Stability of Linear Time-Delay Systems

- Linear time-delay system:
  \[ x(t) = Ax(t) + Bx(t - \tau), \quad \tau \geq 0 \]
  is stable if the zeros of
  \[ \det(sI - A - B e^{-s\tau}) = 0 \]
  are in \( \mathbb{C}_- \) (open left half s-plane, good region).

**Lemma 1:** The system (1) is stable if there exists \( R = R' \) and \( 0 < P_1 = P_3, P_5 \) that satisfy the following LMI:

\[
\begin{bmatrix}
Y_1 & Y_2 & \tau P_B P_B' & \tau P_B P_B' + \tau R' \\
Y_2 & Y_1 & \tau P_B P_B' & -\tau R' \\
\tau B' P_5 & \tau B' P_5 & \tau P_5 & 0 \\
\tau B' P_5 & \tau B' P_5 & 0 & \tau P_5 - \tau R' \end{bmatrix} < 0,
\]

where \( Y_1 = (A + B + P_3 + P_5 A), Y_2 = P_1 - P_3 + (A + B)P_5, \) and \( Y_2 = P_3 - P_5 + \tau R \).

**Robust Stability**

- Structured perturbation:
  \[ x(t) = (A + \Delta_0 A_0) x(t) + (B + E_0 B_0) x(t - \tau) \]
  with \( \Delta_0 \in \mathbb{R}^{n \times n}, \Delta_0 \in \mathbb{R}^{n \times n}, E_0 \in \mathbb{R}^{m \times n}, \) and \( F_0 \in \mathbb{R}^{m \times n} \).

**Theorem 1:** If there exist \( P > 0, Q > 0, X, Y, Z, \) and positive scalars \( e \) such that

\[
\begin{bmatrix}
Y_1 & Y_2 & -Y + PB + RZ_1 P E & PE \\
-Y' + BP & -Q_Z_2 & Q Z_2 - E Z & PE \\
R Z_1 & E Z & -E Z & 0 \\
E P & 0 & 0 & -E Z \\
X & 0 & 0 & 0 \\
Y & Z & 0 \end{bmatrix} > 0,
\]

where \( Y_1 = A'P + AP + X + Z, Y = Y + E + P_2 Z, \) and \( Q_Z_2 = -E Z F E \). The system (2) is asymptotically stable for any time delay \( \tau \) satisfying \( 0 \leq \tau \leq \tau_0 \) and all admissible uncertainties.

**Stability Radius Preliminaries**

- For the regular dynamical system with perturbation
  \[ x(t) = (A + E \Delta F) x(t) \]
  the stability radius is defined as
  \[ r_{r}(A, E, F) = \inf \{ \sigma(M) : \Delta \in \mathbb{R}^{n \times n}, \| I - \Delta M(s) \| = 0 \} \]

**Stability Radius of Single-Delay Systems**

- The real stability radius of the system (2) is given by
  \[ r_{r}(A, B, E, F_0, F_B) = \inf_{\sigma(M)} \| I - \Delta M(s) \| = 0 \]

**Stability Radius of Two-Delay Systems**

- Consider the class of positive delay dynamic systems
  \[ x(t) = A x(t) + \sum_{i=1}^{n} \Delta_i x(t - \tau_i), \quad 0 \leq \tau_i \leq \tau \]
  the system (3) is stable if the zeros of
  \[ \det(sI - A - \sum_{i=1}^{n} \Delta_i e^{-s\tau_i}) = 0 \]

**Stability Radius of Metzlerian Systems**

- Shafiab, and Chen, (1993):
  \[ x(t) = (A + E \Delta F_0) x(t) \]
  where \( A \) is metzlerian or metzlerian such that \( A + B \) is also metzlerian.

**Theorem 2:** The real and complex stability radii coincide

\[ r_{r}(A, E, F) = r_{r}(A, E, F) = \frac{1}{\| F A + B E \|} \]

**Stability of Linear Two-Delay Systems**

- Consider the two-delay retarded type linear system
  \[ x(t) = A x(t) + \sum_{i=1}^{n} \Delta_i x(t - \tau_i), \quad 0 \leq \tau_i \leq \tau \]
  the system (4) is stable if the zeros of
  \[ \det(sI - A - \sum_{i=1}^{n} \Delta_i e^{-s\tau_i}) = 0 \]

**Stability of Linear Time Delay Systems**

- The two LMI’s are satisfied for
  \[ A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E = F = I \]

**Definiton 1:** Fundamental Stability Region

The two-time delay system be zero delay stable. Then a subset of kernel and offspring curves such that any point on these curves can be connected to the origin by a continuous path in the stable region is called fundamental kernel and offspring curves. Furthermore, the region of stability bounded by \( \gamma \) and \( \gamma \) axes and the fundamental kernel and offspring curves is called the fundamental stability region (FSR).

![Fundamental Stability Region](Image)