Problem Formulation
- Delay Differential Equation (DDE) with multiple Delays, \( \tau = (\tau_1, \tau_2, \ldots, \tau_j)^T \).
- Leads to \( \ell \)-toples infinite dimensional spectrum due to \( \ell \) number of delays.
- Stability boundary: The boundary separating STABLE vs. UNSTABLE in \( \tau \).
- Analyzing stability robustness of DDE in presence of parametric uncertainties is a challenge.
- Some respectable techniques include: Hermite-Bieler Theorem, Zero-Exclusion Principle, Edge Theorem, generalization of Kharitonov’s theorem.

Inherent Challenges
- Detection of the stability boundary is a challenge.
- Effects of parametric uncertainties in \( p \in \mathbb{R}^n \) complicates the analysis further.
- Complications in existing techniques in treating singular points on the stability boundary.

Objective
For a nominal \( p = p_0 \), and given \( \tau = \tau_0 \), on the stability boundary of the characteristic function, \( \lambda(p, \tau_0, \tau_0) = \lambda(p, \tau_0, \tau_0) \).

Stability Boundaries without Delays
A recent work by A.A. Mailybaev in 2000 presents a systematic approach to the geometry of the stability boundaries of \( P(\lambda, p) = \sum_{i=0}^{M} a_i(p) \lambda^i \).

Next, a differential operator is introduced,
\[
\nabla = \left( \frac{\partial}{\partial \tau_1}, \ldots, \frac{\partial}{\partial \tau_\ell} \right)
\]
where the partial derivatives are taken at point \( p = p_0 \). Following from (4), the expression \( \nabla \lambda(p, \tau_0, \tau_0) \) is given by
\[
\nabla \lambda(p, \tau_0, \tau_0) = \sum_{i=0}^{M} C_{M-i-1} \lambda^{M-i-1} \nabla a_{M-i-1}
\]
Define the \( \nu \)-dimensional real vectors \( \lambda(\lambda) \) and \( g(\lambda) \) that correspond to \( \lambda_j \),
\[
\lambda_j(\lambda) + ig_j(\lambda) = \lambda_j(\lambda) + ig_j(\lambda) = 0, \quad j = 1, \ldots, \ell - 1
\]
and
\[
\lambda_j(\lambda) = \sum_{i=0}^{M} a_i(\lambda) / b_i(\lambda), \quad b_i(\lambda) \neq 0,
\]
where \( b_i(\lambda) \) is the value of \( b_i(\lambda) \) for \( p = p_0 \).

Mailybaev’s Work - Geometric Approach
\[
P(\lambda, p) = \sum_{i=0}^{M} a_i(p) \lambda^i = \sum_{i=0}^{M} b_i(\lambda) \lambda^i = 0.
\]
By differentiating (3) \( \ell \) times with respect to \( \lambda \) and setting \( \lambda = 0 \), we have
\[
b_j = \frac{\partial \lambda^j}{\partial \tau_j} = \sum_{i=0}^{M} C_{M-i-1} \lambda^{M-i-1} \frac{\partial b_i}{\partial \tau_j}
\]
Main Result
- Mailybaev’s approach is not constructed to consider presence of delays in the characteristic function.
- Mailybaev’s work is adapted to cover the cases \( \tau \neq 0 \).
- This is a non-trivial task to accomplish.

Example – Singular Points
Characteristic function: \( \psi(\lambda) = \lambda^2 + \lambda(-1 + \alpha) + \beta = 0 \).
Nominal PI gains: \((\alpha, \beta) = (1.5, 2)\).

The Approach with Delays, cont’d.
Step 2: When \( k_{\omega} \neq 0 \), the approach of Mailybaev extends to the analysis of DDE. In such a case, the approach modifies \( k_{\omega} \) as \( b_j \).

Comprehensive study for all \( \ell \) is left for future work.