COMPRESSIVE SENSING
IN COHERENT IMAGING AND ANTENNA SYNTHESIS

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Abstract

The classical sampling theory is based on Shannon-Nyquist theorem: the sampling rate must be at least twice higher than the maximum frequency in the signal. Recently a new sampling theory called compressive sensing asserts that in some cases the sampled signal can be well recovered far below the Nyquist rate. This applies to signals that are sparse in a given basis. Compressive sensing can be applied in many areas, such as signal precessing, imaging, earth probing, radar, and holography. In this thesis we introduce the basic concepts of compressive sensing and its application in coherent imaging and antenna synthesis. In coherent imaging we point out that we can acquire not only the amplitude but also the phase information of a phase object via compressive sensing. We also show that in antenna synthesis, compressive sensing can optimize the antenna design.
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Chapter 1

Introduction

Although Compressive sampling (CS) is relatively new theory, it already shows great potential not only in signal processing but also in many other areas. It attracts much research attention. The idea of compressive sampling has long been raised by seismologists in the 1970s. Experiments consisted of probing the structure of the earth by transmitting acoustic waves and receiving the reflected signals. Since the underlying structure does not change rapidly, the researches realized that the reflected signals could be considered as sparse. And indeed they discovered that they could reconstruct images of earth layers with data that did not seem to satisfy the Nyquist-Shannon criterion. However, compressive sensing theory did not begin until 2004 when Emmanuel Candes, a mathematician at Caltech, was working on a problem in magnetic resonance imaging. Candes discovered that a test image could be reconstructed exactly even with data below Nyquist-Shannon rate, which had generally been considered to be the lowest rate at which one could safely sample any general signal without loss.

In contrast, compressive sensing theory asserts that it is possible to recover certain signals from samples which are far below the Nyquist rate with perfect accuracy. This is why CS theory has attracted so much recent attention. An excellent example of the practical implications of CS theory is the single pixel camera, which acquires a sequence of single-pixel measurements of random projections of a scene instead of collecting all the pixels as with a standard imaging array such as a CCD. In particular, many fewer such single-pixel snapshots need to be acquired than the number of samples suggested by the Sampling Theorem. The camera employs a digital micro-mirror array to
project the scene. Since the camera only uses a single photo detector, it has been described as a single pixel camera.

Since the CS projections themselves only involve incoherent light, all the calculation are real, and moreover, the idea of single pixel camera consider only the case where the object is much larger than the wavelength, or the DMD array is far from the photon detector, therefore the diffraction becomes negligible. However, when the wavelength is comparable with the object or the DMD array is very close to the photon detector, we must consider the diffraction and near field imaging. It is better to capture the data coherently, via interferometer or holographic experiments. Otherwise we will lose important phase information. In this thesis we develop the fully coherent imaging version of the compressive sensing optical imaging.
Chapter 2

Review of Compressive Sensing

Compressive sensing is not magic; in particular it is not successful unless two conditions are satisfied: sparsity of the signal sampled and incoherence of a basis used in the sampling procedure. The first condition ensures that the signal bandwidth is significantly smaller than the sampling bandwidth specified by the Nyquist rate. The latter condition ensures that we can acquire that information completely. These two conditions will be summarized separately in the following sections.

2.1 Sparsity

Sparsity means that only a small part of the entries in a signal sequence contain useful information. In other words, if we sample a signal at the Nyquist rate, all the information in the signal will be acquired for sure, but most of it is useless. We waste time and bandwidth to sample a sparse signal by classical sampling theory.

Most natural and man made signals are sparse. They can be represented sparsely when expressed in a well selected basis. For example, let us consider a finite length, one dimensional, discrete time signal $f$ which can be written as an $N \times 1$ column vector with elements $f[n], n = 1, 2, ..., N$. If we properly choose a orthogonal basis : $\Psi = [\psi_1 \psi_2 ... \psi_N]$, the signal can be expressed in the following form:

$$f = \sum_{i=1}^{N} x_i \psi_i$$

(2.1)

where $x = [x_1, x_2, x_3...x_N]$ is a $N \times 1$ column vector coefficient sequence of $f$, 

\[ x_i = \langle f, \psi_i \rangle \]. Equation 2.1 can be rewritten as:

\[ f = \Psi x \]  

(2.2)

where \( \Psi \) is the \( N \) by \( N \) matrix with \( \psi_1, \psi_2, ..., \psi_N \) as columns.

Figure 2.1: Example of image with wavelet transform from [1]: (a) original image with 1024 \( \times \) 768 pixels, (b) the wavelet transform coefficients of original image, (c) the zoomed in part of figure (b).

If there are only \( k \) entries of a sequence that are nonzero or much larger than zero, we can say that the signal is \( k \) sparse, in other words the sparse number is \( k \). When a signal has a sparse expansion, it is compressible which means one can sample the signal less at than the Nyquist rate. One can keep the largest \( k \) entries the same but drop those small coefficients without the much loss of useful information. This is also the basic idea of how transform coding works. For example, the JPEG and JPEG2000 compression standards are based on the discrete cosine transform (DCT) and wavelet bases respectively. These two coding methods are widely used in image processing systems such as digital cameras. In these methods all full \( N \) samples are acquired in the beginning, then transform coefficients \( x_i \) are computed by discrete cosine transform or represented by wavelet basis. Next one needs to Locate the \( k \) largest coefficients, discard the rest small ones, and encode the remaining co-
The difference between transform coding and CS theory is that the former one samples a signal with or more than the Nyquist rate then compresses it, while CS theory compresses and samples the signal simultaneously. The "sample-first-then-compress" method must start with a large number of samples $N$, even when the number of the largest entries $k$ is small. The precious time, bandwidth and store spaces may be wasted during the process of data transfer. Figure 2.1 shows an example of original image with $1024 \times 768$ pixels, and its wavelet transform coefficients. The lower right picture clearly shows that most of the coefficients are insignificant and therefore can be discarded.

2.2 Incoherent Sampling

Before any samples were taken, the sample basis must be chosen properly. The good choice of basis allows us to gather as much information as possible per sample to make the needed sample number small.

This takes us to another important condition in CS: incoherent Sampling. If we define $\Psi$ as the basis to represent signal $f$, and $\Phi$ as the sampling basis, the coherence between $\Psi$ and $\Phi$ is defined as [4]:

$$\mu(\Psi, \Phi) = \sqrt{n} \cdot \max_{1 \leq k,j \leq n} |\langle \psi_k, \phi_j \rangle|.$$ (2.3)

The coherence measures the similarity between the basis $\Psi$ and $\Phi$. If $\Psi$ and $\Phi$ contain correlated elements, $\mu$ is large, otherwise it is small. In compressive sensing, low coherence pairs are optimal.

Among the bases used in compressed sensing, random matrices may be the most common and convenient basis. They are largely incoherent with any other fixed bases, and they can be easily generated and applied.

2.3 Under-sampled Measurement and Recovery

In the process of compressive sensing only limited number of the samples are taken. Under-sampled measurement represents the condition that only $M$ out of $N$ samples were observed and collected. The recovery of the original signal
becomes the major problem to be solved. Define the data collected as:

\[ y_k = \langle f, \phi_k \rangle, k \in M \quad \text{or} \quad y = \Phi f \quad (2.4) \]

where \( M \in 1, ..., N \).

Since \( f = \Psi x \), we can rewrite:

\[ y = \Phi f = \Phi \Psi x = \Theta x \quad (2.5) \]

where \( \Theta = \Phi \Psi \) is an \( M \times N \) matrix. Figure 2.2 shows the relations of the matrices \( \Phi, \Psi \) and \( \Theta \).

![Figure 2.2: Compressive sensing measurement process with measurement matrix \( \Phi \) and represent basis \( \Psi \). The transformed measurement basis become \( \Theta = \Phi \Psi \).](image)

To recover the signal \( f \) with an incomplete set \( y \), we decide to apply \( l_1 \)-norm minimization method under the constraint of \( y = \Phi f \). An optimization problem is considered:

\[ \min ||x||_{l_1} \quad \text{subject to} \quad y = \Theta x \quad (2.6) \]

Among all possible signal sets \( x \) that obey the constraint \( y = \Theta x \), we pick that whose \( l_1 \) norm is minimum.

\( l_1 \) norm and \( l_2 \) norm are defined as follows:

\[ ||x||_{l_1} = \sum_{k=1}^{n} |x_k| \quad (2.7) \]

\[ ||x||_{l_2} = \sqrt{\sum_{k=1}^{n} |x_k|^2} \quad (2.8) \]

Here raise a question. Why do we recover the signal with \( l_1 \) minimization
not $l_2$ minimization? The later one is also known as the minimum energy solution of the linear inverse problem in 2.5. It has been solved properly for long. Figure 2.3 shows the $l_1$ and $l_2$ unit balls in 2 dimensional space. The straight line is the possible signal sets satisfying $H = \{a : y = \Theta a\}$. If we expand the $l_1$ unit ball until it hits $H$, the intersection point is the solution of the optimum problem. The result is sparse with high probability, since the intersection points are always on the axes. In the least square problem, instead of the diamond-shaped $l_1$ unit ball the $l_2$ unit ball is a perfect circle. If we expand the $l_2$ unit ball again, the intersection points will not be on the axes, and thus the solution does not have the sparse characteristic [5]. In order to have a sparse solution, we need to apply $l_1$ minimization method instead of $l_2$ minimization. In high dimensional spaces the situations are almost the same.

![Figure 2.3: Geometry of $l_1$ recovery. The left side is the unit ball for $l_1$ norm, while the right side is the unit ball for $l_2$ norm.](image)

In the next part, we will address another method to solve the inverse problem based on $l_1$ minimization.

### 2.4 Total Variation Minimization Method

Total variation (TV) minimization is another typical method in compressive sensing. It has been proved to be very efficient in the reconstruction problem dealing with 2D imaging. Since image data often has small differences
between adjacent pixels until it reaches the edge of patterns, the variation of an image signal is often sparse. Total variation minimization method utilizes this property to reconstruct the image from its measurements. To continue our discussion we first introduce the discrete gradient operator. Let’s consider a $N \times N$ 2-dimensional matrix: $u$. The gradient $\nabla u$ is defined as [6]:

$$(\nabla u)_{i,j} = ((\nabla u)_{i,j}^1, (\nabla u)_{i,j}^2)$$  \hspace{1cm} (2.9)$$

with:

$$(\nabla u)_{i,j}^1 = \begin{cases} u_{i+1,j} - u_{i,j} & \text{if } i < N \\ 0 & \text{if } i = N \end{cases}$$

$$(\nabla u)_{i,j}^2 = \begin{cases} u_{i,j+1} - u_{i,j} & \text{if } j < N \\ 0 & \text{if } j = N \end{cases}$$

Then, the total variation of $u$ is defined by:

$$||u||_{TV} = \sum_{i\leq N, j\leq N} |(\nabla u)_{i,j}^1| + |(\nabla u)_{i,j}^2|$$  \hspace{1cm} (2.10)$$

Suppose our image is piecewise smooth image $u_0$ that has been transformed via a linear operator $A$ and to which a random noise $n$ has been added [7]:

$$u = Au_0 + n$$  \hspace{1cm} (2.11)$$

Since $u_0$ is smooth and piecewise constant, the total variation of $u$, $||u||_{TV}$, is very sparse. Figure 2.4 shows an example of an image and its total variation. It is clear that only the pixels at the edge of the object have large gradients while the gradients of other pixels are very close to zero.

Consider an image $u$ with $M$ pixels, we can vectorize it to a column vector $x$ with length $M$. The total variation can be expressed as [8]:

$$||u||_{TV} = ||Tx||_1$$  \hspace{1cm} (2.12)$$

where $T$ is a $2M \times M$ matrix with $T(i,j) \in \{-1, 0, 1\}$. To reconstruct the image from $N$ samples out of $M$, the inversion can be based on total variation.
Figure 2.4: Original image and its total variation: (a) original image. (b) the corresponding total variation of the original image.

minimization:

\[
\min ||u_0||_{TV} \quad \text{subject to} \quad |u - Au_0|_2 \leq \epsilon
\]  

(2.13)

where \( \epsilon \) is the error. In the next chapter, we will review some optics background.
Chapter 3

Holography

Before holography was invented, pictures were taken by recording light on a radiation-sensitive medium, such as photographic film. These generated two dimensional pictures contain amplitude and frequency information of the light, but they lack information such as direction and phase. This situation changed in late forties 20th century when holography was invented. There are two phases of holography: recording and reconstruction. In the recording phase, hologram recorded physical wavefront of light from a 3D scene. In the reconstruction phase, a reconstruction beam was illuminated on the hologram. Viewing a hologram is like viewing the original 3D scene at certain position. In the next part we will begin the discussion with some history of the holography.

3.1 Brief History of Holography

Holography was invented in 1947 by Dennis Gabor while he was trying to improve the resolution of electron microscopes[9]. Gabor invented the term hologram which means whole drawing literally wishing to contain all the light information from the object. After Gabor’s first paper published, holography immediately attracted many scientists’ attention all over the world. At that time the most coherent light source scientists can make use of is mercury lamp. Because of it’s low coherency of the light, holography was limited in the fifties 20th century.

In 1960, after the invention of laser which can provide highly pure, intense and coherent light, scientists finally found an ideal light source for making holograms. The holography techniques boomed in the next decades, several
types of holograms were made during that period of time. Among those inventions the most important one may be Dr. Stephen A. Benton’s white light transmission hologram also known as the rainbow hologram. This type of hologram can be viewed in the ordinary white light rather than the coherent laser light required previously. Benton’s hologram made the mass production of holograms possible.

The recent advances of hologram may be the utilization of the cheap semiconductor lasers. These cheap, compact lasers help holograms to be smaller and more accessible to many ordinary people. In the next section we will talk more about holography theory.

3.2 Optical Holography Principles

3.2.1 Diffraction

If classical geometry explain the phenomena which occur when light pass through obstacles, the opaque structure should cast a shadow with the same outline of its structure. However this is not true. There isn’t anything called the boundary of a shadow, because it is changing gradually: some of the light can cross into the shadow area and some of the shadow can propagate into the lit region. [10] This type of phenomenon is called diffraction. Diffraction occurs when all types of waves, such as electromagnetic waves, sound waves, ocean waves, encounter an obstacle.

Because a hologram is the microscopic grating of diffraction pattern formed by the incoming light bent at the object, diffraction is the basic concept of holography. Another key concept is Interference.

3.2.2 Interference

When we talk about holography, we are assuming the light source is single frequency coherent light. If the light source is plane wave, we can define it as following:

\[ E(r, t) = E_0 e^{i(k \cdot r - \omega t)} \] (3.1)

An alternative phasor representation for a wave at a single frequency \( \omega \) is:

\[ E(r, t) = \text{Re}\{ \overline{E(r)} e^{j \omega t} \} \] (3.2)
where phasor \( \mathbf{E} = E_0 e^{-jkr} = E_0 e^{j\phi - jkr} \), \( \phi \) is called the phase of the wave at certain point \( r \).

Consider two plane waves with the same frequency, they are called as coherent waves. the phasor for each one is:

\[
\mathbf{E}_1 = E_{10} e^{-jkr} = |E_{10}| e^{j\phi_1 - jkr} \tag{3.3}
\]

\[
\mathbf{E}_2 = E_{20} e^{-jkr} = |E_{20}| e^{j\phi_2 - jkr} \tag{3.4}
\]

\( |E_{10}| \) and \( |E_{20}| \) are the amplitudes for \( E_1 \) and \( E_2 \). The total wave will be the sum of \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \) at any position of the space [14]:

\[
\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = |E_{10}| e^{j\phi_1 - jkr} + |E_{20}| e^{j\phi_2 - jkr} \tag{3.5}
\]

The total amplitude will be:

\[
|I_0| = |\mathbf{E}_1 + \mathbf{E}_2|^2 = |E_{10}|^2 + |E_{20}|^2 + 2|E_{10}||E_{20}|\cos(\phi_2 - \phi_1) \tag{3.6}
\]

This phenomenon is known as interference. If the two waves are in phase which means \( \phi_1 = \phi_2 \), their troughs and peaks line up and the interferential amplitude will be \( |I_0| = |E_{20} + E_{10}|^2 \). If the two waves are out of phase, in other words the difference between the phase equal to \( \pi \) radius, then one wave’s crests will coincide with another wave’s troughs and so will tend to cancel out: \( |I_0| = |E_{20} - E_{10}|^2 \).

All the unknown in equation 3.7 can be acquired by detectors in serval different experiments except the phase angle \( \phi_1 \) and \( \phi_2 \). Instead of sensing the phase directly, we can easily calculate \( \cos(\phi_2 - \phi_1) \). If we define \( \mathbf{E}_2 \) as reference beam, where we have assumed the initial reference phase to be zero: \( \phi_2 = 0 \), one can find two possible solutions for \( \phi_1 \) with one possible solution in the first or the fourth quadrant and the other one in the second or the third quadrant.

One can do another experiment using a different reference beam with the same amplitude but certain phase shift \( \theta \): \( \phi'_2 = \phi_2 + \theta = \theta \). To do this and compare the result from the previous experiment, one can uniquely determine the phase information of the original field \( \phi_1 \) [15]. This is also the idea of phase shifting holography.
In the next chapter we will compare several different compressive sensing code available online and introduce a method to manipulate a complex signal to real.
Chapter 4

Compressive Sensing with Complex Data

4.1 Operator Notation

There are many available compressive sensing codes on line, some are suitable for our purpose while some are not. In order to implement those codes it is necessary to compare them and choose one. In our simulation we deal with coherent optics involving complex signals. We can not apply many codes directly since they were designed for real signals only. It is necessary to change our complex problem to real case. One approach to handle linear complex elements is to represent the input vector $x$, the linear system matrix $N$ and output vector $y$ as combination of several real vectors or matrices. If we define $y = \mathcal{R}(y) + i\mathcal{I}(y)$, in which $\mathcal{R}(y)$ is the real part of $y$, and $\mathcal{I}(y)$ is the imaginary part of $y$, also $x = \mathcal{R}(x) + i\mathcal{I}(x)$, $N = \mathcal{R}(N) + i\mathcal{I}(N)$. We can rewrite the equation

$$y = Nx \quad (4.1)$$

as

$$\mathcal{R}(y) + i\mathcal{I}(y) = [\mathcal{R}(N) + i\mathcal{I}(N)] \times [\mathcal{R}(x) + i\mathcal{I}(x)]$$
$$= \mathcal{R}(x)\mathcal{R}(N) - \mathcal{I}(x)\mathcal{I}(N) + i\mathcal{R}(x)\mathcal{I}(N) + i\mathcal{I}(x)\mathcal{R}(N) \quad (4.2)$$

we can separate the real part and the imaginary part of the equation as
\[ R(y) = R(x)R(N) - I(x)I(N) \]
\[ I(y) = R(x)I(N) + I(x)R(N) \]  

(4.3)

Equation 4.1 can be rewrite in matrix form as:

\[
\begin{bmatrix}
R(y) \\
I(y)
\end{bmatrix} = \begin{bmatrix}
R(N) & -I(N) \\
I(N) & R(N)
\end{bmatrix} \begin{bmatrix}
R(x) \\
I(x)
\end{bmatrix}
\]  

(4.4)

For the purpose of simplification of notations, we can define a operator \( \mathcal{C} \) for vector and matrix respectively:

\[
\mathcal{C}\{ [x] \} = \begin{bmatrix}
Rx \\
Ix
\end{bmatrix}, \quad \mathcal{C}\{ [N] \} = \begin{bmatrix}
R(N) & -I(N) \\
I(N) & R(N)
\end{bmatrix}
\]  

(4.5)

Also the opposite operator \( \mathcal{C}^{-1} \) is

\[
\mathcal{C}^{-1}\{ [R_x] \} = [x], \quad \mathcal{C}^{-1}\{ [R(N) - I(N)I(N)] \} = [N]
\]  

(4.6)

4.2 The Analysis of Solutions

Now let’s consider the following two minimization problems:

\[
\min_{x \in \mathbb{C}^n} |x|_{l^1}, \quad \text{subject to} \quad y = Nx
\]  

(4.7)

\[
\min_{x' \in \mathbb{R}^{2n}} |x'|_{l^1}, \quad \text{subject to} \quad y' = N'x'
\]  

(4.8)

in which \( y \) and \( x \) are \( n \) by 1 complex vectors, \( N \) is \( n \) by \( n \) complex matrix, \( y' = \mathcal{C}\{ [y] \}, \ N' = \mathcal{C}\{ [N] \}, \) and \( x' = \mathcal{C}\{ [x] \} \).

If \( x_1 \) is the solution of the minimization problem 4.7 for the complex case, is it true that: \( x_1' = \mathcal{C}\{ [x_1] \} \) will also be the solution of the alternated real minimization problem 4.8?

To answer this question, we should first check the definition of the norms:

\[
||x||_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}
\]  

(4.9)
Since \( x_i = R(x_i) + I(x_i) \),

\[
||x||_2 = \left( \sum_{i=1}^{n} \left( \sqrt{R(x_i)^2 + I(x_i)^2} \right)^2 \right)^{\frac{1}{2}} = \left( \sum_{i=1}^{n} R(x_i)^2 + I(x_i)^2 \right)^{\frac{1}{2}} = ||x'||_2 \quad (4.10)
\]

It is clear that \( x_1 \) and \( x'_1 \) have the same \( l_2 \) norms. However the \( l_1 \) norms of them are not equal.

\[
||x||_1 = \sum_{i=1}^{n} \sqrt{R(x_i)^2 + I(x_i)^2} \quad (4.11)
\]

\[
||x'||_1 = \sum_{i=1}^{n} R(x_i) + I(x_i) \quad (4.12)
\]

Since \( ||x||_1 \neq ||x'||_1 \), the solutions of optimum problems 4.7 and 4.8 are not equal to each other, but how different are these two solutions?

### 4.3 Examples

Let’s consider 200 different signal sets: \( x_1 \) to \( x_{200} \), each signal set is a 300 by 1 complex column vector. We can apply the method introduced in the previous section to change these complex signals to be real. The corresponding real signal sets are: \( x'_1 \) to \( x'_{200} \) which should be 600 by 1 real column vectors as:

\[
x'_k = \begin{bmatrix} \Re x_k \\ \Im x_k \end{bmatrix} = \mathcal{C}\{ x_k \} \quad k \in \{1, 2, 3, ..., 200\} \quad (4.13)
\]

For each complex signal set \( x_k \), we calculate its \( l_1 \) norm. To do this we can easily make a 200 by 1 vector as:

\[
c_n = [ ||x_1||_1 ||x_2||_1 ||x_3||_1 ||x_4||_1 ... ||x_{200}||_1 ] \quad (4.14)
\]

whose elements are the \( l_1 \) norms of different signal sets. We can have another vector \( r_n \) from the \( l_1 \) norms of corresponding real signal sets as:

\[
r_n = [ ||x'_1||_1 ||x'_2||_1 ||x'_3||_1 ||x'_4||_1 ... ||x'_{200}||_1 ] \quad (4.15)
\]

Then we plot \( c_n \) and \( r_n \) in figure 4.1 and compare them. It is obvious that these two vectors are very similar. To mathematically determine the similarity
we introduce correlation coefficient as [17]:

$$r_{xy} = \frac{n \sum xy - (\sum x)(\sum y)}{\sqrt{n(\sum x^2 - (\sum x)^2)} \sqrt{n(\sum y^2 - (\sum y)^2)}}$$

(4.16)

The correlation coefficient $r_{xy}$, also called the cross-correlation coefficient or linear correlation coefficient, measures the strength and the direction of a linear relationship between two signal sets: $x$ and $y$. The value of $r_{xy}$ is between $-1$ to $+1$. The + and - signs are used for positive linear correlations and negative linear correlations respectively. If $r_{xy}$ equals 1, it indicates a perfect positive fit, which means signals $x$ and $y$ are the same. If $r_{xy}$ equals 0, it indicates no linear correlation, in other words signals $x$ and $y$ have a random, non-linear relationship.

In this example the calculated correlation coefficient of $cn$ and $cr$ is equal to 0.9734, which indicates that $cn$ and $cr$ are almost identical. In other words, although the $l_1$ norms of complex signals are different with the $l_1$ norms of their converted real signals, the changing trend is the same. When the complex optimum problem has a solution via $l_1$ minimization method, It is highly possible that the corresponding converted real problem also has a solution, and moreover, these two solutions positive fit. In conclusion, the solutions of $l_1$ minimization problems for complex signals and it's corresponding converted real signals are very similar. We can utilize this property to solve a complex
minimization problem by converting it to a corresponding real problem.

To prove this conclusion, we refer to different minimization codes to solve the same reconstruction problem and compare the results. First we generate a set of random sparse signals $x$, keep signal length $n = 300$ but change sparse number $s$ from 1 to 50. Then we keep the number of the linear projection $y$ equal to 50. We use the code called "spgl1" [18] and the "cvx" code [19] to solve the l1 minimization problem both in complex and corresponding real cases. These codes are all based on l1 minimization methods but written by different authors. We also use total variation code called "TVlogbarrier" [20] to solve the same problem in real case only.

Figure 4.2 shows The correlation coefficient $r$ between original signal and recovered signal with the increase of sparse number. When $r$ equals or close to 1, it means the reconstruction is good. From figure 4.2 we can tell that when the signal is very sparse, the "spgl1" code works well for both complex and corresponding converted real case. The "cvx" code for real case also has good performance. When the signal is not highly sparse, none of the methods perform well. The total variation "logbarrier" code works fine in very sparse cases, but not as well as the other three codes mentioned above. The "cvx" code does not work in complex case at all.

Figure 4.3 shows the energy difference between original signals and recovered signals. The smaller of the energy difference is, the better the reconstruction is. The conclusion from figure 4.3 is the same with figure 4.2.

Next we keep the signal the same. Instead of changing sparse number of the original signals we change the number of the sampled linear projection $y$ from 60 to 180.

Figure 4.4 shows how does correlation coefficient change between original complex and recovered real signals with increasing of sample rates. Figure 4.5 shows the relation between energy difference and sample rate. From these two figures, we can tell when the signal is very sparse, the "cvx" code for real signals works the best. the "spgl1" code also have good performance for both complex and real signals. The total variation "logbarrier" code can only be applied in real case. The "logbarrier" code is not suitable for short random signals neither. It is because total variation "logbarrier" method is designed for two dimensional signals. The "cvx" code can not be applied for complex signals even in the very sparse and high sample rate case.

In the next chapter we will continue discussion about simulation method.
Figure 4.2: The correlation coefficient between original complex signal and recovered real signal with different sparse number.

Figure 4.3: The energy difference between original complex signal and recovered real signal with different sparse number.
Figure 4.4: The correlation coefficient between original complex signals and recovered real signals with different sample rates.

Figure 4.5: The energy difference between original complex signals and recovered real signals with different sample rates.
Chapter 5

Simulation Method

In our simulation we apply MATLAB, which has permitted the numerical simulation of two-dimensional fast Fourier transform (FFT) method. Although there are many other simulation methods, FFT method is very fast and powerful, therefore we will apply it in our simulation.

5.1 Diffraction Simulation

If light is illuminated on an arbitrary aperture, the field after the aperture is defined as $u_0(\xi, \eta)$. The Fresnel diffraction field at $(x, y, z)$ is

$$u(x, y; z) = e^{j k z} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u_0(\xi, \eta) e^{j k \frac{2}{z} [(x - \xi)^2 + (y - \eta)^2]} d\xi d\eta \quad (5.1)$$

If we define system impulse response $h$ as:

$$h(x, y; z) = e^{j k z} \exp\left[j \frac{k}{2z} (x^2 + y^2)\right] \quad (5.2)$$

we can rewrite equation 5.1 as [21]:

$$u(x, y; z) = u_0(x, y) \bigotimes h(x, y; z) \quad (5.3)$$

in which $\bigotimes$ means convolution.

By taking two-dimensional Fourier transform of $h(x, y; z)$ [22]

$$H(k_x, k_y; z) = \mathcal{F}_{xy}\{h(x, y, z)\} = \exp(j k z) \exp\left[j \frac{k}{2z} (k_x^2 + k_y^2)z\right] \quad (5.4)$$
If we define
\[
U(k_x, k_y; z) = \mathcal{F}_{xy}\{u(x, y; z)\}
\] (5.5)
and
\[
U_0(k_x, k_y) = \mathcal{F}_{xy}\{u_0(x, y)\}
\] (5.6)

From equation 5.3 we have
\[
U(k_x, k_y; z) = U_0(k_x, k_y)H(k_x, k_y; z)
\] (5.7)

This is diffraction of a point source which is represented by \(u_0(x, y) = \delta(x, y)\). From equation 5.3, the field at a distance \(z\) is given by
\[
u(x, y; z) = \delta(x, y) \otimes h(x, y; z) = \frac{e^{jkz}}{j\lambda z} \exp[j\frac{k}{2z}(x^2 + y^2)]
\] (5.8)

Let’s place a transparent object whose transmission function is \(t(x, y)\). If the incident wave is \(u_{i,0}\), the field immediately after the transparency is \(u_0(x, y) = u_{i,0}(x, y)t(x, y)\). The object can of course be illuminated by a non-planar wave, but for simplicity we assume the incident field is uniform brightness plane wave, and it is incident normally on the transparency. The field passed through the transparency is \(u_0(x, y) = 1 \times t(x, y) = t(x, y)\). From equation 5.3 we have:
\[
u(x, y; z) = t(x, y) \otimes h(x, y; z)
\] (5.9)

\[
U(k_x, k_y; z) = T(k_x, k_y)H(k_x, k_y; z)
\] (5.10)
in which \(T(k_x, k_y)\) is two-dimensional Fourier transform of \(t(x, y)\). We can finally derive the equation as:
\[
u(x, y; z) = \mathcal{F}_{xy}^{-1}\{T(k_x, k_y)H(k_x, k_y; z)\}
\] (5.11)
in which \(\mathcal{F}_{xy}^{-1}\) is inverse two-dimensional Fourier transform.

Here we define chirp function \(q\) as:
\[
q(x, y; \frac{k}{2z}) = \exp[j\frac{k}{2z}(x^2 + y^2)]
\] (5.12)
\[ h(x, y; z) \text{ equal to a constant multiply } q(x, y; \frac{k}{2z}) \]

\[ h(x, y; z) = \frac{e^{jkz}}{j\lambda z} q(x, y; \frac{k}{2z}) \] \hspace{1cm} (5.13)

we can rewrite equation 5.11 as:

\[ u(x, y; z) = \frac{e^{jkz}}{j\lambda} \mathcal{F}^{-1}_{xy} \{T(k_x, k_y)Q(k_x, k_y; \frac{k}{2z})\} \] \hspace{1cm} (5.14)

where \( Q(k_x, k_y; \frac{k}{2z}) \) is the Fourier transform of \( q(x, y; \frac{k}{2z}) \)

Another way to solve the fresnel diffraction problem is to take \( x, y \) outside the integral sign in equation 5.1, we have:

\[ u(x, y; z) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{kz}{2z}(x^2+y^2)} \int \int_{-\infty}^{+\infty} \{u_0(\xi, \eta)e^{j\frac{k}{2z}(\xi^2+\eta^2)}\} e^{j\frac{k}{2z}(x\xi + y\eta)} d\xi d\eta \] \hspace{1cm} (5.15)

Then, equation 5.15 can be rewritten as:

\[ u(x, y; z) = \frac{e^{jkz}}{j\lambda z} q(x, y; \frac{k}{2z}) \mathcal{F}_{xy}[u_0(\xi, \eta)q(\xi, \eta; \frac{k}{2z})] \] \hspace{1cm} (5.16)

If the transparency is \( t(\xi, \eta) \), the diffraction field at distance \( z \) is:

\[ u(x, y; z) = \frac{e^{jkz}}{j\lambda z} q(x, y; \frac{k}{2z}) \mathcal{F}_{xy}[t(\xi, \eta)q(\xi, \eta; \frac{k}{2z})] \] \hspace{1cm} (5.17)

We can use both equation 5.14 and equation 5.17 to do simulation. Theoretically they will provide the same result. However with the different coefficients, input field or it’s spatial frequency, the amount of computation will be different. One method may be easier to compute than the other. In our case we assume the input field is single frequency planer wave, we choose to simulate the diffraction field with equation 5.17.
Chapter 6

Recovery Phase Information with Reference Beam

Figure 6.1: The rectangular phase object mask from reference [3]. (a) a photo of the mask, (b) a schematic graph of the mask.

Figure 6.1 shows an object mask which is made of two rectangular holes in an opaque screen. Both of the holes are covered with transparent plastic plates with slightly different thickness. It is shown in the schematic graph by different color. The amplitudes of the light passed through these two holes are almost the same, but the phases are different. We can certainly apply compressive sensing idea to reconstruct the image with much fewer sensors such as one pixel camera. However as these ordinary algorithms, the reconstruction will exploit the amplitude information only. Some researchers utilize terahertz imaging systems to acquire the phase information [3]. Since pulsed terahertz imaging systems can provide spectroscopic phase information [23],
they take the knowledge of the phase into account and improve the quality of the reconstruction quality relative to phaseless imaging systems. Although their experiments are conducted on pulsed terahertz imaging system, the idea of utilizing phase information can be extended to other similar imaging modalities. In our simulation we are interested in visible light of wavelength 700nm. With such a light source the phase information can not be acquired directly, therefore we decide to apply holography idea to reconstruct the phase and the image.

6.1 Sampling Requirements

Let us concentrate on the computer implementation of algorithm discussed in chapter 5 and consider the wave front passed though the phase object to be properly sampled on a linear grid. The spatial frequency of the wave front must be less than half of the sampling frequency so that it can be correctly reconstructed from the samples later. We also need to consider the sampling of chirp function in equation 5.17. If we sample the wave front and the chirp function separately, the required minimum sample rates are not the same, but the overall sample rates should be set as twice as the largest one. When the object is small, the relevant spatial frequency will be very large. In this case, the Fourier transform of the object dominates. A similar result arises when the object is fixed but other coefficients are used. The spatial frequency of the chirp function may be even larger. The sample rates should be set accordingly.

6.1.1 Sampling of The Wave Front

If we apply the fast Fourier transform algorithm, the sample rates for space domain and frequency domain will be the same. The Whittaker-Shannon sampling theorem implies that if the highest spatial frequency of the input is $B$, the sampling rate should be equal to or greater than $2B$. However this sampling rate may not be high enough in our simulation, since it does not consider the oscillations of the phase term [24].

Figure 6.2 is a schematic graph illustrating diffraction from the transparent object. Assume $Q_1$ and $Q_2$ are the two neighbor sampling points and $h$ is the distance between them. If we apply the numerical integration method, $h$ must be sufficiently small to make the phase oscillation correctly sampled, otherwise
substantial error may occur. As we are dealing with diffraction, the phase oscillation period only depends on the light path difference between $P_0Q_1$ and $P_0Q_2$, where $P_0$ is a single point on image plane. When the point Q is moved on the object plane from $Q_1$ to $Q_2$ by an increment $h$, the path length changes by:

$$\delta = P_0Q_2 - P_0Q_1 = \sqrt{(x_1 - x_0 + h)^2 + z_0^2} - \sqrt{(x_1 - x_0)^2 + z_0^2}$$

$$= z_0 \sqrt{1 + \left(\frac{x_1 - x_0 + h}{z_0}\right)^2} - z_0 \sqrt{1 + \left(\frac{x_1 - x_0}{z_0}\right)^2}$$

$$= z_0 [1 + \left(\frac{x_1 - x_0 + h}{z_0}\right)^2] - z_0 [1 + \left(\frac{x_1 - x_0}{z_0}\right)^2] \approx h \frac{x_1 - x_0}{z_0} \quad (6.1)$$

In equation 6.1 we assume that $z_0 \gg x_1 - x_0$ thus $\frac{z_0}{x_1 - x_0} \approx 0$, and $h^2$ term is sufficiently small to be ignored. It follows from the Rayleigh quarter-wave criterion [25](also called Rayleigh tolerance) that the numerical integration will be valid when $h$ is chosen to make $\delta \leq \frac{\lambda}{4}$. We set the extent of the input as $D$. $n$ is the number of sampling points of the input, so the step size $h = \frac{D}{n}$.

$x_1$ and $x_0$ can be any point on object plane and image plane respectively. Equation 6.1 should be satisfied even when $x_1 - x_0$ has a maximum value,
therefore we set \( x_1 - x_0 = D \frac{m}{n} \). A reasonable setting for \( m \) is \( 2n \), that means the image window is twice of the dimension of the input window. we have:

\[
\delta = h \frac{x_1 - x_0}{z_0} = \frac{D^2 m}{n^2 z} \leq \frac{\lambda}{4} \tag{6.2}
\]

\[
n \geq \sqrt{\frac{8D^2}{\lambda z}} \tag{6.3}
\]

In our simulation \( D = 0.01m \), \( \lambda = 700nm \), the diffraction distance \( z = 0.3m \), so the number of the sampling points should be larger than 100.

### 6.1.2 Sampling The Chirp Function

Let us consider how many samples are needed for the chirp function defined in equation 5.12. One property of the chirp is that the spatial frequency of the chirp increases in spatial frequency domain with distance from the origin. According to the Shannon sampling theorem, the sampling frequency of the chirp function should be twice the largest spatial frequency at the farthest point from the origin in the spatial frequency domain. If the chirp spatial frequency is larger than the sampling frequency, the aliasing will occur and the result will be corrupted.

Figure 6.3 shows the chirp functions with different step sizes in space domain. All of these chirp functions are sampled with extent \( D=0.01m \), wavelength \( \lambda = 700nm \), diffraction distance \( d = 0.3m \). Figures (a) and (b) are the real and imaginary components with sampling number \( n = 800 \), step size \( h = 1.25 \times 10^{-5}m \). In this case, the spatial frequency satisfies the sampling theorem. Figures (c) and (d) are the real and imaginary components with sampling number \( n = 200 \), step size \( h = 5 \times 10^{-5}m \). In this case the chirp function is under sampled, therefore spatial aliasing occurs. It is clear that when under-sampled, the central portion will be reproduced as sets of aliases. Directly using this under-sampled chirp function in simulation will produce errors.

### 6.2 A Discrete Model

Let us consider the experiment setting in figure 6.4 with the interference beam at CCD.
Figure 6.3: Two dimensional chirp function with different step sizes. (a) real and (b) imaginary components of a two dimensional chirp function with step size $h = 1.25 \times 10^{-5} m$. The number of sample points is 800. When the chirp function is under sampled, the spatial aliasing will occur, as shown in (c) real and (d) imaginary components of the chirp function with step size $h = 5 \times 10^{-5} m$. The number of sample points is 200.

The object used in this and the following numerical experiments has a transparent region in the shape of letter T as shown in figure 6.5. The transparent region of the object consists of two parts with different thickness, which are represented by different colors in the schematic graph. For the purpose of simulation, we assume such difference equal to 1/4 of the source light wave length in the transparent material. Theoretically the light passed through these two parts should have a $\pi/2$ phase difference.

In this experiment, the transparent object is illuminated by coherent plane wave. The light diffracts through the optics system $H_1$ and will be detected by CCD camera behind a DMD.

The dimension of the object is 0.01$m$ including the opaque region. The
distance $z$ from the object to the CCD is 0.3$m$. The wave length $\lambda = 700nm$. As the discussion in the previous sections, the sample rate should be more than 800 in order to avoid spatial aliasing. Figure 6.6 shows different diffraction simulation results with sampling rates $n = 200, 400, 800$. It is clear that when $n = 200$ and $n = 400$, the input is under sampled. However, increasing the sample rate, the computation time will also increase. It takes about 10 minutes to do one diffraction simulation for $n = 800$. Moreover, There are
\[ n^2 \times n^2 = 4.096 \times 10^{11} \text{ elements in the matrix of impulse response } H_1 \text{ totally.} \]

Such a big matrix exceeds the ability of our computers, therefore it is not applicable to do a practical simulation in our numerical experiment.

For the purpose of simplification, the object is considered as a 32 by 32 point sources array. The distance between neighbor point sources is 0.00032m. Figure 6.7 shows the field immediately passed though the transparency. The diffracted field at the CCD is shown in figure 6.8. Since the phase information can not be directly detected by CCD, we have to apply holography to capture the phase.

![Figure 6.6: The simulation diffraction image at image plane for experiment setting shown in figure 6.4 with different sample rates: (a) diffraction image with sample rate equals 200, (b) diffraction image with sample rate equals 400 (c) diffraction image with sample rate equals 800.](image)

In the simulation we utilize DMD to create \( k \) different measurement states. Since we assume the DMD is placed closely in front of the CCD, from simulation’s point of view they are all on the image plane where the field is calculated as \( \varphi_c \). With the pattern of the DMD changing, the field captured by CCD, \( f_k \), also changes. It is an inner product of the DMD mask function \( M_k \) and the filed on the image plane: \( f_k = M_k \varphi_c \). The minimization problem is:

\[
\min_{\varphi_c \in \mathbb{C}^n} ||\varphi_c||_1, \quad \text{subject to } f = M\varphi_c
\]  

(6.4)

Since \( \varphi_c \) is the product of the impulse response of the optical system and the field passed though the transparent object: \( \varphi_c = H\varphi_0 \). The minimization problem 6.4 can be rewritten as:

\[
\min_{\varphi_0 \in \mathbb{C}^n} ||\varphi_0||_1, \quad \text{subject to } f = MH\varphi_0
\]  

(6.5)
Figure 6.7: The discrete model of phase object mask: (a) the discrete point sources array, (b) the amplitude of the field passed through the transparent object, (c) the field with phase angle $\phi_0 = 0$, (d) the field with phase angle $\phi_0 = \frac{\pi}{2}$.

Figure 6.8: The diffraction field at CCD: (a) the real component of the field, (b) the imaginary component of the field.

In equation 6.5, the product of the DMD mask matrix $M$ and the system impulse response $H$ can be rewritten as a new basis: $\Psi$. The minimization
Figure 6.9: The calculated phase angle at CCD.

problem is changed as:

\[
\min_{\varphi_0 \in \mathbb{C}^n} ||\varphi_0||_1, \quad \text{subject to} \quad f = \Psi \varphi_0
\] (6.6)

By solving the minimization problem, we can directly recover the original field passed though the transparent object: \(\varphi_0\), which is shown in figure 6.10 with measurement states \(k = 150\). Figure 6.11 shows the recovered field with measurement states \(k = 200\)

Figure 6.10: The recovered field with \(k=150\): (a) the real components of the field, (b) the imaginary components of the field.

Next let us consider the experiment setting in figure 6.12. In this experiment the DMD becomes a source of diffraction while in the previous simulation DMD is used for creating different sampling states only.

As shown in figure 6.13, \(P_2\) is one point source on the object plane, \(P_0\) is a detector on the CCD. \(r21\) and \(r01\) are the vectors from \(P_2\) to \(P_1\) and form \(P_0\) to \(P_1\) respectively.
Figure 6.11: The recovered field with $k=200$: (a) the real component of the field, (b) the imaginary component of the field.

Figure 6.12: Second experiment setting.

If the DMD is at $k_0$ state, The field at point $(x, y)$ on the CCD plane can be calculated using equation 6.7:

$$f(x, y)_{k_0} = \int_{-\infty}^{+\infty} e^{jk_{101}} \varphi_d(x', y') dx' dy'$$

$$= \int_{-\infty}^{+\infty} e^{jk_{101}} \frac{1}{r_{01}} \int_{-\infty}^{+\infty} \varphi_0(x'', y'') \frac{1}{\lambda r_{21}} \left[ \cos(\vec{n}, \vec{r}_{01}) - \cos(\vec{n}, \vec{r}_{21}) \right] dx'' dy''$$

$$= \int_{-\infty}^{+\infty} \varphi_0(x'', y'') h(x'' - x, y'' - y) dx'' dy''$$

(6.7)
Figure 6.13: Point source illumination.

where \( \varphi_0 \) is the original field passed though the transparent object, \( \varphi_d \) is the field on the plane of DMD, \( h \) is the impulse response of the optical system:

\[
h(x'' - x, y'' - y) = \frac{1}{j\lambda} e^{jkr_21} \left\{ \cos(\vec{n}, \vec{r}_{01}) - \cos(\vec{n}, \vec{r}_{21}) \right\}.
\]

Let us assume the detector is located at \((0, 0)\) on the plane of CCD, therefore equation 6.7 can be rewritten in the discrete form as [27]:

\[
f(0, 0)_{k_0} = c \sum_{m_1} \sum_{m_2} M_{k_0} (m_1' \Delta_1', m_2' \Delta_2') \sum_{m_1''} \sum_{m_2''} \varphi_0 (m_1'' \Delta_1'', m_2'' \Delta_2'') H(m_1'' \Delta_1'', m_2'' \Delta_2'') \tag{6.8}
\]

where \( M_{k_0} \) is the mask function of the DMD at \( k_0 \) state, \( \Delta_1' \) and \( \Delta_2' \) are the step lengths of the sampling grid for DMD plane on \( x \) axis and \( y \) axis respectively, while \( \Delta_1'' \) and \( \Delta_2'' \) define the size of sampling grid for object plane. The step lengths are not necessarily the same, but in our simulation we make \( \Delta_1' = \Delta_2' = \Delta_1'' = \Delta_2'' \).

With two different reference beam illuminating at the CCD plane, the complex data \( f(0, 0)_{k_0} \) can be acquired. If we change the pattern of the DMD, the field captured by CCD will also change. We can acquire a set of data with
\(k\) different states of the DMD. We can rewrite equation as:

\[
f = MH\varphi_0
\]

(6.9)

where \(f\) is a \(k\) by one column vector.

Then if we solve the minimum total variation problem we can estimate \(\varphi_0\) as:

\[
\varphi_0 = min||f||_{TV} \text{ such that } f = \Phi\varphi_0
\]

(6.10)

where \(\Phi = MH\)

We use the same transparent object in the previous experiment shown in figure 6.6. The distance between the object plane and the DMD plane is \(z_1\), and the distance between the DMD plane and the CCD plane is \(z_2\). They are all equal to 0.1m. The object is 32 \(\times\) 32 point sources array, the distance between neighbor points is 0.00032m. The dimension of the DMD is 0.02m. We sampled the DMD with the sample rate 64 \(\times\) 64, therefore \(H\) is the matrix with dimension 4096 \(\times\) 1024. We take 200 different measurements to make \(M\) with dimension 200 \(\times\) 4096. The recovered result is shown in figure 6.14

![Figure 6.14: The recovered field: (a) the real components of the field, (b) the imaginary components of the field.](image)

To evaluate the reconstruction quality with different number of the \(f\) sampled. We keep all the optical settings and the object the same, but change the number of the sampled measurement \(f\) from 80 to 400. At each case we calculate the correlation coefficient between \(\varphi_0\) and the recovered field as shown in figure 6.15. We can tell when the number of the sampled measurement \(f\) is
more than 200, the correlation coefficient is larger than 0.9, which means the quality of the recovered field is acceptable. When the number of the sampled measurement $f$ is more than 300, the field is almost perfectly recovered.

Figure 6.15: The correlation coefficient with the increasing of the sampled measurement number.
Chapter 7

Application of Compressive Sensing to Antenna Design

7.1 Antenna Design with Dolph-Chebyshev method

Another area where CS can be applied is antenna synthesis. There are many analytical methods for solving antenna problems to achieve better directivity and small beam width with fewer antenna elements. The conventional methods include Dolph-Chebyshev method, Taylor method and so on. Most of these methods require the antenna array to be equally spaced. However, such a constraint is not obligatory: an antenna array with non-uniform distribution may provide same array factor but with fewer antennas. Notice that the positions of the antenna elements in the array are always sparse, we can apply compressive sensing to optimize antenna designation [28].

Let us first consider a typical antenna synthesis problem: design a broadside Dolph-Chebyshev array of 20 elements with distance between the neighbor elements equal to half of the wavelength, and with major to minor lobe ratio of 30 dB. This problem is a classical antenna synthesis problem and can be solved using Dolph-Chebyshev method easily.

The result is shown in figure 7.1. It is clear that the antenna factor pattern does meet the requirement of the designation.
7.2 Antenna Design with l1 minimization method

If we consider the positions and the coefficients of the antenna as the signals to be recovered, these signals are highly sparse and can be recovered by compressive sensing.

The antenna synthesis problem can be described as:

$$AF(\theta_m) = \sum_{i} a_i \cos(kd_i \cos(\theta_m))$$  \hspace{1cm} (7.1)

where $a_i$ is the excitation coefficient of the $i$th antenna element located at $d_i$. $k$ is the wave number. $\theta_m$ is the angle between the z axis and the position of the sensor. $AF(\theta_m)$ is one sample of the radiation pattern taken at space angle $\theta_m$. The linear mapping from the positions and exciting coefficients of the antennas to the array factors can be described as:

$$AF^T = A \times a^T$$  \hspace{1cm} (7.2)
where

\[ AF = [AF(\theta_1) \ AF(\theta_2) \ AF(\theta_3) \ ... \ AF(\theta_m)] \]

\[ a = [a_1 \ a_2 \ a_3 \ ... \ a_m] \]

\[ A_{i,j} = \cos(kd_i\cos(\theta_j)) \]

\( m \) is the number of the samples taken from radiation pattern, while \( n \) is the number of the possible positions of the antenna element. To make this problem solvable we set \( n \) larger than \( m \). This problem becomes an under-determined problem with infinite number of solutions. A classical method to solve such a problem is least square method. In LS method a solution with minimum energy will be found.

![Figure 7.2: The solution of the possible antenna positions using least square method.](image)

Figure 7.2 shows the solution of this least square problem. As we can see the solution is continuous, which means an infinite number of antennas are needed. Clearly this solution has no practical value. To find the solution with minimum number of antennas, the antenna synthesis problem can be further
considered as a L1 minimization problem:

\[
\min \|a\|_1 \quad \text{such that} \quad AF^T = Aa^T
\]  \hfill (7.3)

Figure 7.3: The solution of the possible antenna position using l1 minimum method.

The solution of this l1 minimization problem consists of a few impulses with small width. It is not strictly sparse. However, if we integrate the impulses, we can consider the integrations as the antenna coefficients, and the antennas locate at the center of the impulses impulses. The result is shown in figure 7.3.

Figure 7.4 shows the reconstruction array factor pattern using L1 minimization method in contrast with the Dolph-Chebyshev array factor pattern. From the reconstruction result, it is clear that the L1 minimization method works very well. There is only small distortion from reconstruction pattern to the pattern from Dolph-Chebyshev antenna array. By compressive sensing method the total antenna elements needed is only 15 which is enough to achieve the desired radiation pattern. The compressive ration is 75% of the uniformly spaced antenna array.
Figure 7.4: The solution of the antenna factor using l1 minimum method.
Chapter 8

Conclusion

In this thesis we introduce the basic concepts of compressive sensing and its application in coherent imaging and antenna synthesis. In coherent imaging we simulate an object with unit amplitude but large variation in phase. We point out that we can acquire not only the amplitude but also the phase information via compressive sensing. This demonstrates a new research direction of compressive holography. We also propose an effective sparse reconstruction method for antenna synthesis. As seen from the simulation results, compressive sensing algorithm can significantly reduce the number of antennas in antenna array while keeping the array factor almost the same. These two applications have shown that compressive sensing algorithm can be widely used in many areas.
Bibliography


