots of CRB\((\tau_{(r)1})\) versus \(d\) for sensing schemes RRT and RRTT, under the Born approximation and the multiple scattering model. The CRB of the multiple scattering and Born models are comparable. Each one outperforms the other in certain interval of \(d\). The CRB is noticeably much better for the RRTT case than for the RRT case due to the extra transmissive data. Even though this case assumes known support, both models still exhibit intervals of \(d\) (near \(\lambda/2\)) for which the CRB of the scattering strengths is too high to enable reliable estimation. In both the multiple scattering and Born approximation...
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Figure 3.7. CRB(d) for case RRTT, and RRT vs scatterers separation distance d for Born approximation and multiple scattering model with $\sigma_0^2 = 10^{-4}$ for the “known material” case.

curves, the plots are flatter than the no a priori knowledge counterpart, figure 3.4. This translates into a more robust situation (thanks to the additional support data), wherein the CRB is less dependent on the particular separation d than in the no a priori case.

3.3.4. Experiment Set 4: Multiple Frequencies. We conclude with the multi-frequency case of $\lambda \in \{1/100, 1/99, \cdots, 1\}$ ($\lambda_{\text{max}} = 1$). Values for the system parameters (scattering strengths, noise, etc.) are the same adopted in the single-frequency case. In the CRB plots, we show the CRB per frequency sample, in particular we plot the CRB multiplied by the number of frequencies. We also ran other simulations for 100 or 1000 randomly picked frequencies in the same interval (results not shown) and found similar overall behavior as in the CRB plots discussed next.

Figure 3.9 shows the CRB(d) for Born and multiple scattering for the RRT and RRTT cases. The CRB for multiple scattering is smaller in the entire range of d shown, and also unlike in Born, in multiple scattering the RRTT case exhibits slight
Figure 3.8. CRB((τ(ᵣ)₁)) for case RRTT, and RRT vs scatterers separation distance d for Born approximation and multiple scattering model with σ₀² = 10⁻⁴ for the “known support” case.

Figure 3.9. CRB(d) per frequency sample for the no a priori knowledge case. σ₀² = 10⁻⁴.
enhancement over the RRT case. In fact we found that there is clear enhancement from R to RT to RRT and so on in the multiple scattering case (results not shown).

Figure 3.10 shows the results for CRB(τ(1)) which shows that there is a crossover point near \( \lambda_{\text{min}} = 1 \) beyond which Born outperforms multiple scattering. In both cases this means that beyond this crossover point the Born approximation is optimistic relative to the real (multiple scattering) estimation model, which we associate to the greater linear simplicity of the Born model. This crossover point becomes further reduced in the “known support” case in figure 3.11.

Figure 3.12 shows the CRB(d) for the “known material” case. In this case multiple scattering clearly outperforms, which further illustrates one of the main points of the work which is that the main enhancement due to multiple scattering is in the localization of the targets, while once the target positions are known, then the linear Born model appears to have the advantage due to simplicity in the linear inversion. Multiple scattering also gives greater flexibility in being able to extract information in some cases.
in which the Born case does not provide any information (as in the reflective geometry). Finally, since multiple scattering is the real physical model, its predictions are the fundamental ones, and our interest in comparatively examining the Born model versus the multiple scattering model was motivated mostly to understand situations under which Born is too pessimistic (due to the possibility of better performance thanks to multiple scattering) or on the contrary too optimistic (thereby giving false expectation of what can be achieved, as was the tendency in the known support case studies).
Figure 3.12. CRB\( (d) \) per frequency sample for the “known material” case. \( \sigma_0^2 = 10^{-4} \).
CHAPTER 4

CRB for Elastic Scattering Regime

4.1. Generalities

Using the signal model defined in Section 3.1, one collects noisy data which depend on the remote sensing configuration (R, T, RT, etc.). In Chapter 2 we considered different vectors $\xi$ corresponding to different combinations of the following parameters: The position $X_1$ of target 1, the target separation $d$, the real and imaginary parts $\tau_{(r)m}$ and $\tau_{(i)m}$ of target strength $\tau_m$, $m = 1, 2$, and the unknown noise variance $\sigma_0^2$. This large parameter space gave complicated expressions that had to be solved numerically, thereby rendering limited insight into the connection between the estimation performance and conditions of our system (parameters and sensing configuration associated to the scattering matrix $K$). In the present treatment this work emphasizes in elastic scattering regime, which reduces the dimensionality of the parameter space. Further reduction of the parameter space is defined by assuming known noise variance $\sigma_0^2$. In particular, then $\tau_1 = \cos \theta_1 \exp \{i \theta_1\}$, $\tau_2 = \cos \theta_2 \exp \{\theta_2\}$ and $\sigma_0^2$ is known, so that the parameter vector

$$\xi = [x_1, d, \theta_1, \theta_2]^T$$

(4.28)

The corresponding Fisher information matrix (FIM) $I(\xi)$ and the CRB lower bound (CRB[\xi(i)]) for the variance var[\xi(i)] of any unbiased estimator for the parameter $\xi(i)$s are defined by equation (3.22).
4.2. Fisher Information for Elastic Scattering

In the following, rather than computing the full FIM above, we focus on the diagonal elements of that matrix. This defines a conditional FIM for each parameter, and the associated CRB, under prior knowledge of the other parameters. Furthermore, from the properties of the FIM, the resulting diagonal-approximated FIM, and resulting FIM, also renders an approximation to the CRB resulting from the full FIM (including the off-diagonal, cross-information terms).

Importantly, due to statistical independence, the FIM of R and/or T data sets, e.g., RRT (both reflective experiments, and one transmissive experiment), is given by the sum of the individual (R,R,T) FIM’s, e.g., \( I_{RRT} = I_{R(-,-)} + I_{R(+,+)} + I_{T(+,-)} \). Therefore our analysis of the single-R and single-T data cases applies to larger sets of R and/or T data.

The fact of estimate just one parameter at the time reduces the complexity associated to this problem and provides insight into the relation between targets locations, reflectivities and frequency. This method represents the exact FIM just for one parameter.

Note that the FIM of one parameter quantifies the significance of choosing some characteristics of the system to enhance the estimation for single or multiple frequency cases. This special case also allow us to distinguish the conditions under each model enhances or diminishes the estimation of one parameter. Therefore, the optimal next experiments are identified.

4.2.1. Diagonal Entry \( I(\xi)_{1,1} \): Estimation of \( X_1 \). The CRB of the first scatterer position \( X_1 \) goes infinity when the case T is used for the estimation of the position of the first scatterer because the transmissive experiment has no information about \( X_1 \).

It will be convenient to gain insight plus facilitate interpretation to introduce \( q_m = i\hat{\tau}_m = \tau_m/2k \) for \( m = 1, 2 \), natural parameters that appears repeatedly in the present
4.2. FISHER INFORMATION FOR ELASTIC SCATTERING

CRB analysis of scattering systems.

For the case $R$ under the Born approximation model, the FIM is obtained using the definition in Eqs.(3.22) and simplifying terms, we get the result:

\[
I(X_1)_{R,,-}^{\text{Born}} = \frac{2}{\sigma_0^2} \left\{ |q_1|^2 + |q_2|^2 + 2 \Re[ q_1^* q_2 \exp(i2kd) ] \right\} = I(X_1)_{R,+,+}^{\text{Born}}
\]

(4.29)

In the multiple scattering model, one obtains

\[
I(X_1)_{R,,-}^{\text{MS}} = \frac{2}{\sigma_0^2} \left\{ |q_1|^2 + |q_2|^2 + 2 \Re[ q_1^* q_2 \exp(i2kd) ] \right\}
\]

\[
+ \frac{2}{\sigma_0^2} \left\{ 2 \Re[ q_1^* q_2 \exp(i2kd) ] \frac{1 + 2 \Im[ q_1 ] + |q_1|^2}{(1 + q_1 q_2 \exp(i2kd))} \right\}
\]

(4.30)

where $\ast$ denotes complex conjugation. This result also gives the corresponding result for the reflective $R_{+,+}$ case via the substitutions $q_1 \to q_2$ and $q_2 \to q_1$.

4.2.2. Diagonal Entry $I(\xi)_{2,2}$: Estimation of $d$. In this case of the estimation of the separation between the two targets “$d$”, the Born approximation model again presents same result for the two different sensing points. The FIM under Born for the reflective case $R$ is defined by:

\[
I(d)_{R,,-}^{\text{Born}} = \frac{2}{\sigma_0^2} |q_2|^2 = I(d)_{R,+,+}^{\text{Born}}
\]

(4.31)

From the Eq.(4.31), it can be seen that the CRB for the scatterers separations "$d$" under the Born approximation model is linear dependent of the natural parameter $q_2$ and the noise of the system.

For multiple scattering model, the reflective data point $R_{-,+}$ produces the following FIM:
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\[ I(d)_{MS}^{R_{+,-}} = \frac{2}{\sigma_0^2} |q_2|^2 \left\{ \frac{(1 + 2 \Im \{ q_1 \})^2 + |q_1|^2 |q_2|^2}{(1 + 2 \Re \{ q_1 q_2 \exp(i2kd) \})^2} \right\} \]

The FIM for the reflective case \( R_{+,-} \) is defined by:

\[ I(d)_{MS}^{R_{+,+}} = \frac{2}{\sigma_0^2} |q_2|^2 \left\{ 1 + 2 \Im \left[ \frac{(1 - iq_2)^2 q_1^2 \exp(i4kd)}{(1 + q_1 q_2 \exp(i2kd))^2} \right] \right\} \]

\[ + \frac{2}{\sigma_0^2} |q_2|^2 \left\{ \frac{|q_1|^4 (1 + 2 \Im \{ q_2 \} + |q_2|^2)^2}{(1 + 2 \Re \{ q_1 q_2 \exp(i2kd) \} + |q_1|^2 |q_2|^2)^2} \right\} \]

(4.33)

For the transmissive case under Born approximation model, the first derivative with respect to "d" vanish, therefore the FIM for the case T is zero. Contrarily, the multiple scattering model presents the following FIM:

\[ I(d)_{MS}^{T_{+,-}} = \frac{2}{\sigma_0^2} |q_2|^2 \left\{ \frac{1 + 2 \Im \{ q_1 \}^2 + |q_1|^2 |q_2|^2}{1 + 2 \Re \{ q_1 q_2 \exp(i2kd) \} + |q_1|^2 |q_2|^2} \right\} \]

(4.34)

where \( I(d)_{T_{+,-}} = I(d)_{T_{+,+}}^{MS} \) due to non dispersive media property again.

The expressions for \( X_1 \) and \( d \) above are written in function of \( q_1 = \tau_1/2k \) and \( q_2 = \tau_2/2k \). Those expressions also apply to any type scattering not only elastic.

The Born approximation and the multiple scattering models for the reflective cases have some similar characteristics in each components of the expressions of FIM in the estimation of scatterers positions (\( d \) and \( X_1 \)). In most of the cases, the multiple scattering model have a multiplicative factor that depends on the natural parameters \( q_1, q_2, \) and \( \exp(i2kd) \) which illustrate the difference between them.

In the expressions of the FIM for parameters \( d \) and \( X_1 \) above, the frequency plays a determinant role being part of the "natural parameters" of the systems. As the fre-
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frequency increase the FIM become small then the CRB will be larger. Consequently \( q_m \ll 1 \) and both models will have an equivalent result in the estimation of the one parameter however with large level of errors. This in turn indicates scattering conditions under which the Born-approximation-based estimation performance estimates are pessimistic (optimistic). In \( d \) estimation, it appears that the scattering strength of the scatterer that is closer to the receiver has a significant role. In particular, numerical results based on eqs. (4.29) through (4.34) (results not shown) show that FIM under multiple scattering tends to increase as the scattering strength of the scatterer that is closer to the receiver increases.

The transmissive case "T" for the estimation of \( X_1 \) will not contribute information for both models. This situation remains for estimation of the scatterers separation \( d \) under the Born approximation model. The transmissive case \( T \) for multiple scattering contributes noticeably in the estimation of \( d \) if both natural scatterers reflectivities \( q_m \) are quite large.

4.2.3. Diagonal Entries \( I(\xi)_{3,3} \) and \( I(\xi)_{4,4} \): Estimation of \( \theta_1 \) and \( \theta_2 \). In the estimation of the scatterer's reflectivity, without loss of generality the variable to estimate will be \( \theta_n \), which immediately can be related with error bounds of the real and imaginary part of the strength of the target \( \tau_m \) for elastic case.

The FIM for case \( R \) under the Born approximation model is given by:

\[
I(\theta_1)_{\text{Born} R} = \frac{2}{\sigma_0^2} \left\{ \frac{1}{(2k)^4} \right\}
\]

The equation (4.35) holds for any reflective and transmissive case for the estimation of strength of the target. This expression also remains in case of estimation of any scatterer reflectivity under the Born approx. model. Thus \( I(\theta_1)_{\text{Born} R} = I(\theta_1)_{\text{Born} T} \) and \( I(\theta_1)_{\text{Born} R or T} = I(\theta_2)_{\text{Born} R or T} \).
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The equation (4.35) indicates that under Born approximation model, the error bounds for the estimation of the scatterer’s reflectivities only depends on the noise variance and the wavelength instead of the reflectivity and position of the targets.

In the multiple scattering model, each reflective case R will have a particular expressions depending of the sensing data point used, so that:

\[
I(\theta_1)_{MS}^{R_{-,+}} = \frac{2}{\sigma_0^2} \frac{1}{(2k)^4} \left\{ \frac{(1 + 2\Im \{q_2 \exp(i2kd)\} + |q_2|^2)^2}{[1 + 2\Re \{q_1q_2 \exp(i2kd)\} + |q_1|^2 |q_2|^2]^2} \right\}
\]

In the case of the reflective case \(R_{+,+}\), the following expression is achieved

\[
I(\theta_1)_{MS}^{R_{+,+}} = \frac{2}{\sigma_0^2} \frac{1}{(2k)^4} \left\{ \frac{(1 + 2\Im |q_2| + |q_2|^2)^2}{[1 + 2\Re |q_1q_2 \exp(i2kd)| + |q_1|^2 |q_2|^2]^2} \right\}
\]

When the transmissive sensing point is used to estimate the FIM becomes

\[
I(\theta_1)_{T_{-,+}}^{MS} = \frac{2}{\sigma_0^2} \frac{1}{(2k)^4} \left\{ \frac{(1 + 2\Im |q_2| + |q_2|^2)(1 + 2\Im |q_2 \exp(i2kd)| + |q_2|^2)}{[1 + 2\Re |q_1q_2 \exp(i2kd)| + |q_1|^2 |q_2|^2]^2} \right\}
\]

where \(I(\theta_1)_{T_{-,+}}^{MS} = I(\theta_1)_{T_{+,+}}^{MS}\).

From Eqs. (4.36), (4.37), and (4.38) we note that three natural factors \(q_1\), \(q_2\) and \(\exp(i2kd)\) affect the limit of the error bound under the multiple scattering model. The choices of those natural parameters of the system will determine the conditions under the multiple scattering model can enhance the estimation or contrarily can be a destructive interference. Like \(I(d)\), similarly \(I(\theta_1)\) under multiple scattering is given by the product of FIM under the Born approximation and a multiplication factor that depends on the natural parameters \(q_1\), \(q_2\), and \(\exp(i2kd)\). If this factor is higher than (lower than) 1 then FIM under multiple scattering is correspondingly higher than (lower than) FIM under the Born approximation, and the Born-approximation-based performance
estimates are correspondingly pessimistic (optimistic).

The result for the error bound for the second reflectivity \( \theta_2 \) for the multiple scattering model are no presented because those result are complementary with the error bounds of the estimation of the first natural scatterer \( \theta_1 \).

4.2.4. Simultaneous Estimation of \( X_1 \) and \( d \). The preceding results for \( I(X_1) \) and \( I(d) \) can be further generalized by considering the more general FIM submatrix \( I_{X_1,d} \) related to unknown \( X_1 \) and \( d \) and known scattering strengths. After some manipulations, one obtains

\[
I_{X_1,d} = \frac{2}{\sigma_0^2} \Re \left[ \begin{array}{cc} I_{X_1} & I_{X_1,d} \\ I_{d,X_1} & I_d \end{array} \right]
\]

(4.39)

where \( I_{X_1} \) and \( I_d \) are the diagonal entries of the total FIM that were defined in sections 3.1 and 3.2. The off-diagonal element \( I_{d,X_1} = I_{X_1,d}^* \). For transmissive data, \( I_{X_1,d} = 0 \). For reflective data,

\[
(I_{X_1,d})^{MS}_{R_{-+}} = \left[ q_2 \right]^2 + \left| q_2 \right|^2 \left( 1 + 2 \Re q_1 + \left| q_1 \right|^2 \right) \left[ 1 + 2 \Re q_1 q_2 \exp(i2kd) + \left| q_1 \right|^2 \left| q_2 \right|^2 \right]
\]

(4.40)

and

\[
(I_{X_1,d})^{MS}_{R_{++}} = \left\{ \left| q_2 \right|^2 + q_1^* q_2 (1 + i q_2^*) \exp(-i2kd) + q_1 \left| q_2 \right|^2 (1 - i q_2^2) \exp(i4kd) \right\}
\]

(4.41)

which for \( |q_1| << 1 \) and \( |q_2| << 1 \) reduce to the Born approximation results,

\[
(I_{X_1,d})^{Born}_{R_{-+}} = \left[ q_2 \right]^2 + q_1^* q_2 \exp(i2kd)
\]

(4.42)
and

\[(I_{X_1,d})^\text{Born}_{R+} = \left[|q_2|^2 + q_1^* q_2 \exp(-i2kd)\right].\]

### 4.2.5. Complete FIM matrix for case RRTT under the Born approximation model.

Under elastic scattering, the parameter vector (4.28) is reduced. Both the Born approximation model and the more general multiple scattering model are considered. The CRB for the Born approximation model can be given in closed-form after long manipulations. In contrast, generally the multiple scattering case does not facilitate closed-form results. As an illustrative example of the CRB calculations under the Born model, consider the case of four measurements (RRTT) corresponding to the full scattering matrix including for the 1D geometry. In this case the FIM is of the form:

\[(4.44)\]

\[I^\text{Born}_{\text{RRTT}}(\xi) = \frac{2}{\sigma_0^2} \Re \begin{bmatrix} I_{X_1,d}(\xi) & I_{X_1,d,\theta_1,\theta_2}(\xi) \\ I_{\theta_1,\theta_2,X_1d}(\xi) & I_{\theta_1,\theta_2}(\xi) \end{bmatrix},\]

where the elements in (4.44) are submatrices that represent special cases. The diagonal 2 X 2 submatrices \(I_{X_1,d}(\xi)\) and \(I_{\theta_1,\theta_2}(\xi)\) represent the FIM under the a priori knowledge of the scatterers’ strength and the scatterers’ position respectively. The off diagonal submatrices \(I_{X_1,d,\theta_1,\theta_2}(\xi)\) and \(I_{\theta_1,\theta_2,X_1d}(\xi)\) are the cross information when there is no a priori information of the material and the position of the points targets.

From section 4.2.4, \(I_{X_1,d}(\xi) = \begin{bmatrix} J_{X_1} & J_{X_1,d} \\ J_{d,X_1} & J_d \end{bmatrix}\), whose components for case RRTT are defined by:

\[J_{X_1} = \frac{2}{(2k)^2} \left\{|\tau_1|^2 + |\tau_2|^2 + 2|\tau_1||\tau_2|\cos(2kd)\Re\left\{\exp i(\theta_1 - \theta_2)\right\}\right\}\]

\[J_{X_1,d} = J_{d,X_1}^\ast = \frac{2}{(2k)^2} \left\{|\tau_2|^2 + |\tau_1||\tau_2|\cos(2kd)\exp i(\theta_1 - \theta_2)\right\}.$$
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\[ J_d = \frac{2}{(2k)^2} \{ |r_2|^2 \} \]

Similarly, \( I_{\theta_1, \theta_2} = \begin{bmatrix} J_{\theta_1} & J_{\theta_1, \theta_2} \\ J_{\theta_2, \theta_1} & J_{\theta_2} \end{bmatrix} \), where

\[ J_{\theta_1} = J_{\theta_2} = \frac{4}{(2k)^4} \]

\[ J_{\theta_1, \theta_2} = J_{\theta_2, \theta_1}^* = \frac{4}{(2k)^4} \{ \cos^2 (kd) \exp i (2\theta_1 - 2\theta_2) \} \]

where * denotes the complex conjugate. The “cross-information” matrices \( I_{X_1 d, \theta_1 \theta_2} \) and \( I_{\theta_1 \theta_2, X_1 d} \) are given by

\[ I_{\theta_1 \theta_2, X_1 d} = \begin{bmatrix} \frac{-i^2}{(2k)^3} \{ |r_2| \sin (2kd) \exp i (2\theta_1 - \theta_2) \} & \frac{-i^2}{(2k)^3} \{ |r_2| \sin (2kd) \exp [i (2\theta_1 - \theta_2)] \} \\ \frac{-i^2}{(2k)^3} \{ |r_1|^2 \sin (2kd) \exp - [i (\theta_1 - 2\theta_2)] \} & 0 \end{bmatrix} \]

where \( I_{\theta_1 \theta_2, X_1 d} = I_{X_1 d, \theta_1 \theta_2}^H \), where \( H \) indicates the hermitian.

In general the combined multistatic data set (case RRTT for Born approximation) in Eqs (4.44) is not easy to connect via Eq.(3.23) with the effect of one parameter in the computation of the error bounds. Additionally the significance of the use of reflective or transmissive data is not clear in the estimation of one parameter. It is also too complicated to find a direct relation between the parameters and the error bounds due to the complexity of the cross information terms of FIM. Therefore to carry out analysis and interpretation, the diagonal method applied in the previous sections allow us to define the characteristic for the best optimal experiments to reduce the error in the estimation process.
4.3. Numerical Illustration

In the following numerical experiments, we consider the wavenumber \( k = 2\pi/\lambda \) for unit-value \( \lambda = 1 \). In presenting the results for different values of \( \theta_1 \) or \( \theta_2 \), in some of the plots we consider only the interval \((0, \pi/2)\) because the corresponding results are symmetrical to those for \((-\pi/2, 0)\).

Figure (4.1) shows plots of CRB\((kd)\) versus scattering separation \((kd)\), all under the multiple scattering model, for different data. In this example, \( \tau_1 = -0.5 + i0.5 \) and \( \tau_2 = -0.9 + i0.3 \). Clearly the reflective data dictates the estimation limits. The transmissive data contributes minimally.

The CRB for \( \theta_1 \) is computed from (4.35), (4.36), (4.37), and (4.38). In addition, one can also obtain from the same results and the method in [18, pp. 229-233] the CRB for the real and imaginary parts of \( \tau_1 \). Figure (4.2) shows the corresponding CRB\((\Re(\tau_1)/k))\) versus \( kd \). As in figure (4.1), \( \tau_1 = -0.5 + i0.5 \) and \( \tau_2 = -0.9 + i0.3 \). A quasi-resonance appears in the plot of the CRB as a function of \( kd \). We observe that the cases R and T
individually have, in average, the same error bounds as function of the natural scatterer separation $kd$, although the plot reveals that for $kd \epsilon (.596 \pi - 1.007 \pi)$ the case $R(-,-)$ is better. Similar characteristics remains for case $R(+,+)$ not shown. As more sensing data are added, the error bound for the first scatterer reflectivity decreases. The “next optimal data” (R or T) to estimate the scatterer reflectivity can be either reflective or transmissive because the CRB’s with the same number of samples show comparable results, see, e.g., for example, the cases RR, RT and TT in the figure.

To elucidate the regions in scattering parameter space where multiple scattering enhances or diminishes the CRB relative to the Born approximation reference model, we plot in the following the ratio of the Fisher information of the two models, $I(\xi)^{MS}/I(\xi)^{Born}$. When this quantity is above 1 the Born approximation estimation performance estimates are pessimistic, while when it is below 1 the Born approximation estimates are optimistic. When $I(\xi)^{MS}/I(\xi)^{Born} \simeq 1$ both models coincide in information-carrying capacity about $\xi$.

Figures (4.3) and (4.4) show the ratio $I(\delta)^{MS}/I(\delta)^{Born}$ for different reflective data. In
4.3. NUMERICAL ILLUSTRATION

Figure (4.3a) the $I(d)$ ratio is shown as a function of the scatterers strength $\theta_1$ and $\theta_2$, for $d = 0.25\lambda$ for data set $R_{(-,-)}$. The dark regions in the plot correspond to regions where both models coincide in estimation capabilities. The other areas illustrate the enhancement of multiple scattering in the estimation for scatterers separation $d$. Figure (4.3b) shows the ratio of the $I(d)$ for reflective data $R_{(+,+)}$ for $d = 0.15\lambda$. We can identify regions with ratio less than one. This means the Born approximation model presents a smaller error bound than the real multiple scattering model.

Figure (4.4) illustrates the behavior of $I(d)^{MS}/I(d)^{Born}$ versus $\theta$ and $d$, in the case when both scatterers have the same reflectivity, $\theta_1 = \theta = \theta_2$. Each data set shows a different behavior. For the reflective data set $R_{(-,-)}$, the scatterers reflectivity $\theta$ is dominant in controlling when multiple scattering enhances or diminishes estimation. From figure (4.4a), approximately for $\theta \gtrsim 0$ multiple scattering enhances information about $d$. In contrast, as seen in figure (4.4b), for the data set $R_{(+,+)}$ both parameters $d$ and $\theta$ play an important role in the estimation. Approximately, for $(-0.2\pi \lesssim \theta \lesssim 0.2\pi) \cap (0.22\lambda \lesssim d \lesssim 0.35\lambda)$ multiple scattering enhances the estimation.

The behavior of the ratio $I(\theta_1)$ versus $\theta_2$ and $d$ is shown in figure (4.5). The top (a), middle (b), and bottom (c) plots forming this figure correspond to cases $R_{(-,-)}$, $R_{(+,+)}$, and $T$, respectively. Each data set shows a different behavior. The dark region in (a), (b), and (c) corresponds to diminished estimation due to destructive multiple scattering. On the other hand, the regions where multiple scattering significantly enhances estimation are, approximately: $0.25\lambda \lesssim d \lesssim 0.5\lambda$ for $R_{(-,-)}$ in part (a), $\theta_2 \gtrsim 0$ for $R_{(+,+)}$ in part (b), and $\{\theta_2 \geq 0\} \cap \{0.25\lambda \lesssim d \lesssim 0.5\lambda\}$ for $T$ in part (c).
Figure 4.3. $I(d)^{MS}/I(d)^{Born}$ for reflective data versus the scattering strengths $\theta_m$ for $m=1,2$, under scatterer separation $d = 0.25\lambda$ in (top plot) and $d = 0.15\lambda$ (bottom plot), with $\sigma_0^2 = 10^{-4}$. The top plot corresponds to $R_{(-,-)}$, while the bottom plot corresponds to $R_{(+,+)}$. 
Figure 4.4. $I(d)^{MS}/I(d)^{Born}$ for reflective data versus $\theta_2$ and $d$, when both scatterers have the same reflectivity. The top and bottom plots correspond to the cases $R_{(-,-)}$ and $R_{(+,+)}$, respectively.
Figure 4.5. $I(\theta_1)^MS/I(\theta_1)^{Born}$ versus $\theta_2$ and $d$. The top left, top right, and bottom plots correspond to the cases $R(-,-)$, $R(++,+)$ and $T$, respectively.
CHAPTER 5

Conclusions

This work investigated scattering systems in 1D space in the framework of the CRB associated with the problem of estimating scattering parameters from the response of the system as measured using given excitations (transmit experiments) and given receivers. We adopted the fundamental CRB measure which is algorithm-independent, i.e., it is a bound on estimation error for any unbiased estimator applied to the given data. The 1D geometry is an idealization of certain fully 3D problems which is relevant to certain radar and tomographic experiments for which we considered the full set of bistatic data: reflective and transmissive experiments. Among other results, we studied the information content of different data sets, corresponding to different sensing configurations. Under the general scatterer regime, the Born model gives only low-spatial-frequency information in the transmissive geometry. It gives more information in the reflective geometry. The multiple scattering model tends to be more informative in general (more parameters or at least the same number as in Born can be estimated, hence we can talk about more degrees of freedom thanks to multiple scattering). The two reflective geometries (RR) tend to add important new information relative to the single R. One of the transmissive experiments (single T) also adds noticeable new information. However, the transmissive geometry is reciprocal under no noise and the second T experiment, corresponding to the other direction (TT), usually gives only marginal new information relative to the first T experiment. It was shown that multiple scattering can either enhance or reduce estimation performance, depending on the system parameters and the particular data used for estimation. Since (for fixed scattering system parameters) multiple scattering is the correct model, this means that
performance characterizations based on the Born approximation can be optimistic or pessimistic. Thus multiple scattering may give enhanced resolution relative to what one would expect under the Born model, but multiple scattering also implies a more complex nonlinear signal model which can reduce our ability to estimate parameters. To further maintain the focus, we considered the special case of two scatterers for which the total number of parameters is four (two positions and two angles describing the individual scattering strengths). This focus allowed closed form expressions for special cases. Of much interest are the diagonal elements of the FIM, for which closed form expressions were derived. Quite often the inverse of the FIM which defines the CRB is essentially dominated by the diagonal elements of FIM. This calculation is also relevant in special cases when one has a priori information in scatterer position and/or scattering strength, and the different diagonal elements correspond to different physical situations or applications. It was found that certain natural parameters, in particular, \( q_i = \frac{\tau_i}{2k} \), consistently appear in all the CRB calculations. It was found that reflective (R) data is usually better than transmissive (T) data, particularly in position or separation distance estimations, but that they are comparable in certain cases, particularly in scattering strength estimation. We developed our results for both signal models, to comparatively study the role of multiple scattering in either enhancing or diminishing estimation performance from scattering data. This gives an indication of the goodness of rule-of-thumb performance estimates based on the Born approximation model and establishes parametric conditions under which such estimates are applicable, or in contrary, are either clearly optimistic or pessimistic. Optimistic and pessimistic regions in parameter space were illustrated. Thus contrary to the focus in several recent papers [1, 2, 8], which emphasize the positive role of multiple scattering in enhancing imaging, we have provided counterexamples where multiple scattering is detrimental relative to weak scattering imaging despite the greater contrast of highly scattering targets. For realistic applicability of these general findings, study in the full 3D case
is necessary and is proposed as an interesting future avenue. But clearly the simpler 1D case has provided a new corroboration of known facts in this area as well as a lot of new intuition into the role of multiple scattering and of different configurations in yielding information about scattering systems from their response matrix. Additionally, this work provides light on the role of multiple scattering among different cross sections of an object under investigation via tomographic experiments for cross sections that exhibit slow spatial variation relative to the wavelength (as in x-ray tomography) but where unlike most (linearizing) models the multiple scattering interactions including backscattering by different cross section planes are not neglected. Also, even though we have emphasized for conceptual simplicity the case of point scatterers, in reality a similar formulation applies to extended objects having uniform constitutive properties after substitution of our scattering strength coefficients for reflection and transmission coefficients that depend on the mismatch between the region of interest and the free space medium, as is well known (for example, see in [17], p. 220-223).
Bibliography


[12] F. Simonetti. Multiple scattering: The key to unravel the subwavelength world from the far-field


APPENDIX A

Elastic Scattering Condition

It is shown next that, under energy-conservative, elastic scattering, the scattering strengths $\tau_m$ obey the condition

$$\text{(A.45)} \quad [\text{Re}(\tau_m)]^2 + \text{Re}(\tau_m) + [\text{Im}(\tau_m)]^2 = 0$$

where $\text{Re}$ and $\text{Im}$ denote real and imaginary part, respectively. To arrive at this result, one notes that in the present scalar wave context, the quantity that is analogous to the Poynting vector for electromagnetic waves is the energy flux vector (\text{[19]}, p. 717) which in our 1D framework can be expressed in terms of the field $\psi(x)$ as

$$\text{(A.46)} \quad F(x) = \beta \text{Im} \left\{ \psi^*(x) \frac{\partial}{\partial x} \psi(x) \right\}$$

where $\beta$ is a positive constant. Consider a point target having strength $\tau$. Let $\exp(ikx)$ be the incident field, so that the field to the left of the target is $\exp(ikx) + \tau \exp(-ikx)$ while the field to the right of the target is $(1 + \tau) \exp(ikx)$. Then the energy flux to the left is $F(x < 0) = \beta k(1 - |\tau|^2)$, while the energy flux to the right is $F(x > 0) = \beta k[1 + 2\text{Re}(\tau) + |\tau|^2]$. In elastic scattering the flux remains continuous at the scattering interface so that $k(1 - |\tau|^2) = k[1 + 2\text{Re}(\tau) + |\tau|^2]$, which gives (A.45), as desired. Furthermore, it can be shown that, consequently, the most general scattering strength $\tau$ which gives rise to elastic (non-dissipative) scattering is of the form

$$\text{(A.47)} \quad \tau = -\cos(\theta) \exp(i\theta)$$

where $\theta \in [-\pi/2, \pi/2]$. 

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