ABSTRACT

Effects of Singapore's Model Method on Elementary Student Problem-Solving Performance: Single Case Research

This research investigation examined the effects of Singapore’s Model Method, also known as “model drawing” or “bar modeling” on the word problem-solving performance of third and fourth grade students. Employing a single-case design, a researcher-designed teaching intervention was delivered to a child in third grade over the course of 8 teaching sessions. Using researcher-designed assessment probes, repeated measures of the dependent variable (percentage of problems solved correctly) were taken throughout the experiment through three different phases: baseline, intervention, and maintenance. The design was repeated across four different third and fourth grade participants. The results demonstrated the existence of a positive functional relationship between the independent variable (the model drawing intervention) and the participant’s problem-solving performance. The percentage of problems solved correctly rose significantly as soon as the intervention phase began and the child employed Singapore’s Model Method in solving complex word problems. The pattern was repeated across two different problem types, multiplicative comparison word problems and fraction word problems. The validity of the findings was strengthened considerably when the results showed a very similar functional relationship across four different subjects in grades 3 and 4. The existence of a functional relationship between model drawing and word problem performance has implications for research, policy and teaching practice in elementary mathematics education. Limitations of this study as well as implications will be discussed.
DEDICATION

For Finn and Owen for their patience and understanding at such young ages.

For Arthur, for his patience, humor and wisdom. You helped me keep my eyes on the prize.

For Polly, for the sunshine I needed in the darkest days.
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Chapter 1: Introduction

Statement of Problem

Among the topics commonly taught to math students, word problems pose significant challenges to teaching and learning for elementary students and their teachers. Skillful problem solving is widely considered an essential asset not only for elementary mathematics students but also for any person who wishes to succeed in the global marketplace (Reed, 1999; Stigler & Hiebert, 1999; U.S. Department of Education, 2008). In addition, children's ability to analyze and interpret word problems directly impacts the results of mathematics achievement tests (Bhattacharjee, 2004; National Research Council, 2001; National Council of Teachers of Mathematics, 2000; National Governors Association, 2011), with implications for national and international comparisons.

The National Council of Teachers of Mathematics (NCTM) is considered to be one of America’s most important mathematics education institutions. As recently as 1994, the editor of the NCTM’s *Journal of Mathematics Research in Education* considered problem solving "the most researched and least understood topic in the U.S. mathematics curriculum." (Lester, 1994, p.17). Theorists such as Richard Mayer have gradually mapped the cognitive complexities of problem solving and researchers such as Asha Jitendra have explored effective strategies for problem-solving instruction.

Recently, the nation of Singapore developed an innovative teaching strategy for problem solving through its acclaimed elementary math curriculum. Known as the *Model Method* in Singapore, or as *model drawing* in the United States, this teaching strategy asks
students to draw schematic diagrams as part of a solution process for word problems (see Appendix A for examples of this strategy).

The purpose of this study is to investigate the effects of model drawing on the word problem-solving performance of elementary level children. Because this study seeks to gain understanding of a potentially powerful new methodology for improving word problem performance in students, the results of this study may have implications for educational policy, academic research and theory, as well as mathematics teaching practice. Singapore's Model Method has the potential to improve the teaching and learning of problem solving in the United States.

**Significance for Policy**

Around the country, educators, policy makers and laypeople have called for improved achievement in our mathematics students (National Governors Association, 2011; Stigler & Hiebert, 1999; U.S. Department of Education, 2008). International comparisons of mathematics achievement consistently put U.S. students at the middle of the pack or worse (Jones, 1988; Kilpatrick, Swafford, Findell, & National Academy of Sciences, 2001; Stigler & Hiebert, 1999). In fact, the United States' lackluster performance in comparisons of mathematics achievement has been well documented over the past two decades (Koretz, 2009; Stigler & Hiebert, 1999; Tatsuoka, 2004), and has driven fears about economic competition since the Sputnik era. As the international economy becomes increasingly interconnected, most policymakers in the United States agree that improving STEM (science, technology, engineering and mathematics) education, specifically boosting facility and performance in
mathematics, will allow American businesses and workers to continue to compete more effectively in the global marketplace (National Academy of Sciences, 2005).

In response to the chorus of calls for action in mathematics education, the National Governors Association published its *Common Core Standards for Mathematics* in 2011 (National Governors Association, 2011). As of this writing, 42 states have adopted these common standards for math education, taking the United States one step closer to a national curriculum in mathematics. This important document will have implications in U.S. mathematics education for decades to come. Textbooks, testing, standards and teaching will all be driven by these learning standards for mathematics. More importantly for this study, the *Common Core Standards for Mathematics* cites problem solving as a core skill in mathematics, especially at the elementary level (Porter, McMacken, Hwang, & Yang, 2011). If the *Common Core Standards for Mathematics* place problem solving at the center of its curriculum, then problem solving will only increase in importance in mathematics education of the future. The *Common Core Standards* also hold up Singapore's problem solving-centered elementary program as a model for excellence in math curricula (National Governors Association, 2011).

Indeed, Singapore's math curriculum has received much international acclaim over the past 20 years, as educators and researchers worldwide are attracted by that nation’s consistent placement at the top of most international comparisons of mathematics achievement, even at the elementary levels (Bhattacharjee, 2004). With problem solving at its core, Singapore's curriculum employs the Model Method, a diagramming technique developed in Singapore for computational word problems. The Model Method joins two problem-solving strategies asking students to both categorize word problems by type as well as to draw schematic diagrams of the
mathematical relationships depicted in the problem (Foong, 2009) (see Appendix A for examples of problem solving through model drawing). Singapore's curriculum and instructional strategies have captured the attention of researchers in the United States, and over 2,000 U.S. schools have adopted Singapore's math curriculum as their own (Ginsburg, Leinwand, Anstrom, Pollock, & American Institutes of Research, 2005).

In 2008, President Bush's National Math Advisory Panel (NMAP) presented the results of its investigation into research-based practices for improving mathematics education in the United States (U.S. Department of Education, 2008). This report echoed the findings of the previous AIR report as it suggested looking to Singapore as a model for math instruction programs (U.S. Department of Education, 2008). More recently, the Common Core Standards for School Mathematics offers Singapore's curriculum as worthy of attention and modeling, also citing the report findings of the American Institute of Research (National Governors Association, 2011). In addition, The Common Core Standards join the recent calls for research-based teaching strategies that will improve performance in problem solving (Porter et al., 2011).

It is clear that mathematical problem solving is a major issue in mathematics education in the United States. The ability of mathematics students to solve problems effectively is certainly in question, a situation that has wider relevance in concerns over participation of Americans in the STEM fields. Calls for improving problem-solving skills of school-aged Americans are also based, in part, upon fear and the US’s place in global economic competition. In investigating the potential of Singapore's Model Method, this study seeks to
add to the existing body of knowledge concerning ways to address these national concerns over performance in mathematical problem solving.

**Significance for Research**

This study is relevant not only to issues of policy but to issues of research. The study will add to what is known about teaching techniques for word-problem solving. Investigation by researchers into instructional techniques for problem solving has ranged quite widely over the past 40 years (Lester, 1994). The literature review of this study will show there exists a substantial body of evidence demonstrating that students improve their problem-solving performance when they use visual representations depicting the problem at hand (Kintsch & Greeno, 1985). Along with visual representations, schema-based instruction, or SBI, will be shown to be an effective teaching technique that asks students to categorize problems by type before they begin to actually solve the problem (A.K. Jitendra, 2010).

Several minor gaps, however, occur through this literature on problem solving and SBI. Subjects for studies are most often learning disabled students of either the early elementary grades (second and third grades) or the middle school level. Perhaps due to the focus on the learning disabled (LD) child, comparatively little research exists on the more complex operations and concepts of elementary mathematics, such as multiplicative relationships or fractions. In order to address these research needs, this study will focus on average-ability elementary students in the upper elementary grades (four and five) as they attempt to solve more complex word problems involving multiplicative comparisons and fractions.

The most prominent gap in research-based knowledge on model drawing is its effectiveness as an instructional strategy. As mentioned earlier, Singapore's Model Method is
founded on the use of schematic visual representations, representing a combined utilization of visual representations and SBI. A small handful of studies have been conducted in Singapore on the Model Method, and no empirical studies have been conducted anywhere else in the world. This technique must be studied further to determine its impact on problem-solving performance in elementary aged children.

A further gap exists within the research base connected to schematic diagrams and problem solving. Investigators have used schematic diagrams with subjects according to one or two problem types or arithmetic operations. Single researchers have not developed a unified system for diagramming word problems, but years of development in Singapore has produced a systematic, unified teaching technique for connecting representations for all four arithmetic operations as well as fractions and other rational numbers.

This study has clear implications for research. Chapter Two will define what is known about visual representations in problem solving, outlining the most effective techniques within this literature stream. In addition, the research gaps on Singapore's Model method will be addressed, potentially increasing the knowledge base on the effectiveness of model drawing as an instructional strategy.

**Significance for Practice**

An investigation of representational teaching strategies, and of the Model Method in particular, has significance for teaching practice. Related to the previously mentioned policy concerns around mathematics problem solving, teaching methods for problem solving in U.S. elementary schools lack cohesion, depth and focus, despite the important position given to problem solving by state and national mathematics standards (National Council of Teachers of
Most teachers know that problem solving is a vitally important skill in mathematics, but many lack well-defined techniques for teaching students to analyze and interpret word problems. If model drawing can be proven effective through this study and others like it, American elementary math teachers could have a powerful new strategy in their teaching tool kits, unique in its schematic approach, its systematic coverage of word problem situations and its purposeful attention to algebraic reasoning.

Model drawing may also help teachers by improving their own mathematics content knowledge. Visual representations may help develop the problem-solving skills and algebraic reasoning abilities of teachers themselves, which could further improve overall mathematics teaching across the nation.

Improved teaching practices could lead to improved results in achievement, national and international standardized testing, and cultural perceptions of mathematics and the STEM fields. It is not too farfetched to say that a young population whose confidence in mathematics has increased could be a population more likely to enter the STEM fields, which in turn potentially alleviates concerns over global economic competition in the longer term.

**Statement of Purpose and Research Question**

The purpose of this research study is to determine what, if any, effect model drawing, a schematic representational system, has on the problem-solving performance of elementary school students. The results of the study will potentially contribute to policy concerns around mathematics achievement, understanding of Singapore's Model Method, and techniques for elementary mathematics teaching. This purpose then dictates the following research question:
What is the functional relationship between the application of the Model Method schematic diagramming teaching technique and the performance of non-classified elementary students on multiplicative comparison and fraction word problems?

**Definition of Key Terms**

*Functional Relationship* - a relationship of causation, demonstrated when the dependent variable, in this case percentage of problems solved correctly, changes as a result of the implementation of the independent variable, in this case Singapore's Model Method.

*Model Method* - The teaching strategy developed in Singapore for diagramming arithmetic word problems. The Model Method is also known as model drawing or bar modeling in the United States. The method consists of first classifying a word problem according to type, then teaching students to draw various configurations of rectangles, or bar models, that form schematic representations of quantities and relationships among quantities in the word problem (see Appendix A for a closer look at problem solving with the Model Method).

*Non-Classified elementary students* - Students enrolled in an elementary school who have not been diagnosed with or otherwise classified as learning disabled or gifted/talented in any academic discipline.

*Multiplicative comparison word problems* - These word problems involve the comparison of two or more quantities through multiplication. For example, John has 23 cookies. Mary has 4 times as many cookies. How many cookies does Mary have? These problems can involve seeking a variety of unknown variables within the comparison context (see Appendix A for further explanation and examples). This type of problem was chosen
because multiplicative comparison word problems are quite challenging for elementary aged children, and testing all problem types would have made the scope of this study far too wide.

**Fraction word problems** - These word problems ask the solver to find a fraction or fraction of a quantity. For example, John has 48 cookies. Three-fourths of the cookies are chocolate. How many cookies are not chocolate? As with multiplicative comparison problems, these problems can ask the solver to find a variety of variables within the problem context (see Appendix A for further explanation and examples). Again, this type of problem was chosen because multiplicative comparison word problems are quite challenging for elementary aged children, and testing all problem types would have made the scope of this study far too wide.

**Key Constructs**

The research question for this study comprises two primary constructs, schematic visual representations, and problem-solving performance. The first construct is schematic visual representations,-- diagrams drawn by students to help them understand a given word problem. Accurate visual representations make plain the quantities in the story context and the relationships that exist amongst them (Bishop, 1989). Schematic diagrams also include numerical or textual information taken directly from the text of the word problem in the form of labels (Jitendra 2010). This construct is supported by theory on schema and on visual representations in problem solving, and will be presented in detail in the theoretical framework section of this chapter. Schematic visual representations will be operationalized in this study through the teaching and student use of model drawing in solving word problems.

The second construct in the research question is problem-solving performance. This construct is defined as solving a word problem accurately (Jitendra,. Solving word problems
correctly is a complex cognitive process. The process is well represented through researcher and theorist Richard Mayer's two-phase model of problem solving, which will be detailed in the subsequent theoretical framework section of this study. This construct will be operationalized in the study through collecting data on the percentage of problems solved correctly by students. If students learn to use model drawing as a problem-solving tool, it is possible that problem-solving performance will increase over the course of the study. These two constructs are supported by research and theory, and the interaction between these constructs forms the foundational action investigated in this study.

**Theoretical Framework**
Figure 1.1. Researcher designed theoretical model depicting the interaction of elements making up Singapore’s Model Method. Schema Theory and Problem Solving Theory make up Mayer’s two-phase model of problem solving. Mayer’s model is operationalized in Singapore’s Model Method.

A theoretical framework, designed by the researcher and illustrated in Figure 1.1, supports this research study. As can be seen in Figure 1.1, schema theory, a central model of cognitive psychology and information processing, helps describe the internal cognitive processing engaged when we solve problems. Problem-solving theory is intertwined with
schema theory, and describes domain-specific models of mathematical problem solving. These two fields of knowledge are utilized in Richard Mayer's two-phase model of problem solving, which provides a picture of the problem-solving process. Model drawing is an enactment of Mayer's problem-solving model, serving as a systematic way to generate and utilize schematic diagrams, and adds other elements from theory on learning and instruction. Each of these components will be explored in depth through the remainder of this chapter.

**Schema Theory**

Schema are interconnected knowledge frameworks used by the mind to process information. One of the earliest schema theorists was Frederick Bartlett, writing in 1932. Bartlett defined schema, and then posited that schema are accessed and used by the mind to categorize our knowledge (Bartlett, 1932). For example, knowledge about bicycle riding can serve as an explanatory example of the role of schema in information processing. Initial observations of bicycle riding form early, more primitive schema. The framework for bicycle riding is accessed as we learn to actually ride a tricycle. Two-wheeled bicycle experiences further develop and expand the internal schema for bicycle riding. Bike-riding schema are accessed every time we ride a bike, and are further developed as we attach schema for using hand brakes and gears in expansion of the overarching schema for riding a bicycle.

Bartlett addressed representation in assisting schema development. He proposed that schema build with experience, and that success in solving problems was based in part upon access to previous schema on similar problems (Bartlett, 1932).

Later authors expanded schema theory through research (Bransford and Johnson, 1973 as cited in Banerjee, 2010). This team found that relating new information to relevant aspects of
prior knowledge is a crucial part of the comprehension process. The authors related schema to problem solving in the form of diagrams. Bransford and Johnson (1973) (as cited in Banerjee, 2010) suggested that the comprehension of text can improve if a diagram of the context is given or drawn. This finding connects to the tasks inherent in word-problem solving (Banerjee, 2010). Silver refined a definition of schema as a "prototypical abstraction of a complex and frequently encountered concept or phenomenon, and it is usually derived from past experience with numerous exemplars of this concept." (Silver, 1982, p. 15)

Kintch and Greeno applied schema theory to mathematical problem solving more directly in 1985. In their investigations of the processes of problem solving, this team posited that two kinds of representations are formed during mathematical problem solving, one for the numerical quantities and relationships in the problem, and another for the context of the story itself (Kintsch & Greeno, 1985). This work was later verified by researchers who characterized this dual representation as the "problem model" and the "situation model" (Coquin-Viennot & Moreau, 2007). The complexity of word-problem solving is captured in this schematic model.

Kintsch and Greeno went on to categorize the finite number of word problem situations that elementary students typically face. Two key classifications were part to whole situations and comparison situations (Kintsch & Greeno, 1985). Such classification of word problem schema has been helpful to many researchers in the field (Christou & Philippou, 1999; L. S. Fuchs et al., 2010; Fuson & Willis, 1989; A. K. Jitendra, George, Sood, & Price, 2010). In model drawing, students are asked to categorize a word problem as a part-whole situation or a comparison situation. Different models are drawn for each type of word problem (Yeap, 2010).
Schoenfeld (1979) proposed that direct instruction of schematic thinking, or metacognition, could have beneficial effects on student performance. For Schoenfeld, prior knowledge ingredients for successful problem solving included knowledge of mathematical procedures, experience with drawing diagrams, knowledge of operation choices, and knowledge of problem categories and problem heuristics (Schoenfeld, 1979).

Schema theory has developed over the decades since Bartlett's initial ideas. At this point, a considerable literature has accumulated on helping students solve word problems through accessing and developing their internal schema for mathematical word problems (Powell, 2011). Schema theory is enacted in the technique of using schematic diagrams to access schema knowledge. Schematic diagrams are central to Singapore's Model Method, and thus schema theory informs the fundamental constructs of this study. This literature will be further explored in the literature review for this study.

**Problem-Solving Theory**

Since Descartes, mathematicians and educators have considered ways to effectively understand and teach the problem-solving process (Banerjee, 2010). For example, George Polya, considered the founder of modern theory in mathematical problem solving, developed a detailed treatise on general heuristics for solving mathematical problems in his 1945 book titled *How To Solve It*. (Polya, 1945). Widely cited as a seminal work in problem solving, Polya's work breaks down mathematical problem solving into four simple steps: understand the problem, devise a plan, carry out the plan, and look back (see Table 1.1). Many current elementary textbooks use a version of Polya's plan in their problem-solving instruction (Cai, 2004).
Table 1.1

*Polya's Four Steps for Problem Solving*

<table>
<thead>
<tr>
<th>Step</th>
<th>Primary Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Understand the problem</td>
<td>Reading, interpreting, comprehension</td>
</tr>
<tr>
<td>Devise a plan</td>
<td>Making a representation, considering past problems, choosing an operation</td>
</tr>
<tr>
<td>Carry out the plan</td>
<td>Operations on numbers</td>
</tr>
<tr>
<td>Look back</td>
<td>Checking on computation and reasonableness of solution</td>
</tr>
</tbody>
</table>

Note: (Polya, 1945)

Polya's first step, understanding the problem, asks the solver to identify the unknown in the problem, differentiating the unknown from the known data in the story context (Polya, 1945). Polya suggests separating the condition of the problem into smaller parts if necessary. Polya also includes advice to draw a diagram of the problem in order to facilitate understanding (Polya, 1945), early advice germane to this study.

In Polya's second step, making a plan, the solver is advised to consider previous knowledge before actually solving the problem (Polya, 1945). Polya encourages the solver to relate the problem at hand to similar previous problems. The same search for previous knowledge is undertaken for the unknown in the problem. This step has clear connections to schema theory, as the solver accesses schema on problem solving and problem types as a method for choosing the correct mathematical operations (Schoenfeld, 1992).

Polya's third step, carrying out the plan, relates directly to mathematical computation and heuristics for computation. Polya encourages checking computation carefully, relating it to the understanding of the problem generated in the first and second steps (Polya, 1945). Polya spends relatively little effort in developing this step, as he considered that most difficulties in
problem solving emanated from misunderstandings arising from understanding the problem or making a plan. (Schoenfeld, 1992).

Once the solution is reached, Polya advises the solver to look back. This step encompasses checking calculations but also includes deriving a solution through an alternative method, and checking the context of the problem to make sure that the unknown has been made known (Polya, 1945).

Polya's four steps provide an early model for the problem-solving process. Many later researchers and theorists in problem solving began their investigations with references to this foundational work (Leong, Tay, Toh, Quek, & Dindyal, 2011; Mayer, 1983; NCTM, 2000; Porter et al., 2011; Schoenfeld, 1992; Stigler & Hiebert, 1999; U.S. Department of Education, 2008). In the next section, Richard Mayer's two-stage model of problem solving is presented as a modern expansion of Polya's initial theoretical framework for mathematical problem solving.

**Mayer's Two-Stage Model of Problem Solving**

Beginning in the 1980s, Richard Mayer has made significant contributions to theory on word problem solving. It is the Mayer’s work that provides the backbone of the theoretical foundations of this study. Further expanding theory on schema, Mayer showed that students do compare problems at hand to schema for previously solved problems (Mayer, 1985). Furthermore, when students lack a schema for a problem they are facing, the student's representation of the problem is far more likely to be incorrect (Mayer, 1983). Incorrect representation of a problem is likely to produce an incorrect solution. In contrast, Mayer points out the fact that typical problem-solving instruction tends to focus on facts and algorithms rather than on correct representation (Mayer, 1989). Mayer's emphasis on the importance of the
representation phase is echoed in this study, as model drawing is primarily an action taken in the representation phase of problem solving.

Mayer outlines several different types of knowledge required in the act of problem solving (Mayer, 1982). Linguistic and factual knowledge concerns how to encode sentences, such as grammar, along with background knowledge connected to the problem context. Schematic knowledge is access to problem types, such as the aforementioned combine, change and comparison problem types in elementary mathematics (Kintsch & Greeno, 1985). Algorithmic knowledge concerns how to perform common repetitive procedures in computation, such as the steps followed in column addition, long division or multiplying fractions (Mayer, 1982).

Mayer then proposes a two-stage process for problem solving that incorporates when and where these types of knowledge are applied within the process of solving a word problem. This model is consolidated in the table below (Table 1.2).
Table 1.2

*Types of Knowledge Used in Each of Mayer's Stages of Problem Solving*

<table>
<thead>
<tr>
<th>Stage</th>
<th>Step</th>
<th>Knowledge</th>
<th>Examples from sample problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>Translation</td>
<td>Linguistic and factual</td>
<td>“Pete has 3 more cents than John”</td>
</tr>
<tr>
<td>Representation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Integration</td>
<td>Schema</td>
<td>A nickel equals 5 cents.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>This is a “comparison” problem,</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>consisting of two subsets and a</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>superset.</td>
</tr>
<tr>
<td>Problem Solution</td>
<td>Planning</td>
<td>Strategic</td>
<td>The goal is to add 3 + 5.</td>
</tr>
<tr>
<td></td>
<td>Execution</td>
<td>Algorithmic</td>
<td>Counting-on procedure.</td>
</tr>
</tbody>
</table>

Note: (Mayer, 1985, p.131)

**Problem Representation Stage.** Mayer's first stage, *problem representation*, captures the work done by the problem solver before computation is attempted. In this stage, the solver "converts a problem from words into an internal representation." (Mayer, 1985 p. 130). In many cases, this work involves translating the words and context of a problem into an abstract equation. It is in this stage that schema for similar problems are accessed, and Mayer integrates schema theory. Actions to move the internal representation to an external representation are also taken during this stage. For example, during the representational phase of problem solving, blocks might be manipulated to show the actions of a problem, or a solver might draw a picture or a diagram of the elements of the problem (Mayer, 1983). Polya's steps of "understanding the
problem" and "making a plan" (Polya, 1945) are encapsulated by Mayer's problem representation stage (see Table 1.1).

Both of Mayer's two stages are further broken down into steps. The problem representation stage consists of problem translation followed by problem integration. Mayer's problem translation step consists of decoding the text of the word problem, and then translating each proposition of the problem into an internal representation (Mayer, 1985). This step taps into the linguistic knowledge of the solver. The solver must be able to decode and comprehend the words of the text. In addition, factual knowledge comes into play in this step, where the solver must apply background knowledge that may not be explicit in the context of the story. In the example given in Table 1.2, the fact that a nickel is equal to five cents is considered to be factual knowledge assumed to be held by the solver. Mayer points out that problem solvers can differ here in terms of their ability to comprehend linguistic expressions in terms of relational mathematical propositions (Mayer, 1985).

The problem integration step takes the basic understanding of the text and then generates an internal representation of both the problem model and the situation model (Coquin-Viennot & Moreau, 2007) taken from the word problem (Mayer, 1985). The problem integration step is where schematic knowledge is most needed. A solver must access prior knowledge about similar problem types, discerning how the problem at hand is like problems solved in the past and illuminating a path towards a solution. Efforts to bring the internal representation to the external realm are made here (Mayer, 1989). At this step, students can vary widely in the level of experience they have with different types of word problems (Mayer, 1985).
**Problem Solution Stage.** The second stage of Mayer's model is the *problem-solution* stage, where the solver acts upon the representations generated in the first stage. It is here that mathematics is applied and equations are solved for numerical quantities. Algorithms or other strategies are chosen, applied and worked in order to arrive at the desired numerical solution. Finally, a numerical solution is interpreted by the problem solver within the story context.

Polya's "carry out the plan" and "look back" steps map to Mayer's problem-solution stage (Mayer, 1982). Singapore's Model Method is primarily an action that develops and supports Mayer's problem representation phase, although the models are also useful in Mayer's problem-solution phase, especially in aiding algebraic reasoning and determining the reasonableness of solutions (see Appendix A for examples of model drawing).

The problem solution stage is also broken down into two steps; *problem planning* followed by *problem execution*. Problem planning requires the solver to be strategically knowledgeable. The solver chooses an operation, and considers how to carry out that operation (Mayer, 1985). Mental arithmetic, counting strategies, and pencil-paper algorithms could be applied, depending on the computation abilities of the solver coupled with the solver's level of strategic knowledge. In this step, students can differ from each other in terms of their familiarity with general problem-solving strategies (Mayer, 1985).

The second step, problem execution demands algorithmic knowledge, the understanding of the steps required to carry out the computation involved in the word problem (Mayer, 1985). In the example from Table 1.2, the solver could accurately move through all previous steps, but unless the solver has some knowledge of how to actually add 3 and 5, the solver cannot execute a final solution for the problem. Again, Mayer suggests that wide variation can occur among
students in the sophistication, accuracy and automaticity of their algorithms for basic
operations (Mayer, 1985).

Mayer's two-stage model of problem solving points out the complex and interconnected
nature of the task of problem solving. There are many points along the road at which a problem
solver can become lost. Mayer suggests that much of the research done on problem solving has
focused on the second, problem solution stage of his model (Mayer, 1983). Perhaps this is true
because it is easier to examine final solutions when compared to the complexities of how
students actually conceive of the problem itself. Following Mayer's call for further research, the
primary focus of this study is on the first stage of Mayer's model, problem representation.

Singapore's Model Method follows all four of Mayer's problem-solving steps, but because the
method utilizes a unique schematic representational strategy, inquiry into Mayer's
representation stage is of the utmost importance. As Mayer's theory provides a model for
understanding the entire problem-solving process, the work of Mayer informs and guides the
inquiry of this study, providing the backbone for the dissertation.

Theory Framing Singapore's Model Method

Singapore's Model Method is a word problem-solving process taught to elementary
aged children working with Singapore's elementary mathematics curriculum. The Model
Method was developed in part through consultation of two theorists in cognition and
mathematics education, Jerome Bruner and Zoltan Dienes (Yeap, 2010).

Jerome Bruner's theory of Phases of Intellectual Development was integral to the
creation process of Singapore's Model Method (Yeap, 2010). Bruner posited that human
learning moves through a continuum of three phases. First, the learner understands at the
enactive stage (Bruner, 1961). In young children, this phase of intellectual development is observed when the learner must act upon objects in order to make sense of them. Bruner's first phase is alternatively referred to as the "concrete" stage of learning (Krawec, 2010). According to Bruner's theory, learners then move through an iconic stage, where they are more able to express their understanding in more abstract terms, through generalizations or diagrammatic representations (Bruner, 1973). The final phase of intellectual development is the symbolic stage, where concepts are understood at their most abstract level and can most easily be connected to other ideas and knowledge (Bruner, 1973). In mathematics education, the phase most often ignored by educators is the iconic phase. This "pictorial" stage acts as a bridge between the concrete and the abstract. Mathematics educators, especially at the elementary level, make frequent use of concrete materials in order to access Bruner's enactive phase of learning. Unfortunately, the intermediary pictorial stage is often skipped, with the teacher moving students directly from the concrete to the abstraction of symbolic representation, especially equations and algebraic expressions. Singapore's Model Method was developed as a distinct intermediary activity that would activate the iconic phase of learning (Yeap, 2010). This concept is embodied in the “concrete-pictorial-abstract” approach detailed in monographs by Singapore’s Ministry of Education (Hong, Mei, & Lim, 2009) and discussed further in subsequent chapters of this dissertation.

The work of mathematics educational theorist Zoltan Dienes was also considered to be influential in the development of Singapore's Model Method (Yeap, 2010). Dienes proposed that multiple representations were crucial to mathematical understanding (Dienes & Others, 1980). For example, fraction concepts might be taught to a child using a circle to represent the
whole. Dienes posited that fraction concepts are best developed in children who have been exposed to fractions represented as circles, squares, groups, sets and 3-dimensional objects (Dienes, 1971).

Singapore's Model Method asks students to construct diagrams of word problems, thereby providing them with multiple representations (Dienes, 1971) of problems and further developing their understanding of mathematical operations and relationships at Bruner's (Bruner, 1973) iconic or pictorial level of intellectual development (Yeap, 2010). Theory surrounding model drawing demonstrates that the pedagogy is based upon previous work in cognitive psychology, and helps to demonstrate how the instructional strategy operationalizes what we know about learning.

Conclusions: Theoretical Framework Model

The theoretical framework supports this investigation by providing an explanation of how and why model drawing potentially affects problem-solving performance. Schema theory is an integral part of Mayer's two-phase model of problem solving, accessed by the problem solver specifically during Mayer's integration step. Singapore's Model Method is potentially effective because it provides a helpful bridge between the representation phase and the solution phase of Mayer's model. The Model Method enhances problem solving by allowing the problem solver to physically actualize Mayer's first phase through a schematic diagram. This drawing also facilitates solution planning in Mayer's second phase through depicting relationships and pointing the solver towards the correct mathematical operation. In other words, when a child is asked to draw a model for a word problem, the child creates a physical schematic representation that enhances understanding of the situation and then facilitates
correct choice of a mathematical operation. This enhanced problem-solving process is depicted in Figure 1.2, a diagram developed by the researcher as a practical representation of how Singapore’s Model Method interacts with the central theory of mathematical word-problem solving as posited by Mayer (1985).

Figure 1.2

*Mayer's Two-Phase Problem Solving Model Facilitated by Singapore's Model Method*

*Figure 1.2.* Singapore's Model Method acts as a bridge between Mayer's problem representation stage and his problem solution stage, enhancing the work done in both stages.

The three major theories detailed in Figure 1.2 guide the intentions and actions of this study. The model was designed by the researcher as a practical representation of how Singapore’s Model Method interacts with the central theory of word problem solving as posited by Mayer (1985). These theories explain the potential of Singapore's Model Method. Schema theory and theory on problem solving are key features of Mayer's problem-solving model. Mayer's model helps us understand the potential of model drawing as an instructional strategy.
Model drawing is potentially an effective teaching technique because it asks students to first categorize word problems by schematic type, then asks students to draw a schematic diagram, encapsulating Mayer's representation phase of problem solving. The diagram that the student generates can then be used to choose a solution pathway, thus potentially enhancing Mayer's solution phase of problem solving.

The material that follows will be organized into two chapters: the Literature Review, and Research Design. Chapter 2, The Literature Review will give the reader a picture of what is known about word problem instruction, particularly in the area of visual representation and its connection to schema theory. Chapter 3, Research Design, outlines the experimental design and connects the design to both theory and the literature on problem solving and Singapore's Model Method.
Chapter 2: Literature Review

Problem solving is a fundamental task of mathematics, and therefore central to mathematics instruction. The specific inquiry of this review is based upon the following research question:

*What is the functional relationship between the application of the Model Method schematic diagramming teaching technique and the performance of non-classified elementary students on multiplicative comparison and fraction word problems?*

The research question incorporates two fundamental constructs, *problem-solving performance* and *schematic visual representations*. This review attempts to encapsulate what is known about the representational phase of the problem-solving process and how this knowledge is operationalized in improving problem-solving performance. Schematic visualizations are defined here as diagrams made by the problem solver through the use of Singapore’s model drawing technique. These diagrams are schematic in nature because, rather than depicting actual objects from the problem context, they represent quantities and relationships. In addition, schematic visual representations incorporate numbers and other information from the story text into the diagram itself. For example, if a child is asked to make some kind of representation while solving a word problem about ducks, it is likely that the child will make likenesses of ducks. A schematic diagram asks the child to represent quantities and relationships with abstractions such as circles or rectangles, labeling these parts with numbers from the problem. Different types of diagrams are used for different types of problems, which encourages sorting problems by schema before interacting with the
computation needed to solve the problem. Problem-solving performance is defined here as the percentage of arithmetic word problems solved correctly.

The Argument of Discovery will form the foundation for an Argument of Advocacy (Machi & McEvoy, 2009). The Argument of Discovery in this review will demonstrate what is known about representation as a helpful technique in teaching children to solve word problems. The argument will be broken down into a logical series of claims chained together to make a case for what is currently known in the field. First, the broad body of literature on representation in problem solving will be reviewed. Next, the narrower field of schema and representations will be investigated. Finally, knowledge about Singapore's Model Method will be reviewed as a confluence of the first two bodies of literature. In all three bodies of literature, connections to methodology, theory and literature limitations will be examined and analyzed.

This review is not comprehensive of the entire field of mathematical problem solving. For example, this review excludes the body of research done on reading comprehension and word problems. Reading is an intrinsic part of solving word problems, and many researchers have investigated the connections between reading text and word problem performance (Lester, 1994; Schurter, 2002). The extensive knowledge body surrounding mathematical computation and procedures is also excluded from the scope of this review. Both of these concepts are important to mathematical problem solving. Reading the word problem, however, is incorporated in Mayer's translation stage of problem solving, and mathematical computation is part of Mayer's execution stage of problem solving. As shown in Table 1.1, model drawing is primarily an enactment of Mayer's representation phase. Thus this review focuses primarily on
the complexities of this first phase of problem solving, as this is where problem conceptions are
developed (Mayer, 1989).

The Argument of Advocacy will demonstrate that given what is known about schematic
visual representations in problem solving, a clear case exists for the proposed study. The study
will add significantly to several bodies of knowledge including problem solving and the
effectiveness of Singapore's Model Method. Literature gaps uncovered in the literature will also
inform the mandate for the investigation.

Argument of Discovery

Claim 1: visual representations are important aids to problem solving.

Visual imagery has been used in mathematics since the origins of the field of study. Van
Garderen & Montague, leading researchers in the field of visual representations in problem
solving, define visual representation as, "the construction and formation of internal images and/
or external images." (van Garderen & Montague, 2003, p. 247). Inquiry into the effectiveness
of visual images in mathematics education has continued for at least the past century (Bishop,
1989). In fact, visual representation in mathematics is widely regarded as essential to the
problem-solving process. Through their research in understanding mathematical representation
learning and the value of visual-spatial thinking increasingly have been acknowledged as
essential to mathematics education" (Edens & Potter, 2008, p. 3).

The National Council of Teachers of Mathematics (NCTM) considers representation
one of its core processes, a foundational building block in its Principles and Standards. For the
NCTM, representation refers to both a process and a product. The act of generating an accurate
representation of a mathematical relationship or idea is a key standard of K-12 work in mathematics (NCTM, 2000). The NCTM goes further, arguing that representations have too frequently been taught as goals of instruction rather than as tools for understanding. Instead, representations should be treated as essential elements of teaching for understanding in mathematics (NCTM, 2000).

...Claim 2: there are many types of visual representation in problem solving, and some are more effective than others.

Visual representation is important to problem solving, yet visualization can take numerous forms. Manipulatives are common physical materials used in elementary mathematics classrooms. They take the form of blocks, counters, chips or tiles that represent quantities and relationships between quantities. In a 1996 study, Marsh and Cooke showed that interaction with these representations helped students more accurately choose the correct operation in word problems (Marsh & Cooke, 1996).

Informal drawings are another form of mathematical representation sometimes used in classrooms. These drawings by students are most often focused on non-mathematical details, or represent objects from the word problem without necessarily containing more information about mathematical relationships (Hegarty & Kozhevnikov, 1999). Drawings are usually concrete as opposed to abstract in nature and are commonly used by children in problem solving (Hegarty & Kozhevnikov, 1999). For example, in a word problem involving ducks, concrete drawings by children would include depictions of ducks as opposed to abstractions that represent the number of ducks in the problem.
Diagrams constitute another category of visual representations in math education, a step higher up the ladder of representational abstraction. Hegarty and Kozhevnikov (1999) describe this type of imagery as "representing the spatial relationships between objects and imagining spatial transformations" (Hegarty & Kozhevnikov, 1999, p. 29). With a diagram, a student can more easily organize, record and communicate mathematical ideas, a core component of the NCTM's representation standard (NCTM, 2000). From the perspective of this study's theoretical framework, diagrams fit neatly into Bruner's iconic stage of intellectual development, bridging the gap between the concrete and the abstract.

Within the imagery commonly used in elementary mathematics, equations and algebraic variable expressions are the most abstract forms of representation. These depictions demonstrate the most efficient mathematical procedures used in solving a word problem. The ability to generate an equation does not necessarily translate to a demonstration of understanding of mathematical relationships (Pape & Tchoshanov, 2001).

Over time, researchers and theorists have begun to categorize and order mathematical representations. In 1986, Norma Presmeg created the following taxonomy of representations in mathematics problem solving: concrete imagery or mental images; pattern imagery showing pure relationships depicted in a visual-spatial scheme; kinesthetic imagery involving movement and gestures; dynamic imagery involving transformations of geometric figures; and memory images of formulae in which problem solvers typically visually recall a mathematical formula that then leads to problem solution (Presmeg, 1986). This taxonomy was foundational for further research in representation in mathematical problem solving (Montague & Jitendra, 2006).
In an often-cited investigation, Hegarty and Kozhevnikov (1999) reliably categorized students' representations of word problems as either primarily pictorial or primarily schematic in nature. Pictorial representations generate accurate imagery and detail, while primarily schematic representations are much more focused on depicting abstractions that convey mathematical and spatial relationships, referring to internal schema (Hegarty & Kozhevnikov, 1999). This research showed that students who utilized schematic representations were more likely to solve word problems accurately than users of pictorial representations. The authors surmised that schematic representations were more powerful for problem solvers because they contained more pertinent and relational information from the problem, generating an external image of an internal schema (Hegarty & Kozhevnikov, 1999). The importance of schematic representations readily connects to Mayer's problem representation stage, where problem translation and integration occur.

In 1992, research on problem representation and problem solving was encapsulated by Ray Hembree in his meta analysis of 487 research reports (Hembree, 1992). Although Hembree's work encompassed 4 different categories of research on problem solving, the results from his meta-analysis on the topic of representations in problem solving were some of the most significant. Hembree found that of all the instructional strategies studied, only teaching formats coupled problem-solving statements with the use of diagrams, figures or sketches that produced large effect sizes (Hembree, 1992). Hembree (1992) also showed that training for skill in representations such as diagrams provided the largest problem-solving performance improvement. The evidence presented in this claim underscores the idea that visual
representations can take many forms, but diagrams are one of the most effective forms of visual representation.

Claim 3: empirical studies show that visual representations are usually effective.

Enhancing the research base described in Claim 2, a number of empirical studies have been done that point to the value of visual representation in mathematic problem solving. There are also several studies that point out the limitations. In a group study design, Edens and Potter (2008) confirmed earlier work on the categorization and effects of pictorial and schematic representations in problem solving. During art classes, 214 fourth and fifth graders were given a series of art and math challenges to solve. Researchers utilized correlation analysis to uncover relationships among spatial visualization ability, problem solving, schematic and pictorial representations and drawing skill (Edens & Potter, 2008). Consistent with prior research, these researchers found a statistically significant relationship between the use of schematic visualizations and problem-solving performance (Bishop, 1989; Hegarty & Kozhevnikov, 1999). Also of interest were the findings that a slightly greater percentage of girls than boys used schematic representations, and that drawing ability was not correlated with problem-solving performance (Edens & Potter, 2008). In other words, artistic skills did not make one better at utilizing schematic diagrams effectively.

Van Garderen and Montague, more recent contributors to the field, studied students with learning disabilities’ (LD), average-achieving students’, and gifted students’ use of visual representations while solving word problems (van Garderen & Montague, 2003). Their group design of 66 sixth graders in urban Florida elementary and middle schools assessed students across several variables, including math achievement, mathematical problem-solving, visual-
spatial representations and spatial visualization (van Garderen & Montague, 2003). The subjects were assessed using the Woodcock-Johnson Tests of Achievement subtests of calculation, math fluency and applied problems. The Mathematical Processing Instrument (MPI) test was also used to measure problem solving in students.

The results of the study showed that students who used imagery were more likely to answer word problems correctly. LD and average-achieving students used significantly fewer images for problem solving and performed more poorly on measures of spatial visualization ability than the gifted students (van Garderen & Montague, 2003). In addition, 5 types of images were coded from the results: concrete imagery, action imagery, kinesthetic imagery, number forms, and pattern imagery. The spatial-visual images used by students were coded as either pictorial, representing objects or characters in the problem, or schematic, representing relationships within the problem. Van Garderen & Montague showed a positive correlation between success in problem solving and the use of schematic representations. Furthermore, those students who used pictorial representations were more likely to solve word problems incorrectly (van Garderen & Montague, 2003). This work expanded research on how students use imagery in problem solving, although much of the study focused on the work of learning disabled students.

In 2007, van Garderen returned to the topic of visual representations and word problems with a single subject design. Her use of a multiple baseline across participants design improved the validity of the results. In her study, 3 eighth graders were given instruction in diagram generation for solving word problems and were assessed throughout the treatment. The results indicated all three students improved in several significant ways. They used more diagrams in
solving word problems, they improved their ability to create diagrams for word problems, and their performance on solving word problems increased (van Garderen, 2007). Van Garderen tested her subjects for both maintenance and generalization of the strategy to non-routine word problems, and found that performance remained stable under both conditions (van Garderen, 2007).

Visual representations for problem solving can be problematic in themselves. In 1995, Hegarty, Mayer and Monk wrote, "Most problem solvers have more difficulty constructing a useful problem representation than in performing the computations necessary to solve the problem" (Hegarty, Mayer, & Monk, 1995, p. 19). It is clear that the process of translating internal schema and understanding to a visual representation is not a simple one.

Presmeg's 1986 study introduced categories of visualization and also further developed definitions of visual images in problem solving to "allow for the possibility that verbal, numerical or mathematical symbols may be arranged spatially" (Presmeg, 1986, p. 299). Presmeg used qualitative methods to investigate the problem-solving styles of seven high school seniors who were identified as “star pupils” by their math teachers. The results showed that very few of the students could be characterized as "visualizers". The author suggested these results reflected the way the students may have been taught, without an emphasis on or teacher interest in visual proofs over numerical explanations. It is also conceivable that star pupils do not see visual models as helpful in their problem-solving process. It may be that excellent problem solvers, especially students at the secondary school level, do not need to convert from internal schema to a drawn expression of that schema. Instead, they may be able to convert
from an internal schema directly to an external arithmetic or algebraic alphanumerical expression of the internal schema.

Walker & Poteet (1989) generated a group comparison design study of middle school students and problem solving. Seventy students, all of whom had some type of learning difficulty, were randomly assigned to either a word strategy group or a diagramming group. The word strategy group was given instruction typical of many American math classrooms, where students are taught to seek out key words in the text of word problems to help them choose the correct operation for the solution. The second group was given instruction in diagramming instruction that included students drawing a diagram, writing an equation, and then solving the equation. Results showed no significant differences between the two groups from pretest to posttest (Walker & Poteet, 1990). Walker and Poteet explained that the diagrams used by instructors were not labeled clearly enough with information from the problem, reducing their potency as external representations (Walker & Poteet, 1990). If the teacher diagrams were not truly schematic, then the diagrams may not have been as helpful to students (Hegarty & Kozhevnikov, 1999).

In general, the large majority of the studies investigated here demonstrate the potential for visual representations as helpful tools to be used by mathematical problem solvers. Concrete, pictorial representations have been shown to be less effective than more abstract schematic representations, especially when these types of diagrams include information from the word problem itself. Relatively few of these studies chose elementary aged children as their subjects, instead focusing primarily on middle school or secondary school populations. In addition, no evidence could be found of a dynamic or flexible diagram system that could be
adapted for the various types of problem-solving situations categorized in previous research (Kintsch & Greeno, 1985) or (Schoenfeld, 1985).

**Claim 4: schematic diagrams are effective because of the information and relationships they help to develop for the problem solver.**

Claims one through three demonstrate that visual representations can improve problem-solving performance. More specifically, schematic drawings are perhaps the most beneficial types of representations for word-problem solving (van Garderen & Montague, 2003). Schematic diagrams essentially become external representations of internal knowledge frameworks (Schoenfeld, 1979).

Schematic diagrams include germane information from the problem that supports an accurate problem solution (Edens & Potter, 2008). These diagrams, as opposed to pictorial representations, are positively correlated with improved problem-solving performance (Hegarty & Kozhevnikov, 1999; Pape & Tchoshanov, 2001; Stylianou & Silver, 2004). Schematic representations also help the student identify important information in the problem (Schurter, 2002).

Many word problems include extraneous information, or distract the student from the mathematical relationships with engaging problem context. A schematic diagram allows the student to keep track of what steps have already been taken and to make connections between Kintsch and Greeno’s (1985) situation model and problem model (A. Jitendra, 2002). Students can also make use of the diagram to help them reorganize confusing problem information (Xin & Zhang, 2009).
Alongside identifying and organizing important information, schematic diagrams help the problem solver see and understand the problem model more clearly (Rockwell, Griffin, & Jones, 2011). Mathematical relationships, usually obscured by the abstraction of the situation, are presented clearly as the real focus of the problem (Dixon, 2005). In a sense, schematic diagrams serve as placeholders for thinking.

Educators and researchers have also found an assessment advantage to using schematic representations with their subjects: the schematic drawings give the reader or teacher a glimpse into the thought process of students. The drawn diagrams can become evidence of skill and understanding in problem solving (Willis & Fuson, 1988). Students are often asked to explain their thinking, usually in narrative form, in problem-solving solutions. Schematic drawings can be used by students as an alternative to written narratives of their problem-solving process (Willis & Fuson, 1988).

In Mayer's two-stage model of problem solving, schematic diagrams would fall into the problem integration step of the problem-solving process. Drawing a schematic diagram asks the solver to develop a physical representation of internal schema, expressing on paper the problem integration detailed by Mayer (Mayer, 1985). The literature clearly supports the claim that schematic diagrams are effective because of the informative representation they generate for use by the problem solver.

Claim 5: schema-based instruction (SBI) combines schema theory with diagram representations.

Schema-based instruction (SBI) is an instructional strategy that embodies some of the ideas in Claims 1-4. SBI has developed in the last two decades as an instructional approach to
teaching word-problem solving (A. K. Jitendra, Star, Rodriguez, Lindell, & Someki, 2011) that integrates schema theory with the effectiveness of relational diagrams. The strategy centers on the mathematical structure found in word problems as critical to problem comprehension and representation (Leh, 2011). SBI proposes that the connection of the algorithmic procedure to the context of the word problem is more powerful, and more problematic, than knowing how to carry out the particular procedure itself (Coquin-Viennot & Moreau, 2007). In other words, SBI provides a bridge between Mayer's problem representation and problem solution phases of word-problem solving (R. Mayer, 1989).

In most iterations of the SBI approach, students are taught to recognize word problems as falling into one of several types (schema). The student then creates a visual representation incorporating the standard schema for the problem type. For example, the student sorts the word problem as fitting into a “part-whole” problem type, and draws the corresponding diagram for that problem type. Next, particular information from the context is added to the diagram. Using the diagram, the student then chooses a procedure for calculation. Once comprehension and representation are secure, students can more frequently correctly choose an operation and apply their procedural knowledge to a given word problem (Powell, 2011).

In schema-based instruction, students are asked to relate individual word problems to a small number of schematic types, fundamentally changing the problem-solving process (Christou & Philippou, 1999). Focusing on schema attempts to move the student's thinking beyond the particular details of the problem in front of the student, in order to pay deeper attention to how the particular problem relates to a body of other problems with similar solution patterns (A. K. Jitendra et al., 2010).
In a 2011 survey of the literature on SBI, Sarah Powell found that SBI has been shown to be a valid method for helping students at risk for or with learning disabilities to become more proficient problem solvers (Powell, 2011). Powell (2011) reviewed and synthesized separate studies incorporating schematic word problem instruction. Two different approaches were identified for elementary aged students at risk for or with LD: schema-based instruction and schema-broadening instruction. Schema-broadening instruction introduced learners to a wider variety of problem schema, as opposed to focusing on categorizing a smaller number of problem types (Powell, 2011). Both types of instruction were shown to enhance the word-problem solving skills of the participants. The review was limited by the small number of studies investigated (12) and to a relatively small segment of the elementary population, namely LD students in grades 2 and 3.

**Claim 6: representative empirical studies show that SBI positively affects problem-solving performance.**

As shown in claim five, SBI combines the categorization action from schema theory with the power of schematic diagrams. A distinct body of evidence has been produced around the effects of SBI on problem-solving performance. A representative selection of work, including that of the most important contributors is presented in this section.

Willis and Fuson (1988) conducted an earlier study on instruction based on schematic diagrams. In a group design, the researchers taught two classes of second graders problem-solving strategies involving schematic drawings for several different types of addition and subtraction word problems (Willis & Fuson, 1988). Subjects were of average or above-average
The children were taught to enter quantities from the stories into the diagrams and then used the drawing to choose the correct operation on the quantities.

The results demonstrated that indeed, children can be taught to draw schematic diagrams (Willis & Fuson, 1988). The participants usually inserted the numbers from the problem into the diagram correctly, and usually selected the correct solutions strategy for the problem. In only about 6% of cases did a student's diagram mismatch the computation strategy chosen (Willis & Fuson, 1988). The pretest to posttest scores for both groups rose from 53% correct to 70% correct on average. The study's significance is in its use of schematic diagrams that continued to be used by future researchers, and in verification that schema-based instruction is effective even with younger children, a relatively rare age group for study within this body of literature.

L. Fuchs, Fuchs, and Prentice, (2004) conducted a more recent study that further expanded on the effects of SBI. The purpose of this study was to assess the effects of schema-based instruction in promoting mathematical problem solving. The research team also tested the effects of problem-sorting instruction on problem-solving performance.

The research team split 24 third-grade female teachers from six southeastern urban schools into three groups. The groups were: schema-based instruction, schema-based instruction plus sorting practice, and a contrast group which included teacher-designed and implemented instruction on the four problem types. Each of the groups was comprised of about 120 third-grade students who were given a pretest three weeks prior to instruction. After 16 weeks of instruction, each group of students was post-tested on mathematical problem solving and schema development (L. Fuchs et al., 2004).
Results from this study show that the schema-based instruction groups outperformed the contrast group for both schema development and successful problem solving. In order to isolate the effects of SBI, researchers taught basic problem-solving strategy instruction with all three groups, such as focused reading, lining up computation numerals correctly and checking solutions (L. Fuchs et al., 2004). Schema development was higher for the SBI groups, although the effects of problem-sorting practice were deemed inconclusive by the authors, suggesting further research into this technique (L. Fuchs et al., 2004).

Asha Jitendra is perhaps the most prolific researcher into the effects of SBI, with leadership of or contributions to at least 20 studies in the past two decades. In an early study, (A. K. Jitendra, Hoff, & Beck, 1999), the research team investigated the effects of a schema-based approach on the problem-solving performance of LD middle school students. Single-subject methodology was used with a multiple baseline design across four students and two different behaviors to improve internal and external validity. The experimental behaviors tested were schema strategy instruction in one-step word problems and schema-based instruction in two-step word problems.

The results of the study indicated that schema-based instruction was effective for improving the word problem performance of the four LD middle school students. Previous research on SBI effectiveness was supported. Of note was the fact that during the baseline phase of the study, the subjects never used diagrams of any kind. After the intervention of SBI training, these four students outperformed the average-achieving peers in problem-solving assessments, utilizing diagrams for most of the problems attempted (A.K.Jitendra et al., 1999). Limitations to the study included that these learning disabled subjects were being taught skills
(addition and subtraction) for problems far below their actual grade level, which may have had an effect on the performance data.

By 2007, A.K. Jitendra was investigating SBI more deeply. A.K. Jitendra and her colleagues conducted a comparison study on the relative effects on problem solving and mathematical achievement of SBI compared with a set of multiple traditional strategies grouped together as general strategy instruction (GSI) (A.K. Jitendra et al., 2007). In a group design, 88 third graders and their teachers were assigned randomly to one of the two conditions (SBI or GSI). The research team measured performance with pre- and post-test data backed up by data from the Pennsylvania System of School Assessment Mathematics test (A.K. Jitendra et al., 2007).

The comparison condition, or GSI instruction, followed the same problem types and general routines of teacher modeling, think-alouds, guided practice and independent practice (A.K. Jitendra et al., 2007). Where the two instructional conditions differed was in the use of the schema instruction phase in the SBI condition. Here, students were instructed to classify problems by types and by the linked diagrams that accurately portrayed that problem type. This group of students was then taught to map story quantities onto the diagrams.

The adjusted mean scores indicated that the SBI group significantly outperformed the GSI group. In addition, this significance was carried forward into the maintenance assessments for both groups. The effect sizes comparing the SBI with the GSI groups were 0.52 on the immediate posttest and 0.69 six weeks later on the maintenance test (A.K. Jitendra et al., 2007). The researchers also confirmed their hypothesis that the work by students on problem solving enhanced both groups’ performance on computation skills. The authors noted limitations to the
study, including a small sample size, and difficulties in staffing during the instructional phases of the study (A.K. Jitendra et al., 2007).

A third study by A.K. Jitendra, Star, Rodriguez, Lindell, and Someki (2011), provides the last, most current sample from her large body of work. In this study, the researchers studied the effectiveness of SBI on seventh graders’ performance on proportion word problems. The study investigated a population of 482 students and six teachers in 21 classrooms from two different Minnesota middle schools. The two schools used very different mathematics textbook curricula (A.K. Jitendra et al., 2011). The group design randomly assigned the 21 classrooms and their teachers to one of two conditions: SBI and control. The control group followed the regular instruction in ratios, proportions and percents as detailed in their district-adopted textbook materials. The SBI group replaced that regular instruction with a different treatment on ratios, proportions and percents (A.K. Jitendra et al., 2011).

The SBI program used by A.K. Jitendra et al. (2011) reflects the accumulation of research-based knowledge on SBI, resulting in a more fully formed instructional program and sequence. As usual, the instruction for the SBI group focused on the underlying mathematical structure of word problems through the use of schematic diagrams. A four-step problem-solving procedure, similar in scope and sequence to Polya's original problem-solving schema (Polya, 1945), was laid over this instruction as a way to support and monitor problem solving. Teachers were trained in the flexible use of alternative solution strategies based on the problem situation. Additionally, students were taught using complete stories, without any missing variables, to give complete schema to students at the outset of their instruction. Later, new stories were presented as problems, each with one or two missing pieces of numerical information that
required discovery (A. K. Jitendra et al., 2011). These facets of the instructional treatment evolved over the past 20 years of research. Two major findings surfaced from the results of statistical analysis in the study. First, confirming A.K. Jitendra's prior research, the SBI group significantly outperformed the control group on measures of problem-solving performance. Second, no significant differences were found between the two conditions after a one-month delayed posttest. The researchers surmised that the second finding might have been caused by the fact that the intervening unit of instruction was based on geometry, which may have interfered with recall and application of the previous treatment.

At this point, schema-based instruction is a well-defined and investigated instructional strategy for word-problem solving. The contributions of Asha Jitendra and many others have shown that focusing students on classifying word problems and then applying particular problem quantities to pre-established schematic diagrams is an effective method for analyzing and solving simple and complex arithmetic word problems. This body of evidence has been conducted with variety of subjects, age groups, methodologies and instructional models used to teach SBI, thereby enhancing the reliability of the findings in this body of literature. Mayer's two-stage model of problem solving has been cited and validated through the work of these researchers (Griffin & Jitendra, 2009; A. K. Jitendra et al., 2011; Xin & Jitendra, 2006).

There are also gaps in this body of literature. Most of the studies conducted focused on the LD population of middle school students. Although elementary aged students were certainly represented, more attention needs to be paid to this age group, as this is where most instruction in word problems begins, and it is at this age level that internal schema for word problems are certainly developing (Van de Walle, 2007). Although it is logical to apply SBI to LD
populations, some research has shown the instructional strategy is also effective with regular education math students (Griffin & Jitendra, 2009). SBI, or similar schematic representational strategies must be further explored with students of this age and achievement level.

As mentioned previously in the introduction, a gap in the literature base exists in the lack of a systematic diagram structure for schematic drawings. No researcher has described or proposed a broad system of schematic diagrams for problem solving. In each case, authors have used or modified diagrams from other authors (A. K. Jitendra et al., 2011), or have devised their own schematic forms (Powell, 2011). A system of schematic diagrams, useful in all four operations and fractions, then linked together in form and function, could improve the effects of schematic representations and SBI on problem-solving performance for students. Such a system could also prove beneficial to teachers as they struggle to understand the mathematics that they must teach to their students (U.S. Department of Education, 2008).

Claim 7: Singapore's Model Method is a form of SBI with a comprehensive system of representation created to develop problem-solving skill and algebraic reasoning at the elementary level.

The first six claims dealt with instructional strategies for math problem solving in general. Now, we consider a very specific strategy, Singapore's Model Method. In the early 1980s, as a response to poor word problem performance in its elementary school system, Singapore's Ministry of Education began to develop and implement what it came to call its "Model Method for Problem Solving", an instructional strategy of schematic diagramming for arithmetic word problems (Hong, Mei, & Lim, 2009). The strategy was developed over the decade and was in full use in Singaporean schools by the early 1990s. The Model Method is
also known in Singapore and elsewhere as "bar modeling" and in the United States as "model drawing".

The Model Method first asks students to classify word problems according to schema such as "part-whole model" and "comparison model". Based on this classification, students draw rectangular "bars" or rectangles representing the whole quantity and any quantitative parts in the story context. Besides the part-whole schema, the bars can also be drawn to represent comparison and change problems. In fact, this visual system is used to model all (Riley, 1984) word problem types for addition and subtraction, and are also used to model multiplication, division, and rational numbers (see Appendix A for model drawing examples). Monographs from Singapore's Ministry of Education describe this teaching tool as but one example of the “concrete-pictorial-abstract” (c-p-a) teaching approach integrated throughout its elementary mathematics curriculum (Hong, Mei, & Lim, 2009). Typical math instruction at the elementary level asks students to engage with concrete materials before progressing to abstractions of math concepts. In the c-p-a approach, diagrams are used as a bridge, or intermediary step, between learning from manipulating concrete objects and the abstractions of mathematical concepts and algorithms.

The models generated by this method are not the simple pictorial renderings warned against by Hegarty and others (Hegarty & Kozhevnikov, 1999; Presmeg, 1986; Lowrie, 2001). These models are schematic in nature, as they refer to internal schema for the variety of problem types found in word problems (Yee, 2010). The models do not depict actual objects from the word problem, but depict quantities and relationships between quantities (Swee &
Lee, 2009a). The bars drawn by problem solvers are labeled with numerical information found in the word problem text.

Singapore's Model Method is intended to help elementary aged children use schemas to differentiate between word problem types and, at the same time, see and understand mathematical relationships within word problems (Swee & Lee, 2009a; Swee & Lee, 2009b; Yeap, 2010). The models allow students to identify known and unknown variables in word problems (Foong, 2009). Once the model is generated, a student can use the model to refer to relationships between quantities as described by Schurter and others (Schurter, 2002). Using the few model types, students can identify problems as part-whole, change or comparison, sorting problems by schema as in A.K. Jitendra's versions of SBI (Griffin & Jitendra, 2009; A. K. Jitendra & Hoff, 1995; A. K. Jitendra, Sczesniak, Griffin, & Deatline-Buchman, 2007).

Along with the stated purposes mentioned above, Singapore's Model Method was designed with another goal in mind: improving algebraic reasoning in elementary students in order to prepare them for algebra courses in the later grades (Looi & Lim, 2009; Swee & Lee, 2009b). The schematic diagrams are designed so that students can use the representation to depict and analyze multiplicative and ratio relationships, allowing students to reason algebraically without the formal abstractions of alphanumeric variables commonly found in algebra courses (Looi & Lim, 2009). Thus, the Model Method has also been conceived as a bridge to formal algebra instruction in later years. In fact, some students in Singapore persist in using the model method over more formal algebraic notation even while studying formal algebra (Looi & Lim, 2009).
The evidence presented in this claim shows that model drawing is a form of SBI that incorporates schema categorization with schematic visual representations. This instructional strategy embodies the theoretical framework of this study, bringing together schema theory with the representation phase of Mayer's problem-solving model. This confluence occurs within a well-formed visual system that can be adapted for all four arithmetic operations, rational numbers, and algebraic reasoning.

**Claim 8: empirical studies on Singapore's Model Method highlight its features, but do not demonstrate its effectiveness convincingly.**

Claim seven established the potential for Singapore's Model method, while through claim eight, the literature stream on Singapore's math curriculum, model drawing in particular, will be detailed. Several studies conducted in the United States have examined Singapore's elementary math curriculum. In 2005, the American Institutes for Research (AIR) undertook a study for the U.S. Department of Education to investigate Singapore's math curriculum in comparison to American textbook programs (Ginsburg et al., 2005). This exploratory study incorporated comparison analysis of Singapore's program, textbooks, teachers and lessons to those of three different U.S. elementary math textbook curricula (Ginsburg et al., 2005). Also included in the report was an exploratory implementation study conducted at two separate sites in the U.S. where Singapore's math curriculum had been implemented in American public schools (Ginsburg et al., 2005).

The results showed Singapore's math curriculum and textbooks to be more focused, coherent and mathematically challenging than any of the American counterparts studied (Ginsburg et al., 2005). One of the strengths of the program emphasized in the report was the
central focus on problem solving promoted by the curriculum. With problem solving as its core goal, AIR researchers showed the powerful use of multiple representations in Singapore's textbooks and teacher-delivered lessons, including the Model Method (Ginsburg et al., 2005). The principal investigators on the AIR report later wrote more specifically about their observations of the Model Method:

Singapore's program consistently uses the bar or strip model as a pictorial model for parts and wholes in addition, subtraction, multiplication, division, fractions, ratios and percentages. This consistent use of a single powerful model provides a unifying pedagogical structure entirely missing in U.S. mathematics classrooms. (Leinwand & Ginsburg, 2007p. 35)

Singapore's math curriculum has received much attention in the US and abroad over the past decade, and many schools in the US have begun to adopt Singapore's curriculum. Despite the attention to Singapore’s success, these investigations and reports are broad in scope, and do little to explore the effects of the Model Method. The only glimpse by international researchers into use of the Model Method came in 2010, with the publication of an informal study by Lisa Englard in a peer-reviewed journal (Englard, 2010). Englard taught the Model Method to one out of three of her school's third grade classes and compared pretest and posttest scores for the entire group. The remaining two-thirds of the group was not given the treatment, instead progressing with the problem-solving instruction provided by the school's textbook series. She also assessed the fourth and fifth grade students at her school with the same researcher-designed measure of performance (Englard, 2010). Englard found that the students in the test group were the only third graders to increase their problem-solving performance. The average
number of test problems solved correctly by the test group after the intervention was also higher than the averages for both the fourth and fifth grade students (Englard, 2010). Although published under peer review, unfortunately this work was an informal study by a practitioner, and the inquiry would not meet academic standards for validity or reliability.

The Model Method has been examined in Singapore from the perspective of cognitive psychology, with inquiry into how students think and behave when using the strategy (Foong, 2009). Through qualitative methodology in interviews, investigators found that, "Oscillations between the text and the model drawing allowed for checking of the accuracy of the latter product. Checking the accuracy of the arithmetic expressions was achieved by oscillating between it and the model drawing." (Foong, 2009). In other words, the findings showed that the model served as a reference point for the student, enabling the child to check the accuracy of his or her understanding of relationships and his understanding of the mathematical solution (Foong, 2009). This finding is consistent with the previously detailed findings on the usefulness of schematic diagrams in claims four and five.

Swee & Lee, (2009) citing a "dearth of research on the delivery of the model method and on children's use of the model method" (Swee & Lee, 2009b, p. 294) conducted the only other investigation of the Model Method in Singapore. Their exploration consisted of two different studies. The first study, using qualitative methodology, asked teachers and school administrators to answer questionnaires on student use of the Model Method while asking their students to solve a set of increasingly difficult word problems. This section of the investigation focused on perceptions of teachers around the Model Method. Unfortunately, the investigators did not actually observe teachers working with the Model Method, nor were they able to
observe students using the method (Swee & Lee, 2009b). This lack of direct observation limited the validity of their qualitative findings.

In the second portion of the study, quantitative analysis was used on the results of testing 151 fifth grade students on a series of five increasingly difficult word problems. The problems ranged from less complex arithmetic word problems up to algebraic word problems involving the solving of a system of equations. Through this second study, the authors were able to show that the Model Method is an effective strategy for solving word problems in children of average ability and above (Swee & Lee, 2009b). Models were less effective for solving the most complex algebraic word problems that students were likely to encounter in higher grades. In addition, the research team shed light on typical difficulties faced by users of the Model Method. Error analysis detected several types of errors, including mislabeling parts of the model, changing generators (referents) midway, and lack of conceptual knowledge (Swee & Lee, 2009b). In what they considered their most important finding, Ng & Lee (?) demonstrated that even partially correct models could aid students in problem solving (Swee & Lee, 2009b). Other findings replicated previous results of research on problem-solving representations in Claim 3, specifically the idea that errors were more likely due to a mistaken representation of the problem than with procedural computation (de Corte & Verschaffel, 1996; Hegarty, Mayer, & Monk, 1995; Hegarty & Kozhevnikov, 1999).

In summary, the body of research on Singapore's math curriculum, especially the Model Method, is primarily descriptive in nature and not yet fully developed. Investigations into the Model Method have focused almost entirely on the ways in which students use the strategy to solve word problems. No research could be found on the effectiveness of the instructional
strategy. No research could be found on particular problem types or with particular age groups. Perhaps this "dearth of research" (Swee & Lee, 2009a) can be explained by the fact that the Model Method has been in use in Singapore for so many years that its effectiveness is relatively unquestioned. Because Singapore uses a uniform state curriculum, it would be difficult to find subjects who have never seen the Model Method in a classroom. Perhaps when this form of SBI is removed from its original environment and implemented in schools elsewhere, the more natural questions around the effectiveness of model drawing will arise.

**Summary of Argument of Discovery**

As established in Chapter 1, helping students better learn to solve word problems is important. Improving student word problem performance has clear implications for national policy in education, theory and research on problem solving, and teaching practice in elementary mathematics. Since improving problem-solving performance is important, researchers and educators must continually work towards developing effective instructional strategies that help students excel in math.

The Literature Review presented here shows what is known about instructional strategies for problem solving in general and visual representations in particular. Various investigators have shown that the representation phase of problem solving is a vital and decisive portion of the problem-solving process (Marsh & Cooke, 1996; Pape & Tchoshanov, 2001; Presmeg, 1986; van Garderen & Montague, 2003). It is in this phase of problem solving that students come to comprehend how the quantities in the problem context are related to each other (Edens & Potter, 2008).
Of the various forms of visual representations in problem solving, the review then established that schematic diagrams were the most effective types. Schematic diagrams were proven to be potent in assisting children in successfully solving word problems (Dixon, 2005; Jitendra, Hoff, & Beck, 1999; A.K. Jitendra, 2011). Allowing students to "unpack" the content of word problems, schematic diagramming also encourages the organization and documentation of thinking as the student accesses and represents internal schema on paper (Edens & Potter, 2008; Jitendra et al., 2007).

The literature review thus demonstrates that SBI is an instructional strategy derived from research, combining schematic diagrams with schema theory (Schoenfeld, 1979; Fuson & Willis, 1989; Powell, 2011). SBI first asks students to categorize word problems into types and then draw schematic diagrams for each problem, thereby activating Mayer's representation phase of problem solving (Mayer, 1985). Empirical data from the field establishes the claim that SBI is an instructional strategy that improves word problem performance in students.

The first six claims establish two fundamental facts: first, that schematic visual representations are effective in helping children decipher word problems; second, one of the most effective instructional strategies for word problems is schema-based instruction (SBI) because it combines the activities of categorization (from schema theory) and schematic diagramming. The research reviewed through the first six claims also exposes several important literature gaps:

1. Both of these bodies of literature focus primarily on the LD population in middle schools, with some attention paid to the LD populations of elementary schools. Focus on the average-achieving elementary-aged student is far less common.
2. Following from the first gap, the more complex problem types found in the upper elementary grades are not represented in empirical studies.

3. No single researcher was able to employ or generate a systematic approach to the representations used. The missing element here is a single visual language that categorizes problem schemas while also modeling the most common topics in elementary math classrooms: the four arithmetic operations, fractions, ratios, and percentages. The use of a systematic schematic diagramming approach could make both the teaching and learning of word problems more successful for teachers and students.

**Argument of Advocacy**

Given what is known from the claims of the Argument of Discovery, a clear need for further research on Singapore's Model Method can be seen. An investigation of Singapore's Model Method is needed because the method is a form of SBI that provides a system of schematic diagramming flexible enough to encompass the major topics of elementary computation. Despite the potential apparent in model drawing, the effectiveness of this instructional strategy has not been investigated deeply. These claims will be detailed below (for examples of word problems worked through model drawing, see Appendix A).

**Claim 1: model drawing is a form of SBI.**

As outlined in the Literature Review, SBI is an instructional strategy that first asks students to categorize word problems by type and then draw schematic diagrams of the word problem for use in determining solution strategies (Jitendra, 2011). Students using model drawing are first asked to determine what kind of model to draw based on problem type. The student then draws the form of the model that matches the problem type or schema, labeled
with specific information from the given problem context (Yeap, 2010). Singapore's Model Method is a form of SBI.

**Claim 2: model drawing provides a systematic approach to word problem modeling.**

A second compelling reason for further investigation into model drawing is the wide versatility of Singapore's Model Method. Researchers included in this Literature Review have developed and then taught various forms of schematic diagrams for problem solving (Jitendra, 1996; Dixon, 2005; Fuchs 2010). No researcher has developed a flexible system of schematic diagramming that can model situations in the four arithmetic operations as well as rational numbers. Singapore's Model method teaches students to use the same basic shapes (rectangles) to represent all four operations and rational numbers in word problems. Furthermore, students are taught to model several different problem contexts such as part-whole relationships, comparisons and change models. These model types translate to the schematic categorization inherent in SBI (see Appendix A for examples of model drawing). Researchers in Singapore have also identified model drawing as an important pictorial representation for algebraic reasoning. It is clear that Singapore's Model Method is a fully-fledged SBI strategy broad in scope and versatile in design.

**Claim 3: Singapore's Model Method has not been thoroughly investigated.**

The first two claims establish the fact that model drawing is a form of visual instructional strategy for word problems that is well developed in terms of research on problem solving and in the completeness of its scope. To date, the actual effectiveness of this well-formed teaching method has not been adequately investigated. As outlined in the Literature Review, Singaporean researchers have investigated problem-solving behaviors connected with
model drawing. International researchers have only touched upon model drawing as they compared Singapore's entire elementary math curriculum with that of other nations. No studies have been published that investigate any changes in problem-solving performance that may or may not occur with the implementation of Singapore's Model Method.

Given these three claims, the effectiveness of model drawing must be investigated. Claims one and two outline the potential of the method, while the third claim establishes a weak knowledge base on how model drawing changes problem-solving performance. The research gaps outlined in the Argument of Advocacy add further detail to this call for investigation. In order to address these gaps, the study design should include focus on non-identified upper elementary students. Problem types should include the more challenging mathematics of upper elementary school such as multiplicative and fractional relationships. The need for understanding of these aspects of problem solving is clear, and the mandate for the research question is readily supported by research and theory.

The proposed study will provide needed insight into the effectiveness of model drawing, especially among the underrepresented upper elementary student population. The study will help fill knowledge gaps in problem-solving representation and will test the effectiveness of a systematic diagramming approach. This approach has the potential to greatly affect problem-solving instruction for elementary-aged children in the United States.
Chapter 3: Research Design

Research Perspective

This study used a single case research design, employing multiple baselines across three participants and across two problem-solving types in order to establish internal validity and experimental control. Firmly grounded in determinism, single case research designs originated in the field of experimental psychology with behavioralists such as Thorndike, Watson, B.F. Skinner and others (Kennedy, 2005). Seeking to investigate human behavior through observation of physical events, the behavioralists established the methods of Applied Behavior Analysis (ABA). The work of these behavioralists over decades has brought single case research to a level of acceptance as a common methodology in fields as diverse as education, clinical psychology, developmental disabilities, business and pediatrics (O'Neill, McDonnell, Billingsley, & Jenson, 2011).

As opposed to group designs, single case research designs follow an inductive approach, where researchers formulate general principles based on results from particular sets of results and data. In single case research, multiple experiments may be conducted over time, exploring various aspects and effects of particular interventions (O'Neill et al., 2011).

While group designs collect data from a single moment in time, single case designs rely on the principle of repeated measures to establish validity (Kennedy, 2005). Single case designs repeatedly measure behaviors of interest throughout several different phases of an experiment, thus deepening internal validity. The primary effort in a single case design is to establish the existence of experimental control, or the convincing demonstration that the dependent variable
changes as a result of the intervention, or independent variable (Kennedy, 2005). When experimental control can be established across different conditions, validity is deepened. For example, in the study described herein, if experimental control is established across two different problem types, validity of the treatment is said to be further improved. Furthermore, if experimental control is shown across several different subjects receiving treatment separately, internal and external validity is shown.

**Design Overview**

The study utilized an experimental strategy to determine the existence of variance in the dependent variable caused by the independent variable. This strategy included creating an experimental design that isolates the Model Method and its effects on student word problem performance. If experimental control could be demonstrated, it would be shown that model drawing was the cause of the existence of variance in problem-solving performance.

Following the typical structure of a single case design, the study was conducted in several phases. First, participants were screened and selected. Next, the baseline phase began. During the baseline phase, the first participant was given repeated measures of problem-solving performance consisting of pencil-and-paper assessments called probes. These assessments tested problem-solving performance across both kinds of word problems: multiplicative comparison problems and fraction problems (see Appendix D). Once a stable baseline was established for performance (i.e. percentage of problems solved correctly), this subject entered the intervention phase where the Model Method was taught to the student in a series of sessions. Each teaching session ended with a unique version of the same probes given during the baseline phase.
In the intervention phase, the participant was taught the Model Method through a scripted instructional sequence designed to teach the participant to categorize the given problem by type, and then draw an appropriate schematic diagram (model) that accurately depicts the quantities and relationships given in the problem (see Appendix B for session procedures). This model would then be used to select the appropriate computation needed to solve the problem. Again, multiple probes were given repeatedly across time to determine the level of understanding and accuracy of representation by the student. After the four sessions of instruction on the multiplicative comparison model type, the student was then moved on to four instructional sessions dealing with the second type of word problem: fraction situations. The remaining students entered the baseline phase and continued through all phases of the study individually at asynchronous times.

The maintenance phase began a week after the final intervention phase. A maintenance probe was given twice to each participant, once one week after the intervention phase, and again after three to four weeks after the end of the intervention phase. Throughout the study, collected data was entered into a line graph depicting the three distinct phases: baseline, intervention and maintenance.

**Researcher Bias and Validity**

Researcher bias is a reasonable threat to the validity of this experiment. The researcher works as the Math Curriculum Coordinator at an independent elementary school. This person has implemented Singapore's math curriculum in the school and has trained teachers and students extensively in the use of Singapore's Model Method. The researcher has seen the
method as an effective aid in solving word problems at many different grade levels with many different types of learners.

.....Given this bias, the design of this study was built around protection against several different threats to its validity. The subject selection and sampling procedures detailed in the procedures section of this chapter defended the study against much of the researcher bias depicted here. In addition, multiple defenses of validity will be employed throughout the experimental design. As mentioned previously, single case designs rely upon the principle of repeated measures to establish patterns of behavior over time, as opposed to the single-point-in-time approach of a group study. The multiple baselines of this study exist to ensure the data collected was not the result of simply one attempt at group intervention, but that the data reflects the repeated application of the treatment to each unique subject.

Validity was further protected through the use of multiple baselines across students and problem types. If functional control could be established through the data collected on one participant, validity was strengthened if a similar pattern could be established repeatedly with different students. The design also carried the baseline across two different problem types. The multiplicative comparison and fraction problem types were chosen for two reasons; first, both of these types of problems had been identified in the literature as particularly difficult for elementary aged students (Beitzel, Stally, & DuBois, 2011; de Corte & Verschaffel, 1996; Fan & Zhu, 2007; Lianghuo, 2007). Second, most research on word-problem solving has focused on addition and subtraction situations, leaving a gap in knowledge regarding the more complex situations encountered by upper elementary level students in multiplicative and fraction word
problem situations. Carrying the baseline across these two problem types also improved validity by potentially demonstrating functional control across two separate behaviors.

In general, single case designs effectively protect against several common threats to validity. History effects are controlled through the use of careful selection techniques that screen a small number of participants for experiences that may impact performance on the dependent variable (Kennedy, 2005). Diffusion of treatment effects is controlled for through the staggered entrance of each participant into the intervention phase, thereby shielding each participant from learning the model method before he or she had established a baseline for performance. Selection bias effects usually do not occur in single case designs because groups are not part of the design (O'Neill et al., 2011).

The particular design of this study protects validity against two more threats, testing effects and maturation effects. Through the use of non-identical multiple probes, the possibility of a participant learning to solve word problems more effectively simply through testing is not likely. In addition, the bulk of the study would take place over the course of two months, thereby controlling for the longer-term effects of maturation and cognitive development on word problem solving.

Replication within this study strengthened its internal validity. Replication of effects both within and across studies allowed for the establishment of generality or external validity of interventions (O'Neill et al., 2011). Thus, this study established very clear experimental procedures, including scripted instructional lessons, in order to allow the design to be more easily replicated by other researchers (see Appendix D to examine the teaching script).
Social validity is also a common consideration in single case research. Single subject researchers commonly include some assessment about how their subjects reacted to the utility or practicality of the interventions being studied. Due to the applied nature of this research, it is important to report on the experiences of the students involved. Following the social validity features of many single case researchers (A.K. Jitendra et al., 2007), the questionnaire given to students at the end of intervention in this study gave a picture of student reactions to learning model drawing and provides some measure of social validity for the study.

**Research Question**

*What is the functional relationship between the application of the Model Drawing teaching technique and the performance of regular education elementary students on multiplicative comparison and fraction word problems?*

This variance-oriented research question examined the effect of an independent variable (model drawing instruction) on a dependent variable (word problem-solving performance). This question investigated the two major constructs (word problem-solving performance and model-drawing instruction) of the study directly. The single case design demonstrated the extent to which model-drawing instruction allowed students to more successfully solve traditionally difficult word problems.

**Hypothesis**

A positive functional relationship would be shown to exist between the model-drawing treatment and percentage of word problems correct. The percentage of word problems solved correctly would increase after the application of the model drawing treatment. It was expected that the treatment for the first type of problem would result in an increase in the number of
multiplicative comparison problems solved correctly, while the number of fraction problems solved correctly would remain low. Only after introducing the treatment for fraction problems would the percentage correct of these types of problems increase. A functional relationship would be said to exist if a similar reaction to the treatment could be observed in the data from four different students taught separately at three different times. This result would be due to the enhancement of Mayer's problem representation phase through the use of the Model Method's schematic diagrams. The participants who applied models to their solution process would solve challenging multiplicative comparison and fraction word problems more easily and more accurately.

**Data**

**Descriptive Data.** Data collected on participants would show they met the requirements for participation. Age, sex, race, grade, and evidence that the child is not receiving services for any kind of learning disability would make up the bulk of the table provided on the subjects in the final report of the study. Regional demographic data describing the rural New Hampshire towns in which the participants lived would give further contextual detail to the participants and the study.

**Sampling Plan.** A sample of convenience of five subjects was to be taken from the following population: Grade 4 students living in or near a particular small rural town in New Hampshire. The small town (population: 2,000) is in the rural Northern half of the state. The town population consists of mostly lower middle-class families and small farms, with tourism and summer-only residents increasing the population during the summer months. Solicitations for subjects were conducted through the town's internet forum, subscribed to by most residents.
These students would likely have no prior knowledge of the Model Method, as no local schools, public or independent, utilized this curriculum in the area. Children who had used or seen model drawing before would be excluded from the study.

From this population, five students were be chosen for the study if they meet the following further criteria: (a) they were currently in the fourth grade (either entering or leaving), (b) they did not have any known learning disabilities, and (c) they were certified by their parents or classroom teachers to be of average ability in mathematics. In addition, each of these possible subjects needed to have permission from their parents to participate, with the parent acknowledging that the child would be able to attend all sessions. From this pool, four subjects would be randomly chosen, with at least one girl and one boy in the final selection group.

**Independent Variable Data.** Data was collected on the intervention itself. Each baseline or maintenance session consisted of the subject completing a 10-question assessment probe detailed in the instrumentation section. Eight instructional sessions, each lasting 60 minutes, was given during the intervention phase outlined above. The instructional intervention itself is depicted in the Appendices B, C and D, which detail the instructional plan for every intervention session, a teaching script, and the topics taught in every intervention session.

**Dependent Variable Data.** The percentage of word problems of two distinct types solved correctly was the dependent variable in this study. This data measured the primary construct for the study, problem-solving performance. Data consisted of percentage of correctly solved problems on 10-problem pencil-and-paper assessment probes. These assessment probes are represented in Appendix D with a representative sample.
In other single case designs, data can be collected along several different dimensions, including duration, latency and inter-response time (Kennedy, 2005). None of these data collection types fit the design of this study. The dependent variable data collected in this particular study measured frequency. More specifically, data collected on the dependent variable measured the frequency of correct solutions to two different types of word problems.

This study utilized permanent product recording as its primary measurement recording technique. Single case research can employ a wide variety of measurement recording techniques, including event recording, latency recording, or duration recording (Kennedy, 2005). The permanent product measurement technique was most germane to this study because it recorded the results of an event at the end of a period of time or after some task is completed. Permanent product recording is most often used as a way of documenting frequency as a dimensional quality (Kennedy, 2005). In this case, probes given to students at the end of each session were kept and scored as opposed to observations of the student actually solving the problem. The diagrams and written work produced (the permanent products for this study) were used to measure frequency, but because a product remains after the assessment, these probes also served as evidence of accurate model drawing, schema representation and mathematical reasoning.

These permanent products were used to gather additional data on the problem-solving performance of the subjects. First, the percentage of correctly chosen schema was collected. Although a subject may not solve a problem correctly, it was be helpful to know what percent of the time the subject chose the correct schema, which was observed through the drawings made by the student. The two schemas dealt with in this study were the part-whole problem...
type and the comparison type. The correlation between correct schema choice and correct solution was be determined from the permanent products.

The percentage of problems modeled correctly was also be gathered from each permanent product. Correctly modeled problems were defined as those that depicted the correct relationship between variables, that placed numerical labels correctly, and that placed the question mark correctly (for examples of correctly modeled problems, see Appendix A). Correctly modeled problems may not yield correct solutions, but the frequency of a correct solution accompanied by a correctly drawn model helped shed light on how the student was able to successfully solve the problem.

Finally, the percentage of correct equations was gathered as well. Research has shown that when children can deepen their representation of a problem, they are more likely to choose the correct operation to solve the problem (Mayer, 1989). The percentage of correct equations measured the subject's understanding of the mathematics required to reach a solution, and the correlation between the percentage of correct equations shed additional light on the percentage of problems solved correctly (see Appendix E to examine the data collection sheet used for every probe).

During both the baseline and maintenance phases of the study, the same data was collected using the same protocols. The only difference was that in the maintenance phase, data was collected only twice: at one week after the end of the intervention, and again at three weeks after the end of the intervention phase (see Appendix E for a sample of the data sheet to be used for every assessment probe).
**Instrument Description and Development.** Designed by the researcher, the probes used throughout the study consisted of 10 problems each. Five problems presented multiplicative comparison situations, and five problems presented fraction situations. These word problems were based on typical problem situations found in most elementary math textbooks. No extraneous information was included in the problem texts. All problems required only one or two computational steps, and the two problem types were randomly placed within the group of 10 problems in each probe.

Although it would have been preferable to create an assessment with more than 10 items, completing even this task took between 20 and 40 minutes for a typical student, which elongates the training session and taxes the participant. As with most teaching situations, it was important to keep implementation sessions to a reasonable length of time in order to maintain participant interest and enthusiasm for the intervention.

The multiplicative comparison situations carried Kintsch & Greeno's categorization of addition and subtraction situations (Kintsch & Greeno, 1985) into the area of multiplication. Three situations were depicted in these multiplicative comparison problems. In the first situation, the sum of the two compared variables was missing. In the second situation, the value of one of the two compared variables was missing, and in the third situation, the difference between the two compared variables was missing. Each probe will contained at least one of these three situations within the five questions allotted to multiplicative comparisons (see Appendix D for a sample assessment probe).

Like the multiplicative comparison situations in each probe, the fraction situations depicted consisted of two types: (a) finding the fraction of a whole number, and (b) comparison
problems involving fractions. In the fraction of a whole number situation, the solver was asked to find the value of fractional parts given the value of the whole quantity. In the second situation, the solver was asked to compare one variable to another through fractional relationships. From that comparison, the solver would have to find the value of one of the variables or the sum of the two variables. In the assessment probe, both fraction situations were represented at least twice out of the five fraction problems offered (see Appendix D).

**Instrument Validity and Reliability.** The probes used in the experiment had been used in classroom situations successfully with groups of third through fifth grade students who had already been taught the model method. In this way, the difficulty of questions, balance of situations, and readability were all validated.

The instruments used to gather data are fair, reasonable, and delivered in ways that used replication as a means to deepen validity. The study was conducted with students that the researcher had never met, coming from educational situations where the Model Method had never been seen. In addition, the study was conducted in a state other than that where the researcher works, thus guarding against any conflicts between the researcher's role as a professional and as a researcher.

Reliability of the instrumentation was enhanced in several ways. The probes were scored twice, once by the researcher and again by one other educator. In order to further protect against bias, the educator did not know the full purpose of the probes, in order to further protect against bias. Inter-rater reliability for both percentage correct and error analysis was established. Inter-rater reliability, a common method in single case research for enhancing reliability of instruments, is measured in this instance by taking the total of the cases of correct
problem-solution agreement between the researcher and the other educator, and then dividing that number by the total number of problems checked. This percentage measures the degree to which two separate scorers agreed upon the scoring on the instrument (Kennedy, 2005). Social validity was determined by a simple five-question survey given at the end of the intervention phase. The questions employed a 5-point Likert scale to measure attitude towards solving word problems and the usefulness of the model method in solving word problems (see Appendix F for the social validity instrument). In the interests of enhancing validity, the data collection structures outlined here closely followed the single-case designs employed by Asha Jitendra in her studies of schema-based instruction (A.K. Jitendra et al., 2007; A.K. Jitendra & Star, 2011; A. Jitendra & Star, 2009).

**Procedures**

**Participant Selection Phase Data Collection Procedures.** Dependent variable data collection occurred through the sampling process, used to determine eligibility for the study. The data was collected through parent questionnaires given over the phone or in person. Additional information about the math background of each participant was collected via the parent questionnaire (see Appendix G for the parent questionnaire).

**Baseline Phase Data Collection Procedures.** During the baseline phase, dependent variable data was collected through a series of probes detailed previously. A session consisted simply of the subject working through the assessment probe. At each testing session, the probe was administered to the subject for 30 minutes. During that time, no help was given to the subject. Calculator use was allowed to control dependent variable results for computational errors (see Appendix B for further detail on session procedures). A minimum of
three data points is required to establish pattern and stability during the baseline phase (Kennedy, 2005).

**Intervention Phase Data Collection Procedures.** Intervention sessions were one hour in duration, consisting of a 30 minute lesson in model drawing, followed by a 30 minute assessment period where the subject completed another probe. Data was collected through scoring each probe and recording the percentage of problems solved correctly. Eight sessions were conducted with each student in the intervention phase.

The instructional portion of each session consisted of a scripted lesson on how to draw the models for the given problem type (For examples of worked model drawing word problems, see Appendix A). The first activity of the lesson was an instructor demonstration of how to solve a typical problem using model drawing. Next, a series of at least 3 problems were solved together between the instructor and the subject. Last, the subject was asked to solve problems using models independently. This "I do, we do, you do" pattern was applied during every instructional session. The problems chosen for instructional sessions were stories of varied context, but the mathematics applied remained the same. Variations of variables within the problem type were mixed as described above. (See Appendix B for a more detailed description of the instructional steps followed in every intervention session. Appendix C depicts the teaching script used in each session.)

**Maintenance Phase Data Collection Procedures.** Maintenance probes were given one week after intervention and again three weeks after the intervention phase has ended. The probes were of the same makeup as the ones used during the baseline and intervention phases. No instruction was given, so each session lasted approximately 30
minutes. The social validity instrument was administered during the one-week maintenance session (see Appendix B).

**Data Handling and Storage Procedures.** Data was entered into a computer spreadsheet application as soon as it was collected and scored. Scoring to determine inter-rater reliability was entered into the spreadsheet at end of the intervention phase. Graphs and charts were generated using protocols recommended for single case research (O'Neil et al., 2011).

**Data Analysis and Presentation Procedures.** The primary mode of analysis in single case research is through graphic analysis (O'Neil et al., 2011). Creation and visual analysis of line graphs are common in single case research designs because graphs communicate the overall experimental design, the sequence of experimental conditions or phases, identify the dependent and independent variables, and provide a compact and detailed picture of the relationships between dependent and independent variables. In addition, graphic data analysis allows for independent evaluation of data by individuals not involved in the study (O'Neil et al., 2011).

Analysis of the data from this study was applied both within phases and across phases. The primary visual analysis tools used within phases of single case research are level, variability, trend and stability (O'Neil et al., 2011). Across phase analysis tools include immediacy of effect, percentage of non-overlapping data, and similarity of data (O'Neil et al., 2011).

Visual analysis for level looks at the average of the data within the condition (baseline, intervention, maintenance) and is usually calculated as the mean or median (Kennedy, 2005). Particular attention is to be paid to the last data points before a phase change. Visual analysis of
trend refers to a best-fit line that can be placed over the data during a phase. Both the slope (positive or negative) of the best-fit line and the magnitude (extent of the slope) were considered in this study. Connected to trend, visual analysis for variability relates to the degree to which individual data points differ from the trend line. Both trend and variability are most often described using qualitative terminology of high, medium or low (Kennedy, 2005). All of these tools are combined in a statement of stability for a data line within a phase. For each subject in this study, observations were recorded for the trend, magnitude and variability of the data within each phase.

The single case researcher is most concerned with the functional relationship between the dependent and independent variables. Of crucial importance is the change (or lack thereof) in the dependent variable between the baseline and intervention phases. Different visual analysis tools are applied to these changes between phases. *Immediacy of effect* refers to the rapidity of change between phases, considering any changes in level or trend in the data line. *Percentage of non-overlapping data* (PND) considers the ratio of data points in a phase that do not similar quantitative values with an adjacent phase (Kennedy, 2005). PND in this study was calculated by deriving the ratio of problems solved correctly above the highest data point in the previous phase. This ratio was expressed as a percentage. A high PND would be evidence of experimental control, as this number measures the degree of change between two phases. In addition, the similarities in stability across participants were used to demonstrate the validity of experimental control. Validity of the findings would be considered to be improved if it could be shown that the changes between phases for one participant were similar to those found in the other participants (O'Neill et al., 2011). In this study, any changes in performance (percentage
of word problems solved correctly) were be noted. Of particular interest were any observable differences between the baseline and intervention phases. The immediacy of the effect was determined, as well as the percentage of overlapping data.

Social validity data was tabulated and averaged to indicate overall reactions to the intervention from participants. In addition, any changes in attitude towards problem solving from the baseline phase to the end of the intervention condition was noted.

**Schedule.** The entire study took eight weeks. Within this period, the following steps or phases were completed.

1. Sampling and participant enlistment - 2 weeks in June, 2012
2. Baseline phase - 1 week per student, three sessions in June, 2012
3. Intervention - 4 weeks per student, 8 sessions in July, 2012
4. Maintenance phase - 2 sessions per student in July and August, 2012

**Ethics and Human Subjects Considerations**

By its very nature, this study involved human subjects, and ethical considerations were made for each of the participants. The study was completely voluntary, and participants were allowed to withdraw at any time without reason. Data collected from participants, their families and their schools was treated as completely confidential. Names and other personal data were altered or concealed to protect the confidentiality of each participant and his or her family, and were not shared through publication.

Participation in this study may have given several benefits to students. Regardless of the outcomes of the intervention, the time and effort spent on practicing problem solving most likely proved beneficial to both math concept and skill acquisition. The intervention did not
interfere in any way with the participant's current classroom instruction, and potentially served to broaden the problem-solving perspective of each student. It is possible that students left the study with an improved interest in and attitude towards problem solving in mathematics.

Researcher bias was a distinct threat to the validity of the study, as the researcher conducted all teaching and assessment sessions. Efforts such as repeated measures, inter-rater reliability, and visual analysis tools helped to combat the possible effects of researcher bias on the results of the study.
Chapter 4: Results

This chapter presents the data generated by the experimental design. Four children, aged 9-10 years old, were chosen to participate in the study. Three boys and one girl met all requirements for participation and completed all three phases of the study. The names of the participants have been changed to protect confidentiality. Each participant's data will be presented and treated with visual analysis within phases for level, variability, trend and general stability. Each participant's data will also be analyzed across phases for immediacy of change and percentage of non-overlapping data (PND). Visual analysis of the problem type data will also be conducted in this way. Through this analysis, a case demonstrating the existence of experimental will be established.

Data Across Participants

Jeff (participant 1).

Jeff, aged 9, was entering grade 4 at his local public school in the fall. The within phase results for Jeff begin with the baseline phase. Jeff's baseline level is 3.3% problems solved correctly, derived from Jeff's 0% performance on the first two probes coupled with one correct
answer on the final baseline probe. The final probe of the phase indicates a slight increase from the first two probes, but there is not enough further data to indicate the trend of the line would continue upwards independently of the intervention protocol.

Within the intervention phase, significant variability can be observed. The level increases from the baseline of 3% problems solved correctly to an intervention phase level of 55% problems solved correctly. A best-fit line for the intervention phase would show an almost linear progression in problems solved correctly with the exception of two data points, which
were not far from the best fit line. Although variability is high because of the significant increase in scores from the beginning of the intervention phase to the end, the trend is quite positive and reflects the incremental nature of the intervention phase instruction. The score for probe 10 (the 7th intervention session out of 8) altered the linear pattern slightly, but then the probe 11 score of 100% correct brought the line back to its positive trend. Taken together, the variability and trend characteristics show that the data for Jeff's intervention phase has a high degree of stability.

Jeff's maintenance data line is more variable, but remains within the 70% correct range. The level for this phase was 85% higher than that of the intervention phase level. Of note is the positive trend of the two data points within the maintenance phase. One would expect the maintenance phase data to remain level or perhaps decrease as the student spends more time away from the work of the intervention instruction.

Along with the data within phases, the data between phases is equally important to the single subject study (Kennedy, 2005). The interval between the baseline and intervention phases is the most significant for this study. In the case of Jeff's data, the immediacy of effect is readily apparent in the graph (figure 4.1). As soon as the intervention phase begins, Jeff's percentage of problems solved correctly increases at a consistent rate. The PND comparing these first two phases is 100%, showing that as soon as the intervention phase began, the dependent variable changed and did not return to the baseline level. The high PND underlines the significant differences between the baseline and intervention phases.

Between Jeff's intervention and maintenance phases, The PND moves to 0%, indicating the decrease in performance during the maintenance phase compared with Jeff's 100% score on
the final assessment probe of the intervention phase. The level dropped to the previous level within the intervention phase before the high score. The effect was immediate as shown by the drop in both maintenance scores. Overall, the data graphed over three phases of the study show a stable baseline phase followed by the dramatic increase in the dependent variable that began with the onset of the intervention phase. The maintenance phase was marked by a slight decrease in the dependent variable. Jeff’s performance in solving difficult word problems increased significantly after receiving the intervention.

**Harvey (participant 2).**

Having been homeschooled for his entire childhood, Harvey (age 9), had a basic understanding of multiplication and division, but lacked the understanding of and facility with algorithms and fraction concepts typical of a student his age educated in a public school setting. The baseline phase for Harvey was level, stable and lacked any variation. The level remained at 0% problems solved correctly throughout the phase, thus the data does not show any variation and is perfectly stable through the condition. The child could not answer a single problem correctly in any of the three baseline probes. Due to the complete lack of variation and the consistent stability, the trend line indicates that this condition would have continued unless an intervention of some kind occurred.
The intervention phase for Harvey shows a remarkably different set of data than the baseline phase. Beginning with the fourth probe, where the first intervention lesson was taught, the number of problems solved correctly by Harvey rises. In contrast to the baseline phase (level 0% solved correctly), the level for Harvey's intervention phase was 61% problems answered correctly. Rising from one problem solved correctly to eight problems solved correctly by the conclusion of the intervention phase, the best fit line for this phase displays an

*Figure 4.2. Participant 2 (Harvey) Assessment Probe Results*
almost linear path, with very low variability and a decidedly positive trend. The stability of the line during the intervention phase is thus considered to be high (Kennedy, 2005).

The maintenance phase for Harvey is characterized by a negative trend with medium variability. The level stabilized at 70% problems answered correctly, higher than the level of the previous phase. The trend line is generally negative, although at a much less aggressive slope than the positive growth observable in the intervention phase. The overall maintenance phase results indicate that skills were generally maintained, but that some loss in performance did occur with the passage of time. In general, within-phase data for Harvey show dramatic differences within the intervention phase, with much less variability within both the baseline and maintenance phases.

The interval between the baseline and intervention phases is most crucial for determining experimental control. The interval between the intervention and maintenance phase will determine the extent to which skills and concepts taught during the intervention phase were retained over time (O'Neill et al, 2011).

Harvey's data line shows an immediate effect directly after the onset of the intervention. The percentage of problems that Harvey solved correctly rose immediately after the first intervention session. As discussed previously, the trend line increased with much regularity after the initial session, with the participant able to solve one more problem correctly with almost every session. The effect shown by this data is definitely immediate. The growth in the number of problems solved correctly was consistent and incremental, it was not of great magnitude immediately. This trend line reflects the nature of the intervention whereby a new type of word problem was introduced to a participant with every intervention session, thus it
logically follows that Harvey was not able to solve all types of word problems after the first intervention session, but performance increased as the intervention sessions accumulated.

The second form of visual analysis considered for Harvey is the PND. 100% of the data points in the intervention phase were above those of the baseline phase. In other words, the probe scores for all of Harvey's intervention sessions showed more problems answered correctly than any of the baseline scores, whose level was 0% answered correctly. A PND of 100% is considered quite high in single subject research (O'Neill et al., 2011).

Between the intervention and maintenance phases, there is a clearly different immediacy of effect. Bridging from the intervention phase to the maintenance phase, Harvey's trend line remains stable at 80% solved correctly, and then decreases slightly to 70% solved correctly one month after the intervention phase ended. This transition across phases produced a PND of 0%, indicating no increases in performance after the intervention phase. Taken together, Harvey's performance in solving challenging word problems increased significantly through the intervention phase and into the maintenance phase of the study.

Amy (participant 3).

Amy was aged 9 at the time of the study, having just completed grade 3. The baseline data for Amy, presented in figure 4.3, shows a level of zero problems solved correctly for the phase. The data line here has no variability and high stability. It is likely that Amy's baseline level would have remained at zero given more probes of the same nature.

The intervention phase for Amy showed significant change in the dependent variable. Amy's probe scores rose from 0% problems solved correctly in the first intervention phase probe to 80% problems solved correctly on the final intervention phase probe. In contrast to
Amy's baseline level of 0% problems solved correctly, Amy's intervention phase level rose to 37.5% problems solved correctly. A best-fit line for Amy's intervention performance data does not quite follow a linear pattern. For the first three probe scores, a decidedly positive trend can easily be seen. The next two scores remain at 30% correct. After the fifth intervention session, the scores again rise sharply, taking on a similar trend line to the first three session scores. Thus, Amy's data shows some variability and instability. The trend of the data line remains distinctly positive for the phase, although it is clear that Amy's acquisition and mastery of

Figure 4.3. Participant 3 (Amy) Assessment Probe Results
model drawing was not complete. Perhaps some of this variability can be attributed to the fatigue noted in Amy during her late afternoon sessions, but there is not enough information to make a definitive statement on the cause of the instability of Amy's intervention phase data.

Amy's performance data during the maintenance phase shows far more stability. Amy answered six problems correctly on both probes, giving the data a level of 60%, significantly higher than the level of her intervention phase data. The trend of the data is neutral with linear regularity. As a whole, the data line for this phase of Amy's performance during the study is highly stable.

Amy's data is interesting when between-phase analysis is applied. The interval between baseline and intervention is of note because the trend of the baseline continued for one more probe into the intervention phase. The immediacy of effect was not high. In other words, the first intervention session did not appear to improve Amy's performance, as it did with all other participants. The sharp slope of the next three sessions indicates that the interventions were perhaps still having an effect, although it was more cumulative, resulting in sharp rises in the dependent variable twice during the intervention phase. The PND for the two phases is 87.5%, still considered to be high for the measure (Kennedy, 2005), and suggesting that the dependent variable was increased through the intervention.

Between the phases of intervention and maintenance, the PND is 0%, demonstrating that no new skill was developed over the intervening time without the intervention. The effect was immediate, with the percentage of problems solved correctly dropping to 60% with the first maintenance probe. Amy's best performance certainly was not maintained completely after the intervention phase ended, but the data suggests that skills were maintained when compared
to the baseline. Amy's performance in solving challenging word problems improved significantly, though with some variability, through the course of the intervention and into the maintenance phase of the study.

Bruce (participant 4).

Bruce, a hard working 10 year-old who had just completed grade 4, was initially accepted into the study, as he appeared to have met all criteria for participation. After the third session of the study, Bruce's mother disclosed that although Bruce did not receive support for learning difficulties of any kind during the past school year, he had received support in math in second and third grades. Bruce had difficulty in "not getting lost and confused while doing all of the steps in math problems" according to his mother. Since Bruce was not currently receiving support in math, the researcher decided to retain Bruce in the study and observe the results of the intervention.

Bruce's baseline data, represented graphically in figure 4.4, shows a level of 0% problems solved correctly. The slope is neutral, and the variability of the line is quite low. It is reasonable to assume that the trend would continue without some sort of intervention. Bruce did attempt many of the problems presented in the baseline probes, but his solutions usually consisted of multiplying the numbers presented in the problem without regard for the numerical relationships detailed in the text.

In contrast to Bruce's baseline data, the intervention phase shows significant variation throughout the phase. The first intervention probe continued Bruce's performance trend of zero problems solved correctly. By the end of the phase, Bruce was able to solve 80% of problems correctly. The 0% level demonstrated in Bruce's baseline rose to 42.5% problems solved.
correctly during the intervention phase. The trend for the intervention phase is decidedly positive overall, with a degree of variability through the middle probe scores in the phase. By the end of the phase, an upward trend was again established, similar to the variability of Amy's data in this phase. The data during this phase suggests that like Amy, Bruce's acquisition and mastery of model drawing was not complete.

*Figure 4.4. Participant 4 (Bruce) Assessment Probe Results*
Within the maintenance phase, a drop in performance occurred for Bruce. The drop was not significant, and the second probe score, after a greater period of time away from the intervention, actually represented a return to the 80% level of the final intervention score. The level of Bruce's maintenance performance was 75% problems solved correctly, higher than Bruce's intervention phase level.

Bruce's performance across phases must now be investigated. Between the baseline and intervention phases, Bruce continued the 0% performance through the transition. In other words, the first intervention appeared to have no immediate effect. The intervention effect is noted only after the second intervention session, when Bruce solved his first problem correctly, and then Bruce's probe scores jumped 40% across two consecutive probes. The data line shows a lower immediacy of effect for Bruce than for other participants, although the slope of the best-fit (trend) line is more sharply positive. The PND across the two phases is 87.5%, taking into account the first intervention phase score of zero problems solved correctly. This level of PND is still considered significant for purposes of this study (O'Neill et al., 2011).

Across the intervention and maintenance phases, Bruce's problem-solving scores decreased and then rebounded, producing a PND of 50%. The effect was immediate, reflecting the performance losses associated with time spent away from the practice of problem solving. The degree of effect was similar to all other participants, suggesting a small loss of performance while still maintaining much of the skills acquired during the intervention. Overall, Bruce's data shows that the baseline performance increased significantly by the intervention, and that the gains made during the intervention phase were maintained with some loss over time.
Table 4.1 *Summary Visual Analysis Data for All Participants*

<table>
<thead>
<tr>
<th>Participant</th>
<th>Final Intervention Probe Score</th>
<th>Intervention level (mean)</th>
<th>Intervention Stability</th>
<th>Baseline to Intervention PND</th>
<th>Baseline to Intervention Immediacy of Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jeff</td>
<td>100%</td>
<td>55%</td>
<td>High</td>
<td>100%</td>
<td>High</td>
</tr>
<tr>
<td>Harvey</td>
<td>80%</td>
<td>61%</td>
<td>High</td>
<td>100%</td>
<td>High</td>
</tr>
<tr>
<td>Amy</td>
<td>80%</td>
<td>37.5%</td>
<td>Medium</td>
<td>87.5%</td>
<td>Medium</td>
</tr>
<tr>
<td>Bruce</td>
<td>80%</td>
<td>42.5%</td>
<td>Medium</td>
<td>87.5%</td>
<td>Medium</td>
</tr>
</tbody>
</table>

*Note.* The table summarizes the most salient visual analysis for the four participants.

**Group Data**

**Percentage of Non-overlapping Data (PND).**

Taking all four participants as a group, the PND between the baseline and intervention phases was 93.75%. This measure indicates a very consistent change across the entire group between baseline and intervention phases. The mean percentage correct for the final intervention probe was 85%, indicating a dramatic increase in percentage of problems solved correctly when compared to the final baseline mean score of .25% problems solved correctly. This comparison underscores the PND findings for the group, corroborating the high degree of change visible during the intervention phase.

**Problem Type Analysis.**

In order to provide further evidence of experimental control in the study, data was collected across two fundamental problem types, multiplicative comparison problems and fraction problems (see Appendix A for examples). As detailed in Chapter 3, various levels of complexity were designed within each problem type as well. Single subject research designs
depend upon the concept of repeated measures to demonstrate that results are repeated through the design (Kennedy, 2005). In the case of this study, data collected across two problem types was examined to determine if similar action in the dependent variable was replicated across problem types and participants.

Jeff’s case (see Figure 4.5) shows a repeated pattern across problem type. Over the first 7 assessment probes, the only problem type taught to all participants was model drawing for the multiplicative comparison type (MC). The number of problems of this type that Jeff solved correctly rises as the number of intervention session increases, peaks with the last intervention
session that focused on multiplicative comparison problems, and then remains relatively stable for the remainder of the intervention and maintenance phases. The data tells a variation of the same story with fraction problems. The number of fraction problems solved correctly remains at zero through the baseline phase, just like with the MC problem type. Unlike the MC problem type, the number of fraction problems solved correctly remains at zero throughout the first part of the intervention, where instruction focused only on drawing models for the MC problem type. The number of fraction problems solved correctly only rises at assessment probe eight, where the first instruction on model drawing for fraction problems occurred. From this point on, the number of fraction problems solved correctly rises quickly until it stabilizes after the four intervention sessions dedicated to drawing models for fraction problems. The pattern for increase in problems solved correctly repeats across two different problem types.

Harvey's data in Figure 4.6 shows a very similar repeated pattern across problem type. First, the number of MC problems increases with the beginning of the teaching intervention. Just like with Jeff's data, the number of fraction problems solved correctly remains at zero until assessment probe eight, when the first instruction on model drawing for fractions occurred. After this point, the number of fraction problems solved correctly rises quickly until it stabilizes at the final intervention phase assessment probe.
Again, this pattern is repeated within Amy's data, although as detailed in Figure 4.7, Amy's performance was somewhat more variable than the first two participants. Nonetheless, similarities surrounding the data pattern between problem types are quite evident. Amy's performance data rises for MC problems at the outset of the intervention phase. Just like the other participants, performance on fraction problems remains at zero until assessment probe eight, where the number of fraction problems she solved correctly climbs steeply, and with greater variation, to a more stable higher level by the end of the intervention phase.

Figure 4.6. Participant 2 (Harvey) Problem Type Comparison Data
Bruce's data (Figure 4.8) is the fourth example of the same pattern. Bruce could not solve a single fraction problem correctly for the first seven probes. Only after the instruction session on drawing models for fractions did the number of correctly solved fraction problems rise. Like Amy, Bruce's performance on both types of problems showed greater variation than the first two participants, but the sharp increase followed by some stability is still evident.

Bruce's fraction performance data shows the greatest variability among the four participants, but rises to its highest level only at the final maintenance probe. At minimum, this data line shows that Bruce had more difficulty with fraction problems than the other three participants.

Figure 4.7. Participant 3 (Amy) Problem Type Comparison Data

Note: The graph plots the total percentage of problems solved correctly for each problem type within the given probe. Because there were only 5 MC problems on each probe, the maximum score possible for MC problems was 50% of the total assessment probe problems solved correctly.
When considered together, a similar pattern emerges for all four participants. For both MC problems and fractions, performance began to change with the application of the model drawing strategy. This repeated measure indicates the presence of experimental control in the study.

**Additional Group Data From Permanent Products.**

As detailed in Chapter 3, additional data was collected from participants to provide further detail around the essential constructs of the study. Although not a part of typical single case research, the data collected from the permanent products was not treated using the visual...
analysis techniques detailed in this study but was analyzed using simple group statistical analysis. The connections between schema choice and correct answers will be explored. In addition, the connections between correctly drawn models and correct equations will be exposed. This data will be analyzed from a group perspective.

From Chapter 2, it is clear that schematic diagrams are effective tools for mathematical problem solving (Jitendra, 2011). Through the intervention teaching sequence, participants were taught to choose between a *part-whole* schema or a *comparison* schema. Both schema were represented in the problem sets of each assessment probe, with the majority of problems being of the more complex comparison schema. As a group, when participants chose the correct schema for the problem, they generated a correct model for the problem 83.75% of the time.

Correctly drawn models were also measured and recorded in order to shed more light on the confirmation of the theoretical model introduced in Chapter 1. In general, once the intervention phase began, participants did not attempt to solve a problem without using a model. Every participant attempted to use a model to solve a problem 100% of the time after the intervention began. As a group, when participants constructed a correct model, they generated a correct equation 92.7% of the time. Correct equations led to correct problem solutions quite frequently, most likely due to the availability of calculators for computation. Had these children been asked to work all computation by hand, there would likely have been more errors in computation. When participants constructed a correct model, they reached a correct solution 95.7% of the time.
Social Validity Data.

Table 4.2 Social Validity Survey Results

<table>
<thead>
<tr>
<th>Statement</th>
<th>Jeff</th>
<th>Harvey</th>
<th>Amy</th>
<th>Bruce</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>I like using model drawing.</td>
<td>5</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>4.5</td>
</tr>
<tr>
<td>I enjoyed learning how to draw models for word problems.</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4.5</td>
</tr>
<tr>
<td>I think other children should be taught model drawing.</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>4.5</td>
</tr>
<tr>
<td>This is how difficult it was for me to learn how to draw models for word problems.*</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3.75</td>
</tr>
<tr>
<td>Drawing models for word problems helps you solve the problem.</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Note. * A Likert scale of 1 to 5 was used for each question with 1 = very difficult and 5 = very easy for this question and 5 = strongly agree, 1 = strongly disagree for each of the other questions.

Single subject researchers often include the perceptions of their subjects as part of their research to improve the social validity of the study (Jitendra, A., 2002, Jitendra A.K., 2011; Jitendra & Star, 2009). In this study, it is evident the participants had strong opinions about each of the questions presented to them. Those questions and the responses are identified in Table 4.2.

In general, the social validity demonstrates that solving word problems with model drawing was a positive experience for the participants. The first question shows no negative views on solving word problems with model drawing. The second and fourth questions examine the participants' perceptions of learning how to draw models. Though enjoyable, the participants showed that learning the technique was not "very easy" for any of them. Of note is the final question, which produced complete consensus among participants. The unanimous perception among participants was that model drawing was useful to them in solving word problems.
Findings: Research Question

What is the functional relationship between the application of the Model Drawing teaching technique and the performance of regular education elementary students on multiplicative comparison and fraction word problems?

This variance-oriented research question examines the effect of an independent variable (model drawing instruction) on a dependent variable (word problem-solving performance). The results of this study show that the independent variable brought about a positive change in the dependent variable. In more specific terms, model drawing instruction brings about an increase in the percentage of multiplicative comparison and fraction word problems solved correctly. Utilizing the single subject analysis concept of repeated measures, the content of this chapter focuses on how the study results prove the existence of experimental control within the study design.

Hypothesis

The hypothesis stated in Chapter 3 suggests that a positive functional relationship would be found between the percentage of problems solved correctly and the application of the model drawing treatment. The data proves that within this study a positive functional relationship does exist between the model-drawing treatment and percentage of word problems correct. The percentage of word problems solved correctly increased only after the application of the model drawing treatment.

This effect is demonstrated in the data generated by Jeff (see figure 4.1). The percentage of problems solved correctly during the baseline phase was very close to zero for the entire phase. Only after the treatment began did the percentage of multiplicative comparison (MC)
problems solved correctly begin to increase rapidly. The immediacy of effect across these two phases is high, and the PND between these two phases was 100%. These characteristics demonstrate that the functional relationship between the dependent and independent variables, establishing experimental control. The increase continued through the intervention phase as Jeff learned more and more about how to draw schematic diagrams and apply them to different problem variations within the MC problem type.

In contrast to Jeff's MC problems solved correctly, the percentage of fraction problems that Jeff solved correctly remained at zero past the baseline phase and halfway into the intervention phase (figure 4.5). This percentage of fraction problems solved correctly increased as soon as Jeff was taught how to model fraction problems, providing corroboration of the existence of experimental control. Again, the immediacy of effect was high, and the PND was 100% for this type of problem. Just like the data for MC problems, the dependent variable did not change until the treatment was applied. Jeff could not solve the challenging fraction word problems presented until he learned how to draw schematic diagrams for these problems. The similarity of the trend line for MC problems when compared to fraction problems indicates that the intervention had a similar effect on Jeff's problem-solving performance under both conditions. Thus, the functional relationship between the dependent and independent variables was corroborated by the fraction problem data.

The stability of the trend line for Jeff also indicates that Jeff learned model drawing incrementally, not all at once. The vast majority (87.5%) of intervention phase teaching sessions resulted in an increase in the number of problems Jeff solved correctly. Jeff's accuracy
in modeling different MC and fraction situations improved with the instruction and practice woven into the intervention protocol.

Jeff's maintenance phase data shows that he was able to retain the new skills he learned through the intervention phase. Although his data indicates a minor drop in performance, Jeff was certainly better able to solve challenging word problems when his maintenance phase data (70% level) is compared with his performance during the baseline phase (0.3% level). Taken as a whole, the data generated by Jeff indicate that the hypothesis put forth in Chapter 3 was validated. Visual analysis of the data both within and across phases consistently demonstrates experimental control. Jeff learned how to solve complex word problems much more effectively through learning to use Singapore's Model Method.

The same functional relationship between dependent and independent variables can be seen in the case of Harvey's problem-solving performance (see figure 4.2). The percentage of problems that Harvey solved correctly during the baseline phase was 0% for the entire phase. Harvey could not solve any of the challenging word problems presented to him. His work during the baseline phase showed frequent attempts to multiply the numbers presented together, demonstrating a limited understanding of the mathematical relationships involved in these more complex problems. Only after the treatment began did Harvey's percentage of multiplicative comparison problems solved correctly begin to increase. The immediacy of effect across these two phases was high, and the PND between these two phases was 100%. These characteristics across phases indicate the presence of experimental control. As was the case with Jeff, Harvey's increase continued through the intervention phase as he learned more and more about how to draw schematic diagrams and apply them to different problem variations within the MC
problem type. Harvey's data line was slightly more variable than that of Jeff, but the trend line is strongly positive and demonstrates an obvious change in problem-solving performance across the first two phases.

Looking at Harvey's performance across problem type, we can see a pattern quite similar to that of Jeff. Harvey's performance on the MC problem type began to change only once the intervention phase began (see figure 4.6). The percentage of MC problems that Harvey solved correctly then increased incrementally after that time. The percentage of fraction problems that Harvey solved correctly began at 0%, just like his performance on MC problems. As was the case with Jeff, Harvey's fraction problem performance remained at 0% across the beginning of the intervention phase, right up until session 7, at which point Harvey was introduced to models for solving fraction problems. At that point, Harvey's fraction problem-solving performance rose steadily. This repeated pattern across both problem types is further evidence of the functional relationship between model drawing instruction and the percentage of problems solved correctly. The repetition of this pattern across two different participants taught at separate times underscores the existence of experimental control.

Harvey's maintenance phase data demonstrates similar stability to that of Jeff, with a minor decrease in performance over time. Again, the maintenance phase level of 75% of problems solved correctly is clearly a major change from the baseline performance level of 0% of problems solved correctly. The improvement across these two phases is further evidence of the existence of the intervention bringing about a change in the number of problems solved correctly by Harvey. When considered together, Harvey's problem-solving performance
demonstrates a second and independent case of experimental control, corroborating the findings generated by Jeff's data.

The data generated by Amy's problem-solving performance further corroborates the case for experimental control (see figure 4.3). Like the first two participants, Amy could not solve any of the challenging word problems presented to her in the baseline phase. In the problems that Amy attempted to solve during this phase, she was most likely to multiply two quantities together and present it as a solution without further consideration, demonstrating, like Harvey, her lack of understanding of the actual relationships represented in the problems presented to her. Amy's problem-solving performance did not change until the intervention began, and then the number of problems she could solve successfully climbed throughout the course of the intervention.

Amy's data line shows a distinctly positive trend, although the trend is marked by less stability than the first two participants. Amy's intervention sessions were scheduled according to her working parent's schedule, and occasionally began as late as 5:00 pm, a time of day when Amy appeared quite fatigued after a full day of summer activities. While fatigue was likely a factor that affected Amy's ability to focus and persevere through challenging math work, there is not enough data to determine the extent of the effect or even its definite existence. Although Amy's data line demonstrated greater variability than that of the first two participants, the same general patterns exist in trend. It is still evident that the treatment produced a dramatic increase in problem-solving performance.

Across phases, Amy's data provides even further evidence of experimental control. Between baseline and intervention phases, Amy's 87.5% PND is quite high, again suggesting
that the intervention brought about a significant change in problem-solving performance. The maintenance phase data from Amy's performance is relatively stable in comparison with the other participants, suggesting a slower pace of learning, but perhaps leading to greater retention of skills. The change recorded from the baseline to the maintenance phase (0% level to 75% level) underscores the learning that occurred for Amy during the intervention.

In Amy's case, data patterns across problem types also repeated in similar ways to those of the other participants. Just like Jeff and Harvey, Amy's percentage of MC problems solved correctly began to rise almost immediately after the intervention for that problem type began (medium immediacy of effect), and then stayed relatively high for the remainder of the experiment (figure 4.7). In contrast, Amy's performance on fraction problems remained at 0% correct until the intervention introduced models for fractions. Just like the other two participants, Amy's performance on fraction problems only changed when instruction for fraction model drawing was introduced. The data patterns for all three participants across problem types are quite similar, providing further corroboration of the existence of experimental control.

The data generated from Bruce's work with problem solving provides a fourth case that is again similar to those of the first three participants (see figure 4.4). This similarity is further evidence for the existence of experimental control and the presence of a functional relationship within the data, where the intervention of model drawing instruction brought about a marked improvement in Bruce's ability so solve MC and fraction problems successfully.

Like all the other participants, Bruce demonstrated no facility with solving these challenging word problems during the baseline phase. With the same baseline level of Amy and
Harvey of 0% problems solved correctly, it is safe to say that Bruce could not solve either type of problem successfully. Bruce, like Harvey, demonstrated observable discomfort at his inability to solve any of the word problems during the baseline phase, but exhibited a similar pride in accomplishment by the end of the study.

Just like all the other participants, Bruce's problem solving improved only after the introduction of the intervention, where MC models were presented. The same one-session delay in performance occurred in Bruce's data line as in Amy's data line. Bruce's percentage of correctly solved problems was somewhat more variable than that of Jeff or Harvey, but was closer to the variability present in Amy's data line (see figure 4.4). The immediacy of effect across these two phases was medium, but the change in level was still large, changing from 0% for the entire baseline level to 41% for the entire intervention level. The PND of 87.5% takes into account Bruce's 0% performance on the first intervention phase assessment probe, and provides more pointed evidence of change in problem-solving performance brought about by the intervention of model drawing instruction. The trend line for Bruce's performance data shows the same steady positive growth as the other three participants.

The maintenance phase data for Bruce is also similar to that of the other three participants. Bruce's stability within the maintenance phase was not as high as the others, scoring a 50% and then rising to an 80% in the final maintenance phase assessment probe. It is not clear why this variability exists. Perhaps data taken from more maintenance assessment probes would help to clarify the trends here. Despite the lower level of the maintenance data, the change from baseline level (0%) to maintenance level (65%) still provides solid evidence of
the functional relationship between model drawing instruction and word problem-solving performance.

Further evidence for experimental control is provided by Bruce's performance across problem types (figure 4.8). Following the same pattern as all of the other participants, the percentage of MC problems Bruce could solve correctly did not improve until after the intervention began. At the same time, performance on fraction problems remained at 0% until the models fractions were taught. At that point, performance on fraction problems rose in a pattern similar to that for MC problems. This data suggests once again that the intervention brought about a change in problem-solving performance, not another factor in the experimental design.

Across all four participants, similar patterns in data demonstrate the functional relationship between model-drawing instruction and problem-solving performance on MC and fraction problem types. The immediacy and scale of the change between the baseline and intervention phases is an important piece of evidence, demonstrating that problem-solving performance did not increase until the model-drawing instruction was introduced. The fact that this pattern occurred not just once, but across four different participants at four different times is significant evidence of experimental control.

Adding to the case for experimental control is the high PND for all four participants. After the intervention began, performance for all participants rose and never returned to baseline levels, even well after the intervention phase had ended. In every case, the effect of the intervention took place very quickly, immediately in two of the four cases.
In addition to the changes across phases, the changes across problem types further
demonstrate the functional relationship between model-drawing instruction and problem-
solving performance. In every case while the percentage of MC problems began to rise, the
performance on fraction problems remained at 0% until the participant had been taught models
for solving fraction problems. So, for all participants, their ability to solve these challenging
types of word problems changed dramatically with the introduction of model-drawing
instruction. The individual data patterns were similar across four different participants taught
independently of each other.

Alongside the major data sources for this study, additional data was detailed in Chapter
4 that helps to corroborate the conclusions of experimental control that can be drawn from the
major data sources. A correct schema choice led to a correct problem solution among the four
participants 83.75% of the time. In addition, the fact that 95% of correctly drawn models led to
correct solutions is further convincing evidence of the effect of model drawing on problem-
solving performance.

The social validity data generated by the study does not serve to prove or disprove a
Instead, this data helps us understand how children who learn how to use model drawing
perceive the technique. Looking at the data from table 4.2, the mean of 4.5 out of a possible 5
points demonstrates that all of the participants found model drawing to be enjoyable, both to
learn and apply to new problems. Participants also found that model drawing was useful to
them. They strongly agreed that other children would benefit from learning model drawing, and
they unanimously agreed that model drawing helps one to solve word problems. This data
shows that Singapore's Model Method is a technique that makes sense to the user. The technique is not perceived as an abstraction that has no value or use within work typically encountered by elementary students. Users of the technique can see the value of the technique in the development of their own problem-solving skills.

The primary conclusion of this study is an answer to the research question. A positive functional relationship exists between model-drawing instruction and the percentage of problems solved correctly. Within the context of this single-subject research design, it is clear that model-drawing instruction helps children to solve complex word problems with much success and even enjoyment. The primary basis for this conclusion is the dependent variable data collected and analyzed across four different participants taught independently. This conclusion is further corroborated through analysis of the data across two different problem types. Additional data from the permanent products was also shown to corroborate the primary conclusion of the study, and social validity data proved that participants understood the intervention to be enjoyable, effective and helpful in mathematical word-problem solving.

Presented in this chapter were the data generated by the experiment and an initial analysis of that data. Data was analyzed across participants and across problem types in order to establish the presence of experimental control. Additional data from the work of the participants was presented in order to provide further evidence of experimental control. The next chapter presents interpretations and conclusions that can be reasonably drawn from the results detailed above.
Chapter 5: Conclusions

This chapter details the conclusions that can be drawn from the outcomes of the experimental control presented in Chapter 4. Implications for research and for practice will be explored.

Implications for Research.

The results of this study confirm the conceptual model outlined in Chapter 1. The conceptual model posits that model drawing serves as a bridge between Mayer's problem representation phase and the problem solution phase, providing a practical application of the work done in both phases. The baseline data patterns observable across all four participants show that before the model drawing intervention, these subjects could not solve the challenging word problems set before them. Only the intervention of model drawing instruction allowed them to form schematic problem representations, deepening the quality of the Mayer’s representation phase cognition. The drawing of these representations allowed participants to understand the relationships between quantities and variables in the problem, which then allowed participants to choose the correct operations to perform. A correctly drawn model led to a correct equation 92% of the time. Without this bridge, it is highly unlikely that the participants would have made such large gains in performance on their own.

The data on schema choice corroborates the research presented in Chapter 2. A body of research on schematic diagrams and SBI show that teaching students about schematic categorization and diagrams is helpful in improving word problem performance (Christou & Philippou, 1999; Bishop, 1989; Jitendra, 2001MISSING REF., Dixon, 2005). The functional relationship between model drawing and problem-solving performance in this study
corroborates the findings on SBI because model drawing is a form of SBI, as shown in the argument of discovery in Chapter 3. For the four participants in this study, a correct schema choice led to a correct problem solution 83.75% of the time, and correctly drawn models led to a correct problem solution 95% of the time. These findings also support the body of positive evidence on the effectiveness of SBI and schematic diagrams on problem-solving performance.

The results of this study also verified previous research on Singapore's Model Method. In terms of Singapore’s pedagogy, the findings confirm the importance and utility of the concrete-pictorial-abstract approach developed in Singapore through the model provided by Jerome Bruner. As presented in Chapter 2, researchers have shown that model drawing helps students depict the mathematical relationships within a given word problem. (Swee & Lee, 2009a; Swee & Lee, 2009b; Yeap, 2010). Researchers also have shown that models allow students to identify known and unknown variables in word problems (Foong, 2009). The work of the participants in this study highlights these aspects of model drawing, as their performance increased with their ability to correctly draw models for complex mathematical relationships. The functional relationship demonstrated between learning model drawing and correctly solving challenging word problems is strong evidence supporting these previous findings. The results of this study provide some of the first evidence that Singapore's Model Method is effective outside of the Singaporean education system, with children who have never used the method previously.

**Study Limitations.**

*Design.* The baseline phase of the study consisted of three separate assessment probes, providing a baseline of only three data points. Although three data points in a baseline is still
considered adequate among single subject researchers (O'Neill et al, 2011), a baseline of greater than three data points would improve the accuracy of the assessment of what each participant could do before the intervention began.

The intervention phase was eight sessions long, with four sessions being given to each problem type, one session for each problem variation. Representing the minimum possible time spent on each topic, this design did not allow for more complete teaching and practice within each problem type and variation. Thus, performance scores would likely have been even higher with greater stability if there had been more instructional time spent on each topic. Another limitation to the performance of students was the time of day chosen for intervention sessions. Because of the varied needs of working families, intervention sessions were scheduled when the parent was most easily able to schedule instructional time for their child. This meant that sessions could be scheduled any time of the day. As indicated in some cases, conducting an intervention session in the late afternoon or early evening may have resulted in the child working on challenging math problems at the end of a long and energetic summer's day. This fatigue is likely to have affected the performance of some of the participants during some of their sessions.

The maintenance phase of the study consisted of two sessions, the first coming one week after the end of the intervention phase, the second coming three weeks after the end of the intervention phase. A third session, scheduled 2 months after the end of the intervention phase, would have given three data points for the maintenance phase, and would have given a better indication about the level of retention achieved by the participant.
Among the five children chosen for initial inclusion in the study, only four were actually able to meet the scheduling requirements of full participation. Unfortunately, these four children were not balanced in terms of gender. Although results would likely have been similar, it is preferable to represent each gender equally in such a study.

Because the researcher also conducted all of the intervention sessions, researcher bias is also a realistic threat to validity in this study. To protect against this threat, a sample of 20% of the assessment probes was given to another educator to be scored in order to establish inter-rater reliability. The assisting educator was not given the full context of the study, merely asked to score the student work given to him. Inter-rater reliability was established at 98%. This score was determined by taking the number of scoring agreements divided by the total number of scoring opportunities. This high inter-rater reliability deepens the validity of the findings, protecting against researcher bias in data collection. Using other educators to deliver the intervention sessions would also have further protected the results from researcher bias.

**Context.** Further limitations to the study exist within the context used for the experimental design. The design was carried out in a public library or in private homes during the summer vacation with children working one-on-one with the researcher. This setting was realistic given the timing of the study, but does not adequately simulate a normal classroom environment. Conducting the study within the school building and school day may have given a more accurate picture of student performance in a typical classroom setting.

**Opportunities For Further Research.**

A functional relationship was found to exist between model drawing instruction and the percentage of problems solved correctly for the four participants in this single-subject research
study. This result does not predict the results for the entire population of elementary-aged math students, nor does it establish the model method as a universally effective method for helping children to solve word problems. Instead, these results break promising new ground in understanding the effectiveness of Singapore's Model Method. These results can be used to inform further research into model drawing under new student profiles, problem types, and study designs. It is only through continued research that the effectiveness of Singapore's Model Method can truly be established.

This study was conducted with mainstream education students entering grade 4. Other student populations must be considered in further research, such as grade 1 or grade 6, students receiving academic support for mathematics, or students populating gifted and talented programs in schools. In addition, further study could be made exploring the concept of learning styles and model drawing. Model drawing depends heavily upon the visual representation of mathematical abstraction. Further research may show that model drawing is not an effective intervention for students who have difficulties with visualization or who have more verbal learning styles. All of these results would help to broaden the knowledge base on this teaching technique.

Although not the focus of this study, the results obtained from Harvey are of interest because he was a homeschooled child who had no experience with education within a school setting. The almost linear demonstration of his progress is certainly intriguing, with potential positive implications for homeschool education. Singapore’s math textbooks are a popular choice among homeschooling families in the United States. It is possible that the schematic representations involved in the concrete-pictorial-abstract approach could also be effective
tools for parents to use with their homeschooled children. Deeper exploration of the effects of the Model Method on the homeschooled population could be a fruitful avenue of research.

Multiplicative comparison and fraction problems were the only problem types explored by this study. Addition, subtraction and division problems all must be investigated, and the two primary schema of part-whole and comparison problem types must be investigated across all four operations. Only through investigating a broadening array of problem schemas and types can it be established that model drawing is a universally effective teaching technique for mathematical word problems.

Multiple experimental designs should be applied to the study of Singapore's Model Method. Single subject designs within distinct student populations would give further evidence around these particular educational sub-groups. Group designs could further establish the effectiveness of the technique across larger populations, and within more realistic classroom contexts while negating some of the limitations of single-subject designs. Group designs using multiple teachers should also be pursued, as the Model Method is new to most teachers as well as students. The ways that teachers use model drawing may vary according to the depth of the teacher's mathematical background knowledge. Standard measures of problem solving could also be used in conjunction with group designs, potentially shedding light on the subject from a wider angle.

This study established that Singapore's Model Method could be an effective teaching tool for helping elementary aged children solve more word problems successfully. It is clear that before model drawing can be established as a research-proven technique, further research
must be conducted. This research should focus on broadening the scope of contexts, designs, problem types and student populations.

**Implications for Practice**

If the results of this study can be corroborated through the variety of research strategies delineated above, the case for model drawing as an instructional strategy would indeed be strong. As discussed in Chapter 1, mathematics is an important focus in U.S. educational reform, and performance in mathematics remains lackluster in this country. If the findings of this study could be replicated in a wider variety of educational settings, model drawing could be used as a potent instructional strategy in the United States. This study shows that model drawing has the potential to raise understanding, performance and empowerment in elementary mathematics students. Boosting achievement in elementary mathematics could realistically have a positive ripple effect throughout K-12 math education, and perhaps improve engagement in the STEM. (science, technology, engineering, mathematics) fields as a whole.

Among the topics addressed in the teaching of elementary mathematics, problem solving is often cited as an area of particular difficulty and frustration for teachers, as discussed in Chapter 1. Singapore’s Model Method. (WORDING?) Most U.S. teachers are generalists at the elementary level, and their own math knowledge is an area of concern for policymakers and teacher educators nationwide (Cai, 2004; Stigler & Heibert, 1999; Englard, 2010). Model drawing is certainly useful in teaching children to solve word problems. At the same time, bar models by their very nature illustrate mathematical ideas in a clear and meaningful way. Teachers in Singapore use models to teach problem solving, but they also use models to teach math concepts. (Yeap, 2010). This practice also could be used in the United States, improving
math instruction beyond the topic of problem solving. In addition, simply by learning to teach through model drawing, teachers may find they gain a better understanding of the mathematics they teach and that Singapore’s Model Method deepens their own understanding of mathematical concepts.

The effort to unify mathematics instruction through the Common Core State Standards is a major development in the history of mathematics education in America. As mentioned in Chapter 1, The Common Core State Standards for mathematics already endorse curricula from Singapore as a model for U.S. math instruction. Research results echoing the findings of this study would certainly add to the credibility of Singapore’s elementary math curriculum as a whole, and generate further practitioner and researcher interest into this exemplary mathematics program. A need exists in the United States for a greater understanding of productive mathematics teaching practices around the world (Stigler & Heibert, 1999). Awareness of helpful techniques such as model drawing should be part of pre-service and in-service teacher training. Such cross-cultural exchanges have the potential to improve mathematics instruction greatly, and not just in the United States. As the world shares its knowledge of best teaching practices, children worldwide can reap the benefits.

Helping children improve their understanding of mathematics is a national goal. Any tool that leads to greater success in problem solving by elementary students also has the potential to unlock the mathematical abilities of future generations while enhancing the teaching practices of the thousands of elementary math teachers who instruct them. In addition, the results highlighting the potential of Singapore’s Model Method can positively influence the literature streams of academic research and debates in education policy.
References


Appendix A

Modeling Four Sample Word Problems with Singapore's Model Method

Part-Whole Model - addition

Polly bought a book for $32.95. Then she bought a poster $15.25. She has $4.50 left. How much money did she begin shopping with?

Step 1: basic part/whole model is drawn

Polly's money

Step 2: the whole is split into parts detailed in the text. Labels are added.

Polly's money

$32.95 $15.25 $4.50

Step 2: the question mark is added, indicating that all three quantities must be added together to find the answer to the question.
Comparison Model - subtraction

Jill has 34 marbles.
Jack has 85 marbles.
How many more marbles does Jack have than Jill?

Step 1: The basic comparison model is drawn.

Step 2: Jack’s marbles are extended with an added bar, and both bars are labeled.

Step 3: The question mark is placed indicating that the difference between the two quantities is sought.
Comparison Model - multiplicative comparison

Jeff's cards

Abby's cards

---

Step 1: The basic comparison model is drawn.

Step 2: The multiplicative aspect of Kyla's goals is added to match with the relationship from the text. The bracket shows that the total of both variables is 275.

Step 3: The question mark and bracket is added, indicating that quotient of all 5 parts must be obtained to solve the problem. Note the algebraic reasoning afforded by the model.

\[
\begin{align*}
5\square &= 275 \text{ cards} \\
1\square &= 55 \text{ cards} \\
4\square &= 220 \text{ cards}
\end{align*}
\]
Part-Whole Model - fraction multiplication

232 students went on a field trip.
3/8 of the students brought their own lunch.
How many students did not bring their own lunch?

Step 1: The basic part-whole model is drawn.

Step 2: The whole is divided into eighths and the part describing those who brought lunch is indicated.

Step 3: The whole is divided into eighths and labels are added. Note the algebraic reasoning.

1 \[= 29 \text{ students } (232 \div 8)\]

5 \[= 145 \text{ students } (29 \times 5)\]
Appendix B

Session Procedures

Baseline Session Procedures

1. **Explain procedures to student.** Here, the instructor tells the student that he or she will be given 10 word problems to solve independently. This work is done so that the instructor and the student can have a clear view of what kinds of problems the student already knows how to do, and what kinds of problems the student still needs to learn about. The word problems should be read carefully. Space for work or drawings is provided on the page. If the student is not able to solve a problem, the space can be left blank.

2. **Administer Probe.** The assessment probe will last for 30 minutes or until the student indicates that he or she is finished with the work, whichever comes first. After the directions are read to the student, no help will be given to the student. At the conclusion of the assessment probe, the work will be collected and saved.

Treatment Session Procedures

Each of the eight treatment sessions will follow the same teaching sequence; Warm-up, Demonstration Problem, Guided Practice and Assessment Probe. During the Warm-Up portion, the student and teacher will review problems from previous sessions in order to aid in retention of previous learning. The Demonstration Problem portion of the treatment session is designed to allow the student to see the day's problem type solved by the instructor, followed by short
questions and discussion. The Guided Practice portion of the treatment session is designed to allow the student a chance to solve the new type of problem with help from the instructor. It is here that the student and instructor can discuss strategies, model types, and computation procedures, thereby developing the student's understanding of the new problem type. The practice problems are designed to increase independence by the student, so that by the time the assessment probe is administered for that session, the student is already solving problems of this type independently. Each portion of the treatment session is further detailed below:

1. **Warm-Up.** Using two different problem types from the previous sessions, student will solve two problems. After finishing the work, student and instructor will discuss the modeling and solution of these problems. Discussion must include consideration of model type, unique features of the model used, and computation strategies.

2. **Demonstration Problem.** Here the instructor will demonstrate drawing the model and choosing a solution operation for a new type of problem. Instruction will follow these steps:

   1. *Read the problem aloud.* The problem is read aloud to the student, making sure the student understands all of the words used in the text.

   2. *Write the solution sentence.* With the numerical quantity left blank at the bottom of the provided work space, the instructor writes a sentence that answers the problem. The actual numerical solution is written as a blank line to be filled in once a solution is reached. This is done to call attention to what quantity the problem is really asking the solver to find.

   3. *Determine comparison vs. part whole model.* - Here the instructor models asking the question, "Is this a part-whole model or a comparison model?" Models that ask for two
variables to be compared (fewer, shorter, longer, more, larger) are generally modeled with comparison models that include two variables. Models without such comparisons only require depicting the part or parts out of a larger whole. This question is a key activity for accessing schema and sorting problems according to type, one of the hallmarks of SBI.

4. Ask clarifying questions. The instructor asks clarifying questions about the quantities and relationships in the text. Questions like, "who has more?", "How big is the whole?" and "what are we really looking for here" point the solver to more clear conceptions of the key relationships in the text.

5. Write labels for variables. Before the bars are drawn, the teacher models writing labels for the variable or variables. This labeling focuses the solver on the key objects in the text.

6. Draw bars for model. A basic bar or bars are drawn to represent the whole or, in the case of a comparison model, to represent the two variables being compared.

7. Elaborate on basic model. Once the basic model is complete, the instructor models reading the problem sentence by sentence, adding numerical information to the model or altering the model to depict the relationships described in the problem text.

8. Label parts of model. The instructor adds any final word labels that may be helpful in understanding problem contexts, including the very important question mark. Placing a question mark in the model demonstrates the problem solver has a clear understanding of what unknown quantity is actually being sought in the problem.
9. Choose operation for computation and write equation. Using the model as a reference point, the instructor then models the choice of operation, speaking his thoughts aloud as he looks at the parts of the model. Steps 4 through 9 generate a problem representation as described by Mayer (Mayer, 1985).

10. Carry out computation plan. Once the operation is chosen, the instructor then models the computation required to find the unknown in the problem. Calculators are allowed here in order to control for arithmetic errors, so the computation work will be demonstrated on a calculator.

11. Fill in solution blank. Now that the solution has been reached, the instructor models filling in the blank he created at the outset of the problem solving sequence. Filling in this answer blank encourages the solver to consider the reasonableness of the solution.

12. Check solution for reasonableness. The instructor thinks aloud again, considering if the numerical solution is realistic and appropriate for the problem context. Estimation and the model itself can be used in testing the reasonableness of answers. This step refers back to Polya's "check back" step of problem solving.

3. Guided Practice.

Three to five problems of the same type as the demonstration problem will be worked together with the student. The first problem will be worked in tandem, with the instructor and student generating identical models and equations. For the second problem, the student will lead the discussion, dictating the steps of drawing the model, writing the equation and resolving the computation. In the third problem, the student moves through the solution process independently, with a discussion of the problem with the instructor after a solution is found. If
the instructor deems that further practice is necessary, two additional problems will be given to
the student. In this way, the student works with increasing independence through the series of
guided practice problems.

4. Assessment Probe.

1. Explain procedures to student. Here, the instructor tells the student that he or she
will be given 10 word problems to solve. The word problems should be read carefully. Space
for work or drawings is provided on the page. If the student is not able to solve a problem, the
space can be left blank.

2. Administer Probe. The assessment probe will last for 30 minutes or until the student
indicates that he or she is finished with the work, whichever comes first. After the directions are
read to the student, no help will be given to the student. At the conclusion of the assessment
probe, the work will be collected and saved.

Maintenance Session Procedures

1. Explain procedures to student. Here, the instructor tells the student that he or she
will be given 10 word problems to solve by themselves. The word problems should be read
carefully. Space for work or drawings is provided on the page. If the student is not able to solve
a problem, the space can be left blank.

2. Administer Probe. The assessment probe will last for 30 minutes or until the student
indicates that he or she is finished with the work, whichever comes first. After the directions are
read to the student, no help will be given to the student. At the conclusion of the assessment
probe, the work will be collected and saved.
Appendix C

**Intervention Phase Teaching Script**

What follows is the teaching script for one of the intervention phase teaching sessions on the topic of multiplicative comparison problems. This script is representative of all of the intervention phase teaching sessions. All sessions will follow the same structure, number of problems, pedagogy and organization depicted in this appendix.

**Setting:**

Empty study room at the town library. A table and two chairs are in the room. Materials are available on the table. The instructor and subject sit side by side at the table, working from teaching problems and practice problems for the session.

**Materials:**

Pencils

blank paper

calculator

teaching problems for this session number, copies for both the instructor and the student. The teaching problems for the session are presented together, with room for work, in a paper packet. The instructor and subject each have their own packets.

1. **Warm up.**
INSTRUCTOR: *Good morning. Today we are going to review some problems of the kind we did before as a warm-up. Then, we are going to learn to model and solve a new kind of problem. After you have worked with these new problems together with me, you will take another assessment all by yourself. Are you ready to work?*

The instructor gives the problem set for the session to the student, and keeps a copy of the set for himself.

INSTRUCTOR: *Here are two problems like the ones we did last time. I would like you to try and solve these two problems by yourself. Then I'll check them over with you and we'll see if we agree on the models and the solutions. O.K., you can get started now.*

The instructor observes the subject solving the two problems. He will be observing for correct schema choice, correctly drawn models, correct equations and correct solutions. If there are any errors, they will be corrected with help from the instructor. The two problems are as follows: Jane is 4 times as tall as her little brother. Her little brother is 20 inches tall. How tall are both kids together?

Maggie is 5 times heavier than her little brother. Maggie weighs 65 lbs. How much do both kids weigh together?

2. Demonstration Problem
INSTRUCTOR: O.K., now you understand any mistakes you might have made with those two problems. Now you are all warmed up and ready to take on a new kind of problem. I hope you notice that this new kind of problem is only a little bit different than the kind you just practiced. Now, first, I am going to solve a problem while you watch me and ask any questions you might have. Then, you and I will solve some problems together. Watch me solve this next problem.

The instructor then reads the following problem to the subject:

*Amy has 4 times as many books as Bob does. If they have 35 books altogether, how many books does Bob have?*

INSTRUCTOR: Now that I've read the problem, I am wondering if you notice anything that sounds similar to the problems you just practiced?

Instructor listens to observations from subject, conversing on the similarities between the problems.

INSTRUCTOR: O.K. Now, just like we have always done, I am going to start this problem by writing my solution sentence at the bottom of the work space like this...

Instructor writes a solution sentence, substituting the numerical solution with an empty answer blank for the time being. It will be filled in as one of the final steps to solving the problem. The sentence would read something like this:
Bob has _____ books.

INSTRUCTOR: O.K, now that I have written that sentence, I know I am trying to find out how many books Bob has. I have to keep that in mind when I try and place my question mark. But first, I am going to set up the basic model. I can see that I am comparing Amy's books to Bob's books. I can see that because it says here in the problem, "Amy has 4 times as many books as Bob does." We are comparing Amy's books to Bob's books, so I am going to draw a comparison model. Do you remember how to start a basic comparison model?

Instructor waits to hear what the subject remembers, then draws the basic comparison model on the paper for the subject to observe. He makes sure to label boxes for Amy's books and Bob's books.

INSTRUCTOR: Now, I know that this model needs some work, because right now it looks like Amy and Bob have the same amount of books. It says in the problem that. "Amy has 4 times as many books as Bob does," so I am going to make 3 more equal boxes for Amy because I know she has more. Now I can see Amy has 4 boxes all together, and Bob has only one box. Can you see how I showed that Amy has 4 times as many as Bob?

Instructor checks with the subject for understanding and asks if the subject has any questions so far.
INSTRUCTOR: *O.K., now that I have drawn my model, I need to put in labels on some of these boxes. I read the first sentence, and that is now in the model. Now, the next sentence says the two kids have 35 books altogether. I can put that information in the model by drawing a bracket. The bracket shows both Amy and Bob's books together add up to 35.\*

Instructor then draws the bracket to the right of both sets of boxes, encompassing the two variables and showing that the sum of both variables is 35. He then answers any questions that the subject may have at that time.

INSTRUCTOR: *Now I am almost done with my model. The last sentence also asks, "how many books does Bob have?" I have to put a question mark in here to point out what part I am looking for. I will draw in a question mark above Bob's Box to show this number is what I want to find. Now, the model is finished. Look at the model. Does the model show you what to do to find out what goes in that missing box?\*

Instructor then elicits thinking from the subject on what arithmetic procedures might help to solve the problem.

INSTRUCTOR: *O.K. I want you to notice there are not just four boxes here, for Amy, but ALTOGETHER there are five boxes. You can see now that there are five boxes in all. Now, five equal boxes all add up to 35. I know if I have 35 of something, and I want to share it equally into five groups, I need to divide. Can you see the five equal groups in the model?*
Instructor then elicits thinking from the subject on how the model points the problem solver towards division.

INSTRUCTOR: Good, now all I need to do is to divide 35 by 5, and I get 7. That equation is important. I write that down on my paper, even if I can do that division in my head. That way, I show everybody who looks at this what I was thinking at the time. I also do that on my calculator to check it. So, $35 \div 5 = 7$. That means that each box has 7 in it. Watch me put this information into the model.

Instructor then puts the number 7 into each of the five empty boxes.

INSTRUCTOR: So, I want you to see that Bob has seven books. Now that I have that information, I write the number seven down here in the solution sentence. I read that sentence again, and I make sure that the answer really does make sense. If Bob has seven books, and Amy has 4 times as many books, how many books does Amy have? Well, I can see that Amy has four boxes, with seven in each box. That is 4 groups of 7, or 4 times 7. I can check that on my calculator to make sure. So, if Amy has 28 books and Bob has seven books, does that add up to 35 books altogether, like the problem says? Yes, it does. Now I know that my solution is right.

Does this make sense to you? Tell me, now that I have done one of these problems, what do you notice is different about this kind of problem than the problems we warmed up
with? Do you notice that this time, the problem told us what both parts added up to together?

That information was what we had to find out on those previous problems.

Instructor then elicits thinking from subject about the subject's observations of differences between problems.

3. Guided Practice

INSTRUCTOR: Now that I have done a problem and you watched, it is time for us to do a few problems together. Just like before, my goal for you is that you practice enough together with me so that you feel comfortable doing the problems by yourself. Let's work on this first one side by side. So, we'll both draw models and solve this next one step by step together. O.K.?

Instructor then reads the next problem together with the subject:

Ken has 6 times as many mosquito bites as Meg has. Altogether, they have 63 mosquito bites. How many bites does Ken have?

INSTRUCTOR: O.K., just like before, we are going to write our solution sentences at the bottom of the work space. You write yours, and I'll write mine.

Instructor writes the sentence, "Ken has ____ bites." He checks with the subject to make sure the sentences are similar.
INSTRUCTOR: Good, now we have to start building a model for this problem. First, we need to decide if this is a part-whole or a comparison situation. I am thinking this might be a comparison situation because it seems like we are comparing Ken's mosquito bites to Meg's mosquito bites. What do you think about that?

Instructor elicits thinking from subject. If disagreement occurs, then instructor convinces subject that this is a comparison situation.

INSTRUCTOR: Now, we need to draw our basic model. I am going to do that on my paper, and you do that on your paper, then we'll compare what we both did.

Instructor draws the basic two-variable comparison model, with each name receiving one identical box next to it. The instructor then waits until the subject is finished. A short conversation comparing the work of both people ensues. If there are any major omissions or errors, the instructor helps the subject correct them.

INSTRUCTOR: Good. Now, we are ready to look back into the problem and add information to our model. The first sentence says "Ken has 6 times as many mosquito bites as Meg has." I can show that by giving Ken five more equal boxes. You do that on your model, and I'll do the same on mine. Next, I'll use a bracket to show that both kids' mosquito bites add up to 63 altogether. You show that on your model too.
Instructor then adds this visual information to his model. The instructor waits until the subject is finished. A short conversation comparing the work of both people ensues. If there are any major omissions or errors, the instructor helps the subject correct them.

INSTRUCTOR: O.K. We have a good model here, and I can easily tell that poor Ken has way more bites than Meg. Now, the next sentence is the question sentence. It asks us “how many mosquito bites does Ken have?” We have to put a question mark into our model, and it has to go right above Ken's blocks.

Instructor then draws in a line above Ken's blocks, indicating the distance from beginning to the end of Ken's quantity. That line then is labeled with a question mark, indicating that this quantity, encompassing Ken's six blocks, is unknown. The instructor waits until the subject is finished. A short conversation comparing the work of both people ensues. If there are any major omissions or errors, the instructor helps the subject correct them.

INSTRUCTOR: Now we know what we need to discover. The model tells me that there are 63 things altogether, and that these 63 things need to be spread out equally over six, no, seven boxes. Ken has six boxes, but Meg has one too, so there are seven equal boxes altogether. I know that I need to divide the 63 things up evenly into those seven boxes, so I write my equation, 63 ÷ 7. Now, I am pretty sure of the answer there, but I will check it with my
calculator and then write down the whole equation including the answer. You do the same on your model.

The instructor then waits until the subject is finished. A short conversation comparing the work of both people ensues. If there are any major omissions or errors, the instructor helps the subject correct them.

INSTRUCTOR: So, we finally know that each equal box holds nine mosquito bites. Now, I know that Ken has six of those boxes, so I need to figure out what six boxes with nine in each comes to. I write another equation down, just to show everybody what I was thinking. I write 6 x 9 = 54 on my paper, and check it on my calculator to make sure. You do the same.

The instructor waits until the subject is finished. A short conversation comparing the work of both people ensues. If there are any major omissions or errors, the instructor helps the subject correct them.

INSTRUCTOR: We know that Ken got 54 mosquito bites. That is a lot! Let's write the solution into the solution sentence, and then read that sentence to make sure that it makes sense. Ken was supposed to have lots more bites than Meg. Does it still look that way now, with our solution of 54 bites? How many bites does Meg have, then? Do they both add up to make 63 bites altogether? You check yours, and I will check mine.
The instructor then waits until the subject is finished. A short conversation comparing the work of both people ensues. If there are any major omissions or errors, the instructor helps the subject correct them.

INSTRUCTOR: *Now we'll move on to our second practice problem. For the last problem, you and I worked side by side. For this problem, you are going to do all the work on your paper, but you are going to talk to me about every step you take, sort of like you are now teaching me. Let's start with you reading the problem out loud to me.*

The instructor listens to the problem being read by the subject:

*Over the past two years, it has snowed 246 inches altogether. Last year, we got 5 times as much snow as we got this year. How many inches of snow did we get this year?*

INSTRUCTOR: *O.K., now it is time for you to tell me what you are going to do next.*

The instructor listens to the subject, asking clarifying questions and correcting any misconceptions as the subject orally describes each step to the instructor. Here, the instructor is monitoring specific areas of correct schema choice, correct model drawing, correct equation writing, and correct solution. The subject's models may include some differences in terms of order or style, but these four categories must be accurate, and the instructor will interrupt the subject's descriptions if misconceptions arise in the crucial areas of representation. After this
second guided-practice problem is successfully solved, the instructor gives the third problem to be completed independently by the subject.

INSTRUCTOR: We have finished two practice problems, and now it is time for you to complete one more. This time, you are going to do the problem while I watch you do it. You do not have to explain what you are doing, like you did with the last problem. You just need to work the problem while I watch you. Don't be surprised if I interrupt your work to ask you a question about what you are doing, but otherwise, just work by yourself at solving this next problem.

There are 365 apples in the farmer's crate. There were 4 times as many red apples as there were green apples in the crate. How many red apples were in the crate?

The instructor listens to the subject, asking clarifying questions and correcting any misconceptions as the subject orally describes each step to the instructor. Here, the instructor is monitoring specific areas of correct schema choice, correct model drawing, correct equation writing, and correct solution. The subject's models may include some differences in terms of order or style, but these four categories must be accurate, and the instructor will interrupt the subject's descriptions if misconceptions arise in the crucial areas of representation.

INSTRUCTOR: Well, you have worked hard through these practice problems. Now, it is time to take a break. Go ahead and stretch your legs inside the library. I will walk upstairs with you. Perhaps you might use the bathroom or get a drink of water. We'll come back in 5 minutes or
so, and we'll get to the last part of our work together for the day.

INSTRUCTOR: Now, I have ten problems for you to solve. You will recognize some types of problems, and others you may not recognize. Solve each of the ten problems on this sheet. This sheet helps me know what you can do all by yourself, so the only kind of help I can give you is help with reading the words on the page. Take your time and show all of the work that you can, just like we practiced. If you come across a problem you cannot solve by yourself, just write a question mark in the answer section and move on to another problem that you feel like you can solve. Go ahead and begin your work. You have 30 minutes to finish.

The instructor then observes the child working on the assessment probe for that session. The instructor answers only questions about reading, not about mathematics or problem solving. When the student is finished, the instructor asks the subject to check their work one more time, and then takes the assessment probe from the subject. The subject is dismissed for the session with praise and thanks for their effort during the session.
### Assessment Probe : 1

<table>
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<tr>
<th>Assessment Probe : 1</th>
<th>Student Number:</th>
<th>Date:</th>
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1. Jane is 5 times as tall as her little brother. Her little brother is 25 inches tall. How tall are both kids altogether?  

<table>
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<tr>
<th>Work Space</th>
<th>Answer sentence</th>
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2. Amy has 4 times as many books as Bob does. If they have 35 books altogether, how many books does Bob have?  

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<tr>
<th>Work Space</th>
<th>Answer sentence</th>
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3. Abby has 4 times as many cards as Jeff. If they have 275 cards altogether, how many cards does Abby have?

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<th>Work Space</th>
<th>Answer</th>
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<tbody>
<tr>
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4. Last summer, there were 5 times as many sunny days than cloudy days. If there were 72 days counted altogether, how many more sunny days than cloudy days were there?

5. The dog and the cat weighed 144 lbs. altogether. The dog weighed 5 times as much as the cat. How much lighter was the cat than the dog?
6. At the game, \( \frac{2}{3} \) of the fans wore red clothes. If there were 375 fans, how many wore red clothes?

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sentence
7. There are 245 neighbors on a street. 4/7 of them said that they smelled a skunk last night. How many neighbors did not smell a skunk last night?

8. Today, Mr. Popcorn sold 330 bags of popcorn. Yesterday, he sold 2/3 as many bags. How many bags of popcorn did he sell over both days together?
9. Spongebob and Patrick caught 448 jellyfish altogether. Patrick caught $\frac{1}{3}$ of the number of jellyfish that Spongebob caught. How many jellyfish did Patrick catch?
| 10. Joe and Bertha raised some tadpoles together. Joe was able to hatch \(\frac{2}{5}\) of the number of tadpoles that Bertha hatched. If Bertha hatched 210 tadpoles, how many fewer tadpoles did Joe hatch? |
|---|---|---|
| Work Space | Answer sentence |
## Sample Data Sheet

Subject Name:

<table>
<thead>
<tr>
<th>Probe #</th>
<th>% correct solution</th>
<th>% correct schema choice</th>
<th>% accurate model</th>
<th>% correct equation</th>
<th>% model/solution agreement</th>
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Appendix F

Social Validity Survey

Name:

Date:

Directions: Use the number lines below to show how much you agree or disagree with each of the statements below. Circle a number that best shows your opinion.

1. I like solving word problems using model drawing.
   
   1  2  3  4  5
   strongly disagree  strongly agree

2. I enjoyed learning how to draw models for word problems.
   
   1  2  3  4  5
   strongly disagree  strongly agree

3. I think that other kids my age should be taught how to draw models for word problems.
   
   1  2  3  4  5
   strongly disagree  strongly agree

4. This is how difficult it was for me to learn how to draw models for word problems.
   
   1  2  3  4  5
   very difficult  very easy

5. Drawing models for word problems helps you solve the problem.
   
   1  2  3  4  5
   strongly disagree  strongly agree